Orthogonal filterbanks

Recall that a perfect reconstruction filterbank can be designed by first constructing $G_0(z)$ and $G_1(z)$ satisfying

$$G_0(z)G_1(-z) - G_1(z)G_0(-z) = 2z^{-m}. \qquad \text{(No distortion)}$$

Once we have such a $G_0(z)$ and $G_1(z)$, we can then define the filters

$$H_0(z) = G_1(-z) \qquad \text{and} \qquad H_1(z) = -G_0(-z),$$

and automatically form a perfect reconstruction filterbank.

Thus our central challenge is to construct $G_0(z)$ and $G_1(z)$ satisfying the no distortion condition. Here we will discuss one possible solution which satisfies some particularly nice properties. Suppose that $G_0(z)$ is given. Then the **alternating flip** construction is to set

$$G_1(z) = -z^{-m}G_0(-z^{-1}),$$

where m is odd and will correspond to the total delay of the system. What does $G_1(z)$ look like in this case? Suppose that

$$G_0(z) = \alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \alpha_3 z^{-3}.$$

Then for $m = 3$ we have

$$\begin{aligned} G_0(z^{-1}) &= \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 \\ G_0(-z^{-1}) &= \alpha_0 - \alpha_1 z + \alpha_2 z^2 - \alpha_3 z^3 \\ -G_0(-z^{-1}) &= -\alpha_0 + \alpha_1 z - \alpha_2 z^2 + \alpha_3 z^3 \\ G_1(z) &= \alpha_3 - \alpha_2 z^{-1} + \alpha_1 z^{-2} - \alpha_0 z^{-3}. \end{aligned}$$

Note that with this construction (when m is odd), $G_1(-z) = z^{-m}G_0(z^{-1})$, and thus the no distortion condition reduces to

$$z^{-m}\left[G_0(z)G_0(z^{-1}) + G_0(-z)G_0(-z^{-1})\right] = 2z^{-m}.$$

Alternatively, if we set $P(z) = G_0(z)G_0(z^{-1})$, then this simply reduces to

$$P(z) + P(-z) = 2. \qquad \text{(No distortion (v2))}$$

Recalling that

$$P(z) = \sum_{n=-\infty}^{\infty} p[n] z^{-n},$$

we can see that the above condition on $P(z)$ reduces to

$$p[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0, n \text{ even} \\ \text{anything} & n \text{ odd.} \end{cases}$$

But what exactly is $p[n]$, and what does it tell us about $g_0[n]$? Using the fact that $P(z) = G_0(z)G_0(z^{-1})$, we have

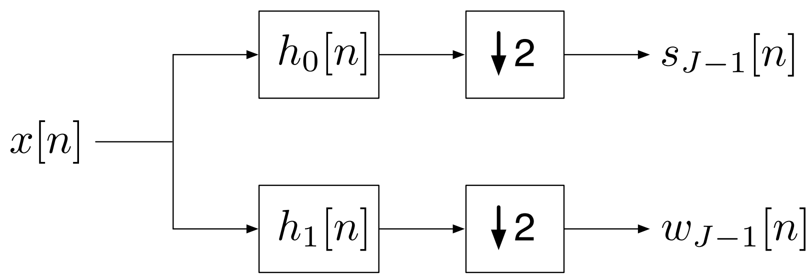
$$\begin{aligned} \sum_{n=-\infty}^{\infty} p[n] z^{-n} &= \left(\sum_{k=-\infty}^{\infty} g_0[k] z^{-k} \right) \left(\sum_{m=-\infty}^{\infty} g_0[m] z^{m} \right) \\ &= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g_0[k] g_0[m] z^{m-k} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} g_0[k] g_0[k-n] z^{-n}. \end{aligned}$$

Thus, we can conclude that

$$p[n] = \sum_{k=-\infty}^{\infty} g_0[k] g_0[k-n],$$

i.e. $p[n]$ is just the autocorrelation function of $g_0[n]$, and the perfect reconstruction condition reduces to a simple constraint on this function.

To summarize, the procedure we have described to design a filterbank consists of designing $g_0[n]$ so that the autocorrelation function $p[n]$ is zero for all even n except $n = 0$, and then using the "alternating flip" construction of $g_1[n]$, which together dictate $h_0[n]$ and $h_1[n]$. In this context, the constraint on $p[n]$ has significant consequences. Specifically, for *any* filterbank designed in this way, the filterbank architecture



can be thought of as computing a representation of $x[n]$ in an orthogonal basis.

