# Midterm 1: Patrick Kim

## Section 1.1: Systems of Linear Equations

### Definitions

- Linear equation
  - $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$
  - Organization of coefficients and variables with a solution 'b'
- System of linear equations
  - Collection of multiple linear equations
- Solution of a system
  - $(s_1, s_2, \ldots, s_n)$
  - List of numbers that make each equation a true statement when the s values are substituted for the x variables
- Solution set
  - Set of all possible solutions of a linear system
- Equivalent linear systems
  - 2 linear systems with the **same solution set**
- Consistent system
  - 1 solution or infinitely many solutions
- Inconsistent system
  - No solution for a **specific input**
- Existence
  - Does a solution set exist?
- Uniqueness
  - If a solution exists, is there more than one solution?

### Key Notes

- A system of linear equations has either:
  - No solution
  - Exactly one solution
  - Infinitely many solutions
- Matrix notation
  - Rectangular format that contains info of a linear system
  - Example system
    - $1x_1 - 2x_2 + x_3 = 0$
    - $0x_1 + 2x_2 - 8x_3 = 8$
    - $5x_1 + 0x_2 - 5x_3 = 10$
  - Coefficient matrix

        - $$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$
    - Augmented matrix
        - $$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix}$$
    - Size of a matrix
        - $m$ x $n$
        - $m$: rows
        - $n$: columns
- Row reduction operations
    - Replacement
        - Eliminating elements (making them 0) by comparing two rows and scaling one of them
    - Interchange
        - Swapping rows
    - Scaling
        - Usually done to make a leading entry into one
- Goal of row reduction: to create an **echelon form** or **RREF**
    - Triangle of 0's

## Section 1.2: Row Reduction and Echelon Forms

### Definitions

- Non-zero row/column
    - Row or column with **at least one** nonzero entry
- Zero row/column
    - Row or column with **all zeros**
- Leading entry
    - Leftmost nonzero entry in a row
- Row reduced echelon form (RREF)
    - A simplified matrix that represents a potential solution set for a linear system
    - Each matrix has only one RREF
- Pivot position
    - Location in a matrix that corresponds to a leading 1 in RREF
- Pivot column
    - Column that contains a pivot position
- Basic/leading variables
    - Variables that correspond to a pivot
    - Basic variables have an exact value for a solution set

- Free variables
  - Variables that do not correspond to any pivots and pivot columns
  - Can be assigned **any value** for a consistent linear system
- Overdetermined system
  - # of rows > # of columns
  - System of linear equations with more equations than unknowns
    - Can be consistent
    - Can have a unique solution
  - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Underdetermined system
  - # of columns > # of rows
  - System of linear equations with more unknowns than equations
  - Can **never** have a unique solution (always a **free variable**)
    - If system is consistent -> infinite solutions
    - If system is inconsistent -> no solution
  - $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

## Key Notes

- Echelon Form of a Matrix
  - 3 Properties:
    1. All zero rows are **at the bottom**
    2. Each leading entry (non-zero entry) of a row is to the **right** of any leading entries in the row above it (if any)
    3. Below a leading entry, all entries are 0
- RREF
  - All leading entries are 1's
  - There are 0's **above and below** each leading 1
- A matrix can be in **neither** echelon form nor RREF
  - This means that **more row reduction** needs to be done
- Uniqueness of the RREF
  - Each matrix is **row equivalent** (has same solution set) to **one and only one** reduced echelon matrix
    - A matrix has **only one RREF** matrix
- **Inconsistent** systems have **empty** solution sets

- Existence and Uniqueness Theorem
    - A linear system is consistent if and only if the rightmost column of the augmented matrix is **not** a pivot column
        - No row of the form:
            - [0 0 0 0 0 | b] with b non-zero
    - If a linear system is consistent, the solution set has either:
        - Unique solution (no free variables)
        - Infinitely many solutions (at least one free variable)

## Section 1.3: Vector Equations

### Definitions

- Vector
    - An ordered list of numbers
- $R^n$
    - R: collection of all lists of $n$ real numbers
    - n: number of entries (rows) in the vector
- Zero vector: 0
    - Vector with all entries 0
- Linear combination
    - Given vectors $\{v_1, v_2, ..., v_p\}$ in $R^n$ and given scalars $\{c_1, c_2, ..., c_p\}$, a vector $y$ defined by $y = c_1v_1 + c_2v_2 + ... + c_pv_p$ is a linear combination
- Span$\{v_1 ... v_p\}$
    - Collection of all vectors that can be written in the form
    $$c_1v_1 + c_2v_2 + ... + c_pv_p$$

