Number Properties

ODDS & EVENS

Even numbers are integers that are divisible by 2. Odd numbers are integers that are not divisible by 2. All integers are either even or odd.

Evens: 0, 2, 4, 6, 8, 10, 12... Odds: 1, 3, 5, 7, 9, 11...

Consecutive integers alternate between even and odd: 9, 10, 11, 12, 13...
O, E, O, E, O...

Negative integers are also either even or odd:

Evens: −2, −4, −6, −8, −10, −12... Odds: −1, −3, −5, −7, −9, −11...

Arithmetic Rules of odds and evens

The GRE or GMAT test your knowledge of how odd and even numbers combine through addition, subtraction, multiplication and division. Rules for these operations between add and even numbers can be derived by simply picking numbers and testing them out. While this is certainly a valid strategy, it also pays to memorize the following rules for operating with odds and evens, as they are extremely useful for certain GMAT or GRE math questions.

<u>Addition and Subtraction:</u>
Add or subtract 2 odds or 2 evens, and the result is EVEN. $7 + 11 = 18$ and $8 + 6 = 14$
Add or subtract an odd with an even, and the result is ODD. $7 + 8 = 15$

<u>Multiplication:</u>
When you multiply integers, if ANY of the integers is even, the result is EVEN. $3 \times \mathbf{8} \times 9 \times 13 = 2{,}808$
Likewise, if NONE of the integers is even, then the result is ODD.

If you multiply together several even integers, the result will be divisible by higher and higher powers of 2. This result should make sense from our discussion of prime factors. Each even number will contribute at least one 2 to the factors of the product.

For example, if there are TWO even integers in a set of integers being multipled together, the result will be divisible by 4. $\mathbf{2} \times 5 \times \mathbf{6} = 60$ (divisible by 4)

If there are THREE even integers in a set of integers being multipled together, the result will be divisible by 8. $\mathbf{2} \times 5 \times \mathbf{6} \times \mathbf{10} = 600$ (divisible by 8)

To summarize so far:

Odd ± Even = ODD	Odd × Odd = ODD
Odd ± Odd = EVEN	Even × Even = EVEN (and divisible by 4)
Even ± Even = EVEN	Odd × Even = EVEN

The Sum of Two Primes

Notice that all prime numbers are odd, except the number 2. (All larger even numbers are divisible by 2, so they cannot be prime.) Thus, the sum of any two primes will be even ("Add two odds . . ."), unless one of those primes is the number 2. So, if you see a sum of two primes that is odd, one of those primes must be the number 2. Conversely, if you know that 2 CANNOT be one of the primes in the sum, then the sum of the two primes must be even.

If a and b are both prime numbers greater than 10, which of the following CANNOT be true?

I. ab is an even number.
II. The difference between a and b equals 117.
III. The sum of a and b is even.

(A) I only
(B) I and II only
(C) I and III only
(D) II and III only
(E) I, II and III

Do your work here

Testing Odd & Even Cases

Sometimes multiple variables can be odd or even, and you need to determine the implications of each possible scenario. In that case, set up a table listing all the possible odd/even combinations of the variables, and determine what effect that would have on the question.

If a, b, and c are integers and $ab + c$ is odd, which of the following must be true?

I. $a + c$ is odd
II. $b + c$ is odd
III. abc is even

(A) I only
(B) II only
(C) III only
(D) I and III only
(E) II and III only

Do your work here

Positives and negatives

The variable X is not necessarily positive, nor is –X necessarily negative.

Remember that when you multiply or divide signed numbers, the **NUMBER** of negative signs determines the **SIGN** of the result.

Consider the following data sufficiency problem:

Is the product of all of the elements in Set *S* negative?

(1) All of the elements in Set *S* are negative.
(2) There are 5 negative numbers in Set *S*.

Do your work here

Some positive negative problems deal with multiple variables, each of which can be positive or negative.

Consider the following problem:

If $ab > 0$, which of the following must be negative?

(A) $a + b$ (B) $|a| + b$ (C) $b - a$ (D) $\frac{a}{b}$ (E) $-\frac{a}{b}$

Do your work here

CONSECUTIVE INTEGERS

Consecutive integers are integers that follow one after another from a given starting point, without skipping any integers. For example, 4, 5, 6, and 7 are consecutive integers, but 4, 6, 7, and 9 are not. There are many other types of consecutive patterns. For example:

Consecutive Even Integers: 8, 10, 12, 14
(8, 10, 14, and 16 is incorrect, as it skips 12)

Consecutive Primes: 11, 13, 17, 19
(11, 13, 15, and 17 is wrong, as 15 is not prime)

Evenly Spaced Sets

To understand consecutive integers, we should first consider sets of consecutive integers **evenly spaced sets**. These are sequences of numbers whose values go up or down by the same amount (the **increment**) from one item in the sequence to the next. For instance, the set {4, 7, 10, 13, 16} is evenly spaced because each value increases by 3 over the previous value.

