Using a table of derivatives

11.2

Introduction

In Section 1 you were introduced to the idea of a derivative and you calculated some derivatives from first principles. Rather than always calculate the derivative of a function from first principles it is common practise to use a **table of derivatives**. This Section provides such a table and shows you how to use it.

Prerequisites

Before starting this Section you should . . .

- understand the meaning of the term 'derivative'
- understand what is meant by the notation $\frac{dy}{dx}$

Learning Outcomes

After completing this Section you should be able to . . .

✓ use a table of derivatives

1. Table of derivatives

Table 1 lists some of the common functions used in engineering and their corresponding derivatives. Remember, that in each case the function in the right-hand column tell us the rate of change, or the gradient of the graph, of the function on the left at a particular value of x.

N.B. The angle must always be in radians when differentiating trigonometric functions.

Table 1
Common functions and their derivatives
(In this table k, n and c are constants)

Function	Derivative
constant	0
x	1
kx	k
x^n	nx^{n-1}
kx^n	knx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x$	$1/x$
$\ln kx$	$1/x$

For the following the angle is in radians

Function	Derivative
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\sin(kx + c)$	$k \cos(kx + c)$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\cos(kx + c)$	$-k \sin(kx + c)$
$\tan x$	$\sec^2 x$
$\tan kx$	$k \sec^2 kx$
$\tan(kx + c)$	$k \sec^2(kx + c)$

Key Point

Particularly important is the rule for differentiating powers of functions.

$$\text{If} \quad y = x^n \quad \text{then} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$$

For example, if $y = x^3$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

Example Use Table 1 to find $\frac{dy}{dx}$ when y is given by (a) $7x$ (b) 14 (c) $5x^2$ (d) $4x^7$

Solution

(a) We note that $7x$ is of the form kx where $k = 7$. Using Table 1 we then have $\frac{dy}{dx} = 7$.

(b) Noting that 14 is a constant we see that $\frac{dy}{dx} = 0$.

(c) We see that $5x^2$ is of the form kx^n, with $k = 5$ and $n = 2$. The derivative, knx^{n-1}, is then $10x^1$, or more simply, $10x$. So if $y = 5x^2$, then $\frac{dy}{dx} = 10x$.

(d) We see that $4x^7$ is of the form kx^n, with $k = 4$ and $n = 7$. Hence the derivative, $\frac{dy}{dx}$, is given by $28x^6$.

Use Table 1 to find $\dfrac{dy}{dx}$ when y is (a) $\sqrt{x}$ (b) $\dfrac{5}{x^3}$ (c) $\dfrac{7}{x}$

Your solution

(a) Write $\sqrt{x}$ as $x^{\frac{1}{2}}$, and use the result for differentiating x^n with $n = \frac{1}{2}$.

$\frac{dy}{dx} = nx^{n-1} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$. This may be written as $\frac{1}{2\sqrt{x}}$.

Your solution

(b) Write $\frac{5}{x^3}$ as $5x^{-3}$ and use the result for differentiating kx^n with $k = 5$ and $n = -3$.

$5(-3)x^{-3-1} = -15x^{-4}$

(c)

Your solution

$$\frac{dy}{dx} =$$

$7(-1)x^{-1-1} = -7x^{-2} = -\frac{7}{x^2}$

Use Table 1 to find $\frac{dz}{dt}$ given (a) $z = e^t$ (b) $z = e^{8t}$ (c) $z = e^{-3t}$

Although Table 1 is written using x as the independent variable, the Table can be used for any variable.

(a) From Table 1, if $y = e^x$, then $\frac{dy}{dx} = e^x$. Hence if $z = e^t$ then $\frac{dz}{dt} = e^t$.

(b) Using the result for e^{kx} in Table 1 we see that when $z = e^{8t}$,

Your solution

$$\frac{dz}{dt} =$$

$8e^{8t}$

(c) Using the result for e^{kx} in Table 1 we see that when $z = e^{-3t}$,

Your solution

$$\frac{dz}{dt} =$$

$-3e^{-3t}$

Find the derivative, $\frac{dy}{dx}$, when y is:- (a) $\sin 2x$ (b) $\cos\frac{x}{2}$ (c) $\tan 5x$

(a) Using the result for $\sin kx$ in Table 1, and taking $k = 2$, we see that when $y = \sin 2x$

Your solution

$$\frac{dy}{dx} =$$

(a) Using the result for $\sin kx$, and taking $k = 2$, we see that when $y = \cos\frac{x}{2}$ $\frac{dy}{dx} = 2\cos 2x$

(b) Note that $\cos\frac{x}{2}$ is the same as $\cos\frac{1}{2}x$. From the result for $\cos kx$ in Table 1, and taking $k = \frac{1}{2}$ we see that when $y = \cos\frac{x}{2}$

Your solution

$$\frac{dy}{dx} =$$

$-\frac{1}{2}\sin\frac{x}{2}$

(c) From the result for $\tan kx$ in Table 1, and taking $k = 5$ we see that when $y = \tan 5x$

