Homework 1
ME5659 Spring 2024

Due: See Canvas, turn in on Gradescope

Problem 1 (6 points)

Describe the dynamical systems in state-space representations.

(a) **3 points.** For the following system described by the given transfer function, derive valid state-space realization (define the state variables and derive the state-space representation):

$$G(s) = \frac{Y(s)}{U(s)} = \frac{4s^4 - s^2 + 5s - 1}{2s^4 + 2s^2 - 4s + 6}$$

(b) **3 points.** Given the following differential equations, derive valid state-space realization with $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ (define the state variables and derive the state-space representation):

$$\ddot{y}_1(t) + 2\dot{y}_1(t) - 5(y_2(t) - y_1(t)) = u_1(t)$$
$$\ddot{y}_2(t) + \dot{y}_1(t) - 4\dot{y}_2(t) - 3(y_2(t) - y_1(t)) = u_2(t)$$

SOLUTIONS

a) METHOD 1:

Note that the transfer function G(s) is a 4th-order property transfer function, meaning that the highest orders of the polynomials in both the numerator and the denominator are the same. Due to this property, we cannot directly convert G(s) back into the time domain to find y(t) in terms of u(t), as this would introduce derivatives of u(t) in the terms. One viable approach to deriving the state-space representation is to introduce an intermediate variable, such as X(s). By multiplying X(s) into both the numerator and the denominator of G(s), the equation can be reformulated as:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Y(s)\,X(s)}{U(s)\,X(s)} = \frac{Y(s)}{X(s)} \cdot \frac{X(s)}{U(s)} = \frac{4s^4 - s^2 + 5s - 1}{2s^4 + 2s^2 - 4s + 6}$$

This allows us to rewrite the relationship as follows:

$$Y(s) = (4s^4 - s^2 + 5s - 1)\,X(s)$$
$$Y(s) = 4s^4 X(s) - s^2 X(s) + 5s\,X(s) - X(s)$$
$$U(s) = (2s^4 + 2s^2 - 4s + 6)\,X(s)$$
$$U(s) = 2s^4 X(s) + 2s^2 X(s) - 4s\,X(s) + 6\,X(s)$$

Into the time domain yields:

$$y = 4\overset{IV}{x} - \ddot{x} + 5\dot{x} - x$$

$$u = 2x^{IV} + 2\ddot{x} - 4\dot{x} + 6x$$

Now, let's define the state variables in vector form as $x = [x_1, x_2, x_3, x_4]^T$ where $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $x_4 = \dddot{x}$. With the defined state variables, the previous equations can be written as follows:

$$y = 4\dot{x}_4 - x_3 + 5x_2 - x_1$$

$$u = 2\dot{x}_4 + 2x_3 - 4x_2 + 6x_1$$

$$\Rightarrow \quad \dot{x}_4 = \frac{u - 2x_3 + 4x_2 - 6x_1}{2} = -3x_1 + 2x_2 - x_3 + \frac{u}{2}$$

Substiting the latter in the previous one, we obtain

$$y = 2(u - 2x_3 + 4x_2 - 6x_1) - x_3 + 5x_2 - x_1$$

$$y = 2u - 4x_3 + 8x_2 - 12x_1 - x_3 + 5x_2 - x_1$$

$$y = 2u - 5x_3 + 13x_2 - 13x_1$$

So, we can build the state-space representation as:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} u \qquad \text{STATE EQUATION}$$

$$y = \begin{bmatrix} -13 & 13 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 2u \qquad \text{OUTPUT EQUATION}$$

METHOD 2)

Since G(s) is a proper transfer function, one can initially use long division to reduce the order of the numerator. As a result, the transfer function can be reexpressed in the following form:

$$G(s) = \frac{Y(s)}{U(s)} = \underset{\substack{\downarrow \\ \text{D matrix}}}{2} + \frac{-5s^2 + 13s - 13}{2s^4 + 2s^2 - 4s + 6}$$

