Bayesian Cognitive Modeling applied to Signal Detection Theory: the Mirror Effect in a Perceptual Task as a special guest

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Abstract:
Signal Detection Theory constitutes one of the most well-known and broadly applied statistical models within cognitive science, to describe a wide range of phenomena where a decision-making system is faced with the task of detecting the presence of a specific signal, within a noisy environment. Within recognition memory studies where subjects' responses is compared between classes of stimuli that are deferentially recognized, it has been found systematically that this difference is reflected in the identification of both targets and lure stimuli, as measured by hit and false alarm rates in Signal Detection Theory (the implied order of the signal and noise distributions has led to the identification of this pattern of response as "the mirror effect"). Since this phenomena has been predominantly tested for recognition memory tasks, most attempts to explain the observed pattern of response involves theorizing about high-level processes engaged in the study phase. To test the generalizability of this pattern to other domains where signal detection theory has been applied, we designed a perceptual task with two levels of discriminability which were defined by manipulating an optical illusion. After conducting a step by step replication of the mean-performance based analysis reported in the literature, we present evidence of the mirror effect outside recognition memory. We then developed a more detailed model based analysis, using signal detection theory and hierarchical Bayesian methods to assess the existence of the mirror effect at both the group and individual level.

Keywords: Signal Detection Theory; Bayesian Modeling; Perception; Recognition Memory; Mirror Effect; Ebbinghaus Illusion.

1 Introduction

We live in a world full of noisy information, changes and uncertainty. Every decision-making system is constantly exposed to a huge variety of stimulation that does not always provide relevant information about the current state of their environment and the currently contingency rules operating. Once organisms have defined the most relevant contingency rules within their environment, the detection of specific stimuli (i.e. signals) while discarding all non-informative stimuli (i.e. noise), becomes a major priority task to determine the actual state of the world and the most adaptive way to interact with it. We can think of a huge variety of examples to illustrate the importance of this signal-detection kind of task, from the vulnerable animal that has to determine whether or not the sound it has heard in the woods represents a threat or not to decide between keep on feeding or start running; to the physician that has to determine if the tomography scan she's inspecting contains evidence of a cancerous tumor or not, to decide whether she's going to submit her patient to a very invasive treatment.

Signal Detection Theory (SDT) has provided a very intuitive framework to understand the problem faced by any decision-making system that has to decide whether or not a signal is present in their current environment. The statistical model provided by SDT has been widely explored, applied and developed within and outside Psychological Science to account for a huge variety of phenomena related to this kind of dilemma.

1.1 Signal Detection Theory Model

Signal Detection Theory appeared for the first time in 1954. Just like many other scientific and technological developments that were born around that time, its purpose was to contribute to the solution of a necessity that aroused with the Second World War: the correct functioning of radars (Peterson, Birdsall y Fox, 1954). However, not long after that, Signal Detection Theory was brought to the Psychological domain to study organic perception (Tanner y Swets, 1954; Swets, Tanner y Birdsall, 1961).

The SDT model works as a descriptive model that embodies the problem faced by any decision-making system who finds itself in the necessity to decide whether or not a signal is present within its immediate environment, in order to produce the most appropriate response to the contingencies its presence would be announcing, (Tanner y Swets, 1954; Swets, 1973). The core contribution of the model is that it captures the idea that there's always a certain level of uncertainty involved in this kind of tasks, a notion that is represented by using probability distributions (typically, normal distributions) to describe the variability that both signal and noise stimuli present in terms of the values with which they can be associated along a "decision axis" that contains whatever kind of information the system is sensing and processing to produce a detection judgement ("Yes, the signal is there" / "No, the signal is absent"). This variability can be explained either by assuming stimuli vary in the way they appear in the world (e.g. there's a lot of variability in terms of animals that could be considered a threat) or because there's always variability in the way decision-making systems read their environment (as observed in Psychophysics). Uncertainty comes from the notion that a proportion of the noise distribution overlaps the signal distribution, causing that a range of values on the decision axis are associated with either kind of stimuli. This is what makes the task complicated, decision-making systems can no longer trust merely on the evidence they perceive to emit a judgement. Detection tasks do not depend on the efficiency of the detection system alone; it also requires the tested system to consider all the information it has about its environment, (Killeen, 2014; Wickens 2002).

