Homework 4
ME5659 Spring 2024

Due: See Canvas, turn in on Gradescope

Problem 1 (4 points)

Consider the following LTI system

$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} -2 & 3 \end{bmatrix} x - 2u$$

(a) **1 points.** Is the system asymptotically stable?

(b) **3 points.** Is the system BIBO stable? Does BIBO stability imply asymptotic stability for this system? Why or why not?

SOLUTIONS)

a) $A = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix}$ Let's check the eigenvalues $\det(\lambda I - A) = 0$

$\det \begin{bmatrix} \lambda+1 & -10 \\ 0 & \lambda-1 \end{bmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-1) = 0$

$\Rightarrow \lambda^2 - 1 = 0,\ \lambda^2 = 1 \Rightarrow \lambda = \pm 1$

the system is NOT Asymptotically stable. It is unstable since $Re(1) > 0$

b) To check BIBO STABILITY

$h(t) = \mathcal{L}^{-1}(H(s))$

$H(s) = C(sI - A)^{-1}B + D$

$H(s) = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2$

$H(s) = \begin{bmatrix} -2 & 3 \end{bmatrix} \frac{1}{s^2-1} \begin{bmatrix} s-1 & 10 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2$

$H(s) = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{10}{s^2-1} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2$

$H(s) = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{-2}{s+1} \\ 0 \end{bmatrix} - 2u = \frac{4}{s+1} - 2$

IN TIME

$h(t) = 4e^{-t} - 2\delta(t) \quad t \geq 0$ IMPULSE response

$\int_0^{\infty} \| h(t) \| d\tau = \int_0^{\infty} 4e^{-t} - 2\delta(t)\, dt = 4 < \infty$ it IS FINITE

$\Rightarrow$ the system IS BIBO STABLE

If the system is A.S then is also BIBO stable (as the poles of the H(s) are a subset of. The poles of te system). However BIBO stability doesn't generally imply internal stability.. BIBO stability implies internal stability only when the system has no transmission zeros (i.e when the number of poles of the H(s) is equal to thee number of poles of the state-space representation of the system, or, in other words , when the state variable system is a minimal representation of the transfer function). The transfer function of this system is not a minimal representation, and s even if it is BBO stable, the system is not A.S

Problem 2 (9 points)

Controllability test. Consider the following linear systems, where

(i) $A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

(ii) $A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$

(iii) $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$

(a) **3 points.** Assess the controllability using the controllability matrix.

(b) **3 points.** Use the Popov-Belevitch-Hautus tests for controllability assessment.

(c) **3 points.** Compute the controllable canonical form.

SOLUTIONS

a) CONTROLLABILITY matrix $P = \begin{bmatrix} B & AB \end{bmatrix}$ if $\text{rk } P = n$ then the system is controllable

i) $A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$

$\det P = \det\begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} = 0 \implies \text{rk } P = 1 < n \implies$ the system is NOT controllable

ii) $A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\det P = -1 \neq 0 \implies \text{rk } P = 2 = n$ the system is controllable

iii) $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $AB = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix} \Rightarrow \det P = 0 \Rightarrow rk\, P = 1 < n$ the system is not controllable.

b) RANK TEST

(A,B) is controllable iff the rank of $([\lambda I - A \mid B]) = n$ for all $\lambda \in \mathbb{C}$

i) $A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda + 4 & 0 \\ 0 & \lambda + 5 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_1 = -5 \\ \lambda_2 = -4 \end{array}$

1^{st} case $\lambda_1 = -4$

$rk([\lambda I - A \mid B]) = n$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow rk = 2$

both have to be $rk = n = 2$

2^{nd} case $\lambda_2 = -5$

$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rk = 1$

$\Rightarrow$ the system is not controllable

ii) $A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 10 & \lambda + 2 \end{bmatrix}$ $\det \begin{bmatrix} \lambda & -1 \\ 10 & \lambda + 2 \end{bmatrix} = \lambda(\lambda + 2) + 10 = \lambda^2 + 2\lambda + 10$

$\lambda_{1,2} = -1 \pm 3i$

1st case $\lambda_1 = -1+3i$

$rk([\lambda I - A \mid B]) = n$

$$\begin{bmatrix} -1+3i & -1 & 0 \\ 10 & 1+3i & 1 \end{bmatrix} \Rightarrow rk = 2$$

both have $rk = 2$

2nd case $\lambda_2 = -1-3i$

$$\begin{bmatrix} -1-3i & -1 & 0 \\ 10 & 1-3i & 1 \end{bmatrix} \Rightarrow rk = 2$$

