Problem -1 State Space representation, linearization and simulation

a) Non linear state space description

given, $-m_2 L \ddot{\omega} \cos\theta + m_2 L^2 \ddot{\theta} - m_2 g L \sin\theta = 0$ — (1)

$(m_1+m_2)\ddot{\omega} - m_2 L \ddot{\theta}\cos\theta + m_2 L \dot{\theta}^2 \sin\theta = f(t)$ — (2)

State variables: $x = [\theta, \dot{\theta}, \omega, \dot{\omega}]^T$

Solving for $\ddot{\theta}$: From 1,

$$m_2 L^2 \ddot{\theta} = m_2 L (g\sin\theta + \ddot{\omega}\cos\theta)$$

$$\ddot{\theta} = \frac{\cancel{m_2}\cancel{L}(g\sin\theta + \ddot{\omega}\cos\theta)}{\cancel{m_2} L^{\cancel{2}}}$$

$$\ddot{\theta} = \frac{g}{L}\sin\theta + \frac{\ddot{\omega}}{L}\cos\theta \quad \text{— (3)}$$

Solving for $\ddot{\omega}$: From 2,

$$\ddot{\omega} = \frac{m_2 L \ddot{\theta}\cos\theta - m_2 L \dot{\theta}^2 \sin\theta + f(t)}{(m_1+m_2)}$$

From 3,

$$\ddot{\omega} = \frac{m_2 \cancel{L} \cdot \frac{1}{\cancel{L}}(g\sin\theta + \ddot{\omega}\cos\theta)\cos\theta - m_2 L \dot{\theta}^2 \sin\theta + f(t)}{(m_1+m_2)}$$

$$\ddot{\omega} = \frac{f(t) + m_2 g \sin\theta\cos\theta - m_2 L \dot{\theta}^2 \sin\theta}{m_1 + m_2 - m_2\cos^2\theta} \quad \text{— (4)}$$

State-Space Representation

With state variables $x = [\theta, \dot{\theta}, \omega, \dot{\omega}]^T$ and input $u = f(t)$, we can define state derivatives:

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\omega} \\ \ddot{\omega} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{g}{L}\sin\theta + \frac{\ddot{\omega}}{L}\cos\theta \\ \dot{\omega} \\ \dfrac{f(t) + m_2 g \sin\theta\cos\theta - m_2 L \dot{\theta}^2 \sin\theta}{m_1 + m_2 - m_2\cos^2\theta} \end{bmatrix}$$

Moreover, if we assume θ to be small angle i.e $\theta \approx 0°$, $\sin\theta = \theta$ and $\cos\theta = 1$:

$$\ddot{\theta} \approx (g\theta + 1)/L$$

$$\ddot{\omega} \approx \frac{f(t) + m_2 g\theta - m_2 L \dot{\theta}^2 \theta}{m_1 + \cancel{m_2} - \cancel{m_2}}$$

b) Linearization about equilibrium points

$$\Delta\dot{x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 & \partial f_1/\partial x_4 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 & \partial f_2/\partial x_4 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 & \partial f_3/\partial x_4 \\ \partial f_4/\partial x_1 & \partial f_4/\partial x_2 & \partial f_4/\partial x_3 & \partial f_4/\partial x_4 \end{bmatrix} \Delta x + \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \partial f_1/\partial u_3 & \partial f_1/\partial u_4 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \partial f_2/\partial u_3 & \partial f_2/\partial u_4 \\ \partial f_3/\partial u_1 & \partial f_3/\partial u_2 & \partial f_3/\partial u_3 & \partial f_3/\partial u_3 \\ \partial f_4/\partial u_1 & \partial f_4/\partial u_2 & \partial f_4/\partial u_3 & \partial f_4/\partial u_4 \end{bmatrix} \Delta u$$

4.b) BIBO Stability

The system is BIBO stable if $g(t) = \mathcal{L}^{-1}\{G(s)\}$ satisfies

$$\int_0^{\infty} \| g(\tau) \| \, d\tau < \infty \quad (\text{is finite})$$

$$G(s) = C(sI - A)^{-1} B$$

$$G(s) = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1 & 0 & 0 \\ 40 & s & 0 & 0 \\ 0 & 0 & s & 1 \\ 5 & 0 & 0 & s \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 2/3 \\ 0 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1/s & 0 & 1 & 0 \\ 0 & -1/s & 0 & 1 \\ -\frac{\sqrt{5/2}}{4} & 1/16 & 0 & 0 \\ \frac{\sqrt{5/2}}{4} & 1/16 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2/3 \\ 0 \\ 1/2 \end{bmatrix}$$