Splines

Recall the definition of the B-spline functions:

$$b_0(t) = \begin{cases} 1, & -1/2 \le t \le 1/2, \\ 0, & \text{otherwise}, \end{cases}$$

$$b_1(t) = (b_0 * b_0)(t),$$

$$b_2(t) = (b_1 * b_0)(t),$$

$$\vdots$$

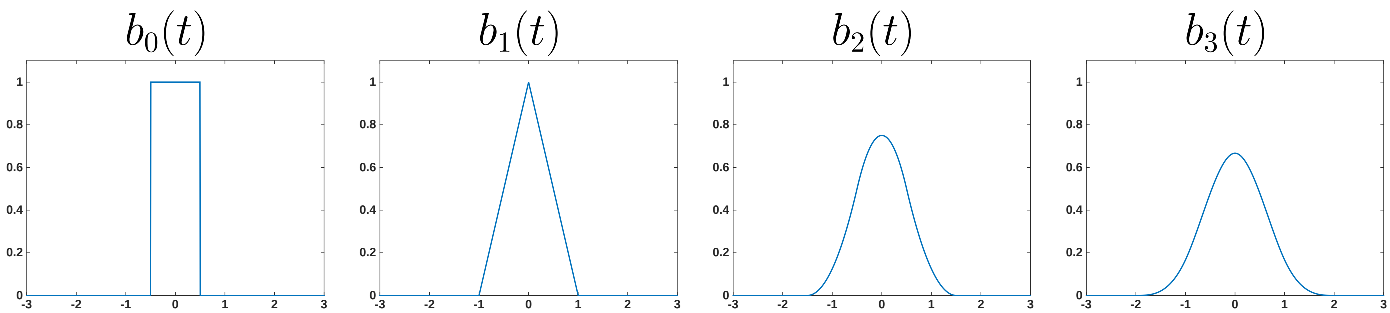
$$b_L(t) = (b_{L-1} * b_0)(t).$$

By computing the convolutions above, we can easily compute the first three:

$$b_0(t) = \begin{cases} 1, & -1/2 \le t < 1/2, \\ 0, & \text{else}. \end{cases} \quad b_1(t) = \begin{cases} t+1, & -1 \le t \le 0, \\ 1-t, & 0 \le t \le 1, \\ 0, & \text{else}. \end{cases}$$

$$b_2(t) = \begin{cases} (t+3/2)^2/2, & -3/2 \le t \le -1/2, \\ -t^2 + 3/4, & -1/2 \le t \le 1/2, \\ (t-3/2)^2/2, & 1/2 \le t \le 3/2, \\ 0, & |t| \ge 3/2. \end{cases}$$

Here are pictures of the first four:



Some key properties of these functions:

1. $b_L(t)$ is supported (non-zero) on $[-(L+1)/2, (L+1)/2]$

2. The $b_L(t)$ have $L+2$ *knots* at locations

$$\tau_k = -(L+1)/2 + k, \quad k = 0, \ldots, L+1.$$

   For L odd, these knots are at integer values,
   for L even, the knots are halfway in between the integers

3. Between the τ_k, the $b_L(t)$ are polynomials of order L

4. At the knots τ_k, the $b_L(t)$ are continuous (for $L \geq 1$) and have $L-1$ continuous derivatives (for $L \geq 2$).

5. As a special case of the above, note that at the end points, $t = \pm(L+1)/2$, the $b_L(t)$ are equal to zero and have $L-1$ derivatives equal to zero.

For fixed L, the superposition

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n b_L(t-n),$$

is a polynomial spline: $x(t)$ is an L^{th} order polynomial in between equally spaced *knots*. When L is odd, these knots are at the integers (at $\{\ldots,-1,0,1,2,\ldots\}$), and when L is even, these knots are shifted over by $1/2$ (at $\{\ldots,-3/2,-1/2,1/2,3/2,\ldots\}$). At these knots, $x(t)$ is continuous and has $L-1$ continuous derivatives. In between the knots $x(t)$ has an infinite number of derivatives (since it is polynomial there).

