Lecture 1 Pigeonhole Principle

Applications:

Lemma: At a party with n people, there are 2 people who know the same number of other people.

Pf: By contradiction

$\{0, 1, \ldots, n-1\}$, 0 & n-1 can't happen in the same time.

Lemma: Given 20 4-digits positive numbers $x_1, x_2 \ldots x_{20}$

there are 2 disjoint not empty sets with same sum

$$\exists I, J \subseteq \{1, 2 \ldots 20\} \quad I, J \neq \phi, \; I \cap J = \phi, \; \sum_{i \in I} x_i = \sum_{j \in J} x_j.$$

pigeons: nonempty subsets $n = 2^{20} - 1 >> 200000$.

holes: possible sums.

$$m \leq \sum_{i=1}^{20} x_i \leq 20 \times 10^4 = 200000$$

now we can't make sure A & B are disjoint.

but A & B don't necessarily be disjoint!

remove the common part in A and B, create two disjoint subsets

$I = A \setminus B$, $J = B \setminus A$, still same sum. D.

Thm: In any permutation of $1, 2, 3 \ldots nm+1$, there is an increasing subsequence of length $m+1$ or a decreasing subseq of length $n+1$.

This thm is very tight.

eg. $m=n=3$ $\quad 3, 2, 1, 6, 5, 4, 9, 7, 8, 10$

Pf: consider the s^{th} element in sequence, and let

i_s = length of longest increasing subsequence ending at s^{th} location

d_s = ... decreasing ....

Claim: For $s \neq t$, $(i_s, d_s) \neq (i_t, d_t)$

Pf: WLOG $s > t$

if element at s^{th} location > element at t^{th} location

$\Rightarrow i_s \geq i_t + 1$

if .. < ... $\Rightarrow d_s \geq d_t + 1$ $\square$.

returning, suppose the thm is false

there are $\leq nm$ possible values for (i_s, d_s) "holes"

but there are $nm+1$ "pigeons" impossible $\square$.

Lemma: Everyone's family tree contains someone whose parents are blood relatives

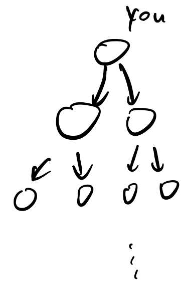
Setup: (1) Everyone has 2 bio-parents

(2) no one has children after 100

(3) human race > 4000 years old

(4) $< 10^{12}$ human ever lived in history

trace back family tree



depth ≥ 40

$\Rightarrow \geq 2^{40}$ nodes

but $2^{40} \geq (10^3)^4 = 10^{12} \Rightarrow$ not a tree. $\square$

Why study discrete probability?

① counting

② existance of objects (prob. method)

③ to design algorithms (prob testing)

④ to model communication

Sample space S

points in sample space $\Leftrightarrow$ possible outcomes

to each point $x \in S$, assign $p(x)$

① $p(x) \geq 0, \ \forall x \in S$

② $\sum_{x \in S} p(x) = 1$

Events: $T \subseteq S$, $p(T) = \sum_{x \in T} p(x)$

Complements: $p(\bar{T})$ $\quad \bar{T} = \neg T = S \setminus T$

Fact: $p(\bar{T}) = 1 - p(T)$