The Logarithmic Function

6.4

Introduction

In this section we consider the logarithmic function $y = \log_a x$ and examine its important characteristics. We see that this function is only defined if x is a positive number. We also see that the log function is the inverse function of the exponential function and vice versa. We show, through numerous examples, how equations involving logarithms and exponentials can often be solved.

Prerequisites

Before starting this Section you should ...

① have knowledge of inverse functions

② have knowledge of the laws of logarithms and of the laws of indices

③ be able to solve quadratic equations

Learning Outcomes

After completing this Section you should be able to ...

✓ understand the relation between the logarithm and the exponential function

✓ solve equations involving exponentials and logarithms

✓ change bases in logarithms

1. The logarithmic function

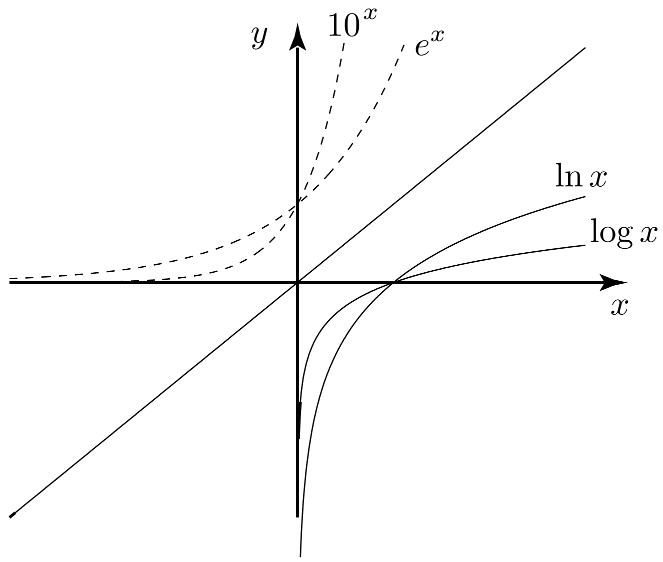
In section 3 we introduced the operation of taking logarithms which essentially reverses the operation of exponentiation.

$$\text{If } a > 0 \text{ and } a \neq 1 \text{ then } \quad x = a^y \quad \text{implies} \quad y = \log_a x$$

In this section we consider the function $\log_a x$ in more detail. We shall concentrate only on the functions $\log x$ (i.e. to base 10) and $\ln x$ (i.e. to base e) as these are, invariably, the only logarithmic functions used in practice. In any case the logarithmic function to any other base can be re-written in terms of $\log x$ or $\ln x$ since:

$$\log_a x = \frac{\log x}{\log a} \qquad \text{and also} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Both of the functions $y = \log x$ and $y = \ln x$ have similar characteristics. We can never choose x as a negative number since 10^y and e^y are each always positive. The graphs of $y = \log x$ and $y = \ln x$ are shown in the following diagram.



From the graphs we see that both functions are one-to-one so each has an **inverse function**. In fact, as is easy to demonstrate, the inverse function of $\log_a x$ is a^x. Let us do this for logs to base 10.
Let $f(x) = \log x$ then $f^{-1}(x)$ is, by definition, precisely that function which takes $\log x$ as the input to produce an output x. We claim that $f^{-1}(x) = 10^x$. To see this we replace x by $\log x$ (as the input) then

$$f^{-1}(\log x) = 10^{\log x}$$

The expression $10^{\log x}$ can be simplified. To simplify let $y = 10^{\log x}$ and take logs of both sides.

$$\log y = \log(10^{\log x}) = (\log x) \log 10 = \log x \qquad \text{so} \quad y = x$$

That is $x = 10^{\log x}$.
This result proves that the inverse function to $\log x$ is 10^x. In a similar way the inverse function to $\ln x$ is e^x.

2. Solving equations involving logarithms and exponentials

To solve equations which involve logarithms and/or exponentials we need to be aware of the basic laws which govern both of these mathematical concepts. We illustrate by considering some examples.

Example Solve for the variable x: (a) $3 = 10^x$, (b) $10^{x/4} = \log 3$, (c) $\frac{1}{17-e^x} = 4$

Solution

(a) Here we take logs (to base 10 because of the term 10^x) of both sides to give

$$\log 3 = \log 10^x = x \log 10 = x$$

where we have used the general property that $\log_a A^k = k \log_a A$ and the specific property that $\log 10 = 1$. Hence $x = \log 3$ or, in numerical form, $x = 0.47712$.