$\Rightarrow$ the system is controllable

iii) $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda-2 & 0 \\ 1 & \lambda-1 \end{bmatrix}, \quad \det\begin{bmatrix} \lambda-2 & 0 \\ 1 & \lambda-1 \end{bmatrix} = (\lambda-2)(\lambda-1) \Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 1 \end{matrix}$$

1st case $\lambda_1 = 2$

$rk([\lambda I - A \mid B]) = n$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow rk = 2$$

both have to be $rk = n = 2$

2nd case $\lambda_2 = 1$

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow rk = 1$$

$\Rightarrow$ the system is not controllable

c) we can compute the CCF of the second system only since we can write CCF iff and only iff the system is controllable.

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

we need to find the system characteristic polynomial

$|sI - A|$

$$\det\begin{vmatrix} s & -1 \\ 10 & s+2 \end{vmatrix} = s(s+2)+10 = s^2+\underline{2}s+\underline{10}$$

$$s^2+\underline{a_1}s+\underline{a_0}$$

$$\Rightarrow a_0 = 10$$
$$a_1 = 2$$

$$P_{CCF}^{-1} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

let's find T_{CCF}

$$T_{CCF} = P\,P_{CCF}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

now, we finally find the CCF realization

$$A_{CCF} = T_{CCF}^{-1} A\, T_{CCF} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}$$

$$B_{CCF} = T_{CCF}^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_{CCF} = C\,T_{CCF} = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$D = D_{CCF} = 0$$

Problem 3 (4 points)

The equations of motion of a satellite, linearized around a steadystate solution, are given by $\dot{x} = Ax + Bu$, where x_1 and x_2 denote the perturbations in the radius and the radial velocity, respectively, x_3 and x_4 denote the perturbations in the angle and the angular velocity, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The input vector consists of a radial thruster u_1 and a tangential thruster u_2.

(a) **(2 points)** Show that the system is controllable from u.

(b) **(2 points)** Can the system still be controlled if the radial thruster fails? What if the tangential thruster fails?

SOLUTIONS

a)

$$P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2w \\ 0 & 1 \\ -2w & 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2w \\ 0 & 1 \\ -2w & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2w \\ -w^2 & 2w \\ -2w & 1 \\ -2w & -4w^2+1 \end{bmatrix}$$

$$A^3B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2w \\ -w^2 & 2w \\ -2w & 1 \\ -2w & -4w^2+1 \end{bmatrix} = \begin{bmatrix} -w^2 & 2w \\ -4w^2 & 6s^3-12s^3+2s \\ -2w & -4w^2+1 \\ 2w^3-2w & 1-8w^2 \end{bmatrix} = \begin{bmatrix} -w^2 & 2w \\ -4w^2 & -2w^3+2w \\ -2w & 1-4w^2 \\ 2w^3-2w & 1-8w^2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 2w & -w^2 & 2w \\ 1 & 0 & 0 & 2w & -w^2 & 2w & -4w^2 & -2w^3+2w \\ 0 & 0 & 0 & 1 & -2w & 1 & -2w & 1-4w^2 \\ 0 & 1 & -2w & 1 & -2w & -4w^2+1 & 2w^3-2w & 1-8w^2 \end{bmatrix}$$

rank P = 4

=> the system is controllable

b) If the radial thruster fails $U_1 = 0$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1, U_2 \end{bmatrix} = \begin{matrix} 0U_1 + 0U_2 \\ 1U_1 + 0U_2 \\ 0U_1 + 0U_2 \\ 0U_1 + 1U_2 \end{matrix}$$

$U_1 = 0$

B is only given by U_2

$$U_1 \text{ fails} \Rightarrow B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2w \\ 1 \\ 1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2w \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2w \\ 2w \\ 1 \\ -4w^2+1 \end{bmatrix}$$

$$A^3B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix} \begin{bmatrix} 2w \\ 2w \\ 1 \\ -4w^2+1 \end{bmatrix} = \begin{bmatrix} 2w \\ -2w^3+2w \\ 1-4w^2 \\ 1-8w^2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 2w & 2w \\ 0 & 2w & 2w & 2w-2w^3 \\ 0 & 1 & 1 & 1-4w^2 \\ 1 & 1 & 1-4w^2 & 1-8w^2 \end{bmatrix}$$

$\operatorname{rank} P = 4$

$\Rightarrow$ the system is controllable

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1, U_2 \end{bmatrix} = \begin{matrix} 0U_1 + 0U_2 \\ 1U_1 + 0U_2 \\ 0U_1 + 0U_2 \\ 0U_1 + 1U_2 \end{matrix}$$

