# CSE 167 (FA22)
# Computer Graphics:
# 3D Rotations

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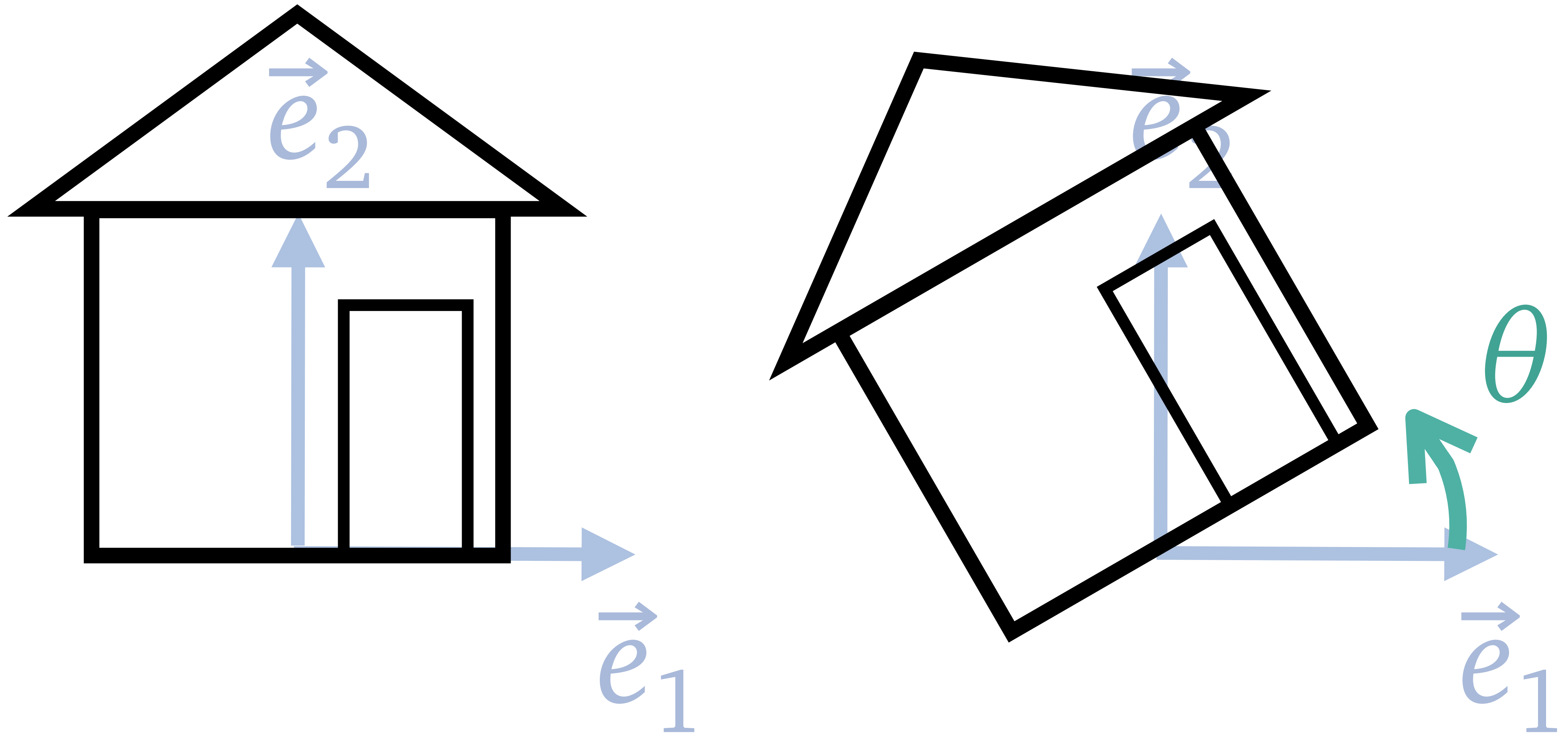
# Recall 2D rotations

- The 2D rotation matrix is

$$\mathbf{R}^{\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



# 2D rotation using complex number

$$\begin{array}{c} \text{real part} \\ \text{imaginary part} \end{array} \begin{bmatrix} \boxed{\cos\theta} & -\boxed{\sin\theta} \\ \boxed{\sin\theta} & \boxed{\cos\theta} \end{bmatrix} \begin{bmatrix} \boxed{a} \\ \boxed{b} \end{bmatrix} = \begin{bmatrix} \boxed{\cos(\theta)a - \sin(\theta)b} \\ \boxed{\sin(\theta)a + \cos(\theta)b} \end{bmatrix} \begin{array}{c} \text{real part} \\ \text{imaginary part} \end{array}$$

- We can view **2D vectors** as complex numbers

  and **rotation matrices** also as complex numbers

$$(\cos\theta + \mathbb{i}\sin\theta)(a + \mathbb{i}\,b) = \left(\cos(\theta)a - \sin(\theta)b\right) + \mathbb{i}\left(\sin(\theta)a + \cos(\theta)b\right)$$

# 2D rotation using complex number

*real part*

*real part*

$$\begin{bmatrix} \boxed{\cos\theta} & -\boxed{\sin\theta} \\ \boxed{\sin\theta} & \boxed{\cos\theta} \end{bmatrix} \begin{bmatrix} \boxed{a} \\ \boxed{b} \end{bmatrix} = \begin{bmatrix} \boxed{\cos(\theta)a - \sin(\theta)b} \\ \boxed{\sin(\theta)a + \cos(\theta)b} \end{bmatrix}$$

*imaginary part*

*imaginary part*

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$$\boxed{(\cos\theta + \mathbb{i}\sin\theta)}\boxed{(a + \mathbb{i}b)} = \big(\cos(\theta)a - \sin(\theta)b\big) + \mathbb{i}\big(\sin(\theta)a + \cos(\theta)b\big)$$

*this "rotor" must have length=1*

*arbitrary vector being rotated*

# 2D rotation using complex number

$$\boxed{(\cos\theta + \mathbb{i}\sin\theta)}\boxed{(a + \mathbb{i}b)} = \big(\cos(\theta)a - \sin(\theta)b\big) + \mathbb{i}\big(\sin(\theta)a + \cos(\theta)b\big)$$

*this "rotor" must have length=1*

*arbitrary vector being rotated*

- Euler formula $e^{\mathbb{i}\theta} = \cos\theta + \mathbb{i}\sin\theta$
- 2D rotation $\boxed{e^{\mathbb{i}\theta}}\boxed{(a + \mathbb{i}b)}$

*the rotor*

*arbitrary vector being rotated*

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \implies \theta = \arctan(\frac{v}{u})$$

- Rotor–rotation matrix conversion $\begin{bmatrix} \mathrm{Re}(e^{\mathbb{i}\theta}) & -\mathrm{Im}(e^{\mathbb{i}\theta}) \\ \mathrm{Im}(e^{\mathbb{i}\theta}) & \mathrm{Re}(e^{\mathbb{i}\theta}) \end{bmatrix}$

# What about 3D rotations

# 3D rotation

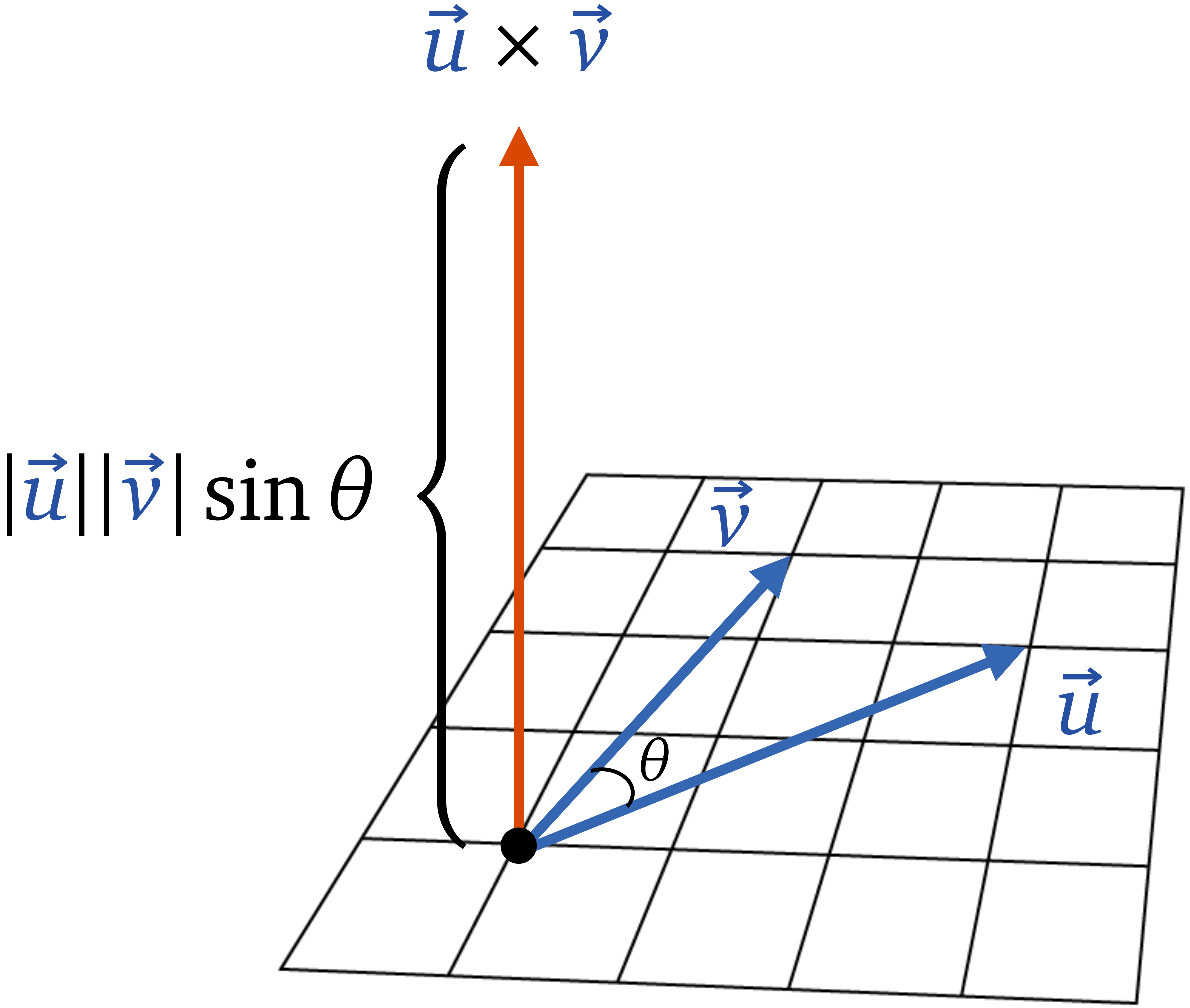
- Each 2D rotation is described by an angle → • Each 3D rotation is described by an **angle** and an **rotation axis**
- 2D rotation matrix → • Rotation matrix (Rodrigues formula)
  → • Euler angle system
- Complex numbers → • Quaternions

• Geometric algebra

# Cross Product

- Cross product
- Rodrigues formula
- More representations
- Quaternions
- Euler angles
- Geometric algebra

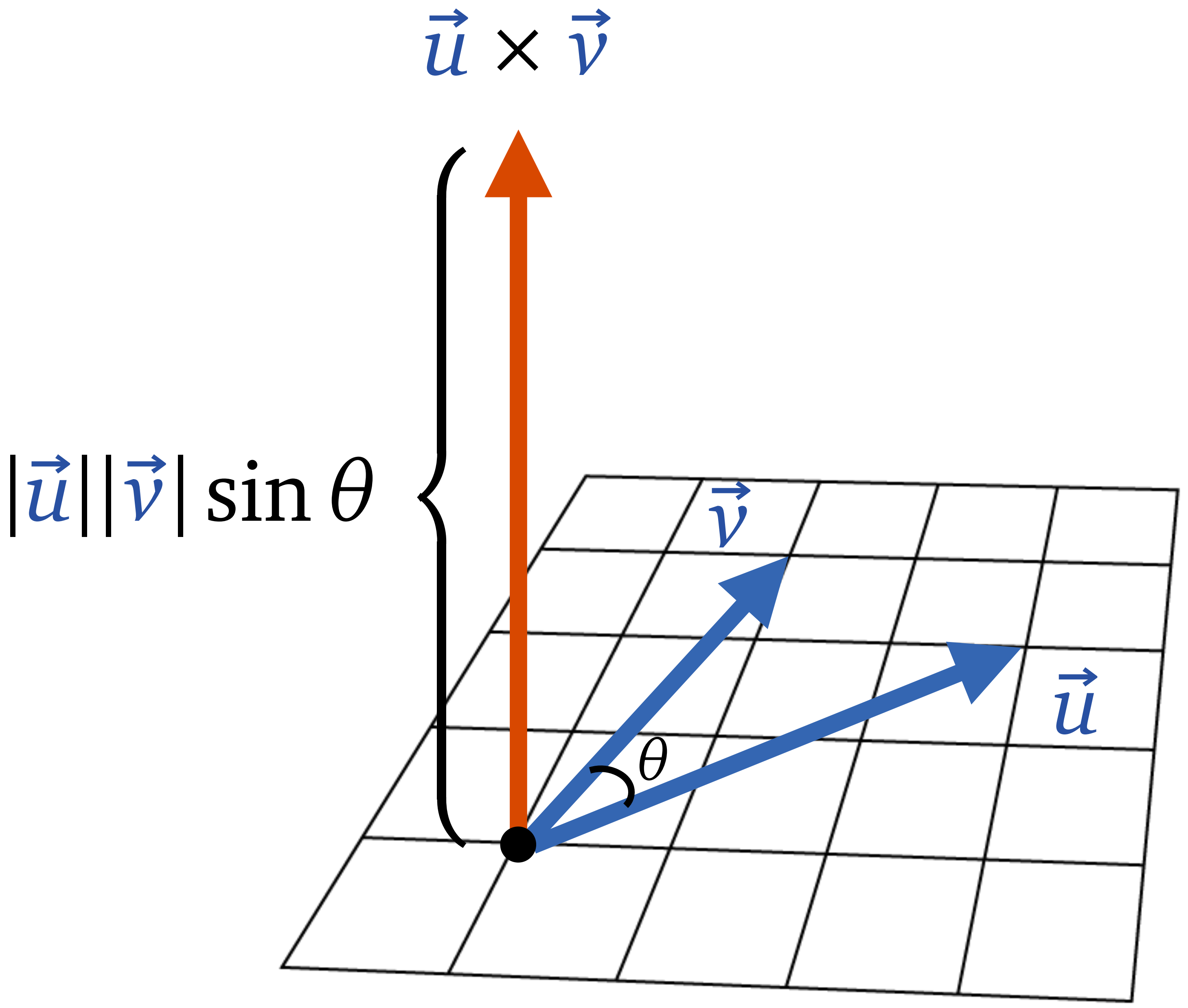
# Cross product (geometric)



