Partial fractions

3.6

Introduction

It is often helpful to break down a complicated algebraic fraction into a sum of simpler fractions. For example it can be shown that $\frac{4x+7}{x^2+3x+2}$ has the same value as $\frac{1}{x+2}+\frac{3}{x+1}$ for any value of x. We say that

$$\frac{4x+7}{x^2+3x+2} \quad \text{is identically equal to} \quad \frac{1}{x+2}+\frac{3}{x+1}$$

and that the **partial fractions** of $\frac{4x+7}{x^2+3x+2}$ are $\frac{1}{x+2}$ and $\frac{3}{x+1}$. The ability to express a fraction as its partial fractions is particularly useful in the study of Laplace transforms, of z-transforms, in Control Theory and in integration. In this section we explain how partial fractions are found.

Prerequisites

Before starting this Section you should ...

- be familiar with addition, subtraction, multiplication and division of algebraic fractions

Learning Outcomes

After completing this Section you should be able to ...

✓ understand the distinction between proper and improper fractions

✓ express an algebraic fraction as the sum of its partial fractions

1. Proper and improper fractions

Frequently we find that an algebraic fraction appears in the form

$$\text{algebraic fraction} = \frac{\text{numerator}}{\text{denominator}}$$

where both numerator and denominator are polynomials. For example

$$\frac{x^3 + x^2 + 3x + 7}{x^2 + 1}, \qquad \frac{3x^2 - 2x + 5}{x^2 - 7x + 2}, \qquad \text{and} \qquad \frac{x}{x^4 + 1},$$

The **degree** of the numerator, n say, is the highest power occurring in the numerator. The degree of the denominator, d say, is the highest power occurring in the denominator. If $d > n$ the fraction is said to be **proper**; the third expression above is such an example. If $d \le n$ the fraction is said to be **improper**; the first and second expressions above are examples of this type. Before calculating the partial fractions of an algebraic fraction it is important to decide whether the fraction is proper or improper.

For each of the following fractions state the degrees of the numerator and denominator. Hence classify the fractions as proper or improper.

a) $\dfrac{x^3 + x^2 + 3x + 7}{x^2 + 1}$, b) $\dfrac{3x^2 - 2x + 5}{x^2 - 7x + 2}$, c) $\dfrac{x}{x^4 + 1}$, d) $\dfrac{s^2 + 4s + 5}{(s^2 + 2s + 4)(s + 3)}$

Your solution

a) The degree of the numerator, n, is 3. The degree of the denominator, d, is 2. Because $d \le n$ the fraction is improper.

b) Here $n = 2$ and $d = 2$. State whether this fraction is proper or improper.

Your solution

$d \le n$; the fraction is improper.

c) Noting that $x = x^1$ we see that $n = 1$ and $d = 4$. State whether this fraction is proper or improper.

Your solution

$d > n$; the fraction is proper.

d) Find the degree of the numerator and denominator.

Your solution

Removing the brackets in the denominator we see that it has degree 3. The degree of the numerator is 2 and so this fraction is proper.

Exercises

1. For each fraction state the degrees of the numerator and denominator, and hence determine which are proper and which are improper.

   a) $\frac{x+1}{x}$, b) $\frac{x^2}{x^3-x}$, c) $\frac{(x-1)(x-2)(x-3)}{x-5}$

Answers 1. a) $n=1, d=1$, improper, b) $n=2, d=3$, proper, c) $n=3, d=1$, improper.

The denominator of an algebraic fraction can often be factorised into a product of linear and quadratic factors. Before we can separate algebraic fractions into simpler (partial) fractions we will completely factorise the denominators into linear and quadratic factors. Linear factors are those of the form $ax+b$; for example $2x+7$, $3x-2$ and $4-x$. Quadratic factors are those of the form ax^2+bx+c such as x^2+x+1, and $4x^2-2x+3$, which cannot be factorised into linear factors (these are quadratics with complex roots).

