GEORGIA INSTITUTE OF TECHNOLOGY
School of Electrical and Computer Engineering

Quiz #1

Date: September 29, 2017 Course: ECE 6250

Name: ______________________________
Last, First

- Closed book, closed notes, one $8\frac{1}{2}'' \times 11''$ handwritten sheet (front and back) is allowed. There is a fifty (50) minute time limit.
- No calculators. None of the problems require involved calculations.
- There are three problems; subparts are weighted equally. The quiz is graded out of a total of 75 points.
- All work should be performed on the quiz itself. If more space is needed, use the backs of the pages.
- This quiz will be conducted under the rules and guidelines of the Georgia Tech Honor Code and no cheating will be tolerated.
- **Write your final answers in the boxes provided.**

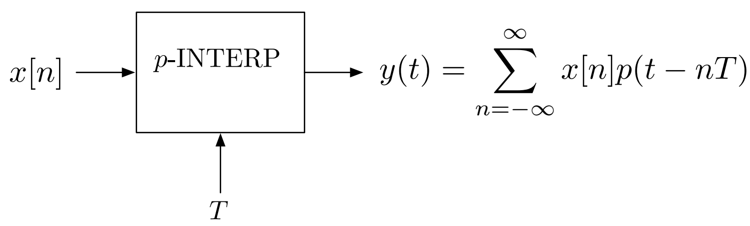
Problem 1 (15 points): Say that the signal $x_c(t)$ is bandlimited in that its Fourier transform

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t)e^{-j\Omega t}\ \mathrm{d}t$$

is zero outside of the interval $[-B, B]$. Suppose that we have taken samples of this signal with spacing T:

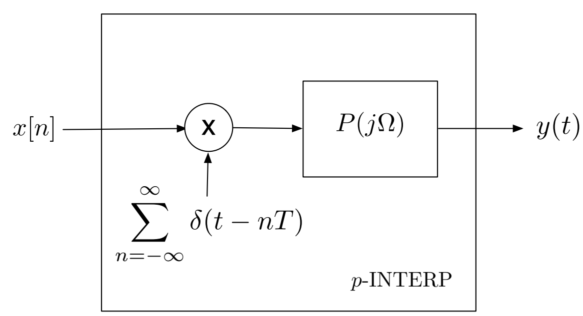
$$x_d[n] = x_c(nT).$$

From these samples, we would like to reconstruct $x_c(t)$. Consider the interpolator, or non-ideal $D \to C$ converter, pictured below:



where $p(t)$ is some fixed, known function. This system converts a discrete sequence of numbers into a continuous-time signal.

Recall that this type of system can be modeled as converting $x[n]$ to a spike train, then passing the result through an analog filter whose frequency response is the continuous-time Fourier transform of $p(t)$:



Suppose that $T = \frac{\pi}{2B}$. In order to ensure that the resulting $y(t)$ will be equal to $x_c(t)$ when given $x_d[n]$ as an input, $P(j\Omega)$ must satisfy certain conditions.

(a) What conditions, if any, must $P(j\Omega)$ satisfy for $|\Omega| \leq \frac{\pi}{2T}$?

(b) What conditions, if any, must $P(j\Omega)$ satisfy for $\frac{\pi}{2T} < |\Omega| \leq \frac{\pi}{T}$?

(c) What conditions, if any, must $P(j\Omega)$ satisfy for $|\Omega| > \frac{\pi}{T}$?

Problem 2 (30 points): Given a 2×2 matrix W, we define

$$\langle x, y \rangle_W = x^T W y \qquad \text{for all} \quad x, y \in \mathbb{R}^2.$$

(a) Let $W = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Set

$$T = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right), \quad \text{and} \quad x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find the closest point $\widehat{x}$ in T to x where the distance is measured by the norm induced by the inner product $\langle \cdot, \cdot \rangle_W$.

$\widehat{x} =$

(b) For each W below, determine whether or not $\langle \cdot, \cdot \rangle_W$ is a valid inner product. For those which are *not* a valid inner product, find specific x and y for which one of the required properties is not satisfied.

(i) $W = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

(ii) $W = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$

(iii) $W = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Problem 3 (30 points):

(a) Consider the following two functions defined on the interval $[-1, 1]$:

$$v_1(t) = t, \quad v_2(t) = t^2.$$

Use the Gram-Schmidt algorithm to find an orthonormal basis $\{u_1(t), u_2(t)\}$ for the same space spanned by $\{v_1(t), v_2(t)\}$.[1]

$u_1(t) =$	$u_2(t) =$

[1] Use the standard inner product given by $\langle x(t), y(t)\rangle = \int_{-1}^{1} x(t)y(t)\, dt$.

(b) Let $x(t) = e^{-t}$. Find the minimizer of

$$\min_{y(t) \in T} \int_{-1}^{1} |x(t) - y(t)|^2 \, dt$$

where $T = \text{span}\{v_1(t), v_2(t)\}$.[2]

$y(t) =$

[2] You may use the facts that

$$\int te^{-t} \, dt = -e^{-t}(t+1) \qquad \text{and} \qquad \int t^2 e^{-t} \, dt = -e^{-t}(t^2 + 2t + 2)$$

Additional workspace:

Additional workspace:

Additional workspace:

Problem	*Score*
1 (30)	
2 (40)	
3 (30)	
Total	