1) $y'' - 9y = 20\,\text{sen}\,x + 3x^2$

Solucion Homogenea = $y'' - 9y = 0$; $y = e^{mx}$

$e^{mx}(m^2-9) = 0 \iff m^2 - 9 = 0 \Rightarrow$ $m^2 = 9$, $m = \pm\sqrt{9}$, $m = \pm 3$ ($m_1 = 3$, $m_2 = -3$)

$$\boxed{y_h(x) = C_1 e^{3x} + C_2 e^{-3x}}$$

Solucion Particular: operador anulador de: $20\,\text{sen}\,x + 3x^2$

$D^3(D^2+1)(20\,\text{sen}\,x + 3x^2) = 0$, $D = \frac{d}{dx}$

$m^3(m^2+1) = 0 \iff m_3 = m_4 = m_5 = 0$ (multiplicidad 3)

$m^2 + 1 = 0 \Rightarrow m \pm \sqrt{-1} = \pm\sqrt{i^2} = \pm i$

$m_5 = i$; $m_6 = -i \rightarrow$ Raiz compleja conjugada

$y_p(x) = C_3 + C_4 x + C_5 x^2 + C_6\,\text{sen}\,x + C_7 \cos x$

Sustituyendo $y_p(x)$ en la ecuacion diferencial inicial:

$y'_p(x) = C_4 + 2C_5 x + C_6 \cos x - C_7\,\text{sen}\,x$

$y''_p(x) = 2C_5 - C_6\,\text{sen}\,x - C_7 \cos x$

$y''_p - 9y_p = 20\,\text{sen}\,x + 3x^2$

$2C_5 - C_6\,\text{sen}\,x - C_7\cos x - 9(C_3 + C_4 x + C_5 x^2 + C_6\,\text{sen}\,x + C_7 \cos x) = 20\,\text{sen}\,x + 3x^2$

$2C_5 - C_6\,\text{sen}\,x - C_7 \cos x - 9C_3 - 9C_4 x - 9C_5 x^2 - 9C_6\,\text{sen}\,x - 9C_7 \cos x = 20\,\text{sen}\,x + 3x^2$

$2C_5 - 9C_3 - 10C_6\,\text{sen}\,x - 10C_7 \cos x - 9C_4 x - 9C_5 x^2 = 20\,\text{sen}\,x + 3x^2$

T.I = $2C_5 - 9C_3 = 0 \Rightarrow 2C_5 = 9C_3 \Rightarrow C_5 = \frac{9}{2}C_3$ (I) $= C_3 = \frac{2}{9}C_5$

$x = -9C_4 = 0 \Rightarrow C_4 = 0$ (II)

$C_3 = \frac{2}{9}\left(-\frac{1}{3}\right)$

$x^2 = -9C_5 = 3 \Rightarrow C_5 = -\frac{3}{9} = -\frac{1}{3} \Rightarrow C_5 = -\frac{1}{3}$ (III)

$C_3 = -\frac{2}{27}$

$\text{sen}\,x = -10C_6 = 20 \Rightarrow C_6 = -\frac{20}{10} = -2 \Rightarrow C_6 = -2$ (IV)

$\cos x = -10C_7 = 0 \Rightarrow C_7 = 0$ (V)

$\Rightarrow y_p(x) = -\frac{2}{27} - \frac{1}{3}x^2 - 2\,\text{sen}\,x$

Solucion general = $\boxed{Y_g = C_1 e^{3x} + C_2 e^{-3x} - \frac{2}{27} - \frac{x^2}{3} - 2\,\text{sen}\,x}$

② $y'' - 4y' + 2y = 5e^{x}$

solucion homogenea= $y'' - 4y' + 2y = 0$; $y = e^{mx}$

$$e^{mx}(m^2 - 4m + 2) = 0 \iff m^2 - 4m + 2 = 0$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} = \frac{4 \pm \sqrt{16+8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$m_1 = 2 + \sqrt{2}$ $\quad m_2 = 2 - \sqrt{2}$

$$\boxed{y_p(x) = C_1 e^{(2+\sqrt{2})x} + C_2 e^{(2-\sqrt{2})x}}$$

solucion Particular: Operador anulador de: $5e^{x}$: $(D-1)(5e^{x}) = 0$, $D = \frac{d}{dx}$

