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Individual differences in algebraic cognition: Relation to the approximate number and semantic memory systems

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ABSTRACT

The relation between performance on measures of algebraic cognition and acuity of the approximate number system (ANS) and memory for addition facts was assessed for 171 ninth graders (92 girls) while controlling for parental education, sex, reading achievement, speed of numeral processing, fluency of symbolic number processing, intelligence, and the central executive component of working memory. The algebraic tasks assessed accuracy in placing *x,y* pairs in the coordinate plane, speed and accuracy of expression evaluation, and schema memory for algebra equations. ANS acuity was related to accuracy of placements in the coordinate plane and expression evaluation but not to schema memory. Frequency of fact retrieval errors was related to schema memory but not to coordinate plane or expression evaluation accuracy. The results suggest that the ANS may contribute to or be influenced by spatial–numerical and numerical-only quantity judgments in algebraic contexts, whereas difficulties in committing addition facts to long-term memory may presage slow formation of memories for the basic structure of algebra equations. More generally, the results suggest that different brain and cognitive systems are engaged during the learning of different components of algebraic competence while controlling for demographic and domain general abilities.

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Introduction

Competence with algebra is the foundation for learning the more complex mathematics that is demanded in science, technology, engineering, and mathematical (STEM) fields (National Mathematics Advisory Panel, 2008) and contributes to employability and wages in many blue-collar occupations (Bynner, 1997). Thus, it is not surprising that improving students' learning of algebra is an educational priority, but this learning has proven to be difficult to achieve for many students (Stein, Kaufman, Sherman, & Hillen, 2011). Efforts to improve algebraic learning have included the identification of poor prerequisite knowledge, such as competence with fractions, to serve as a focus of remedial efforts (Fuchs et al., 2013; Siegler et al., 2012) as well as the development of different instructional approaches for students with different levels of algebraic expertise (Rittle-Johnson & Star, 2009; Rittle-Johnson, Star, & Durkin, 2009).

The latter may be influenced by individual differences in the more basic cognitive systems that support algebraic learning directly or the learning of prerequisite skills, including arithmetic. Intervening with these systems may provide a useful adjunct to interventions that focus on specific algebraic content (see Park & Brannon, 2013, 2014). We have taken a first step in this direction by examining the relation between specific aspects of algebraic cognition and basic cognitive systems that are correlated with individual differences in arithmetic learning, specifically the approximate number system (ANS) and the semantic memory system involved in learning basic arithmetic facts (De Smedt, Holloway, & Ansari, 2011; Geary, Hoard, & Bailey, 2012; Halberda, Mazzocco, & Feigenson, 2008; Mazzocco, Feigenson, & Halberda, 2011a, 2011b; Qin et al., 2014).

ANS and memory system

The ANS is an inherent system for representing, comparing, and combining the magnitudes of collections of objects (see Feigenson, Dehaene, & Spelke, 2004; Geary, Berch, & Mann Koepke, 2015), and there is some evidence that poor acuity of this system contributes to difficulties in learning mathematics (Piazza et al., 2010) and to individual differences in mathematics achievement more generally (Chen & Li, 2014; Fazio, Bailey, Thompson, & Siegler, 2014; Kibbe & Feigenson, 2015; Libertus, Halberda, & Feigenson, 2011; Starr, Libertus, & Brannon, 2013). Other studies, however, suggest that children's and adults' formal mathematical competencies, whether or not they have learning difficulties, are largely independent of ANS acuity and that individual differences in mathematics achievement are more consistently related to the fluency of processing symbolic numerical and arithmetical information (e.g., Bugden & Ansari, 2011; De Smedt et al., 2011; De Smedt, Noël, Gilmore, & Ansari, 2013; Iuculano, Tang, Hall, & Butterworth, 2008; Rousselle & Noël, 2007) or to more basic processes, such as inhibitory control, that influence performance on both ANS tasks and mathematics achievement tests (Fuhs & McNeil, 2013; Gilmore et al., 2013; but see Keller & Libertus, 2015). To further confuse the matter, Fazio and colleagues (2014) found that ANS acuity and symbolic number knowledge independently contributed to fifth graders' mathematics achievement.

The focus on overall mathematics achievement may have contributed to these mixed results by obscuring potentially more nuanced relations between ANS acuity and mathematical competence (Lourenco, Bonny, Fernandez, & Rao, 2012; Lyons & Beilock, 2011). Early in development, it is possible that ANS acuity contributes to learning some aspects of symbolic mathematics, such as the cardinal value of number words, and once these are understood mathematical learning that builds on this knowledge proceeds independently of the ANS (Chu, vanMarle, & Geary, 2015; Nieder, 2009; vanMarle, Chu, Li, & Geary, 2014). Lourenco and colleagues (2012) found that adults' competence with symbolic arithmetic was related to the acuity of ANS representations of discrete collections of items, whereas competence with symbolic geometry was related to sensitivity to variation in area. The implication is that the relation between ANS acuity and mathematical competence may continue into adulthood but may be specifically related to symbolic competencies in number and arithmetic (see also Park & Brannon, 2013, 2014). It is also the case that people's mathematical education can influence ANS acuity (Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Piazza, Pica, Izard, Spelke, & Dehaene, 2013), but whether this relation is specific to some aspects of mathematics education is unknown. We explored whether ANS acuity is specifically related to algebraic competencies that involve spatial–numerical (coordinate plane) or numerical-only (expression evaluation) representations of whole-number magnitude (below) while controlling for fluency of symbolic number processing.

Difficulty in representing and retrieving basic arithmetic facts from long-term memory is a cardinal feature of mathematical learning disabilities (De Visscher & Noël, 2014; Geary, 1993), and more generally speed and accuracy of fact retrieval contribute to individual differences in arithmetic learning and performance (Geary & Widaman, 1987; Jordan, Hanich, & Kaplan, 2003; Siegler & Shrager, 1984). The ANS may be engaged during the early learning of these facts, but so is the semantic memory system that is dependent on the hippocampus (De Smedt et al., 2011; Qin et al., 2014; Squire & Zola-Morgan, 1991). The learning of arithmetic facts, and addition and multiplication in particular, has a clear associative basis (Siegler, 1987, 1988; Siegler & Shrager, 1984); thus, it is not surprising that recent studies have found engagement of the hippocampal memory system during initial fact learning (De Smedt et al., 2011; Qin et al., 2014). Although we do not assess hippocampal engagement directly, we note its critical importance in children's fact learning because it is also involved in the formation of schema memory—memory for the general pattern of relations—and not simply specific facts (Tse et al., 2007). These relations led us to hypothesize that difficulties with addition fact retrieval would predict slow formation of schema memories for algebra equations.

