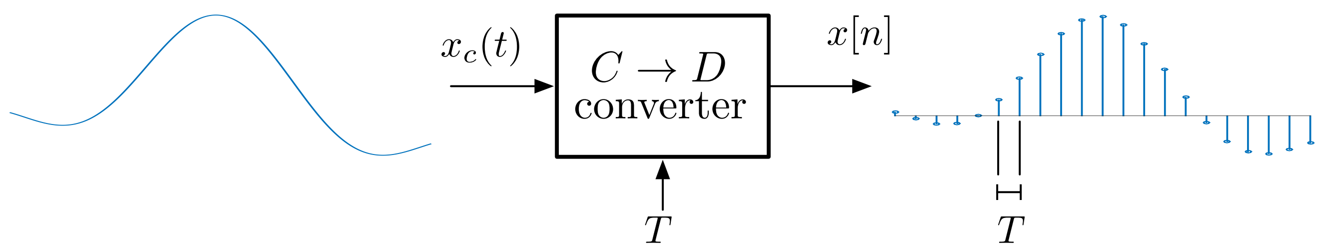
I. Signal Discretization using Basis Decompositions

We will start by reviewing one of the foundational results of digital signal processing: the Shannon-Nyquist sampling theorem. We will use this result as a first example of how continuous-time signals can be systematically discretized (translated into a discrete list of numbers).

The Shannon-Nyquist sampling theorem

Sampling turns a continuous-time signal $x_c(t)$ into a discrete list of numbers simply by evaluating it at equally spaced points:



$$x[n] = x_c(nT)$$

($C \to D$=continuous-to-discrete.)

The constant T is the **sampling interval**
(the amount of time that passes between each sample).

This is a very common practice, and there exists very sophisticated hardware that implements it. Examples:

- Texas Instruments makes an ADC, the 12DL3200, that takes 6.4 *billion* samples per second ($T \approx 0.16$ nanoseconds) at a (reported) resolution of 12 bits. Cost: $\approx$ \$3000.
- Another ADC from TI, the TIADS1261, takes 40,000 samples per second ($T = 25$ microseconds) at a (reported) resolution of 24 bits. Cost: $\approx$ \$11

Questions:

1. When can you reconstruct $x_c(t)$ perfectly from its samples?
2. How do you do it?

Answers:

1. When $x_c(t)$ is **bandlimited**, i.e. when

$$X_c(j\Omega) = 0 \quad \text{for all} \quad |\Omega| \geq \pi/T$$

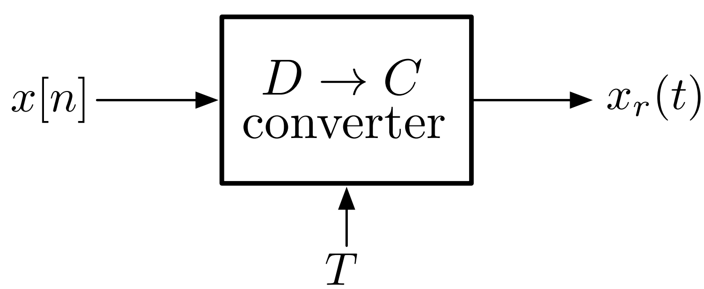
where $X_c(j\Omega)$ is the continuous time Fourier transform (CTFT) of $x_c(t)$:

$$X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t}\, \mathrm{d}t.$$

In other words, the sampling rate ($= 1/T$ in Hz, or $2\pi/T$ in rad/sec) must be larger than **twice** the maximum frequency present in the signal.

This is known as the **Nyquist criterion**.

2. We reconstruct the continuous time signal from the discrete sample sequence using **sinc interpolation**:



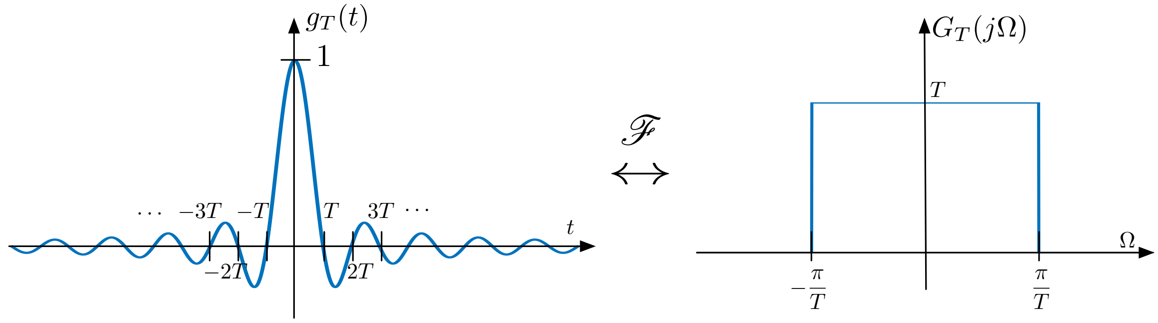
($D \to C$ = discrete-to-continuous.)

