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ON THE UNIDENTIFIABILITY OF THE FIXED-EFFECTS 3PL MODEL

The paper offers a general review of the basic concepts of both statistical model and parameter identification, and revisit the conceptual relationships between parameter identification and both parameter interpretability and properties of parameter estimates. All these issues are then exemplified for the 1PL, 2PL and 1PL-G fixed-effects models. For the 3PL model, however, we provide a theorem proving that the item parameters are not identified, do not have an empirical interpretation and that it is not possible to obtain consistent and unbiased estimates of them.

Key words: Consistent Estimator, Identified Parameters, Parameters of Interest, Statistical Model, Unbiased Estimator.

1. Introduction

In the field of educational measurement, item response theory (IRT) models are powerful tools aimed at providing an appropriate representation of student test-taking behavior, and to produce accurate estimates of student ability. Clearly it is necessary that IRT models capture the underlying response processes. One such response process that needs to be incorporated in IRT models is the so-called guessing behavior. For multiple-choice tests, there is empirical evidence (and therefore it is reasonable to assume) that respondents guess when they do not know the correct response. This type of behavior seems to be especially prevalent in a low-stakes test, where students are asked to take a test for which they receive neither grades nor academic credit (Lord, 1980; Baker & Kim, 2004; Woods, 2008; Cao & Stokes, 2008). The omnipresence of multiple choice low stakes tests is arguably the reason for which the 3PL model (Birnbaum, 1968) has been commonly used in many applications of IRT in the measurement industry (e.g., equating, standard setting, DIF, etc).

In spite of being widely used, there are still basic unanswered questions regarding the 3PL model, and the answers to these questions are likely to have an impact on both theory and practice. Parameter identification is one of those basic questions. Its relevance is not only due to the relationships between parameter identification and parameter estimation, but also because identifiability ensures that the underlying statistical model is meaningful.

The aim of this paper is twofold: we first offer a general overview of a statistical model and show how the concepts of parameter identifiability, interpretability and estimability play a central role within this general description. Second, based on these concepts, we offer a formal proof establishing the unidentifiability of the item parameters in a fixed-effects 3PL model.

The paper is organized as follows: In Section 2 we introduce the general definition of a statistical model and discuss the basics concepts of parameter identifiability, interpretability and estimability. Section 3 conceptualizes these concepts for various IRT models by showing the interpretability of parameters that follows after their identification. In Section 4 we offer a formal proof that establishes the unidentifiability of the fixed-effects 3PL model and accordingly conclude the nonexistence of unbiased and consistent item parameter estimation. The paper finishes with a discussion.

2. Basic Concepts

The statistical analysis of a phenomenon of interest is based on the three fundamental problems that characterize statistical methods (Fisher, 1922): the specification problem, the estimation problem and the distribution problem. Broadly speaking, the specification problem consists of specifying the mathematical form of the probability distribution generating the observations; this probability distribution depends on parameters, which describe specific features of the phenomenon of interest. The estimation problem involves the choice of methods for calculating sample statistics (i.e., functions of the observations) which estimate the unknown parameters of the population distribution. Finally, the distribution problem essentially reduces to the distributional properties of the estimates; for more details, see also Fisher (1973).

According to this general framework, the statistical meaning of the parameters depends only on the specification problem. The choice of the probability distribution generating the observations "is not arbitrary, but requires an understanding of the way in which the data are supposed to, or did in fact, originate" (Fisher, 1973, p.5). A criterion according to which the statistical meaning of a parameter is established is then necessary. This criterion corresponds to parameter identification. In order to explain this last statement, let us start by reviewing the basic concepts of *statistical model* and *parameter identification*, and thereafter recall the conceptual relationships between parameter identification and both parameter interpretability and properties of parameter estimates.

2.1. Statistical model

The statistical description of a phenomenon of interest involves three elements:

1. The *sample space* $(S, \mathcal{S})$, where S is the set including all the possible observations related to the phenomenon of interest, and the class $\mathcal{S}$ is a σ-field of subsets of S. The latter contains not only all the events of interest, but also their permissible combinations.
2. A family of probability distributions $P^{\boldsymbol{\alpha}}$ defined on the sample space $(S, \mathcal{S})$ indexed by a parameter $\boldsymbol{\alpha}$. These probabilities describe the sampling process and are accordingly called

sampling probabilities. These probabilities can be represented through cumulative distribution functions or through density functions, when possible. Note that the sampling probability $P^{\boldsymbol{\alpha}}$ coincides with the likelihood function and, as it will be seen, constitutes a key element in the formal definition of a statistical model. These three terms are used interchangeably in this paper.

3. A *parameter space* $\boldsymbol{A}$, which corresponds to the set of possible values that the parameter $\boldsymbol{\alpha}$ can take.

These three elements constitute a *statistical model*, which can be compactly written as

$$\mathcal{E} = \{(S, \mathcal{S}),\ P^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \boldsymbol{A}\}. \tag{2.1}$$

It should be said that the parameter space $\boldsymbol{A}$ can be a finite-dimensional space (as it is the case for parametric models), an infinite-dimensional space (as it is the case for nonparametric models), or a cartesian product of both (as it is the case for semi-parametric models). Therefore, (2.1) summarizes the basic structure of all kinds of statistical models. For details, see Fisher (1922), Basu (1975), Raoult (1975), Cox and Hinkley (1974), Bamber and Van Santen (2000) and McCullagh (2002).

Often, a researcher is not interested in the parameter $\boldsymbol{\alpha}$, but in a function of it, namely $\boldsymbol{\psi} \doteq h(\boldsymbol{\alpha})$ where $h : \boldsymbol{A} \longrightarrow \boldsymbol{\Psi}$. The function $h(\cdot)$ is called the *parameter of interest*. Typically, parameters are of interest because they are directly related to theories regarding the phenomenon under study that the researcher wishes to test (Engle, Hendry, & Richard, 1983). If the function h is bijective, $\boldsymbol{\psi}$ corresponds to a *reparametrization of* $\boldsymbol{\alpha}$.

Example 1. Consider a one-way ANOVA model specified as follows:

$$(Y_{ij} \mid \mu_j, \sigma^2) \sim \mathcal{N}(\mu_j, \sigma^2), \quad \mu_j = \omega + \theta_j,$$

with $i = 1, \ldots, n_j$ and $j = 1, \ldots, J$. The statistical model is indexed by $\boldsymbol{\alpha} = (\mu_1, \mu_2, \ldots, \mu_J, \sigma^2) \in \mathbb{R}^J \times \mathbb{R}_+$, while one is some times interested in $\boldsymbol{\psi} = (\omega, \theta_1, \ldots, \theta_J, \sigma^2) \in \mathbb{R}^{J+1} \times \mathbb{R}_+$. This is an example of a finite dimensional parameter space.

Example 2. The Factor Analysis Model can be specified through a marginal-conditional decomposition of the joint probability distribution generating (Y, η), where Y is an observable random vector and η is a latent random vector; that is,

$$(Y \mid \eta, \Lambda, \Sigma) \sim \mathcal{N}_p(\Lambda\eta, \Sigma), \quad (\eta \mid \Omega) \sim \mathcal{N}_q(0, \Omega),$$

where $q < p$. After integrating out the latent variables η, the statistical model is given by

$$(Y \mid \Lambda, \Sigma, \Omega) \sim \mathcal{N}_p(0, \Lambda\Omega\Lambda' + \Sigma),$$

where the parameters of interest are $\boldsymbol{\alpha} = (\Lambda, \Sigma, \Omega) \in \mathbb{R}^{p\times q} \times \mathcal{C}_p \times \mathcal{C}_q$; here $\mathbb{R}^{p\times q}$ denotes the space of $p \times q$ matrices, and $\mathcal{C}_p$ is the cone of positive definite matrices of order $p \times p$; for details on cones, see Berman and Ben-Israel (1971) and Berman and Plemmos (1994). Although much more complex, this too is an example of a finite dimensional parameter space.

For examples of semi-parametric statistical models, see San Martín, Jara, Rolin, and Mouchart (2011).

