# Homework 7
# ME5659 Spring 2024

**Due: See Canvas, turn in on Gradescope**

## Problem 1 (7 points)

Consider a linear state-space model

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = 0$$

(a) **1 points.** Is it possible to arbitrarily place the closed-loop eigenvalues with state-feedback control $u = -Kx$? Why or why not?

(b) **2 points.** Can one choose the desired closed-loop eigenvalues to be $\lambda_1 = -2, \lambda_2 = -3, \lambda_3 = -4$? If possible, determine the necessary feedback gain matrix $K$ by a hand calculation.

(c) **4 points.** Consider a state-feedback control law $u = -Kx + k_g r$, where $r(t) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is the reference input. Compute the $k_g$ such that the system outputs $y_1, y_2$ will track the given reference input. Use Matlab to verify your answers by plotting two trajectories $y_1$ vs. time $t$ and $y_2$ vs. time $t$.

a)

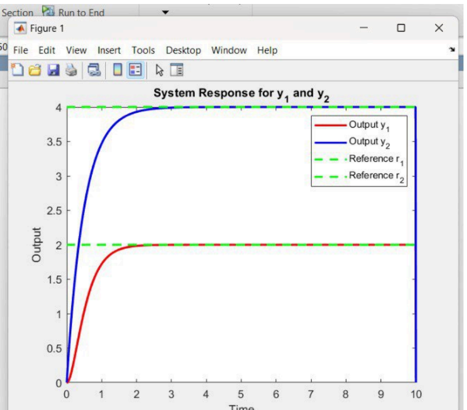
To arbitrarey place the closed loop eigenvalue we need to check the controllabrery of the system:

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 1 & 0 & -2 & 0 & 4 \\ 1 & 0 & 4 & 0 & 16 & 0 \end{bmatrix}$$

rk P = 3 the system is controllable
∴ it's possible to place eigenvalues

b)

```
clear all; clc; close all;
A = [-2 1 0; 0 -2 0; 0 0 4];
B = [0 0; 0 1; 1 0];
C = [1 0 0; 0 0 1];
x0 = [0; 0; 0];
D = zeros(2,2);
p=[-2; -3; -4;];
K=place(A,B,p);
kg = -inv(C*inv(A-B*K)*B);
t = 0:0.01:10;
r = [2; 4];
x = zeros(3, length(t)); % State trajectory
y = zeros(2, length(t)); % Output trajectory
for i = 1 : length(t)-1
    u = -K * x(:, i) + kg * r; % Control input
    x_dot = A * x(:, i) + B * u;
    x(:, i+1) = x(:, i) + x_dot * (t(i+1) - t(i));
    x(:, i+1) = max(min(x(:, i+1), 10), -10);
    y(:, i) = C * x(:, i);
end
%plot
plot(t, y(1,:), 'r', t, y(2,:), 'b', t, r(1)*ones(size(t)), 'g--', t, r(2)*ones(size(t)), 'g--','LineWidth',2);
xlabel('Time');
ylabel('Output');
legend('Output y_1', 'Output y_2', 'Reference r_1', 'Reference r_2');
title('System Response for y_1 and y_2');
```




c)

## Problem 2 (10 points)

Consider a single-input single-output rotational mechanical system described by:

$$J\ddot{\theta} + b\dot{\theta} + k\theta = \tau$$

where the single input is an externally applied torque $\tau(t)$, and the output is the angular displacement $\theta(t)$. $J = 1\,kgm^2$ is the system inertia, $b = 1\,Nms/rad$ is the rotational viscous damping coefficient, and $k = 2\,Nm/rad$ is the torsional spring constant. We assume that a unit step external torque is applied. The initial condition is given by: $\theta(0) = 0\,rad$ , $\dot{\theta}(0) = 0\,rad/s$. (Feel free to use MATLAB)

(a) **4 points.** Evaluate the percent overshoot and the settling time. Plot $\theta(t)$ and $\dot{\theta}(t)$ of the open-loop system (with unit step input) for $t \in [0, 4]\,s$.

