ME 5659 Midterm

Friday, March 1, 2024

YOUR NAME HERE (Last, First):

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Instructions

You have 100 minutes to complete this exam.

You are permitted TWO 8.5"x11" (US letter or A4) sheet of paper (both double-sides) for handwritten notes to complete this exam.

Please write your answers ONLY on the pages labeled for each question (front and back). Please do not attach your formula sheet, scratch work pages, or any other pages to this packet as this may cause problems with scanning/grading.

If you do not want a certain part of your work to be graded, please make sure you cross that part out. All submitted work that is not crossed-out will be graded.

This exam has **5** problems. Please make sure you answer all parts of all problems.

Problem 1:	/ 30
Problem 2:	/ 60
Problem 3:	/ 60
Problem 4:	/ 60
Problem 5:	/ 90

Total Score: / 300

DO NOT WRITE ON THIS PAGE
NO WORK HERE WILL BE GRADED

Question 1 (30 points)

(a) $\dot{x} = ax - bxy$

$\dot{y} = dxy - cy$

For equilibrium,

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad ; \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}$$

As there is trivial equilibrium when $x=0$ and $y=0$,

~~As~~ By linearizing at $\partial x = x - x^*$, $\partial y = y - y^*$

let $x^* = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\dot{x}^* = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$

Here $x_1 = x$, $x_2 = \dot{x}_1 \Rightarrow \dot{x} \Rightarrow ax - bxy$.

$$x^* = \begin{pmatrix} x \\ ax - bxy \end{pmatrix} \qquad \dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

equilibrium:

$$A^* = \begin{pmatrix} 1 & 0 \\ a-by & -bx \end{pmatrix}, \quad B^* = \begin{pmatrix} -bx \\ -c \end{pmatrix}$$

$$X^* = \begin{bmatrix} 1 & 0 \\ a-by & -bx \end{bmatrix} x + \begin{bmatrix} -bx \\ -c \end{bmatrix} y.$$

$$y^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - cy$$

(b) For $a = 4$, $b = 1$, $c = 2$, $d = 1$

$\dot{x} = 4x - xy$

$\dot{y} = xy - 2y$

$$x^* = \begin{bmatrix} 1 & 0 \\ 4-y & -x \end{bmatrix} x + \begin{bmatrix} -x \\ -2 \end{bmatrix} y \quad , \text{ From (a)}$$

$$y^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - 2y.$$

TO check stability:

$$a = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = 4 I_{2\times2} .$$

eigen values: $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-4 & 0 \\ 0 & \lambda-4 \end{vmatrix} = 0 \quad \rightarrow \quad (\lambda-4)^2 = 0$$

$$\Rightarrow \quad \lambda = 4$$

$Re(z) > 0$.

and the eigenvalues are not distinct.

Hence, the system is unstable.

Question 2 (60 points)

(a) $\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$y = \begin{bmatrix} 2 & 4 \end{bmatrix} x$

<u>eigen values</u> : $|\lambda I - A| = 0$

$\begin{vmatrix} \lambda+1 & 0 \\ -2 & \lambda-3 \end{vmatrix} = 0 \Rightarrow (\lambda+1)(\lambda-3) = 0$

$\therefore \lambda = -1$ or $\lambda = 3$.

Here the roots are distinct, Hence it is diagniosable.

<u>DCF</u> :

$[\lambda_1 I - A] \Rightarrow \begin{bmatrix} 0 & 0 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix} = 0$.

$0V_{12} + 0V_{11} = 0$

$-2V_{11} - 4V_{12} = 0 \Rightarrow \quad -V_{11} = 2V_{12}$

For simplicity let : $V_{11} = -2,\ V_{12} = 1$

$[\lambda_2 I - A] \Rightarrow \begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} V_{21} \\ V_{22} \end{bmatrix} = 0$.

$\left.\begin{matrix} 4V_{21} + 0V_{22} = 0 \\ -2V_{21} + 0V_{22} = 0 \end{matrix}\right\}$ $V_{22} = \alpha \rightarrow$ For simplicity, $V_{22} = 1$.