2.2. Parameter Identifiability

From the definition of a statistical model (2.1), it follows that parameters and sampling probabilities are related through a mapping $\mathbf{\Phi} : \boldsymbol{A} \longrightarrow \mathcal{P}(S, \mathcal{S})$ such that $\mathbf{\Phi}(\boldsymbol{\alpha}) = P^{\boldsymbol{\alpha}}$, where $\mathcal{P}(S, \mathcal{S})$ denotes the set of sampling probabilities defined on the sample space $(S, \mathcal{S})$. Thus, when a parameter $\boldsymbol{\alpha} \in \boldsymbol{A}$ is given, the sampling probability $P^{\boldsymbol{\alpha}}$ is fully determined and the probability of an event in $\mathcal{S}$ can be computed using the knowledge of $P^{\boldsymbol{\alpha}}$.

The problem of parameter identifiability (Koopmans, 1949; Koopmans & Reiersøl, 1950; Rothenberg, 1971; Manski, 1995) arise when two different parameters are associated with the same sampling probability. For instance, in Example 1, $\boldsymbol{\alpha} = (\omega, \theta_1, \ldots, \theta_J)$ and $\boldsymbol{\alpha}' = (\omega + c, \theta_1 - c, \ldots, \theta_J - c)$ for $c \neq 0$ are different parameters associated with the same sampling probability. In Example 2, $\boldsymbol{\alpha} = (\Lambda, \Sigma, \Omega)$ and $\boldsymbol{\alpha}' = (\Lambda T', \Sigma, T\Omega T')$ for a $q \times q$ orthonormal matrix T are different parameters associated with the same sampling distribution. In these cases, the interpretation of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$is different, but the observations (through the sampling probability) do not provide enough information to empirically distinguish between them.

This indeterminacy is avoided if the mapping $\mathbf{\Phi}$ is injective, that is,

$$\boldsymbol{\alpha} \neq \boldsymbol{\alpha}' \Longrightarrow P^{\boldsymbol{\alpha}}(E) \neq P^{\boldsymbol{\alpha}'}(E) \quad \forall\, E \in \mathcal{S}.$$

If this is the case, then the parameter $\boldsymbol{\alpha} \in \boldsymbol{A}$ is said to be *identified by the observations in the statistical model (2.1)*.

2.3. Parameter Identifiability and Parameter Interpretability

Only parameters that are identified can statistically be interpreted with respect to the sampling process. In fact, an identified parameter is always related to only one sampling probability. According to the likelihood principle, the observations are completely characterized by the likelihood function or statistical model (Birnbaum, 1962; Basu, 1975). Therefore, the identified parameter completely characterizes the data generating process and, consequently, it possesses an empirical interpretation.

We illustrate these ideas using the one-way ANOVA model introduced in Example 1. *Before* identifying the parameters of interest $\psi = (\omega, \theta_1, \ldots, \theta_J)$, it is *not* possible to endow them with a statistical meaning. Only $\mu_j = \omega + \theta_j$ is statistically meaningful: it corresponds to a functional of the sampling process, namely the expected value of Y_{ij}. Now, one of the following two identification restrictions are typically considered: $\theta_1 = 0$ or $\sum_{j=1}^{J} \theta_j = 0$. Under the first identification restriction, $\omega = \mu_1$, so ω corresponds to the mean of the first population, whereas $\theta_j = \mu_j - \mu_1$ represents the deviation of the jth population mean with respect to the mean of the first population. Under the second identification restriction, $\omega = \frac{1}{J}\sum_{j=1}^{J} \mu_j$, and represents the overall mean, whereas $\theta_j = \mu_j - \frac{1}{J}\sum_{j=1}^{J} \mu_j$ corresponds to the deviation of the jth population mean with respect to the overall mean. For similar examples, see Revuelta (2010); Van der Linden (2010) and Castro, San Martín, and Arellano-Valle (2013).

The main conclusion of these examples is that the interpretation of the parameters depends on the identifying restrictions; once a specific restriction is adopted, the information provided by the sampling process gives the meaning to the parametrization. Note that the relationship between parameter identification and parameter interpretation was recognized in classical seminal works that faced for the first time an identification problem, that is, *before* the concept of identification was formalized; see Thurstone (1935, p. 74) and Haavelmo (1944, section 18).

Remark 1. Although identification ensures parameter interpretability with respect to the data generating process, this does not necessarily mean that each sampling probability $P \in \mathcal{P}(S, \mathcal{S})$ may be fully characterized by the parameters of interest. In practical terms, this means that a researcher may postulate a sampling probability $P \in \mathcal{P}(S, \mathcal{S})$ which is not characterized by a specific family of sampling probabilities $P^{\boldsymbol{\alpha}}$. As a matter of fact, identifiability is defined in terms of the injectivity of the mapping $\boldsymbol{\Phi}$, and does not require surjectivity. This means that there could exist a sampling probability $P \in \mathcal{P}(S, \mathcal{S})$ such that $P^{\boldsymbol{\alpha}} \neq P$ for all $\boldsymbol{\alpha} \in \boldsymbol{A}$. In the example of the one-way ANOVA model, this means that not all normal distributions $\mathcal{N}(\mu_j, \sigma^2)$ are characterized by the one-way ANOVA identified parametrization. When the mapping $\boldsymbol{\Phi}$ is bijective (that is, injective and surjective), then not only are the parameters empirically meaningful, but all sampling probabilities are fully characterized by the identified parameters.

2.4. Parameter Identifiability and Parameter Estimability

Identifiability is not only related with parameter interpretability, but also with parameter estimation in the sense that unidentified parameters cannot be unbiasedly estimated, nor consistently estimated (Koopmans & Reiersøl, 1950; Gabrielsen, 1978; San Martín & Quintana, 2002). To see this, in the context of the statistical model (2.1), let b be a real-valued parameter (which is a real function of $\boldsymbol{\alpha}$), and let $b(\boldsymbol{\alpha})$ represent the value of b at point $\boldsymbol{\alpha}$. The parameter function $b(\boldsymbol{\alpha})$ is identified by the observations in the statistical model (2.1) if

$$b(\boldsymbol{\alpha}) \neq b(\boldsymbol{\alpha}') \Longrightarrow P^{\boldsymbol{\alpha}}(E) \neq P^{\boldsymbol{\alpha}'}(E) \quad \forall\, E \in \mathcal{S};$$

for details, see Le Cam and Schwartz (1960) and Oulhaj and Mouchart (2003).

A sequence of random variables $\{s_n : n \in \mathbb{N}\}$ is *an unbiased estimator* for the real-valued parameter b if $E^{\boldsymbol{\alpha}}(s_n) = b(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \boldsymbol{A}$; here $E^{\boldsymbol{\alpha}}(\cdot)$ denotes the expectation with respect to the sampling probability $P^{\boldsymbol{\alpha}}$. Similarly, the sequence of random variables $\{s_n : n \in \mathbb{N}\}$ is *a consistent estimator* for the real-valued parameter b if $s_n \longrightarrow b(\boldsymbol{\alpha})$ with respect to $P^{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \boldsymbol{A}$.

The following theorem states that the identifiability of a parameter is a necessary condition for an unbiased and consistent estimator to exist; for proofs, see Bunke and Bunke (1974); Gabrielsen (1978); Paulino and Pereira (1994) and San Martín and Quintana (2002).

Theorem 1. Consider the statistical model (2.1).

1. If there exists an unbiased estimator s_n of the parameter $b(\boldsymbol{\alpha})$, then $b(\boldsymbol{\alpha})$ is identified by the observations in the statistical model (2.1).
2. If there exists a consistent estimator s_n of the parameter $b(\boldsymbol{\alpha})$, then $b(\boldsymbol{\alpha})$ is identified by the observations in the statistical model (2.1).