We will show below that

1. The space $\overline{\text{Span}}(\{b_L(t-n)\}_{n\in\mathbb{Z}})$ is the collection of all polynomial splines (with knots at $\mathbb{Z}$ or $\mathbb{Z}+1/2$) of order L.

2. The B-splines $\{b_L(t-n)\}_{n\in\mathbb{Z}}$ are a Riesz basis for this space.

In the process, we will show how to compute the basis constants A, B and how to construct the dual basis.

The set of shifted B-splines is a Riesz basis

First, we will show that $\{b_L(t-n)\}_{n\in\mathbb{Z}}$ are a Riesz basis for $\overline{\mathrm{Span}}(\{b_L(t-n)\}_{n\in\mathbb{Z}})$. We start by computing the continuous-time Fourier transform of an arbitrary linear combination of basis functions:

$$\begin{aligned}\mathscr{F}\left(\sum_{n=-\infty}^{\infty}\alpha_n b_L(t-n)\right) &= \sum_{n=-\infty}^{\infty}\alpha_n B_L(j\Omega)e^{-j\Omega n}\\ &= B_L(j\Omega)\sum_{n=-\infty}^{\infty}\alpha_n e^{-j\Omega n}\\ &= B_L(j\Omega)\,A(e^{j\Omega}),\end{aligned}$$

where $A(e^{j\Omega})$ is the discrete-time Fourier transform of $\{\alpha_n\}$. Using the classical Parseval's theorem, we have

$$\left\|\sum_{n=-\infty}^{\infty}\alpha_n b_L(t-n)\right\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}|B_L(j\Omega)|^2\,|A(e^{j\Omega})|^2\;\mathrm{d}\Omega$$

Since $A(e^{j\Omega})$ is 2π-periodic, we can rewrite this integral as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty}|B_L(j(\Omega+2\pi k))|^2\,|A(e^{j\Omega})|^2\;\mathrm{d}\Omega.$$

Let

$$\begin{aligned}A &= \min_{-\pi\le\Omega\le\pi}\sum_{k=-\infty}^{\infty}|B_L(j(\Omega+2\pi k))|^2,\\ B &= \max_{-\pi\le\Omega\le\pi}\sum_{k=-\infty}^{\infty}|B_L(j(\Omega+2\pi k))|^2.\end{aligned}$$

Then

$$\left\| \sum_{n=-\infty}^{\infty} \alpha_n b_L(t-n) \right\|_2^2 \;\geq\; A \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(e^{j\Omega})|^2 \, \mathrm{d}\Omega$$

$$\left\| \sum_{n=-\infty}^{\infty} \alpha_n b_L(t-n) \right\|_2^2 \;\leq\; B \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |A(e^{j\Omega})|^2 \, \mathrm{d}\Omega,$$

and thus by the (classical) Parseval's theorem for the DTFT:

$$A \cdot \sum_{n=-\infty}^{\infty} |\alpha_n|^2 \;\leq\; \left\| \sum_{n=-\infty}^{\infty} \alpha_n b_L(t-n) \right\|_2^2 \;\leq\; B \cdot \sum_{n=-\infty}^{\infty} |\alpha_n|^2.$$

We need some assurance that $A > 0$ and $B < \infty$. We know exactly what the Fourier transform of the B-splines are, so we can compute these quantities explicitly. We have

$$B_0(j\Omega) = \int_{-1/2}^{1/2} e^{-j\Omega t} \, \mathrm{d}t = \frac{\sin(\Omega/2)}{\Omega/2},$$

