1)

$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u.$$

$$y = \begin{bmatrix} -2 & 3 \end{bmatrix} x - 2u$$

(a) eigen values of A:

$$|\lambda I - A| = 0.$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} \right| \Rightarrow \begin{vmatrix} \lambda+1 & -10 \\ 0 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda^2 - 1) = 0$$

$$\Rightarrow \lambda = \pm 1$$

$$\lambda_1 = 1, \ \lambda_2 = -1. \qquad \lambda_1 > 0$$

AS ONE OF THE EIGENVALUES HAS POSITIVE REAL PARTS, THE SYSTEM IS NOT ASYMPTOTICALLY STABLE but MARGINALLY STABLE.

(b) The system is BIBO stable if $g(t) = \mathcal{L}^{-1}\{G(s)\}$ satisfies

$$\int_0^\infty \|g(\tau)\| \, d\tau < \infty \text{ (is finite)}$$

$$G(s) = C(sI - A)^{-1} B \Rightarrow \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{10}{s^2-1} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{10}{s^2-1} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{-2}{s+1} & \left(\frac{-20}{s^2-1} + \frac{3}{s-1}\right) \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\rightarrow \frac{4}{s+1} \Rightarrow 4e^{-t}$$

$$g(t) = 4e^{-t}$$

$$\int_0^{\infty} 4e^{-t}\,dt = 4 < \infty$$

$\therefore$ The system is BIBO stable.

BIBO stability does not imply that the system is AS. BIBO stability only guarentees that for any bounded input, the output remains bounded. AS refers to the behavior of the system's state variables over time, ensuring that they converge to zero.
Here, the system is BIBO stable because of negative real parts of the poles but it is not AS as it has one positive real part consisting eigen value ($\lambda_i = 1$).

Q) (a) We make use of controllability matrix to check the controllability.

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$, system is controllable iff rank[P] = n.

(i) $A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & -4 & 16 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank} = (1) < (2)$$

"UNCONTROLLABLE"

(ii) $A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 6 \end{bmatrix} \rightarrow \text{rank} = (2) = (2)$$

"CONTROLLABLE"

(iii) $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 2 & 4 \\ -1 & -2 & -4 \end{bmatrix}$$

Perform: $R_2 \rightarrow R_2 + R_1$

Then P becomes: $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ making it's Rank = 1

Rank = (1) < (2)

"UNCONTROLLABLE"

(b) PBH Test

The system is controllable if matrix $[\lambda I - A \quad B]$ has full row Rank at every eigenvalue λ of A.

(i) $A = \begin{pmatrix} -4 & 0 \\ 0 & -5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Finding eigenvalues:

$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda+4 & 0 \\ 0 & \lambda+5 \end{vmatrix} = 0 \Rightarrow (\lambda+4)(\lambda+5) = 0$

$\Rightarrow \lambda = -4, -5$

Rank of $[\lambda_1 I - A \quad B] \Rightarrow R[-4I - A \quad B]$

$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \text{rank} = (2) < (3)$ Full row rank

Rank of $[\lambda_2 I - A \quad B] \Rightarrow R[-5I - A \quad B]$

$\Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow C_1 = C_1 + C_2$

$\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = (1) < (3)$

As the PBH test does not satisfy for every eigenvalue of A i.e as the system does not have full row rank at every eigen value of A, the system becomes 'UNCONTROLLABLE'

(ii) $A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Eigen values:

$$\begin{vmatrix} \lambda & -1 \\ 10 & \lambda+2 \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+2)+10 = 0$$

$$\Rightarrow \lambda^2+2\lambda+10 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-40}}{2}$$

$$= -1 \mp 3i$$

Rank of $\begin{bmatrix} \lambda_1 I - A & B \end{bmatrix} \Rightarrow \begin{bmatrix} -1-3i & -1 & 0 \\ 10 & 1-3i & 1 \end{bmatrix}$

Full row rank

rank $= (3) = (3)$

Rank of $\begin{bmatrix} \lambda_2 I - A & B \end{bmatrix} \Rightarrow \begin{bmatrix} -1+3i & -1 & 0 \\ 0 & 1+3i & 1 \end{bmatrix}$

rank $= (3) = (3)$

The System obeys PBH test rules for Both the eigenvalues of A making the system "CONTROLLABLE"

(iii) $A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

eigen values :

$$\begin{vmatrix} \lambda-2 & 0 \\ 1 & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-2)(\lambda-1) = 0 \rightarrow \lambda = 2 \text{ or } 1$$

Rank of $[\lambda_1 I - A \;\; B] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ $C_2 \rightarrow C_2 - C_1$

