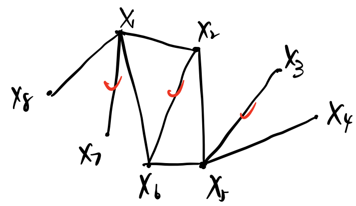
lecture 7

Def: Given graph $G=(V,E)$, a matching is a subgraph of G where every node has degree 1.

Ex:



$\{x_1-x_7,\ x_2-x_6,\ x_3-x_5\}$ is matching of size 3

Def. A matching is perfect if it has size $\frac{|V|}{2}$

Def. The weight of a matching M is the sum of the weights on the edges in M

Def. A mini-weight matching for G is a perfect matching for G with a minimum weight

Stable Marriage Problem

Def: Given a matching M, x & y form a rogue couple if they prefer each other to their mates in M

Def: A matching is stable, if there are no rogue couples.

Goal: Find a perfect matching that is stable.

— N boys & N girls

— each boy has his own ranked list of the girls. girls as the same.

The Mating Algorithm (TMA)

Initial Condition: Each of the N boys has an ordered list of the N girls according to his preferences. Each of the girls has an ordered list of the boys according to her preferences.

Each Day:

- Morning:
  - Each girl stands on her balcony
  - Each boy stands under the balcony of his favorite girl whom he has not yet crossed off his list and serenades. If there are no girls left on his list, he stays home and does 6.042 homework.
- Afternoon:
  - Girls who have at least one suitor say to their favorite from among the suitors that day: "Maybe, come back tomorrow."
  - To the others, they say "No, I will never marry you!"
- Evening:
  - Any boy who hears "No" crosses that girl off his list.

Termination Condition: If there is a day when every girl has at most one suitor, we stop and each girl marries her current suitor (if any).

Need to show:

- Algorithm terminates (quickly)
- Everyone gets married
- No rogue couples.
- fairness

Thm 1: TMA terminates in $\le N^2+1$ days.

Pf: by contradiction. Suppose does not end in N^2+1 days.

Claim: if we don't terminate on a day then some boy crosses a girl off.

then crossed out $\ge N^2+1$ girls ※ $\square$.

progress got made — a very common method in CS.

Lemma: If a girl G ever rejected a boy B. then G has a suitor who she prefers to B.

Pf: By induction on # days. $\square$.

Thm 2: Everyone is married in TMA.

Pf: By contradiction. assume boy B is not married at the end

$\Rightarrow$ B is rejected by every girl.

$\Rightarrow$ every girl has a better suitor (Lemma 1)

$\Rightarrow$ every girl married ※ $\square$.

Thm 3: TMA produces a stable matching

Pf: Let Bob and Gail be any pair that are not married.

Case 1: Gail rejected Bob.

$\Rightarrow$ Gail married someone better than Bob (Lemma 1)

Case 2: Gail did not reject Bob

$\Rightarrow$ Bob never serenaded Gail

$\Rightarrow$ Gail is lower on Bob's list than Bob's wife

$\Rightarrow$ Bob & Gail were not rogue couples. $\square$

Let S: set of all stable matchings $S \neq \phi$

For each person P, we define the <u>realm of possibility</u> for P to be.

$$\{Q \mid \exists m \in S, (P, Q) \in m\}$$

<u>Def</u>: A person's <u>optimal mate</u> is his/her favorite from the realm of possibility.

.. <u>pessimal mate</u> .. least favourite --

<u>Thm 4</u>: TMA marries every boy with his optimal mate

<u>Thm 5</u>: .. girl ... pessimal mate

easy to prove Thm 4 $\Leftrightarrow$ Thm 5 (by contradiction)