Specifically, note that here

$$s_{J-1}[n] = \sum_{k=-\infty}^{\infty} x[k] h_0[2n-k]$$

$$w_{J-1}[n] = \sum_{k=-\infty}^{\infty} x[k] h_1[2n-k].$$

If we define $\boldsymbol{u}_n = h_0[2n-k]$ and $\boldsymbol{v}_n = h_1[2n-k]$, then we can interpret this as

$$s_{J-1}[n] = \langle \boldsymbol{u}_n, \boldsymbol{x} \rangle \qquad \text{and} \qquad w_{J-1}[n] = \langle \boldsymbol{v}_n, \boldsymbol{x} \rangle.$$

The constraint on $p[n]$ turns out to be exactly what we need to ensure that $\mathcal{B} = \{\boldsymbol{u}_n\}_n \cup \{\boldsymbol{v}_n\}$ are orthonormal. This together with the perfect reconstruction property of the filterbank implies that $\mathcal{B}$ forms an orthonormal basis for ℓ_2.

Continuous-time orthonormal wavelet bases

We began our discussion of wavelets by considering the Haar wavelet basis for decomposing continuous-time signals $x(t) \in L_2(\mathbb{R})$, giving us a decomposition of the form

$$x(t) = \sum_{n=-\infty}^{\infty} s_{0,n}\phi_{0,n}(t) + \sum_{j=0}^{\infty}\sum_{n=-\infty}^{\infty} w_{j,n}\psi_{j,n}(t).$$

Recall that the (orthonormal) basis functions are **scaled and shifted** versions of the two template functions $\phi_0(t)$ and $\psi_0(t)$. Moreover, these two functions were linear combinations of shifts of a *contracted* version of $\phi_0(t)$:

$$\phi_0(t) = \phi_0(2t) + \phi_0(2t-1), \qquad \psi_0(t) = \phi_0(2t) - \phi_0(2t-1).$$

This gave us the very nice interpretation of the wavelet coefficients $w_{j,n}$ capturing the **differences** between piecewise-constant approximations of $x(t)$ at different dyadic scales:

$$x(t) = \underbrace{\underbrace{\underbrace{\boldsymbol{P}_{\mathcal{V}_0}[x(t)] + \boldsymbol{P}_{\mathcal{W}_0}[x(t)]}_{=\boldsymbol{P}_{\mathcal{V}_1}[x(t)]} + \boldsymbol{P}_{\mathcal{W}_1}[x(t)]}_{=\boldsymbol{P}_{\mathcal{V}_2}[x(t)]} + \boldsymbol{P}_{\mathcal{W}_2}[x(t)]}_{=\boldsymbol{P}_{\mathcal{V}_3}[x(t)]} + \cdots.$$

Along with this interpretation, we also developed an efficient filter-bank implementation for computing this decomposition from some initial approximation $\hat{x}_J(t) = \boldsymbol{P}_{\mathcal{V}_J}[x(t)]$.

Now that we have seen how to generalize this filterbank structure, it is natural to ask whether these new filterbanks have a similar correspondence to other types of approximation spaces $\mathcal{V}_j$ built using scaling functions $\phi_0(t)$ other than just piecewise-constant functions. Indeed we can, and it leads to a very rich family of **orthonormal wavelet bases**.

As in the Haar case, we will see that essentially all of the properties of any orthonormal wavelet basis will follow from properties of the scaling function $\phi_0(t)$. Before discussing these more general wavelet bases, we will first review some of the key properties of $\phi_0(t)$ that allowed us to interpret the Haar wavelet transform as providing a multiscale approximation.

Multiscale approximation: Scaling spaces

For a given $\phi_0(t)$, the first approximation space $\mathcal{V}_0$ is set of signals we can build up from different linear combinations[1] of the integer shifts of $\phi_0(t)$:

$$\mathcal{V}_0 = \overline{\mathrm{Span}}(\{\phi_0(t-n)\}_{n\in\mathbb{Z}}).$$

The first thing we want is for $\{\phi_0(t-n)\}_{n\in\mathbb{Z}}$ to be an orthobasis, so we ask that

(P1) $$\langle \phi_0(t-k), \phi_0(t-n)\rangle = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases}$$

[1] Technically, this is the set of signals we can approximate arbitrarily well from different linear combinations — this is the closure of the span, which we will denote by $\overline{\mathrm{Span}}$.