Now, the second transfer function on the right-hand side is a strictly proper transfer function. Therefore, we can directly derive the state-space model by utilizing the controller canonical form. This leads us to the following representation:

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} u$$

Given the extra scar term, the output equation in state-space representation will take on the form y=Cx+2u. This can be further expressed as follows:

$$y = \begin{bmatrix} -13 & 13 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 2u$$

This yields results consistent with those obtained using the first method

b) $\ddot{y}_1(t) + 2\dot{y}_1(t) - 5(y_2(t) - y_1(t)) = u_1(t)$

$\ddot{y}_2(t) + \dot{y}_1(t) - 4\dot{y}_2(t) - 3(y_2(t) - y_1(t)) = u_2(t)$

Given that the system incorporates two inputs, u1 and u2, and the highest-order derivative terms in each equation are $\ddot{y}_1$ and $\ddot{y}_2$ respectively. We can then define the state variable as:

$x = [y_1, \dot{y}_1, y_2, \dot{y}_2]^T.$

$u = [u_1, u_2]^T$ INPUT VECTOR

Therefore, the given equations can be reformulated as:

$$\ddot{y}_1 = u_1 - 2\dot{y}_1 + 5(y_2 - y_1)$$

$$\ddot{y}_2 = u_2 - \dot{y}_1 + 4\dot{y}_2 + 3(y_2 - y_1)$$

STATE EQUATION IN ITS STATE-SPACE REPRESENTATION

$$\dot{x} = \begin{bmatrix} \dot{y}_1 \\ \ddot{y}_1 \\ \dot{y}_2 \\ \ddot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -2 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

OUTPUT EQUATION IN ITS STATE-SPACE REPRESENTATION

$$y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}$$

Problem 2 (6 points)

Consider a pendulum as shown in Fig. 1. We assume that the mass m is concentrated at pendulum end, with length as l. Gravity should be considered. The pendulum is driven by a torque input T at the base, and the base rotation joint is subject to rotational damping b. The equation of motion of this pendulum is

$$ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta = T,$$

where T is input and pendulum angle θ is output.

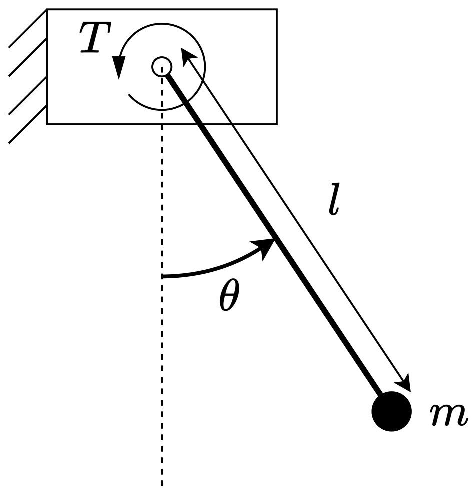

Figure 1: Simple pendulum

(a) **2 points.** Define the 2 state variables of the system. The input is $u = T$. Put the equations of motion in nonlinear state-space form, $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$.

(b) **4 points.** Consider the initial angle of pendulum is θ_0 and initial torque $T_0 = 0$ for a passive system. A small toque input δT is at the base joint, leading to perturbation angle of $\delta\theta$. Linearize the nonlinear state-space model about θ_0 and T_0 to obtain the linear state-space models for $\theta_0 = 0$ and $\theta_0 = \pi$.

