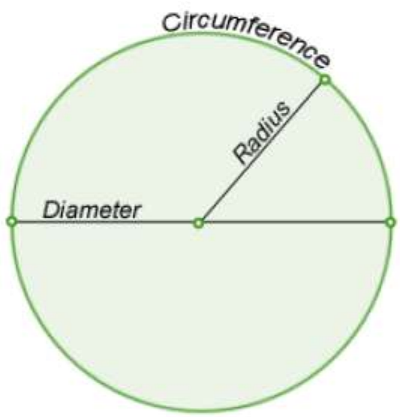
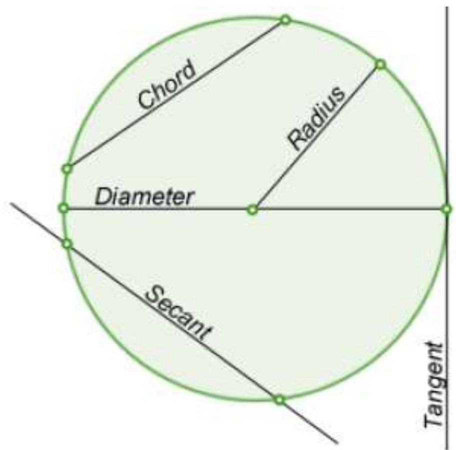
Geometry II

Circles

Popular lines
(you know them for sure)

Not so popular lines
(you may remember them or not)



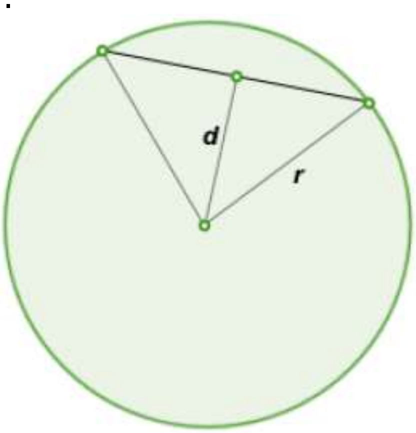


$$Perimeter = p = 2\pi r = \pi D = \sqrt{4\pi A}$$

$$Area = A = \pi r2 \quad = \frac{\pi D^2}{4} = \frac{p^2}{4\pi}$$

Where:
[r] radius
[D] diameter
[A] area
[p] perimeter

Chord: A line that links two points on a circle or curve. The diameter is a chord that passes throw the center.



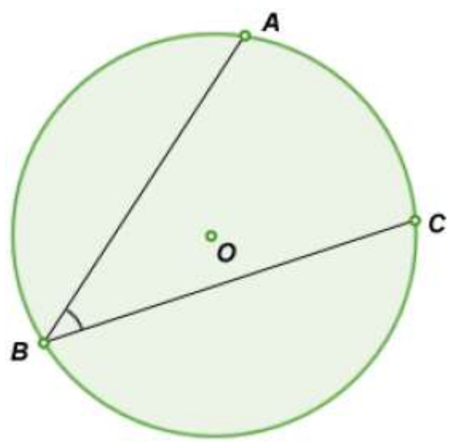
$length = 2\sqrt{r^2 - d^2}$

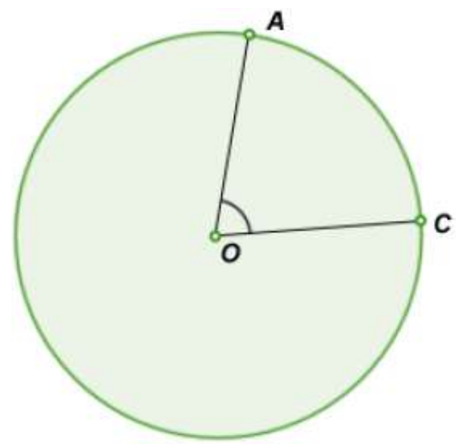
but remember the perfect triangles and the special cases (45-45 and 30-60 triangles)

Angles in a circle

Inscribed angle: Is an angle formed by three points located on circle's circumference.

Central angle: Is an angle with vertex located at the circle's center.

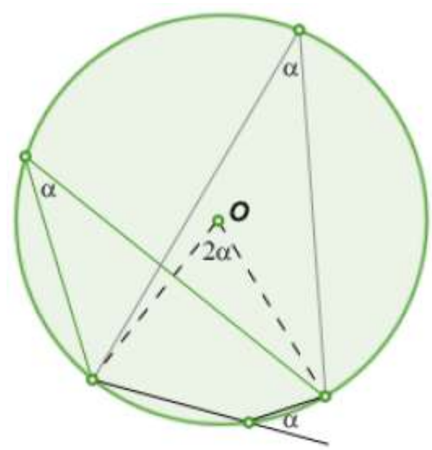




$$angle = \frac{intercepted\ arc}{2}$$

$$angle = \frac{90L}{\pi r}$$

This is based in the arc length formula but the arc length formula assumes the angle is a central angle and because of this the addition of the ratio $\frac{90}{\pi}$

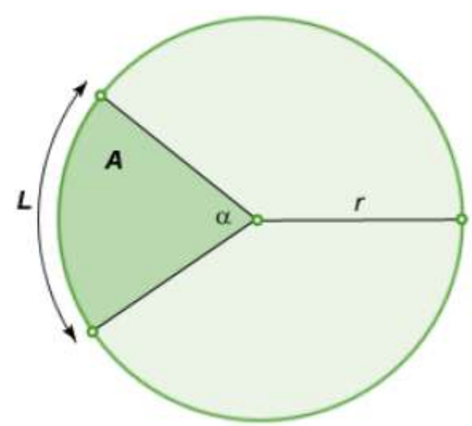


Central angle theorem states that the measure of inscribed angles is always half the measure of the central angle.

