Indices

1.2

Introduction

Indices, or powers, provide a convenient notation when we need to multiply a number by itself several times. In this section we explain how indices are written, and state the rules which are used for manipulating them.
Expressions built up using non-negative whole number powers of a variable – known as polynomials – occur frequently in engineering mathematics. We introduce some common polynomials in this section.
Finally, scientific notation is used to express very large or very small numbers concisely. This requires use of indices. We explain how to use scientific notation towards the end of the section.

Prerequisites

Before starting this Section you should . . .

① be familiar with algebraic notation and symbols

Learning Outcomes

After completing this Section you should be able to . . .

✓ perform calculations using indices

✓ state and use the laws of indices

✓ use scientific notation

1. Index notation

The number $4 \times 4 \times 4$ is written, for short, as 4^3 and read '4 raised to the power 3' or '4 cubed'. Note that the number of times '4' occurs in the product is written as a superscript. In this context we call the superscript 3 an **index** or **power**. Similarly we could write

$$5 \times 5 = 5^2, \text{ read '5 to the power 2' or '5 squared'}$$

and

$$7 \times 7 \times 7 \times 7 \times 7 = 7^5 \qquad a \times a \times a = a^3, \qquad m \times m \times m \times m = m^4$$

More generally, in the expression x^y, x is called the **base** and y is called the index or power. The plural of index is **indices**. The process of raising to a power is also known as **exponentiation** because yet another name for a power is an **exponent**. When dealing with numbers your calculator is able to evaluate expressions involving powers, probably using the x^y button.

Example Use a calculator to evaluate 3^{12}.

Solution

Using the x^y button on the calculator check that you obtain $3^{12} = 531441$.

Example Identify the index and base in the following expressions.

a) 8^{11}, b) $(-2)^5$, c) p^{-q}

Solution

(a) In the expression 8^{11}, 8 is the base and 11 is the index.

(b) In the expression $(-2)^5$, -2 is the base and 5 is the index.

(c) In the expression p^{-q}, p is the base and $-q$ is the index. The interpretation of a negative index will be given in Section 1.4.

Recall from Section 1 that when several operations are involved we can make use of the BODMAS rule for deciding the order in which operations must be carried out. The BODMAS rule makes no mention of exponentiation. Exponentiation should be carried out immediately after any brackets have been dealt with. Consider the following examples.

Example Evaluate 7×3^2

Solution

There are two operations involved here, exponentiation and multiplication. The exponentiation should be carried out before the multiplication. So $7 \times 3^2 = 7 \times 9 = 63$.

Example Write out fully a) $3m^4$, b) $(3m)^4$.

Solution

(a) In the expression $3m^4$ the exponentiation is carried out before the multiplication by 3. So

$$3m^4 \quad \text{means} \quad 3 \times (m \times m \times m \times m)$$

that is

$$3 \times m \times m \times m \times m$$

Solution

(b) Here the bracketed expression is raised to the power 4 and so should be multiplied by itself four times:

$$(3m)^4 = (3m) \times (3m) \times (3m) \times (3m)$$

Because of the associativity of multiplication we can write this as

$$3 \times 3 \times 3 \times 3 \times m \times m \times m \times m \qquad \text{or simply } 81m^4.$$

Note the important distinction between $(3m)^4$ and $3m^4$.

Exercises

1. Evaluate, without using a calculator, a) 3^3, b) 3^5, c) 2^5.
2. Evaluate without using a calculator, a) 0.2^2, b) 15^2.
3. Evaluate using a calculator a) 7^3, b) $(14)^{3.2}$.
4. Write each of the following using index notation:

   a) $7 \times 7 \times 7 \times 7 \times 7$, b) $t \times t \times t \times t$, c) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} \times \frac{1}{7} \times \frac{1}{7}$.
5. Evaluate without using a calculator. Leave any fractions in fractional form.

   a) $\left(\frac{2}{3}\right)^2$, b) $\left(\frac{2}{5}\right)^3$, c) $\left(\frac{1}{2}\right)^2$, d) $\left(\frac{1}{2}\right)^3$, e) 0.1^3.