Algebraic cognition

Rather than focus on individual differences in performance on algebra achievement tests, as in most previous studies (e.g., Rittle-Johnson et al., 2009; Siegler et al., 2012), we focused on three component skills: knowledge of the coordinate plane, fluency in evaluating algebraic expressions, and memory for algebra equations. These, of course, do not cover all components of algebraic competence, but they are core aspects of the algebra knowledge and skills identified by the National Mathematics Advisory Panel (2008).

Graphing functions and equations is foundational to learning algebra, geometry, and other mathematical domains and is dependent on the ability to plot points within the coordinate plane (e.g., Leinhardt, Zaslavsky, & Stein, 1990). The one-dimensional component of the plane is the well-studied linear number line. There is lively debate over how children mentally represent the line (Ashcraft & Moore, 2012; Barth & Paladino, 2011; Cohen & Blanc-Goldhammer, 2011; Rouder & Geary, 2014; Siegler & Opfer, 2003; Slusser, Santiago, & Barth, 2013), but regardless of how they do so, accuracy in placing numerals on it—indicating conceptual understanding of the line (Siegler, Thompson, & Schneider, 2011)—predicts concurrent and future mathematics achievement (Booth & Siegler, 2006; Geary, 2011; Siegler & Booth, 2004). For this study, we developed an analogous coordinate plane task.

We also adapted Walczyk and Griffith-Ross's task (2006) to assess the fluent processing of algebraic expressions (e.g., $2y + 5, y = 1$). As with knowledge of the coordinate plane, this is a critical step toward efficient solving of algebra equations (Walczyk & Griffith-Ross, 2006), but it is one that is not achieved by many high school students (Jansen, Marriott, & Yelland, 2003, 2007; Kirshner, 1989; Ranney, 1987).

De Groot (1966) demonstrated that expertise in chess results in enhanced memory recall for configurations of pieces that would occur during an actual chess game but in no memory advantage for randomly placed pieces. Sweller and Cooper (1985) demonstrated the same effect for recall of valid algebra equations [e.g., $(c + d)(a + e) = (c + d)(b)$] as compared with the recall of invalid equations [e.g.,$)c(+ de = b)(ca + d$]. Students with more experience with algebra correctly recalled more features of valid equations than less experienced students, but there were no recall differences for invalid equations. These results indicate that acquiring expertise in algebra involves the formation of a memory schema for the basic structure of equations.

The current study

We first focused on students' performance on the coordinate plane task developed specifically for this study and on measures of individual differences that can be derived from this task. We then turned to the relation between individual differences in ninth graders' competence on the three algebraic cognition tasks and their overall algebra achievement while controlling for other factors that influence or are correlated with mathematical learning, specifically parental education, sex, reading ability, speed of numeral processing, intelligence, and the central executive component of working memory (Bailey, Littlefield, & Geary, 2012; Clark, Pritchard, & Woodward, 2010; Deary, Strand, Smith, & Fernandes, 2007; Geary, 2011; LeFevre et al., 2010).

After demonstrating that performance on the algebraic tasks does indeed contribute to overall algebra achievement, while controlling for the covariates, we show that measures of ANS acuity and poor arithmetic fact memory are differentially related to performance on these tasks. For these analyses, we included a task that assessed fluency of processing the magnitudes of Arabic numerals as an additional covariate. We added this covariate based on several previous studies suggesting that the relation between ANS acuity and mathematics achievement may be mediated by symbolic number processing, that is, fluency in processing the magnitudes represented by Arabic numerals (e.g., Bugden & Ansari, 2011; De Smedt et al., 2011; Lyons & Beilock, 2011; Rousselle & Noël, 2007).

Method

Participants

The current study is based on a longitudinal assessment of developmental changes in children's mathematical competence from kindergarten to high school (see Geary, Hoard, Nugent, & Bailey, 2012). A total of 288 children were recruited from the public school system in Columbia, Missouri, in the midwestern United States and finished the first year of testing, and 22 additional children were added as a refreshment sample to the study in fifth grade. The latter were identified as at risk for long-term difficulties with mathematics based on standardized mathematics achievement scores below the 30th percentile. The 171 ninth graders (156 from the original sample, 92 girls) who completed all tasks were included in these analyses, and these students had higher intelligence scores ($M = 101$, $SD = 15$) in first grade than the children who did not complete all tasks ($M = 95$, $SD = 14$), $t(286) = 4.2$, $p < .0001$. In ninth grade, the mathematics achievement of the sample was low average (M = 40th national percentile, $SD = 30$) and the reading achievement was average (M = 52nd national percentile, $SD = 28$). The final sample averaged 6 years 2 months of age (SD = 4 months) at the time of the kindergarten assessment and 15 years 1 month (SD = 4 months) at the time of the ninth grade algebra assessment. In terms of race/ethnicity, 4% of the sample identified as Hispanic, with 76% identifying as White, 7% as Black, 6% as Asian, 7% as mixed race, and 4% as other or unknown.

Algebra measures

Coordinate plane

Participants were shown a coordinate plane on a computer screen with perpendicular axes that were each 15 cm in length and labeled with endpoints of −50 and 50. For each trial, a coordinate pair was presented just outside the right edge of the plane, and a button for submission, labeled "Continue," was on the lower right area of the computer screen. One practice item, (25,30), was followed by 24 randomly ordered test item coordinate pairs: (2,4), (14,17), (16,10), (31,32), (32,20), (39,40), (48,43), (−11,12), (−16,4), (−23,27), (−32,20), (−35,37), (−43,35), (−3,−1), (−6,−16), (−22,−29), (−32,−27), (−36,−32), (−41,−48), (7,−2), (21,−22), (41,−40), (41,−44), and (33,−25).

Participants were first given the following practice item: "Over here is the pair of points you want to graph. So, if it says (25,30), you would find where you think 25 would go on this axis and then where you think 30 would go on this axis and put the dot here." They were instructed that for each coordinate pair presented, they should place the cursor and click the mouse at the point where the pair should be positioned. They were allowed to change the placement and were instructed to click the "Continue" button to record the position of the final answer. We examined several measures from the task (below) and determined that percentage absolute error (PAE) for each x and y value was the best measure of individual differences; the measure follows procedures used for the linear number line (Booth & Siegler, 2006; Siegler & Booth, 2004; Siegler & Opfer, 2003). Based on the findings of no effects for axes and small quadrant effects (below), the outcome measure was overall performance based on the sum of PAE across the four quadrants ($\alpha = .82$).