Arc: Is a portion of the circumference of a circle. On GMAT/GRE we assume the **minor arc to be the shortest**.

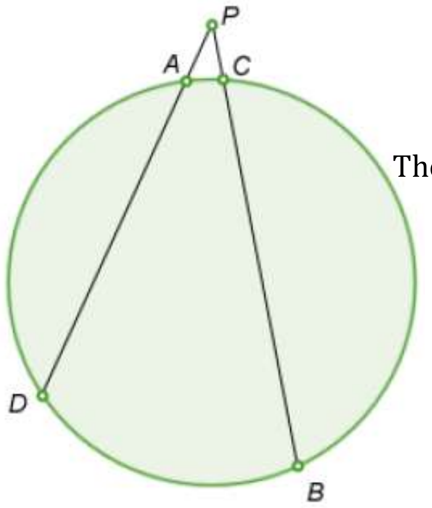
$$L = 2\pi r \frac{\boldsymbol{\alpha}}{\mathbf{360}}$$

$$A = \pi r^2 \frac{\boldsymbol{\alpha}}{\mathbf{360}}$$



$\boldsymbol{2\pi r}$ is the entire circumference so the formula simple reduces it by the ratio of central angle (α) to the full circumference (360°).

$\boldsymbol{\pi r^2}$ This is the same as in the arc length.

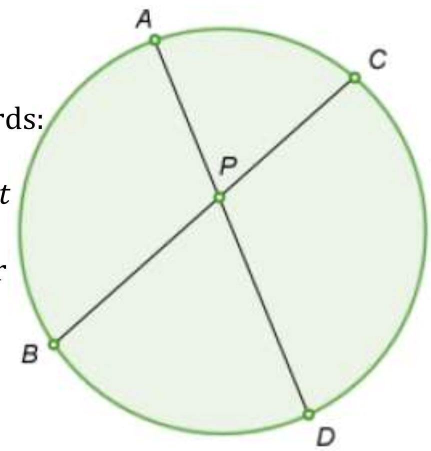


Secants

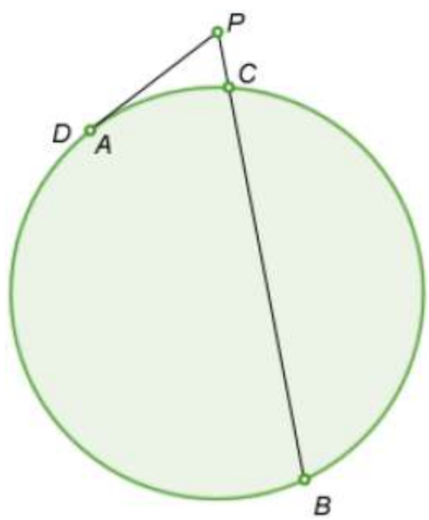
Theorem of intersecting secants/chords:

$$(PA)(PD) = (PC)(PB) = constant$$

It does not matter if P is outside or inside the circle.



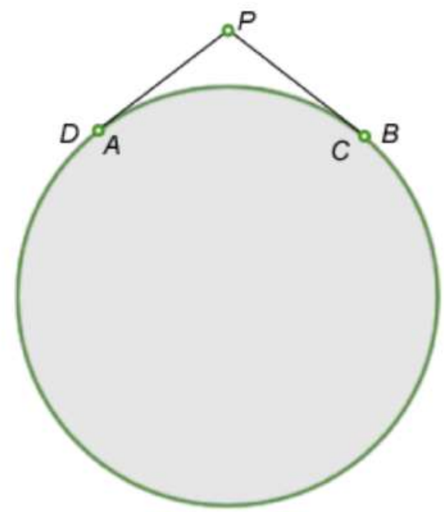
Chords



Secant-tangent

$$(PA)(PD) = (PC)(PB) = constant$$
$$PA^2 = (PC)(PB) = constant$$

Applies the same with:
Secant-tangent/two tangent:



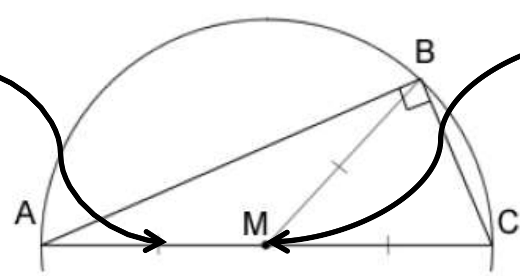
Two tangents

$$(PA)(PD) = (PC)(PB) = constant$$
$$PA^2 = PC^2 = constant$$
$$PA = PC = constant$$

Right triangle inscribed in a circle

Is **AC** the **hypotenuse** of the triangle? If **yes** then it is a **right triangle.**

Remember de definition of inscribed angle.



