NPTEL MOOC

PROGRAMMING, DATA STRUCTURES AND ALGORITHMS IN PYTHON

Week 6, Lecture 5

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Job scheduler

* A job scheduler maintains a list of pending jobs with their priorities.
* When the processor is free, the scheduler picks out the job with maximum priority in the list and schedules it.
* New jobs may join the list at any time.
* How should the scheduler maintain the list of pending jobs and their priorities?

Priority queue

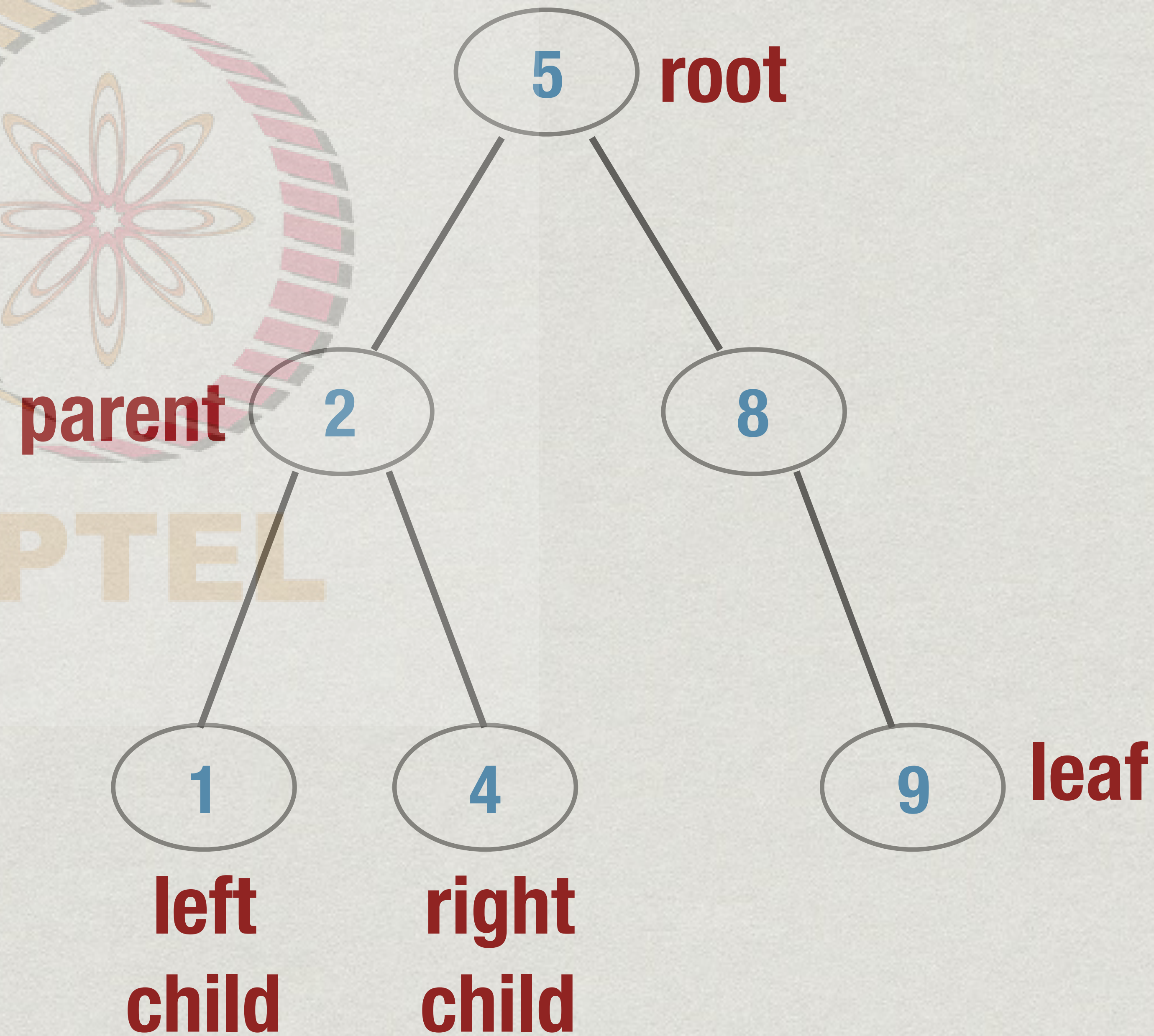
- Need to maintain a list of jobs with priorities to optimise the following operations
  - delete_max( )
    - Identify and remove job with highest priority
    - Need not be unique
  - insert( )
    - Add a new job to the list

Linear structures

- Unsorted list
  - insert( ) takes O(1) time
  - delete_max( ) takes O(n) time
- Sorted list
  - delete_max( ) takes O(1) time
  - insert( ) takes O(n) time
- Processing a sequence of n jobs requires $O(n^2)$ time

Binary tree

- Two dimensional structure
- At each node
  - Value
  - Link to parent, left child, right child

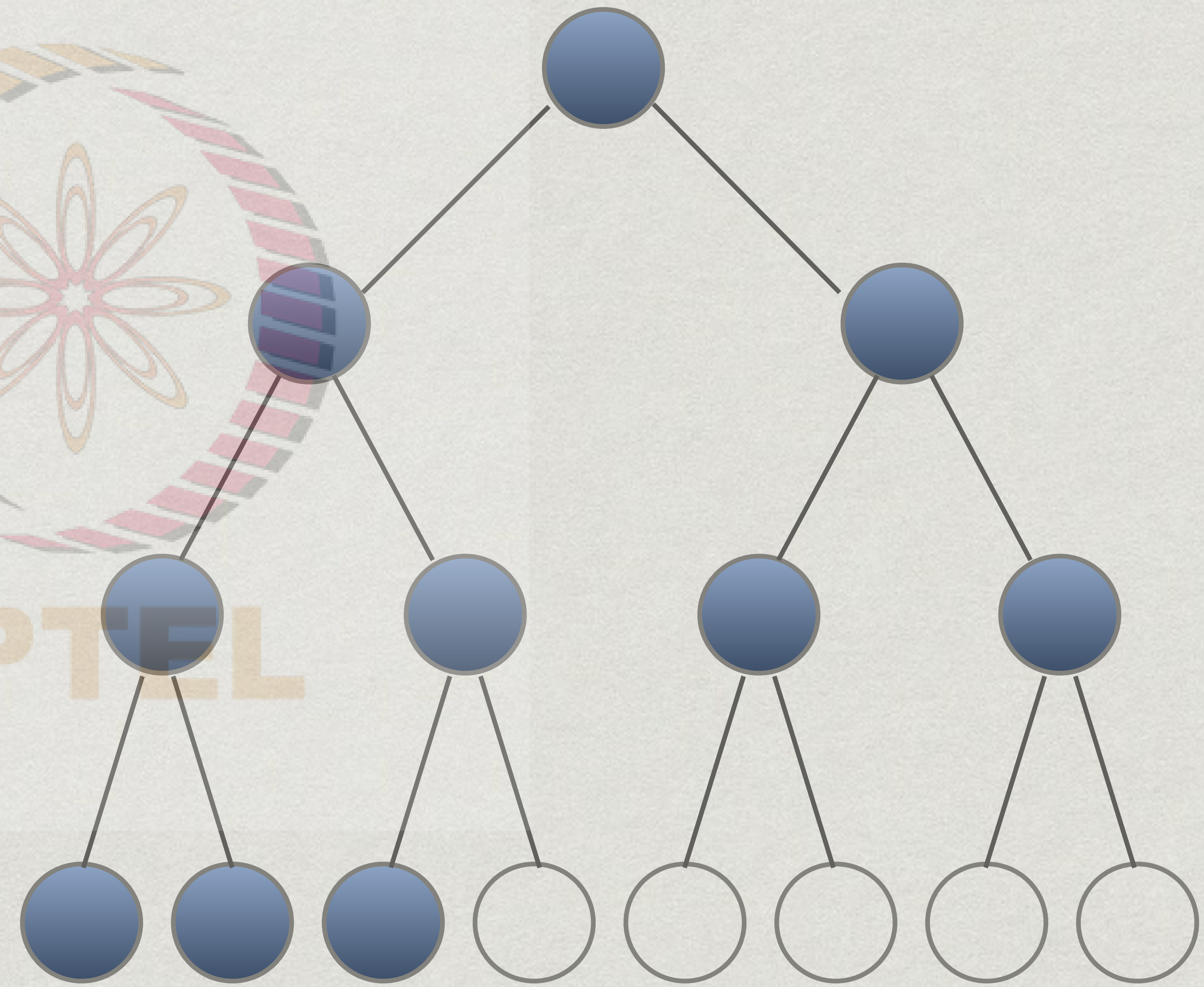


Priority queues as trees

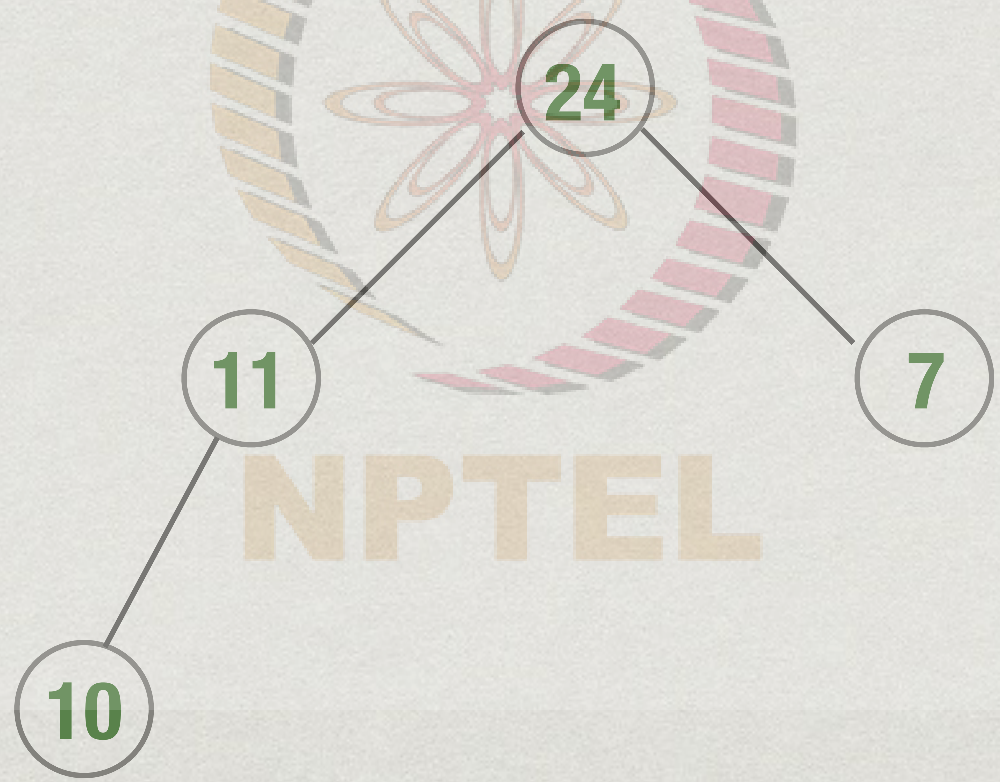
- Maintain a special kind of binary tree called a heap
  - Balanced: N node tree has height log N
- Both insert( ) and delete_max( ) take O(log N)
  - Processing N jobs takes time O(N log N)
- Truly flexible, need not fix upper bound for N in advance

Heaps

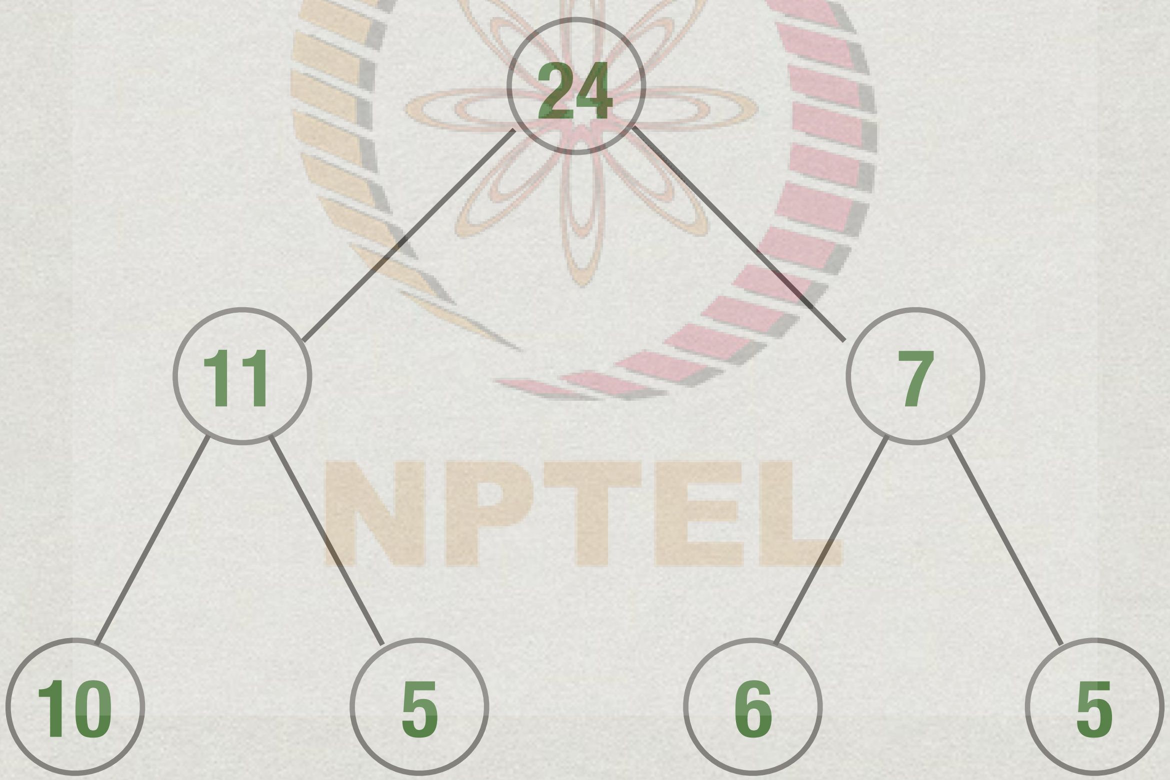
- Binary tree filled level by level, left to right
- At each node, value stored is bigger than both children
  - (Max) Heap PropertyBinary tree filled level by level, left to right



