A comparative study between One-way ANOVA and Kruskal-Wallis test for testing equality of several group means when the underlying population distributions are continuous

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Declaration

I affirm that I have identified all my resources and that no part of my dissertation paper uses unacknowledged materials.

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April, 2023

Abstract

There are hundreds of statistical tests including parametric and non-parametric that help in testing out the proposed hypothesis of a hypothesis testing problem. In parametric tests we assume that sample data comes from some well-known population distribution characterized by a fixed set of parameters. Whereas in a non-parametric test we don't have any prior knowledge about the population distribution and the distribution of a non-parametric test statistic under H0 doesn't depend on the population distribution and hence referred as 'distribution-free'. So it is important to choose an appropriate testing rule according to a proposed hypothesis for further analysis, which is commonly done using "**power comparison**".

Among many parametric tests one-way ANOVA and among many non-parametric tests Kruskal-Wallis are one of the most important tests. One-way ANOVA provides a statistical test of whether two or more population means are equal under parametric setup with the assumptions of normality and homoscedasticity. And Kruskal–Wallis test is a non-parametric method for testing whether two or more independent samples of equal or different sample sizes have been originated from the same population distribution. In this study our main objective is to compare ANOVA and Kruskal-Wallis test in terms of power under the hypothesis problem of testing for equality of several group means.

Monte Carlo simulation is used to calculate powers under different alternatives for different random samples of different sample sizes for both one-way ANOVA and Kruskal-Wallis test and tabulated powers under different cases are used to compare between these two tests when the underlying population distributions are continuous.

Acknowledgement

I would like to acknowledge and give my warmest thanks to my supervisor Prof. Pallabi Ghosh who made this work possible. Her proper guidance and advice carried me through all the stages of writing my project. I have learned so many things under her guidance.

Also, I would like to thank all my other respected Professors of Department of Statistics in St. Xavier's College (Autonomous), Kolkata, Prof. Dr. Surabhi Dasgupta, Prof. Dr. Surupa Chakrabarty, Prof. Dr. Ayan Chandra, Prof. Dr. Durba Bhattacharya, Prof. Debjit Sengupta, Prof. Madhura Dasgupta, Prof. Dr. Sanchari Basak. They all helped me to develop my interest for Statistics which motivated me do such research work.

Also, I would like to thank all the online resources, videos and books which helped me to grow my knowledge in this field and complete my project.

Finally, a big thanks to my parents, without their endless support, love and encouragement I know I could not have accomplished this research work.

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April, 2023

Introduction:

Hypothesis testing is one of the most important tool in inferential statistics. Among many hypothesis testing problem one of the most popular is the test for significant differential effect between different levels or categories of a certain factor, for eg, comparing between different doses of a specific drug, or comparing between different seed varieties of a specific crop etc. In general this type of hypothesis testing problem is called "Multi Sample Problem". Now, there are mainly two broad areas in statistical inference namely, Parametric inference and Nonparametric inference.

Under parametric setup we can rewrite the multi sample hypothesis testing problem as comparison of several group means. The one-way analysis of variance (ANOVA) is one of the most popular parametric test for the comparison of several means. As a classical statistical method, ANOVA requires two major assumptions to produce an optimal result. Those assumptions are - (i) data set has to be normally distributed, (ii) data set has to be homoscedastic with respect to different groups.

On the other hand, Kruskal–Wallis test is a nonparametric method for testing whether two or more independent samples of equal or different sample sizes have been originated from the same population distribution which is basically the multi sample hypothesis testing problem under nonparametric setup. The main advantage of nonparametric method is that, we don't need to make assumptions about the population distribution and even the assumption of homoscedasticity is not needed.

In real-life situations, data sets are not often normally distributed and group variances unequal making the assumptions of ANOVA always unattainable. So after stating a hypothesis problem, the most important step is to choose a proper testing rule.

This study basically aims to compare between ANOVA and Kruskal-Wallis test in terms of power under different situations- (i) equal/unequal group means, (ii) small and large group sizes, (iii) equal/unequal group variances. In this article our focus is the multi-sample problem related to equality of mean of the three independent groups. We will mainly study the tabulated powers of both the test when the underlying population distributions are - Normal, Logistic, Laplace, Log-normal, Exponential. These choices of distributions are completely objective. We will use “Monte Carlo simulation” to generate the powers and also will try to investigate the effects of violations of those assumptions(i.e. normality and homoscedasticity) on the power of the ANOVA and Kruskal-Wallis test under various scenarios.

Now let us define some important associated terminologies.

0.1 Statistical Inference:

To discover new knowledge in the real world, research workers perform experiments and obtain some data. They draw certain conclusions based on that data and may try to generalize the conclusion to the class of all similar types of experiments from that particular experiment. This sort of extension from particular to the general is called “Inductive inference”.

From the point of view of statistics, inductive inference can be referred as to investigating a certain population whose probability distribution is unknown or partly unknown and we try to know about the properties of the population. This population under study is called “Target population” which is defined as the totality of elements which are under discussion and about which information is desired. It is generally impossible to examine the entire population. Hence we have to collect a random sample from the target population, which is used as a representative of the population and using the random sample we try to conclude about the population or process beyond the existing data. For example, suppose we want to know the average height of all students studying in some undergraduate courses in a specific college. Then all students studying in undergraduate in that specific college will be our target population and we can choose all students from 2nd year as our sample.

This inductive process of going from a known sample to the unknown population is called “Statistical Inference”. But some errors are always associated with this procedure. Valid probability statements can be made about the model errors and that also ensures the randomness of error.

Now, statistical inference can be classified into two major areas:

0.1.1 Parametric Inference:

Parametric inference is that area of statistical inference which assumes that sample data comes from a population that can be adequately modelled by some well-known population distribution characterized by a fixed set of constant numerical quantities, called parameters. In real-life problems, we can sometimes assume the mathematical or functional form of the population probability distribution with the parameters being unknown. Hence, we will be interested in finding those parameters and inferring about them.

0.1.2 Non-parametric Inference:

Nonparametric inference is that area of statistical inference that is not based solely on parameterized families of probability distributions. The nonparametric inference is based on either being distribution-free or having a specified distribution that can't be characterized by a finite number of parameters.

As nonparametric methods make fewer assumptions, their applicability is much wider than the corresponding parametric methods, that is, nonparametric methods are more robust. But also, we will see that in cases where a parametric test would be appropriate, nonparametric tests have less power.

Two important problems in statistical inference are:

- Estimation
- Testing of hypothesis

"Estimation" is often used to describe the process of finding an estimate for an unknown quantity and "testing of hypothesis" is related to testing some statements or conjectures about that unknown quantity. Here we will mainly focus on testing of hypothesis. Although, testing of hypothesis is seen to be closely related to the problem of estimation.

0.2 Testing of Hypothesis:

Hypothesis testing is a tool for making statistical inferences about the population data. It is an analysis tool that tests assumptions and determines how likely something is within a given standard of accuracy. Hypothesis testing provides a way to verify whether the results of an experiment are valid.

A null hypothesis and an alternative hypothesis are set up before performing the hypothesis testing. This helps to arrive at a conclusion regarding the sample obtained from the population.

0.2.1 Statistical Hypothesis:

A statistical hypothesis is an assertion or statement about the population distribution of one or more random variables that the statistical process will examine to decide its validity. There are two types of statistical hypothesis:

- **Simple Hypothesis:** The statistical hypothesis completely specifies the population distribution, then it is called a simple hypothesis.
- **Composite Hypothesis:** The statistical hypothesis that does not completely specify the population distribution, then it is called a composite hypothesis.

0.2.2 Null Hypothesis and Alternative Hypothesis:

The null hypothesis is a concise mathematical statement that is used to indicate that there is no difference between two possibilities. In other words, there is no difference between certain characteristics of data. This hypothesis assumes that the outcomes of an experiment are based on chance alone. It is denoted as H_0. Hypothesis testing is used to conclude if the null hypothesis can be rejected or not. Suppose an experiment is conducted to check if girls are shorter than boys at the age of 5. Then we will take the null hypothesis as - boys and girls have same height at the age of 5.

The alternative hypothesis is an alternative to the null hypothesis. It is used to show that the observations of an experiment are due to some real effect. It indicates that there is a statistical significance between two possible outcomes. Alternative hypothesis is denoted by

H_1 or H_a. For the above-mentioned example, the alternative hypothesis would be that girls are shorter than boys at the age of 5.

0.2.3 Test statistic

A test statistic is a statistic (a quantity derived from the sample) used in hypothesis testing. A hypothesis test is essentially specified in terms of a test statistic, considered as a numerical summary of a data-set that reduces the data to one value that can be used to perform the hypothesis test.
An important property of a test statistic is that its sampling distribution under the null hypothesis H_0 must be known to us, either exactly or approximately, which helps to calculate p-values and come to a decision.

0.2.4 Critical region and Acceptance region

A subset (partition) $\mathscr{R}$ of the sample space $\mathscr{X}$ is said to be the critical region if we reject H_0 when sample $(x_1, x_2, \ldots, x_n) \in \mathscr{R}$. That is, A critical region is a set of values for the test statistic for which the null hypothesis is rejected, i.e. if the observed test statistic is in the critical region then we reject the null hypothesis and accept the alternative hypothesis. Critical region is also known as rejection region.

A subset (partition) $\mathscr{A}$ of the sample space $\mathscr{X}$ is said to be the acceptance region if we accept (do not reject) H_0 when sample $(x_1, x_2, \ldots, x_n) \in \mathscr{A}$. That is, A acceptance region is a set of values for the test statistic for which the null hypothesis is accepted.

0.2.5 Errors in hypothesis testing

There are two kinds of errors involved in testing of hypothesis.

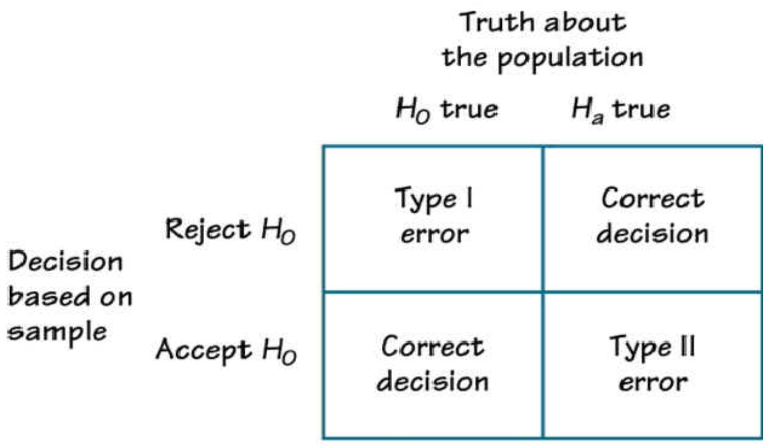

Figure 1: Type I and Type II errors

- **Type I error:** The error committed in rejecting a true null hypothesis is known as Type I error. It is measured as following-

$$\begin{aligned}
\mathbb{P}(\text{Type I error}) &= \mathbb{P}(\text{Reject } H_0, \text{ when it is actually true}) \\
&= \mathbb{P}(\text{Sample } (x_1, x_2, \ldots, x_n) \in \text{ critical region } \mathscr{R} | H_0 \text{ is actually true}) \\
&= \alpha
\end{aligned}$$

- **Type II error:** The error committed in accepting a false null hypothesis is known as Type II error. It is measured as following-

$$\begin{aligned}
\mathbb{P}(\text{Type II error}) &= \mathbb{P}(\text{Reject } H_1, \text{ when it is actually true}) \\
&= \mathbb{P}(\text{Sample } (x_1, x_2, \ldots, x_n) \in \text{ acceptance region } \mathscr{A} | H_1 \text{ is actually true}) \\
&= 1 - \mathbb{P}(\text{Sample } (x_1, x_2, \ldots, x_n) \in \text{ critical region } \mathscr{R} | H_1 \text{ is actually true}) \\
&= 1 - \beta
\end{aligned}$$

  Where, β is the power of the test.

To find the optimal test, our target is to keep both errors at the least possible value. But for a given sample size, simultaneous minimisation of the probability of the two kinds of errors is not feasible. Reduction in the probability of one kind of error leads to the increase in the probability of the other kind and vice-versa.

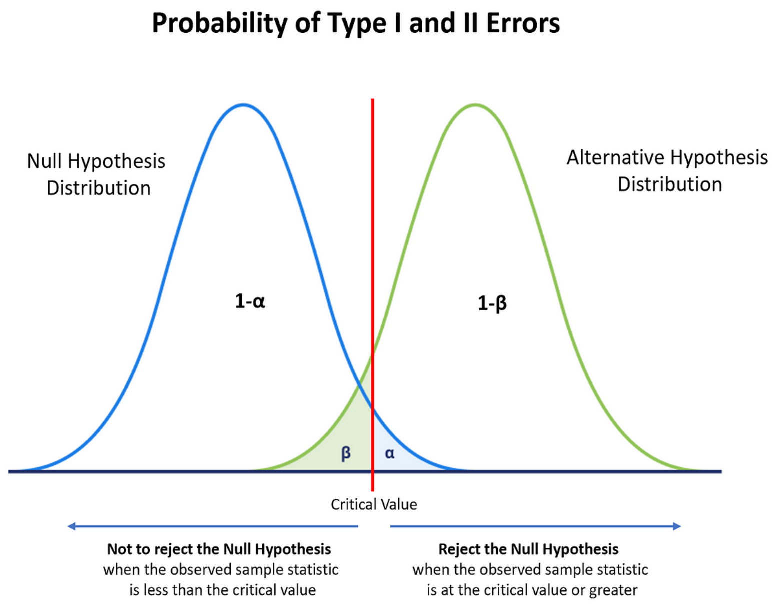

Figure 2: Probability of Type I and Type II errors graphically

Conventionally, Type I error is considered to be more serious. So in practice, we fix the probability of Type I error at a preassigned low value α, $0 < \alpha < 1$ (generally this value is taken to be $0.1, 0.05, 0.01$ etc.) and from among all tests for a certain problem we choose that one for which probability of Type II error is minimum.

0.2.6 Level of Significance and size of the test

The maximum possible probability with which we can choose to reject a true null hypothesis is known as the level of significance. It is denoted by α, $0 < \alpha < 1$ generally.

The actual probability with which we reject a true null hypothesis is known as size of the test. Clearly we have, size of the test $\leq \alpha$, $0 < \alpha < 1$.

0.2.7 Power of the test and Power function

The power of a statistical test is the probability that it will correctly lead to the rejection of a null hypothesis (H0) when it is false. This is one of the correct decisions arising from a testing problem. So, obviously high value of power indicates good test. Power is calculated under H_1

(that is H_1 is true). Hence we have,

$$\text{Power} = \beta = \mathbb{P}(\text{Sample } (x_1, x_2, \ldots, x_n) \in \text{ critical region } \mathscr{R} | H_1 \text{ is actually true})$$

But the concept of power function is more general. It is a function of true parameter $\theta \in \Theta$ (parameter space) and defined as –

$$\beta(\theta) = \mathbb{P}(\text{Rejecting } H_0 | \text{ the true parameter value is } \theta) \text{ , for some } \theta \in \Theta$$

For, θ being parameter value under null hypothesis, $\beta(\theta)$ gives type 1 error probability and for θ being parameter value under alternative hypothesis, $\beta(\theta)$ gives power.

0.2.8 Different type of Hypothesis Testing

- **One Tailed Hypothesis Testing:**

  One tailed hypothesis testing is done when the rejection region is only in one direction. It can also be known as directional hypothesis testing because the effects can be tested in one direction only. This type of testing is further classified into the right tailed test and left tailed test.

  1. **Right Tailed Hypothesis Testing:**

     The right tail test is also known as the upper tail test. This test is used to check whether the population parameter is greater than some value. The null and alternative hypotheses for this test are given as follows:

     $$H_0 : \text{ The population parameter is } \leq \text{ some value}$$

     $$vs$$

     $$H_1 : \text{ The population parameter is } > \text{ some value}$$

     If the test statistic has a greater value than the critical value then the null hypothesis is rejected

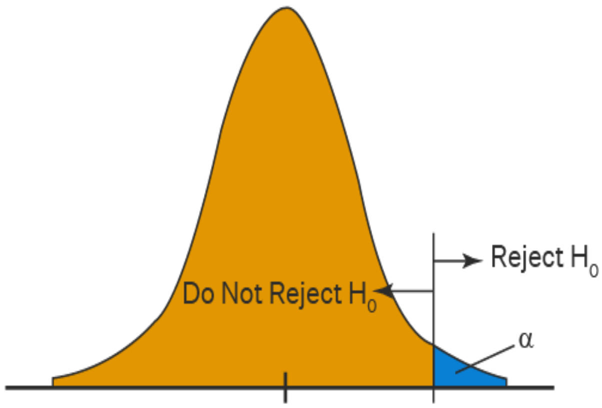

Figure 3: Right tailed test

2. **Left Tailed Hypothesis Testing:**

   The left tail test is also known as the lower tail test. It is used to check whether the population parameter is less than some value. The hypotheses for this hypothesis testing can be written as follows:

$$H_0: \text{ The population parameter is } \geq \text{ some value}$$

$$vs$$

$$H_1: \text{ The population parameter is } < \text{ some value}$$

   The null hypothesis is rejected if the test statistic has a value lesser than the critical value.

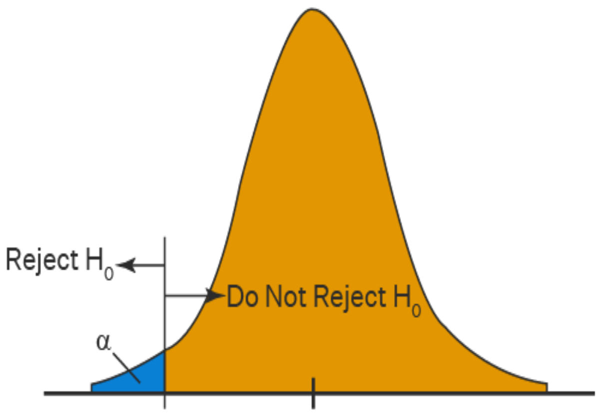

Figure 4: Left tailed test

- **Both Tailed Hypothesis Testing:**
  In this hypothesis testing method, the critical region lies on both sides of the sampling distribution. It is also known as a non - directional hypothesis testing method. The two tailed test is used when it needs to be determined if the population parameter is assumed to be different than some value. The hypotheses can be set up as follows:

$$H_0: \text{ The population parameter is } = \text{ some value}$$

$$vs$$

$$H_1: \text{ The population parameter is } \neq \text{ some value}$$

  The null hypothesis is rejected if the test statistic has a value that is not equal to the critical value.

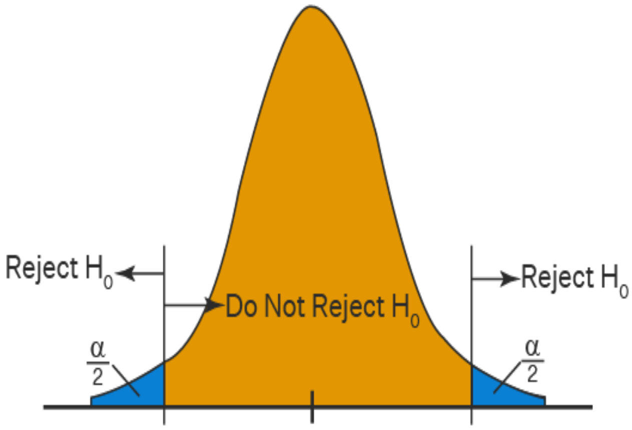

Figure 5: Both tailed test

Types of hypothesis problems:

Depending on different types of random samples from the population, there are different types of hypothesis problems:

- One Sample Problem
- Paired Sample Problem
- Two Sample Problem

- Multiple Sample Problem etc.

Hypothesis Problem regarding this study:

This study focuses on "hypothesis testing problem with **Multiple Sample**". The multiple sample problems are different from two sample problems. In two sample problem the data is consisted of two samples drawn independently from two independent population. Whereas in multiple sample problem, we have more than two independent random samples drawn from independent populations. Some real life examples are as follows:

1. A medical study was conducted to compare three different doses of a certain drug for controlling high blood pressure. One random sample of similar type of patients are taken and divided into three independent groups. Each group is given a specific dose of the drug. After some days blood pressures of each patient from each group is noted. Now, on the basis of this data we will compare three different doses of the drug and will see whether there is any significant difference between these three doses in controlling high blood pressure. Clearly, it is a multiple sample hypothesis problem related.

2. Suppose we want to compare three different seed varieties of wheat. For that the similar type of plots are divided three independent groups. Each group is given a specific seed variety. Then data on yield of wheat of each plot from each group is noted. Now, on the basis of this data we will compare three different seed varieties of wheat and will see whether there is any significant difference between these three seed varieties in yielding wheat. Clearly, it is a multiple sample hypothesis problem related.

Now we will discuss the main parametric and nonparametric tests regarding testing for the equality of several group means. It is quite evident from the title of this study we have considered the parametric test "One-way ANOVA" and the nonparametric test "Kruskal-Wallis" as a part of this study and we will basically compare the empirical size and power of these two tests under different situations. So, let us dig into a brief introduction of these two tests.

Parametric test for testing equality of several group means:

In parametric tests we assume that sample data comes from some well-known population distribution characterized by a fixed set of parameters. Analysis of variance (ANOVA) is one of the most popular parametric tests and it is used to analyze the differences among means. ANOVA is based on the law of total variance, where the observed variance in a particular variable is partitioned into components attributable to different sources of variation. ANOVA provides a statistical test of whether two or more population means are equal, and therefore generalizes the t-test beyond two means.

In this study we are considering test for equality of three group means. So, let us formulate the One-way ANOVA test for three groups that is with three independent populations.

0.3 One-Way ANOVA (Analysis of Variance)

Suppose we have three independent populations X_1, X_2, X_3 with means μ_1, μ_2, μ_3 respectively. We are interested in testing,

$$H_0 : \mu_1 = \mu_2 = \mu_3$$

$$vs$$

$$H_1 : \text{ atleast one } \mu_i \text{ is not identical to atleast one other } \mu_j (i \neq j) \text{ or not } H_0$$

Let we have taken sample of size n_1, n_2, n_3 from population-1, population-2 and population-3 respectively. Let the total sample size be n that is, $\sum_{i=1}^{3} n_i = n$. Also, let y_{ij} denote the jth sample observation corresponding to ith population where $j = 1(1)n_i$ and $i = 1, 2, 3$.

Here we can think these three independent populations as three different levels of a certain factor A and we are interested in testing whether there is any significant differential effect between these different levels of A or not.
Then the corresponding **model** under one-way ANOVA can be written as-

$$y_{ij} = \mu_i + \varepsilon_{ij}$$

$$\Longrightarrow y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad , j = 1(1)n_i \ \& \ i = 1, 2, 3$$

where, μ is the general effect, $\alpha_i = \mu_i - \mu$ is the additional fixed effect due to ith group or ith level of A or due to ith population, $i = 1,2,3$ and ε_{ij} are the random error. Hence, we can rewrite our testing problem as,

$$H_0 : \alpha_1 = \alpha_2 = \alpha_3 = 0$$

$$vs$$

$$H_1 : \text{ atleast one } \alpha_i \text{ is not identical to atleast one other } \alpha_j (i \neq j) \text{ or not } H_0$$

Assumptions:

1. The responses y'_{ij} s, $j = 1(1)n_i$ and $i = 1,2,3$ are independent since the random samples are drawn independently from the three populations.

2. The random errors are independently and identically normally distributed with mean 0 and variance σ^2 (say). That is,

$$e_{ij} \overset{iid}{\sim} N(0,\sigma^2) \ , \forall j = 1(1)n_i \ \& \ i = 1,2,3$$

   Hence error variances are normally distributed and also are homoscedastic (that is constant variance).

Identifiability Constraint:

To get unique estimates of μ and α'_i s, the imposed identifiability constraint is given by,

$$\sum_{i=1}^{3} n_i \alpha_i = 0$$

Least square estimates of μ and α_i:

To find the estimates of μ and α_i' s by method of least squares, we have to minimize the sum of square of errors with respect to μ and α_i' s. Let us denote,

$$E = \sum_{i=1}^{3}\sum_{j=1}^{n_i} \varepsilon_{ij}^2 = \sum_{i=1}^{3}\sum_{j=1}^{n_i} (y_{ij} - \mu - \alpha_i)^2$$

Differentiating E with respect to μ and equating it to 0 we get,

$$\frac{dE}{d\mu} = 0 \Rightarrow \sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \mu - \alpha_i) = 0 \tag{1}$$

Again differentiating E with respect to α_i, $i = 1,2,3$ and equating it to 0 we get,

$$\frac{dE}{d\alpha_i} = 0 \Rightarrow \sum_{j=1}^{n_i}(y_{ij} - \mu - \alpha_i) = 0\ ,\ \forall i = 1,2,3 \tag{2}$$

Solving equation (1) and equations in (2) we get the least square estimates of μ and α_i' s to be,

$$\hat{\mu} = \frac{\sum_{i=1}^{3}\sum_{j=1}^{n_i} y_{ij}}{n} = \overline{y_{00}}$$

$$\hat{\alpha}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} y_{ij} - \hat{\mu} = \overline{y_{i0}} - \overline{y_{00}}\ ,\ \forall i = 1,2,3$$

Orthogonal Split of Total Sum of Square:

The quantity $\sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{00}})^2$ is called the Total sum of square, abbreviated as TSS. Then we have,

$$\begin{aligned}
\text{TSS} &= \sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{00}})^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}} + \overline{y_{i0}} - \overline{y_{00}})^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n_i}(\overline{y_{i0}} - \overline{y_{00}})^2 + \sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2 + 2\sum_{i=1}^{3}\sum_{j=1}^{n_i}(\overline{y_{i0}} - \overline{y_{00}})(y_{ij} - \overline{y_{i0}}) \\
&= \sum_{i=1}^{3} n_i(\overline{y_{i0}} - \overline{y_{00}})^2 + \sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2 \\
&= \text{SSA + SSE}
\end{aligned}$$

The quantity $\sum_{i=1}^{3} n_i(\overline{y_{i0}} - \overline{y_{00}})^2$ is called sum of square due to factor A and abbreviated as SSA. And the quantity $\sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2$ is called sum of square due to residuals and abbreviated as SSE.

Now,

- $\frac{\text{TSS}}{\sigma^2}$ being a quadratic form of n terms has 1 restriction on y'_{ij} s, namely,

$$\sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij}-\overline{y_{00}})=0$$

  Thus $\frac{\text{TSS}}{\sigma^2}$ carries $(n-1)$ degrees of freedom.

- $\frac{\text{SSA}}{\sigma^2}$ being a quadratic form of 3 terms has 1 restriction on $\overline{y_{i0}}'$ s, namely,

$$\sum_{i=1}^{3} n_i(\overline{y_{i0}}-\overline{y_{00}})=0$$

  Thus $\frac{\text{SSA}}{\sigma^2}$ carries $(3-1)=2$ degrees of freedom.

- $\frac{\text{SSE}}{\sigma^2}$ being sum of 3 quadratic forms, the ith form having n_i terms, $\forall i=1,2,3$. It contains 3 restrictions, namely,

$$\sum_{j=1}^{n_i}(y_{ij}-\overline{y_{i0}})=0\ ,\ \forall i=1,2,3$$

  Thus $\frac{\text{SSE}}{\sigma^2}$ carries $(n-3)$ degrees of freedom.

By suitable orthogonal transformation it can be shown that,

$$\frac{\text{SSA}}{\sigma^2}\sim\chi^2_{(2)}\ ,\ \text{independently of}$$
$$\frac{\text{SSE}}{\sigma^2}\sim\chi^2_{(n-3)}$$

Expectation of Mean sum of square:

1. The mean sum of square due to A is given by MSA $=\frac{\text{SSA}}{2}$
2. The mean sum of square due to residuals is given by MSE $=\frac{\text{SSE}}{(n-3)}$

Now we have,

$$1)\ y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

$$2)\ \overline{y_{i0}} = \mu + \alpha_i + \overline{\varepsilon_{i0}}\ ,\ \text{where}\ \ \overline{\varepsilon_{i0}} = \frac{1}{n_i}\sum_{j=1}^{n_i} \varepsilon_{ij}$$

$$3)\ \overline{y_{00}} = \mu + \overline{\varepsilon_{00}}\ ,\ \text{where}\ \ \overline{\varepsilon_{00}} = \frac{1}{n}\sum_{i=1}^{3}\sum_{j=1}^{n_i} \varepsilon_{ij}$$

Now, we can write the sum of square due to residual as-

$$\begin{aligned}
\text{SSE}\ &= \sum_{i=1}^{3}\sum_{j=1}^{n_i}(y_{ij} - \overline{y_{i0}})^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n_i}(\varepsilon_{ij} - \overline{\varepsilon_{i0}})^2 \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n_i}\varepsilon_{ij}^2 - \sum_{i=1}^{3} n_i\overline{\varepsilon_{i0}}^2
\end{aligned}$$

Taking expectation on both sides we get,

$$\begin{aligned}
\mathbb{E}(SSE) &= \sum_{i=1}^{3}\sum_{j=1}^{n_i}\mathbb{E}(\varepsilon_{ij}^2) - \sum_{i=1}^{3} n_i\mathbb{E}(\overline{\varepsilon_{i0}}^2) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n_i}\{Var(\varepsilon_{ij}) + \mathbb{E}^2(\varepsilon_{ij})\} - \sum_{i=1}^{3} n_i\{Var(\overline{\varepsilon_{i0}}) + \mathbb{E}^2(\overline{\varepsilon_{i0}})\} \\
&= \sum_{i=1}^{3}\sum_{j=1}^{n_i}\sigma^2 - \sum_{i=1}^{3} n_i\frac{\sigma^2}{n_i} \qquad\qquad (\text{since } \varepsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2)) \\
&= n\sigma^2 - 3\sigma^2 = (n-3)\sigma^2
\end{aligned}$$

$$\implies \mathbb{E}\left(\frac{\text{SSE}}{(n-3)}\right) = \sigma^2$$

$$\implies \mathbb{E}(\text{MSE}) = \sigma^2$$

Hence, an unbiased estimator of σ^2 is MSE.

Again we can write the sum of square due to factor A as-

$$\begin{aligned}
\text{SSA } &= \sum_{i=1}^{3} n_i(\overline{y_{i0}} - \overline{y_{00}})^2 \\
&= \sum_{i=1}^{3} n_i\{\alpha_i + (\overline{\varepsilon_{i0}} - \overline{\varepsilon_{00}})^2\} \\
&= \sum_{i=1}^{3} n_i\alpha_i^2 + \sum_{i=1}^{3} n_i(\overline{\varepsilon_{i0}} - \overline{\varepsilon_{00}})^2 + 2\sum_{i=1}^{3} n_i(\overline{\varepsilon_{i0}} - \overline{\varepsilon_{00}}) \\
&= \sum_{i=1}^{3} n_i\alpha_i^2 + \sum_{i=1}^{3} n_i\overline{\varepsilon_{i0}}^2 - n\overline{\varepsilon_{00}}^2 + 2\sum_{i=1}^{3} n_i(\overline{\varepsilon_{i0}} - \overline{\varepsilon_{00}})
\end{aligned}$$

Taking expectation on both sides we get,

$$\begin{aligned}
\mathbb{E}(SSA) &= \sum_{i=1}^{3} n_i\alpha_i^2 + \sum_{i=1}^{3} n_i\mathbb{E}(\overline{\varepsilon_{i0}}^2) - n\mathbb{E}(\overline{\varepsilon_{00}}^2) && (\text{since } \mathbb{E}(\overline{\varepsilon_{i0}} - \overline{\varepsilon_{00}}) = 0) \\
&= \sum_{i=1}^{3} n_i\alpha_i^2 + \sum_{i=1}^{3} n_i\{Var(\overline{\varepsilon_{i0}}) + \mathbb{E}^2(\overline{\varepsilon_{i0}})\} - n\{Var(\overline{\varepsilon_{00}}) + \mathbb{E}^2(\overline{\varepsilon_{00}})\} \\
&= \sum_{i=1}^{3} n_i\alpha_i^2 + \sum_{i=1}^{3} n_i\frac{\sigma^2}{n_i} - n\frac{\sigma^2}{n} && (\text{since } \varepsilon_{ij} \overset{iid}{\sim} N(0,\sigma^2)) \\
&= \sum_{i=1}^{3} n_i\alpha_i^2 + (3-1)\sigma^2 = \sum_{i=1}^{3} n_i\alpha_i^2 + 2\sigma^2
\end{aligned}$$

$$\implies \mathbb{E}\left(\frac{\text{SSA}}{2}\right) = \sigma^2 + \frac{1}{2}\sum_{i=1}^{3} n_i\alpha_i^2$$

$$\implies \mathbb{E}(\text{MSA}) = \sigma^2 + \frac{1}{2}\sum_{i=1}^{3} n_i\alpha_i^2$$

Note that, $\mathbb{E}(\text{MSA}) = \mathbb{E}(\text{MSE})$ *iff*

$$\sum_{i=1}^{3} n_i\alpha_i^2 = 0$$

$i.e,\ n_i\alpha_i^2 = 0\ ,\ \forall i = 1,2,3$

$i.e,\ \alpha_i = 0\ ,\ \forall i = 1,2,3$

$i.e,\ H_0$ is true

Otherwise we have, $\mathbb{E}(\text{MSA}) > \mathbb{E}(\text{MSE})$

Test statistic:

The test statistic of one-way ANOVA is given by,

$$F = \frac{\text{MSA}}{\text{MSE}} \overset{H_0}{\sim} F_{2,(n-3)}$$

Since, $\mathbb{E}(\text{MSA}) > \mathbb{E}(\text{MSE})$ other than H_0 is true, hence higher values of the test statistics F indicate departure from H_0. That's why for this testing problem a right tailed test will be appropriate.

