USC UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
DEPARTAMENTO DE MATEMÁTICA APLICADA
Área de Astronomia e Astrofísica

Matemáticas III

Grau em Robótica

EXERCÍCIOS DE AVALIAÇÃO CONTÍNUA

30 de novembro de 2022

Recomendações: Os cálculos deverão ir acompanhados de breves comentários explicando os passos seguidos e, de ser necessário, de figuras que ilustrem a resolução do problema. **Entrega:** Alunas/os que tenham assistido no mínimo 80% de horas de docência.

Nome: RESOLUÇÃO

1 Variável complexa

Calcula o valor principal de Cauchy da integral

$$\int_{-\infty}^{\infty} \frac{4\, e \cos x}{x^4 + 3x^2 + 2} dx.$$

2 Análise de Fourier

Acha a série de Fourier da função

$$f(x) = \begin{cases} -1, & -\pi \le x \le \dfrac{-\pi}{2}, \\ 0, & \dfrac{-\pi}{2} < x \le \dfrac{\pi}{2}, \\ 1, & \dfrac{\pi}{2} < x \le \pi. \end{cases}$$

(1)

$$\int_{-\infty}^{\infty} \frac{4e\cos x}{x^4+3x^2+2}\,dx$$

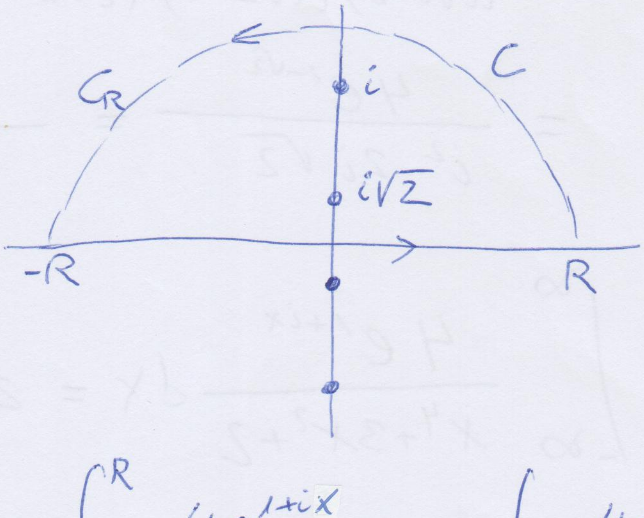
$$x^4+3x^2+2=0$$

$$t^2+3t+2=0 \Rightarrow t=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm 1}{2}$$

$$t=\begin{cases}-1\\-2\end{cases}$$

$$x^2=t \Rightarrow \begin{cases}\pm\sqrt{-1} & \begin{matrix} i \\ -i\end{matrix}\\ \pm\sqrt{-2} & \begin{matrix} i\sqrt{2} \\ -i\sqrt{2}\end{matrix}\end{cases}$$

$$\int_{-\infty}^{\infty} \frac{4e\cos x}{(x-i)(x+i)(x-i\sqrt{2})(x+i\sqrt{2})}\,dx$$



$$I=\oint_C \frac{4e\cdot e^{iz}}{z^4+3z^2+2}\,dz=\underbrace{\int_{-R}^{R}\frac{4e^{1+ix}}{x^4+3x^2+2}\,dx}_{I_1}+\underbrace{\int_{C_R}\frac{4e^{1+iz}}{z^4+3z^2+2}\,dz}_{I_2}$$

Teorema:

$$f(z) = \frac{p(z)}{q(z)} \; ; \; m \geq n+1 \Rightarrow \lim_{R\to\infty} \int_{C_R} f(z)\, e^{i\alpha z}\, dz \longrightarrow 0$$

$$I = I_1 = 2\pi i \left[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), i\sqrt{2}) \right]$$

$$\operatorname{Res}(f(z), i) = \lim_{z\to i} (z-i) \frac{4e^{1+iz}}{(z-i)(z+i)(z-i\sqrt{2})(z+i\sqrt{2})}$$

$$= \frac{4e^{1+i^2}}{2i\,(i-i\sqrt{2})(i+i\sqrt{2})} = \frac{4}{-2i\cdot(-1)} = -2i$$

$$\operatorname{Res}(f(z), i\sqrt{2}) = \lim_{z\to i\sqrt{2}} (z-i\sqrt{2}) \frac{4e^{1+iz}}{(z-i)(z+i)(z-i\sqrt{2})(z+i\sqrt{2})}$$

$$= \frac{4e^{1+i\cdot i\sqrt{2}}}{(i\sqrt{2}-i)(i\sqrt{2}+i)(i\sqrt{2}+i\sqrt{2})} = \frac{4e^{1-\sqrt{2}}}{i(\sqrt{2}-1)\,i(\sqrt{2}+1)\,2i\sqrt{2}}$$

$$= \frac{4e^{1-\sqrt{2}}}{i^2\, 2i\sqrt{2}} = \frac{4e^{1-\sqrt{2}}}{-i2\sqrt{2}} = i\sqrt{2}\, e^{1-\sqrt{2}}$$