### Key Notes

- Vectors in $R^x$
    - $R^2$ vector: $\begin{bmatrix} a \\ b \end{bmatrix}$
    - $R^3$ vector: $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
- Vectors in $R^2$ can be represented as a line to a point in a 2D space
- Vectors in $R^3$ can be represented as a line to a point in a 3D space
- Graphically adding vectors in $R^2$
    - Add "tip to tail"
- Algebraic properties of $R^n$
    - For all u, v, w in $R^n$ and all scalars c & d:
    i. $u + v = v + u$ v. $c(u + v) = cu + cv$

ii. $(u + v) + w = u + (v + w)$ vi. $(c + d) u = cu + du$
iii. $u + 0 = 0 + u = u$ vii. $c (du) = (cd) u$
iv. $u + (-u) = -u + u = 0$ viii. $1u = u$

- A **vector equation** $\mathbf{x_1a_1 + x_2a_2 + \ldots + x_na_n = b}$ has the same solution set as the linear system whose augmented matrix is $[a_1\ a_2\ ..\ a_n \mid b]$
  - b can be generated by a linear combination of $a_1, \ldots, a_n$ if and only if there exists a **solution (weights: $\mathbf{x_1, \ldots, x_n}$)** to the linear system corresponding to the matrix
- If $v_1 \ldots v_p$ are in $R^n$, then the set of all linear combinations of $v_1 \ldots v_p$ is denoted by Span$\{v_1 \ldots v_p\}$ and is called the subset of $R^n$ spanned by $v_1 \ldots v_p$
  - Span$\{v_1 \ldots v_p\}$: collection of all vectors that can be written in the form
$$c_1v_1 + c_2v_2 + \ldots + c_pv_p$$
- Is vector b in Span$\{v_1, \ldots, v_p\}$? == does $x_1v_1 + x_2v_2 + \ldots + x_pv_p = b$ have a solution?
  - Solve $[v_1 \ldots v_p \mid b]$
  - Is b a linear combination of the vectors in $\{v_1 \ldots v_p\}$?
    - **Is there a pivot in every row?**

## Section 1.4: The Matrix Equation Ax=b

### Definitions

- Identity matrix
  - The "one" of multiplying matrices
  - Outputs the same input
- $x \in R^n$
  - x is a vector with *n* elements

### Key Notes

- Star Equation
  - $Ax = \begin{bmatrix} a1 & a2 & \ldots & an \end{bmatrix} \cdot \begin{bmatrix} x1 \\ x2 \\ \ldots \\ xn \end{bmatrix} = x_1a_1 + x_2a_2 + \ldots x_na_n$
    - x: weights
    - Ax is defined only if the number of **columns in A** == number of **entries in x**
- If A is an *m* x *n* matrix, with columns $a_1, \ldots a_n$, and if *b* is in $R^m$:
  - Matrix equation == vector equation == augmented matrix for a linear system
  - $(Ax = b) == (x_1a_1 + x_2a_2 + \ldots + x_na_n = b) == ([a_1\ a_2 \ldots a_n \mid b])$
- Ax = b as a linear combination has **two parts**
  1. A vector
  2. x vector

    - Span of the columns essentially means **multiplying** these **two parts**
- Ax = b has a solution if and only if *b* is a linear combination of the columns of A
- Logically equivalent statements for an $m$ x $n$ matrix A (all true or all false)
    - For each b in $R^m$, the equation Ax = b has a solution
    - Each b in $R^m$ is a linear combination of the columns of A
    - The columns of A span $R^m$
    - A has a pivot position in every row
- If A is an $m$ x $n$ matrix, u & v are vectors in $R^n$, and c is a scalar, then:
    - A(u + v) = Au + Av
    - A(cu) = c(Au)

## Section 1.5: Solution Sets of Linear Systems

### Definitions

- Homogeneous linear system
    - System of linear equations written in the form: A**x** = 0
- Trivial solution
    - x vector = 0
- Nontrivial solution
    - x vector that satisfies Ax = 0 and has **at least one** non-zero element
- Nonhomogeneous linear system
    - System of linear equations written in the form: A**x** = **b**
    - Where b != 0