Examples

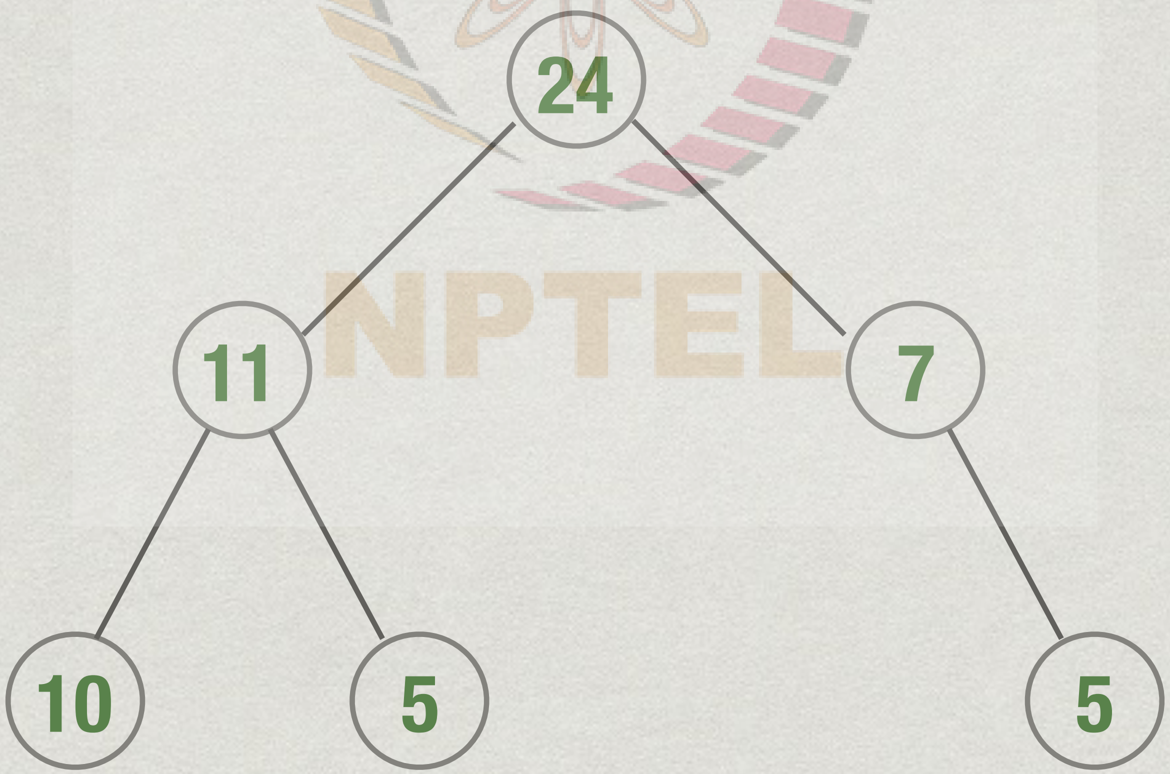


Examples



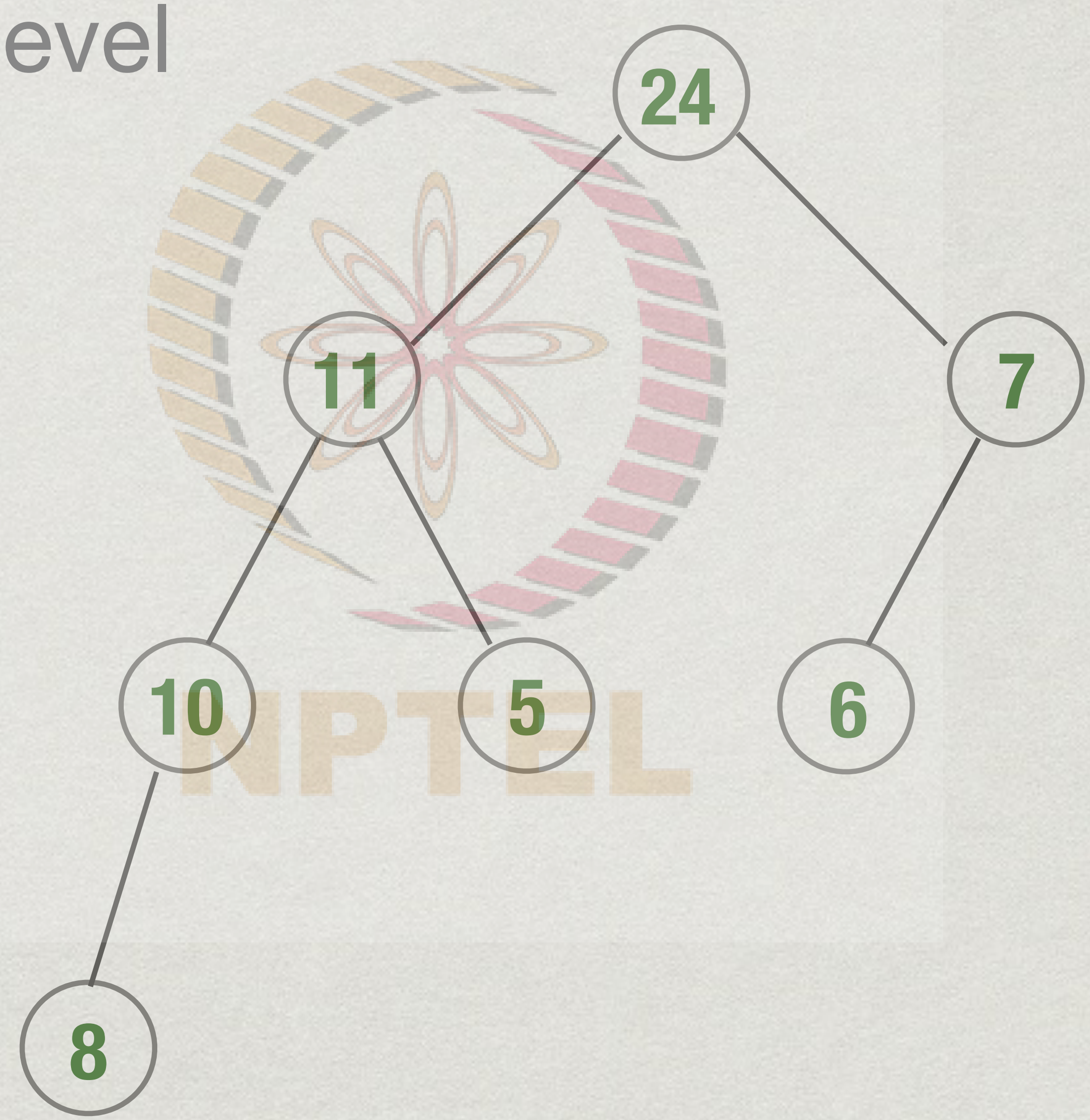
Non-examples

- No “holes” allowed



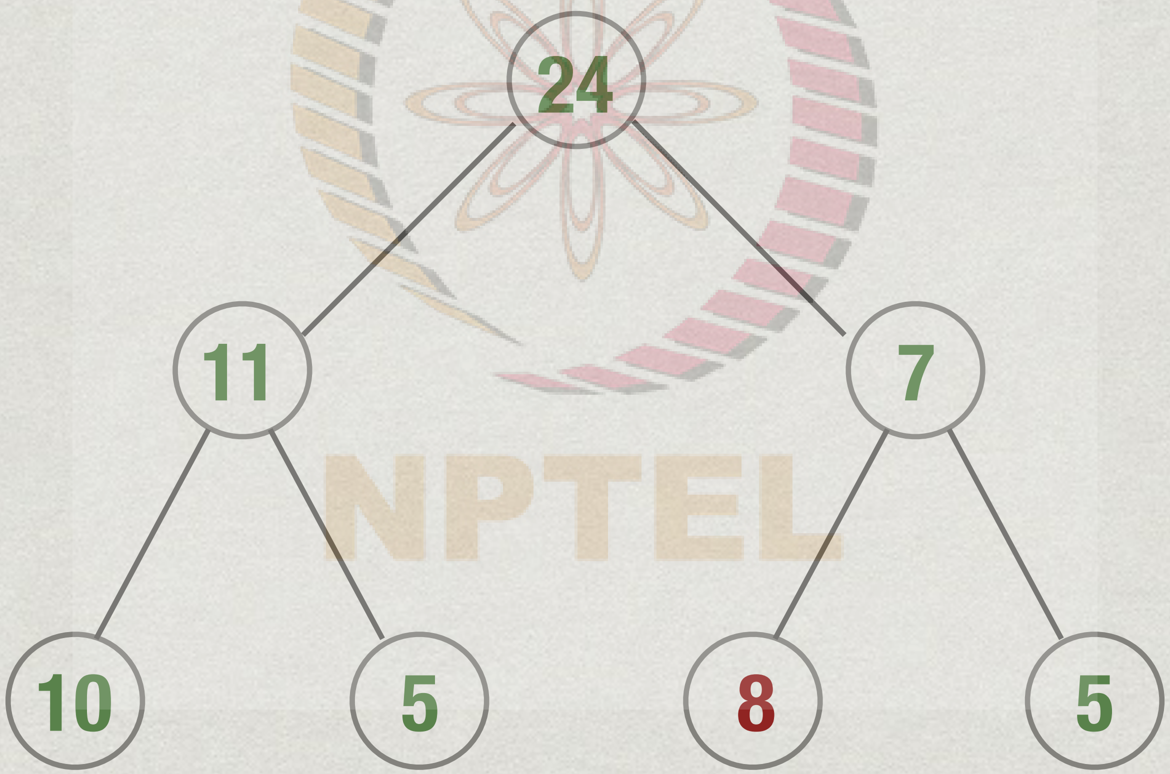
Non-examples

- Can’t leave a level incomplete



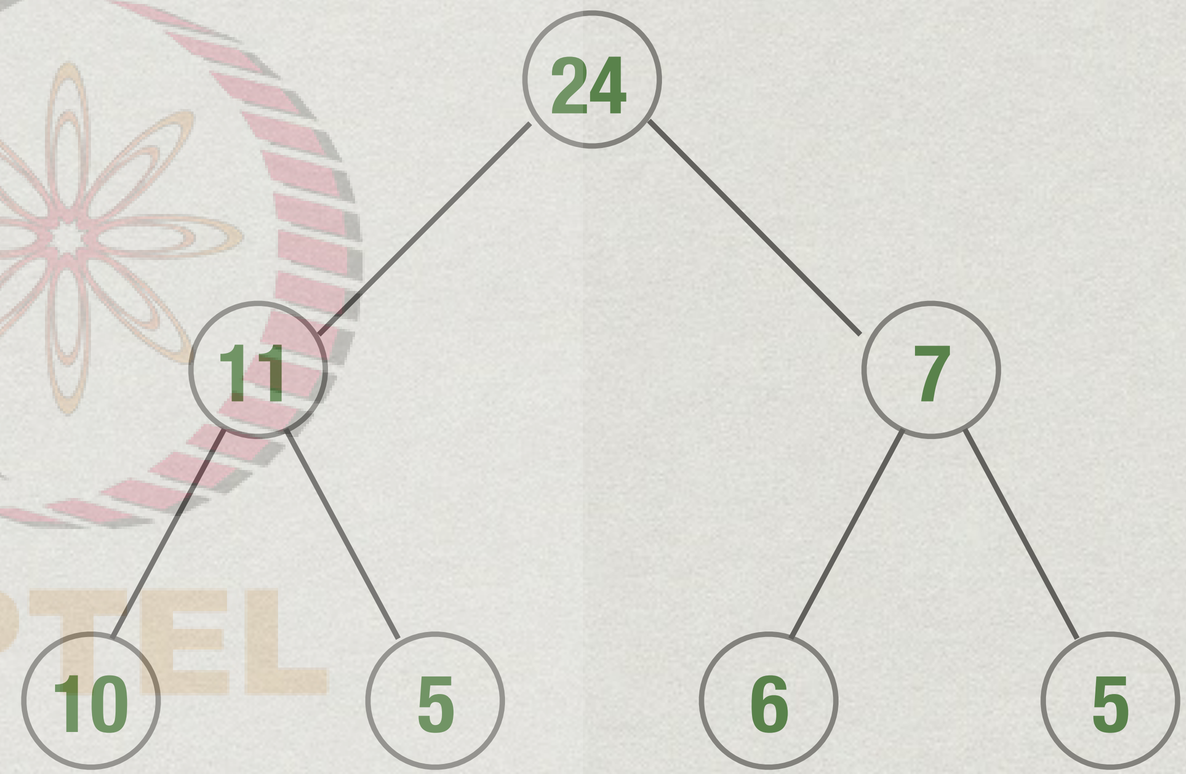
Non-examples

* Violates heap property



insert( )

- insert 12
- Position of new node is fixed by structure
- Restore heap property along the path to the root



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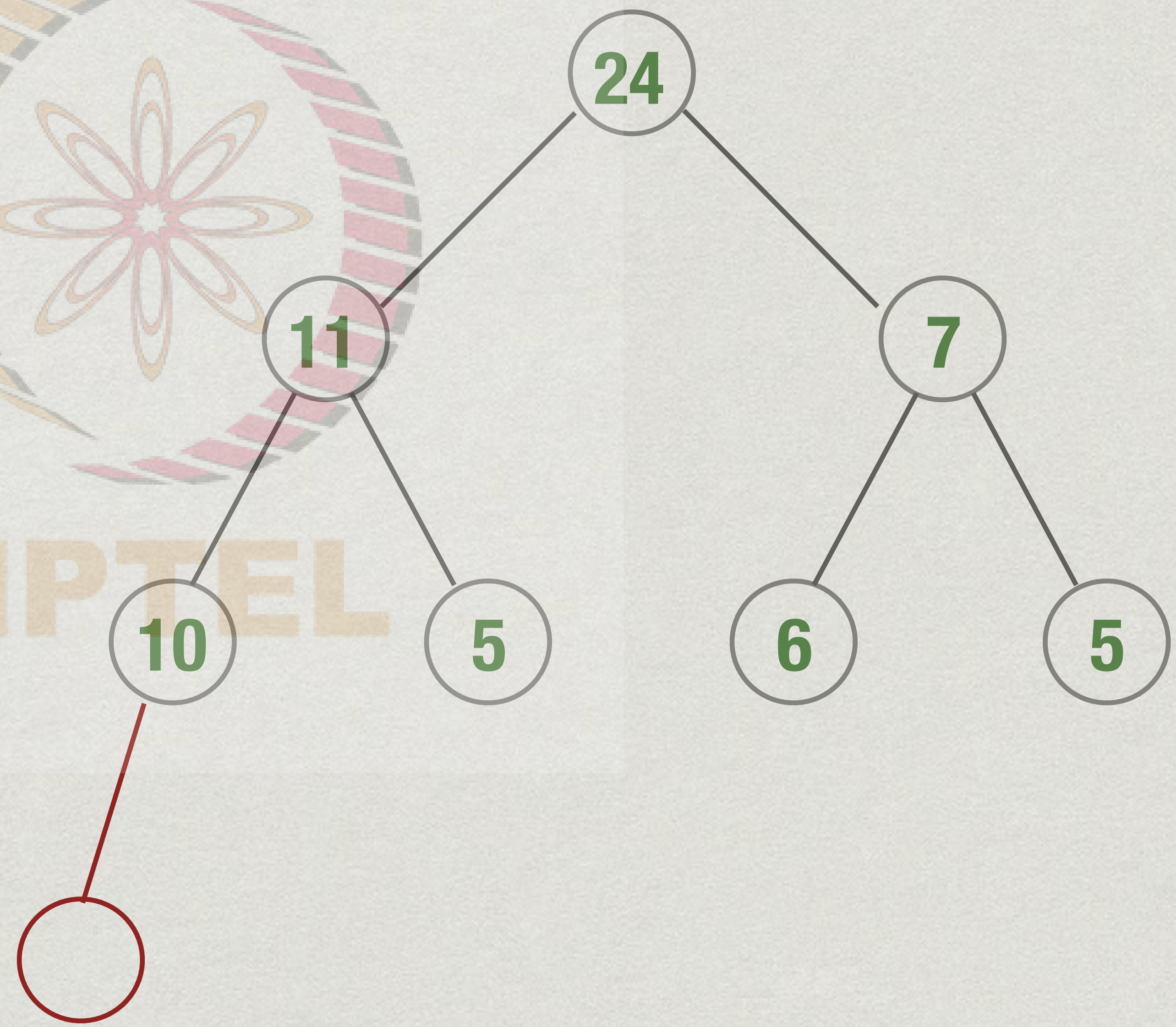