SOLUTIONS

a) $m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = T$ θ output, T input $T = u$

The state variable are θ and $\dot{\theta}$ In fact, the mass m storages kinetic energy.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} ; \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$$

Rewriting the equation using the state variables defined above, we obtain:

$$\ddot{\theta} = \frac{T - b\dot{\theta} - mg\ell\sin\theta}{m\ell^2}$$

$$\dot{x}_2 = \frac{1}{m\ell^2} u - \frac{b}{m\ell^2} x_2 - \frac{\cancel{m}g\cancel{\ell}}{\cancel{m}\ell^{\cancel{2}}} \sin x_1$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{1}{m\ell^2} u - \dfrac{b}{m\ell^2} x_2 - \dfrac{g}{\ell} \sin x_1 \end{cases}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{me^2} u - \frac{b}{me^2} x_2 - \frac{g}{e} \sin x_1 \end{bmatrix} ; \quad \dot{x} = f(x, u)$$

where $\sin x_1$ is the term that makes our system non linear

b) θ_0 and $T_0 = 0$

EQUILIBRIUM POINTS $\theta_0 = 0$ and $\theta_0 = \pi$

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} ; \quad \dot{\tilde{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ noting changes anymore

$$\dot{\tilde{x}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ \frac{1}{me^2} \tilde{u} - \frac{b}{me^2} \tilde{x}_2 - \frac{g}{e} \sin \tilde{x}_1 \end{bmatrix}$$

$\tilde{x}_{1,1} = 0$ pendulum hanging directly down

$\tilde{x}_{1,2} = \pi$ pendulum POINTING directly up.

LET'S linearize about $\theta_0 = 0$

$$\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \tilde{u} = 0$$

A matrix is given by the JACOBIAN and so:

$$A = \frac{\partial f}{\partial x}\Big|_{\tilde{x}, \tilde{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\tilde{x}, \tilde{u}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{e} \cos x_1 & -\frac{b}{me^2} \end{bmatrix}_{\tilde{x}, \tilde{u}}$$ where $\tilde{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\tilde{u} = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{e} & -\frac{b}{me^2} \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\Big|_{\tilde{x}, \tilde{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{me^2} \end{bmatrix}$$

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{e} & -\frac{b}{me^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1/me^2 \end{bmatrix} \delta u$$

LET'S LINEARIZE about $\theta_0 = \pi$

$$\tilde{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \quad \tilde{u} = 0$$

A matrix is given by the JACOBIAN and so:

$$A = \frac{\partial f}{\partial x}\Big|_{\tilde{x},\tilde{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\tilde{x},\tilde{u}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{g}{e}\cos x_1 & -\frac{b}{me^2} \end{bmatrix}_{\tilde{x},\tilde{u}}$$ where $\tilde{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ and $\tilde{u} = 0$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{e} & -\frac{b}{me^2} \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\Big|_{\tilde{x},\tilde{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{me^2} \end{bmatrix}$$

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{e} & -\frac{b}{me^2} \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 1/me^2 \end{bmatrix} \delta u$$

Problem 3 (6 points)

A single-wheel chair cart (unicycle) moving on the plane with linear velocity v and angular velocity ω can be modeled by the nonlinear system

$$\dot{p}_x = v cos\theta, \quad \dot{p}_y = v sin\theta, \quad \dot{\theta} = \omega,$$

where (p_x, p_y) denote the Cartesian coordinates of the wheel and θ its orientation. Regard this as a system with input $u = \begin{bmatrix} v & \omega \end{bmatrix}^T$

(a) **(3 points)** Construct a state-space model for this system with state

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} p_x cos\theta + (p_y - 1)sin\theta \\ -p_x sin\theta + (p_y - 1)cos\theta \\ \theta \end{bmatrix}$$

and output $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

(b) **(3 points)** Compute a linearization for this system around the equilibrium point $x_{eq} = 0, u_{eq} = 0$.

a)

$$(*)\begin{cases} \dot{x}_1 = \dot{P}_x \cos\theta - P_x \sin\theta\,\dot{\theta} + \dot{P}_y \sin\theta + P_y \cos\theta\,\dot{\theta} - \cos\theta\,\dot{\theta} \\ \dot{x}_2 = -\dot{P}_x \sin\theta - P_x \cos\theta\,\dot{\theta} + \dot{P}_y \cos\theta - P_y \sin\theta\,\dot{\theta} + \sin\theta\,\dot{\theta} \\ \dot{x}_3 = \dot{\theta} \end{cases}$$

$$(**)\begin{cases} \dot{P}_x = v\cos\theta \\ \dot{P}_y = v\sin\theta \\ \dot{\theta} = \omega \end{cases}$$

