Homework 3
ME5659 Spring 2024

Due: See Canvas, turn in on Gradescope

Problem 1 (9 points)

Consider the following linear systems $\dot{x} = Ax, x(0) = x_0$, where

(i) $A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}.$

(ii) $A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}.$

(iii) $A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

(a) **3 points.** Characterize the stability of the equilibrium point from the eigenvalues of A.

(b) **3 points.** Use Lyapunov stability analysis to determine whether the system equilibrium state $x_{eq} = 0$ is asymptotically stable. Use $Q = I$ in the Lyapunov equation. (Do all calculations by hand.)

(c) **3 points.** Use MATLAB to plot the state trajectories $x(t)$ vs. time t with the initial condition x_0. Each plot has two trajectories $x_1(t), x_2(t)$. Hand in your plots and your code.

SOLUTIONS

a)

By the theorem we know that: the equilibrium state $x_{eq}=0$ of $\dot{x} = Ax$, $x(0) = x_0$ it is:

1- Stable, in the sense of Lyapunov, if all the eigenvalues of A matrix are $(Re(\lambda_i)) < 0$. So if they have a non-positive real part and for all eigenvalues with zero real part (i.e on imaginary axis), their geometric multiplicity is equal to their algebraic multiplicity. Note: when A has no repeated eigenvalues λ's with $Re(\lambda) = 0$ $\dot{x} = Ax$ is (marginally) stable system.

2- Unstable: if A has any λ's with positive real part or if any eigenvalue with zero real part (i.e on imaginary axis) does not have geometric multiplicity equal to the algebraic multiplicity. or has geometric multiplicity $\neq$ algebraic multiplicity

3- Asymptotically stable if all λ's of A are strictly in the left half plane (have strictly negative real parts)

i) Let's find the eigenvalues of A

$\det(\lambda I - A) = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}\right) = 0 \qquad \det\begin{bmatrix} \lambda & -1 \\ 14 & \lambda+4 \end{bmatrix} = 0 \qquad \lambda(\lambda+4) + 14 = 0 \qquad \lambda^2 + 4\lambda + 14 = 0$$

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16-56}}{2} = \frac{-4 \pm \sqrt{-40}}{2} = \frac{-4 \pm i\,2\sqrt{10}}{2}$$

thus system is asymptotically stable since $Re(-2) < 0$ LHP

ii) $$A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}\right) = 0 \qquad \det\begin{bmatrix} \lambda & -1 \\ 14 & \lambda-4 \end{bmatrix} = 0 \qquad \lambda(\lambda-4) + 14 = 0$$

$$\lambda^2 - 4\lambda + 14 = 0$$

$$\lambda_{1,2} = \frac{+4 \pm \sqrt{16-56}}{2} = \frac{4 \pm i\,2\sqrt{10}}{2}$$

$Re(\lambda_1) = 2 > 0$ $Re(\lambda_2) = 2 > 0$ Right half plane the system is unstable

iii) $$A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}\right) = 0 \qquad \det\begin{bmatrix} \lambda & -1 \\ 14 & \lambda \end{bmatrix} = 0 \quad \Rightarrow \lambda^2 + 14 = 0 \qquad \lambda^2 = -14$$

$$\lambda = \pm i\sqrt{14}$$

$\Rightarrow$ the system is marginally stable.

b) LYAPUNOV STABILITY analysis

i) $$A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}$$

- first of all we need to choose a positive definite $Q = I$ identity matrix

secondly we want to solve $A^T P + PA = -Q$ for P with $P_{12} = P_{21}$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & -14 \\ 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -14 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} & -14P_{22} \\ P_{11}-4P_{21} & P_{12}-4P_{22} \end{bmatrix} + \begin{bmatrix} -14P_{12} & P_{11}-4P_{12} \\ -14P_{22} & P_{21}-4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21}-14P_{12} & -14P_{22}+P_{11}-4P_{12} \\ P_{11}-4P_{21}-14P_{22} & P_{12}-4P_{22}+P_{21}-4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad P_{12}=P_{21}$$