If the tangential thruster faies
$U_2 = 0$
B is oney given by U_1

$$U_2 \text{ fails} \Rightarrow B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3W^2 & 0 & 0 & 2W \\ 0 & 0 & 0 & 1 \\ 0 & -2W & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3W^2 & 0 & 0 & 2W \\ 0 & 0 & 0 & 1 \\ 0 & -2W & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2W \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3W^2 & 0 & 0 & 2W \\ 0 & 0 & 0 & 1 \\ 0 & -2W & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2W \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega^2 \\ -2W \\ -2W \end{bmatrix}$$

$$A^3B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3W^2 & 0 & 0 & 2W \\ 0 & 0 & 0 & 1 \\ 0 & -2W & 0 & 1 \end{bmatrix} \begin{bmatrix} 2W \\ 2W \\ 1 \\ -4\omega^2+1 \end{bmatrix} = \begin{bmatrix} -\omega^2 \\ -4W^2 \\ -2W \\ 2W^3 - 2W \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & -4W^2 \\ 0 & 0 & -2W & -2W \\ 0 & -2W & -2W & 2W^3 - 2W \end{bmatrix}$$

rank $P = 3$
the system is Not controllable

Problem 4 (8 points)

Consider the following the state equation

$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 4 \\ 2 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

(a) **(2 points)** Show that there exists an input $u(t)$ that drives $x(0)$ to 0 in finite time.

(b) **(2 points)** Compute (by hand) the particular input $u(t)$ that achieves this in 4 seconds.

(c) **(2 points)** Verify that the input achieves this by plotting the state trajectories $x(t)$ vs. time t with the initial condition $x(0)$ in Matlab. The plot should have two trajectories $x_1(t), x_2(t)$.

(d) **(2 points)** If possible, compute the controllable canonical form (CCF) of the above state equation. If you cannot, why not?

SOLUTIONS

a) we only need to check wheter the system is controllable or not.

$$A = \begin{bmatrix} -1/2 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad X_0 = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & -2 \\ 2 & -2 \end{bmatrix} \qquad AB = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$\det P = 8-4 = 4 \neq 0 \Rightarrow rk P = 2$ the system is controllable

b)

$$W(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_0-\tau)} B B^T e^{A^T(t_0-\tau)} d\tau$$

CONTROLLABILITY GRAMIAN: OPTIMUM w.r.t the MINIMIZATION of the energy.

$$u(t) = B^T e^{A^T(t_0-t)} W^{-1}(t_0, t_f)\left(e^{A(t_0-t)} x_f - x_0\right)$$

X_f is the origin.

$$W(0,4) = \int_0^4 e^{-A\tau} BB^T e^{-A^T\tau} d\tau \quad \text{where } A = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$e^{-A\tau} = \begin{bmatrix} e^{0.5\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix}$$

$$e^{-A\tau} = \begin{bmatrix} e^{0.5\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix}$$

$$W(0,4) = \int_0^4 \begin{bmatrix} e^{0.5\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} e^{0.5\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix} d\tau$$

$$W(0,4) = \int_0^4 \begin{bmatrix} 4e^{0.5\tau} \\ 2e^{\tau} \end{bmatrix} \begin{bmatrix} 4e^{0.5\tau} & 2e^{\tau} \end{bmatrix} d\tau$$

$$W(0,4) = \int_0^4 \begin{bmatrix} 16e^{\tau} & 8e^{1.5\tau} \\ 8e^{1.5\tau} & 4e^{2\tau} \end{bmatrix} d\tau \Rightarrow W(0,4) = \begin{bmatrix} 16e^{\tau} & \frac{16}{3}e^{1.5t} \\ \frac{16}{3}e^{1.5\tau} & 2e^{2\tau} \end{bmatrix} \Bigg|_0^4$$

$$W(0,4) = \begin{bmatrix} 16e^{4} - 16 & \frac{16}{3}e^{6} - \frac{16}{3} \\ \frac{16}{3}e^{6} - \frac{16}{3} & 2e^{8} - 2 \end{bmatrix}$$

$$W(0,4) = \begin{bmatrix} 857.6 & 2146 \\ 2146 & 5959 \end{bmatrix}$$

This is defined controllability GRAMIEN

and it is a symmetric matrix and all the eigenvalues are $\in \mathbb{R}$ and >0.

when $x(t_f)=0 \Rightarrow u(t) = -B^T e^{A^T(t_0-t)} W^{-1}(t_0,t_f)x_0$ controllable to the origin.

