Solving inequalities

3.5

Introduction

An inequality is an expression involving one of the symbols $\geq, \leq, >$ or $<$. This section will first show how to manipulate inequalities correctly. Then the solution of inequalities, both algebraically and graphically, will be described.

Prerequisites

Before starting this Section you should ...

- be able to solve linear equations

Learning Outcomes

After completing this Section you should be able to ...

✓ re-arrange expressions involving inequalities

✓ solve inequalities

1. The inequality symbols

Recall the meaning of the following symbols:

Key Point

The symbols $>$, $<$, $\geq$, $\leq$ are called **inequalities**

$>$ means: 'is greater than', $\geq$ means: 'is greater than or equal to'

$<$ means: 'is less than', $\leq$ means: 'is less than or equal to'

So, we may state, for example,

$$8 > 7 \qquad 9 \geq 2 \qquad -2 < 3 \qquad 7 \leq 7$$

A number line is often a helpful way of picturing inequalities. Given two numbers a and b, if $b > a$ then b will be to the right of a on the number line as shown in Figure 1.

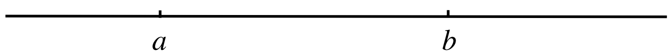

Figure 1. If $b > a$, b will be to the right of a on the number line.

Note from Figure 2 that $-3 > -5$, $4 > -2$ and $8 > 5$.

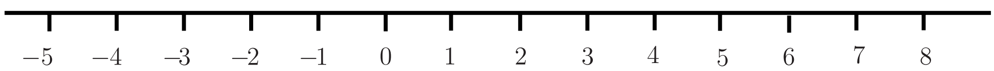

Figure 2

Inequalities can always be written in two ways. For example in English we can state that 8 is greater than 7, or equivalently, that 7 is less than 8. Mathematically we write $8 > 7$ or $7 < 8$. Similarly if $b > a$ then $a < b$. If $a < b$ then a will be to the left of b on the number line.

Example Rewrite the inequality $-\frac{2}{5} < x$ using only the 'greater than' sign, $>$.

Solution

$-\frac{2}{5} < x$ can be written as $x > -\frac{2}{5}$

Example Rewrite the inequality $5 > x$ using only the 'less than' sign, $<$.

Solution

$5 > x$ can be written as $x < 5$.

Sometimes two inequalities are combined into a single statement. Consider for example the statement $3 < x < 6$. This is a compact way of writing '$3 < x$ and $x < 6$'. Now $3 < x$ is equivalent to $x > 3$ and so $3 < x < 6$ means x is greater than 3 but less than 6.

Inequalities obey simple rules when used in conjunction with arithmetical operations.

Key Point

1. Adding or subtracting the same quantity from both sides of an inequality leaves the inequality sign unchanged.
2. Multiplying or dividing both sides by a **positive** number leaves the inequality sign unchanged.
3. Multiplying or dividing both sides by a **negative** number reverses the inequality.

For example, since $8 > 5$, by adding k to both sides we can state

$$8 + k > 5 + k$$

for any value of k. For example (with $k = -3$) $8 - 3 > 5 - 3$.
Further, by multiplying both sides of $8 > 5$ by k we can state

$$8k > 5k$$

provided k is positive.
However

$$8k < 5k$$

if k is negative. We emphasise that the inequality sign is reversed when multiplying both sides by a negative number. A common mistake is to forget to reverse the inequality sign when multiplying inequalities by a negative number. For example $8 > 5$, but multiplying both sides by -1 gives $-8 < -5$.

Find the result of multiplying both sides of the inequality $-18 < 9$ by the number -3.

Your solution

$54 > -27$

The **modulus or magnitude sign** is sometimes used with inequalities. For example $|x| < 1$ represents the set of all numbers whose actual size, irrespective of sign, is less than 1. This means any value between -1 and 1. Thus

$$|x| < 1 \text{ implies } -1 < x < 1$$

Similarly $|x| > 4$ means all numbers whose size, irrespective of sign, is greater than 4. This means any value greater than 4 or less than -4. Thus

$$|x| > 4 \text{ implies } x > 4 \text{ or } x < -4$$

In general, if k is a positive number,

Key Point

$$|x| < k \quad \text{means} \quad -k < x < k$$

$$|x| > k \quad \text{means} \quad x > k \quad \text{or} \quad x < -k$$

Exercises

1. State which of the following statements is true and which is false.

(a) $4 > 9$, (b) $4 > 4$, (c) $4 \geq 4$, (d) $0.001 < 10^{-5}$, (e) $|-19| < 100$,
(f) $|-19| > -20$, (g) $0.001 \leq 10^{-3}$

In questions 2-9 rewrite each of the statements without using a modulus sign:

2. $|x| < 2$, 3. $|x| < 5$, 4. $|x| \leq 7.5$, 5. $|x - 3| < 2$,
6. $|x - a| < 1$, 7. $|x| > 2$, 8. $|x| > 7.5$, 9. $|x| \geq 0$.

