UNIT-4 [PART-B]

1a) If $z = f(x+ay) + \phi(x-ay)$, prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Sol) Given that $z = f(x+ay) + \phi(x-ay)$

Differentiate z partially w.r.t x

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi(x-ay)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay) \quad \text{--- (1)}$$

Differentiate z partially w.r.t y.

$$\frac{\partial z}{\partial y} = f'(x+ay)a + \phi'(x-ay)(-a)$$

$$= a\left[f'(x+ay) - \phi(x-ay)\right]$$

$$\frac{\partial^2 z}{\partial y^2} = a\left[f''(x+ay)a + \phi(x-ay)(-a)\right]$$

$$= a^2\left[f''(x+ay) + \phi(x-ay)\right]$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

LHS = R.H.S

$\therefore$ Hence proved

1b) Verify Euler's theorem for $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$.

Sol By the Euler's theorem we have.

If $z = f(x,y)$ is a homogeneous function of degree n, the $x\frac{\partial z}{dx} + y\frac{dz}{\partial y} = nz$ for all (x,y) in the domain of the function.

Given that $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$

It is a homogenous function of degree 2.

$\therefore$ By the theorem

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$$

<u>Verification:-</u>

Given that $u = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$ — ①

$$\frac{\partial u}{\partial x} = x^2 \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) + \tan^{-1}(y/x)(2x) - y^2\left(\frac{1}{1+(x/y)^2}\right)\left(\frac{1}{y}\right)$$

$$= x^2 \left(\frac{-y/x^2}{\frac{x^2+y^2}{x^2}}\right) + 2x\tan^{-1}(y/x) - y\left(\frac{y^2}{1+x^2}\right)$$

$$\boxed{\frac{\partial u}{\partial x} = x^2\left(\frac{-y}{x^2+y^2}\right) + 2x\tan^{-1}(y/x) - \left(\frac{y^3}{1+x^2}\right)}$$

$$\frac{\partial u}{\partial x} = 2x \tan^{-1}(y/x) - \frac{y}{x^2+y^2}(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = 2x \tan^{-1}(y/x) - y$$

Multiple x O.B.S

$$x \frac{\partial u}{\partial x} = 2x^2 \tan^{-1}(y/x) - xy \quad \text{— ②}$$

$$\frac{\partial u}{\partial y} = x^2\left(\frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right)\right) - y^2\left(\frac{1}{1+(x/y)^2}\left(\frac{-x}{y^2}\right)\right) - 2y\tan^{-1}(x/y)$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} + \frac{\boxed{x x^2 y^2 x}}{} - 2y\tan^{-1}\left(\frac{x}{y}\right)$$

$$= \frac{x\cancel{(x^2+y^2)}}{\cancel{(x^2+y^2)}} - 2y\tan^{-1}(x/y).$$

$$y\frac{\partial u}{\partial y} = xy - 2y^2\tan^{-1}\left(\frac{x}{y}\right)$$

$$y\frac{\partial u}{\partial y} = xy - 2y^2 t\left(\frac{x}{y}\right) \quad -③$$

Add eq ② & ③

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= 2x^2\tan^{-1}(y/x) - xy + xy - 2y^2\tan^{-1}(x/y)$$

$$= 2\left(x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)\right)$$

$$= 2u \quad [\text{from eq } 1]$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x - 2y\tan^{-1} x/y\right)$$

$$= 1 - 2y\left(\frac{1}{1+(x/y)^2}\left(\frac{1}{y}\right)\right)$$

$$= 1 - 2y\left(\left(\frac{y^2}{x^2+y^2}\right)\left(\frac{1}{y}\right)\right)$$

$$= \frac{x^2+y^2-2y^2}{x^2+y^2}$$

$$= \frac{x^2-y^2}{x^2+y^2}$$

$\therefore$ Hence proved

2a) Prove that the function $u = x+y+z$, $V = x^2+y^2+z^2-2xy-2yz-2zx$, $w = x^3+y^3+z^3-3xyz$ are functionally dependent

a) Given that $u = x+y+z$

$V = x^2+y^2+z^2-2xy-2yz-2zx$

$w = x^3+y^3+z^3-3xyz$

Now $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-y-x) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & z-y-x \\ (x-y)(x+y+z) & (y-z)(x+y+z) & z^2-xy \end{vmatrix}$$