Sets of **consecutive multiples** are special cases of evenly spaced sets: all of the values in the set are multiples of the increment. For example, {12, 16, 20, 24} is a set of consecutive multiples because the values increase from one to the next by 4, and each element is a multiple of 4. Note that sets of consecutive multiples must be composed of integers.

Sets of **consecutive integers** are special cases of consecutive multiples: all of the values in the set increase by 1, and all integers are multiples of 1. For example, {12, 13, 14, 15, 16} is a set of consecutive integers because the values increase from one to the next by 1, and each element is an integer.

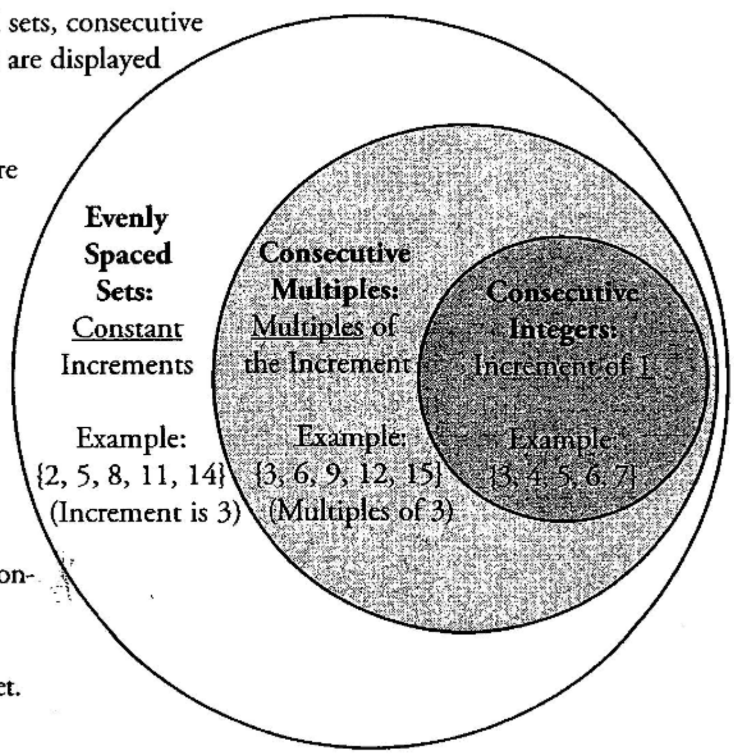
The relations among evenly spaced sets, consecutive multiples, and consecutive integers are displayed in the diagram to the right:

- All sets of consecutive integers are sets of consecutive multiples.
- All sets of consecutive multiples are evenly spaced sets.
- All evenly spaced sets are fully defined if the following 3 parameters are known:

  (1) The smallest (**first**) or largest (**last**) number in the set

  (2) The **increment** (always 1 for consecutive integers)

  (3) The **number of items** in the set.



Properties of Evenly Spaced Sets

The following properties apply to **all** evenly spaced sets.

(1) The **arithmetic mean** (average) and **median** are equal to each other. In other words, the average of the elements in the set can be found by figuring out the median, or "middle number."

What is the arithmetic mean of 4, 8, 12, 16, and 20?

In this example we have 5 consecutive multiples of four. The median is the 3rd largest, or 12. Since this is an evenly spaced set, the arithmetic mean (average) is also 12.

What is the arithmetic mean of 4, 8, 12, 16, 20, and 24?

In this example we have 6 consecutive multiples of four. The median is the arithmetic mean (average) of the 3rd largest and 4th largest, or the average of 12 and 16. Thus the median is 14. Since this is an evenly spaced set, the average is also 14.

(2) The **mean** and **median** of the set are equal to the **average** of the FIRST and LAST terms.

What is the arithmetic mean of 4, 8, 12, 16, and 20?

In this example, 20 is the largest (last) number and 4 is the smallest (first). The arithmetic mean and median are therefore equal to $(20 + 4) \div 2 = 12$.

What is the arithmetic mean of 4, 8, 12, 16, 20, and 24?

In this example, 24 is the largest (last) number and 4 is the smallest (first). The arithmetic mean and median are therefore equal to $(24 + 4) \div 2 = 14$.

Thus for all evenly spaced sets, just remember: the average equals **(First + Last) ÷ 2.**

(3) The **sum** of the elements in the set equals the **arithmetic mean** (average) number in the set times the **number of items** in the set.

