The University of Sydney

MOOC Introduction to Calculus

Notes for 'Real line, decimals and significant figures'

Important Ideas and Useful Facts:

(i) Sets and elements: A *set* is a collection of objects, referred to as *elements*. A set may be represented, for example, by a list of elements surrounded by curly brackets and separated by commas, or using *set builder notation* $\{\ldots \mid \ldots\}$, where the vertical line is an abbreviation for "such that". For example, $\{\, x \mid x \text{ is a natural number less than } 5 \,\}$ and $\{0, 1, 2, 3, 4\}$ represent the same set, whose elements are precisely 0, 1, 2, 3 and 4.

(A *natural number* is a whole counting number, including zero. Note that some people do not count zero as a counting number, but we do in this course!)

(ii) Element symbol: The symbol $\in$ is an abbreviation for "is an element of", and $\notin$ is an abbreviation for "is not an element of". For example, if

$$A = \{\, x \mid x \text{ is a natural number less than } 5 \,\} ,$$

then $2 \in A$, but $5 \notin A$.

(iii) Subset symbols: If A and B are sets and we write $A \subseteq B$ or $B \supseteq A$, then we mean that every element of A is also an element of B, and say that A is a *subset* of B. For example $\{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$ and $\{1, 2, 3, 4\} \supseteq \{1, 2, 3\}$, but $\{1, 2, 3, 4\} \not\subseteq \{1, 2, 3\}$.

(iv) Equality of sets: If A and B are sets then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$, that is, A and B have precisely the same elements. Order and repetition are not important. For example, $\{1, 2, 3, 4\} = \{4, 1, 3, 2\} = \{4, 1, 3, 1, 2, 3\}$.

(v) Intersection, union and slash: If A and B are sets then put

(a) $A \cap B = \{\, x \mid x \in A \text{ and } x \in B \,\}$, called the *intersection* of A and B.

(b) $A \cup B = \{\, x \mid x \in A \text{ or } x \in B \,\}$, called the *union* of A and B.

(c) $A \backslash B = \{\, x \mid x \in A \text{ and } x \notin B \,\}$, called A *slash* B, the result of removing from A all elements from B.

(vi) Natural numbers: The set $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ of *natural numbers* forms a number system, closed under addition and multiplication, by which we mean that if m and n are natural numbers, then $m+n$ and mn (the result of multiplying m by n) are also natural numbers.

(vii) Integers: The set $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ of *integers* forms a number system, closed under addition, subtraction and multiplication.

(viii) Rationals: The set $\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$ of fractions, also called *rational numbers* (derived from the word *ratio*), forms a number system, closed under addition, subtraction, multiplication and division by nonzero elements. To add and multiply rational numbers, use the rules

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} .$$

(ix) Decimal expansion of a real number: A nonnegative *real number* α has a decimal expansion

$$\alpha \;=\; b_n b_{n-1} \ldots b_2 b_1 \cdot a_1 a_2 a_3 \ldots$$

where the b_i and a_j are digits from the set $\{0, 1, 2, \ldots, 9\}$, and the dot between b_1 and a_1 represents the decimal point.

(x) Recurring decimal expansions: A real number is rational if and only if it has a *recurring* decimal expansion, which means the pattern of digits repeats forever from some point onwards. For example,

$$\frac{6}{7} \;=\; 0.857142857142\ldots \;=\; 0.\dot{8}5714\dot{2},$$

where the two dots indicate the start and finish of the pattern of digits that gets repeated.

(xi) The real number line: The real numbers form a number system $\mathbb{R}$ that is closed under addition, subtraction, multiplication and division by nonzero elements. We visualise $\mathbb{R}$ as a continuous number line, called the *real line*, with zero in the middle, negative numbers to the left and positive numbers to the right.

(xii) The number π: The real number $\pi = 3.14159\ldots$ is the ratio of the perimeter of a circle to its diameter. The fact that π is the same for all circles is an advanced argument involving *limits* (a concept explained in **Module 3**) that relies on the fact that circles can be approximated arbitrarily well using similar triangles and the fact that ratios of corresponding sides of similar triangles are equal.

