E10.4

iii.

In this part we are tasked with implementing the LMS algorithm and run it for 40 steps for a stable learning rate value as obtained in part (ii) of this problem.

We will set $\alpha = 0.1$ for this problem.

In order to check for convergence, we will compare the squared errror with a convergence tolerance value which we have set to 10^{-9} here.

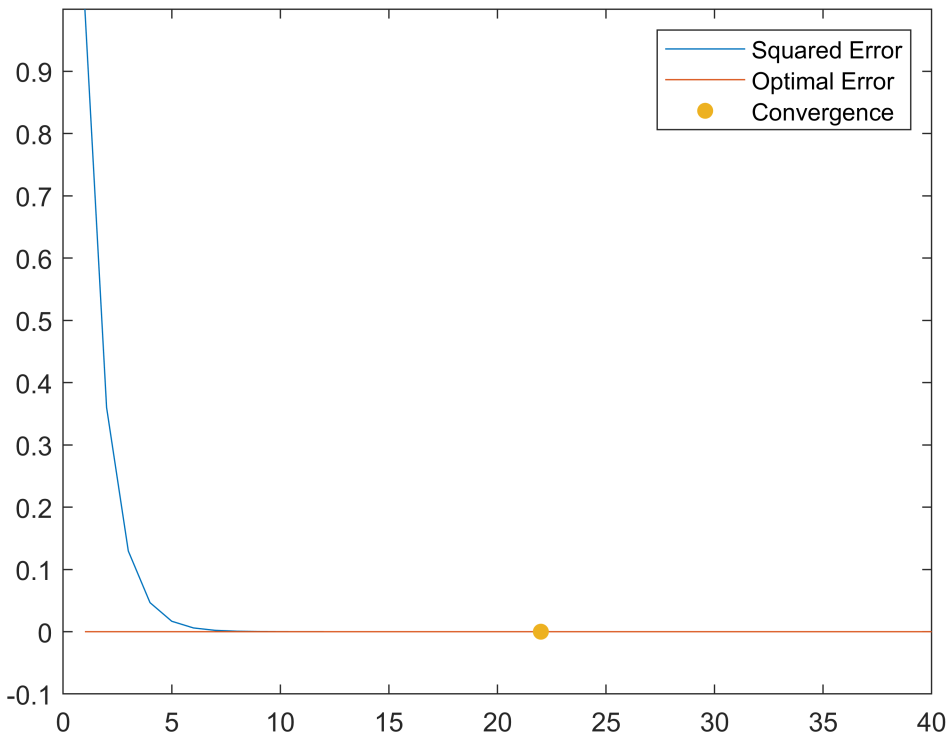
```
% LMS Implementation
x_star = [0; 0];
x_hist = zeros([2, 41]);
x_hist(:, 1) = x_star;
error_hist = zeros([1, 40]);
num_steps = 40;
alpha = 0.1;
convergence_tol = 1e-9;
% repeat the two patterns 20 times to for 40 steps of the algorithm
patterns = [[1; 1], [-1; -1]];
targets = [1, -1];
num_repeats = num_steps / numel(targets);
inputs = repmat(patterns, [1, num_repeats]);
outputs = repmat(targets, [1, num_repeats]);
for iteration=1:num_steps
    iteration
    error = outputs(iteration) - x_star'*inputs(:, iteration)
    error_hist(iteration) = error^2;
    x_star = x_star + 2*alpha*error*inputs(:, iteration)
    x_hist(:, iteration+1) = x_star;
    if error_hist(iteration) < convergence_tol
        fprintf('#####\n\nConvergence reached at iteration %d.', iteration)
        error
        x_star
        break
    end
end
```

```
iteration = 1
error = 1
x_star = 2×1
    0.2000
    0.2000
iteration = 2
error = -0.6000
x_star = 2×1
    0.3200
    0.3200
iteration = 3
error = 0.3600
x_star = 2×1
    0.3920
    0.3920
```

