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Problem Set 4

Problem 1. [15 points] Let $G = (V, E)$ be a graph. A *matching* in G is a set $M \subset E$ such that no two edges in M are incident on a common vertex.

Let M_1, M_2 be two matchings of G. Consider the new graph $G' = (V, M_1 \cup M_2)$ (i.e. on the same vertex set, whose edges consist of all the edges that appear in either M_1 or M_2). Show that G' is bipartite.

Helpful definition: A *connected component* is a subgraph of a graph consisting of some vertex and every node and edge that is connected to that vertex.

Problem 2. [20 points] Let $G = (V, E)$ be a graph. Recall that the *degree* of a vertex $v \in V$, denoted d_v, is the number of vertices w such that there is an edge between v and w.

(a) [10 pts] Prove that

$$2|E| = \sum_{v \in V} d_v.$$

(b) [5 pts] At a 6.042 ice cream study session (where the ice cream is plentiful and it helps you study too) 111 students showed up. During the session, some students shook hands with each other (everybody being happy and content with the ice-cream and all). Turns out that the University of Chicago did another spectacular study here, and counted that each student shook hands with exactly 17 other students. Can you debunk this too?

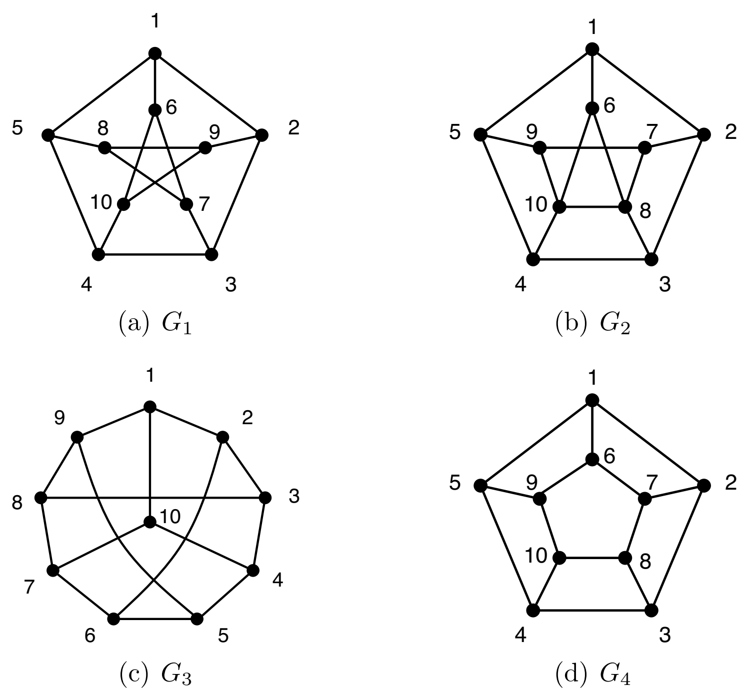
(c) [5 pts] And on a more dull note, how many edges does K_n, the complete graph on n vertices, have?

Problem 3. [15 points] Two graphs are isomorphic if they are the same up to a relabeling of their vertices (see Definition 5.1.3 in the book). A property of a graph is said to be *preserved under isomorphism* if whenever G has that property, every graph isomorphic to G also has that property. For example, the property of having five vertices is preserved under isomorphism: if G has five vertices then every graph isomorphic to G also has five vertices.

(a) [5 pts] Some properties of a simple graph, G, are described below. Which of these properties is *preserved under isomorphism*?

1. G has an even number of vertices.
2. None of the vertices of G is an even integer.
3. G has a vertex of degree 3.
4. G has exactly one vertex of degree 3.

(b) [10 pts] Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).



(a) G_1 (b) G_2 (c) G_3 (d) G_4

Figure 1: Which graphs are isomorphic?

Problem 4. [15 points] Recall that a ***coloring*** of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A k***-coloring*** is a coloring that uses at most k colors.

False Claim. *Let G be a (simple) graph with maximum degree at most k. If G also has a vertex of degree less than k, then G is k-colorable.*

(a) [5 pts] Give a counterexample to the False Claim when $k = 2$.

(b) [10 pts] Consider the following proof of the False Claim:

Proof. Proof by induction on the number n of vertices:

Induction hypothesis: $P(n)$ is defined to be: Let G be a graph with n vertices and maximum degree a most k. If G also has a vertex of degree less than k, then G is k-colorable.

Base case: (n=1) G has only one vertex and so is 1-colorable. So $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices and maximum degree at most k. Also, suppose G_{n+1} has a vertex, v, of degree less than k. We need only prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Removing v reduces the degree of all vertices adjacent to v by 1. So in G_n, each of these vertices has degree less than k. Also the maximum degree of G_n remains at most k. So G_n satisfies the conditions of the induction hypothesis $P(n)$. We conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. □

Identify the exact sentence where the proof goes wrong.