Now set
$$\phi_{j,n}(t) = 2^{j/2}\phi_0(2^j t - n),$$
so the function $\phi_0(2^j t - n)$ is formed by **contracting** $\phi_0(t)$ by a factor of 2^j, then shifting the result on a grid with spacing 2^{-j}. For a fixed scale j, define

$$\mathcal{V}_j = \overline{\text{Span}}(\{\phi_{j,n}(t)\}_{n\in\mathbb{Z}}).$$

Following the Haar case, there are two more key properties we ask of this sequence of approximation spaces; we would like these spaces to be nested,

$$\textbf{(P2)} \qquad \mathcal{V}_j \subset \mathcal{V}_{j+1}, \quad \text{so } x(t) \in \mathcal{V}_j \Rightarrow x(t) \in \mathcal{V}_{j+1},$$

and we also want these approximation spaces to cover all of $L_2(\mathbb{R})$ in their limit:

$$\textbf{(P3)} \qquad \lim_{j\to\infty} \mathcal{V}_j = L_2(\mathbb{R}), \text{ so } \lim_{j\to\infty} \boldsymbol{P}_{\mathcal{V}_j}[x(t)] = x(t) \text{ for all } x(t) \in L_2(\mathbb{R}).$$

Now the question is: What properties does $\phi_0(t)$ have to have to ensure (**P1**)–(**P3**) hold? While the answer is not straightforward, this question was answered completely in the late 1980s/early 1990s. The conditions on $\phi_0(t)$ are actually most easily expressed in terms of the inter-scale relationships between the $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$ and $\{\phi_{j+1,n}\}_{n\in\mathbb{Z}}$, which you may recall is exactly what gave rise to the filterbank structure for computing the Haar wavelet transform.

Specifically, given a $\phi_0(t)$, define the sequence of numbers $g_0[n]$

$$g_0[n] = \langle \phi_0(t), \sqrt{2}\phi_0(2t - n)\rangle. \tag{1}$$

It turns out that whether properties (**P1**)–(**P3**) hold depends entirely on properties of this sequence of numbers. Let $G_0(e^{j\omega})$ be the discrete-time Fourier transform of $g_0[n]$. Then we have following major result:

If $g_0[n]$ obeys the following three properties, then the approximation spaces $\{\mathcal{V}_j\}_{j\geq 0}$ obey properties (**P1**)–(**P3**):

(**G1**) $$|G_0(e^{j\omega})|^2 + |G_0(e^{j(\omega+\pi)})|^2 = 2, \quad \text{for all} \quad -\pi \leq \omega \leq \pi$$

(**G2**) $$G_0(e^{j0}) = \sum_n g_0[n] = \sqrt{2},$$

(**G3**) $$|G_0(e^{j\omega})| > 0 \quad \text{for all} \quad |\omega| \leq \frac{\pi}{2}.$$

The proof of this result is long and complicated.[2] Note, however, that Condition (**G1**) is somewhat familiar. Specifically, using the orthogonal filterbank construction described above, (**G1**) is simply what we obtain by plugging $z = e^{j\omega}$ into v2 of the "No distortion" filterbank condition. The remaining conditions are more technical requirements that allow us to construct $\phi_0(t)$ from knowledge of $g_0[n]$ (see Technical Details at end of notes.)

[2]There are a few good references here. I will recommend Chapter 7 of A Wavelet Tour of Signal Processing, by S. Mallat, and Daubechies' book *Ten Lectures on Wavelets*.