Expression evaluation

Following the expression evaluation task from Walczyk and Griffith-Ross (2006), 18 simple algebraic expressions containing one unknown (X) were horizontally presented, one at a time, in a standard order in the center of a computer screen (see items in Appendix). Above each expression was an equality assigning X to a specific value. The items were of four types: addition without multiplication ($X = 3$, $2 + X$), addition with multiplication ($X = 4$, $2X + 1$), subtraction without multiplication ($X = 6$, $9 - X$), and subtraction with multiplication ($X = 2$, $4X - 5$). X values were randomly sequenced, and numerals for the expression were also randomly generated, with the constraint that the same numeral did not appear twice in one problem. For the subtraction items, all answers were greater than 0.

Participants were asked to solve the expression, given the provided value of X, and to speak the answer into a microphone interfaced with the computer that recorded reaction time (RT) from onset of problem presentation to microphone activation; paper and pencil were made available to participants. The answer, use or not of paper and pencil, and notation for spoiled RT were recorded by the experimenter, and the next problem appeared when the experimenter pressed a key. The Spearman–Brown prediction equation was used to estimate reliability based on split half correlations. The reliability was high for both accuracy ($\alpha = .90$) and RT for correctly solved expressions ($\alpha = .85$). Accuracy and RT were not correlated ($r = .12$, $p = .1081$), indicating that there was no speed–accuracy trade-off. RTs were transformed (square root) to correct skew.

Equation memory

Valid and invalid equations containing 15 or 19 elements (i.e., numerals, letters, parentheses, equal sign, and arithmetic symbols) were developed following Sweller and Cooper (1985; items in Appendix). Each item was presented on a computer screen, one at a time, for 5 s. A mask was presented, and participants then attempted to recreate "as much as you can remember of" the equation with pencil and paper, in order, within 30 s. One practice item of each type was presented [valid: $ab + yz = w$; invalid: $) - ax = b +)$]. Test items were presented in a sequence of two valid, two invalid, two valid, two invalid; element size alternated between 19 and 15, beginning with 19. The score was the longest continuous string of correct symbols/numerals recalled, allowing for substitutions of variable names (scoring manual available from first author). Schema memory for algebra equations should be reflected in the difference in recall scores for valid and invalid equations (Sweller & Cooper, 1985). We assessed these differences across all items and for the short items (i.e., 15 elements) and long items (i.e., 19 elements). The difference scores for long items were the most reliable ($\alpha = .60$) and, thus, the best outcome measure for individual differences analyses (for short items: $\alpha = .27$; for all items: $\alpha = .54$).

Algebra achievement

The measures included 25 multiple-choice problems from Star and colleagues' (2015) test that has been shown to be sensitive to individual differences in algebra learning (Rittle-Johnson & Star, 2009; Rittle-Johnson et al., 2009). The items included standard solve-for-x problems, systems of equations, factoring, determining equation slope, and concept questions (e.g., definition of a nonvertical line). The sum of the item scores created a highly reliable composite ($\alpha = .86$), and the final score was the number correct minus a fraction of the number incorrect to control for guessing.

ANS and memory measures

ANS

The task was completed using the Panamath program (Halberda et al., 2008), a commonly used measure of ANS acuity. Following Mazzocco and colleagues' (2011a) study of ninth graders, each trial consisted of two computer-presented non-overlapping clusters of blue and yellow dots, and students indicated which color set was more numerous by pressing a key on the computer keyboard. All dot displays consisted of 5 to 16 dots and were displayed for only 200 ms to discourage counting. The average size of dots was 36 pixels and varied up to 20%. Dot size was varied to keep the total area of the dots constant across the two colors for half of the trials. The 80 experimental trials were presented following 10 practice trials. A total of 20 experimental items were randomly selected from each of four ranges of larger to smaller dot cluster ratios, specifically ratio ranges of 1.12 to 1.22, 1.22 to 1.34, 1.39 to 1.52, and 2.34 to 2.57.

Overall performance was consistent with Mazzocco and colleagues' (2011a) results; specifically, 77% (SD = 7.34, median = 78) of the items were responded to correctly and yielded a mean Weber fraction value of .35 (SD = .19, median = .31). The distribution of Weber scores was skewed and, thus, transformed (square root). Despite concerns about the reliability of the Weber fraction (an estimate of the underlying acuity of the ANS) for younger students (Inglis & Gilmore, 2014), we found that standardized Weber fraction scores and percentage correct were significantly correlated ($r = .85$, $p < .0001$); thus, their mean was used as a highly reliable ANS composite score ($\alpha = .92$).

Memory retrieval

A retrieval-only task was used to assess addition fact memory (Jordan & Montani, 1997). Fourteen simple addition problems were horizontally presented, one at a time, at the center of a computer monitor (i.e., 3 + 6, 5 + 3, 7 + 6, 3 + 5, 8 + 4, 2 + 8, 9 + 7, 2 + 4, 9 + 5, 7 + 2, 9 + 8, 4 + 7, 2 + 5, and 3 + 9). The problems consisted of the integers 2 through 9, with the constraint that the same two integers (e.g., 2 + 2) were never used in the same problem; half of the problems summed to 10 or less, and the smaller valued addend appeared in the first position for half of the problems. Participants were instructed to solve the problems only by retrieving the answers from memory. They were instructed to try to remember the answers as quickly as they could and were instructed not to count or use any other type of problem-solving strategy; if they could not remember, they were told that it was okay to guess. This task has been used in previous research to assess the extent of mastery of basic facts and retrieval deficits (Geary, Hamson, & Hoard, 2000; Jordan et al., 2003; Jordan & Montani, 1997; Russell & Ginsburg, 1984; Siegler & Shrager, 1984). Based on experimenter observation and participants' reports of how they solved the problems, 93% of the problems were solved by means of retrieval, and 28% (median = 21) of these were errors. The distribution of errors was skewed and, thus, was transformed (square root).

Control measures

Intelligence

Verbal intelligence and nonverbal intelligence were assessed using the Vocabulary and Matrix Reasoning subtests of the Wechsler Abbreviated Scale of Intelligence (WASI; Wechsler, 1999), respectively. Intelligence was estimated based on these scores following standard procedures.