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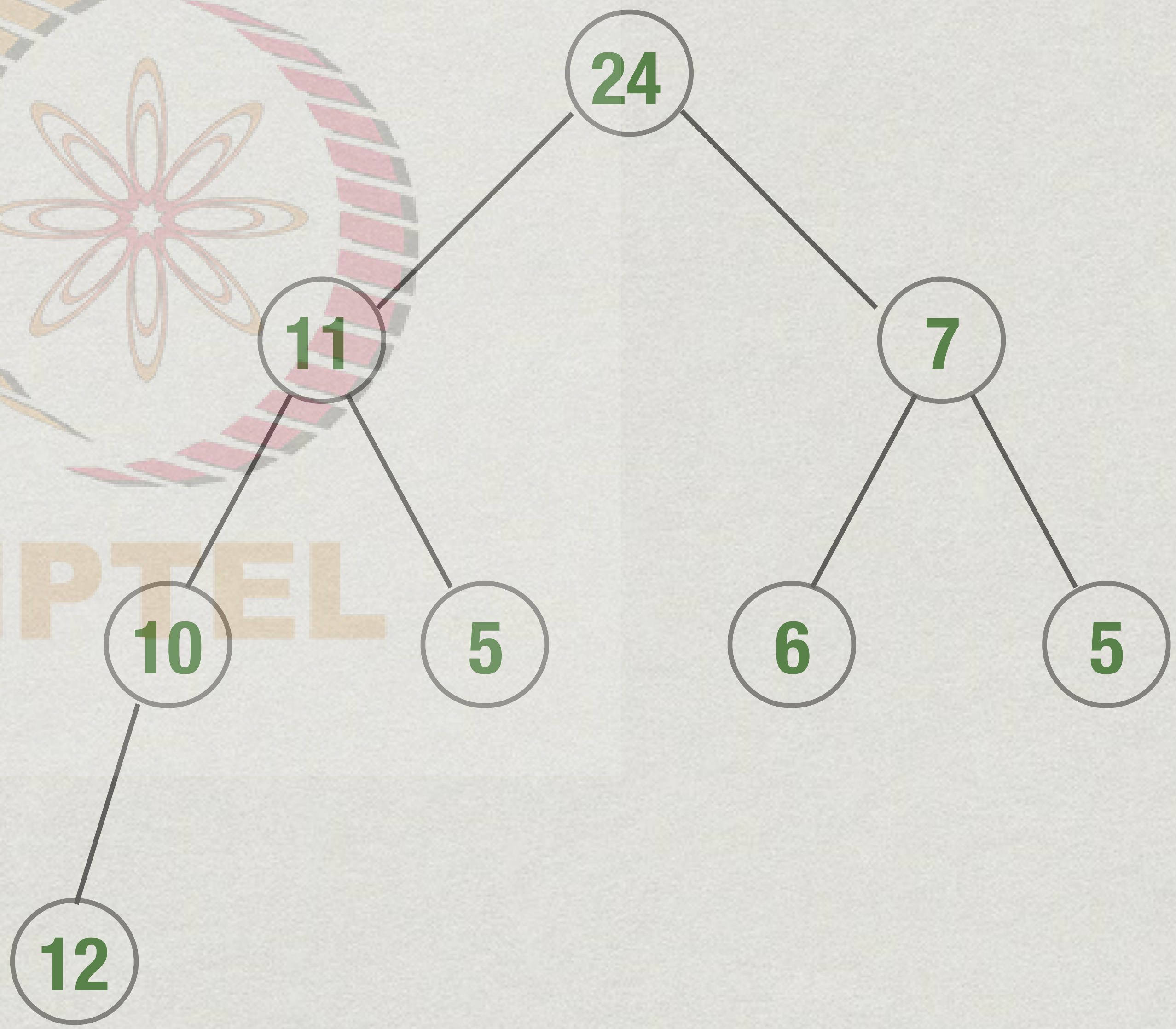

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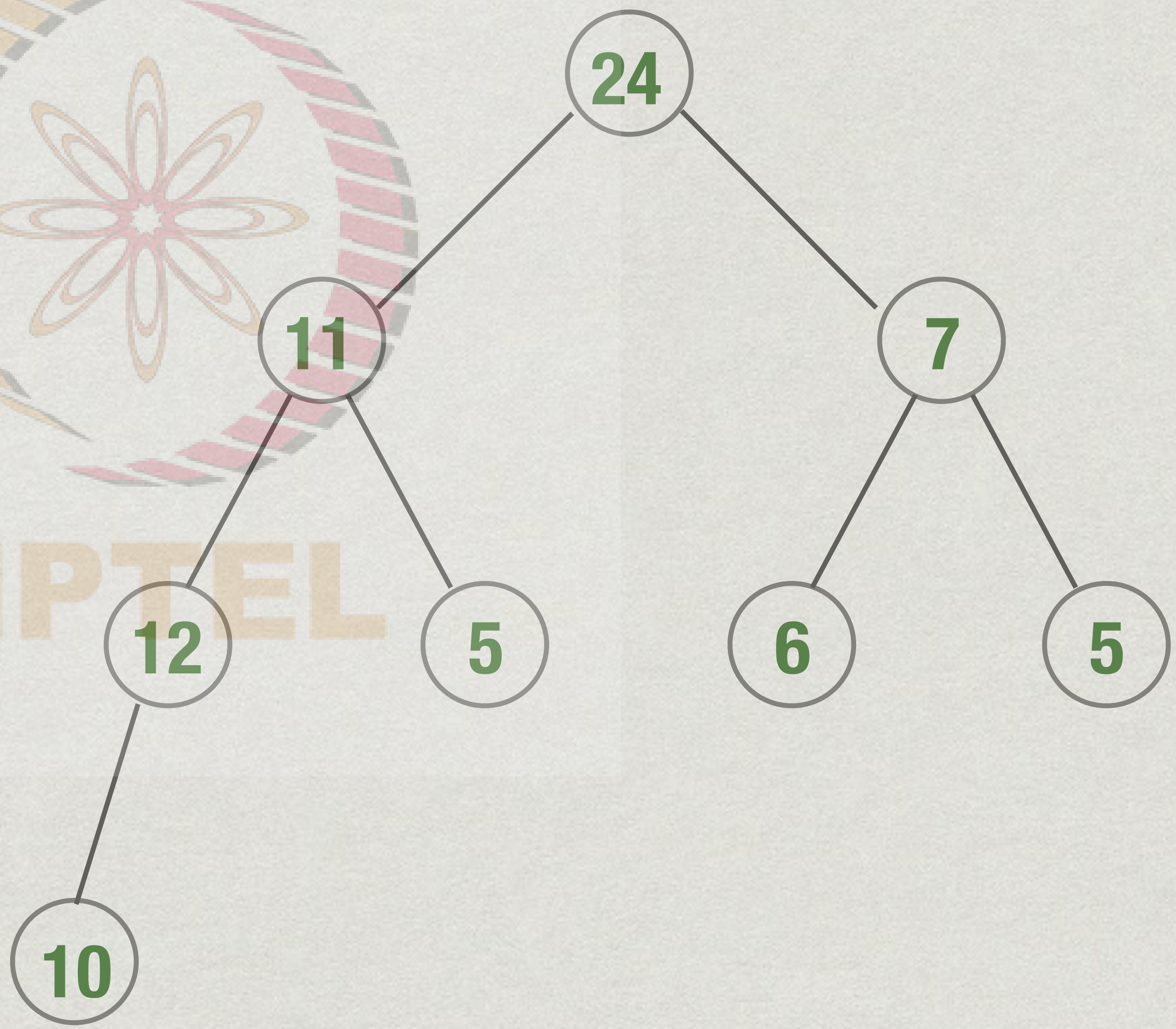

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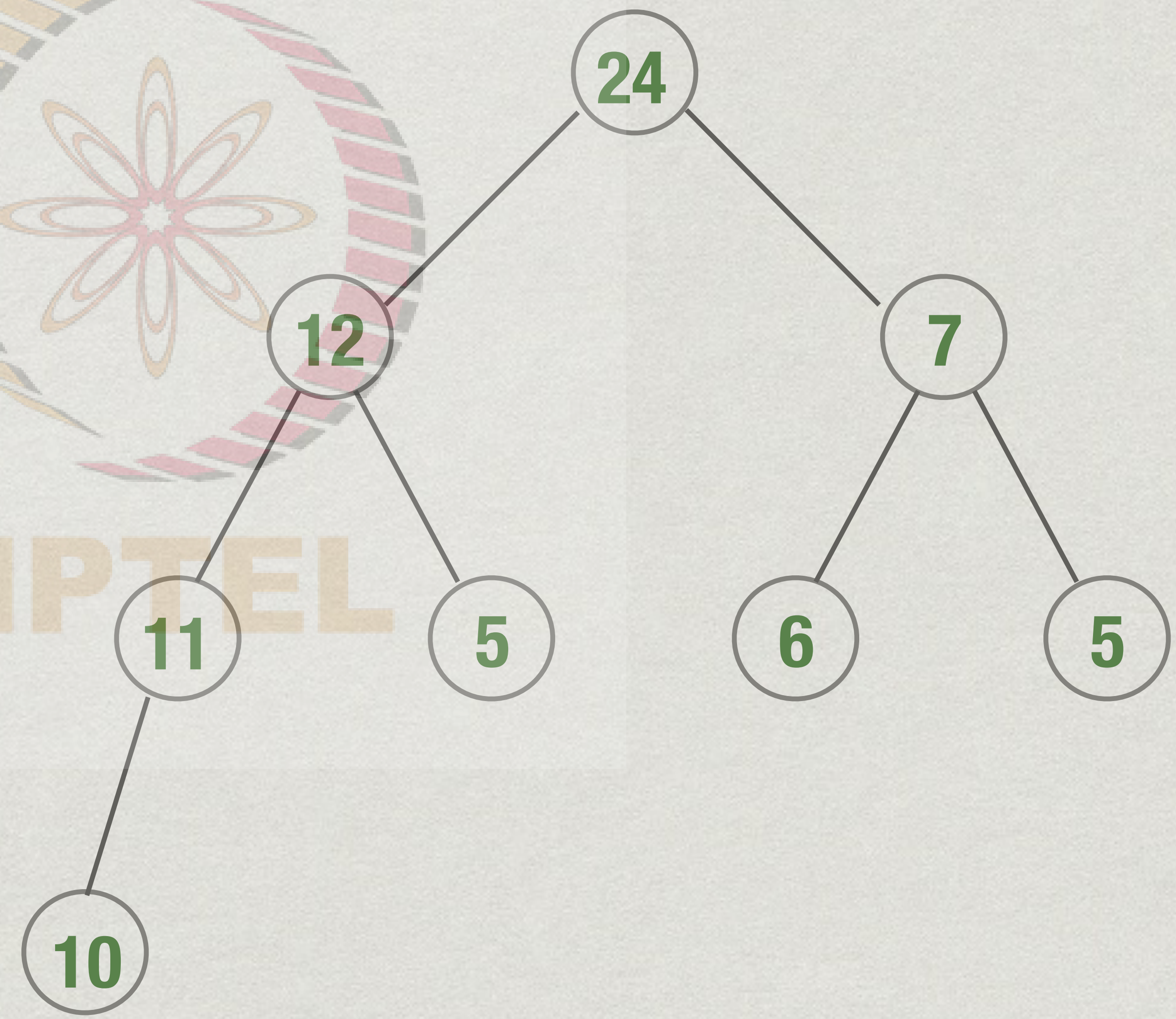
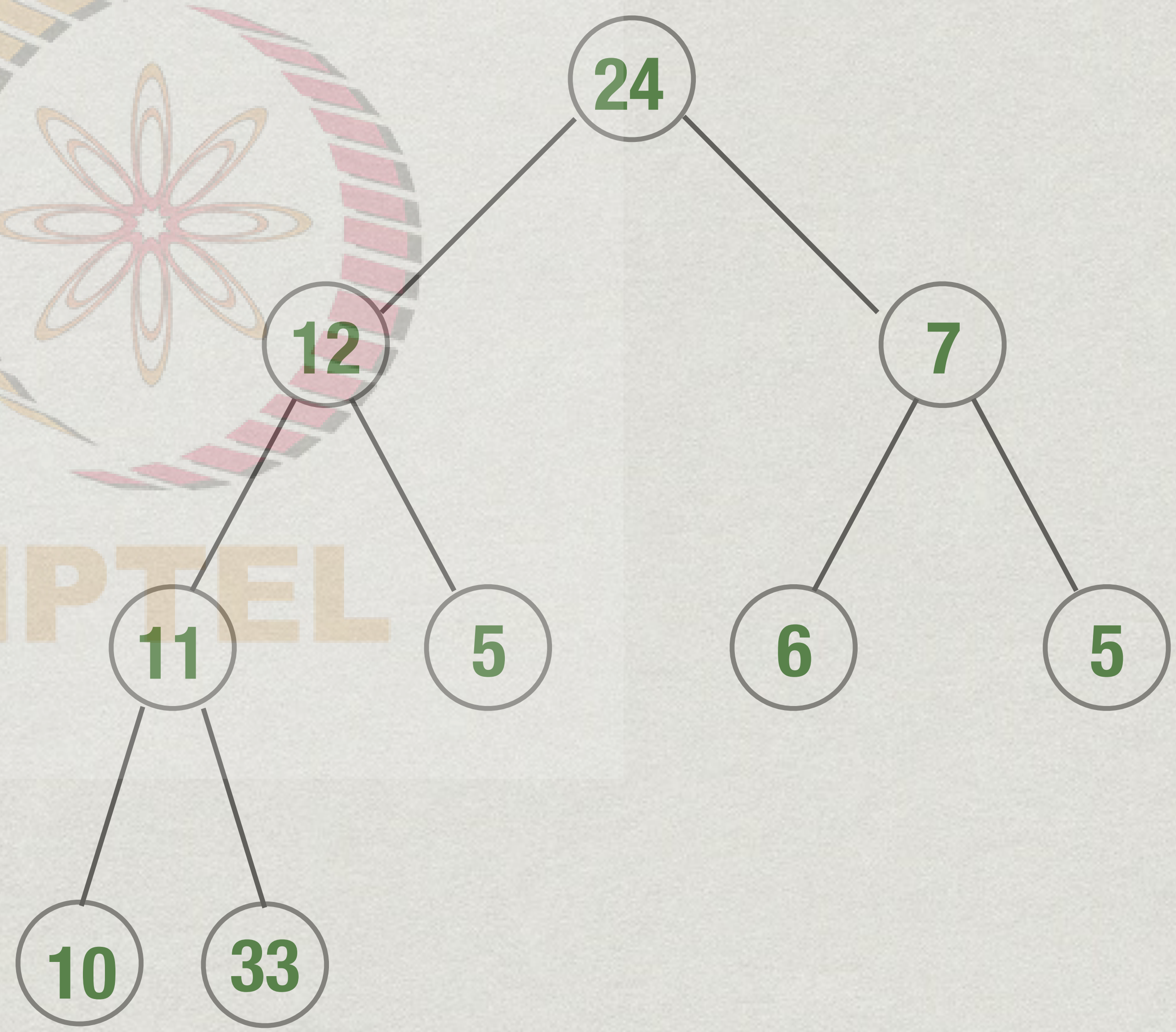


insert( )