$V_{21} = 0$.

$\therefore \quad T = \begin{bmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$

$T^{-1} = \dfrac{\text{adj } A}{\text{Det } A} \Rightarrow$ Adj $A = (\text{Cofactor})^T \longrightarrow$ $C_{11} = (-1)^2 \cdot 1$, $C_{12} = (-1)^3 \cdot 1$

$C_{21} = (-1)^3 \cdot 0$, $C_{22} = (-1)^4 \cdot (-2)$

$C = \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix} \Rightarrow \therefore \text{adj} = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$

Det $= (-2) - (0) \Rightarrow \underline{\underline{-2}}$

$= \dfrac{\begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}}{-2} \Rightarrow \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

$$A^* = T^{-1}AT \Rightarrow \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow (\ )\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B^* = T^{-1}B \Rightarrow \begin{bmatrix} -1/2 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C^* = CT \Rightarrow [2 \ 4]\begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \cancel{\ } \cancel{\begin{pmatrix} 0 \\ 4 \end{pmatrix}} [0 \ 4]$$

$$D^* = 0$$

DCF: $$x^* = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y^* = \begin{bmatrix} 0 \\ 4 \end{bmatrix} x + 0$$

(b) Controllability:

Controllability matrix: $P = [B \ \ AB \ \ A^2B \cdots]$

Here, $P = [B \ \ AB]$

$$AB = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} \rightarrow \text{rank} = 1 < 2$$

Here, rank is less than the State Space Dimension. Hence, System is uncontrollable.

Transformed input matrix: $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

DCF: $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda+1 & 0 \\ 0 & \lambda-3 \end{vmatrix} = 0 \rightarrow \lambda = -1 \text{ or } \lambda = 3.$$

(c) $Y(s) = C(SI-A)^{-1}x_0 + (C(SI-A)^{-1}B + D)\, u(s)$

given $\Rightarrow$ $u(s) = 1$

$x_0 = [2 \ -1]^T$

From (a), $C = [0 \ 4]$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$B = [0 \ 1]^T$$

(c) Continuation

$$(SI-A) = \begin{bmatrix} S+1 & 0 \\ 0 & S-3 \end{bmatrix}$$

$$\cancel{(SI-A)x_0 = \begin{bmatrix} (S+1) & 0 \\ 0 & (S-3) \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$\begin{bmatrix} \cancel{2(S+1)} \\ \cancel{-(S-3)} \end{bmatrix}$$

$$\cancel{C(SI-A)x_0 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}\begin{bmatrix} 2(S+1) \\ -(S-3) \end{bmatrix}}$$

$$\Rightarrow (SI-A)^{-1} = \frac{\begin{bmatrix} S-3 & 0 \\ 0 & S+1 \end{bmatrix}}{(S-3)(S+1)}$$

$$\Rightarrow (SI-A)^{-1} = \begin{bmatrix} \frac{1}{S+1} & 0 \\ 0 & \frac{1}{S-3} \end{bmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$(SI-A)^{-1}x_0 = \begin{bmatrix} 2/(S+1) \\ -1/(S-3) \end{bmatrix}$$

$$C \cdot (SI-A)^{-1} x_0 = \cancel{\ } [0 \ 4]\begin{bmatrix} 2/S+1 \\ -1/(S-3) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-4}{S-3} \end{bmatrix}$$

$$(SI-A)^{-1}B = \cancel{\ } \begin{bmatrix} 0 \\ 1/(S-3) \end{bmatrix}$$

$$C(SI-A)^{-1}B = \begin{bmatrix} \cancel{\ } \\ 4/(S-3) \end{bmatrix}$$

$D = 0.$

$$Y(S) = \frac{-4}{S-3} + \left(\left(\frac{4}{S-3}\right) + 0\right) 1$$

$$= \underline{\underline{0}}$$

$y(t) = 0.$

$\underline{e^{At} = 1}$

Question 3 (60 points)

(a) $|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda+2 & 2 \\ -1 & \lambda+4 \end{vmatrix} = 0.$$

$(\lambda+2)(\lambda+4)+2 = 0.$

$\lambda^2+6\lambda+8+2=0 \Rightarrow \lambda^2+6\lambda+10=0.$

$\dfrac{-6 \pm \sqrt{36-40}}{2} \Rightarrow -3 \pm (\ )i$

$Re(z) < 0 \longrightarrow$ Hence ASYMPTOTICALLY STABLE!

i.e $Re(\lambda_1, \lambda_2) < 0$

let $Q = I$.

