The Chain Rule

11.5

Introduction

In this Section we will see how to obtain the derivative of a composite function (these are often referred to as 'functions of a function'). To do this we use the **chain rule**. This rule can be used to obtain the derivatives of functions such as e^{x^2+3x} (the exponential function of a polynomial); $\sin(\ln x)$ (the sine function of the logarithmic function); $\sqrt{x^3+4}$ (the square root function of a polynomial).

Prerequisites

Before starting this Section you should ...

① be able to differentiate standard functions

② be able to use the product and quotient rule for finding derivatives

Learning Outcomes

After completing this Section you should be able to ...

✓ differentiate a function of a function using the chain rule

✓ differentiate a power function

1. What is a function of a function?

When we use a function like $\sin 2x$ or $e^{\ln x}$ or $\sqrt{x^2+1}$ we are in fact dealing with composite functions or **functions of a function**.

$\sin 2x$ is the sine function of $2x$. This is, in fact, how we 'read' it:

$$\sin 2x \text{ is read 'sine of } 2x\text{'}$$

Similarly $e^{\ln x}$ is the exponential function of the logarithm of x:

$$e^{\ln x} \text{ is read '} e \text{ to the power of } \ln x\text{'}$$

Finally $\sqrt{x^2+1}$ is also a composite function. It is the square root function of the polynomial x^2+1:

$$\sqrt{x^2+1} \text{ is read as the 'square root of } (x^2+1)\text{'}$$

When we talk about functions of a function in a general setting we will use the notation $f(g(x))$ where both f and g are functions.

Example Specify the functions $f,\ g$ for the composite functions
(a) $\sin 2x$ (b) $\sqrt{x^2+1}$ (c) $e^{\ln x}$

Solution

(a) Here f is the sine function and g is the polynomial $2x$. We often write:

$$f(g) = \sin g \qquad \text{and} \qquad g(x) = 2x$$

(b) Here $f(g) = \sqrt{g}$ and $g(x) = x^2+1$

(c) In this case $f(g) = e^g$ and $g(x) = \ln x$

In each case the original function of x is obtained when $g(x)$ is substituted into $f(g)$.

Specify the functions $f,\ g$ for the composite functions
(a) $\cos(3x^2-1)$ (b) $\sinh(e^x)$ (c) $(x^2+3x-1)^{1/3}$

Your solution

(a)

$f(g) = \cos g \qquad g(x) = 3x^2 - 1$

Your solution

(b)

$f(g) = \sinh g \qquad g(x) = e^x$

Your solution

(c)

$f(g) = g^{1/3} \qquad g(x) = x^2 + 3x - 1$

2. The Derivative of a function of a function

To differentiate a function of a function we use the following key point:

Key Point

If $y = f(g(x))$, that is, a function of a function, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

This is called the **chain rule**.

Example Find the derivatives of the following composite functions and check the result using other methods

(a) $(2x^2 - 1)^2$ (b) $\ln e^x$

Solution

(a) Here $y = f(g(x))$ where $f(g) = g^2$ and $g(x) = 2x^2 - 1$. Thus

$$\frac{\mathrm{d}f}{\mathrm{d}g} = 2g \qquad \text{and} \qquad \frac{\mathrm{d}g}{\mathrm{d}x} = 4x \qquad \therefore \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 2g.(4x) = 2(2x^2 - 1)(4x) = 8x(2x^2 - 1)$$

This result is easily checked by using the rule for differentiating products:

$$y = (2x^2 - 1)(2x^2 - 1) \quad \text{so} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 4x(2x^2 - 1) + (2x^2 - 1)(4x) = 8x(2x^2 - 1) \quad \text{as obtained above}$$

Solution

(b) Here $y = f(g(x))$ where $f(g) = \ln g$ and $g(x) = e^x$. Thus

$$\frac{\mathrm{d}f}{\mathrm{d}g} = \frac{1}{g} \qquad \text{and} \qquad \frac{\mathrm{d}g}{\mathrm{d}x} = e^x$$

$$\therefore \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{g} \cdot e^x = \frac{1}{e^x} \cdot e^x = 1$$

This is easily checked since, of course,

$$y = \ln e^x = x$$

and so, obviously $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ as above.

Obtain the derivatives of the following functions

(a) $(2x^2 - 5x + 3)^9$ (b) $\sin(\cos x)$ (c) $\left(\dfrac{2x+1}{2x-1}\right)^3$

(a) What are f, g in this case?

Your solution

(a) $f(g) =$ $g(x) =$

$f(g) = g^9$ $g(x) = 2x^2 - 5x + 3$

Now obtain the derivative using the chain rule

Your solution

$9(2x^2 - 5x + 3)^8(4x - 5)$. Can you see how to obtain the derivative without going through the intermediate stage of specifying f, g?

(b) Again, specify f and g

Your solution

(b)

$f(g) = \sin g$ $g(x) = \cos x$

Now use the chain rule to obtain the derivative

Your solution

$-[\cos(\cos x)] \sin x$

Your solution

(c)

$-\dfrac{12(2x+1)^2}{(2x-1)^4}$

3. A Power function

An example of a function of a function which often occurs is the so-called power function $[g(x)]^k$ where k is any rational number. This is an example of a function of a function in which

$$f(g) = g^k$$

Thus, using the chain rule: if

$$y = [g(x)]^k$$

then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x} = k\, g^{k-1} \frac{\mathrm{d}g}{\mathrm{d}x}.$$

For example, if $y = (\sin x + \cos x)^{1/3}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}(\sin x + \cos x)^{-2/3}(\cos x - \sin x)$.

Find the derivatives of the following power functions

(a) $y = \sin^3 x$ (b) $y = (x^2+1)^{1/2}$ (c) $y = (e^{3x})^7$

(a) Note here that $\sin^3 x$ is the conventional way of writing $(\sin x)^3$. Now find the derivative.

Your solution

(a)

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3(\sin x)^2 \cos x$ which we would normally write as $3 \sin^2 x \cos x$

Your solution

(b)

$$\frac{dy}{dx} = \frac{1}{2}(x^2+1)^{-1/2}2x = \frac{x}{\sqrt{x^2+1}}$$

Your solution

(c)

$$\frac{dy}{dx} = 7(e^{3x})^6(3e^{3x}) = 21(e^{3x})^7 = 21e^{21x}$$

note that $(e^{3x})^7 = e^{21x}$ $\quad\therefore\quad$ $\dfrac{dy}{dx} = 21e^{21x}$ directly.

Exercises

1. Obtain the derivatives of the following functions:

   (a) $\left(\frac{2x+1}{3x-1}\right)^4$

   (b) $\tan(3x^2+2x)$

   (c) $\sin^2(3x^2-1)$

Answers

1. (a) $-\frac{20(2x+1)^3}{(3x-1)^5}$

   (b) $2(3x+1)\sec^2(3x^2+2x)$

   (c) $6x\sin(6x^2-2)$: (remember $\sin 2x = 2\sin x\cos x$)