CONTROLS HOMEWORK - II

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1) Given, $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -12 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

at $t = 0$, $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

What would $x(4) = ?$ look like?

We know that, $\dot{x} = Ax + Bu$

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -12 \end{bmatrix}, \quad B = 0$$

$$e^{-A(t-t_0)} = e^{-A(T-t_0)} \cdot (Ax)$$

At $t = 0$,

$$e^{-At} = e^{-A(T)} (Ax)$$

we know that,

$$x(t) = e^{At} x_0 + \int_{t_0}^{t} e^{A(t-T)} B u(T)\, dT$$

$$\Rightarrow \quad x(t) = e^{At} x_0$$

$$e^{At} = (SI - A)^{-1}$$

$$= \left(\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -12 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} S & -1 \\ 6 & S+12 \end{bmatrix}^{-1}$$

$$= \frac{1}{[S(S+12)+6]} \begin{bmatrix} S+12 & 1 \\ -6 & S \end{bmatrix}$$

$$= \begin{bmatrix} \frac{S+12}{S^2+12S+6} & \frac{1}{S^2+12S+6} \\ \frac{-6}{S^2+12S+6} & \frac{S}{S^2+12S+6} \end{bmatrix}$$

After calculating the inverse and converting it w.r.t exponential terms,

and put $t = 4$ at e^{At} to obtain

$$x(4) = e^{4A} x(0)$$

At $t = 4$,

Determine using inverse Laplace.

$$x(4) = e^{A(4)} x_0$$

$$\Rightarrow \quad e^{4A} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2)

Given,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$[SI - A] = \begin{bmatrix} S & -1 \\ 1 & S+2 \end{bmatrix}$$

$$e^{At} \Leftrightarrow (SI - A)^{-1}$$

$$\Rightarrow \begin{bmatrix} \frac{(S+2)}{S^2+2S+1} & \frac{1}{S^2+2S+1} \\ \frac{-1}{S^2+2S+1} & \frac{S}{S^2+2S+1} \end{bmatrix}$$

upon partial fractionisation,

a) $$\frac{S+2}{S^2+2S+1} = \frac{S+1}{(S+1)^2} + \frac{1}{(S+1)^2}$$

$$= \frac{1}{S+1} + \frac{1}{(S+1)^2}$$

b) $$\frac{1}{S^2+2S+1} = \frac{1}{(S+1)^2}$$

c) $$\frac{S}{S^2+2S+1} = \frac{1}{S+1} - \frac{1}{(S+1)^2}$$

d) $$\frac{-1}{S^2+2S+1} = \frac{-1}{(S+1)^2}$$

$$\therefore e^{At} = \begin{bmatrix} e^{-t} + e^{-t}t & e^{-t}\cdot t \\ -e^{-t}\cdot t & e^{-t} - e^{-t}\cdot t \end{bmatrix}$$

$$x_{zi}\,t = e^{At} x_0$$

$$= \begin{bmatrix} e^{-t} + e^{-t}t & e^{-t}\cdot t \\ -e^{-t}\cdot t & e^{-t} - e^{-t}\cdot t \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + e^{-t}\cdot t - 2e^{-t}\cdot t \\ -e^{-t}\cdot t - 2e^{-t} + 2e^{-t}\cdot t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} - e^{-t}\cdot t \\ -2e^{-t} + e^{-t}\cdot t \end{bmatrix}$$

$$x_{zi}(S) = (SI - A)^{-1} \cdot B\, V(S)$$

$$u(t) = e^{-2t}$$

$$= \begin{bmatrix} \frac{(S+2)}{S^2+2S+1} & \frac{1}{S^2+2S+1} \\ \frac{-1}{S^2+2S+1} & \frac{S}{S^2+2S+1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{S+2}$$

$$= \begin{bmatrix} \dfrac{1}{(s+2)(s+1)^2} \\[2ex] \dfrac{s}{(s+2)(s+1)^2} \end{bmatrix}$$

$$y_{zi}(t) = C \underbrace{e^{At} x_0}_{x_{zi}(t)}$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ \dfrac{e^{-3t} - e^{-2t}}{5} & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} \\ \dfrac{e^{-3t} - e^{-2t}}{5} \end{bmatrix}$$