Mathematically, we can write the output as:

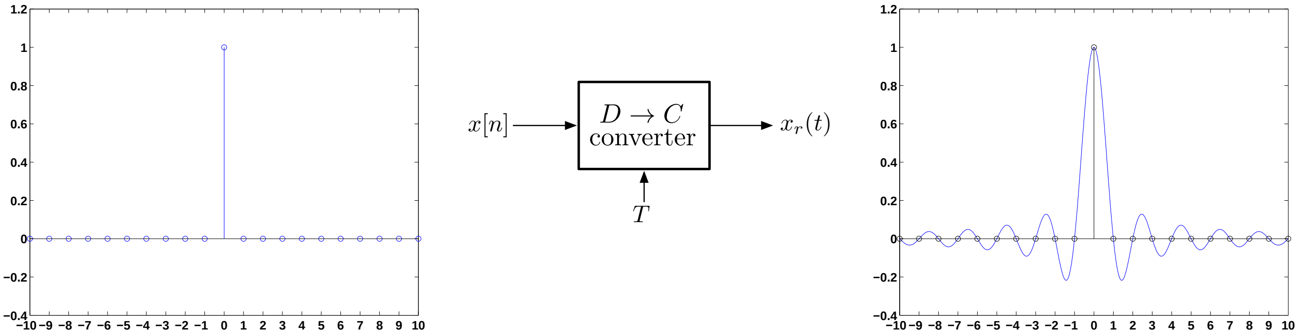
$$\begin{aligned} x_r(t) &= \sum_{n=-\infty}^{\infty} x[n] \, \frac{\sin(\pi(t-nT)/T)}{\pi(t-nT)/T} \\ &= \sum_{n=-\infty}^{\infty} x[n] \, \underbrace{g_T(t-nT)}_{\text{shifts of the sinc}} \end{aligned}$$

Recall that:

$$g_T(t) = \frac{\sin(\pi t/T)}{\pi t/T} \quad \stackrel{\mathscr{F}}{\longleftrightarrow} \quad G_T(j\Omega) = \begin{cases} T, & |\Omega| < \frac{\pi}{T}, \\ 0, & |\Omega| > \frac{\pi}{T}. \end{cases}$$

