HOMEWORK-7

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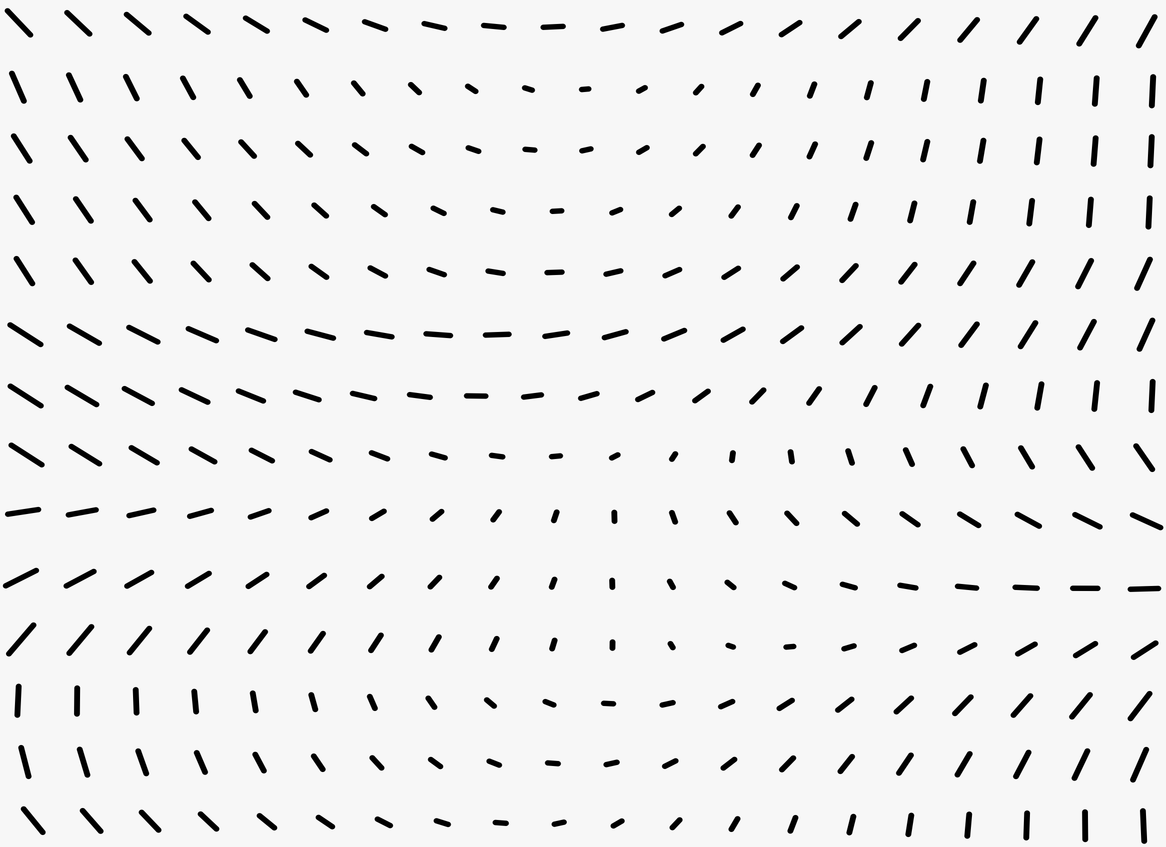


1)

a) If the rank of controllability matrix is equal to the order of the system, we can place the closed loop eigenvalues asbitriarly.

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = 0.$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & -2 \\ 4 & 0 \end{bmatrix}, \quad A^2B = \begin{bmatrix} 0 & -4 \\ 0 & 4 \\ 16 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & -2 & 0 & 4 \\ 1 & 0 & 4 & 0 & 16 & 0 \end{bmatrix}$$

$\rightarrow$ rank $= 3 = 3$ (system order)

As system is controllable, it is arbitrarly place the closed loop eigen values with State feedback control

$u = -Kx$.

