Homework 2
ME5659 Spring 2024

Due: See Canvas, turn in on Gradescope

Problem 1 (6 points)

Solutions to state-space models.

(a) **2 points.** For the given homogeneous system below, subject only to the initial state $x(0) = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$, calculate the matrix exponential e^{At} and the state vector at time $t = 4$.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

(b) **4 points.** Calculate the system response $y(t)$ for the following state equation when $u(t) = e^{-2t}$, $t \geq 0$. Plot the system response $y(t)$ versus t using Matlab commands, e.g., `sys = ss(A,B,C,D)` and `lsim(sys,u,t,x_0)`.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Solutions

a) As we can see the given A matrix is not in DCF so we need to transform it into a DCF. A must be diagonalizable, if A is diagonalizable then the eigenvalues of A will appear on the main diagonal of A. To be diagonalizable A needs to have n linearly independent eigenvectors

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -12 \end{bmatrix}$$

the eigenvalues are given by the characteristic polynomial.
$\det(\lambda I - A) = 0$.

$$(\lambda I - A) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -12 \end{bmatrix} = \begin{bmatrix} \lambda & -1 \\ 6 & \lambda + 12 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda + 12) + 6 = \lambda^2 + 12\lambda + 6$$

the solutions are given by

$$\lambda_{1,2} = \frac{-12 \pm \sqrt{138}}{2} = -6 \pm \sqrt{30}$$

$\lambda_1 = -0.52$
$\lambda_2 = -11.48$

A has n distinct eigenvalues and hence it is DIAGONALIZable.

```
clear all; clc; close all;
t=4;
A = [0 1; -6 -12];
x_0=[2; 1];

[T_dcf,Adcf]=eig(A);

display(T_dcf);

% Now, let's apply the coordinates transformation

A_dcl=inv(T_dcf)*A*T_dcf;

display(A_dcl); %we know that we should obtain a matrix with the eigenvalues in the diagonal

exp_A_dcl=[exp(-0.5228*t) 0; 0 exp(-11.4772*t)];

% Now we want to come back to the original coordinates system to get
% exp(A*t)

exp_A=T_dcf*exp_A_dcl*inv(T_dcf);

display(exp_A);

x_state=exp_A*x_0
```

```
T_dcf = 2x2
    0.8862   -0.0868
   -0.4633    0.9962

A_dcl = 2x2
   -0.5228    0.0000
    0.0000  -11.4772

exp_A = 2x2
    0.1294    0.0113
   -0.0677   -0.0059

x_state = 2x1
    0.2701
   -0.1412
```

b) $u = e^{-2t}, t \geq 0$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$y = \begin{bmatrix} 2 , 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$x(t) = x_{zi}(t) + x_{zs}(t)$$

$x_{zi}(t)$ zero input response $\quad x_{zi}(t) = e^{A(t-t_0)}$

$x_{zs}(t)$ zero state response $\quad x_{zs}(t) = \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) \, d\tau$

So if the initial conditions are zero then our response will only be given by $x_{zs}(t)$. On the contrary if our input is zero the solution is given only by the initial conditions.

If we use the laplace DOMAIN representation

$$\dot{x} = Ax + Bu$$

$$s X(s) - x_0 = A X(s) + B U(s)$$

$$(sI - A) X(s) = x_0 + B U(s)$$

$$X(s) = \underbrace{(sI - A)^{-1} x_0}_{\text{zero input response } x_{zi}(s)} + \underbrace{(sI - A)^{-1} B U(s)}_{\text{zero-state response } x_{zs}(t)}$$

$y(t) = \underbrace{y_{zi}(t)}_{\text{zero input response}} + \underbrace{y_{zs}(t)}_{\text{zero state response}}$ OUTPUT RESPONSE