$$\begin{bmatrix} -28P_{12} & -14P_{22}+P_{11}-4P_{12} \\ -14P_{22}+P_{11}-4P_{12} & 2P_{12}-8P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -28 & 0 \\ 1 & -4 & -14 \\ 0 & 2 & -8 \end{bmatrix}\begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{cases} -28P_{12}=-1 \qquad P_{12}=\frac{1}{28}=0.036 \\ P_{11}-4P_{12}-14P_{22}=0 \\ 2P_{12}-8P_{22}=-1 \end{cases} \quad \begin{cases} P_{11}=14P_{22}+4P_{12} \\ P_{12}=0.036 \\ 8P_{22}=1+2P_{12} \end{cases} \quad \begin{cases} P_{11}=14(0.1340)+4\cdot 0.036 \\ P_{12}=0.036 \\ 8P_{22}=1+2(0.036) \end{cases} \quad \begin{cases} P_{11}=1.8760+0.144 \\ P_{12}=0.036 \\ P_{22}=\frac{1+0.072}{8} \end{cases}$$

$P_{11}=2.020$

$P_{12}=0.036$

$P_{22}=0.1340$

$$P=\begin{bmatrix} 2.020 & 0.036 \\ 0.036 & 0.1340 \end{bmatrix}$$

Now we need to check wheter or not P is positive definite. To do so:

1. $P_{11}>0 \qquad 2.02>0$ ✓
2. $\det P>0 \qquad \det P = 0.1340\cdot 2.020 - 0.036\cdot 0.036 = 0.2694 > 0$

=> Hence P is positive definite

-> the system is asymptotically stable

(ii) $A=\begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}$

- first of all we need to choose a positive definite $Q=I$ identity matrix

Secondly we want to solve $A^TP+PA=-Q$ for P with $P_{12}=P_{21}$

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad A^T=\begin{bmatrix} 0 & -14 \\ 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -14 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} & -14P_{22} \\ P_{11}+4P_{21} & P_{12}+4P_{22} \end{bmatrix} + \begin{bmatrix} -14P_{12} & P_{11}+4P_{12} \\ -14P_{22} & P_{21}+4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21}-14P_{12} & -14P_{22}+P_{11}+4P_{12} \\ P_{11}+4P_{21}-14P_{22} & P_{12}+4P_{22}+P_{21}+4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad P_{12}=P_{21}$$

$$\begin{bmatrix} -28P_{12} & -14P_{22}+P_{11}+4P_{12} \\ -14P_{22}+P_{11}+4P_{12} & 2P_{12}+8P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -28 & 0 \\ 1 & 4 & -14 \\ 0 & 2 & 8 \end{bmatrix}\begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{cases} -28P_{12} = -1 \\ P_{11}+4P_{12}-14P_{22}=0 \\ 2P_{12}+8P_{22} = -1 \end{cases} \quad P_{12} = \tfrac{1}{28} = 0.036 \quad \begin{cases} P_{11} = 14P_{22}-4P_{12} \\ P_{12} = 0.036 \\ 8P_{22} = -1-2P_{12} \end{cases} \quad \begin{cases} P_{11} = 14(-0.1340)-4\cdot 0.036 \\ P_{12}=0.036 \\ 8P_{22} = -1-2(0.036) \end{cases} \quad \begin{cases} P_{11} = -2.020 \\ P_{12} = 0.036 \\ P_{22} = \frac{-1.0720}{8} \end{cases}$$

$P_{11} = -2.020$

$P_{12} = 0.036$

$P_{22} = -0.1340$

$$P = \begin{bmatrix} -2.020 & 0.036 \\ 0.036 & -0.1340 \end{bmatrix}$$

Now we need to check if P is positive definite, to do so we can easily see the first element of the first row and since it is negative P is not positive definite. => not asymptotically stable.

iii)

$$A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 0 & -14 \\ 1 & 0 \end{bmatrix}$$

$$A^T P + PA = -I$$

$$\begin{bmatrix} 0 & -14 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} & -14P_{22} \\ P_{11} & P_{12} \end{bmatrix} + \begin{bmatrix} -14P_{12} & P_{11} \\ -14P_{22} & P_{21} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21}-14P_{12} & -14P_{22}+P_{11} \\ P_{11}-14P_{22} & P_{12}+P_{21} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P_{12} = P_{21}$$

$$\begin{bmatrix} -14P_{21}-14P_{12} & -14P_{22}+P_{11} \\ P_{11}-14P_{22} & P_{12}+P_{21} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -28P_{12} & -14P_{22}+P_{11} \\ P_{11}-14P_{22} & 2P_{12} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -28 & 0 \\ 1 & 0 & -14 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