(b) **6 points.** Shape the dynamic response so that the percent overshoot is 2% and the setting time $t_s = 1\,s$. The steady-state performance should be the same as the one in the open-loop response. Plot $\theta(t)$ and $\dot{\theta}(t)$ of the closed-loop system (with a state-feedback control law and a unit step input) for $t \in [0, 4]\,s$.

SOLUTIONS

a) $J\ddot{\Theta} + b\dot{\Theta} + k_R\Theta = \tau$

$x_2 = \dot{x}_1(t)$

$\ddot{\Theta} = \frac{1}{J}\tau(t) - \frac{b}{J}\dot{\Theta} - \frac{k_R}{J}\Theta \quad \Rightarrow \quad \dot{x}_2 = \frac{1}{J}\tau(t) - \frac{b}{J}x_2 - \frac{k_R}{J}x_1$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{\Theta}(t) \\ \ddot{\Theta}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \frac{1}{J}\tau(t) - \frac{b}{J}x_2 - \frac{k_R}{J}x_1 \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k_R}{J} & -\frac{b}{J} \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/J \end{bmatrix}}_{B}\tau(t) \qquad \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}}_{A} \qquad \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B}$$

$\dot{X} = AX + Bu$ state equation.

Output equation : $y = Cx + Du$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$ because the output is the angular displacement $\Theta(t)$

To evaluate the percent overshoot.

$$\%\,O.S = \underbrace{e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}}_{HP} \cdot 100\%$$

Setting time $(T_s)$ $\quad T_s = \frac{4}{\zeta \omega_n}$

$$H(s) = C(sI-A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \left( \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k_R}{J} & -\frac{b}{J} \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$H(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k_R}{J} & s+\frac{b}{J} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$H(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s(s+\frac{b}{J})+\frac{k_R}{J}} \begin{bmatrix} s+b/J & 1 \\ -\frac{k_R}{J} & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$H(s) = \frac{1}{s^2+\frac{b}{J}s+\frac{k_R}{J}} \begin{bmatrix} s+\frac{b}{J} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

$$H(s) = \frac{1}{s^2+\frac{b}{J}s+\frac{k_R}{J}} \cdot 1/J = \frac{1/J}{s^2+\frac{b}{J}s+\frac{k_R}{J}}$$

$$H(s) = \frac{C\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1/J}{s^2+\frac{b}{J}s+\frac{k_R}{J}}$$

$$\omega_n^2 = \frac{k_R}{J} \Rightarrow \omega_n = \sqrt{\frac{k_R}{J}}$$

$$\frac{b}{J} = 2\zeta\omega_n \Rightarrow \zeta = \frac{b}{2J\sqrt{\frac{k_R}{J}}} = \frac{b}{2\sqrt{\frac{k_R J^{\cancel{2}}}{\cancel{J}}}} = \frac{b}{2\sqrt{k_R J}}$$

$$C\omega_n^2 = \frac{1}{J} \qquad C = \frac{1}{J}\frac{1}{\omega_n^2} = \frac{1}{J}\cdot\frac{1}{\frac{k_R}{J}} = \frac{1}{k_R}$$