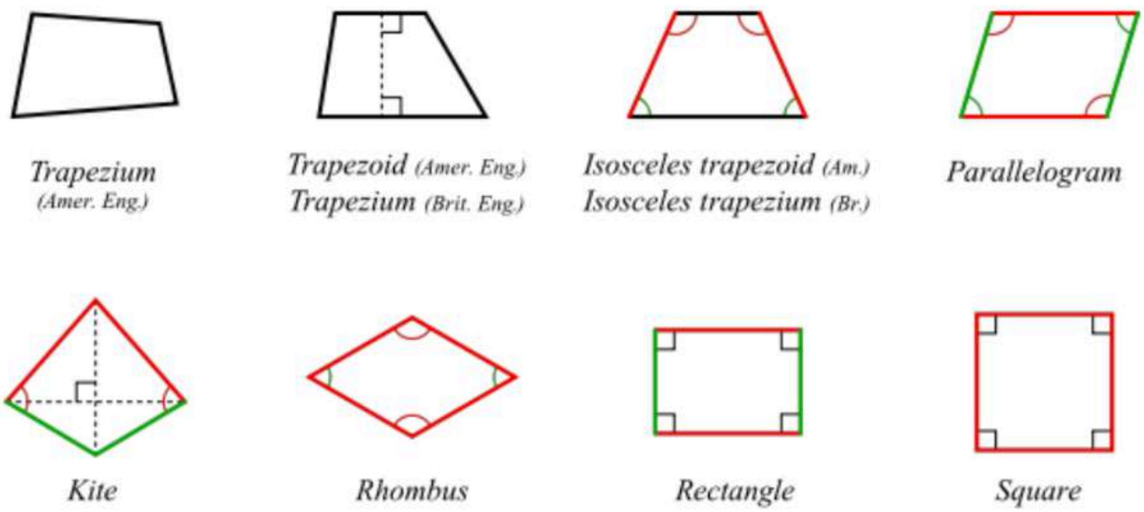
Is M the mid point?
If **yes** then $BM = \frac{AC}{2}$ because BM is a **radius.**

Polygons

$interior\ angles = (n-2)180$

GMAT/GRE is dealing mainly with the following polygons:

Quadrilateral:



Parallelogram:

Opposite sides of a parallelogram are **equal** in **length**.
Opposite angles of a parallelogram are **equal** in **measure**.

- The **diagonals** of a parallelogram **bisect** each other.
- A **rectangle** is a parallelogram but with all **angles** fixed at **90°**
- A **rhombus** is a parallelogram but with all **sides equal** in length
- A **square** is a parallelogram but with all **sides equal** in length and all **angles fixed** at 90°

Rectangle
- The diagonals bisect each other
- The diagonals are congruent
- It is a special case of a parallelogram but with extra limitation; angles are fixed at 90°.
- The two diagonals are congruent (same length).
- Each diagonal bisects the other.
- Each diagonal divides the rectangle into two congruent right triangles.
- A square is a special case of a rectangle where all four sides are the same length.

Squares

• If the diagonals of a rhombus are equal, then that rhombus must be a square. The diagonals of a square are about 1.414 $(\sqrt{2})$ times the length of a side of the square.

• If a circle is circumscribed around a square, the area of the circle is $\frac{\pi}{2}$ (about 1.57) times the area of the square.

• If a circle is inscribed in the square, the area of the circle is $\frac{\pi}{4}$ (about 0.79) times the area of the square.

• A square has a larger area than any other quadrilateral with the same perimeter.

• Like most quadrilaterals, the area is the length of one side times the perpendicular height. So in a square this is

• The "diagonals" method. If you know the lengths of the diagonals, the area is half the product of the diagonals.

Since both diagonals are congruent (same length), this simplifies to: $a = \frac{d^2}{2}$, where d is the length of either diagonal

• Each diagonal of a square is the perpendicular bisector of the other. That is, each cuts the other into two equal parts, and they cross and right angles (90°).

• The length of each diagonal is $s\sqrt{2}$ where **s** is the length of any one side.

• A square is both a rhombus (equal sides) and a rectangle (equal angles) and therefore has all the properties of both these shapes, namely:

The diagonals of a square bisect each other and the diagonals of a square bisect its angles.

• The diagonals of a square are perpendicular.

• The diagonals of a square are equal.

A square can be thought of as a special case of other quadrilaterals, for example

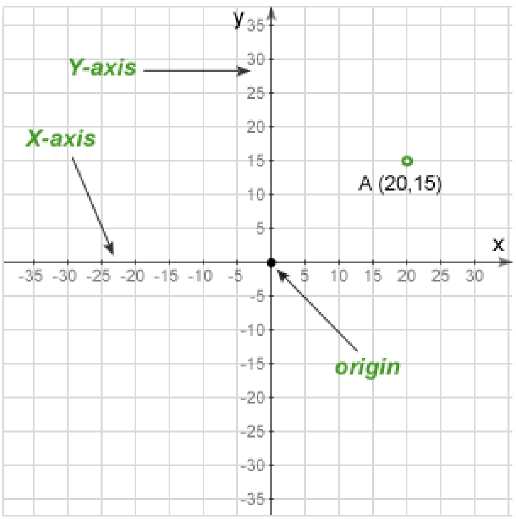
• a rectangle but with adjacent sides equal