Central executive

The central executive was assessed using three dual-task subtests of the Working Memory Test Battery for Children (WMTB-C; Pickering & Gathercole, 2001) in fifth grade. Listening Recall requires the child to determine whether a sentence is true or false and then to recall the last word in a series of sentences. Counting Recall requires the child to count a set of four, five, six, or seven dots on a card and then to recall, in order, the number of dots counted on each card at the end of that series of cards. Backward Digit Recall is a standard format backward digit span task. The subtests consist of span levels ranging from 1 to 6 or 1 to 9 items to remember, and each span level has 6 trials. Failing 3 trials at one span level terminates the subtest, and passing 4 trials moves the child to the next level. The total number of trials answered correctly was used as the central executive measure because these scores are more reliable than span scores ($\alpha = .70$).

Processing speed

A rapid automatized naming (RAN) task assessed speed of processing numerals (Denckla & Rudel, 1976; Mazzocco & Myers, 2003). Although the RAN does not assess all of the multiple components of processing speed (Carroll, 1993), it does assess the educationally relevant facility of serially encoding arrays of visual stimuli, as with multi-digit Arabic numerals. After 5 practice items, the participant was presented with a 5×10 matrix of incidences of numerals and was asked to name them as quickly as possible without making any mistakes. RT was measured via a stopwatch.

Symbolic number processing

To create a control variable for symbolic number processing, we used a subset of items from the number sets test (Geary, Bailey, & Hoard, 2009; Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). The test overall includes two types of stimuli: objects (e.g., stars and circles) in a half-inch square and an Arabic numeral (18-point font) in a half-inch square. These stimuli are combined in domino-like rectangles in the following combinations: objects/same objects, objects/different objects, objects/Arabic numerals, and Arabic numerals/Arabic numerals. The task is to circle any rectangles in which the stimuli sum to match a target number displayed on the top of a page. For practice, 3 items are presented to the participant to choose from using a target number of 4. After the participant's choice(s) has been made and discussed, 5 more practice items, each with two choices, are administered for the target number of 3. In the test, 36 items are presented on each of two pages for the target numbers of 5 and 9. The participant is asked to move across each line of the page from the left to right without skipping any and to "circle any groups that can be put together to make the top number, 5 [or 9]" and to "work as fast as you can without making many mistakes." Using a stopwatch, the participant is given 60 and 90 s per page for the targets 5 and 9, respectively, and is asked to stop at the time limit.

Of the 72 items for each target number, 18 involved the processing of only numerals (e.g., "6 7" or "6 3" and determining whether their sum matched the target), half of which matched the target. We used these items to create a symbolic number processing variable. Following previous studies, we used hits–false alarms for each target number and corrected for RT: (hits – false alarms) * (maximum time limit/actual RT). The RT correction was needed because most participants completed the task before the maximum time of 60 or 90 s per page. The scores for the targets of 5 and 9 were summed to create a composite variable ($\alpha = .92$).

Achievement

Reading achievement and mathematics achievement were assessed with the Word Reading and Numerical Operations subtests of the Wechsler Individual Achievement Test-II: Abbreviated (WIAT-II; Wechsler, 2001), respectively. The easier Word Reading items include matching and identifying letters, rhyming, and beginning and ending sounds, and the more difficult items assess accuracy of reading increasingly difficult words. The Numerical Operations items include number discrimination, rote counting, number production, and basic arithmetic operations. The more difficult items include rational numbers and simple algebra and geometry problems solved with pencil and paper.

Parental education

Participants' parents were asked to complete a survey that included items on their education level, income, and government assistance. Complete or partial information was available for the families of 155 participants. Of these parents, 4% had some schooling but no GED or high school diploma; 33% had a high school diploma or GED; 5% had some college, technical school, or an associate's degree; 32% had a bachelor's degree; and 27% had a post-graduate degree. In the 2007–2011 American Community Survey, the percentages of Missouri residents belonging to these categories were 13%, 32%, 29%, 16%, and 9%, respectively (U.S. Census Bureau, 2011). Total household incomes were as follows: \$0–\$25,000 (8%), \$25,000–\$50,000 (23%), \$50,000–\$75,000 (16%), \$75,000–\$100,000 (16%), \$100,000–\$150,000 (22%), and \$150,000 or higher (16%). In terms of government assistance, 7% of parents reported receiving food stamps and 1% reported receiving housing assistance. To control for parental education, we created two parental education categories based on the highest educational attainment of the participants' primary caregiver: one for college graduates (bachelor's level or higher) and one for nongraduates. The parents with missing data were assigned to the latter category because the mathematics achievement scores of their children (29th national percentile) were more similar to children in the nongraduate group (28th percentile, $p > .05$) than the graduate group (47th percentile, $p < .05$).

Procedure

The WASI was individually administered in the spring of first grade for the original sample and in the spring of fifth grade for the refreshment sample. The WIAT-II was individually administered in the spring of ninth grade, and the algebra test was administered in groups of 1 to 9 (median = 4) students in between the fall and spring ninth grade assessments. The experimental and RAN tasks were administered in the fall of ninth grade. The retrieval-only task and number sets test were administered in the fall of eighth grade. The majority of children were tested in a quiet location at their school site and occasionally on the university campus or in a mobile testing van. Testing on campus or in the van occurred for children who had moved out of the school district or to a nonparticipating school and for administration of the WMTB-C (e.g., on the weekend or after school). The experimental and achievement assessments required between 20 and 40 min, and the WMTB-C required approximately 60 min per assessment.

Analyses

The 1.5% of missing values was replaced by variable means. The first set of analyses presents a detailed assessment of performance on the coordinate plane task, followed by basic analyses for the expression evaluation and equation memory tasks. The third assesses the external validity of the associated measures. The latter involved demonstration that these measures, individually and as a set, predicted individual differences on the algebra achievement test while controlling for the covariates of parental education, sex, word reading achievement, speed of numeral processing, intelligence, and central executive performance. The final set of analyses assessed the relation between the ANS and retrieval scores and performance on the coordinate plane, expression evaluation, and equation memory tasks while controlling for the same set of covariates, symbolic number processing, and algebra test scores. Control of algebra scores enabled a more stringent assessment of the relation between the ANS and retrieval measures and the outcomes of interest. With the exception of parental education and sex, all variables were standardized ($M = 0$, $SD = 1$). Variables for which smaller values (e.g., percentage absolute error) indicate better performance were multiplied by -1 so that higher scores represent better performance for all variables.