$$\%OS = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \cdot 100\%$$

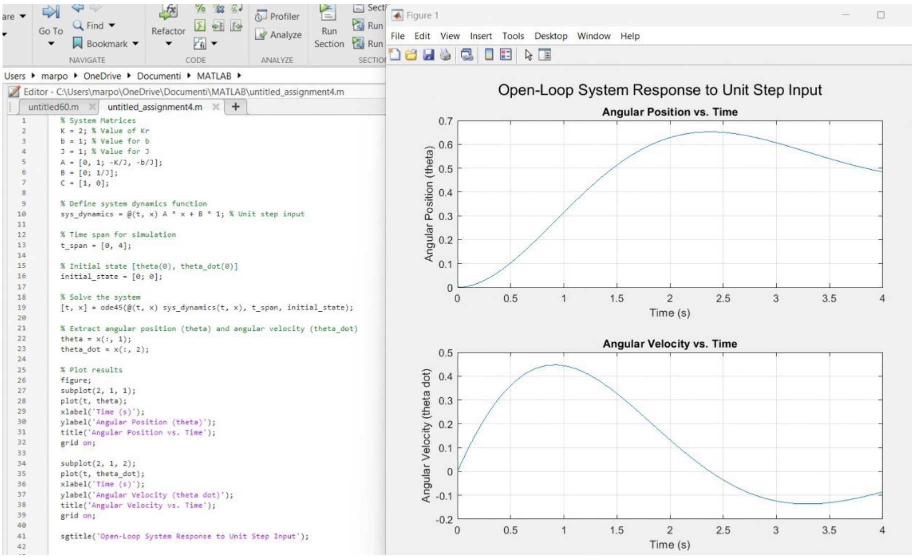
$$\zeta = \frac{b}{2\sqrt{k_R J}} = \frac{1}{2\sqrt{2 \cdot 1}} = \frac{1}{2\sqrt{2}} = 0.353$$ underdamped.

$$\% OS = e^{-\frac{\pi \cdot 0.353}{\sqrt{1-0.353^2}}} \cdot 100\% = 30.57\%$$

settling time

$$T_s = \frac{4}{\omega_n \zeta} \; ; \quad \omega_n = \sqrt{\frac{K_e}{J}} = \sqrt{2} \, \frac{rad}{sec} = 1.41 \, \frac{rad}{sec}$$

$$T_s = \frac{4}{1.41 \cdot 0.353} = 8.036 \text{ sec}$$




b) $\% OS = 2\% \quad T_s = 1 \text{ sec}$

We want to design a control $u = -kx + G \cdot r$

$$y = -\underbrace{\begin{bmatrix} K_p & K_d \end{bmatrix}}_{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Closed loop system

$\dot{x} = Ax + Bu$ but $u = -kx + G \cdot r$

$$\dot{x} = Ax + B(-kx + Gr)$$

$$\dot{x} = \underbrace{(A - BK)}_{A\,closed\,loop} x + \underbrace{BG}_{B\,closed\,loop} r$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{kr}{J} & -\frac{b}{J} \end{bmatrix} \quad B = \underset{2\times1}{\begin{bmatrix} 0 \\ 1/J \end{bmatrix}} \quad K = \underset{1\times2}{\begin{bmatrix} kp & kd \end{bmatrix}}$$

$$BK = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}\begin{bmatrix} kp & kd \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{kp}{J} & \frac{kd}{J} \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -\frac{kr}{J} & -\frac{b}{J} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \frac{kp}{J} & \frac{kd}{J} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{kr}{J} - \frac{kp}{J} & -\frac{b}{J} - \frac{kd}{J} \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{kr}{J} - \frac{kp}{J} & -\frac{b}{J} - \frac{kd}{J} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} G \cdot r$$

eigenvalues of $A - BK$ $\quad [\lambda I - (A - BK)]$

$$\left( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{kr}{J} - \frac{kp}{J} & -\frac{b}{J} - \frac{kd}{J} \end{bmatrix} \right) = \begin{bmatrix} \lambda & -1 \\ \frac{kr}{J} + \frac{kp}{J} & \lambda + \frac{b}{J} + \frac{kd}{J} \end{bmatrix}$$

$$\det[\lambda I - (A - BK)] = 0$$

$$\lambda\left(\lambda + \frac{b}{J} + \frac{kd}{J}\right) + \frac{kr}{J} + \frac{kp}{J} = 0$$

$$\lambda^2 + \frac{b}{J}\lambda + \frac{kd}{J}s + \frac{kr}{J} + \frac{kp}{J} = 0.$$

$$\lambda^2 + \left(\frac{b}{J} + \frac{kd}{J}\right)\lambda + \frac{kr}{J} + \frac{kp}{J} = 0$$ but $J = 1\ kg\,m^2$ $\quad b = 1 \frac{Nms}{rad}$ $\quad kr = 2 Nm/rad$

$\lambda^2 + (1 + kd)\lambda + (2 + kp) = 0$ characteristic polynomial.