- insert 33



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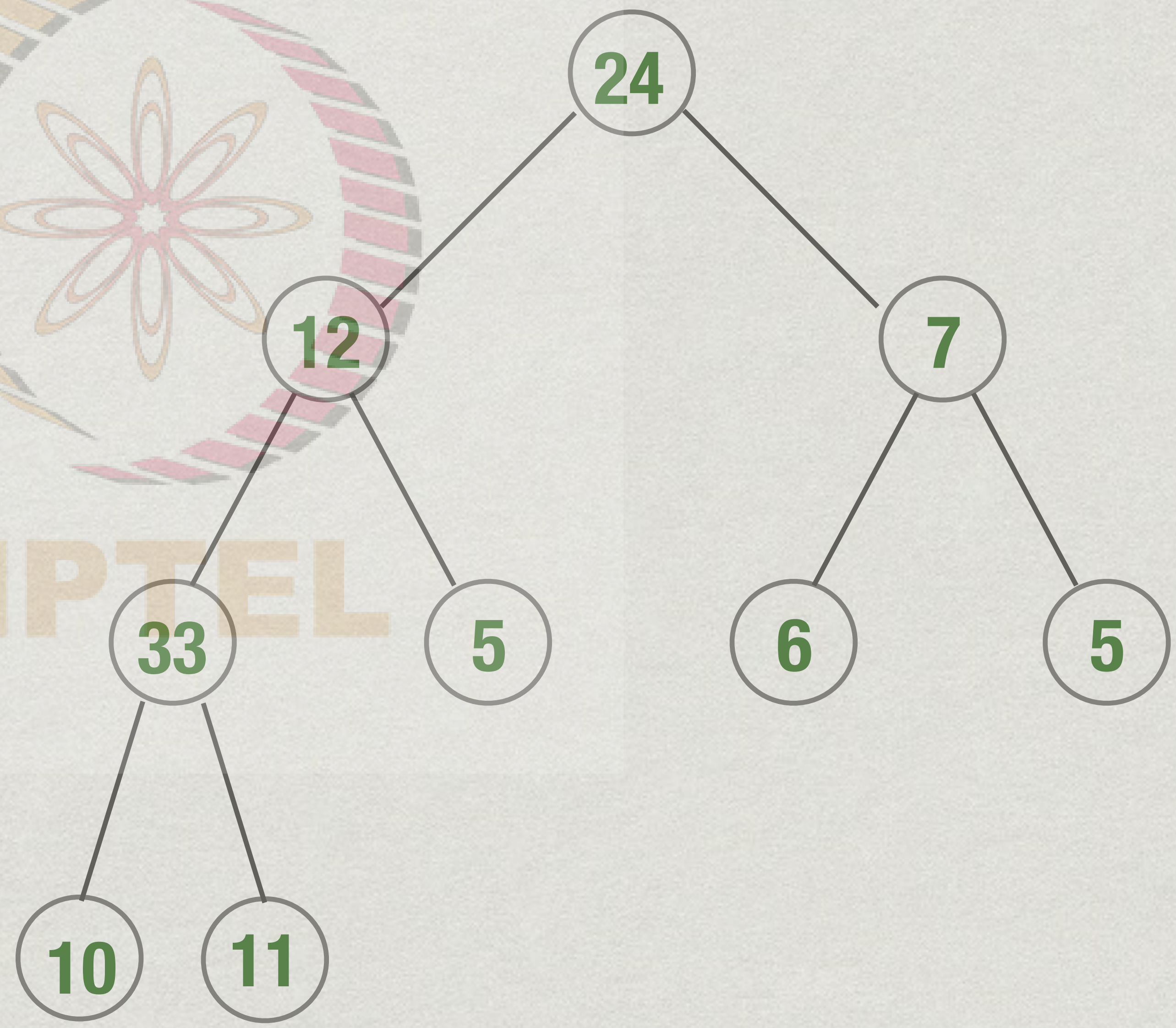

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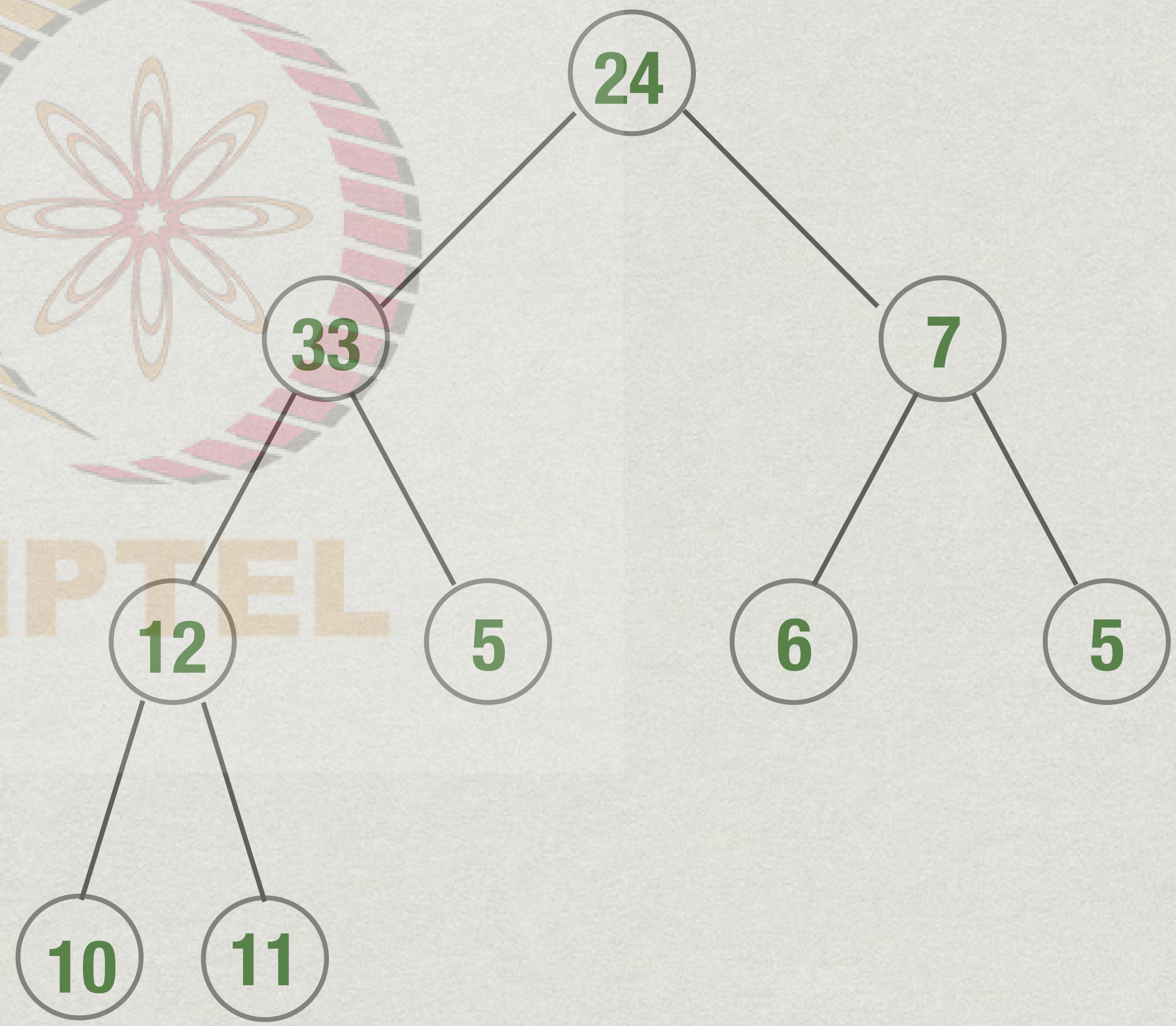

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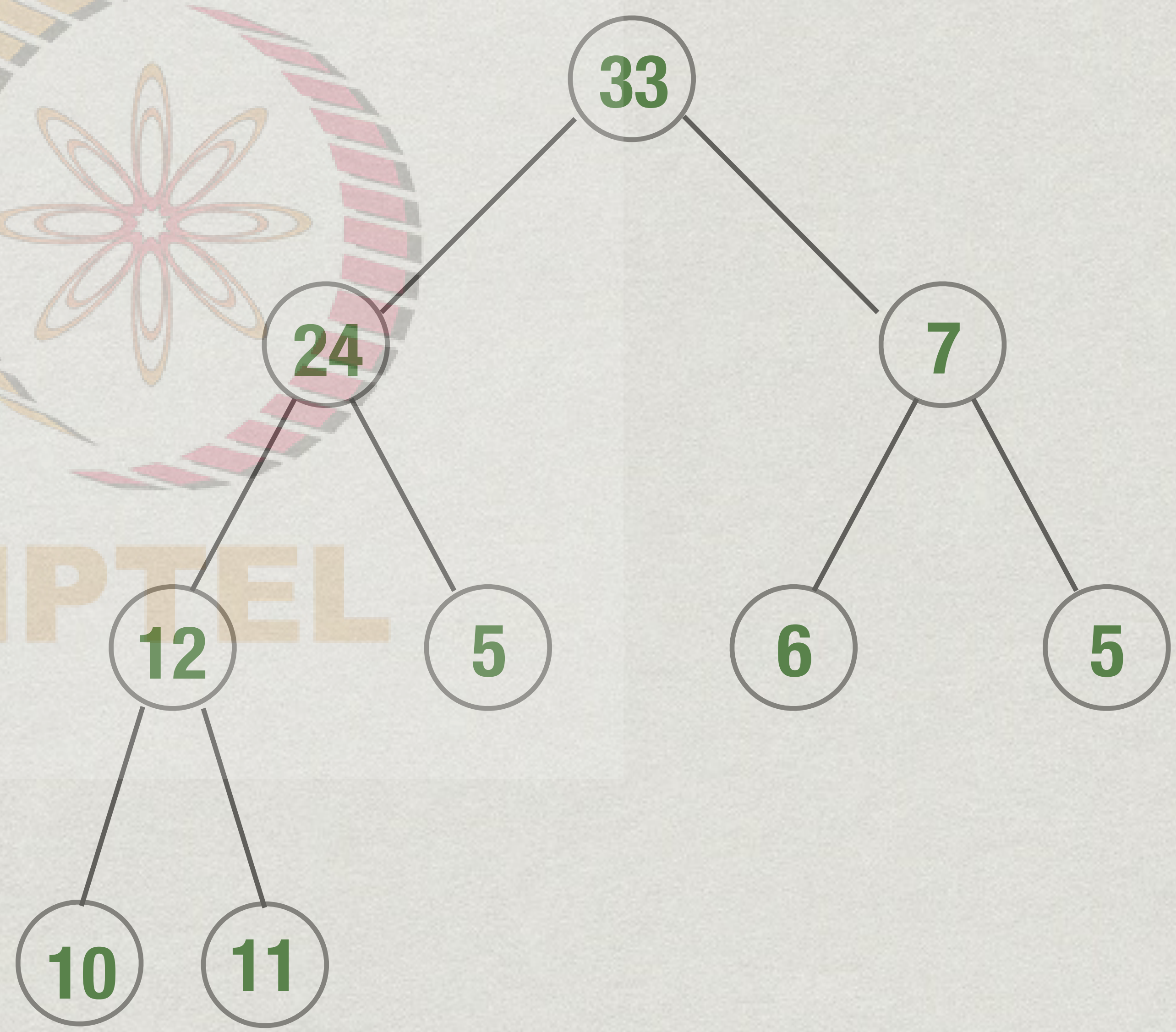
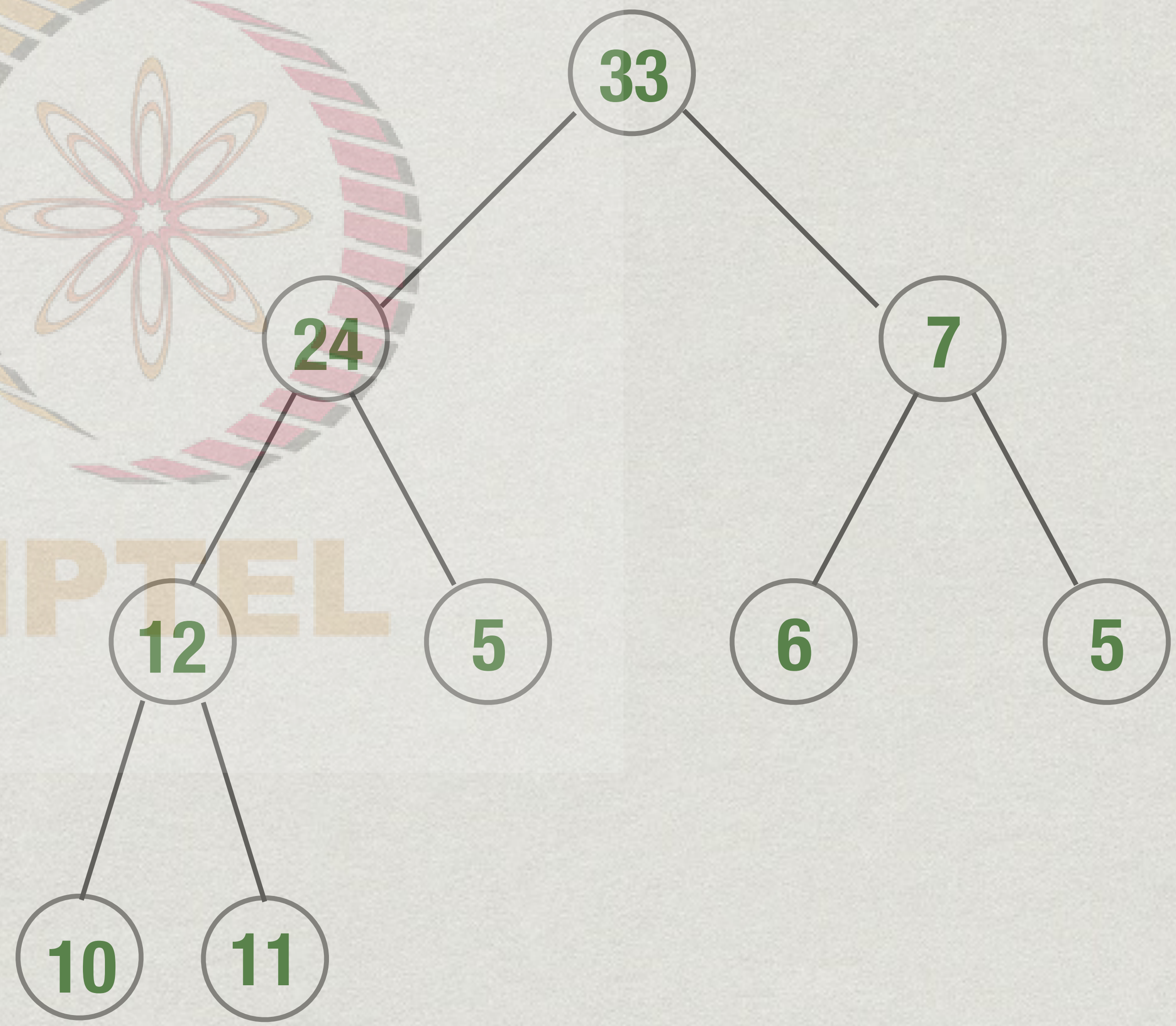