Within Psychological Science, the SDT model has been used as a decision model that describe the underlying mechanisms of the production of binary judgements about the presence/absence of a signal stimulus, weighting the information they're receiving at the current moment with all the previous information they have about the environment's structure. The latter can be, in general terms, classified in two big categories: 1) Information regarding the probabilistic nature of their environment and 2) Information about the consequences at risk, (Killeen, 2014). When organisms are presented with a stimulus that comes from the overlapping area of the signal and noise distributions, they have to decide based either on which is the most likely kind of stimulus to appear in the current situation, or which type of mistake would be the least harmful or expensive. For example, let's think again of the physician who's evaluating some tomography scans: if she's not confident enough about the information being presented in the scan to diagnose the presence of a certain disease, she would have to take into account the information she has about the statistics with which this disease is presented in a person with the same characteristics as the patient, as well as the risks involved in letting a sick patient go without a treatment versus treating someone who's not sick with a very invasive procedure.

(SDT is a decission model that assumes that the system places a decision criterion over the evidence axis, 11)

(One of the most valued features of the SDT model is its ability to differentiate the influence that discriminability and the own system's bias play in)

(To account for the discriminability of the signal vs noise stimuli contained in the detection task, SDT has a d' parameter that represents the distance between the mean of the noise and the signal distributions.)

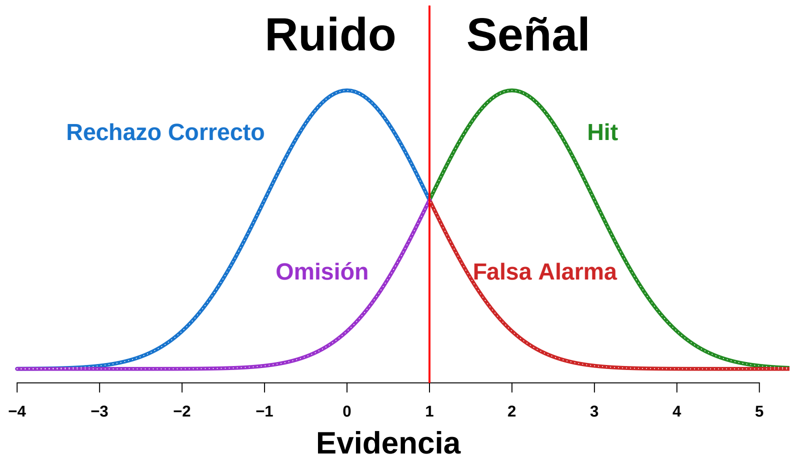


Figure 1: Graphical representation of the Signal Detection Theory model. TRANSLATION PENDING

(It is interesting to note that the SDT model, despite being originally develop as a purely statistical model to account for the performance of radars detecting electrical signals, (Lee, 2016). Its capacity to differentiate between the influence of the decision-making system's bias and the discriminability of the stimuli that compose the task, has proven to be not just extremely useful to...)

To this day, SDT is one of the best-known models in Psychological Science. SDT has been applied to the study of a huge variety of phenomena that revolve around a binary decisions problem and it has been done by using the SDT model both as a tool for data analysis and as a cognitive model to describe the cognitive processes that underlie to the emission of a binary decision judgement. One can find examples of its application to the study of perception (Rosenholtz, 2001; Pessoa, Japee y Ungerleider, 2005; Wallis y Horswill, 2007), where it was first adopted within Psychology; the emission of clinical diagnosis (Grossberg y Grant, 1978; Swets, Dawes y Monahan, 2000; Boutis, Pecaric, Seeto y Pusic, 2010); the syntomathology associated with all kinds of clinical conditions (Westermann y Lincoln, 2010; Bonnel y cols., 2002; Brown, Kosslyn, Breiter, Baer y Jenike, 1994; Naliboff y Cohen, 1981); eyewitness testimonies (Gronlund, Wixted y Mickes, 2014; Wixted y Mickes, 2014; Wixted, Miches, Dunn, Clark y Wells, 2016), and an incredibly ongroing etcetera (Gordon y Clark, 1974; Nuechterlein, 1983; Harvey Jr., Hammond, Lusk y Mross, 1992; Verghese, 2001).