and so

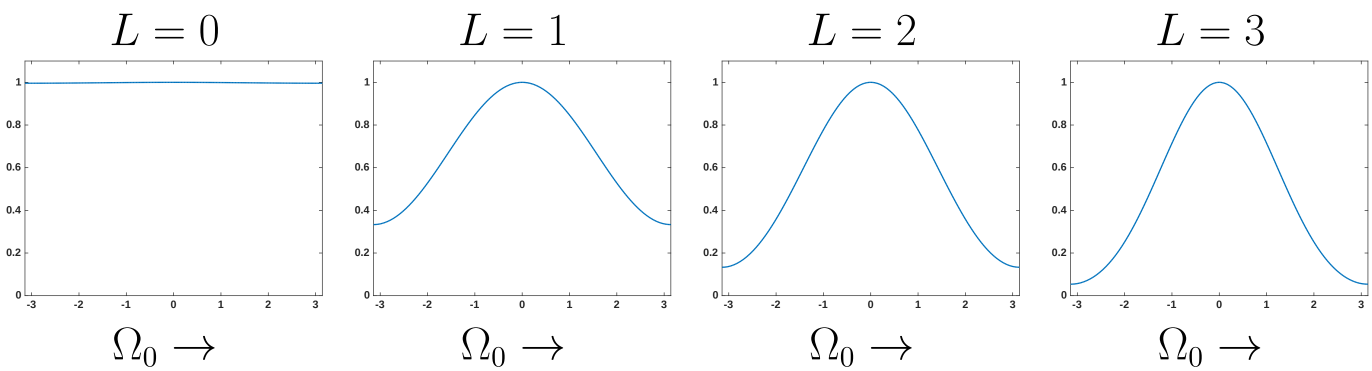
$$B_L(j\Omega) = \left(\frac{\sin(\Omega/2)}{\Omega/2} \right)^{L+1}.$$

For fixed Ω_0 in $[-\pi, \pi]$, we can compute

$$\sum_{k=-\infty}^{\infty} |B_L(j(\Omega_0 + 2\pi k))|^2 = \sum_{k=-\infty}^{\infty} \left| \frac{\sin(\Omega_0/2 + \pi k)}{\Omega_0/2 + \pi k} \right|^{2L+2}$$

When $\Omega_0 = 0$, this expression is equal to 1. Even for small values of L, the terms in the sum above decay quickly in $|k|$ — a pretty good approximation can be calculated for any Ω_0 running the sum from $k = -10, \ldots, 10$.

Here is what this function looks like for $L = 0, \ldots, 3$:



In these examples, we see that the function in the expression above is upper bounded by 1 and can be lower bounded by something greater than zero (but that this something seems to decrease as L increases).

In fact, by analyzing the sum above, you can show that

$$\left(\frac{2}{\pi}\right)^{2L+2} \;\leq\; \sum_{k=-\infty}^{\infty} |B_L(j(\Omega_0 + 2\pi k))|^2 \;\leq\; 1$$

over all $\Omega_0 \in [0, \pi]$. Thus we can take $A = (2/\pi)^{2L+2}$ and $B = 1$ as the Riesz basis constants.

Dual B-splines

The dual basis of a B-spline basis also consists of shifts of a template function. Recall from the previous set of notes that we can write the n^{th} dual basis function as

$$\widetilde{b}_{L,n}(t) = \sum_{\ell=-\infty}^{\infty} H_{n,\ell}\, b_L(t-\ell),$$

where $H_{n,\ell}$ is the "infinite matrix" which specifies the inverse to the linear operator

$$(\mathscr{G}(\boldsymbol{x}))[n] = \sum_{\ell=-\infty}^{\infty} G_{n,\ell}\, x[\ell], \quad \text{with} \;\; G_{n,\ell} = \langle b_L(t-\ell), b_L(t-n)\rangle.$$