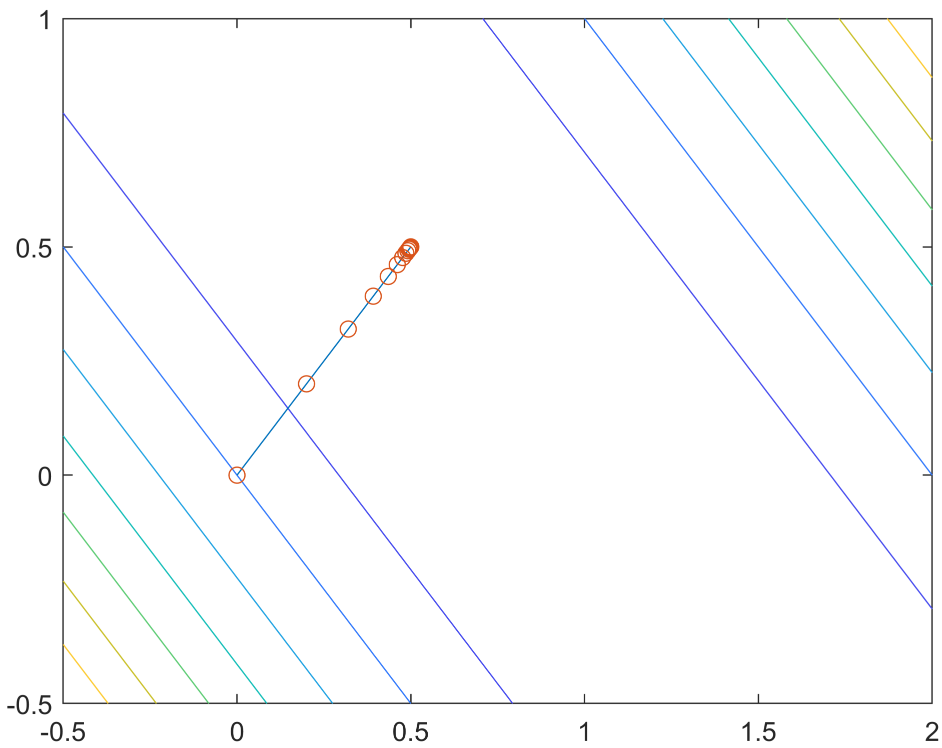
```
iteration = 4
error = -0.2160
x_star = 2×1
    0.4352
    0.4352
iteration = 5
error = 0.1296
x_star = 2×1
    0.4611
    0.4611
iteration = 6
error = -0.0778
x_star = 2×1
    0.4767
    0.4767
iteration = 7
error = 0.0467
x_star = 2×1
    0.4860
    0.4860
iteration = 8
error = -0.0280
x_star = 2×1
    0.4916
    0.4916
iteration = 9
error = 0.0168
x_star = 2×1
    0.4950
    0.4950
iteration = 10
error = -0.0101
x_star = 2×1
    0.4970
    0.4970
iteration = 11
error = 0.0060
x_star = 2×1
    0.4982
    0.4982
iteration = 12
error = -0.0036
x_star = 2×1
    0.4989
    0.4989
iteration = 13
error = 0.0022
x_star = 2×1
    0.4993
    0.4993
iteration = 14
error = -0.0013
x_star = 2×1
    0.4996
    0.4996
iteration = 15
error = 7.8364e-04
x_star = 2×1
    0.4998
    0.4998
iteration = 16
error = -4.7018e-04
x_star = 2×1
    0.4999
```

```
    0.4999
iteration = 17
error = 2.8211e-04
x_star = 2×1
    0.4999
    0.4999
iteration = 18
error = -1.6927e-04
x_star = 2×1
    0.4999
    0.4999
iteration = 19
error = 1.0156e-04
x_star = 2×1
    0.5000
    0.5000
iteration = 20
error = -6.0936e-05
x_star = 2×1
    0.5000
    0.5000
iteration = 21
error = 3.6562e-05
x_star = 2×1
    0.5000
    0.5000
iteration = 22
error = -2.1937e-05
x_star = 2×1
    0.5000
    0.5000
#####

Convergence reached at iteration 22.
error = -2.1937e-05
x_star = 2×1
    0.5000
    0.5000
```

```
% plotting error history
plot(error_hist)
hold on
plot(zeros([1, 40]))
ylim([-0.1 max(error_hist)])
scatter(iteration, error_hist(iteration), 'filled')
legend('Squared Error', 'Optimal Error', 'Convergence')
hold off
```




```
% sketching the trajectory on the contour plot
syms x1 x2 real
X = [x1; x2];
fx = 1 - 2*X'*[1; 1] + X'*[1 1; 1 1]*X;
fcontour(fx,[-0.5 2 -0.5 1])
hold on
plot(x_hist(1,:), x_hist(2,:));
hold on
scatter(x_hist(1,:), x_hist(2,:));
hold off
```



iv.