b) For $\lambda_1 = -2$, $\lambda_2 = -3$, $\lambda_3 = -4$

$$G(s) = \frac{1}{(s+2)(s+3)(s+4)}$$

characteristic eqn $\Rightarrow$ $(s+2)(s+3)(s+4) = 0$

$$(s+2)(s^2+7s+12) = 0$$

$$\Rightarrow s^3 + 9s^2 + 26s + 24 = 0.$$

c) given that $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D = 0$

output eqn: $y = Cx + Dr$

$$= Cx + 0$$

$$y = Cx - CKx + Kg\,r$$

given that $y = r$, we can write

$$r = Cx - CKx + Kg\,r$$

$$(I - C + CKx) = Kg\,[2\ 4]^T$$

$$Kg = (I - C + CKx)^{-1}[2\ 4]^T$$

desired poles : $[-3, -4, -5]$

$$A' = A - BK$$

$$= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & -3 & -4 \end{bmatrix}$$

$$\therefore K_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\therefore K_g = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

```
import numpy as np
import control as ctrl

# Define system matrices
A = np.array([[-2, 1, 0],
              [0, -2, 0],
              [0, 0, 4]])
B = np.array([[0, 0],
              [0, 1],
              [1, 0]])
C = np.array([[1, 0, 0],
              [0, 0, 1]])
D = np.array([[0, 0],
              [0, 0]])

# Desired poles
desired_poles = np.array([-3, -4, -5])

# Calculate feedback gain matrix K
K = ctrl.place(A, B, desired_poles)
print("Feedback Gain Matrix K:")
print(K)
```

MATLAB IMPLEMENTATION

2)

$J\ddot{\theta} + b\dot{\theta} + K\theta = T$

$J = 1,$

$b = 1$

$K = 2$ (given)

$\theta(0) = 0$, $\dot{\theta}(0) = 0$

put them in above eqn.

$$\ddot{\theta} + \dot{\theta} + 2\theta = T$$

→ laplace transformation gives:

$$(s^2 + s + 2)\,\theta(s) = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{s^2+s+2} \Rightarrow \frac{K_{DC}\,w_n^2}{s^2+2\zeta w_n s + w_n^2}$$

$w_n^2 = 2$, $K_{DC} = 1/2$, $2\zeta w_n = 1$

$w_n = \pm\sqrt{2}$, $\zeta = \frac{\pm 1}{2\sqrt{2}}$

as $0 < \zeta < 1$ → underdamped system.

(a) settling time $\Rightarrow \frac{4}{\zeta w_n} \Rightarrow \frac{4}{0.5} \Rightarrow \underline{\underline{8\,s}}$

% overshoot $\Rightarrow e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \cdot 100\%$

$= e^{-\frac{\pi/2\sqrt{2}}{\sqrt{1-1/8}}} \cdot 100\%$

$\Rightarrow e^{-1.18} \cdot 100\%$

$\Rightarrow \underline{\underline{30.51\%}}$

b) % overshoot $= 2\% \Rightarrow e^{-\pi \zeta/\sqrt{1-\zeta^2}} \cdot 100\%$

$$0.02 = e^{-\pi \zeta/\sqrt{1-\zeta^2}}$$

$$(-3.91)\sqrt{1-\zeta^2} = \pi\zeta$$

$$(1.24)^2(1-\zeta^2) = \zeta^2$$

$$1.55 - 1.55\zeta^2 = \zeta^2 \Rightarrow \underline{\zeta = 0.78}$$

$$t_s = 4/\zeta w_n$$

$$w_n = 4/0.78 \Rightarrow \underline{\underline{5.12}}$$

from (a), $\ddot{\theta} + \dot{\theta} + 2\theta = T$

$$\ddot{\theta} = T - 2\theta - \dot{\theta}$$

$$\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = x$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

upon applying control law

$$u = -Kx \Rightarrow \begin{bmatrix} -K_p & -K_d \end{bmatrix} x$$

$$A - Bt = \begin{bmatrix} 0 & 1 \\ -2-K_p & -1-K_d \end{bmatrix}$$

characteristic eqn $\Rightarrow |SI - A| \Rightarrow \begin{bmatrix} S & -1 \\ 2+K_p & S+K_d+1 \end{bmatrix}$

$$\Rightarrow \quad s(s+K_d+1) + 2+K_p = 0$$
$$\Rightarrow \quad s^2 + K_d s + s + 2 + K_p = 0.$$

we require: $s^2 + 2\zeta\omega_n s + \omega_n^2$

$$\Rightarrow s^2 + 2[0.61][5.12]\, s + [5.12]^2 = 0.$$
$$\Rightarrow s^2 + 6.25s + 26.23 = 0.$$