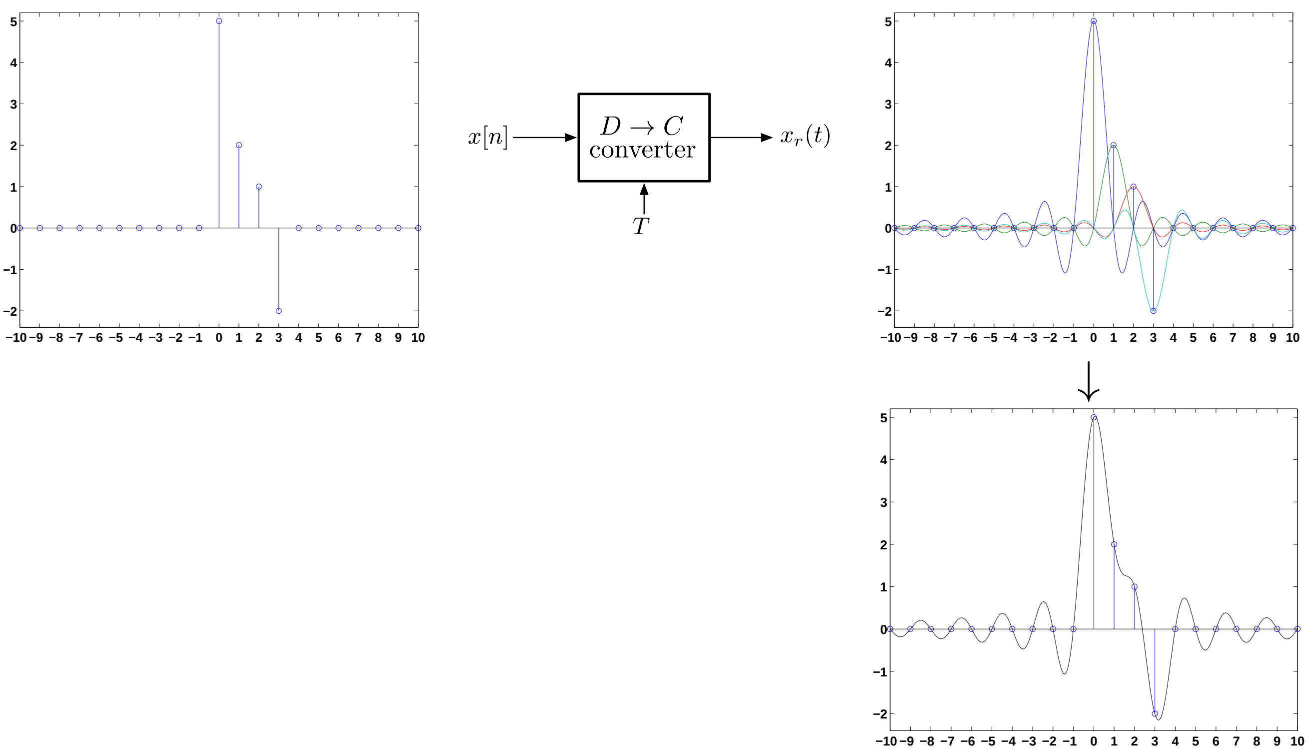


Single sample:



Notice that the sinc function is exactly zero at the other sample locations.

Multiple samples:



In between samples, multiple sincs combine to yield a smooth signal.

The Fundamental Theorem of DSP
If $x_c(t)$ is bandlimited to B ($X_c(j\Omega) = 0$ for $|\Omega| \geq B$), then it can be perfectly reconstructed from samples spaced $T \leq \pi/B$ apart:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]\, g_T(t - nT),$$

where

$$x[n] = x_c(nT), \quad g_T(t) = \frac{\sin(\pi t/T)}{\pi t/T}.$$

1. This is known as the **Shannon-Nyquist sampling theorem**

2. It is the backbone of DSP — it essentially says that we can process $x_c(t)$ by processing its samples

3. The samples are a discrete list of numbers, and hence can be processed **digitally** on a computer, giving us tremendous flexibility.

4. The two equations above are our first example of a **reproducing formula**, which shows how a signal can be written as a discrete combination of linear functionals of that signal (samples, in this case) weighted against a set of fixed "basis" signals. This is a central theme in this first section of the course.

Frequency domain interpretation

Like many things, it is illuminating to look at sampling and reconstruction in the frequency domain.

First, we will relate the **discrete time Fourier transform** (DTFT) of $x[n]$ to the CTFT of $x_c(t)$:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega)\, e^{j\Omega n T}\, \mathrm{d}\Omega \right) e^{-j\omega n} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(j\Omega) \cdot \left(\sum_{n=-\infty}^{\infty} e^{jn(\Omega T - \omega)} \right) \mathrm{d}\Omega.
\end{aligned}
$$

Recall the **Poisson Summation Formula**:

$$
\sum_{n=-\infty}^{\infty} e^{jn\omega} \;=\; 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)
$$