LOOK 1^{st} and 3^{rd} equations $\Rightarrow \nexists P \Rightarrow$ the system is not asymptotically stable.
We can't determine wheter the system is stable or unstable since P it is only a condidate. But we can certaney say that it is not asymptotically stable.

c)

i)

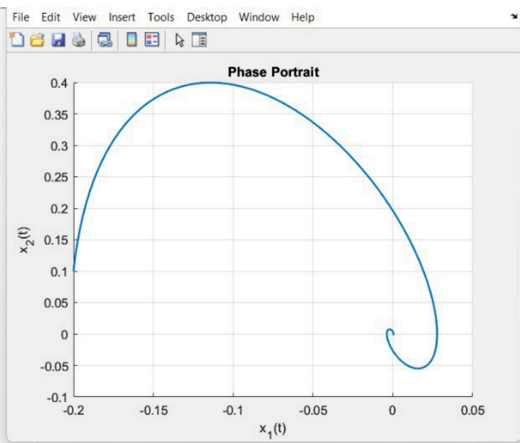
```
clear all;
clc;
close all;

A = [0 1; -14 -4];
x_0 = [-0.2; 0.1];
t = linspace(0, 10, 1000);

figure;
hold on;

x_t = zeros(length(t), 2);
for j = 1:length(t)
    x_t(j, :) = (expm(A * t(j)) * x_0);
end
plot(x_t(:,1), x_t(:,2), 'LineWidth', 1.5);

xlabel('x_1(t)');
ylabel('x_2(t)');
title('Phase Portrait');
grid on;
```



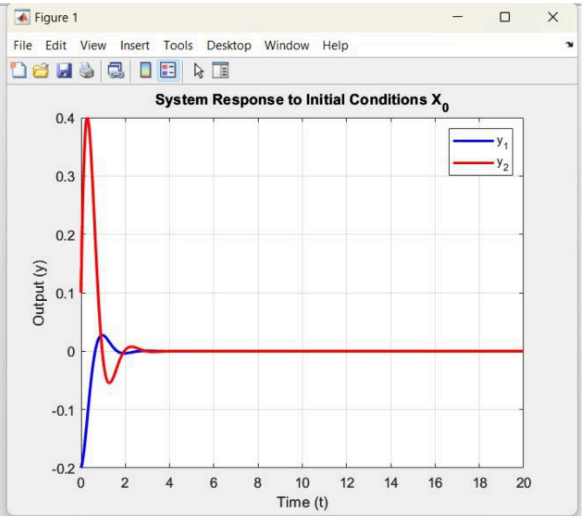
```matlab
clear all;
clc;
close all;

A = [0 1; -14 -4];
B = [0; 0];
C = [1 0; 0 1];
D = [0];

sys1 = ss(A, B, C, D);
t = 0:0.01:20;
u = ones(size(t));
X_0 = [-0.2; 0.1];

% System response
y = lsim(sys1, u, t, X_0);

% Plot of the system output over time
figure;
plot(t, y(:,1), 'b', 'LineWidth', 2); % First state variable
hold on;
plot(t, y(:,2), 'r', 'LineWidth', 2); % Second state variable
xlabel('Time (t)');
ylabel('Output (y)');
title('System Response to Initial Conditions X_0');
legend('y_1', 'y_2');
grid on;
```



ii)

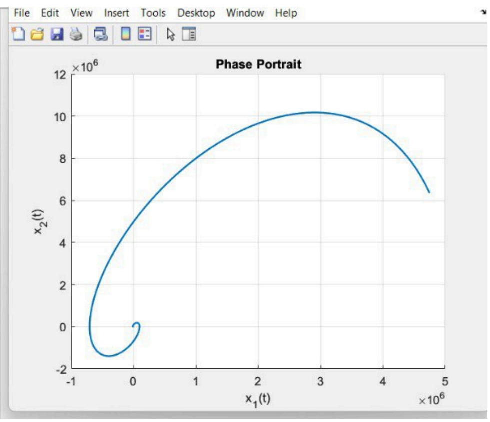
```matlab
clear all;
clc;
close all;

A = [0 1; -14 4];
X_0 = [0.01; 0.02];
t = linspace(0, 10, 1000);

figure;
hold on;

X_t = zeros(length(t), 2);
for j = 1:length(t)
    X_t(j, :) = (expm(A * t(j)) * X_0); % No need to transpose here
end
plot(X_t(:,1), X_t(:,2), 'LineWidth', 1.5);

xlabel('x_1(t)');
ylabel('x_2(t)');
title('Phase Portrait');
grid on;
```