• a parallelogram but with adjacent sides equal and the angles all 90°

• a rhombus but with angles all 90°

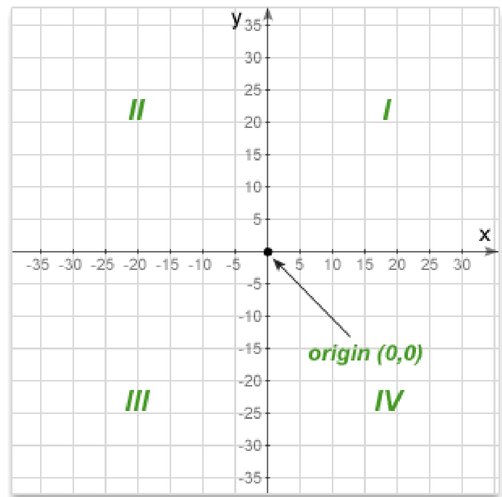
Coordinated geometry

The coordinate plane; just in case you do not remember it:

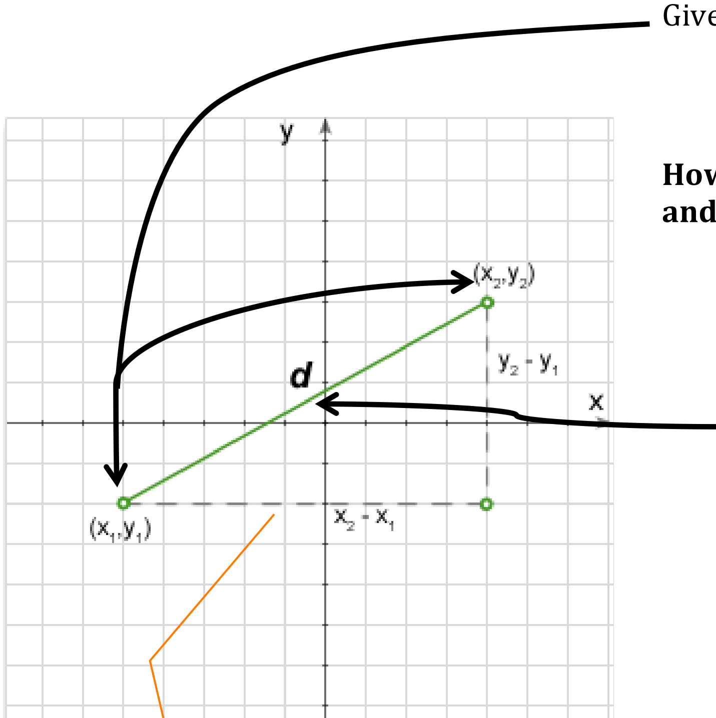


Basics

-X and Y axes
-1 origin
-1 point is given by a Y coordinate and a X coordinate
-4 quadrants



What is the distance between two points?... and the mid point?



Given the X and Y coordinates of two points:

$$d = \sqrt{(x_2 - x_1)^2 (y_2 - y_1)^2}$$

How does this formula change if the horizontal and vertical lines start at the origin?

What is the mid point?

The coordinates of M (mid point) are given by:

$$M(x_m, y_m)$$

Where:

$$x_m = \frac{x_1 + x_2}{2} \quad \text{and} \quad y_m = \frac{y_1 + y_2}{2}$$

Do not forget the perfect right triangles and special cases of right triangles

Lines in coordinate geometry

Must known equations for a line:

1. General form:
$ax + by + c = 0$ from this:

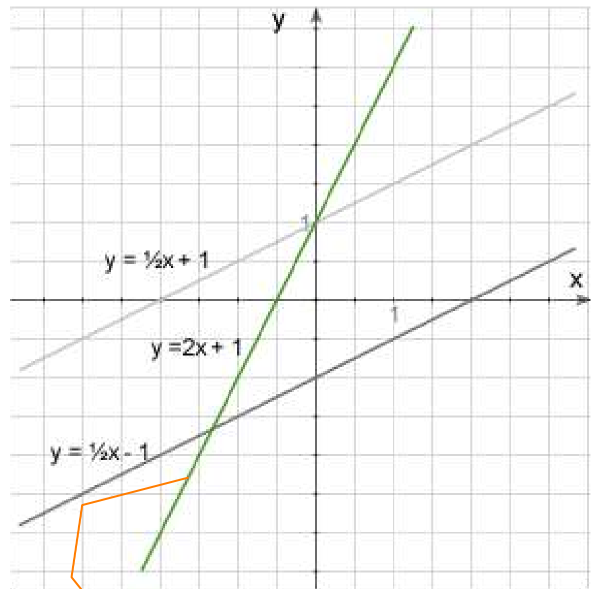
$slope = -\frac{a}{b}$ $y\ intercept = -\frac{c}{b}$

