Solving simultaneous linear equations

3.4

Introduction

Equations often arise in which there is more than one unknown quantity. When this is the case there will usually be more than one equation involved. For example in the two linear equations

$$7x + y = 9, \quad -3x + 2y = 1$$

there are two unknowns: x and y. In order to solve the equations we must find values for x and y that satisfy both of the equations simultaneously. The two equations are called **simultaneous equations**. You should verify that the solution of these equations is $x = 1$, $y = 2$ because by substituting these values into both equations, the left-hand and right-hand sides are equal.

In this section we shall show how two simultaneous equations can be solved either by a method known as **elimination** or by drawing graphs. In realistic problems which arise in mathematics and in engineering there may be many equations with many unknowns. Such problems cannot be solved using a graphical approach (we run out of dimensions in our 3-dimensional world!). Solving these more general problems will require the use of more general elimination procedures or the use of matrix algebra. Both of these topics are discussed in later workbooks.

Prerequisites

Before starting this Section you should . . .

- be able to solve linear equations

Learning Outcomes

After completing this Section you should be able to . . .

✓ solve pairs of simultaneous equations

1. Solving simultaneous equations by elimination

One way of solving simultaneous equations is by **elimination**. As the name implies, elimination, involves removing one of the unknowns. Note that if **both** sides of an equation are multiplied or divided by a non-zero number an equivalent equation results. For example, if we are given the equation

$$x + 4y = 5$$

then by multiplying both sides by 7, say, we find

$$7x + 28y = 35$$

and this modified equation is equivalent to the original one.
Given two simultaneous equations, elimination of one unknown can be achieved by modifying the equations so that the coefficients of that unknown in each equation are the same. By then subtracting one modified equation from the other that unknown is eliminated.
Consider the following example.

Example Solve the simultaneous equations

$$\begin{aligned} 3x + 5y &= 31 \qquad (1) \\ 2x + 3y &= 20 \qquad (2) \end{aligned}$$

Solution

We first try to modify each equation so that the coefficient of x is the same in both equations. This can be achieved if the Equation (1) is multiplied by 2 and the Equation (2) is multiplied by 3. This gives

$$\begin{array}{ccccc} 6x & + & 10y & = & 62 \\ 6x & + & 9y & = & 60 \end{array}$$

Now the unknown x can be removed (or eliminated) if the second equation is subtracted from the first:

$$\begin{array}{rcccccc} & 6x & + & 10y & = & 62 \\ \text{subtract} & 6x & + & 9y & = & 60 \\ \hline & 0x & + & 1y & = & 2 \end{array}$$

The result implies that $1y = 2$ and we see immediately that y must equal 2. To find x we substitute the value found for y into either of the given equations (1) or (2). For example, using Equation (1),

$$\begin{aligned} 3x + 5(2) &= 31 \\ 3x &= 21 \\ x &= 7 \end{aligned}$$

Thus the solution of the given equations is $x = 7$, $y = 2$. You should always check your solution by substituting back into both of the given equations.

Example Solve the equations

$$
\begin{aligned}
-3x + y &= 18 \qquad (3)\\
7x - 3y &= -44 \qquad (4)
\end{aligned}
$$

Solution

We modify the equations so that x can be eliminated. For example, by multiplying Equation (3) by 7 and Equation (4) by 3 we find

$$
\begin{array}{rcrcr}
-21x & + & 7y & = & 126\\
21x & - & 9y & = & -132
\end{array}
$$

If these equations are now added we can eliminate x. Therefore

$$
\begin{array}{rrcrcr}
 & -21x & + & 7y & = & 126\\
\text{add} & 21x & - & 9y & = & -132\\
\hline
 & 0x & - & 2y & = & -6
\end{array}
$$

from which $-2y = -6$, so that $y = 3$. Substituting this value of y into Equation (3) we obtain:

$$-3x + 3 = 18 \qquad \text{so that} \qquad -3x = 15$$

that is $x = -5$

Example Solve the equations

$$
\begin{array}{rcrcr}
5x & + & 3y & = & -74\\
-2x & - & 3y & = & 26
\end{array}
$$

Solution

Note that the coefficients of y differ here only in sign. By adding the two equations we find $3x = -48$ so that $x = -16$. It then follows that $y = 2$.

Solve the equations

$$
\begin{aligned}
5x - 7y &= -80 \qquad (5)\\
2x + 11y &= 106 \qquad (6)
\end{aligned}
$$

First modify the equations so that the coefficient of x is the same in both. This means that if Equation (5) is multiplied by 2 then Equation (6) must be multiplied by

Your solution

5

Write down the resulting equations:

Your solution

$10x - 14y = -160$, $10x + 55y = 530$

Subtract these to eliminate x and hence find y

Your solution

$y =$

10

Finally verify that $x = -2$.

2. Equations with no solution

On occasions we may encounter a pair of simultaneous equations which have no solution. Consider the following example.

Example Show that the following equations have no solution.

$$10x - 2y = -3 \qquad (7)$$
$$-5x + y = 1 \qquad (8)$$

Solution

Leaving Equation (7) unaltered and multiplying Equation (8) by 2 we find

$$\begin{array}{rcrcr} 10x & - & 2y & = & -3 \\ -10x & + & 2y & = & 2 \end{array}$$

Adding these equations to eliminate x we find that y is eliminated as well:

$$\begin{array}{rrcrcr} & 10x & - & 2y & = & -3 \\ \text{add} & -10x & + & 2y & = & 2 \\ \hline & 0x & + & 0y & = & -1 \end{array}$$

The last line '$0 = -1$' is clearly nonsense. We say that equations (7) and (8) are **inconsistent** and they have no solution.