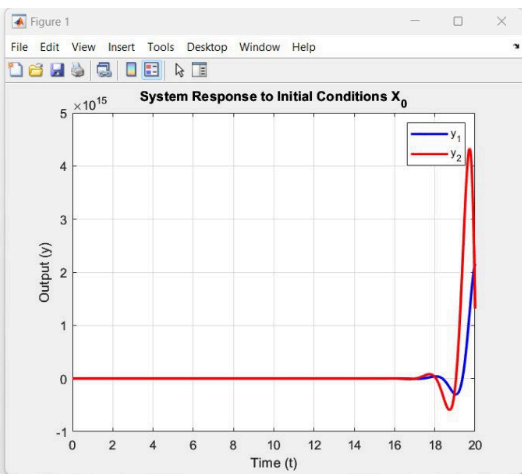

```matlab
clear all;
clc;
close all;

A = [0 1; -14 4];
B = [0; 0];
C = [1 0; 0 1];
D = [0];

sys1 = ss(A, B, C, D);
t = 0:0.01:20;
u = ones(size(t));
X_0 = [0.01; 0.02];

% System response
y = lsim(sys1, u, t, X_0);

% Plot of the system output over time
figure;
plot(t, y(:,1), 'b', 'LineWidth', 2); % First state variable
hold on;
plot(t, y(:,2), 'r', 'LineWidth', 2); % Second state variable
xlabel('Time (t)');
ylabel('Output (y)');
title('System Response to Initial Conditions X_0');
legend('y_1', 'y_2');
grid on;
```



iii) NOTE, the initial conditions are 0,0, so we should obtain an empty graphic.

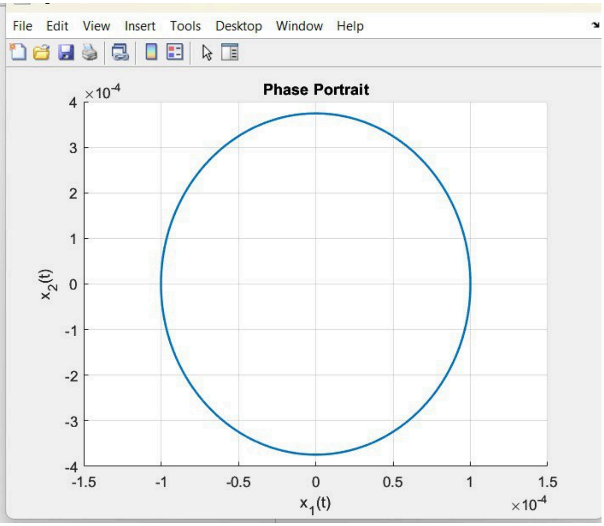
```
clear all;
clc;
close all;

A = [0 1; -14 0];
X_0 = [0.0001; 0.00001];
t = linspace(0, 10, 1000);

figure;
hold on;

X_t = zeros(length(t), 2);
for j = 1:length(t)
    X_t(j, :) = (expm(A * t(j)) * X_0);
plot(X_t(:,1), X_t(:,2), 'LineWidth', 1.5);

xlabel('x_1(t)');
ylabel('x_2(t)');
title('Phase Portrait');
grid on;
```



