# CSE 167 (FA22)
# Computer Graphics: Projective Geometry

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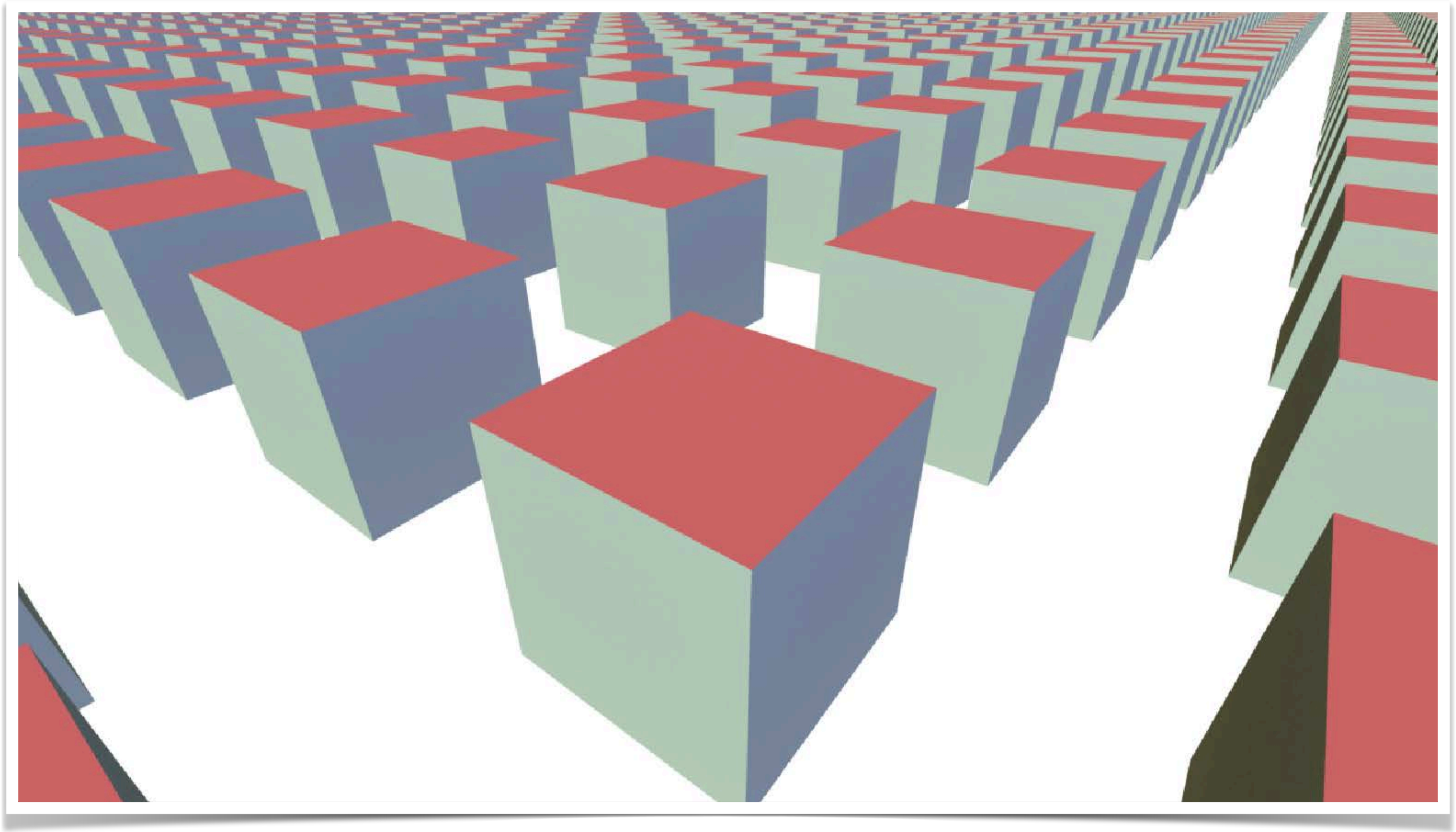
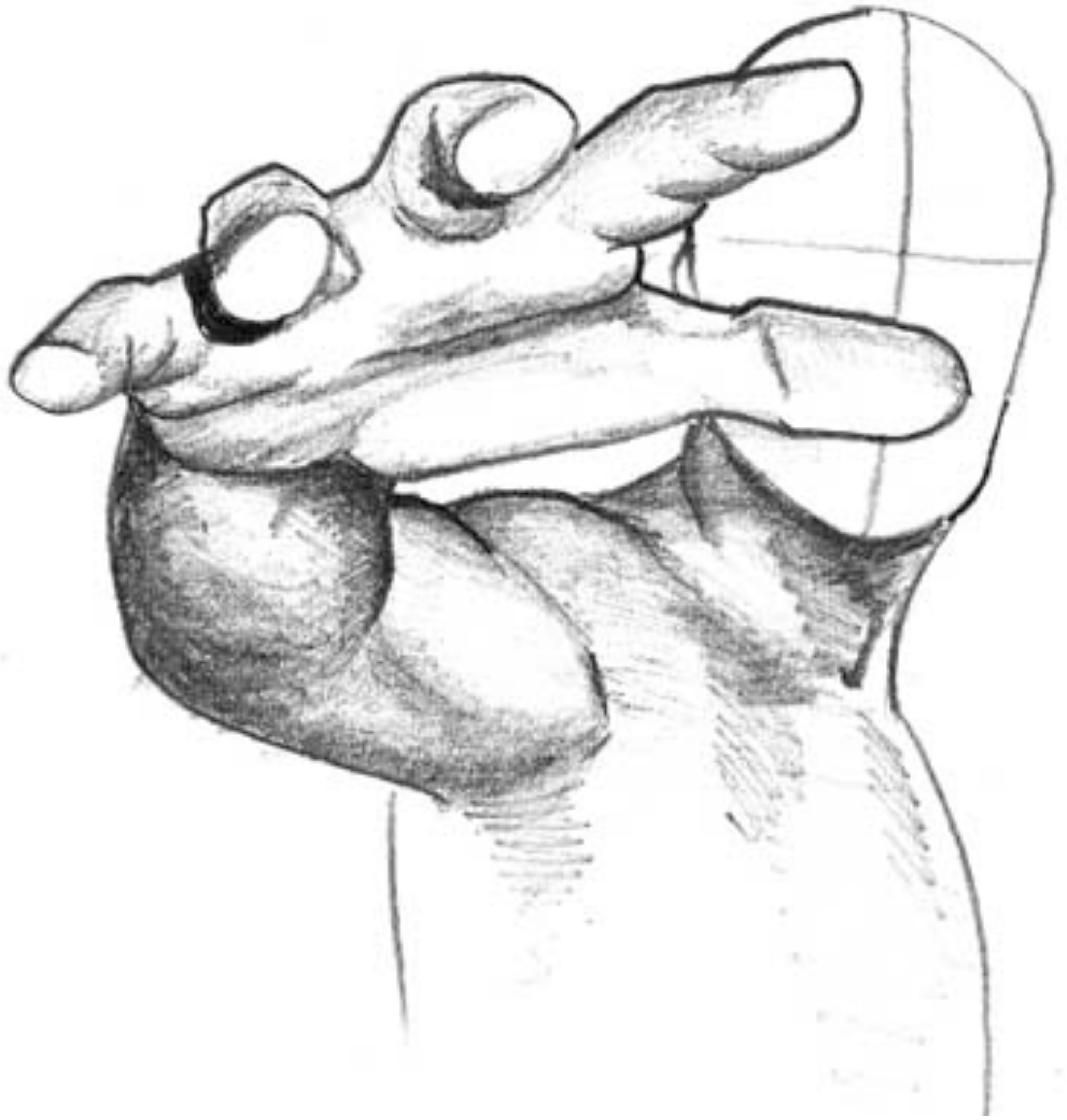
# What are Projective Transformations?

- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

# Perspective

- Foreshortening
  - Further objects look smaller
  - The scaling is non-uniform

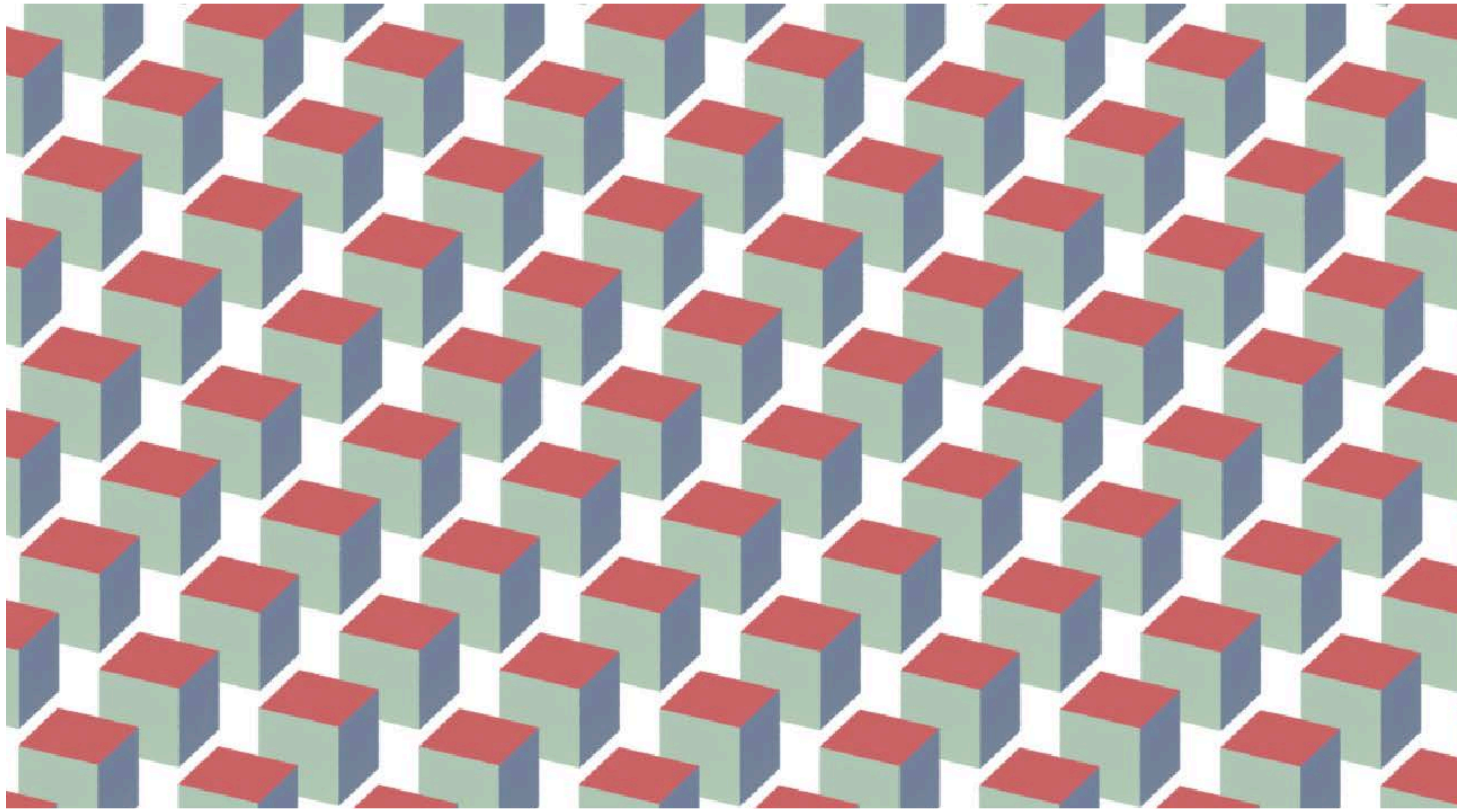


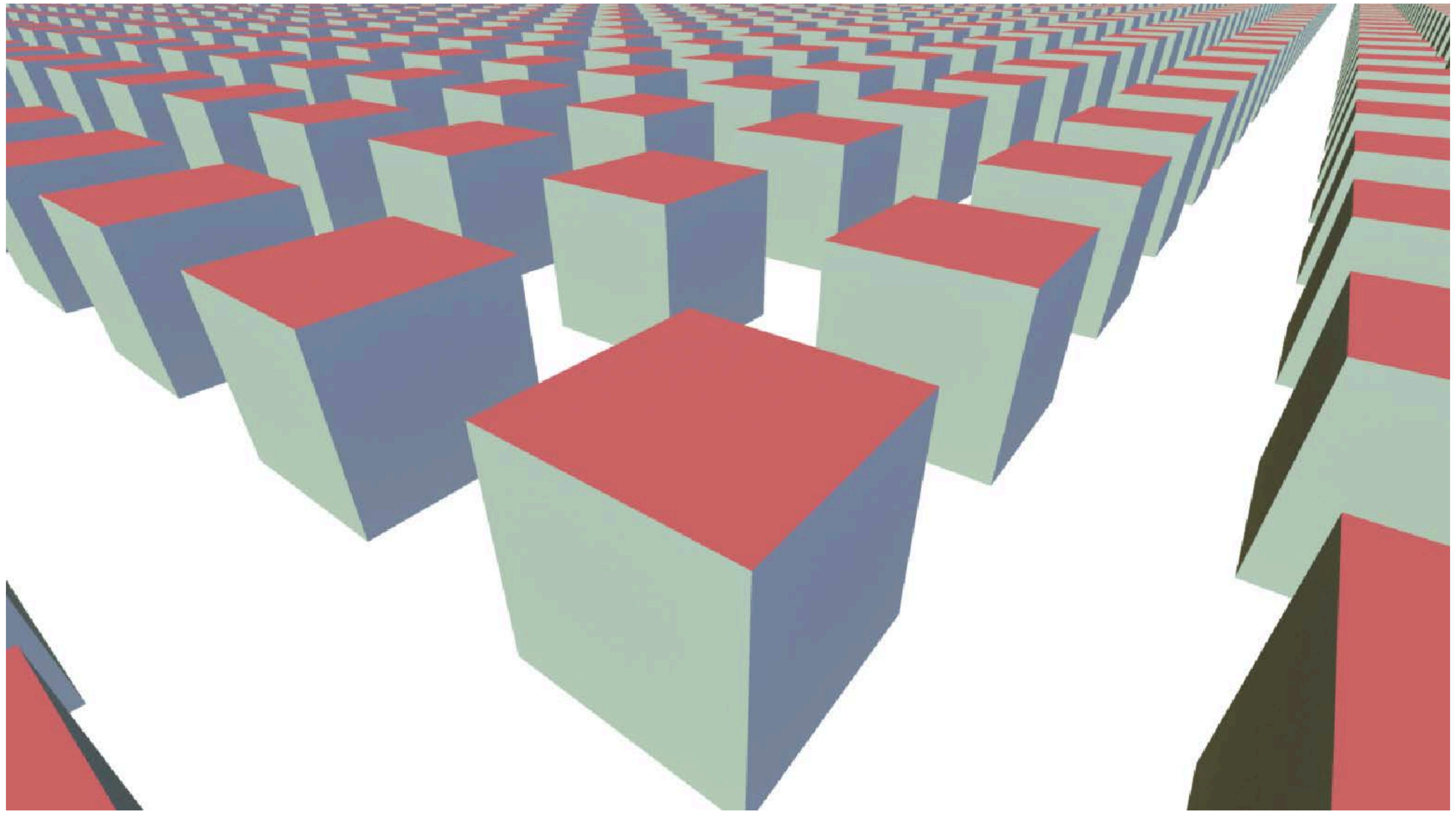


# Perspective

- Orthographic view
  - Telescopic view standing infinitely far away.
  - A view preserving parallelism, proportions
  - Lack a sense of depth

- Perspective view
  - The vantage point is immersed in the scene.
  - Does not preserve parallelism, proportions
  - Provides a sense of depth

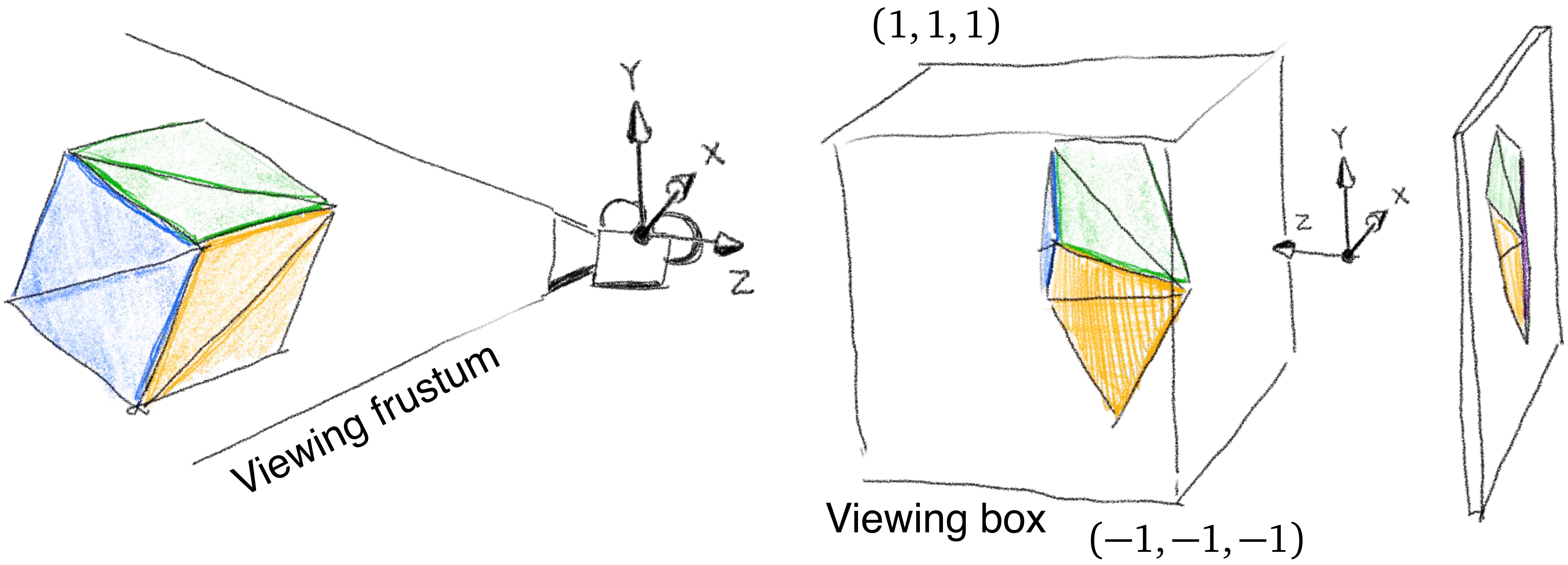




# Goal

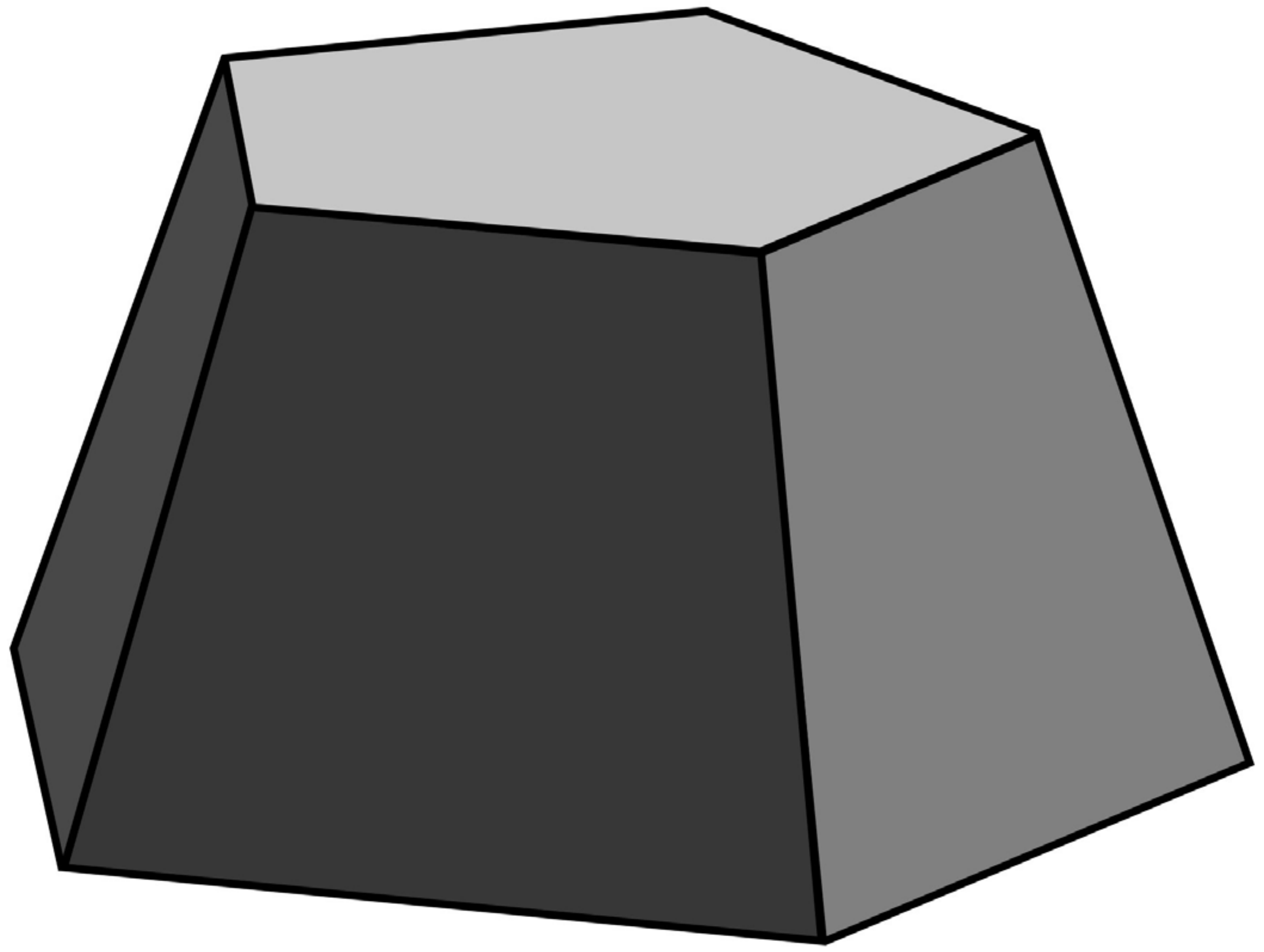
Map points from the **camera coordinate** to the **Normalized Device Coordinate** (NDC) so that

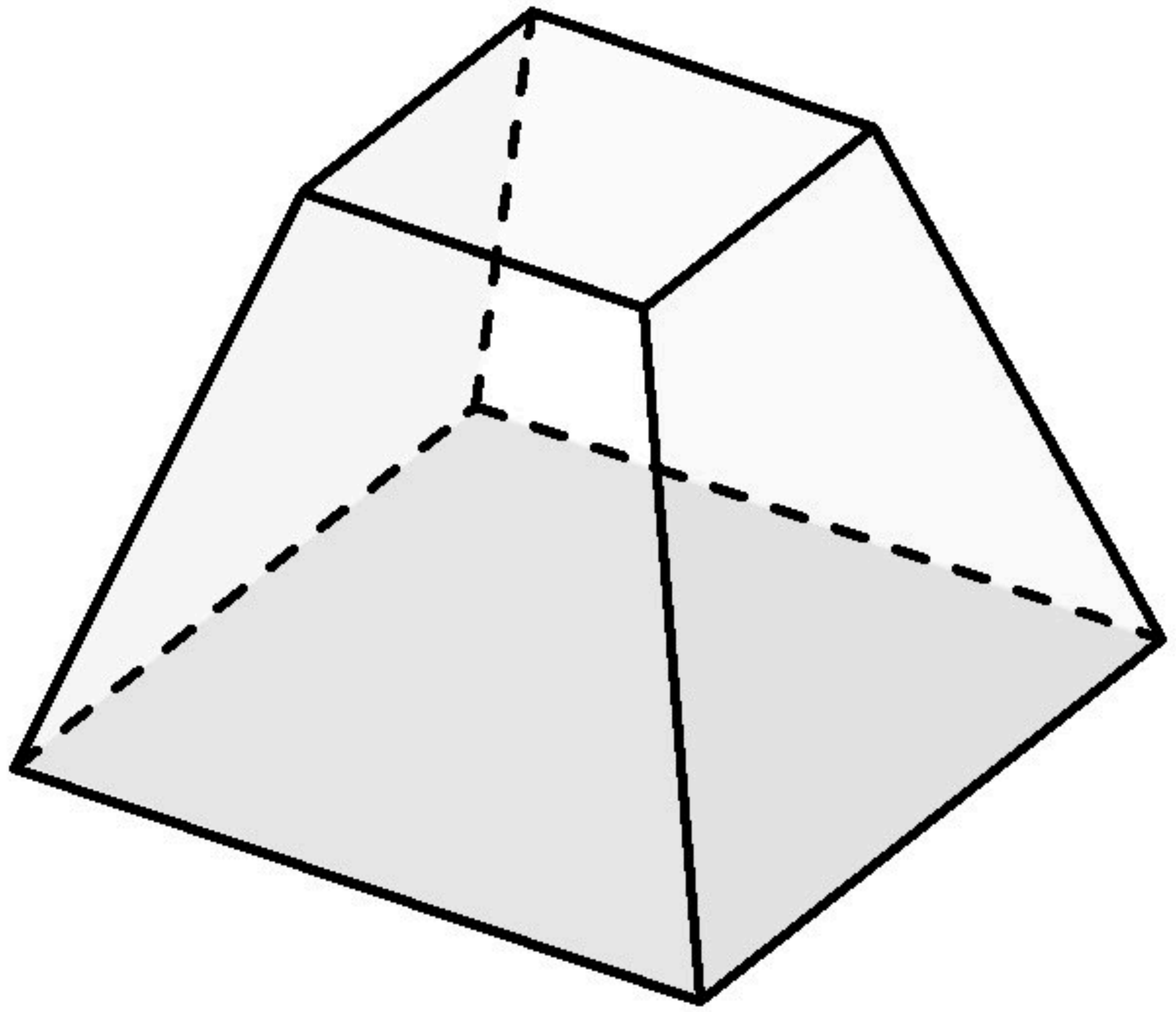
- The **viewing frustum** is mapped to the viewing box in NDC;
- The map is a **collineation**.



# Frustum

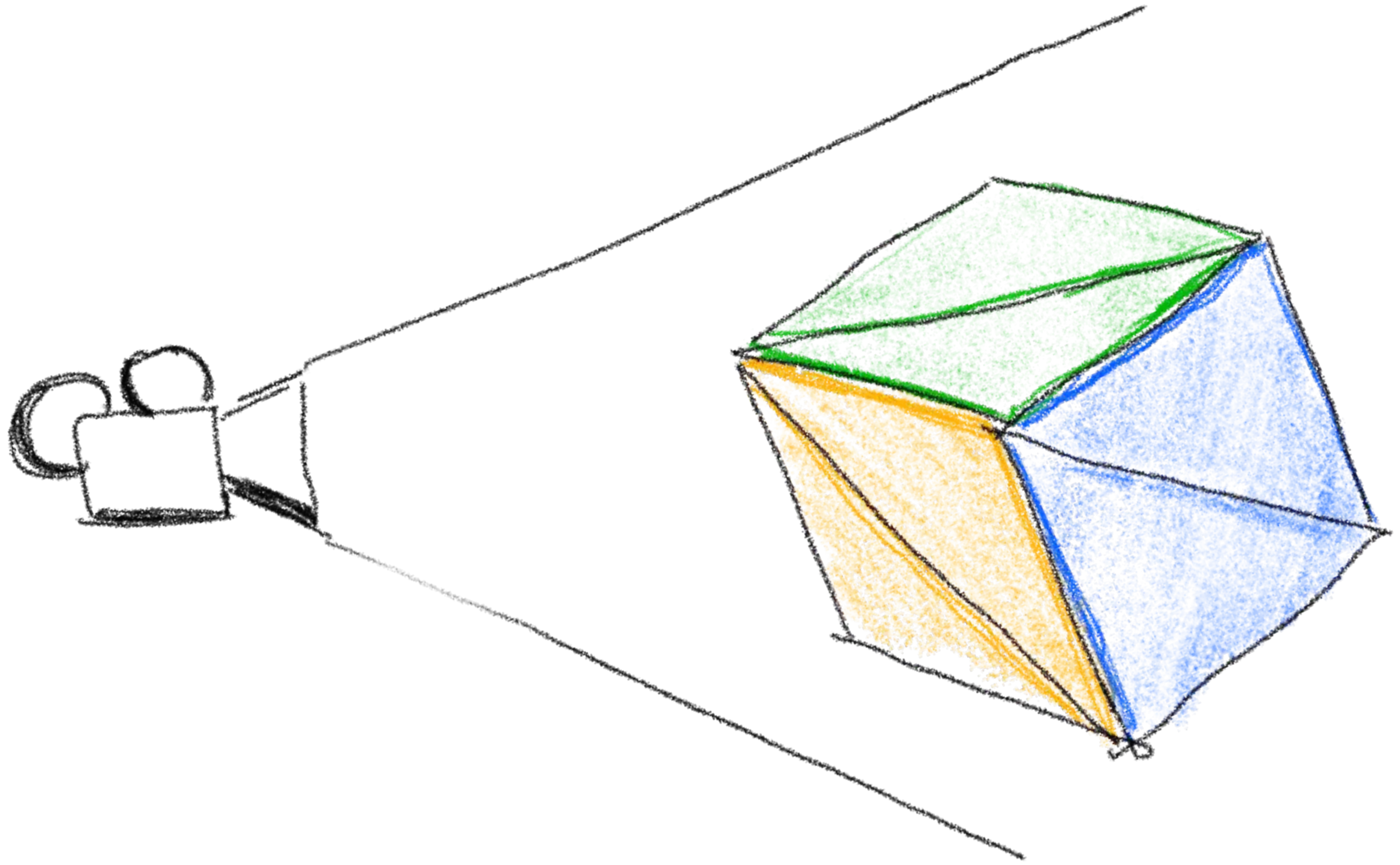
In geometry, a *frustum* is the portion of a pyramid (or cone) that lies between two parallel planes cutting it.

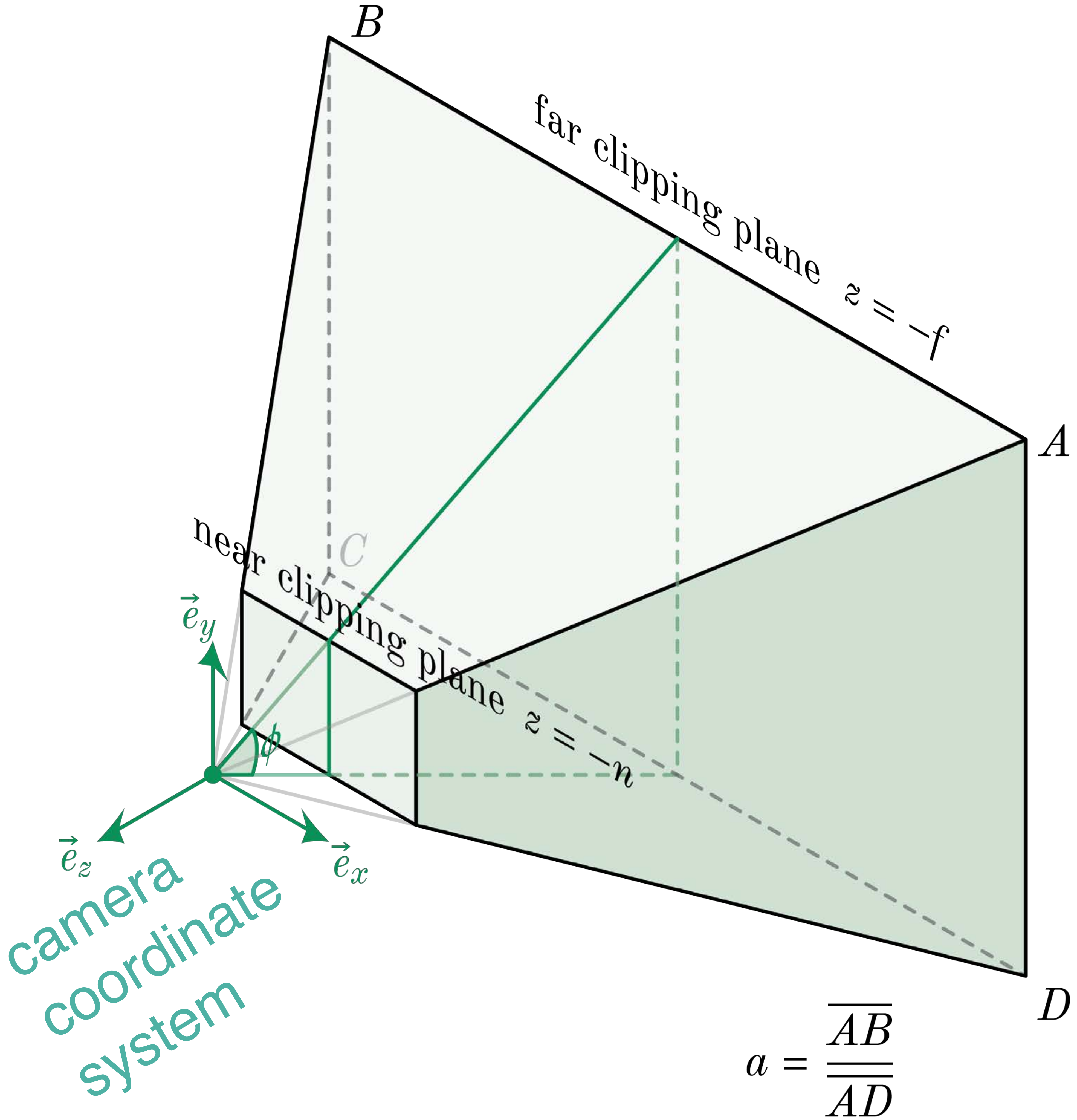




# Viewing frustum

In graphics, a *viewing frustum* is the 3 dimensional region that is visible on the screen.

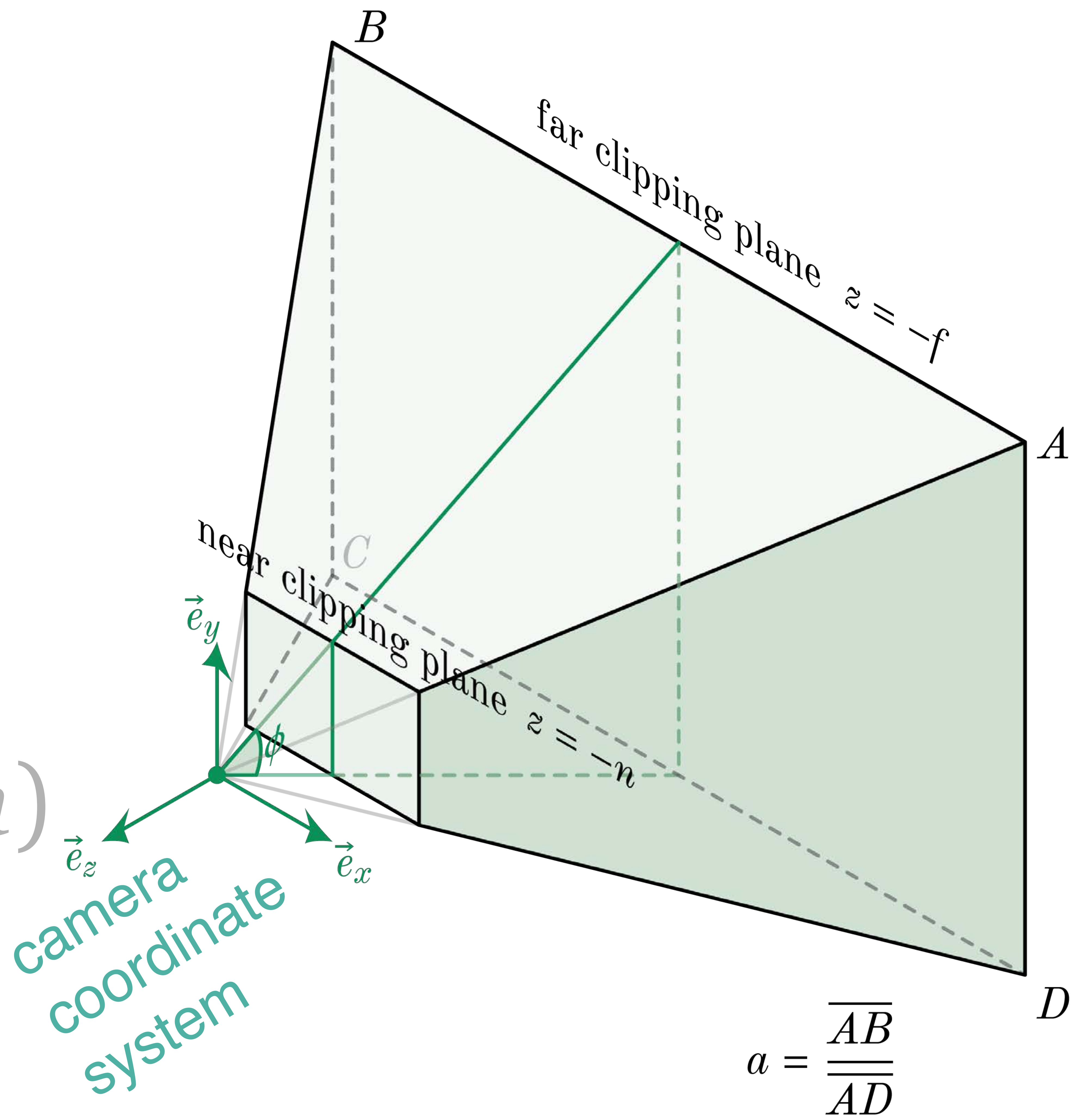




# Viewing frustum

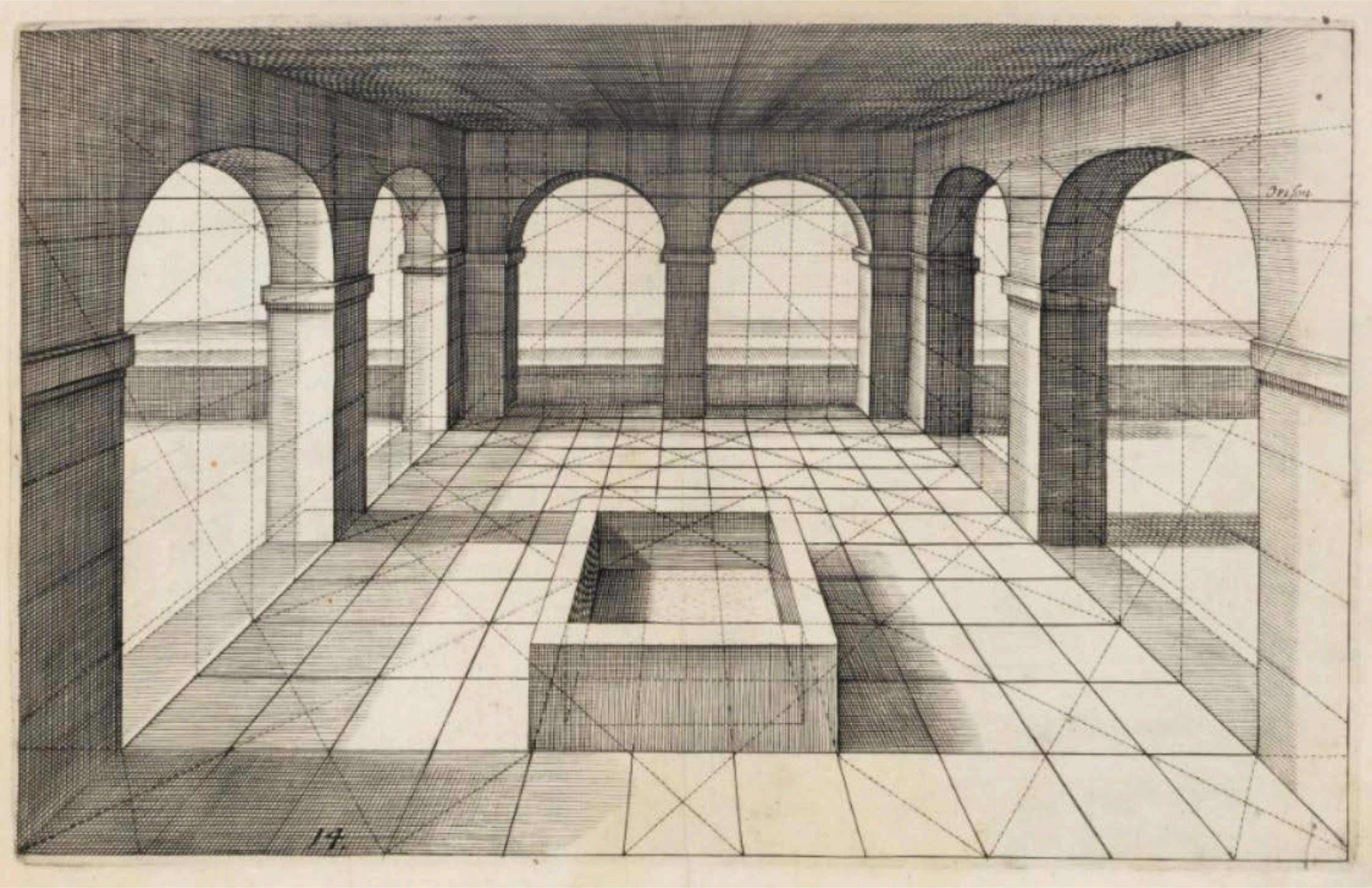
In graphics, a *viewing frustum* is the 3 dimensional region that is visible on the screen.

- Parameters for a viewing frustum:
  - **field of view:** angle $\phi$
  - **aspect ratio:** $a = \frac{\text{width}}{\text{height}}$
  - **near clipping distance** $n > 0$
  - **far clipping distance** $f$ $(f > n)$



# Collineation

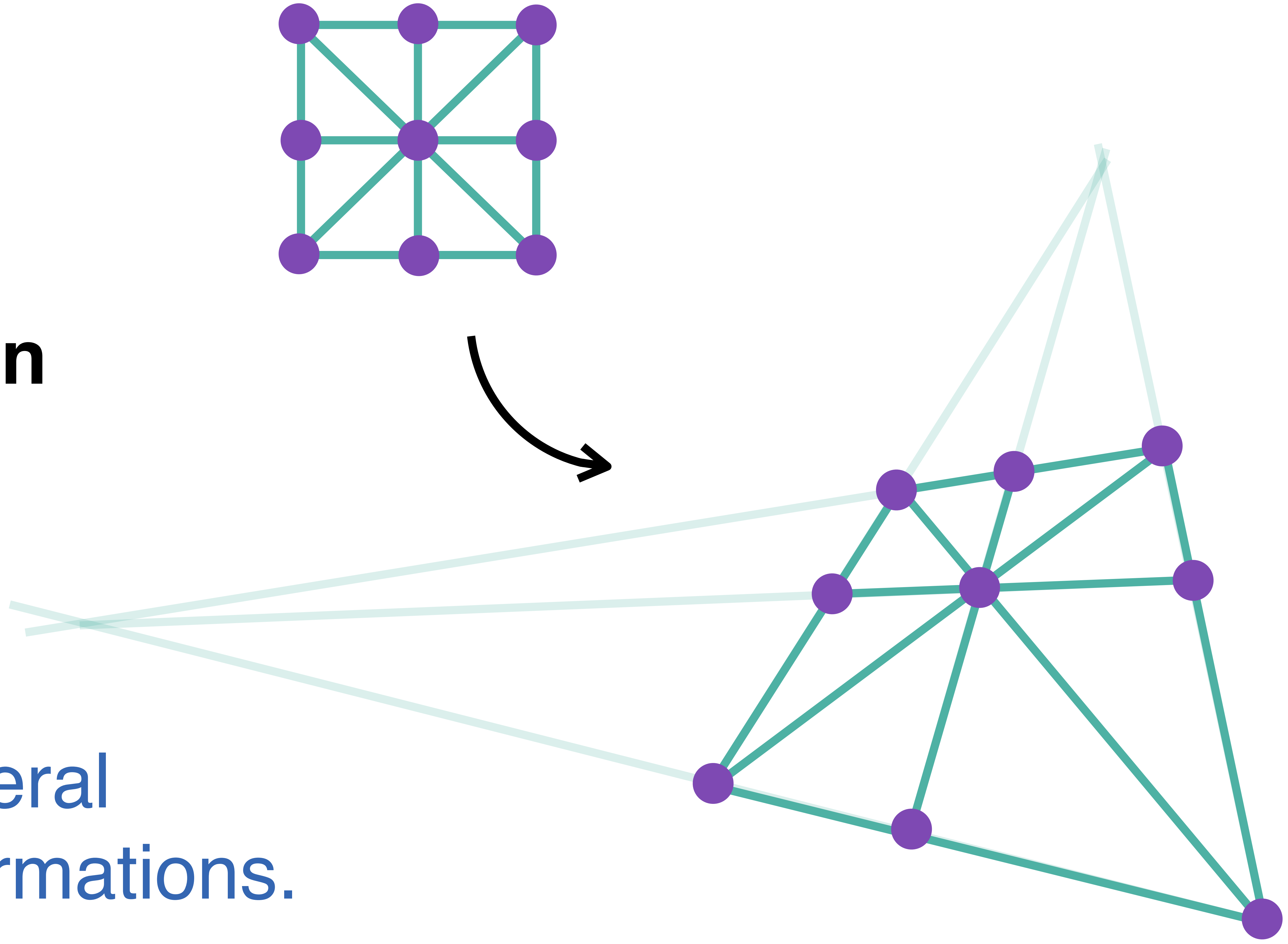
A straight line in the physical space should still be a straight line in the viewing box or on the screen.



# Collineation

A transformation is called a collineation if straight lines are mapped to straight lines.

- Synonyms
  - **Collineation**
  - **Projective transformation**
  - **Homography**



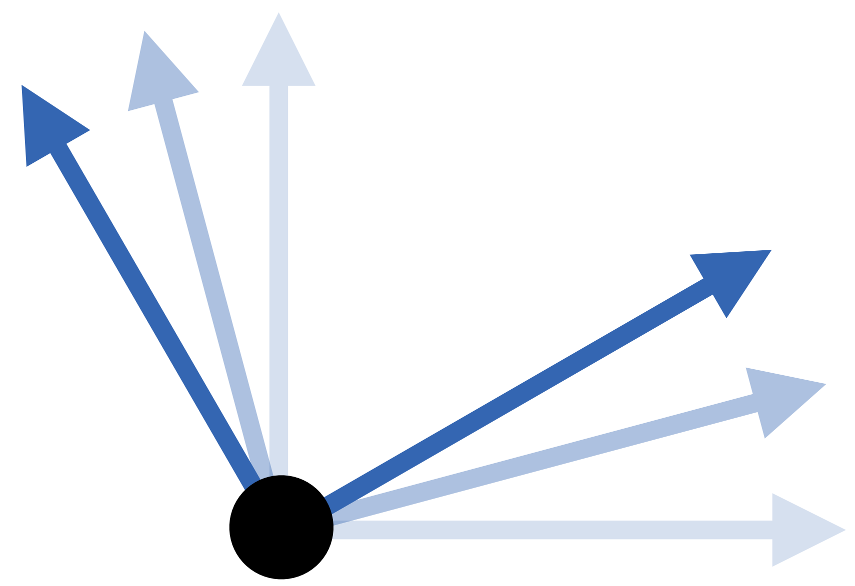
- Parallelism is generally not preserved!  This is more general than linear and affine transformations.

