Assignment-5

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1

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\begin{aligned}
f(x) &= -5x^2 + 5x \\
f^*(y) &= \sup_{x\in\mathbb{R}} \left(yx - f(x)\right) \\
&= \sup_{x\in\mathbb{R}} \left(yx + 5x^2 - 5x\right) \\
&= +\infty,\ (x \to +\infty)
\end{aligned}
$$

Can't plot.

2

$$
\begin{aligned}
&\because a \leq x \leq b,\ \ f \text{ is convex} \\
\therefore f(x) &= f(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b) \\
&= f(\frac{b-x}{b-a}a + (1 - \frac{b-x}{b-a})b) \\
&\leq \frac{b-x}{b-a}f(a) + (1 - \frac{b-x}{b-a})f(b) \\
&= \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)
\end{aligned}
$$

3

$$
\begin{aligned}
\operatorname{dom} h &= \operatorname{dom} f \cap \operatorname{dom} g \\
\because \operatorname{dom} f, \operatorname{dom} g &\text{ are convex} \\
\therefore \operatorname{dom} h &\text{ is convex} \\
\forall (x < y) &\in \operatorname{dom} h, k \in [0,1] \\
h(kx + (1-k)y) &= f(kx + (1-k)y) + g(kx + (1-k)y) \\
&\leq kf(x) + (1-k)f(y) + kg(x) + (1-k)g(y) \\
&= k(f(x) + g(x)) + (1-k)(f(y) + g(y)) \\
&= kh(x) + (1-k)h(y) \\
&\text{Q.E.D.}
\end{aligned}
$$

4

$$
\begin{aligned}
\operatorname{dom} f &= (x, y),\ \ x, y > 0 \\
\therefore \operatorname{dom} f &\ \text{is convex} \\
(x \log x)' &= \log x + 1 > 0,\ \ x > 0 \\
(x \log x)'' &= \frac{1}{x} > 0,\ \ x > 0 \\
\therefore x \log x &\ \text{is convex} \\
\forall (x_1, y_1), (x_2, y_2) &\in \operatorname{dom} f, k \in [0, 1] \\
f(k(x_1, y_1) + (1-k)(x_2, y_2)) &= f(kx_1 + (1-k)x_2, ky_1 + (1-k)y_2) \\
&= (kx_1 + (1-k)x_2) \log (kx_1 + (1-k)x_2) \\
&\quad + (ky_1 + (1-k)y_2) \log (ky_1 + (1-k)y_2) \\
&\le (kx_1 \log x_1 + (1-k)x_2 \log x_2) \\
&\quad + (ky_1 \log y_1 + (1-k)y_2 \log y_2) \\
&= k(x_1 \log x_1 + y_1 \log y_1) \\
&\quad + (1-k)(x_2 \log x_2 + y_2 \log y_2) \\
&= kf(x_1, y_1) + (1-k)f(x_2, y_2) \\
&\text{Q.E.D.}
\end{aligned}
$$

5

$$
\begin{aligned}
{\|y-\theta^T X\|_2}^2 &= (y-\theta^T X)^T (y-\theta^T X) \\
&= y^T y - y^T \theta^T X - X^T \theta y + X^T \theta \theta^T X \\
&= (X^T\theta)^2 - 2y\theta^T X + y^2 \\
&= \theta^T (XX^T)\theta - (2yX^T)\theta + y^2 \\
XX^T &\in S_+^n \\
\|\theta\|_1 &= \sum_{i=1}^{n} |\theta_i| \\
&\geq p^T\theta,\ \ ((\forall p, |p_i| = 1) \wedge (\exists p, p^T\theta = \|\theta\|_1)) \\
\therefore \|\theta\|_1 \leq t &\iff \forall p, |p_i| = 1, p^T\theta \leq t \\
Q &= ({}_1p, {}_2p, {}_3p, \cdots, {}_{2^n}p)^T, \\
(\forall i \neq j &\in [1, 2^n], ({}_ip \neq {}_jp) \wedge (\forall k, |{}_ip_k| = 1))
\end{aligned}
$$

$\therefore$ Lasso regression:

$$
\text{Max.}\ \frac{1}{2}\theta^T(2XX^T)\theta + (-2yX)^T\theta + y^2
$$

$$
\text{s.t.} \begin{cases} Q\theta \leq (t, t, t, \cdots, t)^T, 2XX^T \in {S_+}^n \\ y \in \mathbb{R}, X \in \mathbb{R}^{n+1}, \theta \in \mathbb{R}^{n+1} \end{cases}
$$