where

$$
\delta(\omega) \;=\; \text{“Dirac delta function”}.
$$

Plugging this in, we have

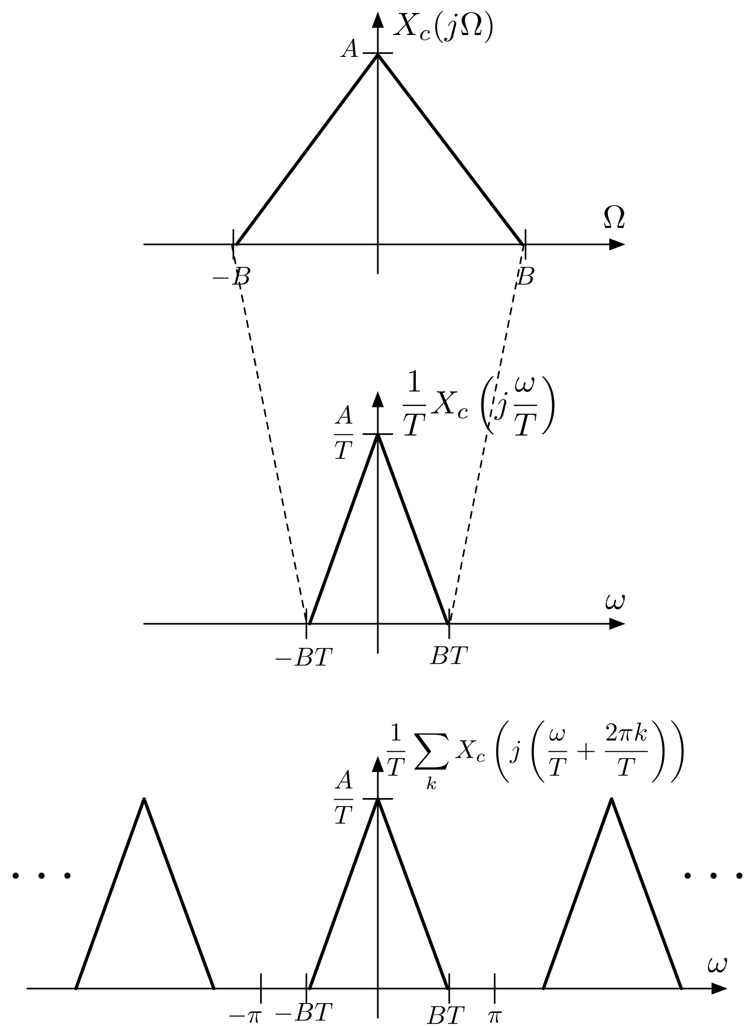
$$
\begin{aligned}
X(e^{j\omega}) &= \int_{-\infty}^{\infty} X_C(j\Omega) \cdot \sum_{k=-\infty}^{\infty} \delta(\Omega T - \omega - 2\pi k)\, \mathrm{d}\Omega \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X_c(j\Omega)\, \delta(\Omega T - \omega - 2\pi k)\, \mathrm{d}\Omega \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j \left(\frac{\omega + 2\pi k}{T} \right) \right)
\end{aligned}
$$

There are essentially two things going on here:

1. $X_c(j\Omega) \longrightarrow \frac{1}{T} X_c\left(j\frac{\omega}{T}\right)$
   dilates the spectrum

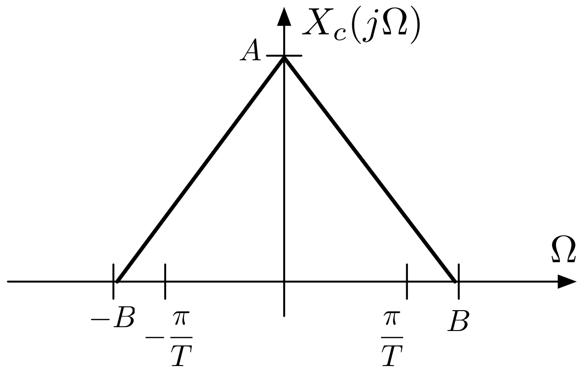
2. $\frac{1}{T} X_c\left(j\frac{\omega}{T}\right) \longrightarrow \frac{1}{T}\sum_k X_c\left(j\left(\frac{\omega}{T}+\frac{2\pi k}{T}\right)\right)$
   makes this dilation **periodic** (w/ period 2π).

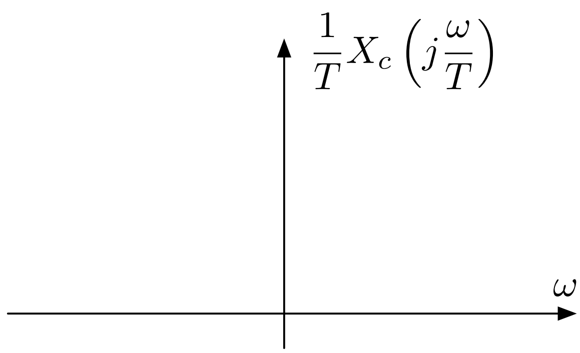
Graphically, this is what happens for $B < \pi/T$:

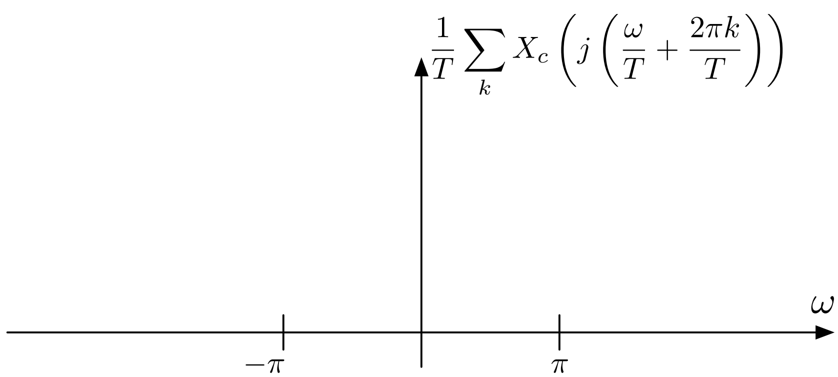


Aliasing

If $T > \pi/B$, there is trouble:







What is another signal with the same samples as $x_c(t)$?

Reconstruction

The reconstructed signal is

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]\, g_T(t - nT),$$

and so

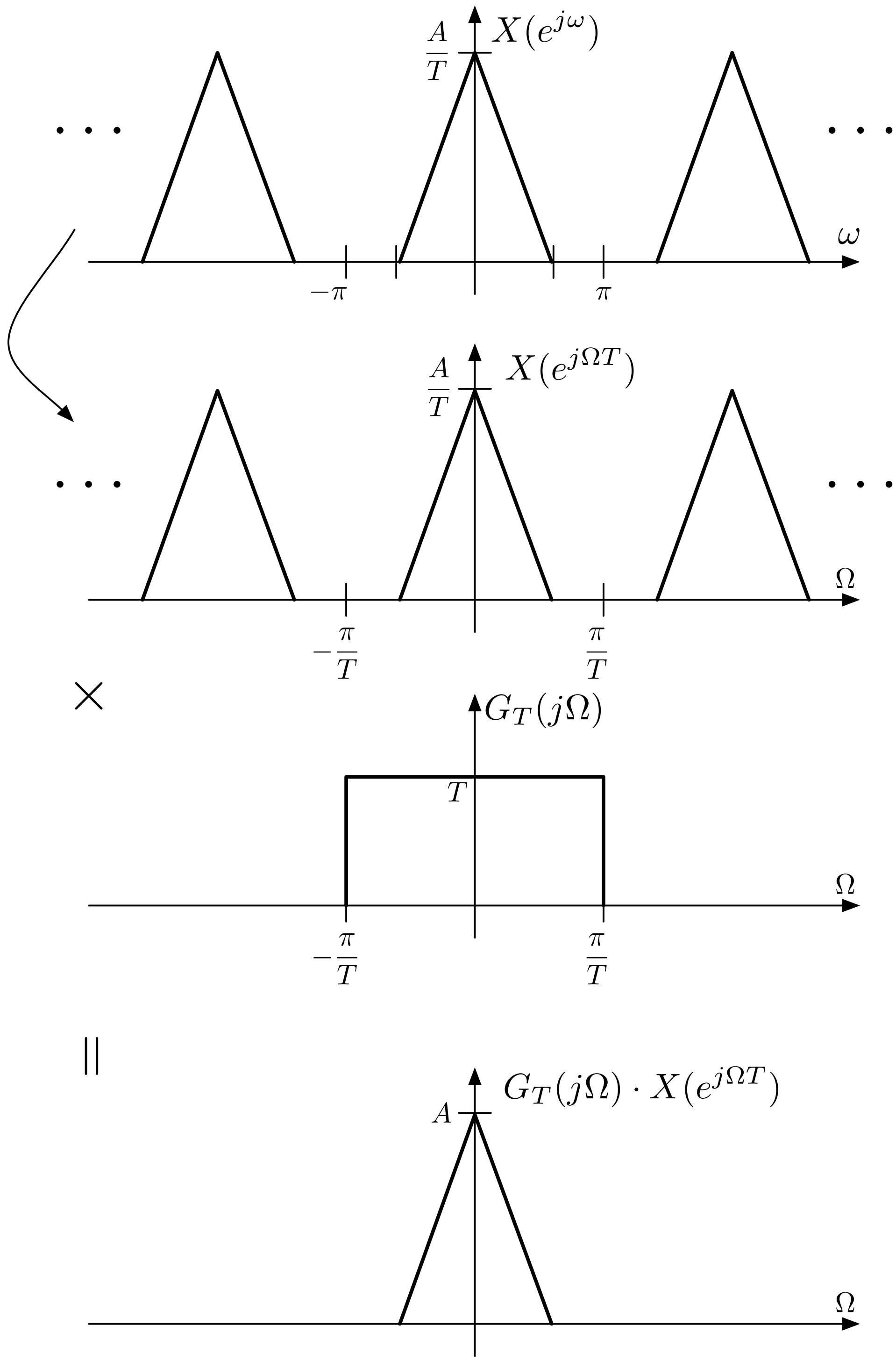
$$\begin{aligned} X_r(j\Omega) &= \sum_{n=-\infty}^{\infty} x[n]\, G_T(j\Omega)\, e^{-j\Omega nT} \\ &= G_T(j\Omega) \cdot \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega nT} \\ &= G_T(j\Omega)\, X(e^{j\Omega T}) \end{aligned}$$