Multiscale approximation: Wavelet spaces

The complementary wavelet spaces and wavelet basis functions can also be generated from the coefficient sequence $g_0[n]$. This is detailed as our second major result:

Suppose $\phi_0(t)$ with corresponding $g_0[n]$ obeys (**G1**)–(**G3**). Set[a]

$$g_1[n] = (-1)^{1-n} g_0[1-n],$$

and

$$\psi_0(t) = \sum_{n=-\infty}^{\infty} g_1[n]\sqrt{2}\,\phi_0(2t-n).$$

Then, along with integer shifts of the scaling function $\phi_{0,n}(t) = \phi_0(t-n)$, the set of all dyadic shifts and contractions of $\psi_0(t)$,

$$\psi_{j,n}(t) = 2^{j/2}\psi_0(2^j t - n), \quad n \in \mathbb{Z}, \quad j = 0, 1, 2, \ldots,$$

form an orthobasis for $L_2(\mathbb{R})$. That is,

$$x(t) = \sum_{n=-\infty}^{\infty} \langle \boldsymbol{x}, \boldsymbol{\phi}_{0,n} \rangle \phi_{0,n}(t) + \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} \langle \boldsymbol{x}, \boldsymbol{\psi}_{j,n} \rangle \psi_{j,n}(t)$$

for all $x(t) \in L_2(\mathbb{R})$.

[a] Note that the choice of $g_1[n]$ here is precisely the "alternating flip" construction we described in the context of filterbanks.

As with the Haar case, the wavelet coefficients at scale j represent the difference between the approximation of a signal in $\mathcal{V}_j$ and the approximation in $\mathcal{V}_{j+1}$. That is, if we set

$$\mathcal{W}_j = \overline{\text{Span}}\left(\{\psi_{j,n}(t)\}_{n\in\mathbb{Z}}\right)$$

then

1. For fixed j, $\langle \psi_{j,n}, \psi_{j,\ell}\rangle = 0$ for $n \neq \ell$. That is, the $\{\psi_{j,n}(t)\}_{n\in\mathbb{Z}}$ are orthobasis for $\mathcal{W}_j$.

2. $\mathcal{W}_j \perp \mathcal{V}_{j'}$ for all $j' \leq j$. Notice that since $\mathcal{W}_j \subset \mathcal{V}_{j+1}$, it follows that the sequence of spaces $\mathcal{V}_0, \mathcal{W}_0, \mathcal{W}_1, \ldots$ are all mutually orthogonal.

3. $\mathcal{V}_{j+1} = \mathcal{V}_j \oplus \mathcal{W}_j$. That is, every $v(t) \in \mathcal{V}_{j+1}$ can be written as

$$v(t) = \boldsymbol{P}_{\mathcal{V}_j}[v(t)] + \boldsymbol{P}_{\mathcal{W}_j}[v(t)].$$

   As the previous property states, these two components are orthogonal to one another.

In summary, this means we can break $L_2(\mathbb{R})$ into orthogonal parts,

$$L_2(\mathbb{R}) = \mathcal{V}_0 \oplus \mathcal{W}_0 \oplus \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \cdots$$

and we have an orthobases for each of these.

Vanishing moments and support size

In addition to forming an orthobasis with a certain multiscale form, there are other desirable properties that wavelet systems often have.

Vanishing moments. We say that $\psi_0(t)$ has p vanishing moments if

$$\int_{-\infty}^{\infty} t^q \psi_0(t)\,\mathrm{d}t = 0, \quad \text{for } q = 0, 1, \ldots, p-1.$$

This means that $\psi_0(t)$ is **orthogonal** to all **polynomials** of degree $p-1$ or smaller. Since shifting a polynomial just gives you another polynomial of the same order, $\psi_0(t-n)$ is also orthogonal to these polynomials. This means that polynomials that have degree at most $p-1$ are completely contained in the scaling space $\mathcal{V}_0$ — all of the wavelet coefficients of a polynomial are zero.

Compact support. The support of $\psi_0(t)$ is the size of the interval on which it is non-zero. If $\psi_0(t)$ is supported on $[0, L]$, then $\psi_{0,n}(t) = \psi_0(t-n)$ is supported on $[n, n+L]$, and

$$w_{0,n} = \langle \boldsymbol{x}, \boldsymbol{\psi}_{0,n} \rangle = \int_n^{n+L} x(t)\psi_{0,n}(t)\,\mathrm{d}t.$$

This means that $w_{0,n}$ only depends on what $x(t)$ is doing on $[n, n+L]$ — the wavelet coefficients are recording **local** information about the behavior of $x(t)$.

These two properties make wavelets very good for representing signals which are smooth except at a few singularities.

