Definite Integrals 13.2

Introduction

When you were first introduced to integration as the reverse of differentiation, the integrals you dealt with were **indefinite integrals**. The result of finding an indefinite integral is usually a function plus a constant of integration. In this section we introduce **definite integrals**, so called because the result will be a definite answer, usually a number, with no constant of integration. Definite integrals have many applications, for example in finding areas bounded by curves, and finding volumes of solids.

Prerequisites

Before starting this Section you should . . .

① understand integration as the reverse of differentiation

② be able to use a table of integrals

Learning Outcomes

After completing this Section you should be able to . . .

✓ find some simple definite integrals

✓ handle some integrals involving an infinite limit of integration

1. What is a Definite Integral

We saw in the previous section that $\int f(x)\mathrm{d}x = F(x) + c$ where $F(x)$ is that function which, when differentiated, gives $f(x)$. That is, $\frac{\mathrm{d}F}{\mathrm{d}x} = f(x)$. For example,

$$\int \sin(3x)\mathrm{d}x = -\frac{\cos(3x)}{3} + c$$

Here, $f(x) = \sin(3x)$ and $F(x) = -\frac{1}{3}\cos(3x)$ We now consider a definite integral which is simply an indefinite integral but with numbers written to the upper and lower right of the integral sign. The quantity

$$\int_a^b f(x)\,\mathrm{d}x$$

is called the definite integral of $f(x)$ from a to b. The numbers a and b are known as the **lower limit** and **upper limit** respectively of the integral. We *define*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

so that a definite integral is usually a number. The *meaning* of a definite integral will be developed in later sections. For the present we concentrate on the *process* of evaluating definite integrals.

2. Evaluating Definite Integrals

When you evaluate a definite integral the result will usually be a number. To see how to evaluate a definite integral consider the following example.

Example Find the definite integral of x^2 from 1 to 4; that is, find $\int_1^4 x^2\,\mathrm{d}x$

Solution

Now

$$\int x^2\mathrm{d}x = \tfrac{1}{3}x^3 + c$$

Here $f(x) = x^2$ and $F(x) = \frac{x^3}{3}$. Thus, according to our definition

$$\int_1^4 x^2\mathrm{d}x = F(4) - F(1) = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

Now writing $F(b) - F(a)$ each time we calculate a definite integral becomes laborious so we replace this difference by the shorthand notation $[F(x)]_a^b$. Thus

$$[F(x)]_a^b \equiv F(b) - F(a)$$

Thus, from now on, we shall write

$$\int_a^b f(x)\mathrm{d}x = [F(x)]_a^b$$

so that, for example

$$\int_1^4 x^2\mathrm{d}x = \left[\frac{x^3}{3}\right]_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = 21$$

Example Find the definite integral of $\cos x$ from 0 to $\frac{\pi}{2}$; that is, find $\int_0^{\pi/2} \cos x\,\mathrm{d}x$.

Solution

Since $\int \cos x\mathrm{d}x = \sin x + c$ then

$$\begin{aligned}\int_0^{\pi/2} \cos x\,\mathrm{d}x &= [\sin x]_0^{\pi/2}\\ &= \sin\left(\frac{\pi}{2}\right) - \sin 0\\ &= 1 - 0\\ &= 1\end{aligned}$$

Always remember, that if you use a calculator to evaluate any trigonometric functions, you must work in radian mode.

Find the definite integral of $x^2 + 1$ from 1 to 2; that is; find $\int_1^2 (x^2 + 1)\mathrm{d}x$

Your solution

First perform the integration

$\left[\frac{1}{3}x^3 + x\right]_1^2$.

Your solution

Now insert the limits of integration, the upper limit first, and hence find the value of the integral.

$\left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right) = \frac{10}{3}$ or 3.333 (3dp).

Find $\int_2^1 (x^2+1)\mathrm{d}x$.

Your solution

(This exercise is very similar to the previous one. Note the limits of integration have been interchanged.)

$-\frac{10}{3}$.

Note from these two exercises that interchanging the limits of integration, changes the sign of the answer.