Your solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} =$$

$5\sec^2 5x$

Exercises

1. Find the derivative of the following functions:

   (a) $9x^2$ (b) 5 (c) $6x^3$ (d) $-13x^4$ (e) $\ln 5t$

2. Find $\frac{dz}{dt}$ when z is given by

   (a) $\dfrac{5}{t^3}$ (b) $\sqrt{t^3}$ (c) $5t^{-2}$ (d) $-\frac{3}{2}t^{\frac{3}{2}}$

3. Find the derivative of each of the following functions

   (a) $\sin 5x$ (b) $\cos 4t$ (c) $\tan 3r$ (d) e^{2v} (e) $\frac{1}{e^{3t}}$

4. Find the derivative of the following

   (a) $\cos\frac{2x}{3}$ (b) $\sin(-2x)$ (c) $\tan \pi x$ (d) $e^{\frac{x}{2}}$ (e) $\ln\frac{2}{3}x$

1. (a) $18x$ (b) 0 (c) $18x^2$ (d) $-52x^3$ (e) $\frac{1}{t}$
2. (a) $-15t^{-4}$ (b) $\frac{3}{2}t^{\frac{1}{2}}$ (c) $-10t^{-3}$ (d) $-\frac{9}{4}t^{\frac{1}{2}}$
3. (a) $5\cos 5x$ (b) $-4\sin 4t$ (c) $3\sec^2 3r$ (d) $2e^{2v}$ (e) $-3e^{-3t}$
4. (a) $-\frac{2}{3}\sin\frac{2x}{3}$ (b) $-2\cos(-2x)$ (c) $\pi\sec^2 \pi x$ (d) $\frac{1}{2}e^{\frac{x}{2}}$ (e) $\frac{1}{x}$

2. Extending the Table of derivatives

We now quote simple rules which enable us to extend the range of functions which we can differentiate. The first two rules are for differentiating sums or differences of functions. The reader should note that all of the rules quoted below can be obtained from first principles using the approach outlined in Section 1.

Key Point

Rule 1: The derivative of $f(x) + g(x)$ is $\dfrac{\mathrm{d}f}{\mathrm{d}x} + \dfrac{\mathrm{d}g}{\mathrm{d}x}$

Rule 2: The derivative of $f(x) - g(x)$ is $\dfrac{\mathrm{d}f}{\mathrm{d}x} - \dfrac{\mathrm{d}g}{\mathrm{d}x}$

These rules say that to find the derivative of the sum (or difference) of two functions, we simply calculate the sum (or difference) of the derivatives of each function.

Example Find the derivative of $y = x^6 + x^4$.

Solution

We simply calculate the sum of the derivatives of each separate function:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 + 4x^3$$

The third rule tells us how to differentiate a multiple of a function. We have already met and applied particular cases of this rule which appear in Table 1.

Key Point

Rule 3: If k is a constant the derivative of $kf(x)$ is $k\dfrac{\mathrm{d}f}{\mathrm{d}x}$

This rule tells us that if a function is multiplied by a constant, k, then the derivative is also multiplied by the same constant, k.

Example Find the derivative of $y = 8e^{2x}$.

Solution

Here we are interested in differentiating a multiple of the function e^{2x}. We simply differentiate e^{2x} and multiply the result by 8. Thus

$$\frac{dy}{dx} = 8 \times 2e^{2x} = 16e^{2x}$$

Example Find the derivative of each of the following functions.

(a) $y = 6\sin 2x$ (b) $y = 6\sin 2x + 3x^2$ (c) $y = 6\sin 2x + 3x^2 - 5e^{3x}$

Solution

(a) From Table 1, the derivative of $\sin 2x$ is $2\cos 2x$. Hence the derivative of $6\sin 2x$ is $6(2\cos 2x)$, that is, $12\cos 2x$.

$$y = 6\sin 2x, \qquad \frac{dy}{dx} = 6(2\cos 2x) = 12\cos 2x$$

(b) The function is the sum of two terms: $6\sin 2x$ and $3x^2$; $6\sin 2x$ has already been differentiated to $12\cos 2x$, so we consider the derivative of $3x^2$. The derivative of x^2 is $2x$ and so the derivative of $3x^2$ is $3(2x)$, that is, $6x$. These derivatives are now summed:

$$y = 6\sin 2x + 3x^2 \qquad \frac{dy}{dx} = 12\cos 2x + 6x$$

(c) We differentiate each part of the function in turn.

$$\begin{aligned} y &= 6\sin 2x + 3x^2 - 5e^{3x} \\ \frac{dy}{dx} &= 6(2\cos 2x) + 3(2x) - 5(3e^{3x}) \\ &= 12\cos 2x + 6x - 15e^{3x} \end{aligned}$$

Find $\frac{dy}{dx}$ where $y = 7x^5 - 3e^{5x}$.

