Probability

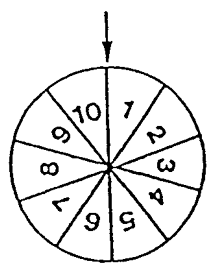
Probability is the numerical measure of the chance of an outcome or event occurring. *When all outcomes are equally likely to occur*, the probability of the occurrence of a given outcome can be found by using the following formula:

$$\text{probability} = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$$

The result of probability is between 0 and 1, "0" meaning there is no probability, and "1" that it is always true certain, to happen.

Examples:

1. Using the spinner below, what is the probability of spinning a 6 in one spin?

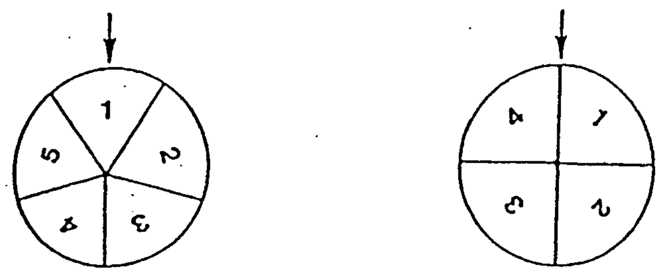


Since there is only *one* 6 on the spinner out of *ten* numbers and all the numbers are equally spaced, the probability is 1/10.

2. Using the spinner above, what is the probability of spinning either a 3 <u>or</u> a 5 in one spin?
   Since there are *two favorable outcomes* out of *ten possible outcomes*, the probability is 2/10, or 1/5. (The word "or" in probability means adding and the word "and" means multiplication.)

When two events are independent of each other, you need to multiply to find the favorable and / or possible outcomes.

3. What is the probability that both spinners below will stop on a 3 on the first spin?



Since the probability that the first spinner will stop on the number 3 is 1/5 and the probability that the second spinner will stop on the number 3 is ¼, and since each event is independent of the other simply multiply.

$$1/5 \times 1/4 = 1/20$$

4. What is the probability that on two consecutive rolls of a die the numbers will be 2 and then 3?

Since the probability of getting a 2 on the first roll is 1/6 and the probability of getting a 3 on the second roll is 1/6, and since the rolls are independent of each other, simply multiply.

$$1/6 \times 1/6 = 1/36$$

5. What is the probability of tossing heads three consecutive times with a two-sided fair coin?

Since each toss in independent and the odds are ½ for each toss, the probability would be ½ x ½ x ½ = 1/8. Some problems ask for the non-probability. In this case you must subtract the probability from 1. What is the probability of **not** tossing heads three consecutive times with a two–sided fair coin? Other problems that use non- probability are the **"at least"** problems.

Example: What is the probability of throwing at least one three in two consecutive rolls of a die?

To solve this problem you must calculate the non-probability for the first throw and multiply it by the non-probability for the second throw. Since the probability of getting out a three is 1/6, the non-probability is 5/6; then the probability of throwing **at least** one three is

$1 - (5/6 \times 5/6) = 1 - 25/36 = 11/36.$

6. What is the probability of getting at least one five in two consecutive throws of a die?

$1 - 1/6 = 5/6$: The probability of not having a five on a die in the first throw.

$1 - 1/6 = 5/6$: The same probability exists for the second throw of the die.

$1-(5/6 \text{ X } 5/6) = 1 - 25/36 = 11/36$

7. What is the probability of rolling two dice in one toss so that they total 5?

Since there are six possible outcomes on each die, the total possible outcomes for two dice is $6 \times 6 = 36$. The favorable outcomes are $(1 + 4)$, $(4 + 1)$, $(2 + 3)$, and $(3 + 2)$. These are all the ways of tossing a total of 5 on two dice. Thus there are four favorable outcomes which gives the probability of throwing a five as

$$4/36 \text{ or } 1/9$$

8. Three green marbles, two blue marbles, and five yellow marbles are placed in a jar. What is the probability of selecting at random a green marble on the first draw?

Since there are ten marbles (total possible outcomes) and three green marbles (favorable outcomes), the probability is 3/10.