# Cross product (algebraic)

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \times \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

# Cross product

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Suppose $\vec{\mathbf{e}}$ is an orthonormal basis, $\vec{u} = \vec{\mathbf{e}}^{\mathsf{T}} \mathbf{u}$, $\vec{v} = \vec{\mathbf{e}}^{\mathsf{T}} \mathbf{v}$.
Then

$$\vec{u} \times \vec{v} = \vec{\mathbf{e}}^{\mathsf{T}} (\mathbf{u} \times \mathbf{v})$$

# Cross product (properties)

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \times \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{bmatrix}$$

- Skew-symmetric $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- Non-associative. In general, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$
- Bilinear. And

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} & -u_z & u_y \\ u_z & & -u_x \\ -u_y & u_x & \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

# 3D Rotations (angle-axis)

- Cross product
- Rodrigues formula
- More representations
- Quaternions
- Euler angles
- Geometric algebra

# 3D Rotation (Rodrigues formula)

- We can describe a 3D rotation by an axis $\mathbf{a} \in \mathbb{R}^3, \ |\mathbf{a}| = 1$ and an angle $\theta \in \mathbb{R}$

- Rodrigues formula

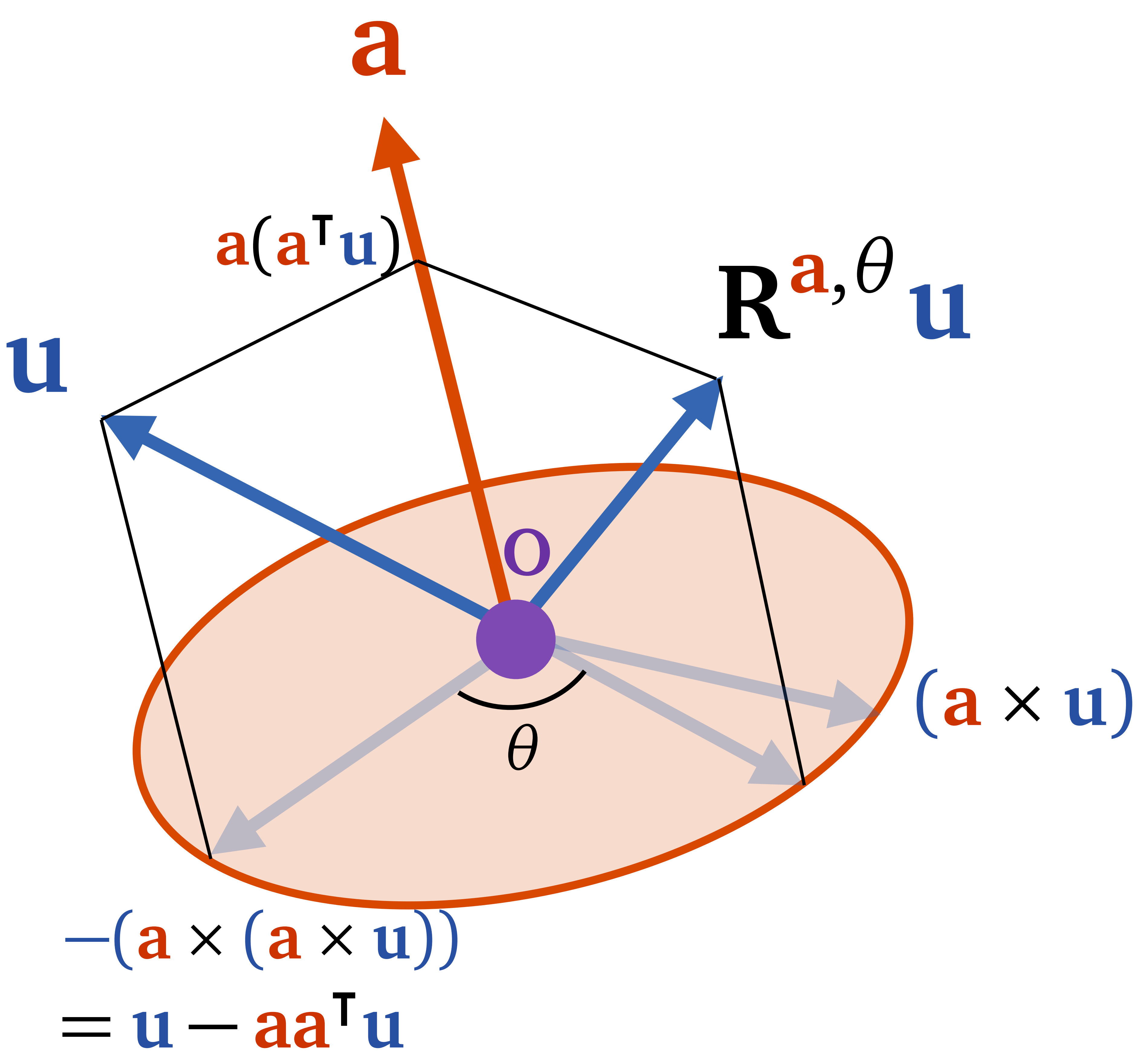
$$\mathbf{R}^{\mathbf{a},\theta}\mathbf{u}$$
$$= \mathbf{a}(\mathbf{a}^\intercal\mathbf{u}) + \cos\theta(\mathbf{u} - \mathbf{a}\mathbf{a}^\intercal\mathbf{u}) + \sin\theta(\mathbf{a} \times \mathbf{u})$$

that is,

$$\boxed{\mathbf{R}^{\mathbf{a},\theta} = \cos\theta\,\mathbf{I}_{3\times3} + (1 - \cos\theta)\mathbf{a}\mathbf{a}^\intercal + \sin\theta[\mathbf{a}\times]}$$

$\mathbf{a}\mathbf{a}^\intercal = \texttt{outerProduct}(\mathbf{a}, \mathbf{a})$

$$[\mathbf{a}\times] = \begin{bmatrix} 0 & -a_2 & a_1 \\ a_2 & 0 & -a_0 \\ -a_1 & a_0 & 0 \end{bmatrix}$$



# Other representations of 3D rotations

- Cross product
- Rodrigues formula
- **More representations**
- Quaternions
- Euler angles
- Geometric algebra

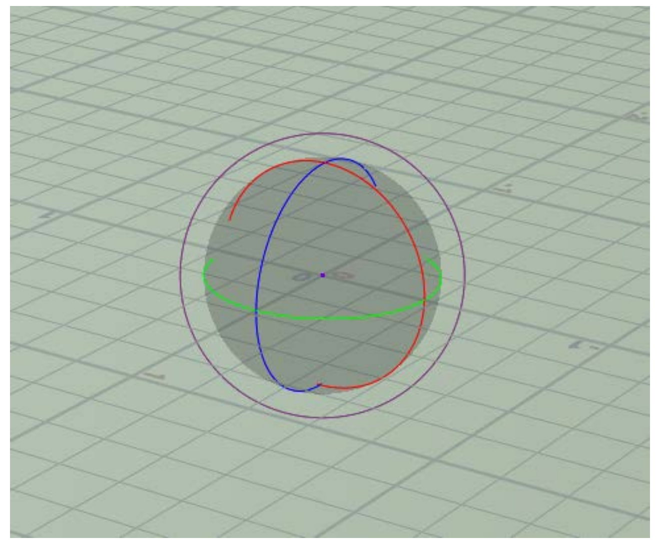
# Various ways for 3D rotations

- Rodrigues formula (angle-axis) (previous slides)

$$\mathbf{R}^{\mathbf{a},\theta} = \cos\theta\,\mathbf{I}_{3\times3} + (1-\cos\theta)\mathbf{a}\mathbf{a}^{\intercal} + \sin\theta[\mathbf{a}\times]$$

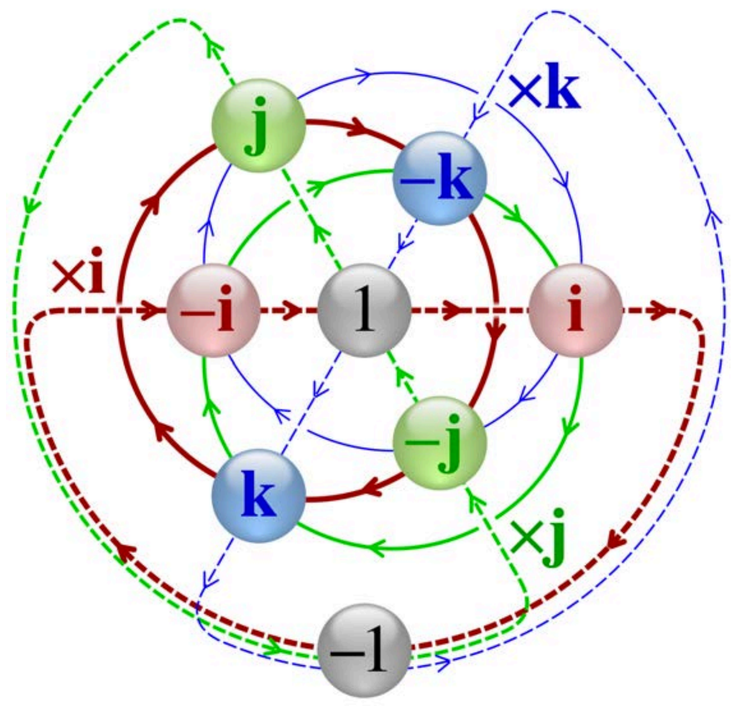
- Euler angles (3 planar rotations)

$$\mathbf{R}^{(\alpha,\beta,\gamma)} = \mathbf{R}^{\vec{e}_y,\alpha}\mathbf{R}^{\vec{e}_z,\beta}\mathbf{R}^{\vec{e}_x,\gamma}$$



- Quaternions (angle-axis)

$$R^{\mathbb{a},\theta}\,\mathbb{w} = e^{\frac{\theta}{2}\mathbb{a}}\,\mathbb{w}\,e^{-\frac{\theta}{2}\mathbb{a}}$$




- Geometric algebra (a way to understand quaternions and general rotors)

# Various ways for 3D rotations

- With any other way of doing 3D rotations, you can convert it to a 3x3 rotation matrix using the following pseudocode

# Rotating vectors is enough

```
vec3 rotate( rotation_parameters ,  vec3  v ){
   …
   return rotated_vector;
}

mat3 rotationMatrix( rotation_parameters ){
   vec3 e1 = (1,0,0); vec3 e2 = (0,1,0); vec3 e3 = (0,0,1);
   vec3 r1 = rotate( rotation_parameters, e1 );
   vec3 r2 = rotate( rotation_parameters, e2 );
   vec3 r3 = rotate( rotation_parameters, e3 );
   mat3 R;  R[0] = r1; R[1] = r2; R[2] = r3; // column major
   return R;
}
```

# Quaternions

- Cross product
- Rodrigues formula
- More representations
- Quaternions
- Euler angles
- Geometric algebra

# What is 3D version of complex number?

- 2D rotation has nice complex algebraic representation

$$\boxed{e^{\mathbb{i}\theta}}\boxed{(a + \mathbb{i}b)}$$

$$= \underbrace{\boxed{(\cos\theta + \mathbb{i}\sin\theta)}}_{\text{the rotor}}\underbrace{\boxed{(a + \mathbb{i}b)}}_{\text{arbitrary vector being rotated}} = \big(\cos(\theta)a - \sin(\theta)b\big)$$

$$+ \mathbb{i}\big(\sin(\theta)a + \cos(\theta)b\big)$$

- What is the 3D version?
  - William Rowan Hamilton (1805–1865): "how do we multiply 3D vectors…?"
  - On 10/16, 1843: Hamilton got it. "4D"

# Discovery of quaternions

Broom Bridge, Dublin, Ireland

William Rowan Hamilton

Helen Maria Bayly

# Quaternions

- A quaternion $q \in \boxed{\mathbb{H}}$ has **1D real part** and a **3D imaginary part**

