CONTROLS HOMEWORK - 3

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a)

i) $A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}$, $x_0 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}$

eigen values of A;

$$|\lambda I - A| = 0.$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix} \right| = 0.$$

$$\begin{bmatrix} \lambda & -1 \\ 14 & \lambda+4 \end{bmatrix} = 0 \Rightarrow \lambda(\lambda+4) + 14 = 0.$$

$$\begin{array}{r} 14 \\ 4 \\ \hline 56 \end{array}$$

$$\lambda^2 + 4\lambda + 14 = 0 \Rightarrow \frac{-4 \pm \sqrt{16-56}}{2}$$

$$\Rightarrow \frac{-4 \pm \sqrt{-40}}{2} \Rightarrow \frac{-4 \pm 2\sqrt{-10}}{2} \Rightarrow \frac{-4 \pm 2\sqrt{10}\, i}{2}$$

As $Re(\lambda_1, \lambda_2) < 0$; THE SYSTEM IS ASYMPTOTICALLY STABLE

ii) $A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}$

eigen values of A;

$$|\lambda I - A| = 0.$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix} \right| = 0.$$

$$\begin{vmatrix} \lambda & -1 \\ 14 & \lambda-4 \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-4)+14=0$$

$$\Rightarrow \lambda^2 - 4\lambda + 14 = 0 \Rightarrow \frac{4 \pm \sqrt{16-56}}{2}$$

$Re(\lambda_1, \lambda_2) = 0$; THE SYSTEM IS UNSTABLE.

iii)

$$A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

eigen values of A;

$$|\lambda I - A| = 0$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} \lambda & -1 \\ 14 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 14 = 0 \Rightarrow \lambda = \pm\sqrt{14}\, i$$

THE SYSTEM IS STABLE.

b)

i) $A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}, \quad Q = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^T P + PA = -Q \qquad (P_{12} == P_{21})$$

$$\begin{bmatrix} 0 & -14 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} & -14P_{22} \\ P_{11} - 4P_{21} & P_{12} - 4P_{22} \end{bmatrix} + \begin{bmatrix} -14P_{12} & P_{11} - 4P_{12} \\ -14P_{22} & P_{21} - 4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} - 14P_{12} & -14P_{22} + P_{11} - 4P_{12} \\ P_{11} - 4P_{21} - 14P_{22} & P_{12} - 4P_{22} + P_{21} - 4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-28P_{21} = -1 \Rightarrow P_{21} = 1/28 \longrightarrow (1)$$

$$-14P_{22} + P_{11} - \cancel{4}\left(\frac{1}{\cancel{28}_{7}}\right) = 0 \Rightarrow -14P_{22} + P_{11} = 1/7 \longrightarrow (2)$$

$14 \cdot 8 = 112$

$15 \cdot 7 = 105$

$$2P_{12} - 8P_{22} = -1$$

From (1) $\frac{2}{28} - 8P_{22} = -1 \Rightarrow P_{22} = \left(\frac{1}{14} + 1\right)/8 \Rightarrow \frac{15}{112}$ $\rightarrow P_{22}$

From (2) $-\cancel{14} \cdot \frac{15}{\cancel{14} \cdot 8} + P_{11} = 1/7 \Rightarrow P_{11} = \frac{1}{7} + \frac{15}{8} \Rightarrow \frac{106}{56} \Rightarrow \frac{53}{28}$ $\rightarrow P_{11}$

$P_{11} > 0$.

$$\det \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \Rightarrow \begin{vmatrix} \frac{53}{28} & \frac{1}{28} \\ \frac{1}{28} & \frac{15}{112} \end{vmatrix} > 0.$$

THE SYSTEM IS ASYMPTOTICALLY STABLE.