$$\text{V.P.} \int_{-\infty}^{\infty} \frac{4e^{1+ix}}{x^4+3x^2+2}\, dx = 2\pi i \left(-2i + i\sqrt{2}\, e^{1-\sqrt{2}} \right)$$

$$= 2\pi \left(2 - \sqrt{2}\, e^{1-\sqrt{2}} \right)$$

Por outro lado:

$$\int_{-\infty}^{\infty} \frac{4e^{1+ix}}{x^4+3x^2+2}\,dx = \int_{-\infty}^{\infty} \frac{4e\cos x}{x^4+3x^2+2}\,dx + i\int_{-\infty}^{\infty} \frac{4e\,\mathrm{sen}\,x}{x^4+3x^2+2}\,dx$$

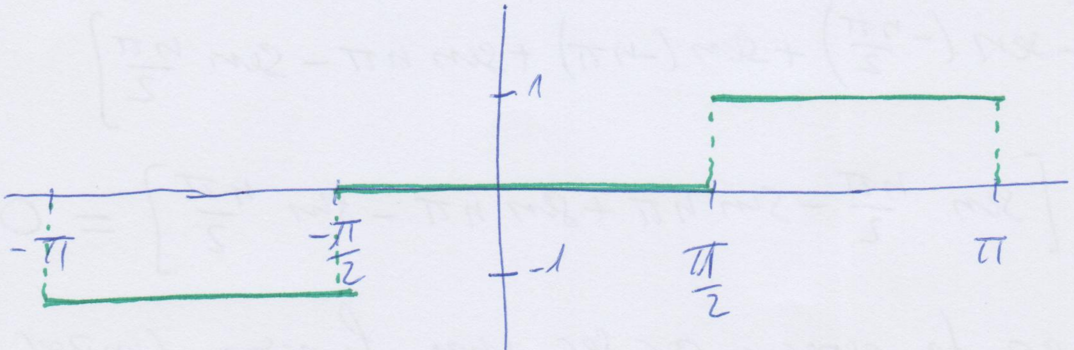
$$= 2\pi\left(2-\sqrt{2}\,e^{1-\sqrt{2}}\right)$$

$$V.P.\int_{-\infty}^{\infty} \frac{4e\,\mathrm{sen}\,x}{x^4+3x^2+2}\,dx = 0$$

$$\boxed{V.P.\int_{-\infty}^{\infty} \frac{4e\cos x}{x^4+3x^2+2}\,dx = 2\pi\left(2-\sqrt{2}\,e^{1-\sqrt{2}}\right)}$$

(2)

$$f(x) = \begin{cases} -1, & -\pi \leq x \leq -\frac{\pi}{2}, \\ 0, & -\frac{\pi}{2} < x \leq \frac{\pi}{2}, \\ 1, & \frac{\pi}{2} < x \leq \pi. \end{cases}$$



função ímpar

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \operatorname{sen} \frac{n\pi x}{p} \right)$$

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$$

$$a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi x}{p}\, dx \ , \quad n \in \mathbb{Z}^+$$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \operatorname{sen} \frac{n\pi x}{p}\, dx \ , \quad n \in \mathbb{Z}^+$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (-1)\,dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 0 \cdot dx + \int_{\frac{\pi}{2}}^{\pi} 1 \cdot dx\right]$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{2} + 0 + \frac{\pi}{2}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (-\cos nx)\,dx + \int_{\frac{\pi}{2}}^{\pi} \cos nx\,dx\right]$$

$$= \frac{1}{\pi n}\left[-\operatorname{sen}\left(-\frac{n\pi}{2}\right) + \operatorname{sen}(-n\pi) + \operatorname{sen} n\pi - \operatorname{sen}\frac{n\pi}{2}\right]$$

$$= \frac{1}{\pi n}\left[\operatorname{sen}\frac{n\pi}{2} - \operatorname{sen} n\pi + \operatorname{sen} n\pi - \operatorname{sen}\frac{n\pi}{2}\right] = 0$$

Como era de esperar por ser uma função ímpar.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen} nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\frac{\pi}{2}} (-\operatorname{sen} nx)\,dx + \int_{\pi/2}^{\pi} \operatorname{sen} nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left(\frac{\cos nx}{n}\right)\Big]_{-\pi}^{-\frac{\pi}{2}} - \left(\frac{\cos nx}{n}\right)\Big]_{\frac{\pi}{2}}^{\pi}\right\} =$$

$$= \frac{1}{\pi n}\left[\cos\left(-\frac{n\pi}{2}\right) - \cos(-n\pi) - \cos n\pi + \cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{\pi n}\left[\cos\frac{n\pi}{2} - \cos n\pi - \cos n\pi + \cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)$$

$$\boxed{f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\cos \frac{n\pi}{2} - (-1)^n \right) \operatorname{sen} nx}$$