2. Proper fractions with linear factors

Firstly we describe how to calculate partial fractions for proper fractions where the denominator may be written as a product of linear factors. The steps are as follows:

- Factorise the denominator.
- Each factor will produce a partial fraction. A factor such as $3x+2$ will produce a partial fraction of the form $\frac{A}{3x+2}$ where A is an unknown constant. In general a linear factor $ax+b$ will produce a partial fraction $\frac{A}{ax+b}$. The unknown constants for each partial fraction may be different and so we will call them A, B, C and so on.
- Evaluate the unknown constants by equating coefficients or using specific values of x.

The sum of the partial fractions is identical in value to the original algebraic fraction for any value of x.

Key Point

A linear factor $ax+b$ in the denominator gives rise to a single partial fraction of the form

$$\frac{A}{ax+b}$$

.

The steps involved are illustrated in the following example.

Example Express $\frac{7x+10}{2x^2+5x+3}$ in terms of partial fractions.

Solution

Note that this fraction is proper. The denominator is factorised to give $(2x+3)(x+1)$. Each of the linear factors produces a partial fraction. The factor $2x+3$ produces a partial fraction of the form $\frac{A}{2x+3}$. The factor $x+1$ produces a partial fraction $\frac{B}{x+1}$, where A and B are constants which we now try to find. We write

$$\frac{7x+10}{(2x+3)(x+1)} = \frac{A}{2x+3} + \frac{B}{x+1}$$

By multiplying both sides by $(2x+3)(x+1)$ we obtain

$$7x+10 \quad = \quad A(x+1) + B(2x+3) \qquad (1)$$

We may now let x take any value we choose. By an appropriate choice we can simplify the right-hand side. Let $x = -1$ because this choice eliminates A. We find

$$\begin{aligned} 7(-1)+10 &= A(0) + B(-2+3) \\ 3 &= B \end{aligned}$$

so that the constant B must equal 3. The constant A can be found by substituting other values for x or alternatively by equating coefficients. Observe that, by rearranging the right-hand side, Equation (1) can be written as

$$7x + 10 = (A+2B)x + (A+3B)$$

Comparing the coefficients of x on both sides we see that $7 = A+2B$. We already know $B = 3$ and so

$$\begin{aligned} 7 &= A + 2(3) \\ &= A + 6 \end{aligned}$$

from which $A = 1$. We can therefore write

$$\frac{7x+10}{2x^2+5x+3} = \frac{1}{2x+3} + \frac{3}{x+1}$$

We have succeeded in expressing the given fraction as the sum of its partial fractions. The result can always be checked by adding the fractions on the right.

Express $\frac{9-4x}{3x^2-x-2}$ in partial fractions.

First factorise the denominator:

Your solution

$3x^2 - x - 2 =$

$(3x+2)(x-1)$

Because there are two linear factors we write

$$\frac{9-4x}{3x^2-x-2} = \frac{9-4x}{(3x+2)(x-1)} = \frac{A}{3x+2} + \frac{B}{x-1}$$

Multiply both sides by $(3x+2)(x-1)$ to obtain the equation from which we can find values for A and B.

Your solution

$9 - 4x =$

$9 - 4x = A(x-1) + B(3x+2)$

By substituting an appropriate value for x obtain B.

Your solution

substitute $x = 1$ and get $B = 1$

Finally by equating coefficients of x obtain the value of A.

Your solution

$-4 = A + 3B$, $A = -7$ since $B = 1$

Finally, write down the partial fractions:

Your solution

$\dfrac{9-4x}{3x^2-x-2} =$

$\frac{-7}{3x+2} + \frac{1}{x-1}$

Exercises

1. (a) Find the partial fractions of $\frac{5x-1}{(x+1)(x-2)}$.
   (b) Check your answer by adding the partial fractions together again.
   (c) Express in partial fractions $\frac{7x+25}{(x+4)(x+3)}$.
   (d) Check your answer by adding the partial fractions together again.