Answers 1. (a) F (b) F (c) T (d) F (e) T (f) T (g) T

2. $-2 < x < 2$ 3. $-5 < x < 5$ 4. $-7.5 \leq x \leq 7.5$

5. $-2 < x - 3 < 2$ 6. $-1 < x - a < 1$ 7. $x > 2$ or $x < -2$

8. $x > 7.5$ or $x < -7.5$ 9. $x \geq 0$ or $x \leq 0$, in fact any x.

2. Solving linear inequalities algebraically

When we are asked to **solve** an inequality, the inequality will contain an unknown variable, say x. Solving means obtaining all values of x for which the inequality is true. In a **linear inequality** the unknown appears only to the first power, that is as x, and not as x^2, x^3, $x^{1/2}$ and so on. It is possible to solve a linear inequality by making the unknown the subject. Consider the following examples.

Example Solve the inequality $4x + 3 > 0$.

Solution

$$\begin{aligned} 4x + 3 &> 0 \\ 4x &> -3, && \text{by subtracting 3 from both sides} \\ x &> -\frac{3}{4} && \text{by dividing both sides by 4.} \end{aligned}$$

Hence all values of x greater than $-\frac{3}{4}$ satisfy $4x + 3 > 0$.

Example Solve the inequality $-3x - 7 \le 0$.

Solution

$$\begin{aligned} -3x - 7 &\le 0 \\ -3x &\le 7 && \text{by adding 7 to both sides} \\ x &\ge -\frac{7}{3} && \text{dividing both sides by } -3 \\ & && \text{and reversing the inequality} \end{aligned}$$

Hence all values of x greater than or equal to $-\frac{7}{3}$ satisfy $-3x - 7 \le 0$.

Solve the inequality $17x + 2 < 4x + 1$.

We try to make x the subject and obtain it on its own on the left-hand side. Start by subtracting $4x$ from both sides to remove quantities involving x from the right.

Your solution

$13x + 2 < 1$

Now subtract 2 from both sides to remove the 2 on the left:

Your solution

$13x < -1$

Finally find the range of values satisfied by x:

Your solution

$x < -1/13$

Example Solve the inequality $|5x - 2| < 4$ and depict the solution graphically.

Solution

$$|5x - 2| < 4 \qquad \text{is equivalent to} \qquad -4 < 5x - 2 < 4$$

We treat each part of the inequality separately:

$$\begin{aligned} -4 &< 5x - 2 \\ -2 &< 5x \qquad \text{by adding 2 to both sides} \\ -\frac{2}{5} &< x \qquad \text{by dividing both sides by 5} \end{aligned}$$

So $x > -\frac{2}{5}$. Now consider the second part: $5x - 2 < 4$.

$$\begin{aligned} 5x - 2 &< 4 \\ 5x &< 6 \qquad \text{by adding 2 to both sides} \\ x &< \frac{6}{5} \qquad \text{by dividing both sides by 5} \end{aligned}$$

Putting both parts of the solution together we see that the inequality is satisfied when $-\frac{2}{5} < x < \frac{6}{5}$. This range of values is shown in Figure 3.

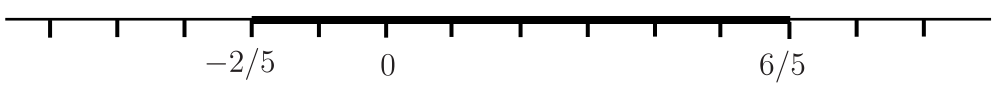

Figure 3. $|5x - 2| < 4$ when $\frac{2}{5} < x < \frac{6}{5}$

Solve the inequality $|1 - 2x| < 5$.
First of all rewrite the inequality without using the modulus sign.

Your solution

$|1 - 2x| < 5$ is equivalent to

$-5 < 1 - 2x < 5$

Then treat each part separately. First of all consider $-5 < 1 - 2x$. Solve this.

Your solution

$x < 3$

The second part is $1 - 2x < 5$. Solve this.

Your solution

$x > -2$

Finally, confirm that the solution is $-2 < x < 3$.

Exercises

In questions 1-16 solve the given inequality algebraically.

1. $4x > 8$ 2. $5x > 8$ 3. $8x > 5$ 4. $8x \leq 5$
5. $2x > 1$ 6. $3x < -1$ 7. $5x > 2$ 8. $2x > 0$
9. $8x < 0$ 10. $3x \geq 0$ 11. $3x > 4$ 12. $\frac{3}{4}x > 1$
13. $4x \leq -3$ 14. $3x \leq -4$ 15. $5x \geq 0$ 16. $4x \leq 0$

17. $5x + 1 < 8$ 18. $5x + 1 \leq 8$ 19. $7x + 3 \geq 0$
20. $18x + 2 > 9$ 21. $14x + 11 > 22$ 22. $1 - 5x \leq 0$
23. $2 + 5x \geq 1$ 24. $11 - 7x < 2$ 25. $5 + 4x > 2x + 1$
26. $7 - 3x > x - 5$

In questions 27-33 solve the inequality.