Results

As shown in Table 1, algebra achievement scores were more highly correlated with Numerical Operations scores, $r = .80$, $p < .001$ ($\beta = .66$, $t = 12.11$, $p < .001$, controlling for Word Reading) than Word Reading scores, $r = .61$, $p < .001$ ($\beta = .22$, $t = 4.08$, $p < .001$, controlling for Numerical Operations), confirming that the algebra test is assessing mathematical competence as measured in a nationally normed test. As noted, we used algebra achievement in subsequent analyses because of our specific interest in individual differences in algebraic competence.

Coordinate plane

We present a more detailed analysis of coordinate plane performance because it is a newly developed task. Using the segment procedure in R, we plotted each student's x,y placement for all 24 items, as shown in Fig. 1. The large circles represent the stimuli, and the smaller circles represent the student's actual placement. The length of the line segment represents the Euclidean distance between the stimuli and the placement, that is, $[(x\text{-axis difference})^2 + (y\text{-axis difference})^2]^{1/2}$. The diagonal lines represent transpositions of the signs of the stimuli such as placing the stimulus (–32,20) at (32,–20). Based on these patterns, we created five coordinate plane variables: error distance, error distance in correct quadrant (i.e., Euclidean distance for x or y values placed in correct quadrant), single transposition (i.e., the sign of either the x or y value is transposed), double transposition (i.e., both values are transposed), and percentage of x or y placements in the correct quadrant.

Table 1
Correlations among study measures.

Variable	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1. Algebra test score	1.00													
2. Numerical Operations	.80	1.00												
3. Word Reading	.61	.58	1.00											
4. Parental education	.34	.31	.38	1.00										
5. Sex	.13	.17	.10	.15	1.00									
6. Processing speed	.26	.35	.39	.08	−.04	1.00								
7. Intelligence	.48	.40	.47	.21	.24	.06	1.00							
8. Central executive	.53	.54	.49	.22	.04	.32	.42	1.00						
9. Equation memory	.41	.41	.36	.17	−.09	.19	.20	.38	1.00					
10. Coordinate plane (PAE)	.53	.64	.43	.17	−.01	.31	.26	.40	.31	1.00				
11. Expression evaluation RT	.46	.57	.34	.20	.12	.34	.13	.37	.26	.20	1.00			
12. Expression evaluation accuracy	.54	.65	.49	.15	.05	.26	.30	.47	.31	.64	.12	1.00		
13. ANS	.38	.40	.43	.17	.08	.28	.31	.37	.18	.41	.21	.48	1.00	
14. Retrieval errors	.50	.54	.39	.20	.13	.20	.18	.36	.39	.37	.32	.40	.40	1.00

Note: RT, reaction time. $p < .05$ for $r > .149$; $p < .01$ for $r > .19$; $p < .001$ for $r > .249$. Sex: boy = 1; girl = 0. Processing speed, PAE, expression evaluation RT, and retrieval errors were multiplied by −1 so that positive scores indicate better performance.

Following studies of the single-dimensional number line, we also calculated the mean PAE for each *x*- and *y*-axis value. These were then submitted to a 2 (Axis) × 4 (Quadrant) repeated-measures analysis of variance (ANOVA). The results revealed a nonsignificant axis effect, $F(1,169) = 1.68$, $p = .1963$, a significant quadrant effect, $F(3,507) = 9.09$, $p < .0001$, and a nonsignificant interaction, $F(3,507) = 2.30$, $p = .0761$. On the basis of the nonsignificant axis effect, PAE was averaged across the *x*- and *y*-values. As shown in Table 2, error rates were higher for pairs in Quadrant II and Quadrant IV due to transpositions shown in Fig. 1 (negative diagonal line). The contrast of PAE for Quadrant I and Quadrant III was not significant, $t(170) = -0.45$, $p = .1031$, nor was the contrast of PAE for Quadrant II and Quadrant IV, $t(170) = 1.58$, $p = .1157$, but all other pairwise contrasts were significant (*ps* < .024).

Table 3 shows the mean values and correlations among the coordinate plane measures and their correlation with algebra test scores. As shown, the mean Euclidean distances for errors were 9.8 overall and 5.9 for points placed in the correct coordinate. To put these values in perspective, the distance between the origin and the (|50,50|) points is 70.7, indicating that placements were generally accurate. Transpositions were infrequent overall, and the majority of these were made by a small number of students. Just 6 students committed 31% of the single value transpositions and 12 other students committed an additional 23% of these transpositions. The skew was more dramatic for double transpositions, where 58% of these transpositions were committed by 7 students; fully 80% of the students had no (0) double transpositions. All of the measures were significantly correlated (*ps* < .05) with algebra test scores, with PAE showing the strongest correlation.

To determine the best single predictor or set of predictors of individual differences in algebra achievement, all measures were entered into a forward stepwise regression ($p < .05$ to enter and stay). PAE was entered at the first step ($\beta = .66$, $p < .001$), followed by distance at the second step ($\beta = -1.35$, $p < .0001$). No other variables were entered into the regression. However, the results indicated that worse performance on the distance measure was associated with higher algebra scores, but this is difficult to interpret given the extremely high correlation between PAE and distance ($r = .98$, $p < .0001$). For this reason, we determined that PAE was the best measure of individual differences on the coordinate plane task.

Expression evaluation and equation memory

Students correctly solved, on average, 15.6 (*SD* = 3.6) of the 18 expressions in the expression evaluation task, with a mean RT for correct trials of 4.02 s (*SD* = 1.74). Mean recall scores for the equation memory task are shown in Table 2. A 2 (Validity) × 2 (Length) repeated-measures ANOVA revealed substantially higher recall for valid items, $F(1,170) = 506.35$, $p < .0001$, as well as a significant length effect, $F(1,170) = 8.16$, $p = .0048$, and a significant interaction effect, $F(1,170) = 111.68$, $p < .0001$.