$C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 12\begin{vmatrix} x-y & y-z \\ (x-y)(x+y+z) & (y-z)(x+y+z) \end{vmatrix}$$

$$= 12(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y+z & x+y+z \end{vmatrix}$$

$= 12(x-y)(y-z)(0)$ $\quad [\because C_1 \text{ \& } C_2 \text{ are identical}]$

$= 0$

$\therefore$ Hence proved.

2b If $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$. Find $\dfrac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functional dependent. Find the functional relation between them.

Sol Given $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2} \quad ; \quad \frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2} \quad ; \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

Now
$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}.$$

$$= \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\[2ex] \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Hence u and v are functionally dependent

Now $v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}(u)$

$\therefore$ $v = \tan^{-1}(u)$ is the functional relation between u and v. $\left[\because u = \frac{x+y}{1-xy}\right]$

3a Prove that the functions $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ are functionally dependent and find the relation between them.

Sol Given

$$\left.\begin{aligned} u &= xy + yz + zx \\ v &= x^2 + y^2 + z^2 \\ w &= x + y + z. \end{aligned}\right\} - (1)$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2(x+y+z)(0) \qquad (\because R_1 \ \& \ R_3 \text{ are identical})$$

$$= 0$$

$\therefore$ Hence u, v and w are functionally dependent. That is, the functional relationship exists between u, v and w.

Now $w^2 = (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$

$= v + 2u$ [from eq 1]

$\therefore w^2 = 2u + v$ is the functional relationship between u, v & w.

3b Discuss the maxima and minima of $x^2y + xy^2 - axy$

Sol Given $x^2y + xy^2 - axy$

Let $f(x,y) = x^2y + xy^2 - axy$

$$\frac{\partial f}{\partial x} = 2xy + y^2 - ay$$

$$\frac{\partial f}{y} = x^2 + 2xy - ax$$

Equating $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ to zero we get $2xy + y^2 - ay = 0$ — (1) and $x^2 + 2xy - ax = 0$ — (2)

Solving (1) & (2)

we get $x = 0,\ y = 0$

$x = \frac{a}{3},\ y = \frac{a}{3}$

$\therefore$ The stationary point are $(0,0)$; $\left(\frac{a}{3}, \frac{a}{3}\right)$

$l = \frac{\partial^2 f}{\partial x^2} = 2y$; $m = \frac{\partial^2 f}{\partial x \partial y}$; $n = \frac{\partial^2 f}{\partial y^2}$

$= 2x + 2y - a$ $\qquad = 2x.$

So, $ln - m^2 = (2x)(2y) - (2x + 2y - a)^2$ [At $x = 0, y = 0$]

$= -a^2 < 0$

So, $ln - m^2 = (2x)(2y) - (2x + 2y - a)^2$ [At $x = \frac{a}{3},\ y = \frac{a}{3}$]

$= \frac{a^2}{3} > 0$

$\therefore f(x,y)$ is minimum at $\left(\frac{a}{3}, \frac{a}{3}\right)$

4a Find the maximum and minimum $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Sol Given function is $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$ — (1)

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$$

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

The condition for $f(x,y)$ to have min (or) max is $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$$y = \frac{a^3}{x^2} \quad -② \quad ; \quad x = \frac{a^3}{y^2} \quad -③$$

Substituting ③ in ②

$$y = \frac{a^3 y^4}{a^6} = \frac{y^4}{a^3}$$

$$y(y^3 - a^3) = 0$$

$$y = 0 \text{ or } y = a$$

Now, $y = a \Rightarrow x = a$

$\therefore$ The extremum points is (a,a)

$f(x,y)$ will have max (or) min at (a,a)