# Linear, affine and projective

Multiplying 3D vectors by a 3x3 matrix

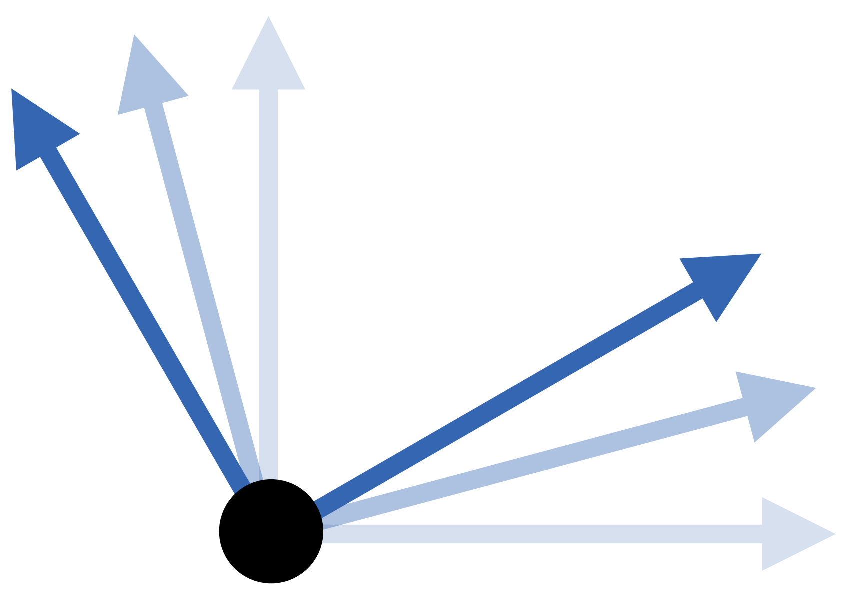
$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

**Linear transformation**



# Linear, affine and projective

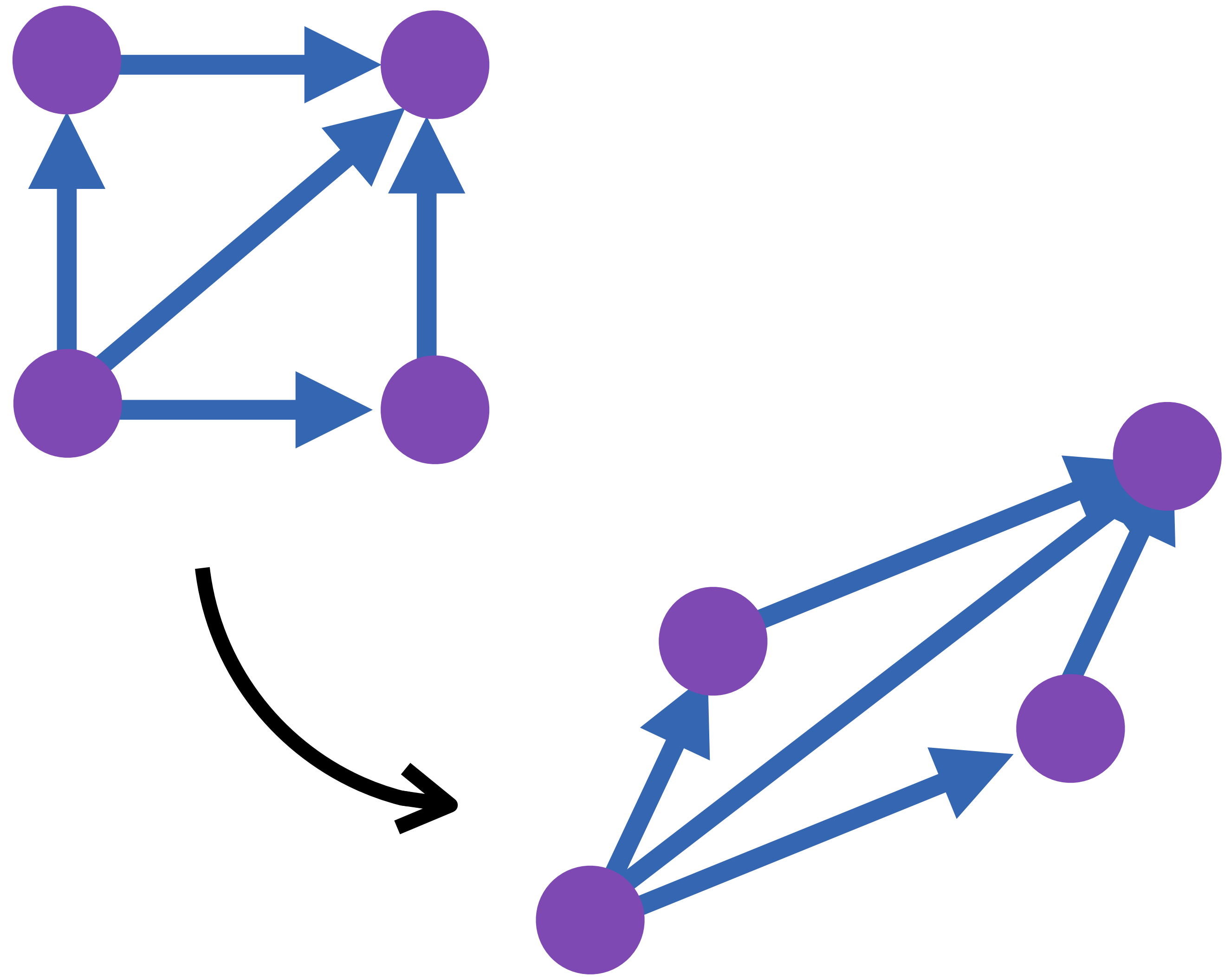
$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



Multiplying 3D positions (in 4D hom. coord.) by a 4x4 matrix with 0,0,0,1 as the last row.

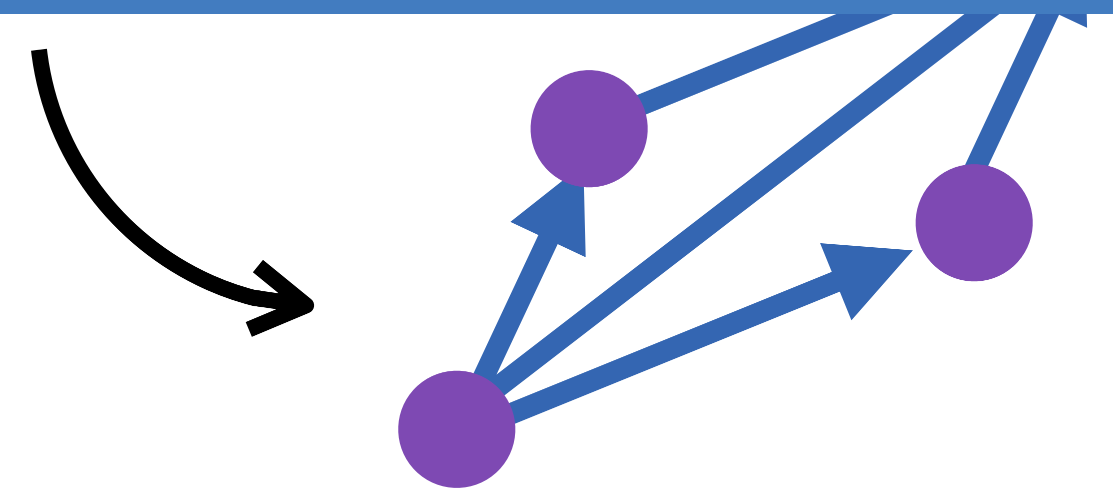
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_x \\ a_{21} & a_{22} & a_{23} & b_y \\ a_{31} & a_{32} & a_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

**Affine transformation**



# Linear, affine and projective

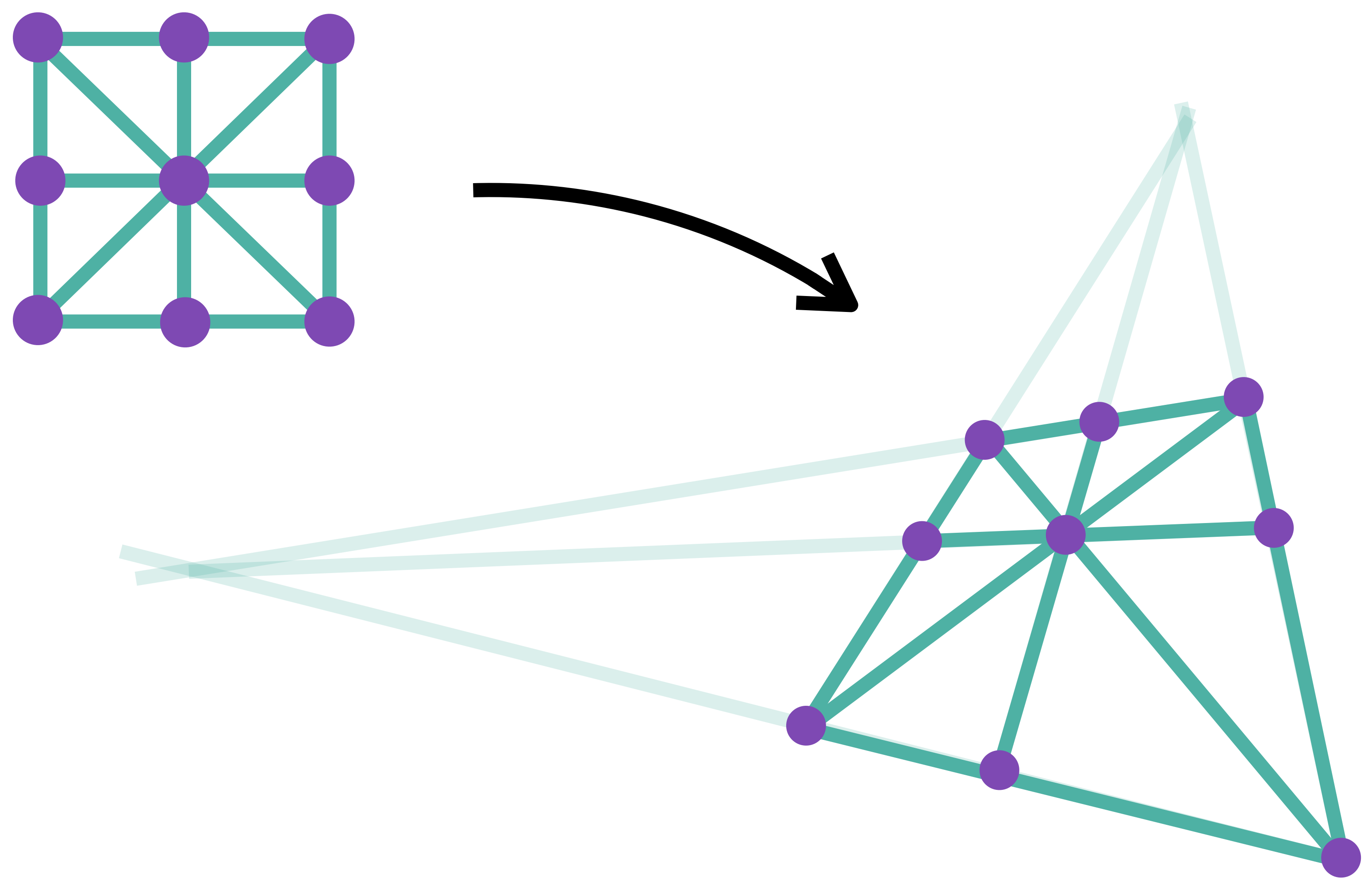
$$\begin{bmatrix} p_z \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a_{31} & a_{32} & a_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_z \\ 1 \end{bmatrix}$$



Multiplying 3D positions (in 4D hom. coord.) by a general 4x4 matrix.

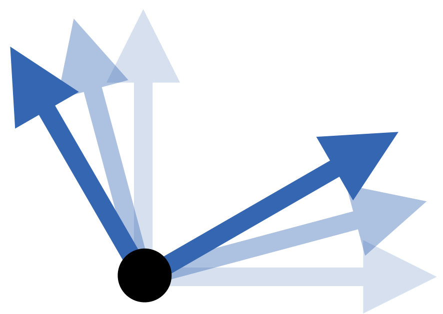
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

**Projective transformation (collineation)**



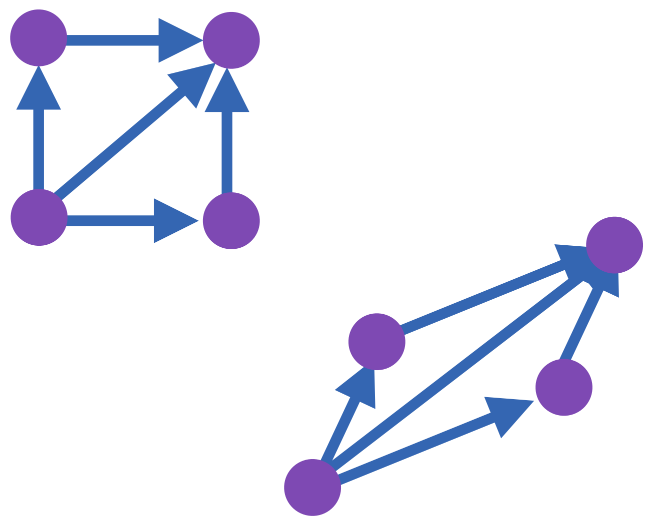
# Linear, affine and projective

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



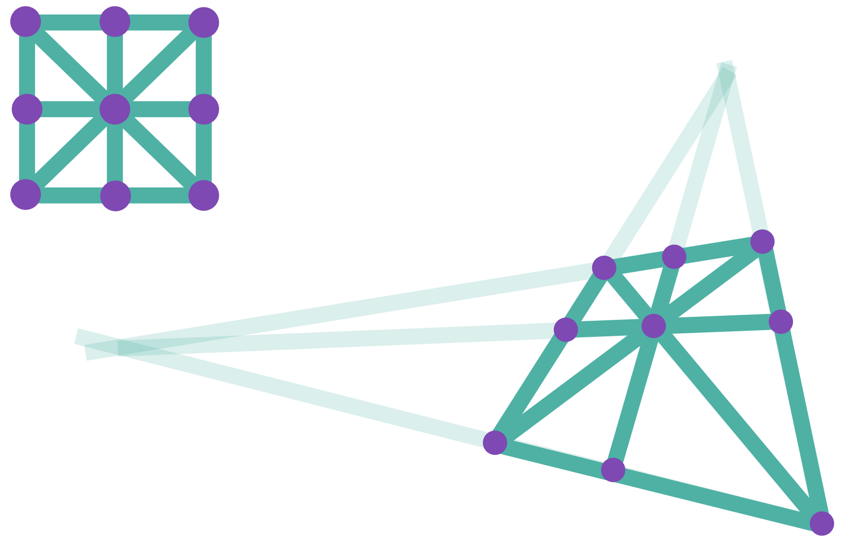
Linear transformations are affine transformations that preserve the origin.

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_x \\ a_{21} & a_{22} & a_{23} & b_y \\ a_{31} & a_{32} & a_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$



Affine transformations are projective transformations that preserve parallelism.

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

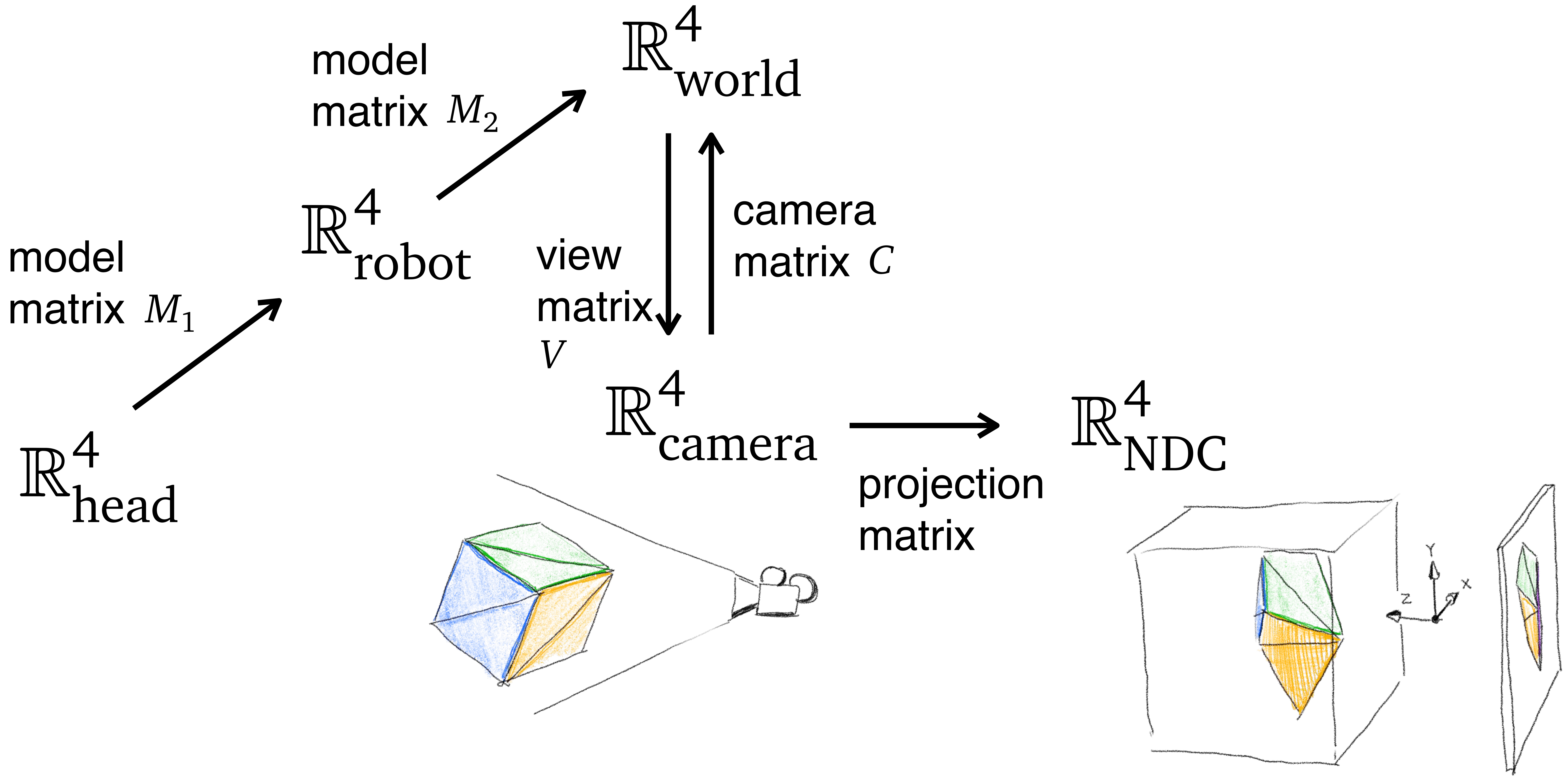


Projective transformations preserve straight lines.

# Final projection after modelview

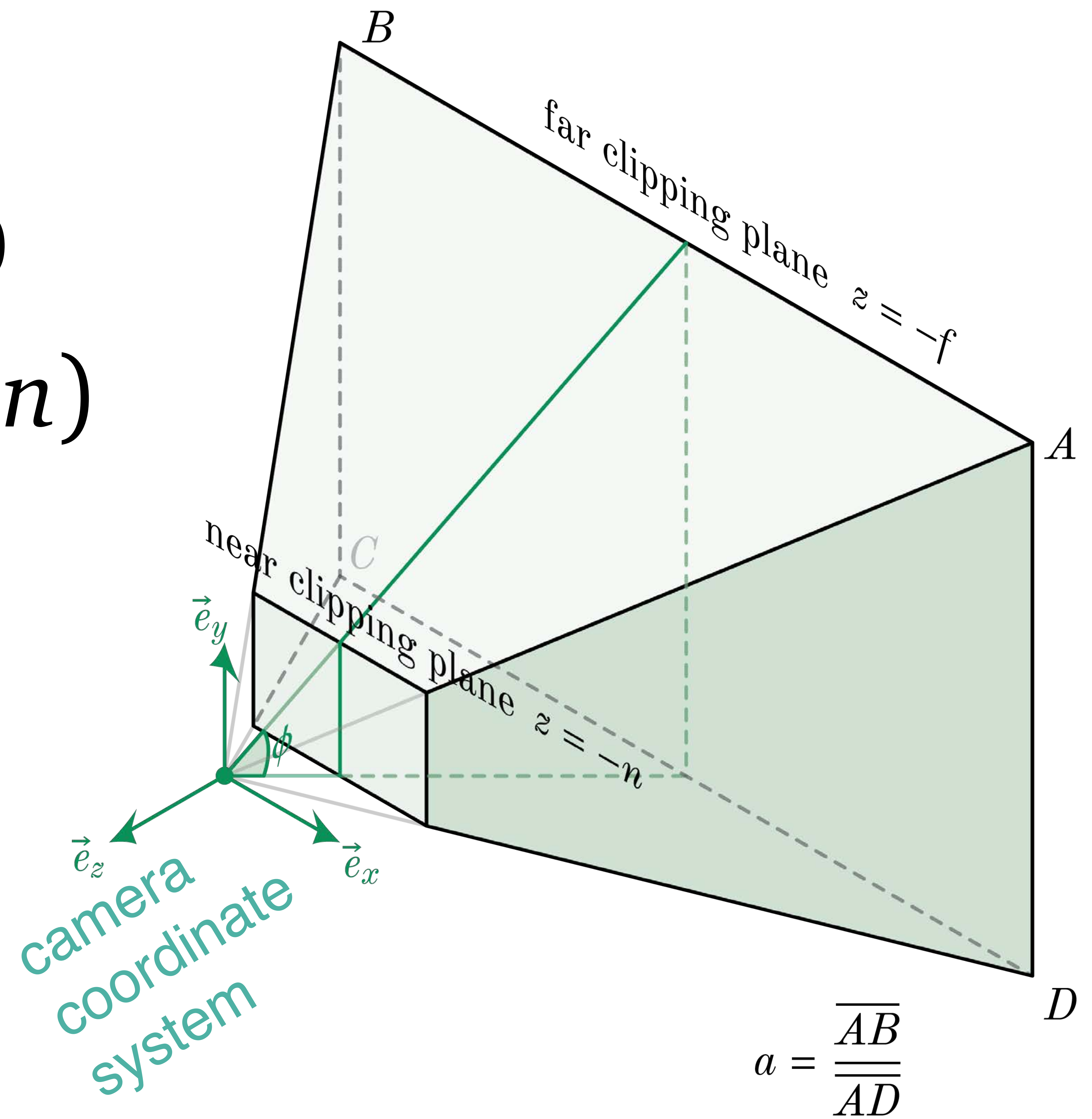
## In the vertex shader:

```
gl_Position = projection * modelview * vec4( vertex_position, 1.0f );
```

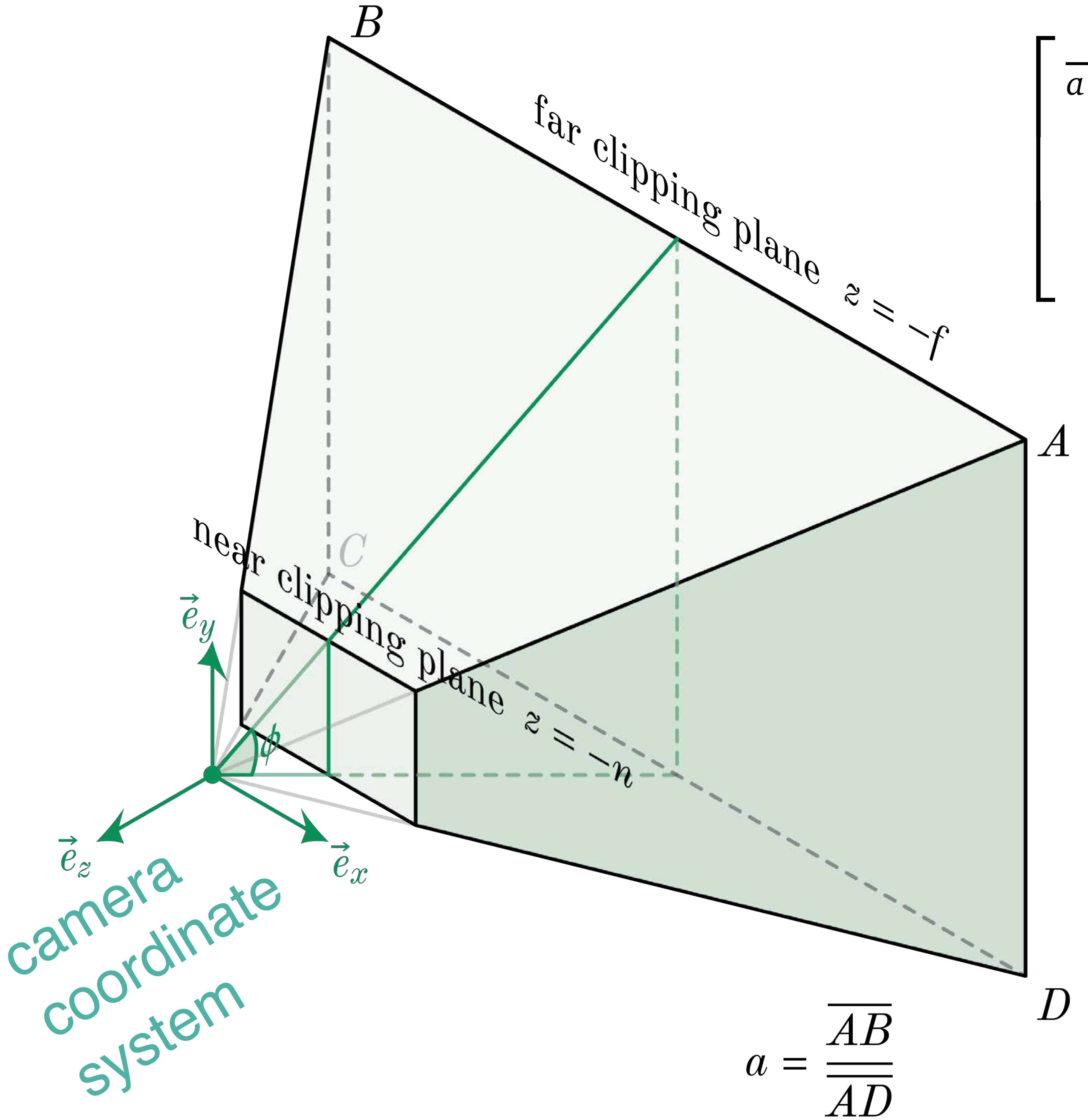


# Projection matrix

- **field of view:** angle $\phi$
- **aspect ratio:** $a = \dfrac{\text{width}}{\text{height}}$
- **near clipping distance** $n > 0$
- **far clipping distance** $f \ (f > n)$

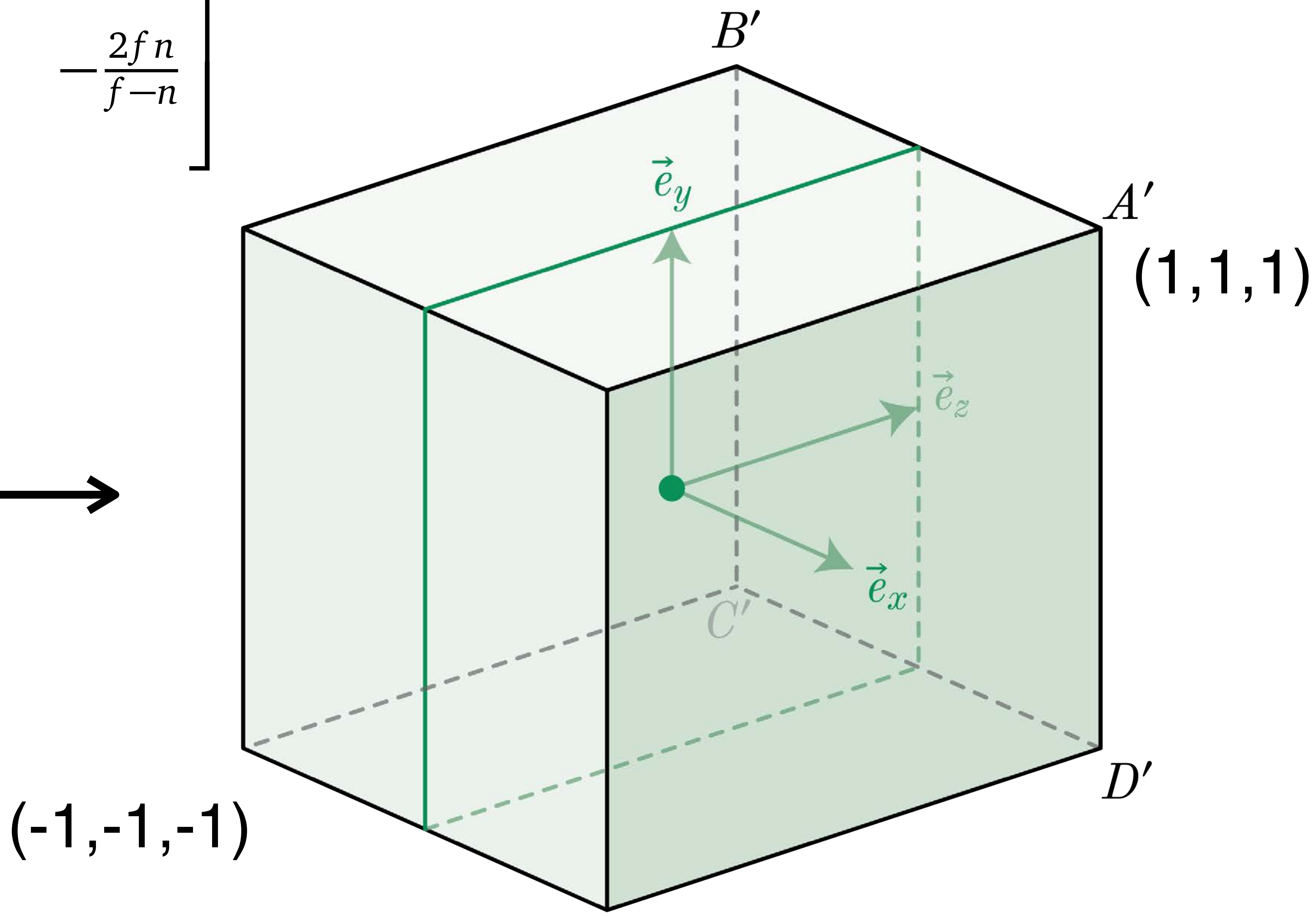


# Projection matrix




$$a = \frac{\overline{AB}}{\overline{AD}}$$

$$\begin{bmatrix} \frac{1}{a\tan(\phi/2)} & & & \\ & \frac{1}{\tan(\phi/2)} & & \\ & & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ & & -1 & \end{bmatrix}$$




normalized device coordinate

# How Homogeneous Coordinates actually work

- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

# How does it work?

Homogeneous coordinates

# Homogeneous coordinates

In the homogeneous coordinate, we view each 4D vector as a 4-proportion.

We write $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \sim \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \\ \lambda w \end{bmatrix}$ to indicate that they have the *same proportion*.

# Homogeneous coordinates

$$\begin{bmatrix} 2 \\ 6 \\ -4 \\ 5 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 3 \\ -2 \\ 2.5 \end{bmatrix} \sim \begin{bmatrix} -100 \\ -300 \\ 200 \\ -250 \end{bmatrix}$$

$$\begin{aligned} & 2 : 6 : -4 : 5 \\ & = 1 : 3 : -2 : 2.5 \\ & = -100 : -300 : 200 : -250 \end{aligned}$$

# Between 3D position & 4D homo. coord.

$$\mathbb{R}^3 \qquad\qquad\qquad \mathbb{R}^4$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \xrightarrow[\text{homogenize}]{} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

# Between 3D position & 4D homo. coord.

$$\mathbb{R}^3 \qquad\qquad \mathbb{R}^4$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \xrightarrow{\text{homogenize}} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$\xleftarrow{\text{dehomogenize}} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

# Between 3D position & 4D homo. coord.

$$\mathbb{R}^3 \qquad\qquad \mathbb{R}^4$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \xRightarrow{\text{homogenize}} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{bmatrix} \xLeftarrow{\text{dehomogenize}} \begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \\ 1 \end{bmatrix} \sim \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

# Between 3D position & 4D homo. coord.

$$\mathbb{R}^3 \qquad\qquad\qquad \mathbb{R}^4$$

$$\begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \end{bmatrix} \xleftarrow{\text{dehomogenize}} \begin{bmatrix} p_x/p_w \\ p_y/p_w \\ p_z/p_w \\ 1 \end{bmatrix} \sim \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

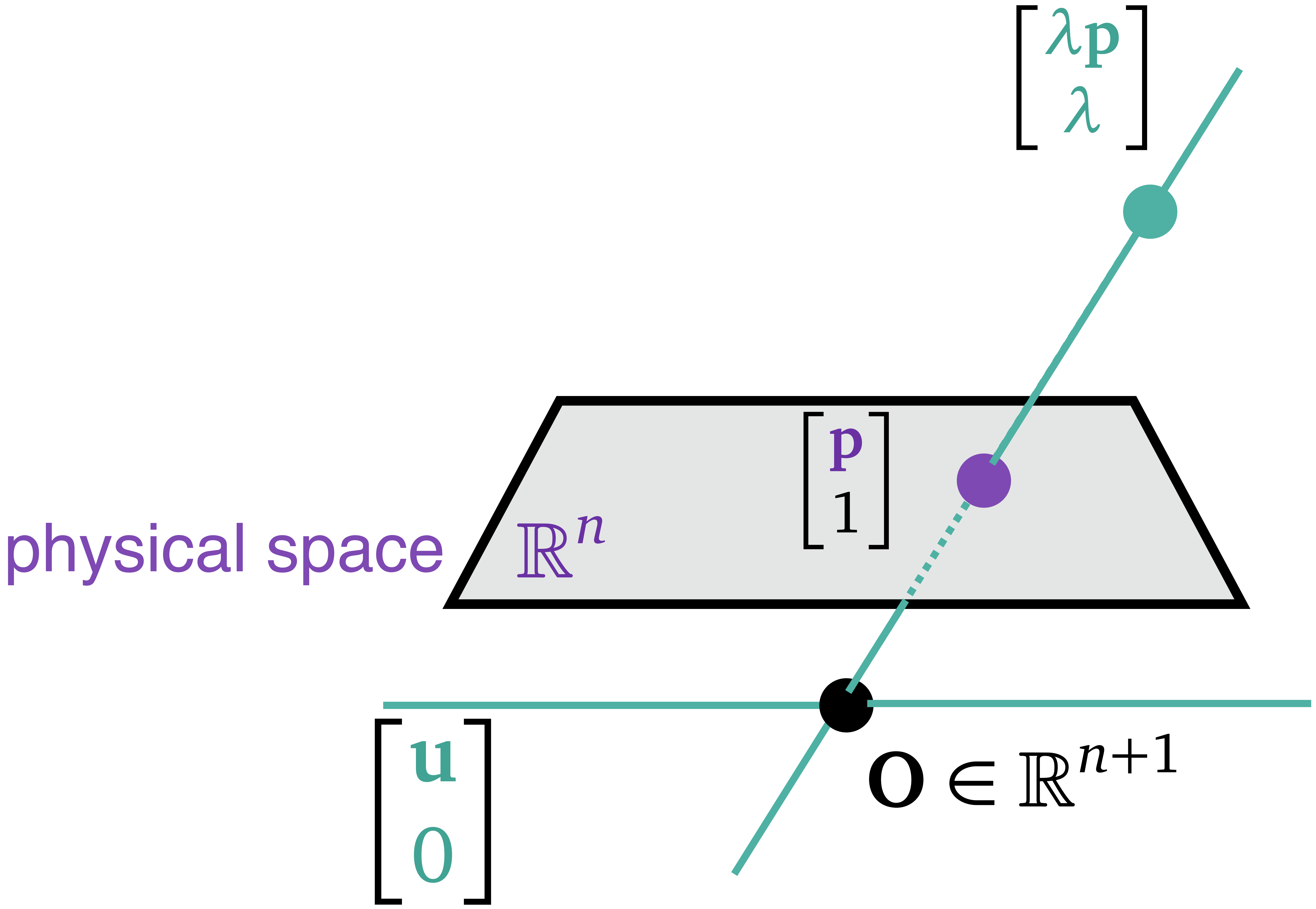
$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 0 \end{bmatrix}$$ represents "points at infinity," (or a pure direction like a vector)

(It is a good idea to keep positions represented in 4D, so that we can talk about points at infinity at ease)

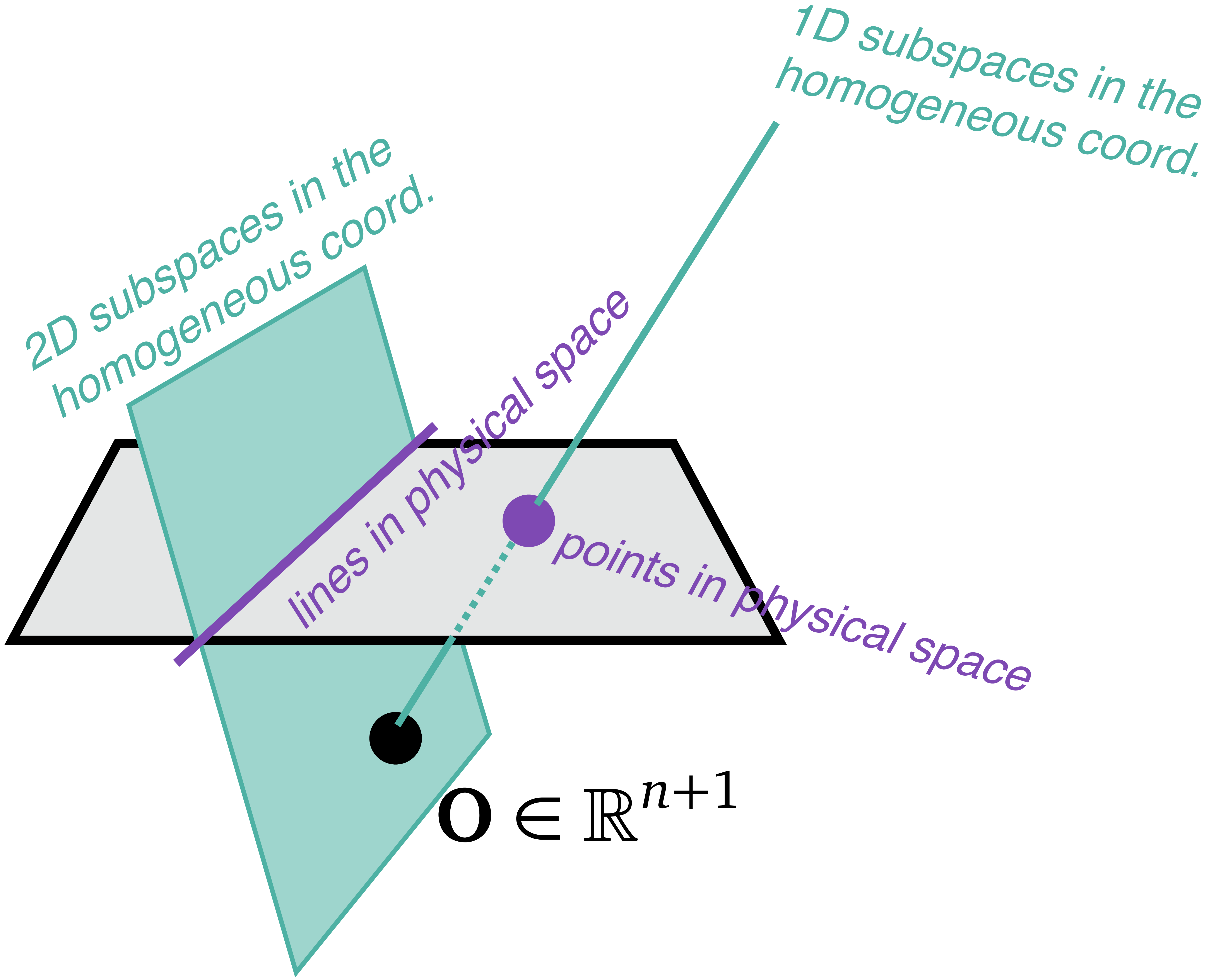
# Between 3D position & 4D homo. coord.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \xrightarrow[\text{dehomogenize}]{} \begin{bmatrix} 0.25 \\ 0.5 \\ 0.75 \end{bmatrix}$$

# Between 3D position & 4D homo. coord.



# Between 3D position & 4D homo. coord.



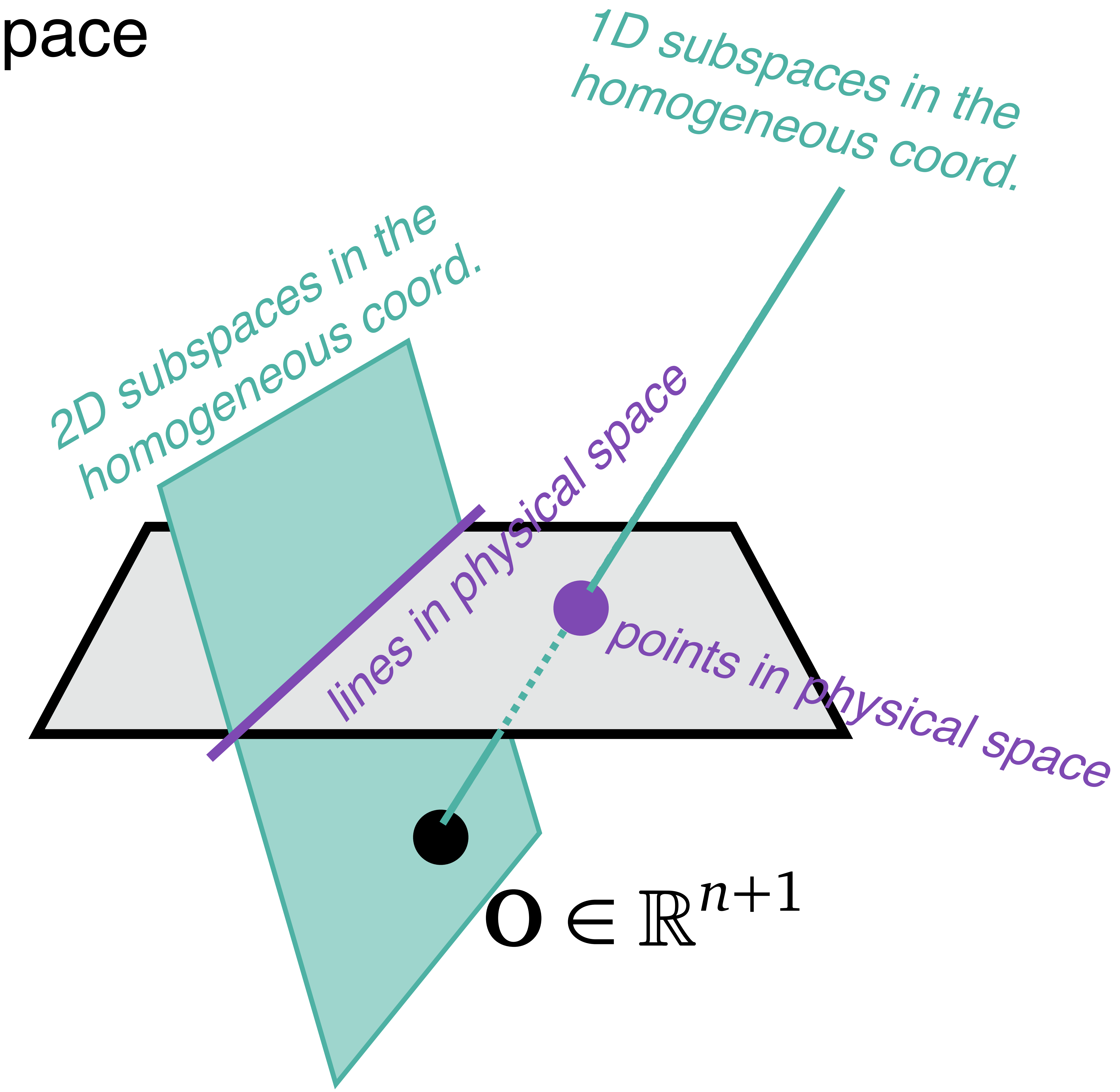
# Between 3D position & 4D homo. coord.

