AM205: Assignment 4 (due 5 PM, November 11)

*For this assignment, first complete problems 1 and 2, and then complete **either** problem 3 (on nonlinear rootfinding) or problem 4 (on data visualization). If you submit answers for both problems 3 and 4, the maximum score from the two will be taken when calculating your grade.*

Program files: A number of program and data files for this homework can be downloaded as a single ZIP file from the course website.

1. **Convergence and stability of a numerical scheme.** Consider the numerical scheme

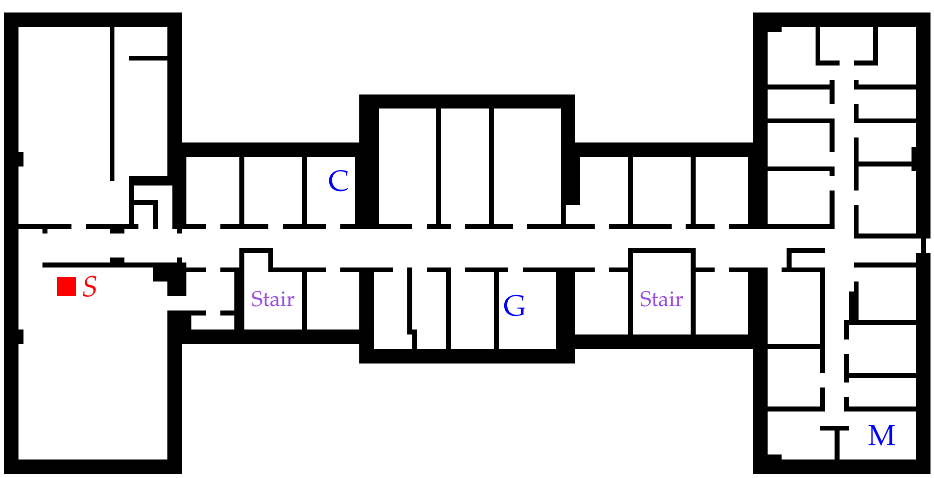
$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} - c^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} = 0 \tag{1}$$

to solve the one-dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$. Here, $c \in \mathbb{R}$ and U_j^n is the numerical approximation of $u(j\Delta x, n\Delta t)$.

   (a) Show that the numerical scheme in Eq. 1 is second-order accurate.

   (b) Use Fourier stability analysis, by subsituting in the ansatz $U_j^n = \lambda(k)^n e^{ijk\Delta x}$, to show that the numerical scheme is stable. Because the wave equation is second-order in time, you will get two solutions $\lambda(k)$ for each k, and both must have magnitude less than or equal to 1 for stability.

2. **Totally rocking out in Pierce Hall.** The image below shows a map of the third floor of Pierce Hall. All the doors are open, apart from those to the main stairwells and the passageway to Maxwell–Dworkin.



A text file `pierce.txt` is provided that encodes this map as a 100×200 matrix using 1 for walls and 0 for empty space. Use the convention that $(j, k) = (0, 0)$ is the top left of the matrix and $(j, k) = (99, 199)$ is bottom right of the matrix. The grid spacing is $h = 36.6$ cm.

A group of students hold an event in Pierce 301, the large room in the bottom left of the map. They set up a loudspeaker shown in red, covering the region S over gridpoints (j, k) for

$57 \le j \le 60, 15 \le k \le 18$. When testing the speaker, they drive it with a 50 Hz sine wave,[1] creating sound pressure waves that travel throughout the building, disturbing the occupants. The pressure field $p(x,y,t)$ satisfies the wave equation

$$\frac{\partial^2 p}{\partial t^2} - c^2 \nabla^2 p = 0, \tag{2}$$

where $c = 3.43 \times 10^4$ cm s^{-1} is the speed of sound. At $t = 0$ when the speaker is switched on, the pressure satisfies the initial conditions $p(x,y,t) = p_t(x,y,t) = 0$. At each wall, the pressure satisfies the boundary condition

$$\mathbf{n} \cdot \nabla p = 0, \tag{3}$$

where $\mathbf{n}$ is a unit vector normal to the wall. In the region S the pressure is driven to satisfy

$$p(x,y,t) = f(t) = p_0 \sin \omega t, \tag{4}$$

where the angular frequency is $\omega = 100\pi$ s^{-1} and the pressure constant[2] is $p_0 = 10$ Pa.