Critical Region:

Reject H_0 at $100\alpha\%$ level of significance if,

$$F_{\text{obs}} > F_{2,(n-3);\alpha}$$

Example:

For example we consider the *iris* dataset from *R* software. We consider the data on *Sepal.Length* for 50 flowers from each of 3 species of iris. The species are *Iris setosa, versicolor, and virginica*. We want to test whether *Sepal.Length* of the flowers differ significantly for different species of iris. Here, $n_1 = n_2 = n_3 = 50$ and $n = 50+50+50 = 150$. Let us denote let y_{ij} denote the *Sepal.Length* of jth flower corresponding to ith species where $j = 1(1)50$ and $i = 1,2,3$. Then the corresponding model is given by-

$$y_{ij} = \mu_i + \varepsilon_{ij}$$
$$\implies y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \quad , j = 1(1)50 \ \& \ i = 1,2,3$$

where, μ is the general effect, $\alpha_i = \mu_i - \mu$ is the additional fixed effect due to ith species of iris, $i = 1,2,3$ and ε_{ij} are the random error. Hence, we are interested in testing, $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = 0$ vs $H_1 :$ atleast one α_i is not identical to atleast one other $\alpha_j (i \neq j)$ or not H_0

Assumptions: y'_{ij} s are independent $\forall j = 1(1)50,\ i = 1,2,3$ and

$$e_{ij} \overset{iid}{\sim} N(0,\sigma^2) \ , \forall j = 1(1)50 \ \& \ i = 1,2,3$$

Identifiability Constraint:

$$\sum_{i=1}^{3} 50\alpha_i = 0$$

Distribution of sum of squares under H_0:

Let SSE be the sum of square due to residuals and SSA be the sum of square due to species of iris. By suitable orthogonal transformation we can show,

$$\frac{\text{SSA}}{\sigma^2} \sim \chi^2_{(2)} \text{ , independently of}$$
$$\frac{\text{SSE}}{\sigma^2} \sim \chi^2_{(147)}$$

Mean sum of square:

1. The mean sum of square due to species of iris is given by MSA $= \frac{\text{SSA}}{2}$
2. The mean sum of square due to residuals is given by MSE $= \frac{\text{SSE}}{147}$

Test statistic:

$$F = \frac{\text{MSA}}{\text{MSE}} \overset{H_0}{\sim} F_{2,147}$$

Critical Region:

Reject H_0 at $100\alpha\%$ level of significance if,

$$F_{\text{obs}} > F_{2,147;\alpha}$$

Calculation:

The ANOVA table for level of significance $\alpha = 0.05$ is given by

(here df = degrees of freedom, SS = sum of square, MS = mean sum of square)-

Sources of variation	degrees	SS	MS	F value	
				F_{obs}	$F_{2,147;0.05}$
Species	2	63.21	31.606	119.3	3.057
Residuals	147	38.96	0.265		
Total	149	102.17			

Decision: Since $F_{\text{obs}} = 119.3 > 3.057 = F_{2,147;0.05}$, hence we reject H_0.

Conclusion:

In the light of the given data, it seems that the *Sepal.Length* of the flowers differ significantly for different species of iris.

Nonparametric test for testing equality of several group means:

In a nonparametric test we don't have any prior knowledge about the population distribution and the distribution of a non-parametric test statistic under H0 doesn't depend on the population distribution and hence referred as 'distribution-free'. Kruskal Wallis test is the most appropriate and popular nonparametric test for testing equality of several group means. It is also sometimes referred as an alternative to the one-way ANOVA when there are more than two groups to compare. Kruskal Wallis test is applied mainly under the following circumstances:

1. We have more than two groups to compare
2. The response values are independent
3. The data do not meet the requirements for a parametric test. (i.e. the assumptions like if the data are not normally distributed; if the variances for the different groups are markedly different, etc).

Since in this study we are considering test for equality of three group means, let us formulate the Kruskal Wallis test for three groups that is with three independent populations.

0.4 Kruskal Wallis Test:

Let we have three independent populations X_1, X_2, X_3 with cumulative distribution function $F_{X_1}(x), F_{X_2}(x), F_{X_3}(x)$ respectively. Also, here we assume that the cumulative distribution functions (*cdf*) only differ with respect to their location. To test,

$$H_0 : F_{X_1}(x) = F_{X_2}(x) = F_{X_3}(x), \forall x \in \mathbb{R}$$

$$vs$$

$$H_1 : \text{ not } H_0$$

Also, *cdf*'s only differ by their location, hence we can write

$$F_{X_i}(x) = F_X(x - \theta_i),\ \forall x \in \mathbb{R},\ \text{for some } \theta_i \in \Theta \subseteq \mathbb{R}$$

where θ_i denotes a location parameter of the ith population, $i = 1,2,3.$ Then the testing problem can be rewritten as,

$$H_0 : \theta_1 = \theta_2 = \theta_3$$

$$vs$$

$$H_1 : \theta_i \neq \theta_j,\ \text{for atleast one } i \neq j$$

Let, $(X_{11}, X_{12}, \ldots, X_{1n_1}), (X_{21}, X_{22}, \ldots, X_{2n_2}), (X_{31}, X_{32}, \ldots, X_{3n_3})$ be three independent random samples of size n_1, n_2, n_3 from the distribution with *cdf* $F_{X_1}, F_{X_2}, F_{X_3}$ respectively and total sample size is n that is, $\sum_{i=1}^{3} n_i = n.$ Under H_0, essentially we have a single sample of size n from same population. We combine these n observations and order them in increasing manner that is from smallest to largest with keeping track of which observation is from which sample. Then, we assign the ranks $1,2,3,,\ldots,n$ to the sequence.

Let R_{ij} denote the rank of the jth observation of the ith sample, $i = 1,2,3$ and $j = 1,2,...,n_i.$ R_i and $\overline{R_{i0}}$ denote the sum of ranks and mean of ranks assigned to the elements in the ith sample, $i = 1,2,3$ respectively, that is

$$R_i = \sum_{i=1}^{3} R_{ij},\ i = 1,2,3$$

$$\overline{R_{i0}} = \frac{1}{n_i} \sum_{i=1}^{3} R_{ij} = \frac{R_i}{n_i},\ i = 1,2,3$$

Test statistic: The test statistic, due to Kruskal and Wallis(1952) is given by,

$$\begin{aligned} Q &= \frac{12}{n(n+1)} \sum_{i=1}^{3} \frac{1}{n_i} \left[R_i - \frac{n_i(n+1)}{2} \right]^2 \\ &= \frac{12}{n(n+1)} \sum_{i=1}^{3} n_i \left[\overline{R_{i0}} - \frac{(n+1)}{2} \right]^2 \end{aligned}$$

Clearly, the larger value of Q rejects a null hypothesis.

Critical Region: Reject H_0 at $100\alpha\%$ level of significance if $Q > Q_\alpha$, where Q_α is such that

$$\mathbb{P}_{H_0}(Q > Q_\alpha) \leq \alpha$$

The exact probabilities for Q can be obtained from the corresponding table of values, but for limited choices of no. of groups and group sizes only like $k = 3,\ n_i \leq 5$ or $k = 4,\ n_i \leq 4$ etc. So, due to these limitations some reasonable approximation of the null distribution of Q is required.

Large sample approximation of the distribution of Q under null hypothesis:

Note that under null hypothesis, the ith sample ranks may be considered as a simple random sample of size n_i drawn without replacement (SRSWOR) from the finite population consisting of the first n natural integers. Then the population mean and variance are given by,

$$\begin{aligned} \mu &= \frac{1}{n} \sum_{i=1}^{n} i = \frac{n+1}{2} \\ \sigma^2 &= \frac{1}{n} \sum_{i=1}^{n} \left(i - \frac{n+1}{2} \right)^2 = \frac{n^2-1}{2} \end{aligned}$$

The sample mean is given by the mean or average rank sum of the ith sample that is $\overline{R_{i0}}$, i=1,2,3 and for sample mean under SRSWOR we have under H_0,

$$\begin{aligned} \mathbb{E}(\overline{R_{i0}}) &= \mu = \frac{n+1}{2} \\ \text{Var}(\overline{R_{i0}}) &= \frac{\sigma^2(n-n_i)}{n_i(n-1)} = \frac{(n+1)(n-n_i)}{12n_i} \\ \text{cov}(\overline{R_{i0}}, \overline{R_{j0}}) &= -\frac{n+1}{2},\ i \neq j \end{aligned}$$

Since $\overline{R_{i0}}$ is the sample mean, by the Central Limit Theorem we have, under H_0, for large $n_i's$,

$$\overline{R_{i0}} \sim N\left(\frac{n+1}{2}, \frac{(n+1)(n-n_i)}{12n_i}\right)$$

Define,

$$Z_i = \frac{\overline{R_{i0}} - \frac{n+1}{2}}{\sqrt{\frac{(n+1)(n-n_i)}{12n_i}}},\ i = 1,2,3$$

Hence by the Central Limit Theorem we have, under H_0, for large $n_i's$,

$$Z_i \sim N(0,1),\ i = 1,2,3$$

Consequently, Z_i^2 is approximately distributed as chi square with one degree of freedom, for $i = 1,2,3$. But note that $Z_i's$ are not independent for the constraint $\sum_{i=1}^{3} n_i\overline{R_{i0}} = \frac{n(n+1)}{2}$. Kruskal showed that under H_0, of no n_i is very small, the random variable $\sum_{i=1}^{3} \frac{(n-n_i)}{n} Z_i^2$ is distributed approximately as chi square with $(3-1) = 2$ degrees of freedom. Hence for ,large sample approximation, the test statistic for Kruskal Wallis test is given by,

$$\begin{aligned} Q &= \sum_{i=1}^{3} \frac{(n-n_i)}{n} Z_i^2 \\ &= \frac{12}{n(n+1)} \sum_{i=1}^{3} n_i \left[\overline{R_{i0}} - \frac{n+1}{2}\right]^2 \qquad \text{(i)} \end{aligned}$$

Approximately, for large n_i' s $Q \overset{H_0}{\sim} \chi^2_{(2)}$. Then we reject H_0 at $100\alpha\%$ level of significance if

$$Q_{\text{obs}} > \chi^2_{(2);\alpha}$$

where, $\chi^2_{(2);\alpha}$ is the upper $\alpha\%$ point of the $\chi^2_{(2)}$ distribution.

Although the test statistic is easier to calculate in the following form which is equivalent to the form in (i):

$$Q = \frac{12}{n(n+1)} \sum_{i=1}^{3} \frac{R_i^2}{n_i} - 3(n+1)$$

We will use this form of the test statistic in our calculation

Tied case:

When two or more observations are tied, the midrank method is generally used. If ties to the extent t are present and are handled by the midrank method, then the variance of the finite population is given by-

$$\sigma^2 = \frac{n^2-1}{12} = \frac{\sum t(t^2-1)}{12}$$

where the sum is over all sets of ties in the population, and using this expression in var$(\overline{R_{i0}})$ for denominator of Z_i, the modified form of the test statistic is given by,

$$\begin{aligned}
Q' &= \sum_{i=1}^{3} \frac{(n-n_i)}{n} \left\{ \frac{\left[\overline{R_{i0}} - \frac{n(n+1)}{2}\right]^2}{\frac{(n+1)(n-n_i)}{12n_i} - \frac{(n-n_i)}{n_i(n-1)} \frac{\sum t(t^2-1)}{12}} \right\} \\
&= \sum_{i=1}^{3} \left\{ \frac{12n_i \left[\overline{R_{i0}} - \frac{n(n+1)}{2}\right]^2}{n(n+1) - \frac{n\sum t(t^2-1)}{(n-1)}} \right\} \\
&= \frac{Q}{1 - \frac{\sum t(t^2-1)}{n(n^2-1)}}
\end{aligned}$$

Hence the correction for tied case is to simply divide Q in (i) by the correction factor $1 - \frac{\sum t(t^2-1)}{n(n^2-1)}$ where the sum is over all sets of t tied ranks.

But in our study we have assumed the population distributions to be continuous to simply avoid the tied cases.

Example:

Suppose, in a mental hospital a study is conducted to know which kind of treatment is most effective for a particular type of mental disorder. A group of 30 patients who were similar as regards diagnosis and also personality, intelligence, physiological factors, were randomly divided into 3 independent groups with 10 patients in each group. For 6 months the respective groups received (1) electroshock, (2) psychotherapy, (3) electroshock plus psychotherapy treatment. At the end of the treatment period, the ranking of all 30 patients on the basis of their relative degree of improvement are noted; rank 1 and rank 2 indicates respectively the highest and the second highest level of improvement and so on.

On the basis of this data we want to test whether there is any difference in effectiveness of the types of treatment.

The table of rankings of patients is given by-

Group - 1	Group - 2	Group - 3
19	14	12
22	21	1
25	2	5
24	6	8
29	10	4
26	16	13
30	17	9
23	11	15
27	18	3
28	7	20

Hence the values of the rank sums are given by $R_1 = 253, R_2 = 122, R_3 = 90$ with $n_1 = n_2 = n_3 = 10 \Rightarrow n = 10 + 10 + 10 = 30$. The value of the test statistic of Kruskal Wallis test is given by-

$$Q = \frac{12}{30(31)(10)}(253^2 + 122^2 + 90^2) - 3(31) = 19.24903$$

Since, here we have group sizes 10, using large sample approximation we have, $Q \overset{H_0}{\sim} \chi_(2)^2$. We will reject H_0 at $100\alpha\%$ level of significance if-

$$Q_{obs} > \chi^2_{2;\alpha}$$

Note that here for level of significance $\alpha = 0.05$, $Q_{obs} = 19.24903 > 5.991465 = \chi^2_{2;0.05}$. Hence we reject H_0 at 5% level of significance.

Conclusion:

In the light of the given data it seems that, there is difference in effectiveness of the types of treatment.

Null and Alternative hypothesis corresponding to our study:

We have already mentioned that our objective is to study the hypothesis of equality of means of three independent groups when the underlying population distributions are - Normal, Logistic, Laplace, Lognormal, Exponential. The means of normal, laplace, logistic distributions are their *Location Parameter* and the mean of exponential distribution is its *Scale Parameter*. Whereas the mean of a *Lognormal*(μ, σ^2) distribution is equals to $e^{\left(\mu+\frac{\sigma^2}{2}\right)}$ and μ is its *Location parameter*. Hence for normal, laplace, logistic and lognormal distributions our hypothesis problem is - testing of equality of location parameters, while for exponential distribution it is - testing of equality of scale parameters. And the variance of *Lognormal*(μ, σ^2) and *Exponential*(μ) are respectively, $[exp(\sigma^2)-1]exp(2\mu+\sigma^2)$ and μ^2. Hence even if we remain σ^2 fixed, since value of μ is changing for different population the assumption of homoscedasticity naturally gets violated when the underlying population distributions are lognormal and exponential.
Now let us state the null and alternative hypothesis of our problem under two different setup.

0.5 Null and Alternative hypothesis of our testing problem under parametric setup:

The null hypothesis are given by,

1. **Population distribution is Normal:**
   Suppose, we have three independent populations denoted by X_1, X_2, X_3. The probability distribution of X_i is *Normal*(μ_i, σ_i^2), $\forall i = 1,2,3$. We want to test, $H_0 : \mu_1 = \mu_2 = \mu_3$.

2. **Population distribution is Laplace:**
   Suppose, we have three independent populations denoted by X_1, X_2, X_3. The probability distribution of X_i is *Laplace*(μ_i, σ_i), $\forall i = 1,2,3$. We want to test, $H_0 : \mu_1 = \mu_2 = \mu_3$.

3. **Population distribution is Logistic:**
   Suppose, we have three independent populations denoted by X_1, X_2, X_3. The probability distribution of X_i is *Logistic*(μ_i, σ_i), $\forall i = 1,2,3$. We want to test, $H_0 : \mu_1 = \mu_2 = \mu_3$.

4. **Population distribution is Lognormal:**
   Suppose, we have three independent populations denoted by X_1, X_2, X_3. The probability distribution of X_i is *Lognormal(*μ_i, σ_i^2), $\forall i = 1,2,3$. We want to test, $H_0 : \mu_1 = \mu_2 = \mu_3$.

5. **Population distribution is Exponential:**
   Suppose, we have three independent populations denoted by X_1, X_2, X_3. The probability distribution of X_i is *Exponential(*μ_i), $\forall i = 1,2,3$. We want to test, $H_0 : \mu_1 = \mu_2 = \mu_3$.

Now, we will define our alternative hypothesis. For all of the null hypothesis we have a number of choices of alternatives.

- $H_1 : \mu_1 < \mu_2 < \mu_3$
- $H_1 : \mu_1 = \mu_2 < \mu_3$
- $H_1 : \mu_1 < \mu_2 = \mu_3$

  $\vdots$
- $H_1 : \mu_1 < \mu_2 > \mu_3$
- $H_1 : \mu_1 > \mu_2 < \mu_3$
- $H_1 : \mu_1 > \mu_2 > \mu_3$

  $\vdots$
- $H_1 : \mu_1 = \mu_2 > \neq \mu_3$
- $H_1 : \mu_1 \neq \mu_2 \neq \mu_3$

Combining all of these choices we simply write our alternative hypothesis as-

$$H_1 : \text{ atleast one } \mu_i \neq \text{ atleast one } \mu_j \text{ where } i \neq j$$

or

$$H_1 : \text{ atleast one inequality in } H_0$$

or

$$H_1 : \text{ not } H_0$$

We will choose $H_1 : \mu_1 < \mu_2 < \mu_3$ as our alternative hypothesis. Because, our objective is to compare parametric test ANOVA and non-parametric test Kruskal-Wallis under different situations.

0.6 Null and Alternative hypothesis of our testing problem under nonparametric setup:

Since nonparametric tests are distribution free, hence we do not need to make any assumption about the exact structure of the underlying population distribution characterized by a fixed set of parameters. Let us suppose we have three independent populations say, X_1, X_2, X_3 with cumulative distribution functions $F_{X_1}(x), F_{X_2}(x), F_{X_3}(x)$ respectively. Also let a random sample of size n_i is drawn from the population X_i, denoted by $(X_{i1}, X_{i2}, X_{i3}, \ldots, X_{in_i}), \forall i = 1,2,3$.

We are to test, $H_0 : (X_{11}, X_{12}, X_{13}, \ldots, X_{1n_1}), (X_{21}, X_{22}, X_{23}, \ldots, X_{2n_2}), (X_{31}, X_{32}, X_{33}, \ldots, X_{3n_3})$, these three independent random samples have been drawn from same population. In other words we can write the null hypothesis as, $H_0 : F_{X_1}(x) = F_{X_2}(x) = F_{X_3}(x), \forall x \in \mathbb{R}$
In general we can write the altenative hypothesis under nonparametric setup as H_1 : Above three independent random samples have been drawn from different populations or simply H_1 : not H_0. But under parametric setup we have chosen the alternative hypothesis $H_1 : \mu_1 < \mu_2 < \mu_3$. So, corresponding to our alternative hypothesis in parametric setup, we have to find an equivalent alternative hypothesis. Let us find the alternative hypothesis when the underlying distribution is normal.
Suppose, $X_i \sim Normal(\mu_i, \sigma_i^2), \forall i = 1,2,3$ and X_1, X_2, X_3 are independent populations. Then we have,

$$F_{X_1}(x) = \mathbb{P}(X_1 \le x) = \Phi\left(\frac{x-\mu_1}{\sigma}\right)$$

$$F_{X_2}(x) = \mathbb{P}(X_2 \le x) = \Phi\left(\frac{x-\mu_2}{\sigma}\right)$$

$$F_{X_3}(x) = \mathbb{P}(X_3 \le x) = \Phi\left(\frac{x-\mu_3}{\sigma}\right)$$

Hence, $F_{X_1}(x) = F_{X_2}(x) = F_{X_3}(x), \forall x \in \mathbb{R}$ essentially implies $\mu_1 = \mu_2 = \mu_3$.

Now if we consider the alternative hypothesis $H_1 : F_{X_1}(x) \geq F_{X_2}(x) \geq F_{X_3}(x),$ with strict inequality for atleast one x, then

$$
\begin{aligned}
& F_{X_1}(x) > F_{X_2}(x) > F_{X_3}(x) \\
\implies & \Phi\left(\frac{x-\mu_1}{\sigma}\right) > \Phi\left(\frac{x-\mu_2}{\sigma}\right) > \Phi\left(\frac{x-\mu_3}{\sigma}\right) \\
\implies & \mu_1 < \mu_2 < \mu_3
\end{aligned}
$$

Thus, the null and alternative hypothesis under nonparametric setup reduces to the null and alternative hypothesis under parametric setup when the underlying population distribution in normal. Similarly, we can find the equivalent alternative hypothesis under nonparametric setup for other distributions also. So, we can conclude that the null and alternative hypothesis regarding our testing problem is given by:

- **Under parametric setup:**

$$
\begin{gathered}
H_0 : \mu_1 = \mu_2 = \mu_3 \\
vs \\
H_1 : \mu_1 < \mu_2 < \mu_3
\end{gathered}
$$

- **Under nonparametric setup:**

$$
\begin{aligned}
& H_0 : F_{X_1}(x) = F_{X_2}(x) = F_{X_3}(x), \forall x \in \mathbb{R} \\
& \qquad\qquad vs \\
& H_1 : F_{X_1}(x) \geq F_{X_2}(x) \geq F_{X_3}(x), \text{ with strict inequality for atleast one } x
\end{aligned}
$$

Simulation and its uses in statistics:

A simulation is the execution of a model, represented by a computer program that gives information about the system being investigated. The simulation approach of analyzing a model is opposed to the analytical approach, where the method of analyzing the system is purely theoretical. As this approach is more reliable, the simulation approach gives more flexibility and convenience. In simulation one uses to create specific situations to study the response of the model to them. So, simulations can be used to carryout experiments which have different constraints like dangerous, difficult, expensive to do in the real world.

However, in statistics we will consider simulation where things are subject to randomness. Models for such systems involve random variables and modelling such systems is called “Probabilistic Modelling or Stochastic Modelling”. Here comes the role of **Stochastic Simulation**.

In probability theory we have studied different types of models and in statistics we have studied different modelling techniques. But sometimes probabilistic and statistics models are so complex that mathematical treatment to them is not possible. Thus, tools of mathematical analysis are not sufficient to answer all the relevant questions about them.

Stochastic Simulation is an alternative approach, which can be used to answer such questions. In stochastic simulation we generate random variables and then insert them into a model of our interest. In our discussion we will only consider stochastic simulations.

In statistics when we use certain models, we have some assumptions regarding the applicability of those models. For example – One of the important assumptions of classical linear model is that the model errors are independently and identically normally distributed with mean zero and constant variance. But what happens when these assumptions are violated? Exact analytical evaluation may not be possible always. Although large sample approximations to properties may be possible, but what would happen if sample size is not large is still a question. In these situations, simulation may be a way out.

Simulation is a numerical technique for conducting experiments on computer. By Simulation statisticians mean “Monte Carlo Simulation”. It is a computer experiment involving random sampling from probability distributions. There are various uses of Monte Carlo Simulation in statistics. Such as –

1. To know different properties of an estimator. Like if the estimator biased in finite samples or not? Is it consistent? How does the sampling variance change as parameter value changes?

2. How does a given test (testing rule) perform when the required assumptions are violated?

3. Does a method for constructing a confidence interval for a parameter achieve the stated confidence interval?

4. What is effect of using non-parametric tests instead of using parametric tests when the underlying probability distribution (e.g., whether it is normal or laplace etc.) of the study variable is known?

To answer such questions, one can use Monte Carlo simulation.

Here we have used use R software to carry out our simulations.

0.7 Uses of Simulation in finding empirical size and power

Sometimes it is not possible to calculate size and power of a test (as per the testing rule) explicitly. Because sometimes the true sampling distribution of the test-statistic is difficult to obtain mathematically. Then, Simulation can be used to get the empirical size and power of the test.

Suppose X be the study variable and $X \sim F_X(x,\theta),\ \theta \in \Theta$ is the unknown parameter and $F_X(x,\theta)$ is the cdf of the probability distribution of X. We consider the testing problem $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. And suppose the testing rule rejects H_0 at $\alpha\%$ level of significance *iff* the test statistic $T > c_\alpha$, where c_α is such that

$$\mathbb{P}_{H_0}(T > c_\alpha) \leq \alpha$$

We will calculate the empirical size and power thorough simulation.

- **Step-1:** First we have to generate R independent random samples under the chosen condition. The condition may be different sample sizes, value of true parameters, different probability distributions of the population etc. For finding empirical size the true parameter value has to be $\theta = \theta_0$ and for empirical power $\theta \in \Theta - \theta_0$.

- **Step-2:** For each of the R data sets obtained in step-1, we calculate the value of the test statistics. Thus we have R many values of the test statistic T. Let us denote them by $T_1, T_2, \ldots, T_R$.

- **Step-3:** Now, we have to find the relative frequency of cases, where the value of the test statistic exceeds c_α. That is we have to calculate,

$$f = \frac{\text{Total no of } T_i' \text{ s such that } T_i > c_\alpha \text{ among } T_1, T_2, \ldots, T_R}{R}$$

  If R is sufficiently large, this measure would give the empirical power and size of the test depending upon the choice of the value of the true parameter.If the critical point c_α is chosen so that level of the test is α. Then, it is expected that the empirical size will be near α. If theoretical results are known, we can verify whether simulated results are near to theoretical results.

Simulations, Tabulated powers, Observations:

In this section we will discuss different simulation studies related to size and power of parametric and non-parametric test, that is One-way ANOVA and Kruskal Wallis test in multiple sample problem. Here, our null hypothesis is equality of means and alternative is mean of group-1 is less than mean of group-2 which is again less than mean of group-3. We wish to compare the empirical powers of ANOVA and Kruskal Wallis test when the underlying population distributions are - normal, laplace, logistic, lognormal, exponential.

Here we will consider different situations. For example- we will take different choices of group sizes, different level of significance, different value of R, that is number of repetitions. For both small and large sample size we take R = 2000. We will consider level of significance $\alpha = 0.10, 0.05$. Sample size is an important factor. For small samples sizes some tests perform very bad, while some perform really well. Thus, we will consider different combination of group sizes. We will take large sample sizes to examine whether a test is consistent for the corresponding hypothesis testing problem or not.

For simulation study we will use R software. For each simulation discussed here, we have taken set. Seed = 987654321 for uniformity of result. To calculate power, we have to consider the parameter values (in our case different group means) under $H_1 : \mu_1 < \mu_2 < \mu_3$. To draw the empirical power curve, we will plot the power values against difference between the population or group means d [$(\mu_2 - \mu_1)$ = $(\mu_3 - \mu_2)$]. But the problem is that the power of all tests will not just depend on difference between population means, it depends on magnitude of μ_1, μ_2, μ_3 also. Hence, it is important to see the study power for different values μ_1, μ_2, μ_3. For that we will use an alternative approach. To calculate power, we will first fix $\mu_1 = \mu$ and then take $\mu_2 = \mu_1 + d$ and $\mu_3 = \mu_2 + d = \mu_1 + 2d$. In our case we will consider different values of $d > 0$ (note that for $d = 0$ we will value of empirical size). To calculate the simulated size and power in different situations for both ANOVA and Kruskal Wallis test, we have used some user defined functions in R.

0.8 Simulation-1: Comparison of Empirical Size and Power of One-way ANOVA and Kruskal Test When Underlying Population Distribution is Normal

0.8.1 Objective

ANOVA and Kruskal Wallis are the most popular tests for testing equality of several group means. We wish to compare empirical size and power of One-way ANOVA and Kruskal Wallis Test when the underlying population distribution is normal. We will compare them for varying sample size, different level of significance (here we have considered $\alpha = 0.05, 0.1$), different values of μ_1, μ_2, μ_3 under alternative hypothesis and also when error variance is homoscedastic and heteroscedatic etc.

0.8.2 Algorithm

At first, we have stored the different values of the group means μ_1, μ_2, μ_3 under null and alternative hypothesis, in a matrix called *tab*. We have defined d as the difference between the group means that is $d = (\mu_2 - \mu_1) = (\mu_3 - \mu_2)$ and taken different choices of d. A fixed value of μ_1 is first chosen and then different values of μ_2 and μ_3 are generated as a function of μ_1 and d. In other words, we have taken,

$$\mu_2 = \mu_1 + d$$

$$\mu_2 = \mu_2 + d = \mu_1 + 2d$$

Then, we have used a user-defined function namely *Power*. The function takes different arguments as input and on the basis that, it calculates empirical size and power using the steps discussed in the previous section.

The different arguments of the function are –

1. R - No. of replications required, that is number of times to repeat the whole simulation process.

2. n_i - A vector with three elements whose elements represent the size of different groups (*i.e.* n_1, n_2, n_3 where n_1, n_2, n_3 are sizes of group-1, group-2 and group-3 respectively.)

3. mu_i - A vector with three elements whose elements represent different group means (*i.e.* μ_1, μ_2, μ_3 where μ_1, μ_2, μ_3 are means of group-1, group-2 and group-3 respectively).

4. Sigma - represents error variances. It is taken as a constant term under homoscedasticity and as a vector with three elements, whose elements represent group wise error variance (say, $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively) under heteroscedasticity.

5. alpha - represents level of significance of the test.

From the function we get empirical size and power as output. We have stored the values of empirical power and size in a data frame called *power.matrix*.Using that output we have drawn the empirical power curve using 'ggplot()' function. We will provide the codes for these functions at the end of the article.

0.8.3 Chosen values of group size, level of significance, different group means, error variances etc. for discussion

Here, for both large sample and small sample we have considered the following values of difference d, $d = 0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1$.

For different combinations of sample size and level of significance (in this study we have only considered 0.05 and 0.10) we will calculate the empirical size and power and will present them using table and graphs. We have used exact critical values from the table of Kruskal Wallis Statistic.

Under homoscedasticity we have chosen the value of error variance to be 3^2 and also we have find empirical size and power for different error variances $= 1.2^2, 5.1^2$. And for calculation under heteroscedasticity we have chose $\sigma_1^2 = 1.2^2, \sigma_2^2 = 3^2, \sigma_3^2 = 5.1^2$, where $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively under heteroscedasticity. Also we have chosen two different values of μ_1 given by $\mu_1 = 1, 5$ and values of μ_2, μ_3 are accordingly generated.

For $n_1 = 3, n_2 = 3, n_3 = 2$, we will construct the table of empirical size and power of Kruskal-Wallis (KW) test and One-way ANOVA side by side for $\mu_1 = 1, 5$ and $\alpha = 0.10, 0.05$.

