Controls Homework - 6

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Problem-1 :

(i)

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) PBH test :

eigenvalues: $|\lambda I - A| = 0$.

$$\begin{vmatrix} \lambda+4 & 0 \\ 0 & \lambda+5 \end{vmatrix} = 0 \Rightarrow (\lambda+4)(\lambda+5) = 0.$$

eigenvalues $\Rightarrow \lambda = -4, -5$.

$v \neq 0$.

$$\begin{bmatrix} C \\ CA \end{bmatrix} = 0$$

$$CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 0 \end{bmatrix}$$

$$O = \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} \rightarrow \text{rank} = 1 < 2$$

System is not observable.

(b)

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \quad R^{-1} = R$$

$$\bar{A} = RAR^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\bar{B} = RB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\bar{C} = CR^{-1} = \begin{bmatrix} 1 & 0 \end{bmatrix} I$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$-5 < 0$$

$\Rightarrow$ Detectable

(ii) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ $\quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$\begin{pmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 1 \end{pmatrix} \longrightarrow \quad \lambda_1 = 2, \ \lambda_2 = 1$$

For $\lambda = 2$

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{rank} = 2$$

For $\lambda = 1$

$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{rank is } 2.$$

<u>System is observable.</u>

b)

$R_1 = C = [1 \quad 1]$

$R_2 = R_1 A + \lambda_1 R_1$

$= [1 \quad 1] \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 2 [1 \quad 1]$

$R_2 = [4 \quad 3]$

$\bar{A} = RAR^{-1} = \begin{bmatrix} -2 & 1 \\ -12 & 5 \end{bmatrix}$

$\bar{B} = RB = (1 \quad 3]^T$

$\bar{C} = CR^{-1} = [1 \quad 0]$

OCF:

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ -12 & 5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(iii) $$\begin{bmatrix} \lambda-3 & 6 & 4 \\ 9 & \lambda-6 & 10 \\ -7 & -7 & \lambda+9 \end{bmatrix}$$

$$\lambda_1 = -3$$
$$\lambda_2 = -2$$
$$\lambda_3 = 5$$

For $\lambda_1 = -3 \longrightarrow$ rank $= 3$
For $\lambda_2 = -2 \longrightarrow$ rank $= 3$
For $\lambda_3 = 5 \longrightarrow$ rank $= 3$

System is not observable.

b)

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} 3 & 6 & 4 \\ 9 & 6 & 10 \\ -7 & -7 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -3 & -3 \\ -6 & -6 & -12 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} -6 & 3 & -3 \\ 12 & 12 & 24 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 0 & -3 & -3 \\ -6 & -6 & -12 \\ -6 & 3 & -3 \\ 12 & 12 & 24 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 1 & 0 & 0 \end{bmatrix} \qquad R^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2/3 & -1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 8 & -10/3 & 3 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 5 & 3 \\ 9 & -7 \\ 1/3 & 4/3 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -2 & 0 \\ 8 & -10/3 & 0 \end{bmatrix} x + \begin{bmatrix} 5 & 3 \\ 9 & 7 \\ 1/3 & 4/3 \end{bmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x$$

$$\underline{5 > 0}$$

<u>System is not detectable</u>

2)

$x = [p \quad r \quad \beta \quad \phi]^T$

$u = [\delta_1 \quad \delta_2]^T$

$$\dot{x} = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

$y = p = [1 \quad 0 \quad 0 \quad 0]\, x$

$C = [1 \quad 0 \quad 0 \quad 0]$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \\ -1000 & -114 & -984.9 & 0 \end{bmatrix}$$

rank $= 3 < 4$

not observable

using gyro.

$$Q' = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \end{bmatrix}$$

rank $= 4$ full rank

$\therefore$ observable using indicator.

3)

(a) $$G(s) = \frac{s-1}{s^3+2s^2-s-2}$$

$$\Rightarrow \frac{s-1}{(s-1)(s+1)(s+2)}$$

$$G(s) = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

upon solving,

$$= \frac{1}{s-1} + \frac{1}{s+1} - \frac{2}{s+2}$$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}, \quad D = 0.$$

$$Q_0 = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ -1 & -1 & 2 \\ 2 & 2 & -4 \end{bmatrix}$$

$$\text{Det}(Q_0) = 1(4-4) - 1(4-4) - 2(-2+2)$$
$$= 0$$

$\Rightarrow$ not observable.

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\det(sI - A) = (s+1)^2(s+2)$$

$$\lambda_1 = -1, \ \lambda_2 = -1, \ \lambda_3 = -2$$

Since all eigenvalues < 0, they lie in left-half eigen plane.

The system is detectable.

These realisations are not fully observable, system is detectable.

(b) As the system is not fully observable,

$\therefore$ The realisation is not fully observable.

$$C = [B \ AB \ A^2B] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ -2 & 4 & 8 \end{bmatrix}$$

rank $= 2 < 3$

so, not controllable.

Also, not stable because stability requires full controllability,

$\therefore$ not stable.

c) In the system, The difference in controllability & Stability arises due to diff dynamics represented by system matrices A, B & C.

For controllability, the controllablity matrix is determined by the system's dynamics represented by Matrix A and the control input matrix B. If certain states are unreachable from certain initial positions, the controllability matrix may not have full rank, indicating lack of full controllablity.

For observatity, the Q_o is determind by system's dynamics represented by matrix A and the output C. If certain states cannot be inferred from the system outputs, the Q_o may not have full rank, indicating lack of full observability.

d)
$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad C_o = \begin{bmatrix} c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

Here the charecteristic polynomial is:

$$s^3+2s^2-s-2$$

$$a_0 = -2$$
$$a_1 = -1$$
$$a_2 = 2$$

CCF:

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$C_c = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$$

observability check:

$$Q_o = \begin{bmatrix} C_c \\ C_c A_c \\ C_c A_c^2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -2 & -1 & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

$$\det = -14 \neq 0.$$

$\therefore$ The system is observable.

The system is in OCF.