# Homework 1
# ME5659 Spring 2024

**Due: See Canvas, turn in on Gradescope**

## Problem 1 (6 points)

Describe the dynamical systems in state-space representations.

(a) **3 points.** For the following system described by the given transfer function, derive valid state-space realization (define the state variables and derive the state-space representation):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4s^4 - s^2 + 5s - 1}{2s^4 + 2s^2 - 4s + 6}$$

(b) **3 points.** Given the following differential equations, derive valid state-space realization with $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ (define the state variables and derive the state-space representation):

$$\ddot{y}_1(t) + 2\dot{y}_1(t) - 5(y_2(t) - y_1(t)) = u_1(t)$$
$$\ddot{y}_2(t) + \dot{y}_1(t) - 4\dot{y}_2(t) - 3(y_2(t) - y_1(t)) = u_2(t)$$

## Problem 2 (6 points)

Consider a pendulum as shown in Fig. 1. We assume that the mass $m$ is concentrated at pendulum end, with length as $l$. Gravity should be considered. The pendulum is driven by a torque input $T$ at the base, and the base rotation joint is subject to rotational damping $b$. The equation of motion of this pendulum is

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = T,$$

where T is input and pendulum angle $\theta$ is output.

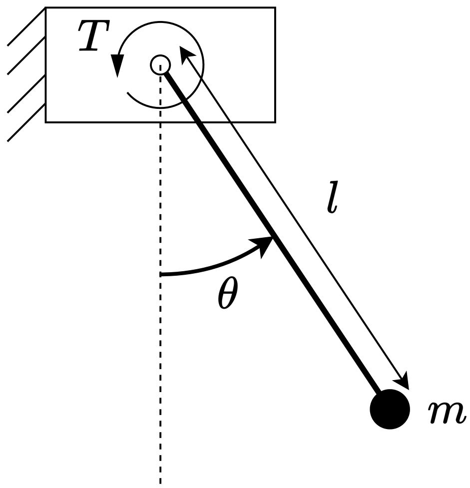


Figure 1: Simple pendulum

(a) **2 points.** Define the 2 state variables of the system. The input is $u = T$. Put the equations of motion in nonlinear state-space form, $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$.

(b) **4 points.** Consider the initial angle of pendulum is $\theta_0$ and initial torque $T_0 = 0$ for a passive system. A small toque input $\delta T$ is at the base joint, leading to perturbation angle of $\delta\theta$. Linearize the nonlinear state-space model about $\theta_0$ and $T_0$ to obtain the linear state-space models for $\theta_0 = 0$ and $\theta_0 = \pi$.

## Problem 3 (6 points)

A single-wheel chair cart (unicycle) moving on the plane with linear velocity $v$ and angular velocity $\omega$ can be modeled by the nonlinear system

$$\dot{p}_x = v cos\theta, \quad \dot{p}_y = v sin\theta, \quad \dot{\theta} = \omega,$$

where $(p_x, p_y)$ denote the Cartesian coordinates of the wheel and $\theta$ its orientation. Regard this as a system with input $u = \begin{bmatrix} v & \omega \end{bmatrix}^T$

(a) **(3 points)** Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} p_x cos\theta + (p_y - 1) sin\theta \\ -p_x sin\theta + (p_y - 1) cos\theta \\ \theta \end{bmatrix}$$

and output $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

(b) **(3 points)** Compute a linearization for this system around the equilibrium point $x_{eq} = 0, u_{eq} = 0$.

## Problem 4 (7 points)

Consider the following model for a DC motor:

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} = K_t i \tag{1}$$

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + K_b\frac{\mathrm{d}\theta}{\mathrm{d}t} = V_s, \tag{2}$$

where $J$ is the mass-moment of inertia of the load on the motor, which is damped by rotary damper with linear damping constant $b$. The torque delivered by the motor is $K_t i$, where $K_t$ is the motor torque constant and $i$ is the motor current. The motor has internal series resistance $R$ and inductance $L$, and a motor speed constant $K_b$. The voltage supplied to the motor is $V_s$. *In all parts, consider the input $u = V_s$ and the output $y = \theta$, motor shaft angle.*

(a) **(2 points)** In a coupled system it may not be clear at first what the order of the system is. In this problem, we have, effectively, a first-order system in $i$ and a second-order system in $\theta$, giving us three states. Making the choice for states $x_1 = \theta$, $x_2 = \dot{\theta}$, and $x_3 = i$, calculate the $\mathbf{A}$ and $\mathbf{B}$ matrices for a state-space representation.

(b) **(2 point)** To better illustrate that the system is third-order, find a single third-order differential equation in terms of $\theta$ and its derivatives (the current will not appear in the equation). Laplace transforming the ODEs for manipulation, or using a differential operator will make this easier.

(c) **(2 points)** State-space representations of dynamical systems are *not unique*. Making the choice for states $x_1 = \theta$, $x_2 = \dot{\theta}$, and $x_3 = \ddot{\theta}$, calculate the $\mathbf{A}$ and $\mathbf{B}$ matrices for a state-space representation.

(d) **(1 point)** While state-space representations of systems are not unique, *they represent the same systems* if originating from the same set of differential equations. Assume that all constant parameters are equal to 1, and use MATLAB to calculate the eigenvalues of both $\mathbf{A}$-matrices (from parts (a) and (c) ), and show that they are the same. *List your MATLAB code, and the program/command outputs.*