GEORGIA INSTITUTE OF TECHNOLOGY

SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Fall 2018

Problem Set #3

Assigned: 5-Sep-18
Due Date: 12-Sep-18

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

All assignments will be due at the begining of class on the due date.

PROBLEM 3.1:

Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 3.2:

Let $\|\cdot\|_p$ be the ℓ_p norm for vectors in $\mathbb{R}^N$ as defined on page I.45 of the notes.

1. Prove that for any integer $p \geq 1$

$$\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_1 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^N$$

2. Prove that for *any* $1 \leq q \leq p \leq \infty$

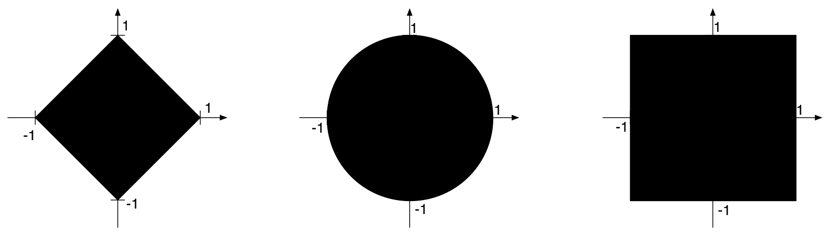
$$\|\boldsymbol{x}\|_p \leq \|\boldsymbol{x}\|_q \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^N.$$

   Hint: It is a fact that if a_1 and a_2 are non-negative real numbers, then

$$(a_1 + a_2)^\alpha \geq a_1^\alpha + a_2^\alpha \quad \text{for } \alpha \geq 1.$$

3. *Optional bonus question*: Prove the fact in the hint above.

PROBLEM 3.3:
One way to visualize a norm in $\mathbb{R}^2$ is by its *unit ball*, the set of all vectors such that $\|\boldsymbol{x}\| \leq 1$. For example, we have seen that the unit balls for the ℓ_1, ℓ_2, and ℓ_∞ norms look like:



$B_1 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_1 \leq 1\}$ $\quad B_2 = \{\boldsymbol{x} : \|\boldsymbol{x}\|_2 \leq 1\}$ $\quad B_\infty = \{\boldsymbol{x} : \|\boldsymbol{x}\|_\infty \leq 1\}$

Given an appropriate subset of the plane, $B \subset \mathbb{R}^2$, it might be possible to define a corresponding norm using

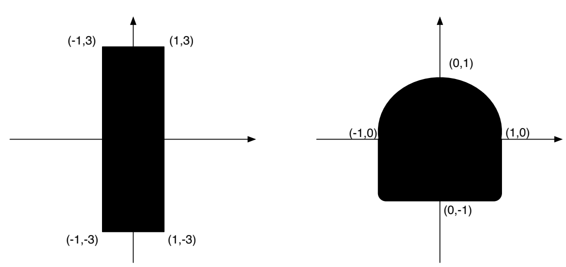
$$\|\boldsymbol{x}\|_B = \text{minimum value } r \geq 0 \text{ such that } \boldsymbol{x} \in rB, \tag{1}$$

where rB is just a scaling of the set B:

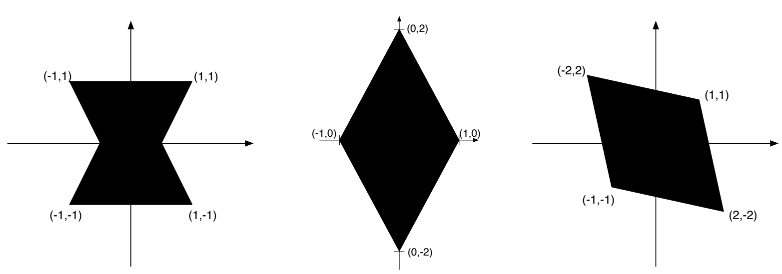
$$\boldsymbol{x} \in B \quad \Rightarrow \quad r \cdot \boldsymbol{x} \in rB.$$

1. Let $\boldsymbol{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. For $p = 1, 2, \infty$, find $r = \|\boldsymbol{x}\|_p$, and sketch $\boldsymbol{x}$ and rB_p (use different axes for each of the three values of p).