Every collineation on the physical space

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \mapsto \begin{bmatrix} p_x^{\mathrm{new}} \\ p_y^{\mathrm{new}} \\ p_z^{\mathrm{new}} \end{bmatrix}$$

can be written as the result of

- Step1: homogenize
- Step2: 4D linear transformation
- Step3: dehomogenize



# An example projective transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$ will perform the following transformation:

Each point $\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ is mapped to $\begin{bmatrix} p_x^{\text{new}} \\ p_y^{\text{new}} \\ p_z^{\text{new}} \end{bmatrix} = ?$

# An example projective transformation

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ 1 \\ p_z \end{bmatrix} \sim \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1/p_z \\ 1 \end{bmatrix}$$

$$\xRightarrow{\text{dehomogenize}} \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1/p_z \end{bmatrix}$$

# An example projective transformation

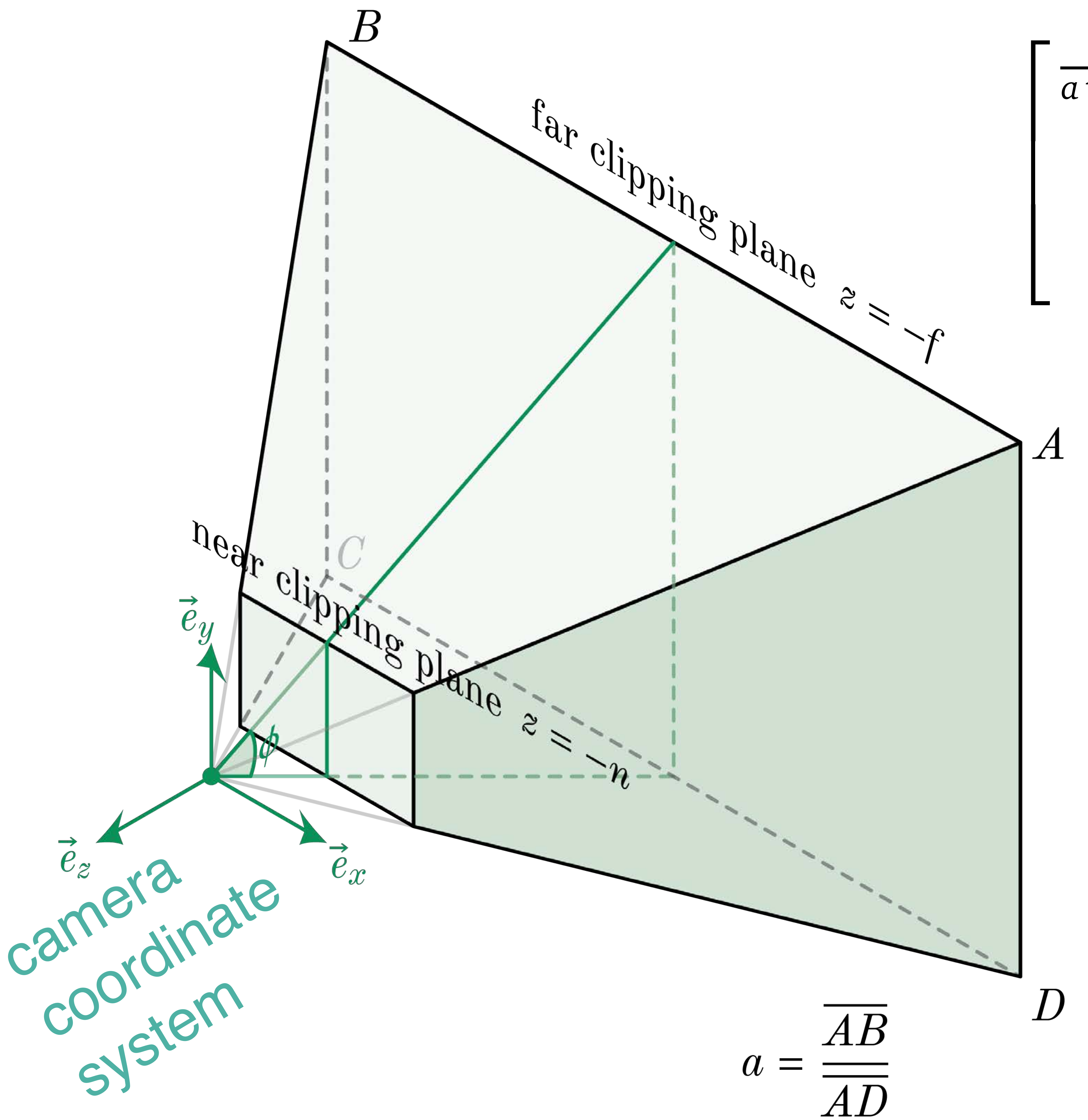
$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$ will perform the following transformation:

Each point $\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ is mapped to $\begin{bmatrix} p_x^{\text{new}} \\ p_y^{\text{new}} \\ p_z^{\text{new}} \end{bmatrix} = \begin{bmatrix} p_x/p_z \\ p_y/p_z \\ 1/p_z \end{bmatrix}$

# Derivation of the Projection Matrix

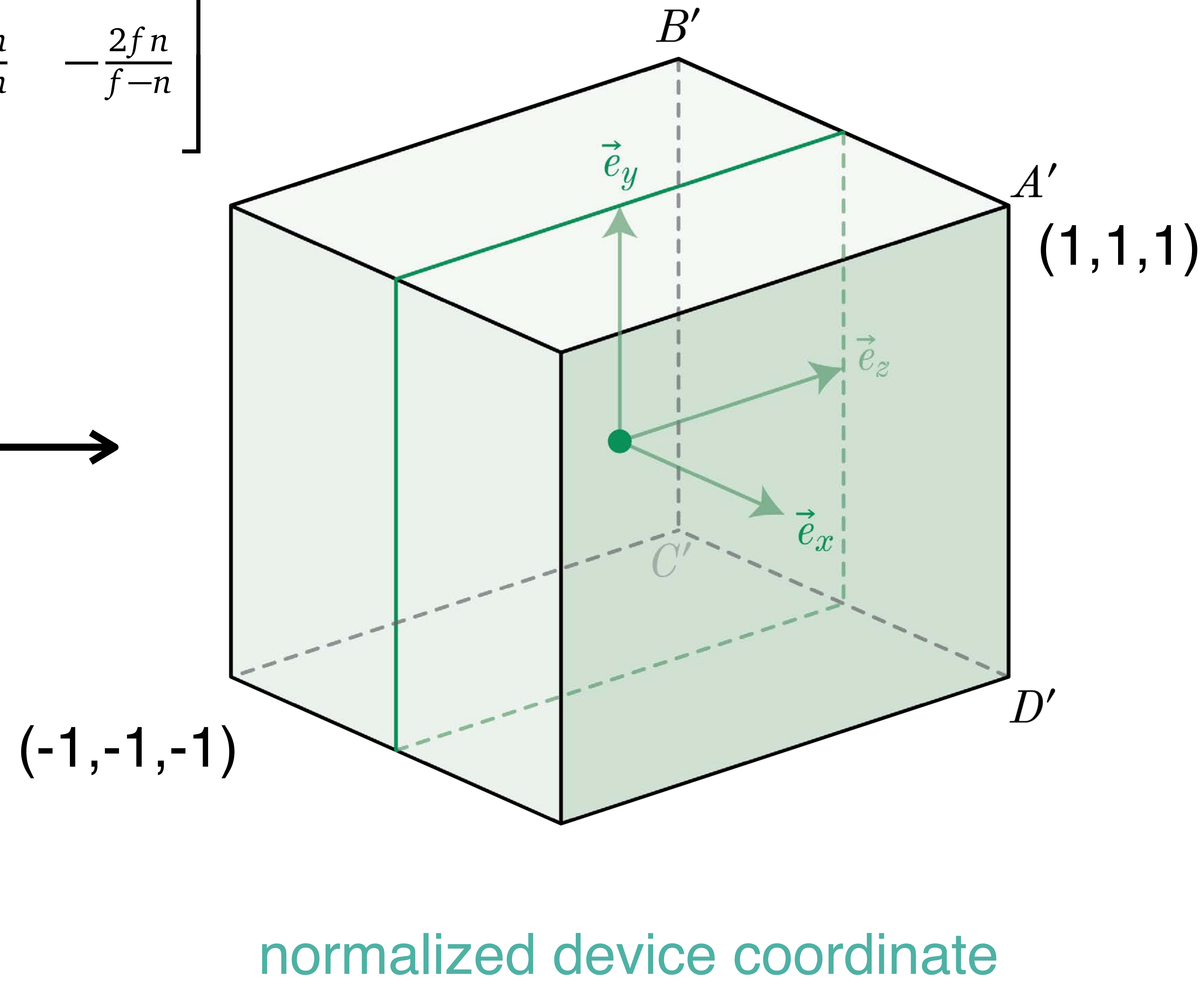
- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

# Projection matrix




$$\begin{bmatrix} \frac{1}{a\tan(\phi/2)} & & & \\ & \frac{1}{\tan(\phi/2)} & & \\ & & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ & & -1 & \end{bmatrix}$$

$$a = \frac{\overline{AB}}{\overline{AD}}$$



# Projection matrix

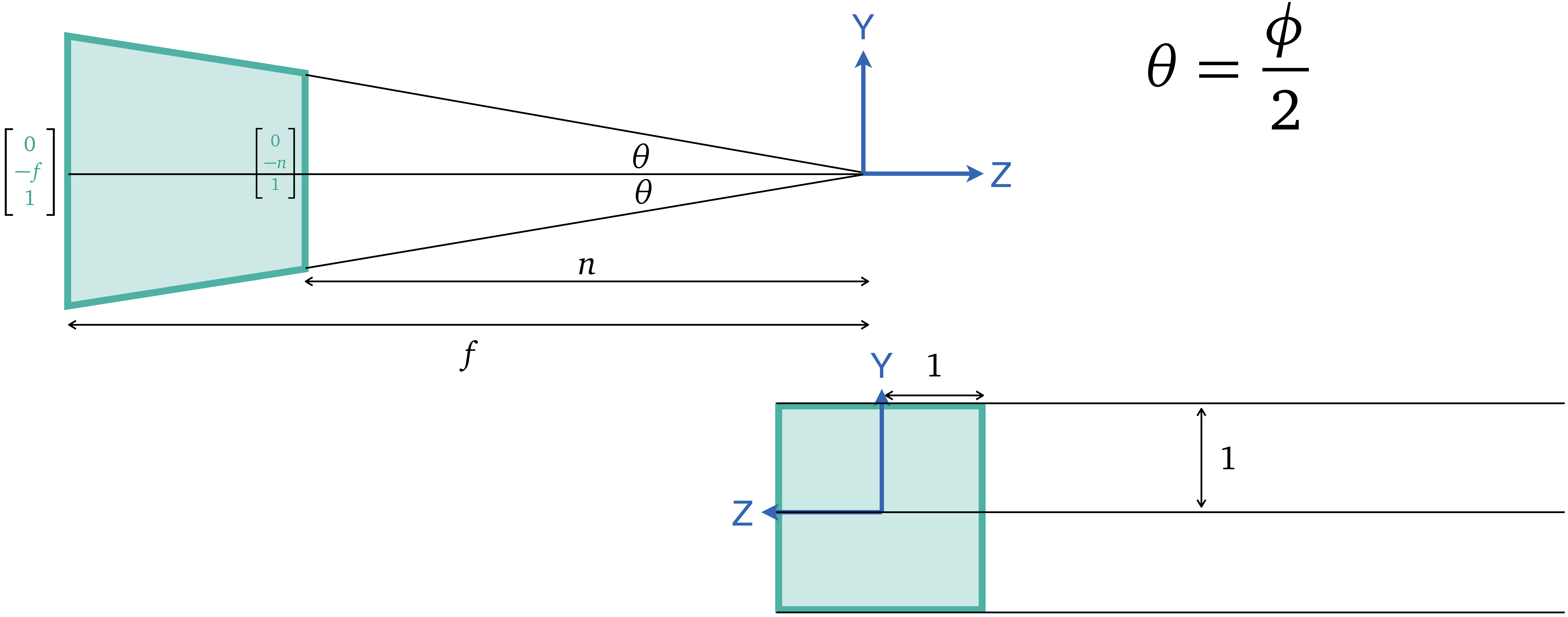

- We will derive it in 2D for visual simplicity
- Notice: in camera coordinate, z points to the back of camera
  in NDC, z represents depth and points away from the viewer

(The basis in NDC is left-hand oriented.)

# Trapezoid to square

- Design a 3x3 matrix (applied to homogeneous coordinates) that deforms the trapezoid to the square.

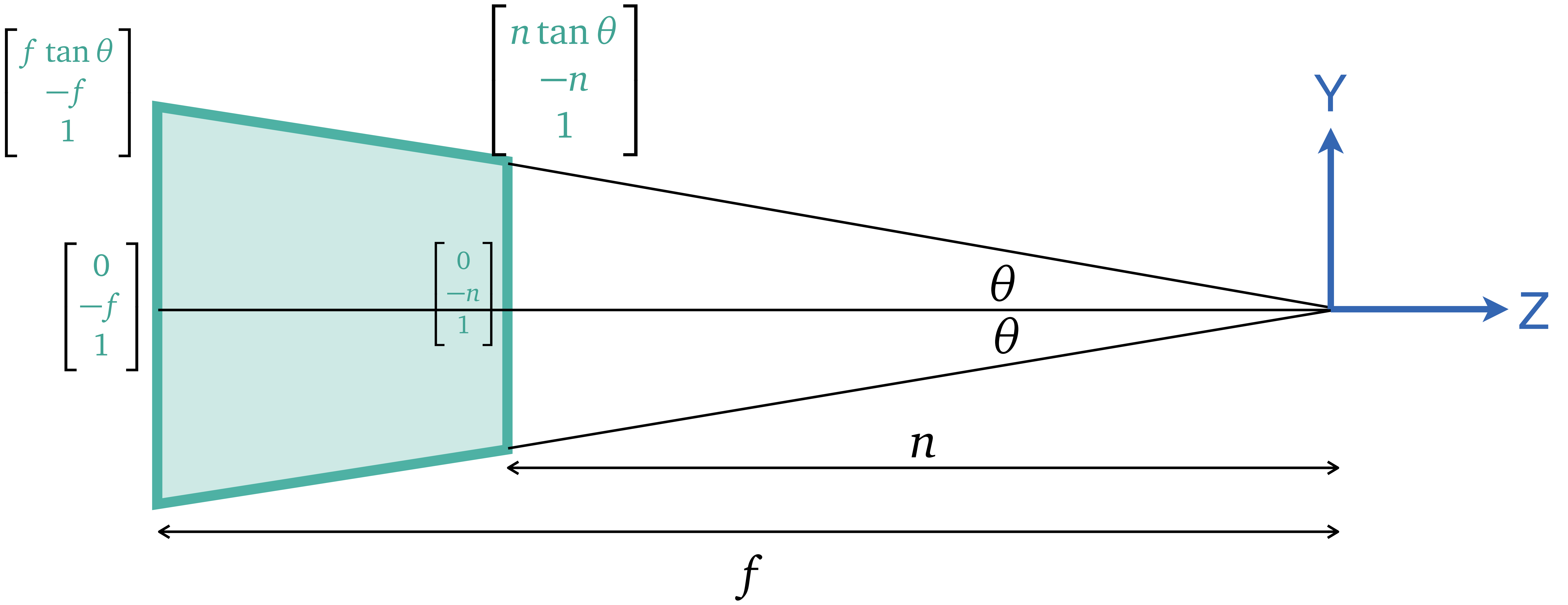
$$\theta = \frac{\phi}{2}$$

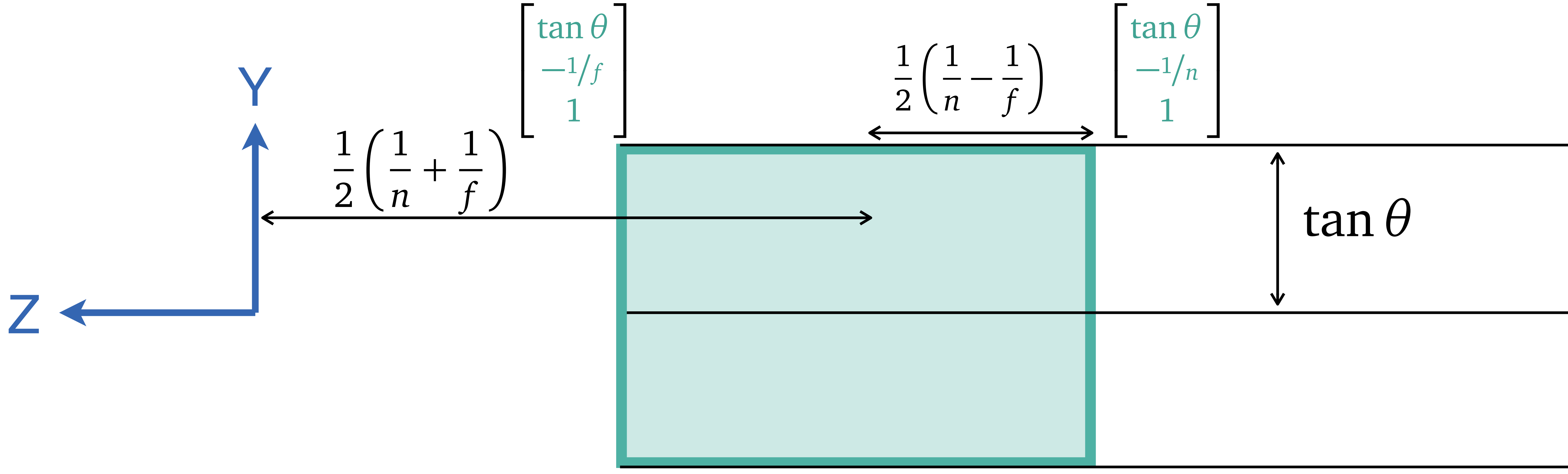


# Trapezoid to square

- Inversion

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y \\ -1 \\ -z \end{bmatrix} \sim \begin{bmatrix} -y/z \\ 1/z \\ 1 \end{bmatrix}$$

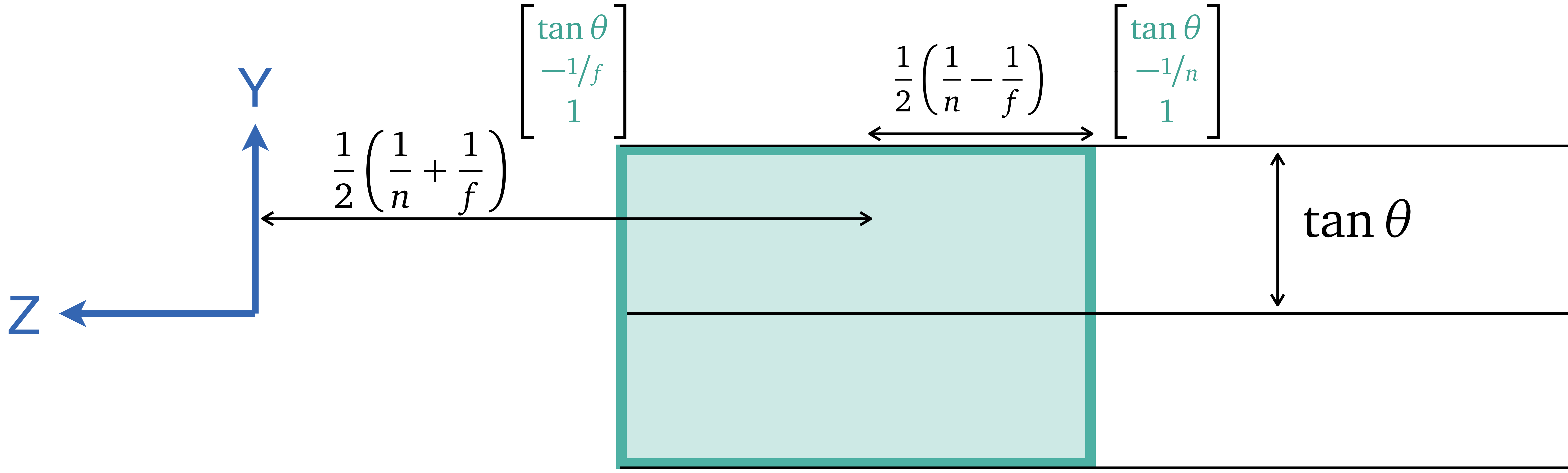


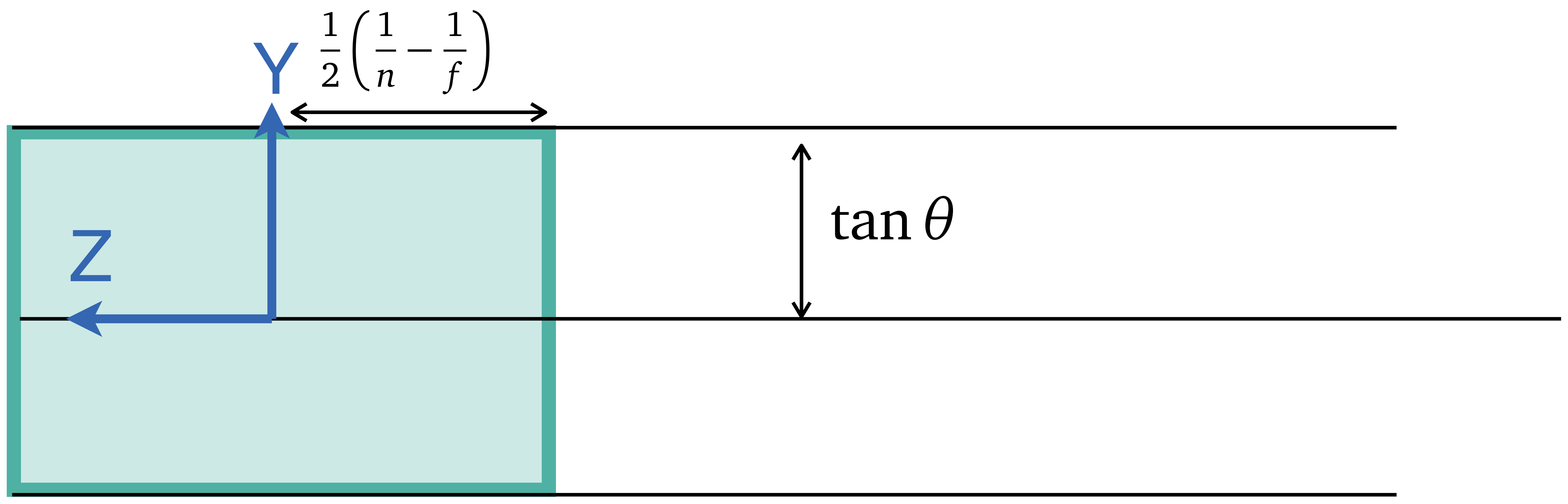
# Trapezoid to square

- Inversion

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y \\ -1 \\ -z \end{bmatrix} \sim \begin{bmatrix} -y/z \\ 1/z \\ 1 \end{bmatrix}$$



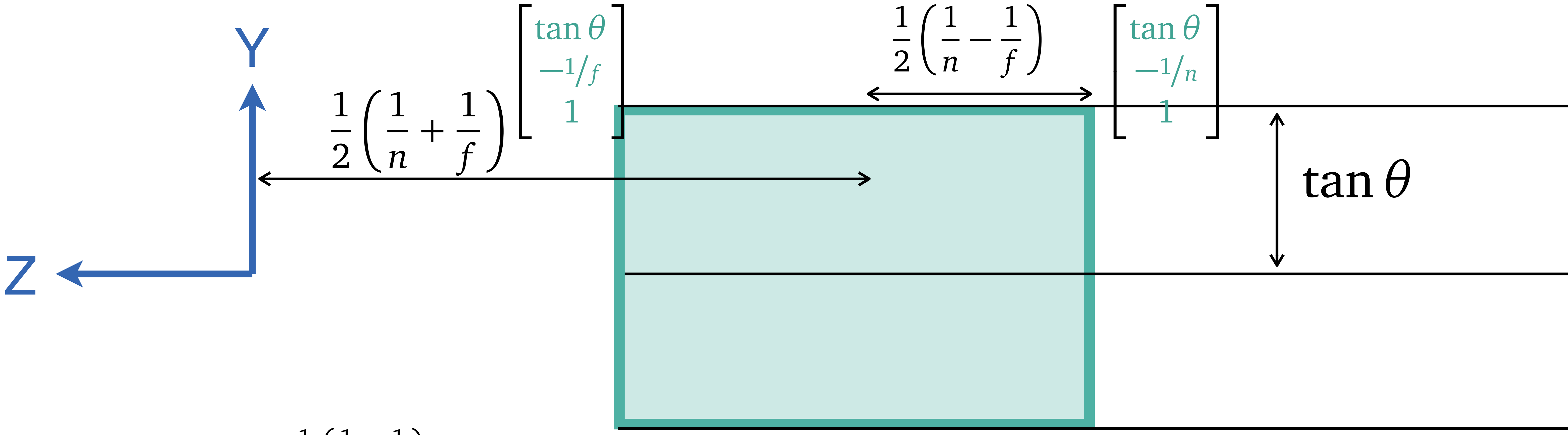

- Translation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{f+n}{2fn} \\ 0 & 0 & 1 \end{bmatrix}$



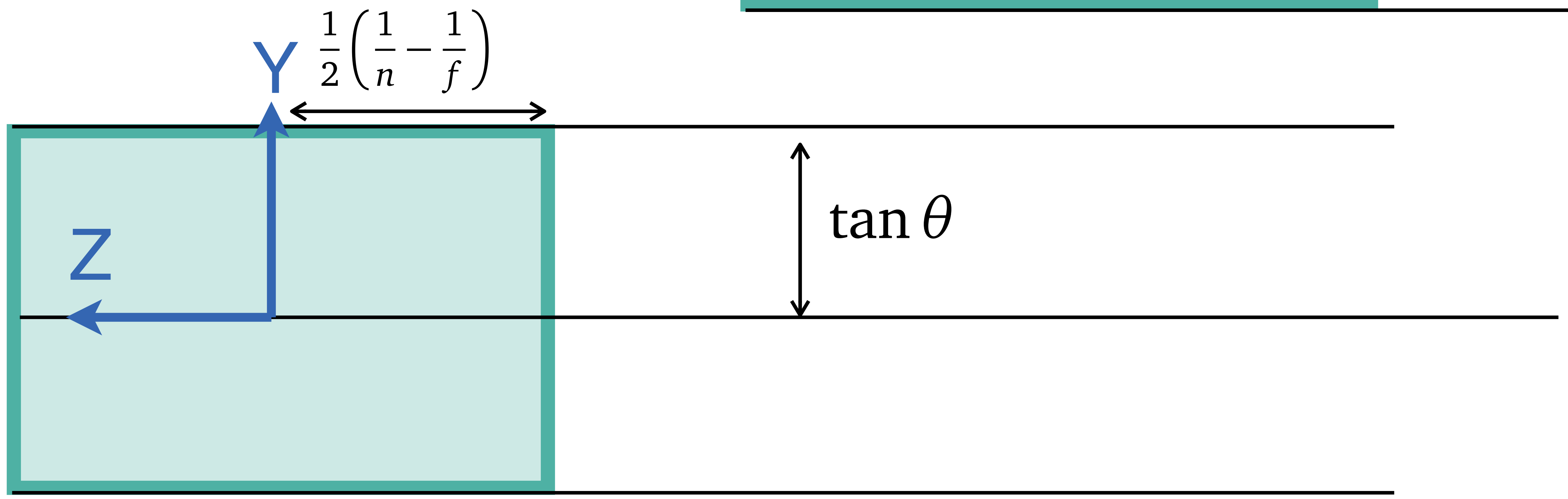
# Trapezoid to square

- Inversion

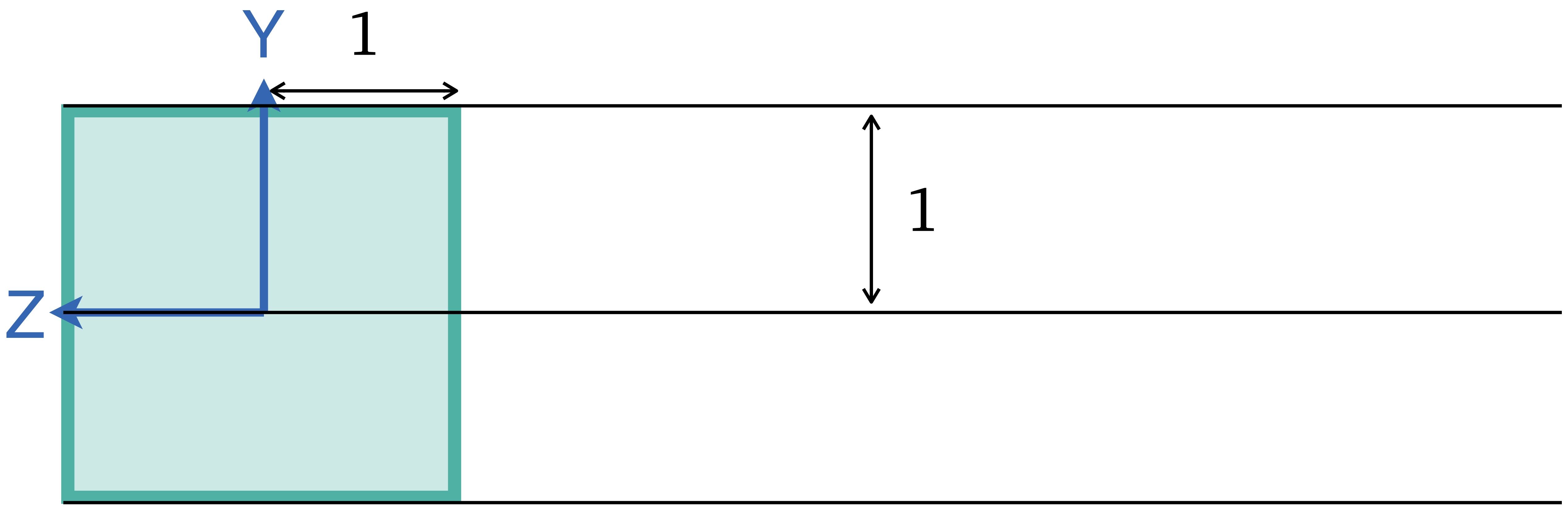
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} y \\ -1 \\ -z \end{bmatrix} \sim \begin{bmatrix} -y/z \\ 1/z \\ 1 \end{bmatrix}$$




- Translation $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{f+n}{2fn} \\ 0 & 0 & 1 \end{bmatrix}$




- Scaling $\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & \frac{2fn}{f-n} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

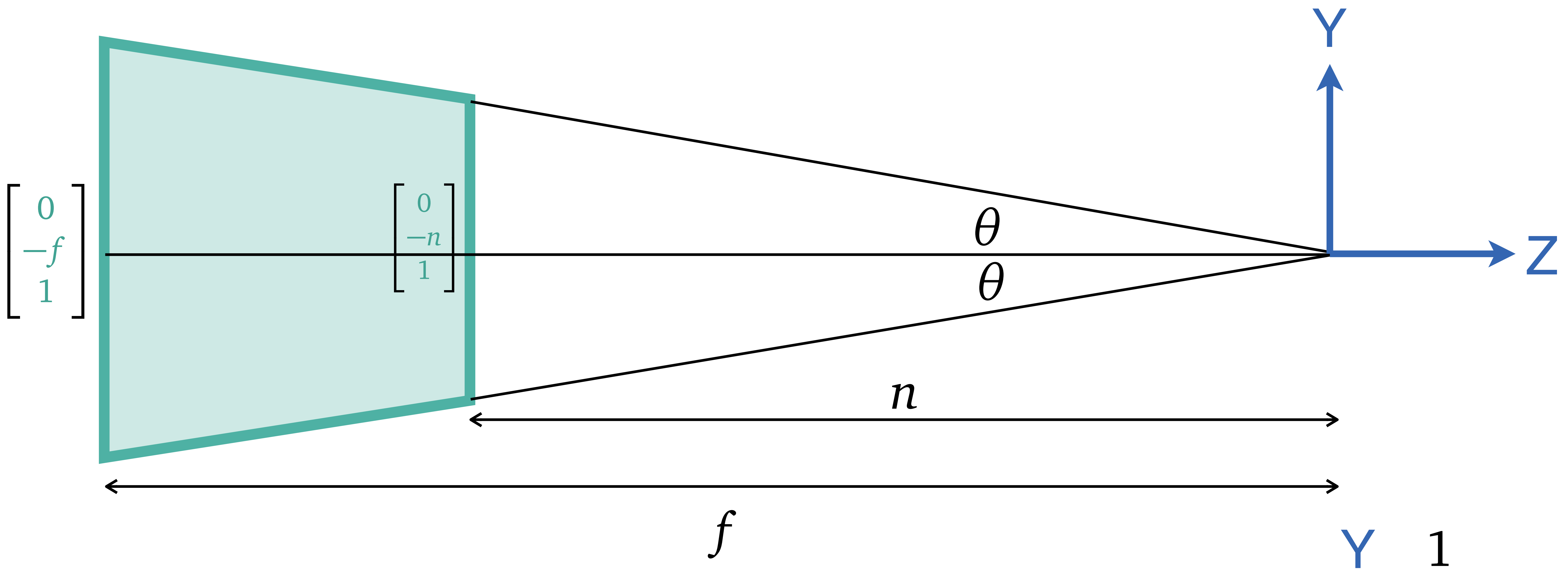


# Trapezoid to square

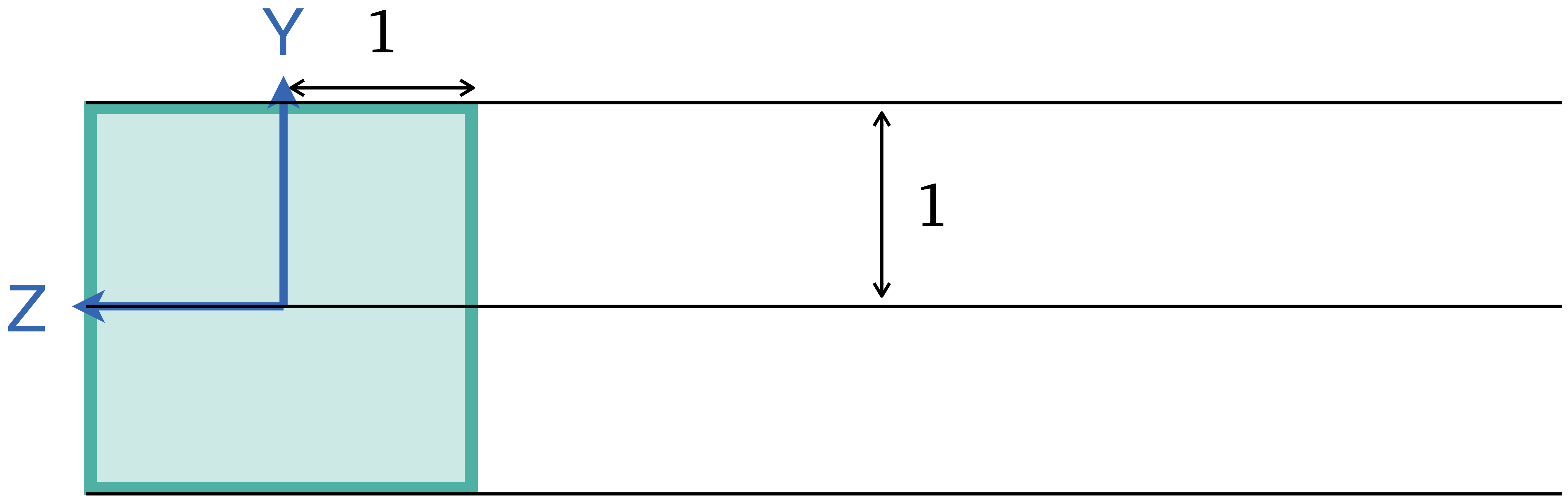
$$\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & \frac{2fn}{f-n} & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{f+n}{2fn} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{f+n}{2fn} & -1 \\ 0 & -1 & 0 \end{bmatrix}} = \begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & -1 & 0 \end{bmatrix}$$

# Trapezoid to square

- Design a 3x3 matrix (applied to homogeneous coordinates) that deforms the trapezoid to the square.




$$\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & -1 & 0 \end{bmatrix}$$



# Add x coordinate

$$\begin{bmatrix} \frac{1}{\tan\theta} & 0 & 0 \\ 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & -1 & 0 \end{bmatrix}$$

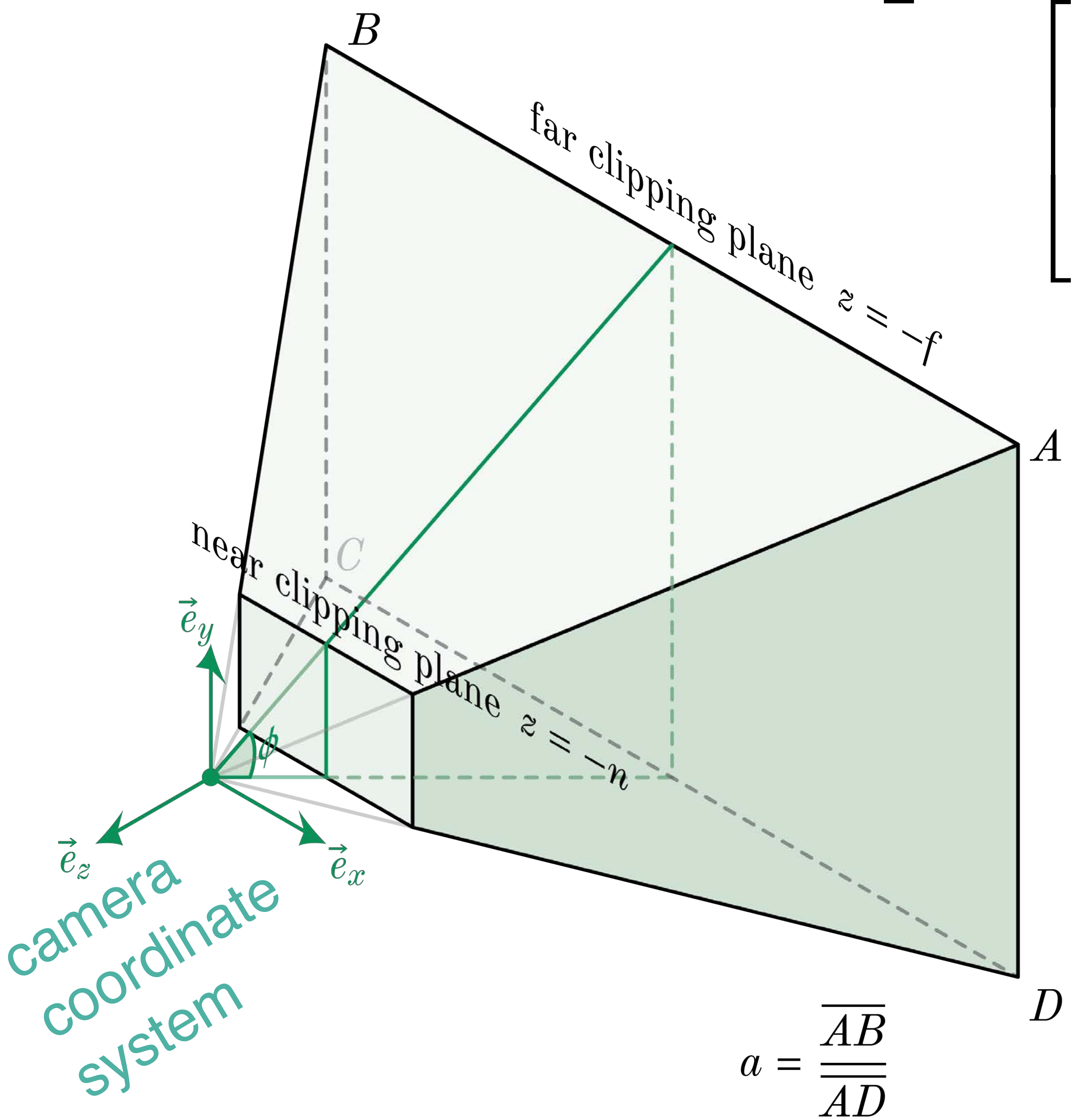
# Add x coordinate

$$\begin{bmatrix} \frac{1}{a\tan\theta} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\theta} & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

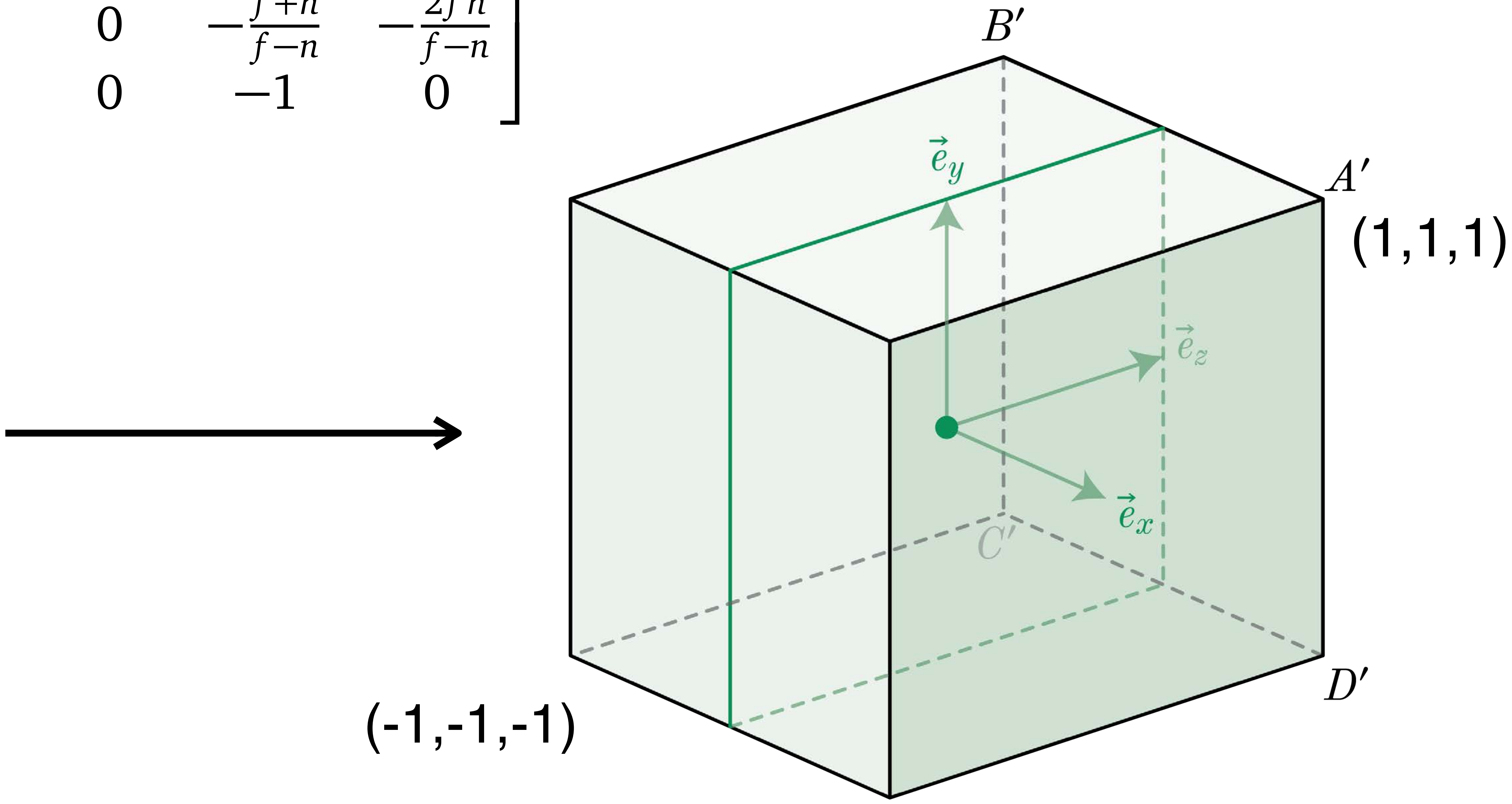
# Projection matrix

$$\theta = \frac{\phi}{2}$$

$$\begin{bmatrix} \frac{1}{a\tan\theta} & 0 & 0 & 0 \\ 0 & \frac{1}{\tan\theta} & 0 & 0 \\ 0 & 0 & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{bmatrix}$$




$$a = \frac{\overline{AB}}{\overline{AD}}$$




normalized device coordinate

# Art of Projective Geometry

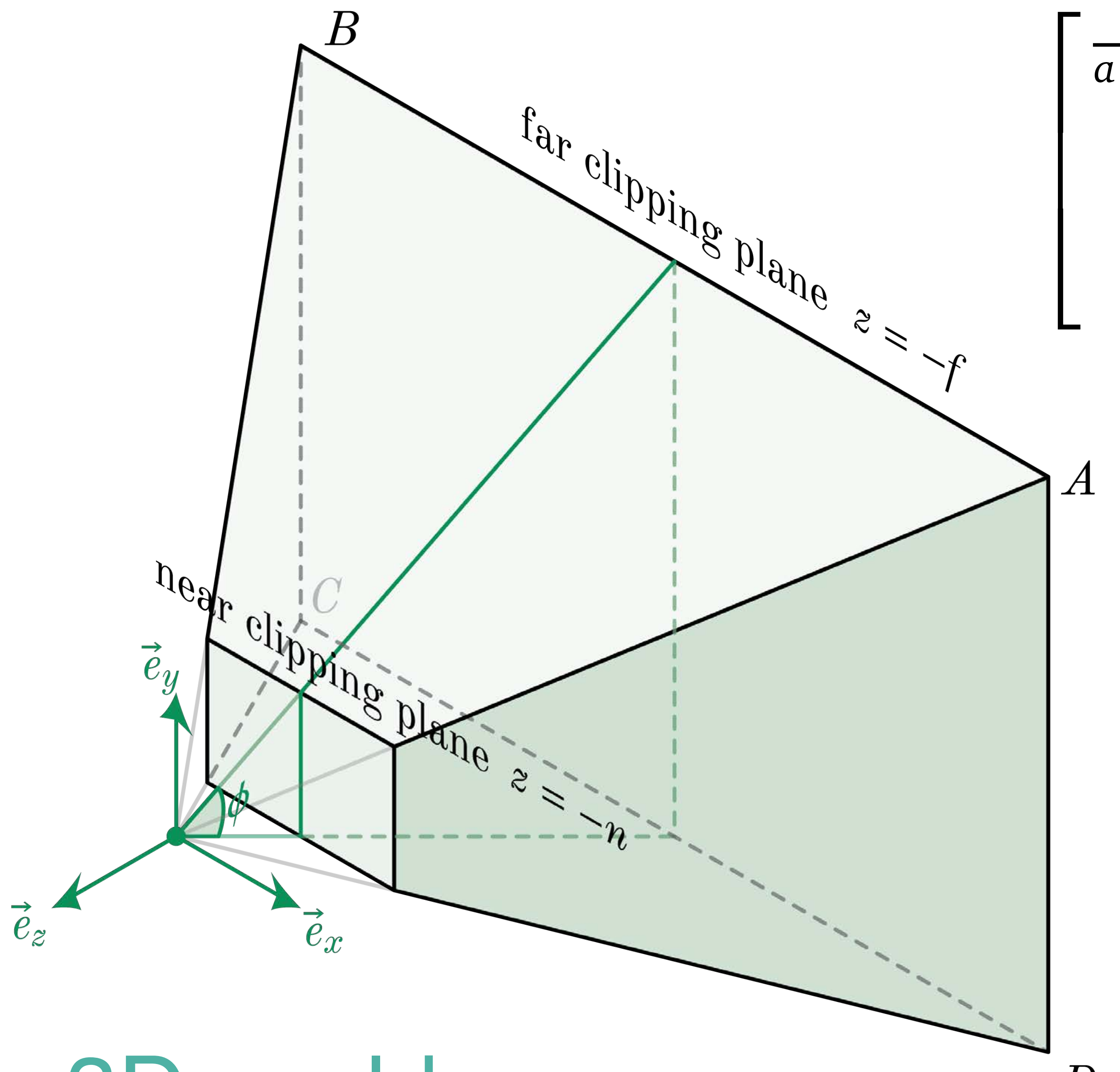
- What are projective transforms?
- How homogeneous coordinates work
- Derivation of the projection matrix
- Art of projective geometry

# Art of Projective Geometry

- Art of projective geometry
  - Perspective drawings
  - Cross ratio
  - From Renaissance Art to Computer graphics

# 2D/3D projective transformations

**Computer graphics**



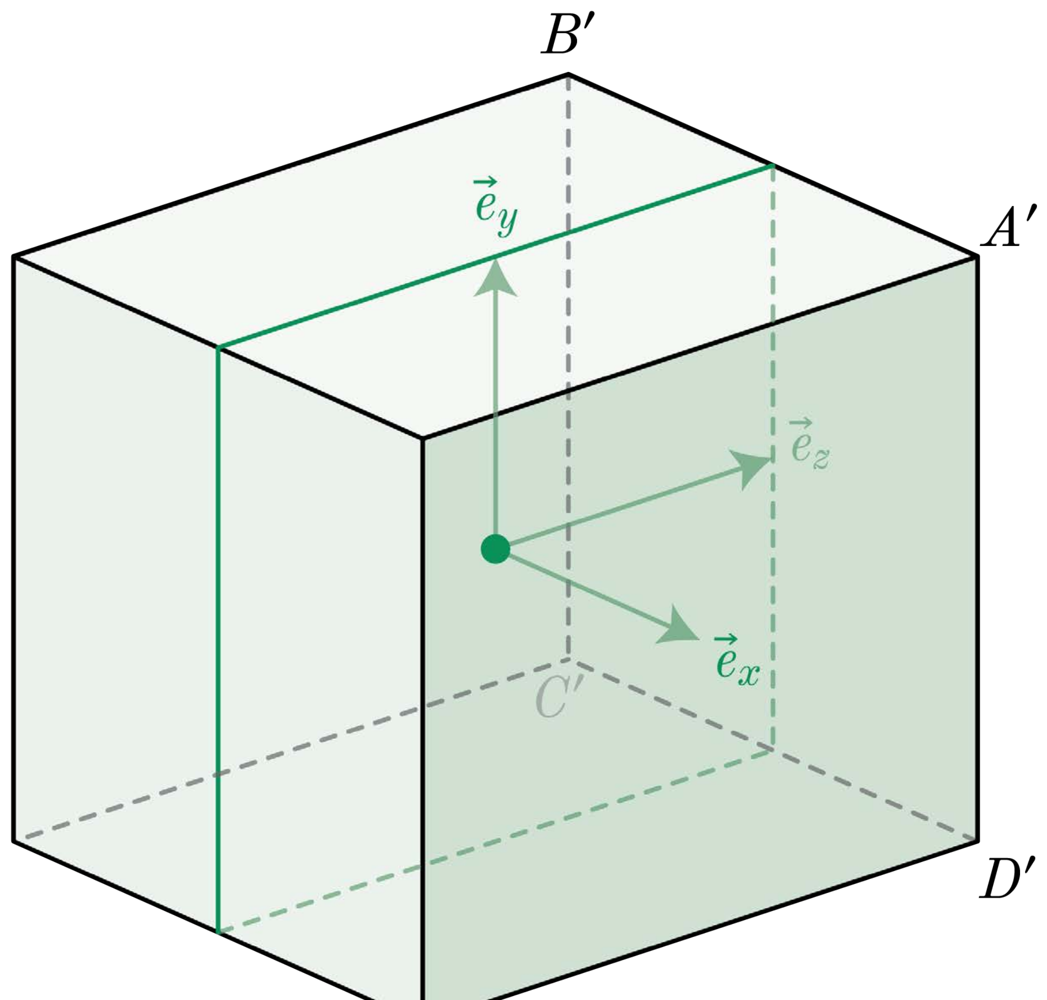

3D world

$a = \frac{\overline{AB}}{\overline{AD}}$

$$\begin{bmatrix} \frac{1}{a\tan(\phi/2)} & & & \\ & \frac{1}{\tan(\phi/2)} & & \\ & & -\frac{f+n}{f-n} & -\frac{2fn}{f-n} \\ & & -1 & \end{bmatrix}$$

