Homework 5
ME5659 Spring 2024

Due: See Canvas, turn in on Gradescope

Problem 1 (12 points)

Consider the following linear systems, where

$$\text{(i) } \dot{x} = \begin{bmatrix} -6 & -3 & -5 \\ 0 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} u, \quad y = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} x.$$

$$\text{(ii) } \dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x.$$

(a) **4 points.** Are these linear systems controllable? Please justify your answers.

(b) **4 points.** Find the Kalman controllable canonical form for the above linear systems.

(c) **4 points.** Are these linear systems stabilizable? Please justify your answers.

SOLUTIONS

a) i)

$$\dot{x} = \begin{bmatrix} -6 & -3 & -5 \\ 0 & -3 & 1 \\ 2 & 2 & 0 \end{bmatrix} x + \begin{bmatrix} -2/3 & 1/3 \\ 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix} u \qquad y = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix} x \qquad P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$P = \begin{bmatrix} -2/3 & 1/3 & 4/3 & -\frac{5}{3} & -8/3 & 19/3 \\ 1/3 & -2/3 & -2/3 & 7/3 & 4/3 & -\frac{23}{3} \\ 1/3 & 1/3 & -2/3 & -2/3 & 4/3 & 4/3 \end{bmatrix}$$

```
P_1 = 3×6
   -0.6667    0.3333    1.3333   -1.6667   -2.6667    6.3333
    0.3333   -0.6667   -0.6667    2.3333    1.3333   -7.6667
    0.3333    0.3333   -0.6667   -0.6667    1.3333    1.3333
```

The system is uncontrollable

$rk\,Q = 2 < n$

ii)

$$A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \qquad P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 & -3 & -5 & 9 & 25 \\ 1 & -1 & -3 & 5 & 9 & -25 \\ 1 & 0 & -3 & 0 & 9 & 0 \end{bmatrix}$$

$rk\,Q = 2 < n$ The system is not controllable

b) KALMANN

i)

$$T = \begin{bmatrix} -2/3 & 1/3 & 1 \\ 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \end{bmatrix} \qquad \det T = 0.33 \neq 0 \qquad \hat{A} = T^{-1} A T = \begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 2 \\ 0 & 0 & -4 \end{bmatrix} \qquad \hat{B} = T^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\tilde{P} = (A_{11}, B_1) = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -3 \end{bmatrix} \quad rk = 2 = n\ (\tilde{P})$$

$$\dot{z}_1 = -2z_1 + z_2 + 4z_3 + u_1$$
$$\dot{z}_2 = -3z_2 + 2z_3 + u_2$$
$$\dot{z}_3 = -4z_3$$

ii) $$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \det T = 1 \neq 0$$

$$\hat{A} = T^{-1} A T = \begin{bmatrix} -3 & 0 & -2 \\ 0 & -5 & -4 \\ 0 & 0 & -1 \end{bmatrix} \quad \hat{B} = T^{-1} B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\tilde{P}(\tilde{A}_{11}, \tilde{B}_1) = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -5 \end{bmatrix} \quad rk = 2 = n$$

$$\dot{z}_1 = -3z_1 - 2z_3 + u_1$$
$$\dot{z}_2 = -5z_2 - 4z_3 + u_2$$
$$\dot{z}_3 = -z_3$$

c) i, ii) Both are stabilizzable since $Re(\lambda_i(A_{22})) = -1, -4 < 0$ the system is uncontrollable but it goes to zero and hence it is stabilizable

Problem 2 (13 points)

Consider the following state-space model:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad x(0) = x_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} x$$

(a) **(3 points)** Is the system controllable? If not, please transform the state-space model into a diagonal canonical form (DCF) where the controllable subspace should be separated from the uncontrollable subspace. Please intuitively tell whether the state-space model in DCF is stabilizable or not.

(b) **(4 points)** Please use the PBH eigenvector test and the PBH rank test to justify the stabilizability of the state-space model in DCF.

(c) **(3 points)** If the system is stabilizable, please calculate the input $u(t)$ that drives $x(0)$ to 0 in 5 seconds.