right initial conditions

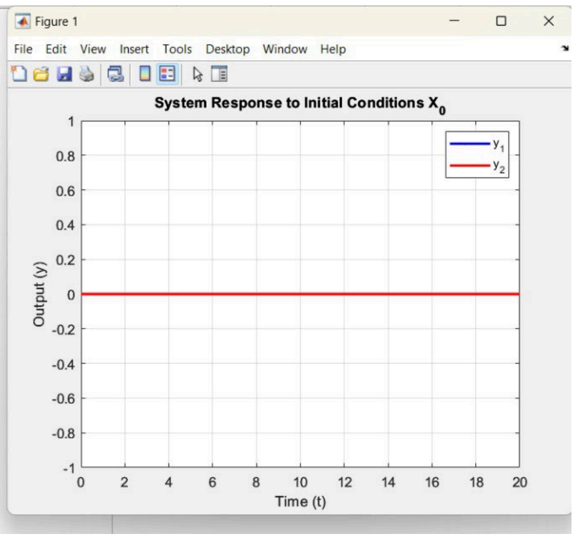
```
clear all;
clc;
close all;

A = [0 1; -14 0];
B = [0; 0];
C = [1 0; 0 1];
D = [0];

sys1 = ss(A, B, C, D);
t = 0:0.01:20;
u = ones(size(t));
X_0 = [0; 0];

% System response
y = lsim(sys1, u, t, X_0);

% Plot of the system output over time
figure;
plot(t, y(:,1), 'b', 'LineWidth', 2); % First state variable
hold on;
plot(t, y(:,2), 'r', 'LineWidth', 2); % Second state variable
xlabel('Time (t)');
ylabel('Output (y)');
title('System Response to Initial Conditions X_0');
legend('y_1', 'y_2');
grid on;
```



Problem 2 (6 points)

Consider the inverted pendulum (Figure 1) which is characterized by

$$ml^2\ddot{\theta} = mglsin\theta - b\dot{\theta} + T$$

where T denotes a torque applied at the base and g is the gravitational acceleration. We assume that $u = T$ and $y = \theta$ are its input and output, respectively.

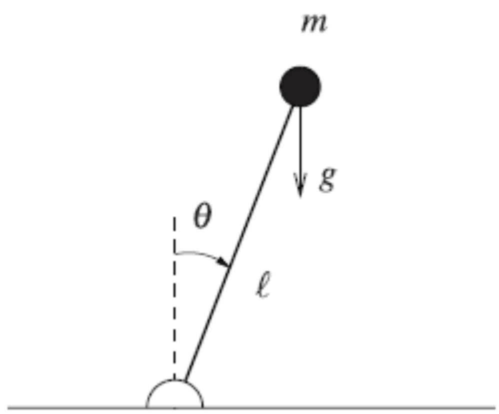


Figure 1: Simple pendulum

(a) **(3 points)** Perform local linearization of this system around the equilibrium point $\theta = \pi$, derive the linear state-space models and determine whether it is stable or not.

(b) **(3 points)** Perform local linearization of this system around the equilibrium point $\theta = 0$, derive the linear state-space models and determine whether it is stable or not.

a) $\theta = \pi$ equilibrium point $\Rightarrow x_1 = \pi$

$$m e^2 \ddot{\theta} = mge\sin\theta - b\dot{\theta} + T$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ state variables $\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{mge}{me^2}\sin x_1 - \frac{b x_2}{me^2} + \frac{T}{me^2} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{e}\sin x_1 - \frac{b}{me^2} x_2 + \frac{T}{me^2} \end{bmatrix}$

Taylor expansion

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x = x_{eq}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{e}\cos\pi & -\frac{b}{me^2} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\Bigg|_{u = u_{eq}} = \begin{bmatrix} 0 \\ \frac{1}{me^2} \end{bmatrix}$$

$\Rightarrow$ Linearized system

$\Delta\dot{x} = A\Delta x + B\Delta u$ where $A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{e} & -\frac{b}{me^2} \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ \frac{1}{me^2} \end{bmatrix}$

To determine whether the system is stable or not we need to find the eigenvalues and see the real part.

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -g/e & -\frac{b}{me^2} \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ g/e & \lambda + \frac{b}{me^2} \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^2 + \frac{b}{me^2}\lambda + \frac{g}{e} = 0$$

$$\lambda_{1,2} = \frac{-\frac{b}{me^2} \pm \sqrt{\left(\frac{b}{me^2}\right)^2 - 4\frac{g}{e}}}{2} = \frac{-\frac{b}{me^2} \pm \sqrt{\frac{b^2 - 4gm^2e^3}{m^2e^4}}}{2} = \frac{-b \pm \sqrt{b^2 - 4gm^2e^3}}{2me^2}$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4gm^2e^3}}{2me^2}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4gm^2e^3}}{2me^2}$$