$$m - 1 = 0 \Rightarrow m_3 = 1$$

$$\Rightarrow y_p(x) = C_3 e^{x}$$

$$y'_p(x) = C_3 e^{x}$$

$$y''_p(x) = C_3 e^{x}$$

Sustituyendo en la ecuacion diferencial inicial:

$$y_p'' - 4y'_p + 2y_p = 5e^{x}$$

$$C_3 e^{x} - 4C_3 e^{x} + 2C_3 e^{x} = 5e^{x}$$

$$-C_3 e^{x} = 5e^{x} \Rightarrow C_3 = -5$$

$$y_p(x) = -5e^{x}$$

Solucion general= $\boxed{y_g = C_1 e^{(2+\sqrt{2})x} + C_2 e^{(2-\sqrt{2})x} - 5e^{x}}$

③ $y'' - 4y' + 4y = t^3 \qquad y(0)=1, \quad y'(0)=0$

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3\}$$

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) + 4Y(s) = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$s^2Y(s) - s - 4sY(s) + 4 + 4Y(s) = \frac{6}{s^4}$$

$$(s^2-4s+4)Y(s) - s + 4 = \frac{6}{s^4}$$

Fracciones Parciales

$$Y(s) = \frac{6}{s^4(s^2-4s+4)} + \frac{s}{(s^2-4s+4)} - \frac{4}{(s^2-4s+4)} = \frac{6+s^5-4s^4}{s^4(s^2-4s+4)}$$

$(s-2)(s-2)$

$(s-2)^2$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s^4} + \frac{E}{s-2} + \frac{F}{(s-2)^2}$$

$$Y(s) = \frac{As^3(s-2)^2 + Bs^2(s-2)^2 + Cs(s-2)^2 + D(s-2)^2 + Es^4(s-2) + Fs^4}{s^4(s-2)^2}$$

$$As^3(s^2-4s+4) + Bs^2(s^2-4s+4) + Cs(s^2-4s+4) + D(s^2-4s+4) + Es^4(s-2) + Fs^4$$

$$= As^5 - 4As^4 + 4As^3 + Bs^4 - 4Bs^3 + 4Bs^2 + Cs^3 - 4Cs^2 + 4Cs - Ds^2 - 4Ds + 4D + Es^5 - 2Es^4 + Fs^4$$

$$= (A+E)s^5 + (B-4A-2E+F)s^4 + (4A-4B+C)s^3 + (4B-4C+D)s^2 + (4C-4D)s + 4D$$

$\text{T.I} = 4D = 6 \Rightarrow D = \frac{6}{4} = \frac{3}{2}$

$s = 4C - 4D = 0 \Rightarrow 4C = 4D = C = 3/2$

$s^2 = 4B - 4C + D = 0 \Rightarrow 4B = 4C - D = 4(\frac{3}{2}) - \frac{3}{2} = \frac{9}{2} \Rightarrow B = 9/8$

$s^3 = 4A - 4B + C = 0 \Rightarrow 4A = 4B - C = 4(9/8) - \frac{3}{2} = 3 \Rightarrow A = 3/4$

$s^4 = B - 4A - 2E + F = -4 \Rightarrow -2E + F = -4 - B + 4A$

$-2E + F = -4 - \frac{9}{8} + 4(\frac{3}{4}) = \frac{-17}{8} \Rightarrow -2E + F = \frac{-17}{8}$

$s^5 = A + E = 1 \Rightarrow E = 1 - A = 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow E = \frac{1}{4}$

$F = -13/8 \Rightarrow F = -\frac{17}{8} + 2E = -\frac{17}{8} + 2(\frac{1}{4}) = -13/8$

$$Y(s) = \frac{3}{4s} + \frac{9}{8s^2} + \frac{3}{2s^3} + \frac{3}{2s^4} + \frac{1}{4(s-2)} - \frac{13}{8(s-2)^2}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{9}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}$$
$$+ \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{13}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$\boxed{y(t) = \frac{3}{4} + \frac{9}{8}t + \frac{3}{4}t^2 + \frac{1}{4}t^3 + \frac{1}{4}e^{2t} - \frac{13}{8}te^{2t}}$$