2. Point-intercept form:
$y = mx + b$

$slope = m$ $y\ intercept = b$

3. Intercept form:
$\frac{x}{a} + \frac{y}{b} = 1$

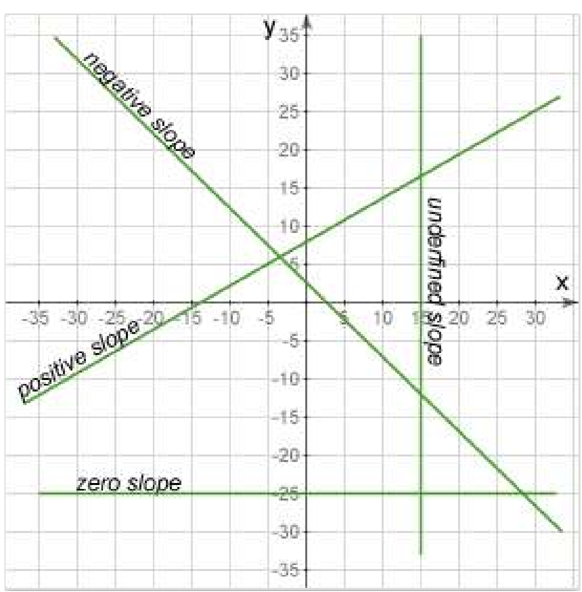
Where x and y intercepts are a and b respectively



Remember the slope of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Slope of a line



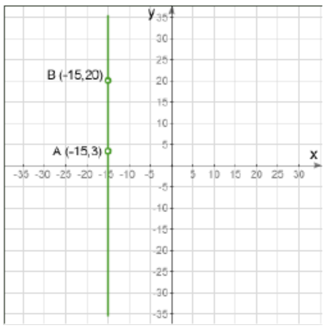
What are the **signs** of the **slopes** to the left?

How many **quadrants intersect** a line with **positive** slope?

And what about a **negative slope**?

Are there any **exceptions**? Lines that crosses the **origin**, lines that have slope = **0** or **undefined** crosses **2 quadrants.**

Lines with **slopes=1** form a **45°** angle



What is the equation of the **vertical** line?

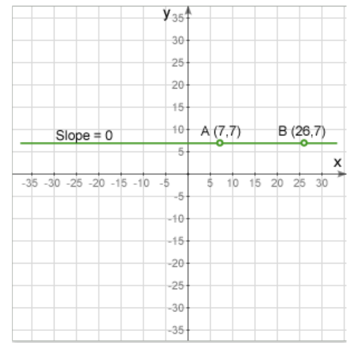
$$x = a$$

Any point on the **vertical** line **satisfy** the equation

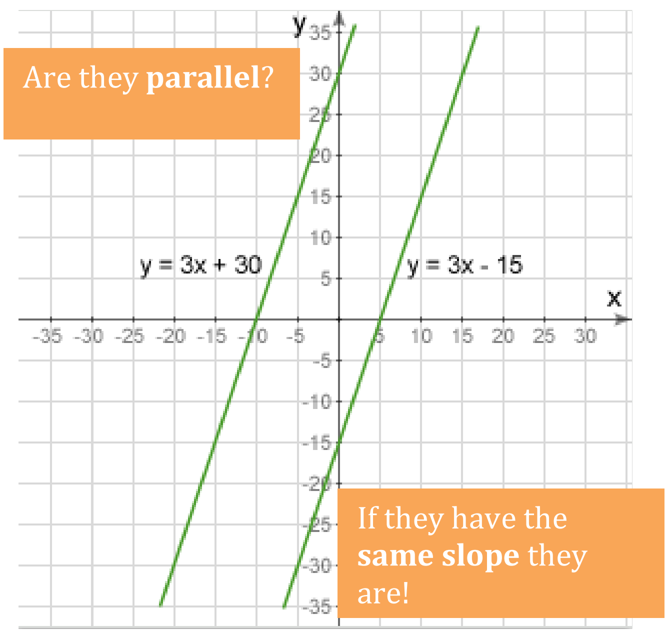
What is the equation of the **horizontal** line?

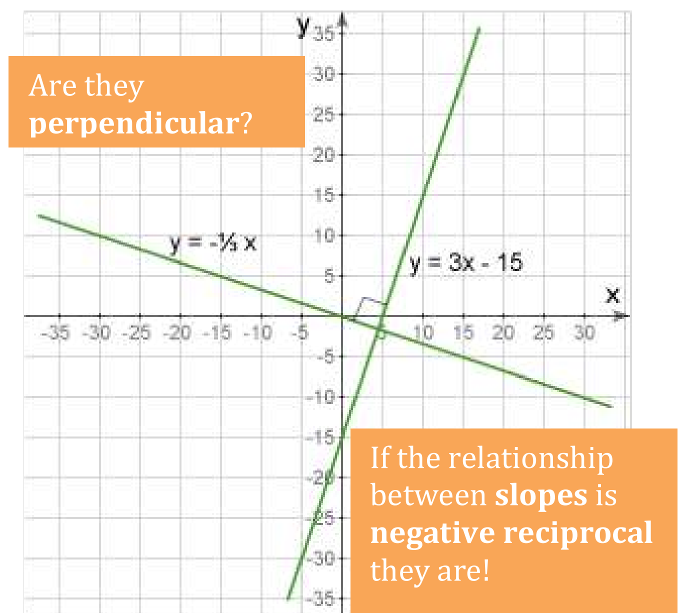
$$y = b$$

Any point on the **horizontal** line **satisfy** the equation



Parallel and perpendicular lines





Remember: you know if two numbers are reciprocal when the product of them is equal to 1

