General Article

REDISCOVERING THE PAST: Gustav Fechner and Signal Detection Theory

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The origins of experimental psychology are found in the theory of discrimination created by Gustav Fechner. The theory advanced by Fechner was the first theory of mental judgment that applied the ideas of Gaussian error to human discrimination. Despite its powerful theoretical and empirical results, the theory vanished. Later embellishments, such as the addition of response bias found in signal detection theory, gave such theories new life, but Fechner never received the credit due to the founder of psychological detection theory.

The great Irish historian Eugene O'Curry spent the bitter Dublin winter of 1855 revising his famous course on the manuscript materials of ancient Irish history. O'Curry opened his lecture of March 13, 1855, with a declaration:

> I believe that the tendency may be called a law of our nature, which induces us to look back with interest and reverence to the monuments and records of our progenitors; and that the more remote and ancient such monuments and records are, the greater the interest which we feel in them. (1861, p. 1)

This historical maxim, a psychological statement if ever there was one, supposes that people assign subjective values to ancient events and discriminate between such subjective values. Unknown to O'Curry, nearly 1,400 km to the east, in Leipzig, Germany, former Professor of Physics Gustav Fechner spent the bitter German winter testing his newly developed theory of human discrimination—a theory that would provide an empirical means for assessing the truth of O'Curry's maxim.

Fechner's theory postulated the existence of subjective values such as sensations, but went further to propose that sensations were perturbed by the same form of measurement error that the great 19th-century mathematician Carl Gauss had already proposed to exist in physical measurements. Extending Gauss, Fechner created the first theory of how sensations are used to make mental judgments about the physical world.

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GAUSSIAN THEORY OF ERROR

Gauss's theory (1809/1963, 1821/1880) postulated that an observed measure, O, depended on two components. The first component was an unknown but true value of the phenomenon that the scientist sought to measure. The second was an unknown amount of error that interfered with the measurement of the true value. Furthermore, the observed measure, O_i, resulted from the addition of these two components:

$$O_i = \mathrm{T} + E_i,$$

where i indicates the i-th observation, T is a constant true value, and E_i is the error associated with the i-th observation.

For the form of the errors, Gauss (1809/1963) developed a simple model based on the idea that the arithmetic mean affords the most probable value for a set of observations and the assumption that deviations from the mean value should be minimized. By translating these principles into mathematics, Gauss derived the form of the probability distribution for errors, a form, incidentally, that Gauss credited to Pierre Laplace.

As a consequence of these deductions, Gauss measured the spread of the errors around the mean by a variable h, which in today's notation equals $1/(\sqrt{2}\sigma)$, where σ is the standard deviation of a Gaussian probability distribution. A small value of h indicated poor measurement because the errors were generally too large (i.e., there was too much variability). When h was large, accuracy in the measurement was good. Because h increases as the standard deviation decreases, h may be viewed as an index of the precision of measurement, a measure of the sensitivity of the measuring device.

FECHNER'S EXTENSION OF GAUSS

To turn psychology toward the rigors of scientific measurement, Fechner (1860/1966) extended the Gaussian theory to the body's measurement of sensation. Fechner assumed that the purpose of the sensory system was to provide a measure of the physical world. But, like any measuring device, the sensory system added an unknown amount of error to the true sensation generated by a fixed stimulus. A fixed weight produced different sensations of heaviness because the sensory system combined a variable amount of error with the constant true heaviness to produce a variable sensation of heaviness. A weight failed to feel the same from one lifting to another because heaviness, like any other measurement, was burdened by the fickle influence of invisible sensory error.

Fechner, the physicist, set out to measure the size of the sensory system's error and to make the first measure of the precision of this human measuring device. By measuring precision, one could determine whether the error of the sensory system remained constant during the day; varied from person to person, or from one sense modality to another; or depended on such factors as amount of practice and number of experimental trials. The measure of precision, h, provided a window on the operation of the mind and the first measure of a property of the mind.

Fechner's problem was how to obtain a measure of the invisible sensory error. Going a step beyond Gauss, Fechner argued that the variability in sensation caused errors in judging whether one weight weighed more than another. Then, Fechner created a theory of errors in mental judgment that depended on the invisible sensory error. The theory showed how the relative frequency of errors of judgment could be used to determine the size of the invisible sensory system error.

Specifically, Fechner postulated that when a person is comparing two weights, the average heaviness of the two weights serves as a criterion for deciding which of the two weights feels heavier. When two weights are compared, the lighter weight, S_A, gives rise to a sensation of heaviness W_A. In today's terms, W_A is a random variable that consists of the constant true heaviness of S_A, denoted a, and an error that is distributed as a Gaussian random variable with mean zero and standard deviation σ. Similarly, the heavier weight, S_B, generates a sensation W_B having a true value $b > a$, plus an additive Gaussian error with mean zero and standard deviation σ.

