GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Fall 2018
Problem Set #4

Assigned: 12-Sep-18
Due Date: 19-Sep-18

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

All assignments will be due at the begining of class on the due date.

PROBLEM 4.1:
Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 4.2:
The vector space $L_2([0,1]^2)$ is the space of signals of two variables, $x(s,t)$ with $s,t \in [0,1]$ such that

$$\int_0^1 \int_0^1 |x(s,t)|^2 \, \mathrm{d}s \, \mathrm{d}t \;<\; \infty.$$

Let $\{\psi_k(t),\ k \geq 0\}$ be an orthobasis for $L_2([0,1])$. Define

$$v_{k,\ell}(s,t) = \psi_k(s)\,\psi_\ell(t), \quad k,\ell \geq 0.$$

Show that $\{v_{k,\ell}(s,t),\ k,\ell \geq 0\}$ are an orthobasis for $L_2([0,1]^2)$.

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PROBLEM 4.3:
Let $\mathcal{E}$ be the space of signals on $[-1,1]$ that are even:

$$x(t) \in \mathcal{E} \quad \Leftrightarrow \quad x(t) = x(-t), \quad t \in [-1,1],$$

and let $\mathcal{O}$ be the space of signals on $[-1,1]$ that are odd:

$$x(t) \in \mathcal{O} \quad \Leftrightarrow \quad x(t) = -x(-t), \quad t \in [-1,1].$$

1. Given an arbitrary $x(t) \in L_2([-1,1])$ what is the closest even function to $\boldsymbol{x}$? That is, solve

$$\min_{\boldsymbol{y} \in \mathcal{E}} \ \|\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$

   A good way to do this is to use the *orthogonality principle* — we know that for the optimal $\hat{\boldsymbol{y}} \in \mathcal{E}$

$$\langle \boldsymbol{x} - \hat{\boldsymbol{y}}, \boldsymbol{z} \rangle = 0 \quad \text{for all} \quad \boldsymbol{z} \in \mathcal{E}.$$

   You might consider filling in the blanks in the following line of reasoning:

$$\begin{aligned}
\langle \boldsymbol{x} - \hat{\boldsymbol{y}}, \boldsymbol{z} \rangle &= \int_{-1}^{1} [x(t) - \hat{y}(t)] z(t) \ \mathrm{d}t \\
&= \int_{0}^{1} [x(t) - \hat{y}(t)] z(t) + \cdots \ \mathrm{d}t \\
&= \cdots \\
&= 0 \quad \text{for all } \boldsymbol{z} \in \mathcal{E} \text{ when } \hat{y}(t) = \cdots .
\end{aligned}$$

2. Given an arbitrary $x(t) \in L_2([-1,1])$, solve

$$\min_{\boldsymbol{y} \in \mathcal{O}} \ \|\boldsymbol{x} - \boldsymbol{y}\|_2^2.$$

3. Let $\{\phi_k(t), \ k \geq 0\}$ be an orthobasis for $L_2([0,1])$. How can we use this orthobasis on $[0,1]$ to construct an orthobasis $\{\phi_k^e(t), \ k \geq 0\}$ for $\mathcal{E}$? What about an orthobasis $\{\phi_k^o(t), \ k \geq 0\}$ for $\mathcal{O}$? Is $\{\phi_k^e(t), \ k \geq 0\} \cup \{\phi_k^o(t), \ k \geq 0\}$ an orthobasis for all of $L_2([-1,1])$? Why or why not?

PROBLEM 4.4:
Write a MATLAB function `gramschmidt.m` that takes a $N \times K$ matrix $\boldsymbol{A}$ with $N \geq K$ and returns an $N \times K$ matrix $\boldsymbol{Q}$ such that 1) the span of the columns of $\boldsymbol{Q}$ is the same as that for $\boldsymbol{A}$, and 2) the columns of $\boldsymbol{Q}$ are orthogonal and have unit ℓ_2 norm.

Try your code out on the $N \times K = 1000 \times 50$ matrix `A` in `hw04problem4.mat`. Verify that the two conditions hold by checking that $\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q} = \mathbf{I}$ ($\mathbf{I}$ is the $K \times K$ identity matrix) and that the column rank of the $N \times 2K$ matrix $\begin{bmatrix} \boldsymbol{A} & \boldsymbol{Q} \end{bmatrix}$ is equal to K. That is, show that running the command

```
>> rank([A Q])
```

gives 50 back, and that

```
>> max(max(abs(eye(50)-Q'*Q)))
```

is suitably small (less than 10^{-10}, say).