Problem 5. [15 points] Prove or disprove the following claim: for some $n \geq 3$ (n boys and n girls, for a total of $2n$ people), there exists a set of boys' and girls' preferences such that every dating arrangement is stable.

Problem 6. [20 points]

Let $(s_1, s_2, ..., s_n)$ be an arbitrarily distributed sequence of the number $1, 2, ..., n-1, n$. For instance, for $n = 5$, one arbitrary sequence could be $(5, 3, 4, 2, 1)$.

Define the graph G=(V,E) as follows:

1. $V = \{v_1, v_2, ..., v_n\}$
2. $e = (v_i, v_j) \in E$ if either:
   (a) $j = i + 1$, for $1 \leq i \leq n - 1$
   (b) $i = s_k$, and $j = s_{k+1}$ for $1 <= k <= n - 1$

(a) [10 pts] Prove that this graph is 4-colorable for any $(s_1, s_2, ..., s_n)$.

Hint: First show that a line graph is 2-colorable. Note that a line graph is defined as follows: The n-node graph containing $n-1$ edges in sequence is known as the line graph L_n.

(b) [10 pts] Suppose $(s_1, s_2, ..., s_n) = (1, a_1, 3, a_2, 5, a_3, ...)$ where $a_1, a2....$ is an arbitrary distributed sequence of the even numbers in $1, ..., n-1$. Prove that the resulting graph is 2-colorable.

Problem Set 4

P1: Obviously, we only have to prove for a connected component.

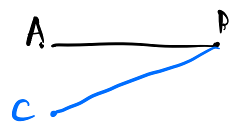
we can piece different c-c together

for one n-node connected component, prove this by induction

$n=2$.  ✓

$n \to n+1$:

- if there is a node A degree is 1



A – B connected in M_1, B–C connected in M_2

consider all nodes apart from A, B

can be divided into 2 rows.

put A in C's, B in no-C's ✓

- if every node has degree 2.

We can put them in one circle

every node is connected to either sides, one in M_1, one in M_2

there must be even edges $\Rightarrow$ there are even points

choose one, mark 1^{th}. in clock-wise mark 2^{th} ... n^{th}

odd ones in row 1. even ones in row 2 $\Rightarrow$ bipartite ✓ □.

P2: (a) d_v means the degree of v

consider that all edges are counted exactly twice

so $\sum_{v \in V} d_v = 2|E|$

(b) If Chicago's study is correct. in this case.

$\sum_{v \in V} d_v = 111 \times 17$ is odd but $2|E|$ must be even ※ □.

(c) by definition. $|E| = C_n^2 = \frac{1}{2} n(n-1)$

P3: (a) all yes

(b) ① $(G_1, G_2), (G_2, G_3), (G_2, G_4)$ are not isomorphic.

the biggest degree in G_1, G_3, G_4 are all 3. while G_2 is 4 (points 8, 10)

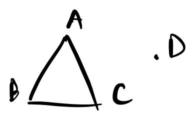
② $(G_1, G_4), (G_3, G_4)$ are not isomorphic.

because the smallest loops (with the least vertices) have different vertices.

③ (G_1, G_3) are isomorphic. $G_3 \xrightarrow{f} G_1$

f: $1 \to 1$ $\quad 6 \to 7$
$2 \to 6$ $\quad 7 \to 3$
$3 \to 10$ $\quad 8 \to 4$
$4 \to 9$ $\quad 9 \to 5$
$5 \to 8$ $\quad 10 \to 2$

P4: (a)



if colored in 2 colors

A, B, C must have 2 colored the same ※

(b) "G_n satisfies the conditions of the induction hypothesis" X

because the vertex V can have degree 0, so that no vertex is adjacent.

P5: impossible: if preference is small, means like

all the preference of the edges (both sides) add together is $2n \cdot \frac{1}{2}(n+1)n = n^2(n+1)$

so the average of the sum of one edge is $\frac{n^2(n+1)}{n^2} = n+1$

so there must be an edge its sum $\leq n+1$

① if all edges have sum $n+1 \Rightarrow 2-(n-1), \ldots$

or

② if there is one's sum $\leq n$

$\Rightarrow$ there must be an edge whose preferences contain no n.

$A \overset{k_1 \qquad k_2}{\textbf{———}} B$ $(k_1, k_2 < n)$

Consider a dating arrangement where $A \overset{n}{\text{———}} B'$

$A' \overset{n}{\text{———}} B$

then $A - B$ are definitely a rogue couple $\square$.

P6:

(a) firstly, line the vertices in the order

$v_1 — v_2 — v_3 — \cdots — v_{n-1} — v_n$

can be 2-colored as: 1, 0

then, line in the order:

$s_1 — s_2 — s_3 \cdots — s_{n-1} — s_n$

can be 2-colored as 1, 0

so that each node is noted in a 2-bit binary code.

11, 10, 01, 00 meaning 4-colored.

adjacent vertices must have at least one character different

so their colors are different $\square$.

(b) Odd in one color, even in another 2-colored $\square$.