Control Systems Engineering - HOMEWORK 1

By,

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Assignment - 1
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1) Given,

$$G(S) = \frac{Y(S)}{U(S)} = \frac{4S^4 - S^2 + 5S - 1}{2S^4 + 2S^2 - 4S + 6}$$

$$= \frac{(2S^2-1)(2S^2+1) + 5S - 1}{2S^2 + 2S^2 - 4S + 6}$$

We can write it as $G(S) = C(SI - A)^{-1} B + D$.

$G(S) \rightarrow$ Transfer function
$Y(S) \rightarrow$ output
$U(S) \rightarrow$ Input
$A, B, C, D \rightarrow$ Matrices representing SS
$I \rightarrow$ Identity Matrix

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t) + D\,u(t)$$

$x(t) \rightarrow$ State vector
$u(t) \rightarrow$ input vector
$y(t) \rightarrow$ output vector

State variables : $x_1(t), x_2(t), x_3(t), x_4(t)$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

$$Y(s) = 4s^4 - s^2 + 5s - 1$$
$$U(s) = 2s^4 + 2s^2 - 4s + 6$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_{n-1} & b_{n-2} & \cdots & b_2 & b_1 & b_0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \end{bmatrix}$$

$b_4 = 4$, $b_3 = 0$, $b_2 = -1$, $b_1 = 5$, $b_0 = -1$
$a_4 = 2$, $a_3 = 0$, $a_2 = 2$, $a_1 = -4$, $a_0 = 6$

Hence,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 4 & -2 & 0 \end{bmatrix} = 2\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 2 & -1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 5 & -1 & 4 \end{bmatrix}$$

$$y(t) = \overset{2}{\cancel{4}}\left[\frac{u - 2z^2 + 4\dot{z} - 6z}{\cancel{2}}\right] - z^2 + 5\dot{z} - 2$$

$$= 2u - 4z^2 + 8\dot{z} - 12z - z^2 + 5\dot{z} - 2$$

$$= 2u - 5z^2 + 13\dot{z} - 13z$$

$$y = \begin{bmatrix} -13 & 13 & -5 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \end{bmatrix} u.$$

(b) $\ddot{y}_1(t) + 2\dot{y}_1(t) - 5\left(y_2(t) - y_1(t)\right) = u_1(t)$ — (1)

$\ddot{y}_2(t) + \dot{y}_1(t) - 4\dot{y}_2(t) - 3\left(y_2(t) - y_1(t)\right) = u_2(t)$ — (2)

$x_1 = y_1, \quad x_2 = \underbrace{\dot{y}_1}_{=\dot{x}_1}, \quad x_3 = y_2, \quad x_4 = \underbrace{\dot{y}_2}_{=\dot{x}_3}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$\ddot{y}_1 + 2\dot{y}_1 - 5(y_2 - y_1) = u_1$

$\ddot{y}_1 - 5y_1 + 5y_2 - 2\dot{y}_1 + u_1$

$= -5y_1 - 2\dot{y}_1 + 5y_2 + u_1$

$\dot{x}_2 = -5x_1 - 2x_2 + 5x_3 + u_1$

$\ddot{y}_2 + \dot{y}_1 - 4\dot{y}_2 - 3(y_2 - y_1) = u_2$

$\ddot{y}_2 = -3y_1 + 3y_2 - \dot{y}_1 + 4\dot{y}_2 + u_2$

$\dot{x}_4 = -3x_1 - x_2 + 3x_3 + 4x_4 + u_2$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -1 & 3 & 4 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \; ; \quad y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

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2 a) we need to express it in the form of $\dot{x} = f(x, u)$

given, $ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = T$

lets say,

$x_1 = \theta$ and $x_2 = \dot{\theta}$

state vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

equations of motion would be:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{T}{ml^2} - \frac{b}{ml^2} x_2 - \frac{g}{l} \sin x_1$$

input vector $u = T$

Therefore,

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{b}{ml^2} x_2 - \frac{g}{l} \sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

b) To linearize the nonlinear SS model about $\theta = 0$ & π, we need to find the equilibrium values of state variables ($x_{1,0}$ and $x_{2,0}$) and input (u_0) at each point. Then we can linearize with the help of Jacobian Matrices.