Answers

1. a) 27, b) 243, c) 32, 2. a) 0.04, b) 225
3. a) 343, b) 4651.7 (1d.p.).
4. a) 7^5, b) t^4, c) $\left(\frac{1}{2}\right)^2\left(\frac{1}{7}\right)^3$
5. a) $\frac{4}{9}$, b) $\frac{8}{125}$, c) $\frac{1}{4}$, d) $\frac{1}{8}$, e) 0.1^3 means $(0.1) \times (0.1) \times (0.1) = 0.001$

2. Laws of Indices

There is a set of rules which enable us to manipulate expressions involving indices. These rules are known as the **laws of indices**, and they occur so commonly that it is worthwhile to memorise them.

Key Point

The laws of indices state:

first law:	$a^m \times a^n = a^{m+n}$	add indices when multiplying numbers with the same base
second law:	$\dfrac{a^m}{a^n} = a^{m-n}$	subtract indices when dividing numbers with the same base
third law:	$(a^m)^n = a^{mn}$	multiply indices together when raising a number to a power

Example Simplify a) $a^5 \times a^4$, b) $2x^5(x^3)$.

Solution

In each case we are required to multiply expressions involving indices. The bases are the same and we can use the first of the given laws.

(a) The indices must be added, thus $a^5 \times a^4 = a^{5+4} = a^9$.

(b) Because of the associativity of multiplication we can write

$$2x^5(x^3) = 2(x^5x^3) = 2x^{5+3} = 2x^8$$

The first law of indices extends in an obvious way when more terms are involved:

Example Simplify $b^5 \times b^4 \times b^7$.

Solution

The indices are added. Thus $b^5 \times b^4 \times b^7 = b^{5+4+7} = b^{16}$

Simplify $y^4y^2y^3$.

Your solution

$y^4y^2y^3 =$

All quantities have the same base. To multiply the quantities together, the indices are added: y^9

Example Simplify a) $\frac{8^4}{8^2}$, b) $x^{18} \div x^7$.

Solution

In each case we are required to divide expressions involving indices. The bases are the same and we can use the second of the given laws.

(a) The indices must be subtracted, thus $\frac{8^4}{8^2} = 8^{4-2} = 8^2 = 64$.

(b) Again the indices are subtracted, and so $x^{18} \div x^7 = x^{18-7} = x^{11}$.

Simplify a) $\frac{5^9}{5^7}$, b) $\frac{y^5}{y^2}$

Your solution

(a) $\frac{5^9}{5^7} =$

The bases are the same, and the division is carried out by subtracting the indices: $5^{9-7} = 5^2 = 25$

Your solution

(b) $\frac{y^5}{y^2} =$

$y^{5-2} = y^3$

Example Simplify a) $(8^2)^3$, b) $(z^3)^4$.

Solution

We use the third of the laws of indices.

(a) $(8^2)^3 = 8^{2\times 3} = 8^6$

(b) $(z^3)^4 = z^{3\times 4} = z^{12}$.

Simplify $(x^2)^5$.
Apply the third law.

Your solution

$(x^2)^5 =$

$x^{2\times 5} = x^{10}$

Simplify $(e^x)^y$

Your solution

$(e^x)^y =$

Again, using the third law, the two powers are multiplied: $e^{x\times y} = e^{xy}$

Two important results which can be derived from the laws state:

Key Point

Any non-zero number raised to the power 0 has the value 1, that is $a^0 = 1$

Any number raised to power 1 is itself, that is $a^1 = a$

A generalisation of the third law states:

Key Point

$$(a^m b^n)^k = a^{mk} b^{nk}$$

Example Remove the brackets from a) $(3x)^2$, b) $(x^3y^7)^4$.

