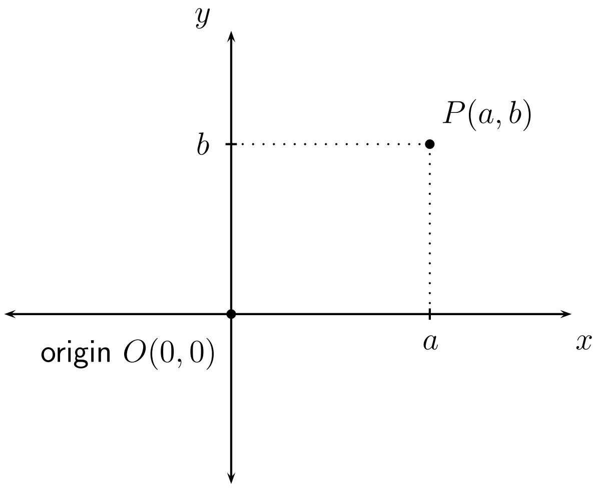
Notes for 'Coordinate systems'

Important Ideas and Useful Facts:

(i) Cartesian plane: The *Cartesian plane* or *xy-plane* consists of all ordered pairs (x, y) as x and y range over all reals numbers, denoted by

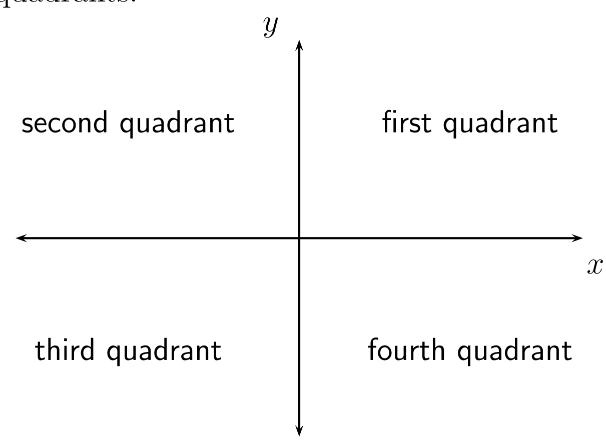
$$\mathbb{R}^2 \;=\; \{(x, y) \mid x, y \in \mathbb{R}\}\ ,$$

containing a *horizontal x-axis* $\{(x, 0) \mid x \in \mathbb{R}\}$ and a *vertical y-axis* $\{(0, y) \mid y \in \mathbb{R}\}$. The axes are perpendicular and intersect at the *origin* $O = O(0, 0)$.



If $P = P(a, b)$ is a point in the xy-plane, then we call a the *x-coordinate* and b the *y-coordinate* of P, obtained by projecting P to the closest points on the x-axis and y-axis respectively. Together, we call the pair (a, b) the *Cartesian coordinates* of P. In the above diagram, the coordinates of P are positive. In general the coordinates can be positive, negative, zero or any mixture of these.

(ii) Quadrants of the Cartesian plane: The Cartesian plane is divided by the coordinate axes into four quadrants.



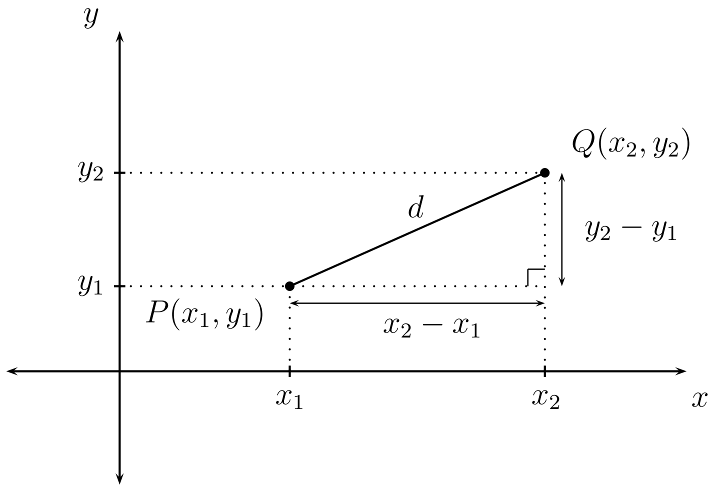
The *first quadrant* consists of points with $(+, +)$ coordinates; the *second* with $(-, +)$ coordinates; the *third* with $(-, -)$ coordinates; the *fourth* with $(+, -)$ coordinates.