1. At $\theta_0 = 0$,

At equilibrium, $\dot{x}_1 = 0$, $\dot{x}_2 = 0$ & $u = 0$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{T}{ml^2} - \frac{b}{ml^2} x_2 - \frac{g}{l} \sin x_1$$

by doing $\dot{x}_1 = \dot{x}_2 = u = 0$ at $\theta_0 = 0$ we get,

$x_{1,0} = 0$, $x_{2,0} = 0$. and $u_0 = 0$

Jacobian Matrices:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_0, u_0} \quad \text{and} \quad B = \left.\frac{\partial f}{\partial u}\right|_{x_0, u_0}$$

$$\therefore \quad A = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} ; \quad B = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial u} \\ \frac{\partial \dot{x}_2}{\partial u} \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 \\ -g/l & -b/ml^2 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}$$

$$\Delta \dot{x} = \begin{bmatrix} 0 & 1 \\ -g/l & -b/ml^2 \end{bmatrix} \partial x + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} \partial u$$

2. At $\theta_0 = \pi$

At equilibrium, $x_{1,0} = \pi$, $x_{2,0} = 0$ and $u_0 = 0$.

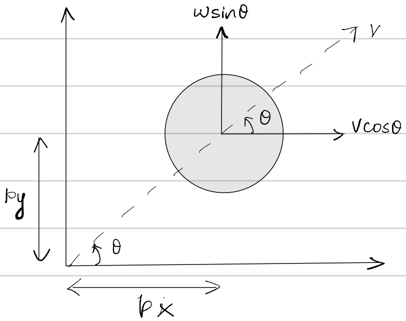
Computing Jacobian in the same way as above we get,

$$\therefore \quad A = \begin{bmatrix} 0 & 1 \\ g/l & -b/ml^2 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix}$$

$$\Delta \dot{x} = \begin{bmatrix} 0 & 1 \\ g/l & -b/ml^2 \end{bmatrix} \partial x + \begin{bmatrix} 0 \\ 1/ml^2 \end{bmatrix} \partial u$$

3)

(a)



$$\dot{P}_x = V\cos\theta \quad \text{--- (1)}$$
$$\dot{P}_y = V\sin\theta \quad \text{--- (2)}$$
$$\dot{\theta} = \omega$$
$$U = \begin{bmatrix} V \\ \omega \end{bmatrix}$$

Given state is represented by,

$$x_1 = P_x\cos\theta + (P_y - 1)\sin\theta$$
$$x_2 = -P_x\sin\theta + (P_y - 1)\cos\theta$$
$$x_3 = \theta$$
$$\dot{x}_1 = P_x\cos\theta + P_y\sin\theta - \sin\theta$$
$$= V\cos^2\theta + V\sin^2\theta + (-P_x\sin\theta + P_y\cos\theta - \cos\theta)\dot{\theta}$$
$$= V + x_2 w \quad \text{--- (1)}$$
$$\dot{x}_2 = -\dot{P}_x\sin\theta - P_x\cos\theta + \dot{P}_y\cos\theta - P_y\sin\theta + \sin\theta$$
$$= -\dot{\theta}\left(P_x\cos\theta + (P_y - 1)\sin\theta\right)$$
$$= -\dot{\theta}x_1 \Rightarrow \underline{\underline{-x_1 w}} \quad \text{--- (2)}$$

$$\dot{x}_3 = \dot{\theta} = \omega \quad \text{--- (3)}$$

Non linear state space representation:

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 w + V \\ -x_1 w \\ \omega \end{bmatrix} \begin{matrix} \leftarrow f_1 \\ \leftarrow f_2 \\ \leftarrow f_3 \end{matrix}$$

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ ← Non linear Representation.