$$u(t) = -B^T e^{-A^T t} W^{-1}(0,3)\, x_0$$

$$e^{-A^T t} = \begin{bmatrix} e^{0.5t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

$$W^{-1}(0,4) = \frac{1}{\det W(0,4)} W(0,4)\, dj.$$

$$W(0,4) = \begin{bmatrix} 857.6 & 2146 \\ 2146 & 5959 \end{bmatrix}$$

$$x_0 = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

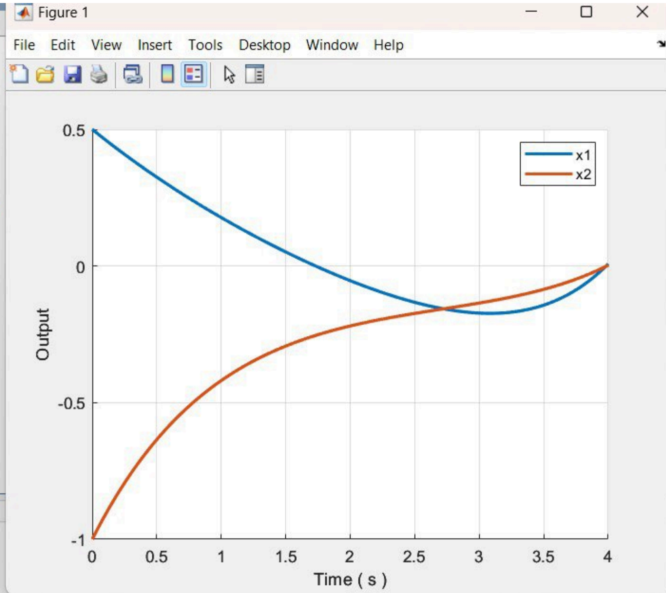
$$W^{-1}(0,4) = \begin{bmatrix} 0.0118 & -0.0043 \\ -0.0043 & 0.0017 \end{bmatrix}$$

$$u(t) = -\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} e^{1/2 t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 0.0118 & -0.0043 \\ -0.0043 & 0.0017 \end{bmatrix} \begin{bmatrix} 0.5 \\ -1 \end{bmatrix} = 0.0077e^{t} - 0.0406e^{t/2}$$

c)

```
clear all; clc; close all;
A =[ -0.5 0; 0 -1];
B =[4;2];
C = eye (2) ;
D =0;
sys = ss (A ,B ,C , D ) ;
t =0:0.01:4;
u =0.0077* exp ( t ) -0.0406* exp (0.5* t ) ;
y = lsim ( sys ,u ,t ,[0.5; -1]) ;

figure ; hold on ; grid on ;
plot (t , y (: ,1) , 'LineWidth'  ,2) ;
plot (t , y (: ,2) , 'LineWidth' ,2) ;
legend ( 'x1 ' , 'x2 ') ;
xlabel ( ' Time ( s ) ') ;
ylabel ( 'Output'  ) ;
```



d) The system is controllable, hence it is possible to compute the CCF

$X(t) = T_{CCF}\, z(t)$ coordinate transformation

$T_{CCF} = P P_{CCF}^{-1}$

Let's find the characteristic polynomial $|sI - A|$

$$\left(\begin{vmatrix} s & 0 \\ 0 & s \end{vmatrix} - \begin{vmatrix} -0.5 & 0 \\ 0 & -1 \end{vmatrix} \right) = \begin{vmatrix} s+0.5 & 0 \\ 0 & s+1 \end{vmatrix} \Rightarrow (s+1)(s+0.5) = 0$$

$$s^2 + 0.5s + s + 0.5 \Rightarrow s^2 + 1.5s + 0.5$$

$$s^2 + a_1 s + a_0$$

$a_0 = 0.5$

$a_1 = 1.5$

$$P_{CCF}^{-1} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix}$$

Finding $T_{CCF} = P \cdot P_{CCF}^{-1} = \begin{bmatrix} 4 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix}$

To find the CCF realization now we need to

$$A_{CCF} = T_{CCF}^{-1} A \cdot T_{CCF} = \frac{1}{4} \begin{bmatrix} 2 & -4 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix}$$

$$A_{CCF} = \begin{bmatrix} 0.5 & -1 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix}$$

$$B_{CCF} = T_{CCF}^{-1} B = \begin{bmatrix} 0.5 & -1 \\ -0.25 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

controller canonical form.

$$\dot{z} = \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

INITIAL CONDITION $z_0 = T_{CCF}^{-1} X_0 = \begin{bmatrix} 1.25 \\ -1.25 \end{bmatrix}$