(xiii) Irrationals: A real number that is not rational (that is, cannot be expressed as a fraction involving a ratio of integers) is called *irrational*. For example, $\sqrt{2}$, $\sqrt{3}$ and π are irrational, though the proof for π is difficult. (A proof that $\sqrt{2}$ is irrational is given in the notes accompanying the next video.)

(xiv) Significant figures: Real numbers may be approximated by rational numbers with finite decimal expansions. The number of digits counted to the right from the leftmost positive digit is called the *number of significant figures*. For example, real numbers represented by 26.103, 0.00304 and 0.003040 are quoted to 5, 3 and 4 significant figures respectively.

(xv) Scientific notation: A positive real number α is expressed in *scientific notation* if it has the form

$$\alpha \;=\; b \cdot a_1 a_2 \ldots a_m \times 10^k$$

where m is nonnegative and k is an integer, and the dot between b and a_1 represents the decimal point. For example, 193.034 and 0.003040 become 1.93034×10^2 and 3.040×10^{-3} respectively in scientific notation. The number of digits used in scientific notation is the number of significant figures being quoted, and this avoids ambiguity in the case of large whole numbers (with zeros as place-holders).

(xvi) Accuracy rule for addition and subtraction: When adding or subtracting numbers, the final answer should be quoted to the least number of decimal places that occurs.

(xvii) Accuracy rule for multiplication and division: When multiplying or dividing numbers, the final answer should be quoted to the least number of significant figures that occurs.

Examples:

1. Consider the following real number

$$\alpha \;=\; 3.\dot{1}\dot{4} \;=\; 3.14141414\ldots\;,$$

which has a recurring decimal expansion. We can find a fraction that represents α, by a trick that expresses α in terms of itself in such a way that the recurring decimal expansion disappears by subtraction:

$$100\alpha - \alpha \;=\; 314.\dot{1}\dot{4} - 3.\dot{1}\dot{4} \;=\; 311\;.$$

Hence $99\alpha = 311$, so $\alpha = \frac{311}{99}$. This trick, though it looks specific to this example, can be turned into a proof that any real number with a recurring decimal expansion can be expressed as a fraction, that is, a ratio of integers.

2. Let $\alpha = 9.4$ and $\beta = 2.13$, considered as measurements, quoted to one and two decimal places respectively. Then

$$\alpha + \beta \;=\; 9.4 + 2.13 \;=\; 11.53 \;\approx\; 11.5$$

and

$$\alpha - \beta \;=\; 9.4 - 2.13 \;=\; 7.27 \;\approx\; 7.3,$$

quoting the answers, in each case, correct to one decimal place, since α is only quoted to one decimal place. Further,

$$\alpha\beta \;=\; \alpha \times \beta \;=\; 9.4 \times 2.13 \;=\; 20.002 \;\approx\; 20 \;=\; 2.0 \times 10$$

and

$$\frac{\alpha}{\beta} \;=\; \frac{9.4}{2.13} \;=\; 4.41314554\ldots \approx 4.4\;,$$

quoting the answer, in each case, correct to two significant figures, since α is only quoted to two significant figures. Note that, to avoid ambiguity, the answer $20 = 2.0 \times 10$ is written in scientific notation, to make clear that the zero digit is significant.

3. The radius of the earth is approximately $6,370,000$ metres correct only to three significant figures (so the zeros that appear are just place holders in the base 10 representation). To avoid ambiguity, one could write the radius of the earth as

$$6,370,000 \;=\; 6.37 \times 10^6 \text{ metres}\;.$$

The radius of the Jupiter however is approximately $70,000,000$ metres. This estimate is intended to be accurate to two significant figures (especially as there is considerable difference measuring the radius of Jupiter from the poles compared with measuring it from the equator). To avoid ambiguity, one could write

$$70,000,000 \;=\; 7.0 \times 10^7 \text{ metres}\;.$$