Remark 2. It is relevant to mention that the relationships between parameter identification and parameter interpretability, and parameter identifiability and statistical inference, are also valid in a Bayesian set-up. As a matter of fact, a Bayesian identified parameter not only fully characterizes the learning-by-observing process, but it can also be exactly estimated (that is, the posterior distribution of the identified parameters converges a.s. and in L^1 to the identified parameters); for details and proofs, see Florens, Mouchart, and Rolin (1999, Chapter 4), San Martín and González (2010), San Martín et al. (2011, section 3) and San Martín, Rolin, and Castro (2013, section 5).

3. Statistical Meaning of the Parameters of Fixed-Effects IRT Models

Unidimensional IRT models specify the probability that a person will correctly answer an item as a function of two parameters: a real valued person parameter θ_i, often called *ability parameter*, and an item parameter which describes properties of the item. Thus, for instance, the 1PL or Rasch model specifies such a probability as $F(\theta_i - \beta_j)$, where F is the standard logistic distribution, and $\beta_j \in \mathbb{R}$ is typically understood as the difficulty of item j. However, *before* analyzing parameter identification, interpretation and estimation, it is *first* necessary to make explicit the sampling process, which leads to asking whether the sampling process is indexed by the ability parameter θ_i. Two possibilities found in the psychometric literature are when the sampling process is indexed by the ability parameter, and when the sampling process is obtained after integrating out θ_i. To distinguish between the two approaches, the first one is called a *fixed-effects IRT model*,

TABLE 1.
Basic structure of fixed-effects IRT models.

Model	$G(\theta_i, \omega_j)$	Item parameter ω_j	Parameter space
1PL model	$F(\theta_i - \beta_j)$	$\omega_j = \beta_j$	$(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J}) \in \mathbb{R}^I \times \mathbb{R}^J$
2PL model	$F(\alpha_j\theta_i - \beta_j)$	$\omega_j = (\alpha_j, \beta_j)$	$(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J}) \in \mathbb{R}^I \times \mathbb{R}_+^J \times \mathbb{R}^J$
1PL-G model	$c_j + (1 - c_j)F(\theta_i - \beta_j)$	$\omega_j = (\beta_j, c_j)$	$(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J}) \in \mathbb{R}^I \times \mathbb{R}^J \times [0,1]^J$
3PL model	$c_j + (1 - c_j)F(\alpha_j\theta_i - \beta_j)$	$\omega_j = (\alpha_j, \beta_j, c_j)$	$(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J}) \in \mathbb{R}^I \times \mathbb{R}_+^J \times \mathbb{R}^J \times [0,1]^J$

Note: $F(\cdot)$ is a c.d.f, $\boldsymbol{\theta}_{1:I} = (\theta_1, \ldots, \theta_I)$ and $\boldsymbol{\omega}_{1:J} = (\omega_1, \ldots, \omega_J)$.

and the second one, a *random-effects IRT model*.

3.1. Specification of IRT models

Fixed-effects IRT models are specified through the following two conditions: (i) mutual independence of Y_{ij}, $i = 1, \ldots, I$, $j = 1, \ldots, J$, where $Y_{ij} = 1$ when person i correctly answer item j, and $Y_{ij} = 0$ otherwise; (ii) $Y_{ij} \sim$ Bernoulli$[G(\theta_i, \omega_j)]$, where $\theta_i \in \mathbb{R}$ corresponds to the person parameter, $\omega_j \in \Omega \subset \mathbb{R}^K$ corresponds to the item parameter, and G is a known cumulative distribution function (cdf), strictly increasing in θ_i for all $\omega \in \Omega$. This last condition, (which incorporates the idea that the probability of a correct answer increases with the person parameter θ_i) can accordingly impose sign restrictions on the item parameter ω_j. Table 1 summarizes the standard fixed-effects IRT models, along with the corresponding parameter spaces; here we rest on the standard suffix PL on the on the word model, although F doest not necessarily corresponds to the standard logistic distribution.

The random-effects IRT models typically assume that the abilities θ_i's are mutually independent, with a common distribution $\mathcal{N}(0, \sigma^2)$. Thus, these type of models are specified through the following conditions:

1. For each person i, his/her responses are mutually independent conditionally on $(\theta_i, \boldsymbol{\omega}_{1:J})$ (this is the so-called Axiom of Local Independence).
2. The conditional distribution of Y_{ij} given (θ_i, ω_j) corresponds to a Bernoulli $[G(\theta_i, \omega_j)]$. The function $G(\theta_i, \omega_j)$ is defined as in Table 1, but now it is a *conditional probability* and not a *marginal probability* as in the fixed-effects framework.
3. The response patterns $\boldsymbol{Y}_i = (Y_{i1}, \ldots, Y_{iJ})$ are mutually independent conditionally on

$(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J})$.

The statistical model is obtained after integrating out the abilities θ_i's. The parameter spaces corresponding to the different conditional specifications are the following: for the 1PL model, $(\boldsymbol{\beta}_{1:J}, \sigma^2) \in \mathbb{R}^J \times \mathbb{R}_+$; for the 2PL model, $(\boldsymbol{\alpha}_{1:J}, \boldsymbol{\beta}_{1:J}, \sigma^2) \in \mathbb{R}_+^J \times \mathbb{R}^J \times \mathbb{R}_+$; for the 1PL-G model, $(\boldsymbol{\beta}_{1:J}, \boldsymbol{c}_{1:J}, \sigma^2) \in \mathbb{R}^J \times [0,1]^J \times \mathbb{R}_+$; and for the 3PL model, $(\boldsymbol{\alpha}_{1:J}, \boldsymbol{\beta}_{1:J}, \boldsymbol{c}_{1:J}, \sigma^2) \in \mathbb{R}_+^J \times \mathbb{R}^J \times [0,1]^J \times \mathbb{R}_+$. For details, see San Martín and Rolin (2013) and San Martín et al. (2013).

The two resulting statistical models are quite different. The parameter space corresponding to the fixed-effects specification depends on the sample size (the number of persons), whereas that of the random-effects does not. Furthermore, in a fixed-effects specification, the J responses for person i are mutually independent, but are positively correlated in the random-effects specification.

The parameters of interest in each of these statistical models should be estimated with a procedure coherent with the underlying sampling process. Thus, both conditional maximum likelihood (CML) and joint maximum likelihood (JML) procedures are coherent with fixed-effects IRT model specifications, whereas the marginal maximum likelihood (MML) procedures are coherent with random-effects IRT model specifications. For details on the estimation procedures, see Molenaar (1995), Baker and Kim (2004), and De Boeck and Wilson (2004).

Remark 3. Note that the Bayesian approach should also be coherent with this perspective by first establishing the sampling process and then assigning priors on the parameters of interest. For example, consider the 1PL model that is characterized by a likelihood of the form $p(Y_{ij} \mid \theta_i, \beta_j)$, with the Y_{ij}'s marginally mutually independent, and assume that, $\theta_i \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\theta^2)$ and $\beta_j \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\beta^2)$, with $\boldsymbol{\theta}_{1:I} \perp\!\!\!\perp \boldsymbol{\beta}_{1:J}$, then this would constitute the Bayesian counterpart of what we call the fixed-effects 1PL model. On the other hand, if we consider the 1PL model that is characterized by a likelihood $p(\boldsymbol{Y}_i \mid \sigma_\theta^2, \boldsymbol{\beta}_{1:J})$, and we assume that, $\sigma_\theta^2 \sim p(\sigma_\theta^2)$, $\boldsymbol{\beta}_{1:J} \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_\beta^2)$ and $\sigma_\theta^2 \perp\!\!\!\perp \boldsymbol{\beta}_{1:J}$, then this constitutes the Bayesian counterpart of what we call random-effects 1PL model.

3.2. *Parameter interpretation*

In spite of the distinction between fixed-effects and random-effects, IRT models are typically introduced using a fixed-effects specification, but the estimation is done in the context of a random-effects specification. The presentation of the models is mainly focused on the meaning of both the person and item parameters (e.g., ability, discrimination, difficulty, etc). Such a meaning, typically explained through item characteristic curves, is depicted *without resorting on an identification analysis*; see, among many others,Van der Linden and Hambleton (1997); Embretson and Reise (2000); Thissen and Wainer (2001); Baker and Kim (2004); De Boeck and Wilson (2004). One then may wonder to what extent the standard parameter interpretation is related with the corresponding sampling process.