SUBSTITUTING $(**)$ INTO $(*)$

$$\dot{x}_1 = v\cos^2\theta - P_x \sin\theta\,\dot{\theta} + v\sin^2\theta + P_y\cos\theta\,\dot{\theta} - \cos\theta\,\dot{\theta}$$

$$\dot{x}_1 = v - P_x\sin\theta\,\dot{\theta} + P_y\cos\theta\,\dot{\theta} - \cos\theta\,\dot{\theta}$$

$$\dot{x}_1 = v + (-P_x\sin\theta + P_y\cos\theta - \cos\theta)\,\dot{\theta} = v + (-P_x\sin\theta + (P_y - 1)\cos\theta)\,\dot{\theta}$$

$$\dot{x}_1 = v + x_2\dot{\theta}$$

$$\dot{x}_2 = -v\cancel{\cos\theta\sin\theta} + v\cancel{\sin\theta\cos\theta} + (-P_x\cos\theta - P_y\sin\theta + \sin\theta)\,\dot{\theta}$$

$$\dot{x}_2 = (-P_x\cos\theta - (P_y - 1)\sin\theta)\,\dot{\theta}$$

$$\dot{x}_2 = -x_1\dot{\theta} \qquad \dot{x}_3 = \omega$$

$$\dot{x}_3 = \omega \qquad \dot{\theta} = \omega$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} v + \omega x_2 \\ -\omega x_1 \\ \omega \end{bmatrix}$$

NON LINEAR

$$y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

b)

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg|_{\substack{u_{ep}=0 \\ x_{ep}=0}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{bmatrix} = \begin{bmatrix} 1 & X_2 \\ 0 & -X_1 \\ 0 & 1 \end{bmatrix} \Bigg|_{\substack{x_{ep}=0 \\ u_{ep}=0}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad D = 0$$

Problem 4 (7 points)

Consider the following model for a DC motor:

$$J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + b\frac{\mathrm{d}\theta}{\mathrm{d}t} = K_t i \tag{1}$$

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri + K_b\frac{\mathrm{d}\theta}{\mathrm{d}t} = V_s, \tag{2}$$

where J is the mass-moment of inertia of the load on the motor, which is damped by rotary damper with linear damping constant b. The torque delivered by the motor is $K_t i$, where K_t is the motor torque constant and i is the motor current. The motor has internal series resistance R and inductance L, and a motor speed constant K_b. The voltage supplied to the motor is V_s. *In all parts, consider the input $u = V_s$ and the output $y = \theta$, motor shaft angle.*

(a) **(2 points)** In a coupled system it may not be clear at first what the order of the system is. In this problem, we have, effectively, a first-order system in i and a second-order system in θ, giving us three states. Making the choice for states $x_1 = \theta$, $x_2 = \dot{\theta}$, and $x_3 = i$, calculate the **A** and **B** matrices for a state-space representation.

(b) **(2 point)** To better illustrate that the system is third-order, find a single third-order differential equation in terms of θ and its derivatives (the current will not appear in the equation). Laplace transforming the ODEs for manipulation, or using a differential operator will make this easier.

(c) **(2 points)** State-space representations of dynamical systems are *not unique*. Making the choice for states $x_1 = \theta$, $x_2 = \dot{\theta}$, and $x_3 = \ddot{\theta}$, calculate the **A** and **B** matrices for a state-space representation.