But notice now that the discrete-time linear system $\mathscr{G}$ is time-invariant, as

$$G_{n,\ell} = \langle b_L(t-\ell), b_L(t-n)\rangle = \langle b_L(t), b_L(t-n+\ell)\rangle = g_L[n-\ell],$$

where

$$g_L[n] = \langle b_L(t), b_L(t-n)\rangle.$$

Thus $\mathscr{G}$ is a convolution operator:

$$(\mathscr{G}(\boldsymbol{x}))[n] = \sum_{\ell=-\infty}^{\infty} g_L[n-\ell]\, x[\ell] = (\boldsymbol{x} * \boldsymbol{g}_L)[n].$$

We can invert this convolution operator (with another time-invariant system) if the DTFT of $g_L[n]$,

$$G_L(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g_L[n]\, e^{-j\omega n},$$

is non-zero for all $\omega \in [-\pi, \pi]$.

We already know that this is true, since

$$\begin{aligned}
G_L(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \langle b_L(t), b_L(t-n)\rangle e^{-j\omega n} \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} B_L(j\Omega)\overline{B_L(j\Omega)} e^{j\Omega n} \ \mathrm{d}\Omega \right) \ e^{-j\omega n} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |B_L(j\Omega)|^2 \left(\sum_{n=-\infty}^{\infty} e^{j(\Omega-\omega)n} \right) \ \mathrm{d}\Omega \\
&= \int_{-\infty}^{\infty} |B_L(j\Omega)|^2 \left(\sum_{k=-\infty}^{\infty} \delta(\Omega - \omega - 2\pi k) \right) \ \mathrm{d}\Omega \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |B_L(j\Omega)|^2 \delta(\Omega - \omega - 2\pi k) \ \mathrm{d}\Omega \\
&= \sum_{k=-\infty}^{\infty} |B_L(j(\omega + 2\pi k)|^2.
\end{aligned}$$

This is exactly the quantity we studied in the previous section to get the basis constants, so we know $G_L(e^{j\omega})$ is real and

$$\left(\frac{2}{\pi}\right)^{2L+2} \ \le \ G_L(e^{j\omega}) \ \le \ 1,$$

and so it is invertible. Set

$$H_L(e^{j\omega}) = \frac{1}{G_L(e^{j\omega})}, \quad h_L[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_L(e^{j\omega}) e^{j\omega n} \ \mathrm{d}\omega$$

Then the n^{th} dual basis function is

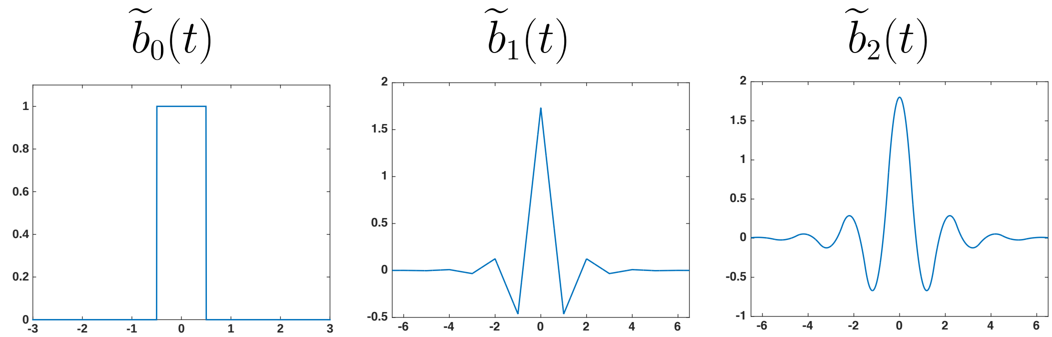
$$\widetilde{b}_{L,n}(t) = \sum_{\ell=-\infty}^{\infty} h_L[n-\ell] b_L(t-\ell) = \widetilde{b}_L(t-n),$$

where

$$\widetilde{b}_L(t) = \sum_{\ell=-\infty}^{\infty} h_L[-\ell] b_L(t-\ell) = \sum_{\ell=-\infty}^{\infty} h_L[\ell] b_L(t-\ell),$$

where the last equality follows from the fact that $h[n]$ is even (its DTFT is real).

