THE UNIVERSITY OF SYDNEY
MOOC INTRODUCTION TO CALCULUS

Notes for 'Distance and absolute value'

Important Ideas and Useful Facts:

(i) Absolute value: If $\alpha \in \mathbb{R}$ then the *absolute value* or *magnitude* of α, denoted by $|\alpha|$, is the distance from α to 0, regarded as points on the real number line, so that

$$|\alpha| \;=\; \begin{cases} \alpha & \text{if } \alpha \geq 0\ , \\ -\alpha & \text{if } \alpha < 0\ . \end{cases}$$

Note that $|\alpha|$ just reproduces itself if α is positive or zero, but takes its negative (which turns out to be positive) if α is negative. For example, $|2| = 2$ and

$$|-2| \;=\; -(-2) \;=\; 2\ .$$

The absolute value is always nonnegative.

(ii) Useful laws, including the triangle inequality: The following laws hold, for all $\alpha, \beta, \gamma \in \mathbb{R}$, with $\gamma \neq 0$:

$$|-\alpha| \;=\; |\alpha| \;=\; \sqrt{\alpha^2}\ , \quad |\alpha\beta| \;=\; |\alpha||\beta|\ , \quad |\alpha/\gamma| \;=\; |\alpha|/|\gamma|\ , \quad |\alpha+\beta| \;\leq\; |\alpha|+|\beta|\ ,$$

the last of which is known as the *triangle inequality*. For example,

$$|2| \;=\; |-2| \;=\; \sqrt{2^2} \;=\; \sqrt{(-2)^2} \;=\; \sqrt{4}\ ,$$

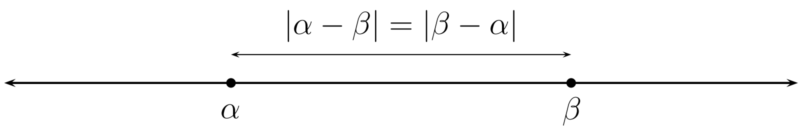
$$|2||-3| \;=\; |2\times(-3)| \;=\; |-6| \;=\; 6 \qquad \text{and} \qquad \frac{|2|}{|-3|} \;=\; \left|\frac{2}{-3}\right| \;=\; \frac{2}{3}\ .$$

Observe further that

$$|2+(-3)| \;=\; |-1| \;=\; 1 \;<\; 5 \;=\; 2+3 \;=\; |2|+|-3|\ ,$$

consistent with the triangle inequality (using "less than" in this particular case, which is included in "less than or equals"). The reason for the terminology "triangle inequality" may seem cryptic, but is related to the fact that the shortest distance between any two points P and Q in the plane or in space is along the straight line joining them. One may think of this straight line as one edge of a triangle made with P and Q and a third point R typically not on this shortest straight line segment. In our context with absolute values, all points are constrained to lie in the real line, so the "triangles" PQR that arise turn out to be degenerate (having no area).

(iii) Distance between two points in the real line: If $\alpha, \beta \in \mathbb{R}$ then $|\alpha - \beta|$ is the distance between α and β, as points on the real number line.



Note that the distance between α and $-\beta$, or between $-\alpha$ and β is

$$|\alpha+\beta| \;=\; |\alpha-(-\beta)| \;=\; |\beta-(-\alpha)|\ .$$

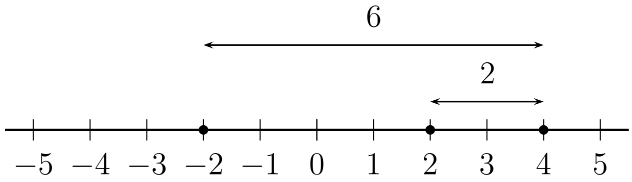
Examples:

1. The distance between 2 and 4 on the real number line is

$$|2-4| = |4-2| = |-2| = |2| = 2\ .$$

By contrast, the distance between -2 and 4 is

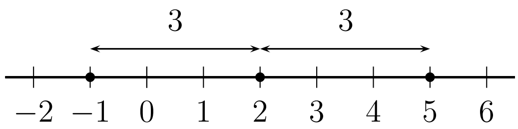
$$|-2-4| \ = \ |4-(-2)| \ = \ |4+2| \ = \ |-6| \ = \ |6| \ = \ 6\ .$$



2. Consider the equation

$$|x-2| \ = \ 3\ .$$

This says that the distance from x to 2 on the real number line is three units.



By inspection, only 5 and -1 lie three units away from 2, so the solution set is $\{-1, 5\}$.

One can also solve this equation algebraically, by observing that

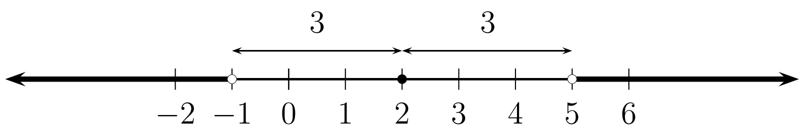
$$3 \ = \ |x-2| \ = \begin{cases} \ \ x-2 & \text{if } x-2 \geq 0 \\ -(x-2) & \text{if } x-2 < 0 \end{cases} \ = \begin{cases} x-2 & \text{if } x \geq 2 \\ 2-x & \text{if } x < 2\ . \end{cases}$$

When $x \geq 2$, we have $3 = x-2$, which rearranges to give $x = 5$. When $x < 2$, we have $3 = 2-x$, which rearranges to give $x = -1$. Combining two cases, we get the solution set $\{-1, 5\}$, as before.