### Key Notes

- Homogeneous linear system (Ax = 0) always has **at least one solution**

- Trivial solution: x = 0
- Homogeneous system has a nontrivial solution if there is **at least one free variable**
- **Implicit** description of a plane
    - $10x_1 - 3x_2 - 2x_3 = 0$
- **Explicit** description of a plane (**Parametric Vector Equation**)
    - x = s**u** + t**v**
        - x: x vector
        - s, t in R
    - x = $x_2$**u** + $x_3$**v**
        - $x_2$ and $x_3$ are free variables
- Parametric Vector Form for a consistent …
    - Ax = b
        - x = u + tv
    - Ax = 0
        - x = tv
- For a consistent Ax = b, the solution set of Ax = b is the set of all vectors of the form

  $$w = p + v_k$$

  where $p$ is a solution and $v_k$ is any solution of Ax = 0
- Writing a solution set in Parametric Vector Form
    1. Row reduce the augmented matrix to RREF
    2. Express each basic variable in terms of any **free variable** appearing in an equation
    3. Write a **typical solution x** as a vector whose entries depend on the free variables
    4. Decompose **x** into a linear combination of vectors using the **free variables as parameters**
        a. Ex: $$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} \rightarrow \underset{[p]}{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}} + x_3 \underset{[v]}{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}$$

## Section 1.7: Linear Independence

### Definitions

- Linearly independent
    - Vector equation $x_1v_1 + x_2v_2 + \ldots + x_pv_p = 0$ has **only the trivial solution**
    - Matrix A has a **pivot in every column**
    - No free variables
- Linearly dependent

- Vector equation $c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$ where weights $c_1, \ldots, c_p$ are **not all zero**
- At least one free variable

**Key Notes**

- If a set of vectors is **linearly independent**, there are **no free variables**
    - **No free variables = pivot in every column**
- If a set of vectors is **linearly dependent**, there is **at least one free variable**
- Quick facts
    - If # of columns > # of rows, then $\{v_1, \ldots, v_p\}$ is linearly dependent
        - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: $x_2$ is free
    - If $\{v_1, \ldots, v_p\}$ is linearly independent, then # of rows ≥ # of columns
        - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$: no free variables
        - $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$: no free variables (there is no $x_3$ here)
    - If $Ax = 0$ has a free variable, then $\{v_1, \ldots, v_p\}$ is linearly dependent
- Sets with **one vector**: $x_1v_1$
    - Linearly independent if and only if $v_1$ is **not the 0 vector**
    - If $v_1$ is the 0 vector -> $x_10$
        - Has infinite nontrivial solutions
        - Linearly dependent
- Sets with **two vectors**: $x_1v_1$ and $x_2v_2$
    - Linearly **dependent** if at least one of the vectors is a **multiple** of the other
    - Linearly **independent** if and only if **neither** of the vectors is a multiple of the other
- Sets with **2 or more vectors**:
    - Linearly dependent if **at least one** of the vectors can be written as a **linear combination** of all the other vectors
    - One vector is **in the span** of the other vectors
        - Vector is a **multiple** of all the other vectors
- If at least one vector is the **zero vector**, then the system is **linearly dependent**

## Section 1.8: Introduction to Linear Transformations

**Definitions**

- Matrix Transformation
  - Assigns (transforms) a vector **x** in $R^n$ to a vector T(**x**) in $R^m$
- Linear Transformation
  - A matrix transformation that preserves the operations of vector addition and scalar multiplication
    - T(cu + dv) = cT(u) + dT(V)
- Domain of transformation $T: R^n \to R^m$
  - Input: set $R^n$
- Codomain of transformation $T: R^n \to R^m$
  - Output: set $R^m$
- Image of **x** under the action of *T*
  - $T(\mathbf{x})$ in $R^m$
- Range of *T*
  - Set of all images $T(\mathbf{x})$
- Principle of superposition
  - $T(c_1\mathbf{v_1} + \ldots + c_k\mathbf{v_k}) = c_1T\mathbf{v_1} + \ldots + c_kT\mathbf{v_k}$

