Determinants

7.3

Introduction

Among other uses, determinants allow us to **determine** whether a system of linear equations has a unique solution or not. The evaluation of a determinant is a key skill in engineering mathematics and this section concentrates on the evaluation of small size determinants. For evaluating larger sizes we can often use some properties of determinants to help simplify the task.

Prerequisites

Before starting this Section you should ...

① understand what a matrix is

Learning Outcomes

After completing this Section you should be able to ...

✓ evaluate a 2×2 determinant

✓ use the method of expansion along the top row to evaluate a determinant

✓ use the properties of determinants to aid their evaluation

1. Determinant of a 2×2 matrix

The **determinant** of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ (note the change from square brackets to vertical lines) and is defined to be the number $ad - bc$. That is:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

We can use the notation $\det(A)$ or $\mid A \mid$ or, Δ to denote the determinant of A.

Find the determinant of the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ -2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 2 \\ -2 & -4 \end{bmatrix}.$$

Your solution

$\mid A \mid = 1 \times 4 - 2 \times 3 = -2$ $\quad \mid B \mid = 4 \times (-3) - (-1) \times (-2) = -12 - 2 = -14$

$\mid C \mid = 0$ $\quad \mid D \mid = 3$ $\quad \mid E \mid = 8$ $\quad \mid F \mid = 3$ $\quad \mid G \mid = -4 + 4 = 0$

Other than when all elements of a 2×2 matrix A are zero, $\mid A \mid = 0$ when either a and d and either b or c are zero; but note matrix G. Its second row is -2 times its first row; alternatively, the second column is twice the first. We shall see in Section 3 that this kind of link between rows and columns allows us to 'spot' if a determinant has value zero without actually evaluating it directly.

2. Laplace expansion along the top row

This is a technique which can be used to evaluate determinants of any order. In principle, this method can use any row or any column as its starting point. We quote one example: using the top row.

Consider $\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$.

First we introduce the idea of a **minor**. Each element in this array of numbers has an associated minor formed by removing the column and row in which the element lies and taking the determinant of the remainder. For example consider element $a_{23} = 3$. We strike out the second row and the third column:

$$\begin{array}{ccc} 4 & 1 & \cancel{1} \\ \cancel{1} & \cancel{2} & \cancel{3} \\ 3 & 1 & \cancel{2} \end{array} \qquad \text{to leave} \qquad \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 4 - 3 = 1.$$

For the element $a_{31} = 3$ we strike out the third row and first column:

$$\begin{array}{ccc} \cancel{4} & 1 & 1 \\ \cancel{1} & 2 & 3 \\ \cancel{3} & \cancel{1} & \cancel{2} \end{array} \qquad \text{to leave} \qquad \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 2 = 1.$$

What is the minor of the element $a_{22} = 2$?

Your solution

$\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} = 8 - 3 = 5$

Next we introduce the idea of a **cofactor**. This is a minor with a sign attached. The appropriate sign comes from the pattern of signs appropriate to a 3×3 array:

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

(i.e. positive signs on the leading diagonal and the signs 'alternate' everywhere else.)
Each element has a cofactor associated with it. The cofactor of element a_{11} is denoted by A_{11}, that of a_{23} by A_{23} and so on.
To obtain the cofactor of an element of a 3×3 matrix we simply multiply the minor of that element by the corresponding sign from the 3×3 array of signs.
Hence the cofactor corresponding to a_{23} is

$$A_{23} = -\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

and the cofactor corresponding to a_{31} is $A_{31} = +\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1.$

What is the cofactor of the element a_{22}?

Your solution

The sign in the position of a_{22} in the array of signs is +
Hence, since the minor of this element is +5 the cofactor is $A_{22} = +5$.

Cofactors are important as it can be shown that the value of the determinant can be found from the formula

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

In words "the determinant of a 3×3 matrix is obtained by multiplying each element of the first row by its corresponding cofactor and then adding the three together". (In fact this rule can be extended to apply to **any** row or **any** column and to **any** order square matrix.)

Key Point

Evaluating General Determinants

If A is an $n \times n$ square matrix then : $\qquad \det(A) = \sum_{j=1}^{n} a_{ij}A_{ij}$

in words:

"the determinant of a square matrix is obtained by multiplying each element of row i by its corresponding cofactor and then adding these products together"

In the case of $\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$ we have $a_{11} = 4,\ a_{12} = 1,\ a_{13} = 1,$

$$A_{11} = +\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4{-}3 = 1 \qquad A_{12} = -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2{-}9) = 7 \qquad A_{13} = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1{-}6 = -5$$

Hence $\quad \Delta = 4 \times 1 \quad + \quad 1 \times 7 \quad + \quad 1 \times -5 = 6.$

Alternatively, choosing to expand along the second row:

$$\begin{aligned} \Delta &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 1\left(-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right) + 2\left(\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}\right) + 3\left(-\begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix}\right) = 6 \qquad \text{as before} \end{aligned}$$

Repeat the procedure above for $\Delta = \begin{vmatrix} 1 & -1 & 3 \\ 0 & 2 & 6 \\ -2 & 1 & 5 \end{vmatrix}$

Your solution

$a_{11} = 1, \quad a_{12} = -1, \quad a_{13} = 3$

$$\begin{aligned} A_{11} &= +\begin{vmatrix} 2 & 6 \\ 1 & 5 \end{vmatrix} = 10 - 6 = 4 \\ A_{12} &= -\begin{vmatrix} 0 & 6 \\ -2 & 5 \end{vmatrix} = -(0 + 12) = -12 \\ A_{13} &= +\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} = 2 + 2 = 4. \end{aligned}$$

Hence $\Delta = 1 \times 4 + (-1) \times (-12) + 3 \times 4 = 4 + 12 + 12 = 28$, as before.