In Laplace domain $Y(s) = \underbrace{C(sI - A^{-1})x_0}_{} + \underbrace{\left(C(sI-A)^{-1}B + D\right)U(s)}_{}$

$$e^{At} = (sI - A)^{-1}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$e^{At} = \left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\right]^{-1} \Rightarrow e^{At} = \begin{bmatrix} s & -1 \\ 1 & s+2 \end{bmatrix}^{-1}$$

$$e^{At} = \frac{1}{(s+2)s+1}\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}^{T} = \frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & 1 \\ -1 & s \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \frac{s+2}{s^2+2s+1} & \frac{1}{s^2+2s+1} \\ \frac{-1}{s^2+2s+1} & \frac{s}{s^2+2s+1} \end{bmatrix} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$y_{zi}(s) = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$y_{zi}(s) = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{s+2}{(s+1)^2} - \frac{2}{(s+1)^2} \\ -\frac{1}{(s+1)^2} - \frac{2s}{(s+1)^2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{s}{(s+1)^2} \\ \frac{-2s-1}{(s+1)^2} \end{bmatrix} = \frac{2\cancel{s}}{(\cancel{s}+1)^2} - \frac{2\cancel{s}}{(\cancel{s}+1)^2} - \frac{1}{(s+1)^2}$$

$$y_{zi}(s) = -\frac{1}{(s+1)^2}$$

Now, we want to go back in time domain:

PARTIAL FRACTION DECOMPOSITION

$$y_{zi}(t) \Rightarrow \frac{R_1}{s+1} + \frac{R_2}{(s+1)^2} = R_1(s+1) + R_2 \Rightarrow R_2 + R_1 s + R_1$$

$$\begin{cases} R_1 s = 0 \\ (R_1 + R_2) = -1 \end{cases} \quad \begin{cases} R_1 = 0 \\ R_2 = -1 \end{cases} \qquad \mathcal{L}^{-1}\left\{-\frac{1}{(s+1)^2}\right\} \Rightarrow -te^{-t}$$

$y_{zi}(t) = -te^{-t}$ ZERO INPUT RESPONSE

$y_{zs}(s) = \left(C(sI-A)^{-1}B + D\right)U(s)$ ZERO STATE RESPONSE

$u(t) = e^{-2t}$ in laplace domain $\Rightarrow U(s) = \frac{1}{s+2}$

$D=0$

$y_{zs} = C(sI-A)^{-1}BU(s)$

$$y_{zs}(s) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{1}{(s+1)^2} \\ -\frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s+2}$$

$$y_{zs}(s) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1/(s+1)^2 \\ s/(s+1)^2 \end{bmatrix} \frac{1}{s+2}$$

$$y_{zs}(s) = \left[\frac{2}{(s+1)^2} + \frac{s}{(s+1)^2}\right] \frac{1}{s+2}$$

$$y_{zs}(s) = \frac{2}{(s+1)^2(s+2)} + \frac{s}{(s+1)^2(s+2)} = \frac{2+s}{(s+1)^2(s+2)}$$

Again applying the PARTIAL FRACTION DECOMPOSITION we obtain:

$$\frac{R_1}{s+2} + \frac{R_2}{s+1} + \frac{R_3}{(s+1)^2} \Rightarrow R_1(s+1)^2 + R_2(s+1)(s+2) + R_3(s+2) =$$

$$= R_1(s^2+1+2s) + R_2(s^2+2s+s+2) + R_3 s + 2R_3 =$$

$$= R_1 s^2 + R_1 + 2R_1 \cdot s + R_2 s^2 + 3R_2 s + 2R_2 + R_3 s + 2R_3 =$$

$$\Rightarrow (R_1+R_2)s^2 + (2R_1+3R_2+R_3)s + R_1 + 2R_2 + 2R_3 = 2+s$$

$$\begin{cases} R_1 + R_2 = 0 \\ 2R_1 + 3R_2 + R_3 = 1 \\ R_1 + 2R_2 + 2R_3 = 2 \end{cases}$$

in matrix notation $\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 2 & 2 \end{array}\right]$