According to the discussion in Section 2, the only way of endowing the parameters of interest $(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J})$ with a statistical meaning is through an identification analysis. Now, it should be noticed that the parameters $\{G(\theta_i, \omega_j) : i = 1, \dots, I; j = 1, \dots, J\}$ are identified since $Y_{ij} \sim \text{Bernoulli}\,[G(\theta_i, \omega_j)]$ and the parameter of a Bernoulli distribution is identified. Moreover, the parameter $G(\theta_i, \omega_j)$'s has a precise statistical meaning: it corresponds to the probability that person i correctly answers item j. Therefore, and this is really crucial, if an injective relationship between the parameters of interest $(\boldsymbol{\theta}_{1:I}, \boldsymbol{\omega}_{1:J})$ and the identified parameters $(G(\theta_1, \omega_1), G(\theta_1, \omega_2), \dots, G(\theta_I, \omega_J))$ is established (under restrictions, if necessary), then not only do the parameters of interest become identified, but it is also possible to endow them with a precise statistical meaning.

In what follows, we revisit the interpretation of the parameters in the fixed-effects 1PL, 2PL and 1PL-G models, with the motivation being to complement the standard presentations of IRT models. In doing so, we illustrate the identification strategy under which the identifiability of the fixed-effects 3PL model will be developed in Section 4.

3.2.1. *Basic structure of the identification analysis*

In a first approach, there are I person parameters and KJ item parameters, where K is the dimension of each item parameter ω_j (see Table 1). Because each person parameter only depends on one item parameter, the number of parameters to be identified can be reduced. As a matter of fact, since the function $G(\theta_i, \omega_j)$ is strictly increasing in θ_i, we have

$$\theta_i = \overline{G}(p_{i1}, \omega_1) \qquad \text{for all } i = 1, \ldots, I, \tag{3.1}$$

where $p_{i1} \doteq P[Y_{i1} = 1]$ and $\overline{G}(p, \omega) = \inf\{\theta : G(\theta, \omega) > p\}$. It follows that the identification of the person parameters depend on the identification of the item parameters. Consequently, the identification analysis of the parameters of interest in a fixed-effects IRT model reduces to the identification of the item parameters.

The preceding relationship leads to establish necessary identification conditions: the number of identified parameters (namely, IJ) should be at least equal to the number of parameters to be identified (namely, KJ), which is equivalent to $K \leq I$. Table 2 summarizes this conditions for the different fixed-effects IRT models. Now, using equation (3.1), it follows that

$$p_{ij} = G\left[\,\overline{G}(p_{i1}, \omega_1),\, \omega_j\,\right], \quad j = 2, \ldots, J;\ i = 1, \ldots, K. \tag{3.2}$$

The identification analysis of fixed-effects IRT models reduces, therefore, to solving this system of equations.

It should be remarked that, for each item j, there are K equations and $2K$ parameters to be identified, namely $(\omega_1, \omega_j) \in \mathbb{R}^K \times \mathbb{R}^K$. Consequently, to obtain an equal number of equations and unknowns, it is necessary to impose a restriction on one (K-dimensional) parameter. Taking into account equation (3.1), this restriction should be imposed on the item parameter ω_1. Under this identification restriction, the person parameters θ_i's becomes identified, and the identification analysis of item parameters rests on the possibility to solve the system (3.2) –that is, to write each item parameter ω_j ($j = 2\ldots, J$) as a function of identified parameters.

In what follows, we revisit not only the identifiability of the 1PL, 2PL and 1PL-G models, but also the statistical meaning of the parameters. This is made for the case where the c.d.f F is the standard logistic distribution.

TABLE 2.

Necessary identification conditions for fixed-effects IRT models

Model	Dimension of the item parameters	Number of unknowns	Number of equations	Necessary identification condition	Number of identification restrictions
1PL	$K = 1$	J	IJ	$1 \leq I$	1
2PL	$K = 2$	$2J$	IJ	$2 \leq I$	2
1PL-G	$K = 2$	$2J$	IJ	$2 \leq I$	2
3PL	$K = 3$	$3J$	IJ	$3 \leq I$	3

3.2.2. Parameter interpretation for the 1PL fixed-effects model

Let us consider the following reparametrization of the 1PL model in terms of the probability of an incorrect response λ_{ij}:

$$\Pr(Y_{ij} = 0) \doteq \lambda_{ij} = \frac{1}{1 + \epsilon_i/\eta_j}, \tag{3.3}$$

where $\epsilon_i = \exp(\theta_i) > 0$ and $\eta_j = \exp(\beta_j) > 0$. In this case, equation (3.1) is rewritten as

$$\epsilon_i = \eta_1 \frac{1 - \lambda_{i1}}{\lambda_{i1}}, \quad i = 1, \ldots, I, \tag{3.4}$$

provided that $\lambda_{i1} \in (0, 1)$. According to the discussion in Section 3.2.1, η_1 needs to be restricted for identifiability. A convenient identification restriction is $\eta_1 = 1$, where the label 1 corresponds to the so-called *standard item* (Rasch, 1960). Under this restriction the person parameters is endowed with a precise meaning, namely

$$\epsilon_i = \frac{1 - \lambda_{i1}}{\lambda_{i1}} = \frac{\Pr(Y_{i1} = 1)}{\Pr(Y_{i1} = 0)}, \qquad i = 1, \ldots, I;$$

that is, ϵ_i ($i = 1, \ldots, I$) represents the "betting odds" of a correct answer to the standard item 1. In particular, if $\epsilon_i > 1$ ($\epsilon_i < 1$), then for person i, the probability of correctly answering the standard item 1 is greater (less) than the probability of answering it incorrectly. This argument is mainly due to Rasch (1960).

Following this argument, San Martín, González, and Tuerlinckx (2009) established the interpretability of what is commonly called the difficulty parameter. As a matter of fact, for each person i, equations (3.3) and (3.4) imply that

$$\eta_j = \epsilon_i \cdot \frac{\lambda_{ij}}{1-\lambda_{ij}} = \frac{1-\lambda_{i1}}{\lambda_{i1}} \cdot \frac{\lambda_{ij}}{1-\lambda_{ij}} = \frac{\Pr(Y_{i1}=1)}{\Pr(Y_{i1}=0)} \cdot \frac{\Pr(Y_{ij}=0)}{\Pr(Y_{ij}=1)}, \qquad j = 2, \ldots, J,$$

provided $\lambda_{ij} \in (0,1)$ for each item j and person i. Thus, what is called the "difficulty of item j" corresponds to an odds ratio between the standard item 1 and the j-th item for each person i. Hence, the difficulty parameter η_j of item j actually measures an association between item j and the standard item. If $\eta_j > 1$, then the odds of person i correctly answering to the standard item 1 are larger than the odds of correctly answering item j.

3.2.3. Parameter interpretation for the 2PL fixed-effects model

Using the same reparametrization as the 1PL model, the 2PL model can be rewritten as

$$\Pr(Y_{ij}=0) \doteq \lambda_{ij} = \frac{\eta_j}{\eta_j + \epsilon_i^{\alpha_j}}. \tag{3.5}$$

In this case, equation (3.1) is rewritten as

$$\epsilon_i = \left[\eta_1 \, \frac{1-\lambda_{i1}}{\lambda_{i1}} \right]^{1/\alpha_1}, \quad i = 1, \ldots, I, \tag{3.6}$$

provided $\lambda_{i1} \in (0,1)$. In agreement with the discussion in Section 3.2.1, it is necessary to restrict (α_1, η_1). For convenience, the identification restriction $(\alpha_1, \eta_1) = (1,1)$ is typically adopted, under which the person parameter is endowed with the same meaning as in the fixed-effects 1PL model, that is, the "betting odds" of a correct answer to the standard item 1.