Complexity of insert( )

- Need to walk up from the leaf to the root
  - Height of the tree
- Number of nodes at level 0,1,…,i is $2^0, 2^1, \ldots, 2^i$
- K levels filled : $2^0+2^1+\ldots+2^{k-1} = 2^k - 1$ nodes
- N nodes : number of levels at most log N + 1
- insert( ) takes time O(log N)

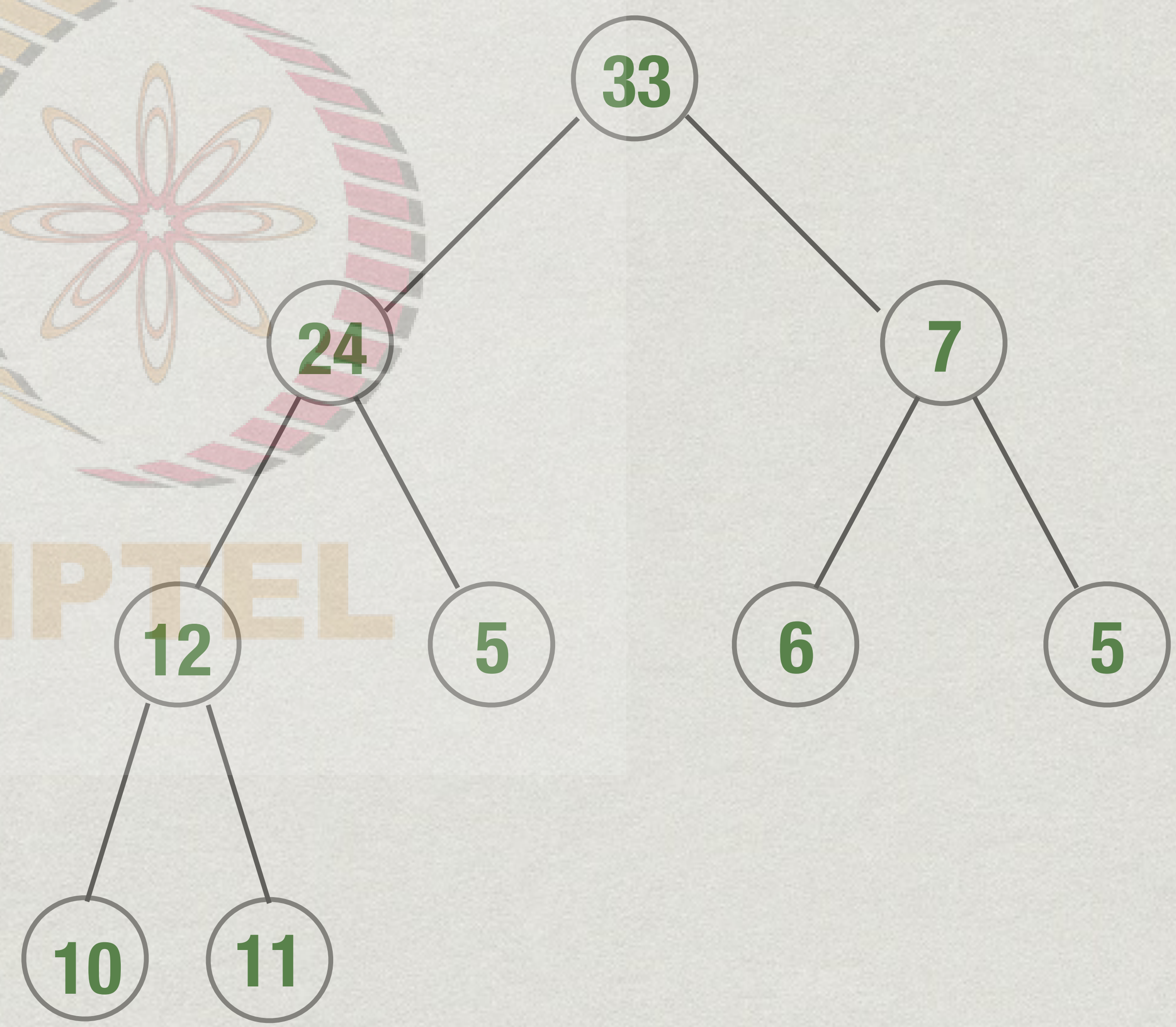
delete_max( )

- Maximum value is always at the root
  - From heap property, by induction
- How do we remove this value efficiently?



delete_max( )

- Removing maximum value creates a “hole” at the root
- Reducing one value requires deleting last node
- Move “homeless” value to root



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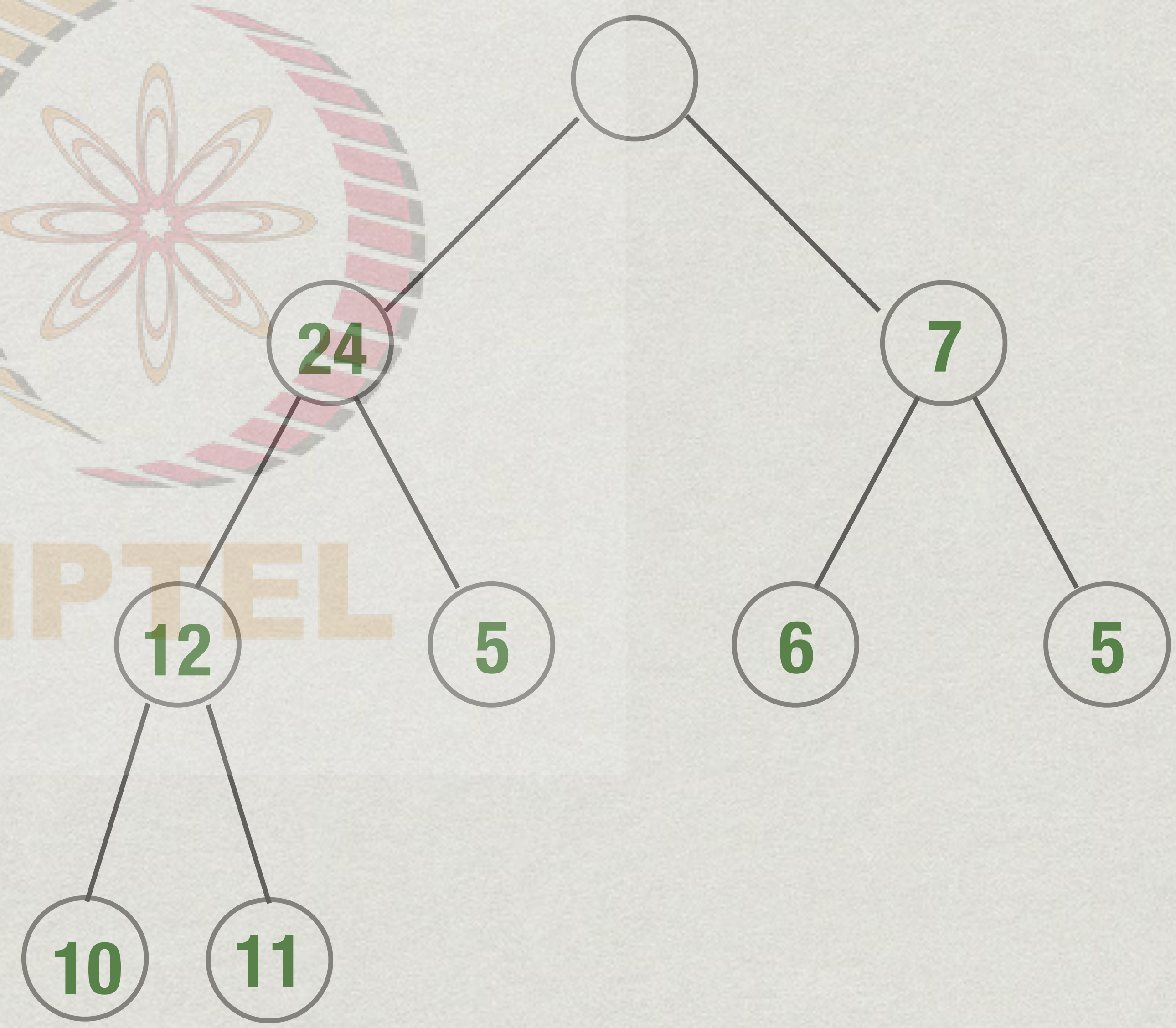

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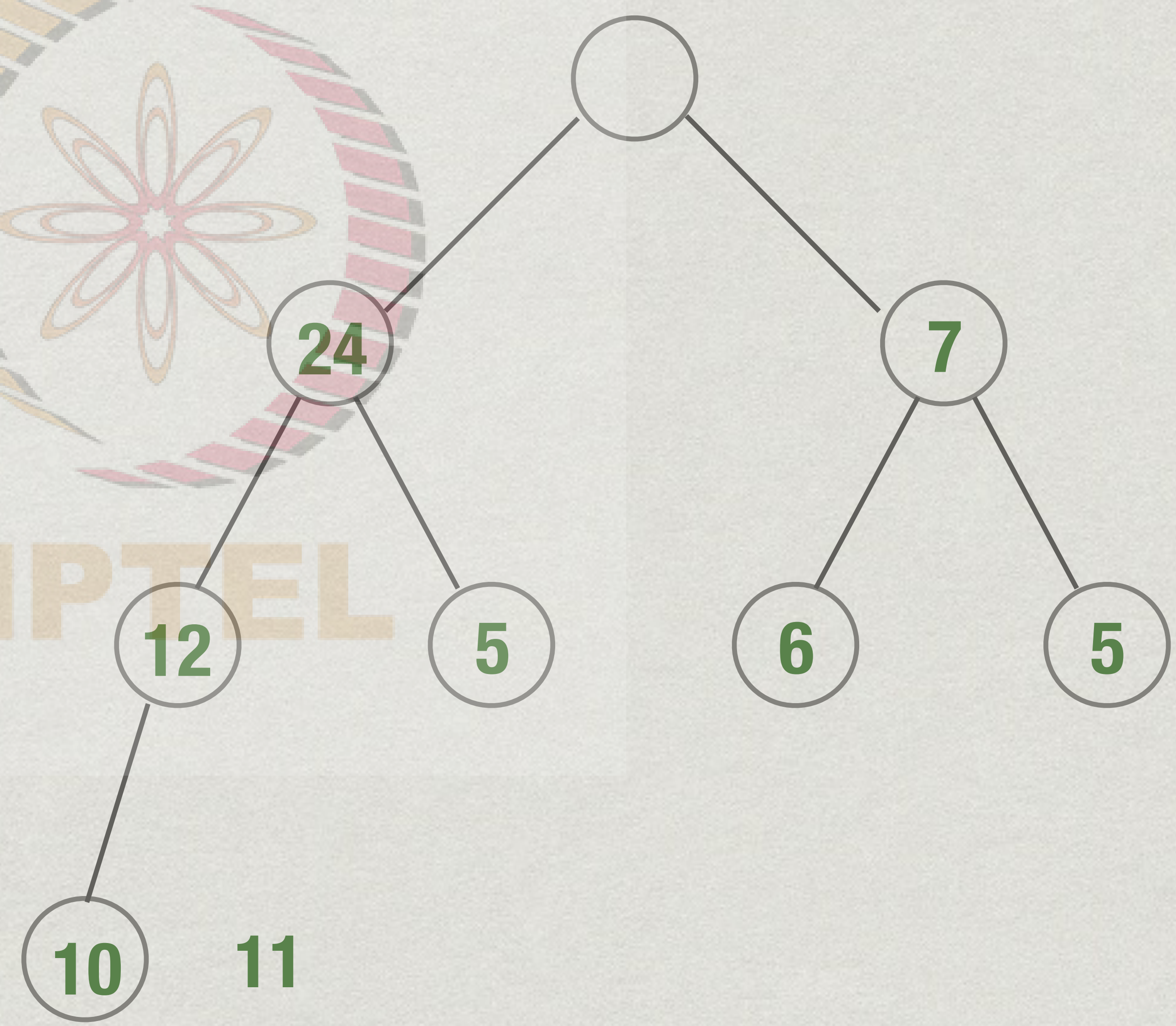