**Key Notes**

- (Solving equation Ax = b) == (finding all vectors **x** in $R^n$ that are transformed into the vector **b** in $R^m$ under the "action" of multiplication by A
- Let A be an $m$ x $n$ matrix -> derive a function:
  - Matrix transformation: $T: R^n \to R^m$, $T(\mathbf{x}) = A\mathbf{x}$
  - Multiplier (A): $m \times n$
  - Domain of T: $R^n$
    - Number of entries in **x**
  - Codomain of T: $R^m$
    - Number of entries in T(**x**): image of **x** under *T*
  - Vector T(**x**)
    - Image of **x** under *T*
  - Range
    - Set of all possible images $T(\mathbf{x})$
- $T: R^y \to R^x$
  - T has $x$ rows and $y$ columns
- A transformation *T* is **linear** if:

  i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of *T*

  ii. $T(c\mathbf{u}) = c(T\mathbf{u})$ for all scalars c and all **u** in the domain of *T*
  - Example:
    - y = 2x
      - f(2 + 3) = f(2) + f(3): linear
    - $y = x^2$

- $f(2 + 3) \neq f(2) + f(3)$: not linear
- Every matrix transformation is a linear transformation
- $T(\mathbf{x}) = r\mathbf{x}$
  - Contraction: $0 \leq r < 1$
  - Dilation: $r > 1$

## Section 1.9: The Matrix of a Linear Transformation

### Definitions

- Standard matrix for a linear transformation $T$
  - $A = [T(e_1) + \ldots + T(e_n)]$
- $e_1$ in $R^2$= $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2$ in $R_2$=$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Onto (existence question)
  - $T: R^n \rightarrow R^m$ is **onto** $R^m$ if each $\mathbf{b}$ in $R^m$ is the image of **at least one** $\mathbf{x}$ in $R^n$
    - At least 1 solution of $T(\mathbf{x}) = \mathbf{b}$
    - Pivot in **every row**
    - Columns of A **spans** $\mathbf{R^m}$
- One-to-one (uniqueness question)
  - $T: R^n \rightarrow R^m$ is **one-to-one** if each $\mathbf{b}$ in $R^m$ is the image of **at most one** $\mathbf{x}$ in $R^n$
    - $T(\mathbf{x}) = \mathbf{b}$ has either **1 solution** or **no solutions**
    - Pivot in **every column**
    - Columns of A are **linearly independent**

### Key Notes

- Every linear transformation from $R^n$ to $R^m$ is also a matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$
  - Finding A: observe what $T$ does to the **standard matrix**
- Geometric Linear Transformations of $R^2$
  - Reflections
    - Reflection through the $x_1$ axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
    - Reflection through the $x_2$ axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
    - Reflection through the line $x_2 = x_1$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
    - Reflection through the line $x_2 = -x_1$: $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$
    - Reflection through the origin: $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
  - Contractions & expansions

        - $0 < k < 1$: contraction
        - $k > 1$: expansion
        - Horizontal contraction & expansion: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
        - Vertical contraction & expansion: $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$
    - Shears
        - Horizontal shear: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
            - $k < 0$: left shear
            - $k > 0$: right shear
        - Vertical shear: $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$
            - $k < 0$: down shear
            - $k > 0$: up shear
    - Projections
        - Projections on the $x_1$-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
        - Projections on the $x_2$-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
    - Rotation
        - CCW rotation: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- Geometric description
    - Onto: can get to any vector with an image
    - One-to-one: cannot have multiple vectors have the same image
- Onto
    - A linear transformation T: $R^n \to R^m$ is **onto** if for all $\mathbf{b} \in R^m$ there is an $\mathbf{x} \in R^n$ so that T(**x**) = A**x** = **b**
        - A**x** = **b** is **always consistent**
        - **At least** one solution
        - Existence property
    - T is onto if and only if its **standard matrix** has a **pivot in every row**
- One-to-one
    - A linear transformation T: $R^n \to R^m$ is one-to-one if for all $\mathbf{b} \in R^m$ there is **at most one** (possible 0) $\mathbf{x} \in R^n$ so that T(**x**) = A**x** = **b**
        - A**x** = **b** has **at most 1** solution
        - No free variables
        - Uniqueness property

- T is one-to-one if and only if the only solution to T(**x**) = **0** is the **trivial solution**
- T is one-to-one if and only if **every column** of A is **pivotal**

## Section 2.1: Matrix Algebra

### Key Notes

- **Theorem 1**
  - Let A, B, and C be matrices of the same size, and let $r$ and $s$ be scalars.

