Homework 6
ME5659 Spring 2024

Due: See Canvas, turn in on Gradescope

Problem 1 (9 points)

Consider the following linear systems, where

(i) $A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

(ii) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

(iii) $A = \begin{bmatrix} 3 & 6 & 4 \\ 9 & 6 & 10 \\ -7 & -7 & -9 \end{bmatrix}$, $B = \begin{bmatrix} 1/3 & 4/3 \\ 4/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}$.

(a) **3 points.** Use the PBH test for observability assessment.

(b) **6 points.** If possible, compute the observer canonical form of these systems. If you cannot, put the systems into Kalman observable canonical form and evaluate their detectability.

SOLUTIONS

a) PBH TEST

$$\text{rank}\begin{bmatrix} C \\ \hline \lambda I - A \end{bmatrix} = n$$

i)

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \begin{matrix} \lambda_1 = -4 \\ \lambda_2 = -5 \end{matrix}$$

$\Rightarrow$ THE SYSTEM IS NOT OBSERVABLE

$$\lambda I - A = \begin{bmatrix} \lambda+4 & 0 \\ 0 & \lambda+5 \end{bmatrix}$$

$$\text{rank}\begin{bmatrix} 1 & 0 \\ \lambda+4 & 0 \\ 0 & \lambda+5 \end{bmatrix} \quad \lambda_1 = -4 \quad \text{rank}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = 2 = n \quad \text{observable}$$

$$\lambda_2 = -5 \quad \text{rank}\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = 1 < n \quad \text{not observale}$$

ii)

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \lambda_1 = 2 \quad \lambda_2 = 1 \quad \text{eigenvalues}$$

$$\text{rank}\begin{bmatrix} 1 & 1 \\ \lambda-2 & 0 \\ 0 & \lambda-1 \end{bmatrix} \quad \lambda_1 = 2 \quad \text{rank}\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = 2 = n \quad \text{observable}$$

$\lambda_2 = 1$

$$\text{rank}\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} = 2 = n \quad \text{observable}$$

THE SYSTEM IS OBSERVABLE

iii) $A = \begin{bmatrix} 3 & 6 & 4 \\ 9 & 6 & 10 \\ -7 & -7 & -9 \end{bmatrix}$ $B = \begin{bmatrix} 1/3 & 4/3 \\ 4/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix}$

$\lambda_1 = 5$
$\lambda_2 = -2$
$\lambda_3 = -3$

$$\begin{bmatrix} \lambda - 3 & -6 & -4 \\ -9 & \lambda - 6 & -10 \\ 7 & 7 & \lambda + 9 \end{bmatrix}$$

$\lambda_1 = 5$

$$\text{rank}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 2 & -6 & -4 \\ -9 & -1 & -10 \\ 7 & 7 & 14 \end{bmatrix} = 2 < 3$$

unobservable

$\lambda_2 = -2$

$$\text{rank}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ -5 & -6 & -4 \\ -9 & -8 & -10 \\ 7 & 7 & 7 \end{bmatrix} = 3 = n \quad \text{observable}$$

THE SYSTEM IS NOT OBSERVABLE

$\lambda_3 = -3$

$$\text{rank}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ -6 & -6 & -4 \\ -9 & -9 & -10 \\ 7 & 7 & 6 \end{bmatrix} = 3 = n \quad \text{observable}$$

b)

i) NOT OBSERVABLE

KALMAN CANONICAL FORM

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = CT$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$A_K = T^{-1} A T$

$$A_K = \left[\begin{array}{c|c} -4 & 0 \\ \hline 0 & -5 \end{array}\right] = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

but SINCE $Re(\lambda_i(A_{22})) < 0$ it is detectable

ii) OBSERVABLE => CCF

$$(sI - A) = \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} s-2 & 0 \\ 0 & s-1 \end{bmatrix} = (s-2)(s-1) = s^2 - 2s - s + 2 = s^2 - \underbrace{3}_{a_1}s + \underbrace{2}_{a_0}$$

$$Q^{-1}_{CCF} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$T_{OCF} = (Q^{-1}_{OCF}\, Q)^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \qquad A_{OCF} = T^{-1}_{OCF}\, A\, T_{OCF} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$

$$C_{OCF} = C\, T_{OCF} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

iii)

$$T^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_{kalman} = \left[\begin{array}{cc|c} -3 & 1 & 0 \\ 0 & -2 & 0 \\ \hline 0 & -2.3 & 5 \end{array}\right]$$

A_{22}

Not detectable since $Re(\lambda_i(A_{22})) > 0$

$$C_{kalman} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

Problem 2 (4 points)

An approximate linear model of the lateral dynamics of an aircraft, for a particular set of flight conditions, has the state and control vectors in the perturbation quantities

$$x = \begin{bmatrix} p & r & \beta & \phi \end{bmatrix}^T, \quad u = \begin{bmatrix} \delta_a & \delta_r \end{bmatrix}^T$$

where p and r are incremental roll and yaw rates, β is an incremental sideslip change, and ϕ is an incremental roll angle. The control inputs are the incremental changes in the aileron angle δ_a and in the rudder angle δ_r, respectively. This linearized model has

$$A = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If you could have either a rate gyro that measures the incremental roll rate or a bank indicator that measures incremental roll angle, which would you choose and why?