  *named after Hamilton*

$$q = \underbrace{\boxed{a}}_{\text{real part}} + \underbrace{\boxed{b_1 \mathbb{i} + b_2 \mathbb{j} + b_3 \mathbb{k}}}_{\text{imaginary part}}$$

$$= a + \mathbb{b} \qquad \text{(scalar + 3D vector)}$$

- Multiplication

$$\mathbb{i}^2 = -1 \qquad \mathbb{j}^2 = -1 \qquad \mathbb{k}^2 = -1$$

$$\mathbb{i}\,\mathbb{j} = \mathbb{k} \qquad \mathbb{j}\,\mathbb{k} = \mathbb{i} \qquad \mathbb{k}\,\mathbb{i} = \mathbb{j}$$

$$\mathbb{i}\,\mathbb{j} = -\mathbb{j}\,\mathbb{i}$$

$$\mathbb{j}\,\mathbb{k} = -\mathbb{k}\,\mathbb{j}$$

$$\mathbb{k}\,\mathbb{i} = -\mathbb{i}\,\mathbb{k}$$

# Quaternions

- Product of imaginary quaternions

$$\mathbb{u} = u_1\,\mathbb{i} + u_2\,\mathbb{j} + u_3\,\mathbb{k} \quad \in \mathbf{Im}\,\mathbb{H}$$

$$\mathbb{v} = v_1\,\mathbb{i} + v_2\,\mathbb{j} + v_3\,\mathbb{k} \quad \in \mathbf{Im}\,\mathbb{H}$$

$$\begin{aligned}\mathbb{u}\mathbb{v} = & -(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ & + (u_2 v_3 - u_3 v_2)\,\mathbb{i} \\ & + (u_3 v_1 - u_1 v_3)\,\mathbb{j} \\ & + (u_1 v_2 - u_2 v_1)\,\mathbb{k} \quad = -(\mathbb{u} \cdot \mathbb{v}) + \mathbb{u} \times \mathbb{v} \quad \in \mathbb{H}\end{aligned}$$

*encapsulates both dot and cross*

# Quaternions

- Product of two quaternions

$$p = a + \mathbb{u} \quad \in \mathbb{H}$$
$$q = b + \mathbb{v} \quad \in \mathbb{H}$$

$$pq = (ab - \mathbb{u} \cdot \mathbb{v}) + (a\mathbb{v} + b\mathbb{u} + \mathbb{u} \times \mathbb{v})$$

# Properties of quaternion

$p = a + \mathbb{u} \quad \in \mathbb{H}$

- Conjugate $\quad \overline{p} := a - \mathbb{u}$
- Norm (absolute value) squared $\quad |p|^2 := \overline{p}p = a^2 + |\mathbb{u}|^2 \quad \in \mathbb{R}$
- Reciprocal $\quad p^{-1} = \dfrac{\overline{p}}{|p|^2} \qquad p^{-1}p = pp^{-1} = 1$
- Associative: $\quad (pq)r = p(qr)$
- Not commutative: In general $\ pq \neq qp$
- Conjugate reverses order $\quad \overline{pq\cdots r} = \overline{r}\cdots\overline{q}\,\overline{p}$
- Norm distributes over product $\ |pq\cdots r| = |p||q|\cdots|r|$

# Unit quaternion

- A unit quaternion: $q \in \mathbb{H}$ with $|q| = 1$

- Every unit quaternion takes the form

$$q = \cos(t) + \sin(t)\mathbb{v} \qquad |\mathbb{v}| = 1$$

$$= e^{t\mathbb{v}}$$

$$\mathbb{v} = \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$$

$$t = \arctan\left(\frac{|\mathrm{Im}(q)|}{\mathrm{Re}(q)}\right)$$

- Conjugate of unit quaternion:

$$\overline{q} = e^{-t\mathbb{v}}$$

# 3D Rotation by unit quaternion

- Treat each 3D vector as imaginary quaternion
- Rotation about axis $\mathbb{a} \in \operatorname{Im}\mathbb{H}$ by angle $\theta$ :

  $|\mathbb{a}| = 1$

$$q = e^{\frac{\theta}{2}\mathbb{a}}$$

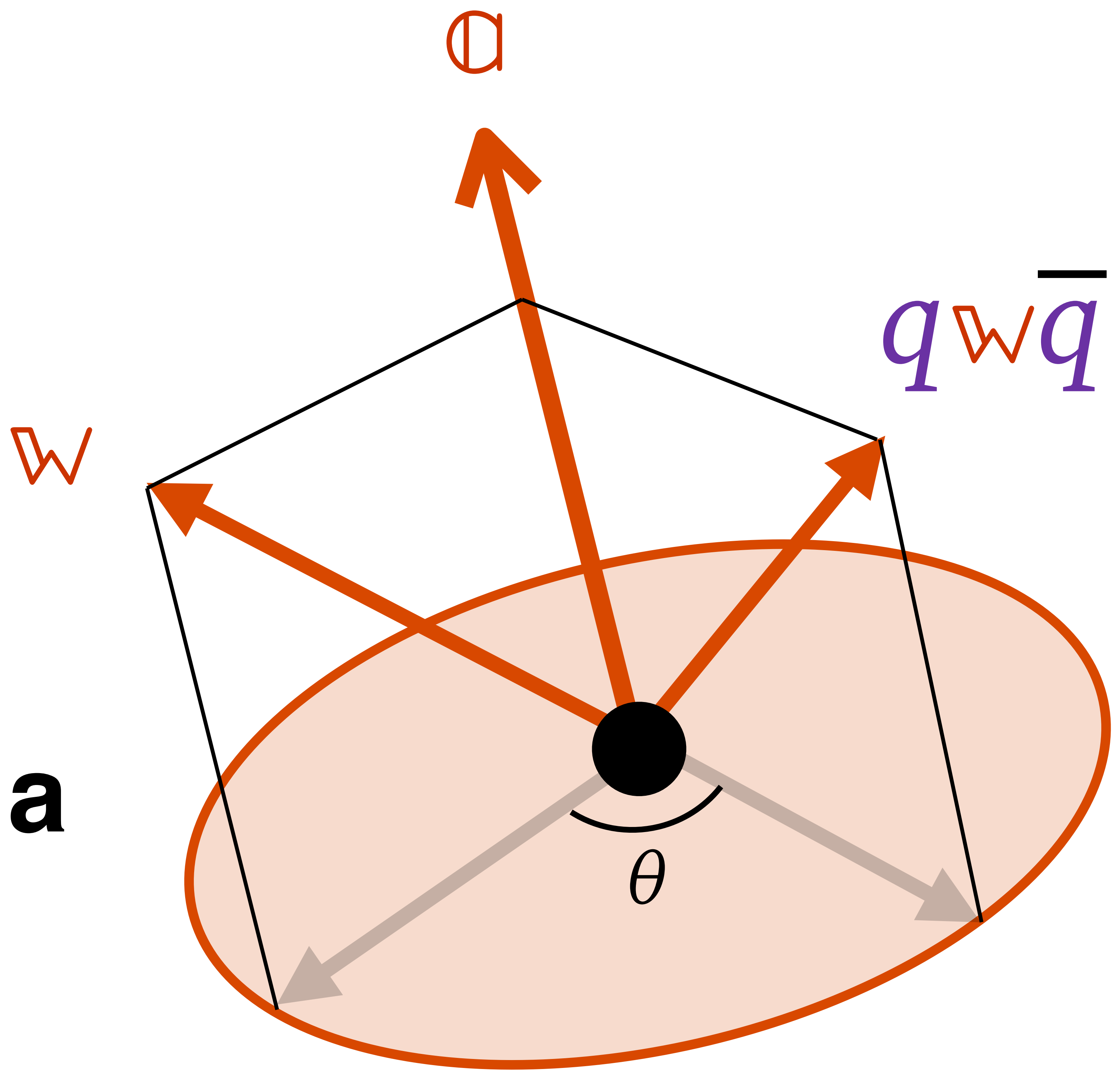
"rotor"
unit quaternion

$$\mathbb{w} \longmapsto q\mathbb{w}\bar{q}$$

before rotation

after rotation

**Quaternion Rotation formula**



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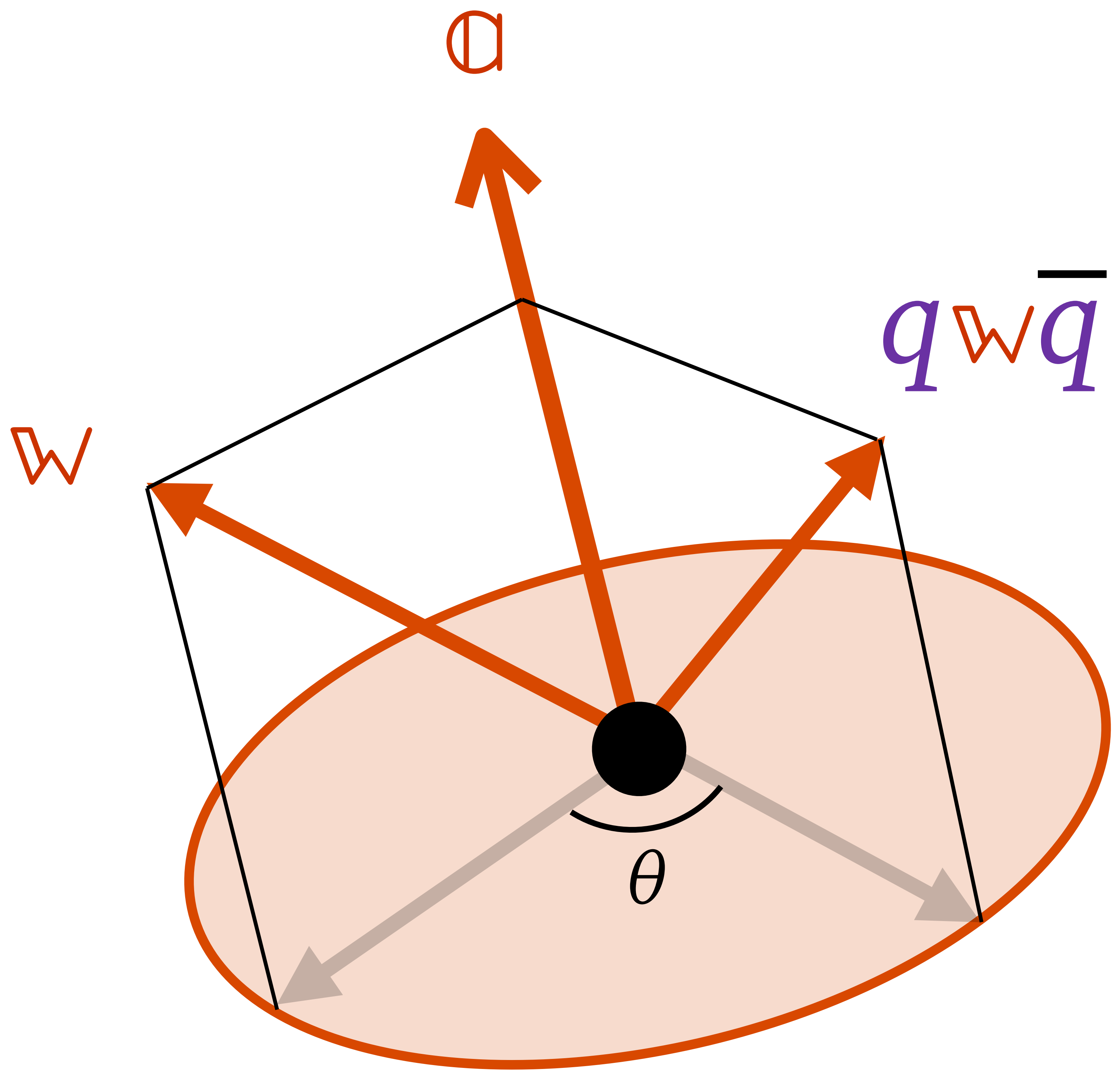


- Notice the **half** angle!
- $q, -q$ yields the same rotation!

  (Every 3D rotation is represented by exactly two unit quaternions in ± pair)

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  (Every 3D rotation is represented by exactly two unit quaternions in ± pair)

# 2D v.s. 3D rotations

- Complex number

$$z = a + b\,\mathbb{i} \in \mathbb{C}$$

- Associative, commutative
- Unit complex number for rotation

$$r = e^{\theta\,\mathbb{i}}$$

- Rotation

$$z \mapsto rz$$

- Quaternion number

$$q = a + u_1\,\mathbb{i} + u_2\,\mathbb{j} + u_3\,\mathbb{k} \in \mathbb{H}$$
$$= a + \mathbb{u}$$

- Associative but not commutative
- Unit quaternion for rotation

$$q = e^{\frac{\theta}{2}\mathbb{a}}$$

- Rotation

$$\mathbb{w} \mapsto q\mathbb{w}\bar{q}$$

# Euler Angles

- Cross product
- Rodrigues formula
- More representations
- Quaternions
- Euler angles
- Geometric algebra

# Euler angles

- Each 3D rotation is parameterized by 3 angles $\alpha, \beta, \gamma$

- Rotation matrix is given by

$$\begin{bmatrix} \cos\alpha & & \sin\alpha \\ & 1 & \\ -\sin\alpha & & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta & \\ \sin\beta & \cos\beta & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \cos\gamma & -\sin\gamma \\ & \sin\gamma & \cos\gamma \end{bmatrix}$$

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*rotation in the zx-plane*

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*rotation in the xy-plane*

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*rotation in the yz-plane*

# Euler angles

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$$\begin{tikzcd} \mathbb{R}^3_{\text{World}} \\ \mathbb{R}^3_{\text{OuterRing}} \arrow[u, "\mathbf{R}^{\mathbf{e}_1,\alpha}"] \\ \mathbb{R}^3_{\text{MiddleRing}} \arrow[u, "\mathbf{R}^{\mathbf{e}_2,\beta}"] \\ \mathbb{R}^3_{\text{InnerRing}} \arrow[u, "\mathbf{R}^{\mathbf{e}_0,\gamma}"] \end{tikzcd}$$

# Euler angles

- Non-intuitive to go from the 3 angles and the final rotation.