2. Find the partial fractions of $\frac{11x+1}{(x-1)(2x+1)}$.

   Express each of the following as the sum of partial fractions:

3. (a) $\frac{3}{(x+1)(x+2)}$, (b) $\frac{5}{x^2+7x+12}$, (c) $\frac{-3}{(2x+1)(x-3)}$,

Answers

1. $\frac{2}{x+1} + \frac{3}{x-2}$,   c) $\frac{3}{x+4} + \frac{4}{x+3}$

2. $\frac{4}{x-1} + \frac{3}{2x+1}$,

3. (a) $\frac{3}{x+1} - \frac{3}{x+2}$, (b) $\frac{5}{x+3} - \frac{5}{x+4}$, (c) $\frac{6}{7(2x+1)} - \frac{3}{7(x-3)}$.

3. Proper fractions with repeated linear factors

Sometimes a linear factor appears more than once. For example in

$$\frac{1}{x^2+2x+1} = \frac{1}{(x+1)(x+1)} \quad \text{which equals} \quad \frac{1}{(x+1)^2}$$

the factor $(x+1)$ occurs twice. We call it a **repeated linear factor**. The repeated linear factor $(x+1)^2$ produces two partial fractions of the form $\frac{A}{x+1} + \frac{B}{(x+1)^2}$. In general, a repeated linear factor of the form $(ax+b)^2$ generates two partial fractions of the form

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

This is reasonable since the sum of two such fractions **always** gives rise to a **proper** fraction:

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2} = \frac{A(ax+b)}{(ax+b)^2} + \frac{B}{(ax+b)^2} = \frac{x(Aa)+Ab+B}{(ax+b)^2}$$

Key Point

A repeated linear factor $(ax+b)^2$ in the denominator produces two partial fractions:

$$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

Once again the unknown constants are found by either equating coefficients and/or substituting specific values for x.

Express $\dfrac{10x+18}{4x^2+12x+9}$ in partial fractions.

First the denominator is factorised.

Your solution

$4x^2+12x+9 =$

$(2x+3)(2x+3) = (2x+3)^2$

We have found a repeated linear factor.

The repeated linear factor $(2x+3)$ gives rise to two partial fractions of the form

$$\frac{10x+18}{(2x+3)^2} = \frac{A}{2x+3} + \frac{B}{(2x+3)^2}$$

Multiply both sides through by $(2x+3)^2$ to obtain the equation which must be solved to find A and B.

Your solution

$10x+18 = A(2x+3) + B$

Now evaluate the constants A and B by equating coefficients. Equating coefficients of x gives

Your solution

$10 = 2A$ so $A = 5$.

Equating constant terms gives $18 = 3A + B$ from which $B = 3$. So, finally, we may write

$$\frac{10x+18}{(2x+3)^2} = \frac{5}{2x+3} + \frac{3}{(2x+3)^2}$$

Exercises

Express the following in partial fractions.

(a) $\dfrac{3-x}{x^2-2x+1}$, (b) $-\dfrac{7x-15}{(x-1)^2}$ (c) $\dfrac{3x+14}{x^2+8x+16}$ (d) $\dfrac{5x+18}{(x+4)^2}$ (e) $\dfrac{2x^2-x+1}{(x+1)(x-1)^2}$

(f) $\dfrac{5x^2+23x+24}{(2x+3)(x+2)^2}$ (g) $\dfrac{6x^2-30x+25}{(3x-2)^2(x+7)}$ (h) $\dfrac{s+2}{(s+1)^2}$ (i) $\dfrac{2s+3}{s^2}$.