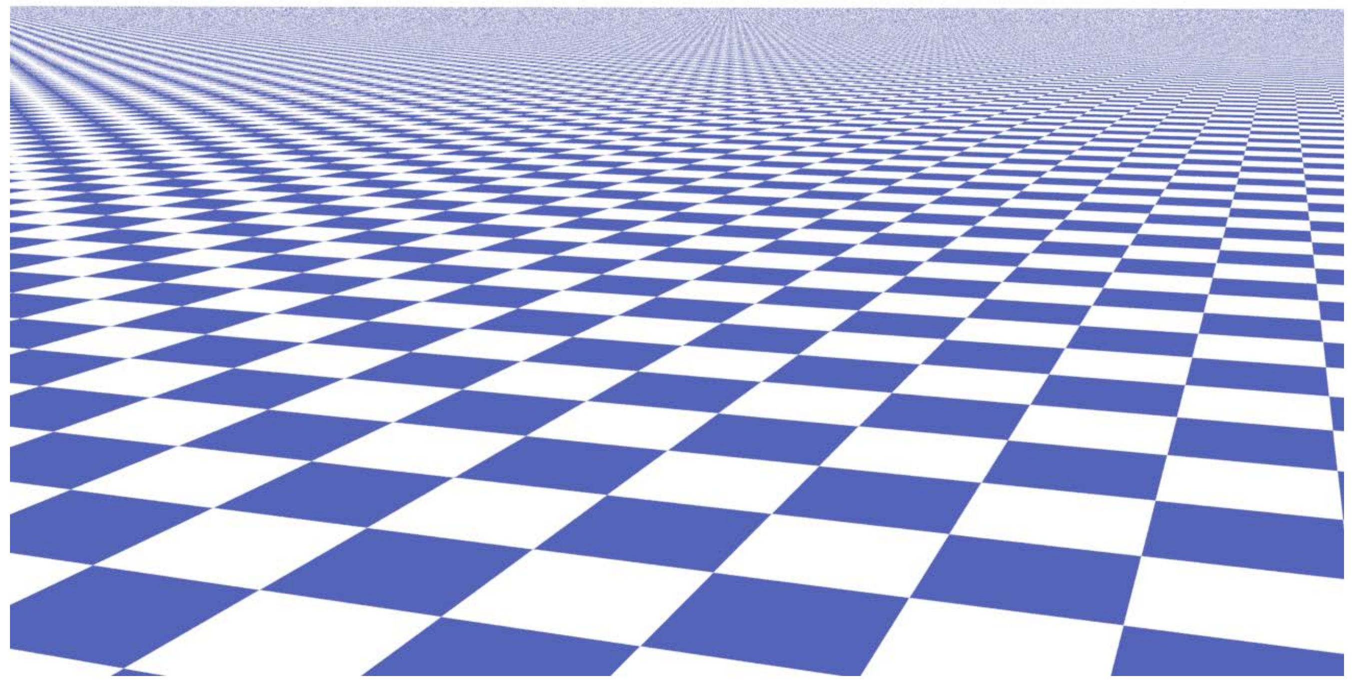
linear transform in 4D

$\longleftrightarrow$




3D NDC

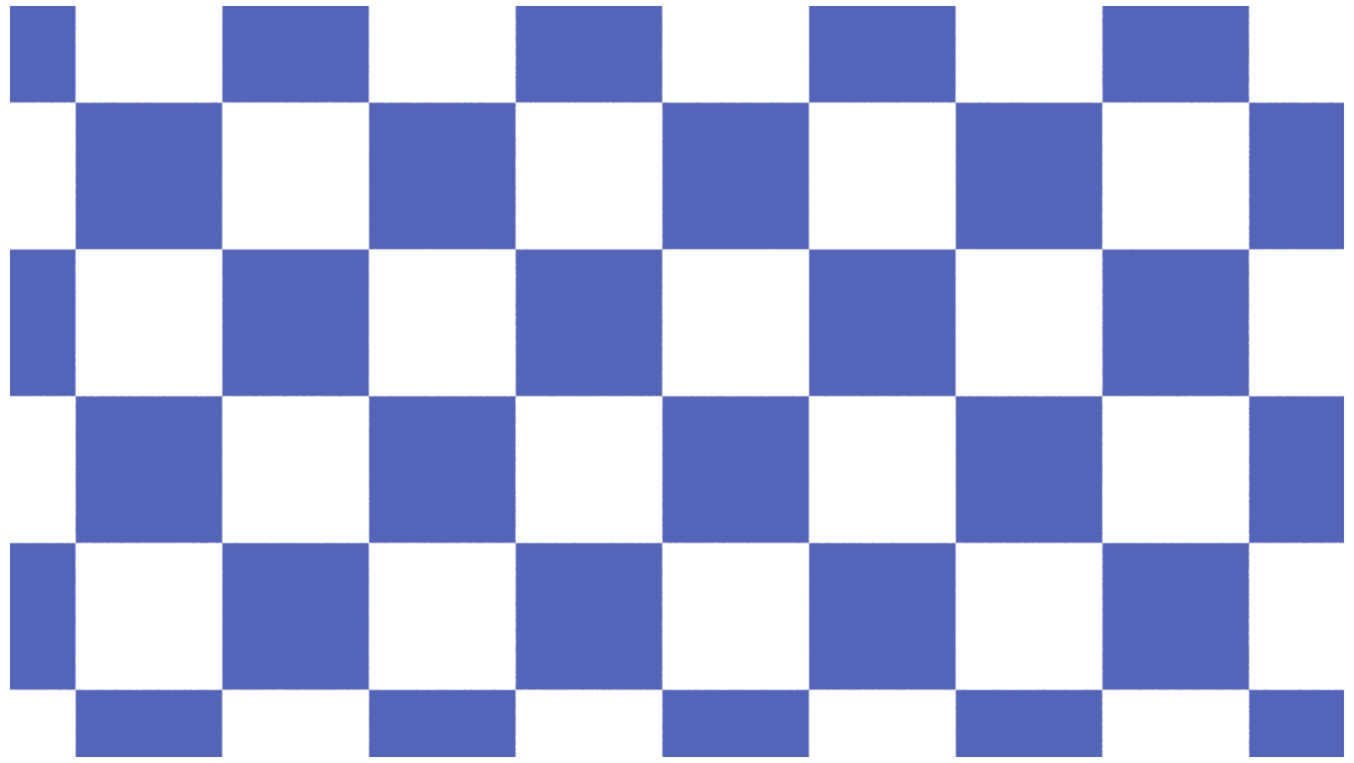
**Renaissance artists:**



2D canvas

linear transform in 3D

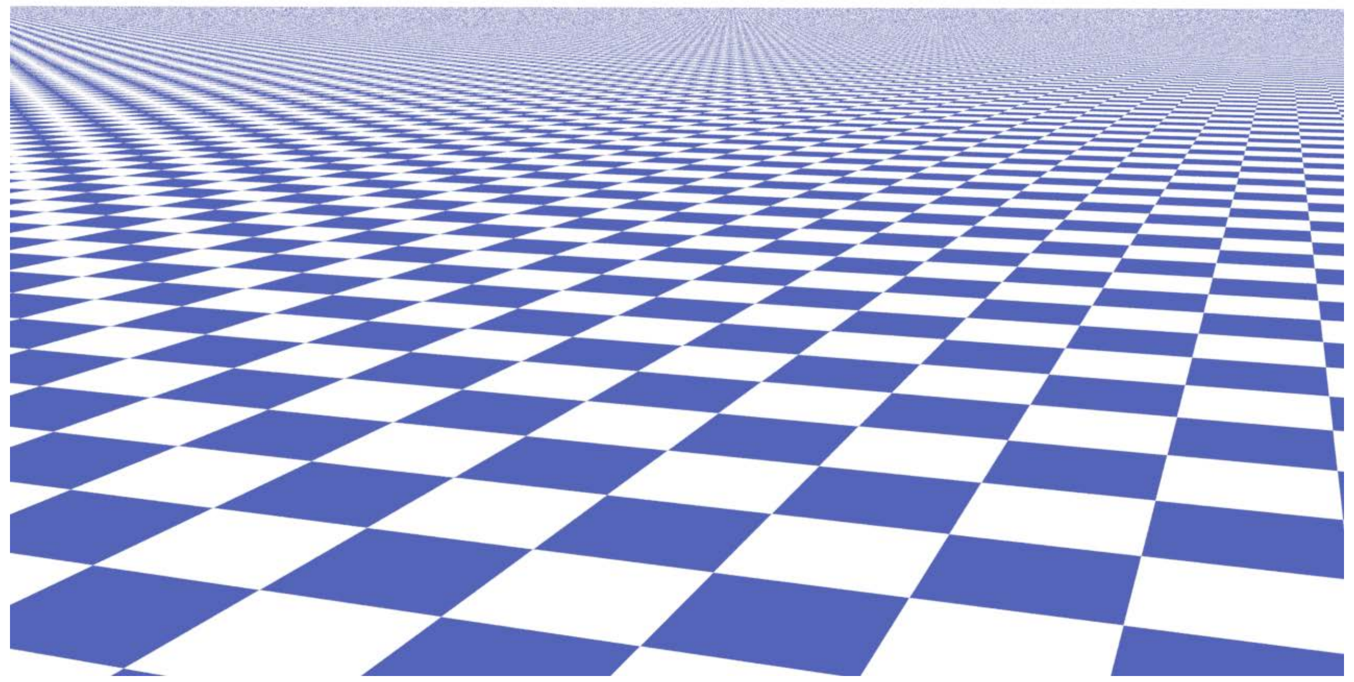
$\longleftrightarrow$



2D ground plane

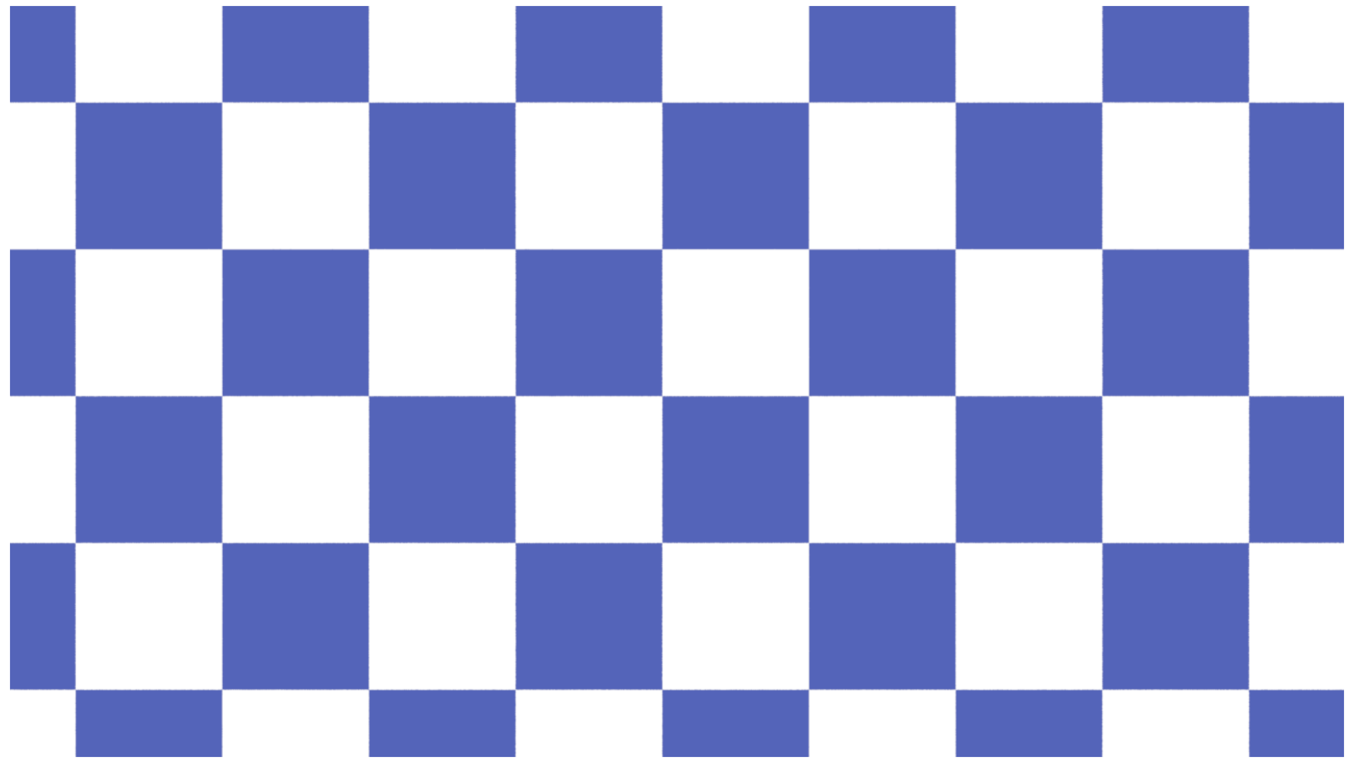
# 2D/3D projective transformations

**Renaissance artists:**



2D canvas

linear transform in 3D

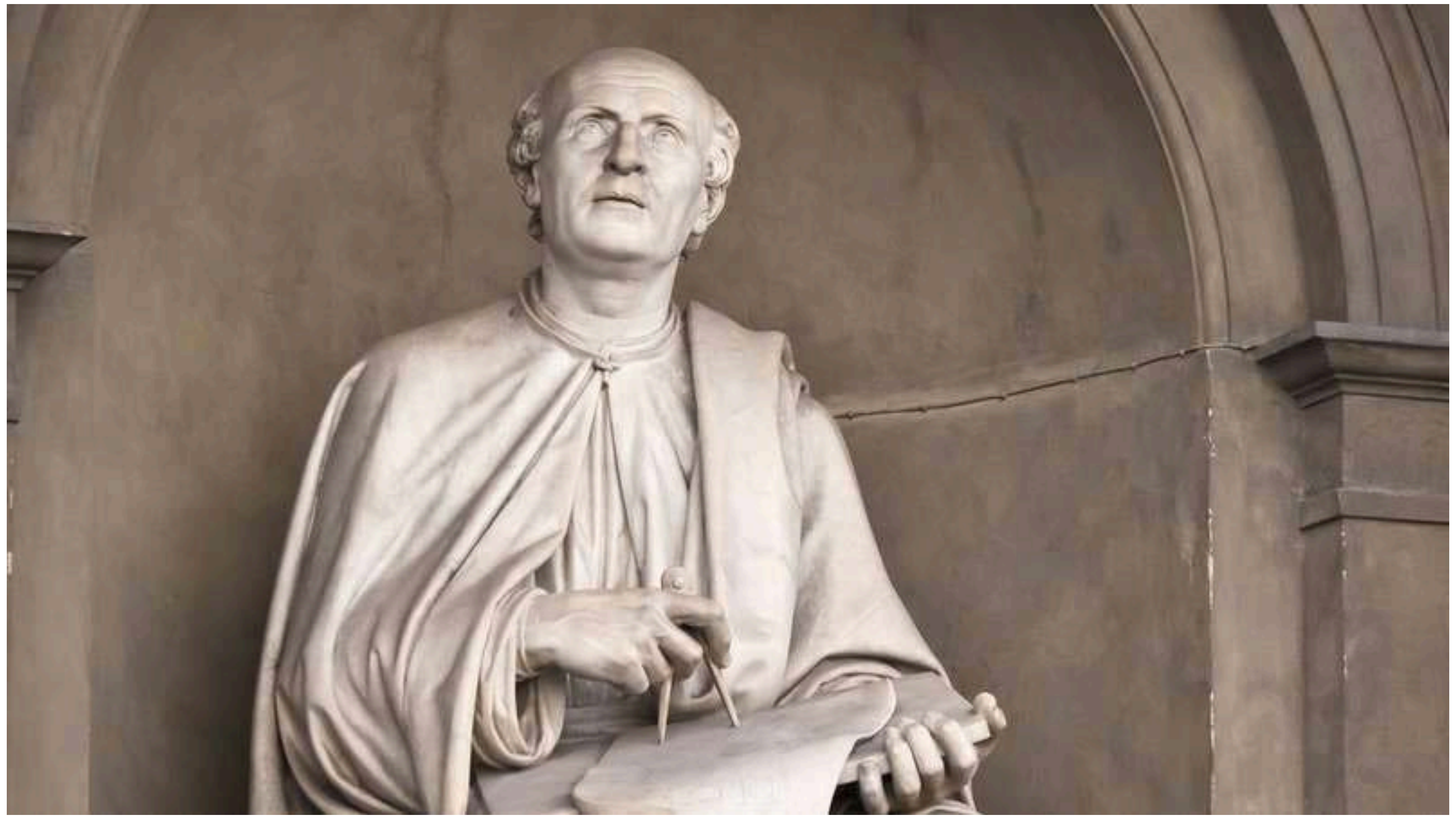


2D ground plane

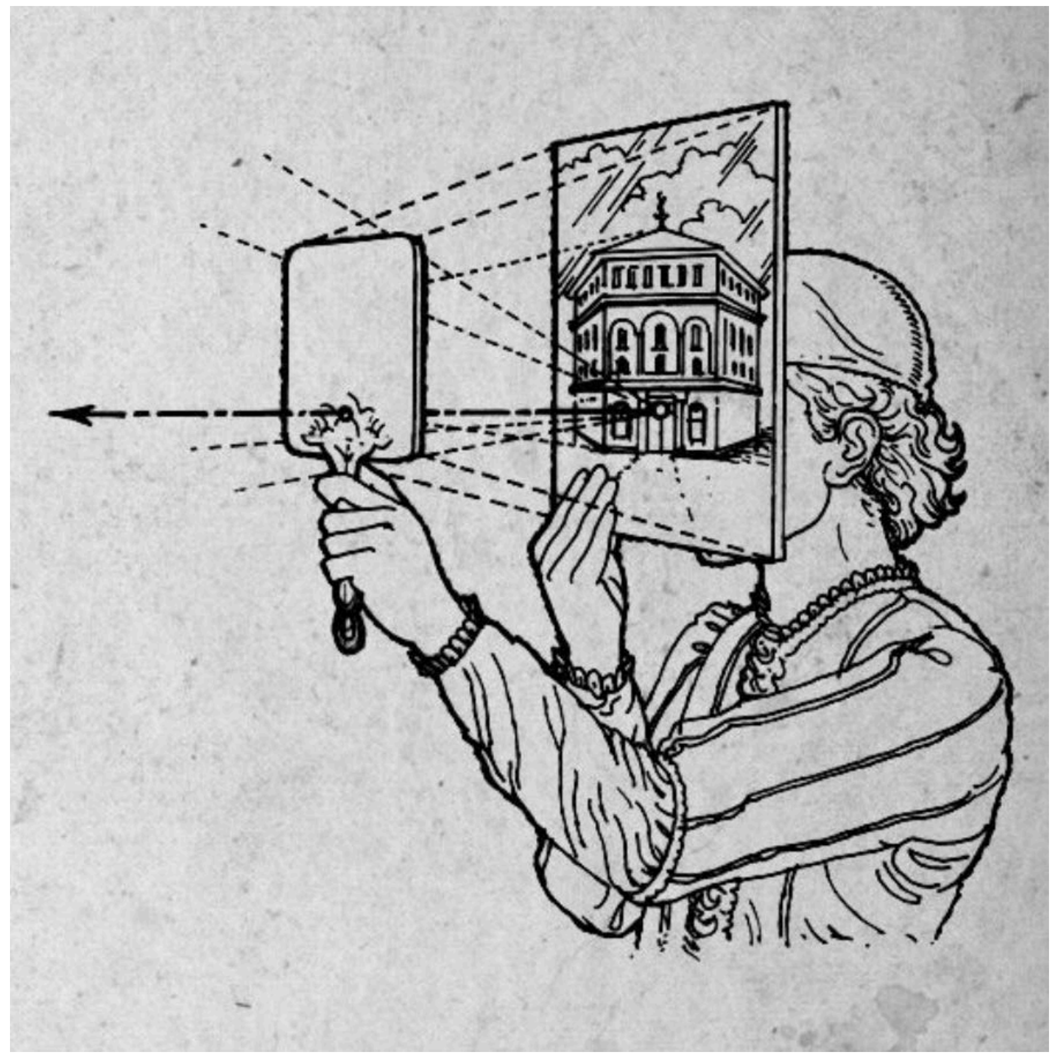
- Case study the 2D projective geometry
- One dimension lower than frustum-to-NDC (easier to grasp)
- Important for
  - History (tools developed for graphics purposes in 15th century)
  - Descriptive/engineering illustration
  - Computer graphics / vision / surveying

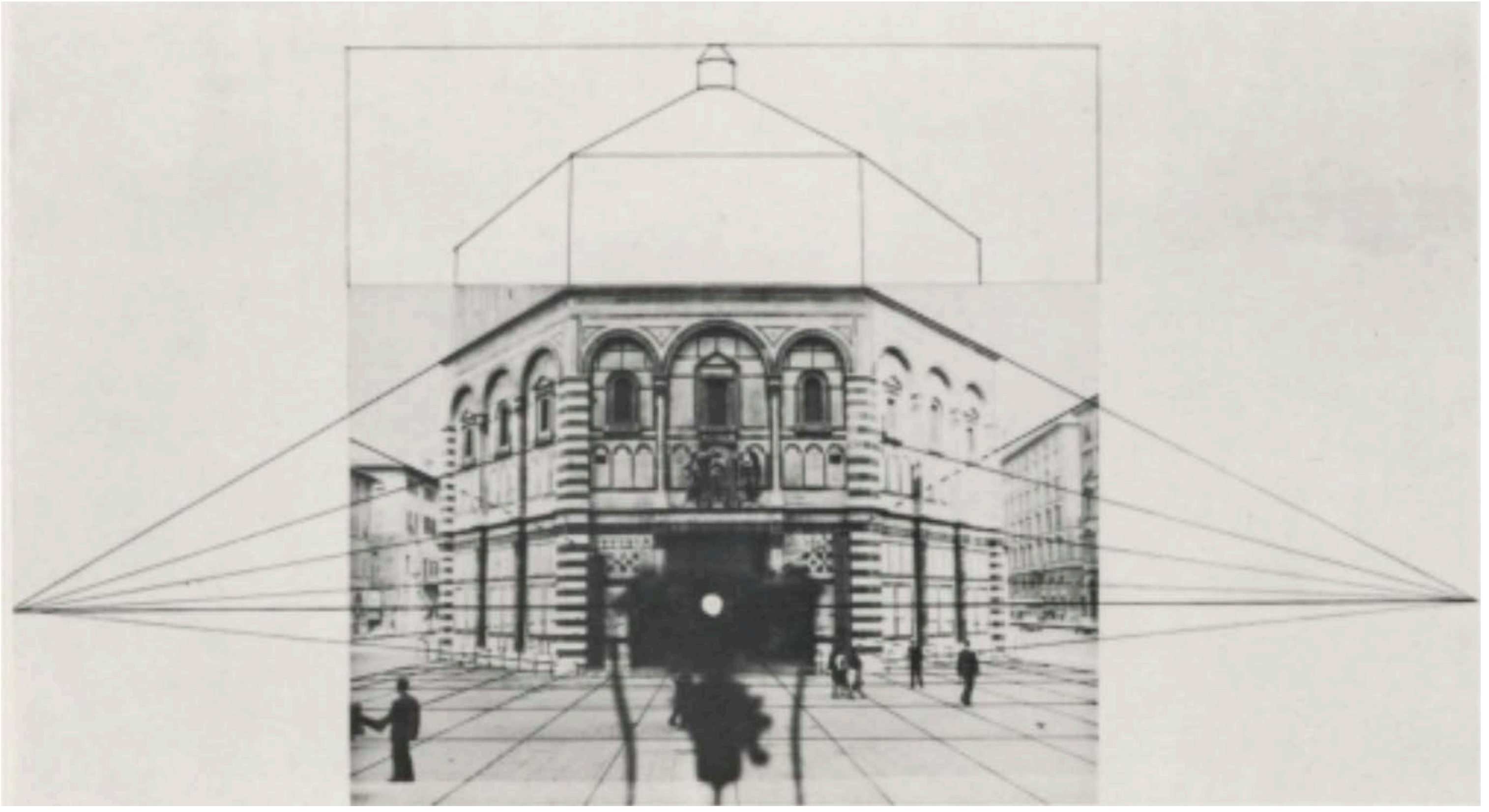
# Renaissance Art

- Filippo Brunelleschi 1415: Linear perspective
  - 2D representation of a 3D object?
  - Found method accurate perspective drawing.
  - Soon after Brunelleschi, most of the artists in Florence adopted the method.



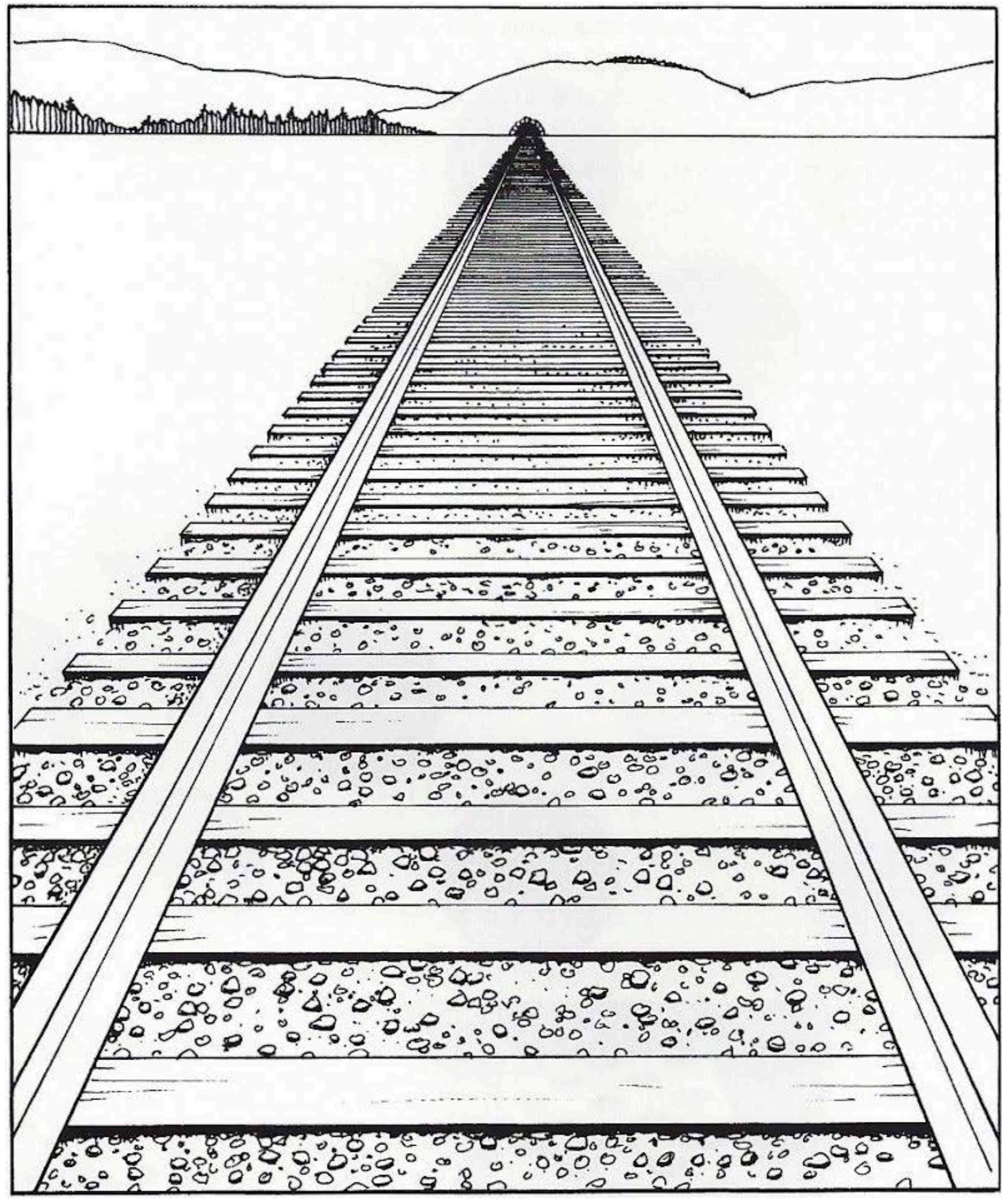
Filippo Brunelleschi 1337–1446

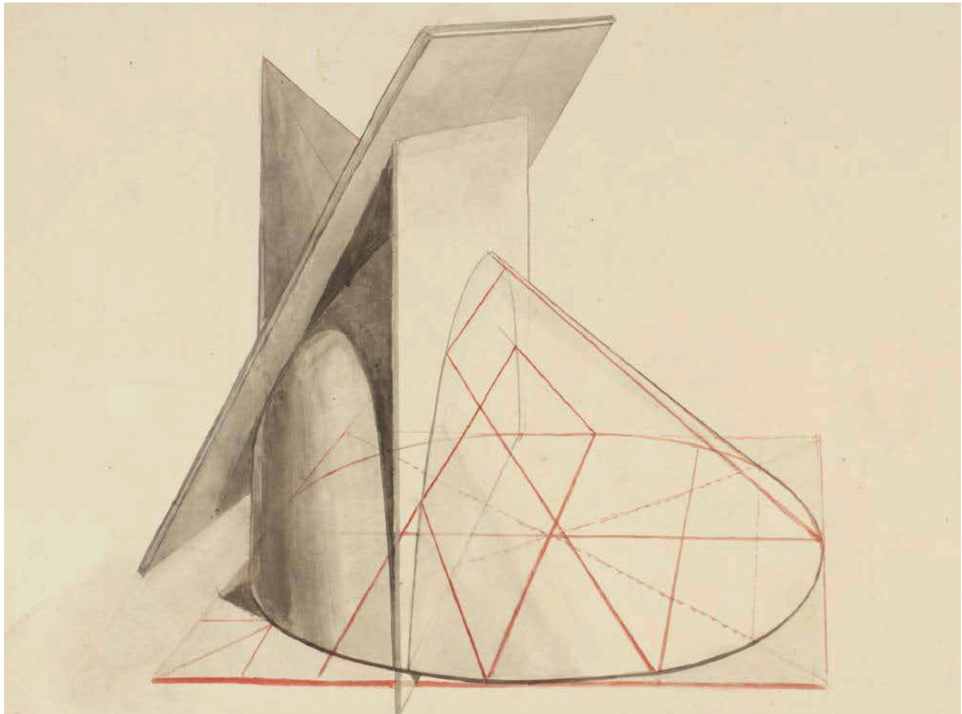




# Rules of perspectivity

- The rules for representing 3D picture on a plane
  - Straight lines must be represented by straight lines.
  - Every pair of lines in a plane will meet exactly once
    - Parallel lines in a plane meet on the horizon of the plane
    - Meeting points of parallel lines are called vanishing points
- Consequences:
  - There is a precise way to determine the *harmonic spacing*
  - From the *anharmonicity* we can measure distances from perspective pictures.
  - Image of a conic is a conic.

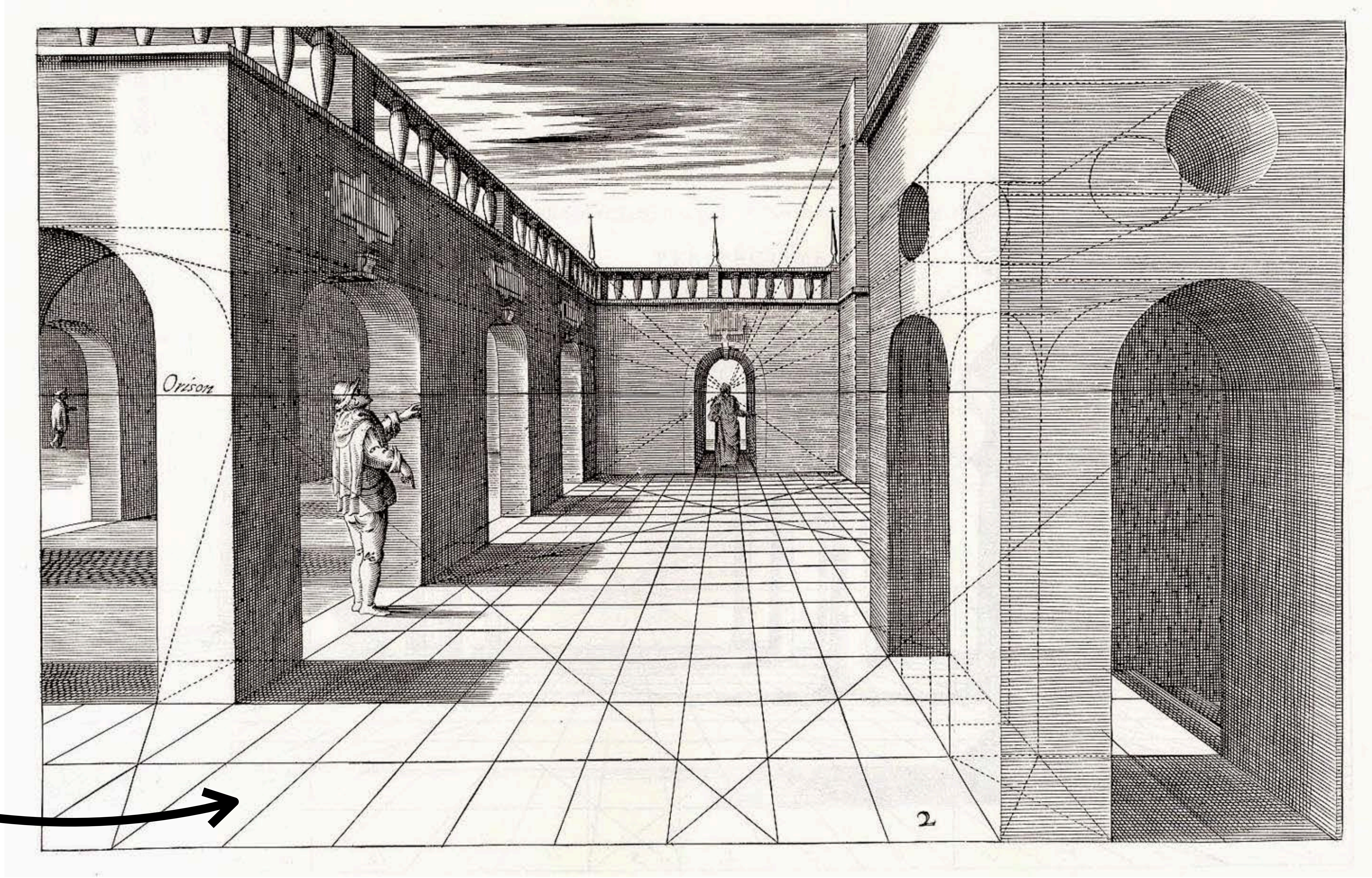




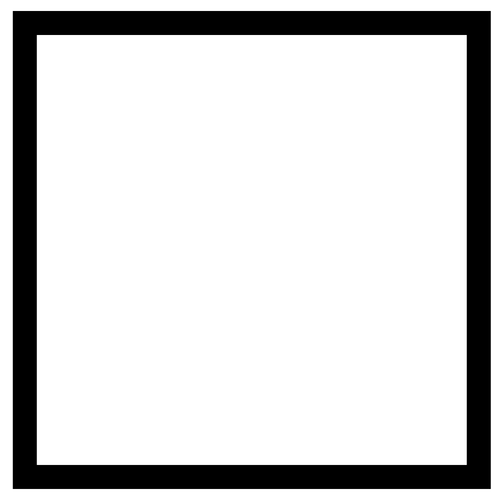
# Perspective grid construction

- Given an arbitrary quadrilateral, we can uniquely extend it to a perspective picture of a square tiling