$\xrightarrow{\text{in RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$R_1 = 0$

$R_2 = 0$

$R_3 = 1$

$$\Rightarrow \quad Y_{zs}(s) = \frac{1}{(s+1)^2}$$

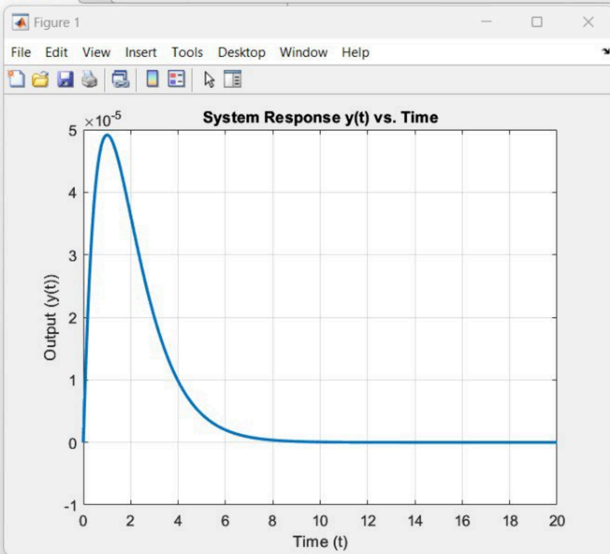
In time domain $\Rightarrow +te^{-t}$

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$$y(t) = -\cancel{te^{-t}} + \cancel{te^{-t}} = 0$$

Prob_3_Part_a.m | Prob_1_Part_a.m | Prob_1_Part_i.m | untitled2 * | Prob_1_Part_b.m

```
close all; clc; clear all;

% Define system matrices
A = [0 1; -1 -2];
B = [0; 1];
C = [2 1];
D = 0;
x0 = [1; -2];

t = linspace(0, 20, 1000);
u = exp(-2*t);

% Create state-space system
sys = ss(A, B, C, D);

% Compute system response
[y, t_out, x] = lsim(sys, u, t, x0);

% Plot
plot(t_out, y,'LineWidth', 2);
xlabel('Time (t)');
ylabel('Output (y(t))');
title('System Response y(t) vs. Time');
grid on
```



Problem 2 (6 points)

Calculate the eigenvalues and eigenvectors for the following matrices, derive the transformation matrix T to rewrite them in diagonal canonical form (DCF) or Jordan canonical form (JCF). Validate your results using the Matlab command `eig(A)`.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

To calculate the eigenvalues we need to compute $\det(\lambda I - A) = 0$

$$\det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right] = 0$$

$$\det\begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda-1 \end{bmatrix} = 0 \quad \Rightarrow \quad (\lambda-1)^2 = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \qquad \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_{1,2} = \frac{+2 \pm \sqrt{4-4}}{2} \Rightarrow \quad \lambda_{1,2} = 1$$

To find the eigenvectors we need to compute $A\vec{e}_i = \lambda_i \vec{e}_i$

eigenvector associated with $\lambda_{1,2} = 1$

$$\begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda-1 \end{bmatrix} = \begin{bmatrix} 1-1 & -2 \\ 0 & 1-1 \end{bmatrix}$$

where α arbitrary

$$\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} -2x_2 = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} x_1 = \alpha \\ x_2 = 0 \end{cases} \quad \vec{v}_1 = \alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Since this matrix is not diagonalizable, we cannot rewrite it in diagonal canonical form. $\lambda_{1,2} = 1$ with algebric mult. $= 2$ and geometric mult. $= 1$

Jordan canonical form (JCF), $AV_1 = \lambda V_2 \quad AV_2 = V_1 + \lambda V_2 \Rightarrow (A - \lambda I)V_2 = V_1$

$$A\begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$V_2 = \begin{pmatrix} x \\ y \end{pmatrix} \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow (A - \lambda I)V_2 = V_1$

where $(A - \lambda I) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2y = 1 \\ 0 = 0 \\ x = \alpha \end{cases} \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

arbitrary

$$T = \begin{array}{cc} V_1 & V_2 \\ \end{array}$$
$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Jordan canonical FORM