Other examples:

In the same problem above the question could be:
What is the probability of drawing 2 green marbles on two consecutive draws (with replacement). To solve this you must multiply

$$3/10 \times 3/10 = 9/100.$$

What is the probability of drawing 2 green marbles on two consecutive draws (without replacement). To solve this you must multiply

$$3/10 \times 2/9 = 1/15$$

Practice: Probability Problems

1. If you have 12 books on a shelf and 9 of them are math books, what is the probability of picking a math book at random?

2. Of the 12 books on a shelf, 4 are math, 5 are history, and 3 are literature. If you choose one book at random, what is the probability that the book you choose is either math or literature?

3. If three pets are chosen at random from a pet shop with 6 dogs and 4 cats, what is the probability that all 3 pets chosen will be dogs?

4. If a fair coin is flipped three times, what is the probability of getting at least one tail?

5. If a fair coin is flipped 5 times, what is the probability of getting exactly 3 heads?

Answer explanations probability problems

1. The total possible options are 12. The favorable outcomes are 9 thus the probability is 9/12=3/4.

2. The probability of picking a math book at random is 4/12. The probability of picking a literature book at random is 3/12. The probability that the chosen book is either math or literature is therefore 4/12+3/12=7/12

3. The probability that the first pet chosen will be dog is 6/10 or 3/5.
After that pet is chosen there are 9 pets remaining, 5 of which are dogs. So the probability of choosing a dog for the second pet is 5/9.
Finally, for the last one, there are 8 pets remaining, 4 of them are dogs. The probability of choosing a dog for the third pet is therefore 4/8=1/2.
The probability that all three pets chosen will be dogs is $\left(\frac{3}{5}\right)\left(\frac{5}{9}\right)\left(\frac{1}{2}\right)=\frac{15}{90}=\frac{1}{6}$

4. Fair means that every outcome is equally possible. A fair coin will land heads up 50 percent of the time.
The favorable outcomes are one tail **or** two tails **or** three tails in three flips. That's a lot of calculation to make. But what if we calculate the undesired part of this situation. This is:
The total - undesired

1-HHH

1-$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

1-$\left(\frac{1}{8}\right)$

$\frac{7}{8}$

5. The favorable outcomes are very clear, 3 heads and 2 tails, but you do not need them in any particular order so your combinations are as follow:

1. HHHTT	2. HHTHT	3. HHTTH	4. HTHTH	5. HTTHH
6. THTHH	7. TTHHH	8. THHHT	9. HTHHT	10. THHTH

This is; 10 favorable outcomes. Then, how many possible outcomes do we have? The answer is 32. Since we have five coins and each coin has 2 options then the total is 2X2X2X2X2=32.

Then the probability is $\frac{Favorable\ outcomes}{Total\ possible\ outcomes}=\frac{10}{32}=\frac{5}{16}$

Combinations

The key words for combinations are CHOOSE, SELECT, NO ORDER, taking ONE AT A TIME.

If there are a *number of successive* choices to make and the choices are *independent of each other* (order makes no difference), the total number of possible choices (*combinations*) is the product of each of the choices at each stage. *For example:*

1. How many possible combinations of shirts and ties are there if there are five different color shirts and three different color ties?

   To find the total number of possible combinations, simply multiply the number of shirts times the number of ties.

   $$5 \times 3 = 15$$

Permutations

If there are a *number of successive choices* to make and the choices are *affected by the previous choice or choices* (dependent upon order), then *permutations* are involved.
The key words for permutations are ARRANGE, ORDER or POSSIBILITY. Note that possibility and probability are not the same. As previously mentioned, we said probability lies between 0 and 1, whereas possibility usually involves larger numbers.

How many ways can you arrange the letters S, T, O, P in a row?
For these types of problems, use N!

# of choices for the first letter		*# of choices for second letter*		*# of choices for third letter*		*# of choices for fourth letter*
4	x	3	x	2	x	1

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

The product 4 x 3 x 2 x 1 can be written 4! *(read 4 factorial or factorial 4)*. Thus there are twenty-four different ways to arrange four different letters.

Another example: In how many ways can you arrange the letters A B C in a row. In this case we have 3! = 3 x 2 x 1 = 6. Counting them off :

ABC	BCA	CAB
ACB	BAC	CBA

As you can see there are only 6 ways to order the letters ABC without repeating any of the possibilities.

Other examples:

Example: There are 6 children at a family reunion, 3 boys and 3 girls. They will be lined up single-file for a photo, alternating genders. How many arrangements of the children are possible for this photo?