↗ Full row rank

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow \text{rank} = (2) < (3)$$

Rank of $[\lambda_2 I - A \;\; B] = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ $R_2 \rightarrow R_2 + R_1$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = (1) < (3)$$

The System does not obey the PBH test rules hence making the system to be "UNCONTROLLABLE"

(c) CCF Test

For the system to satisfy CCF test; we have to write A, B, C, D in CCF and the resulted output must look like:

$$A_{CCF} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix}, \quad B_{CCF} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C_{CCF} = \begin{bmatrix} (b_0 - a_0 b_n) & (b_1 - a_1 b_n) & \cdots & (b_{n-1} - a_{n-1} b_n) \end{bmatrix}, \quad D_{CCF} = b_n$$

(i) $A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Step-1 : Find the controllability matrix :-

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & -4 & 16 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \text{rank}(1) < (2)$$

UNCONTROLLABLE.

If a system is controllable, it can be written in CCF but if it's uncontrollable, it cannot be written in CCF.

(ii) $A = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$

Step-1 : controllability matrix :-

$$P = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 6 \end{bmatrix} \rightarrow \text{rank} = (2) = (2)$$

"CONTROLLABLE"

Step-2: Find the system characteristic polynomial.

$$|SI-A| = \begin{vmatrix} S & -1 \\ 10 & S+2 \end{vmatrix} \Rightarrow S(S+2)+10=0 \Rightarrow S^2+2S+10=0.$$

$$a_0 = 10, \; a_1 = 2$$

Step-3: P_{CCF}^{-1} Finding

$$P_{CCF}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Step-4: Find T

$$T = P \cdot P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step-5: CCF Realization:-

$$\left.\begin{aligned} A_{CCF} &= T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \\ B_{CCF} &= T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ C_{CCF} &= CT = \begin{bmatrix} 1 & 2 \end{bmatrix} \end{aligned}\right\} \text{CCF}$$

(iii) From (a), we know that (ii) is uncontrollable as the rank $P = (1) < (2)$.
Hence, a CCF cannot be commutable.

3) a) we need to check the controllability

Controllability Matrix given by $P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

given

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3w^2 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 \\ 0 & -2w & 0 & 1 \end{bmatrix}$$

given

upon calculating P,

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2w & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2w \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The rank of controllability Matrix is equal to the System's state dimension $\rightarrow$ $(4) = (4)$

Hence, the system is <u>controllable</u>.

b) (i) given that input vectors : radial $\rightarrow u_1$ and Tangential $\rightarrow u_2$

So, If radial thruster fails, we make the corresponding column in B to zero

i.e. $$B_{rf} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

controllability Matrix $P' = \begin{bmatrix} B_{rf} & AB_{rf} & A^2B_{rf} & A^3B_{rf} \end{bmatrix}$

(rf = radial failure)

upon calculation,

$$P' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The rank of P' is $(3) < (4)$ State dimension.

SO, SYSTEM IS NOT CONTROLLABLE If Radial Thruster FAILS.

(ii) If the tangential thruster fails, B becomes:

$$B_{tf} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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The controllability matrix:

$$P'' = \begin{bmatrix} B_{tf} & AB_{tf} & A^2B_{tf} & A^3B_{tf} \end{bmatrix}$$

($tf \rightarrow$ tangential failure)

$$P'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The rank is less than state space dimension: $(3) < (4)$

SO, THE SYSTEM IS UNCONTROLLABLE when Tangential thruster fails.

4)
$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 4 \\ 2 \end{bmatrix} u$$

$$x(0) = \begin{bmatrix} 0.5 & 1 \end{bmatrix}^T$$

a) controllability matrix P:

$$P = [B \;\; AB]$$

$$AB = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & -2 \\ 2 & -2 \end{bmatrix}$$

$$\text{Det } P = -8 + 4 \Rightarrow -4 \quad (\text{non zero})$$

Since the det of P is non-zero, system is controllable. This means there exists an input $u(t)$ that can drive $x(0)$ to 0 in finite time.

b) $$W_c(t) = \int_0^t e^{A(t-\tau)} BB^T e^{A^T(t-\tau)} d\tau$$

$$= \int_0^t \begin{bmatrix} e^{-0.5(t-\tau)} & 0 \\ 0 & e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} e^{-0.5(t-\tau)} & 0 \\ 0 & e^{(t-\tau)} \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} 16a & 8a \\ 8a & 4b \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} d\tau = \int_0^t \begin{bmatrix} 16a^2 & 8ab \\ 8ab & 4b^2 \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} 16e^{t-\tau} & 8e^{-1.5(t-\tau)} \\ 8e^{-1.5(t-\tau)} & 4e^{\tau-t} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 16\left[1-e^{-t}\right] & \frac{8 \cdot 2}{3}\left[1-e^{-1.5t}\right] \\ \frac{8 \cdot 2}{3}\left(1-e^{-1.5t}\right) & 4\left[1-e^{-t}\right] \end{bmatrix}_{t=4}$$

$$W_c = \begin{bmatrix} 15.71 & 5.32 \\ 5.32 & 3.93 \end{bmatrix} \qquad W_c^{-1} = \begin{bmatrix} 0.12 & -0.16 \\ -0.16 & 0.47 \end{bmatrix}$$