LET'S compare it with the standard second order system caracteristic polynomial

$$\lambda^2 + 2\xi wn\lambda + wn^2 = 0$$

$\Rightarrow\ 2\xi wn = 1 + kd \ \Rightarrow\ kd = 2\xi wn - 1$

$\Rightarrow\ wn^2 = 2 + kp \ \Rightarrow\ kp = wn^2 - 2$

2% O.V

$$\xi = \frac{-\ln(MP)}{\sqrt{\pi^2 + \ln^2(MP)}} \qquad \%OS = MP \cdot 100$$

$$\xi = \frac{-\ln(2/100)}{\sqrt{\pi^2 + \ln^2(2/100)}} = 0.78$$

$T_s = 1\,s.$

$$T_s = \frac{4}{\xi w_n} \Rightarrow w_n = \frac{4}{\xi T_s} = \frac{4}{0.78 \cdot 1} = 5.130$$

$$K_d = 2\xi w_n - 1 = 2(0.78)(5.13) - 1 = 7$$

$$K_p = w_n^2 - 2 = (5.13)^2 - 2 = 24.32$$

$$K = [K_p \;\; K_d] = [24.32 \;\; 7]$$

~~Now I want~~

Now I want to design the G matrix. We want to choose G so that the steady-state output y of the closed loop value goes to the yss of the open loop.

transfer function of the open loop

$$H(s)_{OL} = \frac{1/J}{s^2 + \frac{b}{J}s + \frac{K_R}{J}}$$

closed-loop system

$$\dot{x} = (A - BK)x + BGr$$

$$\dot{x} = A_{CL}x + B_{CL}r$$

$$y = (C - DK)x + Gr$$

$$y = C_{CL}x + D_{CL}r \qquad \text{since } D=0 \Rightarrow C_{CL} = C$$

$$u = -Kx + Gr = -[K_p \;\; K_d]\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + Gr$$

$$\tau = -K_p\theta - K_d\dot{\theta} + Gr$$

$$J\ddot{\theta} + b\dot{\theta}(t) + K_r\theta = -K_p\theta - K_d\dot{\theta} + Gr$$

$$J\ddot{\theta} + (b + K_d)\dot{\theta} + (K_p + K_r)\theta = Gr$$

$$(Js^2 + (b + K_d)s + (K_p + K_r))\Theta(s) = G\,R(s)$$

Transfer function closed loop

$$H(s)_{CL} = \frac{\theta(s)}{R(s)} = \frac{G}{(Js^2 + (b + K_d)s + (K_p + K_r))}$$

$$H(s)_{CL} = \frac{G/J}{s^2 + \frac{(b+K_d)}{J}s + \frac{(K_p+K_r)}{J}}$$

final value theorem

$$\theta_{ss}^{OL} = \lim_{s\to 0} \cancel{s} \cdot \frac{1/J}{s^2 + \frac{bs}{J} + \frac{K_r}{J}} \frac{1}{\cancel{s}} = \frac{1/J}{\frac{K_r}{J}} = \frac{1}{K_r}$$

$$\theta_{ss}^{CL} = \lim_{s\to 0} \cancel{s} \cdot \frac{G/J}{s^2 + \left(\frac{b+K_d}{J}\right)s + \left(\frac{K_p+K_r}{J}\right)} \cdot \frac{1}{\cancel{s}} = \frac{G/J}{\frac{K_p+K_r}{J}} = \frac{G}{K_p+K_r}$$

$$\theta_{ss}^{CL} = \theta_{ss}^{OP} \Rightarrow \frac{G}{K_p+K_r} = \frac{1}{K_r} \Rightarrow G = \frac{K_p+K_r}{K_r} = \frac{24.32+2}{2} = \frac{26.32}{2} = 13.16$$