3. Equations with an infinite number of solutions

Some pairs of simultaneous equations can possess an infinite number of solutions. Consider the following example.

Example Solve the equations

$$\begin{aligned} 2x + y &= 8 \qquad (9) \\ 4x + 2y &= 16 \qquad (10) \end{aligned}$$

Solution

If Equation (9) is multiplied by 2 we find both equations are identical: $4x + 2y = 16$. This means that one of them is redundant and we need only consider the single equation

$$2x + y = 8$$

There are infinitely many pairs of values of x and y which satisfy this equation. For example, if $x = 0$ $y = 8$. Similarly, if $x = 1$ $y = 6$, and if $x = -3$, $y = 14$. We could continue like this producing more and more solutions. Suppose we choose a value, say λ, for x. We can then write

$$2\lambda + y = 8 \qquad \text{so that} \qquad y = 8 - 2\lambda$$

The solution is therefore $x = \lambda$, $y = 8 - 2\lambda$ for any value of λ whatsoever. There are an infinite number of such solutions.

Exercises

Solve the given simultaneous equations by elimination:

1. (a) $5x + y = 8$, $-3x + 2y = -10$, (b) $2x + 3y = -2$, $5x - 5y = 20$, (c) $7x + 11y = -24$, $-9x + y = 46$
2. A straight line has equation of the form $y = ax + b$. The line passes through the points with coordinates $(2, 4)$ and $(-1, 3)$. Write down the simultaneous equations which must be satisfied by a and b. Solve the equations and hence find the equation of the line.
3. A quadratic function $y = ax^2 + bx + c$ is used in signal processing to approximate a more complicated signal. If this function must pass through the points with coordinates $(0, 0)$, $(1, 3)$ and $(5, -11)$ write down the simultaneous equations satisfied by a, b and c. Solve these to find the quadratic function.

Answers 1.(a) $x = 2, y = -2$ (b) $x = 2, y = -2$ (c) $x = -5, y = 1$

2. $y = \frac{1}{3}x + \frac{10}{3}$ 3. $y = -\frac{13}{10}x^2 + \frac{43}{10}x$

4. The graphs of simultaneous linear equations

We are aware that each of the equations in a pair of simultaneous linear equations is a linear equation and plotting its graph will produce a straight line. The coordinates (x, y) of the point of intersection of the two lines represent the solution of the equations as this pair of values satisfy both equations simultaneously. If the two lines do not intersect then the equations have no solution (this can only happen if they are distinct and parallel). If the two lines are identical, there are an infinite number of solutions (all points on the line). Although not the most convenient (or accurate) approach it is possible to solve simultaneous equations using this graphical approach. Consider the following examples.

Example Solve the simultaneous equations

$$\begin{aligned} 4x + y &= 9 && (11) \\ -x + y &= -1 && (12) \end{aligned}$$

by plotting two straight line graphs.

Solution

Equation (11) is rearranged into the standard form for the equation of a straight line: $y = -4x + 9$. By selecting two points on the line a graph can be drawn as shown in Figure 1. Similarly, Equation (12) can be rearranged as $y = x - 1$ and its graph drawn. This is also shown in Figure 1.

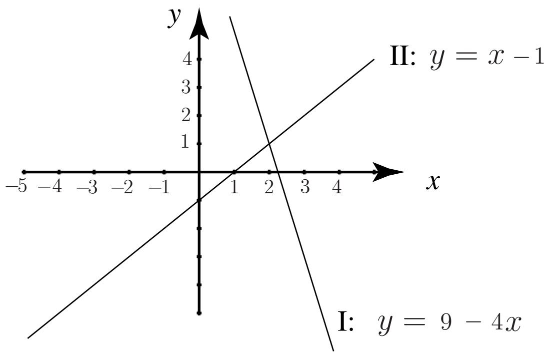

Figure 1. The coordinates of the point of intersection give the required solution

The coordinates of any point on line I satisfy $4x + y = 9$. The coordinates of any point on line II satisfy $-x+y = -1$. At the point where the two lines intersect the x and y coordinates must satisfy both equations simultaneously and so the point of intersection represents the solution. We see from the graph that the point of intersection is $(2, 1)$. The solution of the given equations is therefore $x = 2$, $y = 1$.

Example Solve the equations: $10x - 2y = -3$, $5x - y = -1$.

Solution

Re-writing the equations in standard form we find

$$y = 5x + \frac{3}{2}, \qquad \text{and} \qquad y = 5x + 1$$

Graphs of these lines are shown in Figure 2. Note that these distinct lines are parallel and so do not intersect. This means that the given simultaneous equations do not have a solution; they are inconsistent.

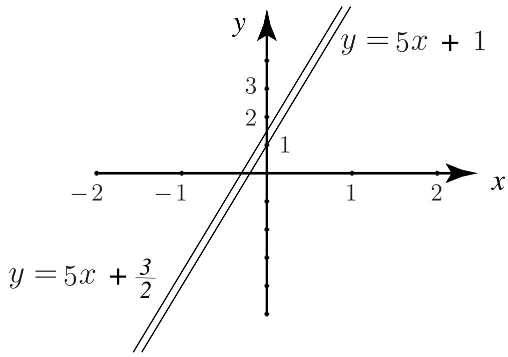

Figure 2.

Exercises

Solve the given equations graphically:

1. $5x - y = 7$, $2x + y = 7$,
2. $2x - 2y = -2$, $5x + y = -9$,
3. $7x + 3y = 25$, $-2x + y = 4$,
4. $4x + 4y = -4$, $x + 7y = -19$.

Answers 1. $x = 2, y = 3$ 2. $x = -5/3, y = -2/3$ 3. $x = 1, y = 6$ 4. $x = 2, y = -3$