(iii) Distance between two points in the Cartesian plane:

The distance d between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \ ,$$

that is, the square root of the sum of the squares of the differences in their coordinates. This follows from the Theorem of Pythagoras, by creating a right-angled triangle with shorter sides parallel to the x and y-axes.



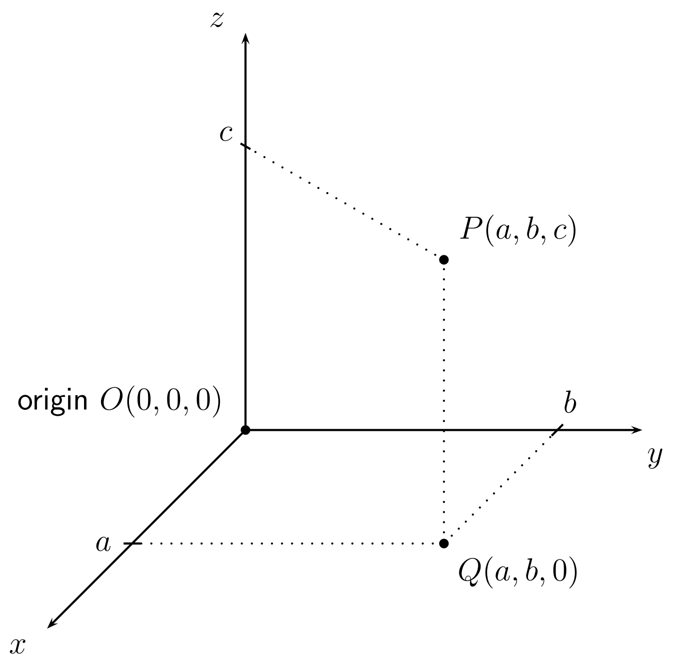
In this diagram, the horizontal side of the triangle has length $x_2 - x_1$ and the vertical side has length $y_2 - y_1$. The length of the hypotenuse is the distance d between P and Q, so that $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, by Pythagoras, and the above formula is immediate by taking positive square roots.

Note that the above diagram is particular in that it assumes the differences in coordinates are both positive. However, the mathematics works and the formula holds in all cases, including when these differences are zero or negative. This is because taking the square of a difference of two real numbers, inside the square root sign, produces the same outcome, regardless of the order in which the difference is taken.

(iv) Coordinates in space: To model positions in space mathematically, we think of the xy-plane now as 'horizontal' (including the y-axis, which we previously thought of as being 'vertical'), and introduce a third z-axis, another copy of the real line, which we think of as the new 'vertical', which is perpendicular to the xy-plane. All axes pass through a common point, again called the *origin*, and denoted by $O = O(0, 0, 0)$. Every point P in space has *coordinates* (a, b, c), for some real numbers a, b and c, which are formed by projecting P to the closest points on the x, y and z-axes respectively. In this way we identify space with the set of triples of real numbers, denoted by