Here are picture of the first three dual functions:



Note that since $\{b_0(t-n)\}$ are orthogonal, they are self-dual.

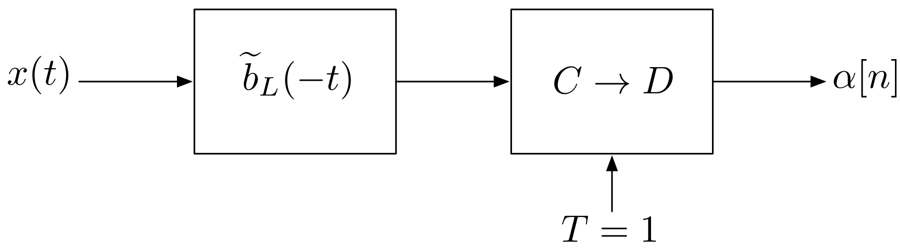
Sampling B-splines

There are sampling and reconstruction architectures for polynomial splines that are very similar to those for bandlimited signals. With the work above, we have shown that given a signal $x(t)$, if we form

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} \langle x(t), \widetilde{b}_L(t-n)\rangle b_L(t-n),$$

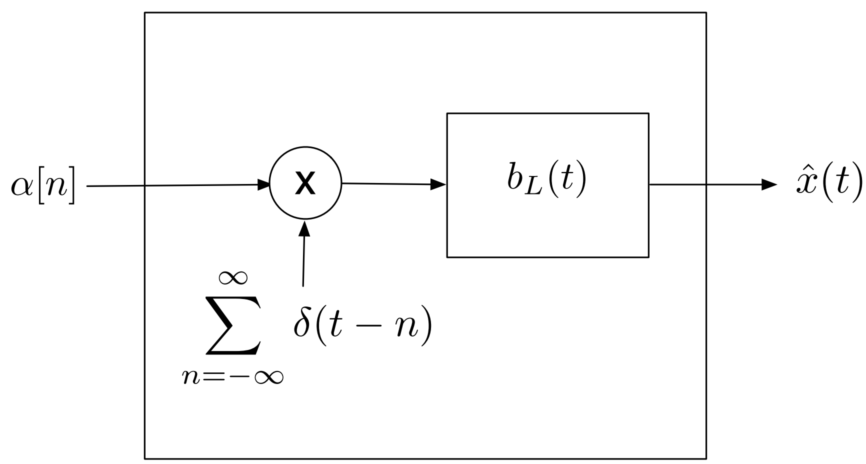
then $\hat{x}(t) = x(t)$ if $x(t)$ is a polynomial spline of order L, and otherwise it is the closest polynomial spline approximation to $x(t)$.

We can compute the coefficients $\alpha[n] = \langle x(t), \widetilde{b}_L(t-n)\rangle$ with a filter-then-sample architecture:



The analog filter on the left above has an impulse response that is the time-reversal of the dual B-spline $\widetilde{b}_L(t)$. You can think of this as the equivalent of the anti-aliasing filter.

To reconstruct the polynomial spline, we have the familiar architecture



So just as it is possible to reconstruct bandlimited signals from uniform samples, splines with equally spaced knots can also be reconstructed from uniform samples (after pre-filtering).

Interpolating with splines

Given a sequence of numbers $\alpha[n]$, the continuous-time signal

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha[n]\, b_L(t-n), \tag{1}$$

will be an L^{th}-order polynomial spline with knots at the integers $\mathbb{Z}$ (if L is odd) or $\mathbb{Z}+1/2$ (if L is even). That this is true follows directly from the properties of the $b_L(t)$ given at the beginning of this set of notes. By construction, in between the knots, $x(t)$ is a sum of L^{th}-th order polynomials, so it is itself an L^{th}-order polynomial in these regions. At knot τ_k, $x(\tau_k)$ is the (weighted) sum of $L+2$ different B-splines. L of these are non-zero, continuous, and have $L-1$ derivatives. The other two will be zero and have $L-1$ derivatives which are zero. Thus the sum at these points is continuous with $L-1$ continuous derivatives.