Let us first tabulate the powers for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0490	0.0325	0.0980	0.0860
1	1.3	1.6	0.0445	0.0310	0.0995	0.0905
1	1.6	2.2	0.0605	0.0365	0.1205	0.1040
1	1.9	2.8	0.0730	0.0565	0.1305	0.1230
1	2.2	3.4	0.0870	0.0540	0.1685	0.1350
1	2.5	4.0	0.1140	0.0725	0.1970	0.1705
1	2.8	4.6	0.1375	0.0755	0.2325	0.1845
1	3.1	5.2	0.1750	0.1035	0.2770	0.2245
1	3.4	5.8	0.2075	0.1225	0.3415	0.2785
1	3.7	6.4	0.2365	0.1410	0.3950	0.2955
1	4.0	7.0	0.2980	0.1890	0.4640	0.3720
1	4.3	7.6	0.3410	0.2135	0.5055	0.4065
1	4.6	8.2	0.4030	0.2535	0.5645	0.4585
1	4.9	8.8	0.4575	0.2940	0.6510	0.5195
1	5.2	9.4	0.5205	0.3480	0.6880	0.5760
1	5.5	10.0	0.5725	0.4170	0.7525	0.6425
1	5.8	10.6	0.6225	0.4615	0.7950	0.6790
1	6.1	11.2	0.6750	0.4870	0.8410	0.7065

Table 1: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

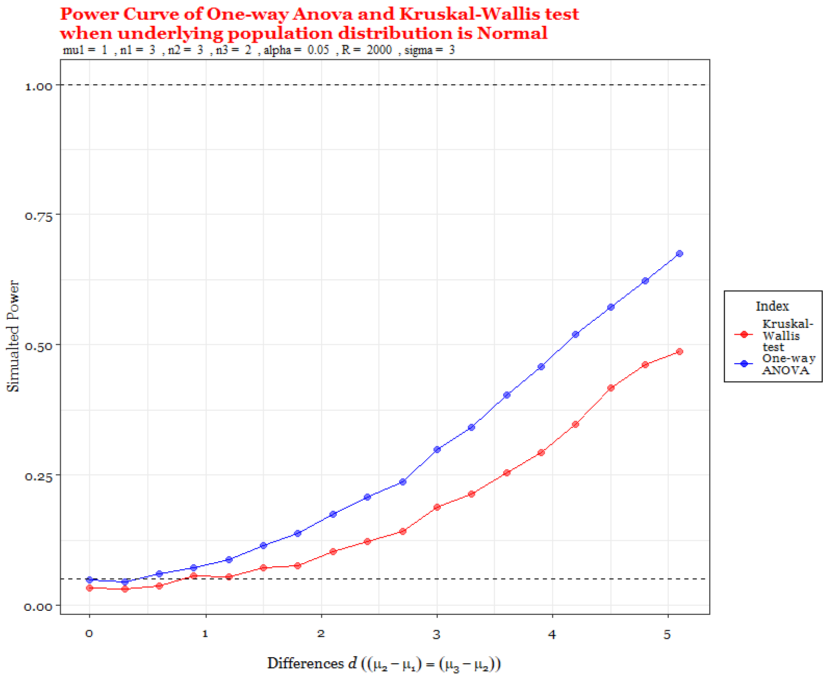

Figure 6: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

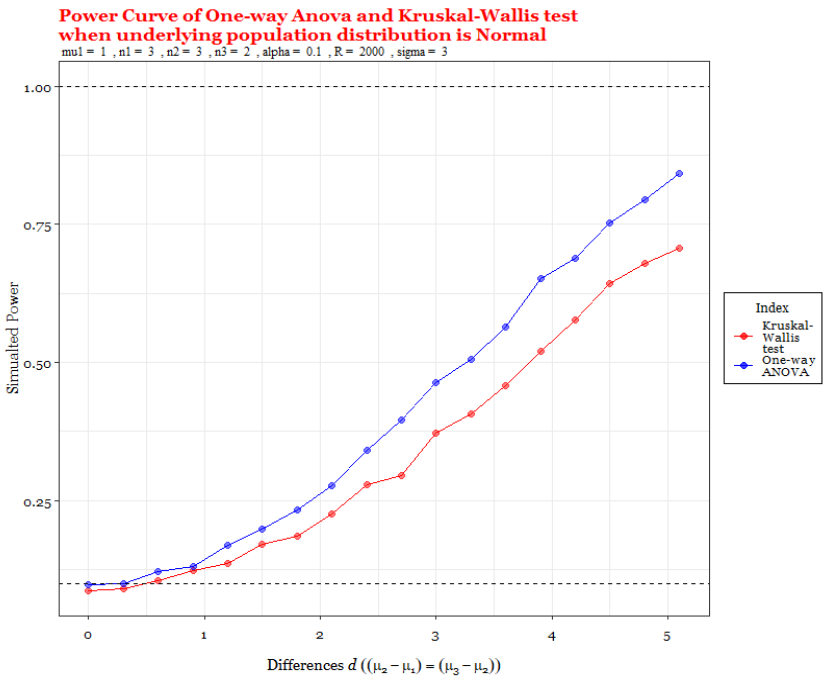

Figure 7: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now we tabulate the powers for $\mu_1 = 5, n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
5	5.0	5.0	0.0490	0.0325	0.0980	0.0860
5	5.3	5.6	0.0445	0.0310	0.0995	0.0905
5	5.6	6.2	0.0605	0.0365	0.1205	0.1040
5	5.9	6.8	0.0730	0.0565	0.1305	0.1230
5	6.2	7.4	0.0870	0.0540	0.1685	0.1350
5	6.5	8.0	0.1140	0.0725	0.1970	0.1705
5	6.8	8.6	0.1375	0.0755	0.2325	0.1845
5	7.1	9.2	0.1750	0.1035	0.2770	0.2245
5	7.4	9.8	0.2075	0.1225	0.3415	0.2785
5	7.7	10.4	0.2365	0.1410	0.3950	0.2955
5	8.0	11.0	0.2980	0.1890	0.4640	0.3720
5	8.3	11.6	0.3410	0.2135	0.5055	0.4065
5	8.6	12.2	0.4030	0.2535	0.5645	0.4585
5	8.9	12.8	0.4575	0.2940	0.6510	0.5195
5	9.2	13.4	0.5205	0.3480	0.6880	0.5760
5	9.5	14.0	0.5725	0.4170	0.7525	0.6425
5	9.8	14.6	0.6225	0.4615	0.7950	0.6790
5	10.1	15.2	0.6750	0.4870	0.8410	0.7065

Table 2: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

One thing to notice is that, for both level of significance $0.1, 0.05$, the power of both tests does not reach 1 for small group sizes for our choices of values. That is, for small sample sizes power of both tests will reach 1 very slowly. Also, the power of ANOVA is higher than Kruskal-Wallis Test for any values of μ_1, μ_2, μ_3 and level of significance. Hence, ANOVA is uniformly powerful than Kruskal-wallis Test when the underlying population distribution is normal and also the assumption of homoscedasticity is valid.

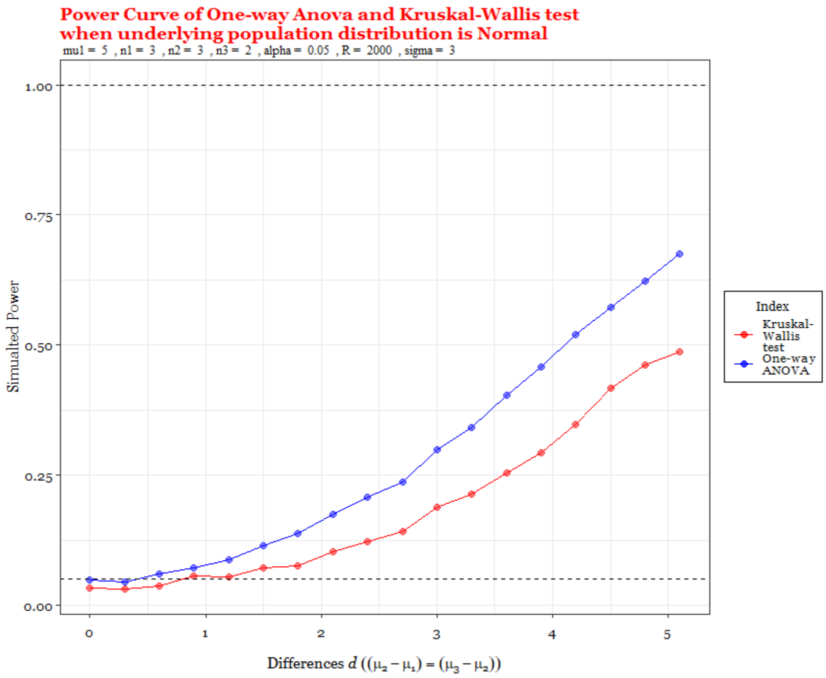

Figure 8: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

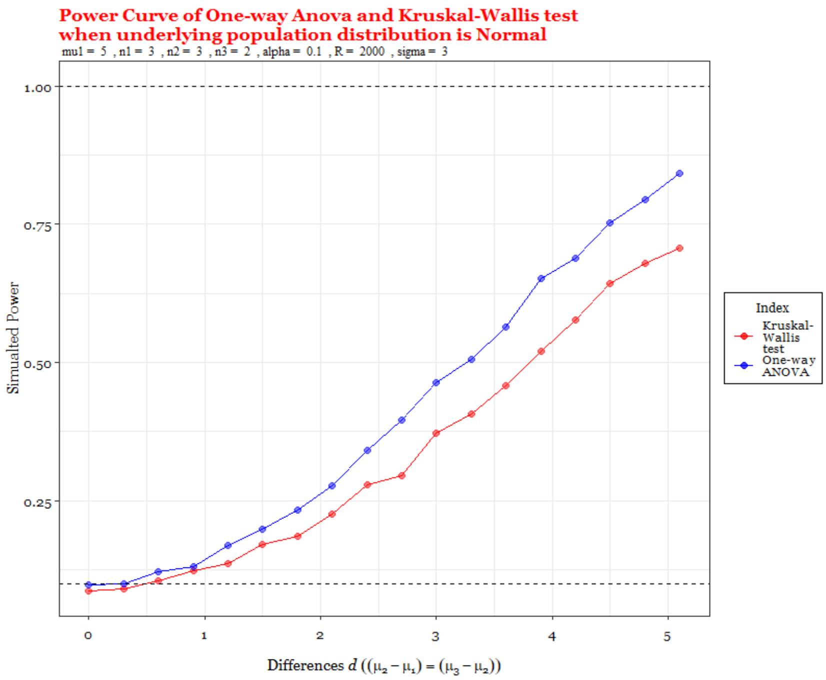

Figure 9: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us tabulate powers for large group sizes, for that we have taken $n_1 = 8, n_2 = 6, n_3 = 7$ and We have only presented the table for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0475	0.0445	0.0955	0.1010
1	1.3	1.6	0.0620	0.0530	0.1150	0.1105
1	1.6	2.2	0.0880	0.0755	0.1650	0.1565
1	1.9	2.8	0.1465	0.1340	0.2315	0.2275
1	2.2	3.4	0.2200	0.1985	0.3385	0.3210
1	2.5	4.0	0.3225	0.2895	0.4610	0.4375
1	2.8	4.6	0.4460	0.4100	0.5905	0.5710
1	3.1	5.2	0.6075	0.5660	0.7365	0.7165
1	3.4	5.8	0.7170	0.6800	0.8300	0.8010
1	3.7	6.4	0.8375	0.8030	0.9130	0.8945
1	4.0	7.0	0.8965	0.8700	0.9560	0.9400
1	4.3	7.6	0.9450	0.9235	0.9745	0.9660
1	4.6	8.2	0.9695	0.9570	0.9885	0.9830
1	4.9	8.8	0.9895	0.9830	0.9960	0.9935
1	5.2	9.4	0.9970	0.9905	0.9995	0.9985
1	5.5	10.0	0.9985	0.9970	0.9990	0.9995
1	5.8	10.6	0.9990	0.9985	0.9995	0.9995
1	6.1	11.2	0.9995	1.0000	1.0000	1.0000

Table 3: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

For large group sizes, powers of both ANOVA and Kruskal-Wallis reaches 1 for both level of significance $0.1, 0.05$. Although the values of powers of both the test are more or less similar, but ANOVA has slightly more power than Kruskal-Wallis in large sample size also. Also, if we interchange the values of n_1, n_2, n_3 we will see that the values of power will change. It may happen due to sample fluctuations.

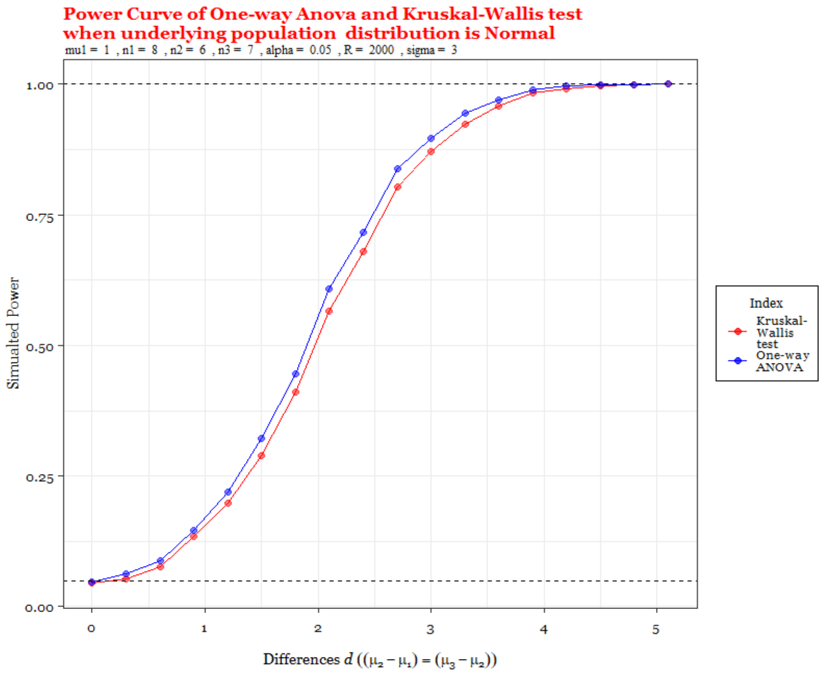

Figure 10: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

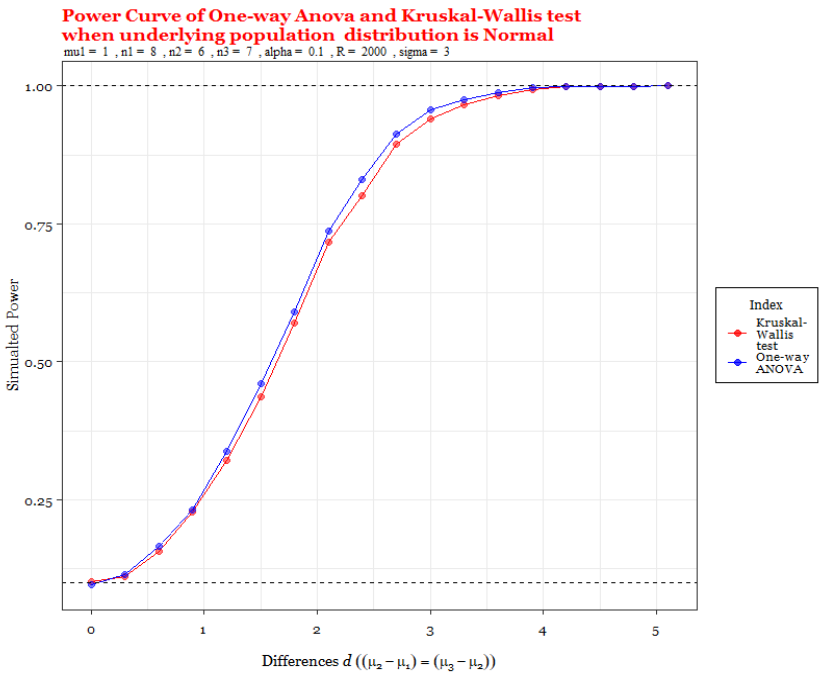

Figure 11: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us choose $n_1 = 50, n_2 = 50, n_3 = 50$ and see how the power values look like when group sizes are very large.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0580	0.0580	0.1180	0.1095
1	1.3	1.6	0.1260	0.1230	0.2105	0.2050
1	1.6	2.2	0.4100	0.3825	0.5475	0.5240
1	1.9	2.8	0.7575	0.7270	0.8360	0.8175
1	2.2	3.4	0.9520	0.9420	0.9745	0.9715
1	2.5	4.0	0.9945	0.9935	0.9985	0.9970
1	2.8	4.6	1.0000	1.0000	1.0000	1.0000
1	3.1	5.2	1.0000	1.0000	1.0000	1.0000
1	3.4	5.8	1.0000	1.0000	1.0000	1.0000
1	3.7	6.4	1.0000	1.0000	1.0000	1.0000
1	4.0	7.0	1.0000	1.0000	1.0000	1.0000
1	4.3	7.6	1.0000	1.0000	1.0000	1.0000
1	4.6	8.2	1.0000	1.0000	1.0000	1.0000
1	4.9	8.8	1.0000	1.0000	1.0000	1.0000
1	5.2	9.4	1.0000	1.0000	1.0000	1.0000
1	5.5	10.0	1.0000	1.0000	1.0000	1.0000
1	5.8	10.6	1.0000	1.0000	1.0000	1.0000
1	6.1	11.2	1.0000	1.0000	1.0000	1.0000

Table 4: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

Here, the power reaches 1 for both the tests more quickly than previous cases. Thus, as we increase the sample size power of both the test reaches 1 more quickly.

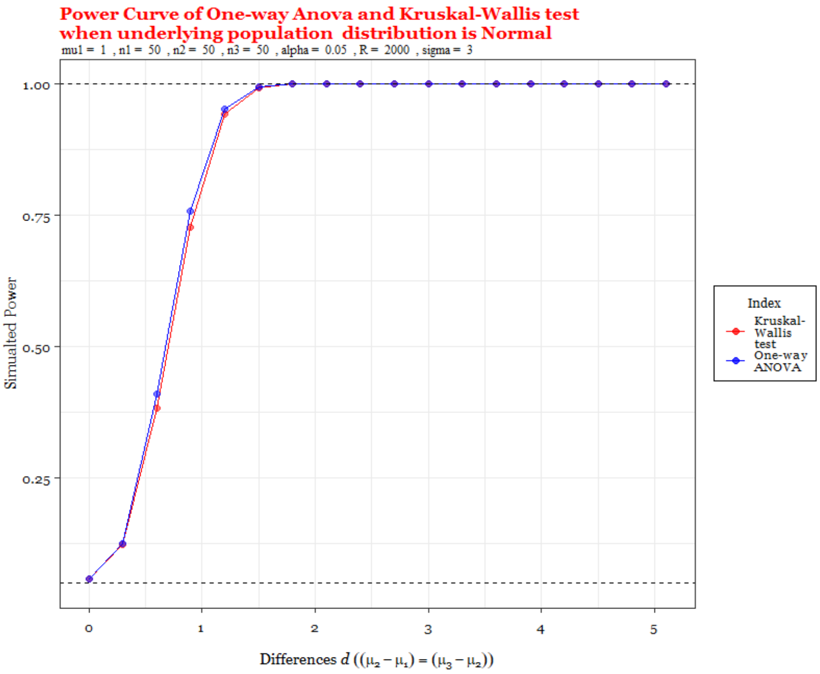

Figure 12: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

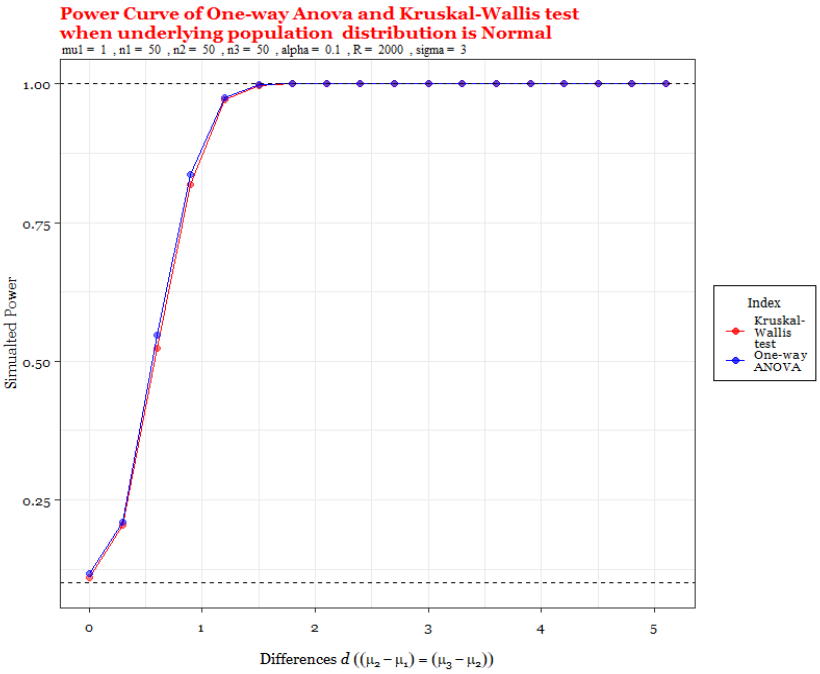

Figure 13: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Again from table-1 and table-2 we can see that remaining other parameters fixed, changing the

value of μ_1 has no seffect on the power. We got same values of empirical power for both $\mu_1 = 1$ and 5. Let us now see how power value changes if we change the value of error variance (σ^2). For that we give the tabulated powers with $\mu_1 = 1, n_1 = 3, n_2 = 3, n_3 = 2$ and $\sigma = 1.2, 5.1$. And we have given the table only for level of significance = 0.05.

μ_1	μ_2	μ_3	Level of Significance ($\alpha = 0.05$)			
			$\sigma^2 = 1.2^2$		$\sigma^2 = 5.1^2$	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0490	0.0325	0.0490	0.0325
1	1.3	1.6	0.0530	0.0365	0.0425	0.0275
1	1.6	2.2	0.1105	0.0665	0.0510	0.0295
1	1.9	2.8	0.1835	0.1210	0.0610	0.0460
1	2.2	3.4	0.3110	0.1845	0.0610	0.0345
1	2.5	4.0	0.4510	0.2960	0.0730	0.0440
1	2.8	4.6	0.5710	0.4085	0.0805	0.0455
1	3.1	5.2	0.7145	0.5460	0.0935	0.0490
1	3.4	5.8	0.8200	0.6590	0.1005	0.0545
1	3.7	6.4	0.8850	0.7395	0.1055	0.0575
1	4.0	7.0	0.9465	0.8485	0.1390	0.0735
1	4.3	7.6	0.9715	0.8950	0.1390	0.0855
1	4.6	8.2	0.9860	0.9460	0.1680	0.1035
1	4.9	8.8	0.9955	0.9670	0.1920	0.1085
1	5.2	9.4	0.9980	0.9790	0.2155	0.1245
1	5.5	10.0	0.9995	0.9910	0.2475	0.1365
1	5.8	10.6	0.9990	0.9930	0.2595	0.1575
1	6.1	11.2	1.0000	0.9990	0.2920	0.1780

Table 5: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 1.2, 5.1$

Hence we see that for both ANOVA and Kruskal Wallis test, value of empirical power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. And as value of error variance increases, value of power decreases.

Same thing we can show for level of signifiance = 0.1.

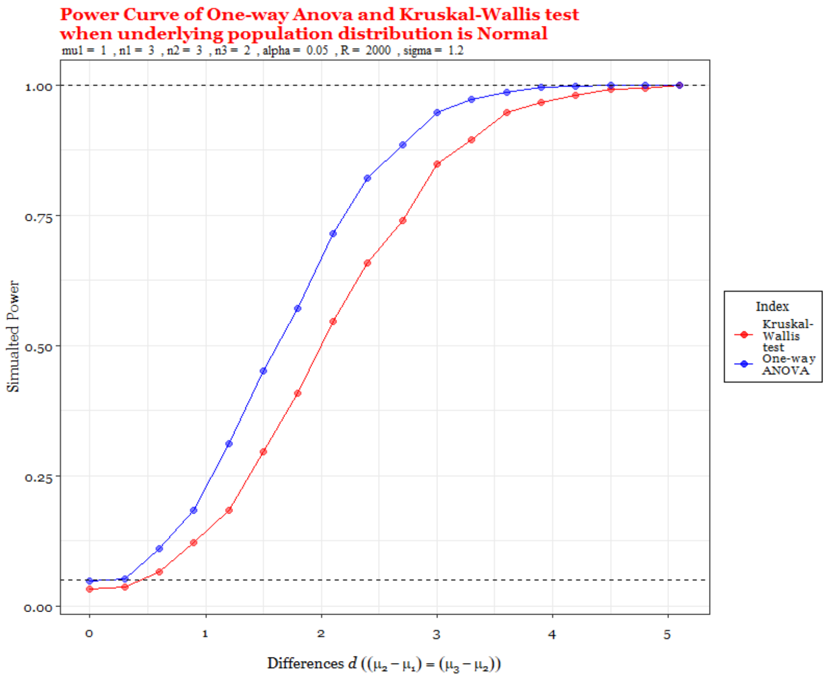

Figure 14: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 1.2$

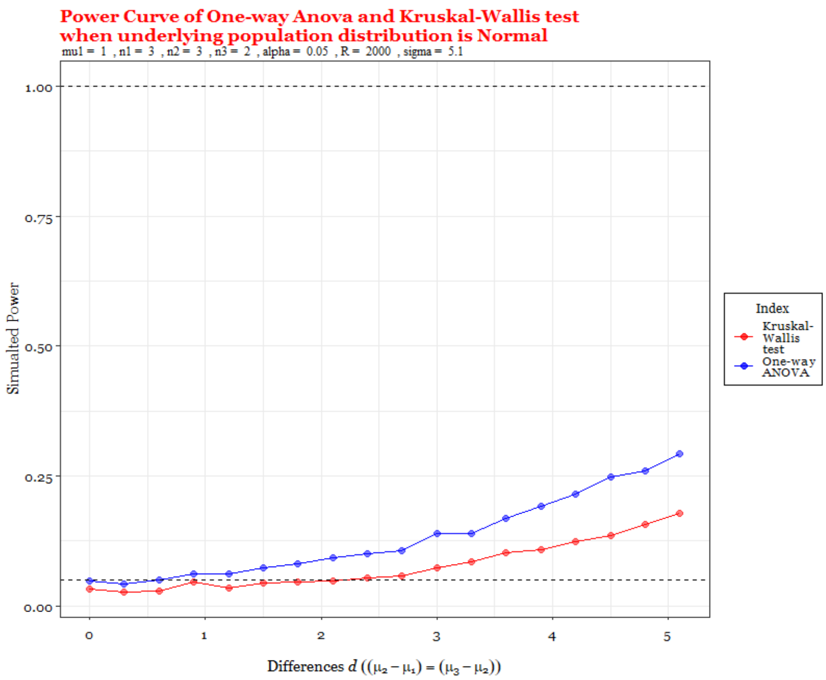

Figure 15: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 5.1$

Now let us try to see how values of empirical power get affected when the assumption of homoscedasticity is violated. For calculation under heteroscedasticity, let us denote σ_i^2 as the error varinace corresponding to ith group, $i = 1,2,3$. We will tabulate the powers for $n_1 = 3, n_2 = 3, n_3 = 2, \mu_1 = 1, \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$ and for both level of significance $\alpha = 0.1, 0.05$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0615	0.0625	0.1160	0.1125
1	1.3	1.6	0.0720	0.0660	0.1255	0.1330
1	1.6	2.2	0.1085	0.1100	0.1815	0.1920
1	1.9	2.8	0.1690	0.1550	0.2535	0.2575
1	2.2	3.4	0.2510	0.2460	0.3425	0.3665
1	2.5	4.0	0.3565	0.3695	0.4830	0.5040
1	2.8	4.6	0.4750	0.4880	0.6055	0.6470
1	3.1	5.2	0.5945	0.6255	0.7230	0.7565
1	3.4	5.8	0.7120	0.7260	0.8235	0.8405
1	3.7	6.4	0.8325	0.8530	0.9090	0.9260
1	4.0	7.0	0.9045	0.9045	0.9550	0.9610
1	4.3	7.6	0.9455	0.9545	0.9810	0.9845
1	4.6	8.2	0.9750	0.9810	0.9910	0.9910
1	4.9	8.8	0.9910	0.9900	0.9990	0.9975
1	5.2	9.4	0.9950	0.9960	1.0000	0.9980
1	5.5	10.0	0.9975	0.9995	0.9995	1.0000
1	5.8	10.6	0.9995	1.0000	1.0000	1.0000
1	6.1	11.2	1.0000	1.0000	1.0000	1.0000

Table 6: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1, \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1, n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.05, 0.1$

We can see under heteroscedasticity, Kruskal-Wallis test and ANOVA has more or less similar values of power. They both perform well under heteroscedasticity also.

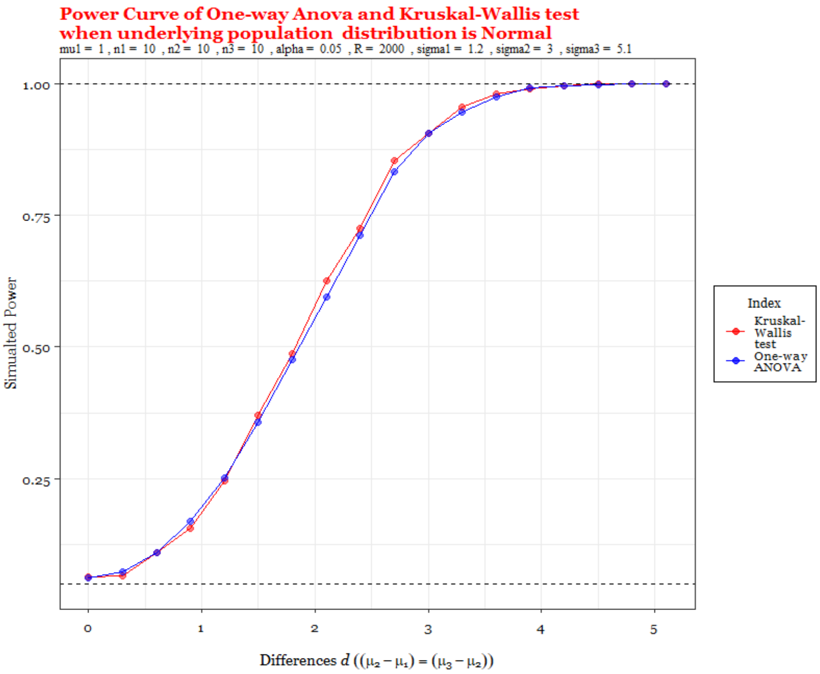

Figure 16: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.05,\ \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$

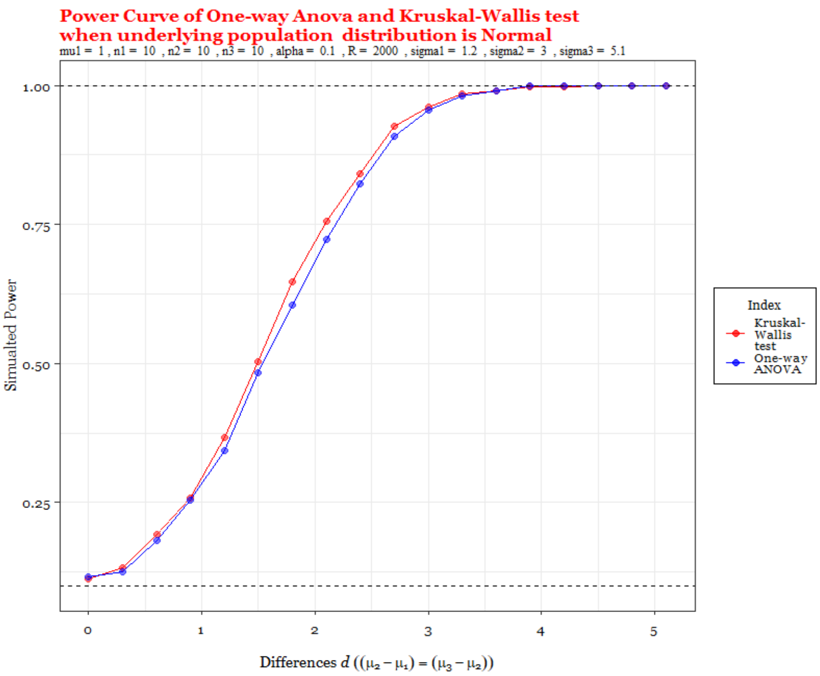

Figure 17: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.1,\ \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$

0.8.4 Observations

Now, we will summarize our observations from the above discussion below –

1. From the above discussion we have observed that when the underlying population distribution is normal, one-way ANOVA performs better than Kruskal-Wallis test in terms of power, i.e., one-way ANOVA is able to detect the same difference between the group means of three independent normal population with same variance more frequently than Kruskal-Wallis test. However, from the table of values of powers, we can notice that the difference between the empirical powers of the two tests are not very high. As sample sizes, that is, group sizes increase the difference between the power of two test decreases. It is clear from the graph also.

2. For small sample sizes (here, $n_1 = 3, n_2 = 3, n_3 = 2$) the power of both the tests reaches 1 very slowly. However, as sample size increases the power of both the tests reaches 1 very quickly. That is both test are consistent for testing our hypothesis problem.

3. The values of empirical power for both tests are slightly higher for higher values of level of significance (α). That is, for small level of significance (α) the power reaches 1 a bit slowly than large level of significance (α). In our discussion, for $\alpha = 0.10$ the power reaches 1 more quickly than $\alpha = 0.05$.

4. The power of both the tests do not depend upon the value of μ_1 (mean of 1st group or population). Other parameters remain fixed, if μ_1 changes, it doesn't change the value of power for both the tests.