The probability of misjudging the smaller weight to be the heavier weight equals the probability that the heaviness of the smaller weight exceeds the average of the two true heavinesses, $(a + b)/2$. Setting the difference between the two true sensations of heaviness to $(b - a) = d$ gave Fechner the following:

$$\begin{aligned} P\{W_A > (a + b)/2\} &= P\{W_A > (a + a + d)/2\} \\ &= P\{W_A > a + d/2\}. \end{aligned}$$

A similar argument for errors in judging the heavier weight to be the lighter weight gives

$$\begin{aligned} P\{W_B < (a + b)/2\} &= P\{W_B < (a + a + d)/2\} \\ &= P\{W_B < a + d/2\}. \end{aligned}$$

Because the distributions of W_A and W_B are reflections of each other with respect to $(a + b)/2$, the two probabilities of error are equal.

The theory is illustrated in Figure 1. The two Gaussian probability distributions show the distribution of error surrounding the true heaviness evoked by two different weights, W_A and W_B, with mean values a and b, respectively. The criterion, or threshold, is positioned midway between these two means at $(a + b)/2$. The theory proposed that when the heaviness of a weight exceeds this "threshold," the subject makes the response "heavier." The lighter of the two weights, S_A, is judged (incorrectly) to be heavier when a sensory error greater than $(b - a)/2$ is added to its true heaviness. The heavier of the two weights, S_B, is judged (incorrectly) to be lighter when a negative sensory error greater than $(b - a)/2$ is added to its true heaviness.

Fechner's genius was to observe that for the lighter weight, the increase in sensation from the true heaviness value to the decision criterion located at $(a + b)/2$ could be measured in units of precision, h, and he used the variable t to denote this distance. As shown in Figure 1, the difference between the two true heaviness values equals $(b - a)$, which is scaled in unknown units of measurement. Not knowing the unit of measurement is a definite problem. But Fechner avoided this problem by noting that the relative distance between b and a, scaled in units of precision, is $2t = (b - a)/\sqrt{2}\sigma$. This ratio is a

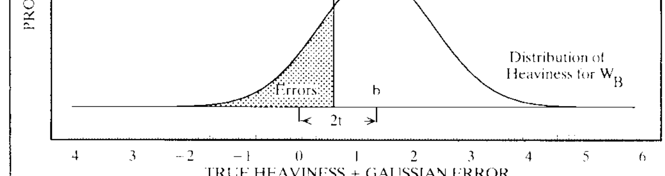

Fig. 1. Fechner's model for discriminating between two stimuli, S_A and S_B. The corresponding sensations of heaviness are W_A and W_B. Sensations greater than the decision criterion, or threshold, at $(a + b)/2$ result in the response "heavier," whereas sensations less than the threshold produce the response "lighter." The lower scale is calibrated in standard deviation units.

dimensionless quantity because the unknown units of measurement in the numerator and denominator are the same and cancel with each other.

Fechner transferred this idea to the physical dimension of the stimuli. Specifically, substituting for $(b - a)$ the physical difference between the two weights, D, gives $2t = D/\sqrt{2}\sigma = Dh$, or $t = (D/2)h$. Now the measure of precision, h, must be in units of the physical stimulus. The value of D is known by the experimenter. If t is known, the unknown value of h is determined. In this way, the previously unknown, invisible precision of the sensory system, a property of the mind, became measured in units of the physical stimuli.

In order to calculate the unknown value of h, Fechner created tables of values of t. For each value of t, there is a unique, fixed probability of an error of judgment. This probability of error equals the area under the Gaussian distribution for W_A (or W_B) to the right (or left) of t. Therefore, given an experimentally fixed value of D, and an experimentally determined error probability, one could look up in Fechner's tables the corresponding value of t. The values of t are similar to today's tables of z for the standard Gaussian ("normal") distribution.