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*rotation in the zx-plane in the world basis*

*rotation in the xy-plane in the outer ring basis*

*rotation in the yz-plane in the middle ring basis*

These planes of rotation are not orthogonal (note the different bases!)

# Euler angles

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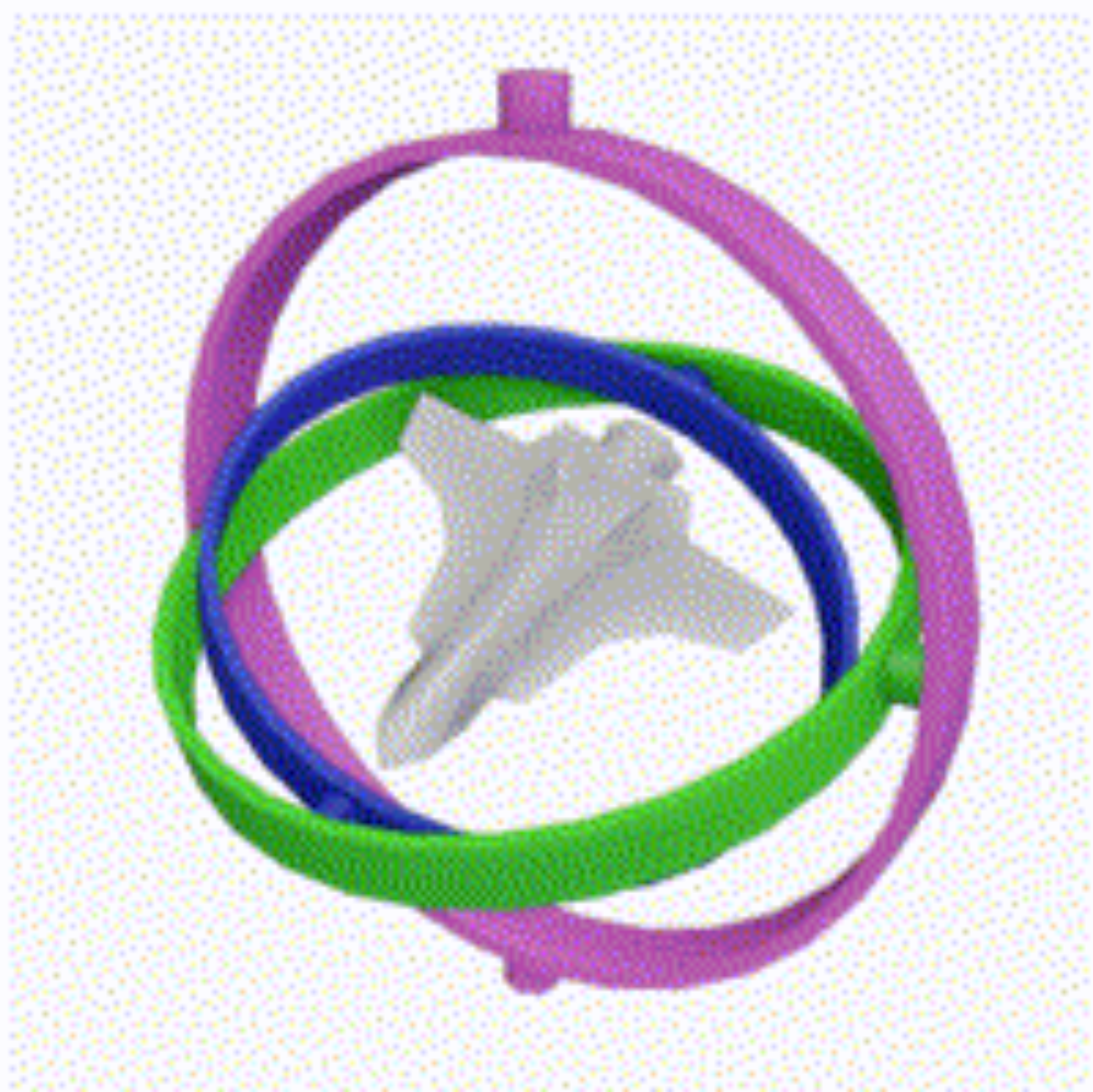
*rotation in the zx-plane in the world basis*

*rotation in the xy-plane in the outer ring basis*

*rotation in the yz-plane in the middle ring basis*

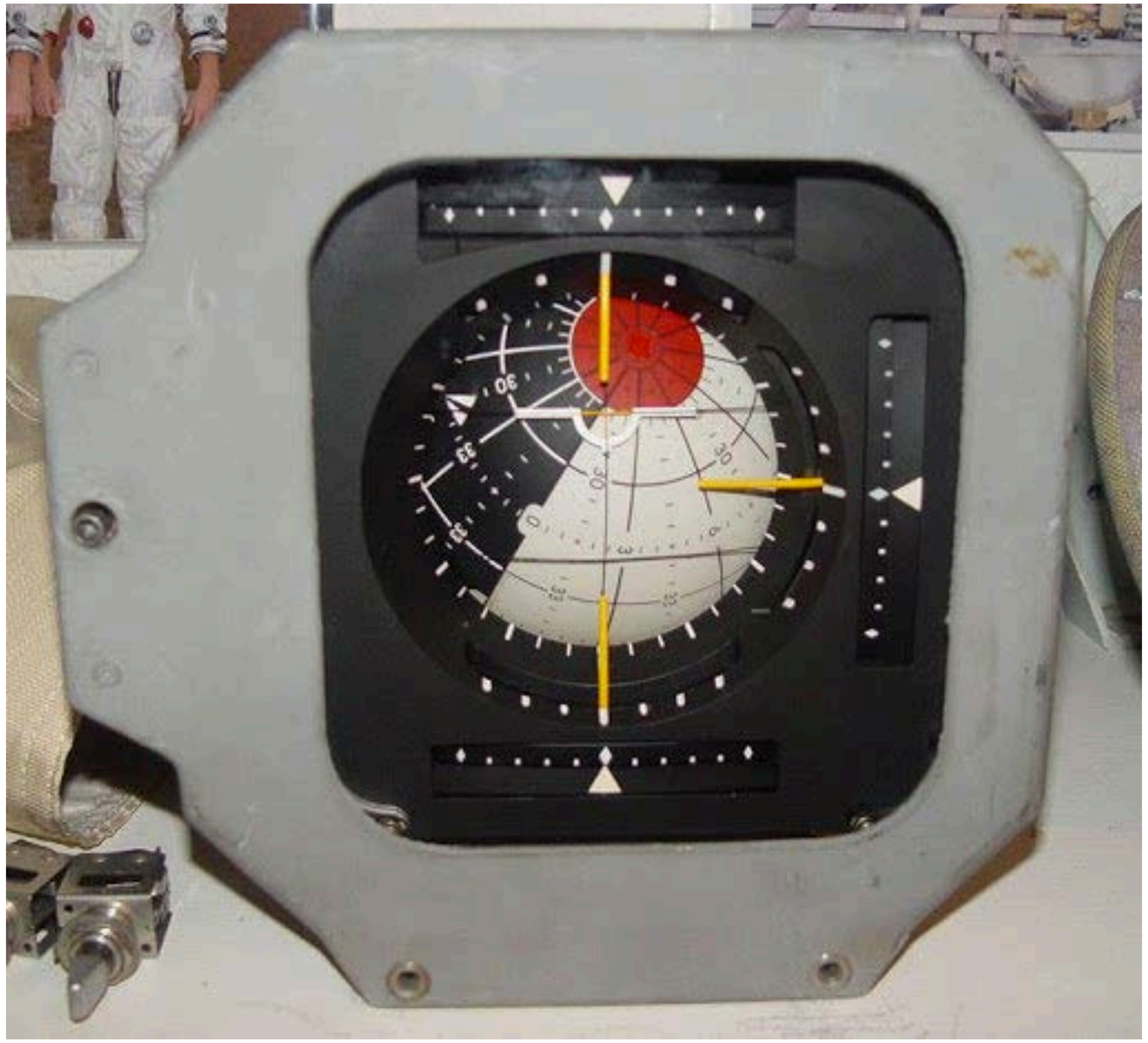
These two planes might even coincide!
In that case, we lost a degree of freedom of rotation.
This is called the **gimbal lock**.

# Gimbal lock

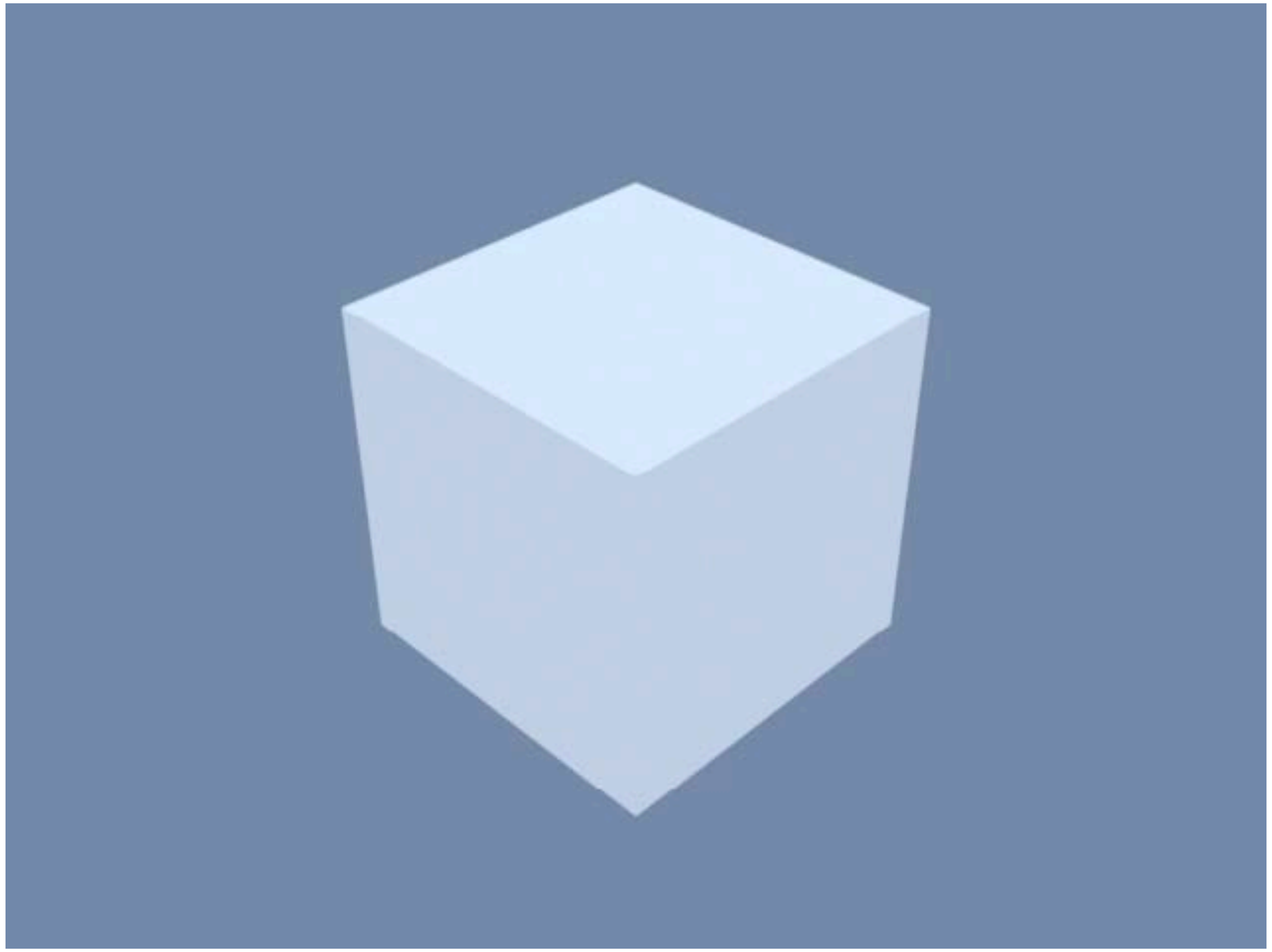


# Gimbal lock

Apollo 11 gimbal lock incident



# Interpolation of Euler angles



# Euler angles v.s. Angle-axis

| **Euler angles** | **Angle-axis / quaternions** |
|---|---|
| Physical gimbal. | Mathematical description. |
| Presence of gimbal lock. | No gimbal lock problem. |
| Unpredictable interpolated animation. | Smooth and robust animation (Interpolate the quaternions, or the angle-axis parameters. Don’t interpolate the 3x3 matrix!) |

# Interactive reading material

https://thenumb.at/Exponential-Rotations/

# Geometric Algebra (optional topic)

- Cross product
- Rodrigues formula
- More representations
- Quaternions
- Euler angles
- Geometric algebra