5. Instead of μ_1, if we change the value of common error variance, i.e, value of σ^2 then we see that the values of empirical size and power for both tests change. Moreover we have seen that, for both ANOVA and Kruskal Wallis test, value of empirical power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. And as value of error variance increases, value of power decreases.

6. Under heteroscedasticity both one-way ANOVA and Kruskal Wallis test performs more or less similarly. Since here only the assumption of homoscedasticity is violated and the

assumption of normality is unaltered, so it does not effect power of one-way ANOVA much. And also since Kruskal Wallis test does not require any basic assumptions, it still performs well under heteroscedasticity.

Now let us compare empirical size and power of One-way ANOVA and Kruskal-Wallis test when underlying population distribution is non-normal. We have chosen the distribution- laplace,logistic,lognormal and exponential. Among them laplace and logistic are symmetric distribution whereas lognormal and exponential are asymmetric distributions.

0.9 Simulation-2: Comparison of Empirical Size and Power of One-way ANOVA and Kruskal Test When Underlying Population Distribution is Laplace

0.9.1 Objective

ANOVA and Kruskal Wallis are the most popular tests for testing equality of several group means. We wish to compare empirical size and power of One-way ANOVA and Kruskal Wallis Test when the underlying population distribution is laplace. We will compare them for varying sample size, different level of significance (here we have considered $\alpha = 0.05, 0.1$), different values of μ_1, μ_2, μ_3 under alternative hypothesis and also when error variance is homoscedastic and heteroscedastic etc.

0.9.2 Algorithm

At first, we have stored the different values of the group means μ_1, μ_2, μ_3 under null and alternative hypothesis, in a matrix called *tab*. We have defined d as the difference between the group means that is $d = (\mu_2 - \mu_1) = (\mu_3 - \mu_2)$ and taken different choices of d. A fixed value of μ_1 is first chosen and then different values of μ_2 and μ_3 are generated as a function of μ_1 and d. In other words, we have taken,

$$\mu_2 = \mu_1 + d$$

$$\mu_2 = \mu_2 + d = \mu_1 + 2d$$

Then, we have used a user-defined function namely *Power*. The function takes different arguments as input and on the basis that, it calculates empirical size and power using the steps discussed in the previous section.

The different arguments of the function are –

1. R - No. of replications required, that is number of times to repeat the whole simulation process.

2. n_i - A vector with three elements whose elements represent the size of different groups (*i.e.* n_1, n_2, n_3 where n_1, n_2, n_3 are sizes of group-1, group-2 and group-3 respectively.)

3. mu_i - A vector with three elements whose elements represent different group means (*i.e.* μ_1, μ_2, μ_3 where μ_1, μ_2, μ_3 are means of group-1, group-2 and group-3 respectively).

4. Sigma - represents error variances. It is taken as a constant term under homoscedasticity and as a vector with three elements, whose elements represent group wise error variance (say, $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively) under heteroscedasticity.

5. alpha - represents level of significance of the test.

From the function we get empirical size and power as output. We have stored the values of empirical power and size in a data frame called *power.matrix*.Using that output we have drawn the empirical power curve using 'ggplot()' function. We will provide the codes for these functions at the end of the article.

0.9.3 Chosen values of group size, level of significance, different group means etc. for discussion

Here, for both large sample and small sample we have considered the following values of difference d, $d = 0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1$.

For different combinations of sample size and level of significance (in this study we have only considered 0.05 and 0.10) we will calculate the empirical size and power and will present them using table and graphs. We have used exact critical values from the table of Kruskal Wallis Statistic.

Under homoscedasticity we have chosen the value of error variance to be 3^2 and also we have find empirical size and power for different error variances $= 1.2^2, 5.1^2$. And for calculation under heteroscedasticity we have chose $\sigma_1^2 = 1.2^2, \sigma_2^2 = 3^2, \sigma_3^2 = 5.1^2$, where $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively under heteroscedasticity. Also we have chosen two different values of μ_1 given by $\mu_1 = 1, 5$ and values of μ_2, μ_3 are accordingly generated.

For $n_1 = 3, n_2 = 3, n_3 = 2$, we will construct the table of empirical size and power of Kruskal-Wallis (KW) test and One-way ANOVA side by side for $\mu_1 = 1, 5$ and $\alpha = 0.10, 0.05$.

Let us first tabulate the powers for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0440	0.0335	0.0875	0.0890
1	1.3	1.6	0.0430	0.0320	0.0935	0.1020
1	1.6	2.2	0.0410	0.0375	0.0910	0.1000
1	1.9	2.8	0.0595	0.0430	0.1065	0.1200
1	2.2	3.4	0.0730	0.0530	0.1420	0.1390
1	2.5	4.0	0.0685	0.0530	0.1440	0.1425
1	2.8	4.6	0.0970	0.0690	0.1905	0.1800
1	3.1	5.2	0.1255	0.0870	0.2245	0.2000
1	3.4	5.8	0.1335	0.0850	0.2405	0.2070
1	3.7	6.4	0.1700	0.1230	0.2785	0.2490
1	4.0	7.0	0.1980	0.1405	0.3255	0.2885
1	4.3	7.6	0.2270	0.1575	0.3755	0.3245
1	4.6	8.2	0.2820	0.1860	0.4260	0.3560
1	4.9	8.8	0.3280	0.2250	0.4750	0.4120
1	5.2	9.4	0.3505	0.2475	0.5000	0.4360
1	5.5	10.0	0.3900	0.2770	0.5320	0.4645
1	5.8	10.6	0.4160	0.2980	0.5860	0.5130
1	6.1	11.2	0.4745	0.3500	0.6425	0.5530

Table 7: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

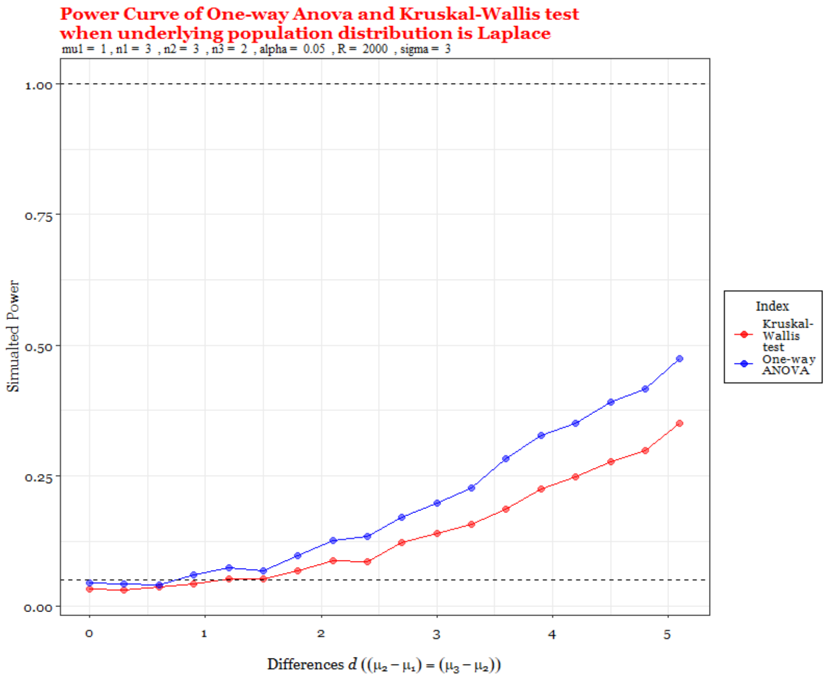

Figure 18: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

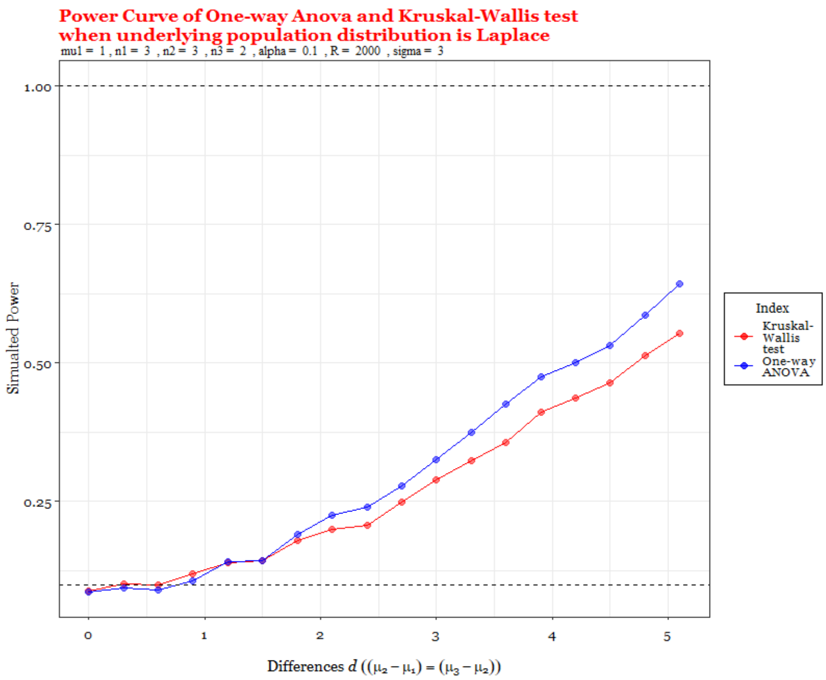

Figure 19: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now we tabulate the powers for $\mu_1 = 5, n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3.$

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
5	5.0	5.0	0.0440	0.0335	0.0875	0.0890
5	5.3	5.6	0.0430	0.0320	0.0935	0.1020
5	5.6	6.2	0.0410	0.0375	0.0910	0.1000
5	5.9	6.8	0.0595	0.0430	0.1065	0.1200
5	6.2	7.4	0.0730	0.0530	0.1420	0.1390
5	6.5	8.0	0.0685	0.0530	0.1440	0.1425
5	6.8	8.6	0.0970	0.0690	0.1905	0.1800
5	7.1	9.2	0.1255	0.0870	0.2245	0.2000
5	7.4	9.8	0.1335	0.0850	0.2405	0.2070
5	7.7	10.4	0.1700	0.1230	0.2785	0.2490
5	8.0	11.0	0.1980	0.1405	0.3255	0.2885
5	8.3	11.6	0.2270	0.1575	0.3755	0.3245
5	8.6	12.2	0.2820	0.1860	0.4260	0.3560
5	8.9	12.8	0.3280	0.2250	0.4750	0.4120
5	9.2	13.4	0.3505	0.2475	0.5000	0.4360
5	9.5	14.0	0.3900	0.2770	0.5320	0.4645
5	9.8	14.6	0.4160	0.2980	0.5860	0.5130
5	10.1	15.2	0.4745	0.3500	0.6425	0.5530

Table 8: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5,$ $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

One thing to notice is that, for both level of significance $0.1, 0.05$, the power of both tests does not reach 1 for small group sizes for our choices of values. That is for both test the value of empirical power will reach 1 very slowly for small sample sizes. Also the values of empirical power of one-way ANOVA is higher than that of Kruskal-wallis Test for any values of μ_1, μ_2, μ_3 under small group sizes when the underlying population distribution is laplace.

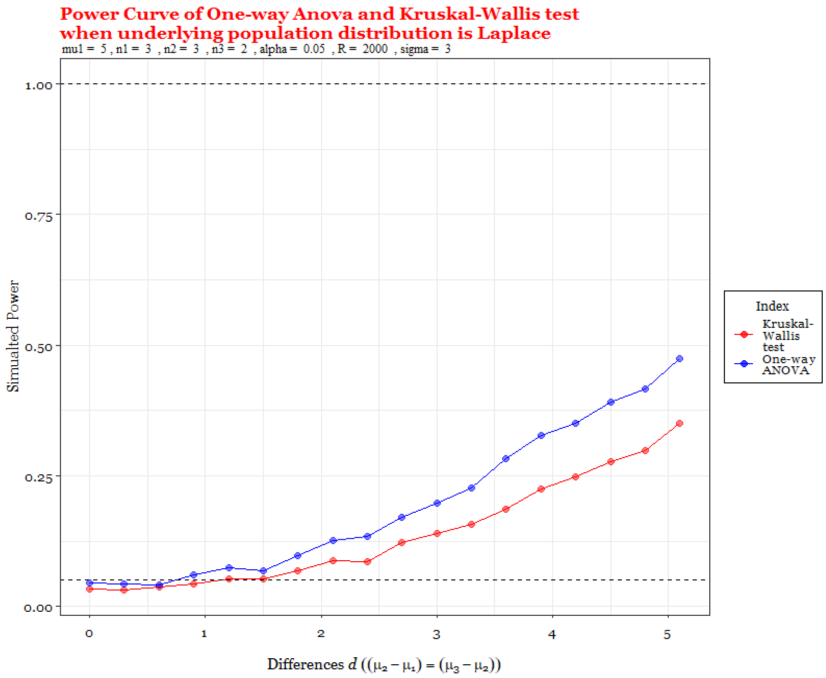

Figure 20: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

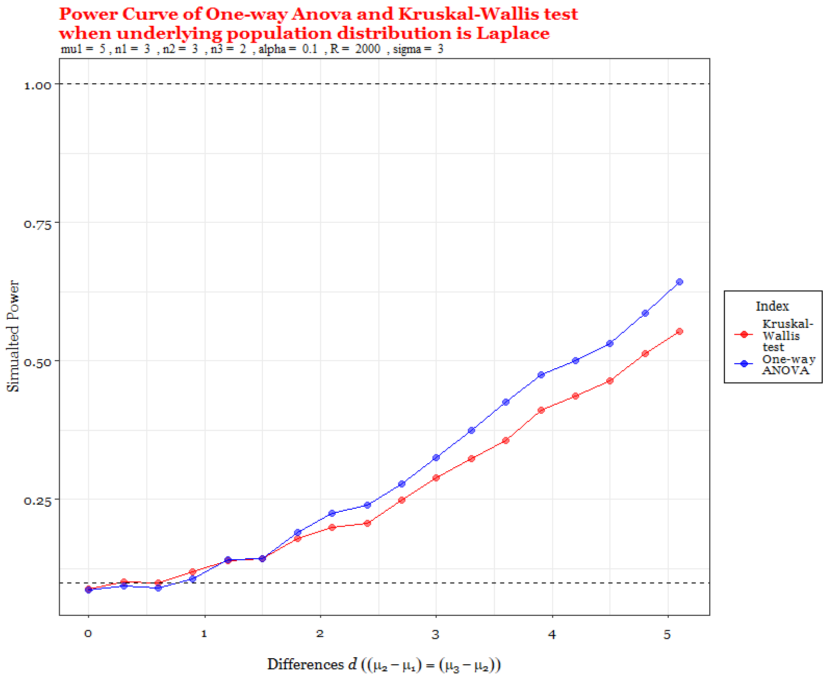

Figure 21: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us tabulate powers for large group sizes, for that we have taken $n_1 = 8, n_2 = 6, n_3 = 7$ and We have only presented the table for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0435	0.0435	0.0860	0.0980
1	1.3	1.6	0.0520	0.0495	0.1065	0.1090
1	1.6	2.2	0.0760	0.0755	0.1355	0.1445
1	1.9	2.8	0.1075	0.1125	0.1960	0.2060
1	2.2	3.4	0.1395	0.1525	0.2410	0.2565
1	2.5	4.0	0.2150	0.2300	0.3235	0.3565
1	2.8	4.6	0.2710	0.3025	0.4060	0.4500
1	3.1	5.2	0.3690	0.4080	0.5110	0.5530
1	3.4	5.8	0.4770	0.5140	0.5995	0.6420
1	3.7	6.4	0.5440	0.5805	0.6715	0.7175
1	4.0	7.0	0.6515	0.6815	0.7565	0.8085
1	4.3	7.6	0.7125	0.7410	0.8105	0.8375
1	4.6	8.2	0.7770	0.8060	0.8700	0.8985
1	4.9	8.8	0.8275	0.8465	0.8995	0.9260
1	5.2	9.4	0.8765	0.8965	0.9295	0.9450
1	5.5	10.0	0.9090	0.9135	0.9490	0.9630
1	5.8	10.6	0.9410	0.9470	0.9730	0.9830
1	6.1	11.2	0.9615	0.9630	0.9790	0.9850

Table 9: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

For large group sizes, powers of both ANOVA and Kruskal-Wallis reaches 1 for both level of significance $0.1, 0.05$. Although the values of powers of both the test are more or less similar, but here Kruskal-Wallis has slightly more power than ANOVA in large sample size in contrary to small sample sizes. Also, if we interchange the values of n_1, n_2, n_3 we will see that the values of power will change. It may happen due to sample fluctuations.

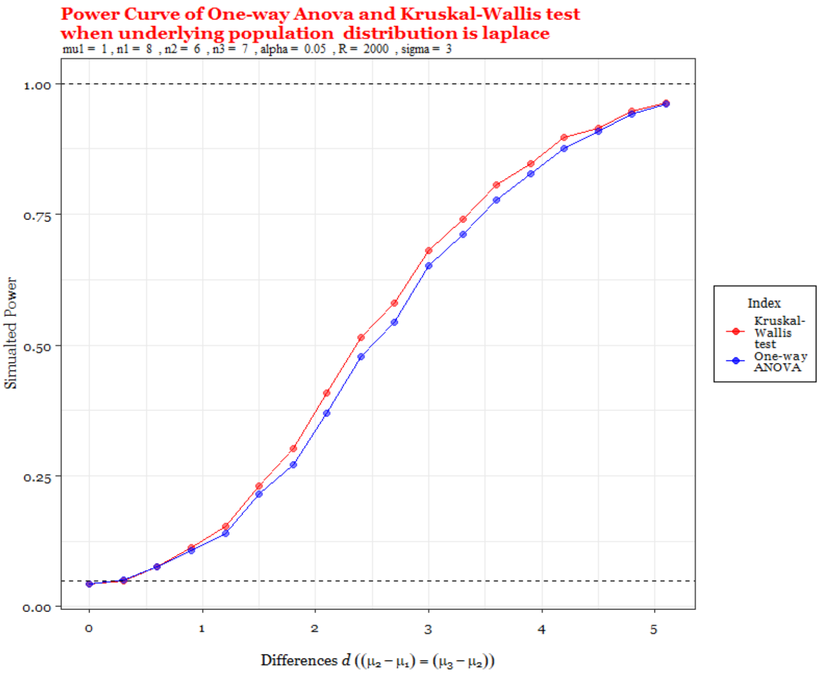

Figure 22: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

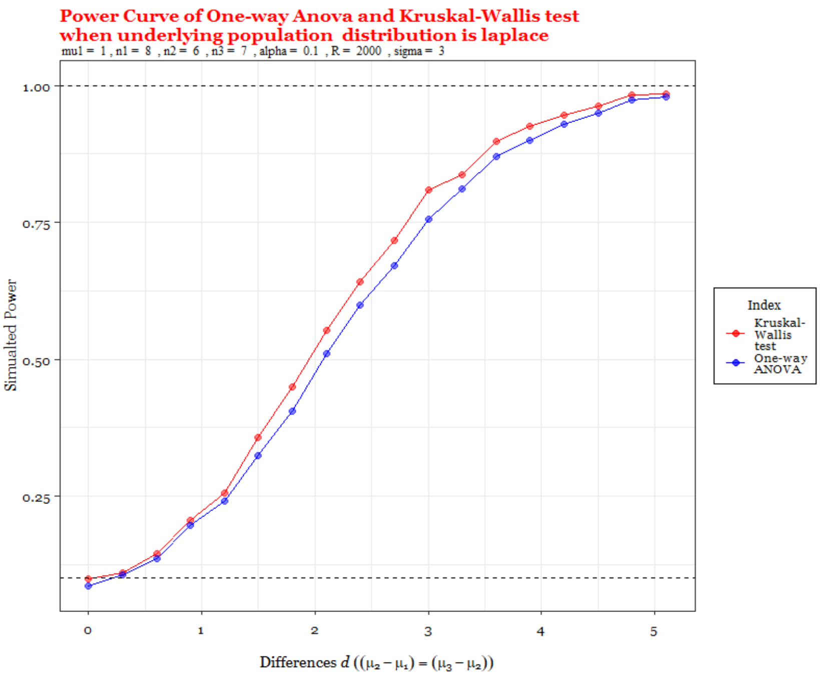

Figure 23: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us choose $n_1 = 50, n_2 = 50, n_3 = 50$ and see how the power values look like when group sizes are very large.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0595	0.0535	0.1100	0.1075
1	1.3	1.6	0.0820	0.1020	0.1590	0.1840
1	1.6	2.2	0.2265	0.3245	0.3340	0.4490
1	1.9	2.8	0.4350	0.5960	0.5655	0.7135
1	2.2	3.4	0.7095	0.8505	0.8050	0.9070
1	2.5	4.0	0.8820	0.9665	0.9340	0.9825
1	2.8	4.6	0.9620	0.9930	0.9825	0.9970
1	3.1	5.2	0.9955	0.9995	0.9990	1.0000
1	3.4	5.8	0.9990	1.0000	0.9995	1.0000
1	3.7	6.4	1.0000	1.0000	1.0000	1.0000
1	4.0	7.0	1.0000	1.0000	1.0000	1.0000
1	4.3	7.6	1.0000	1.0000	1.0000	1.0000
1	4.6	8.2	1.0000	1.0000	1.0000	1.0000
1	4.9	8.8	1.0000	1.0000	1.0000	1.0000
1	5.2	9.4	1.0000	1.0000	1.0000	1.0000
1	5.5	10.0	1.0000	1.0000	1.0000	1.0000
1	5.8	10.6	1.0000	1.0000	1.0000	1.0000
1	6.1	11.2	1.0000	1.0000	1.0000	1.0000

Table 10: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

Here, the power reaches 1 for both the tests more quickly than previous cases. Thus, as we increase the sample size power of both the test reaches 1 more quickly. And here also notice that Kruskal-Wallis test has higher power values than ANOVA for very large group sizes with underlying population distribution laplace.

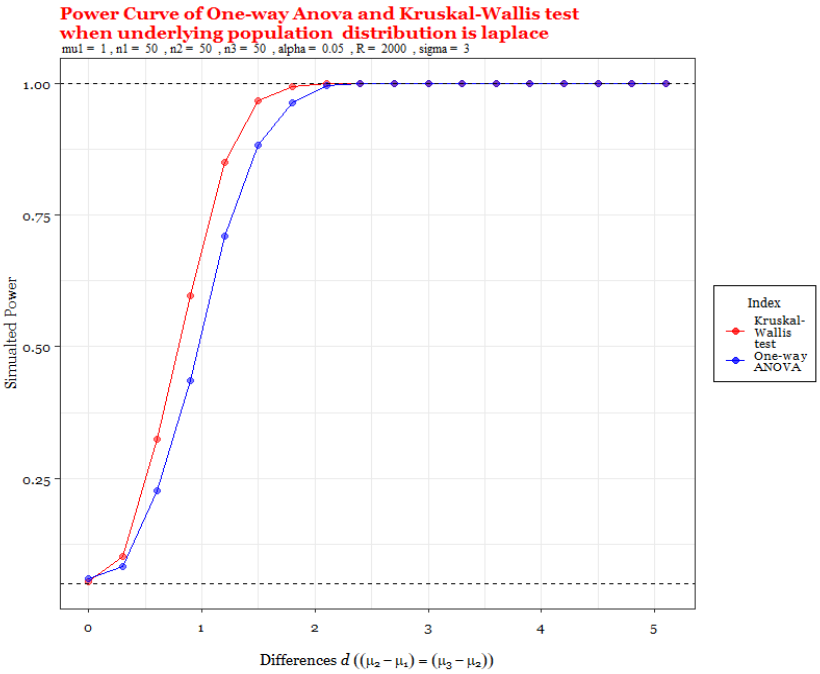

Figure 24: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

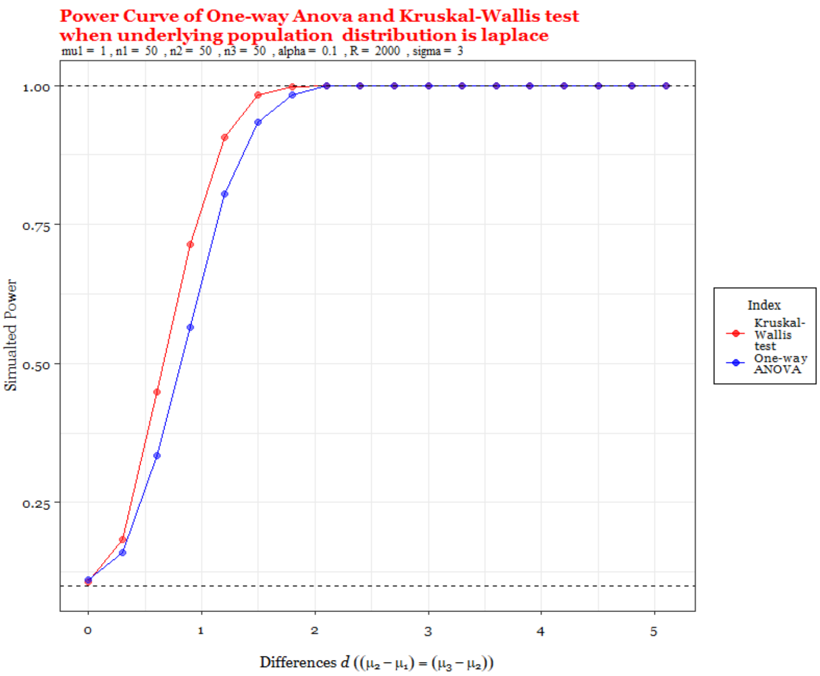

Figure 25: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Again from table-1 and table-2 we can see that remaining other parameters fixed, changing the

values of μ_1 has no effect on the power. We got same values of empirical power for both $\mu_1 = 1$ and 5. Let us now see how power value changes if we change the value of error variance (σ^2). For that we give the tabulated powers with $\mu_1 = 1, n_1 = 3, n_2 = 3, n_3 = 2$ and $\sigma = 1.2, 5.1$. And we have given the table only for level of significance = 0.05.

μ_1	μ_2	μ_3	Level of Significance ($\alpha = 0.05$)			
			$\sigma^2 = 1.2^2$		$\sigma^2 = 5.1^2$	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	.0440	0.0335	0.0440	0.0335
1	1.3	1.6	0.0555	0.0405	0.0415	0.0305
1	1.6	2.2	0.0685	0.0520	0.0370	0.0335
1	1.9	2.8	0.1195	0.0940	0.0485	0.0360
1	2.2	3.4	0.2055	0.1510	0.0620	0.0445
1	2.5	4.0	0.3035	0.2040	0.0445	0.0370
1	2.8	4.6	0.3955	0.2865	0.0595	0.0475
1	3.1	5.2	0.4880	0.3560	0.0720	0.0510
1	3.4	5.8	0.5700	0.4440	0.0615	0.0425
1	3.7	6.4	0.6890	0.5500	0.0810	0.0600
1	4.0	7.0	0.7515	0.6080	0.0925	0.0680
1	4.3	7.6	0.8195	0.6810	0.0925	0.0680
1	4.6	8.2	0.8550	0.7295	0.1220	0.0850
1	4.9	8.8	0.9000	0.7945	0.1300	0.0960
1	5.2	9.4	0.9260	0.8140	0.1400	0.1005
1	5.5	10.0	0.9305	0.8235	0.1665	0.1205
1	5.8	10.6	0.9640	0.8775	0.1680	0.1245
1	6.1	11.2	0.9760	0.9125	0.2025	0.1350

Table 11: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 1.2, 5.1$

Hence we see that for both ANOVA and Kruskal Wallis test, value of power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. Same thing we can show for level of signifiance = 0.1.

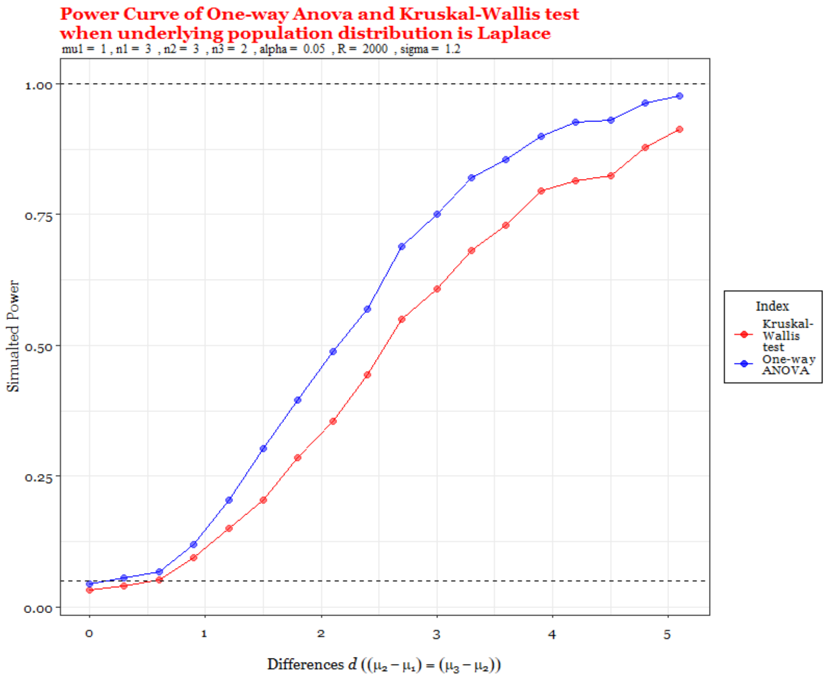

Figure 26: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 1.2$

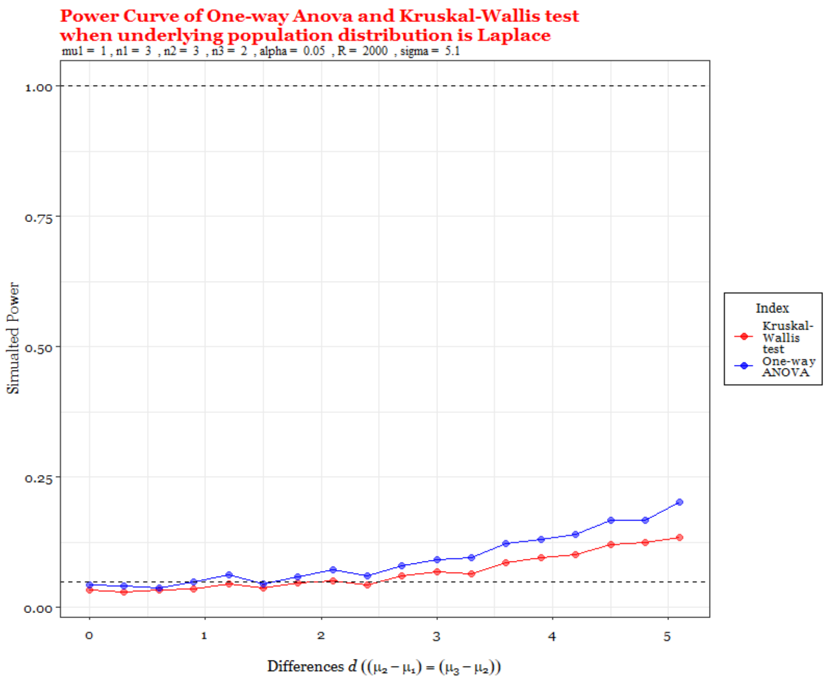

Figure 27: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 5.1$

Now let us try to see how values of power get affected when the assumption of homoscedasticity is violated. For calculation under heteroscedasticity, let us denote σ_i^2 as the error varinace corresponding to ith group, $i = 1,2,3$. We will tabulate the powers for $n_1 = 3, n_2 = 3, n_3 = 2, \mu_1 = 1, \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$ and for both level of significance $\alpha = 0.1, 0.05$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0630	0.0590	0.1150	0.1160
1	1.3	1.6	0.0560	0.0755	0.1100	0.1250
1	1.6	2.2	0.0950	0.0935	0.1470	0.1740
1	1.9	2.8	0.1150	0.1445	0.1840	0.2370
1	2.2	3.4	0.1630	0.2390	0.2495	0.3570
1	2.5	4.0	0.2220	0.3170	0.3265	0.4555
1	2.8	4.6	0.2900	0.4075	0.4010	0.5540
1	3.1	5.2	0.3855	0.5440	0.5060	0.6815
1	3.4	5.8	0.4755	0.6370	0.5885	0.7665
1	3.7	6.4	0.5520	0.7135	0.6550	0.8225
1	4.0	7.0	0.6505	0.8190	0.7580	0.8990
1	4.3	7.6	0.7325	0.8595	0.8200	0.9300
1	4.6	8.2	0.7980	0.9135	0.8695	0.9610
1	4.9	8.8	0.8395	0.9385	0.9015	0.9720
1	5.2	9.4	0.8860	0.9665	0.9325	0.9850
1	5.5	10.0	0.9155	0.9760	0.9490	0.9905
1	5.8	10.6	0.9405	0.9825	0.9705	0.9945
1	6.1	11.2	0.9485	0.9925	0.9735	0.9990

Table 12: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1, \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1, n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.05, 0.1$

Thus, we can see under heteroscedasticity, Kruskal-Wallis test has more power than One-way ANOVA. Hence, if the basic assumption of a parametric test get violated (here assumption of normality and homoscedasticity), then it is better to choose a nonparametric test.