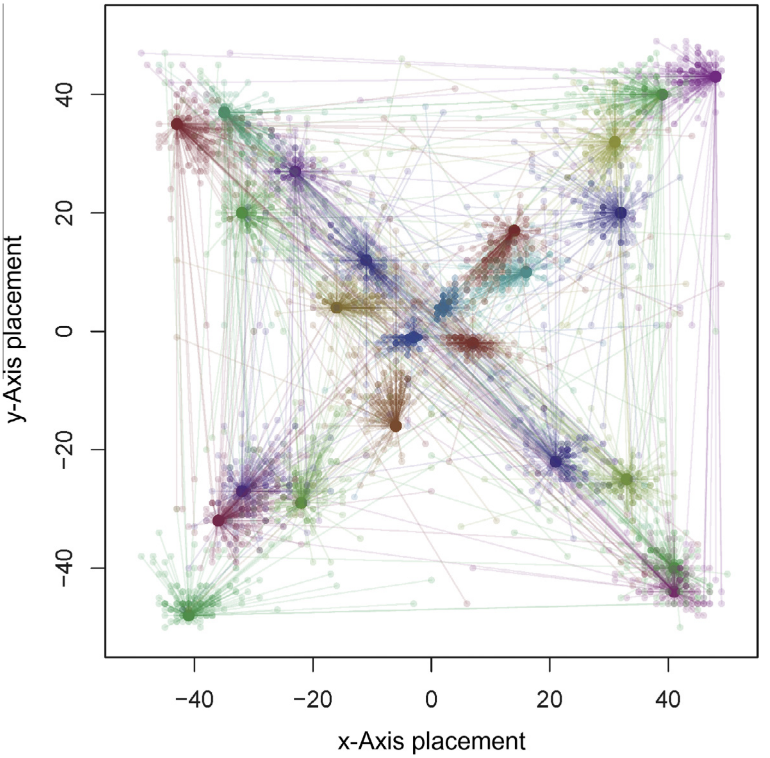

Fig. 1. Plot of student placements of *x,y* pairs in the coordinate plane. The large circles represent the stimuli, and the smaller circles represent the student's actual placement. The length of the line segment represents the Euclidean distance between the stimuli and the placement, that is, $[(x\text{-axis difference})^2 + (y\text{-axis difference})^2]^{1/2}$. The diagonal lines represent transpositions of the signs of the stimuli.

Table 2
Mean performance on coordinate plane and equation memory tasks.

Coordinate plane				
	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
Percentage error	10 (9)	16 (20)	10 (11)	14 (23)
Equation memory				
	Valid long	Valid short	Invalid long	Invalid short
Percentage recalled	55 (20)	44 (15)	22 (7)	28 (7)

Note: Standard deviations are in parentheses. For equation memory, long items had 19 elements and short items had 15 elements.

The validity effects were significant for longer equations, $t(1,170) = 21.49$, $p < .0001$, and shorter equations, $t(1,170) = 13.93$, $p < .0001$. The interaction emerged because a higher percentage of elements was more often correctly recalled for valid longer equations than for valid shorter equations, $t(1,170) = 7.27$, $p < .0001$, whereas the opposite was true for invalid equations, $t(1,170) = -8.81$, $p < .0001$.

As shown in Table 1, with the exception of accuracy on the coordinate plane and expression evaluation measures ($r = .64$, $p < .0001$), the correlations among the algebraic cognition tasks were modest to nonsignificant ($rs = .12$–$.31$). The implication is that they are assessing relatively distinct knowledge and abilities and not a general algebraic competence factor.

Table 3
Means and correlations among coordinate plane measures.

Variable	Mean	*SD*	1	2	3	4	5	6	7
1. Distance	9.8	9.2	1.00						
2. Distance in correct quadrant	5.9	4.5	.74	1.00					
3. Single transposition (%)	5.6	10.2	.57	.71	1.00				
4. Double transposition (%)	6.6	19.6	.78	.33	.08	1.00			
5. Correct quadrant percentage	91	15.5	.94	.77	.78	.66	1.00		
6. PAE	12.6	13.0	.98	.80	.61	.68	.90	1.00	
7. Algebra test score	13.4	5.6	.45	.46	.33	.18	.37	.53	1.00

Note: $p > .05$ for $r < .149$; $p < .05$ for $r > .15$; $p < .001$ for $r > .32$. Distance = $[(x\text{-axis difference})^2 + (y\text{-axis difference})^2]^{1/2}$. Distance in correct quadrant includes only items placed in this quadrant. PAE, percentage absolute error. Distance, distance in correct quadrant, single transposition, double transposition, and PAE were multiplied by –1 so that positive scores indicate better performance.

External validity

Variants of the expression evaluation and equation memory tasks have been used in previous studies, as noted (Sweller & Cooper, 1985; Walczyk & Griffith-Ross, 2006), but their external validity has not been established, nor has that of the coordinate plane task. External validity requires demonstration that the algebraic cognition measures predict, in this case, algebra achievement while controlling for the covariates (Sternberg, 1977) and would provide additional support for the argument that the tasks are assessing different components of algebra skill. Results from the associated regression equations shown in Table 4 confirm that each of the algebraic cognition measures predicts algebra achievement in independent regression equations (βs = .170–.349, *ps* < .007) and when simultaneously included in a single equation (βs = .100–.272, *ps* < .078). Of the covariates, reading achievement (βs = .176–.310, *ps* < .0146) and intelligence (βs = .170–.209, *ps* < .0105) were the only consistent predictors of algebra achievement.

We also examined the relation among ANS acuity, frequency of memory retrieval errors, and algebra achievement. As shown in Table 1, both variables were significantly correlated with overall algebra scores. However, with control of the covariates and simultaneous entry of both of these predictors, ANS acuity was not related to algebra achievement ($\beta = .012$, $p = .8554$), but frequency of retrieval errors was ($\beta = .263$, $p < .0001$).

ANS, semantic memory, and algebraic cognition

As shown in Table 5, after control of the covariates, symbolic number processing, and algebra achievement, frequency of memory retrieval errors when solving addition problems predicted algebra equation memory ($\beta = .250$), $t(160) = 3.03$, $p = .0028$, but was unrelated to PAE on the coordinate plane task ($\beta = .054$), $t(160) < 1$, $p = .3817$, or expression evaluation RT ($\beta = .055$), $t(160) < 1$, $p = .4771$, or accuracy ($\beta = .075$), $t(160) = 1.02$, $p = .3080$. In contrast, the ANS composite score predicted PAE on the coordinate plane task ($\beta = .147$), $t(160) = 2.37$, $p = .0187$, and expression evaluation accuracy ($\beta = .240$), $t(160) = 3.26$, $p = .0014$, but not equation memory ($\beta = -.100$), $t(160) = -1.21$, $p = .2279$, or expression evaluation RT ($\beta = -.053$), $t(160) < 1$, $p = .4966$.

Algebra achievement was a significant predictor of PAE on the coordinate plane task and of expression evaluation accuracy and RT (βs = .236–.250, *ps* < .0153). Girls showed an advantage for equation memory ($\beta = -.342$), $t(160) = -2.42$, $p = .0164$, but there were no sex differences for the three remaining algebraic cognition measures (*ps* > .2617). Finally, fluency of symbolic number processing predicted speed of evaluating expressions ($p < .0002$).