But the question remains: is *every* such polynomial spline in $\overline{\text{Span}}\{b_L(t-n)\}_{n\in\mathbb{Z}}$? A polynomial spline is completely determined by the values at its knots, and of course every different set of values will give you a different polynomial spline. So we can answer this question in the affirmative if given a sequence of numbers $x_d[n]$, we can produce $x(t)$ as in (1) that matches $x_d[n]$ on $\mathbb{Z}$ (or $\mathbb{Z}+1/2$).

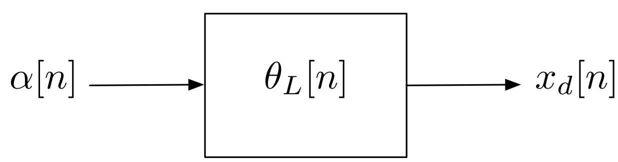
Let's assume that L is odd, just so we can avoid shifts by $1/2$ everywhere — what we say below is easily adapted to L even. We would like to find $\alpha[n]$ such that $x(n) = x_d[n]$ for all $n \in \mathbb{Z}$. Since the $b_L(t-n)$ is non-zero only for L values of n ($n = -(L-1)/2, \ldots, (L-1)/2$), we have

$$x(n) = \sum_{\ell=-(L-1)/2}^{(L-1)/2} \alpha[\ell] b_L(n-\ell).$$

That is, the samples of $x(t)$ at the integers can be computed with the discrete time convolution of $\alpha[n]$ and samples of the B-spline functions

$$\theta_L[n] = b_L(n).$$

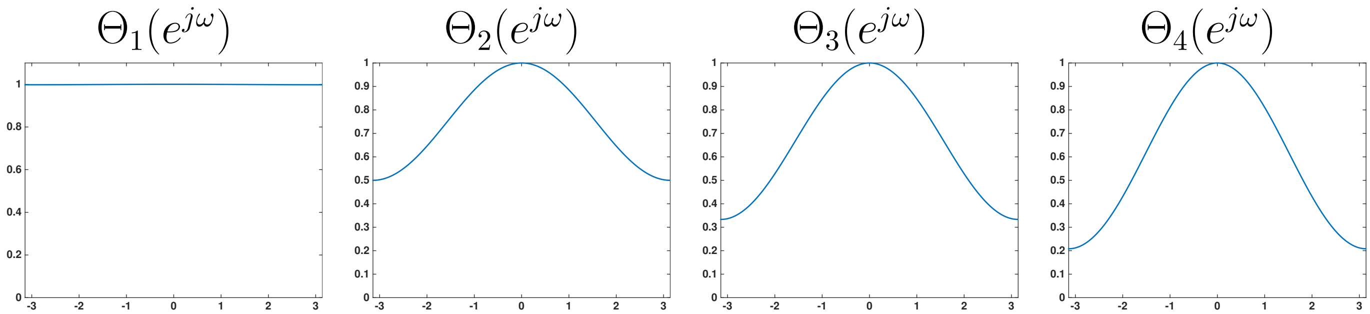
Thus we can imagine $x(n)$ being created by taking the $\alpha[n]$ and passing it through an LTI filter:



Given the sequence $x_d[n]$, we would like to find an $\alpha[n]$ that induces it using the system above. That is, we want the inverse of the filter above. Again, we turn to the frequency domain. Since $\theta_L[n]$ consists of samples of $b_L(t)$, we know that

$$\Theta_L(e^{j\omega}) = \sum_{k=-\infty}^{\infty} B_L(j(\omega + 2\pi k)).$$

This is similar to but not the same as the quantity $G_L(e^{j\omega})$ we looked at when computing the Riesz basis constants and dual functions — we are not squaring the shifts of B_L. Nevertheless, since we know $B_L(j\Omega)$, we calculate this spectrum for the first few values of L:



Notice that the spectrum is flat for $L = 1$ — this is because superimposing the $b_1(t)$ also does linear interpolation. For the other values of L, the inverse is very well conditioned. We set

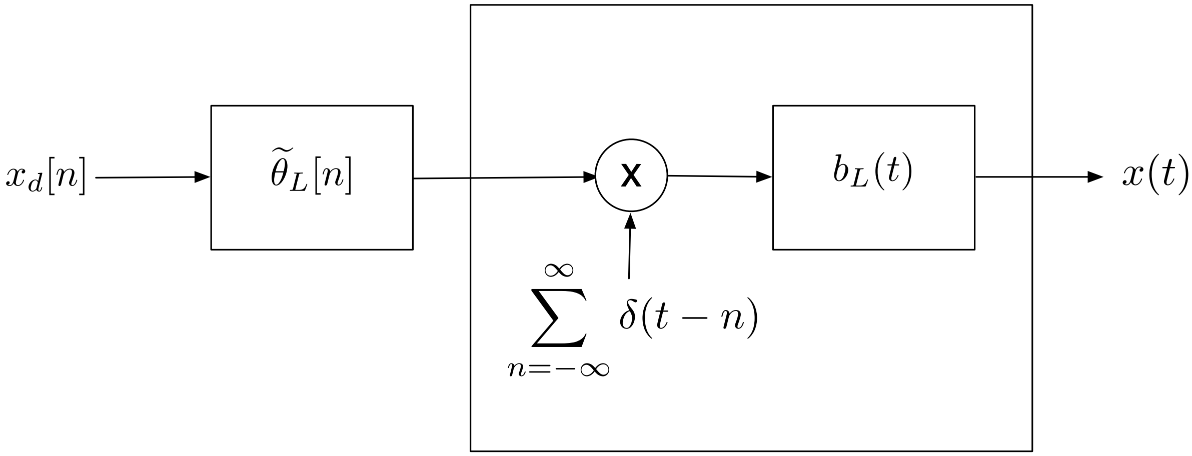
$$\widetilde{\Theta}(e^{j\omega}) = \frac{1}{\Theta_L(e^{j\omega})}, \quad \widetilde{\theta}_L[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{\Theta}_L(e^{j\omega})e^{j\omega n}\ \mathrm{d}\omega,$$

and then given a prescribed $x_d[n]$, we can compute the corresponding $\alpha[n]$ with

$$\alpha[n] = \sum_{\ell=-\infty}^{\infty} \widetilde{\theta}_L[n-\ell]x_d[\ell] = (\boldsymbol{x}_d * \widetilde{\boldsymbol{\theta}}_L)[n].$$

Since we can find such a sequence $\alpha[n]$ for any set of samples at the knots, every polynomial spline is in the (closure of) the span of the $b_L(t-n)$.

The following system takes $x_d[n]$, and interpolates between the integers:



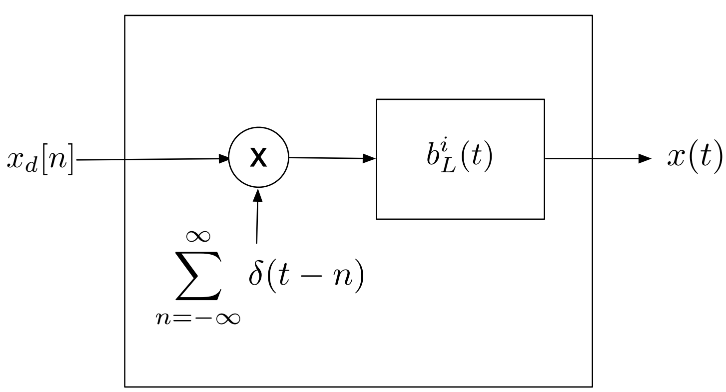
We can combine this into a single block as follows. The impulse response of this system (what comes out when $x_d[n] = \delta[n]$) is

$$b_L^i(t) = \sum_{\ell=-\infty}^{\infty} \widetilde{\theta}_L[\ell] b_L(t-\ell).$$