3. Properties of determinants

Often, especially with determinants of large order, we can simplify the evaluation rules. In this section we quote some useful properties of determinants in general.

1. If two rows (or two columns) of a determinant are interchanged then the value of the determinant is multiplied by (-1).

   For example $\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$ but (interchanging columns) $\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = 3 - 8 = -5$ and (interchanging rows) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = 3 - 8 = -5$.

2. the determinant of a matrix A and its transpose A^T are equal.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2$$

3. If two rows (or two columns) of a matrix A are equal then it has zero determinant For example, the following determinant has two identical rows:

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 1 \times \left(\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \right) + 2 \times \left(- \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \right) + 3 \times \left(\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \right)$$

$$= -3 + 2 \times (6) + 3 \times (-3) = 0.$$

4. If the elements of one row (or one column) of a determinant are multiplied by k, then the value of the resulting determinant is k times the given determinant:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 8 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 3 \\ 7 & 8 & 9 \end{vmatrix}.$$

Note that if one row (or column) of a determinant is a multiple of another row (or column) then the value of the determinant is zero. For example:

$$\begin{vmatrix} 2 & 4 & -1 \\ 4 & 2 & 1 \\ -4 & -8 & 2 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 1 \\ -8 & 2 \end{vmatrix} + 4 \times \left(- \begin{vmatrix} 4 & 1 \\ -4 & 2 \end{vmatrix} \right) - 1 \times \begin{vmatrix} 4 & 2 \\ -4 & -8 \end{vmatrix} = 2(12) + 4(-12) - (-24) = 0$$

This was predictable as the 3rd row is (-2) times the first row.

5. If we add (or subtract) a multiple of one row (or column) to another, the value of the determinant is unchanged.

Given $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$, add ($2 \times$ row 1) to (row 2)

$$\begin{vmatrix} 1 & 2 \\ 4+2 & 5+4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 6 & 9 \end{vmatrix} = 9 - 12 = -3 = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

6. The determinant of a lower triangular matrix, an upper triangular matrix or a diagonal matrix is the product of the elements on the leading diagonal.

As an example, it is easily confirmed that each of the following determinants has the same value $1 \times 4 \times 6 = 24$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{vmatrix}$$

Use property 2 to find a determinant equal to

$$\Delta = \begin{vmatrix} 1 & 4 & 8 & 2 \\ 2 & -1 & 1 & -3 \\ 0 & 2 & 4 & 2 \\ 0 & 3 & 6 & 3 \end{vmatrix}$$

Your solution

$\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 4 & -1 & 2 & 3 \\ 8 & 1 & 4 & 6 \\ 2 & -3 & 2 & 3 \end{vmatrix}$, by transposing the matrix.

Now expand along the top row to express Δ as the sum of two products, each of a number and a 3×3 determinant.

Your solution

$\Delta = 1 \times \begin{vmatrix} -1 & 2 & 3 \\ 1 & 4 & 6 \\ -3 & 2 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 4 & 2 & 3 \\ 8 & 4 & 6 \\ 2 & 2 & 3 \end{vmatrix}$

Use the statement after property 4 to show that the second of the 3×3 determinants is zero.

Your solution

In the second 3×3 determinant, row 2 $= 2\times$row 1 hence the determinant has value zero.

Use the same statement to evaluate the first determinant.

Your solution

In the first 3×3 determinant column 3 $= \frac{3}{2}\times$ column 2. Hence this determinant is also zero. Therefore $\Delta = 0$.

Exercises

1. Use Laplace's expansion along the 1st row to determine

$$\begin{vmatrix} 3 & 1 & -4 \\ 6 & 9 & -2 \\ -1 & 2 & 1 \end{vmatrix}$$

Show that the same value is obtained if you choose any other row or column for your expansion.

2. Evaluate, using any of the properties of determinants to minimise the arithmetic

$$\text{(a)} \quad \begin{vmatrix} 12 & 27 & 12 \\ 28 & 18 & 24 \\ 70 & 15 & 40 \end{vmatrix} \qquad\qquad \text{(b)} \quad \begin{vmatrix} 2 & 4 & 6 & 4 \\ 0 & 4 & 6 & 9 \\ 2 & 1 & 4 & 0 \\ 1 & 2 & 3 & 2 \end{vmatrix}$$

3. Find the cofactors of x, y, z in the determinants

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ x & y & z \end{vmatrix}$$

4. Prove that, no matter what the values of x, y, z,

$$\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Answers

1. $3\begin{vmatrix} 9 & -2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 6 & -2 \\ -1 & 1 \end{vmatrix} - 4\begin{vmatrix} 6 & 9 \\ -1 & 2 \end{vmatrix} = 3(9+4) - 1(6-2) - 4(12+9) = -49$

2. (a) Take out common factors in rows and columns

$$720\begin{vmatrix} 2 & 3 & 1 \\ 7 & 3 & 3 \\ 7 & 1 & 2 \end{vmatrix} = 720\begin{vmatrix} 0 & 0 & 1 \\ 1 & -6 & 3 \\ 3 & -5 & 2 \end{vmatrix} \quad \text{using } (-2C_3 + C_1) \text{ then } (-3C_3 + C_2).$$

The value of the determinant (expand along top row) is then easily found as $720 \times 13 = 9360$.

(b) Zero since (row 1) is $2\times$ (row 4).

3. Cofactors of x, y, z are $1, -2, 1$ respectively.