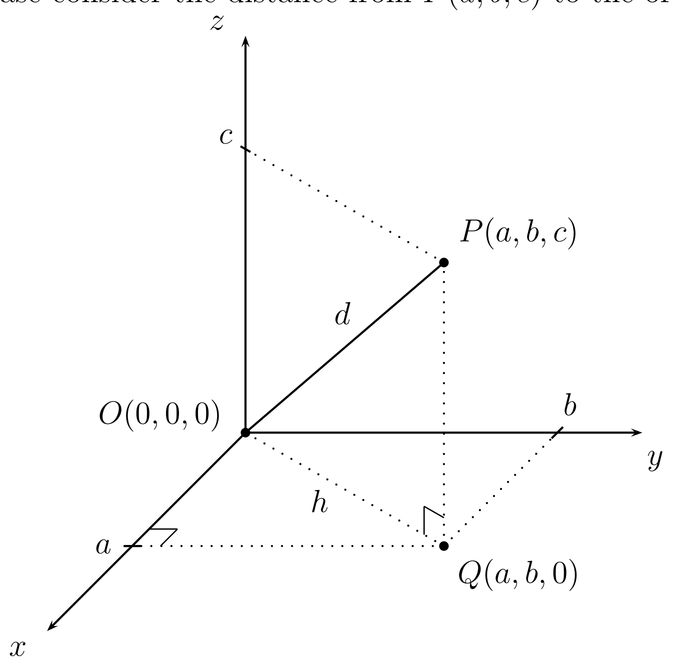
$$\mathbb{R}^3 = \{(a, b, c) \mid a, b, c \in \mathbb{R}\} \ .$$

The x-axis is identified with the subset of triples $\{(x, 0, 0) \mid x \in \mathbb{R}\}$, the y-axis with the subset $\{(0, y, 0) \mid y \in \mathbb{R}\}$ and the z-axis with the subset $\{(0, 0, z) \mid z \in \mathbb{R}\}$. In the following diagram, only the positive parts of the axes emanating out of the origin are drawn, and you have to imagine the negative halves of the axes extending behind the page (for the x-axis), to the far left (for the y-axis) and far below (for the z-axis).



Imagine the point P hovering in space above the (horizontal) xy-plane. We project down (vertically) onto the xy-plane to get to some point Q, say, which has projections a and b on the x and y-axes respectively. The closest point on the z-axis to Q is the origin, with z-value 0, so the coordinates of Q become $(a, b, 0)$. If we move from P directly across (horizontally) to the z-axis then we meet it at some point, say c. Then the triple of real numbers (a, b, c) that we get in this way become the coordinates of P.

(v) Distance between two points in space: The formula for the distance between two points in space generalises the earlier formula for the distance between two points in the xy-plane. As a special case consider the distance from $P(a, b, c)$ to the origin $O(0, 0, 0)$.



Add a line segment joining O to P, whose length is the distance d from P to O. The points O, P and Q form a right-angled triangle with shorter side-lengths c (the closest point to P on the z-axis) and the distance from O to Q, which we denote by h.

By Pythagoras, $d^2 = h^2 + c^2$. But, by Pythagoras again, $h^2 = a^2 + b^2$, since h is the length of the hypotenuse formed by the right-angled triangle using O, Q and the point on the x-axis labelled by a, whose shorter side-lengths are now just a and b. Hence

$$d^2 \;=\; h^2 + c^2 \;=\; a^2 + b^2 + c^2 \,.$$

Taking the square root, we deduce that the distance from $P(a, b, c)$ to the origin $O(0, 0, 0)$ is

$$d \;=\; \sqrt{a^2 + b^2 + c^2} \,.$$

This diagram assumed a, b and c are positive, but the mathematics works regardless of the signs of the coordinates of P.

More generally, the distance d between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by the formula

$$d \;=\; \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \,,$$

that is, the square root of the sum of the squares of the differences in their coordinates. This can be shown using a similar but more elaborate diagram to the above, or alternatively can be deduced from the special case, by translating both of the points parallel to themselves so that one of them goes to the origin. If P is translated to $O(0, 0, 0)$ then Q is translated to the point with coordinates $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$, and the general result now follows from the special case.