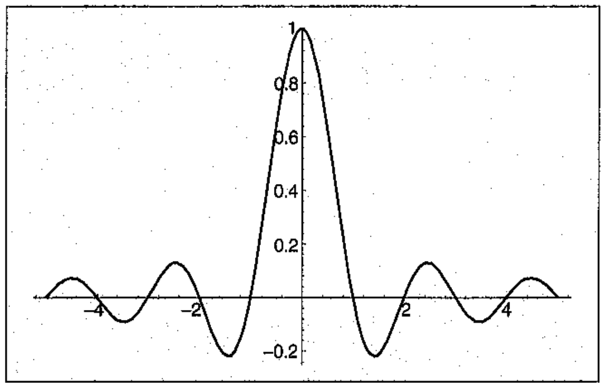
Thus the output is

$$x(t) = \sum_{n=-\infty}^{\infty} x_d[n] b_L^i(t-n).$$

So another architecture is



The $b_L^i(t)$ are called the interpolating splines, or the *cardinal spline functions*. Here is a sketch of the cardinal spline for $L = 3$:

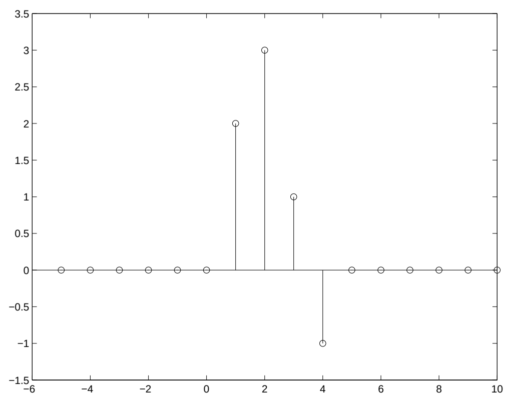


(From Unser, *Splines, A perfect fit ...*)

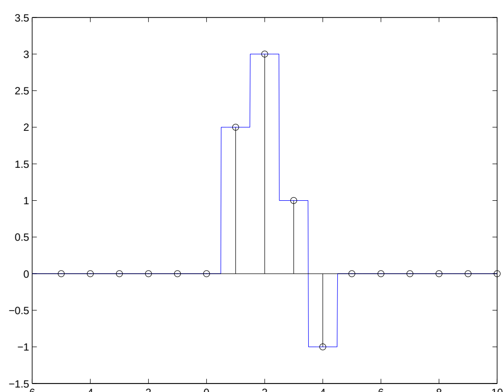
Here is an example that we saw during the first week of class. Let

$$x_d[n] = \begin{cases} 2, & n = 1, \\ 3, & n = 2, \\ 1, & n = 3, \\ 4, & n = 4, \\ 0, & \text{otherwise} \end{cases}$$

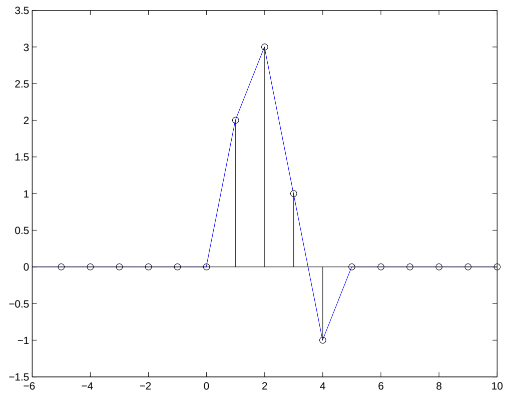
Here is a sketch of $x_d[n]$:



then here is the output of the interpolator for $L = 0$,



then $L = 1$,



now $L = 2$,