$A^T P + PA = -Q \Rightarrow A^T P + PA = -I.$

$$\begin{bmatrix} -2 & 1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} -2 & -2 \\ 1 & -4 \end{bmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{bmatrix} -2P_{11}+P_{21} & -2P_{12}+P_{22} \\ -2P_{11}-4P_{21} & -2P_{12}-4P_{22} \end{bmatrix} + \begin{bmatrix} -2P_{11}+P_{12} & -2P_{11}-4P_{12} \\ -2P_{21}+P_{22} & -2P_{21}-4P_{22} \end{bmatrix} = -I$$

$-4P_{11}+2P_{21} = -1 \Rightarrow P_{21} = \dfrac{4P_{11}-1}{2}$

$-6P_{12}-2P_{11}+P_{22} = 0.$

$-6P_{12}-2P_{11}+P_{22} = 0.$

$-4P_{21} - 8P_{22} = -1$ $\qquad P_{22} = \dfrac{3-8P_{11}}{8}$

$(8P_{11}-2) + 8P_{22} = 1 \Rightarrow 8P_{11}+8P_{22} = 3.$

$-(12P_{11}-3) - 2P_{11} + \dfrac{3-8P_{11}}{8} = 0.$

$-96P_{11}+24-16P_{11}+3-8P_{11} = 0.$

$120P_{11} = 27 \Rightarrow P_{11} = 0.225$

$\therefore P_{21} = -0.05$

$\therefore P_{22} = 0.15$

$$P = \begin{bmatrix} 0.225 & -0.05 \\ -0.05 & 0.15 \end{bmatrix}$$

$V(x) = x^T P x.$

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$

$V(x) = (x_1\ x_2)\begin{bmatrix} 0.225 & -0.05 \\ -0.05 & 0.15 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$V(x) = (x_1\ x_2)\begin{pmatrix} 0.225x_1 - 0.05x_2 \\ -0.05x_1 + 0.15x_2 \end{pmatrix}$

$= 0.225x_1^2 - 0.05x_1x_2 - 0.05x_1x_2 + 0.15x_2^2.$

$V(x) = 0.225x_1^2 - 0.1x_1x_2 + 0.15x_2^2$

$$\dot{V}(x) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{array}{cc} 0 & 1 \end{array}$$
$$\begin{bmatrix} 0.45x_1 - 0.1x_2 & -0.1x_1 + 0.3x_2 \end{bmatrix}$$

(b) $A = \begin{bmatrix} -2 & -2 \\ 1 & -4 \end{bmatrix}$

from a, eigen values are

$\dfrac{-6 \pm \sqrt{36-40}}{2} \Rightarrow \dfrac{-6 \pm 2i}{2}$

$\Rightarrow -3 \pm i$

$\lambda_1 = -3+i, \quad \lambda_2 = -3-i$

As the Real part of both the eigen values is −ve, the system is Asymptotically stable.

(c) $H(s) = C(sI-A)^{-1}B$.

$B = [1 \quad 2]^T$

$$(sI-A) = \begin{bmatrix} s+2 & +2 \\ -1 & s+4 \end{bmatrix}$$

$$(sI-A)^{-1} = \frac{\begin{bmatrix} s+4 & -2 \\ 1 & s+2 \end{bmatrix}}{(s+2)(s+4)+2}$$

$\neq / \cancel{/s+4/}$

$C = [1 \quad 0]$

$$C(sI-A)^{-1}B = [1 \quad 0] \cdot \frac{1}{(s+2)(s+4)+2} \begin{bmatrix} s+4 & -2 \\ 1 & s+2 \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+4)+2} [(s+4) \quad (-2)]$$

As there is no evident pole-zero cancellation, the realization is not minimal.