(d) **(3 points)** Verify that the input achieves this by plotting the state trajectories $x(t)$ vs. time t with the initial condition $x(0)$ in Matlab. The plot should have two trajectories $x_1(t), x_2(t)$.

a)

$P = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}$ $\quad rk\,P = 1 < 2$ the system is not controllable

$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix}\right) = \det\begin{bmatrix} s+1 & 0 \\ -1 & s-4 \end{bmatrix} = (s+1)(s-4)$

$s_1 = -1 \quad s_2 = 4$

eigenvector associated to $s_1 = -1$

$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & -5 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 \\ -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_2 = \alpha$, $+x_1 + 5x_2 = 0$, $x_1 = -5x_2$ $\quad v_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

eigenvector associated to $s_1 = 4$

$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & 0 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = \alpha$ $\quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T = \begin{bmatrix} -5 & 0 \\ 1 & 1 \end{bmatrix}$ $\quad \det T = -5 \neq 0$ ✓

$A_{DCF} = T^{-1} A T = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad B_{DCF} = T^{-1} B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad C_{DCF} = \begin{bmatrix} -9 & 1 \end{bmatrix}$ $\quad z_0 = \begin{bmatrix} -0.4 \\ -0.6 \end{bmatrix}$

$Re(A_{22}(\lambda_i)) = 4$ not stabilizable <u>but</u> there is a non zero component in the second row of B => stabilizable.

b) PBH eigenvector test

$(\lambda_i I - A^T) w_i = 0 \quad A^T = \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}$

$\lambda_1 = -1$

$$\left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 0 & -5 \end{bmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-w_2 = 0$ $\quad$ $w_2 = 0$ $\quad$ $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ $w^T B \neq 0 \Rightarrow \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

$-5w_2 = 0$ $\quad$ $w_1 = \alpha$

so -1 is uncontrollable but since $\lambda = -1 < 0$ it is stabilizable by PBH eigenvector test

$\lambda_2 = 4$

$$\left(\begin{bmatrix} +4 & 0 \\ 0 & +4 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}\right)\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$5w_1 - w_2 = 0$ $\quad$ $w_2 = 5\alpha$ $\quad$ $w = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ $\quad$ $w^T B \neq 0 \Rightarrow \begin{bmatrix} 1 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5$

$w_1 = \alpha$ $\quad$ $w_1 = \alpha$

so 4 is controllable not orthogonal

PBH rank TEST

$\lambda_1 = -1$

$$\text{rank}\left[\lambda I - A_{DCF} \mid B_{DCF}\right] = \text{rank}\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -5 & 1 \end{array}\right] = 1 < 2$$

corresponds to $\lambda_1 = -1$ Not controllable but since $Re(-1) < 0$ stabilizable

$\lambda_2 = 4$

$$\text{rank}\left[\lambda I - A_{DCF} \mid B_{DCF}\right] = \text{rank}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 = n$$

so stabilizable by PBH rank tests

c) $A = A_{DCF}$; $B = B_{DCF}$; $C = C_{DCF}$

CONTROLLABILITY GRAMIAN:

$$W(t_0, t_f) = \int_{t_0}^{t_f} e^{A(t_0 - \tau)} B B^T e^{A^T(t_0 - \tau)} \, d\tau$$

$$u(t) = B^T e^{A^T(t_0 - t)} W^{-1}(t_0, t_f)\left(e^{A(t_0 - t)} x_f - x_0\right)$$

x_f is the origin.

$$W(0,5) = \int_0^5 e^{-A\tau} B B^T e^{-A^T\tau} \, d\tau$$

where $A = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$e^{-A\tau} = \begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{-4\tau} \end{bmatrix}$$

$\dot{x}_1 = -x_1$

$\dot{x}_2 = 4x_2 + u$

$x_{0_{DCF}} = \begin{pmatrix}-0.4\\-0.6\end{pmatrix}$

$x_1(t) = e^{-t}\cdot(-0.4)$

$x_1(t) = e^{-5}(-0.4) = -0.0027 \simeq 0$

$$W(0,5) = \int_0^5 e^{-4\tau} e^{-4\tau}\, d\tau = \int_0^5 e^{-8\tau}\, d\tau = -\frac{1}{8} e^{-8\tau}\Big|_0^5$$

$$W(0,5) = -\frac{1}{8} e^{-40} + \frac{1}{8}$$

$$u(t) = -\frac{3e^{-4t}}{5\left(\frac{e^{-40}}{8} - \frac{1}{8}\right)}$$

d)

3.mlx * | untitled2.mlx * | untitled3.mlx | HW4.mlx | prob_2_part_c

```
clear all; clc; close all;
A=[-1 0; 1 4];
B=[0; 1];
C=[2 1];
D=[0];
x_0=[2; -1];
t=linspace(0, 5, 1000000);
u=(-3*exp(-4*t))/(5*((exp(-40)/8)-1/8));
sys=ss(A, B, C, D);
[y,t_out, x]=lsim(sys,u,t,x_0);
plot(t_out, x, 'LineWidth', 2);
```