This property applies to all sets, but it takes on special significance in the case of evenly spaced sets because the "average" is not only the arithmetic mean, but also the median.

What is the sum of 4, 8, 12, 16, and 20?

We have already calculated the average above; it is equal to 12. There are 5 terms, so the sum equals $12 \times 5 = 60$.

What is the sum of 4, 8, 12, 16, 20, and 24?

We have already calculated the average above; it is equal to 14. There are 6 terms, so the sum equals $14 \times 6 = 84$.

Counting Integers: Add One Before You Are Done

How many integers are there from 6 to 10? Four, right? Wrong! There are actually five integers from 6 to 10. Count them and you will see: 6, 7, 8, 9, 10. It is easy to forget that you have to include (or, in GMAT lingo, **be inclusive of**) extremes. In this case, both extremes (the numbers 6 and 10) must be counted. When you merely subtract ($10 - 6 = 4$), you are forgetting to include the first extreme (6), as it has been subtracted away (along with 5, 4, 3, 2, and 1).

Do you have to methodically count each term in a long consecutive pattern? No. Just remember that if both extremes should be counted, you need to **add one before you are done**.

How many integers are there from 14 to 765, inclusive?

765 – 14, plus 1, yields 752.

Just remember: for consecutive integers, the formula is **(Last – First + 1)**.

This works easily enough if you are dealing with consecutive integers. Sometimes, however, the question will ask about consecutive multiples. For example, "How many multiples of 4..." or "How many even numbers..." are examples of sets of consecutive multiples.

In this case, if we just subtract the largest number from the smallest and add one, we will be overcounting. For example, "All of the even integers between 12 and 24" yields 12, 14, 16, 18, 20, 22, and 24. That is 7 even integers. However, (Last – First + 1) would yield ($24 - 12 + 1$) = 13, which is too large. How do we amend this? Since the items in the list are going up by increments of 2 (we are counting only the even numbers), we need to divide (Last – First) by 2. Then, add the one before you are done:

(Last – First) ÷ Increment + 1 = $(24 - 12) \div 2 + 1 = 6 + 1 = 7$.

Just remember: for consecutive multiples, the formula is **(Last – First) ÷ Increment + 1**. The bigger the increment, the smaller the result, because there is a larger gap between the numbers you are counting.

Sometimes, however, it is easier to list the terms of a consecutive pattern and count them, especially if the list is short or if one or both of the extremes are omitted.

How many multiples of 7 are there between 100 and 150?

Do your work here

The Sum of Consecutive Integers

Consider this problem:

What is the sum of all the integers from 20 to 100, inclusive?

Adding all those integers would take much more time than you have for a GMAT problem. Using the rules for evenly spaced sets mentioned before, we can use shortcuts:

(1) Average the first and last term to find the precise "middle" of the set: $100 + 20 = 120$ and $120 \div 2 = 60$.
(2) Count the number of terms: $100 - 20 = 80$, plus 1 yields 81.
(3) Multiply the "middle" number by the number of terms to find the sum: $60 \times 81 = 4{,}860$.

There are a couple of general facts to note about sums and averages of evenly spaced sets (especially sets of consecutive integers):

- The average of an **odd** number of consecutive integers (1, 2, 3, 4, 5) will always be an integer (3). This is because the "middle number" will be a single integer.
- On the other hand, the average of an **even** number of consecutive integers (1, 2, 3, 4) will never be an integer (2.5), because there is no true "middle number."
- This is because consecutive integers alternate between EVEN and ODD numbers. Therefore, the "middle number" for an even number of consecutive integers is the AVERAGE of two consecutive integers, which is never an integer.

Consider this Data Sufficiency problem:

Is k^2 odd?
(1) $k - 1$ is divisible by 2.
(2) The sum of k consecutive integers is divisible by k.

Do your work here

Products of Consecutive Integers and Divisibility

Can you come up with a series of 3 consecutive integers in which none of the integers is a multiple of 3? Go ahead, try it! You will quickly see that any set of 3 consecutive integers must contain one multiple of 3. The result is that the product of any set of 3 consecutive integers is divisible by 3.

$1 \times 2 \times 3 = 6$	$4 \times 5 \times 6 = 120$
$2 \times 3 \times 4 = 24$	$5 \times 6 \times 7 = 210$
$3 \times 4 \times 5 = 60$	$6 \times 7 \times 8 = 336$

According to the Factor Foundation Rule, every number is divisible by all the factors of its factors. If there is always a multiple of 3 in a set of 3 consecutive integers, the product of 3 consecutive integers will always be divisible by 3. Additionally, there will always be at least one multiple of 2 (an even number) in any set of 3 consecutive integers. Therefore, the product of 3 consecutive integers will also be divisible by 2. Thus, the product of 3 consecutive integers will always be divisible by $3! = 3 \times 2 \times 1 = 6$.