2. Consider the 5 shapes below.



B_a $\qquad B_b$



B_c $\qquad B_d$ $\qquad B_e$

Determine the j for which $\|\cdot\|_{B_j}$ is a valid norm. In the cases where $\|\cdot\|_{B_j}$ is *not* a valid norm, explain why. The most convincing way to do this is to find vectors for which one of the three properties of a valid norm are violated.

PROBLEM 3.4:
Below, $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$.

1. Prove that $|\langle \boldsymbol{x}, \boldsymbol{y} \rangle| \leq \|\boldsymbol{x}\|_\infty \cdot \|\boldsymbol{y}\|_1$.

2. Prove that $\|\boldsymbol{x}\|_1 \leq \sqrt{N} \cdot \|\boldsymbol{x}\|_2$. (Hint: Cauchy-Schwarz)

3. Let B_2 be the unit ball for the ℓ_2 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_2} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector x which achieves the maximum. (Hint: Cauchy-Schwarz)

4. Let B_∞ be the unit ball for the ℓ_∞ norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_\infty} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

5. Let B_1 be the unit ball for the ℓ_1 norm in $\mathbb{R}^N$. Fill in the right hand side below with an expression that depends only on $\boldsymbol{y}$:

$$\max_{\boldsymbol{x} \in B_1} \langle \boldsymbol{x}, \boldsymbol{y} \rangle \quad = \quad ????$$

   Describe the vector $\boldsymbol{x}$ which achieves the maximum. (Hint: Part (a))

(These last two might require some thought. If you solve them for $N = 2$, it should be easy to generalize.)

PROBLEM 3.5:
In this problem, we will develop the computational framework for approximating a continuous-time signal on $[0,1]$ using scaled and shifted version of the classic bell-curve bump:

$$\phi(t) = e^{-t^2}.$$

Fix an integer $N > 0$ and define $\phi_k(t)$ as

$$\phi_k(t) = \phi\left(\frac{t-(k-1/2)/N}{1/N}\right) = \phi\left(Nt - k + 1/2\right)$$

for $k = 1, 2, \ldots, N$. The $\{\phi_k(t)\}$ are a basis for the subspace

$$T_N = \operatorname{span}\left\{\phi_k(t)\right\}_{k=1}^{N}.$$

1. For a fixed value of N, we can plot all of the $\phi_k(t)$ on the same set of axes in MATLAB using:

   ```
   phi = @(z) exp(-z.^2);
   t = linspace(0, 1, 1000);
   figure(1); clf
   hold on
   for kk = 1:N
      plot(t, phi(N*t - kk + 1/2))
   end
   ```

   Do this for $N = 10$ and $N = 25$ and turn in your plots.

2. Since $\{\phi_k(t)\}$ is a basis for T_N, we can write any $y(t) \in T_N$ as

   $$y(t) = \sum_{k=1}^{N} a_k \phi_k(t)$$

   for some set of coefficients $a_1, \ldots, a_N \in \mathbb{R}^N$. If these coefficients are stacked in an N-vector `a` in MATLAB, we can plot $y(t)$ using

   ```
   t = linspace(0,1,1000);
   y = zeros(size(t));
   for jj = 1:N
      y = y + a(jj)*phi(N*t - jj + 1/2);
   end
   plot(t, y)
   ```

   Do this for $N = 4$, and $a_1 = 1, a_2 = -1, a_3 = 1, a_4 = -1$ and turn in your plot.