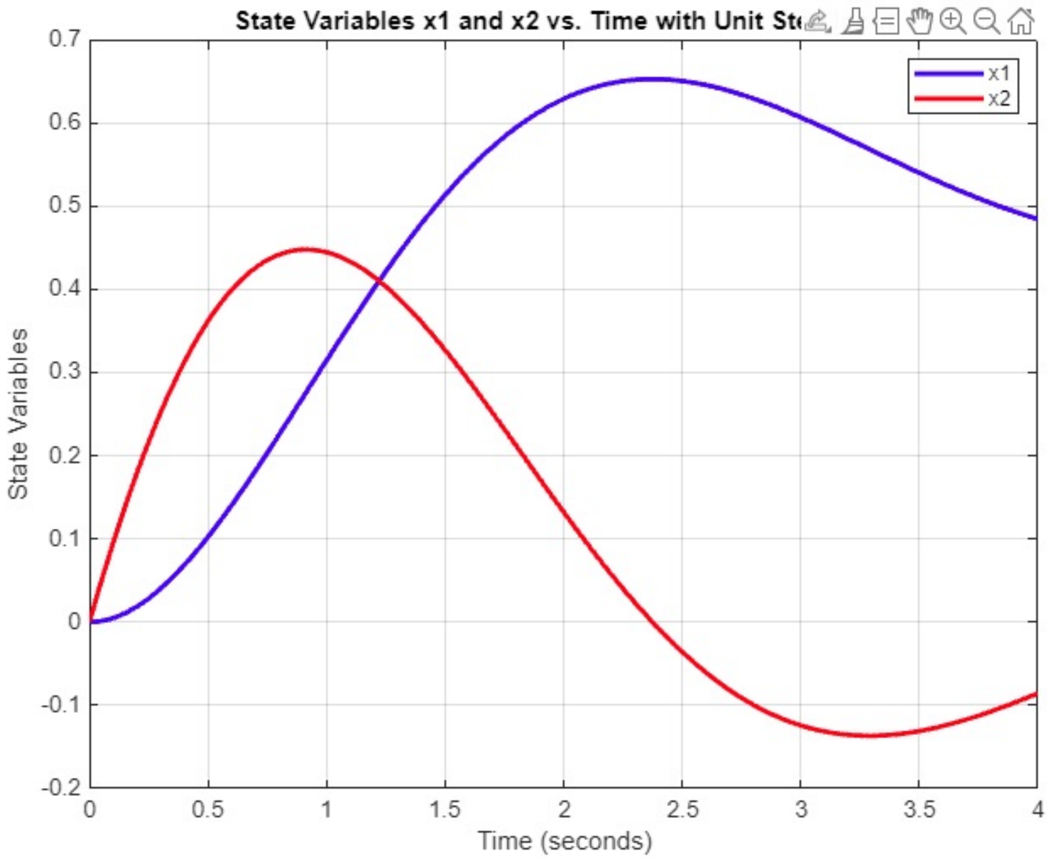
By observing we can say:

$$K_d = 5.25 \rightarrow (6.25 - 1)$$
$$K_p = 24.23 \rightarrow (26.23 - 2)$$

$\therefore$ $K = [24.23 \quad 5.25]$ will modify the system state to required Dynamic response.

c)

```
% Define the state-space matrices
A = [0 1; -2 -1];
B = [0; 1];
C = [1 0];
D = [];

% Define the time span
tspan = 0:0.01:4;

% Define initial conditions
x0 = [0; 0];

% Define the unit step input
u = ones(size(tspan));

% Define the system using the state-space representation
sys = ss(A, B, C, D);

% Simulate the system with the unit step input and initial conditions
[y, t, x] = lsim(sys, u, tspan, x0);

% Plot the state variables x1 and x2 with respect to time
figure;
plot(t, x(:,1), 'b', 'LineWidth', 2);  % x1
hold on;
plot(t, x(:,2), 'r', 'LineWidth', 2);  % x2
xlabel('Time (seconds)');
ylabel('State Variables');
title('State Variables x1 and x2 vs. Time with Unit Step Input');
legend('x1', 'x2');
grid on;
```