(a) Write a program to solve for the pressure field inside the building, using the two-dimensional discretization

$$\frac{P^{n+1}_{j,k} - 2P^n_{j,k} + P^{n-1}_{j,k}}{\Delta t^2} - c^2 \frac{P^n_{j+1,k} + P^n_{j,k+1} - 4P^n_{j,k} + P^n_{j-1,k} + P^n_{j,k-1}}{h^2} = 0. \tag{5}$$

where $P^n_{j,k}$ is the numerical approximation of $p(kh, (99-j)h, n\Delta t)$. Use a timestep of $\Delta t = \frac{h}{2c}$ or smaller. As initial conditions, use

$$P^0_{j,k} = 0, \qquad P^1_{j,k} = \begin{cases} f(\Delta t) & \text{if } (j,k) \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

To account for the boundary condition in Eq. 3, use the ghost node approach: when considering a point (j,k) in Eq. 5 that references an orthogonal neighbor (j^*, k^*) that is a wall, treat $P^n_{j^*,k^*}$ as equal to $P^n_{j,k}$. As an example of this, suppose that at a particular (j,k), the points $(j, k-1)$ and $(j+1,k)$ are within walls. Then, after taking into account the boundary conditions, the appropriate finite-difference relation is

$$\frac{P^{n+1}_{j,k} - 2P^n_{j,k} + P^{n-1}_{j,k}}{\Delta t^2} - c^2 \frac{P^n_{j,k+1} - 2P^n_{j,k} + P^n_{j-1,k}}{h^2} = 0. \tag{7}$$

due to cancellation of some terms. To account for the condition in Eq. 4, enforce throughout the computation that $P^n_{j,k} = f(n\Delta t)$ for $(j,k) \in S$.

(b) Make two-dimensional plots of the pressure field in the building at $t = 0.015$ s, $t = 0.105$ s, $t = 0.505$ s, and $t = 1.005$ s. In the program files, there are some example programs in the `py_2dplot` and `matlab_2dplot` directories that you may find useful, which make plots of a two-dimensional field with the map overlaid. You should expect that your program may take a reasonable amount of wall clock time, possibly up to about three-quarters of an hour to simulate to $t = 1.005$ s. You may wish test your program over smaller intervals of t and consider possible code optimizations if necessary.

[1] The typical human auditory range is from 20 Hz to 20 kHz, so this sound is at the lower end of what is audible.
[2] This pressure is similar to what you might get at a rock concert.

(c) Three people C, G, and M are trying to work at locations $(35, 73)$, $(61, 109)$ and $(91, 188)$, respectively. Find the time t in seconds, accurate to at least two decimal places, when the three people first hear the sound, defined as when $|p(t)|$ at their location exceeds 10^{-3} Pa. Discuss whether your results are reasonable, given the locations of the people in relation to the loudspeaker.

(d) On a single pair of axes, plot $p(t)$ at the three people's locations over the interval $0 \leq t \leq 1$ s. Which person is most likely to be disturbed by the loudspeaker?

(e) **Optional.** Make a movie of p over the time interval $0 \leq t \leq 2.5$ s. In addition, make a movie of the quantity $g = (p^2 + p_t^2 \omega^{-2})^{1/2}$ over the same interval.

(f) **Optional.** Estimate the sound level that C, G, and M hear in terms of decibels. Discuss what modifications could be made to the PDE in Eq. 2 and boundary condition in Eq. 3 to account for sound attenuation.

3. **Solving a nonlinear boundary value problem (BVP).** Consider the nonlinear ordinary differential equation BVP

$$u''(x) = e^{u(x)}, \qquad x \in (-1, 1), \tag{8}$$

with zero Dirichlet boundary conditions $u(-1) = u(1) = 0$, and introduce an n-point grid $x_i = -1 + ih$ where $h = \frac{2}{n-1}$.