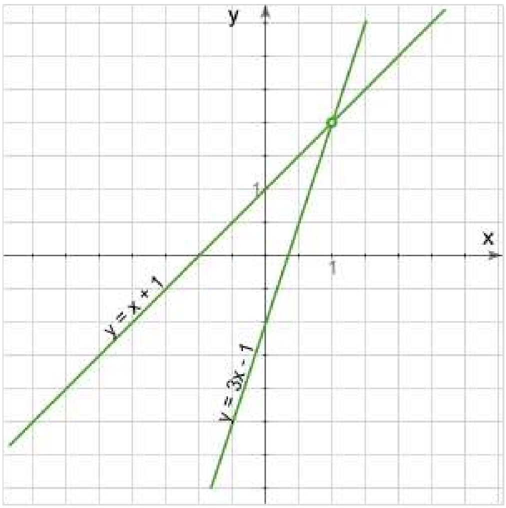
Two lines:

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

Are perpendicular when:

$$a_1a_2 + b_1b_2 = 0$$

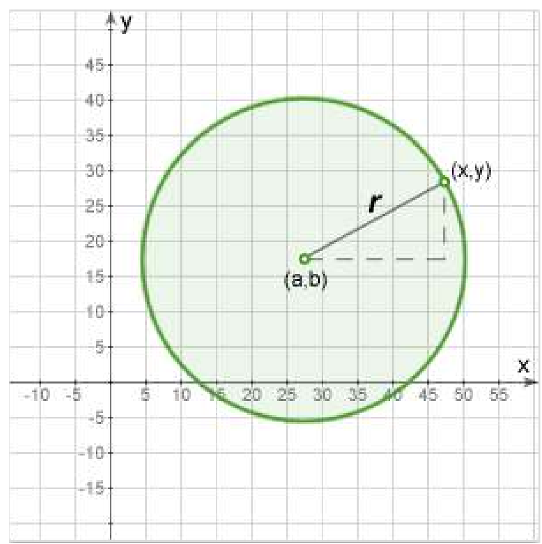
Intersection of two lines:



How do I find the **point** of **intersection**?

Set the **equations equal** to each other and **solve**, usually for y

Circle on a plane



In a circle with **center** (a, b) and **radius** r

$$(x-a)^2 + (y-b)^2 = r^2$$

How does this formula change if the circle is in a different quadrant?

What happen is the center is at $(0,0)$?

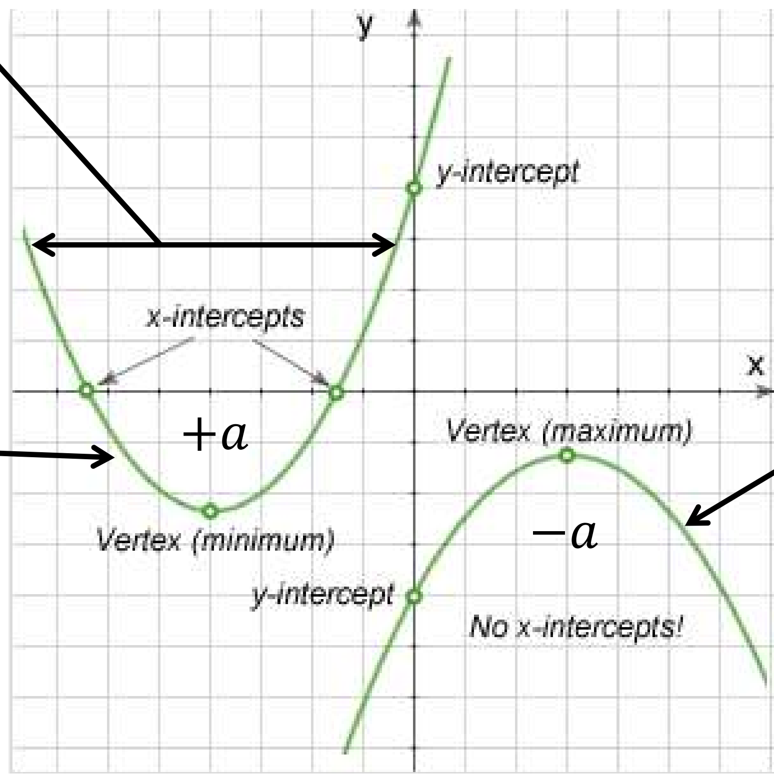
$$x^2 + y^2 = r^2$$

Parabola

Quadratic equation:

$y = ax^2 + bx + c$

The larger the **absolute value** of a, the **thinner** the parabola is



If $+a$ the parabola opens upwards

If $-a$ the parabola opens downwards

To find the roots of this equation:

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

There are **three** cases identifiable using the **discriminant**:

$b^2 - 4ac > 0$ There are **2 solutions** (2 x intercepts)

$b^2 - 4ac = 0$ There is only **1 solution** (1 x intercepts)

$b^2 - 4ac < 0$ There is **no real solution** (No x intercept)

The **vertex** is given by $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$ but typically you compute $-\frac{b}{2a}$ and then plugg it to find y

Kahoot!!