For example, Figure 1 shows that 25% of the sensations produced by the lighter weight lie above the threshold, or decision criterion, and produce errors of judgment—instances in which the subject experiences the lighter weight to be heavier. Fechner's Fundamental Table for the Method of Right and Wrong Cases (1860/1966, p. 90) shows an error probability of .25 occurs when the decision criterion is positioned at $t = 0.4769$ above the mean. Equating t and $h(D/2)$ gives

$$0.4769 = h(D/2).$$

Dividing by $D/2$ gives[1]

$$h = (0.4769)/(D/2).$$

Because $h = 1/\sqrt{2}\sigma$, the estimate of σ is

$$\sigma = \frac{(D/2)}{(0.4769 \times \sqrt{2})} \text{ (grams).}$$

Supposing the physical difference, D, to be 10 g yields a measure of the standard deviation equal to 7.4 g. In this way, the previously unknown measure of σ, the spread of errors generated by the sensory system, became measurable in units of the physical stimulus. A previously invisible property of the mind became observable. Psychology became a science.

EMPIRICAL RESULTS

Fechner used the theory to determine whether performance remained the same at different times of the day, with different orders of stimulus presentation, and across different magnitudes of stimuli. In *Elements of Psychophysics* (hereafter, *Elemente*), after some discussion of experimental design, Fechner (1860/1966) presented studies of the discrimination of brightness, sound intensity, lifted weights, temperature, extensive magnitudes, and illusions. In particular, the data in Table 1 illustrate the method applied to lifted weights in a crucial empirical test. To obtain the observations summarized in Table 1 (p. 159), Fechner and his assistant lifted 24,576 weights using a variety of orders and methods of randomization. Other similar results are analyzed in considerable detail elsewhere (Link, 1992).

The entries in Column 1 of Table 1 are the weights in grams of standard stimuli, P. The second column provides the estimated probability of responding correctly, P_c, when the standard and a comparison stimulus of $0.04P$ more than the standard were lifted, one in each hand, at the same time. That is, P_c is the probability that the comparison was judged heavier than the standard (or that the standard was judged lighter than the comparison). The values of $t = hD$ shown in Column 3 depend on the proportion of errors in Column 2. The fourth column provides the corresponding value for d', or z, given a signal detection theory (Green & Swets, 1966) analysis of these response proportions. The next three columns provide the corresponding estimated response probabilities, values of $t = hD$, and values of d' when the comparison weighed $0.08P$ more than the standard. Note that for the smallest standard weight, $P = 300$ g, the two comparisons weigh 312 g and 324 g, the two values of $t = hD$ are 0.1779 and 0.3792, and the corresponding values of d' or z equal 0.2515 and 0.5363, respectively (obtained from tables of z values in Kendall & Stuart, 1969).

Notice that for each standard weight, ranging to 3,000 g from 300 g, there are two comparison weights, and the heavier comparison weight differs from the standard weight exactly twice as much as does the lighter comparison weight. Although the precision, h, depends on the magnitudes of the judged weights, the value of h should be essentially constant for judgments made against a fixed standard weight. Denote the difference between the first comparison weight and the standard as D_1, and denote the second difference as D_2. Because h is constant for a fixed standard, and noting that $D_2 = 2D_1$, we have

$$h \times D_2 = 2 \times h \times D_1.$$

1. Fechner considered the factor of 2 in $(D/2)$ to be a triviality, noting that "one . . . looks up the corresponding value of $t = hD/2$ in a table . . . and then divides this value by D/2 in order to find h. Alternatively, one may divide by D, if one takes h in the method of right and wrong cases to be only one half as large as it is in the [Gaussian] method of errors (as we shall proceed to do)" (1860/1966, p. 86).

Table 1. *Fechner's results for the two-handed series*

	Size of comparison stimulus					
	.04*P*			.08*P*		
Standard *P* (grams)	P_c	$hD = t$	d' or z	P_c	$hD = t$	d' or z
300	.5993	0.1779	0.2515	.7041	0.3792	0.5363
500	.6044	0.1871	0.2647	.6907	0.3521	0.4978
1,000	.6330	0.2403	0.3403	.7300	0.4333	0.6127
1,500	.6455	0.2639	0.3732	.7778	0.5407	0.7640
2,000	.6581	0.2881	0.4073	.7964	0.5862	0.8289
3,000	.6677	0.3066	0.4336	.8131	0.6285	0.8893

Note. P is the weight in grams of the standard against which comparisons are made; .04*P* and .08*P* are the increases in weight for the two comparison stimuli. P_c is the proportion of correct responses given a comparison weight; *hD* is the physical distance between a standard and comparison weight, *D*, measured in units of precision. Data are from Fechner (1860/1966, p. 159), values of *t* are from Fechner (1860/1966, p. 90), and values of *z* are from Kendall and Stuart (1969). Fechner calculated *t* as equal to *hD* (see footnote 1).

Therefore, if the $h \times D_2$ values of the left-hand side of this equation are plotted against the $h \times D_1$ values on the right-hand side, the resulting graph should be linear with a slope of 2. The second decision criterion should be twice as far from the standard as the first.