The derivative of x^5 is $5x^4$. Hence the derivative of $7x^5$ is

Your solution

$35x^4$

The derivative of e^{5x} is

Your solution

$5e^{5x}$

Hence the derivative of $3e^{5x}$ is

Your solution

$3(5e^{5x}) = 15e^{5x}$

So given $y = 7x^5 - 3e^{5x}$ then

Your solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} =$$

$35x^4 - 15e^{5x}$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ where $y = 4\cos\frac{x}{2} + 17 - 9x^3$.

The derivative of $\cos\frac{x}{2}$ is

Your solution

$-\frac{1}{2}\sin\frac{x}{2}$

The derivative of 17 is 0. The derivative of $9x^3$ is

Your solution

$27x^2$

So given $y = 4\cos\dfrac{x}{2} + 17 - 9x^3$ then

Your solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} =$$

$-2\sin\frac{x}{2} - 27x^2$

Exercises

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when y is given by:

   (a) $3x^7 + 8x^3$ (b) $-3x^4 + 2x^{1.5}$ (c) $\frac{9}{x^2} + \frac{14}{x} - 3x$ (d) $\frac{3+2x}{4}$ (e) $(2+3x)^2$

2. Find the derivative of each of the following functions:

   (a) $z(t) = 5\sin t + \sin 5t$ (b) $h(v) = 3\cos 2v - 6\sin\frac{v}{2}$

   (c) $m(n) = 4e^{2n} + \frac{2}{e^{2n}} + \frac{n^2}{2}$ (d) $H(t) = \frac{e^{3t}}{2} + 2\tan 2t$ (e) $S(r) = (r^2+1)^2 - 4e^{-2r}$

3. Differentiate the following functions.

   (a) $A(t) = (3+e^t)^2$ (b) $B(s) = \pi e^{2s} + \frac{1}{s} + 2\sin\pi s$

   (c) $V(r) = (1+\frac{1}{r})^2 + (r+1)^2$ (d) $M(\theta) = 6\sin 2\theta - 2\cos\frac{\theta}{4} + 2\theta^2$

   (e) $H(t) = 4\tan 3t + 3\sin 2t - 2\cos 4t$

1. (a) $21x^6 + 24x^2$ (b) $-12x^3 + 3x^{0.5}$ (c) $-\frac{18}{x^3} - \frac{14}{x^2} - 3$
   (d) $\frac{1}{2}$ (e) $12 + 18x$
2. (a) $z' = 5\cos t + 5\cos 5t$ (b) $h' = -6\sin 2v - 3\cos\frac{v}{2}$ (c) $m' = 8e^{2n} - 4e^{-2n} + n$
   (d) $H' = \frac{3e^{3t}}{2} + 4\sec^2 2t$ (e) $S' = 4r^3 + 4r + 8e^{-2r}$
3. (a) $A' = 6e^t + 2e^{2t}$ (b) $B' = 2\pi e^{2s} - \frac{1}{s^2} + 2\pi\cos(\pi s)$
   (c) $V' = -\frac{2}{r^2} - \frac{2}{r^3} + 2r + 2$ (d) $M' = 12\cos 2\theta + \frac{1}{2}\sin\frac{\theta}{4} + 4\theta$
   (e) $H' = 12\sec^2 3t + 6\cos 2t + 8\sin 4t$

3. Evaluating a derivative

The need to find the rate of change of a function at a particular point occurs often. We do this by finding the derivative of the function, and then evaluating the derivative at that point. When taking derivatives of trigonometric functions, any angles **must** be measured in radians. Consider a function, $y(x)$. We use the notation $\frac{\mathrm{d}y}{\mathrm{d}x}(a)$ or $y'(a)$ to denote the derivative of y evaluated at $x = a$. So $y'(0.5)$ means the value of the derivative of y when $x = 0.5$.

Example Find the value of the derivative of $y = x^3$ where $x = 2$. Interpret your result.

Solution

We have $y = x^3$ and so $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$.

When $x = 2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 3(2)^2 = 12$, that is, $\frac{\mathrm{d}y}{\mathrm{d}x}(2) = 12$ (Equivalently, $y'(2) = 12$).

The derivative is positive when $x = 2$ and so y is increasing at this point. When $x = 2$, y is increasing at a rate of 12 vertical units per horizontal unit.

Exercises

1. Calculate the derivative of $y = x^2 + \sin x$ when $x = 0.2$ radians.

2. Calculate the rate of change of $i(t) = 4\sin 2t + 3t$ when

   (a) $t = \frac{\pi}{3}$ (b) $t = 0.6$ radians

3. Calculate the rate of change of $F(t) = 5\sin t - 3\cos 2t$ when

   (a) $t = 0$ radians (b) $t = 1.3$ radians

Answers 1. 1.380 2 (a) -1 (b) 5.8989 3 (a) 5 (b) 4.4305