$J = T^{-1} A T$

we should expect

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$J = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

b) $$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

```
clear all; clc; close all;
A=[-2 2 -3; 2 1 -6; -1 -2 0];
eig(A)
[T_dcf,A_dcf]=eig(A)
```

```
ans = 3x1
   -3.0000
    5.0000
   -3.0000

T_dcf = 3x3
   -0.9526    0.4082   -0.0230
    0.2722    0.8165    0.8353
   -0.1361   -0.4082    0.5492

A_dcf = 3x3
   -3.0000         0         0
         0    5.0000         0
         0         0   -3.0000
```

eigenvalues are

$\lambda_1 = -3$ Alg. molt. $= 2$

$\lambda_2 = 5$ Alg molt $= 1$

eigenvectors associated at -3

$$V_1 = \begin{bmatrix} -0.9526 \\ 0.2722 \\ -0.1361 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{bmatrix}$$ geometric molt. $= 2$

eigenvector associated at 5

$$V_3 = \begin{bmatrix} -0.0230 \\ 0.8353 \\ 0.5492 \end{bmatrix}$$ geom molt. $= 1$

$\Rightarrow$ DIAGONALIZABLE $\quad T_{dcf} = [V_1 \ V_2 \ V_3]$

$$A_{DCF} = T_{DCF}^{-1} A \, T_{DCF} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Problem 3 (6 points)

Similar realizations.

(a) **(3 points)** Are the following two state-space models

$$\dot{x} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x$$

and

$$\dot{x} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x$$

equivalent when both $x(0)$ and $u(t)$ are non-zero? Are they equivalent when $x(0) = 0$?

(b) **(3 points)** Verify that

$$\Sigma = \left[\begin{array}{ccc|c} -a_2 & 1 & 0 & b_2 \\ -a_1 & 0 & 1 & b_1 \\ -a_0 & 0 & 0 & b_0 \\ \hline 1 & 0 & 0 & d \end{array}\right]$$

is similar to

$$\Sigma^* = \left[\begin{array}{ccc|c} 0 & 0 & -a_0 & b_0 \\ 1 & 0 & -a_1 & b_1 \\ 0 & 1 & -a_2 & b_2 \\ \hline 0 & 0 & 1 & d \end{array}\right]$$

a) Two systems are equivalent when they have the same response. $Y(s) = C(sI-A)^{-1}x_0 + C(sI-A)^{-1}Bu(s)$
We can calculate the transfer function of A1 and A2

$$A_1 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad C_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

$$(sI-A)^{-1} = \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1/(s-2) & 1/(s-2)^2 & 2/(s-2)^2 \\ 0 & 1/(s-2) & 2/(s-1)(s-2) \\ 0 & 0 & 1/(s-1) \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad C_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

$$(sI-A)^{-1} = \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1/(s-2) & 1/(s-2)^2 & 2/(s-2)^2 \\ 0 & 1/(s-2) & 2/(s-1)(s-2) \\ 0 & 0 & 1/(s-1) \end{bmatrix}$$

$$Y(s) = C(sI-A)^{-1}x_0 + C(sI-A)^{-1}Bu(s) + \cancel{D}u(s), \quad D=0$$

0

$$Y_1(s) = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/(s-2) & 1/(s-2)^2 & 2/(s-2)^2 \\ 0 & 1/(s-2) & 2/(s-1)(s-2) \\ 0 & 0 & 1/(s-1) \end{bmatrix} x_0 + \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/(s-2) & 1/(s-2)^2 & 2/(s-2)^2 \\ 0 & 1/(s-2) & 2/(s-1)(s-2) \\ 0 & 0 & 1/(s-1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(s)$$

