Higher Derivatives

11.3

Introduction

The derivative, $\frac{dy}{dx}$, is more expressly called the **first derivative** of y. By differentiating the first derivative, we obtain the **second derivative**; by differentiating the second derivative we obtain the **third derivative** and so on. These second and subsequent derivatives are known as **higher derivatives**.

Prerequisites

Before starting this Section you should ...

- be able to differentiate standard functions

Learning Outcomes

After completing this Section you should be able to ...

✓ calculate second and other higher derivatives

1. The derivative of a derivative

You have already learnt how to calculate the derivative of a function using a table of derivatives. By differentiating the function, $y(x)$, we obtain the derivative, $\frac{dy}{dx}$. By repeating the process we can obtain higher derivatives.

Example Calculate the first, second and third derivatives of $y = x^4 + 6x^2$.

Solution

The first derivative is $\frac{dy}{dx}$.

$$\text{first derivative} \equiv \frac{dy}{dx} = 4x^3 + 12x$$

To obtain the second derivative we differentiate the first derivative.

$$\text{second derivative} = 12x^2 + 12$$

The third derivative is found by differentiating the second derivative.

$$\text{third derivative} = 24x + 0 = 24x$$

2. Notation

Just as there is a notation for the first derivative so there is a similar notation for higher derivatives. Consider the function, $y(x)$. We know that the first derivative is $\frac{dy}{dx}$ or $\frac{d}{dx}(y)$ which is the instruction to differentiate the function $y(x)$. The second derivative is calculated by differentiating the first derivative, that is

$$\text{second derivative} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

So, using a fairly obvious adaptation of our derivative notation, the second derivative is denoted by $\frac{d^2y}{dx^2}$ and is read as 'dee two y by dee x squared'. This is often written more concisely as y''.

In similar manner, the third derivative is denoted by $\frac{d^3y}{dx^3}$ or y''' and so on. So, referring to the Example above we could have written

$$\begin{aligned} \text{first derivative} = \frac{dy}{dx} &= 4x^3 + 12x \\ \text{second derivative} = \frac{d^2y}{dx^2} &= 12x^2 + 12 \\ \text{third derivative} = \frac{d^3y}{dx^3} &= 24x \end{aligned}$$

Key Point

If $y = y(x)$ then its first, second and third derivatives are, denoted respectively by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \qquad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}$$

In most examples we use x to denote the independent variable and y the dependent variable. However, in many applications, the time t is the independent variable. In this case a special notation is used for derivatives. Derivatives with respect to t are often indicated using a dot notation, so $\frac{\mathrm{d}y}{\mathrm{d}t}$ can be written as $\dot{y}$, pronounced 'y dot'. Similarly, a second derivative with respect to t can be written as $\ddot{y}$, pronounced 'y double dot'.

Key Point

$$\dot{y} \quad \text{stands for} \quad \frac{\mathrm{d}y}{\mathrm{d}t}, \qquad \ddot{y} \quad \text{stands for} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} \quad \text{etc}$$

Calculate $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^3y}{\mathrm{d}t^3}$ given $y = e^{2t} + \cos t$.

Since $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 2e^{2t} - \sin t$, you can now obtain higher derivatives:

Your solution

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} =$$

$4e^{2t} - \cos t$

Your solution

$$\frac{\mathrm{d}^3y}{\mathrm{d}t^3} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(4e^{2t} - \cos t\right) =$$

$8e^{2t} + \sin t$

We could have used the dot notation and written $\dot{y} = 2e^{2t} - \sin t$, and $\ddot{y} = 4e^{2t} - \cos t$.

We may need to evaluate higher derivatives at specific points. We use an obvious notation. The second derivative of $y(x)$, evaluated at say, $x = 2$, is written as $\frac{d^2y}{dx^2}(2)$, or more simply as $y''(2)$. The third derivative evaluated at $x = -1$ is written as $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}(-1)$ or $y'''(-1)$.

Given $\quad y(x) = 2\sin x + 3x^2 \quad$ find (a) $\; y'(1) \;$ (b) $\; y''(-1) \;$ (c) $\; y'''(0)$

We have $\qquad y = 2\sin x + 3x^2 \quad$ and $\quad y' = 2\cos x + 6x$

Your solution

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}(2\cos x + 6x) = \qquad\qquad y''' =$$

$y'' = -2\sin x + 6 \qquad y''' = -2\cos x$

Your solution

(a) $\quad y'(1) =$

(a) $y'(1) = 2\cos 1 + 6(1) = 7.0806$. Remember, in cos 1 the '1' is 1 radian

Your solution

(b) $\quad y''(-1) =$

$-2\sin(-1) + 6 = 7.6829$

Your solution

(c) $\quad y'''(0) =$

$y'''(0) = -2\cos 0 = -2$.

Exercises

1. Find $\frac{d^2y}{dx^2}$ where $y(x)$ is defined by:

   (a) $3x^2 - e^{2x}$ (b) $\sin 3x + \cos x$ (c) $\sqrt{x}$ (d) $e^x + e^{-x}$ (e) $1 + x + x^2 + \ln x$

2. Find $\frac{d^3y}{dx^3}$ where y is given in Q1.

3. Calculate $\ddot{y}(1)$ where $y(t)$ is given by:

   (a) $t(t^2+1)$ (b) $\sin(-2t)$ (c) $2e^t + e^{2t}$ (d) $\frac{1}{t}$ (e) $\cos \frac{t}{2}$

4. Calculate $\dddot{y}(-1)$ of the functions given in Q3.

Answers

1. (a) $6 - 4e^{2x}$ (b) $-9\sin 3x - \cos x$ (c) $-\frac{1}{4}x^{-3/2}$ (d) $e^x + e^{-x}$ (e) $2 - \frac{1}{x^2}$
2. (a) $-8e^{2x}$ (b) $-27\cos 3x + \sin x$ (c) $\frac{3}{8}x^{-5/2}$ (d) $e^x - e^{-x}$ (e) $\frac{2}{x^3}$
3. (a) 6 (b) 3.6372 (c) 34.9927 (d) 2 (e) -0.2194
4. (a) 6 (b) -3.3292 (c) 1.8184 (d) -6 (e) -0.0599