Again, there are two steps:

1. $X(e^{j\omega}) \longrightarrow X(e^{j\Omega T})$
   dilates the (periodic) spectrum

2. $X(e^{j\Omega T}) \longrightarrow G_T(j\Omega) \cdot X(e^{j\Omega T})$
   restricts the spectrum to its fundamental period

Graphically, this is what happens:

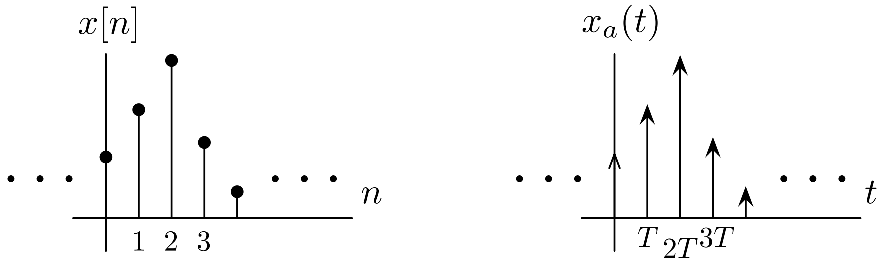


A little more on the $X(e^{j\omega}) \longrightarrow X(e^{j\Omega T})$ step ...

What we are doing is taking a discrete sequence $x[n]$ (with DTFT $X(e^{j\omega})$) and turning it into a function $x_a(t)$ (with CTFT $X_a(j\Omega) = X(e^{j\Omega T})$) of a continuous time variable.

Set

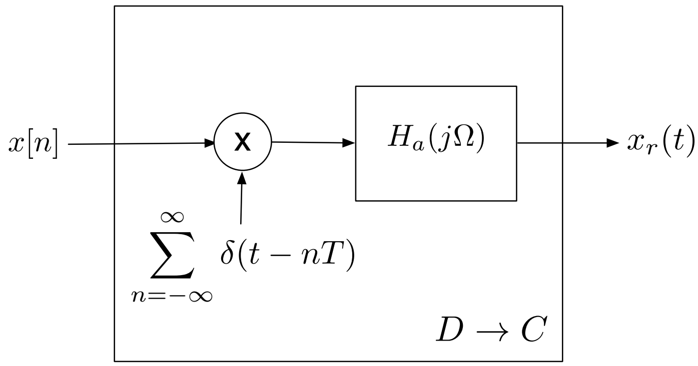
$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t - nT)$$



Then

$$\begin{aligned} X_a(j\Omega) &= \int_{-\infty}^{\infty} \sum_n x[n]\delta(t - nT)\, e^{-j\Omega t}\, \mathrm{d}t \\ &= \sum_n x[n] \int_{-\infty}^{\infty} \delta(t - nT)\, e^{-j\Omega t}\, \mathrm{d}t \\ &= \sum_n x[n]\, e^{-j\Omega T n} = X(e^{j\Omega T}) \end{aligned}$$

So the $D \to C$ converter converts the sample sequence into a **spike train** and then low pass filters it. We can interpret what is inside this block as:

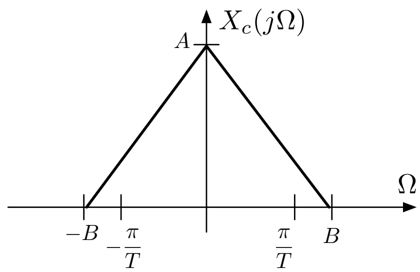


where

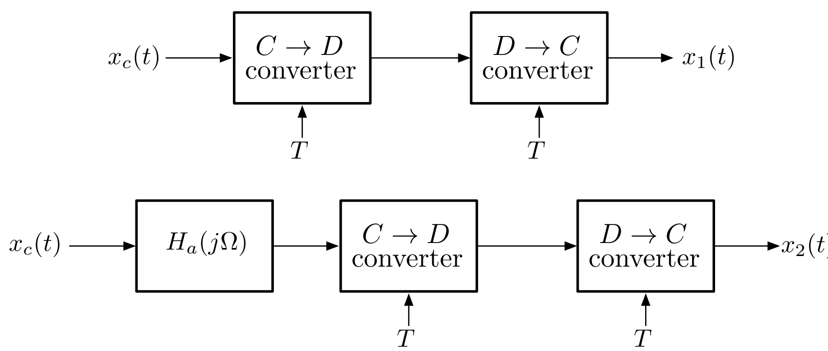
$$H_a(j\Omega) = \begin{cases} T, & |\Omega| < \frac{\pi}{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Anti-aliasing filters

Suppose the spectrum of $x_c(t)$ looks like



Compare the outputs of these two systems:



where

$$H_a(j\Omega) = \begin{cases} 1 & |\Omega| < \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}.$$

Which is closer to $x_c(t)$?

That is, which is smaller:

$$\int |x_c(t) - x_1(t)|^2 \, \mathrm{d}t \quad \text{or} \quad \int |x_c(t) - x_2(t)|^2 \, \mathrm{d}t \quad ?$$

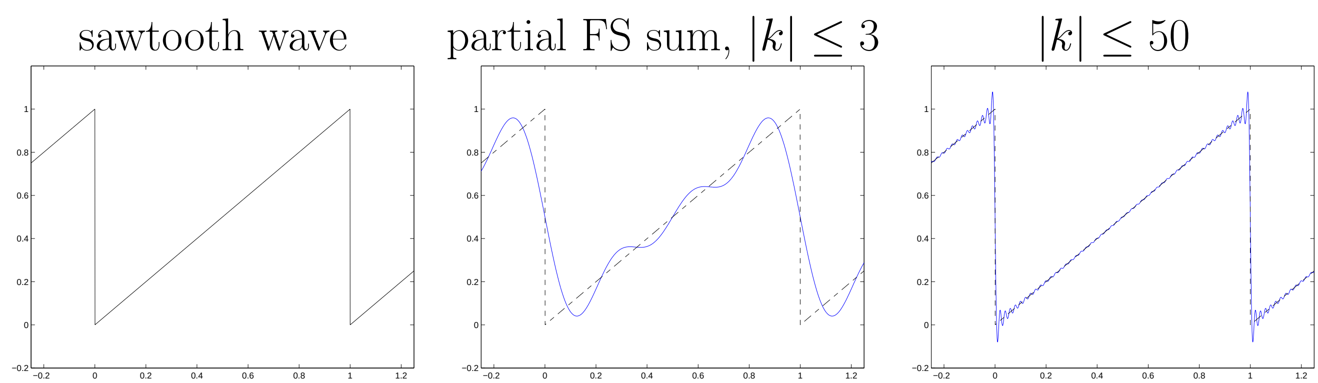
Relationship to Fourier series

Recall that any periodic signal can be written as a (possibly infinite) superposition of **harmonic sinusoids**. If $x(t)$ has period T, we can write

$$x(t) = \sum_{k=-\infty}^{\infty} \alpha_k \, e^{j2\pi kt/T}, \tag{1}$$

$$\text{where} \quad \alpha_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, e^{-j2\pi kt/T} \; \mathrm{d}t. \tag{2}$$

(The integral above can be computed over any interval of length T.) The two equations above are another example of a reproducing formula — (2) shows how to systematically take a signal and map it into a discrete list of numbers, while (1) shows how to take that list of numbers and synthesize the signal.



Equivalently, we can think of the Fourier series as building up a function that is time-limited to $[-T/2, T/2]$. That every ("reasonable") function can be represented this way is a deep result in mathematics, which we will talk a little more about later. But it is **mathematically equivalent to the sampling theorem**, we just switch the roles of time and frequency.