LET'S assume we have a rate gyro that allows us to measure the incremental roll rate $p = \dot{\phi}$

$$\dot{X} = \begin{bmatrix} \dot{p} \\ \dot{r} \\ \dot{\beta} \\ \dot{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix} + \underbrace{\begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B} u$$

$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$ Because we are able to measure p by hypothesis.

LET'S see our Q (observability matrix).

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \\ -1000 & -114 & -984.9 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -10 & 0 \end{bmatrix}$$

$$CA \cdot A = \begin{bmatrix} -10 & 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 10 & 107 & 0 \end{bmatrix}$$

$$CAAA = \begin{bmatrix} 100 & 10 & 107 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1000 & -114 & -984.9 & 0 \end{bmatrix}$$

clearly $\det(Q) = 0$ for the presence of the last column full of zeros

$rk(Q) = 3$ So the system is not fully observable.

LET's see in terms of roll angle ϕ

$$\dot{x} = \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 20 & 2.8 \\ 0 & -3.13 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u$$

where

$$x = \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix} \quad u = \begin{bmatrix} \delta a \\ \delta r \end{bmatrix} \quad \text{so} \quad \dot{x} = \begin{bmatrix} \dot{p} \\ \dot{r} \\ \dot{\beta} \\ \dot{\phi} \end{bmatrix}$$

OUTPUT equation => $y = Cx + Du$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ r \\ \beta \\ \phi \end{bmatrix}$$

we want to measure the incremental roll rate

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ -10 & 0 & -10 & 0 \\ 100 & 10 & 107 & 0 \end{bmatrix}$$

$$CA = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$CA \cdot A = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -10 & 0 \end{bmatrix}$$

$$CAA \cdot A = \begin{bmatrix} -10 & 0 & -10 & 0 \end{bmatrix} \begin{bmatrix} -10 & 0 & -10 & 0 \\ 0 & -0.7 & 9 & 0 \\ 0 & -1 & -0.7 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 10 & 107 & 0 \end{bmatrix}$$

$rk(Q) = ?$ $\quad \det(Q) = -100$ and so the $rk(Q) = 4$ the system is fully observable so we can reconstract every single state from the output.

$\therefore$ pick the bank indicator that measures incremental roll angle, since with that output all the states are observable

Problem 3 (12 points)

Given

$$G(s) = \frac{s-1}{s^3+2s^2-s-2}$$

(a) **3 points.** Find a three-dimensional controllable realization. Is it observable? Is it detectable?

(b) **3 points.** Find a three-dimensional observable realization. Is it controllable? Is it stabilizable?

(c) **2 points.** What causes the apparent difference in controllability and observability of the same system? Derive an irreducible transfer function for $G(s)$.

(d) **4 points.** Please derive a realization that is both controllable and observable, represent it in both controllable canonical form and observable canonial form

a)

$$G(s) = \frac{\overset{b_1}{s} - \overset{b_0}{1}}{s^3 + \underset{a_2}{2}s^2 - \underset{a_1}{s} - \underset{a_0}{2}}$$

$$A_{CCF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \qquad C_{CCF} = \begin{bmatrix} b_0 - a_0 b_3 & b_1 - a_1 b_3 & b_2 - a_2 b_3 \end{bmatrix}$$

$$A_{CCF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix} \qquad B_{CCF} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad C_{CCF} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

$\operatorname{rank} Q = 2$ Not observable

IN kalman canonical form,

$$T^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad A = T^{-1} A T = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 0 \\ \hline -2 & -3 & 1 \end{array}\right]$$

Not detectable since $Re(\lambda_i(A_{22})) = 1 > 0$

$$B = T^{-1} B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad C = CT = \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right]$$

b)

$$G(s) = \frac{\overset{b_1}{s} - \overset{b_0}{1}}{s^3 + \underset{a_2}{2}s^2 - \underset{a_1}{s} - \underset{a_0}{2}}$$

$$A_{OCF} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \qquad C_{OCF} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$B_{OCF} = \begin{bmatrix} b_0 - a_0 b_3 \\ b_1 - a_1 b_3 \\ b_2 - a_2 b_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB & AB^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$ rank $P = 2$ Not controllable

in kalman canonical form:

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad A = T^{-1} A T = \left[\begin{array}{cc|c} 0 & -2 & -2 \\ 1 & -3 & -3 \\ \hline 0 & 0 & 1 \end{array}\right]$$

Not stabilizable since $Re(\lambda_i(A_{22})) > 0$

$$B = T^{-1} B = \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \end{array}\right] \qquad C = CT = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$$

c) Duality

(A, B) is controllable iff $(A^T B^T)$ is observable
(A, C) is observable iff (A^T, C^T) is controllable

$$G(s) = \frac{s-1}{s^3+2s^2-s-2} = \frac{\cancel{s-1}}{\cancel{(s-1)}(s+1)(s+2)} = \frac{1}{(s+1)(s+2)}$$ Not minimal representation

Minimal => irreducible $G(s)$ with n degree of $a(s)$ is minimal iff is both controllable and observable.

d) $G(s) = \dfrac{1}{s^2+3s+2}$ $\quad a_0 = 2 \quad a_2 = 1 \quad a_1 = 3$

$$A_{CCF} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad B_{CCF} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C_{CCF} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$ controllable

$$A_{OCF} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \qquad C_{OCF} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad B_{OCF} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ observable