

1 Problem 1

Convergence and stability of a numerical scheme. In this problem, we can use

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0 \quad (1)$$

to solve the one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$. Here, $c \in \mathbb{R}$ and U_j^n is the numerical approximation of $u(j\Delta x, n\Delta t)$.

1.a

The numerical scheme in Eq. 1 is second-order accurate, this can be seen if we re-write equation (1) through Taylor expansion. Before we expand, the equations should be rewritten to the following form:

$$\begin{aligned} U_j^{n+1} &= U(t_n + \Delta t, x_j) \\ U_j^{n-1} &= U(t_n - \Delta t, x_j) \\ U_{j+1}^n &= U(t_n, x_j + \Delta x) \\ U_{j-1}^n &= U(t_n, x_j - \Delta x) \end{aligned}$$

Then, I Taylor expand these as:

$$\begin{aligned} U_j^{n+1} &= U(t_n + \Delta t, x_j) = U(t_n, x_j) + \Delta t * U_t(t_n, x_j) + \frac{\Delta t^2}{2!} U_{tt}(t_n, x_j) + \frac{\Delta t^3}{3!} U_{ttt}(t_n, x_j) \\ &\quad + \frac{\Delta t^4}{4!} U_{tttt}(t_n, x_j) + O(h^5) \end{aligned}$$

$$\begin{aligned} U_j^{n-1} &= U(t_n - \Delta t, x_j) = U(t_n, x_j) - \Delta t * U_t(t_n, x_j) + \frac{\Delta t^2}{2!} U_{tt}(t_n, x_j) - \frac{\Delta t^3}{3!} U_{ttt}(t_n, x_j) \\ &\quad + \frac{\Delta t^4}{4!} U_{tttt}(t_n, x_j) + O(h^5) \end{aligned}$$

$$\begin{aligned} U_{j+1}^n &= U(t_n, x_j + \Delta x) = U(t_n, x_j) + \Delta x * U_x(t_n, x_j) + \frac{\Delta x^2}{2!} U_{xx}(t_n, x_j) + \frac{\Delta x^3}{3!} U_{xxx}(t_n, x_j) \\ &\quad + \frac{\Delta x^4}{4!} U_{xxxx}(t_n, x_j) + O(h^5) \end{aligned}$$

$$\begin{aligned} U_{j-1}^n &= U(t_n, x_j - \Delta x) = U(t_n, x_j) - \Delta x * U_x(t_n, x_j) + \frac{\Delta x^2}{2!} U_{xx}(t_n, x_j) - \frac{\Delta x^3}{3!} U_{xxx}(t_n, x_j) \\ &\quad + \frac{\Delta x^4}{4!} U_{xxxx}(t_n, x_j) + O(h^5) \end{aligned}$$

Now, I can plug these values in equation (1), and many will cancel out. Hence, the truncation error T will be:

$$T = \frac{2 \frac{\Delta t^2}{2*1} U_{tt}(t_n, x_j) + 2 \frac{\Delta t^4}{4*3*2*1} U_{tttt}(t_n, x_j) + 2 * O(h^5)}{\Delta t^2} - \dots$$

$$\dots - \frac{c^2(2\frac{\Delta x_j^2}{2*1}U_{xx}(t_n, x_j) + 2\frac{\Delta x_j^4}{4*3*2*1}U_{xxxx}(t_n, x_j) + 2 * \frac{O(h^5)}{\Delta x^2}}{\Delta x^2}$$

Cancelling and rearranging other terms leads to:

$$T = \frac{\Delta t^2}{12}U_{tttt}(t_n, x_j) + U_{tt}(t_n, x_j) + 2\frac{O^5}{\Delta t^2} - c^2(U_{xx}(t_n, x_j) + \frac{\Delta x^2}{12}U_{xxxx}(t_n, x_j) + 2*\frac{O(h^5)}{\Delta x^2}\Delta x^2$$

$$T = \frac{\Delta t^2}{12}U_{tttt}(t_n, x_j) + U_{tt}(t_n, x_j) + H.O.T. - c^2(U_{xx}(t_n, x_j) + \frac{\Delta x^2}{12}U_{xxxx}(t_n, x_j) + H.O.T.)$$

From class, we are given $U_{tt}(t_n, x_j) - c^2U_{xx} = 0$, hence:

$$T = \frac{1}{12}(\Delta t^2U_{tttt}(t_n, x_j) - c^2\Delta x^2U_{xxxx}(t_n, x_j)) + H.O.T. \quad (2)$$

These are second order terms, hence this equation is second order accurate.