Finally $u(t) = -Kx(t) + Gr(t)$

$$u(t) = -\begin{bmatrix} 24.32 & 7 \end{bmatrix} x(t) + 13.16 r(t).$$

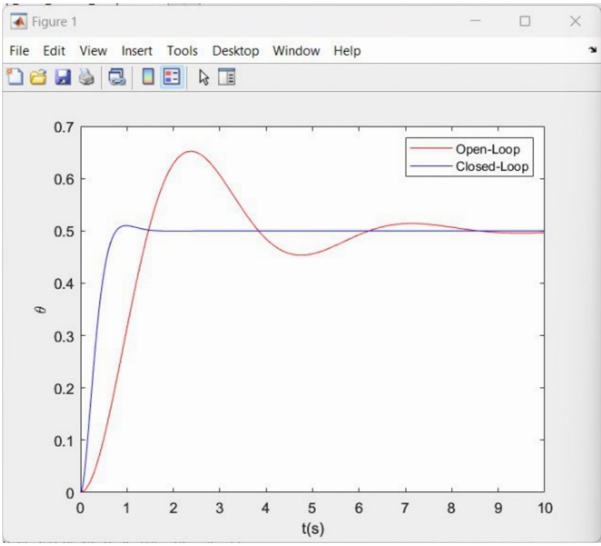
$$\dot{x} = (A - BK)x + BGr$$

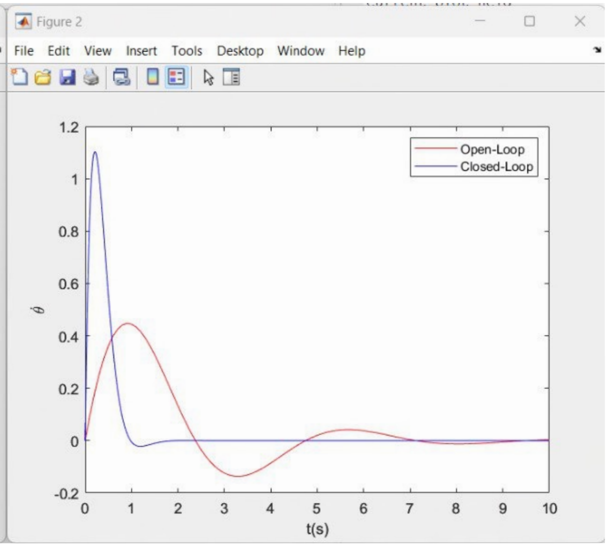
state equation closed loop.

$$\dot{x} = A_{CL}x + B_{CL}r \Rightarrow A_{CL} = A - BK = \begin{bmatrix} 0 & 1 \\ -26.32 & -8 \end{bmatrix} \qquad B_{CL} = BG = \begin{bmatrix} 0 \\ 13.16 \end{bmatrix}$$

$y = C_{CL}x$ output equation closed loop.

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \qquad C_{CL} = \begin{bmatrix} 1 & 0 \end{bmatrix} = C \quad \text{since } D = 0$$