# General construction of rotors

- How do we represent all rotations of n-dimensional space?
- Simple answer: Use det=1 orthogonal matrices

$\mathbf{M} \in \mathbb{R}^{n\times n}$ satisfying $\mathbf{M}^\mathsf{T}\mathbf{M} = \mathbf{I}$ and $\boxed{\det(\mathbf{M}) = 1}$

rule out reflection matrices

Rotation $\mathbf{v} \mapsto \mathbf{M}\mathbf{v}$

- Or we can build an algebraic system where rotors are unit elements.
  - For 2D $\mathbb{C}$ (algebra is 2D) $\quad z \mapsto e^{\frac{\theta}{2}\mathbb{i}} z e^{\frac{\theta}{2}\mathbb{i}}$
  - For 3D $\mathbb{H}$ (algebra is 4D) $\quad \mathbb{w} \mapsto e^{\frac{\theta}{2}\mathbb{a}} \mathbb{w} e^{-\frac{\theta}{2}\mathbb{a}}$

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  - For 2D $\mathbb{C}$ (algebra is 2D) $z \mapsto e^{\frac{\theta}{2}\mathbb{i}} z e^{\frac{\theta}{2}\mathbb{i}}$
  - For 3D $\mathbb{H}$ (algebra is 4D) $\mathrm{w} \mapsto e^{\frac{\theta}{2}\mathbb{a}} \mathrm{w} e^{-\frac{\theta}{2}\mathbb{a}}$
- For nD, there is a simple protocol to construct the algebra that represents **both reflection and rotation**
- For n=2 subset of the algebra gives us complex numbers
- For n=3 subset of the algebra gives us quaternions

# General rule

- Consider a vector space $V$ with an inner product " $\cdot$ "
- We want to equip $V$ with a multiplication $(\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u}\vec{v}$ so that
  1. It is associative $(\vec{u}\vec{v})\vec{w} = \vec{u}(\vec{v}\vec{w})$
  2. Self-multiplication is length squared $\vec{u}\vec{u} = \vec{u} \cdot \vec{u} = |\vec{u}|^2$

*Let's first explain the process with a simpler case*

- We will need to extend $V$ to a bigger space, but that's fine.
- As we will see, the extended space is uniquely determined. This space is called $C\ell(V)$, the Clifford algebra.

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- We want to equip $V$ with a multiplication $(\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u}\vec{v}$ so that

1. It is associative $(\vec{u}\vec{v})\vec{w} = \vec{u}(\vec{v}\vec{w})$

- We will need to extend $V$ to a bigger space, but that's fine.

**2D Example** Let $\vec{e}_1, \vec{e}_2$ be an (orthonormal) basis of $V$

The extended space $\widetilde{V}$ has basis

$$\vec{e}_1, \vec{e}_2, \vec{e}_1\vec{e}_2, \vec{e}_2\vec{e}_1, \vec{e}_1{}^2, \cdots, \vec{e}_1^5\vec{e}_2^2\vec{e}_2^7\vec{e}_1, \cdots$$

*every possible string of multiplication is a new dimension*

By brute force, the extended space is obviously associative

# General rule

- Consider a vector space $V$ with an inner product " $\cdot$ "
- We want to equip $V$ with a multiplication $(\vec{u} \in V, \vec{v} \in V) \mapsto \vec{u}\vec{v}$ so that

1. It is associative $(\vec{u}\vec{v})\vec{w} = \vec{u}(\vec{v}\vec{w})$
2. Self-multiplication is length squared $\vec{u}\vec{u} = \vec{u} \cdot \vec{u} = |\vec{u}|^2$

*This additional rule can simplify some strings of multiplications*

- We will need to extend $V$ to a bigger space, but that's fine.
- As we will see, the extended space is uniquely determined. This space is called $C\ell(V)$, the Clifford algebra.

# 2D

- Let's consider the 2D Euclidean vector space $V_{\mathrm{2D}}$ .
- Let $\vec{e}_x$ , $\vec{e}_y$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1 \qquad \vec{e}_y\vec{e}_y = 1 \qquad \vec{e}_x\vec{e}_y = \mathbb{I} \qquad \vec{e}_y\vec{e}_x = -\mathbb{I}$

$$\begin{aligned}\cancel{2} &= (\vec{e}_x + \vec{e}_y)(\vec{e}_x + \vec{e}_y) \\ &= \vec{e}_x\vec{e}_x + \vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x + \vec{e}_y\vec{e}_y \\ &= \cancel{1} + \vec{e}_x\vec{e}_y + \vec{e}_y\vec{e}_x + \cancel{1}\end{aligned}$$

$\Rightarrow \quad \vec{e}_x\vec{e}_y = -\vec{e}_y\vec{e}_x$ (for orthonormal vectors)

# 2D

- Let's consider the 2D Euclidean vector space $V_{\mathrm{2D}}$ .
- Let $\vec{e}_x$ , $\vec{e}_y$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1$ $\quad$ $\vec{e}_y\vec{e}_y = 1$ $\quad$ $\vec{e}_x\vec{e}_y = \mathbb{I}$ $\quad$ $\vec{e}_y\vec{e}_x = -\mathbb{I}$

  $\vec{e}_x\vec{e}_y\vec{e}_x = -\vec{e}_x\vec{e}_x\vec{e}_y = -\vec{e}_y$

# 2D

- Let’s consider the 2D Euclidean vector space $V_{\mathrm{2D}}$ .
- Let $\vec{e}_x$ , $\vec{e}_y$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1$  $\vec{e}_y\vec{e}_y = 1$  $\vec{e}_x\vec{e}_y = \mathbb{I}$  $\vec{e}_y\vec{e}_x = -\mathbb{I}$

  $\vec{e}_x\vec{e}_y\vec{e}_x = -\vec{e}_y$  $\mathbb{I}^2 = \vec{e}_x\vec{e}_y\vec{e}_x\vec{e}_y = -\vec{e}_x\vec{e}_x\vec{e}_y\vec{e}_y = -1$

# 2D

- Let's consider the 2D Euclidean vector space $V_{2\mathrm{D}}$ .
- Let $\vec{e}_x$ , $\vec{e}_y$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1 \qquad \vec{e}_y\vec{e}_y = 1 \qquad \vec{e}_x\vec{e}_y = \mathbb{I} \qquad \vec{e}_y\vec{e}_x = -\mathbb{I}$

  $\vec{e}_x\vec{e}_y\vec{e}_x = -\vec{e}_y \qquad \mathbb{I}^2 = -1$

$$
\begin{aligned}
\vec{u}\vec{v} &= (u_x\vec{e}_x + u_y\vec{e}_y)(v_x\vec{e}_x + v_y\vec{e}_y) \\
&= (u_x v_x + u_y v_y) + (u_x v_y - u_y v_x)\mathbb{I} \\
&= \mathbf{u}^\mathsf{T}\mathbf{v} + (\mathbf{u}\times\mathbf{v})\mathbb{I} \qquad \textit{(what we would write nowadays)} \\
&= \vec{u}\cdot\vec{v} + \vec{u}\wedge\vec{v} \qquad \textit{(standard notation in geometric algebra)}
\end{aligned}
$$

# 2D

- Let's consider the 2D Euclidean vector space $V_{\text{2D}}$ .
- Let $\vec{e}_x$ , $\vec{e}_y$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1 \qquad \vec{e}_y\vec{e}_y = 1 \qquad \vec{e}_x\vec{e}_y = \mathbb{I} \qquad \vec{e}_y\vec{e}_x = -\mathbb{I}$

  $\vec{e}_x\vec{e}_y\vec{e}_x = -\vec{e}_y \qquad \mathbb{I}^2 = -1 \qquad \vec{u}\vec{v} = \mathbf{u}^\intercal\mathbf{v} + (\mathbf{u}\times\mathbf{v})\mathbb{I}$

  $$= \vec{u}\cdot\vec{v} + \vec{u}\wedge\vec{v}$$
- A general element in $C\ell(V_{\text{2D}})$ is a "multivector"

$$\underbrace{\boxed{a}}_{\text{degree-0}} + \underbrace{\boxed{u_x\vec{e}_x + u_y\vec{e}_y}}_{\text{degree-1}} + \underbrace{\boxed{b\mathbb{I}}}_{\text{degree-2}}$$

*(and we know how to multiply any pair of multivectors!)*

- Products between even-degree elements stay even-degree

# 2D

- Let's consider the 2D Euclidean vector space $V_{2\mathrm{D}}$ .
- $\mathbb{I}^2 = -1$

$$a + b\mathbb{I}$$

- We've just rediscovered **complex numbers**.

# Build your own geometric algebra at home

- Start with your vector space with inner product.
- Build the Clifford algebra.
- Take the even-degree sub-algebra.
- That's it. That will be the space for the scalings and rotations for your space!
    - 2D ⇒ complex numbers
    - 3D ⇒ quaternions (we'll see that next)
    - 4D spacetime (relativistic inner product) ⇒ Dirac spinors.

# 3D

# 3D

- Let's consider the 3D Euclidean vector space $V_{\text{3D}}$.
- Let $\vec{e}_x$ , $\vec{e}_y$ , $\vec{e}_z$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1 \qquad \vec{e}_y\vec{e}_y = 1 \qquad \vec{e}_z\vec{e}_z = 1$

  $\vec{e}_y\vec{e}_z = -\vec{e}_z\vec{e}_y = -\mathbb{i}$

# 3D

- Let's consider the 3D Euclidean vector space $V_{\mathrm{3D}}$.
- Let $\vec{e}_x, \vec{e}_y, \vec{e}_z$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1 \qquad \vec{e}_y\vec{e}_y = 1 \qquad \vec{e}_z\vec{e}_z = 1$

  $\vec{e}_y\vec{e}_z = -\mathbb{i} \qquad \vec{e}_z\vec{e}_x = -\mathbb{j} \qquad \vec{e}_x\vec{e}_y = -\mathbb{k}$

  $\vec{e}_x\vec{e}_y\vec{e}_z = \mathbb{I}$

# 3D

- Let's consider the 3D Euclidean vector space $V_{\mathrm{3D}}$.
- Let $\vec{e}_x$ , $\vec{e}_y$ , $\vec{e}_z$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1$ $\quad$ $\vec{e}_y\vec{e}_y = 1$ $\quad$ $\vec{e}_z\vec{e}_z = 1$

  $\vec{e}_y\vec{e}_z = -\mathbb{i}$ $\quad$ $\vec{e}_z\vec{e}_x = -\mathbb{j}$ $\quad$ $\vec{e}_x\vec{e}_y = -\mathbb{k}$

  $\vec{e}_x\vec{e}_y\vec{e}_z = \mathbb{I}$
- $\mathbb{i}^2 = -1$ $\quad$ $\mathbb{j}^2 = -1$ $\quad$ $\mathbb{k}^2 = -1$

# 3D

- Let's consider the 3D Euclidean vector space $V_{3\mathrm{D}}$.
- Let $\vec{e}_x, \vec{e}_y, \vec{e}_z$ be an orthonormal basis.
- $\vec{e}_x\vec{e}_x = 1 \qquad \vec{e}_y\vec{e}_y = 1 \qquad \vec{e}_z\vec{e}_z = 1$

  $\vec{e}_y\vec{e}_z = -\mathbb{i} \qquad \vec{e}_z\vec{e}_x = -\mathbb{j} \qquad \vec{e}_x\vec{e}_y = -\mathbb{k}$

  $\vec{e}_x\vec{e}_y\vec{e}_z = \mathbb{I}$

- $\mathbb{i}^2 = -1 \qquad \mathbb{j}^2 = -1 \qquad \mathbb{k}^2 = -1$

  $\mathbb{i}\,\mathbb{j} = \mathbb{k} \qquad \mathbb{j}\,\mathbb{k} = \mathbb{i} \qquad \mathbb{k}\,\mathbb{i} = \mathbb{j}$

  $\mathbb{i}\,\mathbb{j} = -\mathbb{j}\,\mathbb{i}$

  $\mathbb{j}\,\mathbb{k} = -\mathbb{k}\,\mathbb{j}$

  $\mathbb{k}\,\mathbb{i} = -\mathbb{i}\,\mathbb{k}$

# 3D

- A general element in $C\ell(V_{3D})$ is a multivector

$$\boxed{a} + \boxed{u_x \vec{e}_x + u_y \vec{e}_y + u_z \vec{e}_z} + \boxed{v_x \mathbb{i} + v_y \mathbb{j} + v_z \mathbb{k}} + \boxed{b\mathbb{I}}$$

degree-0
**scalar**

degree-1
**vector**

degree-2
**bivector**

degree-3
**pseudo-scalar**

- The even-degree elements are called **quaternions**.