27. $|7x - 3| > 1$ 28. $|2x + 1| \geq 3$ 29. $|5x| < 1$ 30. $|5x| \leq 0$
31. $|1 - 5x| > 2$ 32. $|2 - 5x| \geq 3$ 33. $|2x - 1| < 1$

Answers

1. $x > 2$	2. $x > 8/5$	3. $x > 5/8$	4. $x \leq 5/8$
5. $x > 1/2$	6. $x < -1/3$	7. $x > 2/5$	8. $x > 0$
9. $x < 0$	10. $x \geq 0$	11. $x > 4/3$	12. $x > 4/3$
13. $x \leq -3/4$	14. $x \leq -4/3$	15. $x \geq 0$	16. $x \leq 0$
17. $x < 7/5$	18. $x \leq 7/5$	19. $x \geq -3/7$	20. $x > 7/18$
21. $x > 11/14$	22. $x \geq 1/5$	23. $x \geq -1/5$	24. $x > 9/7$
25. $x > -2$	26. $x < 3$	27. $x > 4/7$ or $x < 2/7$	28. $x \geq 1$ or $x \leq -2$
29. $-1/5 < x < 1/5$	30. $x = 0$	31. $x < -\frac{1}{5}$, $x > 3/5$	32. $x \leq -\frac{1}{5}$, $x \geq 1$
33. $0 < x < 1$			

3. Solving inequalities using graphs

Graphs can be used to help solve inequalities. This approach is particularly useful if the inequality is not linear as, in these cases solving the inequalities algebraically can often be very tricky. Graphics calculators or software can help save a lot of time and effort here.

Example Solve graphically the inequality $5x + 2 < 0$.

Solution

We consider the function $y = 5x + 2$ whose graph is shown in Figure 4.

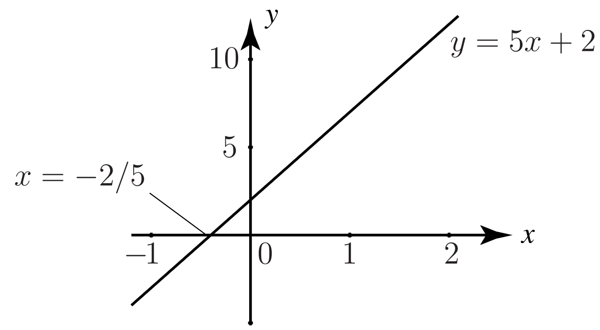

Figure 4. Graph of $y = 5x + 2$.

The values of x which make $5x + 2$ negative are those for which y is negative. We see directly from the graph that y is negative when $x < -\frac{2}{5}$.

Example Find the range of values of x for which $x^2 - x - 6 < 0$.

Solution

We consider the graph of $y = x^2 - x - 6$ which is shown in Figure 5.

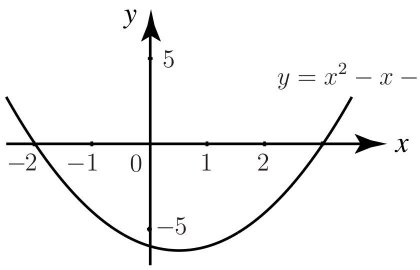

Figure 5. Graph of $y = x^2 - x - 6$.

Note that the graph crosses the x axis when $x = -2$ and when $x = 3$. Now $x^2 - x - 6$ will be negative when y is negative. Directly from the graph we see that y is negative when $-2 < x < 3$.

Find the range of values of x for which $x^2 - x - 6 > 0$.

The graph of $y = x^2 - x - 6$ has been drawn in Figure 5. We require $y = x^2 - x - 6$ to be positive. Use the graph to solve the problem.

Your solution

$x < -2$ or $x > 3$

Example By plotting a graph of $y = 20x^4 - 4x^3 - 143x^2 + 46x + 165$ find the range of values of x for which

$$20x^4 - 4x^3 - 143x^2 + 46x + 165 < 0$$

Solution

A software package has been used to plot the graph which is shown in Figure 6. We see that y is negative when $-2.5 < x < -1$ and when $1.5 < x < 2.2$.

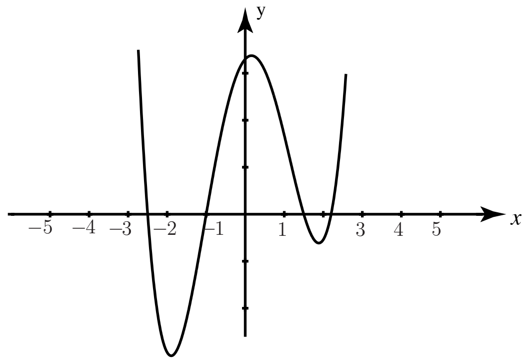

Figure 6. Graph of $y = 20x^4 - 4x^3 - 143x^2 + 46x + 165$

Exercises

In questions 1-4 solve the given inequality graphically:

1. $3x + 1 < 0$ 2. $2x - 7 < 0$ 3. $6x + 9 > 0$, 4. $5x - 3 > 0$

Answers 1. $x < -\frac{1}{3}$ 2. $x < 7/2$, 3. $x > -3/2$ 4. $x > 3/5$