To test this prediction, the $t(.04P)$ and $t(.08P)$ values in Table 1 are plotted against each other in Figure 2. Although the points are somewhat variable, the linear form of the predicted relation is quite evident. The multiplicative increase in the distance of the decision criterion from the mean heaviness of the standard, across standards of increasing magnitude, can be estimated from the slope of the linear regression line. Assuming a zero intercept, the regression analysis provides an estimated slope of 1.999, sufficiently near the predicted value of 2.0. The corresponding regression line, as is shown in Figure 2, provides a good account of the predicted linear relation between *t* values from the two comparison weights.

The importance Fechner attached to this theory is evidenced by the fact that it is the first theoretical development in Volume I of *Elemente* (1860/1966, pp. 85–89). The mathematical derivation was bolstered by the well-known German topologist A.F. Möbius, to whom Fechner gave credit for "substituting a briefer and more precise derivation for my somewhat clumsy one, which in the end leads to the same results" (1860/1966, p. 89).

Fechner's table of values of *hD* was used in two ways. First, if the value of *h* was determined from a previous experiment, then a change in *D* would produce a new value for *hD*. The probability of a correct response could then be determined by locating the value of *hD* and the associated response probability. Second, the table could be used to verify Fechner's generalization of Weber's Law, which stated that the probability of an error in judgment remained constant when the standard and comparison weights were each multiplied by a constant.

The theory performed satisfactorily in a variety of contexts employing different types of stimuli, various methods of presentation, and, for a fixed standard stimulus, several values of *D*. Realizing that the accuracy of the estimation of the area under the Gaussian curve depended on the number of observations used to form a frequency estimate of probability, Fechner provided the experimenter with tables of standardized distances based on $n = 64$ observations. This number of observations gave an exact solution to the binomial approximation to the cumulative Gaussian distribution.

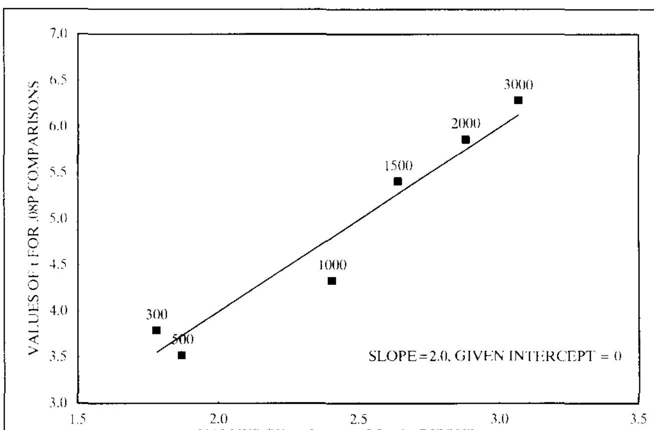

Fig. 2. Examination of a theoretical prediction from Fechner's detection theory. The values of *t* for a fixed standard and two comparison stimuli differing from the standard by amounts D_1 and $D_2 = 2D_1$ should be related by a factor of 2. These data are from Fechner's own lifted-weight experiments run in 1855–1856.

A similar idea for measuring the precision of sensory discrimination is advocated today by signal detection theory. The distance between stimuli is measured by a variable known as d'. The relation between hD and d' can be seen in Table 1: $d'/\sqrt{2} = hD$. Thus, both theories, the Fechnerian theory that founded the field of experimental psychology and the more recent signal detection theory, provide essentially the same measure of sensory variability.

THE LOST THEORY

Given the wide use of d' throughout contemporary psychology, it is easy to wonder why we did not know about Fechner's powerful theory before now. As an astute colleague commented to me, "If Fechner really did this in the 1850s, why don't I know about it?" As I have shown elsewhere (Link, 1992), the critical elements of the theory appeared twice after the original creation in Leipzig. The first reappearance was in 1927, when L.L. Thurstone published his famous paper "The Law of Comparative Judgment." Although the decision mechanism of the law of comparative judgment appears to be different from that specified by Fechner, the two theories make identical predictions under many conditions (Link, 1992). The second reappearance, signal detection theory, comes to psychology from engineering, which adopted the decision-making approach from statistics (Hancock & Wintz, 1966), and extends the theory by adding the important concept of response bias.

What happened to Fechner's decision theory? Stigler (1986), in a detailed evaluation of Fechner's contribution to the history of statistics, noted that "a major portion of *Elemente* is a handbook on experimental design, the most comprehensive treatment of that topic before R.A. Fisher's 1935 *Design of Experiments*" (p. 244). How could a theory of such power, and a book that extends the theory to the design and analysis of experiments, disappear from a discipline that espoused empiricism? I believe the answer is to be found in the fervor of a new psychology that found little time for theoretical developments that depended on at least an elementary knowledge of mathematics such as calculus.