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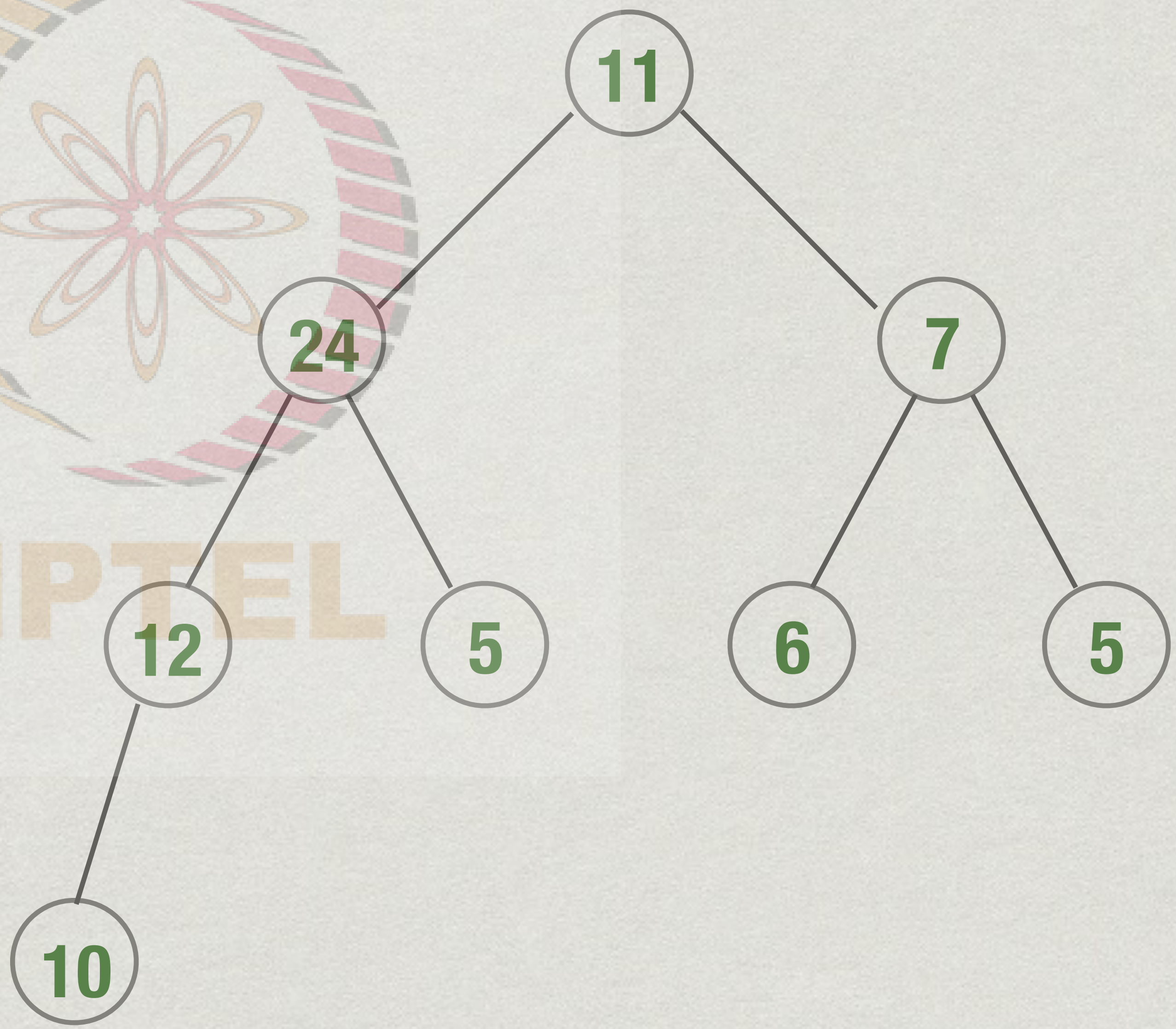
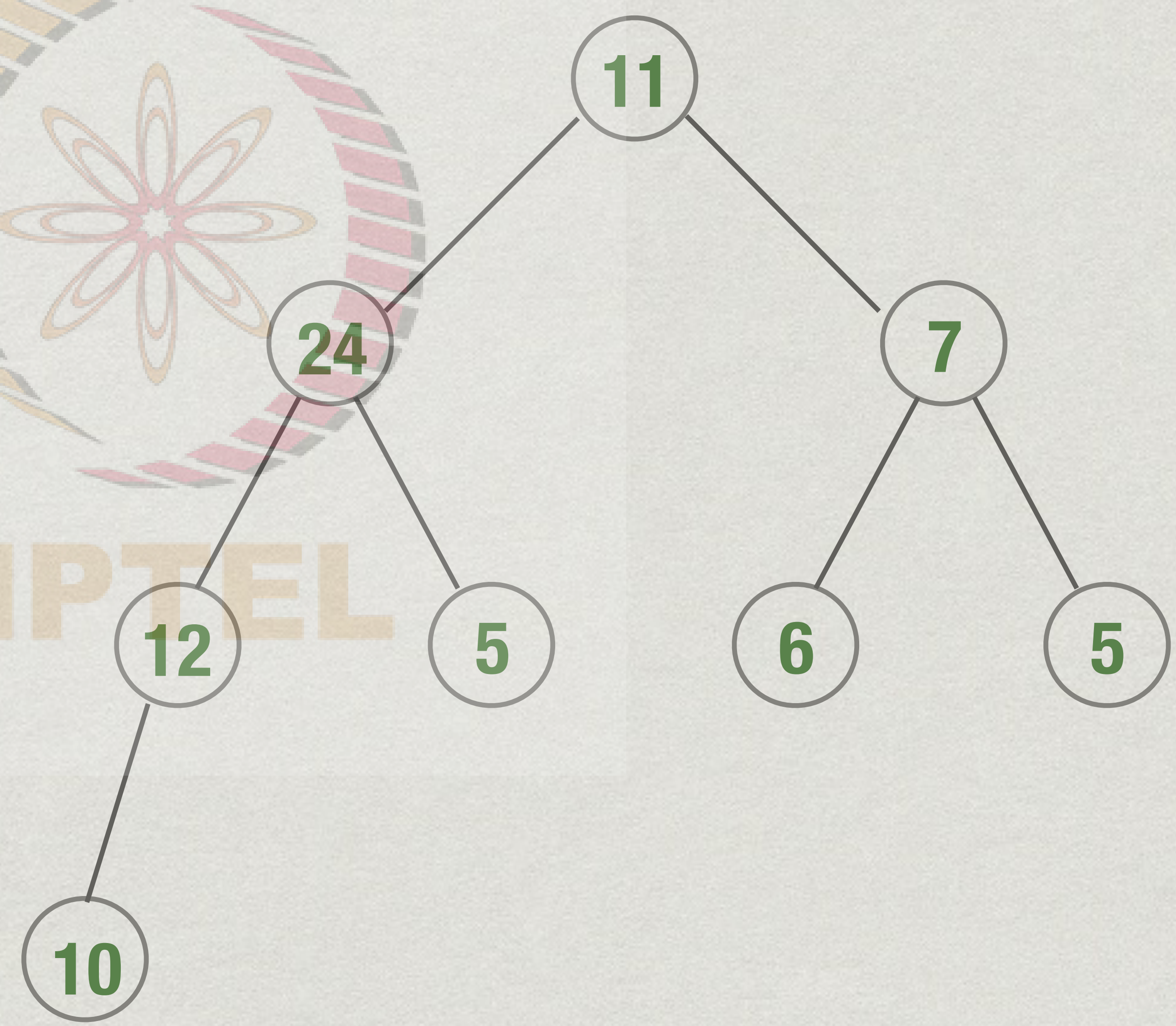


delete_max( )

- Now restore the heap property from root downwards
  - Swap with largest child
- Will follow a single path from root to leaf



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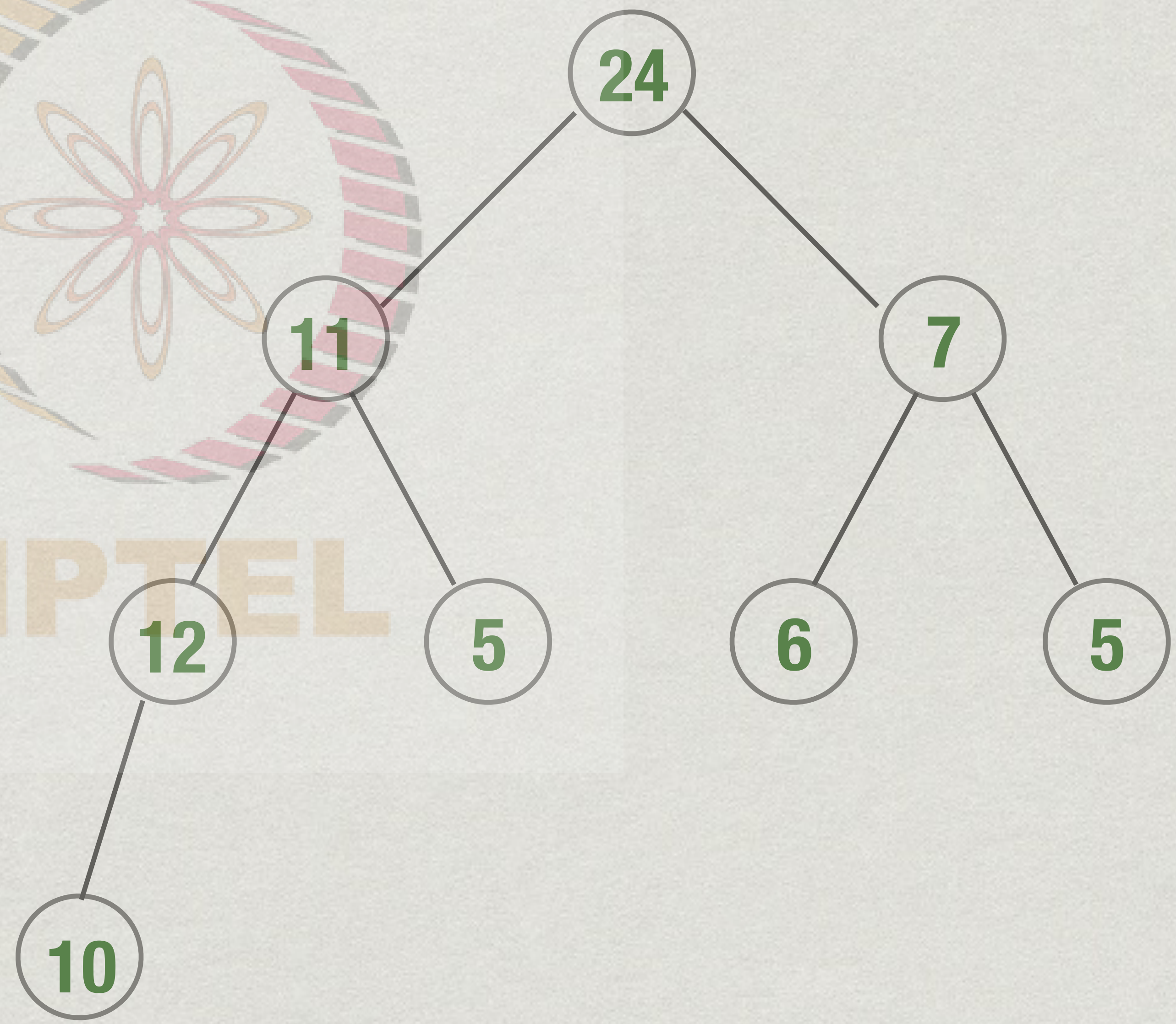