(b) The approach used in (a) is used here. Take logs of both sides:

$$\log(10^{x/4}) = \log(\log 3)$$

$$\text{that is} \qquad \frac{x}{4}\log 10 = \log(\log 3) = \log(0.4771212) = -0.3213712$$

so, since $\log 10 = 1$ we have

$$x = 4(-0.3213712) = -1.2854848$$

(c) Here we simplify the expression before taking logs.

$$\frac{1}{17 - e^x} = 4 \qquad \text{implies} \qquad 1 = 4(17 - e^x)$$

or $4e^x = 4(17) - 1 = 67$ so $e^x = 16.75$. Now taking natural logs of both sides (due to the presence of the e^x term) we have:

$$\ln(e^x) = \ln(16.75) = 2.8183983$$

But $\ln(e^x) = x \ln e = x$ and so the solution to $\frac{1}{17-e^x} = 4$ is $x = 2.8183983$.

Solve the equations (a) $(e^x)^2 = 50$, (b) $e^{2x} = 17e^x$

(a) First solve for e^x by taking square roots of both sides.

Your solution

$(e^x)^2 = 50$ implies $e^x =$

$(e^x)^2 = 50$ implies $e^x = \sqrt{50} = 7.071068$. Here we have taken the positive value for the square root since we know that exponential functions **are always positive**.

Now take logarithms to an appropriate base to find x.

Your solution

$e^x = 7.071068 \quad \text{implies} \quad x =$

$e^x = 7.071068 \quad \text{implies} \quad x = \ln(7.071068) = 1.956012$

(b) Again simplify the expression first as much as possible. In this case divide both sides by e^x and simplify.

Your solution

$e^{2x} = 17e^x \quad \text{implies} \quad \frac{e^{2x}}{e^x} = 17 \quad \text{so}$

$\frac{e^{2x}}{e^x} = 17$ implies $e^{2x-x} = 17$ or $e^x = 17$

Now complete the solution for x.

Your solution

$e^x = 17 \quad \text{implies} \quad x = \ln(17) = 2.8332133$

Example Find x if $\quad 10^x - 5 + 6(10^{-x}) = 0$

Solution

We first simplify this expression by multiplying through by 10^x (this will eliminate the term 10^{-x}).

$$10^x(10^x) - 10^x(5) + 10^x(6(10^{-x})) = 0$$

or

$$10^{2x} - 5(10^x) + 6 = 0 \qquad \text{since} \qquad 10^x(10^{-x}) = 10^0 = 1$$

It is not obvious, at first sight how we might solve the equation $10^{2x} - 5(10^x) + 6 = 0$. However, after a little thought, we realise that this expression is a *quadratic expression*. Let us first put $y = 10^x$ to give

$$y^2 - 5y + 6 = 0$$

in which we have used $10^{2x} = 10^x(10^x) = y^2$. Now, we are familiar with the solution for $y^2 - 5y + 6 = 0$. We can factorise to give

$$(y-3)(y-2) = 0 \qquad \text{so that} \qquad y = 3 \quad \text{or} \quad y = 2$$

For each of these values of y we obtain a separate value for x since $y = 10^x$.

case 1 If $y = 3$ then $3 = 10^x$ implying $x = \log 3 = 0.4771212$

case 2 If $y = 2$ then $2 = 10^x$ implying $x = \log 2 = 0.3010300$

We conclude that the equation $10^x - 5 + 6(10^{-x}) = 0$ has two possible solutions for x: either $x = 0.4771212$ or $x = 0.3010300$.

Solve $2e^{2x} - 7e^x + 3 = 0$.

First write your expression as a quadratic in the variable $y = e^x$:

Your solution

If $y = e^x$ then $2e^{2x} - 7e^x + 3 = 0$ becomes

$2y^2 - 7y + 3 = 0$

Now find the solution to the quadratic for y:

Your solution

$2y^2 - 7y + 3 = 0$ implies $(2y \qquad)(y \qquad) = 0$

$(2y-1)(y-3) = 0$ therefore $y = \frac{1}{2}$ or $y = 3$

For each of your values of y, find x.

Your solution

If $y = \frac{1}{2}$ then $\frac{1}{2} = e^x$ implies $x =$

If $y = 3$ then $3 = e^x$ implies $x =$

$x = -0.693147$ or $x = 1.0986123$

As a final example we consider equations involving the hyperbolic functions.