(d) BIBO stable does not always intend that the System is asymptotically stable. But if a system is asymptotically Stable, it is always BIBO stable. For a ~~function that~~ transfer function existing System whose inverse ~~Lagrane~~ Laplace transform is always definitive need not always have the unique solution i.e Asymptotic stability.

Question 4 (60 points)

(a) For system to be controllable, the controllability matrix must have rank = State space dimension.

$$P = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -10 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 5 & -10 \\ 1 & 1 \end{bmatrix}$$

$\hookrightarrow$ rank = 2 = 2

$\therefore$ CONTROLLABLE

(Continued later)

(b) $|\lambda I - A| = 0$.

$$\begin{vmatrix} \lambda+2 & 0 \\ 0 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda+2)(\lambda-1) = 0$$

$$\underline{\lambda = -2} \text{ or } \underline{\lambda = 1}$$

The system is MARGINALLY STABLE as it has only one root with $\underline{Re(z) \leq 0}$

$$W_c(t) = \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} \, d\tau$$

There must be a $u(t)$ if the system becomes controllable for which the system attains stability at 't' for:

$$u(t) = -B^T e^{At} W_c^{-1} x(0)$$

$$= -\begin{bmatrix} 5 & 1 \end{bmatrix} e^{At} W_c^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) Continuing from (b),

$$u(t) = -\begin{bmatrix} 5 & 1 \end{bmatrix} e^{At} W_c^{-1} \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

We know that $u(t)$ drives $x(0)$ from 0 to 4 seconds. Hence, $\underline{t = 4}$.

$$W_c = \int_0^4 e^{A(4-\tau)} B B^T e^{A^T(t-4)} \, d\tau$$

where $B = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $B^T = \begin{bmatrix} 5 & 1 \end{bmatrix}$

calculation of W_c must be done and put in the above formula to obtain precise answer.

(a) Continuation:

Step2: $|sI - A| = 0$

$$\begin{vmatrix} s+2 & 0 \\ 0 & s-1 \end{vmatrix} = 0$$

$$(s+2)(s-1) = 0$$

$$s^2 + 2s - s - 2 = 0$$

$$s^2 + s - 2 = 0$$

$$a_0 = -2, \; a_1 = 1, \; a_2 = 1$$

Step3: $P_{CCF}^{-1} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Step4: $T = P \cdot P_{CCF}^{-1}$

$$= \begin{bmatrix} 5 & -10 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -5 & 5 \\ 2 & 1 \end{bmatrix}$$

$$A_{CCF} = T A T^{-1} \Rightarrow \begin{bmatrix} -5 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 2 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} 10 & -4 \\ 5 & 1 \end{bmatrix}$$

$$A_{CCF} = \begin{bmatrix} -25 & 25 \\ 25 & -7 \end{bmatrix}$$

$$B_{CCF} = T^{-1} B \Rightarrow \begin{bmatrix} -5 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -23 \\ 26 \end{bmatrix}$$

$$\cancel{C_{CCF} = CT \Rightarrow}$$

$$x^* = \begin{bmatrix} -25 & 25 \\ 25 & -7 \end{bmatrix} x + \begin{bmatrix} -23 \\ 26 \end{bmatrix} u$$

Question 5 (90 points)

(a) A linear Time invariant function is Asymptotically Stable if and only if there exists a negative semi definite P such as $A^T P + PA = -Q$ where $(Q = I_{n\times n})$. When there exists a negative semi definite P, we cannot say anything about it's exponential Stability.

(b) The Statement is correct.

(c) ~~There should be a point~~ It is TRUE.

(d) All systems when they are asymptotically stable, they are BIBO Stable but the viceversa is not true.

So, the statement is correct.

(e) NOPE

Q is considered to be $-I$.

The Stability is said in terms of P.

If for LTI, P is -ve definite, System is stable. (Example)

(f) TRUE.