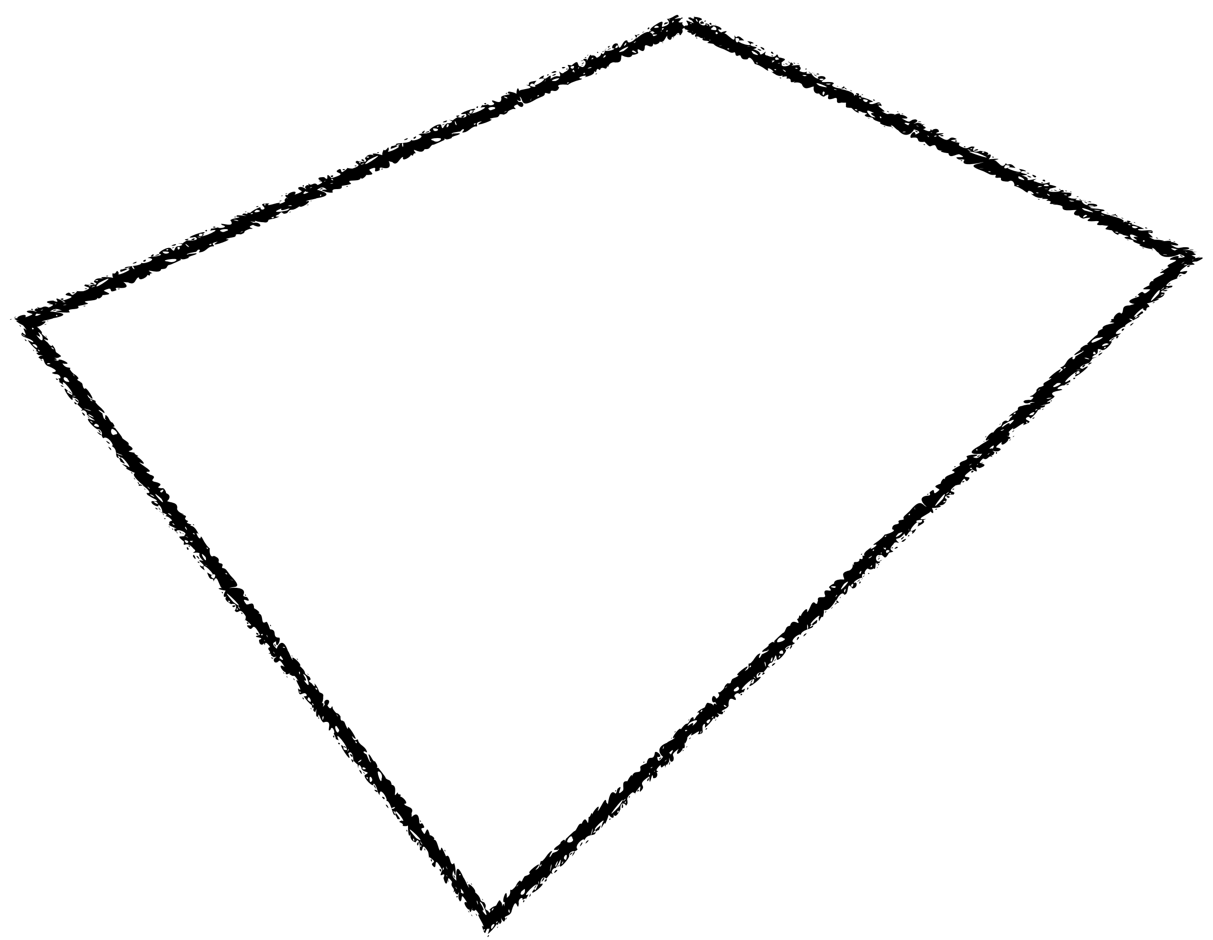
*square floor tiling*



# Perspective grid construction

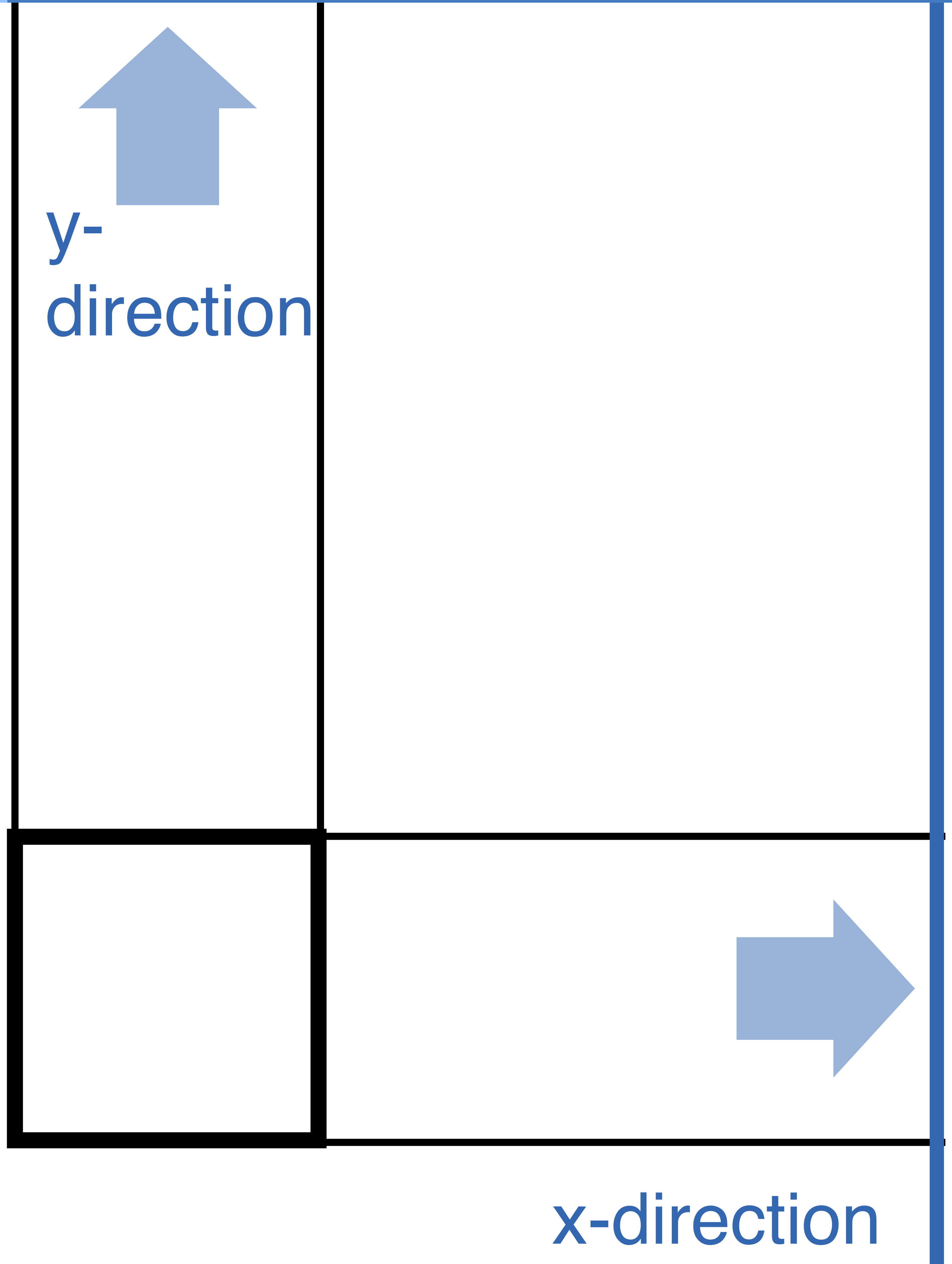


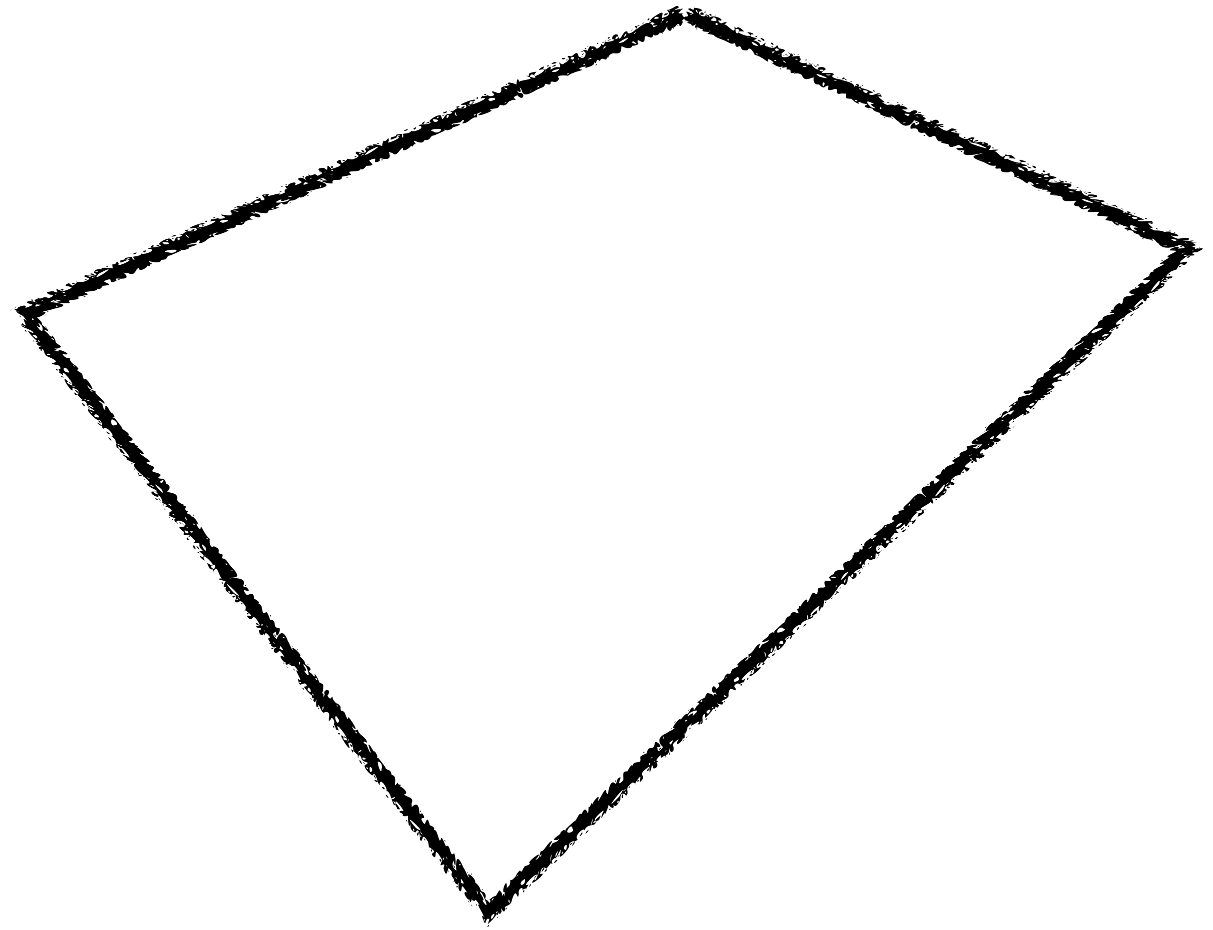
topview floor plan



perspective drawing

# Perspective grid construction

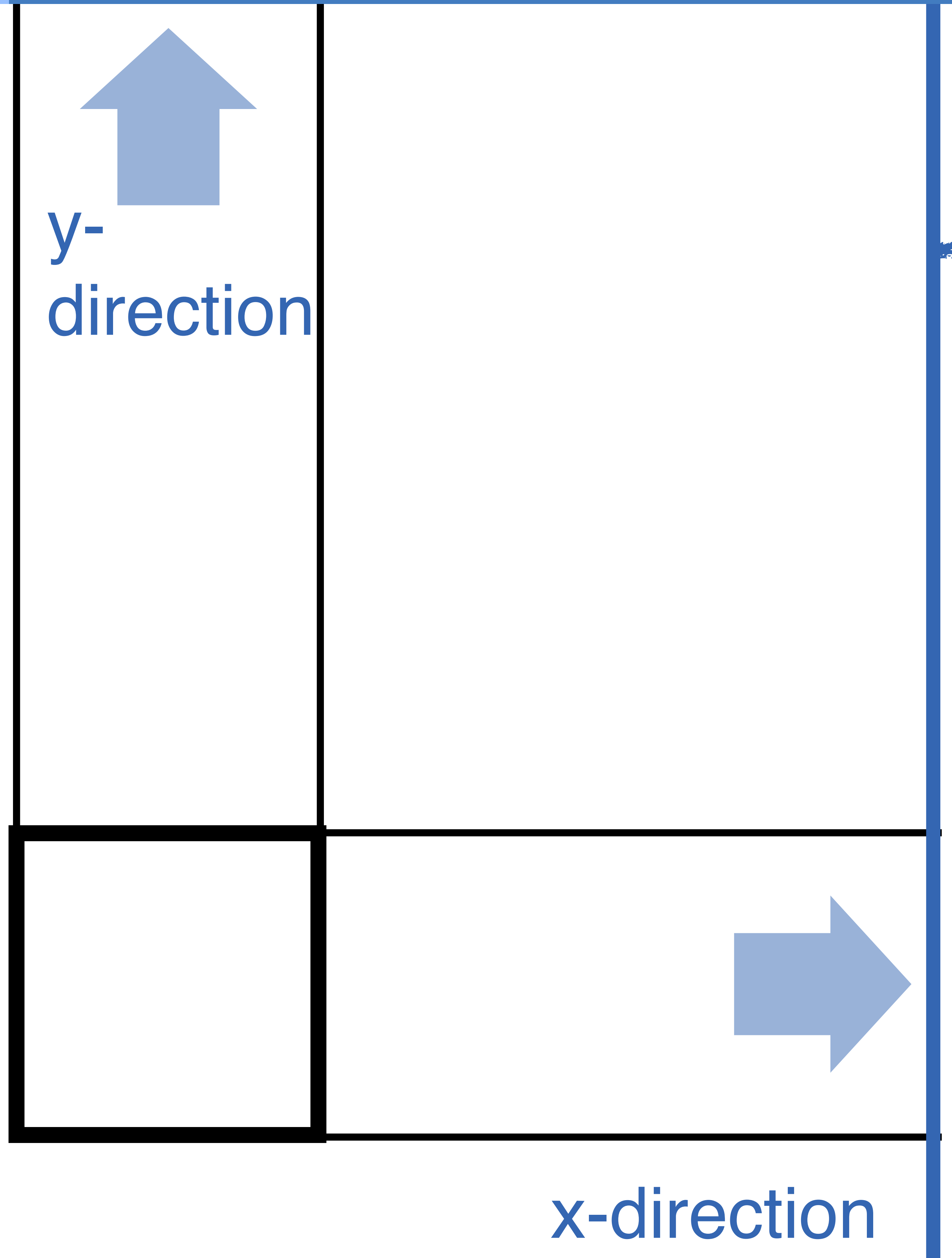


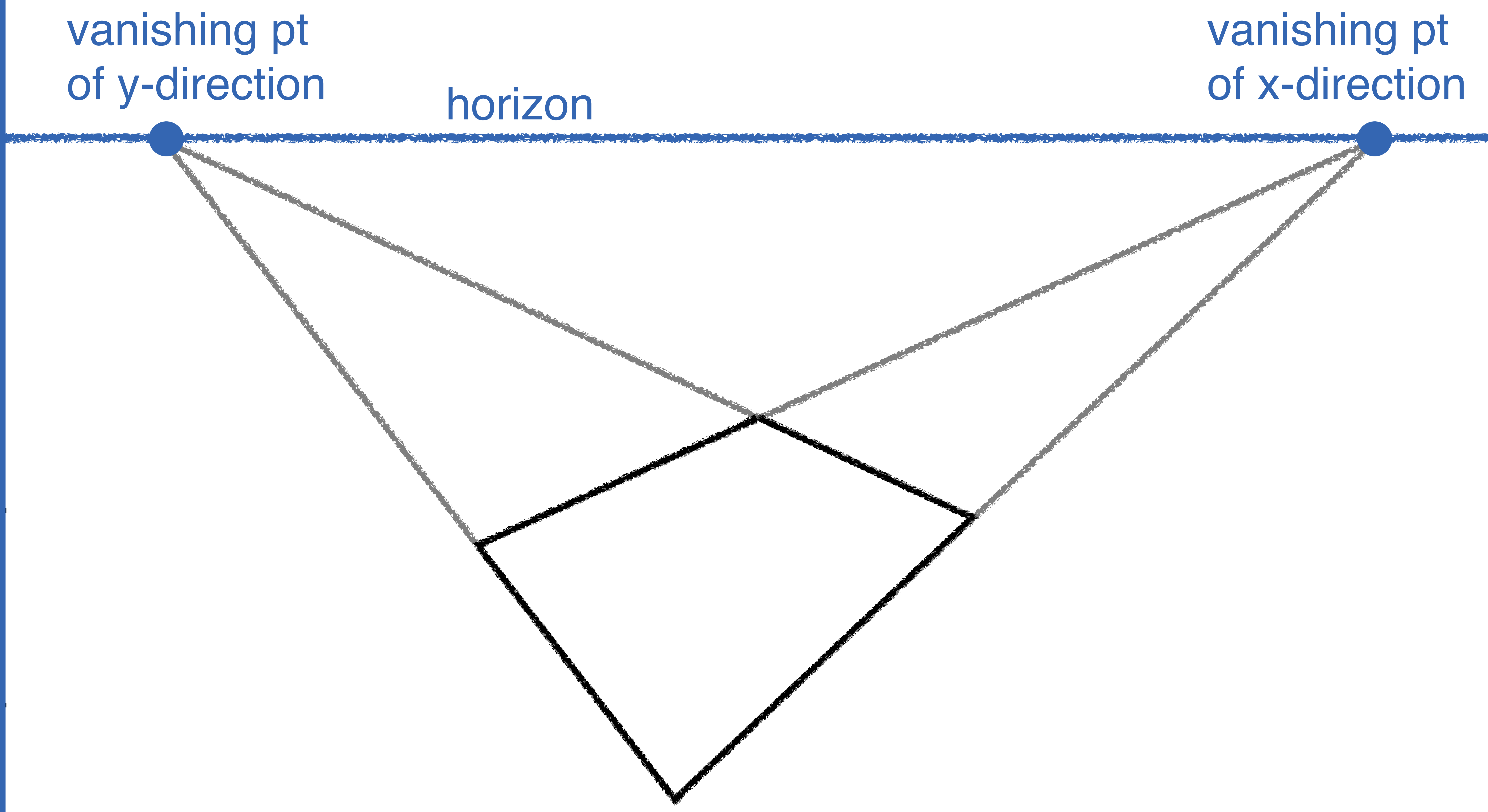
topview floor plan

perspective drawing

# Perspective grid construction




topview floor plan




perspective drawing

# Perspective grid construction

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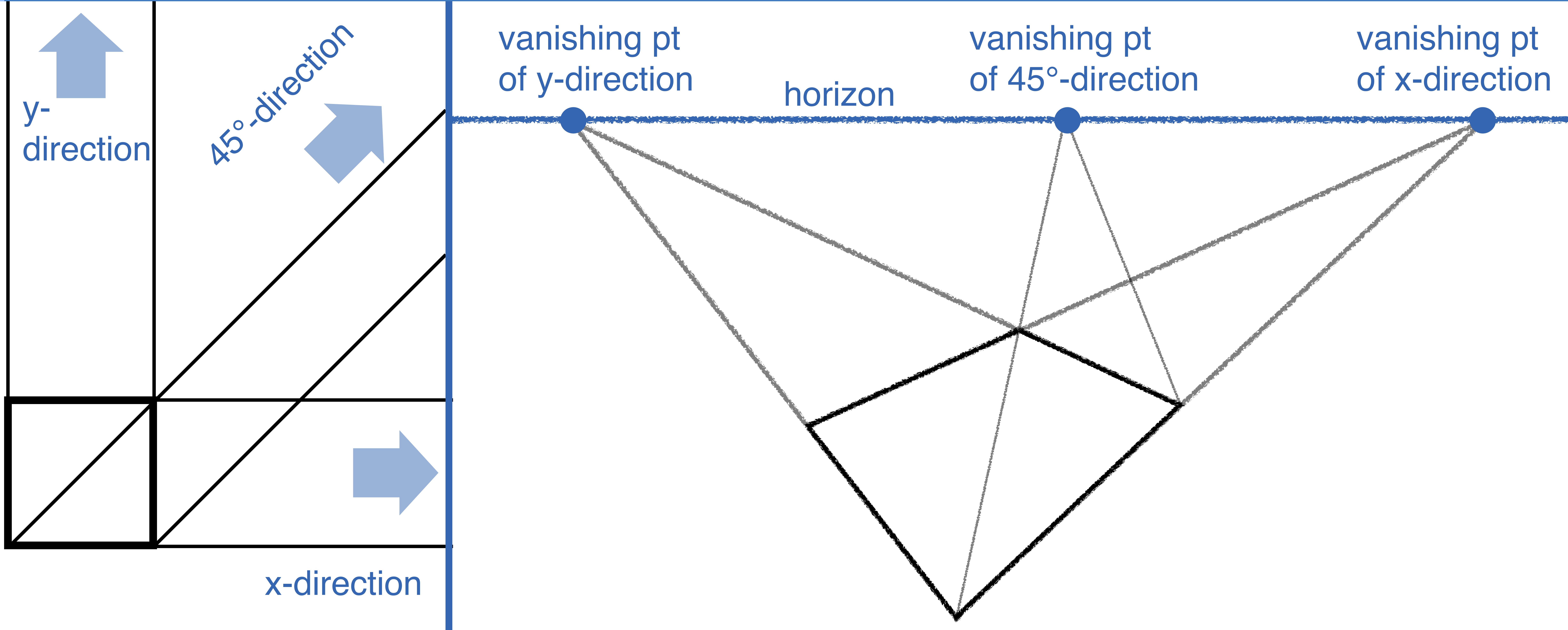
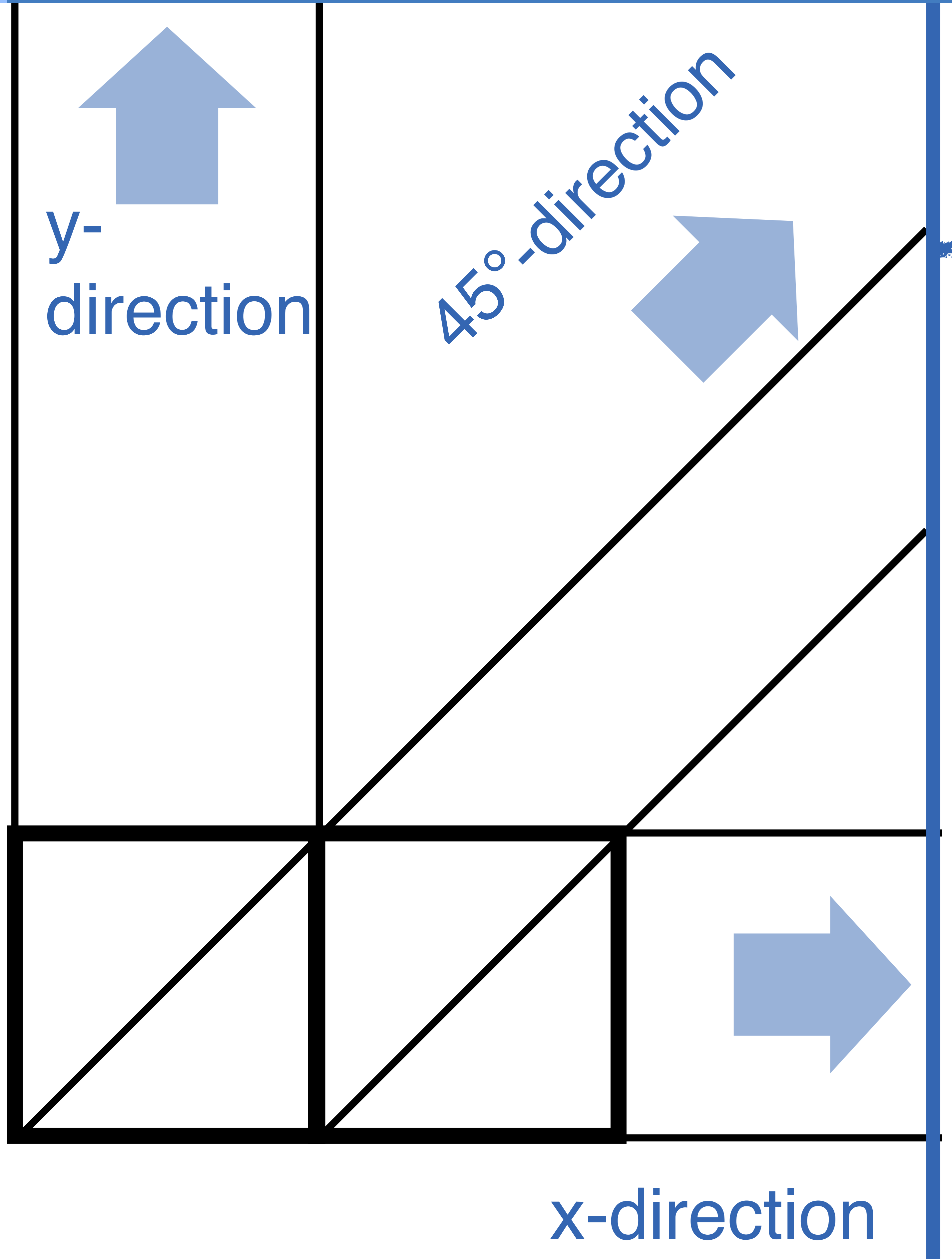


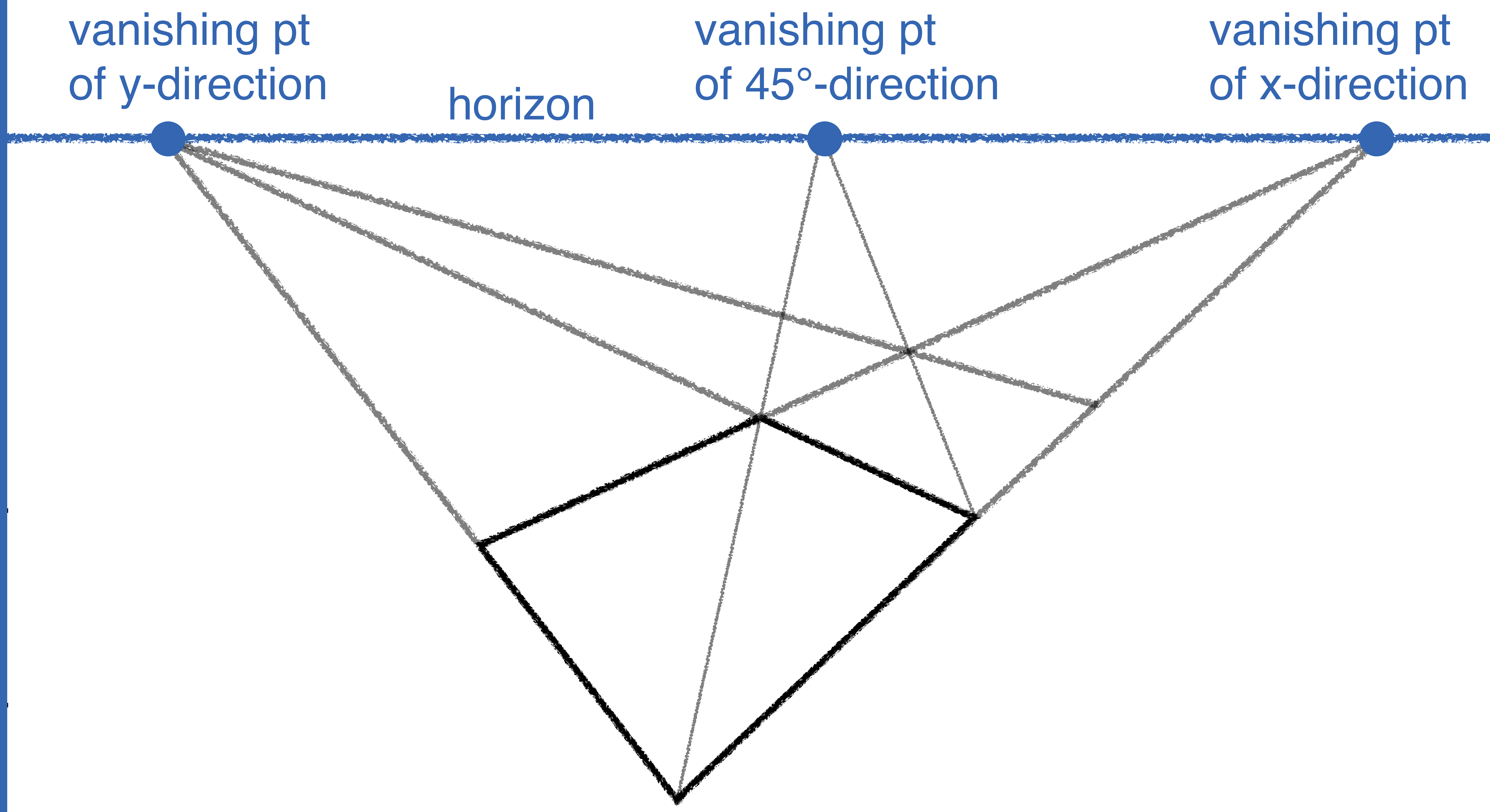

topview floor plan

perspective drawing

# Perspective grid construction




topview floor plan




perspective drawing

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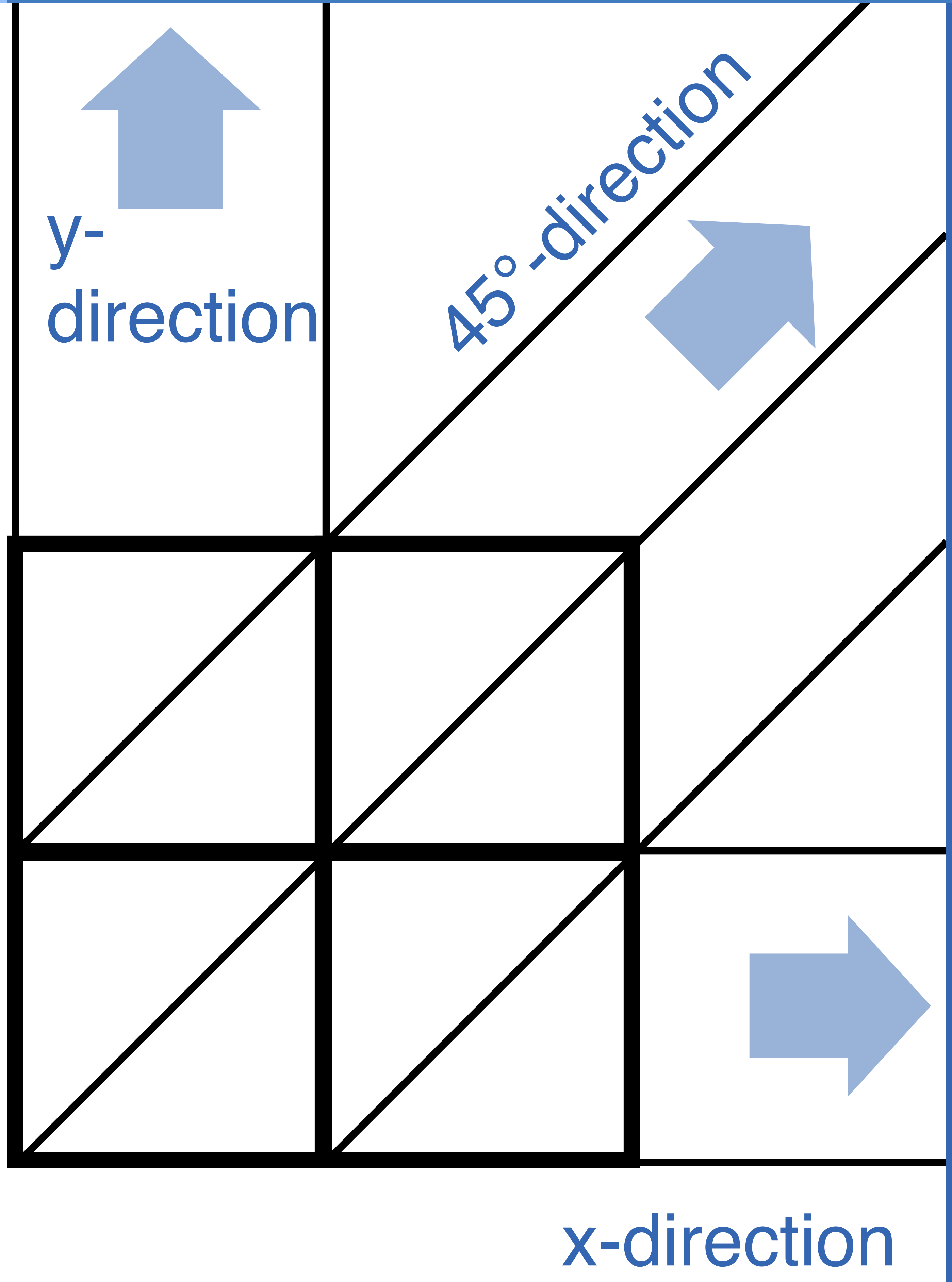
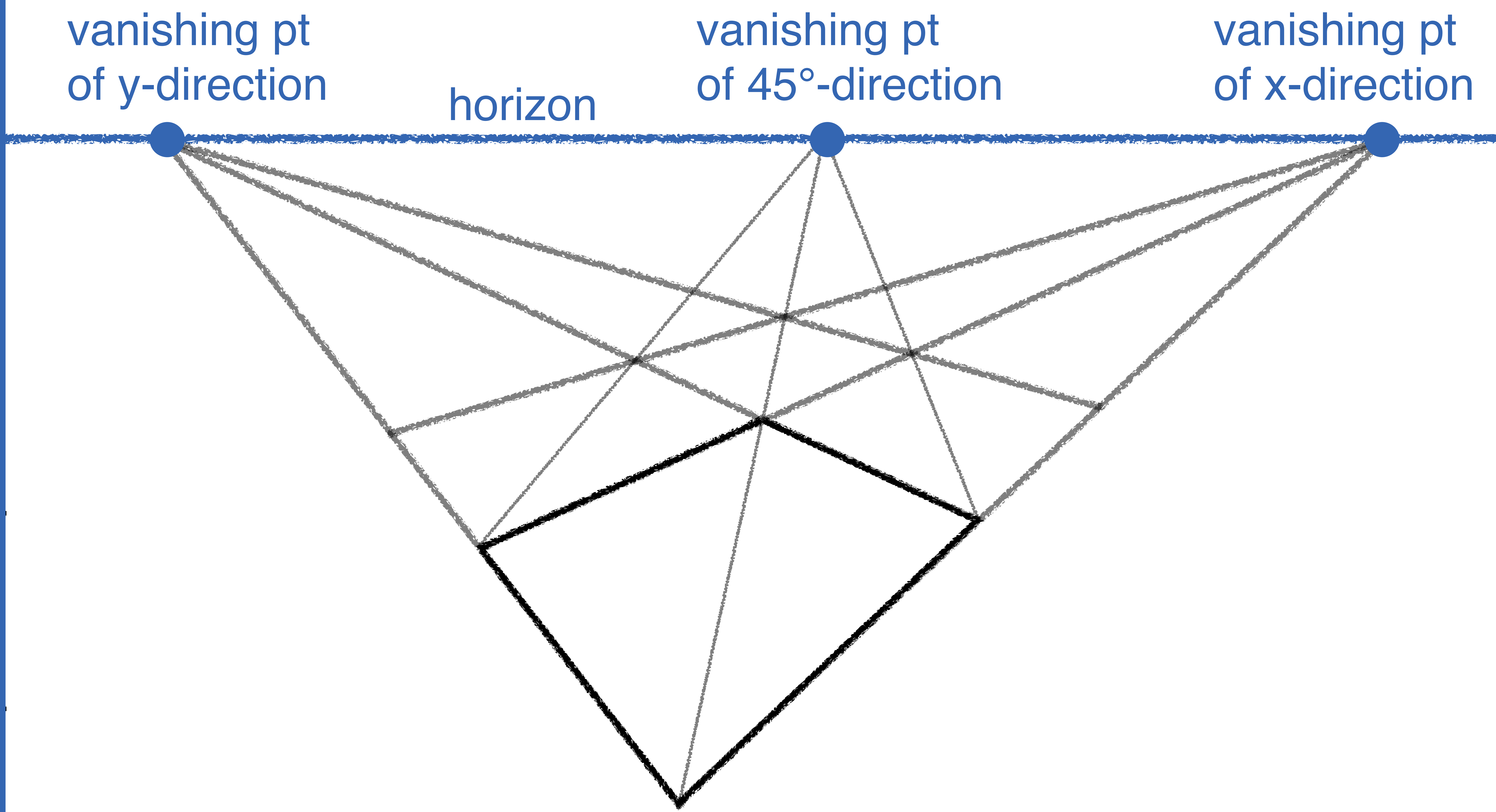
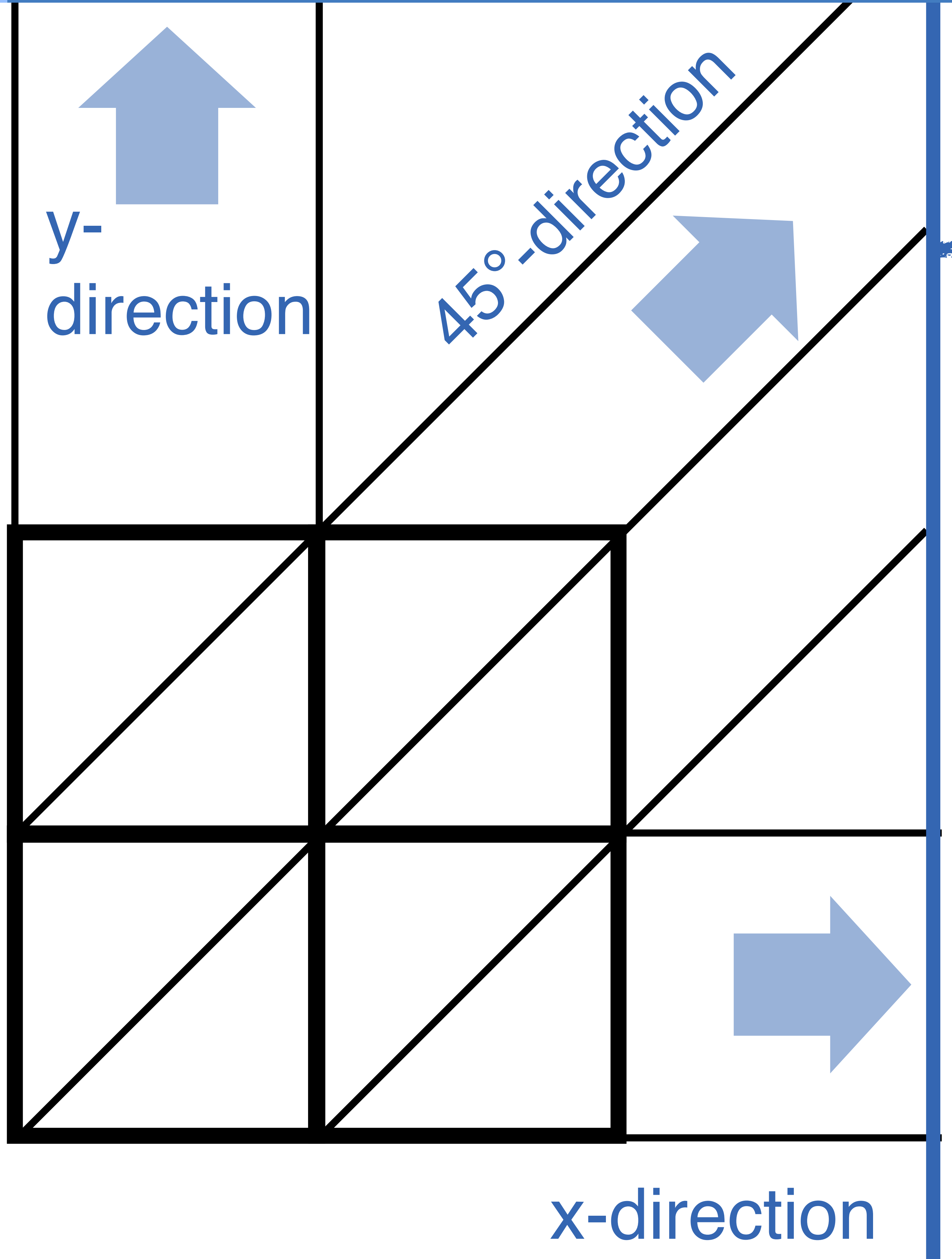


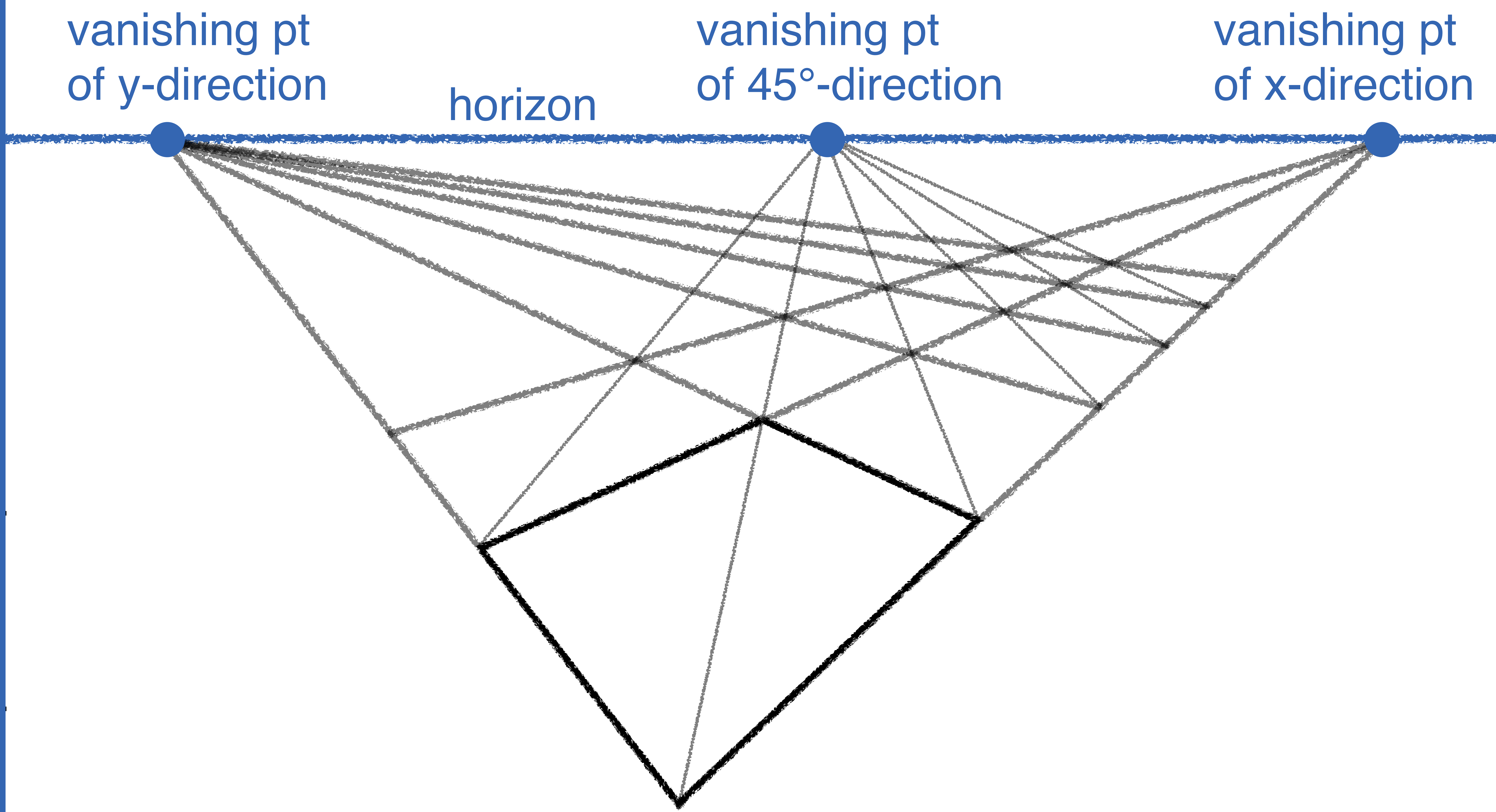

topview floor plan

perspective drawing

# Perspective grid construction

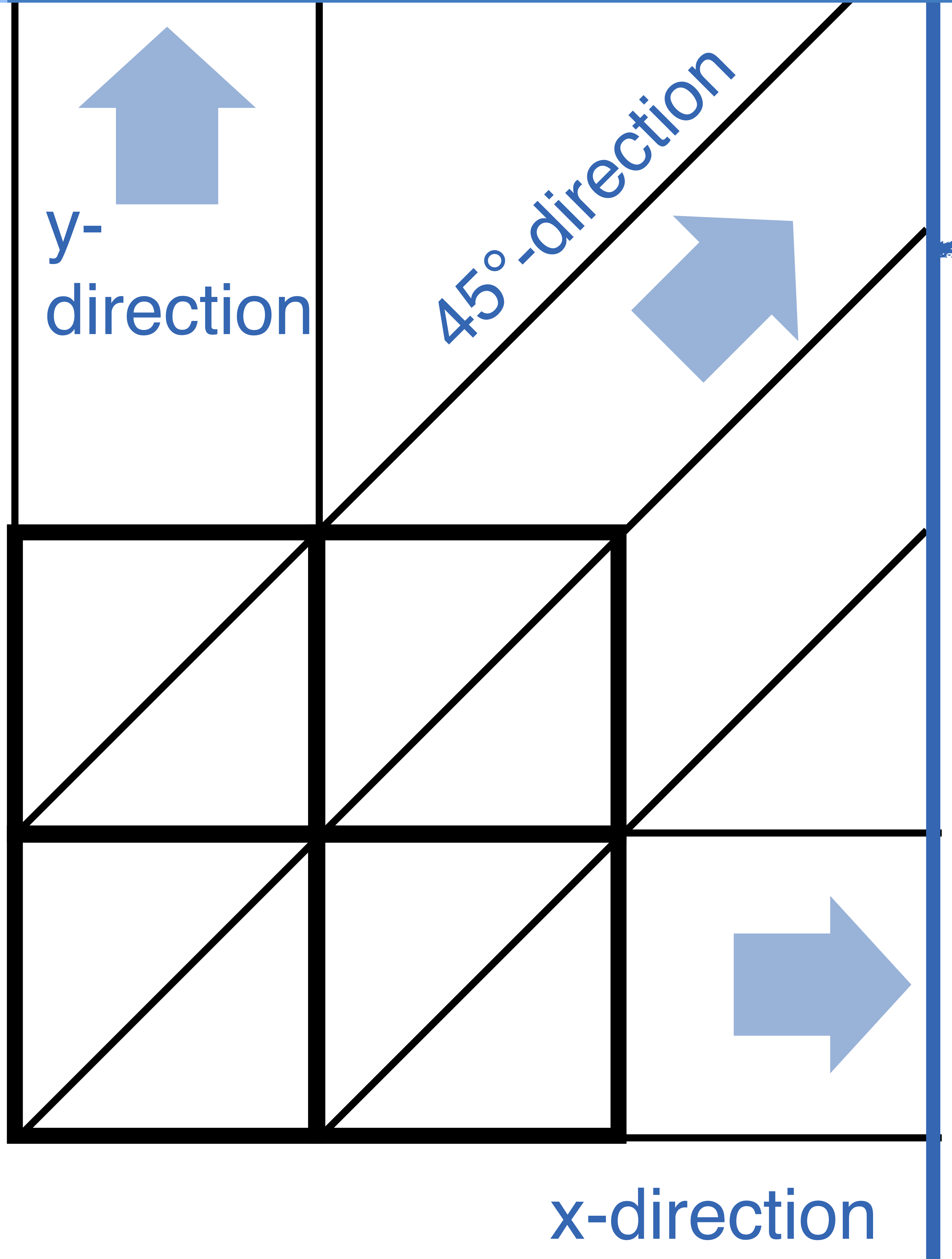



topview floor plan

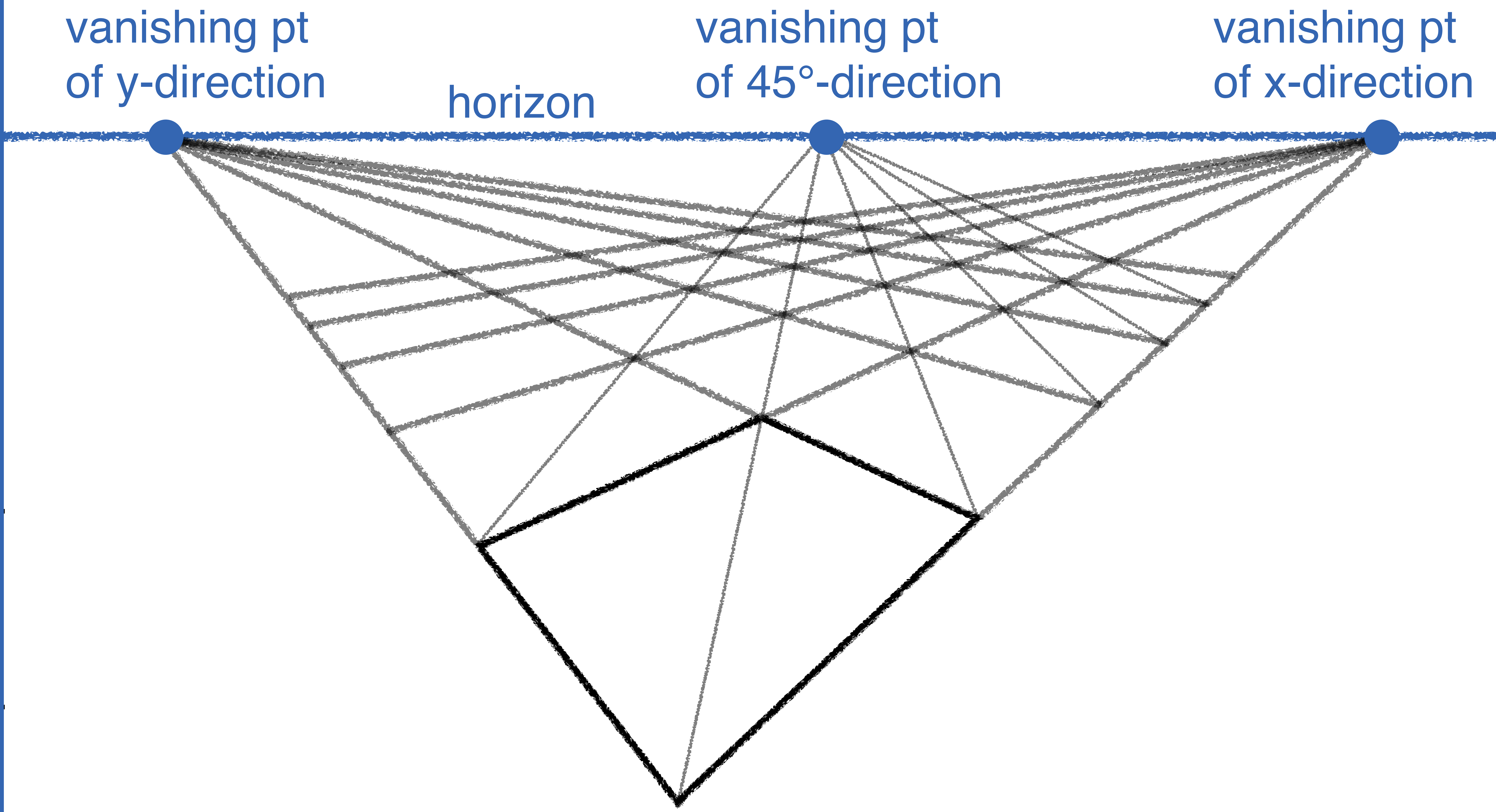



perspective drawing

# Perspective grid construction

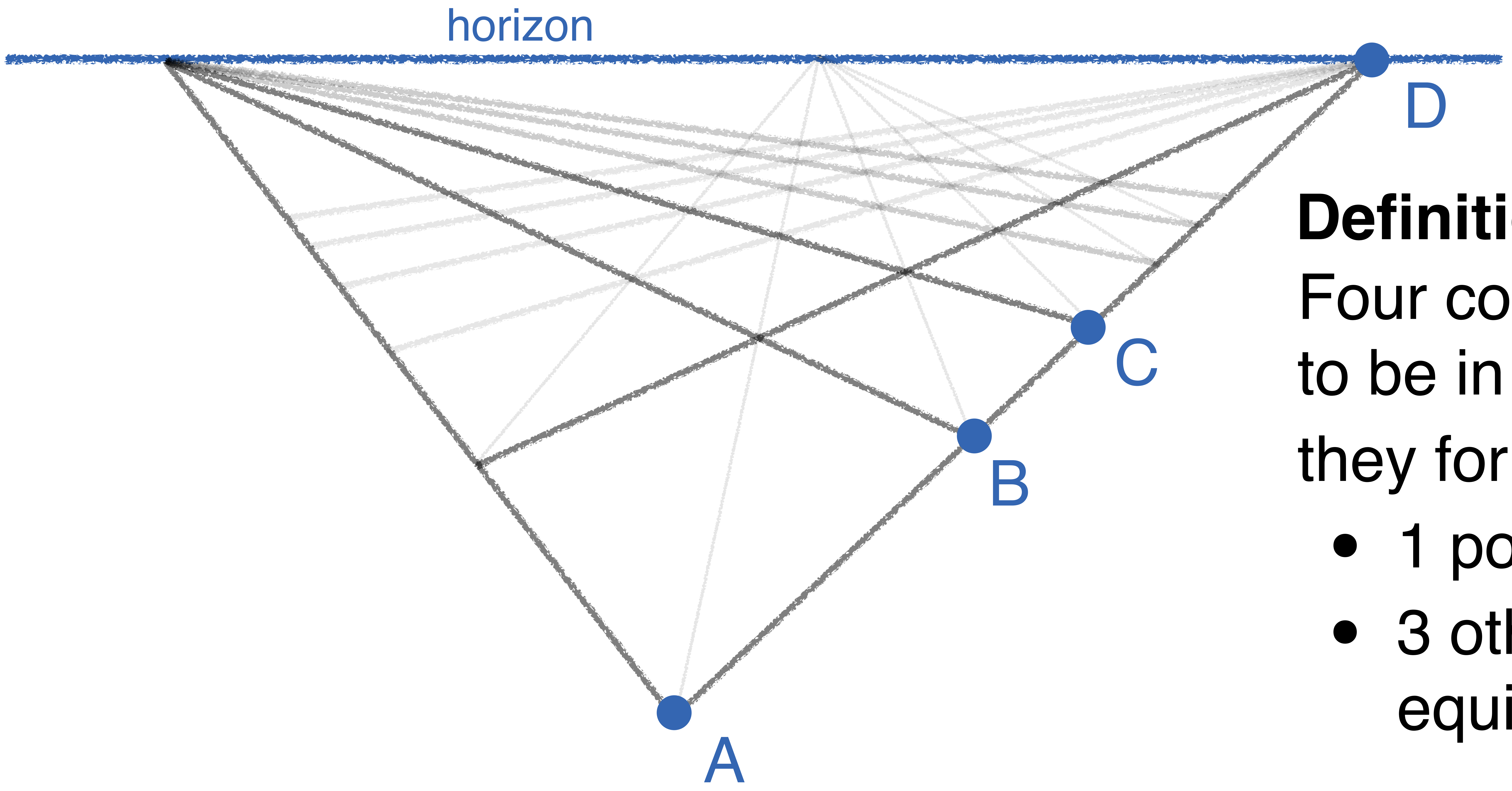



topview floor plan




perspective drawing

# Harmonic range



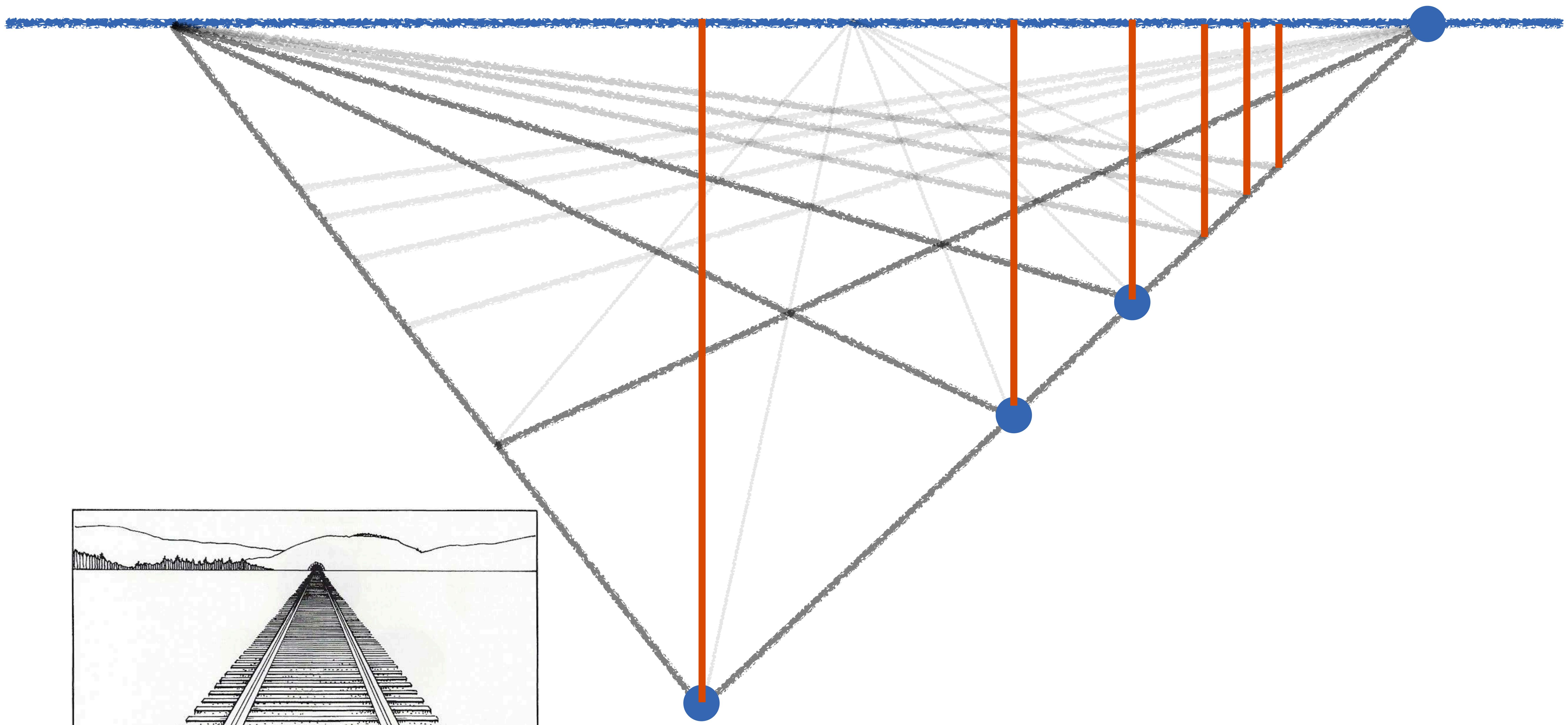

**Definition**

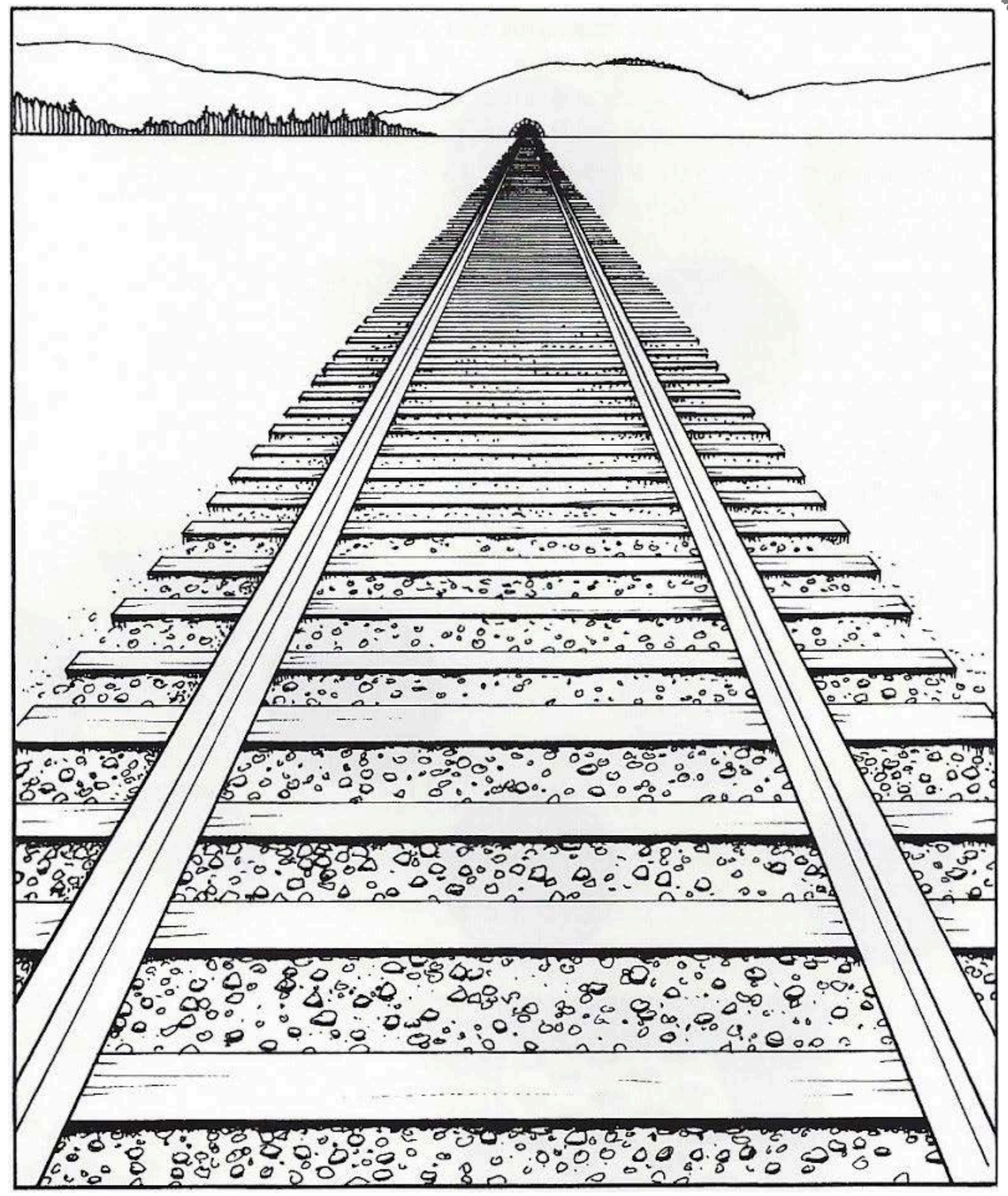
Four collinear points are said to be in a ***harmonic range*** if they form a picture that