Example Solve the equations (a) $\cosh 3x = 1$ (b) $2\cosh^2 x = 3\cosh 2x - 3$

Solution

(a)

$$\cosh 3x = 1 \qquad \text{implies} \qquad \frac{e^{3x} + e^{-3x}}{2} = 1 \qquad \text{or} \qquad e^{3x} + e^{-3x} - 2 = 0$$

Now multiply through by e^{3x} (to eliminate the term e^{-3x}) to give

$$e^{3x}e^{3x} + e^{3x}e^{-3x} - 2e^{3x} = 0 \qquad \text{or} \qquad e^{6x} - 2e^{3x} + 1 = 0$$

This is a quadratic equation in the variable $y = e^{3x}$ so

$$y^2 - 2y + 1 = 0 \qquad \text{implying} \qquad (y-1)^2 = 0 \quad \text{or} \quad y = 1$$

$$\text{so} \qquad e^{3x} = 1 \qquad \text{implying} \qquad x = \frac{1}{3}\ln 1 = 0$$

(b) We first simplify this expression by using the identity: $\cosh 2x = 2\cosh^2 - 1$. Thus the original equation $2\cosh^2 x = 3\cosh 2x - 3$ becomes $\cosh 2x + 1 = 3\cosh 2x - 3$ or, when written in terms of exponentials:

$$\frac{e^{2x} + e^{-2x}}{2} = 3(\frac{e^{2x} + e^{-2x}}{2}) - 4$$

Therefore, multiplying through by $2e^{2x}$ gives $e^{4x} + 1 = 3(e^{4x} + 1) - 8e^{2x}$ or, after simplifying:

$$e^{4x} - 4e^{2x} + 1 = 0$$

Writing this quadratic in the variable y where $y = e^{2x}$ we easily obtain $y^2 - 4y + 1 = 0$ with solution (using the quadratic formula):

$$y = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

so, either $y = 2 + \sqrt{3}$ or $y = 2 - \sqrt{3}$.

If $y = 2 + \sqrt{3}$ then $2 + \sqrt{3} = e^{2x}$ implying $x = 0.65848$

If $y = 2 - \sqrt{3}$ then $2 - \sqrt{3} = e^{2x}$ implying $x = -0.65848$

Find the solution for x if $\tanh x = 0.5$.

First re-write $\tanh x$ in terms of exponentials.

Your solution

$\tanh x =$

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$

Now substitute into $\tanh x = 0.5$

Your solution

$\tanh x = 0.5$ implies $\dfrac{e^{2x} - 1}{e^{2x} + 1} = 0.5$ so, on simplifying, $e^{2x} =$

$\frac{e^{2x}-1}{e^{2x}+1} = 0.5$ implies $(e^{2x} - 1) = \frac{1}{2}(e^{2x} + 1)$ so $\frac{e^{2x}}{2} = \frac{3}{2}$ so, finally, $e^{2x} = 3$

Now complete your solution by finding x.

Your solution

$e^{2x} = 3 \quad$ so $\quad x = \frac{1}{2}\ln 3 =$

$x = 0.549306$

Alternatively, many calculators have a button allowing for the direct calculation of the inverse function $\tanh^{-1}$. If you have such a calculator then

$$\tanh x = 0.5 \qquad \text{implies} \qquad x = \tanh^{-1} 0.5 = 0.549306$$

Example Solve for x if $\quad 3\ln x + 4\log x = 1$.

First express each logarithm in terms of logs to the *same* base, e say. Now

$$\log x = \frac{\ln x}{\ln 10}$$

So $3\ln x + 4\log x = 1$ becomes

$$3\ln x + 4\frac{\ln x}{\ln 10} = 1 \qquad \text{or} \qquad (3 + \frac{4}{\ln 10})\ln x = 1$$

leading to $\ln x = \frac{\ln 10}{3\ln 10 + 4} = \frac{2.302585}{10.907755} = 0.211096$ and so

$$x = e^{0.211096} = 1.2350311$$

Exercises

1. Solve for the variable x: (a) $\pi = 10^x$ (b) $10^{-x/2} = 3$ (c) $\frac{1}{17-\pi^x} = 4$

2. Solve the equations
(a) $e^{2x} = 17e^x$, (b) $e^{2x} - 2e^x - 6 = 0$, (c) $\cosh x = 3$.

Answers 1. (a) $x = \log \pi = 0.497$
(b) $-x/2 = \log 3$ and so $x = -2\log 3 = -0.954$
(c) $17 - \pi^x = 0.25$ so $\pi^x = 16.75$ therefore $x = \frac{\log 16.75}{\log \pi} = \frac{1.224}{0.497} = 2.462$

2. (a) Take logs of both sides: $2x = \ln 17 + x \quad \therefore \quad x = \ln 17 = 2.833$
(b) Let $y = e^x$ then $y^2 - 2y - 6 = 0$ therefore $y = 1 \pm \sqrt{7}$ (we cannot take the negative sign since exponentials can never be negative). Thus $x = \ln(1+\sqrt{7}) = 1.2936$.
(c) $e^x + e^{-x} = 6$ therefore $e^{2x} - 6e^x + 1 = 0$ so $e^x = \frac{6\pm\sqrt{36-4}}{2} = 3 \pm \sqrt{8}$
We have, finally $x = \ln(3+\sqrt{8}) = 1.7627$ or $x = \ln(3-\sqrt{8}) = -1.7627$