Discussion

The current study contributes to our understanding of the sources of individual differences in algebraic competence and provides useful insights into the basic cognitive systems that may influence

Table 4
Standardized regression estimates for the prediction of algebra achievement.

Predictor	Algebra achievement									
	Estimate	*p*	Estimate	*p*	Estimate	*p*	Estimate	*p*	Estimate	*p*
Parental education	.197 (.123)	.1127	.201 (.119)	.0933	.174 (.120)	.1508	.238 (.121)	.0500	.176 (.110)	.1105
Sex	.108 (.117)	.3563	.089 (.112)	.4271	.001 (.113)	.9926	.056 (.113)	.6204	.036 (.105)	.7341
Reading	.306 (.077)	.0001	.277 (.074)	.0003	.310 (.074)	.0001	.258 (.077)	.0010	.176 (.071)	.0146
Processing speed	.025 (.063)	.6949	−.013 (.061)	.8335	−.027 (.062)	.6673	.018 (.061)	.7641	−.064 (.058)	.2662
Intelligence	.182 (.068)	.0081	.170 (.066)	.0104	.209 (.067)	.0020	.178 (.067)	.0082	.200 (.061)	.0014
Central executive	.205 (.069)	.0035	.187 (.066)	.0054	.184 (.067)	.0071	.173 (.069)	.0128	.058 (.065)	.3774
Equation memory	.170 (.062)	.0069	–	–	–	–	–	–	.100 (.056)	.0779
Coordinate plane	–	–	.349 (.077)	.0001	–	–	–	–	.223 (.083)	.0080
Expression RT	–	–	–	–	.254 (.062)	.0001	–	–	.272 (.057)	.0001
Expression accuracy	–	–	–	–			.255 (.067)	.0001	.191 (.070)	.0073
R^2	.497	.0001	.533	.0001	.524	.0001	.519	.0001	.612	.0001

Note: Standard errors are in parentheses. Parental education: college graduate = 1; otherwise = 0. Sex: boy = 1; girl = 0. Processing speed and expression evaluation RT were multiplied by −1 so that positive scores indicate better performance.

Table 5
Regression equations predicting algebraic cognition.

Predictor	Equation memory		Coordinate plane		Expression RT		Expression accuracy	
	Estimate	*p*	Estimate	*p*	Estimate	*p*	Estimate	*p*
Parental education	.038 (.151)	.7991	−.018 (.113)	.8769	.126 (.141)	.3757	−.153 (.134)	.2551
Sex	−.342 (.141)	.0164	−.131 (.106)	.2193	.150 (.133)	.2617	−.042 (.126)	.7419
Reading	.118 (.098)	.2287	.041 (.074)	.5775	−.014 (.092)	.8807	.161 (.087)	.0673
Algebra	.143 (.102)	.1648	.250 (.077)	.0014	.236 (.096)	.0153	.240 (.091)	.0094
Processing speed	−.005 (.081)	.9541	.059 (.060)	.3303	.109 (.076)	.1527	−.037 (.072)	.6126
Intelligence	.017 (.086)	.8440	−.017 (.064)	.7912	−.125 (.081)	.1236	−.041 (.076)	.5957
Central executive	.150 (.088)	.0888	.048 (.066)	.4705	.086 (.083)	.2966	.146 (.078)	.0642
Symbolic number	.076 (.091)	.4038	.095 (.069)	.1666	.334 (.086)	.0002	.106 (.081)	.1954
ANS	−.100 (.082)	.2279	.147 (.062)	.0187	−.053 (.078)	.4966	.240 (.074)	.0014
Retrieval errors	.250 (.082)	.0028	.054 (.062)	.3817	.055 (.078)	.4771	.075 (.073)	.3080
R^2	.284	.0001	.377	.0001	.365	.0001	.430	.0001

Note: Standard errors are in parentheses. Sex: boy = 1; girl = 0. Processing speed, expression evaluation RT, and retrieval errors were multiplied by −1 so that positive scores indicate better performance.

the acquisition of this competence or are influenced by it. We first address the component algebraic competencies assessed in the study and then turn to their relation to addition fact retrieval deficits and ANS acuity.

Individual differences in algebraic competence

Our findings for the expression evaluation and equation memory tasks are consistent with and extend the results of previous studies (Sweller & Cooper, 1985; Walczyk & Griffith-Ross, 2006). The unique contribution here is the demonstration of the external validity of these measures (Sternberg, 1977). Controlling domain general abilities (e.g., intelligence) and demographic factors that can influence mathematics achievement (Clark et al., 2010; Deary et al., 2007; Geary, 2011; LeFevre et al., 2010), speed and accuracy of evaluating expressions, and memory for the structure of algebra equations independently contributed to individual differences in ninth graders' algebra achievement. The coordinate plane measure was developed specifically for this study, and it too contributed to algebra achievement independent of other factors, including other algebraic skills.

We were somewhat surprised that placing coordinate pairs was just as accurate on the *y*-axis as on the more familiar *x*-axis, that is, the one-dimensional number line (Siegler & Booth, 2004; Siegler & Opfer, 2003). The result suggests that most ninth graders have integrated two-dimensional representations of numeral magnitude with their well-developed understanding of magnitude as represented on the number line, that is, the *x*-axis (Siegler et al., 2011). It could be that there are differences in the accuracy of x- and y-axis placements in earlier grades, when children are first learning the coordinate plane, and that any such differences are diagnostic of a poor conceptual understanding of the plane, but this remains to be determined. Even in ninth grade, students were less accurate overall in making placements in quadrant II and quadrant IV, where the signs of the x- and y-values differ. However, as noted, these effects were largely driven by a small percentage of students who consistently transposed the sign for one or both values. For these students, and perhaps other students during the initial learning of the plane, one possibility is that integrating two-dimensional representations of magnitude with their representation of the number line first occurs for x,y pairs that "run in the same direction," that is, up and right for increasing magnitudes on both axes and down and left for decreasing magnitudes. If this interpretation is correct, then integrating magnitudes that differ in direction (sign) into a more general understanding of quantitative magnitude (Siegler et al., 2011) occurs later and suggests that many of these students have not yet mastered the coordinate plane.