To see this, suppose that $x(t)$ is zero outside of $[-T/2, T/2]$, so (1) is building it up only inside this interval. Then its Fourier transform is

$$X(j\Omega) = \int_{-T/2}^{T/2} x(t)\, e^{-j\Omega t}\ \mathrm{d}t.$$

Notice that the Fourier series coefficients α_k in (2) are samples of the Fourier transform spaced $2\pi/T$ apart and scaled by $1/T$:

$$\alpha_k = \frac{1}{T} X\left(j\frac{2\pi k}{T}\right).$$

Now we can write the Fourier transform as a combination of samples α_k:

$$\begin{aligned}
X(j\Omega) &= \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \alpha_k\, e^{j2\pi kt/T}\, e^{-j\Omega t}\ \mathrm{d}t \\
&= \sum_{k=-\infty}^{\infty} \alpha_k \int_{-T/2}^{T/2} e^{j(2\pi k/T-\Omega)t}\ \mathrm{d}t \\
&= \sum_{k=-\infty}^{\infty} \alpha_k\, \frac{2T\sin(\Omega T/2 - \pi k)}{\Omega T - 2\pi k} \\
&= \sum_{k=-\infty}^{\infty} X\left(j\frac{2\pi k}{T}\right) g_{2\pi/T}(\Omega - 2\pi k/T),
\end{aligned}$$

where as before $g_{2\pi/T}(\Omega)$ is a sinc function. This is exactly the same reproducing formula we had for the Shannon-Nyquist sampling theorem. Here it says that the Fourier transform of a signal which is time-limited to T can be reconstructed from samples taken $2\pi/T$ in frequency.

Appendix: Technical Review

The continuous-time Fourier transform (CTFT)

The CTFT of a signal $x(t)$ is

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, \mathrm{d}t,$$

where $j = \sqrt{-1}$. The convention of using $j\Omega$ as the argument (instead of just Ω) is historical, and is common in the signal processing literature.

Anytime you see an integral expression like the one above, it is fair to ask whether or not it converges. If $x(t)$ is *absolutely integrable*, in that

$$\int_{-\infty}^{\infty} |x(t)|\, \mathrm{d}t < \infty,$$

then $X(j\Omega)$ is well-defined for all $\Omega \in \mathbb{R}$. It is also bounded, as in this case

$$|X(j\Omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\Omega t}\, \mathrm{d}t \right| \le \int_{-\infty}^{\infty} |x(t)|\, |e^{-j\Omega t}|\, \mathrm{d}t = \int_{-\infty}^{\infty} |x(t)|\, \mathrm{d}t.$$

If $x(t)$ has *finite energy*, in that

$$\int_{-\infty}^{\infty} |x(t)|^2\, \mathrm{d}t < \infty,$$

then the Fourier transform is also well-defined, but you have to be a little more careful about what it means for two functions to be equal to one another. We will talk a little more about this later, but it is really just a mathematical detail which ends up not affecting our outlook on this topic at all.

The inverse CTFT is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}\,\mathrm{d}\Omega.$$

Parseval's theorem states that the energy in the time- and frequency-domains are equal to one another (to within a constant of $1/2\pi$):

$$\int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\Omega)|^2\,\mathrm{d}\Omega.$$

The discrete-time Fourier transform (DTFT)

The DTFT of the sequence of numbers $\{x[n],\ n \in \mathbb{Z}\}$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

Again, this sum is clearly well-defined (and bounded) when $x[n]$ is absolutely summable,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty,$$

and we can make sense of it when $x[n]$ has finite energy,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty.$$

Notice that $X(e^{j\omega})$ is 2π-periodic, as

$$e^{-j\omega n} = e^{-j(\omega+2\pi\ell)n} \quad \text{for all} \quad \ell \in \mathbb{Z}.$$

The inverse DTFT is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, \mathrm{d}\omega.$$

The DTFT also preserves energy up to a constant, as

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, \mathrm{d}\omega.$$

The Dirac delta function

The Dirac delta is a *generalized function*, defined through the relation

$$\int_{-L}^{L} x(t)\delta(t) \, \mathrm{d}t = x(0), \quad \text{for any} \;\; L > 0.$$

More generally,

$$\int_{t \in \mathcal{T}} x(t)\delta(t - t_0) \, \mathrm{d}t = \begin{cases} x(t_0), & \text{if } t_0 \in \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases}$$

$\delta(t)$ is not a function in the usual sense, but we can manipulate algebraically in much the same way we manipulate standard functions.

The delta function is the "derivative" of the Heaviside step function

$$\mu(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases},$$

in that they obey a relation of the same form as the Fundamental Theorem of Calculus:

$$\mu(t) = \int_{-\infty}^{t} \delta(\tau) \;\, \mathrm{d}\tau.$$

The formalism for $\delta(t)$ and other generalized functions is found the mathematical theory of distributions. A nice overview of this theory can be found in the classic text *Distributions, Complex Variables, and Fourier Transforms*, by Hans Bremermann (1965).