3. Define the continuous-time signal $x(t)$ on $[0,1]$ as

   $$x(t) = \begin{cases} 4t & 0 \le t < 1/4 \\ -4t + 2 & 1/4 \le t < 1/2 \\ -\sin(20\pi t) & 1/2 \le t \le 1 \end{cases}.$$

Write MATLAB code that finds the closest point $\hat{x}(t)$ in T_N to $x(t)$ for any fixed N. By "closest point", we mean that $\hat{x}(t)$ is the solution to

$$\min_{y \in T_N} \|x(t) - y(t)\|_{L_2([0,1])}.$$

Turn in your code and four plots; one of which has $x(t)$ and $\hat{x}(t)$ plotted on the same set of axes for $N = 5$, and then repeat for $N = 10, 20$, and 50.
Hint: You can create a function pointer for $x(t)$ using

```
x = @(z) (z < 1/4).*(4*z) + (z>=1/4).*(z<1/2).*(-4*z+2) - (z>=1/2).*sin(20*pi*z);
```

and then calculate the continuous-time inner product $\langle x, \phi_k \rangle$ with

```
x_phik = @(z) x(z).*phi(N*z - jj + 1/2);
integral(x_phik, 0, 1)
```

You can use similar code to calculate the entries of the Gram matrix $\langle \phi_j, \phi_k \rangle$. (There is actually a not-that-hard way to calculate the $\langle \phi_j, \phi_k \rangle$ analytically that you can derive if you are feeling industrious — just think about what happens when you convolve a bump with itself.)

PROBLEM 3.6:
You want to design an analog filter that has impulse response

$$h(t) = \begin{cases} 0, & t < 0 \\ 1, & 0 \le t < 1 \\ 0, & t \ge 1. \end{cases}$$

The components you have on hand only allow you implement impulse responses of the form

$$g(t) = \begin{cases} 0, & t < 0, \\ \alpha_1 e^{-t} + \alpha_2 t e^{-t} + \cdots + \alpha_N t^{N-1} e^{-t}, & t \ge 0, \end{cases}$$

where the $\alpha_1, \ldots, \alpha_N$ can be controlled through judicious pole-zero placement. Design optimal filters (i.e. calculate optimal $\{\alpha_k\}$) for $N = 2, 5, 10$. By optimal, we mean

$$\int_0^\infty |h(t) - \hat{h}(t)|^2 \, dt$$

is minimized. Plot your results on the same axes, along with $h(t)$. Turn in any code you use to solve this problem.

PROBLEM 3.7:
OPTIONAL–do not turn in

1. A square $N \times N$ matrix $\boldsymbol{G}$ is invertible if for every $\boldsymbol{y} \in \mathbb{R}^N$ there is *exactly one* $\boldsymbol{x} \in \mathbb{R}^N$ such that $\boldsymbol{G}\boldsymbol{x} = \boldsymbol{y}$. Show that $\boldsymbol{G}$ is invertible if and only if its columns are linearly independent and $\boldsymbol{G}\boldsymbol{x} \neq \mathbf{0}$ for all $\boldsymbol{x} \neq \mathbf{0}$.

2. Let $\psi_1(t), \ldots, \psi_N(t)$ be continuous-time signals on $t \in \mathbb{R}$, and let $\langle \cdot, \cdot \rangle$ be an arbitrary inner product. Show that the $N \times N$ Grammian

$$\boldsymbol{G} = \begin{bmatrix} \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_1 \rangle & \langle \boldsymbol{\psi}_2, \boldsymbol{\psi}_1 \rangle & \cdots & \langle \boldsymbol{\psi}_N, \boldsymbol{\psi}_1 \rangle \\ \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \rangle & \langle \boldsymbol{\psi}_2, \boldsymbol{\psi}_2 \rangle & & \langle \boldsymbol{\psi}_N, \boldsymbol{\psi}_2 \rangle \\ \vdots & & \ddots & \vdots \\ \langle \boldsymbol{\psi}_1, \boldsymbol{\psi}_N \rangle & \cdots & & \langle \boldsymbol{\psi}_N, \boldsymbol{\psi}_N \rangle \end{bmatrix},$$

is invertible if and only if the $\{\boldsymbol{\psi}_n\}$ are linearly independent.