Real part of λ_1 and $\lambda_2 < 0$

$\Rightarrow$ the linearized system at the equilibrium point $\theta = \pi$ is asymptotically stable.

b) $\theta = 0$ equilibrium point. $\Rightarrow x_1 = 0$

$$me^2\ddot{\theta} = mge\sin\theta - b\dot{\theta} + T$$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$ state variables $\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{mge\sin x_1}{me^2} - \frac{bx_2}{me^2} + \frac{T}{me^2} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{g}{e}\sin x_1 - \frac{b}{me^2}x_2 + \frac{T}{me^2} \end{bmatrix}$

Tayelor expansion

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x = x_{eq}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{e}\cos 0 & -\frac{b}{me^2} \end{bmatrix} \qquad B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\right|_{u = u_{eq}} = \begin{bmatrix} 0 \\ \frac{1}{me^2} \end{bmatrix}$$

$\Rightarrow$ linearized system

$$\Delta\dot{x} = A\Delta x + B\Delta u \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ \frac{g}{e} & -\frac{b}{me^2} \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ \frac{1}{me^2} \end{bmatrix}$$

To determine whether the system is stable or not we need to find the eigenvalues and see the real part.

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ g/e & -\frac{b}{me^2} \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ -g/e & \lambda + \frac{b}{me^2} \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda + \frac{b}{me^2}\lambda - \frac{g}{e} = 0$$

$$\lambda_{1,2} = \frac{-\frac{b}{me^2} \pm \sqrt{\left(\frac{b}{me^2}\right)^2 + 4\frac{g}{e}}}{2} = \frac{-\frac{b}{me^2} \pm \sqrt{\frac{b^2 + 4gm^2e^3}{m^2e^4}}}{2} = \frac{-b \pm \sqrt{b^2 + 4gm^2e^3}}{2me^2}$$

$$\lambda_1 = \frac{-b + \sqrt{b^2 + 4gm^2e^3}}{2me^2} = \frac{-b + \sqrt{\gamma}}{2me^2}$$

where $\gamma = b^2 + 4gm^2e^3$ now $b, g, m, e > 0$ so it will be a real number

$$\lambda_2 = \frac{-b - \sqrt{b^2 + 4gm^2e^3}}{2me^2} = \frac{-b - \sqrt{\gamma}}{2me^2}$$

Real part of λ_1 and $\lambda_2 > 0$

the linearized system at the equilibrium point $\theta = 0$ is not stable.

Problem 3 (10 points)

Consider the following linear system $\dot{x} = Ax, x(0) = x_0$, where

$$A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$$

(a) **(4 points)** Under what conditions on a, b is the system equilibrium, $x_{eq} = 0$, asymptotically stable? Use Lyapunov stability analysis.

(b) **(3 points)** Write the Lyapunov function $V(x)$ and its time derivative $\dot{V}(x)$. Use the P matrix obtained in (a) to evaluate stability.

(c) **(3 points)** If $b = 0, a < 0$, show that the linear system is stable but not asymptotically stable.

a) Using Lyapunov stability analysis

$$A^T P + PA = -I \qquad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}$$

$$\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} aP_{21} & aP_{22} \\ P_{11} + bP_{21} & P_{12} + bP_{22} \end{bmatrix} + \begin{bmatrix} aP_{12} & P_{11} + bP_{12} \\ aP_{22} & P_{21} + bP_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$P_{12} = P_{21}$

$$\begin{bmatrix} 2aP_{12} & P_{11} + aP_{22} + bP_{12} \\ P_{11} + bP_{21} + aP_{22} & 2bP_{22} + 2P_{12} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2a & 0 \\ 1 & b & a \\ 0 & 2 & 2b \end{bmatrix}\begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{cases} a2P_{12} = -1 \\ P_{11} + bP_{12} + aP_{22} = 0 \\ 2P_{12} + 2bP_{22} = -1 \end{cases} \quad \begin{cases} P_{12} = -\frac{1}{2a} \\ P_{11} - \frac{b}{2a} + \frac{1-a}{2b} = 0 \\ 2bP_{22} = \frac{1}{a} - 1 \end{cases} \quad \begin{cases} P_{12} = -\frac{1}{2a} \\ P_{11} = \frac{b}{2a} - \frac{1-a}{2b} = \frac{b^2 - a + a^2}{2ab} \\ P_{22} = \frac{1-a}{2ab} \end{cases}$$

$$P = \begin{bmatrix} \frac{b^2 - a + a^2}{2ab} & -\frac{1}{2a} \\ -\frac{1}{2a} & \frac{1-a}{2ab} \end{bmatrix}$$