$$Y_1(s) = \begin{bmatrix} \frac{1}{(s-2)} & \frac{1}{(s-2)^2} & \frac{2}{(s-2)^2(s-1)} \end{bmatrix} x_0 + \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-1/(s-2)^2 \\ 1/(s-2) \\ 0 \end{bmatrix} u(s)$$

$$Y_1(s) = \begin{bmatrix} \frac{1}{(s-2)} & \frac{1}{(s-2)^2} & \frac{2}{(s-2)^2(s-1)} \end{bmatrix} x_0 + \frac{1}{(s-2)^2} u(s)$$

but when x_0 and $u(t)$ are non-zero we see $Y_2(s) \neq Y_1(s)$

So $Y_1 = Y_2$ when $x_0 ; u(t) = 0$. Since the zero-input response would become zero. So the two state-space models would be equivalent, since the zero-state response are equivalent.

b)

$$A_\Sigma = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \quad B_\Sigma = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C_\Sigma = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = d$$

$$H_\Sigma(s) = C(sI - A)^{-1} B$$

$$H_\Sigma(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$H_\Sigma(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s+a_2 & -1 & 0 \\ a_1 & s & -1 \\ a_0 & 0 & s \end{bmatrix}^{-1} \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$H_\Sigma(s) = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_2 s^2}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_1 s}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$A_{\Sigma^*} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \quad B_{\Sigma^*} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} \quad C_{\Sigma^*} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad D = d$$

$$H_\Sigma^*(s) = C(sI - A)^{-1} B$$

$$H_\Sigma^*(s) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$H_\Sigma^*(s) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & 0 & +a_0 \\ -1 & s & +a_1 \\ 0 & -1 & s+a_2 \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} =$$

$$H_\Sigma^*(s) = \frac{b_2}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_0 s^2}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_1 s}{s^3 + a_2 s^2 + a_1 s + a_0}$$

the tranfer function are the same $H_{\Sigma^*}(s) = H_{\Sigma}(s)$
and so Σ and Σ^* are similar

```
% A_1=[2 1 2; 0 2 2; 0 0 1];
% B_1=[1;1;0];
% C_1=[1 -1 0];
% syms s
% M_1=inv(s*eye(3)-A)
% A_2=[2 1 1; 0 2 1; 0 0 -1];
% B_2=[1;1;0];
% C_2=[1 -1 0];
% syms s
% M_2=inv(s*eye(3)-A)

syms s a_1 a_2 a_0 b_0 b_1 b_2
A_1=[-a_2 1 0; -a_1 0 1; -a_0 0 0];
B_1=[b_2; b_1; b_0];
C_1=[1 0 0];
M_1=inv(s*eye(3)-A_1)
H_1=C_1*M_1*B_1
A_2=[0 0 -a_0; 1 0 -a_1; 0 1 -a_2];
B_2=[b_0;b_1; b_2];
C_2=[0 0 1];
M_2=inv(s*eye(3)-A_2);
H_2=C_2*M_2*[b_0;b_1;b_2]
```

M_1 =

$$\begin{pmatrix} \frac{s^2}{\sigma_1} & \frac{s}{\sigma_1} & \frac{1}{\sigma_1} \\ -\frac{a_0 + a_1 s}{\sigma_1} & \frac{s\,(a_2+s)}{\sigma_1} & \frac{a_2+s}{\sigma_1} \\ -\frac{a_0 s}{\sigma_1} & -\frac{a_0}{\sigma_1} & \frac{s^2 + a_2 s + a_1}{\sigma_1} \end{pmatrix}$$

where

$$\sigma_1 = s^3 + a_2 s^2 + a_1 s + a_0$$

H_1 =

$$\frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_2 s^2}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_1 s}{s^3 + a_2 s^2 + a_1 s + a_0}$$

H_2 =

$$\frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_2 s^2}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_1 s}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Problem 4 (7 points)

Consider the following state-space model:

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad x(0) = x_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x$$

(a) **(2 points)** Transform the state-space model into diagonal canonical form (DCF).