The equation memory measure produced the expected differences between valid and invalid equations, confirming Sweller and Cooper's (1985) conclusion that expertise in algebra involves the formation of a memory schema for the basic structure of equations. The finding that memory for valid 19-element equations was better than that for valid 15-element equations is surprising at first blush. We suspect that the advantage for the longer equations arose because they included more parenthetical elements, including the familiar "$(a + b)(c - d)$" sequence (see Appendix). The potential advantage associated with such sequences is consistent with the formation of schema memories. In any case, the difference in memory for 19-element valid and invalid equations was the best individual differences measure from a psychometric perspective. We suspect that one of the reasons why the equation memory measure was not as strongly related to algebra achievement as the other measures was its lower reliability, which can be addressed in future studies by increasing the number of 19-element valid and invalid equations in the measure. Finally, we note that girls' advantage on this task is most likely due to their advantage over boys in episodic memory (Herlitz & Rehnman, 2008) given the nature of task demands and the finding of no other sex differences in algebra.

ANS and memory retrieval

Debate regarding the importance of ANS acuity and memory for arithmetic facts for mathematics learning has largely centered on individual differences in mathematics achievement broadly (Geary et al., 2012; Halberda et al., 2008; Libertus et al., 2011) or individual differences in competence with numbers or arithmetic (Chu et al., 2015; Park & Brannon, 2013, 2014). We extend the debate to algebra and demonstrate nuance in the relation between ANS acuity and fact memory and individual differences in this area (see also Kibbe & Feigenson, 2015).

One of our goals was to assess the relation between addition retrieval deficits and equation memory based on the likelihood that the acquisition of both of these competencies is supported in part by the hippocampal-dependent memory system (De Smedt et al., 2011; Qin et al., 2014; Tse et al., 2007) and because retrieval deficits are a cardinal feature of mathematical learning disabilities in children (Geary, 1993). In this view, children who experience difficulties in retrieving basic arithmetic facts from long-term memory may also show deficits or delays in the formation of schema memories for algebra equations. Our results are consistent with this hypothesis. Frequent errors during the retrieval of addition facts in eighth grade predicted poor memory for algebra equations in ninth grade while controlling for domain general abilities, demographic factors, overall algebra achievement, and ANS acuity. The results, however, are not definitive given the correlational nature of the data. If our hypothesis is correct, then future studies will find the same cognitive mechanisms associated with poor fact learning, such as proactive inhibition (De Visscher & Noël, 2014), and the same brain systems that support this learning, such as stability in the functional circuits recruited in the hippocampus and prefrontal cortex when retrieving facts (Qin et al., 2014), will predict individual differences in the ease of forming schema memories for algebra equations.

Consistent with previous studies of mathematics achievement (e.g., Fazio et al., 2014; Libertus et al., 2011), we found that ANS acuity was correlated with algebra achievement test scores, but this relation was not significant while controlling for the covariates. Nor was ANS acuity related to equation memory scores. Nevertheless, ANS acuity did emerge as a significant predictor of accuracy of placements in the coordinate plane and accuracy in evaluating algebraic expressions. The overall pattern suggests more nuanced relations between ANS acuity and mathematical competence than implied in most previous studies, consistent with Lourenco and colleagues' (2012) finding that sensitivity to the quantity of discrete collections of items was more strongly related to arithmetical abilities than to geometric abilities. Our results also suggest that the ANS may be specifically engaged during spatial–numerical (coordinate plane) and numerical-only (expression evaluation) quantity judgments embedded in algebra problems but might not be related to other algebraic skills.

It is possible that acuity of the ANS contributes to students' ability to represent numerical magnitudes in two-dimensional space and to map magnitudes onto algebraic expressions. It is also possible, given the correlational nature of the data, that students' intuitive sense of magnitude only later becomes integrated with their understanding of magnitude within the coordinate plane and associated with algebraic expressions. Experimental studies, such as those conducted by Park and Brannon (2013, 2014) for arithmetic, will be needed to differentiate these alternatives. Future studies are also needed to assess the possibility that these relations are mediated by individual differences in inhibitory control (Gilmore et al., 2013). This is because our central executive measures assess the ability to maintain and update information in working memory and not the ability to inhibit prepotent responses.

Summary and implications

Fluency with high school algebra is a stepping-stone to the more complex mathematics needed for STEM fields, and basic competence with algebra contributes to employment and wage opportunities in many non-STEM occupations (Bynner, 1997; National Mathematics Advisory Panel, 2008). We have provided, to our knowledge, the first study of component algebraic skills that contribute to individual differences in high school students' algebra achievement and showed that these skills are differentially related to more basic cognitive abilities that are related to competence with number and arithmetic. The correlational nature of our study and largely concurrent measurement means that causal relations cannot be inferred from our results; nevertheless, the findings provide direction for future experimental and intervention as well as brain imaging studies. Of course, algebra is a complex domain, and we are not assessing all potential component skills (e.g., factoring, systems of equations). We view our tasks as an important start in the development of measures of specific competencies but by no means the only or last tasks that will be developed for the study of algebra. By analogy, the early studies of arithmetical cognition were almost entirely based on RTs for solving simple addition problems (e.g., 3 + 5), but the field has since moved well beyond this in terms of both topics and cognitive tasks. We see our tasks in the same way, that is, a reasonable start but only the beginning.

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Appendix A.

Equation memory items

$(x)(y - z) = (m + n)(p - q)$
$a + (ab/a) + ae/w = d$
$(\,)\ (\, + w\ (db) + (\,) = e)w + da$

$p\ v =)\ z - (hqh\ s - v//$
$(w + b)(a + e) = (w + d)(b)$
$p = s - (hv/v) - hq/z$
$yz - ((\ = mp +)\ ()\ x)\ qn() -$
$a)a\ d\ a/w + (\ = +\ b/ea$

Expression evaluation items

Addition: No multiplication
1. $X + 9$; $X = 7$
2. $3 + X$; $X = 8$
3. $6 + X$; $X = 5$
4. $X + 5$; $X = 4$

Addition: Multiplication
5. $5 + 9X$; $X = 6$
6. $8 + 7X$; $X = 3$
7. $9 + 5X$; $X = 1$
8. $5X + 7$; $X = 2$
9. $2X + 1$; $X = 9$

Subtraction: No multiplication
10. $X - 8$; $X = 9$
11. $X - 3$; $X = 5$
12. $7 - X$; $X = 8$
13. $4 - X$; $X = 6$

Subtraction: Multiplication
14. $3X - 6$; $X = 7$
15. $4X - 5$; $X = 2$
16. $9X - 7$; $X = 1$
17. $6X - 7$; $X = 3$
18. $9X - 2$; $X = 4$

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