Now we will run the same experiment but we will set $x_0 = [1, 1]^T$.

```
% LMS Implementation
x_star = [1; 1];
x_hist = zeros([2, 41]);
x_hist(:, 1) = x_star;
error_hist = zeros([1, 40]);
num_steps = 40;
alpha = 0.1;
convergence_tol = 1e-9;
% repeat the two patterns 20 times to for 40 steps of the algorithm
patterns = [[1; 1], [-1; -1]];
targets = [1, -1];
num_repeats = num_steps / numel(targets);
inputs = repmat(patterns, [1, num_repeats]);
outputs = repmat(targets, [1, num_repeats]);
for iteration=1:num_steps
    iteration
    error = outputs(iteration) - x_star'*inputs(:, iteration)
    error_hist(iteration) = error^2;
    x_star = x_star + 2*alpha*error*inputs(:, iteration)
    x_hist(:, iteration+1) = x_star;
    if error_hist(iteration)  < convergence_tol
        fprintf('#####\n\nConvergence reached at iteration %d.', iteration)
        error
```

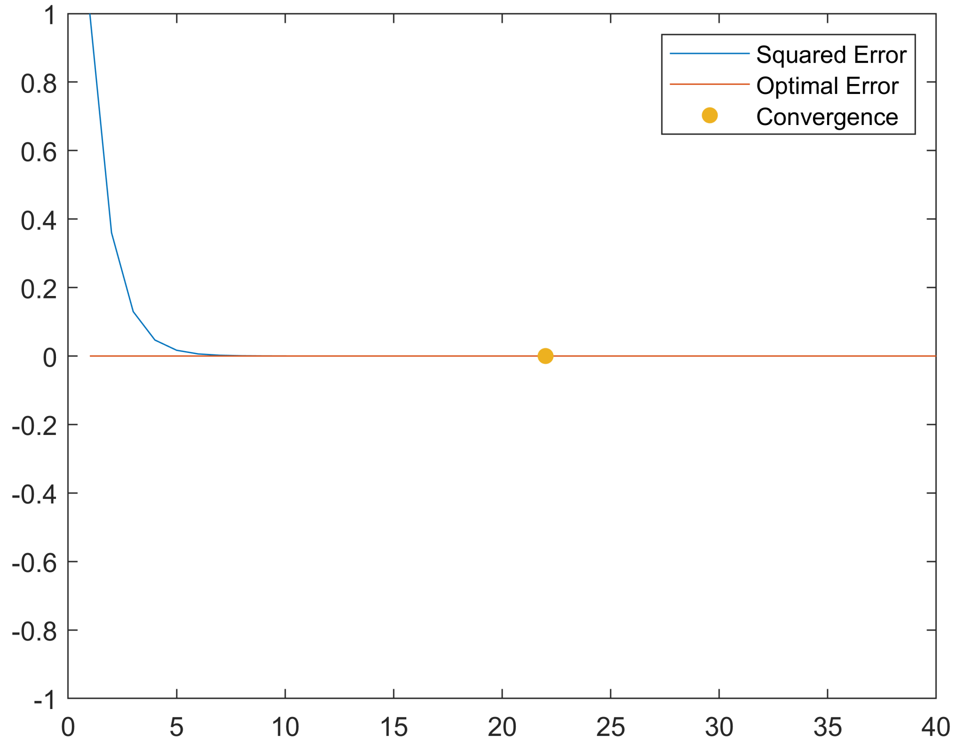
```
        x_star
        break
    end
end
```

```
iteration = 1
error = -1
x_star = 2×1
    0.8000
    0.8000
iteration = 2
error = 0.6000
x_star = 2×1
    0.6800
    0.6800
iteration = 3
error = -0.3600
x_star = 2×1
    0.6080
    0.6080
iteration = 4
error = 0.2160
x_star = 2×1
    0.5648
    0.5648
iteration = 5
error = -0.1296
x_star = 2×1
    0.5389
    0.5389
iteration = 6
error = 0.0778
x_star = 2×1
    0.5233
    0.5233
iteration = 7
error = -0.0467
x_star = 2×1
    0.5140
    0.5140
iteration = 8
error = 0.0280
x_star = 2×1
    0.5084
    0.5084
iteration = 9
error = -0.0168
x_star = 2×1
    0.5050
    0.5050
iteration = 10
error = 0.0101
x_star = 2×1
    0.5030
    0.5030
iteration = 11
error = -0.0060
x_star = 2×1
    0.5018
    0.5018
iteration = 12
error = 0.0036
x_star = 2×1
    0.5011
```