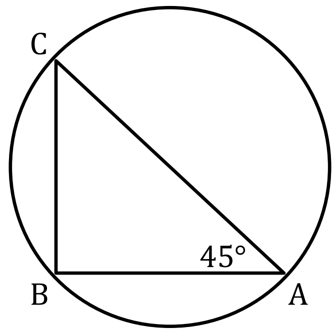


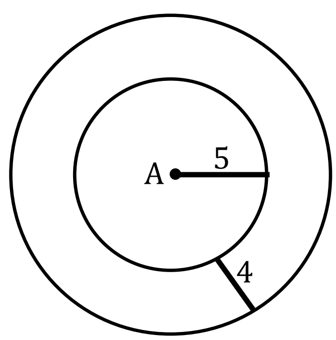
Figure 1, problems 1 and 2.

1. Triangle ABC is inscribed in a circle, such that AC is a diameter of the circle and angle BAC is 45°. If the area of triangle ABC is 72 squared units, how much larger is the area of the circle than the area of triangle ABC?

A) $72\pi - 72$ B) 72 C) 72π D) $52\pi - 72$ E) $72\pi - 52$

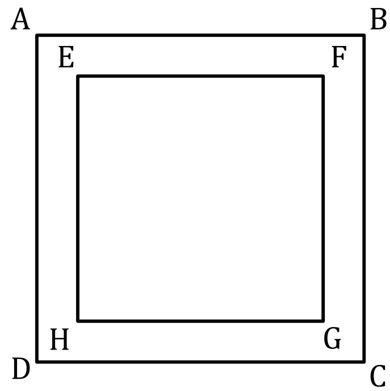
2. Triangle ABC is inscribed in a circle, such that AC is a diameter of the circle and angle BAC is 45°. If the area of triangle ABC is 84.5 squared units, what is the length of arc BC?

A) $3\sqrt{2}\pi$ B) $2\sqrt{13}\pi$ C) 72π D) $\frac{13\sqrt{2}\pi}{4}$ E) $52\sqrt{2}\pi$



3. A circular lawn with a radius of 5 meters is surrounded by a circular walkaway that is 4 meters wide as in the figure above. What is the area of the walkway?

A) $20\pi m^2$ B) $28\pi m^2$ C) $56\pi m^2$ D) $81\pi m^2$ E) $25\pi m^2$



4. ABCD is a squared picture frame, as in the figure above. EFGH is a squared inscribed within ABCD as a space for a picture. The area of EFGH (for the picture) is equal to the area of the picture frame (the area of ABCD minus the area of EFGH). If AB=6, what is the length of EF?

A) $3\sqrt{2}$ B) $6\sqrt{2}$ C) $3\sqrt{3}$ D) $2\sqrt{3}$ E) $4\sqrt{2}$

5. If the length of an edge of cube A is one third the length of an edge of cube B, what os the ratio of the volume of cube A to cube B

A) 1to 6 B) 1 to 9 C) 1 to 18 D) 1 to 27 E) 1 to 36

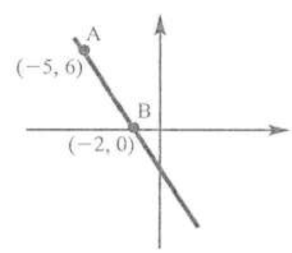
6. The line represented by the equation $y = x$ is the perpendicular bisector of line segment AB. If A has the coordinates $(-3, 3)$, what are the coordinates of B?

A) 3, -3 B) 1.5, -1.5 C) -3, 3 D) 3, 1.5 E) 1.5, 3

7. The line represented by the equation $y = -2x + 6$ is the perpendicular bisector of the line segment AB. If A has the coordinates $(7, 2)$, what are the coordinates for B?

A) 1, -2 B) -2, -2 C) -1, 2 D) -1, -2 E) 1, 1

8. What are the coordinates for the point on Line AB (see figure) that is three times farther from A than B, and that is in between points A and B?



A) 2.75, -1.5 B) -2.75, 1.5 C) -1.75, 2.5 D) 2.75, -1

Answer key

1. **A**
2. **D**
3. **C**
4. **A**
5. **D**
6. **A**
7. **D**
8. **B**

Homework

Circles and Cylinders
Appendix A: D5, D20, D22, 30, 160, 191, 206
Appendix B: 31, 141
Appendix C: 39, 41, 76, 86, 136

Polygons
Appendix A: 3, 13, 16, 105, 112, 134, 238
Appendix B: 12, 22, 139, 175
Appendix C: 38, 47, 102, 117

Coordinated geometry
Appendix A: 7, 23, 36, 89, 199, 227, 248
Appendix B: 19, 123
Appendix C: 15, 78, 85, 124, 140

Answers and explanations

1 . **$72\pi - 72$:** If AC is a diameter of the circle, then angle ABC is a right angle. Therefore, triangle ABC is a 45 - 45 - 90 triangle, and the base and height are equal. Assign the variable x to represent both the base and height:

$$A = \frac{bh}{2} \qquad \frac{x^2}{2} = 72$$

$$x^2 = 144$$

$$x = 12$$

The base and height of the triangle are equal to 12, and so the area of the triangle is $\frac{12 \times 12}{2} = 72$.

