

Assignment-4

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$\because C$ is midpoint convex

$$\therefore \forall a, b \in C, c = \frac{a+b}{2} \in C$$

$$\therefore d = \frac{a+c}{2} = \frac{3a+b}{4} \in C, e = \frac{c+b}{2} = \frac{a+3b}{4} \in C$$

$$\therefore f = \frac{a+d}{2} = \frac{7a+b}{8} \in C, g = \frac{d+c}{2} = \frac{5a+3b}{8} \in C,$$

$$h = \frac{c+e}{2} = \frac{3a+5b}{8} \in C, i = \frac{e+b}{2} = \frac{a+7b}{16} \in C$$

$$\therefore A = \left\{ \frac{xa + (2^n - x)b}{2^n} \mid \forall n, x \in \mathbb{N}, x \leq 2^n \right\} \subset C, \text{ (by induction)} \quad (1)$$

if C is not convex, $\exists k \in [0, 1]$, $p = ka + (1 - k)b \notin C$

$\therefore (1)$

$\therefore \forall n', x' \in \mathbb{N}, x' \leq 2^{n'}, k \neq \frac{x'}{2^{n'}}$

$\therefore C$ is closed

$\therefore \exists \delta \in \mathbb{R}, \delta > 0, U(p, \delta) \cap C = \emptyset$

$q_{t,l} = \frac{xa + (2^t - x)b}{2^t} \prec k, q_{t,r} = \frac{(x+1)a + (2^t - (x+1))b}{2^t} \succ k$

$\therefore q_{t,r} - q_{t,l} = \frac{a-b}{2^t} \succ q_{t,r} - k$

$t' > \log_2 \frac{\|a-b\|}{\delta}, \|q_{t',r} - k\| < \|q_{t',r} - q_{t',l}\| < \delta$

$\therefore q_{t',r} \in U(p, \delta)$

Q.E.D.

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$$\begin{aligned}
& i \in \mathbb{N}^*, \quad x_i' = x_{i'}, \quad \theta_i' = \theta_{i'}, \quad (i+1)' > i' \geq i, \quad \theta_{i'} \neq 0; \quad \max i = t \\
& \therefore \theta_i' > 0, \quad \sum_{i=1}^t \theta_i' = 1, \\
& \sum_{i=1}^k \theta_i x_i = \sum_{i=1}^t \theta_i' x_i' \\
& \because C \text{ is convex} \\
& \therefore t = 2, \quad \sum_{i=1}^k \theta_i x_i = \sum_{i=1}^t \theta_i' x_i' = \theta_1' x_1' + \theta_2' x_2' \in C \\
& \text{if } t = p \ (p < k), \quad \sum_{i=1}^{t=p} \theta_i' x_i' \in C, \tag{2} \\
& t = p + 1, \quad \sum_{i=1}^k \theta_i x_i = \sum_{i=1}^{t=p+1} \theta_i' x_i' \\
& \quad = \theta_{p+1}' x_{p+1}' + \left(\sum_{i=1}^p \theta_i' \right) \left(\sum_{i=1}^p \frac{\theta_i'}{\sum_{j=1}^p \theta_j'} x_j' \right) \\
& \quad = \theta_{p+1}' x_{p+1}' + (1 - \theta_{p+1}') x, \quad (x = \sum_{i=1}^p \frac{\theta_i'}{\sum_{j=1}^p \theta_j'} x_j') \\
& \therefore \sum_{i=1}^p \frac{\theta_i'}{\sum_{j=1}^p \theta_j'} = 1; \tag{2} \\
& \therefore x \in C \\
& \because C \text{ is convex} \\
& \therefore \sum_{i=1}^k \theta_i x_i = \theta_{p+1}' x_{p+1}' + (1 - \theta_{p+1}') x \in C \\
& \text{Q.E.D. (by induction)}
\end{aligned}$$

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