

PM 6 - Understanding Geometric Objects With Computer Algebra System

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WiM Directed Reading Program

2025

Overview

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- ▶ Introduction to definitions/terminology
- ▶ Important theorems and results
- ▶ Application of Gröbner bases to Sudoku puzzles
- ▶ Working algorithm to solve given Sudoku puzzles + live demo

Full Document

Scan here for the full document containing more detailed definitions and proofs:



Multi-variable Monomial and Polynomial

Monomial:

$$x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

for a monomial of n variables, x_1, x_2, \dots, x_n

- ▶ Commonly written as x^α
- ▶ x^3y^2 is a multivariable monomial

Polynomial:

$$f = \sum_{\alpha} a_{\alpha} x^{\alpha}$$

- ▶ $k[x_1, x_2, \dots, x_s]$ is the set of all polynomials with variable x_1, x_2, \dots, x_s
- ▶ $f = 2x^3y^2z - 3xyz + \frac{3}{2}y^3z^3 + y^2$ is a polynomial in $\mathbf{Q}[x, y, z]$

Affine Variety

We commonly write it as:

$$\mathbf{V}(f_1, \dots, f_s)$$

Affine variety is the set of solutions in k^n to a system of equations $f_1, f_2, f_3, \dots, f_s$.

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_1, \dots, a_n) \in k^n \mid f_i(a_1, \dots, a_n) = 0, \forall 1 \leq i \leq s\}.$$

Examples

- ▶ $\mathbf{V}(x^2 + y^2 - 1)$ contains all the values in \mathbf{R} that makes $x^2 + y^2 - 1 = 0$ true, which is all the points in the circle
- ▶ $\mathbf{V}(y - x, y + x)$ contains all the values in \mathbf{R} that makes $y - x = 0, y + x = 0$ true; visualized graphically, it is the intersection of two functions, which is $(0, 0)$

Ideals

Ideal: I the set of polynomials

$$I \subseteq k[x_1, x_2, \dots, x_n]$$

- ▶ $0 \in I$
- ▶ $f, g \in I$ then $f + g \in I$
- ▶ If $f \in I$ and $h \in k[x_1, x_2, \dots, x_n]$, then $hf \in I$

Ideals commonly comes in the form of $\langle f_1, f_2, f_3, \dots, f_s \rangle$ where

$$\langle f_1, f_2, f_3, \dots, f_s \rangle = \left\{ \sum_{i=1}^s h_i f_i \mid h_1, \dots, h_s \in k[x_1, x_2, \dots, x_n] \right\}$$

Ideal Examples

Ideal belonging problem: Does $x^2 - 2x + 2 - y$ belong in $\langle x - 1 - t, y - 1 - t^2 \rangle$?

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Yes! $x^2 - 2x + 2 - y = (x - 1 + t)(x - 1 - t) + (-1)(y - 1 - t^2)$
this is in the form of

$$\sum_{i=1}^s h_i f_i$$

Relationship between Affine Variety and Ideals

When we are trying to find $V(f_1, f_2, \dots, f_n)$ and $\langle f_1, f_2, \dots, f_s \rangle = \langle g_1, g_2, \dots, g_s \rangle$

then...

► $V(g_1, g_2, \dots, g_s) = V(f_1, f_2, \dots, f_n)$

Monomial Ideal

Monomial ideal is a special special polynomial ideal where it can be written in the form of

$$I = \langle x^{a_1}, x^{a_2}, x^{a_3} \dots \rangle$$

Ideal generated by leading terms of an ideal

$$\langle LT(I) \rangle$$

Leading term is a term of a polynomial

- ▶ Determine a monomial ordering, order the terms of the polynomial
- ▶ Leading term is the first term of the ordered polynomials.

Examples

$$f = 2x^3y^2z + \frac{3}{2}y^3z^3 - 3xyz + y^2$$

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Ideal generated by leading terms of an ideal

$$\langle LT(I) \rangle$$

- ▶ $LT(I)$ is the leading term of every polynomial in ideal I
- ▶ $\langle LT(I) \rangle$ is the ideal generated by those leading term

Important Theorems and Results:

Dickson's Lemma:

Let $I = \langle x^\alpha \mid \alpha \in A \rangle \subset k[x_1, \dots, x_n]$ be a monomial ideal. Then I can be written in the form

$$I = \langle x^{\alpha(1)}, \dots, x^{\alpha(s)} \rangle,$$

where $\alpha(1), \dots, \alpha(s) \in A$. In particular, I has a finite basis.

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Results:

- ▶ Necessary for the proof of Hilbert Basis theorem.
- ▶ Ensures finite Gröbner basis exists.

Important Theorems and Results:

Hilbert Basis Theorem:

Every ideal $I \subseteq k[x_1, \dots, x_n]$ has a finite generating set. In other words, $I = \langle g_1, \dots, g_t \rangle$ for some $g_1, \dots, g_t \in I$.

Example:

$I = \langle x^2 + y^3 - z^5, xyz + 1, x^3y^2z^4 - 7xy + z^2, x^2z - y^4, \dots \rangle$
will have a finite generating set.