The huge amount of publications revolving around SDT is not constrained to its application to the study of different phenomena. Its utility to understand the broad range of phenomena already described has motivated the production of several tutorials and manuals, which are aimed provide an introduction to the benefits of the model (Killeen, 2014); a detailed description of its assumptions and implications (McNicol, 2005; Wickens 2002); or a guide to conduct data analysis under a SDT framework (Stainslaw y Todorov, 1999).

1.2 The mirror effect

Among the previously described variety of phenomena where SDT has been applied, its relevance to the study of recognition memory is definitely one of the most noticeable ones.

2 Experiments

To evaluate the generalizability of the Mirror Effect to, we proposed to conduct a perceptual task where participants had to register their response to a binary question involving the detection of a specific type of stimuli. To make our task to work as an analogous of the recognition memory tasks where the mirror effect has been reported, we had two categories of stimuli which would vary in terms of their discriminability.

In our binary decision task, participants had to indicate whether two circles presented on screen for comparison had the same size (signal trials) or not (noise trials). The same task was used in two different settings, referred as Experiment 1 and Experiment 2. On Experiment 1, one of these two circles was used as the center circle of an Ebbinghaus illusion, while in Experiment 2, both circles were part of their own Ebbinghaus illusion. Details about the stimuli design are discussed in a following section.

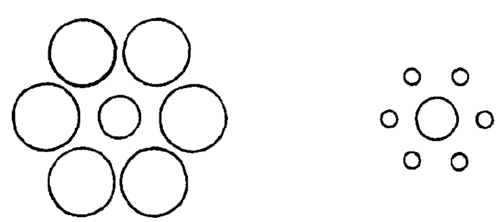


Figure 2: Illustration of the Ebbinghaus Illusion. The center circles have the same size, (illustration taken from Massaro y Anderson, 1971)

(The Ebbinghaus Illusion (see Fig 2), is a well-known and widely studied optical illusion that has been typically explained as a.... La figura de Ebbinghaus -también conocida como Círculos de Titchener- está intrínsecamente vinculada a una ilusión óptica donde la percepción del tamaño de un elemento central es alterada por el contraste que tiene con elementos circundantes (la ilusión de Ebbinghaus). (Coren, 1971; Coren y Miller, 1974; De Fockert, Davidoff, Fagot, Parron y Goldstein, 2007). The Ebbinghaus illusion can produce two different effects....)

In the present study, we used what is known about the variables that have an impact on the intensity of the illusion (Massaro y Anderson, 1971; Girgus, Coren y Agdern, 1972; Roberts, HArris y Yates, 2005) to construct the two levels of discriminability that we require to make our task analogous to those conducted within recognition memory. Figure 3 summarizes the results obtained in a study where the number (x-axis) and size (y-axis) of the external circles of the Ebbinghaus illusion were manipulated to assess their effect on the intensity of the illusion. According to this same Figure, which presents the mean estimations registered for the diameter of the central circle of each Ebbinghaus illusion presented, the discrepancy between the real diameter and participants' estimation increased when more external circles were included in the figures; the vertical distance between every line suggests that increasing the difference between the size of the external circles and the center circle has an impact on the magnitude of the illusion. Both, the Overestimation effect and the Underestimation effect can be seen in the presented figure, (Massaro y Anderson, 1971).

In our study, we used Massaro and Anderson findings (1971), to design perceptual classes A and B:

- **Class A ("Higher discriminability"):** Ebbinghaus illusions with fewer external circles (with two levels: 2 or 3 external circles)
- **Class B ("Lower discriminability"):** Ebbinghaus illusions with fewer external circles (with two levels: 7 or 8 external circles)

In all cases, external circles meant to induce an Overestimation effect were presented with a diameter of $0.5cm$, while external circles meant to induce an Underestimation effect had a constant diameter of $6cm$, to ensure that their diameter will always be either more than $2x$ bigger, or $0.5x$ smaller than the size of the central size (which could vary from $1cm$ to 3, with increments of $0.5cm$).