(d) **(1 point)** While state-space representations of systems are not unique, *they represent the same systems* if originating from the same set of differential equations. Assume that all constant parameters are equal to 1, and use MATLAB to calculate the eigenvalues of both **A**-matrices (from parts (a) and (c)), and show that they are the same. *List your MATLAB code, and the program/command outputs.*

a) $X = \begin{bmatrix} X1 \\ X2 \\ X3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix}$ $\qquad u = Vs$

FIRST of all, LET'S rewrite the matrices in the following way:

$J\ddot{\theta} + b\dot{\theta} = Kt\, i$

$L\dot{i} + R i + Kb\,\dot{\theta} = u$

$\Rightarrow \begin{cases} \ddot{\theta} = \frac{Kt}{J} i - \frac{b}{J}\dot{\theta} \\ \dot{i} = \frac{1}{L} u - \frac{Kb}{L}\dot{\theta} - \frac{R}{L} i \end{cases}$

$$\dot{x} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{Kt}{J} \\ 0 & -\frac{Kb}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u$$

b) LAPLACE TRANSFORMING THE ODES

1ST EQUATION becomes:

$$(Js^2+bs)\Theta(s) = kt\, I(s)$$

2nd EQUATION becomes:

$$(Ls+R)I(s) + kb\, s\, \Theta(s) = \frac{Vs}{s}$$

FROM THE FIRST we get:

$$I(s) = \frac{(Js^2+bs)\Theta(s)}{kt}$$

SUBSTITUTING THE LATTER IN THE 1ST, we obtain:

$$(Ls+R)\left[\frac{(Js^2+bs)\Theta(s)}{kt}\right] + kbs\,\Theta(s) = \frac{Vs}{s}$$

$$\left[\frac{JLs^3 + Lbs^2 + RJs^2 + Rbs}{kt} + kbs\right]\Theta(s) = \frac{Vs}{s}$$

$$(JLs^3 + Lbs^2 + RJs^2 + Rbs + kbkt\, s)\,\Theta(s) = kt\,\frac{Vs}{s}$$

Back in time:

$$JL\dddot{\theta} + Lb\ddot{\theta} + RJ\ddot{\theta} + Rb\dot{\theta} + kbkt\,\dot{\theta} = kt\,Vs \qquad u = Vs$$

Single third-order differential equation in terms of theta and its derivatives

c) $x_1 = \theta$
$x_2 = \dot{\theta}$
$x_3 = \ddot{\theta}$

From b we can rewrite the last equation as follows:

$$\dddot{\theta} = \frac{k_t}{JL} V_s - \frac{Lb}{JL}\ddot{\theta} - \frac{RJ}{JL}\ddot{\theta} - \frac{Rb}{JL}\dot{\theta} - \frac{k_b k_t}{JL}\dot{\theta}$$

$V_s = u \quad x_1 = \theta \quad x_2 = \dot{\theta} \quad x_3 = \ddot{\theta} \quad \dot{x}_3 = \dddot{\theta}$

$$\Rightarrow \dot{x}_3 = \frac{k_t}{JL} u - \frac{b}{J} x_3 - \frac{R}{L} x_3 - \frac{Rb}{JL} x_2 - \frac{k_b k_t}{JL} x_2$$

$$\dot{x}_3 = \frac{k_t}{JL} u + \left(-\frac{b}{J} - \frac{R}{L}\right) x_3 + \left(-\frac{Rb}{JL} - \frac{k_b k_t}{JL}\right) x_2$$

$$\dot{X} = AX + Bu$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{Rb}{JL} - \frac{k_b k_t}{JL} & -\frac{b}{J} - \frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_t}{JL} \end{bmatrix} u$$

d)

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untitled *

```
close all; clc; clear all;
% System 1
A1=[0 1 0; 0 -1 1; 0 -1 -1];
eig(A1)
% System 2
A2=[0 1 0; 0 0 1; 0 -2 -2];
eig(A2)
```

Command Window

```
ans =

   0.0000 + 0.0000i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i


ans =

   0.0000 + 0.0000i
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
```

same EIGENVALUES