At (a,a), $l = \frac{\partial^2 f}{\partial x^2} = 2$; $m = 1$, $n = 2$

$$ln - m^2 = 4 - 1 = 3 > 0 \; ; \; l = 2 > 0$$

$\therefore$ $f(x,y)$ has minimum at (a,a)

The minimum value is $f(a,a) = a^2 + \frac{a^3}{a} + \frac{a^4}{a} = 3a^2$

4b A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Sol Let x ft, y ft, z ft be the dimensions of the box and let S be the surface of the box then from the data we have

$$S = xy + 2(xz) + 2(yz) \text{ and}$$

$$V = xyz = 32 \text{ cubic ft}$$

put $z = \frac{32}{xy}$

Then $f(x,y) = S = xy + 2x\left(\frac{32}{xy}\right) + 2y\left(\frac{32}{xy}\right)$

$$f(x,y) = xy + \frac{64}{y} + \frac{64}{x}$$

$$\frac{\partial f}{\partial x} = y - \frac{64}{x^2} \quad ; \quad \frac{\partial f}{\partial y} = x - \frac{64}{y^2}$$

$$\Rightarrow x^2y - 64 = 0 \ (2) \qquad \Rightarrow xy^2 - 64 = 0 - (3)$$

By solving (2) & (3)

we get $x^2y = xy^2$

$$\boxed{x = y}$$

$$(2) \Rightarrow x^3 - 64 = 0$$

$$x^3 = 64$$

$$\boxed{x = 4}$$

Similarly $\boxed{y = 4}$

$(4,4)$ is a stationary point for the function

Let $l = \frac{\partial^2 f}{\partial x^2}$; $m = \frac{\partial^2 f}{\partial x \partial y}$; $n = \frac{\partial^2 f}{\partial y^2}$

$l = \frac{128}{x^3}$; $m = 1$; $n = \frac{128}{y^3}$

At $(4,4)$ $l = 2, n = 2, m = 1$

$ln - m^2 = 3 > 0$ and $l > 0$

$\Rightarrow$ At $(4,4)$ the function has minimum value

$\therefore$ To construct the box the required least matrix we have to take dimensions. as

$$x = 4 \ ; \ y = 4 \ ; \ z = \frac{32}{xy} = \frac{32}{16} = 2$$

$$(x, y, z) = (4, 4, 2)$$

5) Find the maximum value of $x^2+y^2+z^2$ given $x+y+z=3a$

Sol Given that $x^2+y^2+z^2$ and $x+y+z=3a$.

$f(x,y,z) = x^2+y^2+z^2$ we have to find minimum value of $f(x,y,z)$ subject to condition $\phi(x,y,z) = x+y+z-3a$

By the method of Legranje's multipliers

$$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$$

$$F(x,y,z) = x^2+y^2+z^2+\lambda(x+y+z-3a) \quad -(1)$$

$$\frac{\partial F}{\partial x} = 2x+\lambda(1) = 0 \Rightarrow \lambda = -2x \quad -(2)$$

$$\frac{\partial F}{\partial y} = 2y+\lambda(1) = 0 \Rightarrow \lambda = -2y \quad -(3)$$

$$\frac{\partial F}{\partial z} = 2z+\lambda(1) = 0 \Rightarrow \lambda = -2z \quad -(4)$$

from (2), (3), (4) we can write

$$x = y = z$$

we have $x+y+z = 3a$

$$3x = 3a$$

$$\boxed{x = a}$$

$$x = y = z = a$$

The point (a,a,a) is extreme points

6) Find three positive numbers whose sum is 100 and whose product is maximum.