En ningún experimento se controló la distancia entre los círculos centrales y el halo de círculos externos. Los círculos externos fueron acomodados de acuerdo a los números de círculos externos considerados como parte de la condición difícil (siete y ocho), distribuyéndolos de manera uniforme

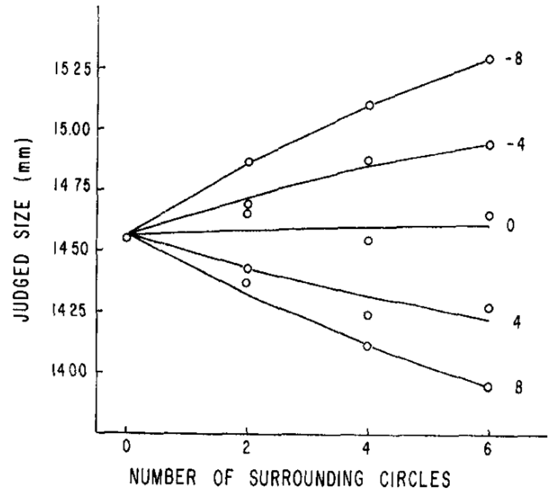


Figure 3: Results obtained in an experiment where the number and size of the external circles of an Ebbinghaus illusion were manipulated to assess their effect on the intensity of the illusion, (Massaro y Anderson, 1971).

y equidistante en torno al círculo central. Estos halos de siete y ocho círculos externos fueron usados como base para la construcción de los estímulos de la condición fácil (con dos o tres círculos externos), eliminando círculos y respetando la ubicación de los restantes. Este procedimiento se realizó para las figuras con efecto de subestimación y sobrestimación. Se procuró que en las figuras con dos círculos externos, estos estuvieran enfrentados en puntos opuestos del círculo central y en las figuras con tres círculos centrales, que rodearan al círculo central en ángulos de ciento veinte grados. La configuración de los halos de círculos externos resultantes permaneció constante para todos los tamaños de círculo central, haciendo que la distancia entre ambos elementos sea distinto para cada valor.

2.1 Experiment 1

On Experiment 1, participants had to compare the size of a constant stimulus (a circle with a diameter of $2cm$, which always appeared at the left side of the screen in the exact same place across trials) to the size of the central circle of a Ebbinghaus figure displayed on the right side of the screen.

Stimuli presented on Experiment 1 were constructed following a 5x2x2 factorial design, illustrated in Figure 4, for each class of stimuli A and B. On Experiment 1, the reference circle with a constant diameter of $2cm$ was compared with the center circle of an Ebbinghaus illusion which could have five possible diameter measures (from $1cm$ to $3cm$ in increments of $0.5cm$); could have been presented to induce either an effect of Overestimation (external circles with a diameter of $0.5cm$) or Underestimation (external circles with a diameter of $6cm$) and two different levels of "Number of external circles" per class (2 or 3 for the A class and 7 or 8 for the B class). Given this design, for each class of stimuli we had 16 different "Noise stimuli" and 4 different "Signal stimuli". To account for the variability of choices made by participants, and to make both signal and noise trials equal in terms of the number of times they were presented to participants, we repeated each of the 16 noise stimuli 10 times, while presenting each of the 4 signal stimuli 40 times. In the end, this lead to a total of 160 noise trials and 160 signal trials per class of stimuli, with a total of 640 trials for the entire experiment.

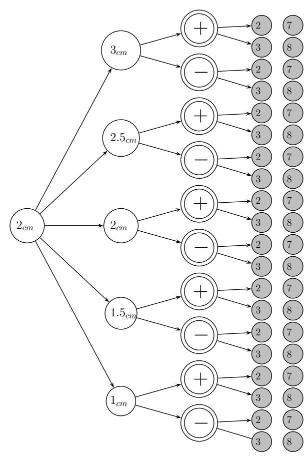


Figure 4: Stimuli design for Experiment 1: A 5x2x2 factorial design was used to construct

2.2 Experiment 2

On Experiment 2, the main detection task remained the same "Are the two central circles the same size?". However, it differ from Experiment 1 because this time, the two circles that participants were asked to compare were used as the central circle of their own Ebbinghaus figure.

