Assignment-4

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1

$$
\begin{gathered}
\because C \text{ is midpoint convex} \\
\therefore \forall a, b \in C,\ c = \frac{a+b}{2} \in C \\
\therefore d = \frac{a+c}{2} = \frac{3a+b}{4} \in C,\ e = \frac{c+b}{2} = \frac{a+3b}{4} \in C \\
\therefore f = \frac{a+d}{2} = \frac{7a+b}{8} \in C,\ g = \frac{d+c}{2} = \frac{5a+3b}{8} \in C, \\
h = \frac{c+e}{2} = \frac{3a+5b}{8} \in C,\ i = \frac{e+b}{2} = \frac{a+7b}{16} \in C \\
\therefore A = \{\frac{xa+(2^n-x)b}{2^n} | \forall n, x \in \mathbb{N},\ x \leq 2^n\} \subset C,\ \text{(by induction)} \qquad (1)
\end{gathered}
$$

if C is not convex, $\exists k \in [0,1],\ p = ka + (1-k)b \notin C$

$$\because (1)$$

$$\therefore \forall n', x' \in \mathbb{N},\ x' \leq 2^{n'},\ k \neq \frac{x'}{2^{n'}}$$

$$\because C \text{ is closed}$$

$$\therefore \exists \delta \in \mathbb{R},\ \delta > 0,\ \mathrm{U}\left(p, \delta\right) \cap C = \varnothing$$

$$q_{t,l} = \frac{xa + (2^t - x)b}{2^t} \prec k,\ q_{t,r} = \frac{(x+1)a + (2^t - (x+1))b}{2^t} \succ k$$

$$\therefore q_{t,r} - q_{t,l} = \frac{a-b}{2^t} \succ q_{t,r} - k$$

$$t' > \log_2 \frac{\|a-b\|}{\delta},\ \|q_{t',r} - k\| < \|q_{t',r} - q_{t',l}\| < \delta$$

$$\therefore q_{t',r} \in \mathrm{U}\left(p, \delta\right)$$

Q.E.D.

2

$$i \in \mathbb{N}^*,\ x_i{}' = x_{i'},\ \theta_i{}' = \theta_{i'},\ (i+1)' > i' \geq i,\ \theta_{i'} \neq 0;\ \max i = t$$

$$\therefore \theta_i{}' > 0,\ \sum_{i=1}^{t} \theta_i{}' = 1,$$

$$\sum_{i=1}^{k} \theta_i x_i = \sum_{i=1}^{t} \theta_i{}' x_i{}'$$

$$\because C \text{ is convex}$$

$$\therefore t = 2,\ \sum_{i=1}^{k} \theta_i x_i = \sum_{i=1}^{t} \theta_i{}' x_i{}' = \theta_1' x_1' + \theta_2' x_2' \in C$$

$$\text{if } t = p\ (p < k),\ \sum_{i=1}^{t=p} \theta_i{}' x_i{}' \in C, \tag{2}$$

$$t = p+1,\ \sum_{i=1}^{k} \theta_i x_i = \sum_{i=1}^{t=p+1} \theta_i{}' x_i{}'$$

$$= \theta_{p+1}{}' x_{p+1}{}' + (\sum_{i=1}^{p} \theta_i{}')(\sum_{i=1}^{p} \frac{\theta_i{}'}{\sum_{i=j}^{p} \theta_j{}'} x_j{}')$$

$$= \theta_{p+1}{}' x_{p+1}{}' + (1 - \theta_{p+1}{}')x,\ (x = \sum_{i=1}^{p} \frac{\theta_i{}'}{\sum_{i=1}^{p} \theta_j{}'} x_j{}')$$

$$\because \sum_{i=1}^{p} \frac{\theta_i{}'}{\sum_{j=1}^{p} \theta_j{}'} = 1;\ (2)$$

$$\therefore x \in C$$

$$\because C \text{ is convex}$$

$$\therefore \sum_{i=1}^{k} \theta_i x_i = \theta_{p+1}{}' x_{p+1}{}' + (1 - \theta_{p+1}{}')x \in C$$

Q.E.D. (by induction)

3

$$A = \{X | X \text{ is convex} \wedge S \subseteq X\}$$

$$\because \operatorname{conv}(S) \text{ is convex; } S \subseteq \operatorname{conv}(S)$$

$$\therefore \operatorname{conv}(S) \in A$$

$$\therefore \bigcap_{X \in A} X \subseteq \operatorname{conv}(S)$$

$$\because \forall X \in A,\ S \subseteq X$$

$$\therefore \forall x, y \in S,\ x, y \in X$$

$$\because X \text{ is convex}$$

$$\therefore \forall \theta_1, \theta_2 \in [0,1],\ \theta_1 x + \theta_2 y \in \operatorname{conv}(S), X$$

$$\therefore \operatorname{conv}(S) \subseteq \bigcap_{X \in A} X$$

Q.E.D.