Answers (a) $-\frac{1}{x-1}+\frac{2}{(x-1)^2}$ (b) $-\frac{7}{x-1}+\frac{8}{(x-1)^2}$ (c) $\frac{3}{x+4}+\frac{2}{(x+4)^2}$

(d) $\frac{5}{x+4}-\frac{2}{(x+4)^2}$ (e) $\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{(x-1)^2}$ (f) $\frac{3}{2x+3}+\frac{1}{x+2}+\frac{2}{(x+2)^2}$

(g) $-\frac{1}{3x-2}+\frac{1}{(3x-2)^2}+\frac{1}{x+7}$ (h) $\frac{1}{s+1}+\frac{1}{(s+1)^2}$ (i) $\frac{2}{s}+\frac{3}{s^2}$.

4. Proper fractions with quadratic factors

Sometimes a denominator is factorised producing a quadratic term which cannot be factorised into linear factors. One such quadratic factor is x^2+x+1. This factor produces a partial fraction of the form $\frac{Ax+B}{x^2+x+1}$. In general a quadratic factor of the form ax^2+bx+c produces a single partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$.

Key Point

A quadratic factor ax^2+bx+c in the denominator produces a partial fraction of the form

$$\frac{Ax+B}{ax^2+bx+c}$$

Express as partial fractions $\frac{3x+1}{(x^2+x+10)(x-1)}$

Note that the quadratic factor cannot be factorised further. We have

$$\frac{3x+1}{(x^2+x+10)(x-1)}=\frac{Ax+B}{x^2+x+10}+\frac{C}{x-1}$$

Multiplying both sides by $(x^2+x+10)(x-1)$ gives

Your solution

$3x+1=$

$(Ax+B)(x-1)+C(x^2+x+10)$

To evaluate C we can let $x = 1$ which eliminates the first term on the right. This gives

$$4 = 12C \quad \text{so that} \quad C = \frac{1}{3}$$

Equate coefficients of x^2 and hence find A. Finally substitute any other value for x or equate coefficients of x to find B.

Your solution

$A =$ $\qquad\qquad$ $B =$

$-\frac{1}{3}$, $\frac{7}{3}$.

Finally
$$\frac{3x+1}{(x^2+x+10)(x-1)} = \frac{-\frac{1}{3}x+\frac{7}{3}}{x^2+x+10} + \frac{\frac{1}{3}}{x-1} = \frac{7-x}{3(x^2+x+10)} + \frac{1}{3(x-1)}$$

Example Admittance, Y, is a quantity which is used in analysing electronic circuits. A typical expression for admittance is

$$Y(s) = \frac{s^2+4s+5}{(s^2+2s+4)(s+3)}$$

where s can be thought of as representing frequency. To predict the behaviour of the circuit it is often necessary to express the admittance as the sum of its partial fractions and find the effect of each part separately. Express $Y(s)$ in partial fractions.

Solution

The fraction is proper. The denominator contains a quadratic factor which cannot be factorised further, and also a linear factor. Thus

$$\frac{s^2+4s+5}{(s^2+2s+4)(s+3)} = \frac{As+B}{s^2+2s+4} + \frac{C}{s+3}$$

Multiplying both sides by $(s^2+2s+4)(s+3)$ we obtain

$$s^2+4s+5 = (As+B)(s+3) + C(s^2+2s+4)$$

To find the constant C we can let $s = -3$ to eliminate A and B. Thus

$$(-3)^2 + 4(-3) + 5 = C((-3)^2 + 2(-3) + 4)$$

so that

$$2 = 7C \qquad \text{and so} \qquad C = \frac{2}{7}$$

Equating coefficients of s^2 we find

$$1 = A + C$$

so that $A = 1 - C = 1 - \frac{2}{7} = \frac{5}{7}$.

Solution

contd.

Equating constant terms gives $\quad 5 = 3B + 4C$

so that $\quad 3B = 5 - 4C = 5 - 4\left(\dfrac{2}{7}\right) = \dfrac{27}{7}$

so $\quad B = \dfrac{9}{7}$

Finally $\quad Y(s) = \dfrac{s^2 + 4s + 5}{(s^2 + 2s + 4)(s + 3)} = \dfrac{\frac{5}{7}s + \frac{9}{7}}{s^2 + 2s + 4} + \dfrac{\frac{2}{7}}{s + 3}$

which can be written as $\quad Y(s) = \dfrac{5s + 9}{7(s^2 + 2s + 4)} + \dfrac{2}{7(s + 3)}$

Exercises

Express each of the following as the sum of its partial fractions.