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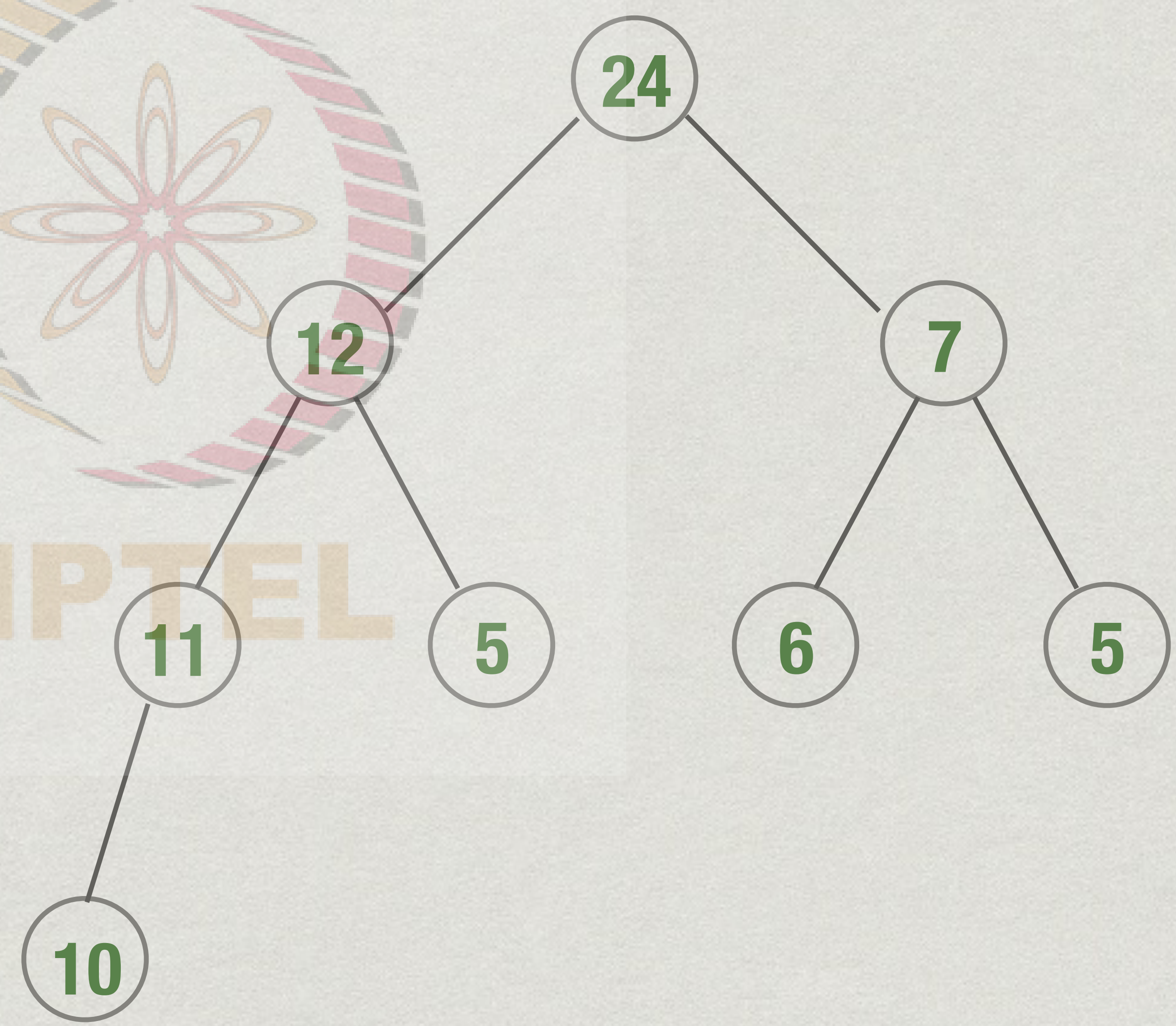
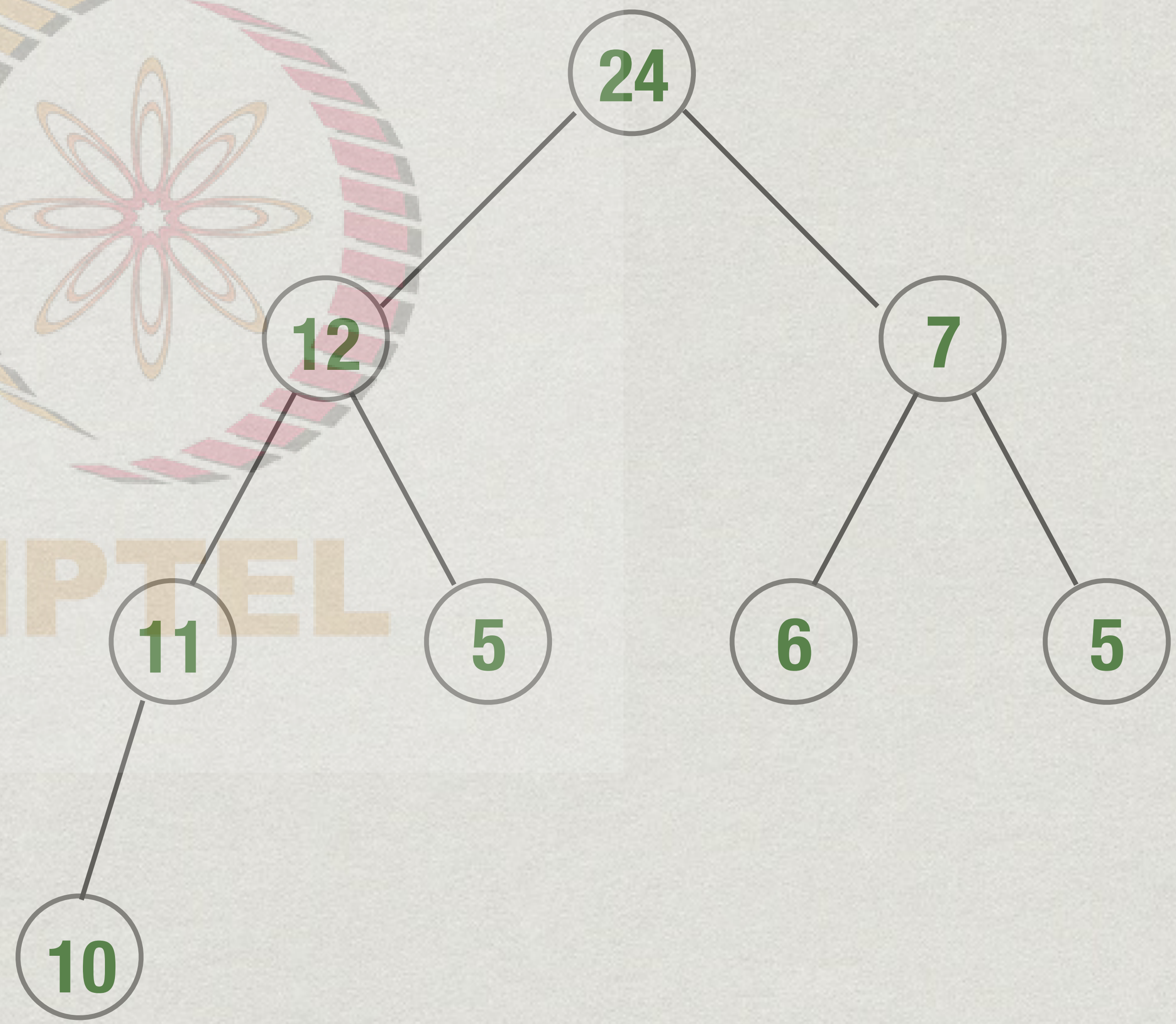


delete_max( )

- Will follow a single path from root to leaf
- Cost proportional to height of tree
- O(log N)



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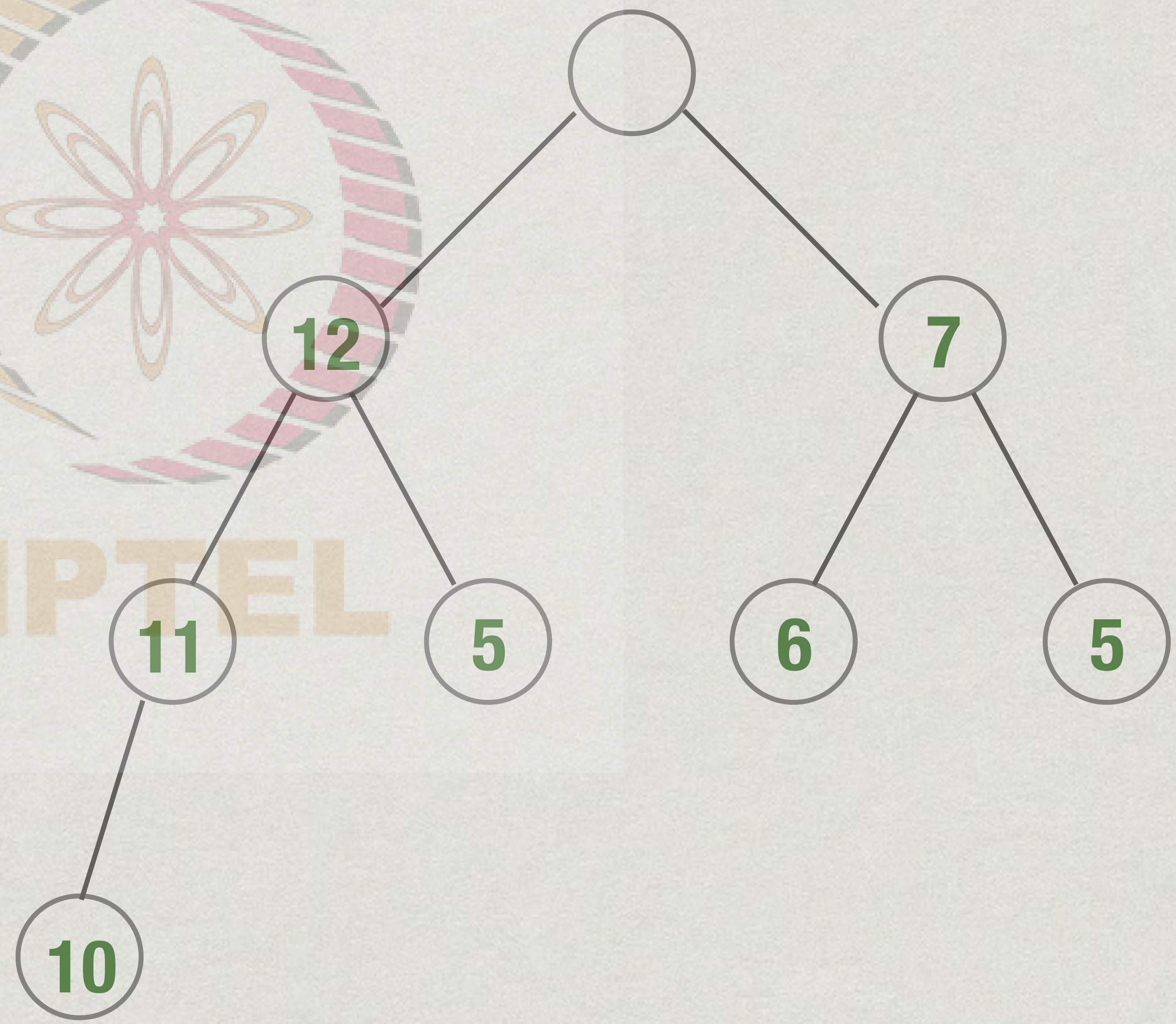

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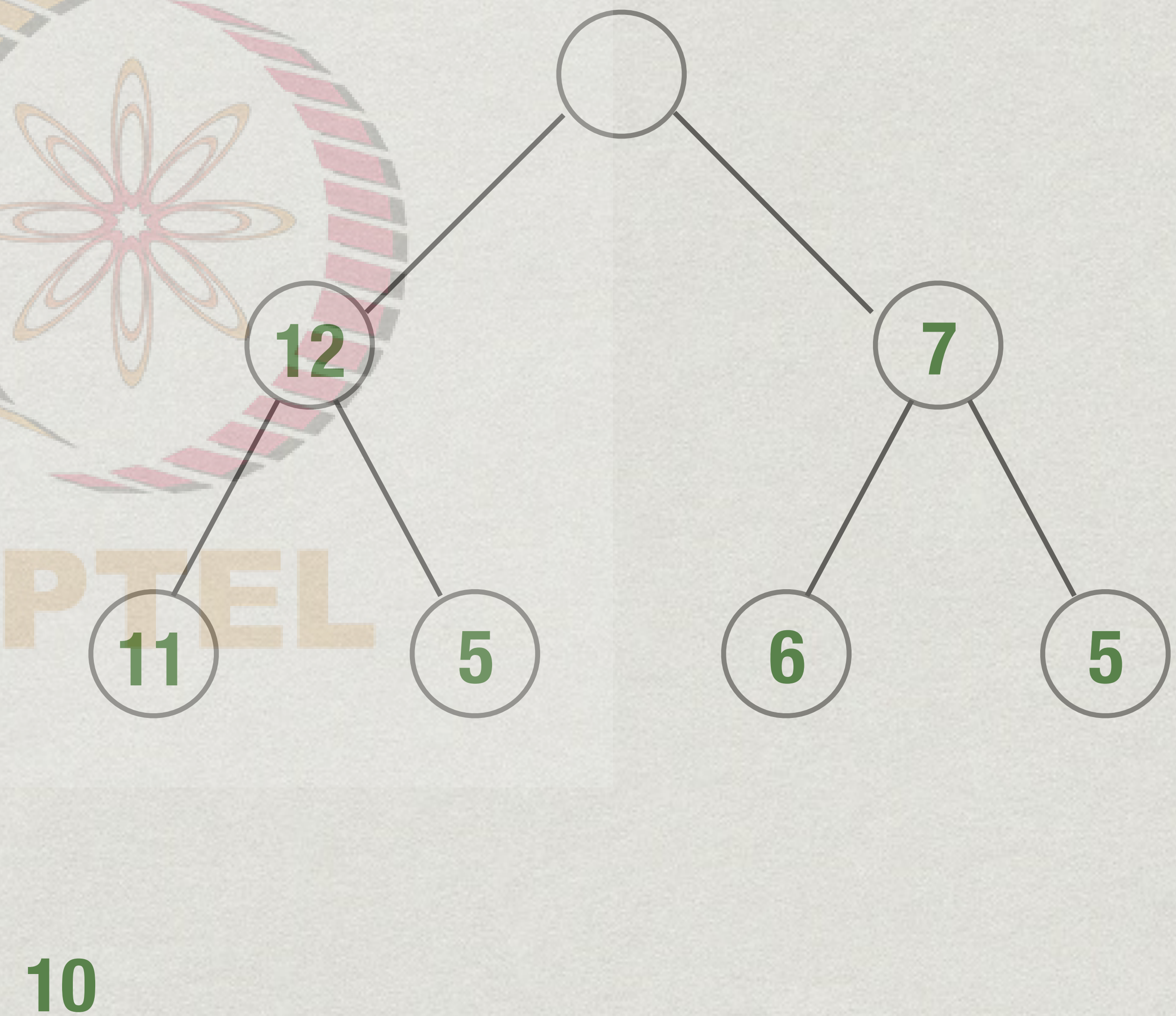

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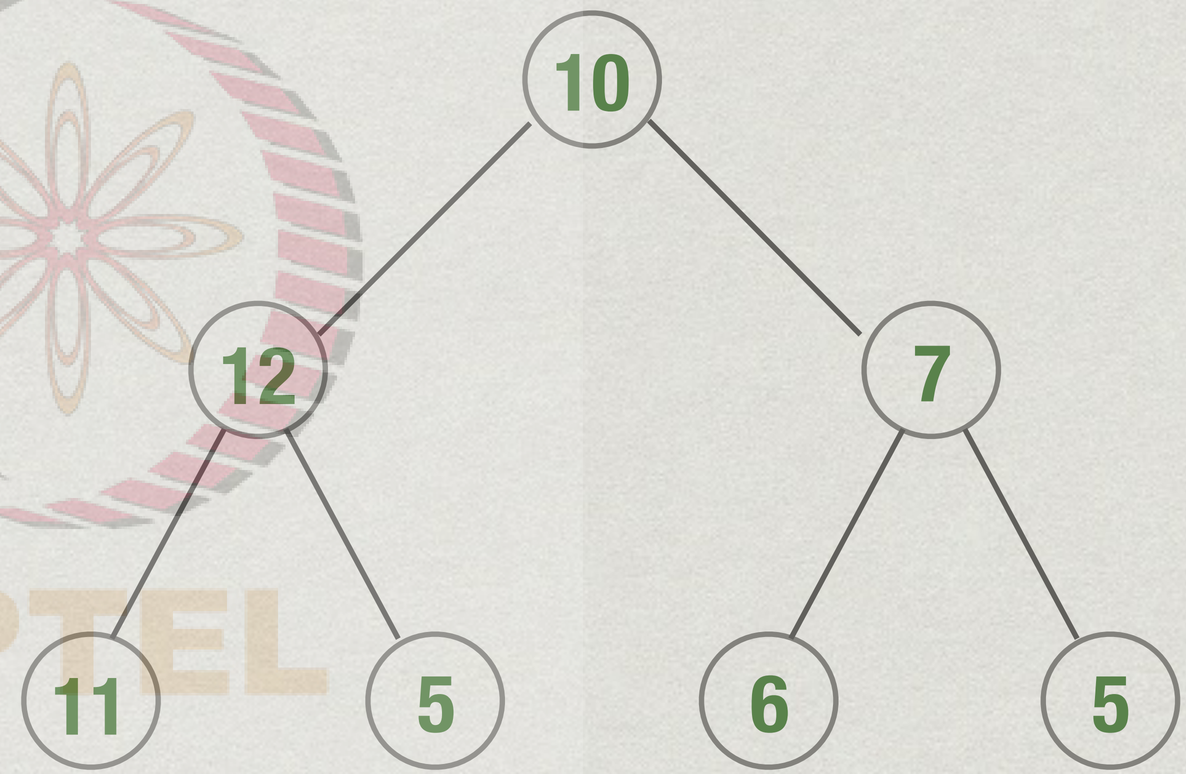