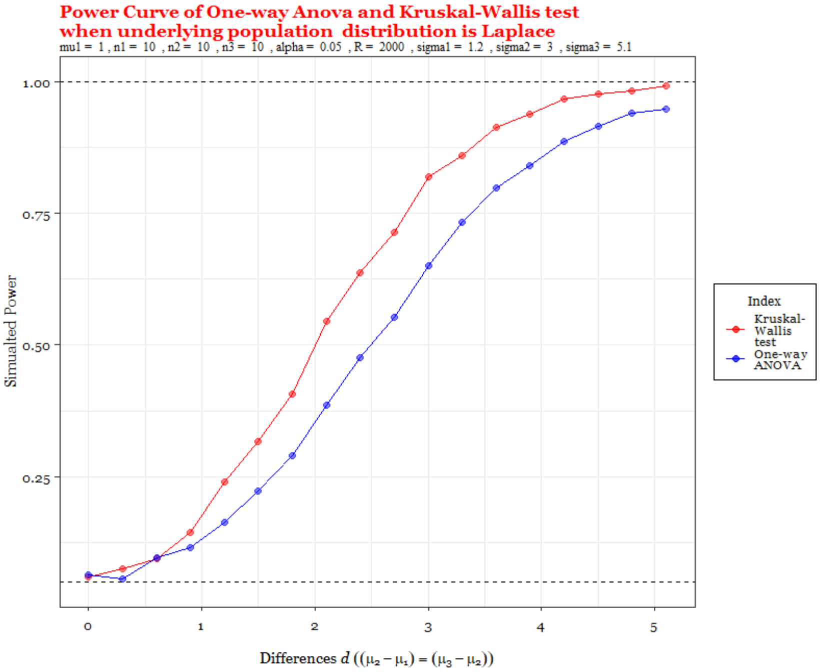

Figure 28: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.05,\ \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$

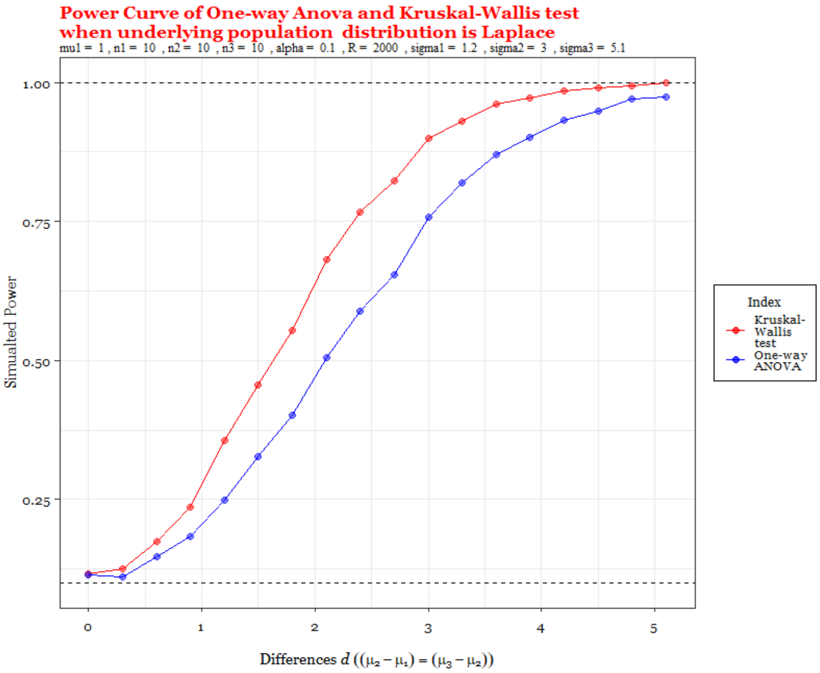

Figure 29: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.1,\ ,\sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$

0.9.4 Observations

Now, we will summarize our observations from the above discussion below –

1. From the above discussion we have observed that when the underlying population distribution is laplace, one-way ANOVA performs better than Kruskal-Wallis test in terms of power when group sizes are small but Kruskal Wallis test performs better than ANOVA when group sizes are large, i.e., Kruskal Wallis test is able to detect the same difference between the group means of three independent laplace population with same variance more frequently than one-way ANOVA under large sample sizes. However, from the table of values of powers, we can notice that the difference between the empirical powers of the two tests are not very high. As group sizes increase, $n_1 = 8, n_2 = 6, n_3 = 7$, the difference between the power of two test decreases, but again the difference increases a bit for $n_1 = n_2 = n_3 = 50$. It is clear from the graph also.

2. For small sample sizes (here, $n_1 = 3, n_2 = 3, n_3 = 2$) the power of both the tests reaches 1 very slowly. However, as sample size increases the power of both the tests reaches 1 very quickly. That is both test are consistent for testing our hypothesis problem.

3. The values of empirical power for both tests are slightly higher for higher values of level of significance (α). That is, for small level of significance (α) the power reaches 1 a bit slowly than large level of significance (α). In our discussion, for $\alpha = 0.10$ the power reaches 1 more quickly than $\alpha = 0.05$.

4. The power of both the tests do not depend upon the value of μ_1 (mean of 1st group or population). Other parameters remain fixed, if μ_1 changes, it doesn't change the value of power for both the tests.

5. Instead of μ_1, if we change the value of common error variance, i.e, value of σ^2 then we see that the values of empirical size and power for both tests change. Moreover we have seen that, for both ANOVA and Kruskal Wallis test, value of empirical power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. And as value of error variance increases, value of power decreases.

6. Under heteroscedasticity Kruskal Wallis performs better than one-way ANOVA. Since for laplace distribution under heteroscedasticity, all the basic requirements of ANOVA i.e. normality and homoscedasticity are violated. But since Kruskal Wallis test does not require any such assumptions, it performs better than ANOVA under heteroscedasticity when underlying population distribution is laplace.

0.10 Simulation-3: Comparison of Empirical Size and Power of One-way ANOVA and Kruskal Test When Underlying Population Distribution is Logistic

0.10.1 Objective

ANOVA and Kruskal Wallis are the most popular tests for testing equality of several group means. We wish to compare empirical size and power of One-way ANOVA and Kruskal Wallis Test when the underlying population distribution is logistic. We will compare them for varying sample size, different level of significance (here we have considered $\alpha = 0.05, 0.1$), different values of μ_1, μ_2, μ_3 under alternative hypothesis and also when error variance is homoscedastic and heteroscedatic etc.

0.10.2 Algorithm

At first, we have stored the different values of the group means μ_1, μ_2, μ_3 under null and alternative hypothesis, in a matrix called *tab*. We have defined d as the difference between the group means that is $d = (\mu_2 - \mu_1) = (\mu_3 - \mu_2)$ and taken different choices of d. A fixed value of μ_1 is first chosen and then different values of μ_2 and μ_3 are generated as a function of μ_1 and d. In other words, we have taken,

$$\mu_2 = \mu_1 + d$$

$$\mu_2 = \mu_2 + d = \mu_1 + 2d$$

Then, we have used a user-defined function namely *Power*. The function takes different arguments as input and on the basis that, it calculates empirical size and power using the steps discussed in the previous section.

The different arguments of the function are –

1. R - No. of replications required, that is number of times to repeat the whole simulation process.

2. n_i - A vector with three elements whose elements represent the size of different groups (*i.e.* n_1, n_2, n_3 where n_1, n_2, n_3 are sizes of group-1, group-2 and group-3 respectively.)

3. mu_i - A vector with three elements whose elements represent different group means (*i.e.* μ_1, μ_2, μ_3 where μ_1, μ_2, μ_3 are means of group-1, group-2 and group-3 respectively).

4. Sigma - represents error variances. It is taken as a constant term under homoscedasticity and as a vector with three elements, whose elements represent group wise error variance (say, $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively) under heteroscedasticity.

5. alpha - represents level of significance of the test.

From the function we get empirical size and power as output. We have stored the values of empirical power and size in a data frame called *power.matrix*.Using that output we have drawn the empirical power curve using 'ggplot()' function. We will provide the codes for these functions at the end of the article.

0.10.3 Chosen values of group size, level of significance, different group means etc. for discussion

Here, for both large sample and small sample we have considered the following values of difference d, $d = 0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1$.

For different combinations of sample size and level of significance (in this study we have only considered 0.05 and 0.10) we will calculate the empirical size and power and will present them using table and graphs. We have used exact critical values from the table of Kruskal Wallis Statistic.

Under homoscedasticity we have chosen the value of error variance to be 3^2 and also we have find empirical size and power for different error variances $= 1.2^2, 5.1^2$. And for calculation under heteroscedasticity we have chose $\sigma_1^2 = 1.2^2, \sigma_2^2 = 3^2, \sigma_3^2 = 5.1^2$, where $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively under heteroscedasticity. Also we have chosen two different values of μ_1 given by $\mu_1 = 1, 5$ and values of μ_2, μ_3 are accordingly generated.

For $n_1 = 3, n_2 = 3, n_3 = 2$, we will construct the table of empirical size and power of Kruskal-Wallis (KW) test and One-way ANOVA side by side for $\mu_1 = 1, 5$ and $\alpha = 0.10, 0.05$.

Let us first tabulate the powers for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0470	0.0335	0.0945	0.0890
1	1.3	1.6	0.0495	0.0310	0.0975	0.0975
1	1.6	2.2	0.0470	0.0335	0.0895	0.0925
1	1.9	2.8	0.0560	0.0375	0.1035	0.1055
1	2.2	3.4	0.0685	0.0455	0.1335	0.1175
1	2.5	4.0	0.0605	0.0390	0.1180	0.1150
1	2.8	4.6	0.0740	0.0490	0.1425	0.1345
1	3.1	5.2	0.0870	0.0530	0.1640	0.1365
1	3.4	5.8	0.0785	0.0495	0.1660	0.1350
1	3.7	6.4	0.1060	0.0695	0.1865	0.1705
1	4.0	7.0	0.1230	0.0775	0.2195	0.1910
1	4.3	7.6	0.1285	0.0815	0.2515	0.1975
1	4.6	8.2	0.1555	0.0980	0.2820	0.2195
1	4.9	8.8	0.1830	0.1170	0.3210	0.2750
1	5.2	9.4	0.2000	0.1260	0.3335	0.2800
1	5.5	10.0	0.2355	0.1570	0.3765	0.3090
1	5.8	10.6	0.2435	0.1590	0.3955	0.3145
1	6.1	11.2	0.2965	0.1900	0.4485	0.3625

Table 13: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

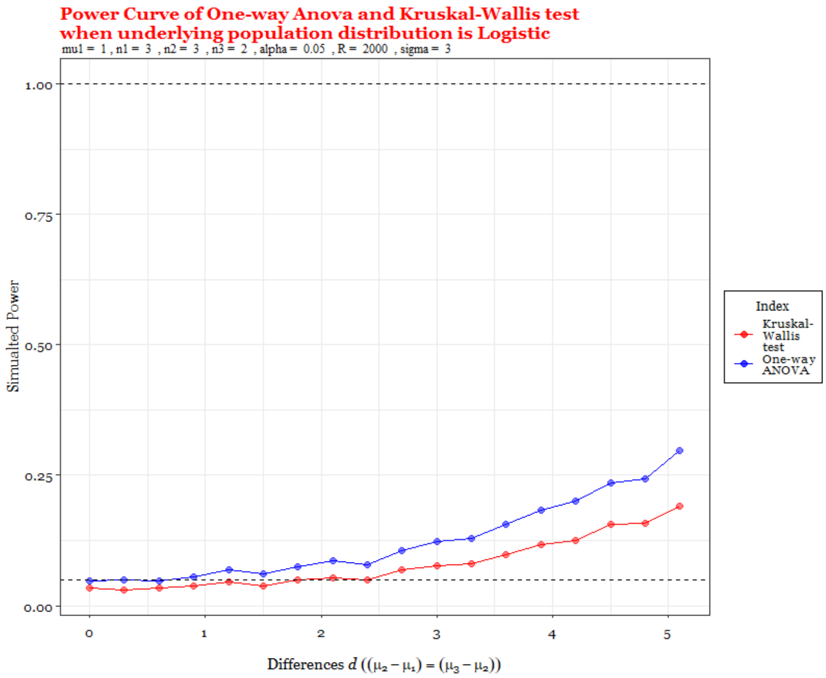

Figure 30: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

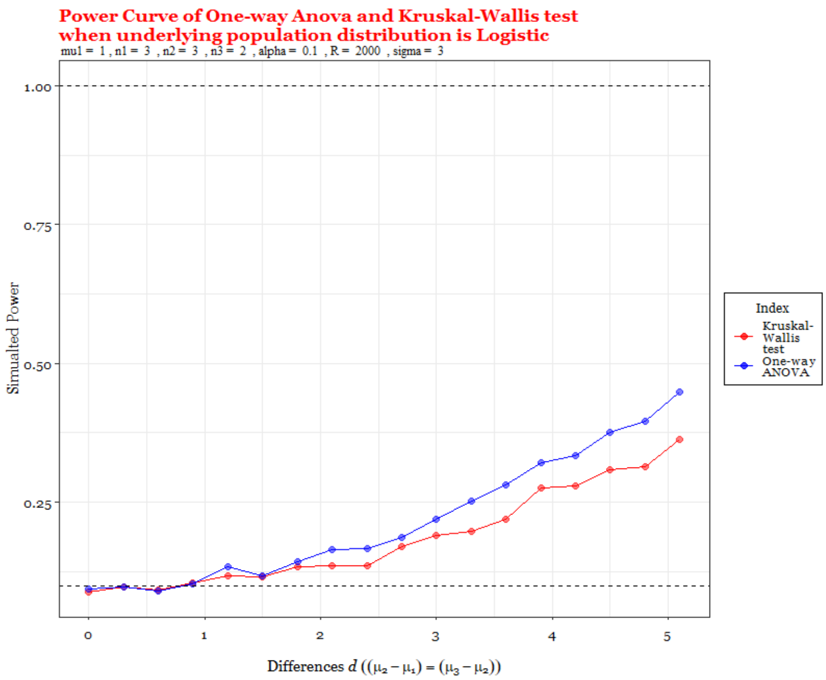

Figure 31: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now we tabulate the powers for $\mu_1 = 5, n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
5	5.0	5.0	0.0470	0.0335	0.0945	0.0890
5	5.3	5.6	0.0495	0.0310	0.0975	0.0975
5	5.6	6.2	0.0470	0.0335	0.0895	0.0925
5	5.9	6.8	0.0560	0.0375	0.1035	0.1055
5	6.2	7.4	0.0685	0.0455	0.1335	0.1175
5	6.5	8.0	0.0605	0.0390	0.1180	0.1150
5	6.8	8.6	0.0740	0.0490	0.1425	0.1345
5	7.1	9.2	0.0870	0.0530	0.1640	0.1365
5	7.4	9.8	0.0785	0.0495	0.1660	0.1350
5	7.7	10.4	0.1060	0.0695	0.1865	0.1705
5	8.0	11.0	0.1230	0.0775	0.2195	0.1910
5	8.3	11.6	0.1285	0.0815	0.2515	0.1975
5	8.6	12.2	0.1555	0.0980	0.2820	0.2195
5	8.9	12.8	0.1830	0.1170	0.3210	0.2750
5	9.2	13.4	0.2000	0.1260	0.3335	0.2800
5	9.5	14.0	0.2355	0.1570	0.3765	0.3090
5	9.8	14.6	0.2435	0.1590	0.3955	0.3145
5	10.1	15.2	0.2965	0.1900	0.4485	0.3625

Table 14: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

One thing to notice is that, for both level of significance $0.1, 0.05$, the power of both tests does not reach 1 for small group sizes for our choices of values. And in case of logistic distribution the value of empirical power for both tests are significantly small than that of under normal and laplace distribution. Also, the power of ANOVA is more or less higher than Kruskal-Wallis Test for any values of μ_1, μ_2, μ_3 and level of significance when the underlying population distribution is normal.

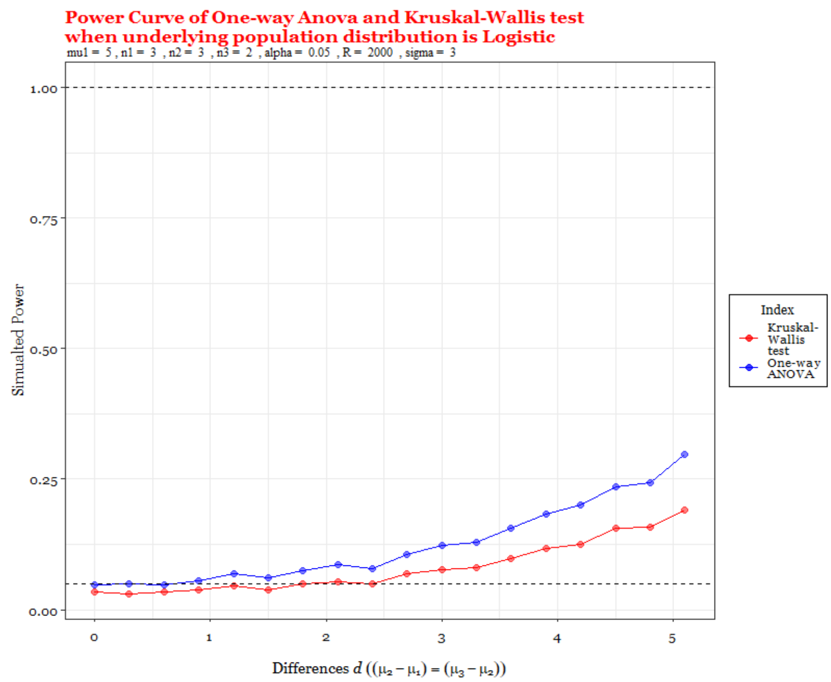

Figure 32: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

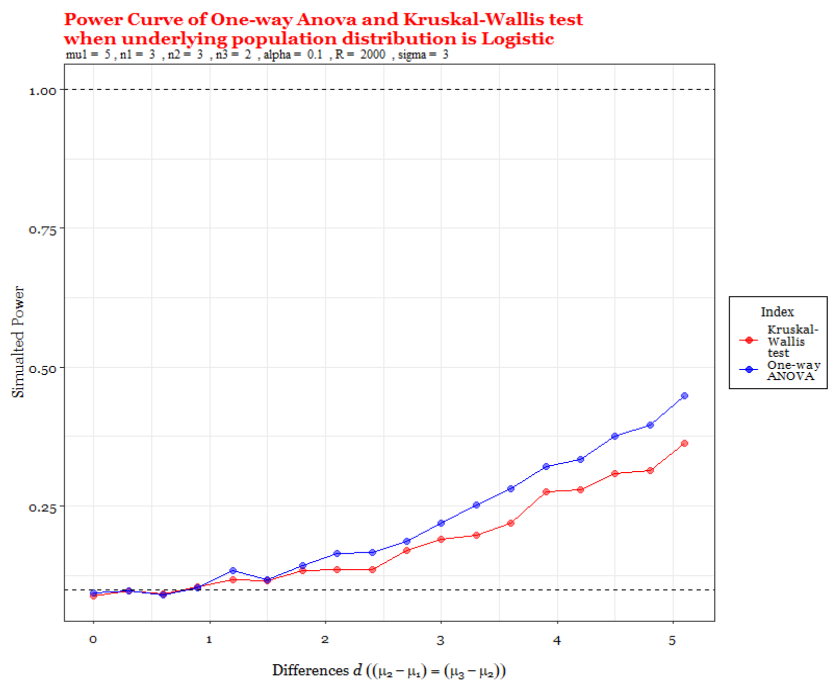

Figure 33: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us tabulate powers for large group sizes, for that we have taken $n_1 = 8, n_2 = 6, n_3 = 7$ and We have only presented the table for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0430	0.0435	0.0895	0.0980
1	1.3	1.6	0.0525	0.0480	0.1100	0.1020
1	1.6	2.2	0.0650	0.0565	0.1195	0.1160
1	1.9	2.8	0.0880	0.0775	0.1530	0.1510
1	2.2	3.4	0.0885	0.0885	0.1685	0.1670
1	2.5	4.0	0.1330	0.1250	0.2340	0.2350
1	2.8	4.6	0.1685	0.1565	0.2725	0.2745
1	3.1	5.2	0.2345	0.2155	0.3375	0.3405
1	3.4	5.8	0.2950	0.2895	0.4250	0.4330
1	3.7	6.4	0.3405	0.3325	0.4780	0.4730
1	4.0	7.0	0.4270	0.4080	0.5750	0.5795
1	4.3	7.6	0.4885	0.4830	0.6300	0.6265
1	4.6	8.2	0.5750	0.5560	0.7025	0.7115
1	4.9	8.8	0.6270	0.6010	0.7495	0.7440
1	5.2	9.4	0.7045	0.6880	0.8115	0.8085
1	5.5	10.0	0.7355	0.7295	0.8430	0.8450
1	5.8	10.6	0.8220	0.8045	0.8945	0.8945
1	6.1	11.2	0.8595	0.8440	0.9190	0.9160

Table 15: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

For large group sizes, powers of both ANOVA and Kruskal-Wallis reaches 1 for both level of significance $0.1, 0.05$. But for logistic distribution it becomes really difficult to distinguish between ANOVA and Kruskal Wallis test in terms of power under large group sizes. Both tests have quite similar values of power for large sample sizes. Also, if we interchange the values of n_1, n_2, n_3 we will see that the values of power will change. It may happen due to sample fluctuations.

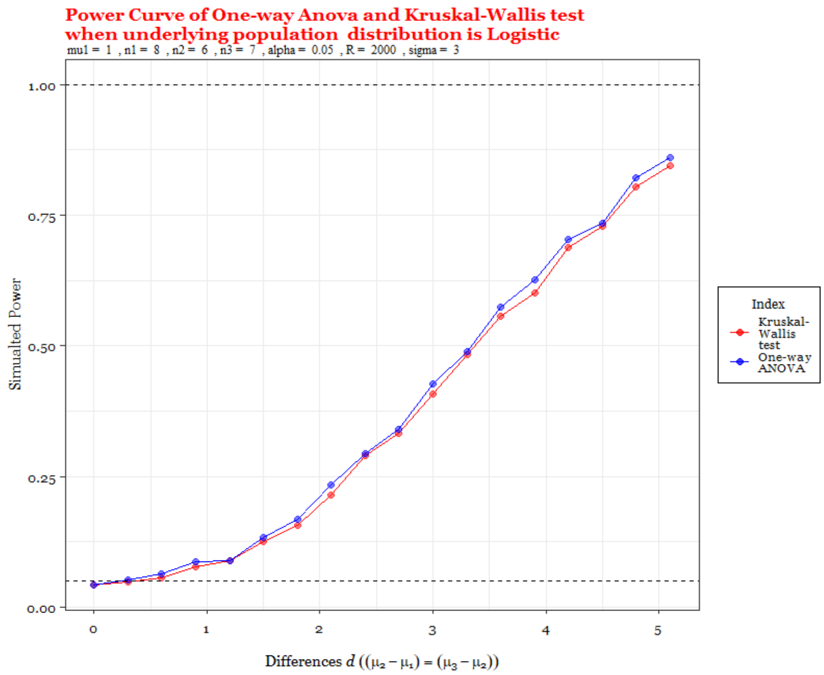

Figure 34: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

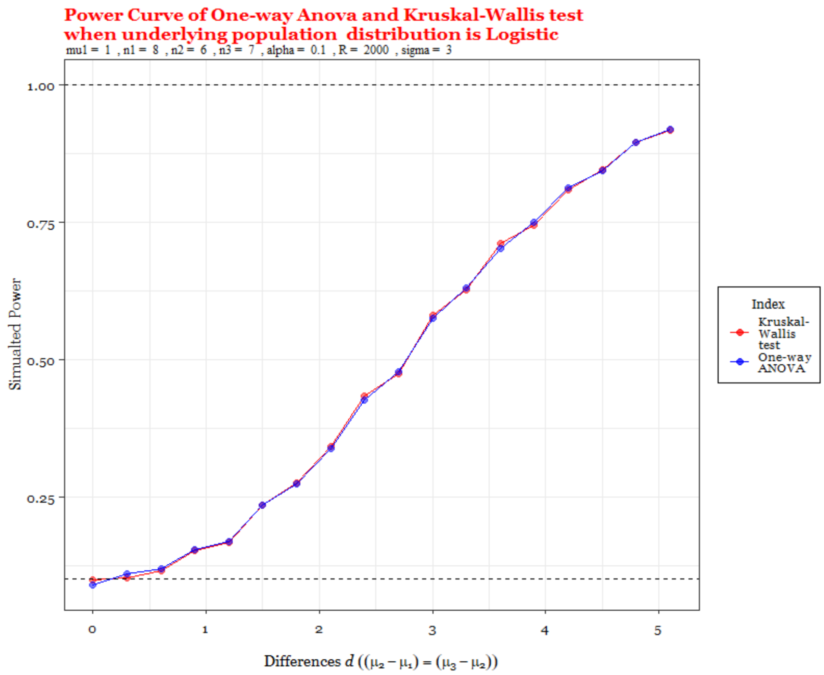

Figure 35: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us choose $n_1 = 50, n_2 = 50, n_3 = 50$ and see how the power values look like when group sizes are very large.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0585	0.0535	0.1105	0.1075
1	1.3	1.6	0.0635	0.0705	0.1285	0.1315
1	1.6	2.2	0.1520	0.1585	0.2455	0.2690
1	1.9	2.8	0.2825	0.3090	0.3990	0.4235
1	2.2	3.4	0.4885	0.5175	0.6150	0.6430
1	2.5	4.0	0.6810	0.7140	0.7795	0.8140
1	2.8	4.6	0.8465	0.8675	0.8970	0.9225
1	3.1	5.2	0.9355	0.9520	0.9685	0.9785
1	3.4	5.8	0.9765	0.9830	0.9870	0.9915
1	3.7	6.4	0.9950	0.9945	0.9980	0.9985
1	4.0	7.0	0.9990	1.0000	1.0000	1.0000
1	4.3	7.6	1.0000	1.0000	1.0000	1.0000
1	4.6	8.2	1.0000	1.0000	1.0000	1.0000
1	4.9	8.8	1.0000	1.0000	1.0000	1.0000
1	5.2	9.4	1.0000	1.0000	1.0000	1.0000
1	5.5	10.0	1.0000	1.0000	1.0000	1.0000
1	5.8	10.6	1.0000	1.0000	1.0000	1.0000
1	6.1	11.2	1.0000	1.0000	1.0000	1.0000

Table 16: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

Here, the power reaches 1 for both the tests more quickly than previous cases. Thus, as we increase the sample size power of both the test reaches 1 more quickly.

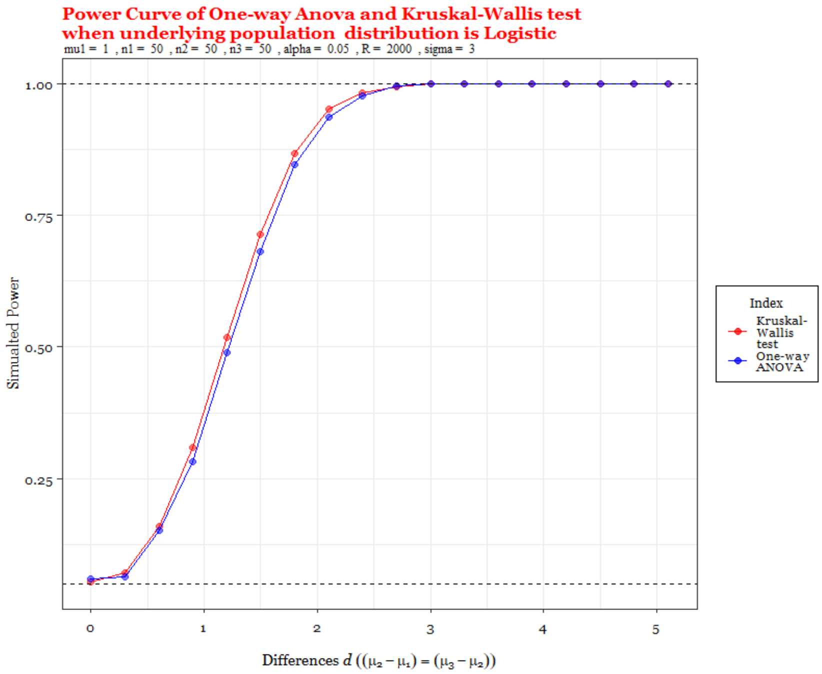

Figure 36: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

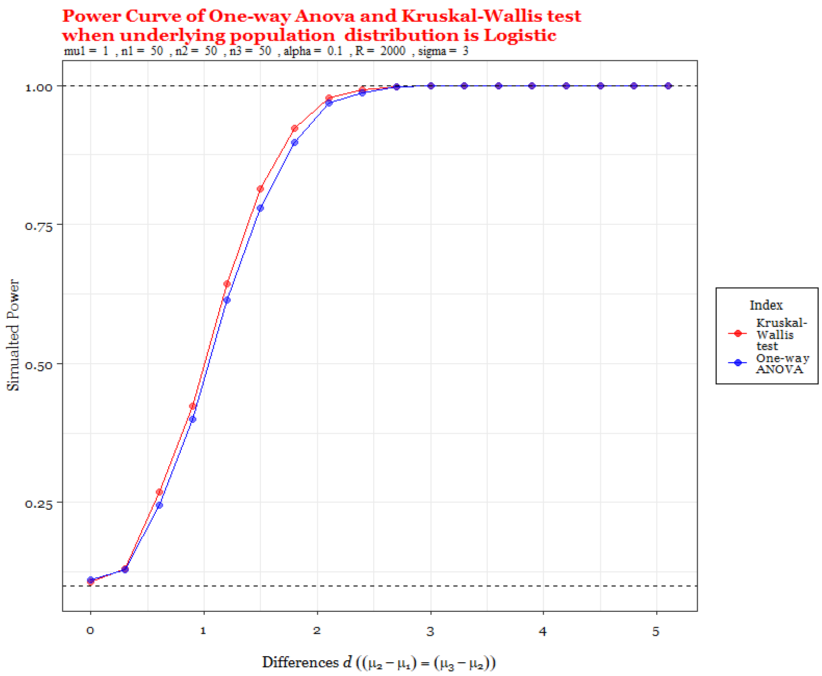

Figure 37: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Again from table-1 and table-2 we can see that remaining other parameters fixed, changing the

values of μ_1 and accordingly μ_2, μ_3 has no significant effect on the power. Let us now see how power value changes if we change the value of error variance (σ^2). For that we give the tabulated powers with $\mu_1 = 1, n_1 = 3, n_2 = 3, n_3 = 2$ and $\sigma = 1.2, 5.1$. And we have given the table only for level of significance = 0.05.

μ_1	μ_2	μ_3	Level of Significance ($\alpha = 0.05$)			
			$\sigma^2 = 1.2^2$		$\sigma^2 = 5.1^2$	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0470	0.0335	0.0470	0.0335
1	1.3	1.6	0.0525	0.0375	0.0480	0.0310
1	1.6	2.2	0.0585	0.0420	0.0445	0.0335
1	1.9	2.8	0.0885	0.0570	0.0500	0.0345
1	2.2	3.4	0.1350	0.0885	0.0640	0.0435
1	2.5	4.0	0.1765	0.1095	0.0470	0.0340
1	2.8	4.6	0.2420	0.1540	0.0585	0.0410
1	3.1	5.2	0.3050	0.1905	0.0610	0.0405
1	3.4	5.8	0.3685	0.2495	0.0495	0.0335
1	3.7	6.4	0.4510	0.3275	0.0610	0.0440
1	4.0	7.0	0.5255	0.3900	0.0715	0.0455
1	4.3	7.6	0.6190	0.4565	0.0660	0.0420
1	4.6	8.2	0.6840	0.5295	0.0925	0.0575
1	4.9	8.8	0.7480	0.6050	0.0880	0.0600
1	5.2	9.4	0.7835	0.6480	0.0950	0.0620
1	5.5	10.0	0.8295	0.6760	0.1085	0.0715
1	5.8	10.6	0.8820	0.7545	0.1095	0.0710
1	6.1	11.2	0.9250	0.8050	0.1315	0.0865

Table 17: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 1.2, 5.1$

Hence we see that for both ANOVA and Kruskal Wallis test, value of power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. Same thing we can show for level of signifiance = 0.1.