- 1 point is at infinity, and
- 3 other points are equidistant

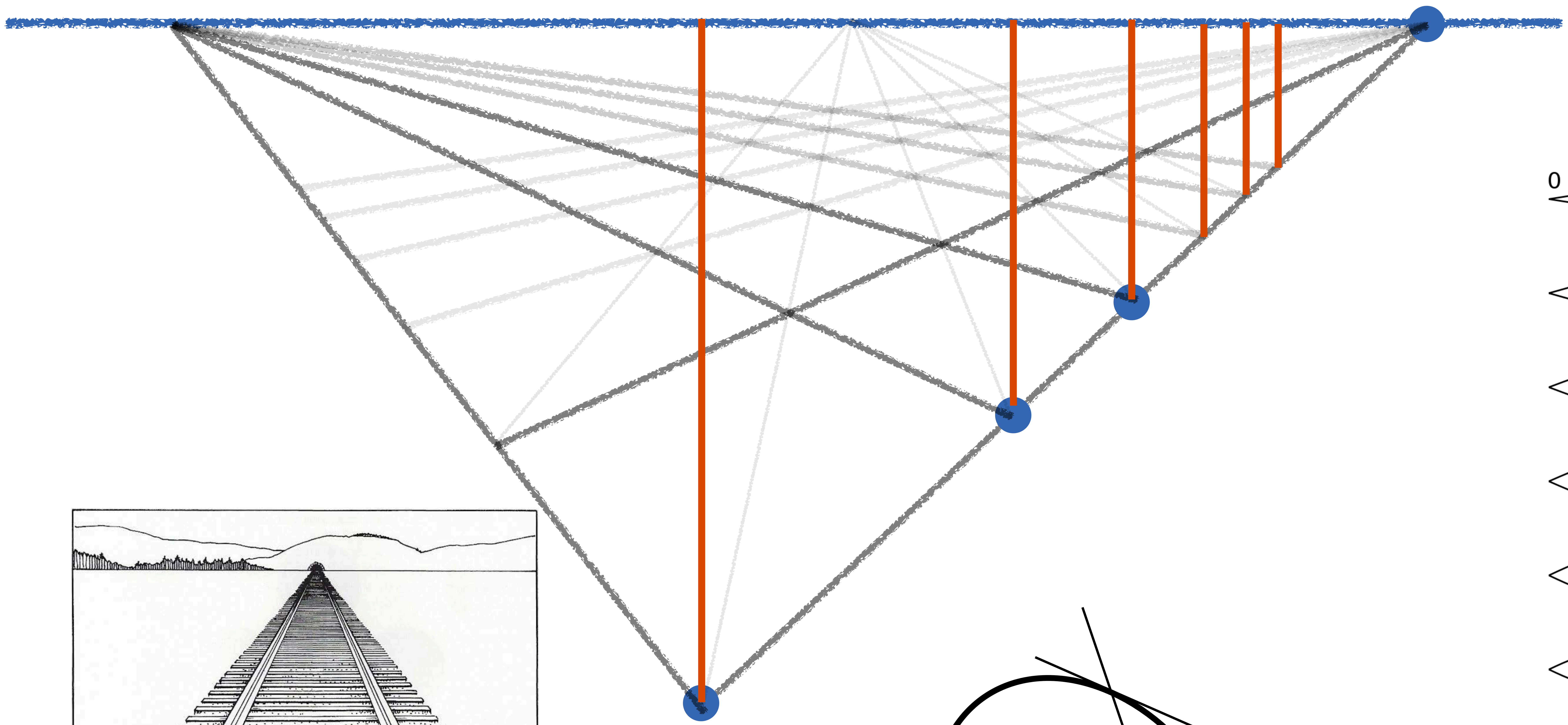
Given 3 collinear points, we can construct "uniquely" the 4th pt for the harmonic range

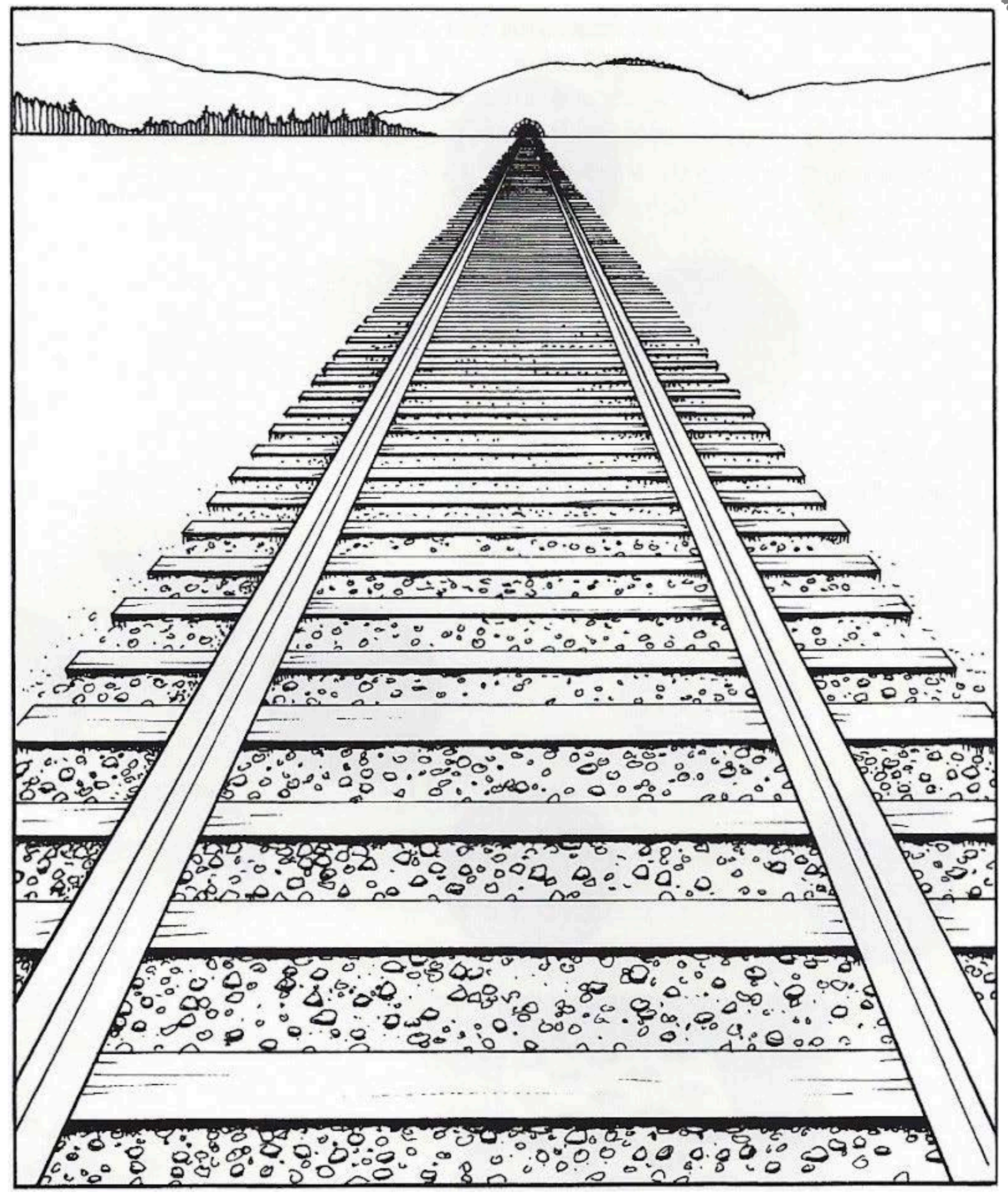
# Harmonic range

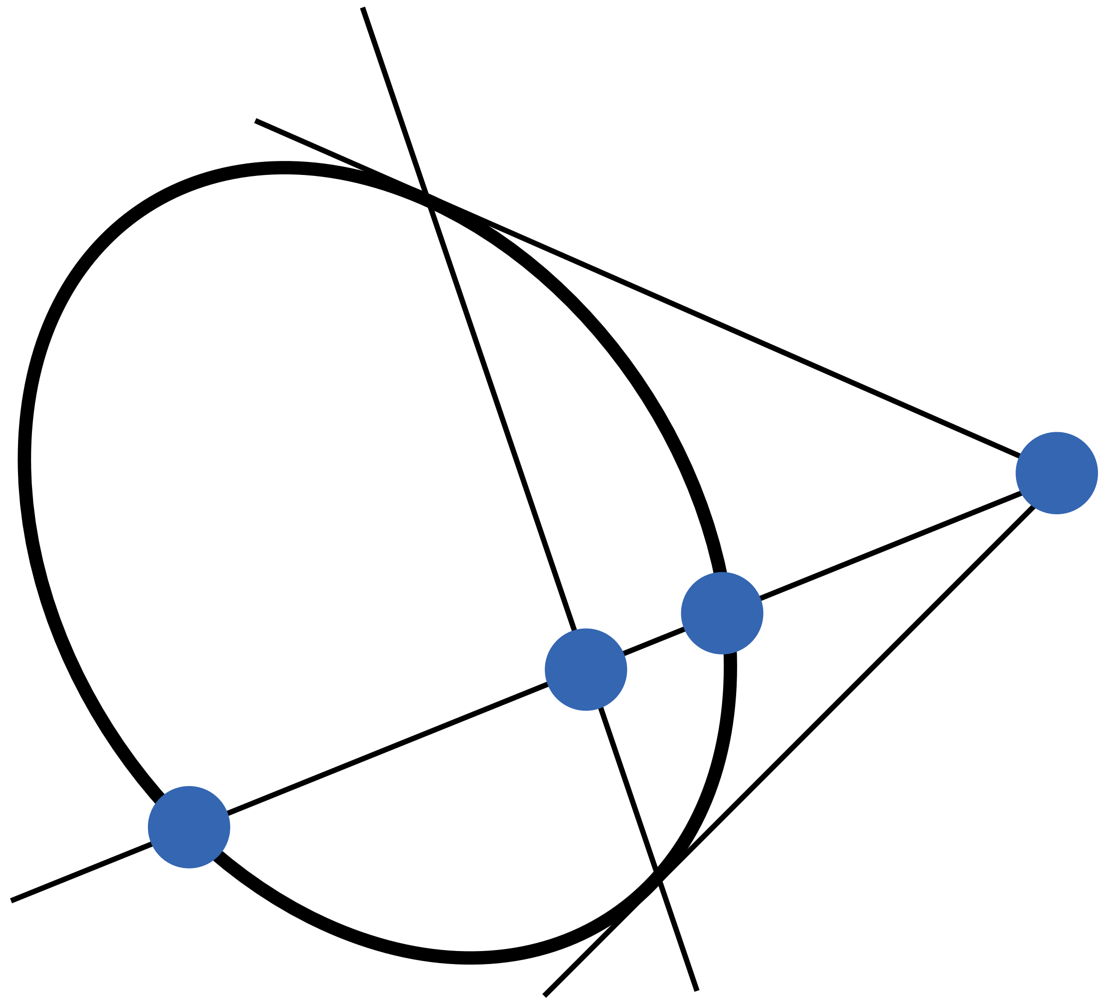


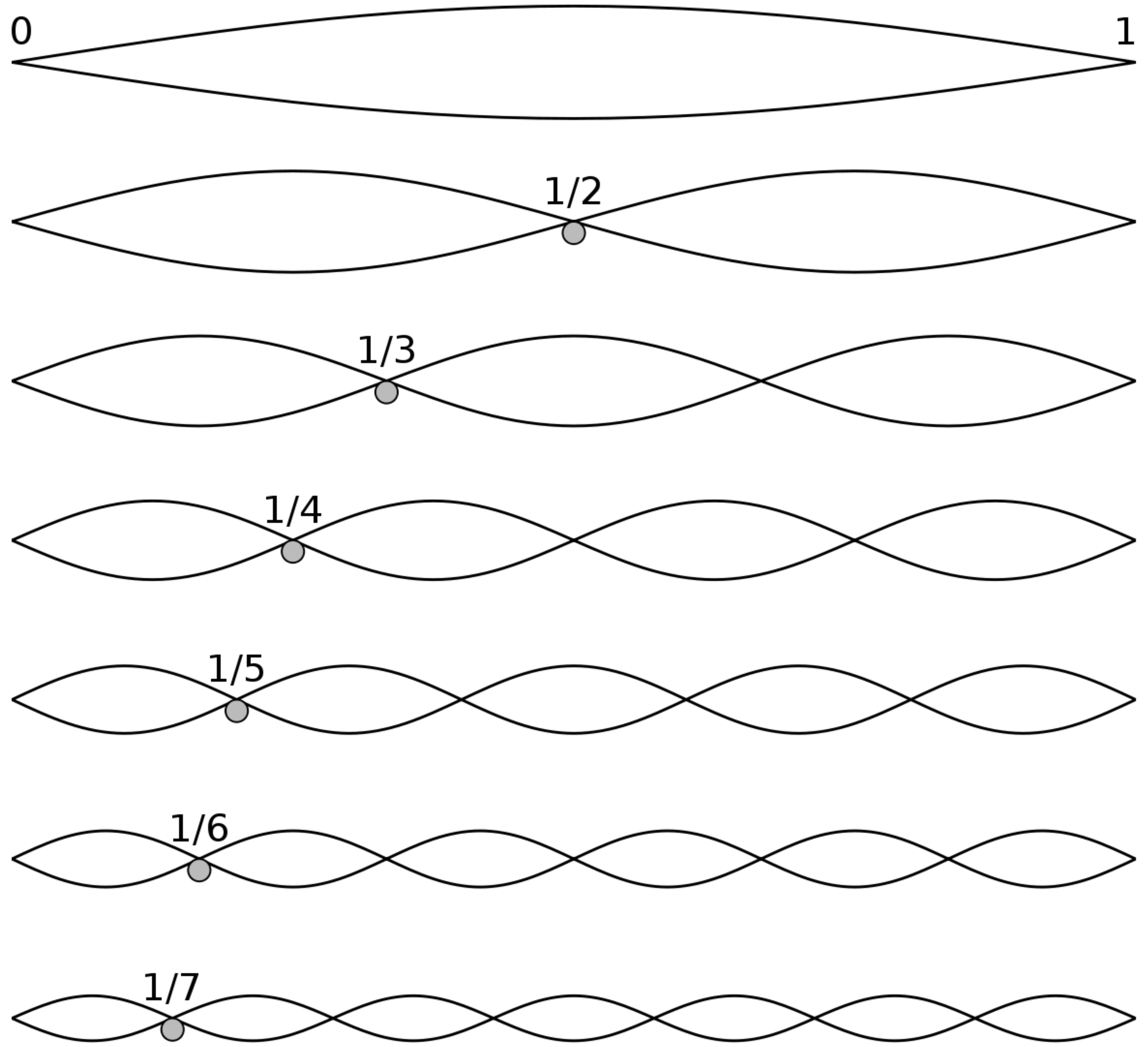


# Harmonic range









# Projectively invariant measurement?

harmonic

not harmonic

- We can tell whether 4 points along a line are harmonic despite perspective distortion
- Given 4 points along a line, there should be some measurement that indicates the deviation from being harmonic. The measurement should be invariant under projective transformations.
- This measurement is the **cross ratio.**

# Cross ratio

- Art of projective geometry
  - Perspective drawings
  - Cross ratio
  - From Renaissance Art to Computer graphics

# Cross ratio (anharmonic ratio)


- Given 4 points A,B,C,D along real line, the **cross ratio** is defined by

$$\mathrm{Cr}(A,B,C,D) = \frac{(B-A)(D-C)}{(C-B)(A-D)} \quad \text{or} \quad \pm\frac{\overline{AB}\cdot\overline{CD}}{\overline{BC}\cdot\overline{AD}}$$

Visit 4 pts cyclicly, and alternate between multiplying and dividing lengths

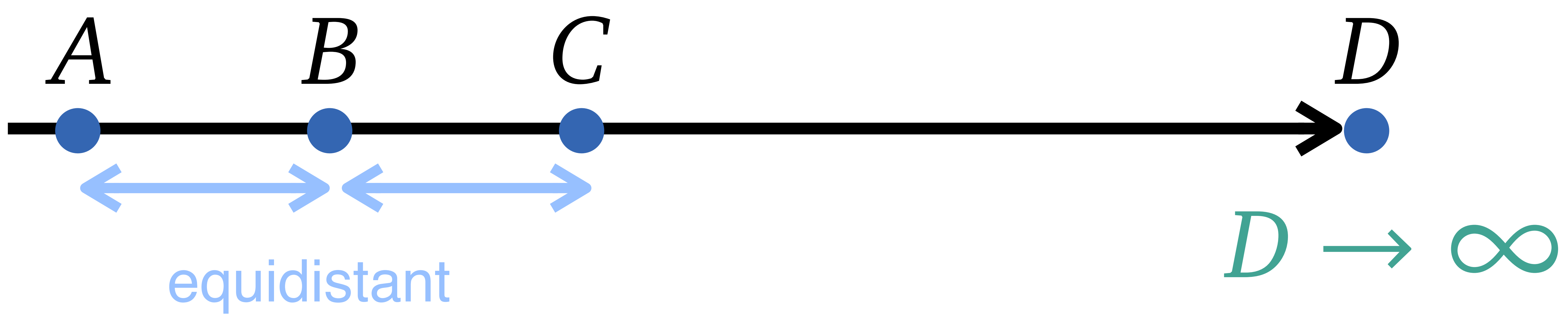
- Other convention:

$$\mathrm{Cr}_{\mathrm{Wikipedia}}(A,B,C,D) = \frac{(C-A)(D-B)}{(B-C)(A-D)}$$

- Cross ratio is projectively invariant. (Very powerful measurement!)
- ABCD are harmonic when Cr(A,B,C,D)= –1

# Example: cross ratio of harmonic range

- ABCD are harmonic when Cr(A,B,C,D)= –1



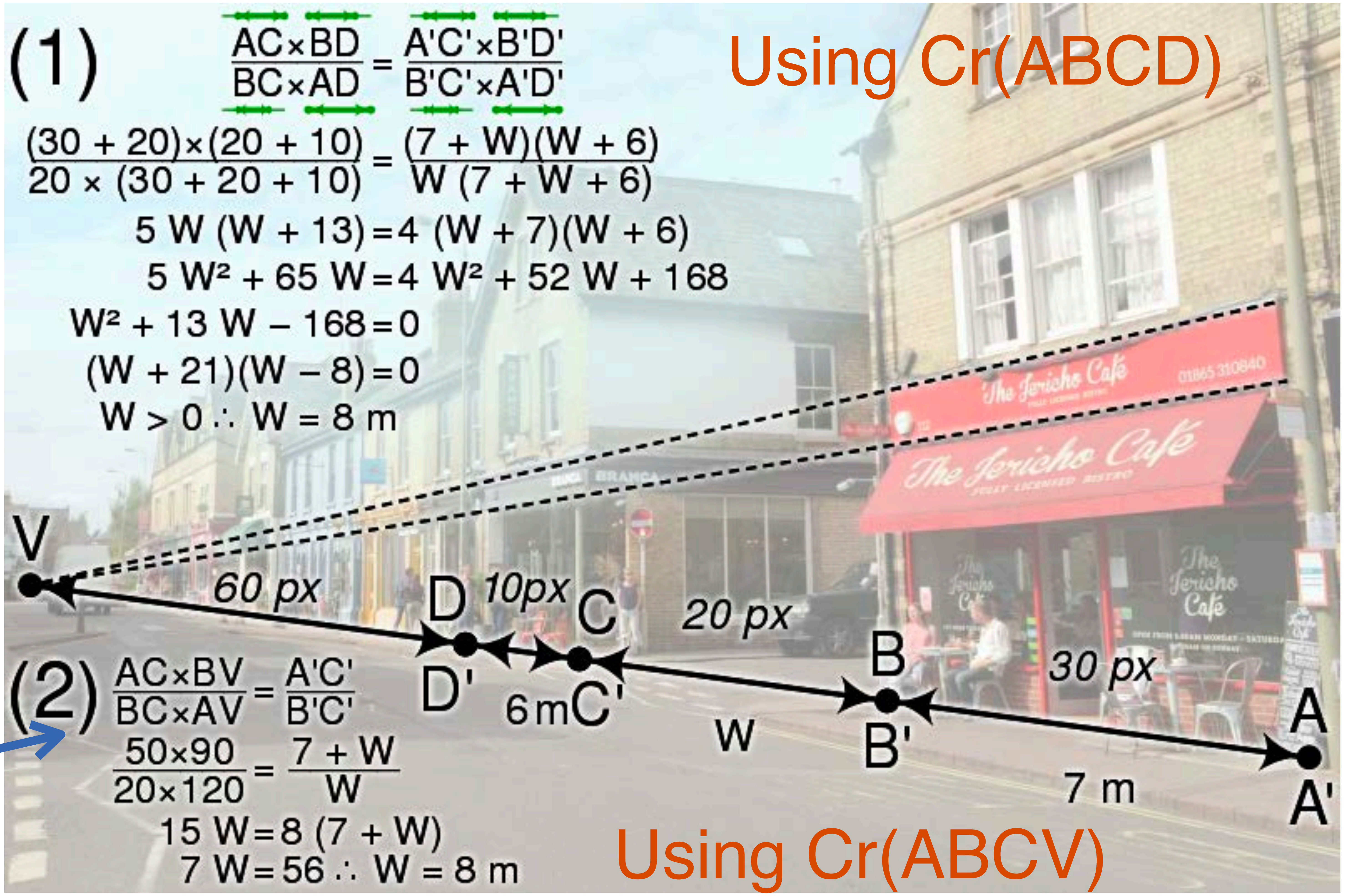

$$\mathrm{Cr}(A,B,C,D) = \frac{(B-A)(D-C)}{(C-B)(A-D)} \qquad \frac{D-C}{A-D} = \frac{1-\frac{C}{D}}{\frac{A}{D}-1} \xrightarrow{D\to\infty} \frac{1-0}{0-1} = -1$$

$$= \underbrace{\boxed{\frac{B-A}{C-B}}}_{=1 \text{ (equidistance)}} \cdot (-1) = -1$$

# Application of cross ratio

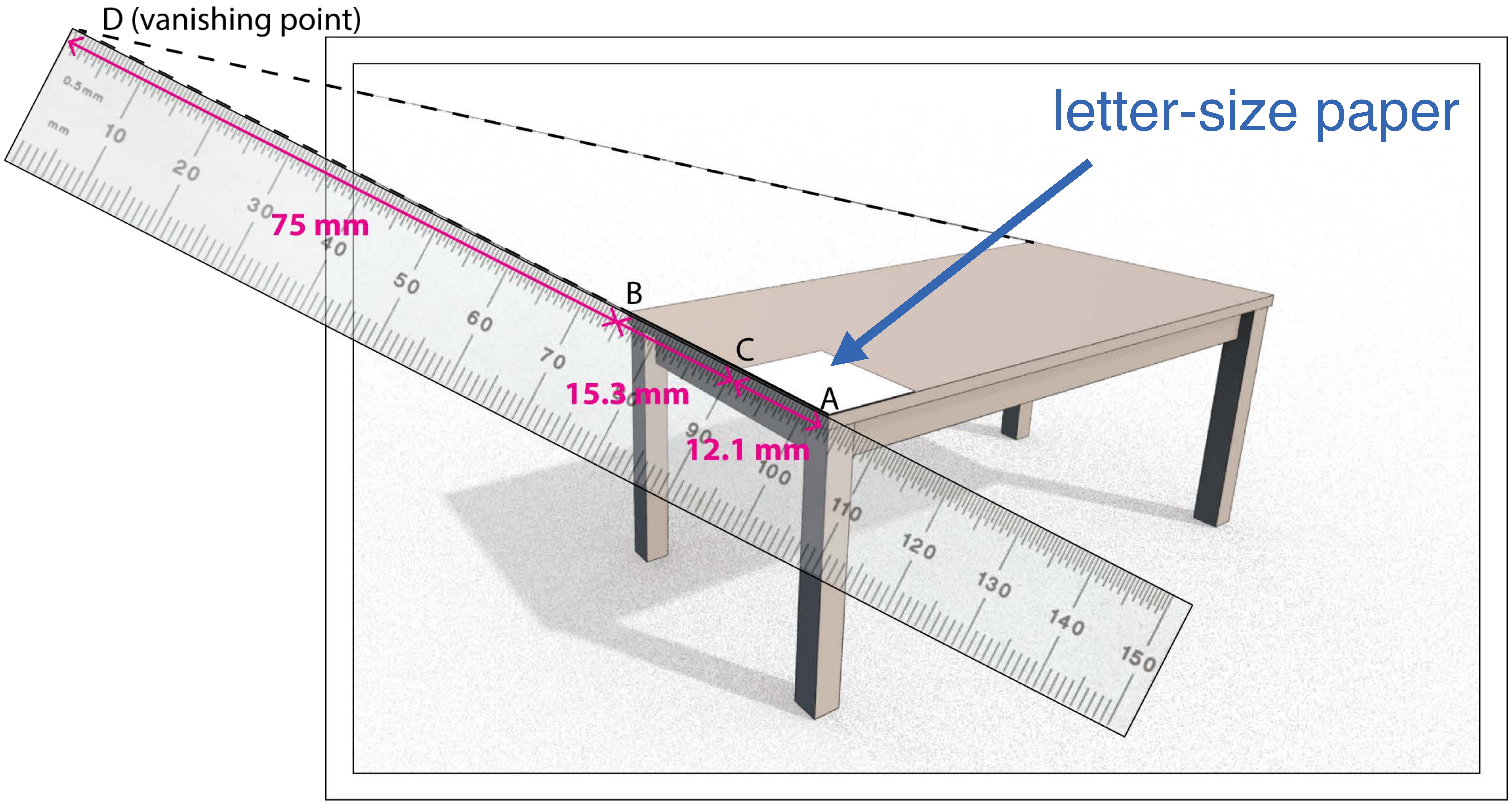
- Let A'B'C'D'V' denote pts in real life, and ABCDV corresponding pts in photograph
- Knowing A'B' = 7m, and all distances between ABCV, we can compute W = distance between B'C'.

When V' is at infinity: $\frac{\overline{B'V'}}{\overline{A'V'}} = 1$




Wikipedia: Cross-ratio

# Application of cross ratio




What's the width of the table in real life?

# From Renaissance Art to Computer Graphics

- Art of projective geometry
  - Perspective drawings
  - Cross ratio
  - From Renaissance Art to Computer graphics

# Development of the mathematical tool

- **1400s**: Brunelleschi, Alberti, da Vinci, …

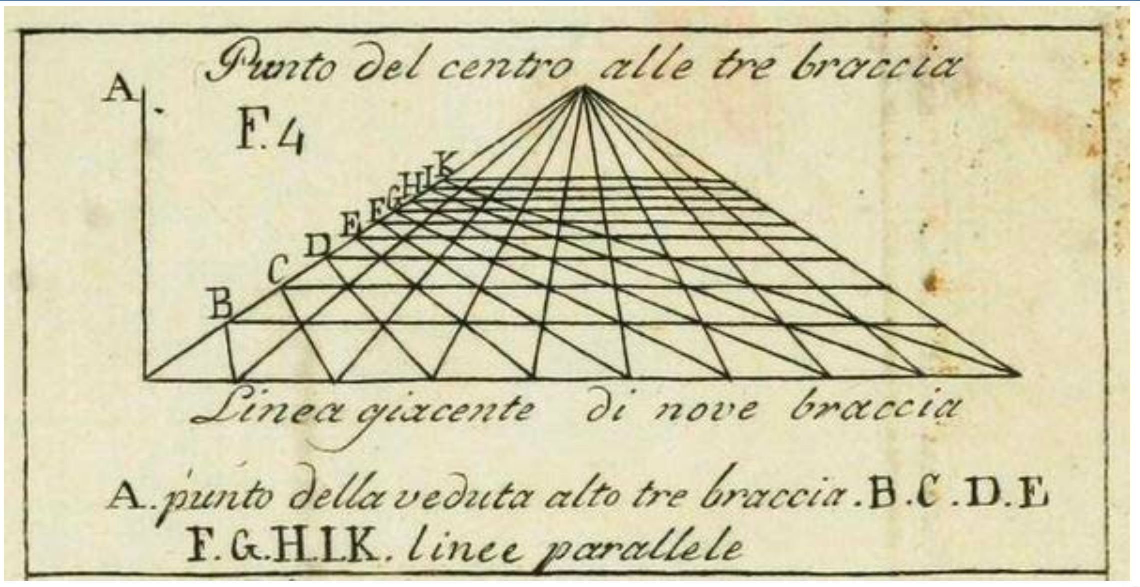
  Principles of lights and drawing
- **1600s**: Desargues

  Projective plane: all pairs of line can meet
- **1700s**: Taylor, Monge

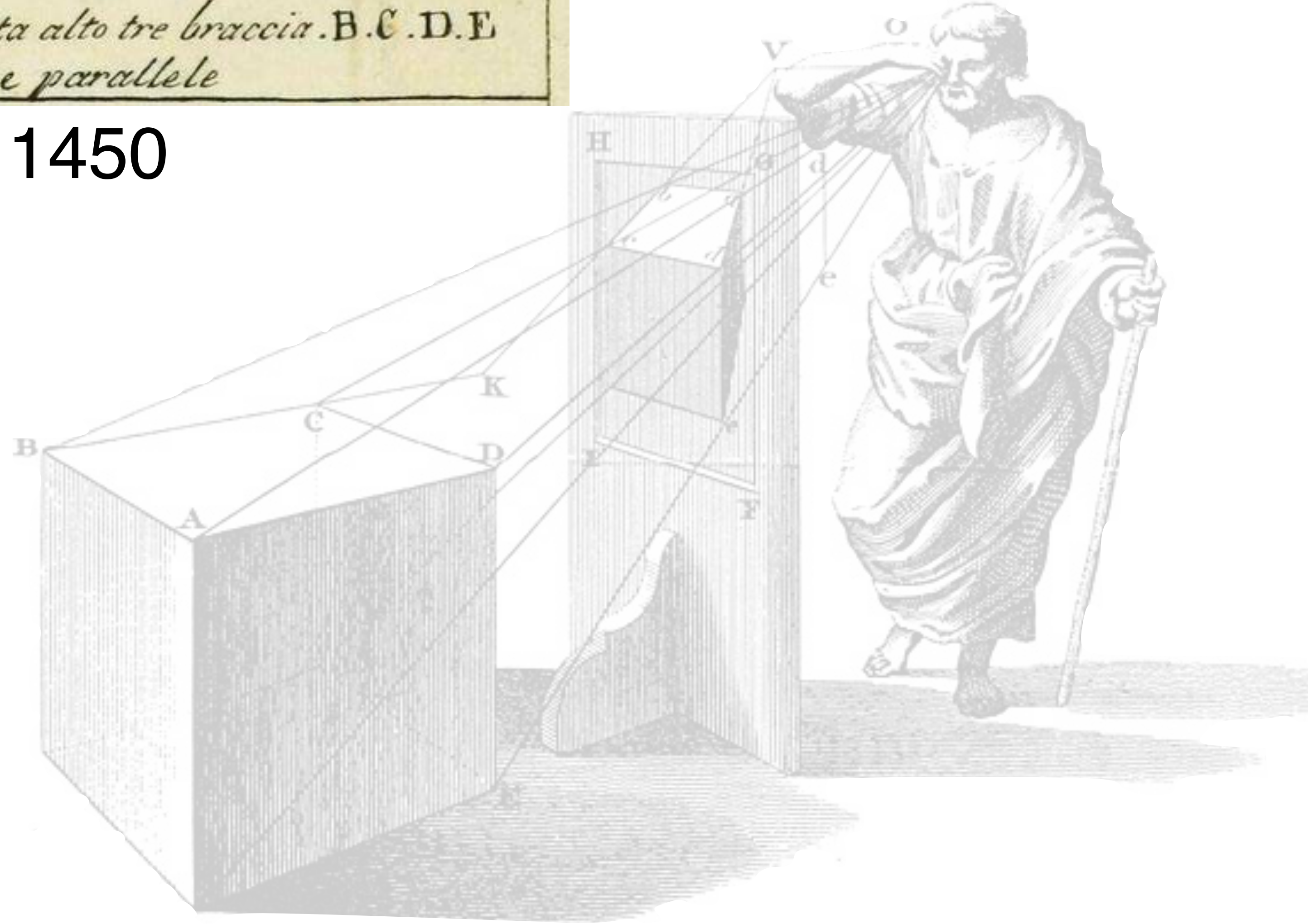
  3D representation. Descriptive geometry
- **1800s**: Möbius, Cayley, …

  Homogeneous coordinates, matrices leading to development of linear algebra
- **20th/21st century**

  Rarely mentioned outside of CS



Alberti 1450


Taylor 1715


Descriptive geometry

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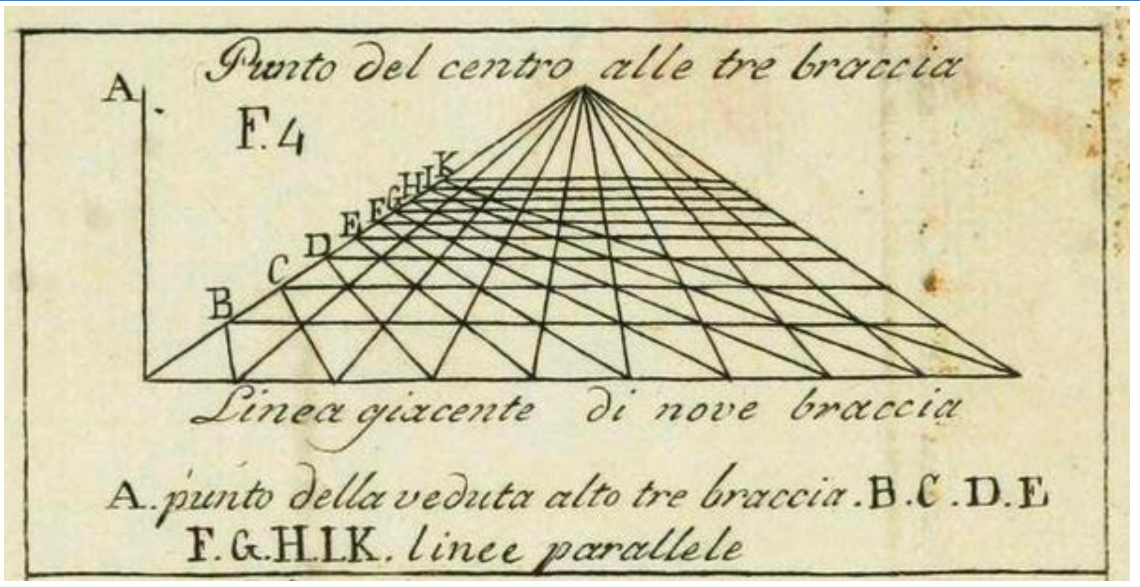

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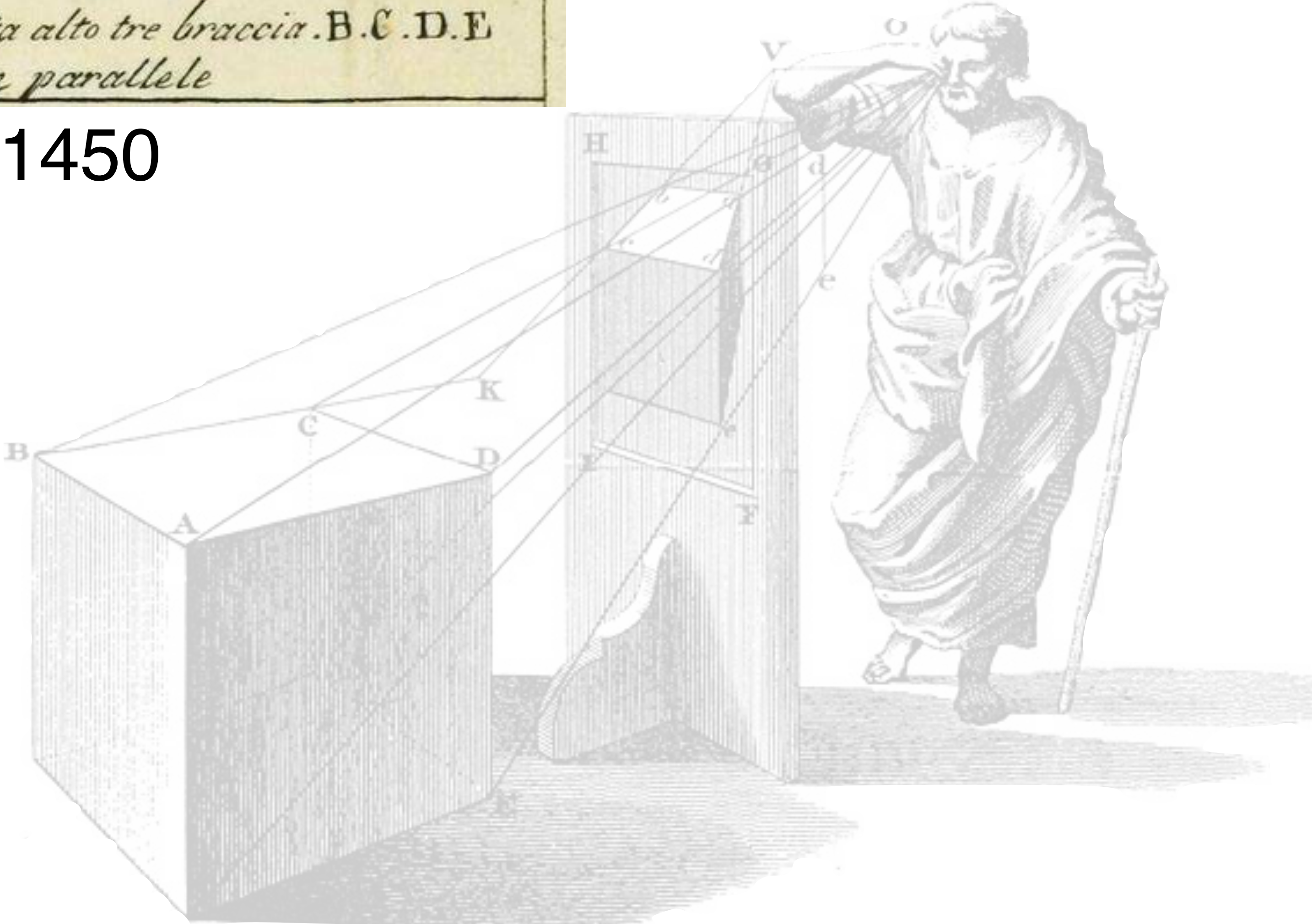
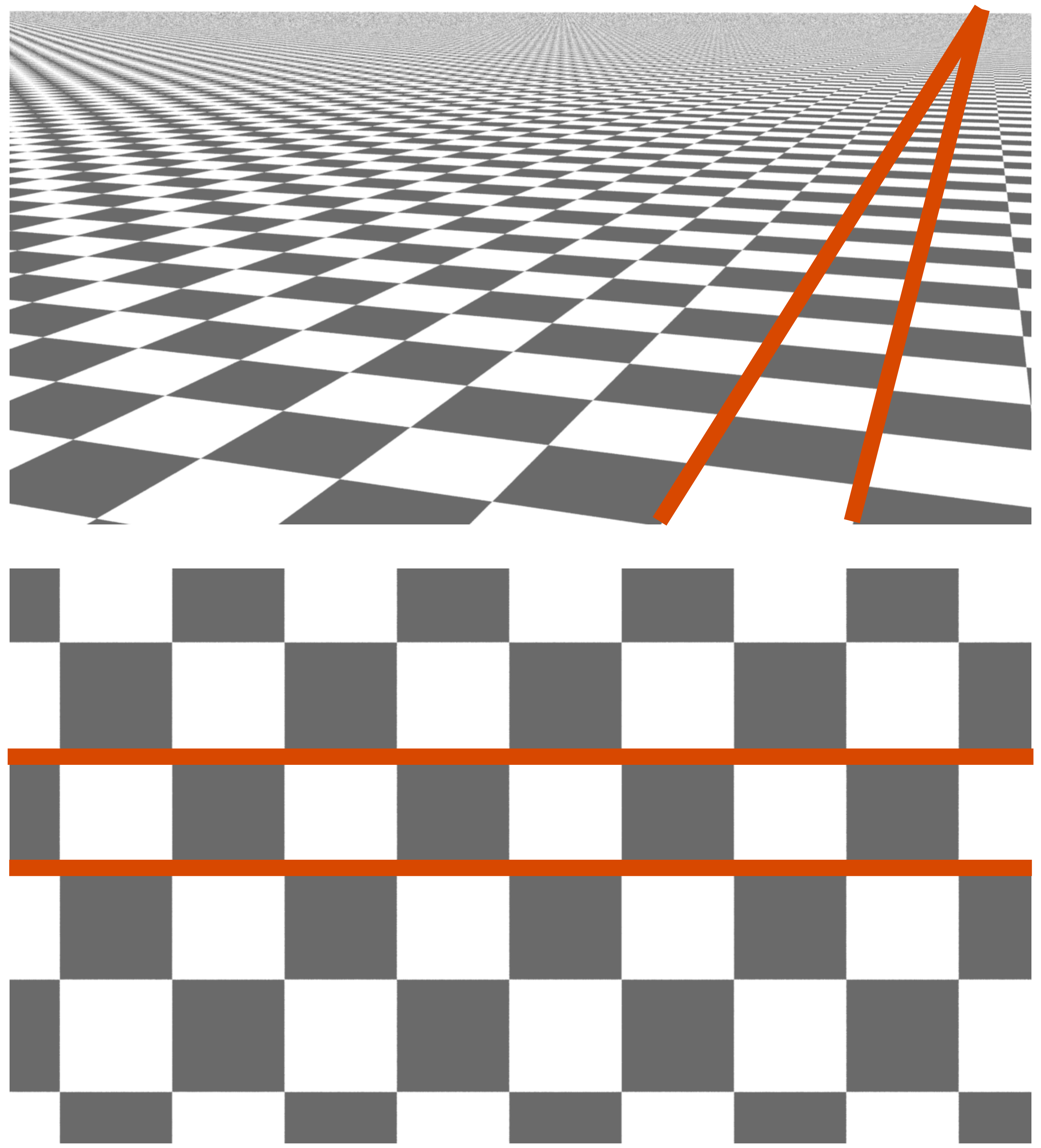

Taylor 1715


Descriptive geometry

# Projective geometry

- Girard Desargues 1600s
  - Projective Geometry: Geometry of straight edges.
  - Insist that every two lines in the plane intersect at a point.
  - We must include “infinity.”

intersecting lines

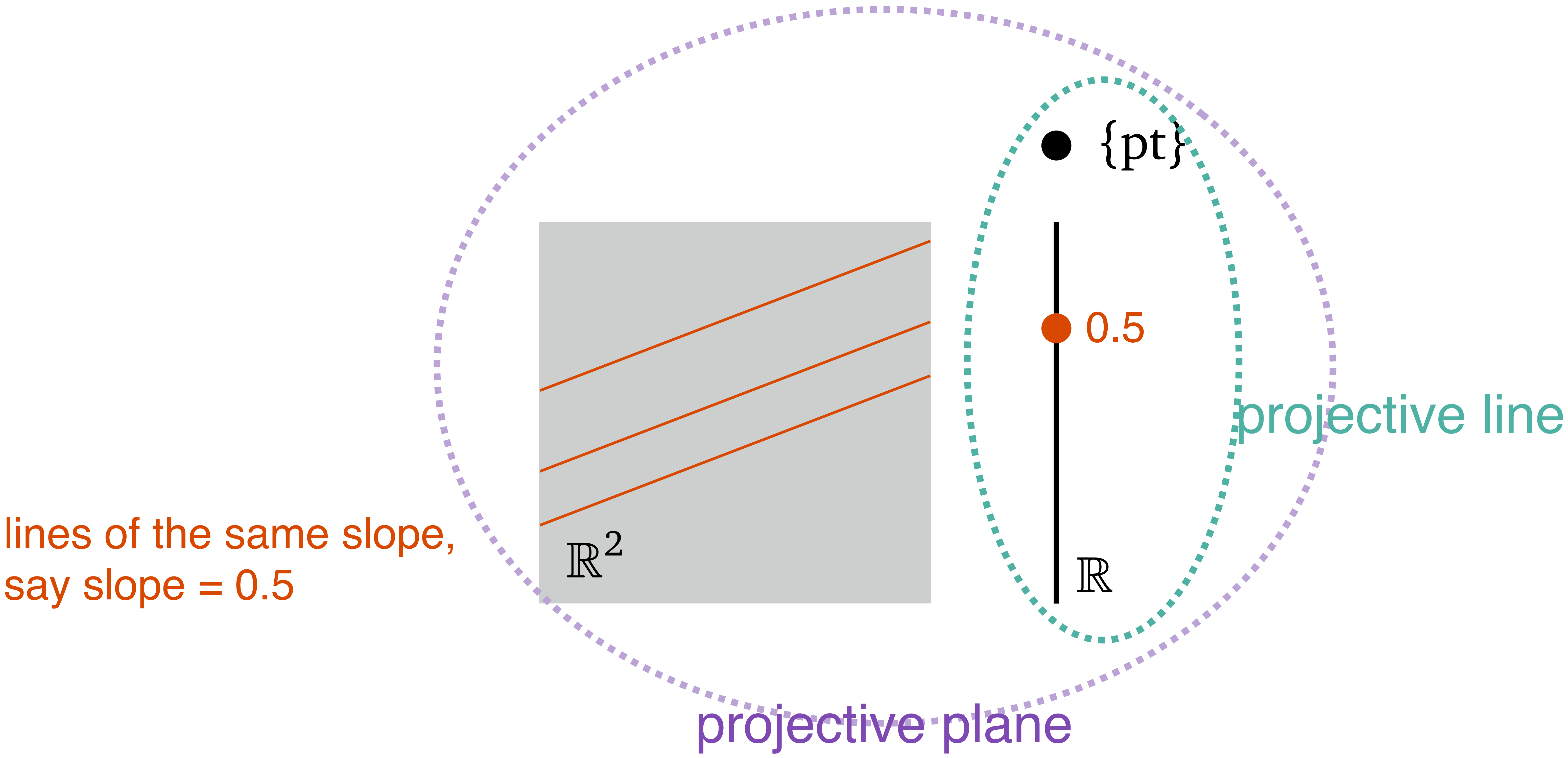


where does the intersection go?