(ii)

$$A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.01 \\ 0.02 \end{bmatrix}$$

$$Q = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$A^T P + PA = -Q$$

$$\begin{bmatrix} 0 & -14 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} & -14P_{22} \\ P_{11}+4P_{21} & P_{12}+4P_{22} \end{bmatrix} + \begin{bmatrix} -14P_{12} & P_{11}+4P_{12} \\ -14P_{21} & P_{21}+4P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-28P_{21} = -1 \Rightarrow P_{12} = 1/28 \quad — \ (1)$$

$$-14P_{22} + P_{11} + 4P_{12} = 0 \Rightarrow -14P_{22} + P_{11} + \frac{1}{7} = 0 \quad — \ (2)$$

$$2P_{12} + 8P_{22} = -1$$

From (1), $\frac{1}{14} + 8P_{22} = -1 \Rightarrow P_{22} = \left(-1 - \frac{1}{14}\right)/8 \Rightarrow \frac{-15}{112} \rightarrow P_{22}$

From (2), $-\cancel{14}\cdot\frac{(-15)}{\cancel{14}\cdot 8} + P_{11} = \frac{-1}{7} \Rightarrow \frac{15}{8} + P_{11} = \frac{-1}{7} \Rightarrow P_{11} = \frac{-1}{7} - \frac{15}{8}$

$$= \frac{-113}{56} \rightarrow P_{11}$$

$P_{11} < 0$, $\quad \mathrm{Det}\begin{bmatrix} -113/56 & 1/28 \\ 1/28 & -15/112 \end{bmatrix} > 0$

THE SYSTEM IS NOT ASYMPTOTICALLY STABLE.

(iii) $A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}$, $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(Q = I)$

$$A^T P + PA = -Q$$

$$\begin{bmatrix} 0 & -14 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -14P_{21} & -14P_{22} \\ P_{11} & P_{12} \end{bmatrix} + \begin{bmatrix} -14P_{12} & P_{11} \\ -14P_{22} & P_{21} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -28P_{21} & P_{11} - 14P_{22} \\ P_{11} - 14P_{22} & 2P_{12} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

NO UNIQUE SOLUTION FOR P.

∴ SYSTEM IS UNSTABLE.

c)

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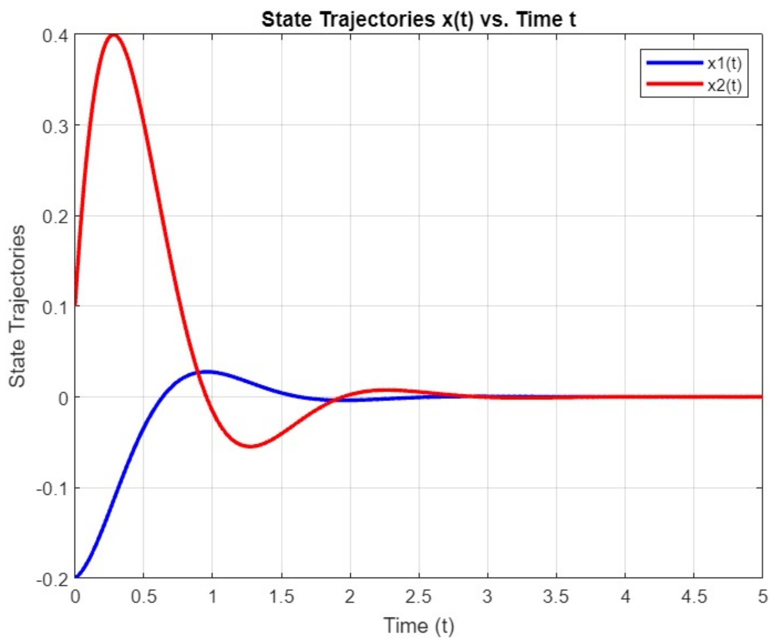
% Define the system matrix and initial condition
A = [0 1; -14 -4];
x0 = [-0.2; 0.1];

% Define the time span for simulation
tspan = 0:0.01:5;

% Define the differential equation function
ode = @(t, x) A * x;

% Solve the differential equation using ode45
[t, x] = ode45(ode, tspan, x0);