Daubechies Wavelets

In the late 1980s, Ingrid Daubechies presented a systematic framework for designing wavelets with vanishing moments and compact support. For any integer p, there is a method for solving for the $g_0[n]$ that corresponds to a wavelet with p vanishing moments and has support size $2p - 1$.

Here are the filter coefficients for $p = 2, \ldots, 10$. ($p = 1$ gives you Haar wavelets.):

	n	$h_p[n]$
$p=2$	0	0.482962913145
	1	0.836516303738
	2	0.224143868042
	3	−0.129409522551
$p=3$	0	0.332670552950
	1	0.806891509311
	2	0.459877502118
	3	−0.135011020010
	4	−0.085441273882
	5	0.035226291882
$p=4$	0	0.230377813309
	1	0.714846570553
	2	0.630880767930
	3	−0.027983769417
	4	−0.187034811719
	5	0.030841381836
	6	0.032883011667
	7	−0.010597401785
$p=5$	0	0.160102397974
	1	0.603829269797
	2	0.724308528438
	3	0.138428145901
	4	−0.242294887066
	5	−0.032244869585
	6	0.077571493840
	7	−0.006241490213
	8	−0.012580751999
	9	0.003335725285
$p=6$	0	0.111540743350
	1	0.494623890398
	2	0.751133908021
	3	0.315250351709
	4	−0.226264693965
	5	−0.129766867567
	6	0.097501605587
	7	0.027522865530
	8	−0.031582039317
	9	0.000553842201
	10	0.004777257511
	11	−0.001077301085
$p=7$	0	0.077852054085
	1	0.396539319482
	2	0.729132090846
	3	0.469782287405
	4	−0.143906003929
	5	−0.224036184994
	6	0.071309219267
	7	0.080612609151
	8	−0.038029936935
	9	−0.016574541631
	10	0.012550998556
	11	0.000429577973
	12	−0.001801640704
	13	0.000353713800
$p=8$	0	0.054415842243
	1	0.312871590914
	2	0.675630736297
	3	0.585354683654
	4	−0.015829105256
	5	−0.284015542962
	6	0.000472484574
	7	0.128747426620
	8	−0.017369301002
	9	−0.04408825393
	10	0.013981027917
	11	0.008746094047
	12	−0.004870352993
	13	−0.000391740373
	14	0.000675449406
	15	−0.000117476784
$p=9$	0	0.038077947364
	1	0.243834674613
	2	0.604823123690
	3	0.657288078051
	4	0.133197385825
	5	−0.293273783279
	6	−0.096840783223
	7	0.148540749338
	8	0.030725681479
	9	−0.067632829061
	10	0.000250947115
	11	0.022361662124
	12	−0.004723204758
	13	−0.004281503682
	14	0.001847646883
	15	0.000230385764
	16	−0.000251963189
	17	0.000039347320
$p=10$	0	0.026670057901
	1	0.188176800078
	2	0.527201188932
	3	0.688459039454
	4	0.281172343661
	5	−0.249846424327
	6	−0.195946274377
	7	0.127369340336
	8	0.093057364604
	9	−0.071394147166
	10	−0.029457536822
	11	0.033212674059
	12	0.003606553567
	13	−0.010733175483
	14	0.001395351747
	15	0.001992405295
	16	−0.000685856695
	17	−0.000116466855
	18	0.000093588670
	19	−0.000013264203

From Mallat, *A Wavelet Tour of Signal Processing*

Here are pictures of some of the scaling functions ($N = 2p$ in the captions below):

