Spring 2024

# 18.200 Homework 1

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**Instructions:** Indicate your recitation and collaborators. We encourage collaboration (you could write for example: 'John Doe, my recitation groupmates, and Peter's office hours'), or state that you worked only on your own. In any case, you must write up your own proofs.

For all problems marked as writing problems, use the homework template provided on this website; you may use this template for all problems if you like. The LaTeX template contains writing guidance, to see it, you will need to download the file and view the tex code. Your grade on the writing problems will be based on both the math and the quality of the writing, so be sure to read and follow the writing guidance provided in the template and in recitation.

1. (10 points each) Consider **two** of the following statements (at your choice) from the recitation and, for each, write the statement as a theorem and prove it. Your proofs should be easily understandable by another 18.200 student.
   (a) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$.
   (b) Prove that any set of 41 integers between 1 and 80 has the property that one of them divides another one.
   (c) Let $a$ and $b$ be integers. Show that $a^2 + b^2$ cannot have a remainder of 3 when divided by 4.
   (d) Show that for a random subset of $\{1, 2, \cdots, n\}$ the sum of the elements of this subset is odd with probability 1/2. Here we choose the set uniformly at random so that each element $i$ is included independently with probability 1/2.
   (e) Show that for any irrational $x$ and positive integer $n$, there is a rational number $p/q$ with $1 \leq q \leq n$ so that $|x - p/q| \leq 1/nq$.
   **Hint:** Try using the pigeonhole principle, and considering values $qx - \lfloor qx \rfloor$, where $\lfloor \alpha \rfloor$ is the *floor* function, i.e., the greatest integer less than or equal to $\alpha$.

2. **Writing Problem** (20 points; 10 points for math, 10 points for writing)

   Let $\pi_1$, $\pi_2$, $\pi_3$ be permutations of the numbers $[n] := \{1, 2, 3, \ldots, n\}$. A *common subword* between $\pi_i$, $\pi_j$, is a subsequence that appears in the same order for both of them. For example, if $\pi_1 = (12345678)$, $\pi_2 = (87654321)$, and $\pi_3 = (54832716)$, then 58 is a common subword of $\pi_1$ and $\pi_3$, and 821 is a common subword of $\pi_2$ and $\pi_3$.

   Let $n = m^3 + 1$ for some positive integer $m$. Prove that for any permutations $\pi_1$, $\pi_2$, $\pi_3$ of $[n]$, we can find two permutations $\pi_i$ and $\pi_j$ which have a common subword of length at least $m + 1$.

1. (a) By induction

$n=1$: LHS $=1$, RHS $=1$ ✓

$n \longrightarrow n+1$: $\Leftrightarrow (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{n(n+1)}{2}\right)^2$

$= \left(\frac{n+1}{2}\right)^2 \cdot \left((n+2)^2 - n^2\right)$

$= \frac{1}{4}(n+1)^2 \cdot 2 \cdot (2n+2)$

$= (n+1)^3$ ✓ D.

(b)
$$\left.\begin{array}{l} \{\boxed{1,}\ 2,\ 4,\ 8,\ 16,\ 32,\ 64\} \\ \{\boxed{3,}\ 6,\ 12,\ 24,\ 48\} \\ \{\boxed{5,}\ 10,\ 20,\ 40,\ 80\} \\ \quad\vdots \end{array}\right\}$$

$\rightarrow$ odd.

40 sets $\longrightarrow$ holes

pick 41 numbers from $1 \sim 80$ $\longrightarrow$ pigeons

two numbers in the same set, divide D.

(c) $\forall n$, if $n \equiv 1 \pmod 4$ : $n^2 \equiv 1 \pmod 4$

elif $n \equiv 2 \pmod 4$ : $n^2 \equiv 0 \pmod 4$

elif $n \equiv 3 \pmod 4$ : $n^2 \equiv 1 \pmod 4$

else $n^2 \equiv 0 \pmod 4$

$\Rightarrow n^2 \equiv 0$ or $1 \pmod 4$

$$\Rightarrow a^2+b^2 \equiv \begin{cases} 0+0 \equiv 0 \\ 1+0 \equiv 1 \\ 1+1 \equiv 2 \end{cases} \pmod 4 \quad \Rightarrow a^2+b^2 \not\equiv 3 \pmod 4 \text{ D.}$$

(d) firstly pick from $\{2, \ldots, n\}$ randomly. assume sum $S'$ odd has probability $p'$

then choose whether to pick element 1, probability $\frac{1}{2}$ to pick

$$\text{the sum} = \begin{cases} \text{Sum}' + 1 & \text{pick } 1 \\ \text{Sum}' & \text{don't pick } 1 \end{cases}$$

sum is odd when ① $S'$ even $\wedge$ pick 1 ② $S'$ odd $\wedge \neg$pick 1 has probability $p = p' \cdot \frac{1}{2} + (1-p') \cdot \frac{1}{2} = \frac{1}{2}$.

(2) by contradiction

assume for all $q$ $(1 \le q \le n)$, any $p \in \mathbb{Z}$ $|x - \frac{p}{q}| > \frac{1}{nq} \Rightarrow |qx - p| > \frac{1}{n}$.

then $\{qx\} > \frac{1}{n}$ & $1 - \{qx\} > \frac{1}{n}$

<u>Claim</u>: for any $1 \le k_1 < k_2 \le n$, $\left|\{k_1 x\} - \{k_2 x\}\right| > \frac{1}{n}$

<u>pf</u>: $k = k_2 - k_1 \Rightarrow 1 \le k \le n \Rightarrow \{kx\} > \frac{1}{n}$

$\{kx\} = |\{k_1 x\} - \{k_2 x\}|$ or $1 - |\{k_1 x\} - \{k_2 x\}|$ ✓

Build holes: $[0, \frac{1}{n}), [\frac{1}{n}, \frac{2}{n}), \ldots [\frac{n-1}{n}, 1)$ # holes $= n$

pigeons: $\{\{kx\} \mid 1 \le k \le n\}$ # pigeons $= n$

Since pigeons must be seperated, the one must go to $[0, \frac{1}{n})$ ※ □.

2. for $k$ that $1 \le k \le n$

let $k_{12}$ = the biggest length of the common subword of $\pi_1$ and $\pi_2$ that end with "$k$"

$k_{23}$ = ... $\pi_2$ .. $\pi_3$ ...

$k_{31}$ = ... $\pi_3$ ... $\pi_1$ ...

<u>Claim</u>: if $k \ne l$, $1 \le k, l \le n$ then $(k_{12}, k_{23}, k_{31}) \ne (l_{12}, l_{23}, l_{31})$

<u>Pf</u>: check the order of "$k$" and "$l$" in $\pi_1, \pi_2, \pi_3$

there must be two permulations that the order of "$k$" and "$l$" are same.

WLOG, assume in $\pi_1, \pi_2$ "$k$" is before "$l$"

then consider the common subword that has the length $k_{12}$ ending with "$k$"

add "$l$" at the end, still a common subword

but its length $= k_{12} + 1 > k_{12}$ so that $l_{12} \ne k_{12}$ □.

returning. by contradiction. assume

pigeons: $\{(k_{12}, k_{23}, k_{31}) \mid 1 \leq k \leq n\}$

holes: $\{(n_1, n_2, n_3) \mid 1 \leq n_i \leq m,\ i = 1, 2, 3\}$

so that every pigeon must be in a seperate hole.

$\#$ holes $= m^3$, $\#$pigeons $= n = m^3+1$

$\Rightarrow$ two pigeons in the same hole $\#$ $\square$.