(b) **(2 points)** Use the state-space model in DCF to find the system transfer function $G(s)$.

(c) **(3 points)** Use the state matrix in diagonal form from (a) to compute the matrix exponential e^{At}, where $A = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$. Calculate the system response $y(t)$ for a unit step $u(t)$.

a) Diagonal canonical form yields a decoupled set of 1^{ST} order ODEs

$\dot{x}_1 = f_1(x_1, u)$

$\dot{x}_2 = f_2(x_2, u)$

$\vdots$

$\dot{x}_n = f_n(x_n, u)$

$$\dot{X} = \underbrace{\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}}_{A} X + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \qquad X(0) = X_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

A must be diagonalizable iff A is diagonalizable then the eigenvalues of A will appear on the main diagonal of $\hat{A}$.

$A = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$ to be diagonalizable A needs to have n linearly independent eigenvectors.

eigenvalues: $\det(\lambda I - A) = 0$

$\det \begin{bmatrix} \lambda+2 & 0 \\ -1 & \lambda-3 \end{bmatrix} = (\lambda+2)(\lambda-3) = 0$

First eigenvalue $\lambda = -2$

second eigenvalue $\lambda = 3$

A has n distinct eigenvalues, then A is diagonalizable.

=> The state space model can be transformed into the diagonal canonical form (DCF)

We need to find the eigenvectors of A.

1st eigenvector associated to $\lambda_1 = 3$

$$A = \begin{bmatrix} \lambda+2 & 0 \\ -1 & \lambda-3 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} x_1 = 0 \\ x_2 = \alpha \end{cases} \qquad \vec{v}_1 = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

second eigenvalue $\lambda = -2$

$$A = \begin{bmatrix} \lambda+2 & 0 \\ -1 & \lambda-3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 0 \\ -1 & -5 \end{bmatrix}$$

α arbitrary

$$A\vec{e}_i = \lambda_i \vec{e}_i \Rightarrow \begin{bmatrix} 0 & 0 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{cases} -x_1 = 5x_2 \\ x_1 = \alpha \end{cases} \qquad \begin{cases} x_2 = -1/5 \\ x_1 = \alpha \end{cases} \qquad \begin{cases} x_1 = \alpha \\ x_2 = -0.2 \end{cases} \qquad \vec{v}_2 = \alpha \begin{pmatrix} 1 \\ -0.2 \end{pmatrix}$$

$x(t) = T\,z(t)$ where $x(t)$ is the original coordinate system and $z(t)$ is the new coordinate system

$z(t) = T^{-1} x(t)$

$\dot{z}(t) = T^{-1} \dot{x}(t)$

$\dot{z}(t) = T^{-1}(Ax + Bu)$

$\dot{z}(t) = T^{-1}A\,x(t) + T^{-1}Bu$

$x(t) = T z(t)$

$\dot{z}(t) = T^{-1}AT\,z(t) + T^{-1}Bu$

$\hat{A} = A_{DCF} = T^{-1}AT$ in the new coordinate system

$\hat{A} = T^{-1}AT$

$\hat{B} = T^{-1}B$

$\hat{B} = B_{DCF} = T^{-1}B$ in the new coordinate system

$$T_{DCF} = [v_1, v_2] = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}$$

$$T_{DCF}^{-1} = \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{DCF} = T_{DCF}^{-1} A\, T_{DCF}$$

$$A_{DCF} = \frac{1}{5}\begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{B} = B_{DCF} = T_{DCF}^{-1} B = \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\dot{z}(t) = A_{DCF}\, z(t) + B_{DCF}\, u(t)$$

$$\dot{z}(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

STATE EQUATION.

$$\dot{z}_1(t) = 3 z_1(t) + u(t)$$
$$\dot{z}_2(t) = -2 z_2(t) - \tfrac{1}{5} u(t)$$