$$y_{zs}(s) = C X_{zs}(s) + D U(s)$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{(s+1)^2(s+2)} \\[2ex] \dfrac{s}{(s+2)(s+1)^2} \end{bmatrix}$$

$$= \frac{2}{(s+2)(s+1)^2} + \frac{s}{(s+2)(s+1)^2}$$

$$= \frac{(s+2)}{(s+2)(s+1)^2} \Rightarrow \underline{\underline{\frac{1}{(s+1)^2}}}$$

$$y_{zi}(t) = e^{-2t}$$

$$y_{zi}(t)\ y_{z}(t) \Rightarrow 2e^{-2t} + \frac{e^{-3t} - e^{-2t}}{5}$$

$$\Rightarrow \boxed{\frac{9e^{-2t} + e^{-3t}}{5}}$$

EVALUATION :

untitled2.m

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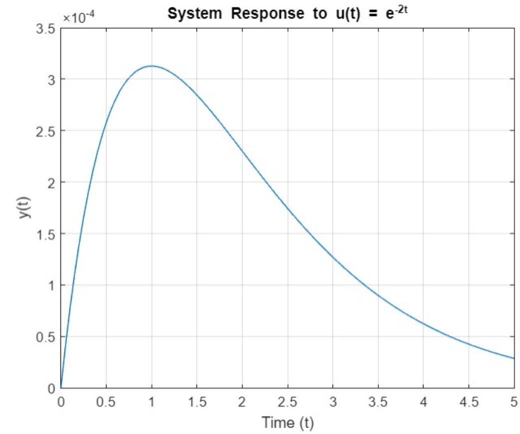
```
% System matrices
A = [0 1; -1 -2];
B = [0; 1];
C = [2 1];
D = 0;

% Create state-space system
sys = ss(A, B, C, D);

% Input function u(t) = e^(-2t)
u = @(t) exp(-2*t);

% Time vector
t = linspace(0, 5, 100);

% Initial condition
x0 = [1; -2];

% Simulate system response
u_values = arrayfun(u, t);
y = lsim(sys, u_values, t, x0);

% Plot system response
plot(t, y);
xlabel('Time (t)');
ylabel('y(t)');
title('System Response to u(t) = e^{-2t}');
grid on;
```

Figure 1



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2) a

The eigenvalues can be given as solutions of :

$$|A - \lambda I| = 0$$

$$\boxed{A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \longrightarrow \text{given}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 = 0$$

$$\Rightarrow \underline{\lambda = 1}$$

For eigenvectors,

$$(A - \lambda I) v = 0.$$

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

we can say that $\beta = 0$ and α can be any non zero value.

$$\therefore \text{ for } \lambda = 1, \quad v_1 = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

$$(\alpha \neq 0)$$

MATLAB VALIDATION:

untitled2.m × code2.m ×

/MATLAB Drive/code2.m

```
% matrix A
A = [1 2; 0 1];

% eigenvalues and eigenvectors using eig() function
[V, D] = eig(A);

% Display eigenvalues and eigenvectors
disp('Eigenvalues:')
disp(diag(D)')
disp('Eigenvectors:')
disp(V)

% Transformation matrix T
T = V;

% Validate using eig() command
eig_A = eig(A);
disp('Eigenvalues using eig():')
disp(eig_A')
```

Command Window

```
>> code2
Eigenvalues:
     1     1

Eigenvectors:
    1.0000   -1.0000
         0    0.0000

Eigenvalues using eig():
     1     1
```

(b)

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

For eigenvalues,

$$|A - \lambda I| = 0.$$

$$\therefore \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0.$$

$$(-2-\lambda)\left[(1-\lambda)(-\lambda) - (-6)(-2)\right] - 2\left((-2)(-\lambda) - (-6)(-1)\right) - (-3)\left(2(-2) - 1(-1)\right) = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda - \lambda^3 - 3\lambda^3 - \lambda^2 - 3\lambda^3 - 2\lambda^3 + 21\lambda + 45$$

$$\Rightarrow -(\lambda-5)(\lambda+3)^2 = 0.$$

$$\therefore \lambda = 5$$

$$(\text{or}) \quad \lambda = -3$$

eigen vectors:

For $\lambda_1 = -3$,

$$(A + 3I)x_1 = 0.$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_{11} + 2x_{21} - 3x_{31} = 0$$

$$2x_{11} + 4x_{21} - 6x_{31} = 0$$

$$-x_{11} - 2x_{21} + 3x_{31} = 0$$

upon solving,

$$x_{11} = -0.9526$$
$$x_{21} = +0.2722$$
$$x_{31} = -0.1361$$

$$\therefore \quad x_1 = \begin{bmatrix} -0.9526 \\ 0.2722 \\ -0.1361 \end{bmatrix}$$

For $\lambda_2 = 5$,

$$(A - 5I)\, x_2 = 0.$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_{12} + 2x_{22} - 3x_{32} = 0$$
$$2x_{12} - 4x_{22} - 6x_{32} = 0$$
$$-x_{12} - 2x_{22} - 5x_{32} = 0$$

upon solving,

$$x_2 = \begin{bmatrix} 0.4082 \\ 0.8165 \\ -0.4082 \end{bmatrix}$$

VALIDATION:

untitled2.m × code2.m × untitled4.m ×

/MATLAB Drive/untitled4.m

```matlab
A = [-2 2 -3; 2 1 -6; -1 -2 0];
[eigenvecs_matlab, eigenvals_matlab] = eig(A);

% Display eigenvalues
disp('Eigenvalues:');
disp(diag(eigenvals_matlab));

% Display eigenvectors
disp('Eigenvectors:');
disp(eigenvecs_matlab);

% Validation
disp('Validation:');
disp(isequal(round(eigenvals, 4), round(diag(eigenvals_matlab), 4)) && isequal(round(eigenvecs, 4), round(eigenvecs_matlab, 4)));
```

Command Window

```
>> untitled4
Eigenvalues:
   -3.0000
    5.0000
   -3.0000

Eigenvectors:
   -0.9526    0.4082   -0.0230
    0.2722    0.8165    0.8353
   -0.1361   -0.4082    0.5492

Validation:
   0
```

3) a

System ONE

$$\dot{x}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_1$$

$$y_1 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x_1$$

System TWO

$$\dot{x}_2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_2$$

$$y_2 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x_2$$

When $x(0) = 0$,
The initial conditions are zero.
We only need to compare matrices $A_1, B_1, C_1, A_2, B_2, C_2$ respectively.
$B_1 = B_2$ and $C_1 = C_2$
But $A_1 \neq A_2$
$\therefore$ G_1 and G_2 are not equivalent for $x(0) = 0$.

For $x(0)$ and $u(t)$ being non zero,
we have to calculate Transfer functions.

$$G_1(s) = C_1(sI-A_1)^{-1}B_1 + D_1$$
$$G_2(s) = C_2(sI-A_2)^{-1}B_2 + D_2$$

$G_1(s)$:

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix}\left(s\begin{pmatrix}-1 & -1 & 2\\ 0 & s-2 & -2\\ 0 & 0 & s-1\end{pmatrix}^{-1}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}+1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$G_1(s) = \frac{1}{(s+1)(s-2)(s-1)}\begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}s-2\\0\\0\end{bmatrix} + \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\boxed{G_1(s) = \frac{(s-2)}{(s+1)(s-2)(s-1)} + (s-1)}$$

$G_2(s)$:

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix}\left(s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}2 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & -1\end{bmatrix}\right)^{-1} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$G_2(s) = \frac{1}{(s+1)(s-2)(s+1)}\begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}s-2\\0\\0\end{bmatrix} + \begin{bmatrix}1 & -1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$$\boxed{G_2(s) = \frac{(s-2)}{(s+1)(s-2)(s+1)} + (s+1)}$$

If we compare G_1, G_2 They are almost equivalent except that they differ in terms of constant term.

4) a) The eigenvalues will be,

$$|\lambda I - A| = 0 \qquad A = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$\begin{vmatrix} \lambda+2 & 0 \\ -1 & \lambda-3 \end{vmatrix} = 0$$

$$(\lambda+2)(\lambda-3) = 0$$

$$\therefore \lambda_1 = -2, \; \lambda_2 = 3.$$

For eigenvectors,

$$(\lambda I - A)\, v = 0.$$

$$\begin{bmatrix} 0 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0 \quad \rightarrow \lambda_1 = -2$$

x_1 can be anything, lets take it as 1 for simplicity.

$$\rightarrow \lambda_2 = 3$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

y_2 can be anything, let it be 1 for simplicity.