A finite-difference approximation gives the nonlinear system $F(U) = 0$, where $U \in \mathbb{R}^{n-2}$ is the finite-difference solution vector after the two boundary terms $U_0 = U_{n-1} = 0$ are dropped since they are already known. The components of the nonlinear function $F : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ are

$$F_1(U) = \frac{U_2 - 2U_1}{h^2} - e^{U_1} = 0, \tag{9}$$

$$F_i(U) = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} - e^{U_i} = 0, \quad i = 2, 3, \ldots, n-3, \tag{10}$$

$$F_{n-2}(U) = \frac{-2U_{n-2} + U_{n-3}}{h^2} - e^{U_{n-2}} = 0. \tag{11}$$

(a) Derive the Jacobian $J_F \in \mathbb{R}^{(n-2)\times(n-2)}$ for the system $F(U) = 0$. Describe the sparsity pattern of J_F.

(b) Use Newton's method to solve this nonlinear ODE BVP for $n = 101$, using the Jacobian matrix derived in (a). Start with an initial guess $U^0 = 0$. Terminate Newton's method once a relative step size, $\|\Delta U^k\|_2 / \|U^k\|_2$, satisfies $\|\Delta U^k\|_2 / \|U^k\|_2 \leq 10^{-10}$, where k refers to the Newton iteration count. Plot the approximate solution U (padded with U_0 and U_{n-1}) of the ODE BVP above on the n-point grid, and report your approximation to $u(0)$ to three significant digits.

4. **Density-equalized map making using the diffusion equation.** In many news articles and publications, it is common practice to distort maps to emphasize certain features. For example, when discussing US politics, one may wish to plot the states so that their area is proportional their population, so highly populated states like California and Texas appear significantly larger than sparsely populated states like Alaska and Wyoming. There are many procedures for doing this, and in a recent paper Gastner and Newman[3] show that this can be achieved

[3] M. T. Gastner and M. E. J. Newman, Proc. Natl. Acad. Sci. **101** 7499–7504 (2004). doi:10.1073/pnas.0400280101

simply by solving the diffusion equation. Newman has many examples on his website,[4] which have be covered in the popular press.

In the Gastner & Newman approach, one first introduces a density field $\rho(\mathbf{x}, t)$ where $\mathbf{x}$ is a two-dimensional vector. This could, for example, be the state-by-state population density in the US. A simple method to equalize the density field is to solve the diffusion equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j} = \nabla^2 \rho. \tag{12}$$

Here $\mathbf{j} = -\nabla \rho$ is the density flux, as given by Fick's law.[5] Any tracers that are carried by this density flux will move with velocity

$$\mathbf{v}(\mathbf{x}, t) = \frac{\mathbf{j}}{\rho} = -\frac{\nabla \rho}{\rho}. \tag{13}$$

If Eq. 12 is solved to steady state, then ρ will have equalized density. The velocity in Eq. 13 induces a deformation of the underlying map. One way to track this is to introduce a *reference map field* $\mathbf{X}(\mathbf{x}, t)$ where $\mathbf{X} = (X, Y)$. The value $\mathbf{X}(\mathbf{x}, t)$ gives the position of the tracer at time 0 that is now located at $\mathbf{x}$. Initially $\mathbf{X}(\mathbf{x}, 0) = \mathbf{x}$ and then the field satisfies the advection equation

$$\frac{\partial \mathbf{X}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{X} = \mathbf{0}. \tag{14}$$

If the original map image is described by a function $\mathbf{C}(\mathbf{x})$ that gives the color of a pixel at $\mathbf{x}$, then the deformed image at some final time T will given by $\mathbf{C}(\mathbf{X}(\mathbf{x}, T))$.