A diferencia del Experimento 1, donde uno de los círculos a comparar era constante, en el Experimento 2 se varió el diámetro de los dos círculos a comparar. Para ello se ello se utilizaron los mismos cinco tamaños de círculo central (de 1 a 3 cm en saltos de 0.5 cm) y por lo tanto, con cinco combinaciones posibles para las Parejas-señal. Así mismo, se formaron cinco Parejas-ruido juntando arbitrariamente valores de círculo central que guardasen una diferencia de 0.5 cm entre sí -1 vs 1.5; 1.5 vs 2; 2 vs 2.5 y 2.5 vs 3 cm- con una quinta pareja con una diferencia de 1 cm entre los valores de círculo central intermedios -1.5 cm vs 2.5 cm-. Por cada una de estas 10 parejas, se crearon cuatro variaciones por condición, de acuerdo con las combinaciones posibles de niveles de *'número de círculos externos'* (2 círculos externos a ambos lados, 3 círculos externos a ambos lados, 2 en izquierdo y 3 en derecho, y 3 en izquierdo y 2 en derecho en la condición fácil; 7 círculos externos a ambos lados, 8 círculos externos ambos lados, 7 círculos del lado izquierdo y 8 en el derecho y 8 círculos en el lado izquierdo y 7 en el derecho en la condición difícil). En total, el Experimento 2 estuvo compuesto por 80 parejas diferentes de figuras de Ebbinghaus cuyos círculos centrales debían compararse, 40 con la señal y 40 con el ruido y 20 de cada uno por condición.

Cada una de las 80 parejas diseñadas para el Experimento 2 se presentó 8 veces, en cuatro colores diferentes (púrpura, anaranjado, azul y verde) para prevenir la fatiga de los participantes, contrabalanceando la posición de las ilusiones de sobrestimación y subestimación a la derecha o izquierda de la pantalla. De tal forma que el Experimento 2 estuvo compuesto por un total de 640 ensayos, 320 por cada tipo de ensayo (ruido y señal) y 160 por cada clase.

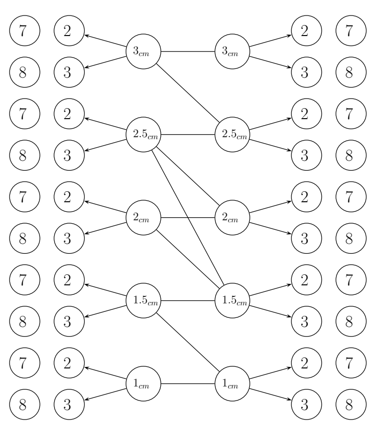


Figure 5: Stimuli design for Experiment 2:

2.3 Materials

The experiment was programmed and executed in **PsychoPy v.12** (Peirce, 2007), using a Mac computer ($59.5cm$ x $34cm$ screen). Experimental sessions were ran inside an isolated room, with participants being sat in a fixed chair $1.10m$ away from the screen.

2.4 Participants

A total of forty-one students (ages 18 to 21) of the Psychology School of the National Autonomous University of Mexico participated in one of the two experiments conducted (20 in Experiment 1 and 41 in Experiment 2). Both experiments were conducted during the same period of time, with participants being assigned to one of these, without them knowing that there was more than one Experiment.

All participants signed an Informed Consent letter prior to the experiment, where we informed that the experiment had a duration of 40 minutes and that they could leave at any minute, emphasizing the importance of they staying until the end of the task.

2.5 Procedure

The detection task was conducted in two different ways: first, as a binary choice task and then, with a Confidence scale in which participants were asked to rank their trust in their binary response in a scale from 1 to 3. The Confidence scale was presented on screen to remind participants that 1 meant "Very insecure" and 3 "Very confident", and it was programmed to translate every registered response into a larger scale (with values from 1 to 6) depending on their previous binary response. According to this new scale, extreme values 1 and 6 referred a greater confidence in their previous response (with 1 being "Very sure it was a Noise trial" and 6 being "Very sure it was a Signal trial"), while values in the middle could serve as indicators of greater uncertainty (so that 2 and 3 would mean "I'm very insecure about the previous trial being a Noise trial" or ""I'm very insecure about the previous trial being a Noise trial). This transformation procedure has been used in recognition memory studies where the mirror effect had been reported (Glanzer y Adams, 1990), with the intention of making the rating task easier for participants, while guaranteeing a match between their binary response and the value assigned in the full scale.

The experiment consisted of 640 trials, with each one of these following the next structure:

- **First part: Binary choice Yes/No**

  At the beginning of each trial, the pair of central circles that participants had to compare appeared on screen along the reminding legends "Do the central circles have the same size?" on top of the screen and the answer keys on the bottom "S = Yes, N = No". After 1.5 seconds, the stimuli disappeared from the screen to prevent habituation, while the reminding legends remained until a response was registered.