Sol Let x, y, z be the three positive numbers whose sum is 100

$$x + y + z = 100$$

Let $f(x, y, z) = xyz$ — ①

$$z = 100 - (x+y)$$

$$f(x, y, z) = xy(100 - (x+y))$$

$$= 100xy - x^2y - xy^2$$

$$\frac{\partial f}{\partial x} = 100y - 2xy - y^2$$

$$= y(100 - 2x - y) = 0 \quad — ②$$

$$\frac{\partial f}{\partial y} = 100x - x^2 - 2yx$$

$$= x(100 - x - 2y) = 0 \quad — ③$$

Solving ② & ③

$$x + 2y = 100$$

$$2x + y = 100$$

$$-x + y = 0$$

$$\boxed{x = y}$$

From eq ② $x = y$

$$100y - 2y^2 - y^2 = 0$$

$$100y - 3y^2 = 0$$

$$y(100 - 3y) = 0$$

$$3y = 100$$

$$\boxed{y = \frac{100}{3}} \quad \text{similarly} \quad \boxed{x = \frac{100}{3}}$$

$$l = \frac{\partial^2 f}{\partial x^2} \quad ; \quad m = \frac{\partial^2 f}{\partial x \partial y} \quad ; \quad n = \frac{\partial^2 f}{\partial y^2}$$

$$= -2y \qquad = \frac{\partial}{\partial x}(100x - x^2 - 2yx) \qquad = -2x$$

$$= 100 - 2x - 2y$$

$$l = -2\left(\frac{100}{3}\right) \quad ; \quad m = 100 - 2\left(\frac{100}{3}\right) - 2\left(\frac{100}{3}\right) \quad ; \quad n = -2\left(\frac{200}{3}\right)$$

$$l = \frac{-200}{3} \quad ; \quad m = \frac{-100}{3} \quad , \quad n = \frac{-200}{3}.$$

$$ln - m^2 = \left(-\frac{200}{3}\right)\left(-\frac{200}{3}\right) - \left(\frac{10000}{9}\right)$$

$$= \frac{40000}{3} - \frac{10000}{9}$$

$$ln - m^2 = \frac{20000}{9} > 0 \quad \text{and} \quad l < 0$$

At $\left(\frac{100}{3}, \frac{100}{3}\right)$ the function f is maximum

eq ① $\Rightarrow$ $f = xyz$

$$= \left(\frac{100}{3}\right)\left(\frac{100}{3}\right)\left(\frac{100}{3}\right)$$

whose sum is 100 and product is $\left(\frac{100}{3}\right)^3$ when the numbers are $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$.

7) Discuss the maximum and minimum $x^2 + y^2 + 6x + 12$

Sol Let $f(x,y) = x^2 + y^2 + 6x + 12$

$$\frac{\partial f}{\partial x} = 2x + 6 \quad , \quad \frac{\partial f}{\partial y} = 2y$$

$l = \frac{\partial^2 f}{\partial x^2}$; $m = \frac{\partial^2 f}{\partial x \partial y}$; $n = \frac{\partial^2 f}{\partial y^2}$

$= 2$ $\quad = 0$ $\quad = 2$

For maxima and minima, the condition is

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

i.e., $2x + 6 = 0$ and $2y = 0$

$x = -3$ and $y = 0$

At $(-3, 0)$, $ln - m^2 = 2 \times 2 - 0 = 4 > 0$ and $l = 2 > 0$

$\therefore$ Hence $f(x, y)$ will be minimum when $x = -3$ & $y = 0$

$\therefore$ Minimum value $= f(-3, 0) = 9 + 10 - 18 + 12 = \underline{\underline{3}}$

8) Find the maximum and minimum value of $f = 3x^4 - 2x^3 - 6x^2 + 6x + 1$.

Sol Given that $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

$$\frac{\partial f}{\partial x} = 12x^3 - 6x^2 - 12x + 6$$

$$= 6(2x^3 - x^2 - 2x + 1)$$

For maxima or minima, $\frac{\partial f}{\partial x} = 0$

i.e., $2x^3 - x^2 - 2x + 1$

$(x-1)(x+1)(2x-1) = 0$

$\therefore x = 1, -1, \frac{1}{2}$

These are the possible extreme points

Now, $\frac{\partial^2 f}{\partial x^2} = 6(6x^2 - 2x - 2)$

$= 12(3x^2 - x - 1)$

when $x = 1$

$$\frac{\partial^2 f}{\partial x^2} = 12(3-1-1)$$

$$= 12 > 0$$

$\therefore$ $f(x)$ has minimum at $x = 1$

when $x = -1$

$$\frac{\partial^2 f}{\partial x^2} = 12(3+1-1)$$

$$= 36 > 0$$

$\therefore$ $f(x)$ has minimum at $x = -1$

when $x = 1/2$

$$\frac{\partial^2 f}{\partial x^2} = 12\left(\frac{3}{4} - \frac{1}{2} - 1\right)$$