According to Table 2, the identification analysis of the remaining item parameters for the 2PL model should be done using at least two persons. Thus, combining equation (3.5) for person 1 and for person 2, it follows that

$$\alpha_j = \ln\left(\frac{\epsilon_1}{\epsilon_2}\right) \, \ln\left[\frac{(1-\lambda_{2j})}{\lambda_{2j}} \cdot \frac{\lambda_{1j}}{(1-\lambda_{1j})}\right], \quad j = 2, \ldots, J; \tag{3.7}$$

and

$$\eta_j = \left(\frac{\lambda_{1j}}{1-\lambda_{1j}}\right) \, \epsilon_1^{\left\{\ln\left(\frac{\epsilon_1}{\epsilon_2}\right) \, \ln\left[\frac{(1-\lambda_{2j})\lambda_{1j}}{(1-\lambda_{1j})\lambda_{2j}}\right]\right\}}, \quad j = 2, \ldots, J, \tag{3.8}$$

provided $\lambda_{ij} \in (0,1)$ for each item i and person j. Since ϵ_1 and ϵ_2 are identified, the item parameters (α_j, β_j)'s are identified.

The statistical meaning of what is commonly called “discrimination parameter” now become precise. As a matter of fact, $\alpha_j > \alpha_1 = 1$ if and only if

$$\frac{(1-\lambda_{2j})}{\lambda_{2j}} \cdot \frac{\lambda_{21}}{(1-\lambda_{21})} > \frac{(1-\lambda_{1j})}{\lambda_{1j}} \cdot \frac{\lambda_{11}}{(1-\lambda_{11})};$$

or, equivalently, $\mathrm{OR}^2_{j1} > \mathrm{OR}^1_{j1}$, where OR^1_{j1} is the odds ratio (OR) between items j and 1 for person 1. Thus if, for instance, the odds ratios were larger than 1, then the chances of a correct answer for item j are larger with respect to the standard item, for both persons. When those chances are greater for person 2 than for person 1, then $\alpha_j > 1$. Similar statements can be made under cases $1 > \mathrm{OR}^2_{j1} > \mathrm{OR}^1_{j1}$ and $\mathrm{OR}^2_{j1} > 1 > \mathrm{OR}^1_{j1}$.

To make the meaning of η_j concrete, consider the inequality $\eta_j > \eta_1 = 1$, which is equivalent to

$$\frac{\lambda_{ij}}{1-\lambda_{ij}} > \left(\frac{\lambda_{i1}}{1-\lambda_{i1}}\right)^{\alpha_j}.$$

When the discrimination parameter of item j is equal to 1, the inequality $\eta_j > 1$ has the same statistical interpretation as in the 1PL model, namely an odds ratio between item 1 and item j for each person i. When $\alpha_j \neq 1$, four cases can be distinguished:

1. If $\alpha_1 > 1$ and $\lambda_{i1}/(1-\lambda_{i1}) > 1$, then $\eta_j > 1$ is equivalent to

$$\frac{\lambda_{ij}}{1-\lambda_{ij}} > \left(\frac{\lambda_{i1}}{1-\lambda_{i1}}\right)^{\alpha_j} > \frac{\lambda_{i1}}{1-\lambda_{i1}} > 1. \tag{3.9a}$$

2. If $\alpha_j > 1$ and $\lambda_{i1}/(1-\lambda_{i1}) < 1$, then $\eta_j > 1$ is equivalent to

$$1 > \min\left\{\frac{\lambda_{i1}}{1-\lambda_{i1}}, \frac{\lambda_{ij}}{1-\lambda_{ij}}\right\} > \left(\frac{\lambda_{i1}}{1-\lambda_{i1}}\right)^{\alpha_j} \tag{3.9b}$$

or

$$\frac{\lambda_{ij}}{1-\lambda_{ij}} > 1 > \frac{\lambda_{i1}}{1-\lambda_{i1}} > \left(\frac{\lambda_{i1}}{1-\lambda_{i1}}\right)^{\alpha_j}. \tag{3.9c}$$

3. If $\alpha_j < 1$ and $\lambda_{i1}/(1-\lambda_{i1}) > 1$, then $\eta_j > 1$ is equivalent to

$$\min\left\{\frac{\lambda_{i1}}{1-\lambda_{i1}}, \frac{\lambda_{ij}}{1-\lambda_{ij}}\right\} > \left(\frac{\lambda_{i1}}{1-\lambda_{i1}}\right)^{\alpha_j} > 1. \tag{3.9d}$$

4. If $\alpha_j < 1$ and $\lambda_{i1}/(1-\lambda_{i1}) < 1$, then $\eta_j > 1$ is equivalent to

$$\min\left\{1, \frac{\lambda_{ij}}{1-\lambda_{ij}}\right\} > \left(\frac{\lambda_{i1}}{1-\lambda_{i1}}\right)^{\alpha_j} > \frac{\lambda_{i1}}{1-\lambda_{i1}}. \tag{3.9e}$$

Consider for a moment the inequality (3.9a) or (3.9e). It can be seen that between the terms used to define "difficulty" in the 1PL sense (that is, $\lambda_{ij}/(1-\lambda_{ij})$ and $\lambda_{i1}/(1-\lambda_{i1})$), there is another term (namely $[\lambda_{i1}/(1-\lambda_{i1})]^{\alpha_j}$) which is used to define "difficulty" in the 2PL sense thus, showing that "difficulty" in the 2PL sense is quite different from "difficulty" in the 1PL sense. For the other inequalities, similar comments can be made.

3.2.4. Parameter interpretation for the 1PL-G fixed-effects model

We reparametrize the model as before and additionally set $\kappa_j = 1 - c_j$. The 1PL-G model can be rewritten as follows:

$$\Pr(Y_{ij} = 0) \doteq \lambda_{ij} = \frac{\kappa_j \, \eta_j}{\eta_j + \epsilon_i}. \tag{3.10}$$

In this case, equation (3.1) is rewritten as

$$\epsilon_i = \eta_1 \frac{\kappa_1 - \lambda_{i1}}{\lambda_{i1}}, \quad i = 1, \ldots, I. \tag{3.11}$$

In agreement with the discussion developed in Section 3.2.1, it is necessary to restrict (η_1, κ_1). By convenience, we adopt the identification restriction $(\eta_1, \kappa_1) = (1, 1)$, under which the person parameters become identified and endowed with the same meaning as in the fixed-effects 1PL model, that is, the "betting odds" of a correct answer to the standard item 1.

According to Table 2, for the 1PL-G model the identification analysis of the remaining item parameters should be done using at least two persons. Thus, combining equation (3.10) for person 1 and for person 2, it follows that

$$\eta_j = \frac{\epsilon_2 \frac{\lambda_{2j}}{\lambda_{1j}} - \epsilon_1}{1 - \frac{\lambda_{2j}}{\lambda_{1j}}} \qquad j = 2, \ldots, J; \tag{3.12}$$

and

$$\kappa_j = \frac{\lambda_{2j} \, (\epsilon_2 - \epsilon_1)}{\epsilon_2 \frac{\lambda_{2j}}{\lambda_{1j}} - \epsilon_1} \qquad j = 1, \ldots, J; \tag{3.13}$$

provided that $\lambda_{1j} \neq \lambda_{2j}$ for all item $j = 1, \ldots, J$. This shows that the item parameters $\{(\eta_j, \kappa_j) : j = 2, \ldots, J\}$ are identified. For additional details, see San Martín et al. (2009) and San Martín et al. (2013).

Let us now turn to what are commonly called the "difficulty parameters", that is, the item parameters η_j's. Using equation (3.12), $\eta_j > \eta_1 = 1$ is equivalent to

$$\frac{\lambda_{2j}}{\lambda_{1j}} > \frac{\lambda_{21}}{\lambda_{11}} \Leftrightarrow \lambda_{2j}\lambda_{11} > \lambda_{1j}\lambda_{21}.$$

(The relation is given here for persons 1 and 2 but it holds for any pair of persons (i, i') with $i \neq i'$). Thus, item j is more difficult than the standard item 1 if the probability that person 2 incorrectly answers the item j and person 1 incorrectly answers item 1 is greater than the probability that person 1 incorrectly answers item j and person 2 incorrectly answer item 1. It is a type of a cross-effect between two items and two persons.