Girard Desargues 1591–1661

# Projective geometry

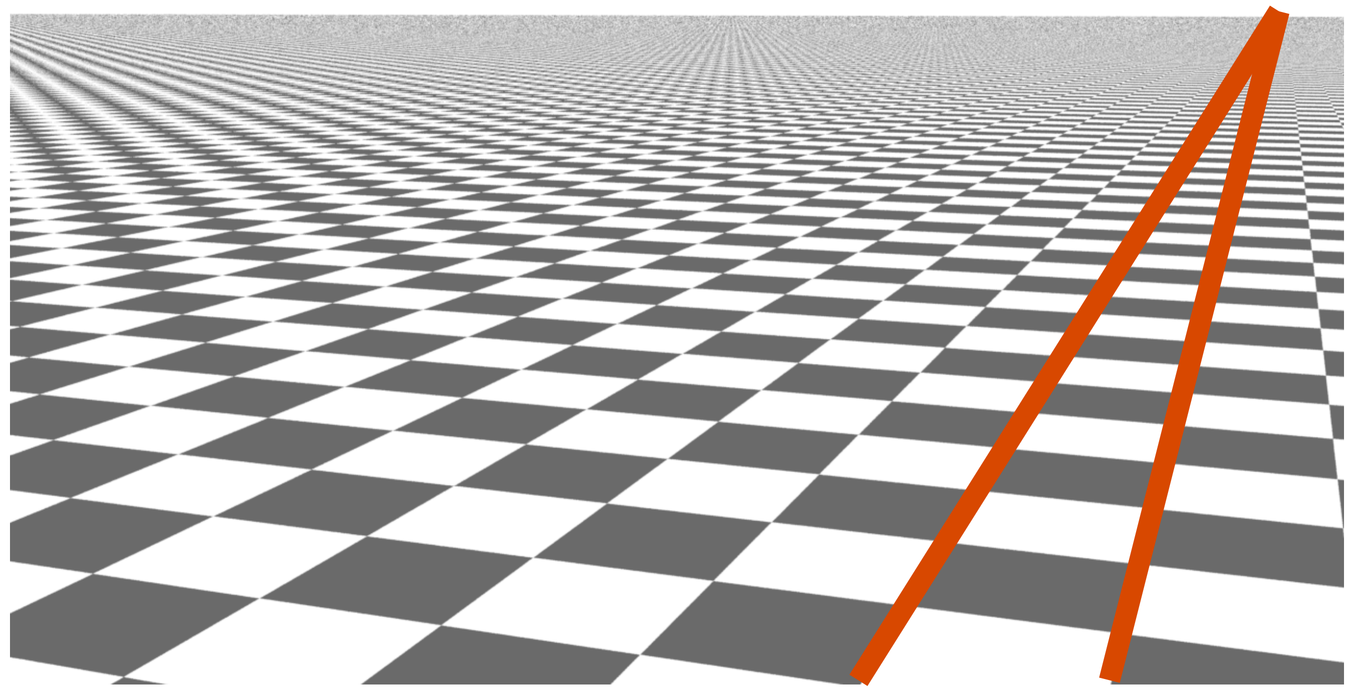
- A **projective plane** is the standard plane union a projective line of infinity.
- A **projective line** is a standard line union a point of infinity.
- Parallel lines in the plane meet at a point the line of infinity.
- Parallel lines of “infinite slope” meet at the “additional point” of the projective line of infinity.

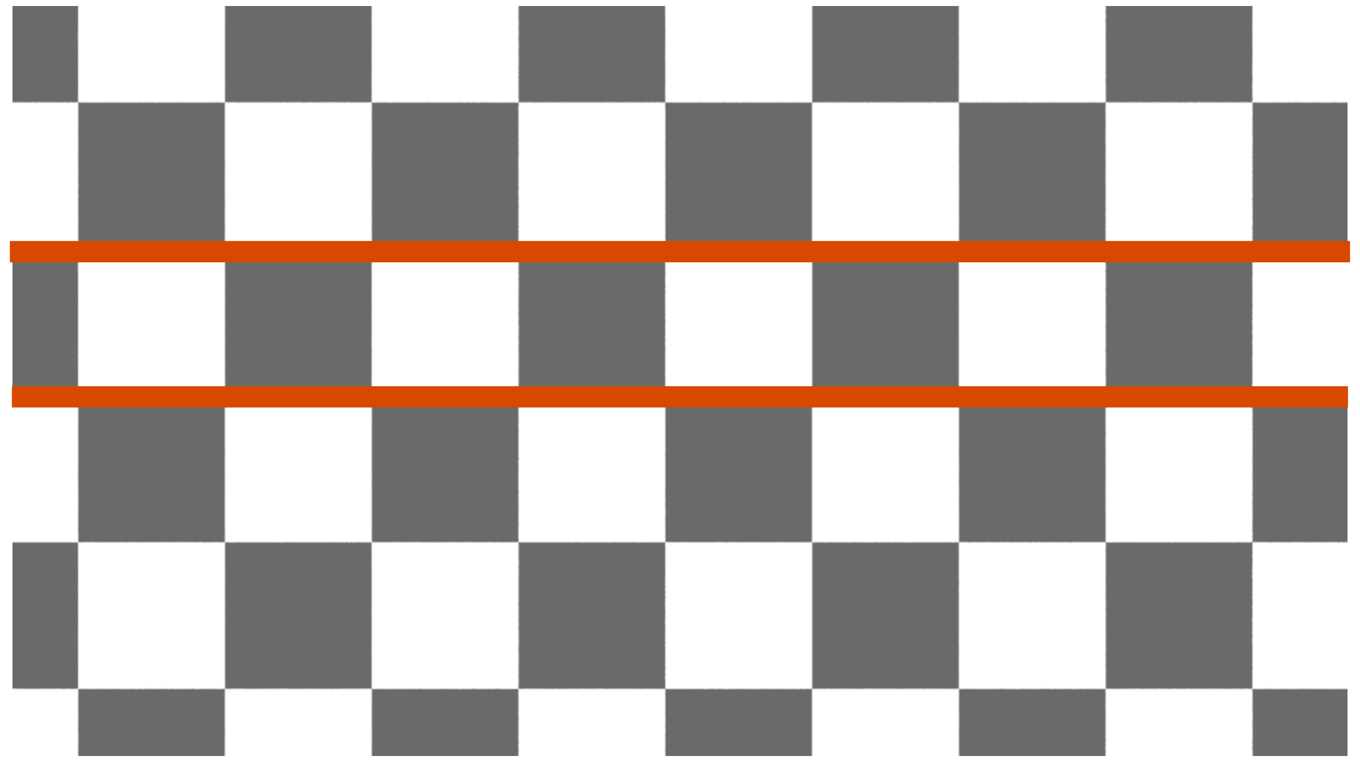


# Projective geometry

- Desargues' projective geometry wasn't appreciated in the 1600's.
- It was not clear how to represent infinities arithmetically.

intersecting lines

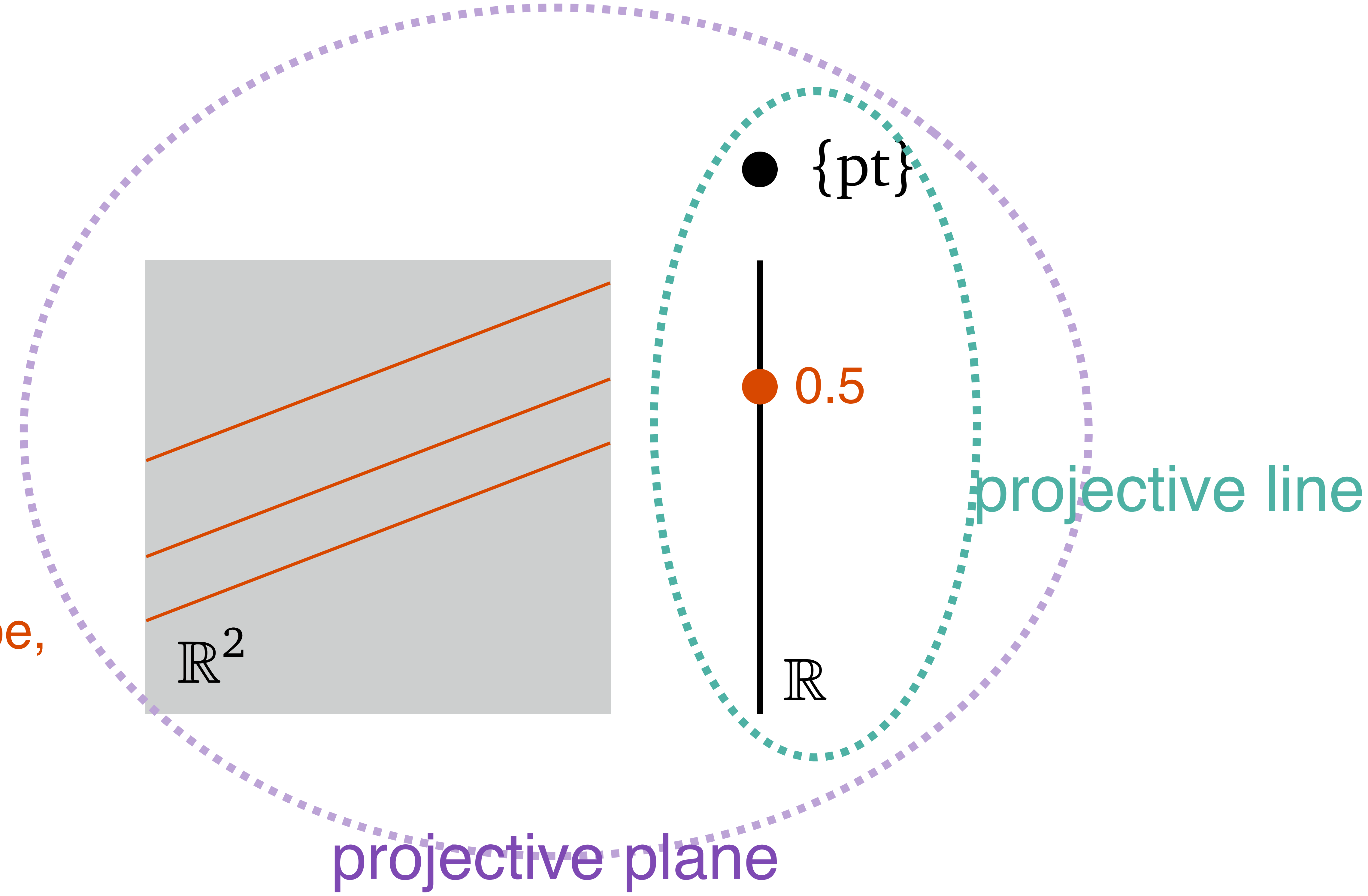




they still intersect

The 2D drawings are snapshots of a projective plane.

lines of the same slope, say slope = 0.5



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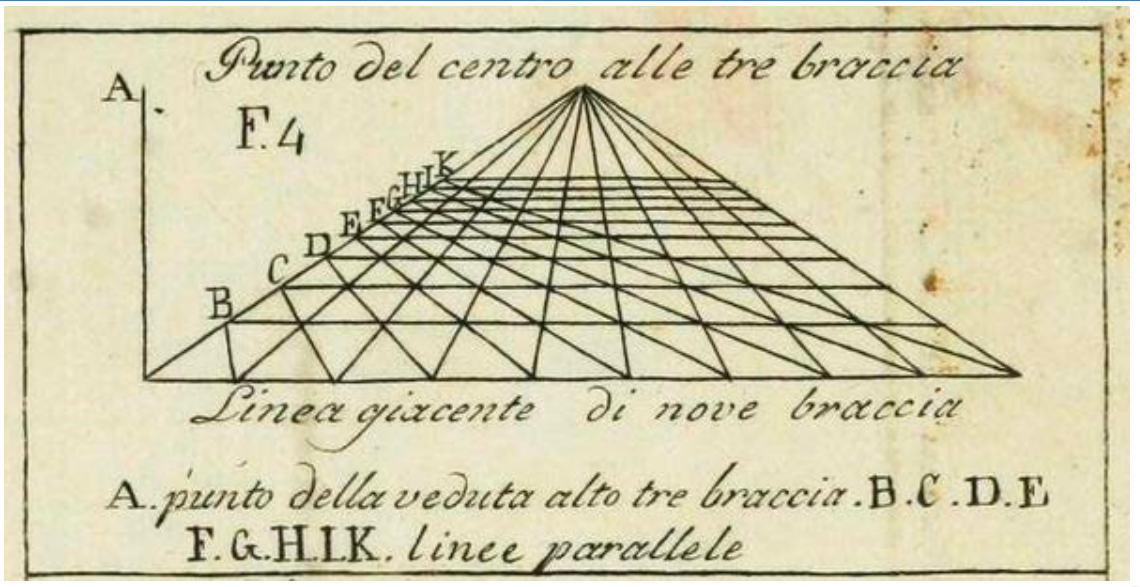

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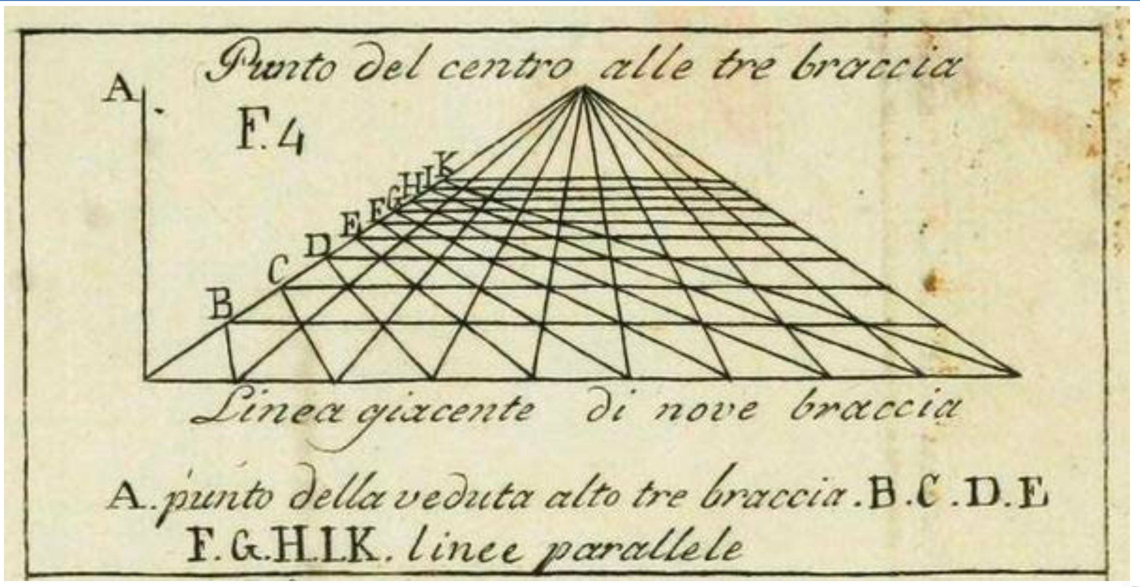
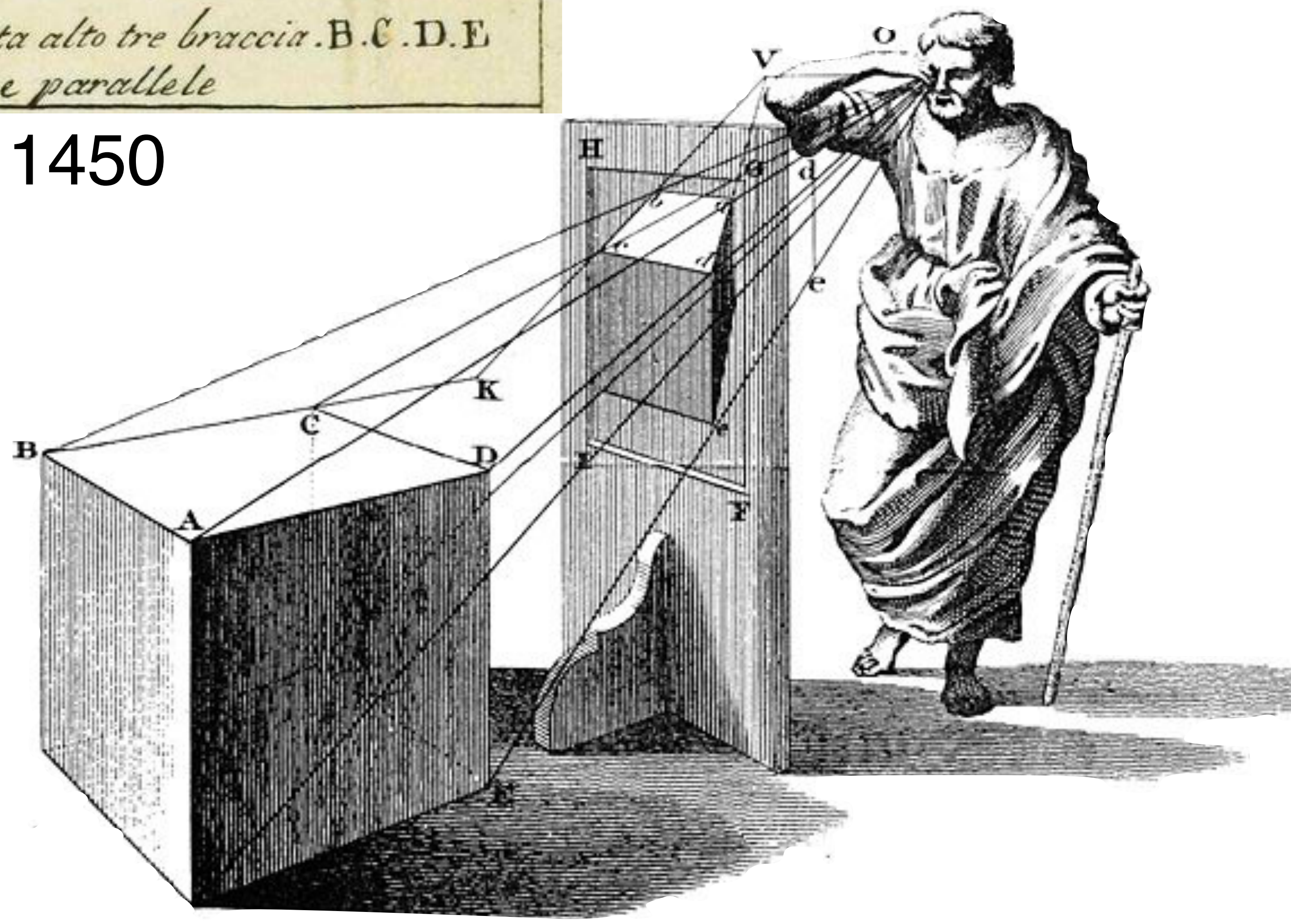
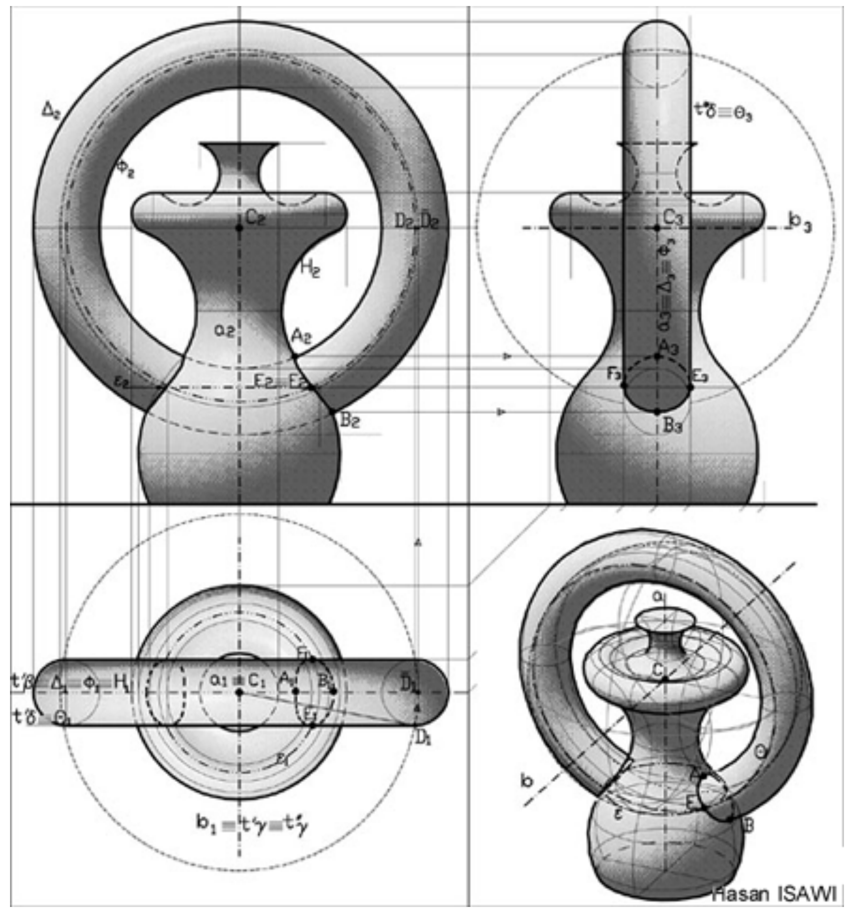


Alberti 1450

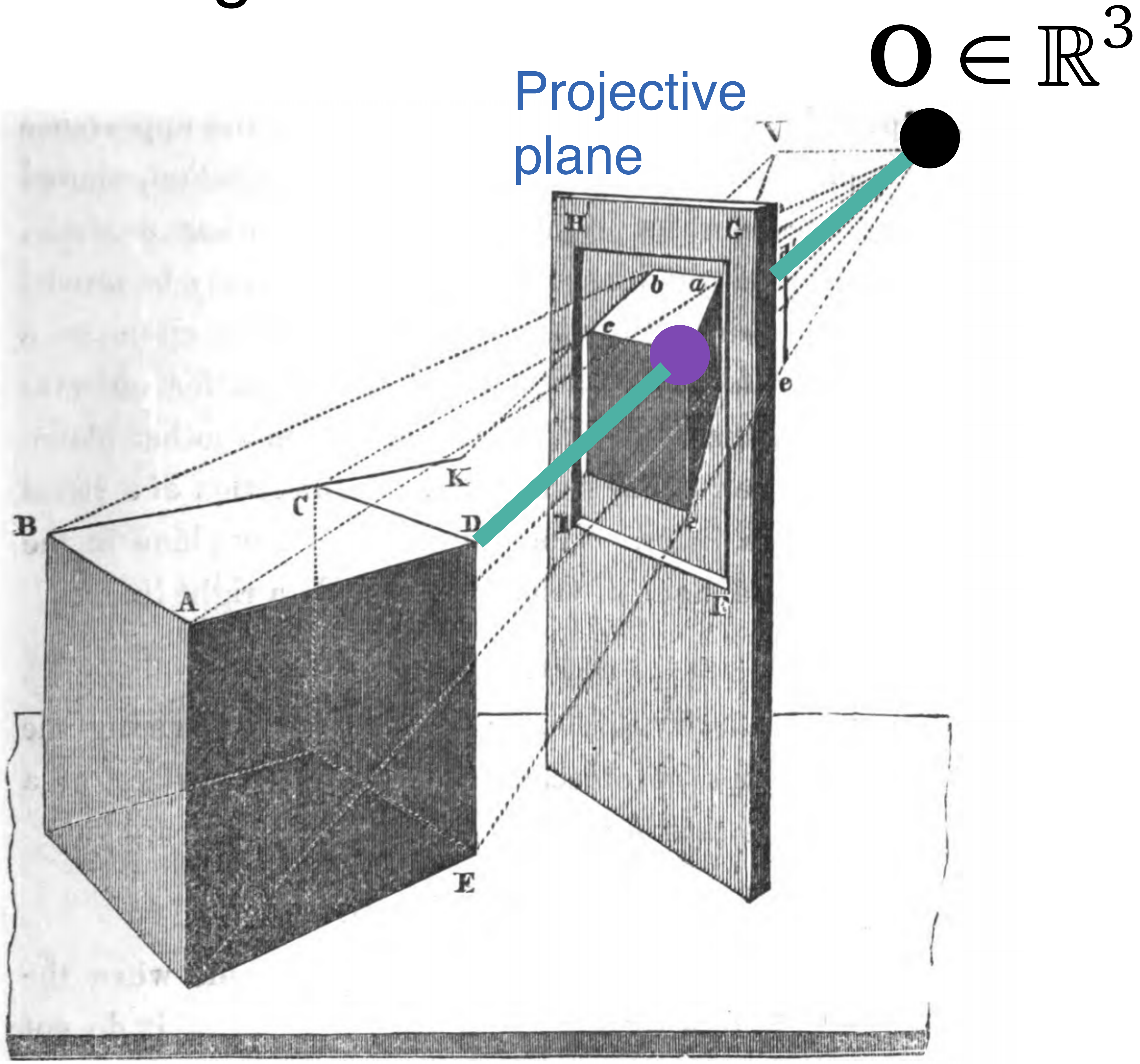

Taylor 1715



Descriptive geometry

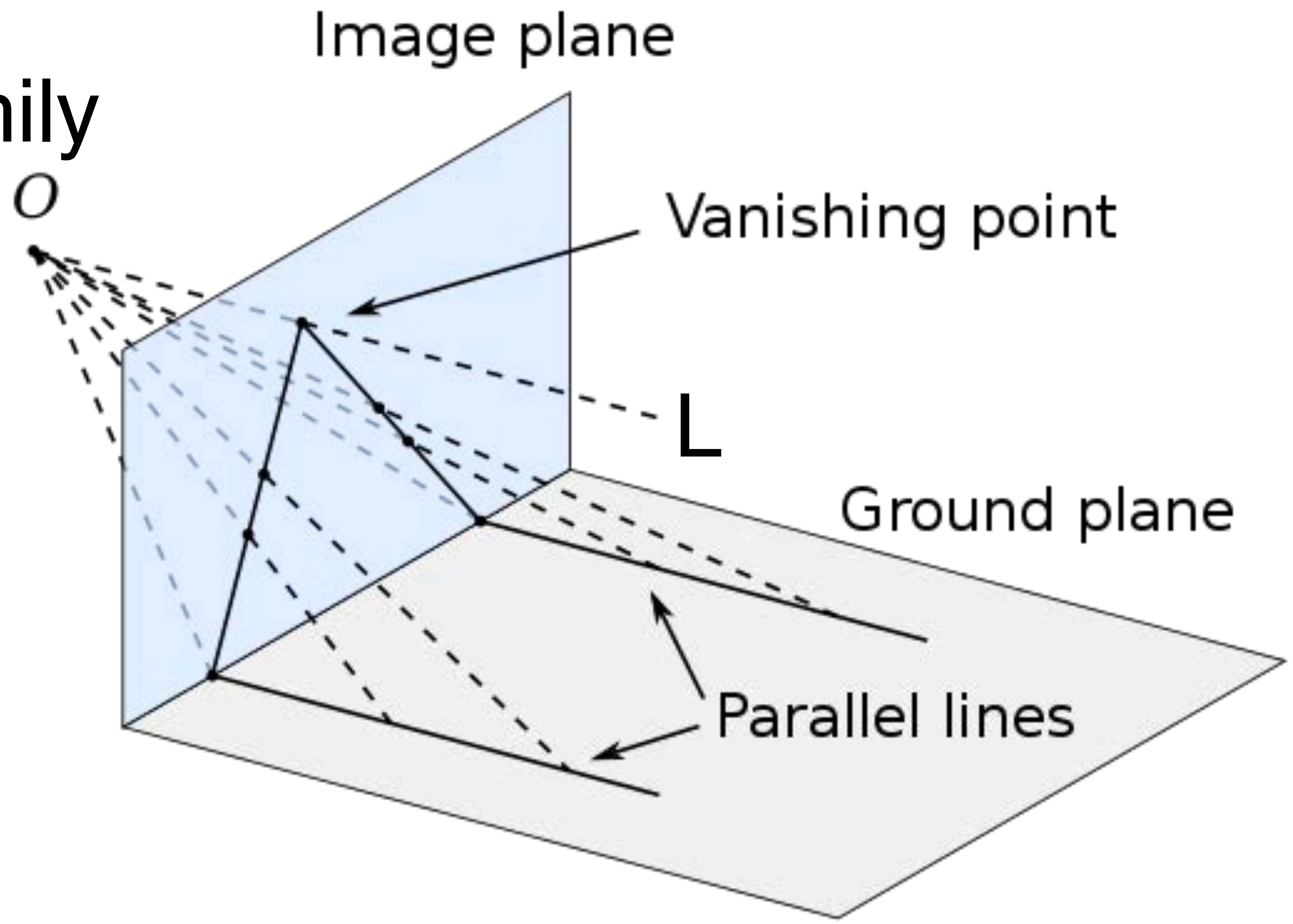
# Taylor's 1715 principles of perspective

- Relation between the 3D object and 2D image
  - Join line from a fixed point (eye) to the 3D object.
  - Intersect the line on the image plane.
  - The image plane is thought of as Renaissance artists' and Desargues' projective plane where geometry of straight edges take place.
  - Each **point** on the image plane is a line in 3D through the origin (eye).
  - A line in 3D through the origin is a **3-proportion**.



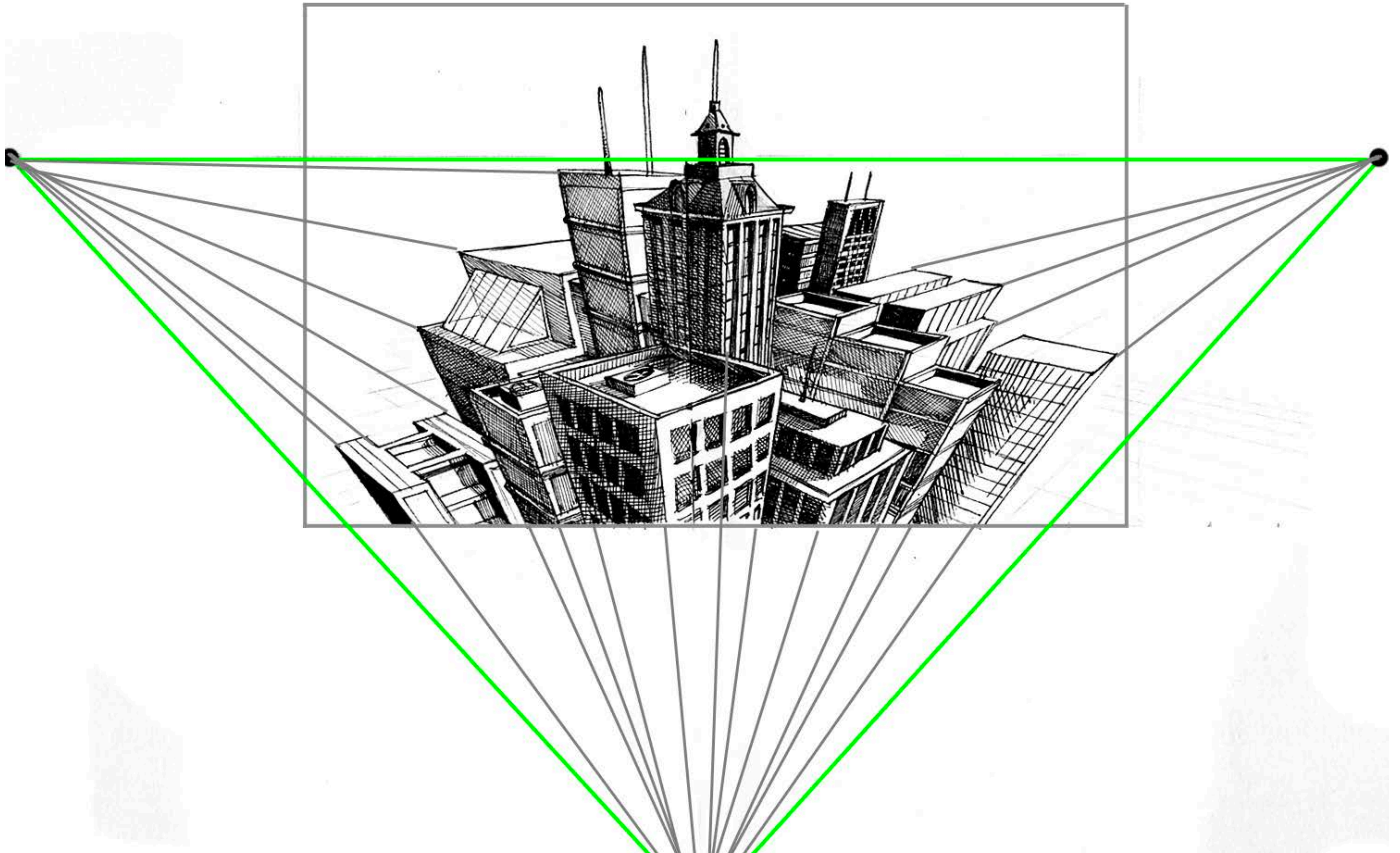
# Vanishing point of parallel lines

- Consider family of parallel lines in 3D
  - There is exactly one line L in the family that goes through the eye (O)
  - The intersection of L and the image plane is the **vanishing point** of the parallel lines
  - Every line in this family of parallel lines corresponds to a line in the image plane that passes through the vanishing point.



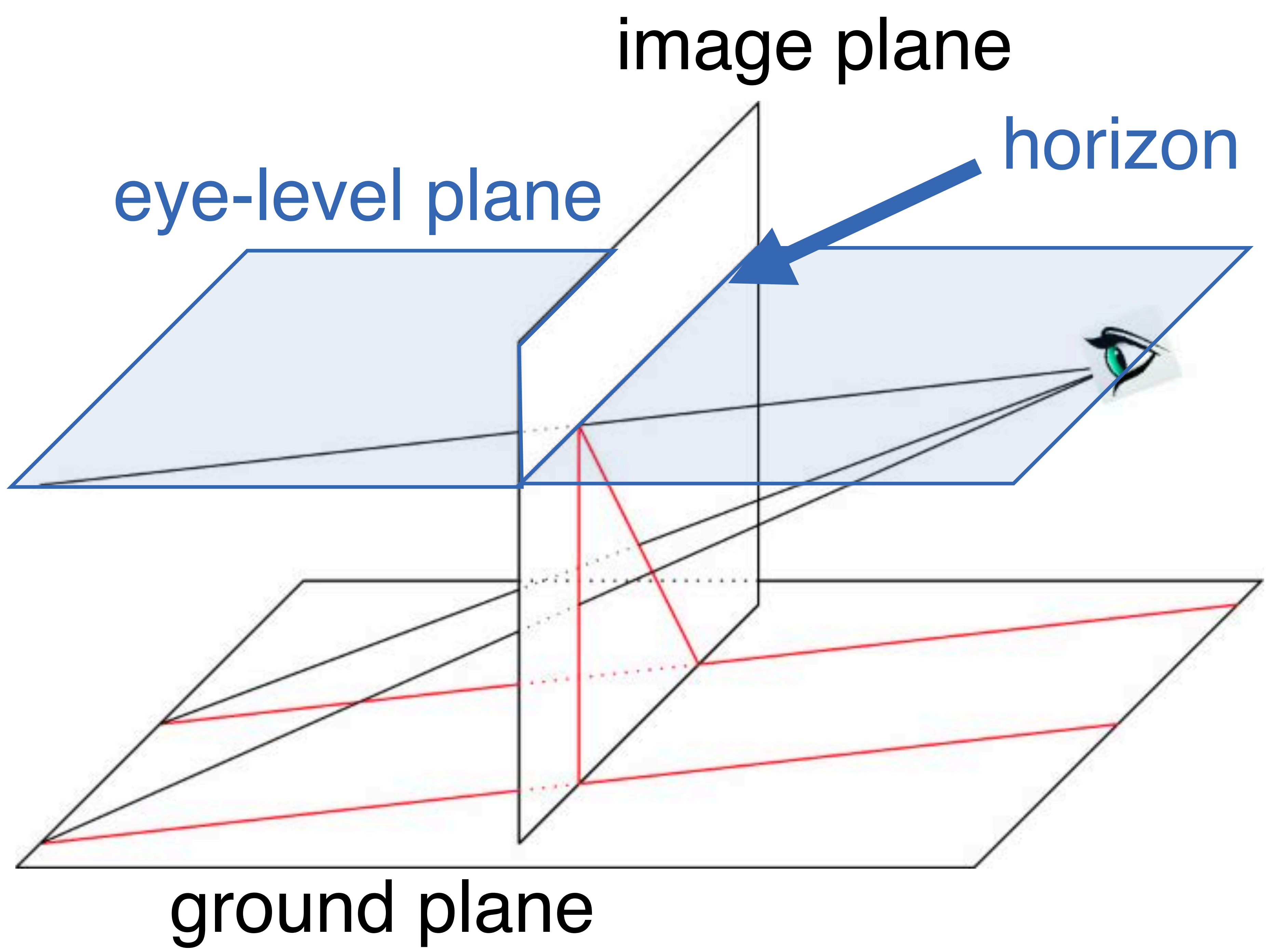
# Vanishing point of parallel lines

- Most scenery of cities or human made objects have 3 families of parallel lines, leading to 3-(vanishing)-point perspective

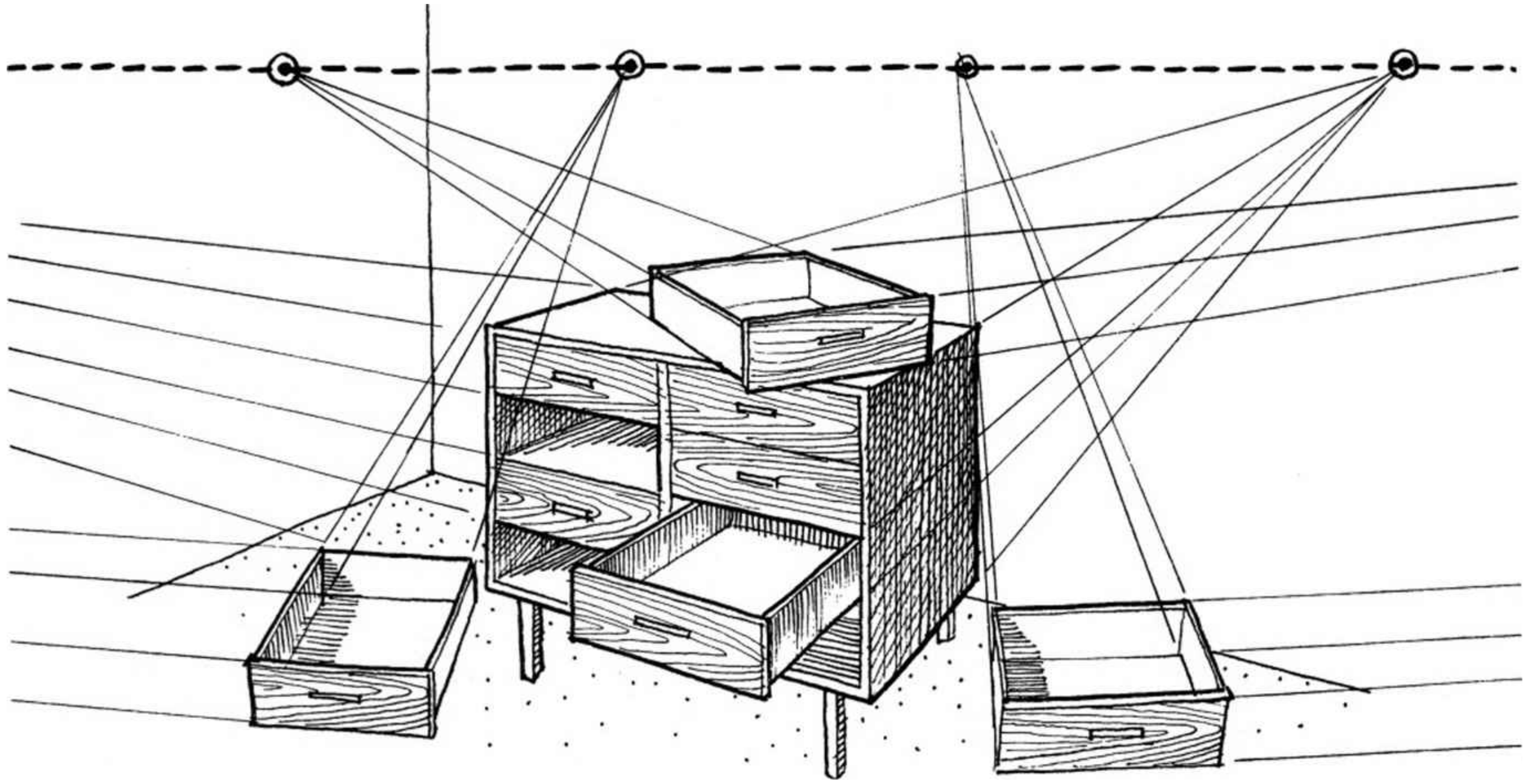


# Horizon of a plane

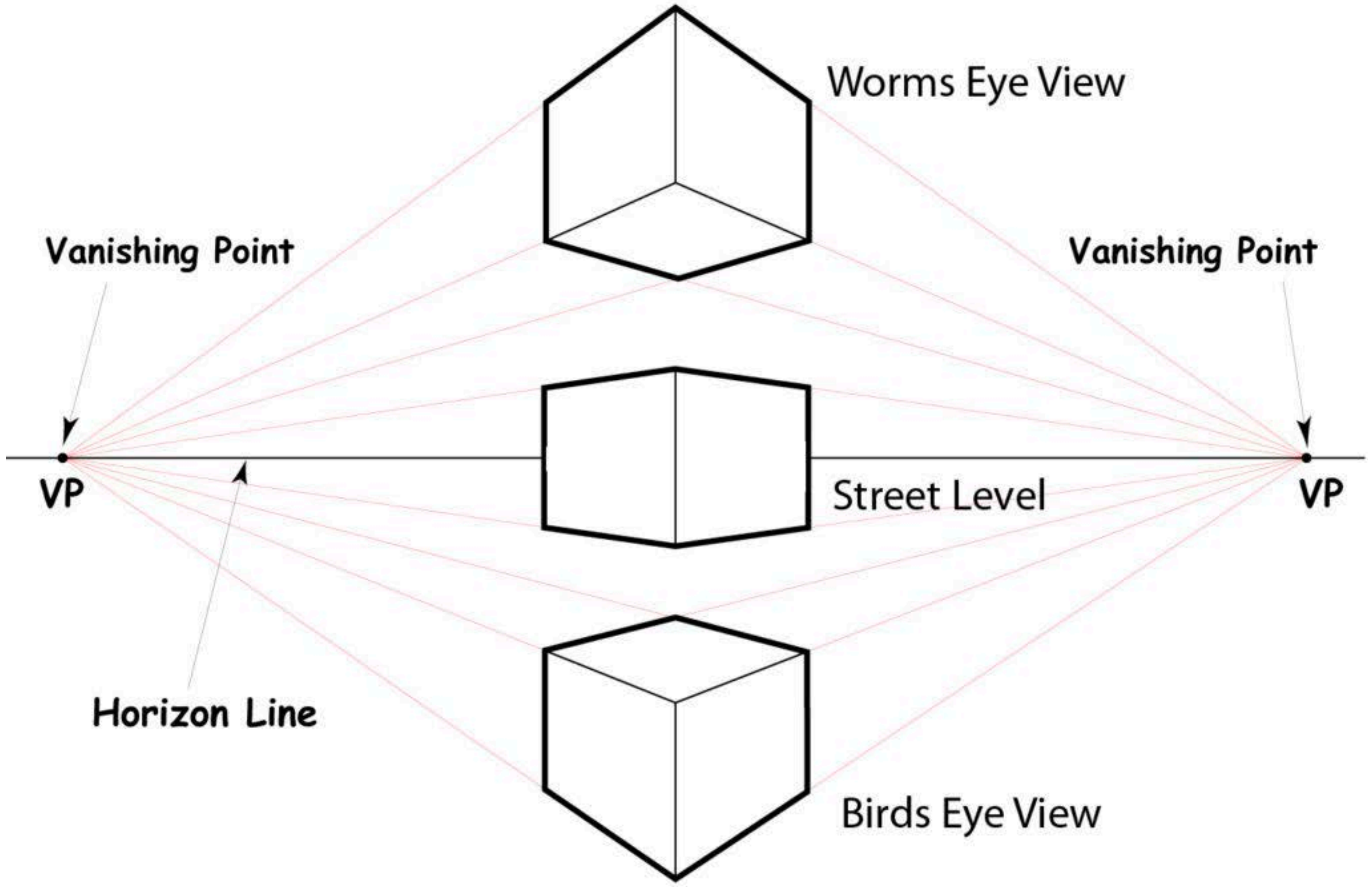
- Consider a plane in 3D (e.g. the ground)
  - There is exactly one plane parallel to this ground that passes through the eye called the **eye-level plane**
  - The intersection of the eye-level and the image plane is the **horizon**
  - Parallel lines in the ground have vanishing point on the horizon