Solution

a) Noting that $3 = 3^1$ and $x = x^1$ then

$$(3x)^2 = (3^1x^1)^2 = 3^2x^2 = 9x^2 \qquad \text{or, alternatively} \qquad (3x)^2 = (3x) \times (3x) = 9x^2$$

b)

$$(x^3y^7)^4 = x^{3\times 4}y^{7\times 4} = x^{12}y^{28}$$

Exercises

1. Show that $(-xy)^2$ is equivalent to x^2y^2 whereas $(-xy)^3$ is equivalent to $-x^3y^3$.
2. Write each of the following expressions with a single index:
   a) 6^76^9, b) $\frac{6^7}{6^{19}}$, c) $(x^4)^3$
3. Remove the brackets from a) $(8a)^2$, b) $(7ab)^3$, c) $7(ab)^3$, d) $(6xy)^4$,
4. Simplify a) $15x^2(x^3)$, b) $3x^2(5x)$, c) $18x^{-1}(3x^4)$.
5. Simplify a) $5x(x^3)$, b) $4x^2(x^3)$, c) $3x^7(x^4)$, d) $2x^8(x^{11})$, e) $5x^2(3x^9)$

Answers

2. a) 6^{16}, b) 6^{-12}, c) x^{12}
3. a) $64a^2$, b) $343a^3b^3$, c) $7a^3b^3$, d) $1296x^4y^4$
4. a) $15x^5$, b) $15x^3$, c) $54x^3$
5. a) $5x^4$, b) $4x^5$, c) $3x^{11}$, d) $2x^{19}$, e) $15x^{11}$

3. Polynomial expressions

An important group of mathematical expressions which use indices are known as **polynomials**. Examples of polynomials are

$$4x^3 + 2x^2 + 3x - 7, \qquad x^2 + x, \qquad 17 - 2t + 7t^4, \qquad z - z^3$$

Notice that they are all constructed using non-negative whole number powers of the variable. Recall that $x^0 = 1$ and so the number -7 appearing in the first example can be thought of as $-7x^0$. Similarly the 17 appearing in the third example can be read as $17t^0$.

Key Point

A polynomial expression takes the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$$

where a_0, a_1, a_2, a_3, $\ldots$ a_n are all constants called the **coefficients** of the polynomial. The number a_0 is also called the **constant term**. The highest power in a polynomial is called the **degree** of the polynomial. Polynomials with low degrees have special names:

Polynomial	**Degree**	**Name**
$ax^3 + bx^2 + cx + d$	3	cubic
$ax^2 + bx + c$	2	quadratic
$ax + b$	1	linear
a	0	constant

Which of the following expressions are polynomials? Give the degree of those which are.

a) $3x^2+4x+2$, b) $\frac{1}{x+1}$, c) $\sqrt{x}$, d) $2t+4$, e) $3x^2+\frac{4}{x}+2$.

Recall that a polynomial expression must contain only terms involving non-negative whole number powers of the variable.

Give your answers by ringing the correct word:

Your solution

(a) $3x^2 + 4x + 2$ yes no

(b) $\frac{1}{x+1}$ yes no

(c) $\sqrt{x}$ yes no

(d) $2t + 4$ yes no

(e) $3x^2 + \frac{4}{x} + 2$ yes no

a) yes: polynomial of degree 2, called a quadratic b) no c) no d) yes polynomial of degree 1, called linear e) no

Exercises

1. State which of the following are linear polynomials, which are quadratic polynomials, and which are constants.

   a) x, b) $x^2 + x + 3$, c) $x^2 - 1$, d) $3 - x$, e) $7x - 2$, f) $\frac{1}{2}$,

   g) $\frac{1}{2}x + \frac{3}{4}$, h) $3 - \frac{1}{2}x^2$.

2. State which of the following are polynomials.

   a) $-\alpha^2 - \alpha - 1$, b) $x^{1/2} - 7x^2$, c) $\frac{1}{x}$, d) 19.