You may not be sure how to approach this with a formula, so draw a picture. You know you'll have 6 "blanks," but you don't know whether to begin with a boy or with a girl. It could be either one. So try both:

bgbgbg or *gbgbgb*

$$\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}+\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}$$

Any of the three boys could go in the first spot, and any of the three girls in the second:

$$\underline{\;3\;}\times\underline{\;3\;}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}+\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}$$

The next spot can be filled with either of the remaining two boys; the one after by either of the two remaining girls. Then the last boy and the last girl take their places:

$$\underline{\;3\;}\times\underline{\;3\;}\times\underline{\;2\;}\times\underline{\;2\;}\times\underline{\;1\;}\times\underline{\;1\;}+\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}\times\underline{\quad}$$

That's the boy-first possibility. The same numbers of boys and girls apply to the girl-first possibility, and so you get:

$$\underline{\;3\;}\times\underline{\;3\;}\times\underline{\;2\;}\times\underline{\;2\;}\times\underline{\;1\;}\times\underline{\;1\;}+\underline{\;3\;}\times\underline{\;3\;}\times\underline{\;2\;}\times\underline{\;2\;}\times\underline{\;1\;}\times\underline{\;1\;}$$

$$9 \times 4 \times 1 + 9 \times 4 \times 1$$

36 + 36 = 72 possible arrangements of 3 boys and 3 girls, alternating genders.

How many ways do you have of seating five boys in 8 chairs?

To solve this last problem, you need the following formula:

$n! / (n - r)!$ $8! / 3! = 8 \times 7 \times 6 \times 5 \times 4$

Repetition of letters or numbers.

In how many different ways can yow arrange the letters in the following words?

House: There is no repetition, hence $5!$

Green: There is repetition of one letter. $5! / 2!$

Level: There is repetition of two letters: $5! / 2!\,2!$

Canada: There is one letter appearing 3 times. $6! / 3!$

Example: A restaurant is hanging 7 large tiles on its wall in a single row. How many arrangements of tiles are possible if there are 3 white tiles and 4 blue tiles?

This problem essentially asks for the arrangements of *WWWBBBB*. Remember that although there are 7 total tiles to arrange, all the white tiles are indistinguishable from one another, as are the blue tiles. Therefore, you will need to divide out the number of redundant arrangements from the 7! total arrangements:

$$\frac{7!}{3!4!} = \frac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1\times4\times3\times2\times1} = 7 \times 5 = 35$$

To calculate **circular permutations**, use the formula $(n - 1)!$

In how many ways can I seat five boys around a circular table?

To solve this problem: $(5 - 1)! = 4! = 24$ ways.

Following is a more difficult type of combination involving permutations.

2. If, from among five people, three executives are to be selected, how many possible combinations of executives are there?
(Notice that the order of selection makes no difference.) The symbol used to denote this situation is C(n, r), which is read the number of combinations of n things taken r at a time. The formula used is

$$C(n, r) = n! / r! (n - r)!$$

(subgroup formula)

Since n -5 and r – 3 (five people taken three at a time), then the equation is as follows:

$$5! / 3! (5 - 3)! = 5! / 3! 2! = 5 \times 4 \times 3! / 2! \times 3! = 10$$

Now solve:

$$\frac{5 . 4 . 3 . 2 . 1}{3 . 2 . 1 (2)!} = \frac{5. 4^2 . 3 . 2 . 1}{3 . 2 . 1 (2 . 1)} = 10$$

Practice: Combinations & Permutations Problems

1.
Example: How many ways are there to arrange the letters in the word ASCENT?

2.
Example: A company is selecting 4 members of its board of directors to sit on an ethics subcommittee. If the board has 9 members, any of whom may serve on the subcommittee, how many different selections of members could the company make?

3.
Example: County X holds an annual math competition, to which each county high school sends a team of 4 students. If school A has 6 boys and 7 girls whose math grades qualify them to be on their school's team, and competition rules stipulate that the team must consist of 2 boys and 2 girls, how many different teams might school A send to the competition?

4.
Example: How many ways are there to arrange the letters in the word ASSETS?

Answer explanations Combinations & Permutations Problems

1.

Example: How many ways are there to arrange the letters in the word ASCENT?