(b)

$$\Delta \dot{x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \partial f_1/\partial x_3 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \partial f_2/\partial x_3 \\ \partial f_3/\partial x_1 & \partial f_3/\partial x_2 & \partial f_3/\partial x_3 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \\ \partial x_3 \end{bmatrix}$$

$$+ \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \partial f_1/\partial u_3 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \partial f_2/\partial u_3 \\ \partial f_3/\partial u_1 & \partial f_3/\partial u_2 & \partial f_3/\partial u_3 \end{bmatrix} \begin{bmatrix} \partial u_1 \\ \partial u_2 \\ \partial u_3 \end{bmatrix}$$

$$\Delta \dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -2v\cos 2x_3 \\ 0 & 0 & 0 \end{bmatrix} \partial x + \begin{bmatrix} 1 & 0 & 0 \\ -\sin 2x_3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \partial u$$

$x_{eq} = 0$
$u_{eq} = 0$

$u_1 = v$
$u_2 = \omega$

$\to u_3$

$$\Delta \dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \partial x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \partial u$$

is the linearization.

4) Model

$$J\ddot{\theta} + b\dot{\theta} = K_t i$$

$$L\dot{i} + Ri + K_b\dot{\theta} = V_s$$

$x_1 = \theta$ $\quad x_3 = i$, $\quad y = \theta$, $\quad u = V_s$

$x_2 = \dot{\theta}$

(a)

$$J\ddot{\theta} + b\dot{\theta} = K_t i$$

$$J\dot{x}_2 = K_t x_3 - b x_2$$

$$\dot{x}_2 = \frac{K_t x_3 - b x_2}{J} \quad — (1)$$

$$L\dot{i} + Ri + K_b\dot{\theta} = V_s$$

$$L\dot{x}_3 + Rx_3 + K_b x_2 = u$$

$$L\dot{x}_3 = u - Rx_3 - K_b x_2$$

$$\dot{x}_3 = \frac{u - Rx_3 - K_b x_2}{L} \quad — (2)$$

From (1) and (2) if $x = \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -b/J & K_t/J \\ 0 & -K_b/L & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(b)

$$J\ddot{\theta} + b\dot{\theta} = K_t i$$

Laplace Transformation:

$$J\theta(s)s^2 + b\theta(s)s - K_t i(s)$$

$$i(s) = \frac{(Js^2 + bs)\,\theta(s)}{K_t} \quad \text{— (1)}$$

$$L\dot{i} + Ri + K_b\dot{\theta} = V_s$$

Laplace Transformation:

$$Li(s)s + Ri(s) + K_b s\theta(s) = V(s)$$

$$(Ls + R)\,i(s) + K_b s\theta(s) = V(s)$$

put (1) in the above eqn.

$$\left[\frac{(Ls+R)(Js^2+bs)}{K_t} + K_b s\right]\theta(s) = V(s)$$

$$\left[\frac{LJs^3 + RJs^2 + bLs^2 + Rbs}{K_t} + K_b s\right]\theta(s) = V(s)$$

$$\left[\frac{LJs^3}{K_t} + \left[\frac{RJ + bL}{K_t}\right]s^2 + \left[\frac{Rb}{K_t} + K_b\right]s\right]\theta(s) = V(s)$$

Laplace eqn is 3rd order represents that the system is 3rd order system.

(c) Inverse laplace, in differential not power

$$\frac{LJ}{K_t}\theta^3 + \left[\frac{RJ+bL}{K_t}\right]\theta^2 + \left[\frac{Rb}{K_t}+K_b\right]\dot{\theta} = V_s$$

$$\theta^3 = V_s - \left[\frac{RJ+bL}{K_t}\right]\theta^2 - \left[\frac{Rb}{K_t}+K_b\right]\dot{\theta}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix}, \quad u = V_s, \quad y = \theta$$

Then $x_3 = V_s - \left[\frac{RJ+bL}{K_t}\right]x_3 - \left[\frac{Rb}{K_t}+K_b\right]x_2$

STATE SPACE REPRESENTATION:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\left[\frac{Rb}{K_t}+K_b\right] & -\left[\frac{RJ+bL}{K_t}\right] \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(d) putting all parameters as 1

from (a) $A_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$

from (c) $A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix}$