(a) $\dfrac{3}{(x^2 + x + 1)(x - 2)}$, (b) $\dfrac{27x^2 - 4x + 5}{(6x^2 + x + 2)(x - 3)}$, (c) $\dfrac{2x + 4}{4x^2 + 12x + 9}$, (d) $\dfrac{6x^2 + 13x + 2}{(x^2 + 5x + 1)(x - 1)}$

Answers (a) $\frac{3}{7(x-2)} - \frac{3(x+3)}{7(x^2+x+1)}$ (b) $\frac{3x+1}{6x^2+x+2} + \frac{4}{x-3}$ (c) $\frac{1}{2x+3} + \frac{1}{(2x+3)^2}$ (d) $\frac{3x+1}{x^2+5x+1} + \frac{3}{x-1}$.

5. Improper fractions

When calculating the partial fractions of improper fractions an extra polynomial is added to any partial fractions that would normally arise. The added polynomial has degree $n - d$ where d is the degree of the denominator and n is the degree of the numerator. Recall that

a polynomial of degree 0 is a constant, A say,

a polynomial of degree 1 has the form $Ax + B$,

a polynomial of degree 2 has the form $Ax^2 + Bx + C$,

and so on.

If, for example, the improper fraction is such that the numerator has degree 5 and the denominator has degree 3, then $n - d = 2$, and we need to add a polynomial of the form $Ax^2 + Bx + C$.

Key Point

If a fraction is improper an additional term is included taking the form of a polynomial of degree $n-d$, where n is the degree of the numerator and d is the degree of the denominator.

Example Express as partial fractions

$$\frac{2x^2-x-2}{x+1}$$

Solution

The fraction is improper because $n=2$, $d=1$ and so $d \le n$. Further, note that $n-d=1$. We therefore need to include an extra term: a polynomial of the form $Bx+C$, in addition to the usual partial fractions. The linear term in the denominator gives rise to a partial fraction $\frac{A}{x+1}$. So

$$\frac{2x^2-x-2}{x+1} = \frac{A}{x+1} + Bx + C$$

Multiplying both sides by $x+1$ we find

$$2x^2-x-2 = A + (Bx+C)(x+1) = Bx^2 + (C+B)x + (C+A)$$

Equating coefficients of x^2 gives $B=2$.
Equating coefficients of x gives $-1 = C+B$ and so $C = -1-B = -3$.
Equating the constant terms gives $-2 = C+A$ and so $A = -2-C = -2-(-3) = 1$.
Finally we have

$$\frac{2x^2-x-2}{x+1} = \frac{1}{x+1} + 2x - 3$$

Exercises

Express each of the following improper fractions in terms of partial fractions.

(a) $\dfrac{x+3}{x+2}$, (b) $\dfrac{3x-7}{x-3}$, (c) $\dfrac{x^2+2x+2}{x+1}$, (d) $\dfrac{2x^2+7x+7}{x+2}$

(e) $\dfrac{3x^5+4x^4-21x^3-40x^2-24x-29}{(x+2)^2(x-3)}$, (f) $\dfrac{4x^5+8x^4+23x^3+27x^2+25x+9}{(x^2+x+1)(2x+1)}$

Answers (a) $1+\frac{1}{x+2}$, (b) $3+\frac{2}{x-3}$, (c) $1+x+\frac{1}{x+1}$ (d) $2x+3+\frac{1}{x+2}$,
(e) $\frac{1}{(x+2)^2}+\frac{1}{x+2}+\frac{1}{x-3}+3x^2+x+2$, (f) $2x^2+x+7+\frac{1}{2x+1}+\frac{1}{x^2+x+1}$