$$= 12\left(\frac{3}{4} - \frac{3}{2}\right)$$

$$= -9 < 0$$

$\therefore$ $f(x)$ has maximum at $x = 1/2$

Hence maximum value is given by $f(1/2)$

$$\therefore f(1/2) = \frac{3}{16} - \frac{2}{8} - \frac{6}{4} + \frac{6}{2} + 1$$

$$= 16(3-4-24+48+16)$$

$$= \frac{39}{16}$$

Minimum value are given by $f(-1)$ & $f(1)$

Minimum value of $f = f(-1) = 3+2-6-6+1 = -6$

$f = f(1) = 3-2-6+6+1 = 2$

9) Examine for minima and maxima values of

$\sin x + \sin y + \sin(x+y)$

Sol Let $f(x,y) = \sin x + \sin y + \sin(x+y)$ — ①

we known that neccessary condition for the $f(x,y)$ to have extreme value of (a,b) are $f_x(a,b)$ and $f_y(a,b) = 0$

Differentiate eq ① w.r.t x and y

$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$ $\qquad$ $\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$

Consider (a,b) is an extreme point of function $f(x,y) \Rightarrow$

$\Rightarrow f_x(a,b) = 0$ $\qquad$ $f_y(a,b) = 0$

$\cos a + \cos(a+b) = 0$ — ② $\qquad$ $\cos b + \cos(a+b) = 0$ — ③

Solve ② & ③

$$\cos a + \cos(a+b) = 0$$
$$\underline{\cos b + \cos(a+b) = 0}$$
$$\cos a - \cos b = 0$$
$$\cos a = \cos b$$
$$\boxed{a = b}$$

If $a = b$

$\Rightarrow \cos a + \cos 2a = 0$

$\because \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$2\cos\left(\frac{a+2a}{2}\right)\cos\left(\frac{a-2a}{2}\right) = 0$$

$$2\cos\left(\frac{3a}{2}\right)\cos\left(\frac{a}{2}\right) = 0$$

To become L.H.S = R.H.S

$\frac{3a}{2} = \pm\frac{\pi}{2}$ or $\frac{a}{2} = \pm\frac{\pi}{2}$

$a = \pm\frac{\pi}{3}$ or $a = \pm\pi$

If $a = \pm\frac{\pi}{3}$ $\qquad$ if $a = \pm\pi$

$b = \pm\frac{\pi}{3}$ $\qquad$ $b = \pm\pi$

$\therefore \left(\pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right), (\pm\pi, \pm\pi)$ are the stationary point

Let us consider

$l = \frac{\partial^2 f}{\partial x^2}$; $m = \frac{\partial^2 f}{\partial x \partial y}$; $n = \frac{\partial^2 f}{\partial y^2}$

$= -\sin x - \sin(x+y)$; $= -\sin(x+y)$; $= -\sin y - \sin(x+y)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$l = -\sin\frac{\pi}{3} - \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \sin\left(\frac{2\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= -\sqrt{3}$

$n = -\sin\frac{\pi}{3} - \sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \sin\left(\frac{2\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= -\sqrt{3}$

$m = -\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2}$

$ln - m^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{-\sqrt{3}}{2}\right)^2$

$= 3 - \frac{3}{4}$

$= \frac{9}{4} > 0$ and $l < 0$

$f(x,y)$ has maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

maximum values of $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{3\sqrt{3}}{2}$

At $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$

$l = -\sin\left(\frac{-\pi}{3}\right) - \sin\left(\frac{-\pi}{3}-\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \sin\left(\frac{2\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