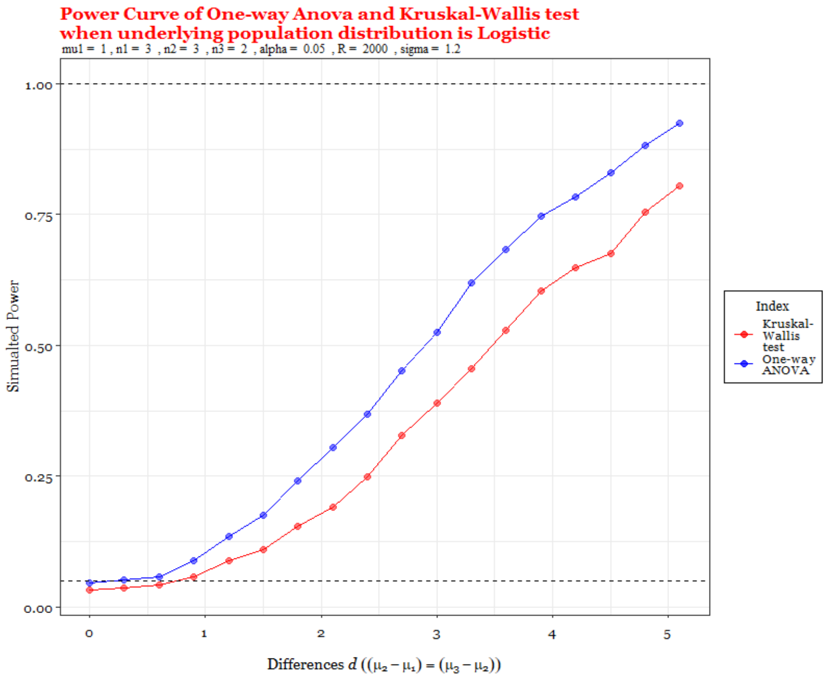

Figure 38: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 1.2$

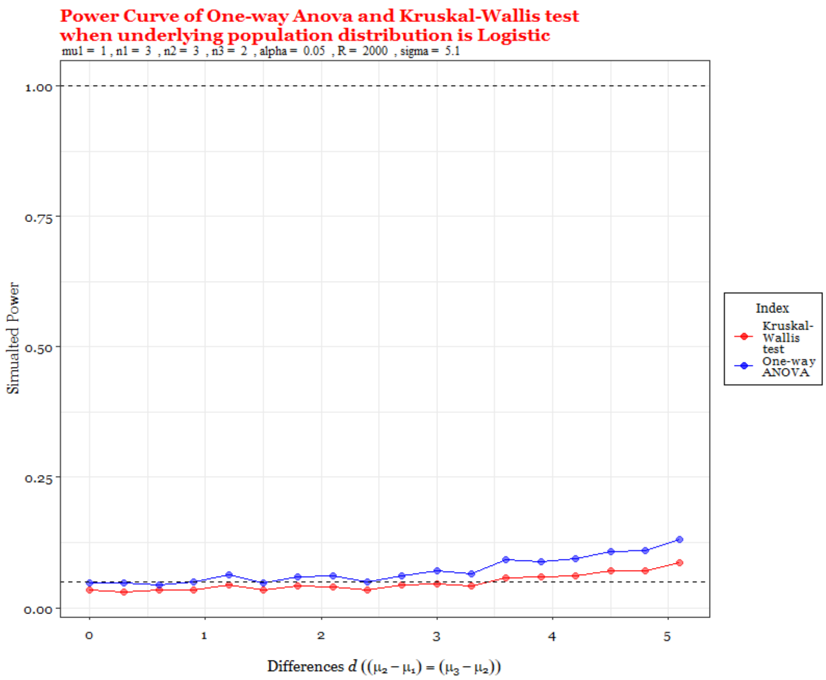

Figure 39: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 5.1$

Now let us try to see how values of power get affected when the assumption of homoscedasticity is violated. For calculation under heteroscedasticity, let us denote σ_i^2 as the error varinace corresponding to ith group, $i = 1,2,3$. We will tabulate the powers for $n_1 = 3, n_2 = 3, n_3 = 2, \mu_1 = 1, \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$ and for both level of significance $\alpha = 0.1, 0.05$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0670	0.0645	0.1175	0.1200
1	1.3	1.6	0.0560	0.0685	0.1145	0.1185
1	1.6	2.2	0.0865	0.0735	0.1315	0.1300
1	1.9	2.8	0.0885	0.0915	0.1510	0.1585
1	2.2	3.4	0.1160	0.1245	0.1820	0.2125
1	2.5	4.0	0.1480	0.1720	0.2285	0.2765
1	2.8	4.6	0.1760	0.1970	0.2760	0.3180
1	3.1	5.2	0.2430	0.2815	0.3480	0.4045
1	3.4	5.8	0.3025	0.3455	0.4085	0.4930
1	3.7	6.4	0.3675	0.4015	0.4730	0.5360
1	4.0	7.0	0.4140	0.4855	0.5550	0.6425
1	4.3	7.6	0.5105	0.5965	0.6460	0.7190
1	4.6	8.2	0.5805	0.6585	0.7025	0.7855
1	4.9	8.8	0.6435	0.7290	0.7545	0.8385
1	5.2	9.4	0.7145	0.7880	0.8080	0.8785
1	5.5	10.0	0.7650	0.8275	0.8530	0.9135
1	5.8	10.6	0.8050	0.8785	0.8890	0.9370
1	6.1	11.2	0.8510	0.9080	0.9110	0.9495

Table 18: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1, \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1, n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.05, 0.1$

Thus, we can see under heteroscedasticity, Kruskal-Wallis test has more power than One-way ANOVA. Hence, if the basic assumption of a parametric test (here assumption of normality and homoscedasticity) get violated, then it is better to choose a nonparametric test.

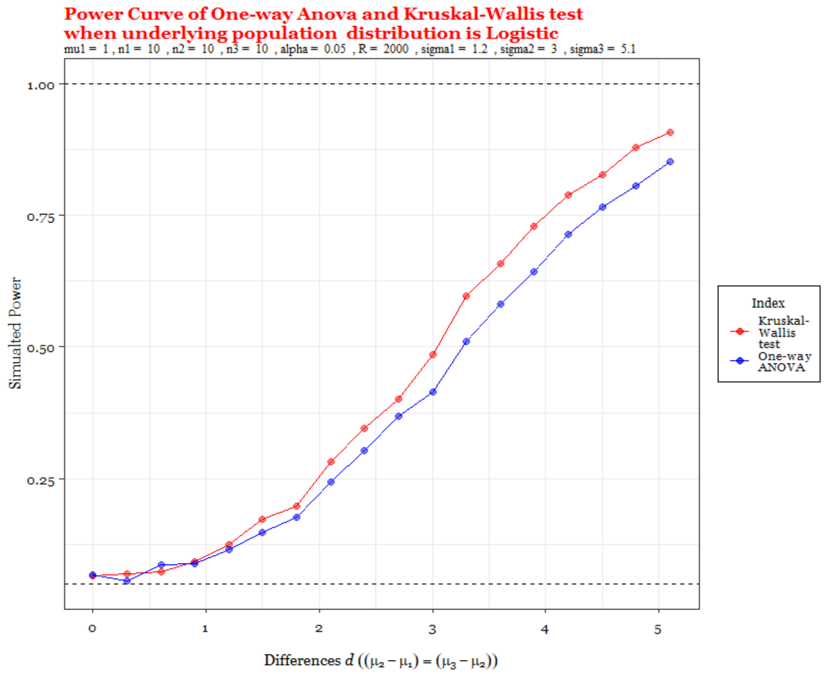

Figure 40: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.05,\ \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$

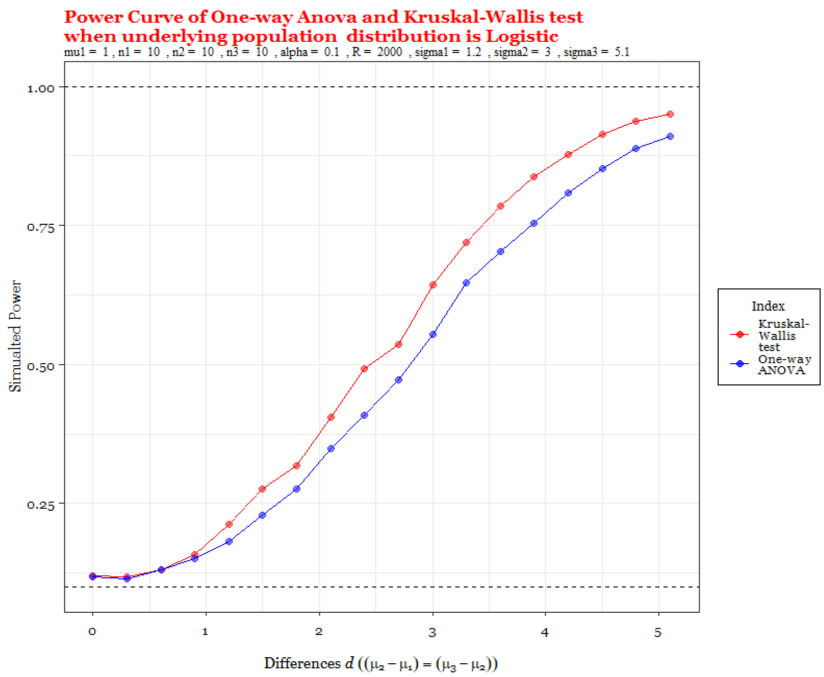

Figure 41: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 10, n_2 = 10,\ n_3 = 10,\ R = 2000,\ \alpha = 0.1,\ \sigma_1 = 1.2, \sigma_2 = 3, \sigma_3 = 5.1$

0.10.4 Observations

Now, we will summarize our observations from the above discussion below –

1. From the above discussion we have observed that when the underlying population distribution is logistic, one-way ANOVA performs better than Kruskal-Wallis test in terms of power for both small and large sample sizes. But under small group sizes, the values of power for both the tests are very small (really smaller than 1). However, from the table of values of powers, we can notice that the difference between the empirical powers of the two tests are not very high. As sample sizes, that is, group sizes increase the difference between the power of two test decreases and even sometimes it becomes really difficult to distinguish between ANOVA and Kruskal Wallis test in terms of power under large group sizes.

2. For small sample sizes (here, $n_1 = 3, n_2 = 3, n_3 = 2$) the power of both the tests reaches 1 very slowly. However, as sample size increases the power of both the tests reaches 1 very quickly. That is both test are consistent for testing our hypothesis problem.

3. The values of empirical power for both tests are slightly higher for higher values of level of significance (α). That is, for small level of significance (α) the power reaches 1 a bit slowly than large level of significance (α). In our discussion, for $\alpha = 0.10$ the power reaches 1 more quickly than $\alpha = 0.05$.

4. The power of both the tests do not depend upon the value of μ_1 (mean of 1st group or population). Other parameters remain fixed, if μ_1 changes, it doesn't change the value of power for both the tests.

5. Instead of μ_1, if we change the value of common error variance, i.e, value of σ^2 then we see that the values of empirical size and power for both tests change. Moreover we have seen that, for both ANOVA and Kruskal Wallis test, value of empirical power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. And as value of error variance increases, value of power decreases.

6. Under heteroscedasticity Kruskal Wallis performs better than one-way ANOVA. Since

for logistic distribution under heteroscedasticity, all the basic requirements of ANOVA i.e. normality and homoscedasticity are violated. But since Kruskal Wallis test does not require any such assumptions, it performs better than ANOVA under heteroscedasticity when underlying population distribution is logistic.

0.11 Simulation-4: Comparison of Empirical Size and Power of One-way ANOVA and Kruskal Test When Underlying Population Distribution is Lognormal

0.11.1 Objective

ANOVA and Kruskal Wallis are the most popular tests for testing equality of several group means. We wish to compare empirical size and power of One-way ANOVA and Kruskal Wallis Test when the underlying population distribution is lognormal. We will compare them for varying sample size, different level of significances (here we have considered $\alpha = 0.05, 0.1$), different values of μ_1, μ_2, μ_3 under alternative hypothesis and also when error variance is homoscedastic and heteroscedatic etc.

0.11.2 Algorithm

At first, we have stored the different values of the group means μ_1, μ_2, μ_3 under null and alternative hypothesis, in a matrix called *tab*. We have defined d as the difference between the group means that is $d = (\mu_2 - \mu_1) = (\mu_3 - \mu_2)$ and taken different choices of d. A fixed value of μ_1 is first chosen and then different values of μ_2 and μ_3 are generated as a function of μ_1 and d. In other words, we have taken,

$$\mu_2 = \mu_1 + d$$

$$\mu_2 = \mu_2 + d = \mu_1 + 2d$$

Then, we have used a user-defined function namely *Power*. The function takes different arguments as input and on the basis that, it calculates empirical size and power using the steps discussed in the previous section.

The different arguments of the function are –

1. R - No. of replications required, that is number of times to repeat the whole simulation process.

2. n_i - A vector with three elements whose elements represent the size of different groups (*i.e.* n_1, n_2, n_3 where n_1, n_2, n_3 are sizes of group-1, group-2 and group-3 respectively.)

3. mu_i - A vector with three elements whose elements represent different group means (*i.e.* μ_1, μ_2, μ_3 where μ_1, μ_2, μ_3 are means of group-1, group-2 and group-3 respectively).

4. Sigma - represents error variances. It is taken as a constant term under homoscedasticity and as a vector with three elements, whose elements represent group wise error variance (say, $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively) under heteroscedasticity.

5. alpha - represents level of significance of the test.

From the function we get empirical size and power as output. We have stored the values of empirical power and size in a data frame called *power.matrix*.Using that output we have drawn the empirical power curve using 'ggplot()' function. We will provide the codes for these functions at the end of the article.

0.11.3 Chosen values of group size, level of significance, different group means etc. for discussion

Here, for both large sample and small sample we have considered the following values of difference d, $d = 0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1$.

For different combinations of sample size and level of significance (in this study we have only considered 0.05 and 0.10) we will calculate the empirical size and power and will present them using table and graphs. We have used exact critical values from the table of Kruskal Wallis Statistic.

Primarily we have chosen the value of σ^2 to be 3^2 and also we have find empirical size and power for different values of $\sigma^2 = 1.2^2, 5.1^2$. Now for lognormal distribution the assumption of homoscedasticity is naturally violated because we have different values of μ_i' s for different groups. Hence, we do not need to consider the case of heteroscedasticity separately when underlying population distribution is lognormal. Also we have chosen two different values of μ_1 given by $\mu_1 = 1, 5$ and values of μ_2, μ_3 are accordingly generated.

For $n_1 = 3, n_2 = 3, n_3 = 2$, we will construct the table of empirical size and power of Kruskal-Wallis (KW) test and One-way ANOVA side by side for $\mu_1 = 1, 5$ and $\alpha = 0.10, 0.05$.

Let us first tabulate the powers for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0220	0.0325	0.0320	0.0860
1	1.3	1.6	0.0245	0.0310	0.0360	0.0905
1	1.6	2.2	0.0340	0.0365	0.0500	0.1040
1	1.9	2.8	0.0360	0.0565	0.0575	0.1230
1	2.2	3.4	0.0530	0.0540	0.0800	0.1350
1	2.5	4.0	0.0615	0.0725	0.0925	0.1705
1	2.8	4.6	0.0745	0.0755	0.1135	0.1845
1	3.1	5.2	0.0875	0.1035	0.1210	0.2245
1	3.4	5.8	0.0900	0.1225	0.1335	0.2785
1	3.7	6.4	0.0985	0.1410	0.1395	0.2955
1	4.0	7.0	0.1055	0.1890	0.1560	0.3720
1	4.3	7.6	0.1100	0.2135	0.1655	0.4065
1	4.6	8.2	0.1300	0.2535	0.1845	0.4585
1	4.9	8.8	0.1325	0.2940	0.2030	0.5195
1	5.2	9.4	0.1475	0.3480	0.2145	0.5760
1	5.5	10.0	0.1560	0.4170	0.2225	0.6425
1	5.8	10.6	0.1790	0.4615	0.2445	0.6790
1	6.1	11.2	0.1845	0.4870	0.2480	0.7065

Table 19: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

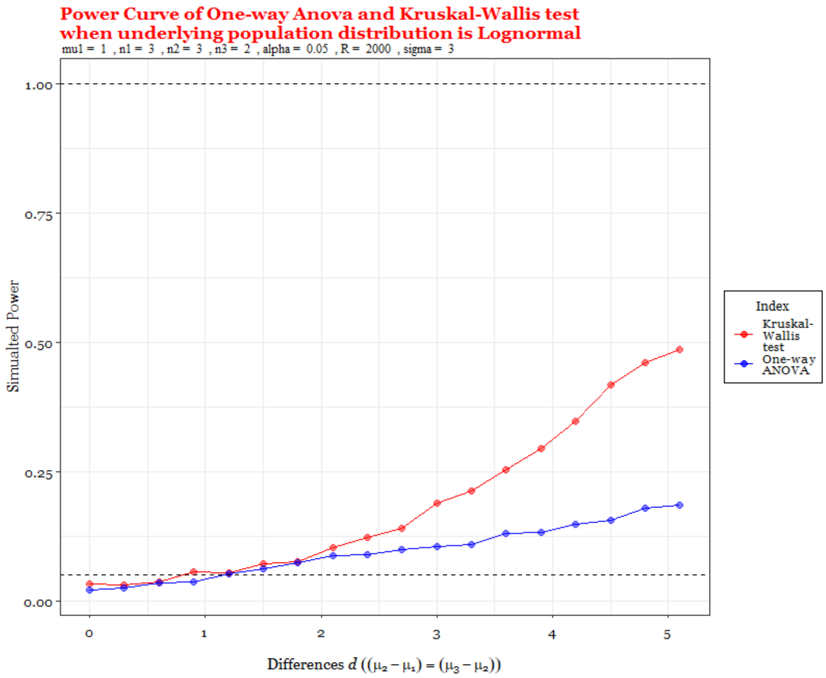

Figure 42: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

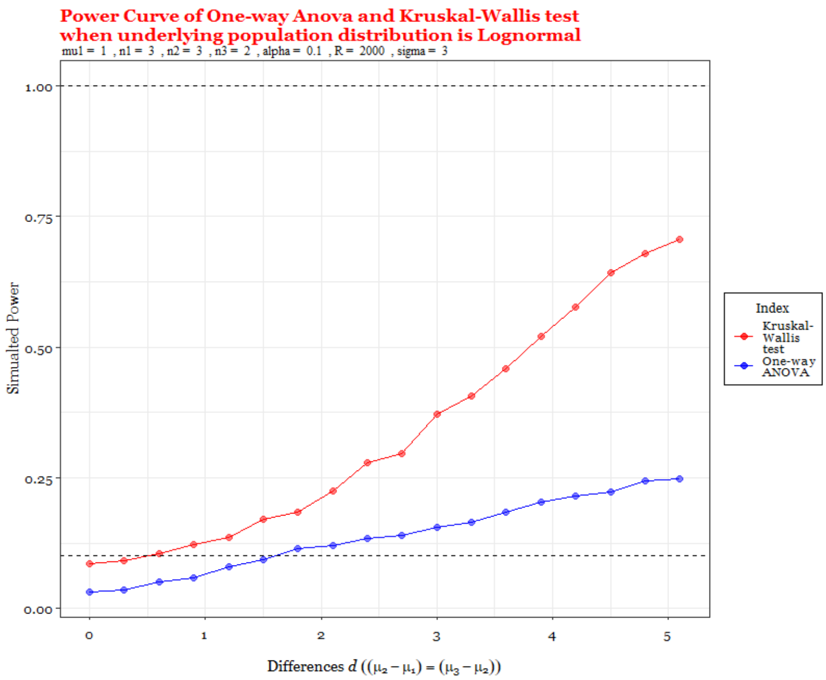

Figure 43: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now we tabulate the powers for $\mu_1 = 5, n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
5	5.0	5.0	0.0220	0.0325	0.0320	0.0860
5	5.3	5.6	0.0245	0.0310	0.0360	0.0905
5	5.6	6.2	0.0340	0.0365	0.0500	0.1040
5	5.9	6.8	0.0360	0.0565	0.0575	0.1230
5	6.2	7.4	0.0530	0.0540	0.0800	0.1350
5	6.5	8.0	0.0615	0.0725	0.0925	0.1705
5	6.8	8.6	0.0745	0.0755	0.1135	0.1845
5	7.1	9.2	0.0875	0.1035	0.1210	0.2245
5	7.4	9.8	0.0900	0.1225	0.1335	0.2785
5	7.7	10.4	0.0985	0.1410	0.1395	0.2955
5	8.0	11.0	0.1055	0.1890	0.1560	0.3720
5	8.3	11.6	0.1100	0.2135	0.1655	0.4065
5	8.6	12.2	0.1300	0.2535	0.1845	0.4585
5	8.9	12.8	0.1325	0.2940	0.2030	0.5195
5	9.2	13.4	0.1475	0.3480	0.2145	0.5760
5	9.5	14.0	0.1560	0.4170	0.2225	0.6425
5	9.8	14.6	0.1790	0.4615	0.2445	0.6790
5	10.1	15.2	0.1845	0.4870	0.2480	0.7065

Table 20: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

One thing to notice is that, for both level of significance $0.1, 0.05$, the power of both tests does not reach 1 for small group sizes for our choices of values. Values of both tests reaches 1 very slowly under small group sizes. Also, the power of Kruskal-Wallis test is higher than one-way ANOVA for any values of μ_1, μ_2, μ_3 and level of significance when the underlying population distribution is lognormal. Also as values of μ_1, μ_2, μ_3 increase, the differences between the values of power of Kruskal Wallis and ANOVA are becoming prominent.

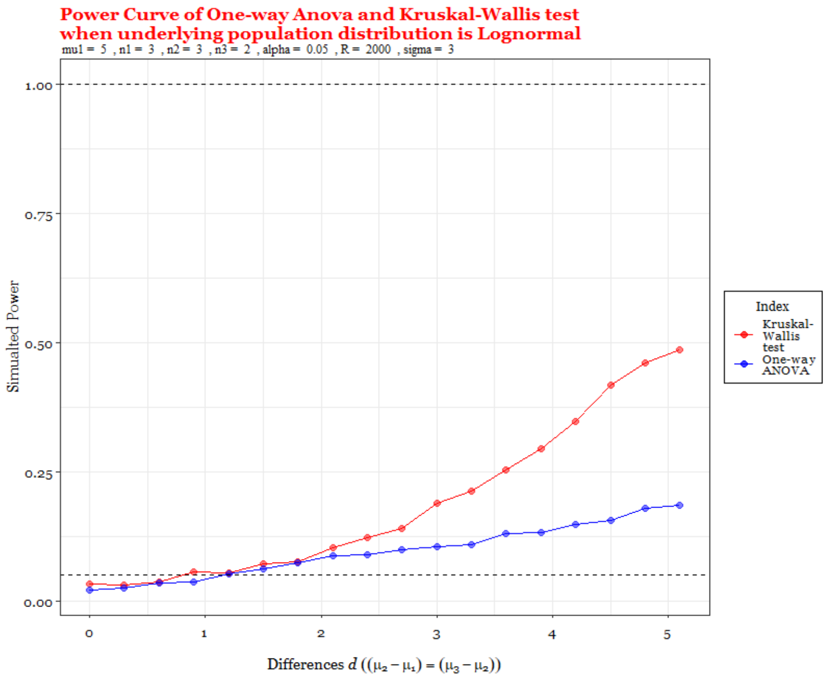

Figure 44: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

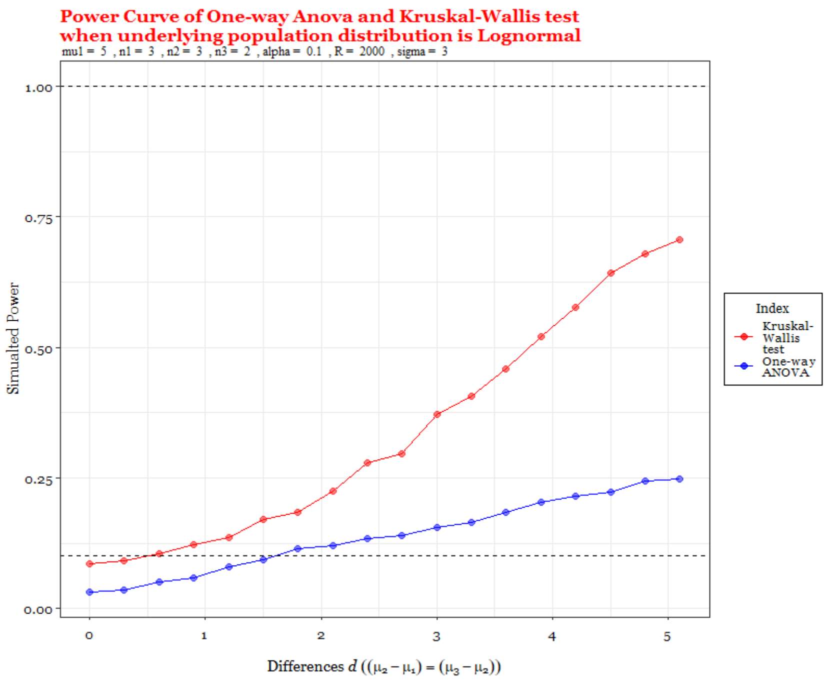

Figure 45: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us tabulate powers for large group sizes, for that we have taken $n_1 = 8, n_2 = 6, n_3 = 7$ and We have only presented the table for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0090	0.0445	0.0355	0.1010
1	1.3	1.6	0.0180	0.0530	0.0470	0.1105
1	1.6	2.2	0.0175	0.0755	0.0470	0.1565
1	1.9	2.8	0.0195	0.1340	0.0650	0.2275
1	2.2	3.4	0.0280	0.1985	0.0760	0.3210
1	2.5	4.0	0.0455	0.2895	0.1120	0.4375
1	2.8	4.6	0.0595	0.4100	0.1225	0.5710
1	3.1	5.2	0.0680	0.5660	0.1525	0.7165
1	3.4	5.8	0.0855	0.6800	0.2020	0.8010
1	3.7	6.4	0.0955	0.8030	0.2030	0.8945
1	4.0	7.0	0.1215	0.8700	0.2340	0.9400
1	4.3	7.6	0.1195	0.9235	0.2480	0.9660
1	4.6	8.2	0.1320	0.9570	0.2515	0.9830
1	4.9	8.8	0.1675	0.9830	0.3015	0.9935
1	5.2	9.4	0.1575	0.9905	0.2900	0.9985
1	5.5	10.0	0.1705	0.9970	0.3250	0.9995
1	5.8	10.6	0.1685	0.9985	0.3015	0.9995
1	6.1	11.2	0.1735	1.0000	0.3160	1.0000

Table 21: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

For large group sizes, power Kruskal-Wallis reaches 1 for both level of significance $0.1, 0.05$ but power of ANOVA still does not reaches 1. And in case of lognormal distribution, we see a significant difference betweem the values of power of the two tests. Also, if we interchange the values of n_1, n_2, n_3 we will see that the values of power will change. It may happen due to sample fluctuations.

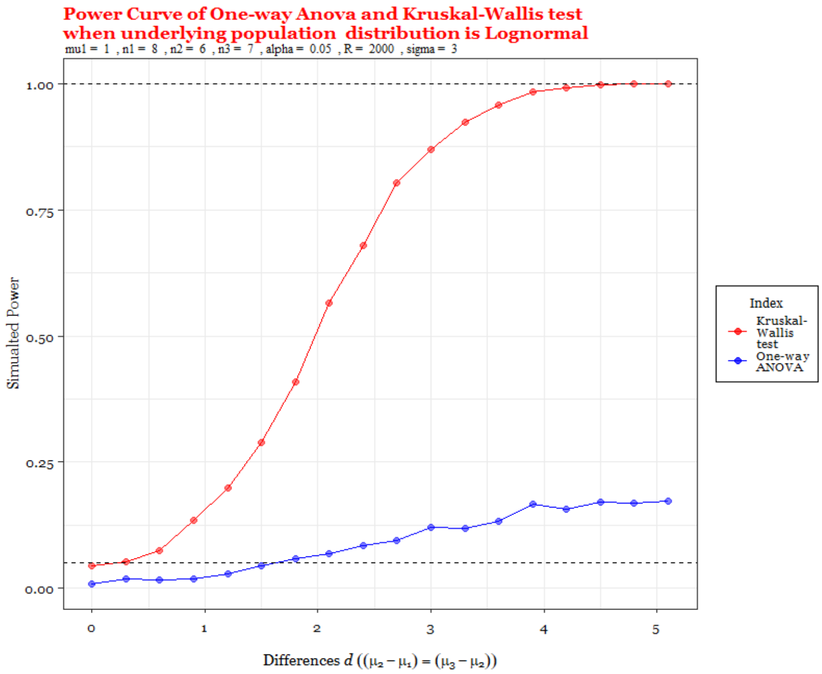

Figure 46: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

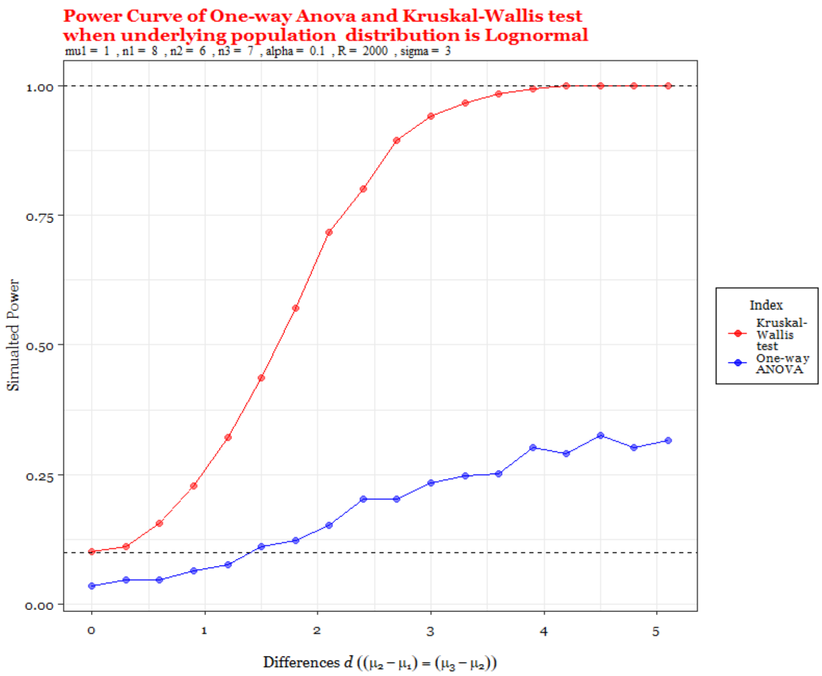

Figure 47: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Now let us choose $n_1 = 50, n_2 = 50, n_3 = 50$ and see how the power values look like when group sizes are very large.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0110	0.0580	0.0510	0.1095
1	1.3	1.6	0.0195	0.1230	0.0630	0.2050
1	1.6	2.2	0.0530	0.3825	0.1170	0.5240
1	1.9	2.8	0.0970	0.7270	0.1770	0.8175
1	2.2	3.4	0.1490	0.9420	0.2740	0.9715
1	2.5	4.0	0.2270	0.9935	0.3665	0.9970
1	2.8	4.6	0.2905	1.0000	0.4565	1.0000
1	3.1	5.2	0.3715	1.0000	0.5230	1.0000
1	3.4	5.8	0.4240	1.0000	0.5820	1.0000
1	3.7	6.4	0.4415	1.0000	0.5870	1.0000
1	4.0	7.0	0.4945	1.0000	0.6290	1.0000
1	4.3	7.6	0.5080	1.0000	0.6445	1.0000
1	4.6	8.2	0.5520	1.0000	0.6780	1.0000
1	4.9	8.8	0.5435	1.0000	0.6955	1.0000
1	5.2	9.4	0.5760	1.0000	0.7125	1.0000
1	5.5	10.0	0.5720	1.0000	0.7005	1.0000
1	5.8	10.6	0.5975	1.0000	0.7155	1.0000
1	6.1	11.2	0.6100	1.0000	0.7200	1.0000

Table 22: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05, 0.1,\ \sigma = 3$

Here, the power reaches 1 for Kruskal Wallis test more quickly than previous cases. But power of ANOVA does not reach 1 for very large group sizes also. Thus, we can say that Kruskal Wallis test is uniformly more powerful than ANOVA for testing equality of several group means with underlying population distribution being lognormal.