# Vanishing points and horizon



Illlustration: Joshua Nava Art



# Development of the mathematical tool

- **1400s**: Brunelleschi, Alberti, da Vinci, …

  Principles of lights and drawing

- **1600s**: Desargues

  Projective plane: all pairs of line can meet

- **1700s**: Taylor, Monge

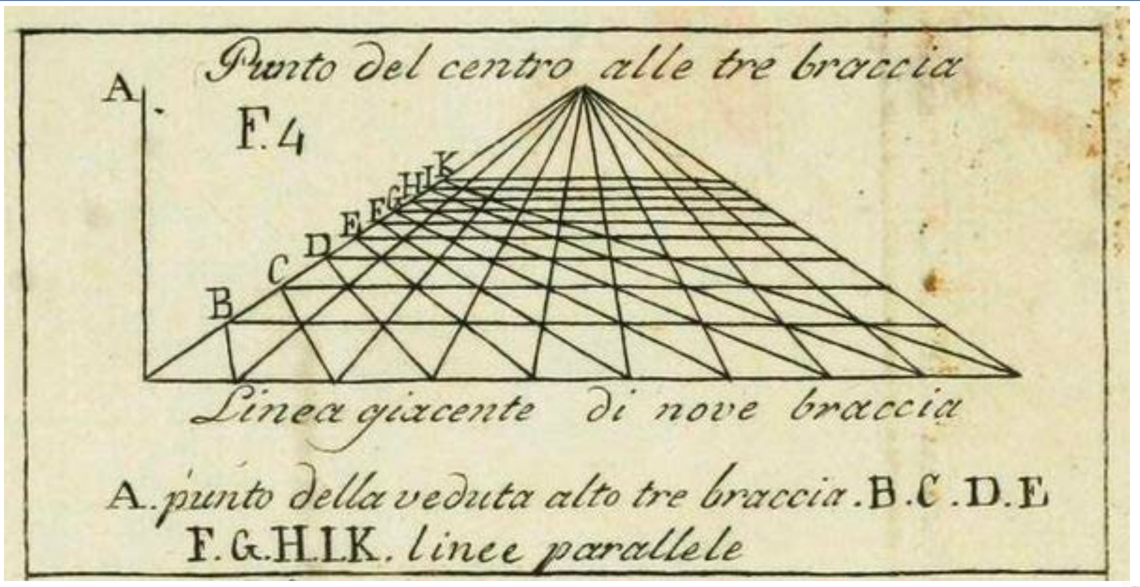
  3D representation. Descriptive geometry

- **1800s**: Möbius, Cayley, …

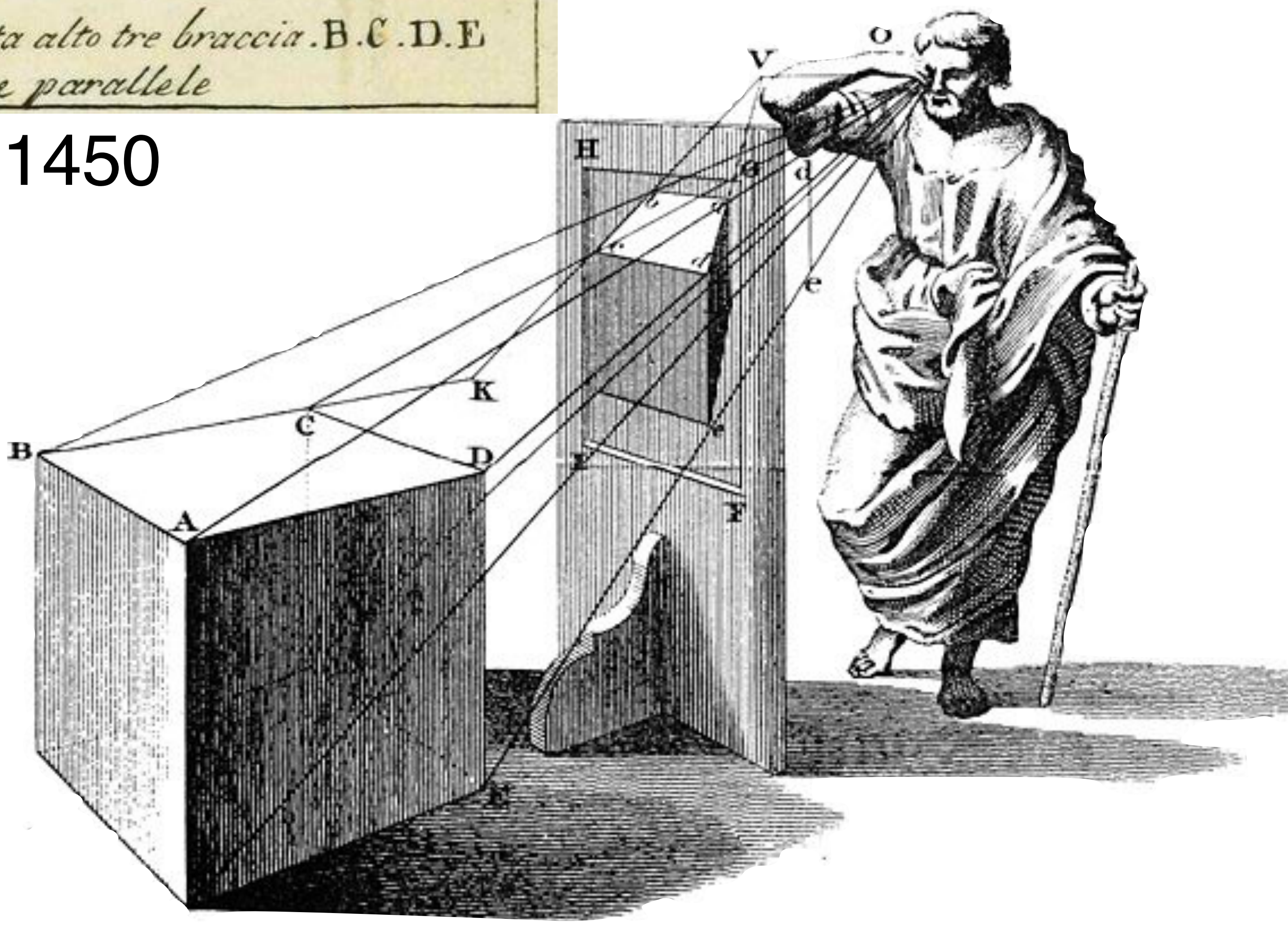
  Homogeneous coordinates, matrices leading to development of linear algebra

- **20th/21st century**

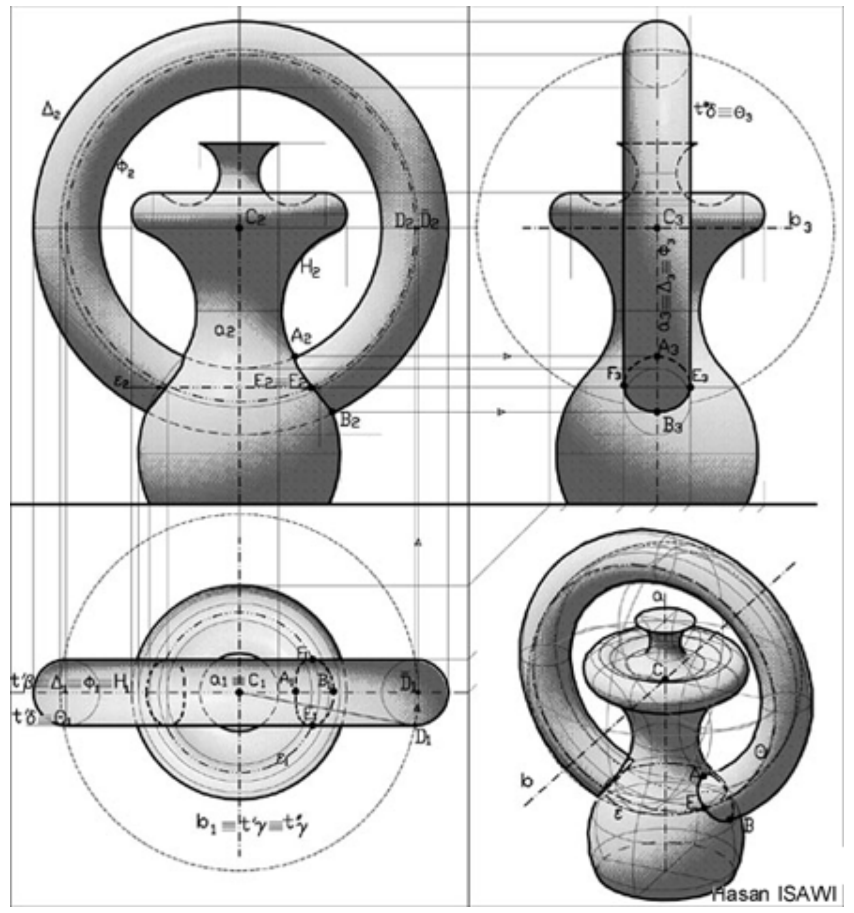
  Rarely mentioned outside of CS




Alberti 1450



Taylor 1715



Descriptive geometry

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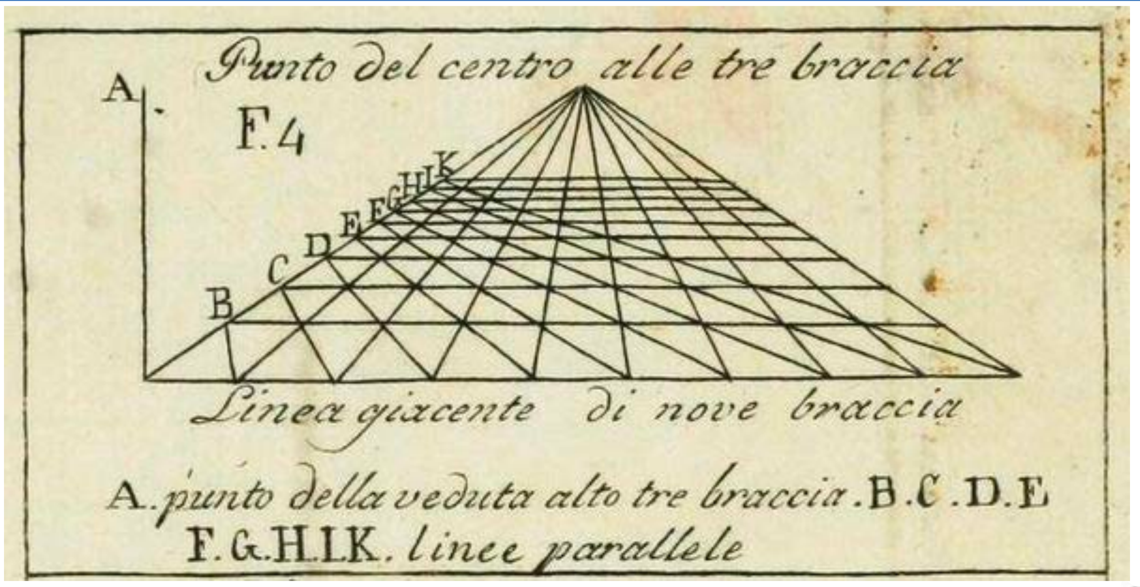

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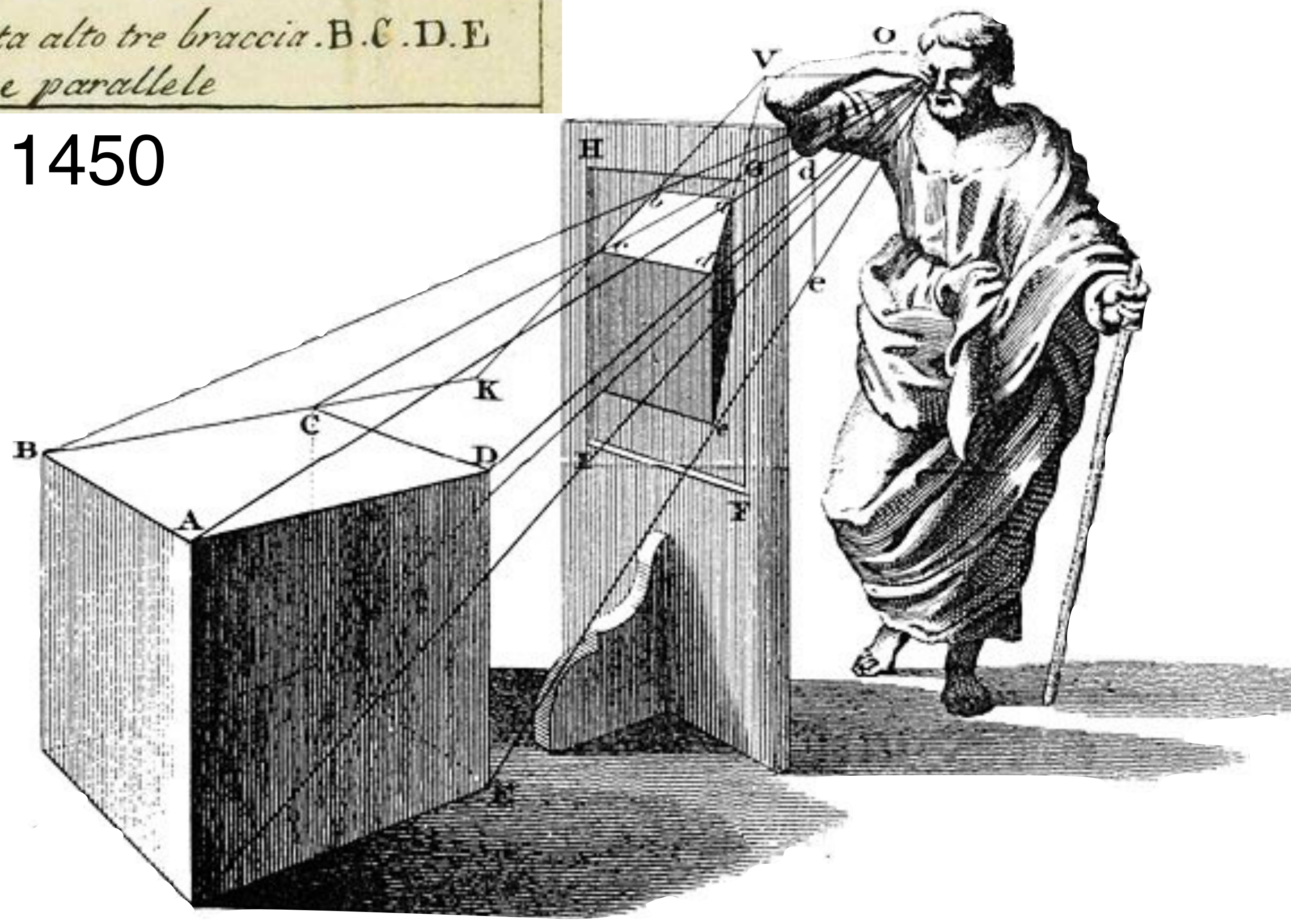
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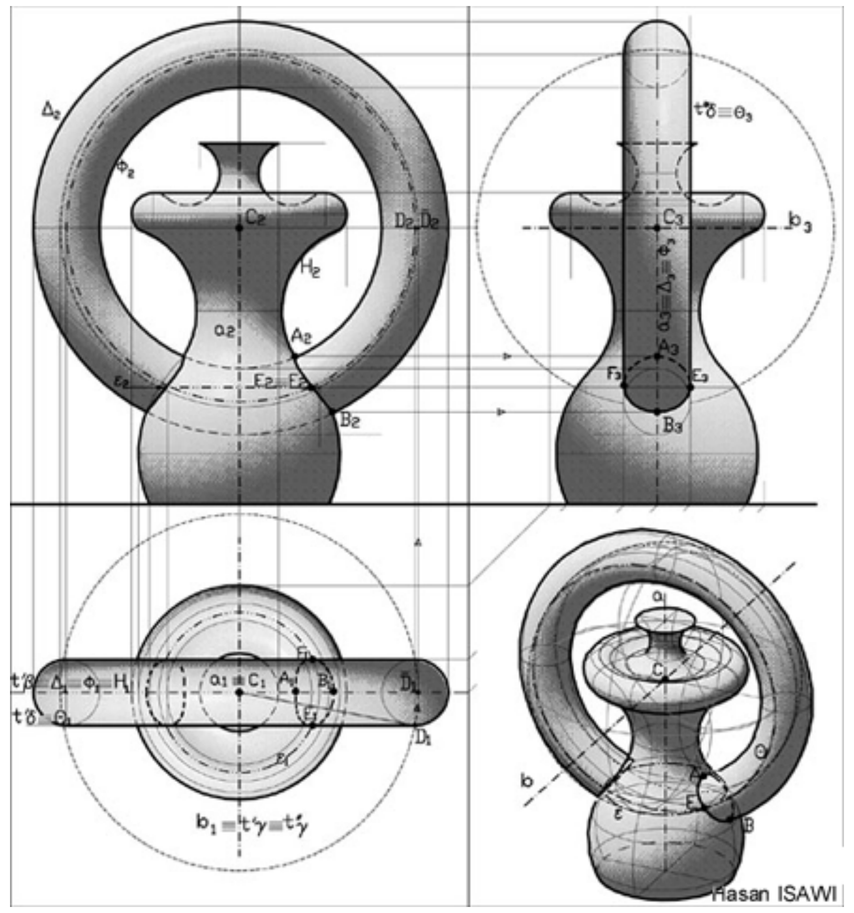
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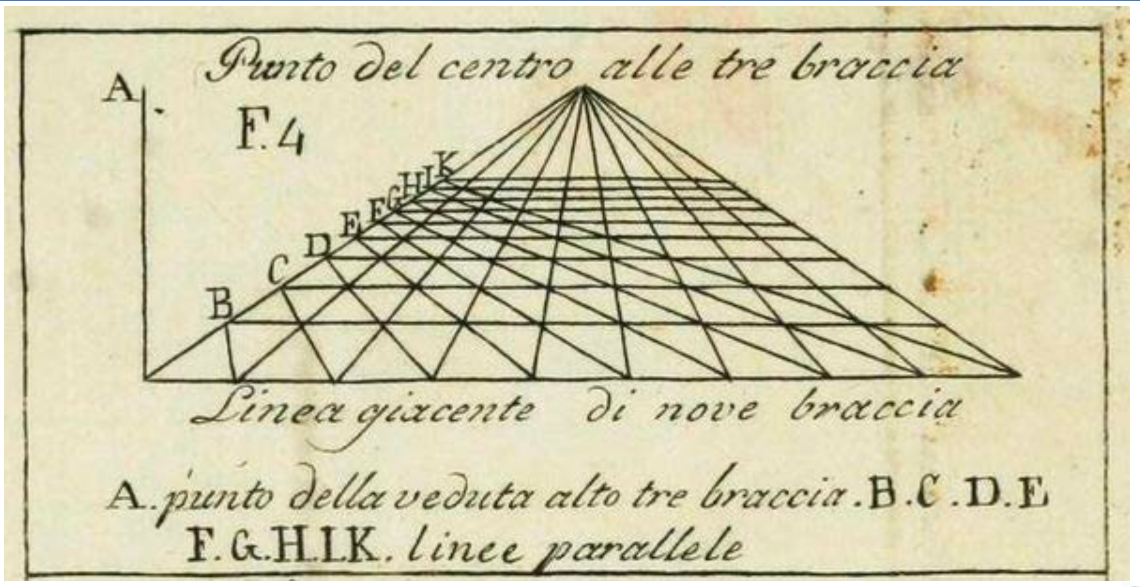

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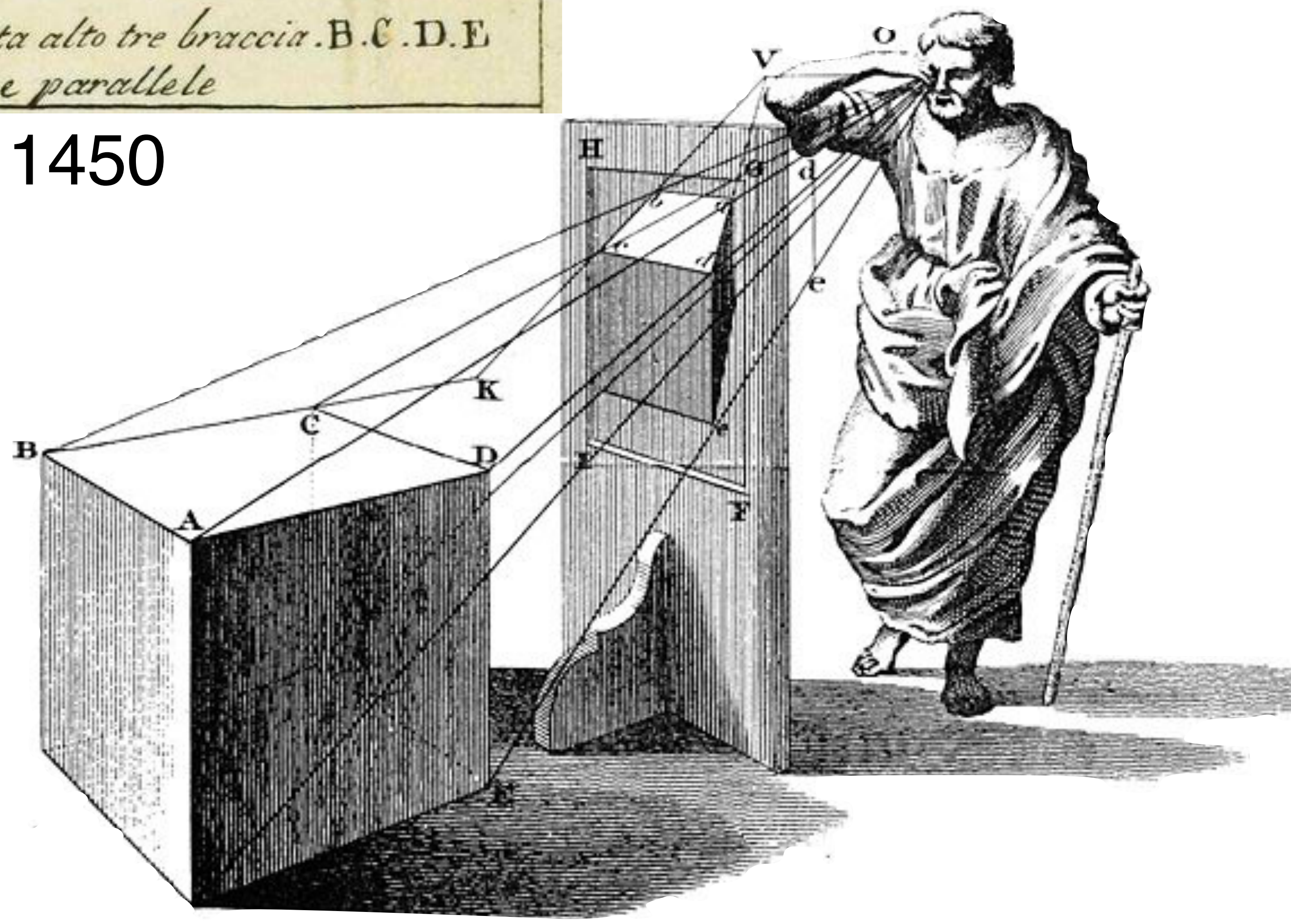
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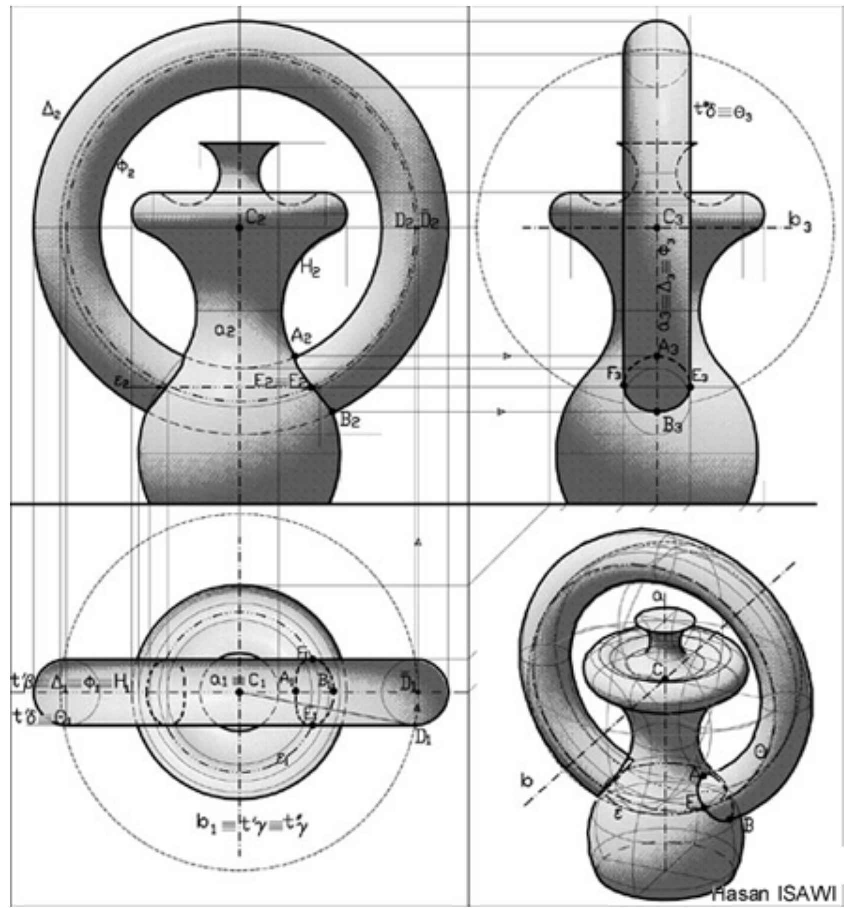
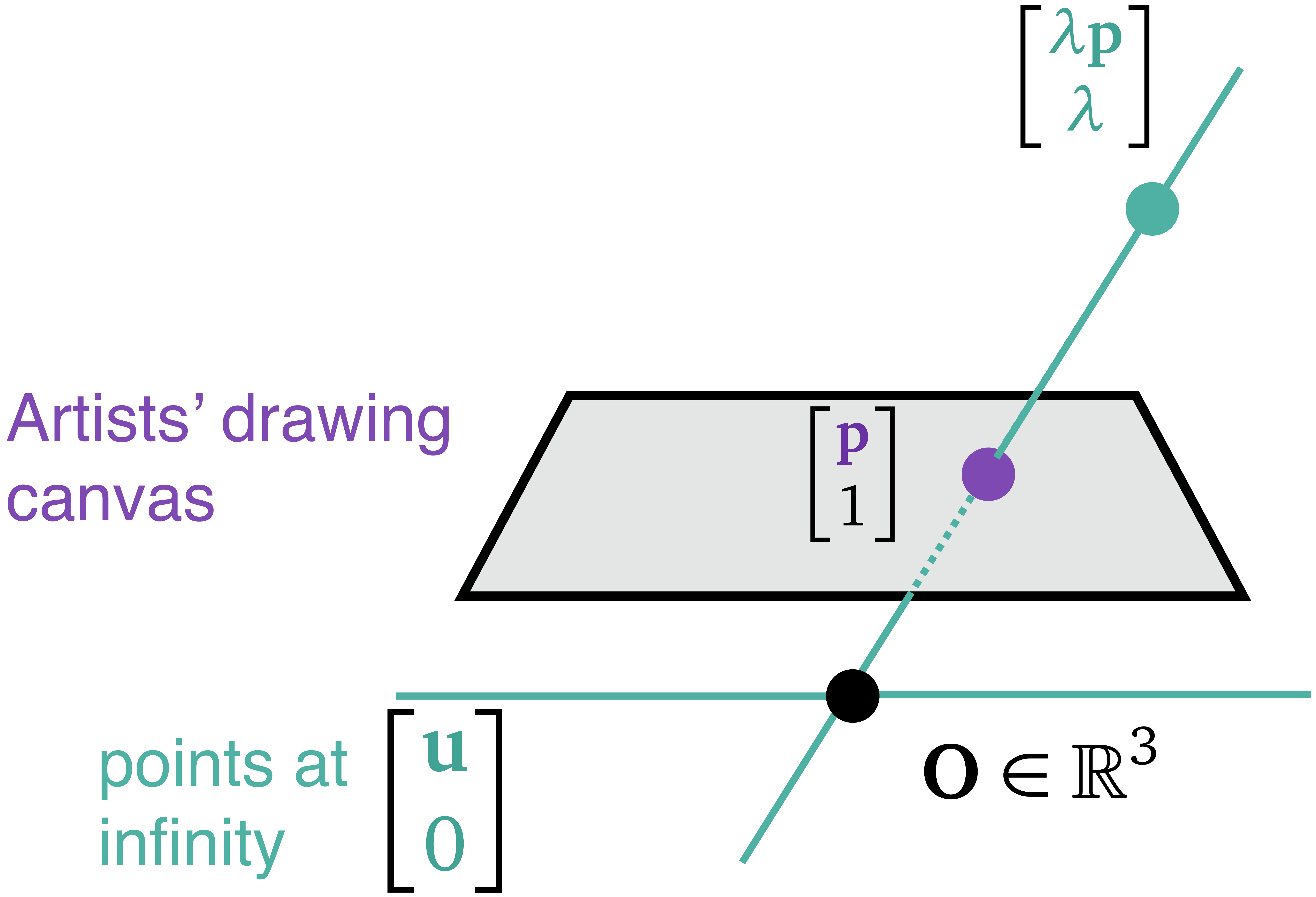
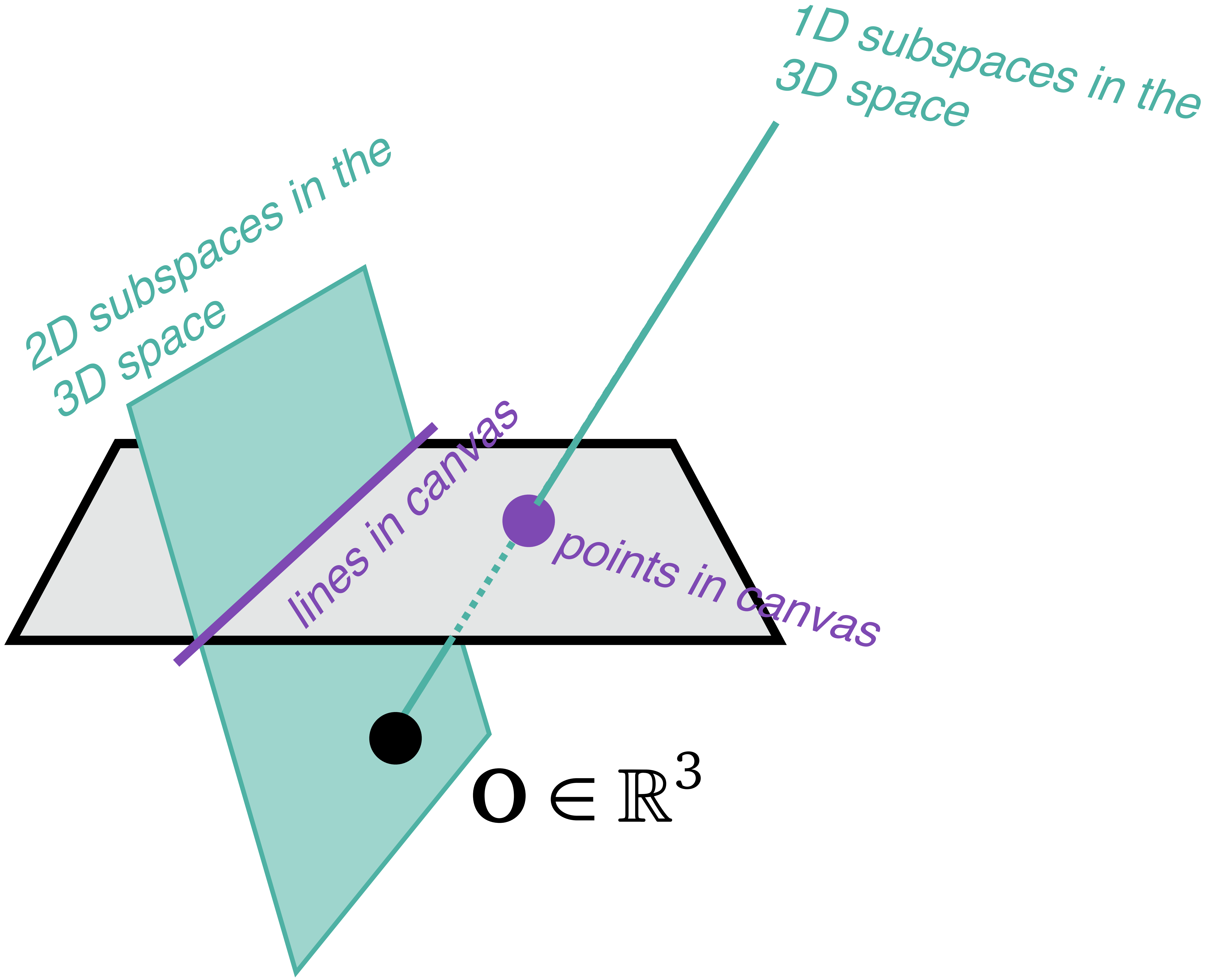

Descriptive geometry

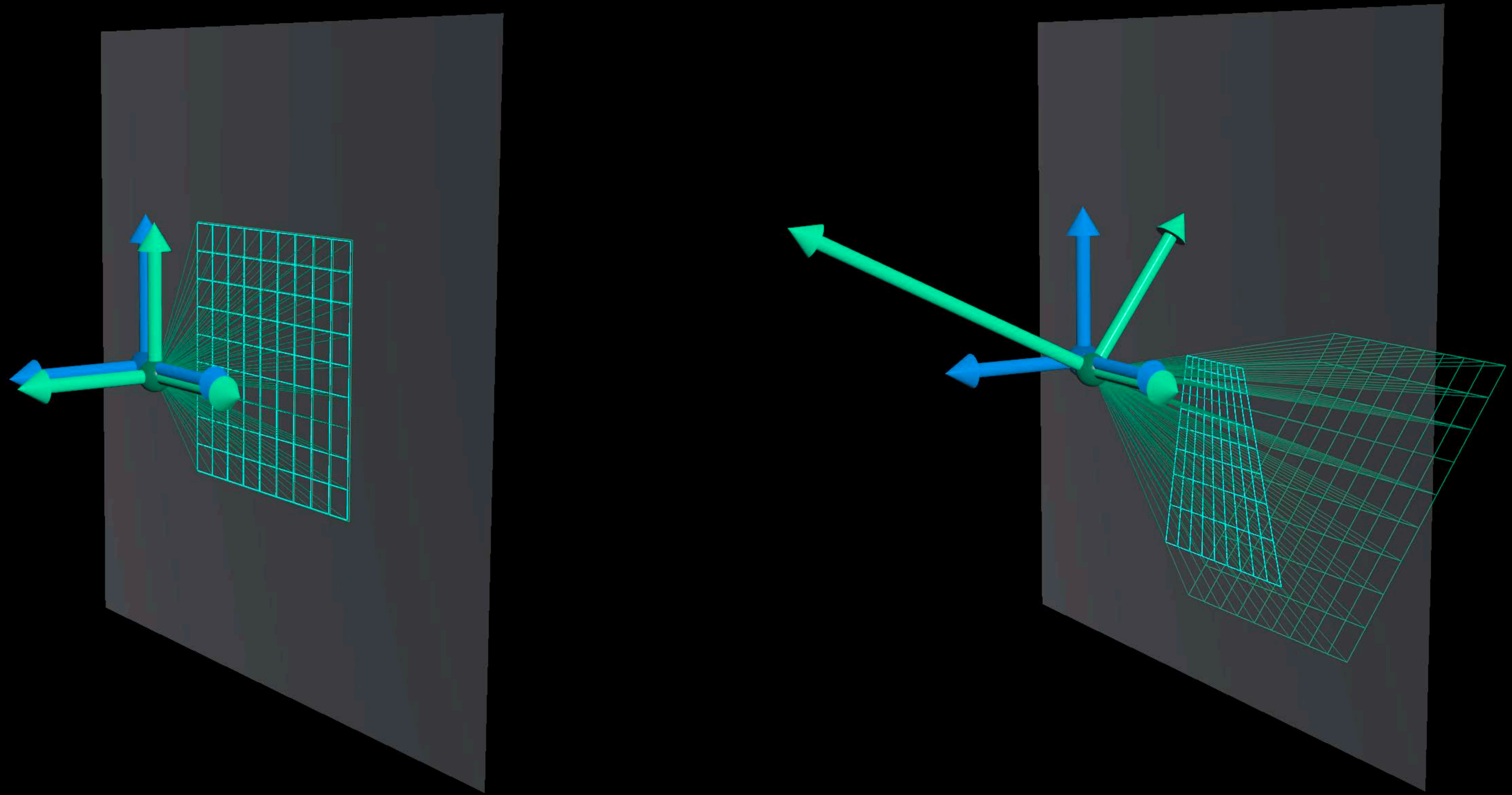
# Between 2D position & 3D position



# Between 2D position & 3D position



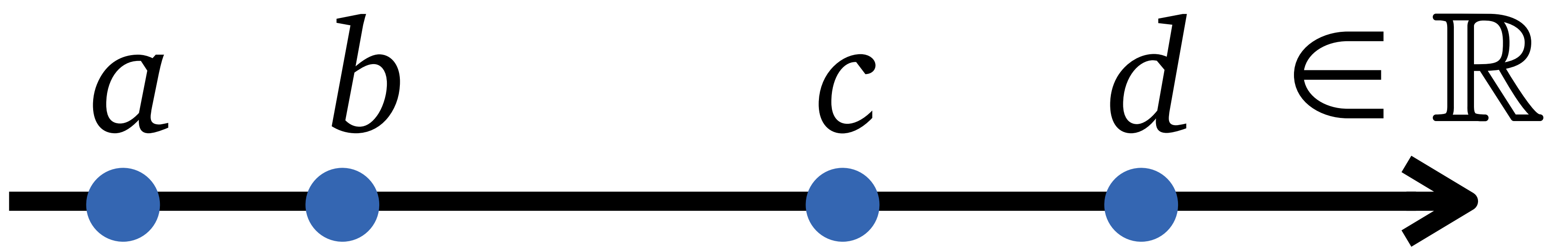
# Linear transforms in hom. coord.



# Appendix: Proof of projective invariance of cross ratio

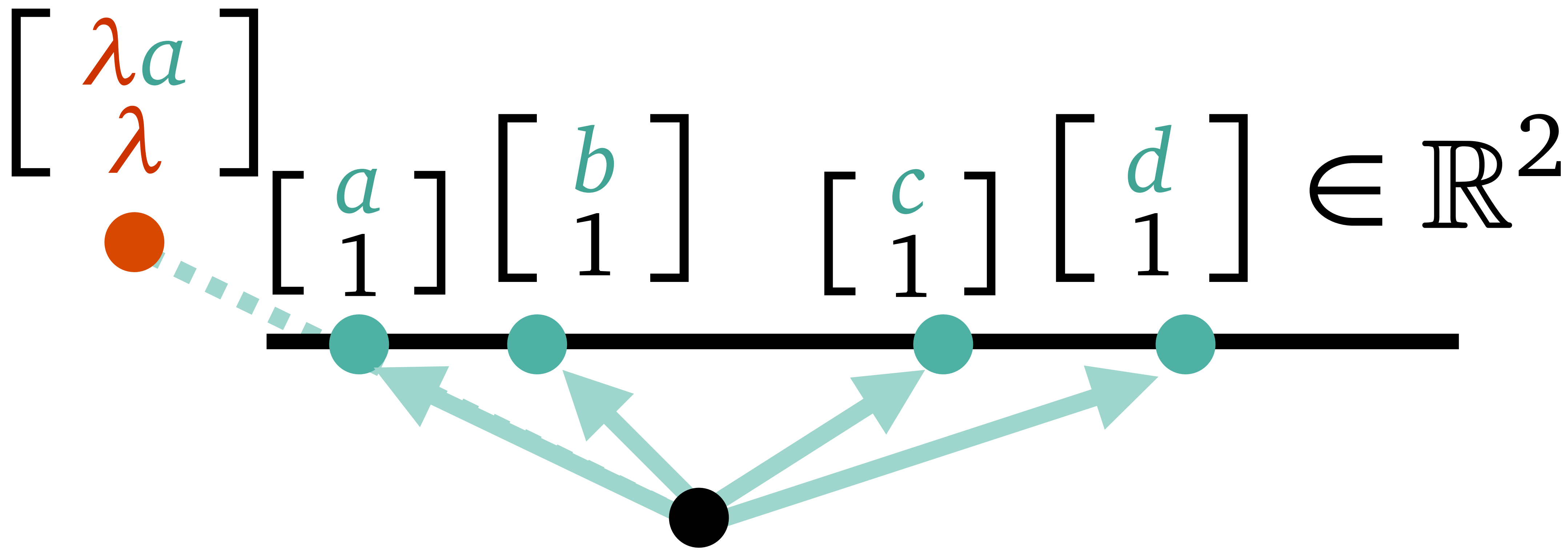
# Proof of proj. invariance of cross ratio

- On 1D, the **cross ratio** of 4 points




$$\mathrm{Cr}(a,b,c,d) = \frac{(b-a)(d-c)}{(c-b)(a-d)}$$

- Lift it to homogeneous coordinate




$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

(2x2 determinants)

- This expression is consistent even if we scale individual point

$$\frac{\begin{vmatrix} b & \lambda a \\ 1 & \lambda \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} \lambda a & d \\ \lambda & 1 \end{vmatrix}} = \frac{\lambda\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\lambda\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

# Proof of proj. invariance of cross ratio

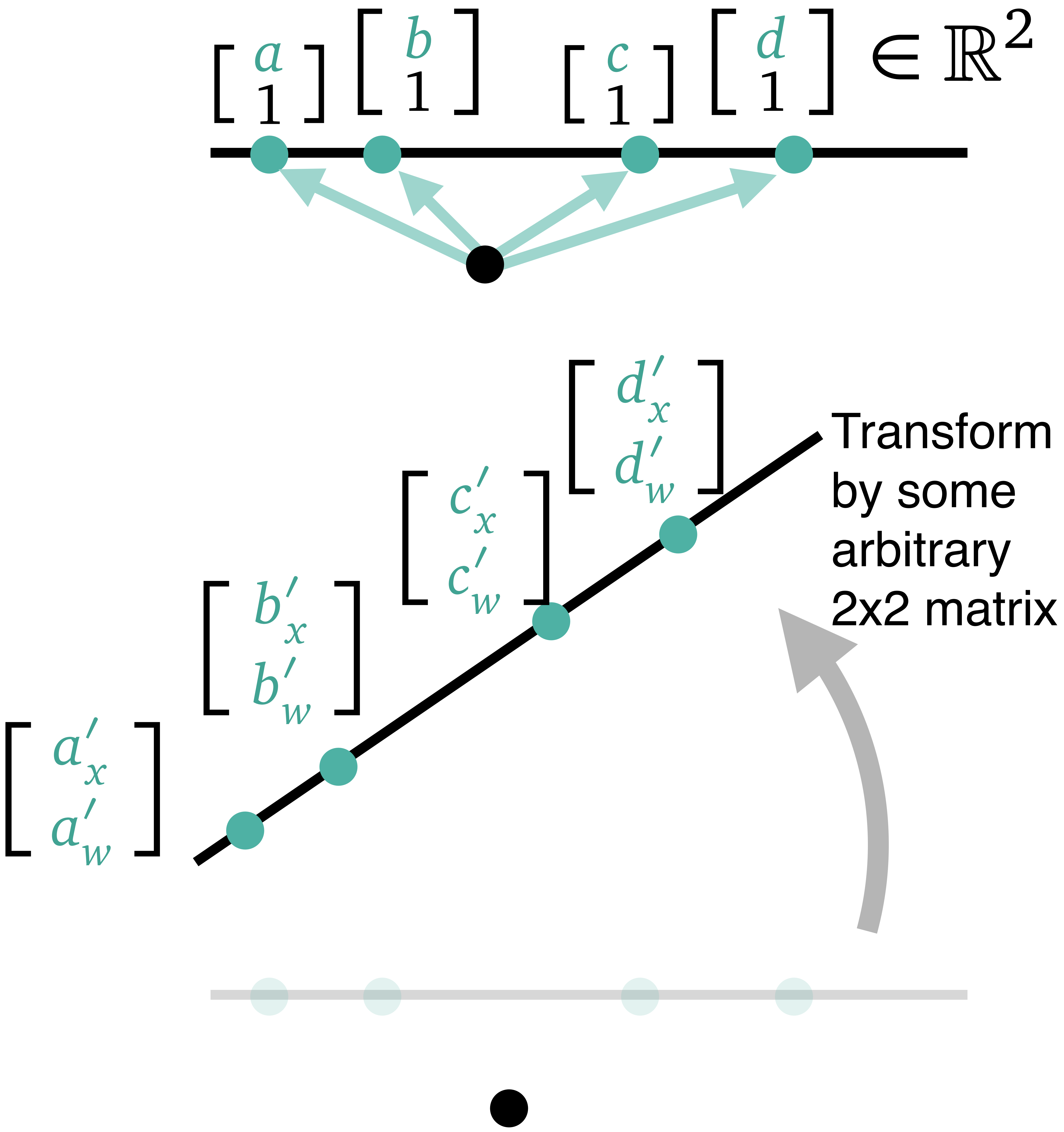
- On 1D, the **cross ratio** of 4 points

$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

- Consider arbitrary transformation

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = \underbrace{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = [\mathbf{M}] \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$



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- On 1D, the **cross ratio** of 4 points

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- Consider arbitrary transformation

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = [\mathbf{M}]\begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- We can slice/swizzle the channels to get, for example,

$$\begin{bmatrix} b'_x & a'_x \\ b'_w & a'_w \end{bmatrix} = [\mathbf{M}]\begin{bmatrix} b & a \\ 1 & 1 \end{bmatrix} \implies \begin{vmatrix} b'_x & a'_x \\ b'_w & a'_w \end{vmatrix} = \underbrace{|\mathbf{M}|}_{=:m}\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}$$

- Fact: determinant distributes over matrix multiplication

# Proof of proj. invariance of cross ratio

$$\mathrm{Cr}(a,b,c,d) = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}}$$

- Consider arbitrary transformation

$$\begin{bmatrix} a'_x & b'_x & c'_x & d'_x \\ a'_w & b'_w & c'_w & d'_w \end{bmatrix} = [\mathbf{M}]\begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix} \implies \begin{vmatrix} b'_x & a'_x \\ b'_w & a'_w \end{vmatrix} = \underbrace{|\mathbf{M}|}_{=:m}\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix} \text{ etc}$$

- The new cross ratio is

$$\mathrm{Cr}' = \frac{\begin{vmatrix} b'_x & a'_x \\ b'_w & a'_w \end{vmatrix}\begin{vmatrix} d'_x & c'_x \\ d'_w & c'_w \end{vmatrix}}{\begin{vmatrix} c'_x & b'_x \\ c'_w & b'_w \end{vmatrix}\begin{vmatrix} a'_x & d'_x \\ a'_w & d'_w \end{vmatrix}} = \frac{m\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}m\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{m\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}m\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} b & a \\ 1 & 1 \end{vmatrix}\begin{vmatrix} d & c \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} c & b \\ 1 & 1 \end{vmatrix}\begin{vmatrix} a & d \\ 1 & 1 \end{vmatrix}} = \mathrm{Cr}$$

QED