3. Which of the following are polynomials ?

   a) $4t + 17$, b) $\frac{1}{2} - \frac{1}{2}t$, c) 15, d) $t^2 - 3t + 7$, e) $\frac{1}{t^2} + \frac{1}{t} + 7$

4. State the degree of each of the following polynomials. For those of low degree, give their name.

   a) $2t^3 + 7t^2$, b) $7t^7 + 14t^3 - 2t^2$, c) $7x + 2$,

   d) $x^2 + 3x + 2$, e) $2 - 3x - x^2$, f) 42

Answers

1. a), d), e) and g) are linear. b), c) and h) are quadratic. f) is a constant.
2. a) is a polynomial, d) is a polynomial of degree 0. b) and c) are not polynomials.
3. a) b) c) and d) are polynomials.
4. a) 3, cubic, b) 7, c) 1, linear, d) 2, quadratic, e) 2, quadratic, f) 0, constant.

4. Negative indices

Sometimes a number is raised to a negative power. This is interpreted as follows:

Key Point

negative powers : $$a^{-m} = \frac{1}{a^m}, \qquad a^m = \frac{1}{a^{-m}}$$

Thus a negative index can be used to indicate a reciprocal.

Example Write each of the following expressions using a positive index and simplify if possible.

a) 2^{-3}, b) $\frac{1}{4^{-3}}$, c) x^{-1}, d) x^{-2}, e) 10^{-1}

Solution

a) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$, b) $\frac{1}{4^{-3}} = 4^3 = 64$, c) $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$, d) $x^{-2} = \frac{1}{x^2}$,
e) $10^{-1} = \frac{1}{10^1} = \frac{1}{10}$ or 0.1.

Write each of the following using a positive index.
a) $\frac{1}{t^{-4}}$, b) 17^{-3}, c) y^{-1}, d) 10^{-2}
Use the previous Key Point.

Your solution

$\frac{1}{t^{-4}} =$

t^4

Your solution

$17^{-3} =$

$\frac{1}{17^3}$

Your solution

$y^{-1} =$

$\frac{1}{y}$

Your solution

$10^{-2} =$

$\frac{1}{10^2}$ which equals $\frac{1}{100}$ or 0.01

Simplify a) $\frac{a^8 \times a^7}{a^4}$, b) $\frac{m^9 \times m^{-2}}{m^{-3}}$

a) Use the first law of indices to simplify the numerator:

Your solution

$\frac{a^8 \times a^7}{a^4} =$

$\frac{a^{15}}{a^4}$

Then use the second law to simplify the result:

Your solution

a^{11}

b) First simplify the numerator using the first law of indices:

Your solution

$\frac{m^9 \times m^{-2}}{m^{-3}} =$

$\frac{m^7}{m^{-3}}$

Then use the second law to simplify the result:

Your solution

$m^{7-(-3)} = m^{10}$

Exercises

1. Write the following numbers using a positive index and also express your answers as decimal fractions:

   a) 10^{-1}, b) 10^{-3}, c) 10^{-4}

2. Simplify as much as possible:

   a) x^3x^{-2}, b) $\frac{t^4}{t^{-3}}$, c) $\frac{y^{-2}}{y^{-6}}$.

Answers 1. a) $\frac{1}{10} = 0.1$, b) $\frac{1}{10^3} = 0.001$, c) $\frac{1}{10^4} = 0.0001$.
2. a) $x^1 = x$, b) $t^{4+3} = t^7$, c) $y^{-2+6} = y^4$.

5. Fractional indices

So far we have used indices that are whole numbers. We now consider fractional powers. Consider the expression $(16^{\frac{1}{2}})^2$. Using the third law of indices, $(a^m)^n = a^{mn}$, we can write

$$(16^{\frac{1}{2}})^2 = 16^{\frac{1}{2}\times 2} = 16^1 = 16$$

So $16^{\frac{1}{2}}$ is a number which when squared equals 16, that is 4 or -4. In other words $16^{\frac{1}{2}}$ is a square root of 16. There are always two square roots of a non-zero positive number, and we write

$$16^{\frac{1}{2}} = \pm 4$$

In general

Key Point

$a^{\frac{1}{2}}$ is a square root of a

Similarly

$$(8^{\frac{1}{3}})^3 = 8^{\frac{1}{3}\times 3} = 8^1 = 8$$

so that $8^{\frac{1}{3}}$ is a number which when cubed equals 8. Thus $8^{\frac{1}{3}}$ is the cube root of 8, that is $\sqrt[3]{8}$, namely 2. Each number has only one cube root, and so

