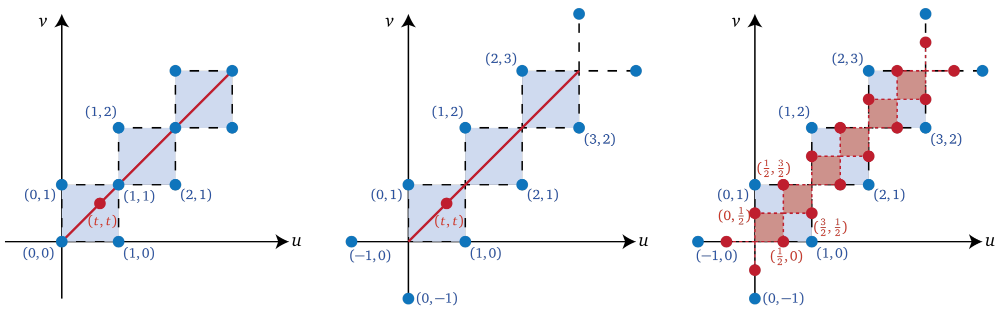
CSE 167 (FA 2022) Exercise 7 — Due 11/9/2022

In the lecture, we learned about cubic blossom representation for cubic Bézier curves and cubic B-splines. In this exercise, we explore its simpler version—quadratic blossom. Instead of considering a symmetric tri-affine function $\mathbf{F}(u, v, w)$, we consider a symmetric bi-affine function $\mathbf{F}(u, v)$.

A position-valued function of two parameters $\mathbf{F}(u, v)$ is said to be *symmetric bi-affine* if

- $\mathbf{F}(u, v) = \mathbf{F}(v, u)$ for all u, v.
- $\mathbf{F}(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 \mathbf{F}(u_1, v) + \lambda_2 \mathbf{F}(u_2, v)$ where $\lambda_1 + \lambda_2 = 1, 0 \leq \lambda_1, \lambda_2 \leq 1$.

The control points are some given values of $\mathbf{F}(u, v)$ at a few tuples (u, v), and the resulting spline is the evaluation of $\mathbf{f}(t) = \mathbf{F}(t, t)$ for a continuous range of t. The value of $\mathbf{F}(t, t)$ is computed by interpolation using the symmetry and the bi-affine property of $\mathbf{F}$. We can visualize these control points (blue) and the spline points (red) on the uv plane as shown in Figure 1: Bilinear interpolate the value at the red point using the values at the blue points.[1]



(a) Bézier curve **(b)** B-spline **(c)** Subdivision for B-Spline

Figure 1 The blue points are the locations where the value of $\mathbf{F}(u, v)$ is given. By bilinear interpolation along u, v directions within the light-blue regions, one can obtain the value of $\mathbf{F}$ at the red points.

For the quadratic Bézier curve, the control points are $\mathbf{F}(0, 0)$, $\mathbf{F}(0, 1) = \mathbf{F}(1, 0)$, $\mathbf{F}(1, 1)$, $\mathbf{F}(1, 2) = \mathbf{F}(2, 1)$, $\mathbf{F}(2, 2)$, $\mathbf{F}(2, 3) = \mathbf{F}(3, 2)$,. . .. As shown in Figure 1(a), the final spline will pass through a few control points, and is broken into pieces where each piece has the bilinear interpolated result computed using the neighboring blue points.

The B-spline (Figure 1(b)) has a different specification for the control points, defining another piecewise quadratic curve. This time, the spline does not pass through any control point.

The subdivision (Figure 1(c)) updates the control points to approximately twice as many points (the red points) where the values of $\mathbf{F}$ are also computed through bilinear interpolation from the blue points. The red control points will define the same B-spline as the blue control points. As one repeat the subdivision process, one gets dense points closely following the diagonal, and hence the subdivision curve converges to the B-spline.

[1]Note that the $\mathbf{F}$ is bi-affine in a way that $\mathbf{F}(u, v)$ is affine in u when v is fixed, and $\mathbf{F}$ is affine in v when u is fixed. So we can only perform linear(affine) interpolation along line segments parallel to the u, v axes. In particular, interpolation along line segment of any other direction is not expected to be linear. You may observe that de Casteljau's algorithm is exactly performing linear interpolations step by step by moving along axes.

Exercise 7.1 — 1 pt. (Bézier)

Let $\mathbf{F}(u, v)$ be a symmetric bi-affine $\mathbb{R}^2$-valued function with

$$\mathbf{F}(0,0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \mathbf{F}(1,0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{F}(1,1) = \begin{bmatrix} 8 \\ 4 \end{bmatrix}. \tag{1}$$

What is $\mathbf{F}(\frac{1}{2}, \frac{1}{2})$? ▪

Exercise 7.2 — 2 pts. (B-spline)

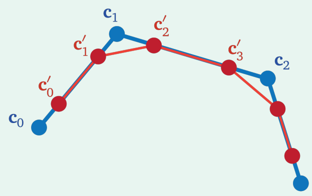
Let $\mathbf{F}(u, v)$ be a symmetric bi-affine $\mathbb{R}^2$-valued function with

$$\mathbf{F}(-1,0) = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \quad \mathbf{F}(0,1) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \quad \mathbf{F}(1,2) = \begin{bmatrix} 8 \\ 4 \end{bmatrix}. \tag{2}$$

What is $\mathbf{F}(\frac{1}{2}, \frac{1}{2})$? ▪

Exercise 7.3 — 1 pt. (Chaikin's corner cutting algorithm)

Look at Figure 1(c). Given 2 consecutive control points $\mathbf{c}_0 = \mathbf{F}(-1, 0), \mathbf{c}_1 = \mathbf{F}(0, 1)$, we can construct 2 control points $\mathbf{c}'_0 = \mathbf{F}(-\frac{1}{2}, 0), \mathbf{c}'_1 = \mathbf{F}(0, \frac{1}{2})$ for the subdivided curve.



The new control points are given by some weighted average of the old control points $\mathbf{c}'_0 = \lambda_0\mathbf{c}_0 + \lambda_1\mathbf{c}_1$, and $\mathbf{c}'_1 = \mu_0\mathbf{c}_0 + \mu_1\mathbf{c}_1$. What are the coefficients $\lambda_0, \lambda_1, \mu_0, \mu_1$?

Hint *You probably have already used these coefficients for Exercise 7.2.*

(This subdivision scheme is Chaikin's algorithm (1974) for generating smooth curve by repeatedly shaving off corners of a polygon (as shown in the figure). Figure 1(c) is a simple way to show that it converges to the quadratic B-spline.) ▪