Results:

- Any affine variety can be defined by a finite number of equations, no infinite constraints are needed.

Important Theorems and Results:

Gröbner Bases: Let I be an ideal from $k[x_1, \dots, x_n]$. Fix a monomial ordering. A finite subset $G = \{g_1, \dots, g_r\} \subseteq I$ is called a *Gröbner basis* for I if:

$$1. \quad I = \langle g_1, \dots, g_r \rangle$$

$$2. \quad \langle \text{LT}(g_1), \dots, \text{LT}(g_r) \rangle = \langle \text{LT}(f) \mid f \in I \rangle$$

Property of G:

If $G = \{g_1, \dots, g_r\}$ is a Gröbner basis for an ideal, then every polynomial $f \in k[x_1, \dots, x_n]$ can be divided by G , and the remainder satisfies:

- ▶ The remainder is **unique**.
- ▶ The remainder is **zero** if and only if $f \in I$.

Important Theorems and Results:

Results:

This helps us to solve the ideal membership problem!!

Example:

Let our polynomial ring be $\mathbb{Q}[x, y]$. Let

$$I = \langle f_1, f_2 \rangle = \langle x^2 + y, xy - 1 \rangle.$$

Question: Does $f = x^3y + x \in I$?

We compute a Gröbner basis using lexicographic order with $x > y$. Then $G = \{y^2 + x, xy - 1, x^2 + y\}$. By performing multivariate polynomial division of f with respect to G , the remainder is 0 hence, $f \in I$.

Sudoku Application: Rules

To begin, we will consider a 4x4 sudoku grid with the following rules:

			3
	4		
1			4
		3	

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3. No two blocks in the same row, column, or 2x2 block can be equal

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- ▶ Use an algorithm (Buchberger's algorithm) to generate Gröbner basis
- ▶ Find the variety (solutions) of the ideal using generated basis

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- ▶ $x_i - a$ for $1 \leq a \leq 4$ (the pre-existing squares)

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- ▶ If the sudoku is proper, each element of this basis is of the form $x_i - a_i$ for $1 \leq a_i \leq 4$
- ▶ The solutions to the variety of I is simply the solution to setting each element of G to zero
- ▶ Thus, each a_i describes the value present in grid x_i (or the solution to the sudoku) since each $x_i = a_i$

Implementation

```
1  import itertools
2
3  #initialize variables x_0,...,x_15
4  var_names = ['x_{}'.format(i) for i in range(0,16)]
5
6  #initialize polynomial ring with x_0,...,x_15
7  R = PolynomialRing(QQ, var_names, order='lex')
8  R.inject_variables()
9
10 #generate the polynomial (x_i-1)(x_i-2)(x_i-3)(x_i-4)
11 #These polynomials restrict that x_i must equals to
12 #1, 2, 3, or 4
13 poly_List = []
14 for x in R.gens():
15     poly_List.append(prod([(x-i) for i in [1..4]]))
```

Implementation

```
16 row_List = [] #generate a list of polynomial for each row
17 for i in range(4):
18     temp = []
19     for j in range(4):
20         temp.append(poly_List[i*4+j])
21     row_List.append(temp)
22
23 col_List = [] #generate a list of polynomial for each column
24 for i in range(4):
25     temp = []
26     for j in range(4):
27         temp.append(poly_List[i+4*j])
28     col_List.append(temp)
29
30 block_List = [] #generate a list of polynomial for each block
31 for i in range(4):
32     start = (i//2)*8 + (i%2)*2
33     temp = []
34     for j in range(4):
35         temp.append(poly_List[start + j%2 + (j//2)*4])
36     block_List.append(temp)
```

Implementation

```
37 # combine all lists
38 adjacent_List = row_List + col_List + block_List
39 diff_List = []
40 for l in adjacent_List:
41     comb = list(itertools.combinations(l,2))
42     # take all possible combinations and eliminate the
43     # factor (x_i-x_j), so x_i = x_j if they are in
44     # the same row, column or block
45     for c in comb:
46         diff = c[0] - c[1]
47         vars = diff.variables()
48         diff_List.append(diff/(vars[0]-vars[1]))
49
50 #clues from sudoku
51 initial_clues = [x_3-3,x_5-4,x_8-1,x_11-4,x_14-3]
52
53 #generate the ideal using the clues, non-repetitive
54 #restrictions and number restriction of each variables
55 I = R.ideal(initial_clues + diff_List + poly_List)
56 G = I.groebner_basis()
57 show(G)
```

Result

```
# the groebner basis of the ideal I is the solution of  
# the sudoku  
[x_0 - 2, x_1 - 1, x_2 - 4, x_3 - 3,  
 x_4 - 3, x_5 - 4, x_6 - 1, x_7 - 2,  
 x_8 - 1, x_9 - 3, x_10 - 2, x_11 - 4,  
 x_12 - 4, x_13 - 2, x_14 - 3, x_15 - 1]
```

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Further applications:

- ▶ Similar usage of Gröbner bases to solve other games (eg. minesweeper) or more complex sudoku puzzles (eg. 9x9)
- ▶ Algebraic Geometry appears in other areas of pure math such as topology, complex analysis, number theory and more.