State Variables x1 and x2 vs. Time with Unit St
x1
x2
State Variables
Time (seconds)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4

3) we should verify the desired poles of $G_{cl}(s)$ are reachable

(a) with state feedback.

we should find state space representation:

$$G_{op}(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)} \Rightarrow \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s+3}$$

$$(s-1)(s+2) = A(s-2)(s+3) + B(s+1)(s+3) + C(s+1)(s-2)$$

$$\frac{-5}{6s+6} + \frac{1}{3s-6} + \frac{1}{2s+6}$$

$$(-15s+30)(2s+6) + (6s+6)(2s+6) + (6s+6)(3s-6)$$

$$-3\cancel{0}s^2 - 9\cancel{0}s + 6\cancel{0}s + 180 + 1\cancel{2}s^2 + 3\cancel{6}s + 1\cancel{2}s + \cancel{36} + 1\cancel{8}s^2 - 3\cancel{6}s + 1\cancel{8}s - \cancel{36}$$

$$A = \frac{-5}{6}, \quad B = \frac{1}{3}, \quad C = \frac{1}{2}.$$

so, state space repn is :

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{+5}{6} & \frac{-1}{3} & \frac{-1}{2} \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$$

$$D = 0$$

closed loop charecteristic eq^n:

$$(u = -Kx)$$

$$\det(SI - (A - BK)) = 0.$$

$K \to$ gain matrix

To get the charecteristic polynomial, pole must be at $S = -2, -3$.

$$\Rightarrow \det(SI - (A - BK)) = (S+2)(S+3)$$

Let $K = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$

$$\det(SI - (A - BK)) = \det\left(\begin{bmatrix} S+1 & 0 & 0 \\ 0 & S-2 & 0 \\ 0 & 0 & S+3 \end{bmatrix} - \begin{bmatrix} -\frac{5}{6}K_1 & -\frac{5}{6}K_2 & -\frac{5}{6}K_3 \\ \frac{1}{3}K_1 & \frac{1}{3}K_2 & \frac{1}{3}K_3 \\ \frac{1}{2}K_1 & \frac{1}{2}K_2 & \frac{1}{2}K_3 \end{bmatrix} \right)$$

$$= (S+2)(S+3)$$

$$\Rightarrow \quad 1 + \frac{5}{6}K_1 = 0, \quad 2 - \frac{1}{3}K_2 = 0, \quad 3 - \frac{1}{2}K_3 = 6.$$

$$\Rightarrow \quad K_1 = -6/5 \qquad K_2 = 6 \qquad K_3 = -6$$

gain matrix: $\begin{bmatrix} -6/5 & 6 & -6 \end{bmatrix}$

(b) We should check the eigenvalues of closed loop matrix $A - BK$ for BIBO stability.

$$A - BK = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 1/3 \\ 1/2 \end{bmatrix} \begin{bmatrix} -6/5 & 6 & -6 \end{bmatrix}$$

$$BK = \begin{bmatrix} -1 & 5 & -5 \\ -2/5 & 2 & -2 \\ -3/5 & 3 & -3 \end{bmatrix}, \quad A - BK = \begin{bmatrix} 0 & -5 & 5 \\ 2/5 & 0 & 2 \\ 3/5 & -3 & 0 \end{bmatrix}$$

$\det\left(sI - (A - BK)\right) = 0 \rightarrow$ for eigen values.

$\lambda_1 = -1.592$

$\lambda_2 = 0.796 + i(\ )$

$\lambda_3 = 0.796 - i(\ )$

As all the eigenvalues do not have negative real part, the system is not BIBO Stable.

(c) Similarly as one of the eigenvalues has a positive real part, the system is not asymptotically stable.