Closed-loop and Open-loop converge to the same value.

```
close all; clc;clear all;
K_r=2;
J=1;
b=1;
A=[0 1; -K_r/J -b/J];
B=[0; 1/J];
C=[1 0];
MP=2/100;
Psi=-log(MP)/(sqrt(pi^2+(log(MP))^2));
T_s=1;
omega=4/Psi*T_s;
% calculation of controller gains
K_p=omega^2-2;
K_d=2*Psi*omega-1;
K=[K_p K_d];
G=(K_p+K_r)/K_r;
% open-loop
s=tf('s');
H_OL=1/(s^2+(b/J)*s+(K_r/J));
[theta_OL,x_1]=step(H_OL,10);
[theta_dot_OL,x_2]=step(s*H_OL,10);
% Closed-loop
H_CL=G/(J*s^2+(b+K_d)*s+(K_r+K_p));
[theta_CL,x_3]=step(H_CL,10);
[theta_dot_CL,x_4]=step(s*H_CL,10);
% Plot theta_OL vs theta_CL
figure
plot(x_1,theta_OL,'r-');
hold
plot(x_3,theta_CL,'b-');
xlabel('t(s)');
ylabel('\theta');
legend('Open-Loop','Closed-Loop');
% Plot Theta_dot_OL vs Theta_dot_CL
figure
plot(x_2,theta_dot_OL,'r-');
hold
plot(x_4,theta_dot_CL,'b-');
xlabel('t(s)');
ylabel('$\dot{\theta}$', 'Interpreter','latex');
legend('Open-Loop','Closed-Loop');
```

## Problem 3 (8 points)

Consider a system with transfer function

$$G_{op}(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$$

(a) **4 points.** Is it possible to change the transfer function to

$$G_{cl}(s) = \frac{(s-1)}{(s+2)(s+3)}$$

by state feedback?

(b) **2 points.** Is the resulting system BIBO stable?

(c) **2 points.** Is the resulting system asymptotically stable?

SOLUTIONS)

a) $(s+1)(s-2)(s+3) = \underset{a_2}{s^3 + 2s^2} \underset{a_1}{- 5s} \underset{a_0}{- 6}$   system 1.

open loop

CCF

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & 5 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = [-2, 1, 1]$$

SYSTEM 2   closed loop

$(s+2)(s+3) = s^2 + 3s + 2s - 6 = \underset{a_2}{s^2} + \underset{a_1}{5s} + \underset{a_0}{6}$

CCF

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [-1 \quad 1]$$

$$K = [k_0 \quad k_1 \quad k_2] = [a_{cl,0} - a_0 \quad a_{cl,1} - a_1 \quad a_{cl,2} - a_2] = [6+6 \quad 5+5 \quad 1-2] = [12 \quad 10 \quad -1]$$

It is possible and the state feedback gain vector is K.

b) BIBO STABLE

$$\int_0^\infty h(\tau)\,d\tau$$

$$h(\tau) = \mathcal{L}^{-1}\left\{\frac{s-1}{(s+2)(s+3)}\right\}$$

PFD

$$\frac{A}{s+2} + \frac{B}{s+3} = A(s+3) + B(s+2) \Rightarrow \begin{cases} (A+B)\cancel{s} = \cancel{s} \\ 3A + 2B = -1 \end{cases} \begin{cases} A+B = 1 \\ 3A+2B = -1 \end{cases} \begin{cases} A = 1-B \\ 3-3B+2B = -1 \end{cases} \begin{cases} A = 1-B \\ -B = -4 \end{cases} \begin{cases} A = -3 \\ B = 4 \end{cases} \quad \frac{-3}{s+2} + \frac{4}{s+3}$$

in time $\Rightarrow -3e^{-2t} + 4e^{-3t}$

$$\int_0^\infty (-3e^{-2t} + 4e^{-3t})\,dt = \left(\frac{3}{2}e^{-2t} - \frac{4}{3}e^{-3t}\right)\Big|_0^\infty = \frac{3}{2} - \frac{4}{3} = \frac{9-8}{6} = \frac{1}{6} \neq 0 \quad \text{BIBO STABLE}$$

c) eigenvalues

$$\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\right) = \begin{bmatrix} \lambda & -1 \\ 6 & \lambda+5 \end{bmatrix} = \lambda^2 + 5\lambda + 6 \Rightarrow \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2} \to -2,\ -3$$

$Re(\lambda_i) < 0$ the closed loop system is stable