# 3D

$$a + v_x \mathbb{i} + v_y \mathbb{j} + v_z \mathbb{k}$$

- The even-degree elements are called **quaternions**.

$$\mathbb{i}^2 = -1 \qquad \mathbb{j}^2 = -1 \qquad \mathbb{k}^2 = -1$$

$$\mathbb{i}\,\mathbb{j} = \mathbb{k} \qquad \mathbb{j}\,\mathbb{k} = \mathbb{i} \qquad \mathbb{k}\,\mathbb{i} = \mathbb{j}$$

$$\mathbb{i}\,\mathbb{j} = -\mathbb{j}\,\mathbb{i}$$
$$\mathbb{j}\,\mathbb{k} = -\mathbb{k}\,\mathbb{j}$$
$$\mathbb{k}\,\mathbb{i} = -\mathbb{i}\,\mathbb{k}$$

# Geometric Picture

- Compute $\vec{u}\vec{v}$

$$
\begin{aligned}
&= (u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z)(v_x\vec{e}_x + v_y\vec{e}_y + v_z\vec{e}_z) \\
&= (u_x v_x + u_y v_y + u_z v_z) + (u_y v_z - u_z v_y)(-\mathbb{i}) \\
&\qquad + (u_z v_x - u_x v_z)(-\mathbb{j}) \\
&\qquad + (u_x v_y - u_y v_x)(-\mathbb{k}) \\
&= \mathbf{u}^\mathsf{T}\mathbf{v} + \begin{bmatrix} -\mathbb{i} & -\mathbb{j} & -\mathbb{k} \end{bmatrix} (\mathbf{u} \times \mathbf{v}) \\
&= \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}
\end{aligned}
$$

*This is **dot product** and **cross product** but the cross product part is a bivector.*

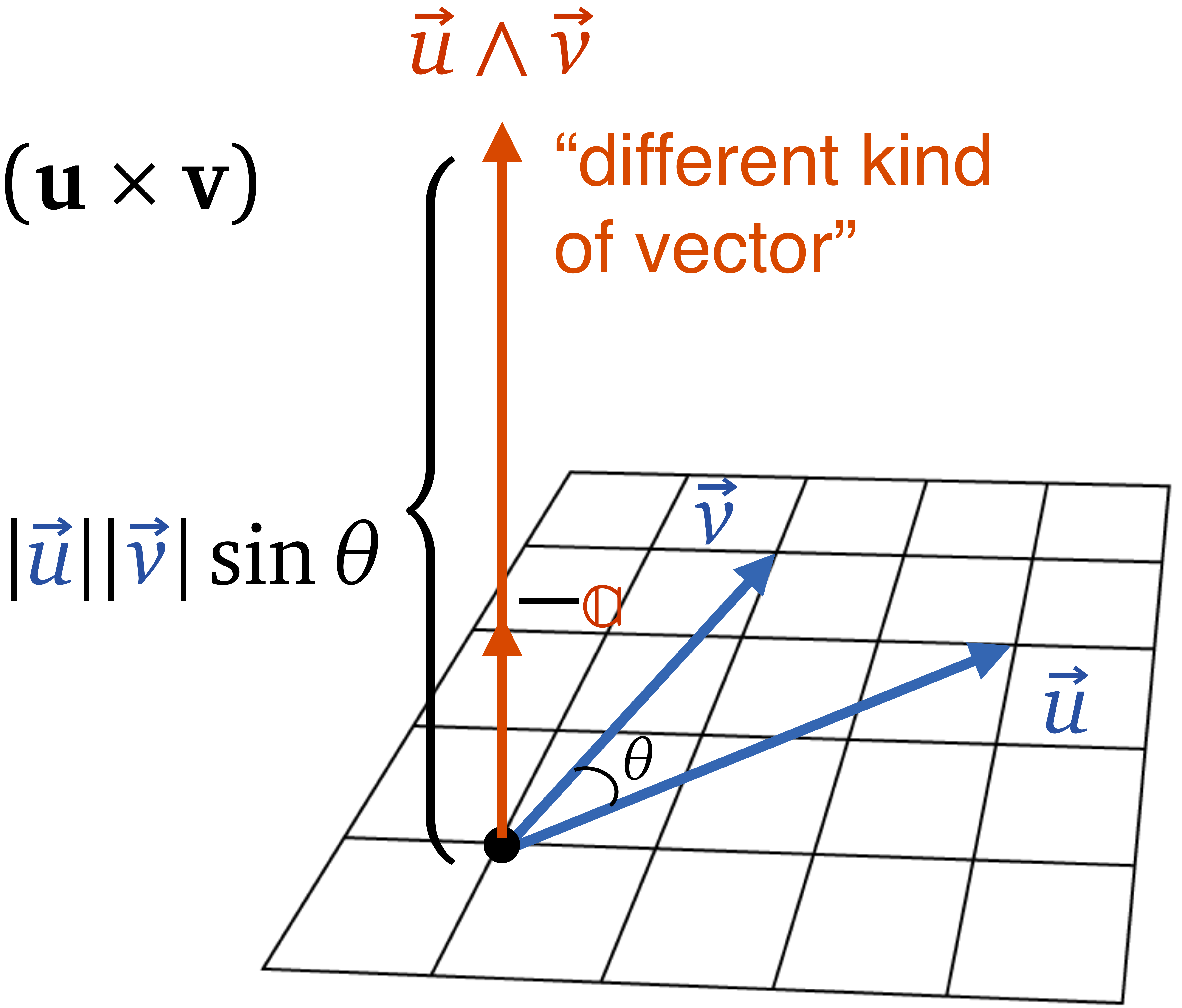
# Geometric Picture

- Compute $\vec{u}\vec{v}$

$$= \mathbf{u}^\mathsf{T}\mathbf{v} + \begin{bmatrix} -\mathbb{i} & -\mathbb{j} & -\mathbb{k} \end{bmatrix} (\mathbf{u} \times \mathbf{v})$$

$$= \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$$

$$= |\vec{u}||\vec{v}| \cos\theta$$
$$+ |\vec{u}||\vec{v}| \sin\theta (-\mathbb{a})$$



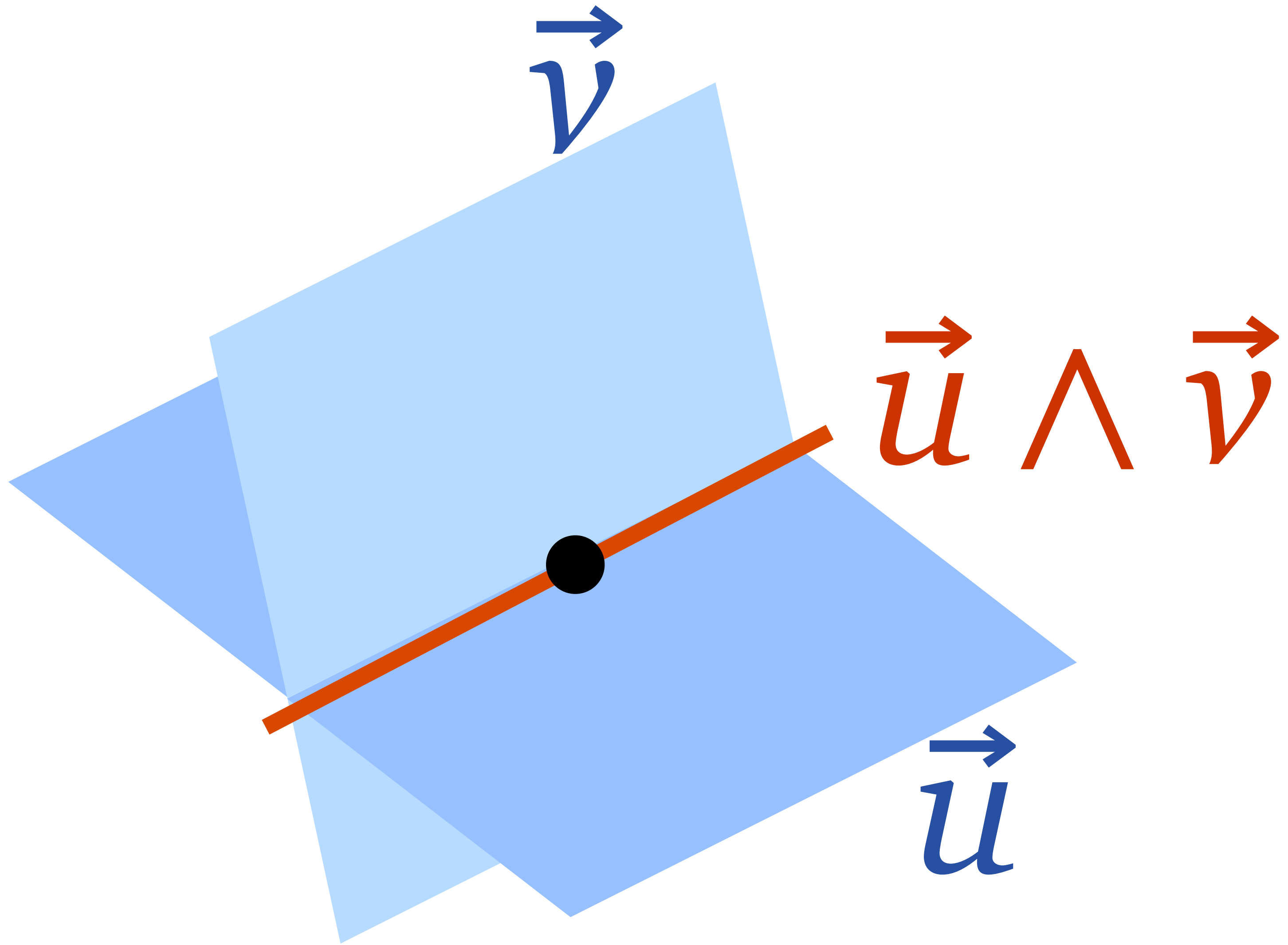
# A Better Geometric Picture

- Each **vector** is thought of as a plane (vector is the plane's normal)
- Each **bivector** is thought of as a line.

*the **meet** between two planes*

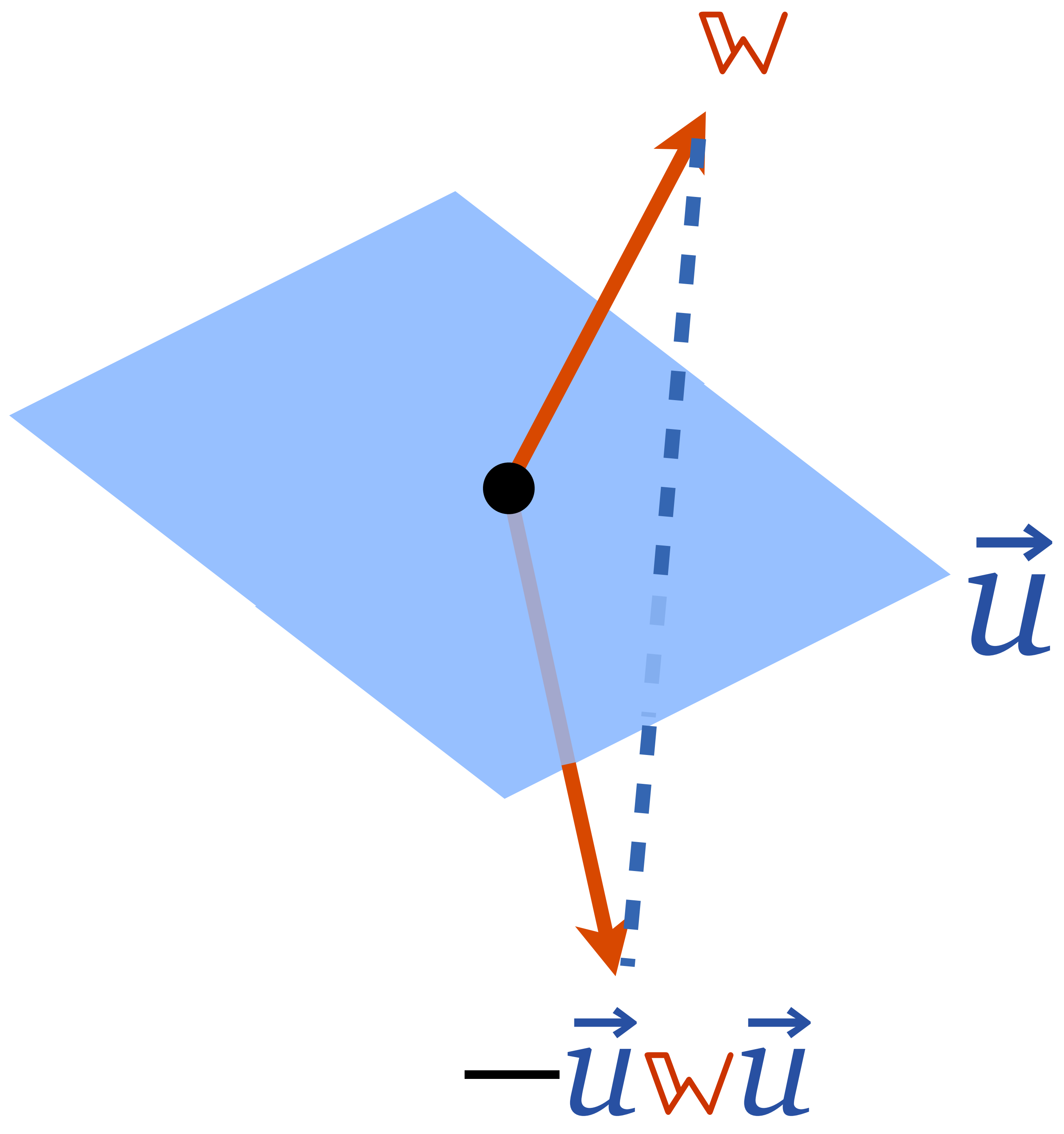
$$\vec{u}\vec{v} = \vec{u}\cdot\vec{v} + \vec{u}\wedge\vec{v}$$

