

Assignment-5

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1

$$\begin{aligned}f(x) &= -5x^2 + 5x \\f^*(y) &= \sup_{x \in \mathbb{R}} (yx - f(x)) \\&= \sup_{x \in \mathbb{R}} (yx + 5x^2 - 5x) \\&= +\infty, \quad (x \rightarrow +\infty)\end{aligned}$$

Can't plot.

2

$\because a \leq x \leq b, f$ is convex

$$\begin{aligned}
\therefore f(x) &= f\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right) \\
&= f\left(\frac{b-x}{b-a}a + \left(1 - \frac{b-x}{b-a}\right)b\right) \\
&\leq \frac{b-x}{b-a}f(a) + \left(1 - \frac{b-x}{b-a}\right)f(b) \\
&= \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)
\end{aligned}$$

3

$$\text{dom } h = \text{dom } f \cap \text{dom } g$$

$\because \text{dom } f, \text{dom } g$ are convex

$\therefore \text{dom } h$ is convex

$$\forall (x < y) \in \text{dom } h, k \in [0, 1]$$

$$\begin{aligned}
h(kx + (1-k)y) &= f(kx + (1-k)y) + g(kx + (1-k)y) \\
&\leq kf(x) + (1-k)f(y) + kg(x) + (1-k)g(y) \\
&= k(f(x) + g(x)) + (1-k)(f(y) + g(y)) \\
&= kh(x) + (1-k)h(y)
\end{aligned}$$

Q.E.D.

4

$$\text{dom } f = (x, y), \quad x, y > 0$$

$\therefore \text{dom } f$ is convex

$$(x \log x)' = \log x + 1 > 0, \quad x > 0$$

$$(x \log x)'' = \frac{1}{x} > 0, \quad x > 0$$

$\therefore x \log x$ is convex

$$\forall (x_1, y_1), (x_2, y_2) \in \text{dom } f, k \in [0, 1]$$

$$\begin{aligned} f(k(x_1, y_1) + (1 - k)(x_2, y_2)) &= f(kx_1 + (1 - k)x_2, ky_1 + (1 - k)y_2) \\ &= (kx_1 + (1 - k)x_2) \log (kx_1 + (1 - k)x_2) \\ &\quad + (ky_1 + (1 - k)y_2) \log (ky_1 + (1 - k)y_2) \\ &\leq (kx_1 \log x_1 + (1 - k)x_2 \log x_2) \\ &\quad + (ky_1 \log y_1 + (1 - k)y_2 \log y_2) \\ &= k(x_1 \log x_1 + y_1 \log y_1) \\ &\quad + (1 - k)(x_2 \log x_2 + y_2 \log y_2) \\ &= kf(x_1, y_1) + (1 - k)f(x_2, y_2) \end{aligned}$$

Q.E.D.

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