$$8^{\frac{1}{3}} = 2$$

In general

Key Point

$a^{\frac{1}{3}}$ is the cube root of a

More generally we have

Key Point

$x^{\frac{1}{n}}$ is an nth root of x, that is $\sqrt[n]{x}$

Your calculator will be able to evaluate fractional powers, and roots of numbers. Check that you can obtain the results of the following examples on your calculator, but be aware that your calculator may give only one root when there may be others.

Example Evaluate a) $144^{1/2}$, b) $125^{1/3}$

Solution

a) $144^{1/2}$ is a square root of 144, that is ± 12.

b) Noting that $5^3 = 125$, we see that $125^{1/3} = \sqrt[3]{125} = 5$

Example Evaluate a) $32^{1/5}$, b) $32^{2/5}$, and c) $8^{2/3}$.

Solution

a) $32^{\frac{1}{5}}$ is the 5th root of 32, that is $\sqrt[5]{32}$. Now $2^5 = 32$ and so $\sqrt[5]{32} = 2$.

b) Using the third law of indices we can write $32^{2/5} = 32^{2\times\frac{1}{5}} = (32^{\frac{1}{5}})^2$. Thus

$$32^{2/5} = ((32)^{1/5})^2 = 2^2 = 4$$

c) Note that $8^{1/3} = 2$. Then

$$8^{\frac{2}{3}} = 8^{2\times\frac{1}{3}} = (8^{1/3})^2 = 2^2 = 4$$

Note the following alternatives:

$$8^{2/3} = (8^{1/3})^2 = (8^2)^{1/3}$$

Example Write the following as a simple power with a single index:
a) $\sqrt{x^5}$, b) $\sqrt[4]{x^3}$.

Solution

a) $\sqrt{x^5} = (x^5)^{\frac{1}{2}}$. Then using the third law of indices we can write this as $x^{5\times\frac{1}{2}} = x^{\frac{5}{2}}$.

b) $\sqrt[4]{x^3} = (x^3)^{\frac{1}{4}}$. Using the third law we can write this as $x^{3\times\frac{1}{4}} = x^{\frac{3}{4}}$.

Example Show that $z^{-1/2} = \frac{1}{\sqrt{z}}$.

Solution

$$z^{-1/2} = \frac{1}{z^{1/2}} = \frac{1}{\sqrt{z}}$$

Simplify $\frac{\sqrt{z}}{z^3 z^{-1/2}}$

Rewrite $\sqrt{z}$ using an index and simplify the denominator using the first law of indices:

Your solution

$\frac{\sqrt{z}}{z^3 z^{-1/2}} =$

$\frac{z^{\frac{1}{2}}}{z^{\frac{5}{2}}}$

Finally, use the second law to simplify the result:

Your solution

$z^{\frac{1}{2}-\frac{5}{2}} = z^{-2}$ or $\frac{1}{z^2}$

Example The generalisation of the third law of indices states that $(a^m b^n)^k = a^{mk} b^{nk}$. By taking $m = 1$, $n = 1$ and $k = \frac{1}{2}$ show that $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Solution

Taking $m = 1$, $n = 1$ and $k = \frac{1}{2}$ gives $(ab)^{1/2} = a^{1/2} b^{1/2}$ and the required result follows immediately.