- **Second part: Confidence Scale**

  Once participants had registered their binary response to the Yes/No task, the Confidence scale was shown below the instructional question "How certain are you about your response?". The Confidence scale shown contained numbers 1, 2 and 3 along with the legends "Very insecure", "More or less sure" and "Very confident".

- **Third part: "Space-bar to continue"**

  Right after a response to the Confidence scale was registered, a third and final screen was shown to indicate the end of the trial with the legend "Press the space-bar to move on to the next trial" . This final screen was included with the explicit intention of avoiding participants' fatigue by giving them the chance to take a break before continuing to be exposed to more Ebbinghaus figures.

At the end of the 640 trials, participants were shown the total number of right and wrong responses registered (which was included just to provide participants' with a sense of ending). We never told participants the purpose of the experiment and more importantly, participants were never told anything about the presence of two distinct classes of stimuli coexisting within the same task.

3 Results

Both experiments shown evidence of the patterns of response identified as part of the mirror effect in more than three fourths of the participants. In Experiment 1, where a total of 20 participants did the task, 17 showed the mirror effect pattern of response in the binary task and 18 in the Confidence scale. In Experiment 2, 18 participants showed evidence of the mirror effect patterns of response in both the binary task and the confidence rate task. These proportions shown to be statistically significant when analyzed with a simple binomial test.

It is important to note that prior to the conduction of any kind of formal data analysis, we examined the responses registered for every participant to make sure that the data obtained was consistent with the task we proposed. While reviewing participants' responses registered in every consecutive trial we found that Participant No.1 in Experiment 2 spent the very first eighty trials responding to the answer key for "No", (view Fig 6). This participant was excluded from any further data analysis.

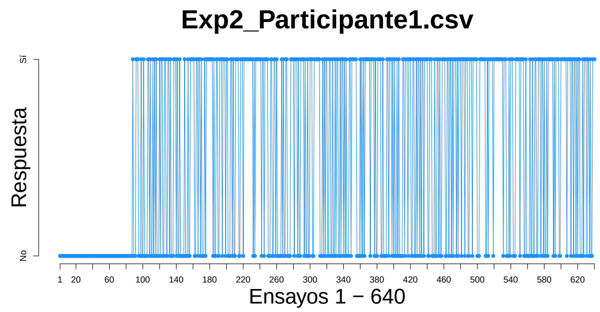


Figure 6: Yes/No responses registered on each of the 640 trials by participant No.1 in Experiment 2. A clear pattern can be observed during the first 80 trials, which suggest that participant's responses were not made according to the instructions or the stimulus presented.

To make the results obtained in the present study comparable to what has been reported in the recognition memory study where the mirror effect has been exposed, we conducted a step by step replication of the data analysis reported by Glanzer and Adams, (1990);

1. Comparing d' across classes of stimuli

   A t-test was conducted to compare the estimations for d' obtained for each class of stimuli, where statistically significant differences were found in the direction that we expected according to our experimental design.

2. Comparing the Hit and False Alarm rates obtained in each class of stimuli

   Two separate t-tests were conducted to compare separately the Hit rates and the False alarm rates registered between each class of stimuli. Statistically significant differences were found in the same direction as reported in the recognition memory literature.

3. Comparing the mean Confidence ratings registered in each class of stimuli

   Two separate t-tests were conducted to compare separately the mean Confidence rates registered for the Signal and Noise trials in each class of stimuli. Statistically significant differences were found in the same direction as reported in the recognition memory literature.

4 Bayesian Cognitive Modeling

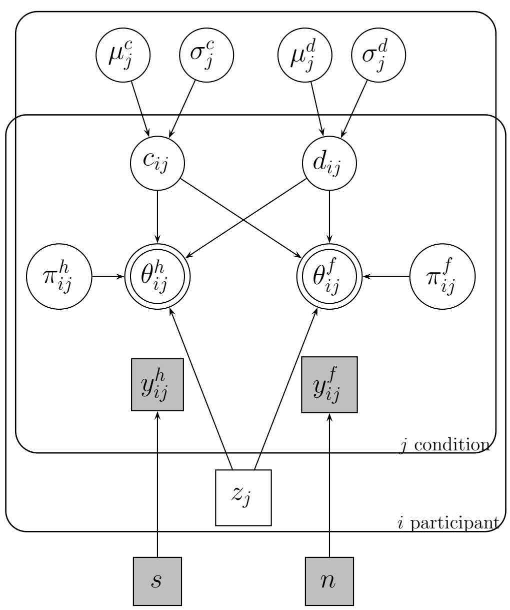
The present study aims to encourage the application of Bayesian methods to improve the .... when working with cognitive models.