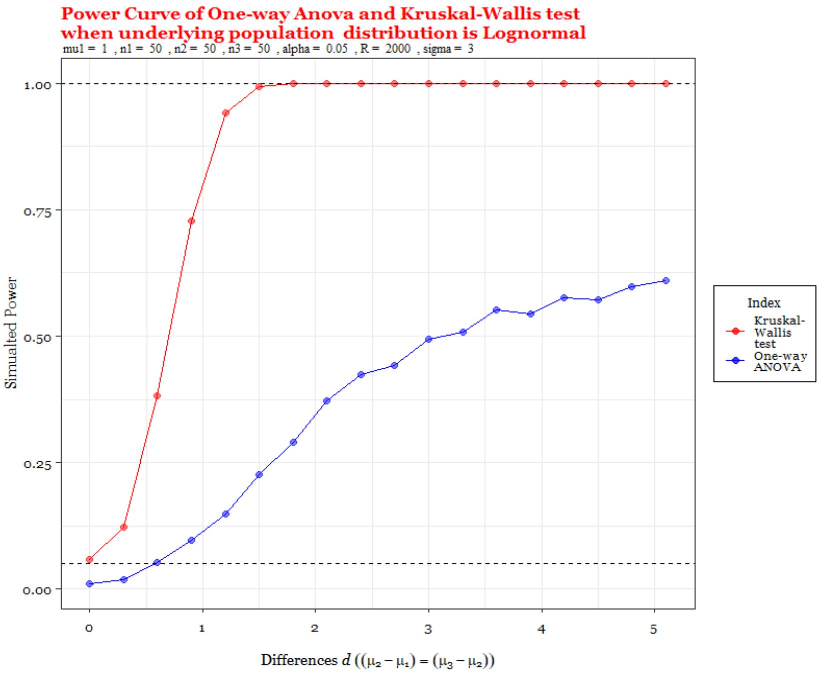

Figure 48: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05,\ \sigma = 3$

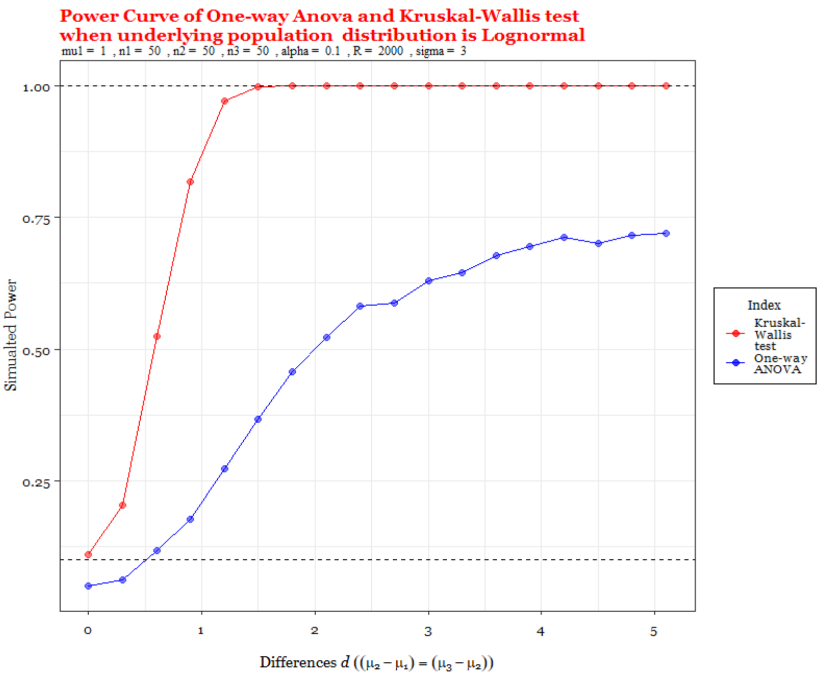

Figure 49: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.1,\ \sigma = 3$

Again from table-1 and table-2 we can see that remaining other parameters fixed, changing the

values of μ_1 has no effect on the power. Let us now see how power value changes if we change the value of error variance (σ^2). For that we give the tabulated powers with $\mu_1 = 1, n_1 = 3, n_2 = 3, n_3 = 2$ and $\sigma = 1.2, 5.1$. And we have given the table only for level of significance = 0.05.

μ_1	μ_2	μ_3	Level of Significance ($\alpha = 0.05$)			
			$\sigma^2 = 1.2^2$		$\sigma^2 = 5.1^2$	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0360	0.0325	0.0145	0.0325
1	1.3	1.6	0.0495	0.0365	0.0145	0.0275
1	1.6	2.2	0.1000	0.0665	0.0175	0.0295
1	1.9	2.8	0.1440	0.1210	0.0185	0.0460
1	2.2	3.4	0.2065	0.1845	0.0235	0.0345
1	2.5	4.0	0.2925	0.2960	0.0245	0.0440
1	2.8	4.6	0.3465	0.4085	0.0350	0.0455
1	3.1	5.2	0.3955	0.5460	0.0365	0.0490
1	3.4	5.8	0.4215	0.6590	0.0355	0.0545
1	3.7	6.4	0.4550	0.7395	0.0425	0.0575
1	4.0	7.0	0.4670	0.8485	0.0400	0.0735
1	4.3	7.6	0.4905	0.8950	0.0410	0.0855
1	4.6	8.2	0.5160	0.9460	0.0545	0.1035
1	4.9	8.8	0.5340	0.9670	0.0530	0.1085
1	5.2	9.4	0.5500	0.9790	0.0555	0.1245
1	5.5	10.0	0.5445	0.9910	0.0550	0.1365
1	5.8	10.6	0.5470	0.9930	0.0650	0.1575
1	6.1	11.2	0.5375	0.9990	0.0720	0.1780

Table 23: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05,\ \sigma = 1.2, 5.1$

Hence we see that for both ANOVA and Kruskal Wallis test, value of power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. Same thing we can show for level of signifiance = 0.1.

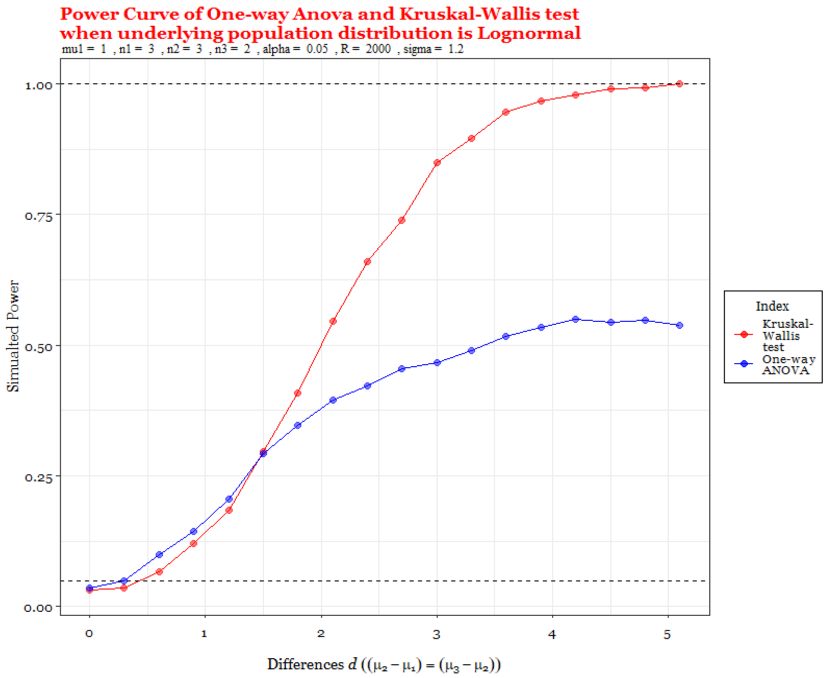

Figure 50: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 1.2$

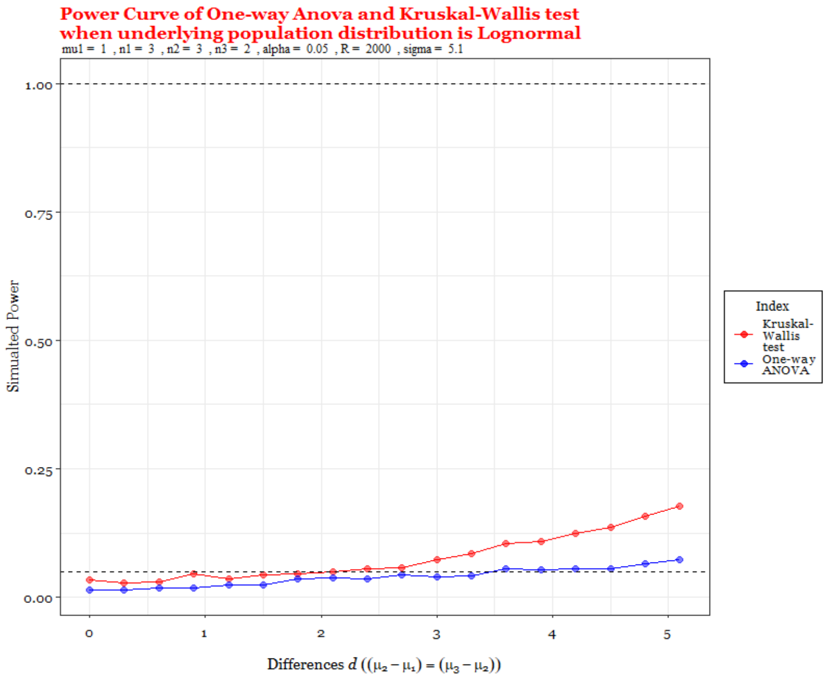

Figure 51: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05\ \sigma = 5.1$

Now we have already mentioned that for lognormal distribution the assumption of homoscedasticity is naturally violated because we have different values of μ_i' s for different groups. Hence, we do not need to consider the case of heteroscedasticity separately when underlying population distribution is lognormal.

0.11.4 Observations

Now, we will summarize our observations from the above discussion below –

1. From the above discussion we have observed that when the underlying population distribution is lognormal, Kruskal Wallis test performs better than one-way ANOVA in terms of power, i.e., Kruskal Wallis test is able to detect the same difference between the group means of three independent lognormal population with same variance more frequently than ANOVA. This is may be happening because in case of lognormal distribution, not only the assumption of normally distributed data is violated but also the assumption of homoscedasticity is also violated and these two requirements are basic requirements of one-way ANOVA test. Also, from the table of values of powers, we can notice a significant difference between the empirical powers of the two tests (power of Kruskal Wallis test $>>$ power of ANOVA). As group sizes increase the difference between the power of two test also increases. It is clear from the graph also.

2. For small sample sizes (here, $n_1 = 3, n_2 = 3, n_3 = 2$) the power of both the tests reache 1 very slowly. However, as sample size increases the power of Kruskal Wallis test reaches 1 very quickly but power of ANOVA does not reach 1 quickly in case of large group sizes also. That is Kruskal Wallis test is consistent for testing our hypothesis problem under lognormal population but ANOVA is not consistent in this case.

3. The values of empirical power for both tests are slightly higher for higher values of level of significance (α). That is, for small level of significance (α) the power reaches 1 a bit slowly than large level of significance (α). In our discussion, for $\alpha = 0.10$ the power reaches 1 more quickly than $\alpha = 0.05$.

4. The power of both the tests do not depend upon the value of μ_1 (mean of 1st group or population). Other parameters remain fixed, if μ_1 changes, it does not change the value

of power for both the tests.

5. Instead of μ_1, if we change the value of common error variance, i.e, value of σ^2 then we see that the values of empirical size and power for both tests change. Moreover we have seen that, for both ANOVA and Kruskal Wallis test, value of empirical power and value of error variance (σ^2) are inversely proportionate. That is, as value of error variance decreases, value of power increases. And as value of error variance increases, value of power decreases.

0.12 Simulation-5: Comparison of Empirical Size and Power of One-way ANOVA and Kruskal Test When Underlying Population Distribution is Exponential

0.12.1 Objective

ANOVA and Kruskal Wallis are the most popular tests for testing equality of several group means. We wish to compare empirical size and power of One-way ANOVA and Kruskal Wallis Test when the underlying population distribution is exponential. We will compare them for varying sample size, different level of significance (here we have considered $\alpha = 0.05, 0.1$), different values of μ_1, μ_2, μ_3 under alternative hypothesis and also when error variance is homoscedastic and heteroscedatic etc.

0.12.2 Algorithm

At first, we have stored the different values of the group means μ_1, μ_2, μ_3 under null and alternative hypothesis, in a matrix called *tab*. We have defined d as the difference between the group means that is $d = (\mu_2 - \mu_1) = (\mu_3 - \mu_2)$ and taken different choices of d. A fixed value of μ_1 is first chosen and then different values of μ_2 and μ_3 are generated as a function of μ_1 and d. In other words, we have taken,

$$\mu_2 = \mu_1 + d$$

$$\mu_2 = \mu_2 + d = \mu_1 + 2d$$

Then, we have used a user-defined function namely *Power*. The function takes different arguments as input and on the basis that, it calculates empirical size and power using the steps discussed in the previous section.

The different arguments of the function are –

1. R - No. of replications required, that is number of times to repeat the whole simulation process.

2. n_i - A vector with three elements whose elements represent the size of different groups (*i.e.* n_1, n_2, n_3 where n_1, n_2, n_3 are sizes of group-1, group-2 and group-3 respectively.)

3. mu_i - A vector with three elements whose elements represent different group means (*i.e.* μ_1, μ_2, μ_3 where μ_1, μ_2, μ_3 are means of group-1, group-2 and group-3 respectively).

4. Sigma - represents error variances. It is taken as a constant term under homoscedasticity and as a vector with three elements, whose elements represent group wise error variance (say, $\sigma_1^2, \sigma_2^2, \sigma_3^2$ be the error variances of group-1, group-2 and group-3 respectively) under heteroscedasticity.

5. alpha - represents level of significance of the test.

From the function we get empirical size and power as output. We have stored the values of empirical power and size in a data frame called *power.matrix*.Using that output we have drawn the empirical power curve using 'ggplot()' function. We will provide the codes for these functions at the end of the article.

0.12.3 Chosen values of group size, level of significance, different group means etc. for discussion

Here, for both large sample and small sample we have considered the following values of difference d, $d = 0, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0, 3.3, 3.6, 3.9, 4.2, 4.5, 4.8, 5.1$.

For different combinations of sample size and level of significance (in this study we have only considered 0.05 and 0.10) we will calculate the empirical size and power and will present them using table and graphs. We have used exact critical values from the table of Kruskal Wallis Statistic.

Here μ_i^2' s are basically the variances corresponding to the model. There is no term involving σ^2, hence we do not need to tabulate values of power for different σ^2.

Also, we have already mentioned that for exponential distribution the assumption of homoscedasticity is naturally violated because we have different values of μ_i' s for different groups. Hence, we do not need to consider the case of heteroscedasticity separately when underlying population distribution is exponential. Also we have chosen two different values of

μ_1 given by $\mu_1 = 1, 5$ and values of μ_2, μ_3 are accordingly generated.

For $n_1 = 3, n_2 = 3, n_3 = 2$, we will construct the table of empirical size and power of Kruskal-Wallis (KW) test and One-way ANOVA side by side for $\mu_1 = 1, 5$ and $\alpha = 0.10, 0.05$.

Let us first tabulate the powers for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0515	0.0325	0.0975	0.1005
1	1.3	1.6	0.0755	0.0365	0.1155	0.1060
1	1.6	2.2	0.0990	0.0495	0.1605	0.1370
1	1.9	2.8	0.1110	0.0655	0.1770	0.1635
1	2.2	3.4	0.1220	0.0800	0.2000	0.1800
1	2.5	4.0	0.1475	0.0825	0.2290	0.2165
1	2.8	4.6	0.1660	0.0990	0.2510	0.2500
1	3.1	5.2	0.1705	0.1175	0.2660	0.2790
1	3.4	5.8	0.1900	0.1415	0.2860	0.3105
1	3.7	6.4	0.1960	0.1295	0.2790	0.3040
1	4.0	7.0	0.1890	0.1500	0.2870	0.3275
1	4.3	7.6	0.2130	0.1610	0.3250	0.3765
1	4.6	8.2	0.2035	0.1775	0.3025	0.3755
1	4.9	8.8	0.2270	0.1830	0.3410	0.4050
1	5.2	9.4	0.2130	0.1770	0.3285	0.4070
1	5.5	10.0	0.2335	0.1945	0.3380	0.4520
1	5.8	10.6	0.2405	0.2220	0.3670	0.4720
1	6.1	11.2	0.2460	0.2170	0.3620	0.4660

Table 24: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1$

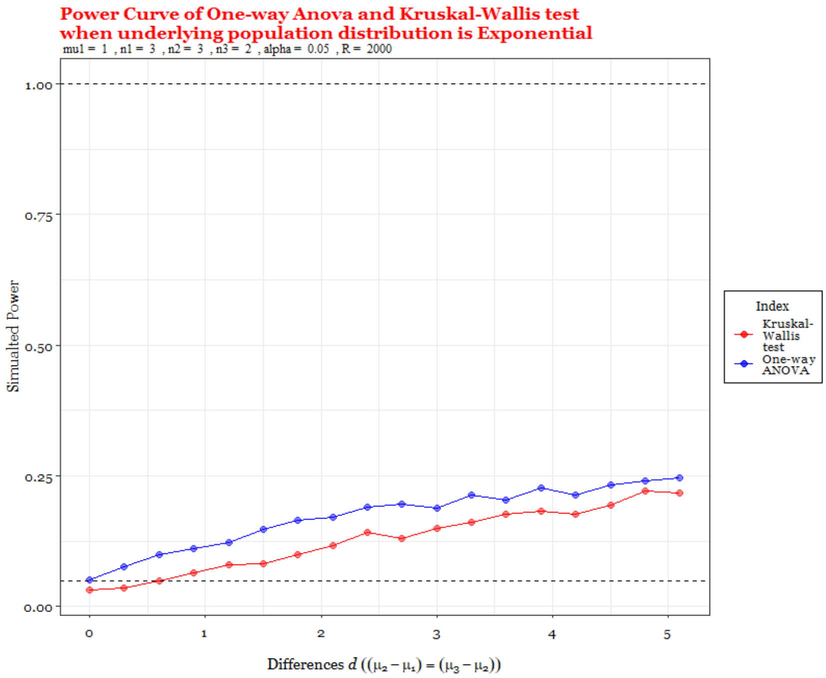

Figure 52: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05$

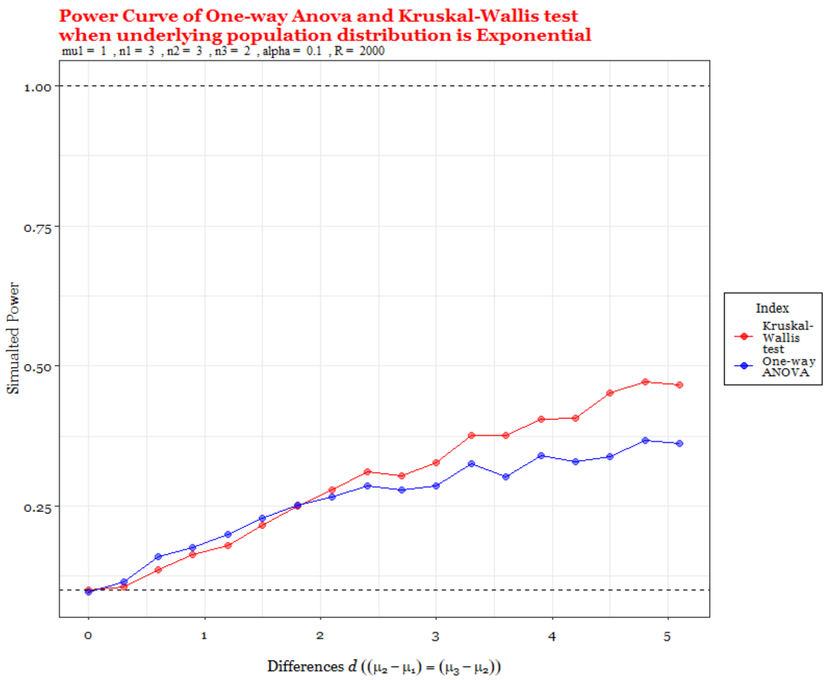

Figure 53: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1$

Now we tabulate the powers for $\mu_1 = 5, n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
5	5.0	5.0	0.0515	0.0325	0.0975	0.1005
5	5.3	5.6	0.0565	0.0320	0.0975	0.1035
5	5.6	6.2	0.0600	0.0350	0.1065	0.1010
5	5.9	6.8	0.0550	0.0360	0.0985	0.1020
5	6.2	7.4	0.0615	0.0345	0.1025	0.0970
5	6.5	8.0	0.0630	0.0405	0.1115	0.1085
5	6.8	8.6	0.0755	0.0465	0.1250	0.1260
5	7.1	9.2	0.0810	0.0360	0.1370	0.1135
5	7.4	9.8	0.0775	0.0480	0.1460	0.1250
5	7.7	10.4	0.0910	0.0350	0.1405	0.1150
5	8.0	11.0	0.0805	0.0445	0.1405	0.1195
5	8.3	11.6	0.0920	0.0550	0.1570	0.1405
5	8.6	12.2	0.0995	0.0535	0.1515	0.1470
5	8.9	12.8	0.1005	0.0575	0.1720	0.1440
5	9.2	13.4	0.1060	0.0515	0.1725	0.1450
5	9.5	14.0	0.1110	0.0610	0.1865	0.1555
5	9.8	14.6	0.1185	0.0740	0.1940	0.1850
5	10.1	15.2	0.1235	0.0685	0.1950	0.1675

Table 25: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05, 0.1$

One thing to notice is that, for both level of significance $0.1, 0.05$, the power of both tests does not reach 1 for small group sizes for our choices of values. Values of both tests reaches 1 very slowly under small group sizes. Also, under $\mu_1 = 1$ the power of one-way ANOVA is higher than Kruskal-Wallis test for level of significance $\alpha = 0.05$ whereas the power of Kruskal-Wallis test is higher than one-way ANOVA for $\alpha = 0.1$. But power of both tests are very low for small group sizes.

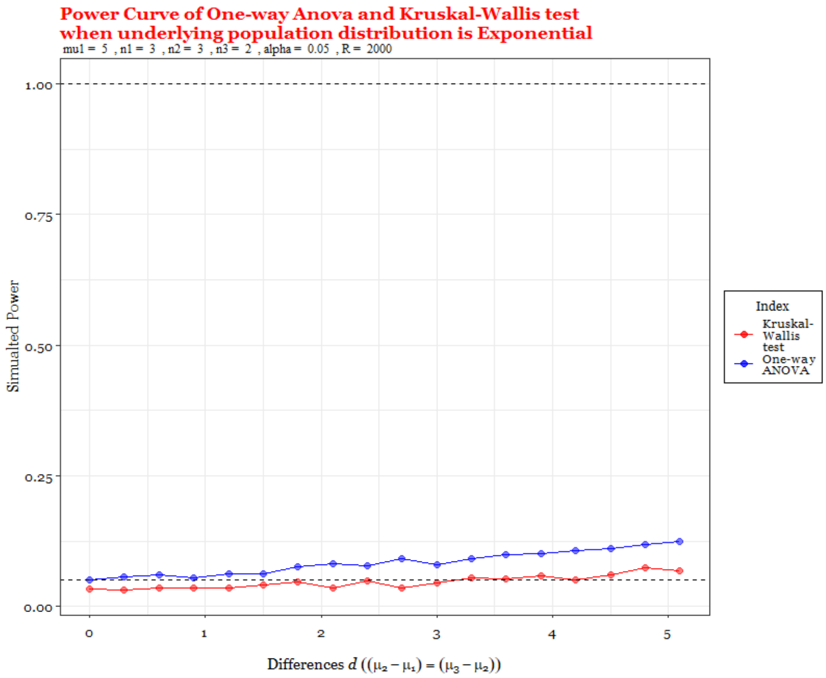

Figure 54: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.05$

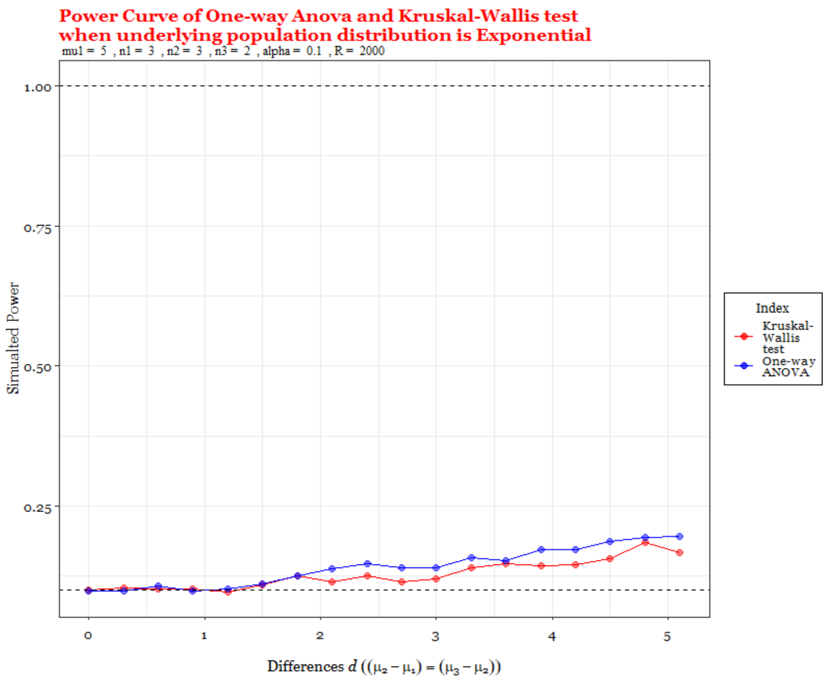

Figure 55: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 5$, $n_1 = 3, n_2 = 3,\ n_3 = 2,\ R = 2000,\ \alpha = 0.1$

Now let us tabulate powers for large group sizes, for that we have taken $n_1 = 8, n_2 = 6, n_3 = 7$ and We have only presented the table for $\mu_1 = 1$.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0415	0.0450	0.0870	0.1000
1	1.3	1.6	0.0905	0.0935	0.1720	0.1720
1	1.6	2.2	0.1575	0.1650	0.2730	0.2760
1	1.9	2.8	0.2355	0.2485	0.3880	0.3860
1	2.2	3.4	0.3330	0.3495	0.4995	0.4985
1	2.5	4.0	0.4035	0.4440	0.5960	0.6080
1	2.8	4.6	0.4500	0.5135	0.6560	0.6650
1	3.1	5.2	0.4880	0.5755	0.6915	0.7280
1	3.4	5.8	0.5505	0.6565	0.7345	0.7910
1	3.7	6.4	0.5660	0.7035	0.7670	0.8255
1	4.0	7.0	0.5960	0.7285	0.7780	0.8505
1	4.3	7.6	0.6270	0.7920	0.8055	0.8820
1	4.6	8.2	0.6525	0.8070	0.8250	0.9055
1	4.9	8.8	0.6580	0.8325	0.8475	0.9155
1	5.2	9.4	0.6950	0.8545	0.8700	0.9300
1	5.5	10.0	0.6945	0.8745	0.8655	0.9485
1	5.8	10.6	0.6855	0.8860	0.8555	0.9550
1	6.1	11.2	0.7335	0.8975	0.8830	0.9600

Table 26: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05, 0.1$

For large group sizes, power of Kruskal-Wallis reaches nearer to 1 for both level of significance $0.1, 0.05$ quickly than one-way ANOVA. And for large group sizes, Kruskal Wallis test has more power than ANOVA in case of exponential distribution. Also we see a significant difference between the values of power of the two tests as values of μ_1, μ_2, μ_3 increases. Also, if we interchange the values of n_1, n_2, n_3 we will see that the values of power will change. It may happen due to sample fluctuations.

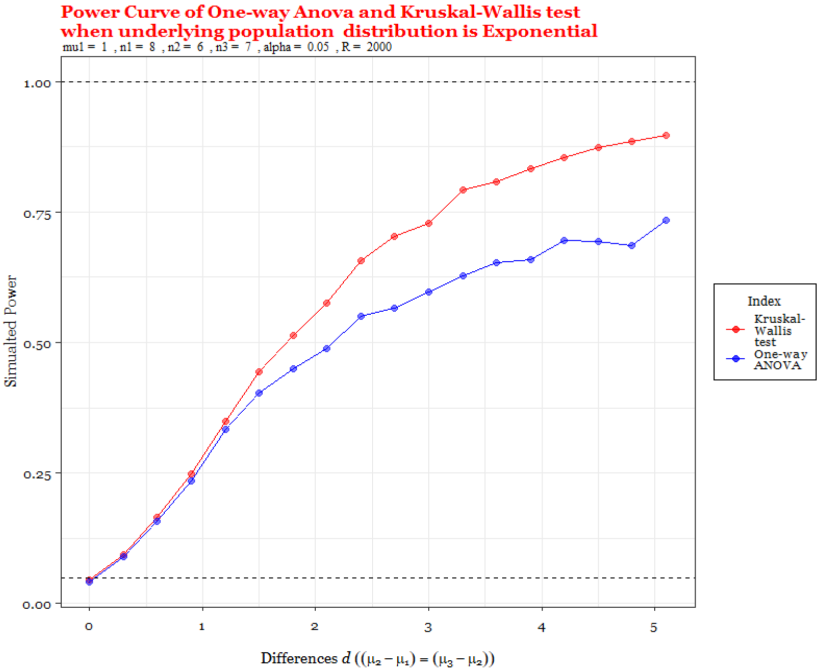

Figure 56: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.05$

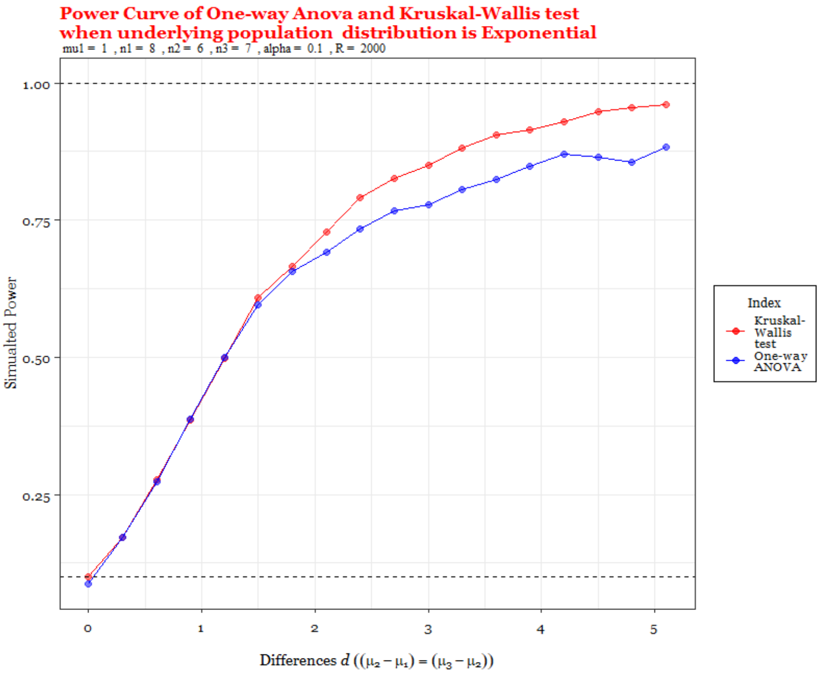

Figure 57: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 8, n_2 = 6,\ n_3 = 7,\ R = 2000,\ \alpha = 0.1$

Now let us choose $n_1 = 50, n_2 = 50, n_3 = 50$ and see how the power values look like when group sizes are very large.

μ_1	μ_2	μ_3	Level of Significance (α)			
			0.05		0.1	
			ANOVA	KW	ANOVA	KW
1	1.0	1.0	0.0505	0.0485	0.0985	0.0935
1	1.3	1.6	0.5080	0.4105	0.6395	0.5415
1	1.6	2.2	0.9335	0.8530	0.9675	0.9295
1	1.9	2.8	0.9980	0.9820	0.9985	0.9930
1	2.2	3.4	0.9995	0.9990	1.0000	1.0000
1	2.5	4.0	1.0000	0.9995	1.0000	1.0000
1	2.8	4.6	1.0000	1.0000	1.0000	1.0000
1	3.1	5.2	1.0000	1.0000	1.0000	1.0000
1	3.4	5.8	1.0000	1.0000	1.0000	1.0000
1	3.7	6.4	1.0000	1.0000	1.0000	1.0000
1	4.0	7.0	1.0000	1.0000	1.0000	1.0000
1	4.3	7.6	1.0000	1.0000	1.0000	1.0000
1	4.6	8.2	1.0000	1.0000	1.0000	1.0000
1	4.9	8.8	1.0000	1.0000	1.0000	1.0000
1	5.2	9.4	1.0000	1.0000	1.0000	1.0000
1	5.5	10.0	1.0000	1.0000	1.0000	1.0000
1	5.8	10.6	1.0000	1.0000	1.0000	1.0000
1	6.1	11.2	1.0000	1.0000	1.0000	1.0000

Table 27: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05, 0.1$

Here, the power reaches 1 for both the tests more quickly than previous cases. Also, for these group sizes we do not see any significant difference between the values of power of ANOVA and Kruskal Wallis. So, for very large group sizes both the tests have more or less similar performances in terms of power.