% Plot the state trajectories
figure;
plot(t, x(:, 1), 'b', 'LineWidth', 2, 'DisplayName', 'x1(t)');
hold on;
plot(t, x(:, 2), 'r', 'LineWidth', 2, 'DisplayName', 'x2(t)');
xlabel('Time (t)');
ylabel('State Trajectories');
title('State Trajectories x(t) vs. Time t');
legend('show');
grid on;
hold off;
```



3) $\dot{x} = Ax$

$x(0) = x_0$.

$$A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$$

a) Lets say $Q = I_{2\times2}$.

We know: $A^T P + PA = -Q$

$$\begin{bmatrix} 0 & a \\ 1 & b \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} aP_{21} & aP_{22} \\ P_{11}+bP_{21} & P_{12}+bP_{22} \end{bmatrix} + \begin{bmatrix} aP_{12} & P_{11}+bP_{12} \\ aP_{22} & P_{21}+bP_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2aP_{21} & P_{11}+bP_{12}+aP_{22} \\ P_{11}+bP_{21}+aP_{22} & 2P_{21}+2bP_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2a & 0 \\ 1 & b & a \\ 0 & 2 & 2b \end{bmatrix}\begin{bmatrix} P_{11} \\ P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

$P_{12} = 0 \Rightarrow P_{11} + 0 + aP_{22} = 0$ for $a \neq 0$

$\Rightarrow P_{11} = -aP_{22}$

$2P_{21} + 2bP_{22} = -1$

$\Rightarrow P_{22} = \frac{-1}{2b}$ $\therefore P_{11} = \frac{a}{2b}$

$$\therefore P = \begin{bmatrix} a/2b & 0 \\ 0 & -1/2b \end{bmatrix}$$

For P to be positively definite and eventually asymptotically stable, the leading principle Minors have to be greater than 0.

$$\frac{a}{2b} > 0 \quad , \quad \frac{-1}{2b} > 0$$

$$\therefore \boxed{b < 0}$$
$$\Rightarrow \boxed{a < 0}$$

b) The Lyapunov function V (x) can be written as:

$$V(x) = x^T P x \Rightarrow [x_1 \; x_2] \begin{bmatrix} a/2b & 0 \\ 0 & -1/2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{a x_1}{2b} & -\frac{x_2}{2b} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \frac{a x_1^2}{2b} - \frac{x_2^2}{2b}$$

$$\dot{V}(x) = -x^T Q x \Rightarrow -(x_1 \; x_2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$-(x_1 \; x_2) \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow -(x_1^2 + x_2^2)$$

For $a < 0, b < 0$; $V(x) = \frac{a x_1^2}{2b} - \frac{x_2^2}{2b} > 0$ which means

$V(x)$ is positively Definite.

$\dot{V}(x) = -x_1^2 - x_2^2 < 0$ for all $x \neq 0$, which means $\dot{V}(x)$ is negatively Definite. By Analysing these two points, we can say that the system is ASYMPTOTICALLY STABLE!

c) If $b=0$, $a<0$,

$$\dot{V}(x) = -x^T Q x \Rightarrow -x^T (A^T P + PA) x$$

$$\Rightarrow \begin{pmatrix} -x_1 & -x_2 \end{pmatrix} \left[\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -x_1 & -x_2 \end{pmatrix} \begin{bmatrix} 0 & (a+1) \\ (a+1) & 2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -x_2(a+1) & -x_1(a+1) - 2x_2 b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow -(a+1)x_1x_2 - (a+1)x_1x_2 - 2x_2^2 b$$

$\Rightarrow -2(a+1)x_1x_2 - 2x_2^2 b$, If $b=0$ and $a<0$

$\Rightarrow \dot{V}(x) = -2(a+1)x_1x_2$. If $x_1, x_2 > 0$, we can't define $\dot{V}(x)$