The same logic applies to a set of 4 consecutive integers, 5 consecutive integers, and any other number of consecutive integers. For instance, the product of any set of 4 consecutive integers will be divisible by $4! = 4 \times 3 \times 2 \times 1 = 24$, since that set will always contain one multiple of 4, at least one multiple of 3, and another even number (a multiple of 2).

This rule applies to any number of consecutive integers: **The product of *k* consecutive integers is always divisible by *k* factorial (*k*!).**

Sums of Consecutive Integers and Divisibility

Find the sum of any 5 consecutive integers:

$4 + 5 + 6 + 7 + 8 = 30$	Notice that both sums are multiples of 5.
$13 + 14 + 15 + 16 + 17 = 75$	In other words, both sums are divisible by 5.

We can generalize this observation. **For any set of consecutive integers with an ODD number of items, the sum of all the integers is ALWAYS a multiple of the number of items.** This is because the sum equals the average times the number of items. For an odd number of integers, the average is an integer, so the sum is a multiple of the number of items. The average of {13, 14, 15, 16, 17} is 15, so $15 \times 5 = 13 + 14 + 15 + 16 + 17$.

Find the sum of any 4 consecutive integers:

$1 + 2 + 3 + 4 = 10$	Notice that NEITHER sum is a multiple of 4.
$8 + 9 + 10 + 11 = 38$	In other words, both sums are NOT divisible by 4.

For any set of consecutive integers with an EVEN number of items, the sum of all the items is NEVER a multiple of the number of items. This is because the sum equals the average times the number of items. For an even number of integers, the average is never an integer, so the sum is never a multiple of the number of items. The average of {8, 9, 10, 11} is 9.5, so $9.5 \times 4 = 8 + 9 + 10 + 11$. That is, $8 + 9 + 10 + 11$ is NOT a multiple of 4.

WEIGHTED-AVERAGE PROBLEMS

You solve *weighted-average* problems using the arithmetic mean (simple average) formula, except you give the set's terms different weights. For example, if a final exam score of 90 receives *twice* the weight of each of two midterm exam scores 75 and 85, think of the final exam score as two scores of 90—and the total number of scores as 4 rather than 3:

$$WA = \frac{75 + 85 + (2)(90)}{4} = \frac{340}{4} = 85$$

Similarly, when some numbers among terms might appear more often than others, you must give them the appropriate "weight" before computing an average.

During an 8-hour trip, Brigitte drove 3 hours at 55 miles per hour and 5 hours at 65 miles per hour. What was her average rate, in miles per hour, for the entire trip?

(A) 58.5

(B) 60

(C) 61.25

(D) 62.5

(E) 66.25

Do your work here

A certain olive orchard produces 315 gallons of oil annually, on average, during four consecutive years. How many gallons of oil must the orchard produce annually, on average, during the next six years, if oil production for the entire 10-year period is to meet a goal of 378 gallons per year?

(A) 240

(B) 285

(C) 396

(D) 420

(E) 468

Do your work here

Kahoot!!!

1. Dan had an average of 72 on his first four math tests. After taking the next test, his average dropped to 70. Which of the following is his most recent test grade?

   (A) 60
   (B) 62
   (C) 64
   (D) 66
   (E) 68

4. 30 students had an average of X, while 20 students had an average of 80. What is the average for the entire group?

   (A) $\frac{X+80}{50}$
   (B) $\frac{X+80}{2}$
   (C) $\frac{50}{X+80}$
   (D) $\frac{3}{5}X+32$
   (E) $\frac{30X+80}{50}$

5. What is the average of the first 15 positive integers?

   (A) 7
   (B) 7.5
   (C) 8
   (D) 8.5
   (E) 9

6. A man travels a distance of 20 miles at 60 miles per hour and then returns over the same route at 40 miles per hour. What is his average rate for the round trip in miles per hour?

   (A) 50
   (B) 48
   (C) 47
   (D) 46

7. A number p equals $\frac{3}{2}$ the average of 10, 12, and q. What is q in terms of p?

   (A) $\frac{2}{3}p-22$
   (B) $\frac{4}{3}p-22$

8. Susan has an average of 86 in three examinations. What grade must she receive on her next test to raise her average to 88?

   (A) 94
   (B) 90
   (C) 92
   (D) 100
   (E) 96

Kahoot Answer key:

1. B   4. D   5. C   6. B   7. C   8. A

Homework

Odds & Evens

Appendix A: 108, 187

Appendix B: 150

Consecutive Integers

Appendix A: D2, 86, 201, 204, 215, 236, 242

Appendix B: 9, 160