The hypotenuse of the triangle, which is also the diameter of the circle, is equal to $12\sqrt{2}$. Therefore, the radius is equal to $6\sqrt{2}$ and the area of the circle, πr^2, $= 72\pi$. The area of the circle is $72\pi - 72$ square units larger than the area of triangle ABC.

2. $\mathbf{\frac{13\sqrt{2}\pi}{4}}$ **:** We know that the area of triangle ABC is 84.5 square units, so we can use the same logic as in the previous problem to establish the base and height of the triangle:

$$A = \frac{bh}{2} \qquad \frac{x^2}{2} = 84.5$$

$$x^2 = 169$$

$$x = 13$$

The base and height of the triangle are equal to 13. Therefore, the hypotenuse, which is also the diameter of the circle, is equal to $13\sqrt{2}$, and the circumference ($C = \pi d$) is equal to $13\sqrt{2}\pi$. Angle A, an inscribed angle, corresponds to a central angle of 90°. Thus, arc BC = 90/360 = 1/4 of the total circumference:

$$\frac{1}{4} \text{ of } 13\sqrt{2}\pi \text{ is } \frac{13\sqrt{2}\pi}{4}.$$

3. **$56\pi\text{m}^2$:** The area of the walkway is the area of the entire image (walkway + lawn) minus the area of the lawn. To find the area of each circle, use the formula:

Large circle: $A = \pi r^2 = \pi(9)^2 = 81\pi$

Small circle: $A = \pi r^2 = \pi(5)^2 = 25\pi$

$$81\pi - 25\pi = 56\pi\text{m}^2$$

4. **$3\sqrt{2}$:** The area of the frame and the area of the picture sum to the total area of the image, which is 6^2, or 36. Therefore, the area of the frame and the picture are each equal to half of 36, or 18. Since EFGH is a square, the length of EF is $\sqrt{18}$, or $3\sqrt{2}$.

5. **1 to 27:** First, let's call the length of one side of Cube A, x. Thus, the length of one side of Cube B is $3x$. The volume of Cube A is x^3. The volume of Cube B is $(3x)^3$, or $27x^3$.

Therefore, the ratio of the volume of Cube A to Cube B is $\frac{x^3}{27x^3}$, or 1 to 27.

6. **(3, −3):** Perpendicular lines have negative inverse slopes. Therefore, if $y = x$ is perpendicular to segment AB, we know that the slope of the perpendicular bisector is 1, and therefore the slope of segment AB is -1. The line containing segment AB takes the form of $y = -x + b$. To find the value of b, substitute the coordinates of A, $(-3, 3)$, into the equation:

$$3 = -(-3) + b$$
$$b = 0$$

The line containing segment AB is $y = -x$.

Find the point at which the perpendicular bisector intersects AB by setting the two equations, $y = x$ and $y = -x$, equal to each other:

$$x = -x$$
$$x = 0;\ y = 0$$

The two lines intersect at (0, 0), which is the midpoint of AB.

Use a chart to find the coordinates of B.

	x	y
A	−3	3
Midpoint	0	0
B	**3**	**−3**

7. **(−1, −2):** If $y = -2x + 6$ is the perpendicular bisector of segment AB, then the line containing segment AB must have a slope of .5 (the negative inverse of -2). We can represent this line with the equation $y = .5x + b$. Substitute the coordinates (7, 2) into the equation to find the value of b.

$$2 = .5(7) + b.$$
$$b = -1.5$$

The line containing AB is $y = .5x - 1.5$.

Find the point at which the perpendicular bisector intersects AB by setting the two equations, $y = -2x + 6$ and $y = .5x - 1.5$, equal to each other.

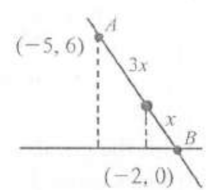
$$-2x + 6 = .5x - 1.5$$
$$2.5x = 7.5$$
$$x = 3;\ y = 0$$

The two lines intersect at (3, 0), which is the midpoint of AB.

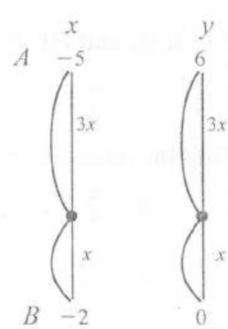
Use a chart to find the coordinates of B.

	x	y
A	7	2
Midpoint	3	0
B	**−1**	**−2**

8. **(−2.75, 1.5):** The point in question is 3 times farther from A than it is from B. We can represent this fact by labeling the point $3x$ units from A and x units from B, as shown, giving us a total distance of $4x$ between the two points. If we drop vertical lines from the point and from A to the x-axis, we get 2 similar triangles, the smaller of which is a quarter of the larger. (We can get this relationship from the fact that the larger triangle's hypotenuse is 4 times larger than the hypotenuse of the smaller triangle.)



The horizontal distance between points A and B is 3 units (from -2 to -5). Therefore, $4x = 3$, and $x = .75$. The horizontal distance from B to the point is x, or .75 units. The x-coordinate of the point is .75 away from -2, or -2.75.



The vertical distance between points A and B is 6 units (from 0 to 6). Therefore, $4x = 6$, and $x = 1.5$. The vertical distance from B to the point is x, or 1.5 units. The y-coordinate of the point is 1.5 away from 0, or 1.5.