$n = -\sin\left(\frac{-\pi}{3}\right) - \sin\left(\frac{-\pi}{3}-\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \sin\left(\frac{2\pi}{3}\right)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

$(x+y)$

$m = -\sin\left(-\frac{\pi}{3} - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$(x,y) \Rightarrow$ $ln - m^2 = (\sqrt{3})(\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{9}{4} > 0$ & $l > 0$

$f(x,y)$ has minimum at $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$

minimum values $f\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right) = +\sin\left(\frac{-\pi}{3}\right) + \sin\left(\frac{-\pi}{3}\right) + \sin\left(\frac{-\pi}{3} - \frac{\pi}{3}\right)$

$= \frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= -\frac{3\sqrt{3}}{2}$

At $(\pm\pi, \pm\pi)$

$ln - m^2 = 0$

$\therefore$ There is need for further investigation

$f(\pm\pi, \pm\pi)$ is not a extreme point and has no extreme values

10) Given that $x+y+z=a$, find the maximum values of $x^m y^n z^p$.

Sol Given that

$f(x,y,z) = x^m y^n z^p$ —(1) we have to find maximum value of $f(x,y,z)$ subject to the condition $\phi(x,y,z) = x+y+z-a$ —(2)

By the method of Lagrange's multipliers.

$F(x,y,z) = f(x,y,z) + \lambda\,\phi(x,y,z)$

$F(x,y,z) = x^m y^n z^p + \lambda(x+y+z-a)$ —(3)

$\frac{\partial F}{\partial x} = m x^{m-1} y^n z^p + \lambda = 0 \Rightarrow \lambda = -m x^{m-1} y^n z^p$ —(4)

$\frac{\partial F}{\partial y} = n y^{n-1} x^m z^p + \lambda = 0 \Rightarrow \lambda = -n y^{n-1} x^m z^p$ —(5)

$$\frac{\partial F}{\partial z} = p z^{p-1} x^m y^n + \lambda = 0 \Rightarrow \lambda = -p z^{p-1} x^m y^n \quad (6)$$

From (4)

$$\lambda = \frac{-m f(x,y,z)}{x}$$

$$x = \frac{-mf}{\lambda}$$

Similarly from (5) & (6)

$$y = \frac{-mf}{\lambda}, \quad z = \frac{-pf}{\lambda}$$

Sub x, y, z in eq (2)

$$x + y + z = a$$

$$-\left[\frac{m}{\lambda} + \frac{n}{\lambda} + \frac{p}{\lambda}\right] f = a$$

$$\frac{-1}{\lambda}(m+n+p) f = a$$

$$\lambda = \frac{-1}{a}(m+n+p) f$$

By sub λ in x, y, z

$$x = \frac{\cancel{-}mf}{\frac{\cancel{-}1}{a}(m+n+p) f} = \frac{ma}{m+n+p}$$

$$y = \frac{\cancel{-}nf}{\frac{\cancel{-}1}{a}(m+n+p) \cancel{f}} = \frac{na}{(m+n+p)}$$

$$z = \frac{\cancel{-}p\cancel{f}}{\frac{\cancel{-}1}{a}(m+n+p) \cancel{f}} = \frac{pa}{(m+n+p)}$$

$\therefore \left(\frac{ma}{m+n+p}, \frac{na}{m+n+p}, \frac{pa}{m+n+p}\right)$ The values of $f(x,y,z)$ is maximum

The maximum values is

$$x^m y^n z^p = \left(\frac{ma}{m+n+p}\right)^m \left(\frac{na}{m+n+p}\right)^n \left(\frac{pa}{m+n+p}\right)^p$$

$$= \frac{a^{m+n+p}}{(m+n+p)^{m+n+p}} m^m . n^n . p^p$$

$$\left(\frac{a}{m+n+p}\right)^{m+n+p} m^m . n^n . p^p$$

$\therefore$ This is the maximum value of given $f(x, y, z)$ subject to condition $\phi(x, y, z)$