OUTPUT EQUATION.

$$y = \underbrace{\begin{bmatrix} 2 & 1 \end{bmatrix}}_{C} x$$

$$y(t) = C\, x(t) + D\, u(t)$$

INITIAL CONDITION

$$x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = x_0$$

$$x(t) = T_{DCF}\, z(t) \qquad y(t) = C\, T_{DCF}\, z(t) + D\, u(t)$$

$$z_0 = T_{DCF}^{-1} x_0$$

$$\hat{C} = C_{DCF} = C\, T_{DCF}$$

$$z_0 = \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ 2 \end{bmatrix}$$

new initial conditions.

$$\hat{C} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 1.8 \end{bmatrix}$$

$$\hat{D} = D_{DCF} = [0]$$

$$y(t) = \begin{bmatrix} 1 & 1.8 \end{bmatrix} z(t)$$

b)

$$A_{DCF} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \quad B_{DCF} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_{DCF} = \begin{bmatrix} 1 & -9 \end{bmatrix} \quad D_{DCF} = [0]$$

$$\dot{z}(t) = A_{DCF}\, z(t) + B_{DCF}\, u(t)$$
$$y(t) = C_{DCF}\, z(t) + D_{DCF}\, u(t)$$

$$G(s) = \frac{Y_{ZS}(s)}{U(s)} \; ; \qquad Y_{ZS}(s) = (C(sI - A)^{-1} B + D)\, U(s)$$

ZERO STATE RESPONSE

$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ the transfer function remains the same, it doesn't change upon the coordinate system picked.

$$G(s) = [1 \ -9]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\right)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + [0]$$

$$G(s) = [1 \ -9]\begin{bmatrix} s-3 & 0 \\ 0 & s+2 \end{bmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$G(s) = [1 \ -9]\frac{1}{(s-3)(s+2)}\begin{bmatrix} s+2 & 0 \\ 0 & s-3 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$G(s) = \overset{1\times 2}{[1 \ -9]}\begin{bmatrix} \frac{1}{s-3} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$G(s) = \left[\frac{1}{s-3}, \frac{-9}{s+2}\right]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s-3}$$

c)

$$A_{DCF} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

As we know, T defines the transformation between the two matrices.

In terms of matrices exponential

$$e^{\hat{A}t} = e^{T^{-1}ATt} = T^{-1}e^{At}T$$

$$A_{DCF} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$e^{A_{DCF}t} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} = e^{\hat{A}t}$$

We want e^{At} so we know that $e^{\hat{A}t} = T^{-1} e^{At} T$

$e^{At} = T e^{\hat{A}t} T^{-1}$

$$e^{At} = \begin{bmatrix} 0 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1/5 & 1 \\ -1/5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/5\, e^{3t} & e^{3t} \\ -\frac{1}{5} e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} e^{-2t} & 0 \\ \frac{1}{5} e^{3t} - \frac{1}{5} e^{-2t} & e^{3t} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} + e^{-2t} & 0 \\ \frac{1}{5} e^{3t} - \frac{1}{5} e^{-2t} & e^{3t} \end{bmatrix}$$

Using A we know that $e^{At} = (sI - A)^{-1} \Rightarrow (sI - A)^{-1} = \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} s+2 & 0 \\ -1 & s-3 \end{bmatrix}^{-1}$

$$e^{At} = \frac{1}{(s+2)(s-3)} \begin{bmatrix} s-3 & 1 \\ 0 & s+2 \end{bmatrix}^T = \frac{1}{(s+2)(s-3)} \begin{bmatrix} s-3 & 0 \\ 1 & s+2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ \frac{1}{(s+2)(s-3)} & \frac{1}{(s-3)} \end{bmatrix}$$