Key Point

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}$$

This result often allows answers to be written in alternative forms. For example we may write $\sqrt{48}$ as $\sqrt{3 \times 16} = \sqrt{3}\sqrt{16} = 4\sqrt{3}$.
Although this rule works for multiplication we should be aware that it does **not** work for addition or subtraction so that

$$\sqrt{a \pm b} \neq \sqrt{a} \pm \sqrt{b}$$

Exercises

1. Evaluate using a calculator a) $3^{1/2}$, b) $15^{-\frac{1}{3}}$, c) 85^3, d) $81^{1/4}$
2. Evaluate using a calculator a) 15^{-5}, b) $15^{-2/7}$
3. Simplify a) $\frac{a^{11} a^{3/4}}{a^{-1/2}}$, b) $\frac{\sqrt{z}}{z^{3/2}}$, c) $\frac{z^{-5/2}}{\sqrt{z}}$, d) $\frac{\sqrt[3]{a}}{\sqrt[2]{a}}$, e) $\frac{\sqrt[5]{z}}{z^{1/2}}$.
4. From the third law of indices show that $(ab)^{1/2} = a^{1/2} b^{1/2}$. Deduce that the square root of a product is equal to the product of the individual square roots.
5. Write each of the following expressions with a single index:

   a) $(x^{-4})^3$, b) $x^{1/2} x^{1/4}$, c) $\frac{x^{1/2}}{x^{1/4}}$

Answers 1 a) 1.7321, b) 0.4055, c) 614125, d) 3
2 a) 0.000001317 (4s.f.), b) 0.4613 (4s.f.),
3 a) $a^{12.25}$, b) z^{-1}, c) z^{-3}, d) $a^{-1/6}$, e) $z^{-3/10}$
5 a) x^{-12}, b) $x^{3/4}$, c) $x^{1/4}$

6. Scientific notation

It is often necessary to use very large or very small numbers such as 78000000 and 0.00000034. **Scientific notation** can be used to express such numbers in a more concise form. Each number is written in the form

$$a \times 10^n$$

where a is a number between 1 and 10. We can make use of the following facts:

$$10 = 10^1, \quad 100 = 10^2, \quad 1000 = 10^3 \qquad \text{and so on}$$

and

$$0.1 = 10^{-1}, \quad 0.01 = 10^{-2}, \quad 0.001 = 10^{-3} \qquad \text{and so on}$$

Furthermore, to multiply a number by 10^n the decimal point is moved n places to the right if n is a positive integer, and n places to the left if n is a negative integer. If necessary additional zeros are inserted to make up the required number of decimal places.

Then, for example,

the number 5000 can be written $5 \times 1000 = 5 \times 10^3$
the number 403 can be written $4.03 \times 100 = 4.03 \times 10^2$
the number 0.009 can be written $9 \times 0.001 = 9 \times 10^{-3}$

Write the number 0.00678 in scientific notation.

Your solution

$0.00678 = 6.78 \times 10^{-3}$

Example *Engineering Constants*

Many constants appearing in engineering calculations are expressed in scientific notation. For example the charge on an electron equals 1.6×10^{-19} coulomb and the speed of light is $3 \times 10^8 \text{ms}^{-1}$. Avogadro's constant is equal to 6.023×10^{26} and is the number of atoms in one kilomole of an element. Clearly the use of scientific notation avoids writing lengthy strings of zeros.

Your scientific calculator will be able to accept numbers in scientific notation. Often the E button is used and a number like 4.2×10^7 will be entered as $4.2E7$. Note that 10E4 means 10×10^4, that is 10^5. To enter the number 10^3 say, you would key in $1E3$. Entering powers of 10 incorrectly is a common cause of error. You must check how your particular calculator accepts numbers in scientific notation.

Use your calculator to find $4.2 \times 10^{-3} \times 3.6 \times 10^{-4}$.

This exercise is designed to check that you can enter numbers given in scientific notation into your calculator. Check that

Your solution

$4.2 \times 10^{-3} \times 3.6 \times 10^{-4} =$

1.512×10^{-6}

Exercises

1. Express each of the following numbers in scientific notation:

   a) 45, b) 456, c) 2079, d) 7000000, e) 0.1, f) 0.034,

   g) 0.09856

2. Simplify $6 \times 10^{24} \times 1.3 \times 10^{-16}$

Answers

1. a) 4.5×10^1, b) 4.56×10^2, c) 2.079×10^3, d) 7×10^6, e) 1×10^{-1}, f) 3.4×10^{-2}, g) 9.856×10^{-2}

2. 7.8×10^8