4.1 Contaminant Models

One of the many advantages that comes with the application of Bayesian methods to the development of cognitive models is that it allows us to build models that test the general assumption that data obtained can be explained as coming from... (Chávez, M, Villalobos, E., Baroja, J.L & Bouzas, A., 2017; Velázquez, C., Villarreal M., & Bouzas, A., 2019)

In the present study we worked with two different Contaminant models, which aimed to look for contaminant data at the level of both the overall responses registered during the whole experiment and the individual responses registered during each trial. The first, general approach allows us to make sure that the observed pattern of Hits and False alarms made by each participants can be explained within a SDT framework, while the second, more specific approach allow us to detect (just like we observed with Participant No.1 in Experiment 2 in Fig. 6).

The general contaminant model is presented in Figure 7



$$\mu_j^c \sim \text{Gaussian}(0, 0.7)$$
$$\mu_j^d \sim \text{Gaussian}(0, 1)_{T(0,6)}$$
$$\sigma_j^c, \sigma_j^d \sim \text{Uniform}(0, 4)$$

$$d_{ij} \sim \text{Gaussian}(\mu_j^d, \sigma_j^d)$$
$$c_{ij} \sim \text{Gaussian}(\mu_j^c, \sigma_j^c)$$

$$\pi_{ij}^h, \pi_i^f \sim \text{Beta}(1, 1)$$

$$\theta_{ij}^h \leftarrow \begin{cases} \pi_{ij}^h & \text{if } z_j = 0 \\ \phi(\frac{1}{2}d_{ij} - c_{ij}) & \text{if } z_j = 1 \end{cases}$$

$$\theta_{ij}^f \leftarrow \begin{cases} \pi_{ij}^f & \text{if } z_j = 0 \\ \phi(-\frac{1}{2}d_{ij} - c_{ij}) & \text{if } z_j = 1 \end{cases}$$

$$z_j \sim \text{Bernoulli}(0.5)$$

$$y_{ij}^h \sim \text{Binomial}(\theta_{ij}^h, s)$$
$$h_{ij}^f \sim \text{Binomial}(\theta_{ij}^f, s)$$

Figure 7: Latent-Mixture model to detect contaminant participants withn each sample, based on their overall performance during the task

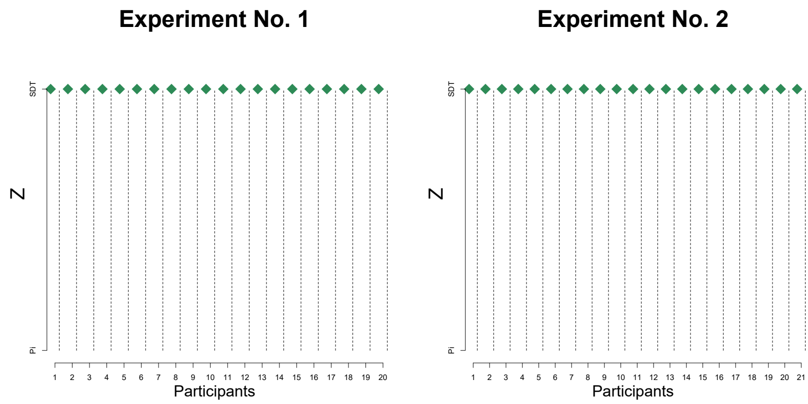


Figure 8: Latent-Mixture model to detect contaminant participants withn each sample, based on their overall performance during the task

4.2 Hierarchical SDT Model

To start with the Bayesian Cognitive modeling of the data we collected, we used a simple Hierarchical extension of the Signal Detection Model, as presented in the Graphical Model.