Hence we get,

$y_1 = -0.2$ and

$x_2 = -0.2$ respectively.

$$T_{DCF} = [v_1 \; v_2] = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{DCF} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

$$B_{DCF} = T^{-1} B$$

$$= \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C_{DCF} = C \cdot T_{DCF}$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1.8 \end{bmatrix}$$

$$D_{DCF} = 0.$$

$\therefore$ The SS model in DCF will be

$$\boxed{\dot{z}(t) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u}$$

$$z(0) = T^{-1} x(0) \Rightarrow \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\therefore \quad z_0 = \begin{bmatrix} -0.6 \\ 2 \end{bmatrix}$$

$$y(t) = C_{DCF}\, z(t).$$

$$y(t) = \begin{bmatrix} 1 & 1.8 \end{bmatrix} z(t)$$

(b)

using the ss in DCF, Transfer func^n $G(S)$ will be:

$$G(S) = C_{DCF}(SI - A_{DCF})^{-1} B_{DCF} + D_{DCF}$$

$$(SI - A_{DCF})^{-1} = \left(\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \right)^{-1}$$

$$= \begin{bmatrix} (S-3) & 0 \\ 0 & (S+2) \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{(S-3)} & 0 \\ 0 & \frac{1}{(S+2)} \end{bmatrix}$$

$$\therefore G(s) = \begin{bmatrix} 1 & 1.8 \end{bmatrix} \begin{bmatrix} \frac{1}{(S-3)} & 0 \\ 0 & \frac{1}{(S+2)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$\Rightarrow \boxed{\frac{1}{S-3}}$$

(c) A_{DCF} is Diagnol matrix.

$$e^{A_{DCF} t} = \left[SI - A_{DCF} \right]^{-1}$$

$$= \begin{bmatrix} S+2 & 0 \\ 0 & S-3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \frac{1}{S+2} & 0 \\ 0 & \frac{1}{S-3} \end{bmatrix}$$

$$e^{A_{DCF} t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

$$e^{At} = T_{DCF}\, e^{A_{DCF} t}\, T_{DCF}^{-1}$$

$$= \begin{bmatrix} -5 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1/5 & 0 \\ 1/5 & 1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ \frac{e^{3t}}{5} - \frac{e^{-2t}}{5} & e^{3t} \end{bmatrix}$$

$$y(t) = C e^{At} \cdot x_0$$

$$= C_{DCF}\, e^{At} z_0$$

$\Longrightarrow$

$$\Rightarrow [-9 \quad +1] \begin{bmatrix} e^{-2t} & 0 \\ \frac{e^{3t}}{5} - \frac{e^{-2t}}{5} & e^{3t} \end{bmatrix} \begin{bmatrix} -0.4 \\ -0.6 \end{bmatrix}$$

$$\Rightarrow [-9 \quad -1] \begin{bmatrix} -0.4 e^{-2t} \\ \frac{-0.4e^{3t} - 3e^{3t} + 0.4e^{-2t}}{5} \end{bmatrix}$$

$$\Rightarrow -9(-0.4e^{-2t}) - \frac{3.4e^{3t} + 0.4e^{-2t}}{5}$$

$$y(t) = 3.6e^{-2t} - \frac{3.4e^{3t} + 0.4e^{-2t}}{5}$$

$$\Rightarrow \frac{18.4e^{-2t} - 3.4e^{3t}}{5}$$

$$\Rightarrow \boxed{3.6e^{-2t} - 0.68e^{3t}}$$

3(b)

given,

$$\Sigma = \left[\begin{array}{ccc|c} -a_2 & 1 & 0 & b_2 \\ -a_1 & 0 & 1 & b_1 \\ -a_0 & 0 & 0 & b_0 \\ \hline 1 & 0 & 0 & d \end{array}\right]$$

$$\Sigma^* = \left[\begin{array}{ccc|c} 0 & 0 & -a_0 & b_0 \\ 1 & 0 & -a_1 & b_1 \\ 0 & 1 & -a_2 & b_2 \\ \hline 0 & 0 & 1 & d \end{array}\right]$$

When we try to apply Transformations,

$$T_\Sigma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and the Transformation for Σ^* correspondingly would be

$$T_{\Sigma^*} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

As the transformation exists for both Σ and Σ^* and it is the same for both, we can say that they both are similar.