1.b

To show that the numerical scheme is stable, we can use Fourier stability analysis, using the ansatz $U_j^n = \lambda(k)^n e^{ijk\Delta x}$. I want to rewrite equation (1). We can now denote $v = \frac{\Delta t c}{\Delta x}$. Consequently, equation (1) becomes:

$$T = U_j^{n+1} - 2U_j^n + U_j^{n-1} - v^2[U_{j+1}^n - 2U_j^n + U_{j-1}^n] = 0$$

Here, we can now plug in: $U_j^n = \lambda(k)^n e^{ijk\Delta x}$.

$$\lambda^{n+1}e^{ikj\Delta x} - 2\lambda e^{ikj\Delta x} + \lambda^{n-1}e^{ikj\Delta x} - v^2\lambda[e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}] = 0 \quad (3)$$

Additionally, I know that $e^{ik(j+1)\Delta x} = e^{ikj\Delta x}e^{ik\Delta x}$. This means we can divide out the $e^{ikj\Delta x}$ in all the terms and then we are just left with:

$$\lambda^{n+1} - 2\lambda^n + \lambda^{n-1} - v^2\lambda[e^{ik\Delta x} - 2 + e^{-ik\Delta x}]\lambda^n = 0$$

Note: $e^{ik\Delta x} = \cos(k\Delta x) + i\sin(k\Delta x)$ and $e^{-ik\Delta x} = \cos(k\Delta x) - i\sin(k\Delta x)$. For our equation above, we can then further reduce using: $e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$. Then, we can divide out by λ^{n-1} to get:

$$\lambda^2 - 2\lambda + 1 - v^2\lambda[2\cos(k\Delta x) - 2] = 0$$

Here we want to solve for λ and we can use the quadratic formula.

$$\lambda^2 - \lambda(2v^2\cos(k\Delta x) - 2v^2 + 2) + 1 = 0$$

Solving for λ it becomes:

$$\begin{aligned} \lambda &= \frac{2(v^2\cos(k\Delta x) - v^2 + 1) \pm \sqrt{4(v^2\cos(k\Delta x) - v^2 + 1)^2 - 4}}{2} \\ &= v^2\cos(k\Delta x) - v^2 + 1 \pm \sqrt{(v^2\cos(k\Delta x) - v^2 + 1)^2 - 1} \end{aligned}$$

Where we know that $(v^2 k \Delta x - v^2 + 1)^2 - 1$ is always negative. Hence λ is a complex number made of a real part a and an imaginary part b , i.e. $a+ib$. Since b is an imaginary part, we can flip the sign now to solve for it and get b is $1 - (v^2 k \Delta x - v^2 + 1)^2$. Now our a is $v^2 \cos(k \Delta x) - v^2 + 1$.

$$\lambda = v^2 \cos(k \Delta x) - v^2 + 1 \pm i \sqrt{1 - (v^2 \cos(k \Delta x) - v^2 + 1)^2}$$

$$\begin{aligned} a &= v^2 \cos(k \Delta x) - v^2 + 1 \\ b &= 1 - (v^2 k \Delta x - v^2 + 1)^2 \end{aligned}$$

Now, we can solve for the roots of λ :

$$|\lambda| = a^2 + ib^2$$

$$\lambda = (v^2 \cos(k \Delta x) - v^2 + 1)^2 \pm (i \sqrt{1 - (v^2 \cos(k \Delta x) - v^2 + 1)^2})^2$$

$$\lambda = (v^2 \cos(k \Delta x) - v^2 + 1)^2 \pm 1 - (v^2 \cos(k \Delta x) - v^2 + 1)^2$$

$$|\lambda| \pm 1$$

$$\lambda = 1$$

Given this, our scheme is stable because it satisfies that $\lambda \leq 1$.