Key Point

If you interchange the limits, you must introduce a minus sign:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

Exercises

1. Evaluate a) $\displaystyle\int_0^1 x^2\mathrm{d}x$, b) $\displaystyle\int_2^3 \frac{1}{x^2}\mathrm{d}x$
2. Evaluate $\displaystyle\int_1^2 \mathrm{e}^x\mathrm{d}x$
3. Evaluate $\displaystyle\int_{-1}^1 (1+t^2)\mathrm{d}t$
4. Find $\displaystyle\int_0^{\pi/3} \cos 2x\mathrm{d}x$
5. Find $\displaystyle\int_0^{\pi} \sin x\mathrm{d}x$
6. Find $\displaystyle\int_1^3 \mathrm{e}^{2t}\mathrm{d}t$

1. a) $\dfrac{1}{3}$, b) $\dfrac{1}{6}$, 2. $e^2 - e^1 = 4.671$. 3. 2.667. 4. $\sqrt{3}/4 = 0.4330$. 5. 2.
6. 198.019

3. Some Integrals with Infinite Limits

On occasions, and notably when dealing with Laplace and Fourier transforms, you will come across integrals in which one of the limits is infinite. We avoid a rigorous treatment of such cases here and instead give some commonly occurring examples.

Example Find the definite integral of e^{-x} from 0 to ∞; that is, find $\displaystyle\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x$.

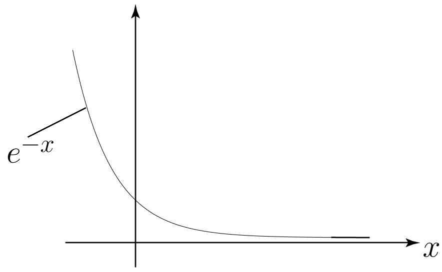
Solution

The integral is found in the normal way:

$$\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}\right]_0^\infty$$

There is no difficulty in evaluating the square bracket at the lower limit. We obtain simply $-\mathrm{e}^{-0} = -1$. At the upper limit we must examine the behaviour of $-\mathrm{e}^{-x}$ as x gets infinitely large. This is where it is important that you are familiar with the properties of the exponential function. If you refer to the graph you will see that as x tends to infinity e^{-x} tends to zero. Consequently the contribution to the integral from the upper limit is zero. So

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x &= \left[-\mathrm{e}^{-x}\right]_0^\infty\\ &= (-\mathrm{e}^{-\infty}) - (-\mathrm{e}^{-0})\\ &= (0) \quad - \quad (-\mathrm{e}^{-0})\\ &= 1\end{aligned}$$



Thus the value of $\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x$ is 1.

Another way of achieving this result is as follows:
We change the infinite limit to a finite limit, b, say and then examine the behaviour of the integral as b tends to infinity, written

$$\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x = \lim_{b\to\infty}\int_0^b \mathrm{e}^{-x}\mathrm{d}x$$

Now,

$$\int_0^b \mathrm{e}^{-x}\mathrm{d}x = \left[-\mathrm{e}^{-x}\right]_0^b = \left(-\mathrm{e}^{-b}\right) - \left(-\mathrm{e}^{-0}\right) = -\mathrm{e}^{-b} + 1$$

Then as b tends to infinity $-\mathrm{e}^{-b}$ tends to zero, and the resulting integral has the value 1, as before.

Many integrals having infinite limits cannot be evaluated in a simple way like this, and many cannot be evaluated at all. Fortunately, most of the integrals you will meet will exhibit the sort of behaviour seen in the last example.

Exercises

1. a) Find $\displaystyle\int_1^{\infty} \mathrm{e}^{-x}\mathrm{d}x$ b) Find $\displaystyle\int_0^{\infty} \mathrm{e}^{-2x}\mathrm{d}x$ c) Find $\displaystyle\int_2^{\infty} \mathrm{e}^{-3x}\mathrm{d}x$

Answers 1. a) e^{-1} or 0.368. b) $\frac{1}{2}$. c) $\frac{1}{3}\mathrm{e}^{-6} = 0.0008$ (4dp).