3. Consider the inequality

$$|x-2| \ > \ 3\ .$$

This says that the distance from x to 2 on the real number line is greater than three units.



Thus x must fall to the right of 5, so within the interval $(5, \infty)$, or to the left of -1, so within the interval $(-\infty, -1)$. The solution set is therefore the union of these two intervals:

$$(-\infty, -1) \cup (5, \infty)\ .$$

One can also solve this equation algebraically, by observing that

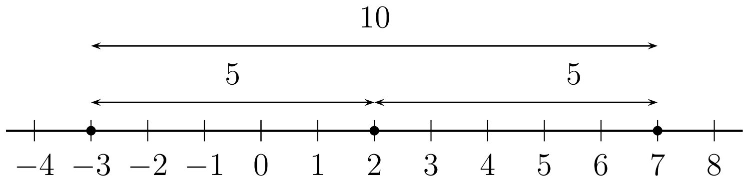
$$3 \ < \ |x-2| \ = \begin{cases} x-2 & \text{if } x \geq 2 \\ 2-x & \text{if } x < 2\ . \end{cases}$$

When $x \geq 2$, we have $3 < x - 2$, which rearranges to give $x > 5$. When $x < 2$, we have $3 < 2 - x$, which rearranges to give $x < -1$. Combining two cases, we get the solution set $(-\infty, -1) \cup (5, \infty)$, as before.

4. Consider the equation

$$|x + 3| \; = \; |x - 7| \; .$$

This says that the distance from x to -3 on the real number line is the same as the distance from x to 7.



Thus x must fall on the midpoint between -3 and 7, which is exactly 5 units from both. By inspection, only 2 satisfies these conditions, so the solution set is $\{2\}$.

One can also solve this equation algebraically, by observing that

$$|x+3| \; = \; \begin{cases} x+3 & \text{if } x+3 \geq 0 \\ -x-3 & \text{if } x+3 < 0 \end{cases} \; = \; \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \; , \end{cases}$$

and

$$|x-7| \; = \; \begin{cases} x-7 & \text{if } x-7 \geq 0 \\ -x+7 & \text{if } x-7 < 0 \end{cases} \; = \; \begin{cases} x-7 & \text{if } x \geq 7 \\ -x+7 & \text{if } x < 7 \; . \end{cases}$$

It is impossible for $x < -3$ and $x \geq 7$ simultaneously, so there are three cases to consider. When $x \geq 7$ then also $x \geq -3$, so that the equation becomes

$$x + 3 \; = \; x - 7 \; ,$$

which has no solution, since it would imply $3 = -7$, which is impossible. When $x < -3$ then also $x < 7$, so that the equation becomes

$$-x - 3 \; = \; -x + 7 \; ,$$

which also has no solution, since it would imply $-3 = 7$, which is impossible. The remaining case is when $-3 \leq x < 7$, so that the equation becomes

$$x + 3 \; = \; -x + 7 \; ,$$

which becomes $2x = 4$, so that $x = 2$, and the solutions set is $\{2\}$, as before.

Clearly the geometric approach leads more directly to the solution, and is less error prone.

5. Consider the inequality

$$|x + 3| \; < \; |x - 7| \; .$$

This says that x is closer to -3 than to 7. From the diagram for the previous example, this says that x must fall to the left of 2, so the solution set is $(-\infty, 2)$.

One can also solve this inequality algebraically. As in the previous example,

$$|x+3| \;=\; \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3\,, \end{cases}$$

and

$$|x-7| \;=\; \begin{cases} x-7 & \text{if } x \geq 7 \\ -x+7 & \text{if } x < 7\,. \end{cases}$$

Again, it is impossible for $x < -3$ and $x \geq 7$ simultaneously, so there are three cases to consider. When $x \geq 7$ then also $x \geq -3$, so that the inequality becomes

$$x+3 \;<\; x-7\,,$$

which has no solution, since it would imply $3 < -7$, which is impossible.

When $x < -3$ then also $x < 7$, so that the inequality becomes

$$-x-3 \;<\; -x+7\,,$$

which holds for all such x, since it is equivalent to $-3 < 7$, which is true. This case yields the interval of solutions $(-\infty, -3)$.

The remaining case is when $-3 \leq x < 7$, so that the inequality becomes

$$x+3 \;<\; -x+7\,,$$

which becomes $2x < 4$, so that $x < 2$, yielding the interval of solutions $[-3, 2)$.

Thus the final solution set is the union of the intervals produced in the last two cases, which is

$$(-\infty, -3) \cup [-3, 2) \;=\; (-\infty, 2)\,,$$

as before.

Clearly the geometric approach is very fast, leading directly to the solution, and is less error prone than the algebraic method.