In order to grasp the meaning of the guessing parameter c_j, let us consider the inequality $\kappa_j < \kappa_1 = 1$ or, equivalently, $c_j > c_1 = 0$. Taking into account that $\kappa_j > 0$, two cases can be distinguished:

1. If $\epsilon_2 > \epsilon_1$, then

$$\frac{1-\lambda_{1j}}{\lambda_{1j}} \frac{\lambda_{11}}{1-\lambda_{11}} > \frac{1-\lambda_{2j}}{\lambda_{2j}} \frac{\lambda_{21}}{1-\lambda_{21}};$$

   that is, $\mathrm{OR}^1_{j1} > \mathrm{OR}^2_{j1}$.

2. If $\epsilon_2 < \epsilon_1$, then

$$\frac{1-\lambda_{1j}}{\lambda_{1j}} \frac{\lambda_{11}}{1-\lambda_{11}} < \frac{1-\lambda_{2j}}{\lambda_{2j}} \frac{\lambda_{21}}{1-\lambda_{21}};$$

   that is, $\mathrm{OR}^1_{j1} < \mathrm{OR}^2_{j1}$.

Thus, the chances of correctly answering an item with positive guessing, relative to correctly answering the standard item (without guessing) are greater for the less able persons than for the more able persons. We remark that a guessing behavior becomes statistically meaningful only when it is compared with a non-guessing behavior (which is the sense of the restriction $c_1 = 0$).

4. The unidentifiability of the 3PL model

In this section, we analyze the identifiability of the fixed-effect 3PL model. Following the general argument depicted in Section 3.2.1, the following theorem is proved:

Theorem 2. Suppose that $I \geq 3$ for the fixed-effects 3PL model. If $\omega_1 = (\alpha_1, \beta_1, c_1)$ is fixed at $(1, 0, 0)$, then

1. The person parameters are identified by the observations.
2. The item parameters $(\boldsymbol{\alpha}_{2:J}, \boldsymbol{\beta}_{2:J}, \boldsymbol{c}_{2:J})$ are unidentified by the observations.

To prove this theorem, we will employ the reparametrization $\epsilon_i = \exp(\theta_i)$, $\gamma_j = \exp(\alpha_j \beta_j)$ and $\delta_j = 1 - c_j$, and rewrite the 3PL model as follows:

$$P(Y_{ij} = 0) \doteq \lambda_{ij} = \frac{\delta_j \, \gamma_j}{\gamma_j + \epsilon_i^{\alpha_j}}. \tag{4.1}$$

4.1. *Identification of the person parameters*

In this case, equation (3.1) is rewritten as

$$\epsilon_i = \left[\gamma_1 \, \frac{(\delta_1 - \lambda_{i1})}{\lambda_{i1}} \right]^{1/\alpha_1}, \quad i = 1, \ldots, I, \tag{4.2}$$

provided that $\lambda_{i1} > 0$. In agreement with the discussion developed in Section 3.2.1, it is necessary to restrict $(\alpha_1, \gamma_1, \delta_1)$. By convenience, we adopt the identification restriction $(\alpha_1, \gamma_1, \delta_1) = (1, 1, 1)$, under which the person parameters become identified and endowed with the same meaning as in the fixed-effects 1PL model, that is, the "betting odds" of a correct answer to the standard item 1.

4.2. *Unidentifiability of the discrimination parameters*

According to Table 2, the identification analysis of the remaining item parameters of the 3PL model should be done using three persons. Thus, combining equation (4.2) for persons 1, 2 and 3, it follows that

$$\gamma_j = \frac{\frac{\lambda_{2j}}{\lambda_{1j}} \, \epsilon_2^{\alpha_j} - \epsilon_1^{\alpha_j}}{1 - \frac{\lambda_{2j}}{\lambda_{1j}}}, \quad j = 2, \ldots, J; \tag{4.3}$$

and

$$\delta_j = \frac{\lambda_{2j}(\epsilon_2^{\alpha_j} - \epsilon_1^{\alpha_j})}{\frac{\lambda_{2j}}{\lambda_{1j}} \epsilon_2^{\alpha_j} - \epsilon_1^{\alpha_j}}, \quad j = 2, \ldots, J, \tag{4.4}$$

provided that $\lambda_{1j} \neq \lambda_{2j}$ for each item j. Using equations (4.1), (4.3) and (4.4), it follows that

$$\epsilon_3^{\alpha_j} = \frac{1}{\lambda_{ij}\left(1 - \frac{\lambda_{2j}}{\lambda_{1j}}\right)} \left[\lambda_{2j}\left(1 - \frac{\lambda_{ij}}{\lambda_{1j}}\right)\epsilon_2^{\alpha_j} + (\lambda_{ij} - \lambda_{2j})\epsilon_1^{\alpha_j}\right], \tag{4.5}$$

provided that $\lambda_{1j} \neq \lambda_{2j}$, $\lambda_{1j} \neq \lambda_{3j}$ and $\lambda_{2j} \neq \lambda_{3j}$.

In order to identify the discrimination parameter α_j (for $j > 1$), it is necessary to write it as a function of ϵ_1, ϵ_2 and ϵ_3. This requires solving equation (4.5) for α_j which can be rewritten as

$$l_j \epsilon_3^{\alpha_j} - m_j \epsilon_2^{\alpha_j} - n_j \epsilon_1^{\alpha_j} = 0, \tag{4.6}$$

where $l_j \doteq \lambda_{3j}(\lambda_{1j} - \lambda_{2j})$, $m_j \doteq \lambda_{2j}(\lambda_{1j} - \lambda_{3j})$ and $n_j \doteq \lambda_{1j}(\lambda_{3j} - \lambda_{2j})$. To determine if this equation has a solution in α_j, the signs of the constants l_j, m_j and n_j need to be examined. As shown in Table 3, eight cases should be considered. For a specific item j, we simplify the notation by writing $l_j = l$, $m_j = m$, $n_j = n$ and $\alpha_j = x$. Thus, (4.6) can be rewritten as

$$l\epsilon_3^x - m\epsilon_2^x - n\epsilon_1^x = 0. \tag{4.7}$$

This equation can not be solved explicitly for x, but it is possible to determine if there exists a unique solution in x. If a solution exists, the discrimination parameter becomes identified and, by equations (4.3) and (4.4), the γ_j's and the δ_j's are also identified

It is therefore necessary to determine if equation (4.7) have a solution in x; if this is the case, the second step is to prove that such a solution is unique. To investigate the existence of a solution we consider the function $g(x) = l\epsilon_3^x - m\epsilon_2^x - n\epsilon_1^x$ (or an equivalent one) and we look for conditions on the constants l, m and n such that the function g crosses the axis at $x = 0$. Since $\epsilon_i^x > 0$ for $i = 1, 2, 3$, it is necessary to distinguish different combinations of signs for the constants l, m and n according to Table 3. It is straightforward to see that for some combinations of signs no solution exists (e.g., $l > 0$, $m < 0$ and $n < 0$). With this in mind, we now study each of the eight cases summarized in Table 3.