(vi) The midpoint between two points in the plane or in space: The midpoint of the line segment joining points $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ in space has coordinates

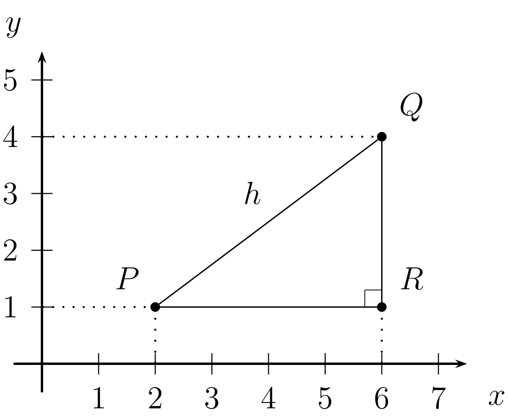
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right),$$

that is, the triple formed by taking the averages of the respective coordinates of P and Q. By ignoring the z-coordinate in the previous formula, we see that the midpoint of the line segment joining points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Examples:

1. Consider the points $P(2, 1)$, $Q(6, 4)$ and $R(6, 1)$ in the xy-plane.



They form vertices of a right-angled triangle with hypotenuse h joining P and Q. The horizontal shorter side-length of this triangle is the distance from P to R, which is 4 units, the difference in their x-coordinates, whilst the vertical shorter side-length is the distance from Q to R, which is 3 units, the difference in their y-coordinates. By Pythagoras (and the distance formula described above),

$$h \;=\; \sqrt{4^2+3^2} \;=\; \sqrt{25} \;=\; 5\,.$$

The midpoint M between P and Q has coordinates which are the averages of the coordinates of P and Q, that is,

$$M \;=\; \left(\frac{2+6}{2}, \frac{1+4}{2}\right) \;=\; \left(4\,, \frac{5}{2}\right)\,.$$

The distance from M to each of P and Q should be the same, namely $h/2 = 5/2$. We can check: the distance from M to P is

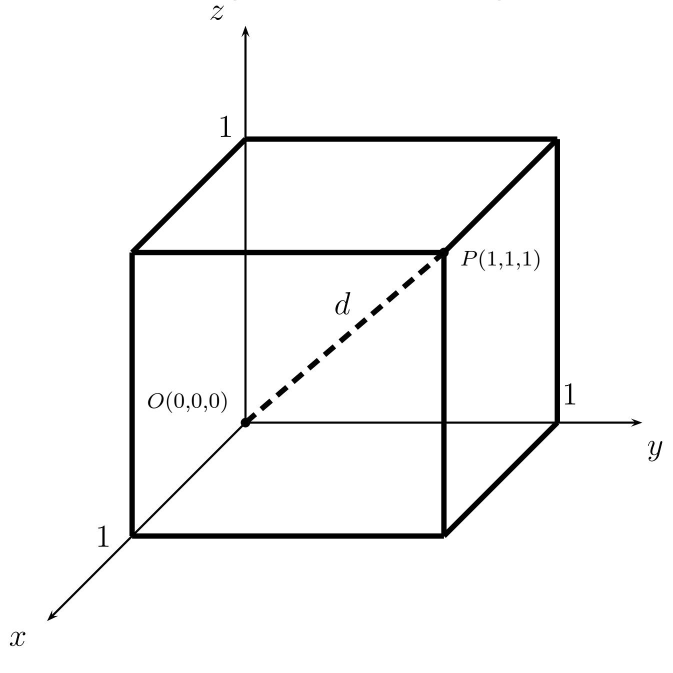
$$\sqrt{(2-4)^2+(1-5/2)^2} \;=\; \sqrt{4+9/4} \;=\; \sqrt{\frac{25}{4}} \;=\; \frac{5}{2}\,,$$

and the distance from M to Q is

$$\sqrt{(6-4)^2+(4-5/2)^2} \;=\; \sqrt{4+9/4} \;=\; \sqrt{\frac{25}{4}} \;=\; \frac{5}{2}\,,$$

both as expected.

2. Consider the *unit cube* formed by taking side lengths all equal to 1 unit. One can see by a direct argument that the length of the main diagonal of the cube is $\sqrt{3}$ units.



This matches our distance formula, because we can position the unit cube with respect to the coordinate axes, so that one vertex is at the origin $O(0,0,0)$ and the opposite vertex at the point $P(1,1,1)$. If we denote by d the length of the diagonal joining O to P, then

$$d \;=\; \sqrt{1^2+1^2+1^2} \;=\; \sqrt{3}\,,$$

as expected.