The importance of Fechner's contributions was emphasized by Wilhelm Wundt, who was the creator of the first psychological laboratory in Leipzig and, as a personal friend of Fechner, was well acquainted with his theory. Even this esteemed psychologist, often credited with establishing the field of experimental psychology, in his voluminous *Lehrbuch der Physiologie des Menschen* (1878), did not describe Fechner's extension of Gauss's theory of error. Yet he drew freely from *Elemente* in describing a logarithmic relation between sensation and stimulus magnitude.

Sully (1884), in his famous *Outlines of Psychology*, the first book in English about the new psychology, described "Weber's or Fechner's Law" relating the amount of sensation to the magnitude of the physical stimulus producing it. But Sully made no mention whatsoever of Fechner's detection theory. Similarly, E.B. Titchener's magnificent series of volumes (1901a, 1901b, 1905) detailing the procedures of experimental psychology transported to America many of the new psychology's principal experimental methods, including those for measuring "thresholds." Yet nowhere in Titchener's volumes is there any discussion of the Fechnerian theory that provided the basis for much of experimental psychology.

Fechner's theory of discrimination simply vanished from textbooks of psychology. By 1958, Krech and Crutchfield's famous *Elements of Psychology*, requisite reading for most graduate students in experimental psychology, acknowledged that "it was with psychophysics that the history of experimental psychology began about one hundred years ago" (p. 49). Their brief summary of psychophysics (pp. 49–50) mentioned Weber's law but made no mention of *Elemente der Psychophysik*, the basis for the title of their own work; no mention of Fechner at all; no mention of the law of sensation that bears his name; and no mention of the decision theory that was the basis for psychophysics and the experimental psychology that followed. This popular graduate-level presentation of experimental psychology did much to obscure the origins of the experimental methods previously adopted by many experimental psychologists and to erase from psychological history even the name of Gustav Fechner.

Fechner published a last, extensive, 300-page work on the theory in 1884 and died in 1888. His theory seems to have followed him to his grave, with only Wundt awarding him credit for a truly outstanding scientific achievement. Wundt read the eulogy at Fechner's graveside:

> Perhaps in none of his other scientific achievements did his rare unity of gifts shine forth more brilliantly than in his psychophysic work. It required a knowledge of the principles of exact physico-mathematical method and also a strong yearning to penetrate the deepest problems of human experience such as up to date he alone has possessed. To this work, too, he brought originality of thought which enabled him to transform the traditional material and methods freely, according to his own needs, and he had no hesitation in striking out on new unexplored paths. The limited observations of E.H. Weber, which were remarkable on account of their general simplicity, and the isolated, often more accidental than planned, methods and results of other physiologists—these were the limited material out of which Fechner built up this new science. What he himself contributed in exact observations by their thoroughness and mathematical completeness had great value. Although with his limitation of means and assistance he could not finally solve

very many questions, the clearness with which he formulated the somewhat scrappy and gappy material; and created and carried through his exact methods is one of the very great achievements that the science of our day has to show. His original application of the principle of probabilities, which had hither to been applied only to objective measurements, to the domain of subjective perception is, of itself and quite apart from the end it had in view, of the very highest theoretical interest. (translated in Hall, 1912, pp. 168–169)

Perhaps, as O'Curry's maxim posits, we might attach more value to our early achievements and learn some lessons from our past. One lesson to be taken from the previous scientific advances in psychology, evidenced in its very foundation, is that the use of mathematical reasoning allows for the creation of very powerful psychological theories. Fechner's theory provided many decades of research in the field of experimental psychology. The reappearance of similar theories, each time with additional embellishments, co-occurred with the reappearance of psychological scientists with mathematical training. Each reappearance led to substantial new developments in psychology that might just as well have happened as a continuous unbroken sequence of theoretical development.

The reason the development failed to continue, it seems to me, is clear in the lack of mathematical training received by psychologists. Titchener (1901a) remarked, "Knowledge of elementary mathematics is part of a man's general scientific outfit: but one may work a lifetime, and with success, in psychology, without needing the knowledge" (p. xxiii). With that reasoning, where would other sciences be today? Would scientific psychology exist today? Without more emphasis on the value of our past accomplishments and the mathematical training of our young researchers, is there any doubt that future psychologists will once again lose, and once again reinvent, their most powerful scientific theories?

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