$$= |\vec{u}||\vec{v}|\cos\theta + |\vec{u}||\vec{v}|\sin\theta(-\mathbb{a})$$



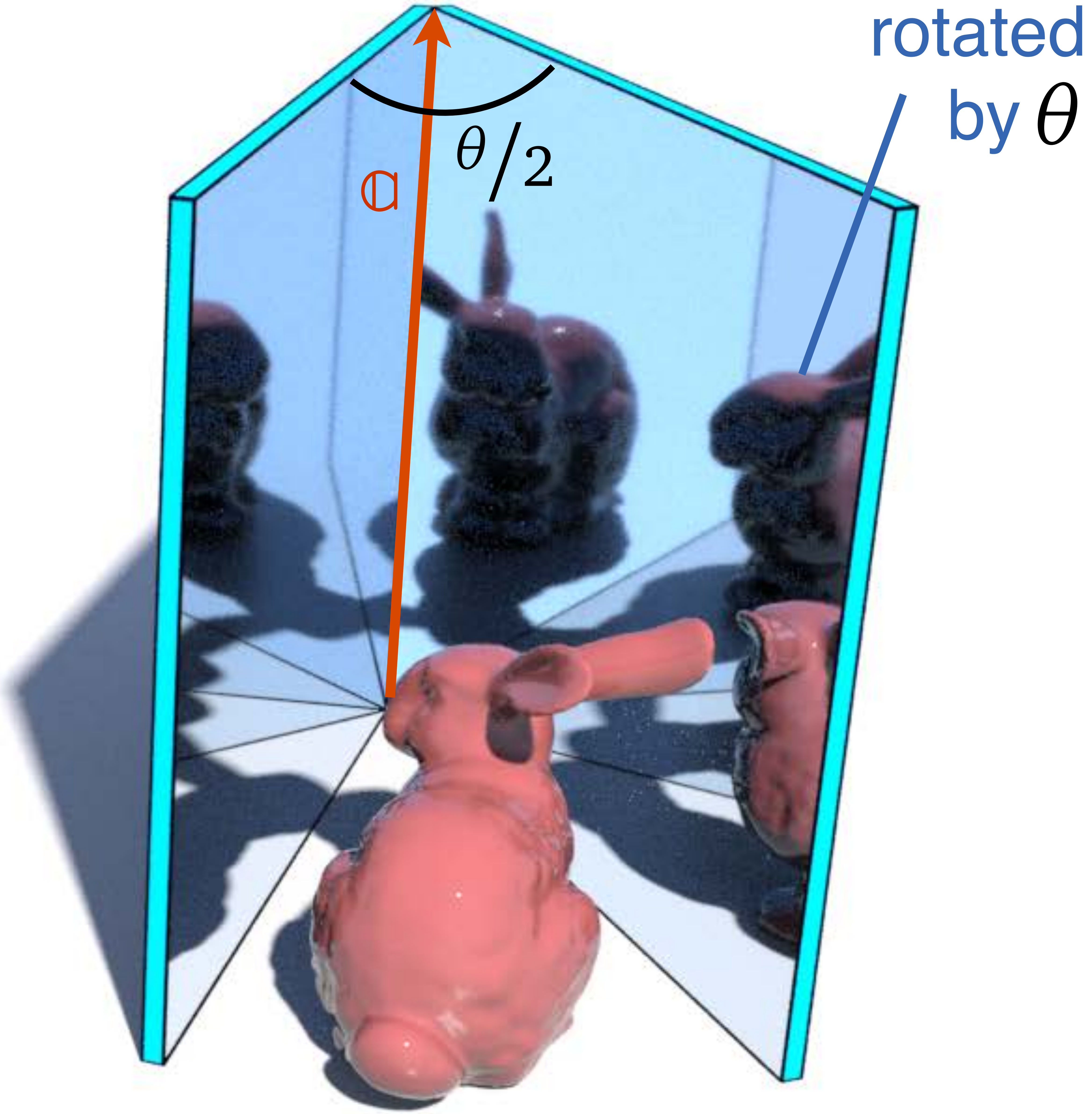
# Mirror reflection

- Suppose $\vec{u}$ is a unit vector representing a plane.
- Then $-\vec{u}\mathrm{w}\vec{u}$ is the mirror reflection of $\mathrm{w}$ with respect to the plane $\vec{u}$.



# Rotation

- To make a rotation about a unit axis $\mathbb{a}$ with an angle $\theta$, take two mirrors and hinge them along $\mathbb{a}$ at an angle $\theta/2$.



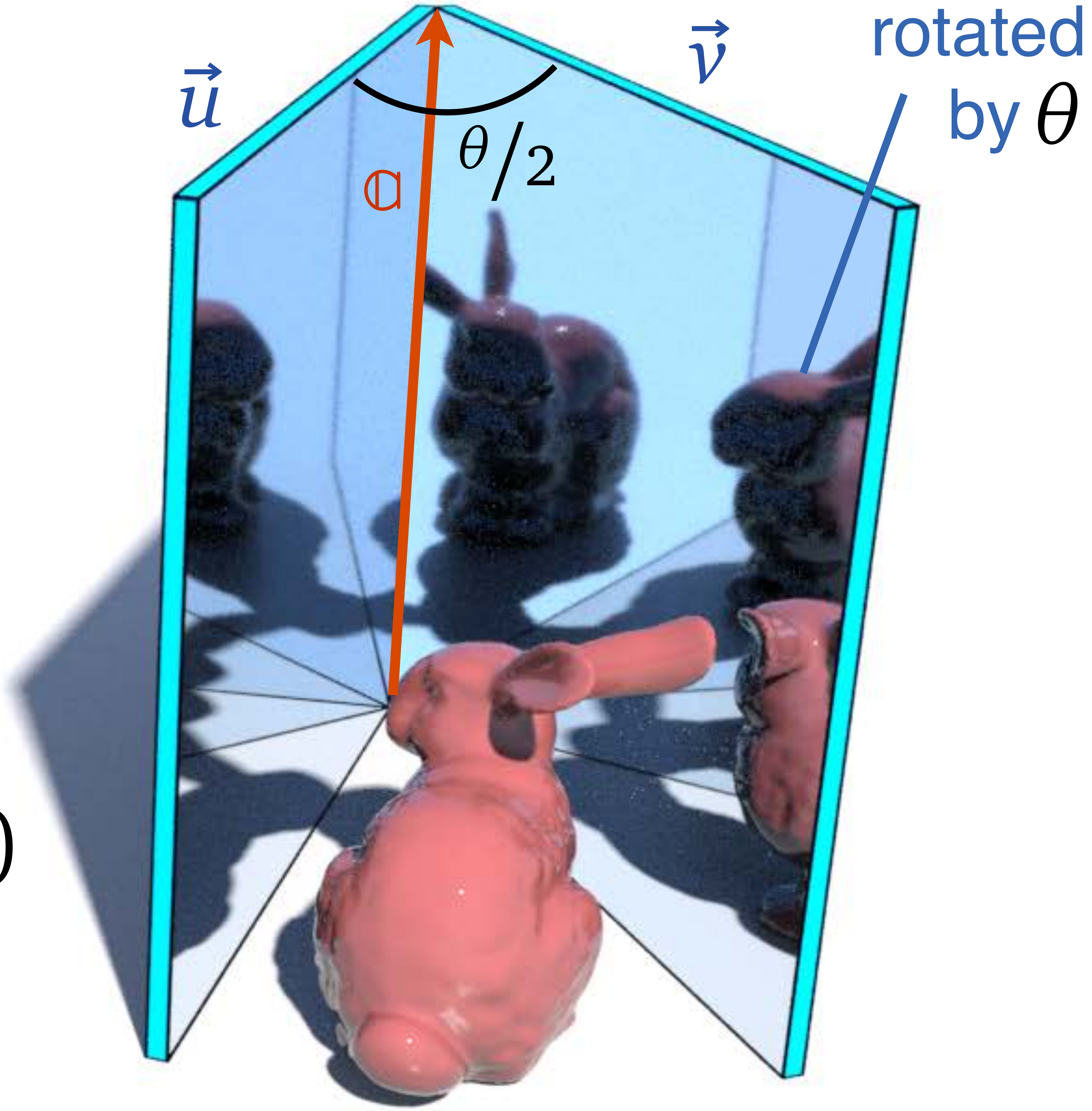
# Rotation

- To make a rotation about a unit axis $\mathbb{a}$ with an angle $\theta$ , take two mirrors and hinge them along $\mathbb{a}$ at an angle ${}^{\theta}/_{2}$ .

- Let the two mirrors be the unit vectors $\vec{u}$ , $\vec{v}$

  Then we know:

$$
\begin{aligned}
\vec{u}\vec{v} &= \vec{u}\cdot\vec{v} + \vec{u}\wedge\vec{v} \\
&= \cos({}^{\theta}/_{2}) + \sin({}^{\theta}/_{2})(-\mathbb{a})
\end{aligned}
$$



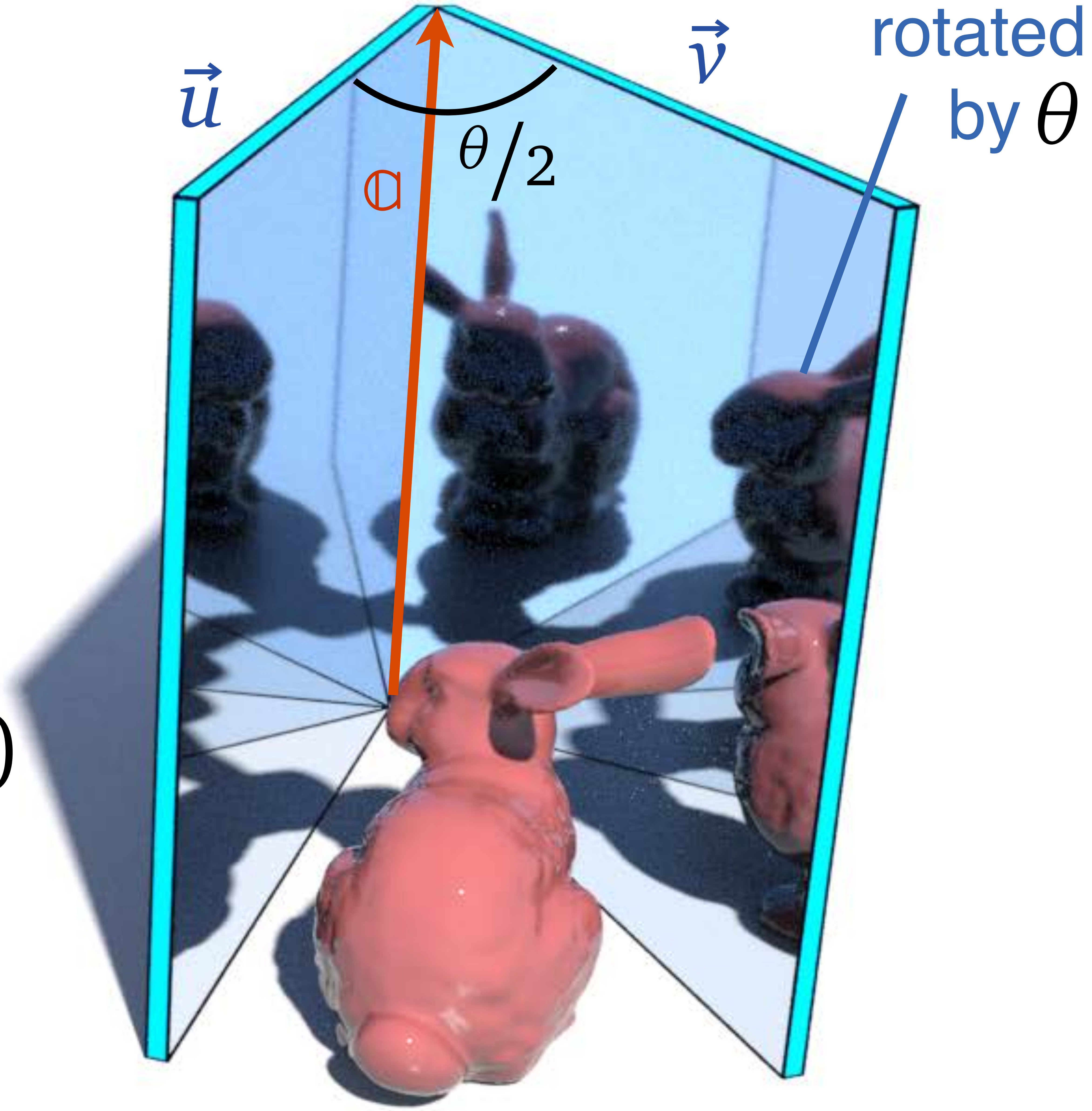
# Rotation

- To make a rotation about a unit axis $\mathbb{a}$ with an angle $\theta$, take two mirrors and hinge them along $\mathbb{a}$ at an angle ${}^{\theta}/_{2}$.