    a. A + B = B + A | d. $r$ (A + B) = $r$A + $r$B
    b. (A + B) + C = A + (B + C) | e. ($r$ + s) A = $r$A + sA
    c. A + 0 = A | f. $r$ (sA) = ($r$s) A
- Matrix multiplication
  - If A is an $m$ x $n$ matrix, and if B is an $n$ x $p$ matrix with columns $b_1, \ldots, b_p$, then the product AB is the $m$ x $p$ matrix whose columns are $Ab_1, \ldots, Ab_p$

    $$AB = A[b_1 \quad b_2 \;\ldots\; b_p] = [Ab_1 \quad Ab_2 \ldots Ab_p]$$
  - # of columns in A == # of rows in B
- **Theorem 2: Properties of Matrix Multiplication**
  - Let A be an $m$ x $n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

    a. A (BC) = (AB) C
    b. A (B + C) = AB + AC
    c. (B + C) A = BA + BC
    d. r (AB) = (rA) B = A (rB)
    e. $I_mA = A = AI_n$
- Matrices that **commute**
  - Matrices A and B commute when AB = BA
- Warnings
  - **Order** when multiplying matrices **matters**
    - In general, AB ≠ BA
  - AB = AC does not suggest B = C
  - If AB is the zero matrix, cannot conclude in general that either A = 0 or B = 0
- Transpose of a Matrix
  - Given an $m$ x $n$ matrix, the transpose of A is the $n$ x $m$ matrix, denoted by $A^T$
  - $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$
  - $B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$
- **Theorem 3**

- Led A and B denote matrices whose sizes are appropriate for the following sums and products
  a. $(A^T)^T = A$
  b. $(A + B)^T = A^T + B^T$
  c. For any scalar r, $(rA)^T = rA^T$
  d. $(AB)^T = B^TA^T$
- The **transpose** of a product of matrices equals the product of their transposes in the **reverse order**

- Powers of Matrices
  - Can **only** be applied to **square matrices**

## Theorems

### Theorem 1: Uniqueness of RREF

- Each matrix is row equivalent to one and only one row reduced echelon matrix.

### Theorem 2: Existence and Uniqueness Theorem

- A linear system is consistent if and only if the rightmost column of the augmented matrix is **not** a pivot column - that is, if and only if an echelon form of the augmented matrix has **no** row of the form:

$$[0 \ \dots \ 0 \ b] \text{ with } b \text{ nonzero}$$

- If a linear system is consistent, then the solution set contains either:
  i. a unique solution (no free variables)
  ii. infinitely many solution (at least one free variable)

### Theorem 3: Matrix, Vector, and Linear Equations

- If A is an $m$ x $n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $R^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

- has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{b}$$

- which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \dots \quad \mathbf{a}_n \mid \mathbf{b}]$$

### Theorem 4: Logically Equivalent Statements

- Let A be an $m$ x $n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.
  a. For each $\mathbf{b}$ in $R^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.
  b. Each $\mathbf{b}$ in $R^m$ is a linear combination of the columns of A.
  c. The columns of A span $R^m$.

*d.* A has a pivot position in every row.

**Theorem 5: Properties of the Matrix-Vector Product Ax**

- If A is an $m$ x $n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $R^n$, and $c$ is a scalar, then:
  - *a.* $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
  - *b.* $A(c\mathbf{u}) = c(A\mathbf{u})$

**Theorem 6: Parametric Vector Form of a Nonhomogeneous System**

- Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = 0$.

**Theorem 7: Characterization of Linearly Dependent Sets**

- An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

**Theorem 8: Linear Dependence based on Matrix Size**

- If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $R^n$ is linearly dependent if the number of columns > than the number of rows.

**Theorem 9: Linear Dependence based on a Zero Vector**

- If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $R^n$ contains the zero vector, then the set is linearly dependent.

**Theorem 10: Using the Standard Matrix to find Columns of A**

- Let $T: R^n \to R^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } R^n$$

- In fact, A is the $m$ x $n$ matrix whose $j$th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the $j$th column of the identity matrix in $R^n$

$$A = [T(\mathbf{e}_1) \ldots T(\mathbf{e}_n)]$$

**Theorem 11: One-to-One using the Homogeneous Equation**

- Let $T: R^n \to R^m$ be a linear transformation. Then $T$ is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution

### Theorem 12: Onto and One-to-One

- Let $T$: $R^n$ -> $R^m$ be a linear transformation, and let A be the standard matrix for $T$. Then:

  a. $T$ maps $R^n$ onto $R^m$ if and only if the columns of A span $R^m$
    - All rows have pivots

  b. $T$ is one-to-one if and only if the columns of A are linearly independent
    - All columns have pivots
    - A has linearly independent columns