$$V(x) = x^T P x. \Rightarrow \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

i.e $V(x) > 0 \ \forall \ x \neq 0$. (Positive Definite)

As $V(x) > 0$, we can say system is stable but we can't comment on $\dot{V}(x)$ Hence making the system ASYMPTOTICALLY UNSTABLE.

2)

(a) $ml\ddot{\theta} = mgl\sin\theta - b\dot{\theta}_{+}T$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix}$$

Now, $\ddot{\theta} = g\sin\theta - \frac{b\dot{\theta}}{ml} + \frac{T}{ml}$

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} \Rightarrow \begin{pmatrix} x_2 \\ g\sin\theta - \frac{bx_2}{ml} \end{pmatrix} + \begin{pmatrix} 0 \\ 1/ml \end{pmatrix} \cdot T$$

f_1

f_2

$$y = [1 \ 0] \ x$$

upon linearization,

$$\Delta\dot{x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix} + \begin{bmatrix} \partial f_1/\partial u_1 \\ \partial f_2/\partial u_2 \end{bmatrix} \partial u$$

$$\Rightarrow \quad \Delta\dot{x} = \begin{bmatrix} 0 & 1 \\ g\cos x_1 & -b/ml \end{bmatrix} \partial x + \begin{bmatrix} 0 \\ 1/ml \end{bmatrix} \partial T$$

$x_1 = \pi$

$\hookrightarrow A$

$$\Delta\dot{x}_{\pi} = \begin{bmatrix} 0 & 1 \\ -g & -b/ml \end{bmatrix} \partial x + \begin{bmatrix} 0 \\ 1/ml \end{bmatrix} \partial T$$

$$\Delta y = [1 \ 0] \ \partial x$$

$|A - \lambda I|$

Finding eigen values:

$$\begin{vmatrix} -\lambda & 1 \\ -g & -\frac{b}{ml} - \lambda \end{vmatrix} \Rightarrow \lambda\left[\frac{b}{ml} + \lambda\right] + g = 0.$$

$$\lambda^2 + \frac{b\lambda}{ml} + g = 0.$$

Finding Roots: $\dfrac{-\frac{b}{ml} \pm \sqrt{\frac{b^2}{m^2l^2} - 4g}}{2}$

$$\Rightarrow \underbrace{\frac{-b}{2ml}}_{a} \pm \underbrace{\sqrt{\frac{b^2}{4m^2l^2} - g}}_{b}$$

if $\frac{b^2}{4m^2l^2} - g < 0$, then roots are complex conjugate

and real part $-\frac{b}{2ml} < 0$.

Hence, it will be stable

if $\frac{b^2}{4m^2l^2} - g < 0$,

comparing a and b

$a^2 \Rightarrow \frac{b^2}{4m^2l^2}$ $\qquad$ $b^2 = \frac{b^2}{4m^2l^2} - g$

b^2 will always be less than a^2.

$b < a$ so roots $= a \pm b$ will always be negative.

Hence it will be stable.

(b) From A,

if $x=0$, putting $\theta = 0$ i.e $\cos\theta = 1$

$$\Delta \dot{x}_1 = \begin{bmatrix} 0 & 1 \\ g & -b/ml \end{bmatrix} \partial x + \begin{bmatrix} 0 \\ 1/ml \end{bmatrix} \partial T$$

$$\Delta y = \begin{bmatrix} 1 & 0 \end{bmatrix} \partial x$$

Roots $\Rightarrow$ $\begin{vmatrix} -\lambda & 1 \\ g & -\frac{b}{ml} - \lambda \end{vmatrix} = 0$

$$\left(\frac{b}{ml} + \lambda\right)\lambda - g = 0$$

$$\lambda^2 + \frac{b\lambda}{ml} - g = 0.$$

roots $\rightarrow$ $$\frac{\overbrace{-\frac{b}{ml}}^{a'} \pm \sqrt{\overbrace{\frac{b^2}{m^2 l^2} + 4g}^{b'}}}{2}$$

$\frac{b^2}{m^2l^2} + 4g$ will always be positive as

$b, m, l, g > 0$.

comparing a' and b'

$(a')^2 = b^2/m^2l^2$ $\quad\quad$ $(b')^2 = \frac{b^2}{m^2l^2} + 4g$

$(b')^2 > (a')^2$ $\quad$ i.e $\quad$ $b' > a'$

as $b' > 0$.

$-a' + b' > 0$

$-a' - b' < 0$ i.e 1 root will be positive.

Hence this system will always be stable.