TABLE 3.
Signs of the constants of equation (4.6)

Case	l_j	m_j	n_j
1	$+$	$+$	$+$
2	$-$	$-$	$-$
3	$+$	$+$	$-$
4	$-$	$-$	$+$
5	$+$	$-$	$+$
6	$-$	$+$	$-$
7	$+$	$-$	$-$
8	$-$	$+$	$+$

4.2.1. Cases 1 and 2

Let $b \doteq \frac{m}{l}$, $e \doteq \frac{n}{l}$, $\eta_2 \doteq \frac{\epsilon_2}{\epsilon_3}$ and $\eta_1 \doteq \frac{\epsilon_1}{\epsilon_3}$. Equation (4.7) becomes

$$b\,\eta_2^x + e\,\eta_1^x = 1, \tag{4.8}$$

Let $f(x) = b\,\eta_2^x + e\,\eta_1^x$. This function is defined from $\mathbb{R}_+$ to $\mathbb{R}_+$. Furthermore,

$$f'(x) = b\,\ln(\eta_2)\,\eta_2^x + e\,\ln(\eta_1)\eta_1^x, \quad f''(x) = b\,[\ln(\eta_2)]^2\eta_2^x + e\,[\ln(\eta_1)]^2\eta_1^x.$$

In Case 1, $b > 0$ and $e > 0$ and, therefore, $f''(x) > 0$ for all $x \in \mathbb{R}_+$. Thus, the function f is convex on $\mathbb{R}_+$. Consider the following subcases:

1. If $\eta_2 < 1$ and $\eta_1 < 1$, then $f'(x) < 0$ and f is, therefore, a decreasing function. Furthermore, $\lim\limits_{x\to+\infty} f(x) = 0$. Therefore, $f(x) = 1$ has a unique solution if and only if $\lim\limits_{x\to 0} f(x) > 1$; that is, $b + e > 1$; see Figure 1, panel (a).
2. If $\eta_2 > 1$ and $\eta_1 > 1$, then f is an increasing function. Furthermore, $\lim\limits_{x\to+\infty} f(x) = +\infty$. Therefore, $f(x) = 1$ has a unique solution if and only if $\lim\limits_{x\to 0} f(x) < 1$, that is, $b + e < 1$; see Figure 1, panel (b).
3. If $\eta_2 > 1$ and $\eta_1 < 1$, then f has a minimum x^* given by

$$x^* = \frac{\ln\left[\frac{-e\ \ln(\eta_1)}{b\ \ln(\eta_2)}\right]}{\ln\left(\frac{\eta_2}{\eta_1}\right)}.$$

Note that the right-hand side of this equality is well defined because $\ln(\eta_1) < 0$. Furthermore, $\lim_{x \to +\infty} f(x) = +\infty$. The convexity of f implies that if $\lim_{x \to 0} f(x) < 1$ (that is, $b + e < 1$), then always there exists a unique solution; see Figure 1, panel (c).

4. The case $\eta_2 < 1$ and $\eta_1 > 1$ is equivalent to the previous one and so details are omitted.

We now determine the conditions under which equation (4.8) has a unique solution:

1. If $\eta_2 < 1$ and $\eta_1 < 1$, then equation (4.8) has a unique solution if and only if $b + e > 1$, that is, $l < m + n$. This condition is equivalent to

$$\lambda_{3j}(\lambda_{1j} - \lambda_{2j}) < \lambda_{2j}(\lambda_{1j} - \lambda_{3j}) + \lambda_{1j}(\lambda_{3j} - \lambda_{2j}) \Longleftrightarrow \lambda_{1j}\lambda_{3j} < \lambda_{1j}\lambda_{3j},$$

which is a contradiction.

2. If $\eta_2 > 1$ and $\eta_1 > 1$, then equation (4.8) has a unique solution if and only if $b + e < 1$, that is, $l > m + n$. This condition also entails a contradiction since

$$\lambda_{3j}(\lambda_{1j} - \lambda_{2j}) > \lambda_{2j}(\lambda_{1j} - \lambda_{3j}) + \lambda_{1j}(\lambda_{3j} - \lambda_{2j}) \Longleftrightarrow \lambda_{1j}\lambda_{3j} > \lambda_{1j}\lambda_{3j}.$$

3. If $\eta_2 > 1$ and $\eta_1 < 1$, then equation (4.8) has a unique solution if and only if $b + e < 1$, which still leads to a contradiction.

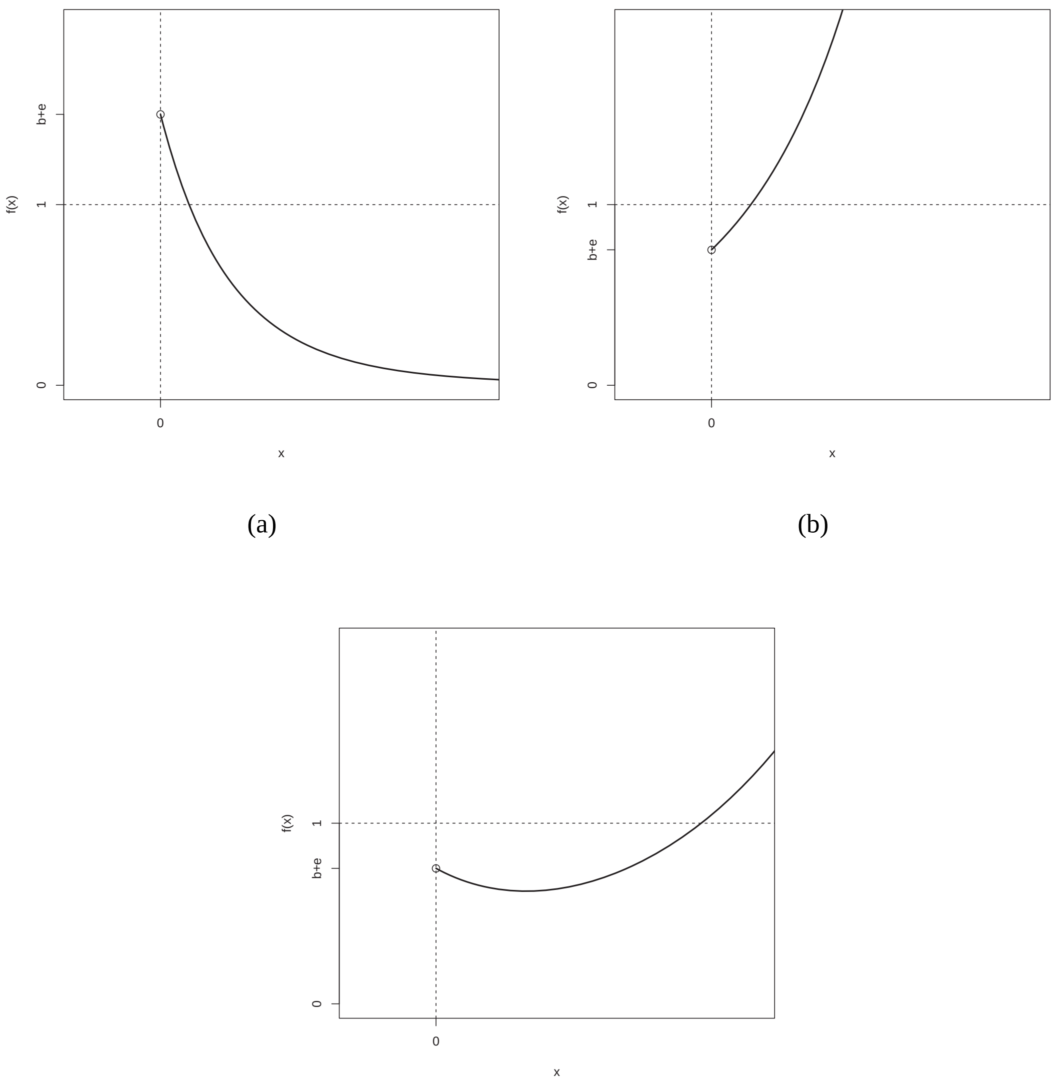
Thus, for the Case 1, the identifiability of α_j (for $j > 1$) leads to a contradiction.

Under Case 2, $b = \frac{m}{l} > 0$ and $e = \frac{n}{l} > 0$ and, therefore, the analysis of the function $f(x) = b\,\eta_2^x + e\,\eta_1^x$ is the same as in Case 1.

4.2.2. Cases 3 and 4

Equation (4.7) can be rewritten as follows:

$$\left(\frac{l}{m}\right)\left(\frac{\epsilon_3}{\epsilon_2}\right)^x + \left(\frac{-n}{m}\right)\left(\frac{\epsilon_1}{\epsilon_2}\right)^x = 1.$$



(a) (b)

(c)

FIGURE 1.

Illustration of the different cases such that equation (4.8) have a solution.