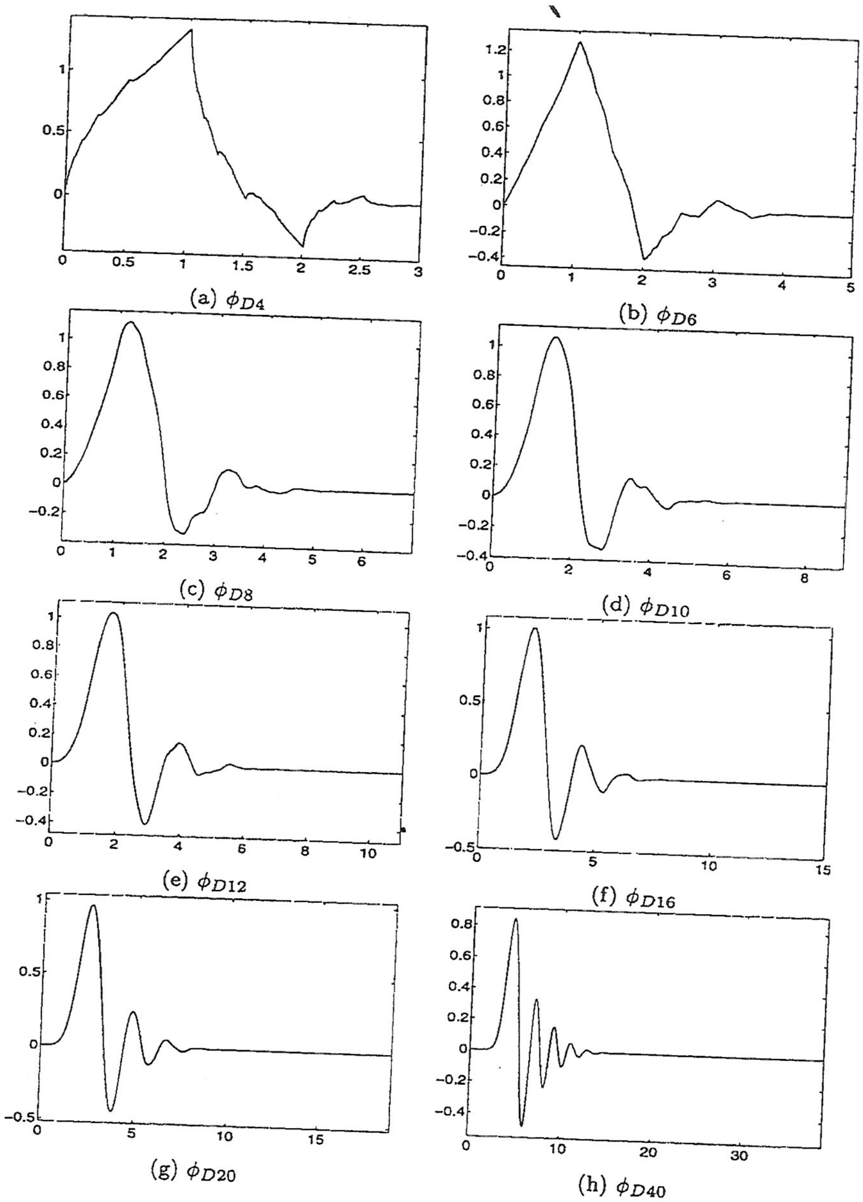


Figure 6.1. Daubechies Scaling Functions, $N = 4, 6, 8, \ldots, 40$

From Burrus et al, *Introduction to Wavelets ...*

Here are pictures of some of the wavelet functions ($N = 2p$ in the captions below):