Clearly order matters here, since ASCENT is different from TNECSA. There are 6 letters in the word, so you must calculate the permutations of 6 items:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

2.

Example: A company is selecting 4 members of its board of directors to sit on an ethics subcommittee. If the board has 9 members, any of whom may serve on the subcommittee, how many different selections of members could the company make?

Since the order in which you select the members doesn't change the composition of the committee in any way, this is a combinations question. The size of the group from which you choose is n, and the size of the selected group is k. So $n = 9$ and $k = 4$.

$${}_9C_4 = \frac{9!}{4!(9-4)!}$$

$$\frac{9!}{4!5!}$$

$$\frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times5\times4\times3\times2\times1}$$

Save yourself some work. There's rarely a need to multiply out factorials, since many of the factors can quickly be canceled.

$$\frac{9\times\cancel{8}^{2}\times7\times\cancel{6}\times\cancel{5\times4\times3\times2\times1}}{\cancel{4}\times\cancel{3\times2}\times1\times\cancel{5\times4\times3\times2\times1}}$$

$$9 \times 2 \times 7 = 126$$

3.

Example: County X holds an annual math competition, to which each county high school sends a team of 4 students. If school A has 6 boys and 7 girls whose math grades qualify them to be on their school's team, and competition rules stipulate that the team must consist of 2 boys and 2 girls, how many different teams might school A send to the competition?

The order of selection doesn't matter here, so you can use the combinations formula. But be careful ... if you lump all the students together in a group of 13 and calculate ${}_{13}C_4$, you'd wind up including some all-boy teams and all-girl teams. The question explicitly says you can only select 2 boys and 2 girls. So you aren't really choosing 4 students from 13 but rather choosing 2 boys from 6 and 2 girls from 7.

$$_6C_2 \text{ and } _7C_2$$

$$\frac{6!}{2!4!}\times\frac{7!}{2!5!}$$

$$\frac{6\times5\times4\times3\times2\times1}{2\times1\times4\times3\times2\times1}\times\frac{7\times6\times5\times4\times3\times2\times1}{2\times1\times5\times4\times3\times2\times1}$$

$$\frac{\cancel{6}^3\times5\times\cancel{4\times3\times2\times1}}{\cancel{2}\times1\times\cancel{4\times3\times2\times1}}\times\frac{7\times\cancel{6}^3\times\cancel{5\times4\times3\times2\times1}}{\cancel{2}\times1\times\cancel{5\times4\times3\times2\times1}}$$

3 × 5 × 7 × 3 = 315 possible teams consisting of 2 boys and 2 girls.

4.

Example: How many ways are there to arrange the letters in the word ASSETS?

Earlier you saw that the rearrangement of ASCENT was a permutation. But what about ASSETS? The order of the E, the A, the T, and the S's matter … but the order of the three S's themselves does not.

Think about it this way: Put a "tag" on the S's … $AS_1S_2ETS_3$. If you just calculated 6! again, you'd be counting $AS_1S_2ETS_3$ and $AS_3S_1ETS_2$ as different words, even though with the tags gone, you can clearly see that they aren't (ASSETS is the same as ASSETS). So you'll need to eliminate all the redundant arrangements from the 6! total.

Since there are 3 S's in the word ASSETS, there are 3! ways to rearrange those S's without changing the word. You need to count every group of 3! within the 6! total as only 1 arrangement.

So instead of 6! arrangements, as there were for ASCENT, the word ASSETS has $\frac{6!}{3!}$.

$$\frac{6!}{3!}=\frac{6\times5\times4\times3\times2\times1}{3\times2\times1}=6\times5\times4=120$$

Last Examples

5.

Example: There are 8 employees including Bob and Rachel. If two employees are to be randomly chosen to form a committee, what is the probability that the committee includes both Bob and Rachel?

6.

Triplets Adam, Bruce, and Charlie enter a triathlon. There are nine competitors in the triathlon. If every competitor has an equal chance of winning, and three medals will be awarded, what is the probability that at least two of the triplets will win a medal?

A) $\frac{3}{14}$ B) $\frac{19}{84}$ C) $\frac{11}{42}$ D) $\frac{15}{28}$ E) $\frac{3}{4}$

Homework

Appendix A: D7, 10, 64, 121, 135, 173, 195, 217, 231
Appendix B: 80, 132, 151
Appendix C: 10, 82