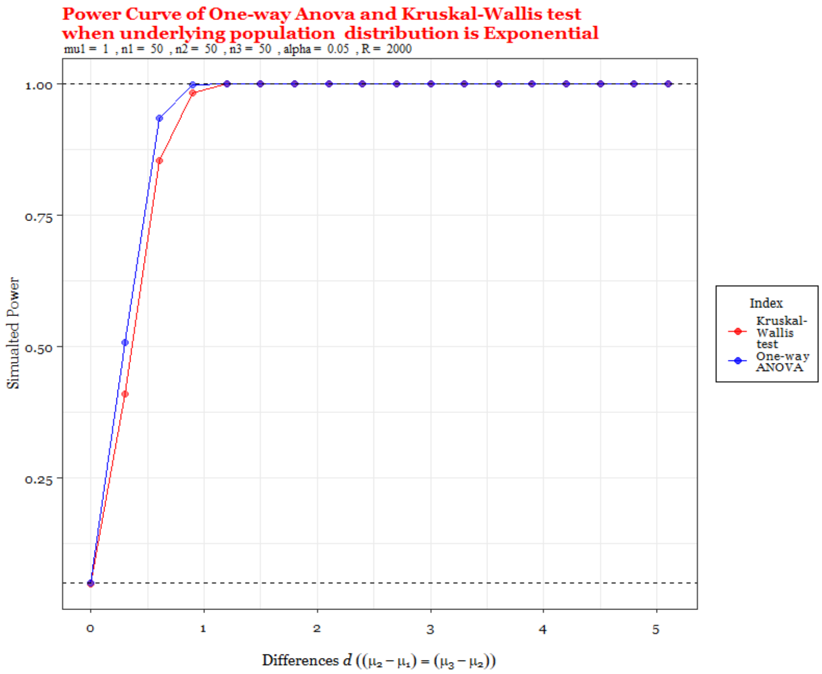

Figure 58: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.05$

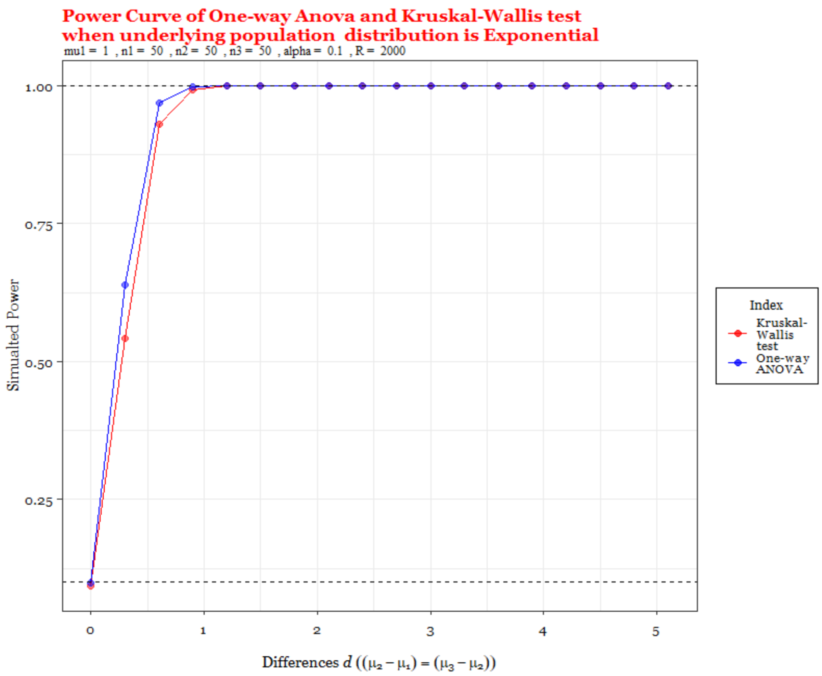

Figure 59: Empirical size and Power of ANOVA and Kruskal Wallis(KW) Test for $\mu_1 = 1$, $n_1 = 50, n_2 = 50,\ n_3 = 50,\ R = 2000,\ \alpha = 0.1$

Again from table-1 and table-2 we can see that remaining other parameters fixed, changing the values of μ_1 has significant effect on the power of both tests. Since for a *Exponential*(μ) distribution μ is the scale parameter, we see that as value of μ_1 and accordingly μ_2, μ_3 increases the value of power for both test decreases. Again as values of μ_1, μ_2, μ_3 decreases, value of power for both the test increases. So, values of μ_1, μ_2, μ_3 and power for both the test are inversely proportionate.

Here μ_i^2' s are basically the variances corresponding to the model. There is no term involving σ^2, hence we do not need to tabulate values of power for different σ^2.

Also, we have already mentioned that for exponential distribution the assumption of homoscedasticity is naturally violated because we have different values of μ_i' s for different groups. Hence, we do not need to consider the case of heteroscedasticity separately when underlying population distribution is exponential.

0.12.4 Observations

Now, we will summarize our observations from the above discussion below –

1. From the above discussion we have observed that when the underlying population distribution is exponential, one-way ANOVA performs better than Kruskal Wallis test in terms of power for $\mu_1 = 1$ and $\alpha = 0.05$ whereas Kruskal Wallis test performs better than one-way ANOVA for $\mu_1 = 1$ and $\alpha = 0.1$ under small group sizes. Again for large group sizes that is, $n_1 = 8, n_2 = 6, n_3 = 7$, Kruskal Wallis test has more power than ANOVA. So, for large group sizes Kruskal Wallis test performs better than ANOVA but in case of very large sample $n_1 = n_2 = n_3 = 50$, both the tests has more or less similar values of power. This is may be happening because in case of exponential distribution, not only the assumption of normally distributed data is violated but also the assumption of homoscedasticity is also violated and these two assumptions are basic requirements of one-way ANOVA test.

2. For small sample sizes (here, $n_1 = 3, n_2 = 3, n_3 = 2$) the power of both the tests reach 1 very slowly. However, as sample size increases the power of Kruskal Wallis test reaches 1 quickly than ANOVA. Again for very large sample size, $n_1 = n_2 = n_3 = 50$, power of

both tests reach 1 very quickly.

3. The values of empirical power for both tests are slightly higher for higher values of level of significance (α). That is, for small level of significance (α) the power reaches 1 a bit slowly than large level of significance (α). In our discussion, for $\alpha = 0.10$ the power reaches 1 more quickly than $\alpha = 0.05$.

4. The power of both the tests depend upon the values of μ_1 and in case of our alternative hypothesis as value of μ_1 increases, values of μ_2, μ_3 also increase. Now for exponential distribution μ_i' s are scale parameter. So, as μ_i' s change, values of variances corresponding to the model also changes. Moreover we have seen that, for both ANOVA and Kruskal Wallis test, value of empirical power and value of μ_1 and accordingly values of μ_2, μ_3 are inversely proportionate. That is, as values of μ_1, μ_2, μ_3 decreases (keeping the condition $\mu_1 < \mu_2 < \mu_3$ fixed), value of power increases. And as values of μ_1, μ_2, μ_3 increases (keeping the condition $\mu_1 < \mu_2 < \mu_3$ fixed), value of power decreases.

Conclusion:

For testing of equality of means under normal distribution, one-way ANOVA is the uniformly most powerful test. Even if only the assumption of homoscedasticity is violated, it does not effect the power of ANOVA much. Under heteroscedasticity, we see that ANOVA and Kruskal-Wallis test give more or less similar performance for normal distribution.

For non-normal symmetric distributions like laplace and logistic, we see still one-way ANOVA is performing better than Kruskal Wallis test for small group sizes. Moreover under logistic distribution, ANOVA has more power than Kruskal Wallis test for large group sizes also, whereas under laplace distribution, as sample size increases, power of Kruskal Wallis test increases. And for non-normal symmetric distribution if the assumption of homoscedasticity is also violated, we see Kruskal Wallis test has more power than one-way ANOVA. This is may be happening because, here both the basic requirements of the parametric test ANOVA are violated and hence the nonparametric test Kruskal Wallis, which does not require any such assumptions, performs better in this case.

For non-normal asymmetric distributions like lognormal and exponential, Kruskal Wallis test outperforms one-way ANOVA in most of the cases. For lognormal distribution, Kruskal Wallis test has more power than ANOVA in every considered situations. Also we have seen that ANOVA is not a consistent test for testing equality of means under lognormal distribution because its power does not reach 1 even if we take the group sizes to be very large. On the other hand, under exponential distribution we see that Kruskal Wallis test has more power than ANOVA moderately large group sizes. Again, for small group sizes and small values of μ_i' s ANOVA performs better than Kruskal Wallis when level of significance is 0.05 and we see the opposite picture under same condition for level of significance 0.1. Although for very large group sizes both the test performs quite similarly under exponential distribution. But in general Kruskal wallis test performs better than ANOVA for non-normal asymmetric distribution. This is may be happening because we have said that the variances of lognormal and exponential depend upon the values of μ_i' s, so as we have different values of μ_i' s for different population, not only the assumption of normality is violated but also the assumption of homoscedasticity is also violated.

And lastly one thing to notice that in case of normal, laplace, logistic and lognormal distribution, location parameter μ_1 does not affect the values of empirical power. Remaining other parameters fixed if we increase the value of μ_1 we still get the same values of powers for both tests. Whereas note that, the values of error variance and empirical power are inversely proportionate for these distributions.

But in case of exponential distribution, μ_i' s are the scale parameters. So if keeping other parameters fixed it we increase the value of μ_1 and accordingly values of μ_2, μ_3 (because we have chosen the alternative hypthesis $\mu_1 < \mu_2 < \mu_3$ and moreover we have taken $\mu_2 = \mu_1 + d,\ \mu_3 = \mu_1 + 2d$), we see a change in the values of power for both the tests. Also, we have noticed that they are inversely proportionate when the underlying population distribution is exponential

Hence, The results of the simulations show that even if one-way ANOVA is a robust test under many situations but an analysis of the data is needed before a test on equality of group means or location parameter is conducted. Since, when both the assumptions (normality and homoscedasticity) of the parametric test ANOVA are violated, nonparametric test Kruskal Wallis performs better.

Reference

1. *'Introduction to the Theory of Statistics'* Book by *Alexander M. Mood, Duane C. Boes* and *Fracklin A. Graybill*

2. *'Nonparametric Statistical Inference'* Book by *Jean D. Gibbons* and *Subhabrata Chakraborti*

3. `https://www.researchgate.net/publication/228457648_Power_study_of_anova_versus_Kruskal-Wallis_test`

4. `https://www.cuemath.com/data/hypothesis-testing/`

5. *'Power comparison of ANOVA and Kruskal–Wallis tests when error assumptions are violated'* Article by *Felix N. Nwobi** and *Felix C. Akanno*, *Imo State University, Department of Statistics, Owerri, Nigeria*

6. `https://dfrieds.com/math/images/errors.jpg` (for Figure-1)

7. `https://miro.medium.com/v2/resize:fit:720/format:webp/1*_dBMMDyMXcd8opVzDEmVLw.png` (for Figure-2)

Appendix

0.13 R codes

0.13.1 Required Packages

```
#required packages
rm(list = ls())                 #removes all data
set.seed(seed = 987654321)      #for uniformity of result
library(tidyverse)
library(ggplot2)                #for graphics
library(wesanderson)            #for colours
library(extrafont)
font_import()
```

```
library(VGAM)                    #for laplace distribution
```

0.13.2 New theme for plots

```
#Define new_theme() function
theme_new <- function(){
font <- "Georgia"   #assign font family up front
theme_bw() %+replace%    #replace elements we want to change
theme(plot.title = element_text(             #title
        family = font,            #set font family
        size = 13,                #set font size
        face = 'bold',            #bold typeface
        hjust = 0,                #left align
        vjust = 2,
        color = "red"),                #raise slightly
      plot.subtitle = element_text(          #subtitle
        family = "serif",            #font family
        size = 9,
        hjust = 0.01,
        vjust = 2),               #font size
      plot.caption = element_text(           #caption
        family = font,            #font family
        size = 12,                 #font size
        hjust = 1),               #right align
      axis.title = element_text(             #axis titles
        family = font,            #font family
        size = 11),               #font size
      axis.text = element_text(              #axis text
        family = font,            #axis famuly
        size = 9),                #font size
      axis.text.x = element_text(            #margin for axis text
        margin=margin(5, b = 10)),
      legend.title = element_text(
        family = font,
        size = 9),
      legend.text = element_text(
        family = font,
        size = 8),
```

```
      legend.background = element_blank(),
      legend.box.background = element_rect(
        colour = "black"))}
```

0.13.3 Tabulated critical values of Kruskal Wallis test

```
#tabulated critical values of kruskal-wallis
tab <- matrix(c(3,  2,  2,  4.500,  4.714,
                3,  3,  1,  4.571,  5.143,
                3,  3,  2,  4.556,  5.361,
                3,  3,  3,  4.622,  5.600,
                4,  2,  2,  4.458,  5.333,
                4,  3,  1,  4.056,  5.208,
                4,  3,  2,  4.511,  5.444,
                4,  3,  3,  4.709,  5.791,
                4,  1,  4,  4.167,  4.967,
                4,  2,  4,  4.555,  5.455,
                4,  4,  3,  4.545,  5.598,
                4,  4,  4,  4.654,  5.692,
                5,  2,  1,  4.200,  5.000,
                5,  2,  2,  4.373,  5.160,
                5,  3,  1,  4.018,  4.960,
                5,  3,  2,  4.651,  5.251,
                5,  3,  3,  4.533,  5.648,
                5,  4,  1,  3.987,  4.985,
                5,  4,  2,  4.541,  5.273,
                5,  4,  3,  4.549,  5.656,
                5,  4,  4,  4.668,  5.657,
                5,  5,  1,  4.109,  5.127,
                5,  5,  2,  4.623,  5.338,
                3,  5,  5,  4.545,  5.705,
                5,  5,  4,  4.523,  5.666,
                5,  5,  5,  4.560,  5.780
) , byrow = T, ncol = 5)
colnames(tab) = c("n1","n2","n3","alpha = 0.1",
                  "alpha = 0.05")
tab
```

0.13.4 Choices of μ_1, μ_2, μ_3 under alternative hypothesis

```
# choices of mui's under alternative
m=matrix(0,ncol=18,nrow=3)
m[1,] <- rep(1,each = 18)
d = c(seq(0,5,0.3),5.1);d
for(j in 1:18){
  for(i in 2:3){
    m[i,j] = m[1,j]+((i - 1)*d[j])}}
m
```

0.13.5 Empirical size and power of ANOVA and Kruskal Wallis test under normal distribution

1. **For small group sizes**

```
#choosing row number for group size choice(1 to 26)
row = 3
#choosing col number for choice of
#between (4 -> alpha = 0.1; 5 -> alpha = 0.05)
col = 5
#choices of group sizes
ni = as.vector(tab[row,1:3])
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
```

```
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rnorm(ni[i]*R,mui[i],sigma)),ncol
   =R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>tab[row,col],1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 5, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(m[1,i],m[2,i],m[3,i],
```

```
                       power(R,ni,m[,i],3,
                             alpha))
}
colnames(power.matrix) = c("mu1","mu2","mu3",
                           "anova_test",
                           "kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(d))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             alpha = 0.5,size = 2)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             alpha = 0.5,size = 2)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu[1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis test \nwhen underlying population distribution is Normal",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
                        " , alpha = ",alpha,
                        " , R = ", R,
                        " , sigma = ", 3))+
  theme_new()
```

2. For large group sizes

```
#choices of group sizes
#ni = c(8,6,7)
ni = c(50,50,50)
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rnorm(ni[i]*R,mui[i],sigma)),ncol
=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
```

```
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],3,alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test"
   )
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             size = 2, alpha = 0.5)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \
   nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
```

```
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
   [1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis
   test \nwhen underlying population  distribution is Normal",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
                        " , alpha = ",alpha,
                        " , R = ", R,
                        " , sigma = ", 3))+
  theme_new()
```

3. For heteroscedasticity

```
#choices of group sizes
ni = c(10,10,10)
#choice of alpha
alpha <- 0.1
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
```

```
    assign(paste0("y_",i),matrix((rnorm(ni[i]*R,mui[i],sigma[i])),
  ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
  sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
  new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],c(1.2,3,5.1),alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test"
```

```
  )
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             size = 2, alpha = 0.5)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu[1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis test \nwhen underlying population  distribution is Normal",
       subtitle = paste("mu1 = ",m[1,1],
                        ", n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
                        " , alpha = ",alpha,
                        " , R = ", R,
                        " , sigma1 = ", 1.2,
                        " , sigma2 = ", 3,
                        " , sigma3 = ", 5.1))+
  theme_new()
```

0.13.6 Empirical size and power of ANOVA and Kruskal Wallis test under laplace distribution

1. **For small group sizes**

```
#choosing row number for group size choice(1 to 26)
row = 3
#choosing col number for choice of
#between (4 -> alpha = 0.1; 5 -> alpha = 0.05)
col = 5
#choices of group sizes
ni = as.vector(tab[row,1:3])
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rlaplace(ni[i]*R,mui[i],sigma)),
   ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
```

```
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>tab[row,col],1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 5, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(m[1,i],m[2,i],m[3,i],
                       power(R,ni,m[,i],3,
                             alpha))
}
colnames(power.matrix) = c("mu1","mu2","mu3",
                           "anova_test",
                           "kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(d))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
```

```
geom_point(aes(y = kruskal_wallis_test,
               col="Kruskal- \nWallis \ntest"),
           alpha = 0.5,size = 2)+
geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
           alpha = 0.5,size = 2)+
scale_color_manual("Index",breaks =
                     c("Kruskal- \nWallis \ntest","One-way \
 nANOVA"),
                   values = c("Kruskal- \nWallis \ntest" = "red",
                              "One-way \nANOVA" = "blue"))+
geom_hline(yintercept=alpha,linetype = 'dashed')+
geom_hline(yintercept = 1,linetype = 'dashed')+
labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
 [1])==(mu[3] - mu[2])))),
     y = "Simualted Power",
     title = "Power Curve of One-way Anova and Kruskal-Wallis
 test \nwhen underlying population distribution is Laplace",
     subtitle = paste("mu1 = ",m[1,1],
                      " , n1 = ",ni[1],
                      " , n2 = ",ni[2],
                      " , n3 = ",ni[3],
                      " , alpha = ",alpha,
                      " , R = ", R,
                      " , sigma = ", 3))+
theme_new()
```

2. For large group sizes

```
#choices of group sizes
#ni = c(8,6,7)
ni = c(50,50,50)
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
```

```r
yi0 = matrix(0,ncol=3,nrow=R)
y00 = 0
msa = 0
mse = 0
f = 0
H = 0
grp_size = matrix(rep(ni,each=R),nrow=R)
n = sum(ni)
rank=1:n
x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
for(i in 1:length(ni)){
  assign(paste0("y_",i),matrix((rlaplace(ni[i]*R,mui[i],sigma)),
 ncol=R))
}
y10 = apply(y_1, 2, mean)
y20 = apply(y_2, 2, mean)
y30 = apply(y_3, 2, mean)
yi0 = matrix(c(y10,y20,y30),ncol=3)
for(i in 1:R){
  #test statistic for anova
  y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
  msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
  mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
 sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
  f[i] = msa[i]/mse[i]
  #test statistic for kruskal-wallis
  y = c(y_1[,i],y_2[,i],y_3[,i])
  d = cbind(y,x)[order(y,x,decreasing = F), ]
  data = data.frame(cbind(d,rank))
  for(j in 1:length(ni)){
    assign(paste0("grp",j),data[data$x==j,3])
  }
  new_y = c(grp1,grp2,grp3)
  new_data=data.frame(cbind(new_y,x))
  rank_sum = aggregate(new_data$new_y,by=list(as.factor(
 new_data$x)),sum)$x
  H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
```

```
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],3,alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             size = 2, alpha = 0.5)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu[1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis test \nwhen underlying population  distribution is Laplace",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
```

```
                              " , n2 = ",ni[2],
                              " , n3 = ",ni[3],
                              " , alpha = ",alpha,
                              " , R = ", R,
                              " , sigma = ", 3))+
  theme_new()
```

3. For heteroscedasticity

```
#choices of group sizes
ni = c(10,10,10)
#choice of alpha
alpha <- 0.1
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rlaplace(ni[i]*R,mui[i],sigma[i])
  ),ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
```

```
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],c(1.2,3,5.1),alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test"
   )
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
```

```r
geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
           size = 2, alpha = 0.5)+
scale_color_manual("Index",breaks =
                     c("Kruskal- \nWallis \ntest","One-way \
nANOVA"),
                   values = c("Kruskal- \nWallis \ntest" = "red",
                              "One-way \nANOVA" = "blue"))+
geom_hline(yintercept=alpha,linetype = 'dashed')+
geom_hline(yintercept = 1,linetype = 'dashed')+
labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
[1])==(mu[3] - mu[2])))),
     y = "Simualted Power",
     title = "Power Curve of One-way Anova and Kruskal-Wallis
test \nwhen underlying population  distribution is Laplce",
     subtitle = paste("mu1 = ",m[1,1],
                      ", n1 = ",ni[1],
                      " , n2 = ",ni[2],
                      " , n3 = ",ni[3],
                      " , alpha = ",alpha,
                      " , R = ", R,
                      " , sigma1 = ", 1.2,
                      " , sigma2 = ", 3,
                      " , sigma3 = ", 5.1))+
theme_new()
```

0.13.7 Empirical size and power of ANOVA and Kruskal Wallis test under logistic distribution

1. **For small group sizes**

```r
#choosing row number for group size choice(1 to 26)
row = 3
#choosing col number for choice of
#between (4 -> alpha = 0.1; 5 -> alpha = 0.05)
col = 5
#choices of group sizes
ni = as.vector(tab[row,1:3])
```

```
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rlogis(ni[i]*R,mui[i],sigma)),
   ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
```

```
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>tab[row,col],1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 5, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(m[1,i],m[2,i],m[3,i],
                       power(R,ni,m[,i],3,
                             alpha))
}
colnames(power.matrix) = c("mu1","mu2","mu3",
                           "anova_test",
                           "kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(d))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             alpha = 0.5,size = 2)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             alpha = 0.5,size = 2)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
```

```
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
   [1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis
   test \nwhen underlying population distribution is Logistic",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
                        " , alpha = ",alpha,
                        " , R = ", R,
                        " , sigma = ", 3))+
  theme_new()
```

2. For large group sizes

```
#choices of group sizes
#ni = c(8,6,7)
ni = c(50,50,50)
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
```

```r
    assign(paste0("y_",i),matrix((rlogis(ni[i]*R,mui[i],sigma)),ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],3,alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test"
```

```r
  )
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             size = 2, alpha = 0.5)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \
  nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
  [1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis
  test \nwhen underlying population  distribution is Logistic",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
                        " , alpha = ",alpha,
                        " , R = ", R,
                        " , sigma = ", 3))+
  theme_new()
```

3. For heteroscedasticity

```r
#choices of group sizes
ni = c(10,10,10)
#choice of alpha
```

```r
alpha <- 0.1
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rlogis(ni[i]*R,mui[i],sigma[i])),
   ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
```

```
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],c(1.2,3,5.1),alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test"
   )
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             size = 2, alpha = 0.5)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \
   nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
   [1])==(mu[3] - mu[2])))),
```

```
      y = "Simualted Power",
      title = "Power Curve of One-way Anova and Kruskal-Wallis
  test \nwhen underlying population  distribution is Logistic",
      subtitle = paste("mu1 = ",m[1,1],
                       ", n1 = ",ni[1],
                       " , n2 = ",ni[2],
                       " , n3 = ",ni[3],
                       " , alpha = ",alpha,
                       " , R = ", R,
                       " , sigma1 = ", 1.2,
                       " , sigma2 = ", 3,
                       " , sigma3 = ", 5.1))+
  theme_new()
```

0.13.8 Empirical size and power of ANOVA and Kruskal Wallis test under lognormal distribution

1. **For small group sizes**

```
#choosing row number for group size choice(1 to 26)
row = 3
#choosing col number for choice of
#between (4 -> alpha = 0.1; 5 -> alpha = 0.05)
col = 5
#choices of group sizes
ni = as.vector(tab[row,1:3])
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
```

```
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rlnorm(ni[i]*R,mui[i],sigma)),
   ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>tab[row,col],1,0))
  return(c(anova.power,kruskal_wallis.power))
}
```

```
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 5, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(m[1,i],m[2,i],m[3,i],
                       power(R,ni,m[,i],3,
                             alpha))
}
colnames(power.matrix) = c("mu1","mu2","mu3",
                           "anova_test",
                           "kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(d))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal-\nWallis\ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal-\nWallis\ntest"),
             alpha = 0.5,size = 2)+
  geom_line(aes(y = anova_test, col = "One-way\nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way\nANOVA"),
             alpha = 0.5,size = 2)+
  scale_color_manual("Index",breaks =
                       c("Kruskal-\nWallis\ntest","One-way\nANOVA"),
                     values = c("Kruskal-\nWallis\ntest" = "red",
                                "One-way\nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu[1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis test \nwhen underlying population distribution is Lognormal",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
```

```
                                    " , alpha = ",alpha,
                                    " , R = ", R,
                                    " , sigma = ", 3))+
  theme_new()
```

2. For large group sizes

```
#choices of group sizes
#ni = c(8,6,7)
ni = c(50,50,50)
#choice of alpha
alpha <- 0.05
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,sigma,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rlnorm(ni[i]*R,mui[i],sigma)),
   ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
```

```
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 3, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(d[i],power(R,ni,m[,i],3,alpha))
}
colnames(power.matrix) = c("mu1","anova_test","kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(mu1,y=value))+
  geom_line(aes(y = kruskal_wallis_test,
              col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
               col="Kruskal- \nWallis \ntest"),
           size = 2, alpha = 0.5)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
```

```
                size = 2, alpha = 0.5)+
  scale_color_manual("Index",breaks =
                        c("Kruskal- \nWallis \ntest","One-way \
nANOVA"),
                      values = c("Kruskal- \nWallis \ntest" = "red",
                                 "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
[1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
       title = "Power Curve of One-way Anova and Kruskal-Wallis
test \nwhen underlying population  distribution is Lognormal",
       subtitle = paste("mu1 = ",m[1,1],
                        " , n1 = ",ni[1],
                        " , n2 = ",ni[2],
                        " , n3 = ",ni[3],
                        " , alpha = ",alpha,
                        " , R = ", R,
                        " , sigma = ", 3))+
  theme_new()
```

0.13.9 Empirical size and power of ANOVA and Kruskal Wallis test under exponential distribution

1. **For small group sizes**

```
#choosing row number for group size choice(1 to 26)
row = 3
#choosing col number for choice of
#between (4 -> alpha = 0.1; 5 -> alpha = 0.05)
col = 4
#choices of group sizes
ni = tab[row,1:3]
#choice of alpha
alpha <- 0.1
#choice of no of repetition
```

```
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))
  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rexp(ni[i]*R,1/mui[i])),ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
    sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
```

```
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>tab[row,col],1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 5, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(m[1,i],m[2,i],m[3,i],power(R,ni,m[,i],alpha))
}
colnames(power.matrix) = c("mu1","mu2","mu3",
                           "anova_test","kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(d))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
  geom_point(aes(y = kruskal_wallis_test,
                 col="Kruskal- \nWallis \ntest"),
             alpha = 0.5,size = 2)+
  geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
  geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
             alpha = 0.5,size = 2)+
  scale_color_manual("Index",breaks =
                       c("Kruskal- \nWallis \ntest","One-way \nANOVA"),
                     values = c("Kruskal- \nWallis \ntest" = "red",
                                "One-way \nANOVA" = "blue"))+
  geom_hline(yintercept=alpha,linetype = 'dashed')+
  geom_hline(yintercept = 1,linetype = 'dashed')+
  labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu[1])==(mu[3] - mu[2])))),
       y = "Simualted Power",
```

```
      title = "Power Curve of One-way Anova and Kruskal-Wallis test \nwhen underlying population distribution is Exponential",
      subtitle = paste("mu1 = ",m[1,1],
                       " , n1 = ",ni[1],
                       " , n2 = ",ni[2],
                       " , n3 = ",ni[3],
                       " , alpha = ",alpha,
                       " , R = ", R))+
  theme_new()
```

2. **For large group sizes**

```
#choices of group sizes
ni = c(50,50,50)
#choice of alpha
alpha <- 0.1
#choice of no of repetition
R <- 2000
#power function for anova and kruskal-wallis test
power = function(R,ni,mui,alpha){
  yi0 = matrix(0,ncol=3,nrow=R)
  y00 = 0
  msa = 0
  mse = 0
  f = 0
  H = 0
  grp_size = matrix(rep(ni,each=R),nrow=R)
  n = sum(ni)
  rank=1:n
  x=c(rep(1:length(ni),c(ni[1],ni[2],ni[3])))

  for(i in 1:length(ni)){
    assign(paste0("y_",i),matrix((rexp(ni[i]*R,1/mui[i])),ncol=R))
  }
  y10 = apply(y_1, 2, mean)
  y20 = apply(y_2, 2, mean)
  y30 = apply(y_3, 2, mean)
  yi0 = matrix(c(y10,y20,y30),ncol=3)
```

```
  for(i in 1:R){
    #test statistic for anova
    y00[i] = sum(grp_size[i,]*yi0[i,])/sum(grp_size[i,])
    msa[i] = sum(grp_size[i,]*(yi0[i,]-y00[i])^2)/(length(ni)-1)
    mse[i] = (sum((y_1[,i]-yi0[i,1])^2)+sum((y_2[,i]-yi0[i,2])^2)+
   sum((y_3[,i]-yi0[i,3])^2))/(sum(ni)-length(ni))
    f[i] = msa[i]/mse[i]
    #test statistic for kruskal-wallis
    y = c(y_1[,i],y_2[,i],y_3[,i])
    d = cbind(y,x)[order(y,x,decreasing = F), ]
    data = data.frame(cbind(d,rank))
    for(j in 1:length(ni)){
      assign(paste0("grp",j),data[data$x==j,3])
    }
    new_y = c(grp1,grp2,grp3)
    new_data=data.frame(cbind(new_y,x))
    rank_sum = aggregate(new_data$new_y,by=list(as.factor(
   new_data$x)),sum)$x
    H[i] = ((12/(n*(n+1)))*sum(rank_sum^2/ni))-(3*(n+1))
  }
  anova.power = mean(ifelse(f>qf(1-alpha,2,n-3),1,0))
  kruskal_wallis.power = mean(ifelse(H>qchisq(1-alpha,2),1,0))
  return(c(anova.power,kruskal_wallis.power))
}
#storing powers in a data-frame
power.matrix = matrix(0,ncol = 5, nrow = 18)
for(i in 1:18){
  power.matrix[i,] = c(m[1,i],m[2,i],m[3,i],power(R,ni,m[,i],alpha)
   )
}
colnames(power.matrix) = c("mu1","mu2","mu3",
                           "anova_test","kruskal_wallis_test")
power.matrix = as.data.frame(power.matrix)
power.matrix %>%
  ggplot(aes(d))+
  geom_line(aes(y = kruskal_wallis_test,
                col="Kruskal- \nWallis \ntest"))+
```

```
geom_point(aes(y = kruskal_wallis_test,
               col="Kruskal- \nWallis \ntest"),
           size = 2, alpha = 0.5)+
geom_line(aes(y = anova_test, col = "One-way \nANOVA"))+
geom_point(aes(y = anova_test, col = "One-way \nANOVA"),
           size = 2, alpha = 0.5)+
scale_color_manual("Index",breaks =
                     c("Kruskal- \nWallis \ntest","One-way \
 nANOVA"),
                   values = c("Kruskal- \nWallis \ntest" = "red",
                              "One-way \nANOVA" = "blue"))+
geom_hline(yintercept=alpha,linetype = 'dashed')+
geom_hline(yintercept = 1,linetype = 'dashed')+
labs(x = expression(paste("Differences"~italic(d)~((mu[2] - mu
 [1])==(mu[3] - mu[2])))),
     y = "Simualted Power",
     title = "Power Curve of One-way Anova and Kruskal-Wallis
 test \nwhen underlying population  distribution is Exponential",
     subtitle = paste("mu1 = ",m[1,1],
                      " , n1 = ",ni[1],
                      " , n2 = ",ni[2],
                      " , n3 = ",ni[3],
                      " , alpha = ",alpha,
                      " , R = ", R))+
theme_new()
```

THE END