It can be shown that for these cases

$$\left(\frac{l}{m}\right) > 0, \qquad \left(\frac{-n}{m}\right) > 0.$$

Therefore, the analysis of the function $f(x) = \left(\frac{l}{m}\right)\left(\frac{\epsilon_3}{\epsilon_2}\right)^x + \left(\frac{-n}{m}\right)\left(\frac{\epsilon_1}{\epsilon_2}\right)^x$ is similar to that developed for Case 1.

4.2.3. *Cases 5 and 6*

Equation (4.7) can be rewritten as follows:

$$\left(\frac{l}{n}\right)\left(\frac{\epsilon_3}{\epsilon_1}\right)^x + \left(\frac{-m}{n}\right)\left(\frac{\epsilon_2}{\epsilon_1}\right)^x = 1.$$

It can be verified that for these cases

$$\left(\frac{l}{n}\right) > 0, \qquad \left(\frac{-m}{n}\right) > 0.$$

Therefore, the analysis of the function $f(x) = \left(\frac{l}{n}\right)\left(\frac{\epsilon_3}{\epsilon_1}\right)^x + \left(\frac{-m}{n}\right)\left(\frac{\epsilon_2}{\epsilon_1}\right)^x$ is similar to that developed for Case 1.

4.2.4. *Cases 7 and 8*

For case 7 and 8, equation (4.7) does not have a solution since its left side is always larger than 0.

4.3. *Comments to the previous result*

The previous results deserve some comments:

1. Most of the time, the didactic presentation of the 3PL model is carried out in the context of a fixed-effects framework. In this context, the item parameters are interpreted using the item characteristics curves. However, as was discussed in Section 2, only identified parameters are accompanied with a statistical meaning. The previous results show that the item parameters of a 3PL model are not identified and, therefore, they are meaningless. This also contradicts what Lord stated about the (un)identifiability of the 3PL model (see, Lord, 1980, p. 185).

Consequently, it is not longer possible to present (and to teach) the 3PL model in its fixed-effects specification.

2. Maris and Bechger (2009) considered a 3PL model, where the discrimination parameters are all equal to an unknown common parameter α. They proved that, after fixing α at 1 and constraining β_1 and c_1 in such a way that $\beta_1 = -\ln(1 - c_1)$, the remaining item parameters are unidentified. If β_1 and c_1 are fixed, then we get the identified 1PL-G model; see Section 3.2.4. Both specifications can be considered as particular cases of the 3PL model, although only one of them is statistically meaningful.

3. According to Theorem 1, parameter identifiability is a necessary condition to ensuring the existence of an unbiased and consistent estimator. Consequently, Theorem 2 implies that there does not exist an unbiased nor consistent estimator for the item parameters of a fixed-effects 3PL model.

4. Suppose that the 3PL model is specified with a common guessing parameter $c \doteq 1 - \delta$. In this case, equation (4.1) becomes

$$P(Y_{ij} = 0) \doteq \lambda_{ij} = \frac{\delta\gamma_j}{\gamma_j + \epsilon_i^{\alpha_j}}.$$

Taking into account that $\gamma_j > 0$, it follows that

$$\frac{\gamma_j}{\gamma_j + \epsilon_i^{\alpha_j}} \leq 1,$$

and, therefore, $\lambda_{ij} \leq \delta$. Thus, $\widetilde{\lambda}_{ij} \doteq \lambda_{ij}/\delta$ is a probability since it belongs to the interval $[0, 1]$, and it satisfies the following equality:

$$\widetilde{\lambda}_{ij} = \frac{\gamma_j}{\gamma_j + \epsilon_i^{\alpha_j}}.$$

This model corresponds to a 2PL model so that $(\boldsymbol{\alpha}_{2:J}, \boldsymbol{\gamma}_{2:J}, \boldsymbol{\epsilon}_{1:I})$ can be written as a function of $\{\widetilde{\lambda}_{ij} : i = 1, \ldots, I;\ j = 1, \ldots, J\}$ under the restriction $(\alpha_1, \gamma_1) = (1, 1)$. Thus, the parameters $(\boldsymbol{\alpha}_{2:J}, \boldsymbol{\gamma}_{2:J}, \boldsymbol{\epsilon}_{1:I})$ are identified if the $\widetilde{\lambda}_{ij}$'s are identified, which is true if δ is fixed at some value.

5. If more parameters are constrained, it is not possible to get the identification of the item parameters, except when the 3PL model reduces to the 1PL-G model. As a matter of fact, if for instance $\alpha_2 = 1$, then items 1 and 2 are specified through a 1PL-G model, which ensures the

identifiability of β_1 and c_2 under the identification restriction $(\gamma_1, c_1) = (1, 0)$. However, for the remaining item parameters $(\boldsymbol{\alpha}_{3:J}, \boldsymbol{\gamma}_{3:J}, \boldsymbol{c}_{3:J})$, the arguments developed in Section 4.2 show their unidentifiability. In a similar vein, if a new person parameter is included , say ϵ_4, then an equation of the form (4.5) is obtained between $\epsilon_1, \epsilon_2, \epsilon_4$ (which are already identified) and α_j and, therefore, the unidentifiability of α_j follows from the non-existence of a solution of this equation.

5. Discussion

Identifiability is relevant in statistical modeling because it is a condition necessary for ensuring coherent inference on the parameters of interest. The parameters of interest are related to the sampling distributions, which describe the data generating process. If an injective relationship does not exist between the parameters and the sampling distributions, they lack any empirical meaning. In this paper we have given a general overview statistical models and the relationships that exist between identification, interpretation and estimation of parameters. All these considerations were then explicitly illustrated for the fixed-effects 1PL, 1PL-G, and 2PL models. Using the same rationale, we have shown that for the fixed-effects 3PL model, after fixing the difficulty, the discrimination and the guessing parameters of an item, the remainder items parameters are still unidentified by the observations, do not have an empirical interpretation, and cannot be unbiasedly and consistently estimated.

Although it might be argued that fixed-effects IRT models are rarely used in practice, didactic presentations of IRT models are typically based on the fixed-effects perspective, exemplifying the meaning of the parameters by mean of characteristic curve functions. In contrast, in this paper we have given the interpretation of the identified parameters in terms of the underlying sampling process. In addition, there exist commonly used software that implements both JML and CML estimation methods which, as it has been argued, are coherent with the sampling process underlying fixed-effects models (see `http://www.rasch.org/software.htm` where at least seventeen out of the fifty two cited software declare to use JML estimation). In this context, we have clarified that the estimation method should be coherent with the corresponding sampling process, and then software that employs JML for the estimation of the 3PL model, should warn the user about the credibility of the obtained parameter estimates because they are not identified. It should be mentioned that, indeed, there are software programs such as LOGIST (Wood, Wingersky, & Lord, 1976; Wingersky, 1983) and Winsteps (Linacre, 2012) that implement JML for the estimation of the 3PL model. Given that the item parameters in the fixed-effects 3PL are not identified, it can be inferred that, despite the restriction $\alpha_1 = 1$, $\beta_1 = 0$, and $c_1 = 0$, only the person parameters have a statistical meaning and the interpretation of α_j, β_j, and c_j is unclear.

Another important finding in this paper relates to the consistency of parameter estimates. As a matter of fact, a well known reason of why fixed-effects IRT models are not widely used in practice is due to a statistical shortcoming, the so-called *incidental problem* which leads to inconsistent estimators even for identified models (see Neyman & Scott, 1948; Lancaster, 2000; Del Pino, San Martín, González, & De Boeck, 2008). Haberman (2004) has shown the joint consistency of estimators for the Rasch model. However, our results show that for the 3PL there is no hope at all to reach consistent estimates because of the unidentifiability of the model.

Obviously, our analysis leads one to question the identifiability of the more commonly used random-effects 3PL model. This is still an open problem. To the best of our knowledge there is not yet a formal proof establishing the (un)identifiability of this model. However, the identifiability of the random-effects 1PL-G model has been obtained restricting one guessing parameter; see San Martín et al. (2013). How this proof can be extended to identify the random-effects 3PL model is still unclear to us.

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