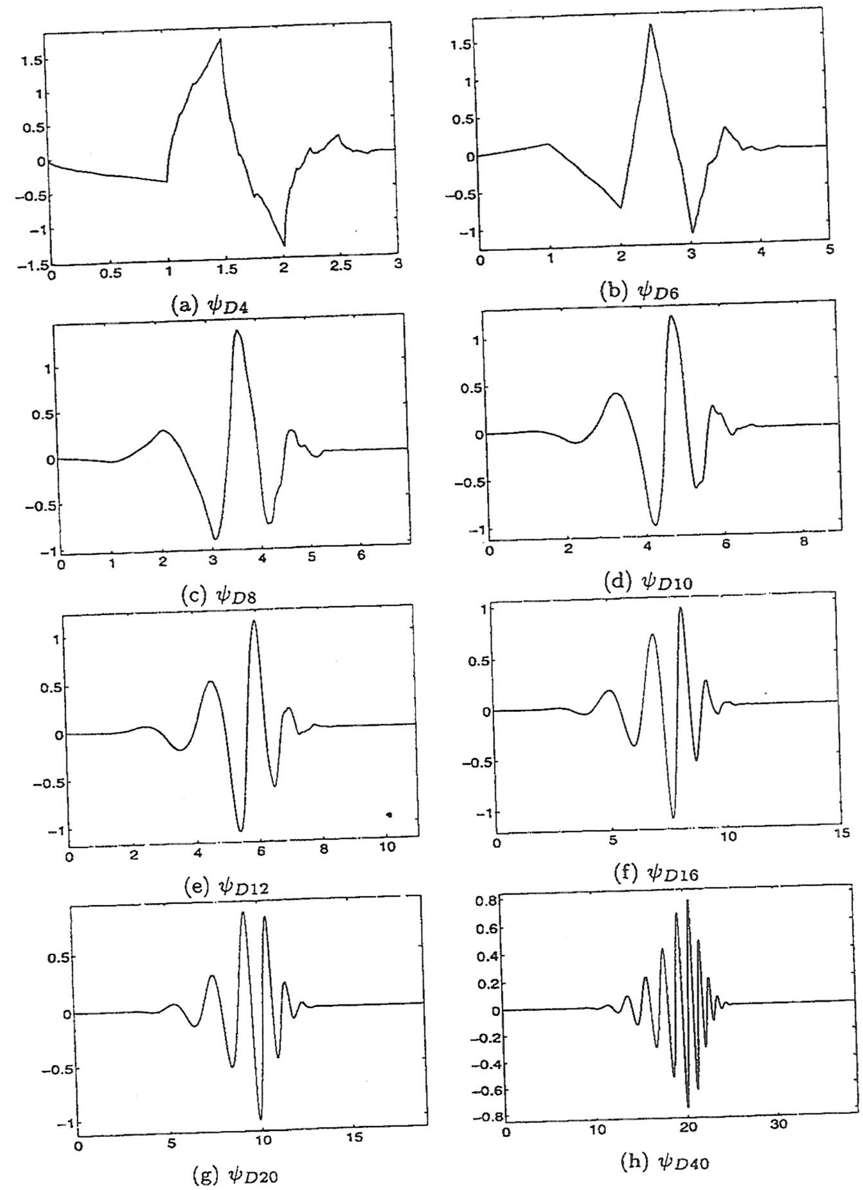


Figure 6.2. Daubechies Wavelets, $N = 4, 6, 8, \ldots, 40$

From Burrus et al, *Introduction to Wavelets ...*

Technical Details: Constructing $\phi_0(t)$

Note that with (**P2**) established, we know that $\phi_0(t) \in \mathcal{V}_1$. This gives us an additional interpretation of the $g_0[n]$; they tell us how to build up $\phi_0(t)$ out of shifts of the contracted version $\phi_0(2t)$:

$$\phi_0(t) = \sum_{n=-\infty}^{\infty} g_0[n]\,\sqrt{2}\phi_0(2t-n). \tag{2}$$

Given a particular $\phi_0(t)$, we can of course generate the $g_0[n]$ using (1) – but we can also go the other way. If we design a sequence $g_0[n]$ that obeys the three properties above, it specifies a unique scaling function $\phi_0(t)$. To get $\phi_0(t)$ from $g_0[n]$, we take the continuous-time Fourier transform of both sides of (2):

$$\begin{aligned}
\Phi_0(j\Omega) &= \sum_{n=-\infty}^{\infty} g_0[n]\sqrt{2}\int_{-\infty}^{\infty} \phi_0(2t-n)e^{-j\Omega t}\ \mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} g_0[n]\,\frac{1}{\sqrt{2}}e^{j\Omega n/2}\Phi_0(j\Omega/2) \\
&= \frac{1}{\sqrt{2}}\overline{G(e^{j\Omega/2})}\Phi_0(j\Omega/2)
\end{aligned}$$

We can again expand $\Phi_0(j\Omega/2) = \frac{1}{\sqrt{2}}\overline{G(e^{j\Omega/4})}\Phi_0(j\Omega/4)$, etc. Condition (**G3**) above means that the limit exists, and we have

$$\Phi_0(j\Omega) = \left(\prod_{p=1}^{\infty} \frac{\overline{G(e^{j2^{-p}\Omega})}}{\sqrt{2}}\right)\Phi_0(j0) = \prod_{p=1}^{\infty} \frac{\overline{G(e^{j2^{-p}\Omega})}}{\sqrt{2}},$$

since $\Phi_0(j0) = 1$ (this follows from integrating both sides of (2) and applying Condition (**G2**) above). Unfortunately, except in special cases it is hard to compute $\Phi_0(j\Omega)$ past the iterative expression above. This is why wavelets are usually specified in terms of their corresponding sequences $g_0[n]$.