4.3 Comparing d' across classes of stimuli

4.4 Comparing c across classes of stimuli

4.5

4.6

5 Discussion

6 Conclusion

7 Acknowledgements

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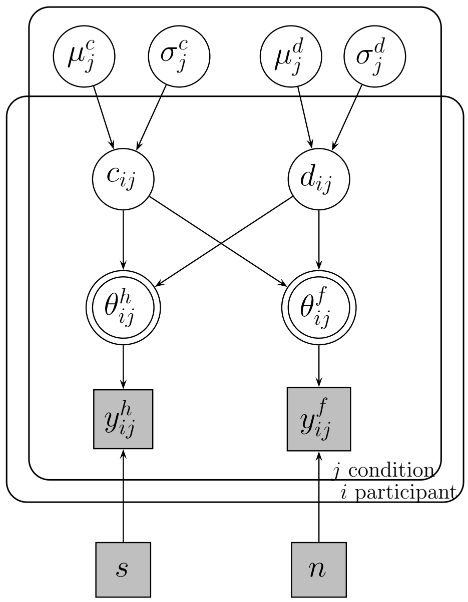

$$\mu_j^d \sim \text{Gaussian}(0, 1)_{T(0,6)}$$
$$\mu_j^c \sim \text{Gaussian}(0, 0.7)$$
$$\sigma_j^c, \sigma_j^d \sim \text{Uniform}(0, 4)$$

$$d_{ij} \sim \text{Gaussian}(\mu_j^d, \sigma_j^d)$$
$$c_{ij} \sim \text{Gaussian}(\mu_j^d, \sigma_j^c)$$

$$\theta_{ij}^h = \phi(\tfrac{1}{2}d_{ij} - c_{ij})$$
$$\theta_{ij}^f = \phi(-\tfrac{1}{2}d_{ij} - c_{ij})$$

$$y_{ij}^h \sim \text{Binomial}(\theta_{ij}^h, s)$$
$$y_{ij}^f \sim \text{Binomial}(\theta_{ij}^f, n)$$

Figure 9: Graphical model for the Hierarchical extension of the SDT model. This model assumes that both individual d' and individual c values are drawn from a group-level normal distribution.

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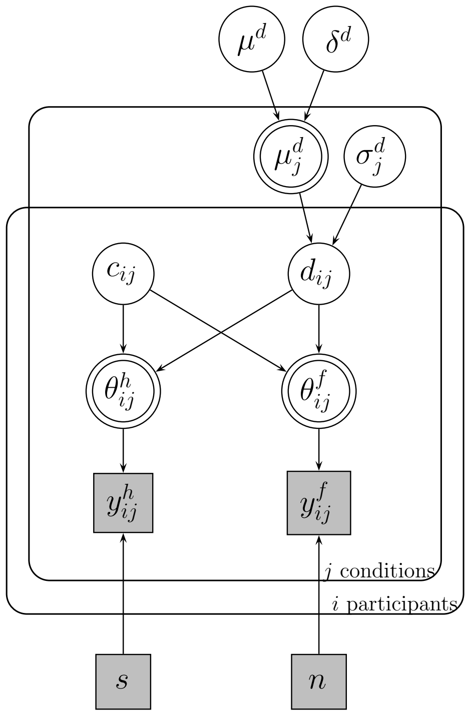

$$\mu^d \sim \text{Gaussian}(0,1)_{T(0,6)}$$
$$\delta^d \sim \text{Gaussian}(0,1)_{T(0,6)}$$

$$\mu^d_A = \mu^d + \tfrac{\delta}{2}$$
$$\mu^d_B = \mu^d - \tfrac{\delta}{2}$$
$$\sigma^d_j \sim \text{Uniform}(0,5)$$

$$d_{ij} \sim \text{Gaussian}(\mu^d_j, \sigma^d_j)$$
$$c_{ij} \sim \text{Gaussian}(0, 0.7)$$

$$\theta^h_{ij} = \phi(\tfrac{1}{2}d_{ij} - c_{ij})$$
$$\theta^f_{ij} = \phi(-\tfrac{1}{2}d_{ij} - c_{ij})$$

$$y^h_{ij} \sim \text{Binomial}(\theta^h_{ij}, s)$$
$$y^f_{ij} \sim \text{Binomial}(\theta^f_{ij}, n)$$

Figure 10: A boat.

linewidth linewidth.

Figure 11: A boat.

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