From grammian theorem:

$$u(t) = B^T e^{A^T t} \left[W_c\right]^{-1} \left[x(t) - e^{At} x(0)\right]$$

At $t = 4s$ & $x(t) = 0$.

$$u(t) = -B^T e^{A^T t} \left[W_c\right]^{-1} e^{At} x(0)$$

$$e^{At} = \begin{bmatrix} e^{-0.5t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$$e^{A^T t} = \begin{bmatrix} 0.14 & 0 \\ 0 & 0.02 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 0.5 & -1 \end{bmatrix}^T$$

$$u(t) = -\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 0.14 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 0.12 & -0.16 \\ -0.16 & 0.47 \end{bmatrix} \begin{bmatrix} 0.14 & 0 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

$$= -0.0052$$

c)

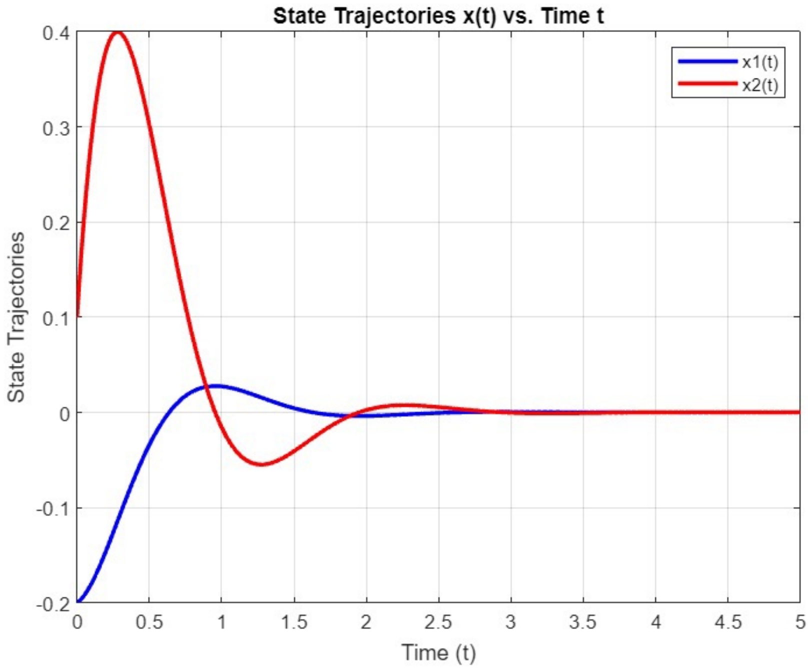
```
% Define the system matrix and initial condition
A = [0 1; -14 -4];
x0 = [-0.2; 0.1];

% Define the time span for simulation
tspan = 0:0.01:5;

% Define the differential equation function
ode = @(t, x) A * x;

% Solve the differential equation using ode45
[t, x] = ode45(ode, tspan, x0);

% Plot the state trajectories
figure;
plot(t, x(:, 1), 'b', 'LineWidth', 2, 'DisplayName', 'x1(t)');
hold on;
plot(t, x(:, 2), 'r', 'LineWidth', 2, 'DisplayName', 'x2(t)');
xlabel('Time (t)');
ylabel('State Trajectories');
title('State Trajectories x(t) vs. Time t');
legend('show');
grid on;
hold off;
```



d) $|SI-A| \Rightarrow \begin{vmatrix} S+0.5 & 0 \\ 0 & S+1 \end{vmatrix} \Rightarrow \underbrace{S^2+1.5}_{a_1}\underbrace{S+0.5}_{a_0} = 0$

$$P_{CCF}^{-1} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P = \begin{bmatrix} 4 & -2 \\ 2 & -2 \end{bmatrix}$$

$$T = P P_{CCF}^{-1} = \begin{bmatrix} 4 & -2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1.5 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix}$$

$$A_{CCF} = T^{-1} A T = \frac{1}{4} \begin{bmatrix} 2 & -4 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -0.5 & -1.5 \end{bmatrix}$$

$$B_{CCF} = [0 \ 1]^T$$