- Let the two mirrors be the unit vectors $\vec{u}, \vec{v}$

  Then we know:

$$\begin{aligned} \vec{u}\vec{v} &= \cos({}^{\theta}/_{2}) + \sin({}^{\theta}/_{2})(-\mathbb{a}) \\ &= e^{-\frac{\theta}{2}\mathbb{a}} \end{aligned}$$



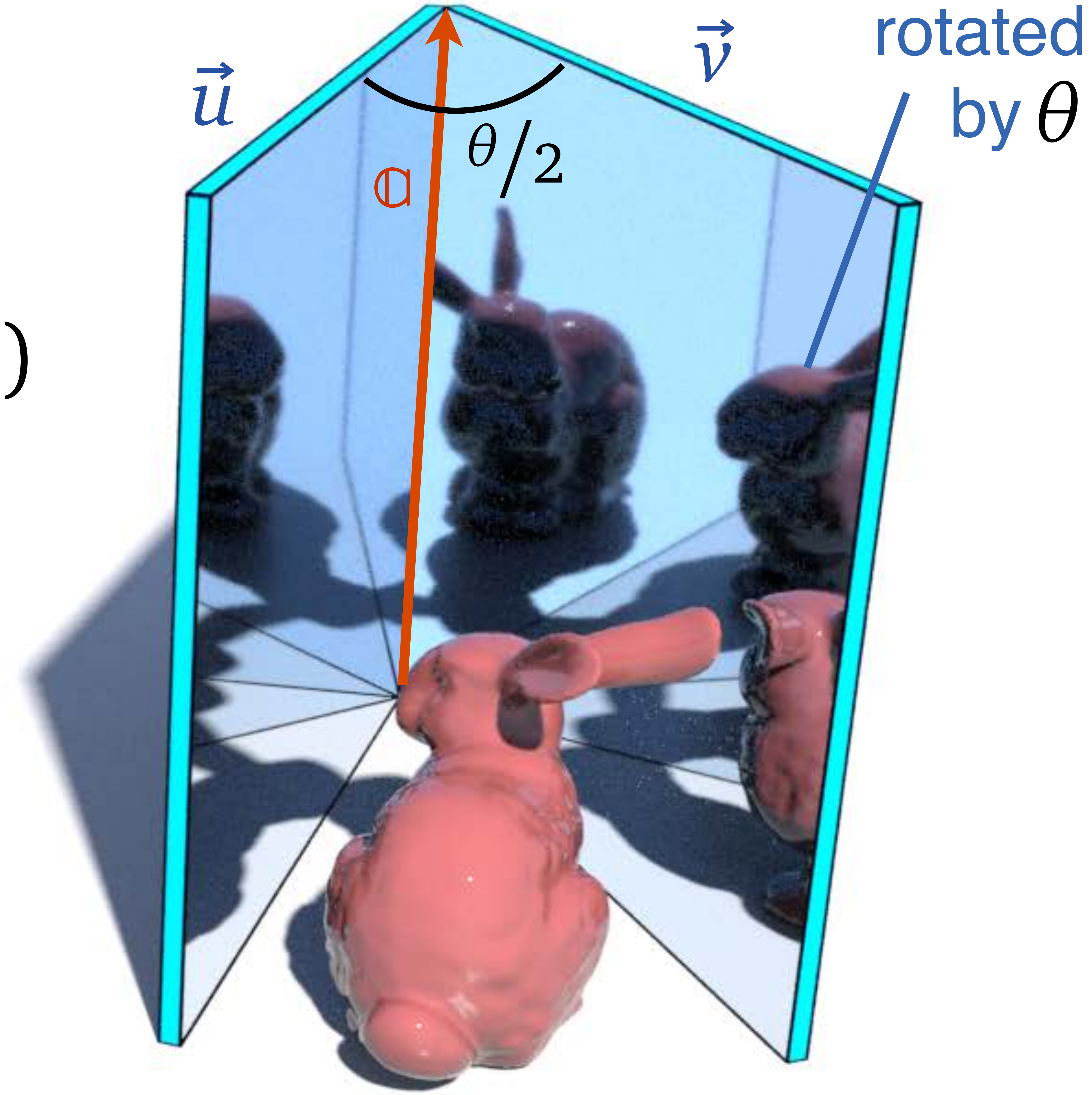
# Rotation

- Let the two mirrors be the unit vectors $\vec{u}\,, \vec{v}$

  Then we know:

$$\vec{u}\vec{v} = \cos(\theta/2) + \sin(\theta/2)(-\mathbb{a})$$
$$= e^{-\frac{\theta}{2}\mathbb{a}}$$

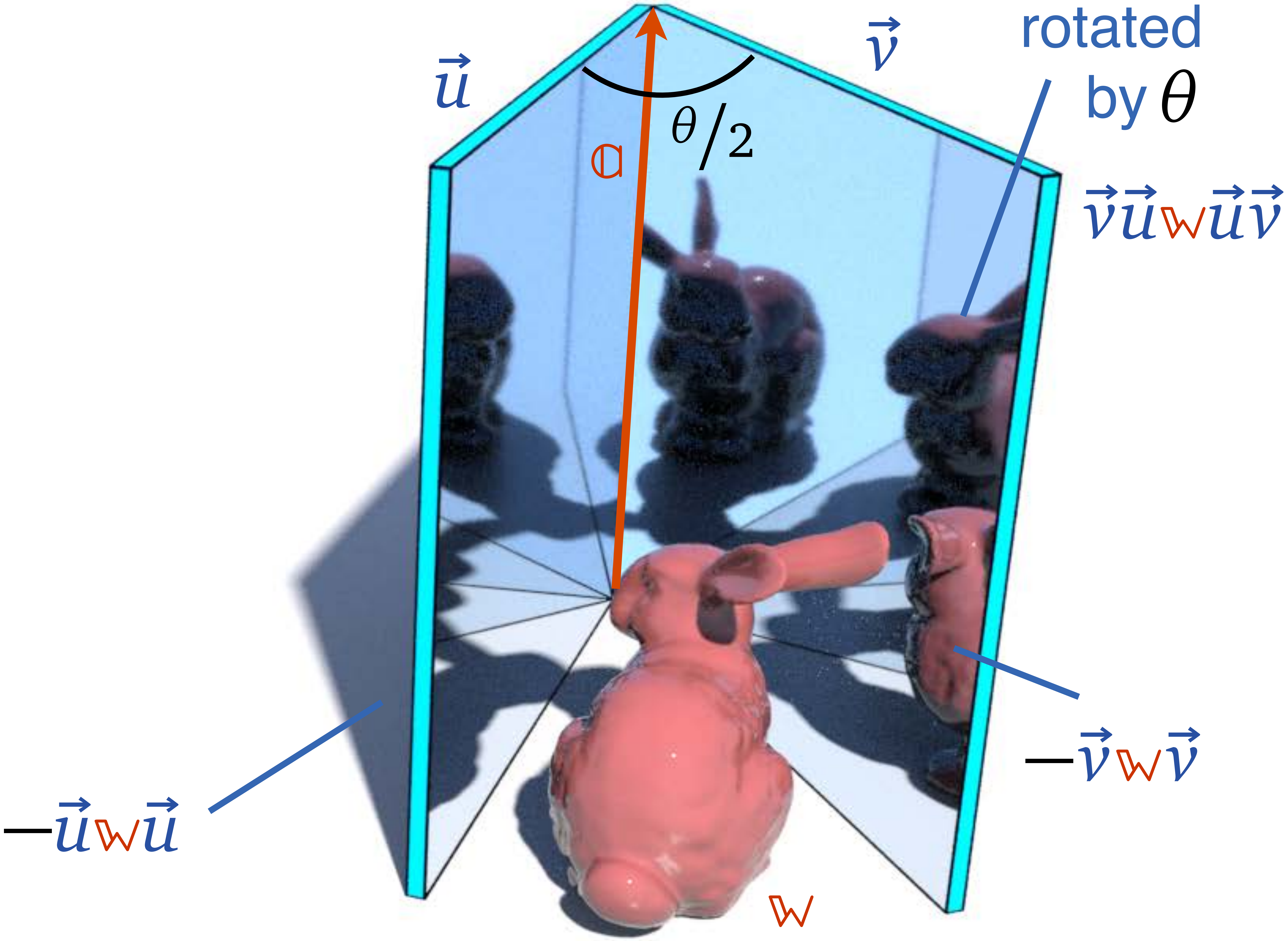
$$\vec{v}\vec{u} = \cos(\theta/2) + \sin(\theta/2)\mathbb{a}$$
$$= e^{\frac{\theta}{2}\mathbb{a}}$$



# Rotation

$$\vec{u}\vec{v} = e^{-\frac{\theta}{2}\mathbb{a}}$$

$$\vec{v}\vec{u} = e^{\frac{\theta}{2}\mathbb{a}}$$



# Rotation

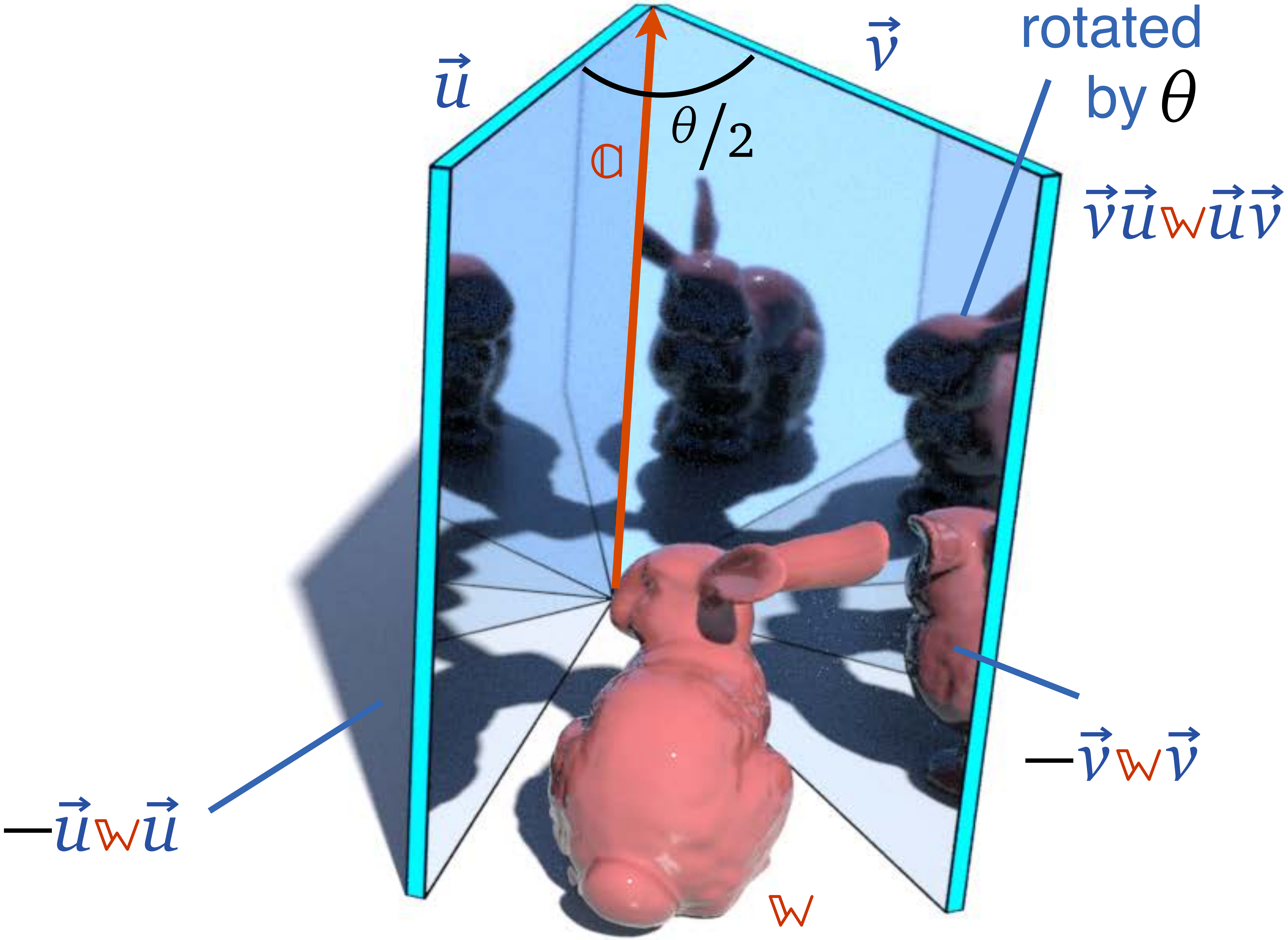
$$\vec{u}\vec{v} = e^{-\frac{\theta}{2}\mathbb{a}}$$

$$\vec{v}\vec{u} = e^{\frac{\theta}{2}\mathbb{a}}$$

**Rotation formula**

$$R^{\mathbb{a},\theta}\,\mathbb{w} = \vec{v}\vec{u}\mathbb{w}\vec{u}\vec{v}$$

$$= e^{\frac{\theta}{2}\mathbb{a}}\,\mathbb{w}\,e^{-\frac{\theta}{2}\mathbb{a}}$$



# Rotation

**Quaternion**
**Rotation formula**

$$R^{\mathbb{a},\theta}\,\mathbb{w} = e^{\frac{\theta}{2}\mathbb{a}}\,\mathbb{w}\,e^{-\frac{\theta}{2}\mathbb{a}}$$

# Rotation

**vec3** Rotate( $\mathbf{a} \in \mathbb{R}^3$, $\theta \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^3$ ){

$$\mathbb{w} = w_x \mathbb{i} + w_y \mathbb{j} + w_z \mathbb{k}$$

$$\mathbb{a} = a_x \mathbb{i} + a_y \mathbb{j} + a_z \mathbb{k}$$

**return** **(vec3)**( $R^{\mathbb{a},\theta} \mathbb{w} = e^{\frac{\theta}{2}\mathbb{a}} \mathbb{w} e^{-\frac{\theta}{2}\mathbb{a}}$ );

}

# Next week:
# Homogeneous Coordinates