Now we need to use partial fraction $\quad \frac{1}{s-a} \Rightarrow e^{at}$

- $\frac{1}{s+2} \quad \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}$
- $0 \quad \mathcal{L}^{-1}(0) = 0.$
- $\frac{1}{(s+2)(s-3)} \Rightarrow \frac{R_1}{s+2} + \frac{R_2}{s-3} = R_1(s-3) + R_2(s+2) = (R_1 + R_2)s - 3R_1 + 2R_2$

$$\begin{cases} R_1 + R_2 = 0 \\ -3R_1 + 2R_2 = 1 \end{cases} \quad \begin{cases} R_1 = -R_2 \\ +3R_2 + 2R_2 = 1 \end{cases} \quad \begin{cases} R_1 = -R_2 \\ 5R_2 = 1 \end{cases} \quad \begin{cases} R_1 = -1/5 \\ R_2 = 1/5 \end{cases}$$

- $-\frac{1}{5} \frac{1}{(s+2)} + \frac{1}{5} \frac{1}{(s-3)} \Rightarrow \mathcal{L}^{-1} \Rightarrow -\frac{1}{5} e^{-2t} + \frac{1}{5} e^{3t}$
- $\frac{1}{s-3} \Rightarrow \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$

$$e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ -\frac{1}{5} e^{-2t} + \frac{1}{5} e^{3t} & e^{3t} \end{bmatrix}$$

same result because, the solution doesn't chang from the coordinate system picked zero input output component $y_{zi}(t)$ initial state $x(0) = x_0$

$$y_{zi}(t) = Ce^{At} x_0 = \underset{1\times 2}{\begin{bmatrix} 2 & 1 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} e^{-2t} & 0 \\ -\frac{1}{5}e^{-2t} + \frac{1}{5}e^{3t} & e^{3t} \end{bmatrix}} x_0$$

$$y_{zi}(t) = \begin{bmatrix} 2e^{-2t} - \frac{1}{5}e^{-2t} + \frac{1}{5}e^{3t} & e^{3t} \end{bmatrix} x_0 = \begin{bmatrix} \frac{9}{5}e^{-2t} + \frac{1}{5}e^{3t} & e^{3t} \end{bmatrix} x_0$$

$$x_0 = x(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$y_{zi}(t) = \underset{1\times 2}{\begin{bmatrix} \frac{9}{5}e^{-2t} + \frac{1}{5}e^{3t} & e^{3t} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \frac{18}{5}e^{-2t} + \frac{2}{5}e^{3t} - e^{3t} = \frac{18}{5}e^{-2t} - \frac{3e^{3t}}{5}$$

$$y_{zsr}(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau$$

$$y_{zsr}(t) = \int_0^t \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-2(t-\tau)} & 0 \\ -\frac{1}{5}e^{-2(t-\tau)} + \frac{1}{5}e^{3(t-\tau)} & e^{3(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\tau$$

$$y_{zsr}(t) = \int_0^t \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{3(t-\tau)} \end{bmatrix} d\tau = \int_0^t e^{3(t-\tau)} d\tau$$

$$y_{zsr}(t) = -\frac{1}{3}e^{3(t-\tau)} \Big|_0^t = -\frac{1}{3} + \frac{1}{3}e^{3t}$$

$$y(t) = \frac{18}{5}e^{-2t} - \frac{3e^{3t}}{5} - \frac{1}{3} + \frac{1}{3}e^{3t} = \frac{18}{5}e^{-2t} - \frac{4}{15}e^{3t} - \frac{1}{3}$$

```
close all; clc; clear all;

% Define system matrices
A = [-2 0; 1 3];
B = [0; 1];
C = [2 1];
D = 0;
x0 = [2; -1];

t = linspace(0, 5, 1000);
u = ones(size(t));

% Create state-space system
sys = ss(A, B, C, D);

% Compute system response
[y, t_out, x] = lsim(sys, u, t, x0);

% Plot
plot(t_out, y,'LineWidth', 2);
xlabel('Time (t)');
ylabel('Output (y(t))');
title('System Response y(t) vs. Time');
```