P has to be positive negative and so P_{11} must be >0 and then $\det P > 0$.
so to verify the 1^{st} condition $a<0$
so

$$P = \begin{bmatrix} \frac{b^2 - a + a^2}{2ab} & -\frac{1}{2a} \\ -\frac{1}{2a} & \frac{1-a}{2ab} \end{bmatrix} \qquad P_{11} > 0 \quad \frac{b^2 - a + a^2}{2ab} > 0$$

and $\det P > 0 \Rightarrow \frac{1-a}{2ab} \cdot \left(\frac{b^2 - a + a^2}{2ab} \right) - \frac{1}{4a^2} > 0$

$$\frac{(b^2-a+a^2)(1-a)-b^2}{4a^2b^2}>0 \qquad \frac{\cancel{b^2}-\cancel{a}+a^2-\cancel{a}b^2+a^2-a^{\cancel{3}\,2}-\cancel{b^2}}{4a^2b^2}>0$$

$$\Rightarrow \frac{-b^2+2a-a^2-1}{4ab^2}>0$$

and $\frac{b^2-a+a^2}{2ab}>0$

Also we know that

$$\det\begin{bmatrix}\lambda & -1\\ -a & \lambda-b\end{bmatrix}=0 \Rightarrow \lambda(\lambda-b)-a=0 \Rightarrow \lambda^2-b\lambda-a=0$$

$$\lambda_{1,2}=\frac{b\pm\sqrt{b^2+4ac}}{2a}$$ so we want $Re(\lambda_i)<0$ so $b<0$ $a<0$

b) $V(x)=x^TPx$ where $P=\begin{bmatrix}\frac{b^2-a+a^2}{2ab} & -\frac{1}{2a}\\ -\frac{1}{2a} & \frac{1-a}{2ab}\end{bmatrix}$

We can emit ourselves to Lyapunov function candidates that are quadratic

$\dot{x}=Ax \quad x(0)=x_0$

$2\times2 \quad 2\times1$

$$V(x)=x^TPx$$

$$V(x)=[x_1\ x_2]\begin{bmatrix}\frac{b^2-a+a^2}{2ab} & -\frac{1}{2a}\\ -\frac{1}{2a} & \frac{1-a}{2ab}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=[x_1\ x_2]\begin{bmatrix}\frac{b^2-a+a^2}{2ab}x_1-\frac{1}{2a}x_2\\ -\frac{1}{2a}x_1+\frac{1-a}{2ab}x_2\end{bmatrix}=\frac{b^2-a+a^2}{2ab}x_1^2-\frac{1}{a}x_1x_2+\frac{1-a}{2ab}x_2^2$$

$$V(x)=\frac{(b^2-a+a^2)x_1^2-2b\,x_1x_2+(1-a)x_2^2}{2ab}$$

$$\dot{V}(x)=x^TP(Ax)+x^TA^TPx=x^T(PA+A^TP)x=x^TQx$$

$$\dot{V}(x)=[x_1\ x_2]\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=[-x_1\ \ -x_2]\begin{bmatrix}x_1\\x_2\end{bmatrix}=-x_1^2-x_2^2$$

For $a, b<0$, it is clear that

$$V(x)=\frac{(b^2-a+a^2)x_1^2-2b\,x_1x_2+(1-a)x_2^2}{2ab}>0$$

indicating that $V(x)$ is positive definite. Furthermore its time derivative $\dot{V}(x)=-x_1^2-x_2^2<0$ for all $x\neq0$, demonstrates that $\dot{V}(x)$ is negative definite. By combining these two, it can be concluded that the system is asymptotically stable.

c) $b=0 \quad a<0$

$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}$

its eigenvalues are given by $\det(\lambda I - A) = 0$

$$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\begin{bmatrix} \lambda & -1 \\ a & \lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 + a = 0 \qquad \lambda_{1,2} = i\sqrt{a} \qquad Re(\lambda_i) = 0$$

$\Rightarrow$ marginally stable but not asymptotically stable.