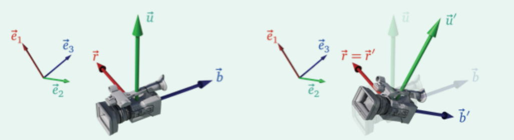
# exer3

**CSE 167 (FA 2022) Exercise 3 — Due 10/12/2022**

**Exercise 3.1 — 2 pts.** Consider an object (such as the camera shown in the figure below), and consider an orthonormal basis $(\vec{r}, \vec{u}, \vec{b})$, where the "right vector" $\vec{r}$ points from left to right of the object, the "up vector" $\vec{u}$ points from bottom to top of the object, and the "back vector" $\vec{b}$ points from front to back of the object.

The rotation operator $R^{\vec{r},\theta}$ **rotates** this object and the orthonormal basis $(\vec{r}, \vec{u}, \vec{b})$ **by angle** $\theta$ **about the right vector** $\vec{r}$, producing a new orthonormal basis $(\vec{r}', \vec{u}', \vec{b}')$. In other words $R^{\vec{r},\theta}\vec{r} = \vec{r}'$, $R^{\vec{r},\theta}\vec{u} = \vec{u}'$, $R^{\vec{r},\theta}\vec{b} = \vec{b}'$.




**(a)** What is the rotation matrix $\mathbf{R}$ for the rotation operator $R^{\vec{r},\theta}$ represented under the basis $(\vec{r}, \vec{u}, \vec{b})$?

$$\begin{bmatrix} R^{\vec{r},\theta}\vec{r} & R^{\vec{r},\theta}\vec{u} & R^{\vec{r},\theta}\vec{b} \end{bmatrix} = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix} \begin{bmatrix} & & \\ & \mathbf{R} & \\ & & \end{bmatrix} \tag{1}$$

Write down the matrix $\mathbf{R}$ in terms of $\theta$ and trigonometry functions.

**(b)** Suppose there is a "world" orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Suppose $\vec{r}, \vec{u}, \vec{b}$ have coefficients $\mathbf{r}, \mathbf{u}, \mathbf{b}$ under this world basis; *i.e.*,

$$\vec{r} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{2}$$

Let $\vec{p} = \begin{bmatrix} \vec{r}' & \vec{u}' & \vec{b}' \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ be an arbitrary point on the rotated object (hence naturally expressed using the basis $(\vec{r}', \vec{u}', \vec{b}')$). If we represent the same vector $\vec{p}$ in terms of the world basis, obviously we get a different list of coefficients $\vec{p} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$.

$\vec{u}' = \cos(\theta)\hat{u} - \sin\theta[\vec{r} \times \hat{u}]$

$\vec{b}' = \cos(\theta)\vec{b} - \sin\theta[\vec{r} \times \vec{b}]$

**(a)** $R^{\vec{r},\theta}_{\vec{r}} = \vec{r}' = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{r}$

$R^{\vec{u},\theta}_{\vec{u}} = \vec{u}' = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix} \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$

$R^{\vec{b},\theta}_{b} = \vec{b}' = \begin{bmatrix} \vec{r} & \vec{u} & \vec{b} \end{bmatrix} \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \vec{r}' & \vec{u}' & \vec{b}' \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$$\Rightarrow R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Find the transformation matrix **M** such that

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} & \mathbf{M} & \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}. \tag{3}$$

**Hint** *You can use the result of (a).*

**Exercise 3.2 — 2 pts.** What is the quaternion $q \in \mathbb{H}$ so that

$$q\mathbb{i}\bar{q} = \mathbb{j}, \quad q\mathbb{j}\bar{q} = \mathbb{k}, \quad q\mathbb{k}\bar{q} = \mathbb{i}, \quad \mathrm{Re}(q) > 0? \tag{4}$$

**Hint** *Think of some 3D rotation and its axis and angle.*

**(b)** $\vec{r}' = [\vec{r} \;\; \vec{u} \;\; \vec{b}] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{u}' = [\vec{r} \;\; \vec{u} \;\; \vec{b}] \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$

$\vec{b}' = [\vec{r} \;\; \vec{u} \;\; \vec{b}] \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$

$$\vec{p} = [\vec{r}' \;\; \vec{u}' \;\; \vec{b}'] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}}_{M} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$



**(3.2)** if $q i \bar{q} = j$ $\quad q j \bar{q} = k$

$q k \bar{q} = i$ $\quad Re(q)$ ?

$i^2 = j^2 = k^2 = ijk = \boxed{-1}$

$ijk$ space $= Re(q)$

$\Rightarrow i^2 + j^2 + k^2 = 1$

$i = jk = -kj$

$i = kj = -jk$

$k = ij = -ji$

$$q = \cos(\theta) + \sin(\theta)[i \ j \ k]$$

$$q = a + bi + cj + dk$$

$$\bar{q} = a - bi - cj - dk$$

where; $a, b, c, d$ are coffecie

$$q = 0 + b(qk\bar{q}) + c(qi\bar{q}) + d(qj\bar{q})$$

$$\Rightarrow q(q(qi\bar{q})\bar{q})\bar{q} + q(q(qj\bar{q})\bar{q})\bar{q} + q(q(qk\bar{q})\bar{q})\bar{q}$$

let: $v = \begin{bmatrix} i \\ j \\ k \end{bmatrix}$, then :-

$$q = q(q(qv\bar{q})\bar{q})\bar{q} \ //$$

$\mathrm{Re}(q) = -1$ which is $\neq\ > 0$ so $\underline{\underline{\text{NO}}}$