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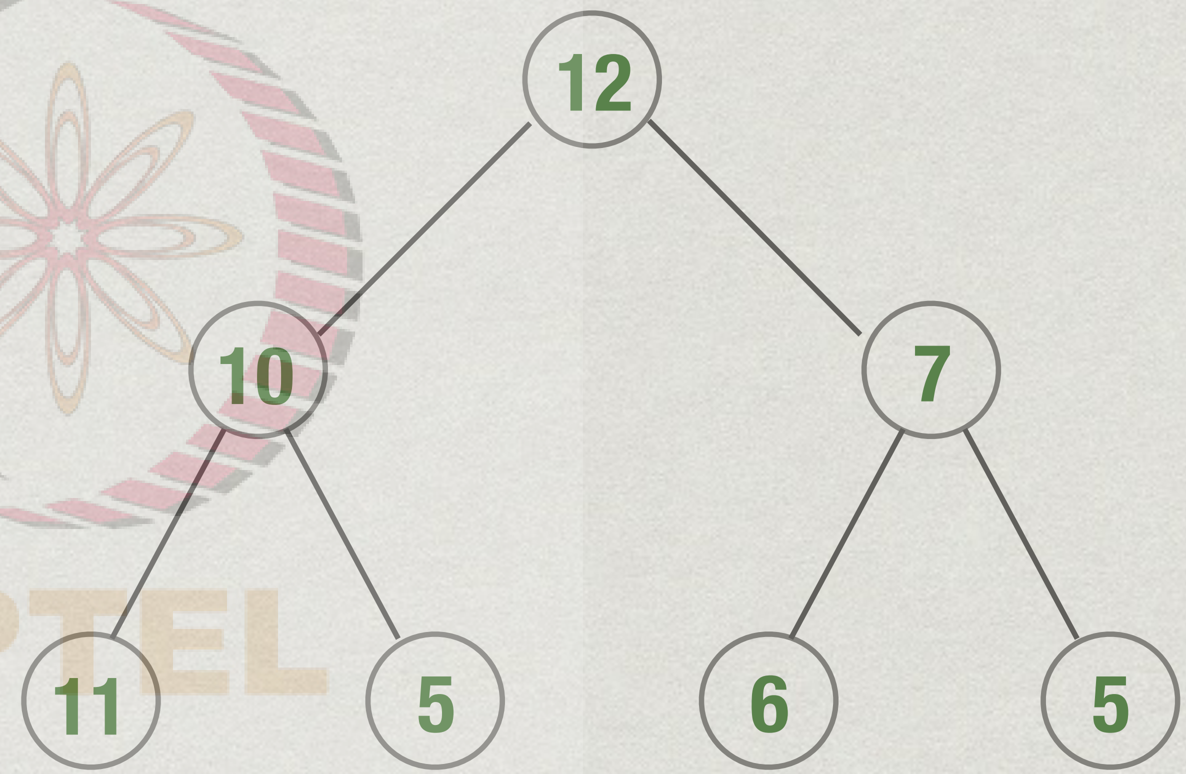

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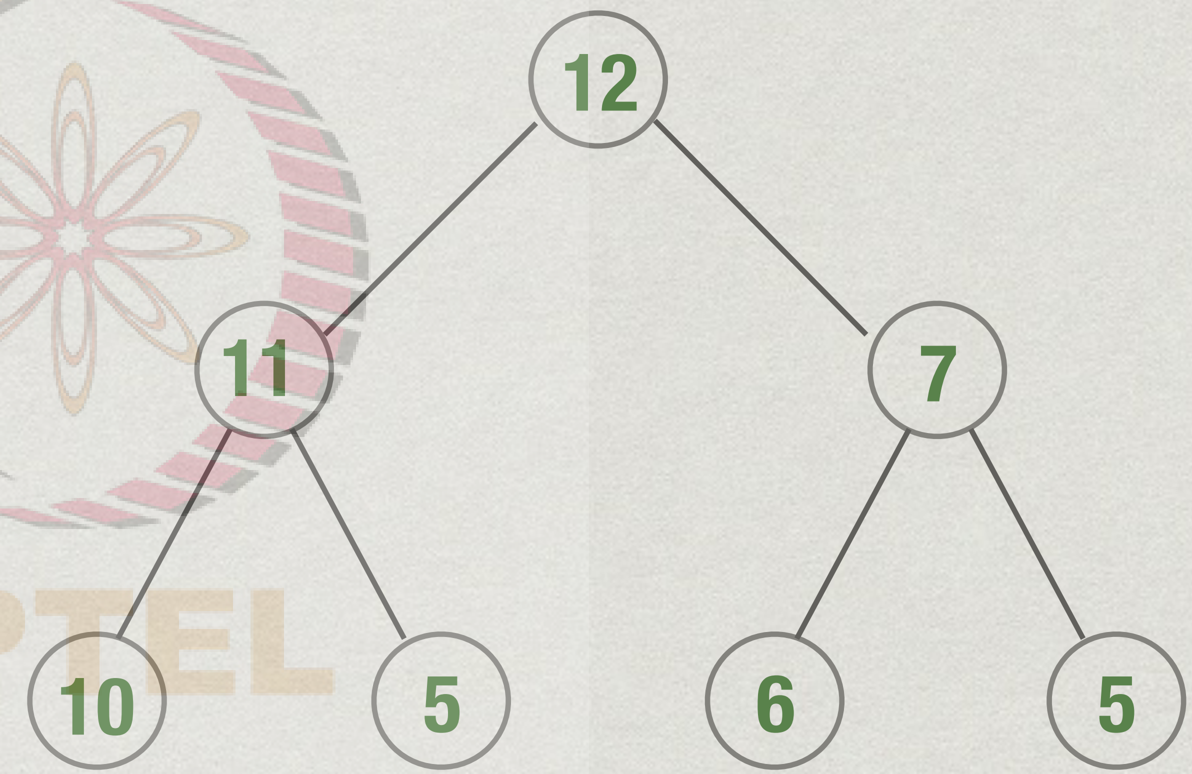
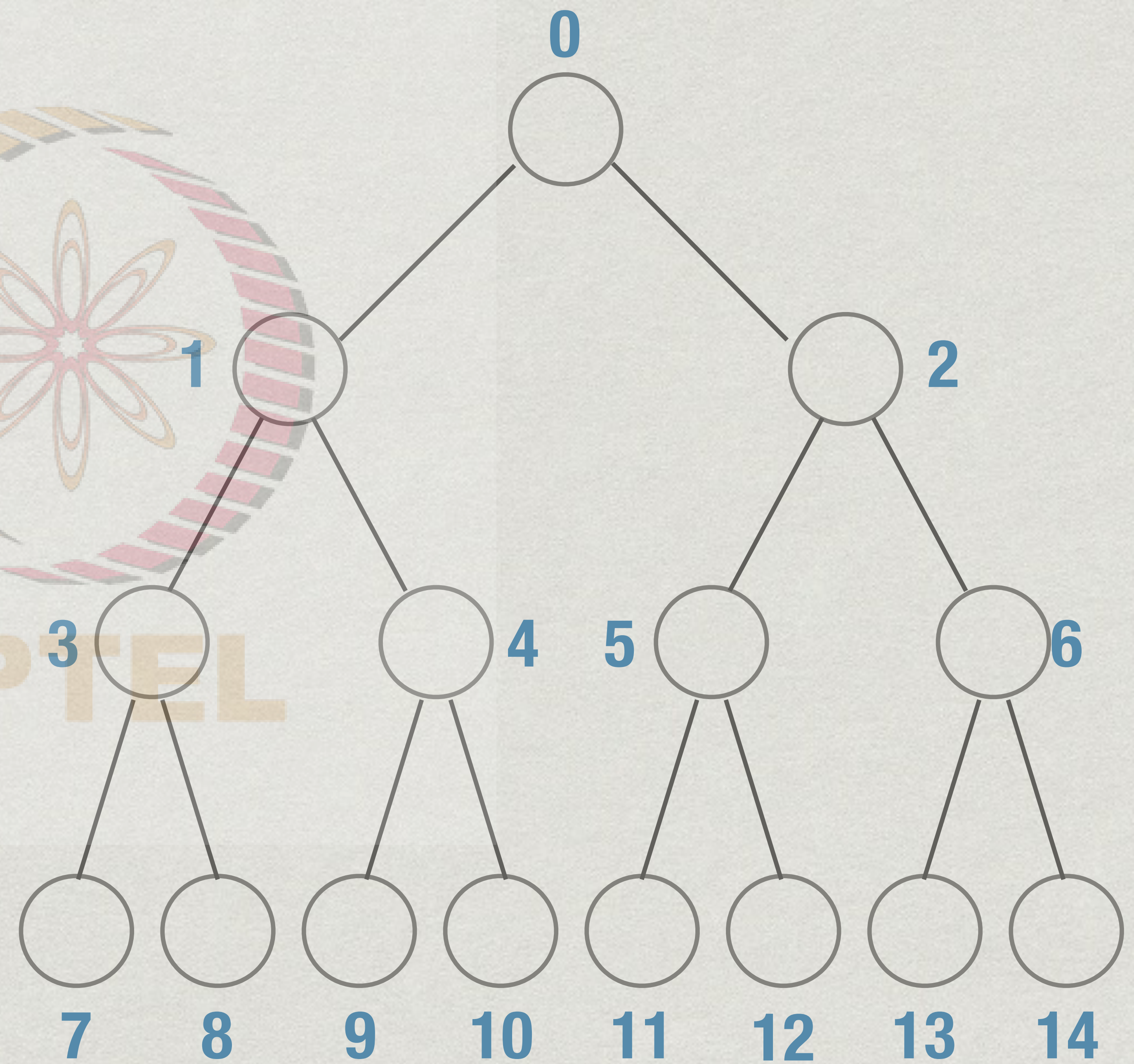


Impementing using arrays

- Number the nodes left to right, level by level
- Represent as an array H[0..N-1]
- Children of H[i] are at H[2i+1], H[2i+2]
- Parent of H[j] is at H[floor((j-1)/2)] for j > 0



Building a heap, heapify( )

- Given a list of values $[x_1,x_2,\ldots,x_N]$, build a heap
- Naive strategy
  - Start with an empty heap
  - Insert each x_j
  - Overall $O(N \log N)$

Better heapify( )

- Set up the array as $[x_1, x_2, \ldots, x_N]$
  - Leaf nodes trivially satisfy heap property
  - Second half of array is already a valid heap
- Assume leaf nodes are at level k
  - For each node at level k-1, k-2, … , 0, fix heap property
  - As we go up, the number of steps per node goes up by 1, but the number of nodes per level is halved
  - Cost turns out to be O(N) overall

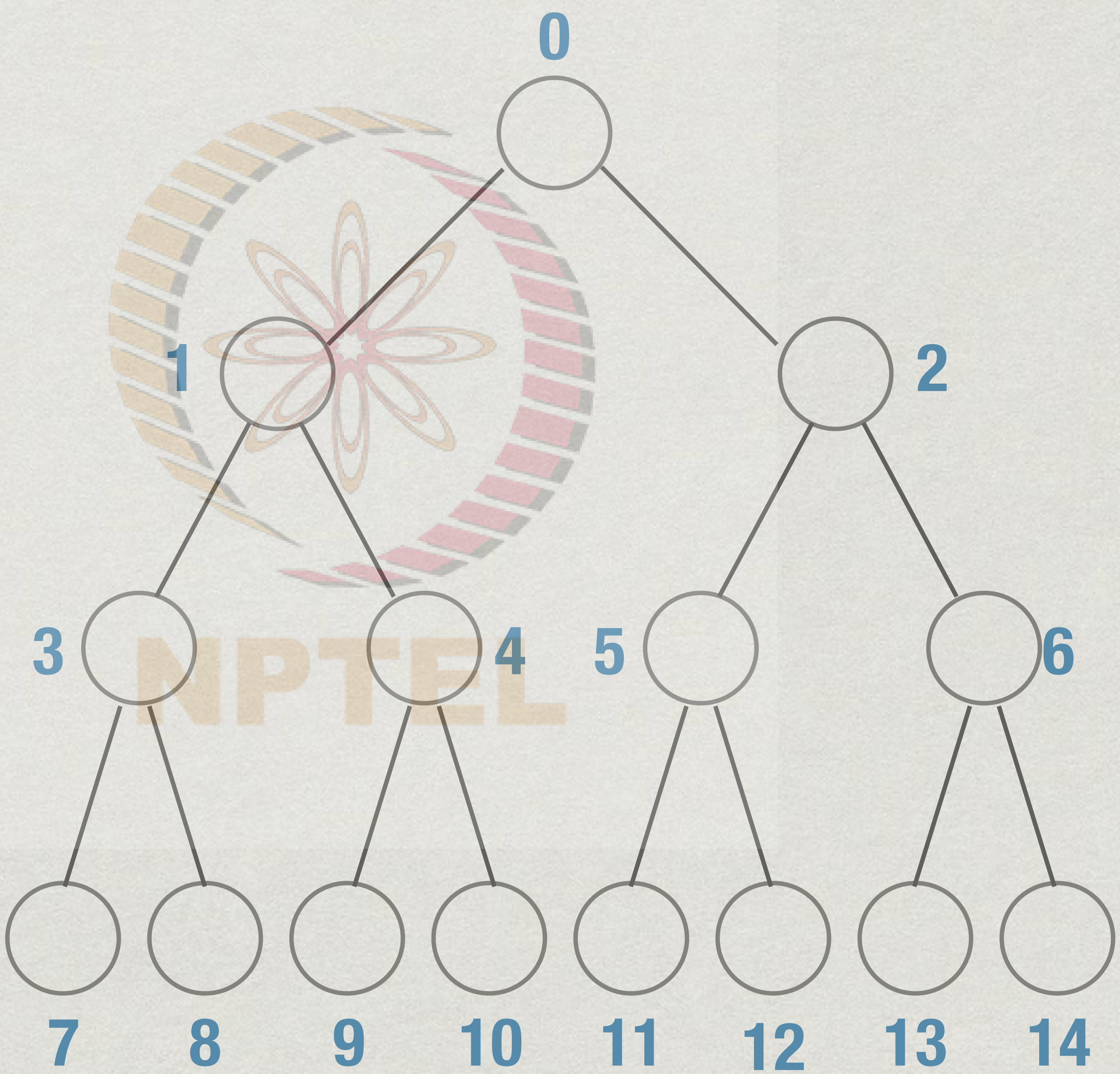
Better heapify( )

1 node,
height 3 repair

2 nodes,
height 2 repair

4 nodes,
height 1 repair

N/2 nodes
already satisfy
heap property



Heap sort

- Start with an unordered list
- Build a heap — O(n)
- Call delete_max( ) n times to extract elements in descending order — O(n log n)
- After each delete_max( ), heap shrinks by 1
    - Store maximum value at the end of current heap
    - In place O(n log n) sort

Summary

- Heaps are a tree implementation of priority queues
  - insert( ) and delete_max( ) are both O(log N)
  - heapify( ) builds a heap in O(N)
  - Tree can be manipulated easily using an array
- Can invert the heap condition
  - Each node is smaller than its children
  - Min-heap, for insert( ), delete_min( )