(a) Write a program to solve Eq. 12 for the density $\rho(\mathbf{x}, t)$ the domain $[0, 31]^2$, discretized 32×32 grid with grid spacing $h = 1$ where $\rho_{i,j}$ corresponds to $\rho(hi, hj)$ for $i, j \in \{0, 1, \ldots, 31\}$. Use the initial condition of

$$\rho(\mathbf{x}, t) = \begin{cases} 5 & \text{if } x < 15.5 \text{ and } y < 15.5, \\ 1 & \text{if } x \geq 15.5 \text{ and } y < 15.5, \\ 2 & \text{if } x < 15.5 \text{ and } y \geq 15.5, \\ 3 & \text{if } x \geq 15.5 \text{ and } y \geq 15.5. \end{cases} \tag{15}$$

Use a timestep of $\Delta t = 0.24$. At the boundaries, apply the no-flux condition $\mathbf{n} \cdot \nabla \rho = 0$ where $\mathbf{n}$ is an outward-pointing unit normal vector. This can be implemented using the same discretization procedure as for the Laplacian term in question 2. Simulate the equation up to $t = 512$ using the forward Euler method,[6] and make a plot showing $\min_{i,j} \rho_{i,j}$ and $\max_{i,j} \rho_{i,j}$ as a function of t.

(b) Extend your program from part (a) to simultaneously solve Eq. 14 using the forward Euler method. Here, $\mathbf{v}$ at each gridpoint should be computed using centered differences of ρ. If not enough points of ρ are available, then the velocity component should be

[4] http://www-personal.umich.edu/~mejn/election/2016/

[5] Here, the diffusion constant in Fick's law is set to 1, which can be achieved through an appropriate rescaling of the time variable.

[6] Note that an integer multiple of Δt does not exactly match 512. To resolve this, you can either use a tiny modification to the timestep of $\Delta t' = 512/(\lceil \frac{512}{\Delta t} \rceil)$, or you can use the given Δt then take one smaller final timestep to get exactly to $t = 512$.

set to zero.[7] The $\nabla \mathbf{X}$ term should use first-order one-sided finite differences, where the direction is chosen based on upwinding considerations.

After solving the combined system for ρ and $\mathbf{X}$, for each of the time points $t = 0$, $t = 16$, $t = 32$, $t = 512$, make a separate plot of the contours of both X and Y superimposed on the same graph. Describe how the figure at $t = 512$ represents a density equalized map, paying particular attention to the contours $X = 15.5$ and $Y = 15.5$.

(c) The program files contain several color-coded images of the US map, `usa_vs.png`, `usa_sm.png`, `usa_md.png`, and `usa_lg.png`, of varying size (m, n). Each state is colored with a unique color, and points exterior to any state are colored white. Begin by considering the smallest one, `usa_vs.png`. The program files also contain a file `colchart.txt` containing the (R, G, B) values for each state, and a file `density.txt` containing the population density of each state, in terms of people per square mile. Write a code to set the initial condition as

$$\rho_{i,j} = \begin{cases} \text{Population density of state } k & \text{if } (i,j) \text{ is in state } k, \\ \bar{\rho} & \text{if } (i,j) \text{ is exterior to any state,} \end{cases} \tag{16}$$

where

$$\bar{\rho} = \frac{\sum_{\text{Interior } i,j} (\rho_{i,j})}{\text{Number of interior gridpoints}}. \tag{17}$$

Use $h = 1$. Solve the system of Eqs. 12 & 14 up to $t = \frac{1}{12}(m^2 + n^2)$. To take into account large velocities during the first few steps, use 24 timesteps of size 0.01, and then switch to timesteps of 0.24 from then on. Make a plot of the deformed map image by coloring (i, j) according to the nearest pixel to the position $\mathbf{X}_{i,j}$.

(d) **Optional.** Determine a procedure to add black outlines to each state in your plot from part (c).

(e) **Optional.** Repeat your calculation for one of the larger images. This may require some code optimization or a faster method such as ADI to do in a reasonable timeframe.

(f) **Optional.** Use your code to plot a different statistic on the US map. Construct a map for a different region of the world.

[7] Specifically, the x component of velocity should zero when $i = 0$ or $i = 31$ and the y component of velocity should zero when $j = 0$ or $j = 31$.