GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 6250 Fall 2018
Problem Set #6

Assigned: 10-Oct-18
Due Date: 17-Oct-18

As stated in the syllabus, unauthorized use of previous semester course materials is strictly prohibited in this course.

All assignments will be due at the begining of class on the due date.

PROBLEM 6.1:
Using your class notes, prepare a 1-2 paragraph summary of what we talked about in class in the last week. I do not want just a bulleted list of topics, I want you to use complete sentences and establish context (Why is what we have learned relevant? How does it connect with other things you have learned here or in other classes?). The more insight you give, the better.

PROBLEM 6.2:

1. Consider the two systems given below:

   $$x[n] \longrightarrow \boxed{\downarrow 2} \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{\downarrow 2} \longrightarrow y[n]$$

   (a) (b)

   Are these systems equivalent? If so, provide a rigorous justification. If not, demonstrate this by a counterexample.

2. Consider the two systems given below:

   $$x[n] \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{\uparrow 2} \longrightarrow y[n]$$

   (a) (b)

   Are these systems equivalent? If so, provide a rigorous justification. If not, demonstrate this by a counterexample.

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PROBLEM 6.3:
Suppose we construct an orthogonal filterbank using the procedure described in class. Specifically, suppose that we are given a filter $g_0[n]$ satisfying

$$p[n] = \sum_{k=\infty}^{\infty} g_0[k]g_0[k-n] = \begin{cases} 1 & n=0 \\ 0 & n \neq 0, n \text{ even} \\ \text{anything} & n \text{ odd.} \end{cases}$$

Suppose that $g_0[n]$ is an N-tap filter, where N is even. The remaining filters are defined in the z-domain by setting

$$\begin{aligned} G_1(z) &= -z^{-N-1}G_0(-z^{-1}) \\ H_0(z) &= G_1(-z) \\ H_1(z) &= -G_0(-z). \end{aligned}$$

1. Provide expressions for $h_0[n]$ and $h_1[n]$ that depend only on $g_0[n]$.
2. Define $\boldsymbol{u}_n = h_0[2n-k]$. Show that $\langle \boldsymbol{u}_n, \boldsymbol{u}_m \rangle = \delta[m-n]$, i.e. show that the $\{\boldsymbol{u}_n\}$ form an orthonormal set.
3. Define $\boldsymbol{v}_n = h_1[2n-k]$. Show that $\langle \boldsymbol{v}_n, \boldsymbol{v}_m \rangle = \delta[m-n]$, i.e. show that the $\{\boldsymbol{v}_n\}$ form an orthonormal set.
4. Now show that $\langle \boldsymbol{u}_n, \boldsymbol{v}_m \rangle = 0$ for all n, m, which demonstrates that $\{\boldsymbol{u}_n\} \cup \{\boldsymbol{v}_n\}$ are orthonormal.

PROBLEM 6.4:
Let $x(t)$ be a function which is zero outside the interval $[0,1]$ and that is equal to different polynomials of degree d on the intervals $[\tau_k, \tau_{k+1}],\ k = 0, \ldots, K-1$, with $\tau_0 = 0$ and $\tau_K = 1$. Let $\psi_0(t)$ be a wavelet with p vanishing moments and which is supported on the interval $[0, L]$ for some $L \geq 1$. Depending on p, compute the number of possible non-zero wavelet coefficients $\langle \boldsymbol{x}, \boldsymbol{\psi}_{n,j} \rangle$ for a given scale j. How should we choose p to minimize this number?

PROBLEM 6.5:

1. Let $b[n]$ be an FIR filter of length L. Show how

$$\sum_{n=1}^{L} n^q b[n] = 0 \quad \text{for all} \quad q = 0, \ldots, p \tag{1}$$

implies

$$b[n] \star x[n] = \sum_{k=-\infty}^{\infty} b[k]x[n-k] = 0,$$

when

$$x[n] = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_1 n + a_0$$

for arbitrary $a_p, \ldots, a_0 \in \mathbb{R}$. That is, show that the discrete sequence $b[n]$ having p vanishing moments really does mean that it annihilates (discrete) polynomials of order p.

2. Now suppose we have a wavelet $\psi_0(t)$, which can be written as a superposition of scaling functions at scale $j = 1$ using

$$\psi_0(t) = \sum_n b[n]\phi_{1,n}(t).$$

Show that if the discrete *sequence* $b[n]$ has p vanishing moments (as in (1)), then the continuous time wavelet $\psi_0(t)$ must also have p vanishing moments, meaning

$$\int_{-\infty}^{\infty} t^q \psi_0(t) \ \mathrm{d}t = 0 \quad \text{for all} \quad q = 0, \ldots, p.$$

Note that the $\phi_{1,n}(t)$ will not in general have vanishing moments — just make the following constant substitutions when you see the integrals below:

$$C_0 = \int_{-\infty}^{\infty} \phi_{1,0}(t) \ \mathrm{d}t, \quad C_1 = \int_{-\infty}^{\infty} t\phi_{1,0}(t) \ \mathrm{d}t, \quad \cdots, \quad C_p = \int_{-\infty}^{\infty} t^p \phi_{1,0}(t) \ \mathrm{d}t.$$

(Hint: Start by showing this for $p = 0$, then $p = 1$, then generalize ... maybe by using the binomial theorem at some point.)

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