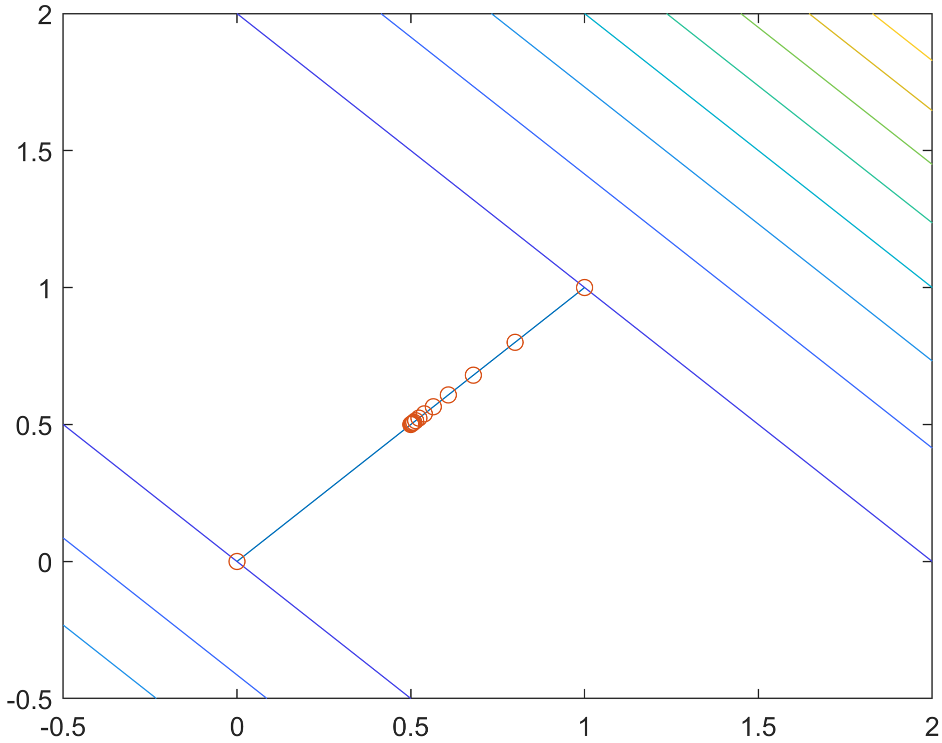
```
    0.5011
iteration = 13
error = -0.0022
x_star = 2×1
    0.5007
    0.5007
iteration = 14
error = 0.0013
x_star = 2×1
    0.5004
    0.5004
iteration = 15
error = -7.8364e-04
x_star = 2×1
    0.5002
    0.5002
iteration = 16
error = 4.7018e-04
x_star = 2×1
    0.5001
    0.5001
iteration = 17
error = -2.8211e-04
x_star = 2×1
    0.5001
    0.5001
iteration = 18
error = 1.6927e-04
x_star = 2×1
    0.5001
    0.5001
iteration = 19
error = -1.0156e-04
x_star = 2×1
    0.5000
    0.5000
iteration = 20
error = 6.0936e-05
x_star = 2×1
    0.5000
    0.5000
iteration = 21
error = -3.6562e-05
x_star = 2×1
    0.5000
    0.5000
iteration = 22
error = 2.1937e-05
x_star = 2×1
    0.5000
    0.5000
#####

Convergence reached at iteration 22.
error = 2.1937e-05
x_star = 2×1
    0.5000
    0.5000
```

```
% plotting error history
plot(error_hist)
hold on
plot(zeros([1, 40]))
```

```
ylim([-max(error_hist) max(error_hist)])
scatter(iteration, error_hist(iteration), 'filled')
legend('Squared Error', 'Optimal Error', 'Convergence')
hold off
```



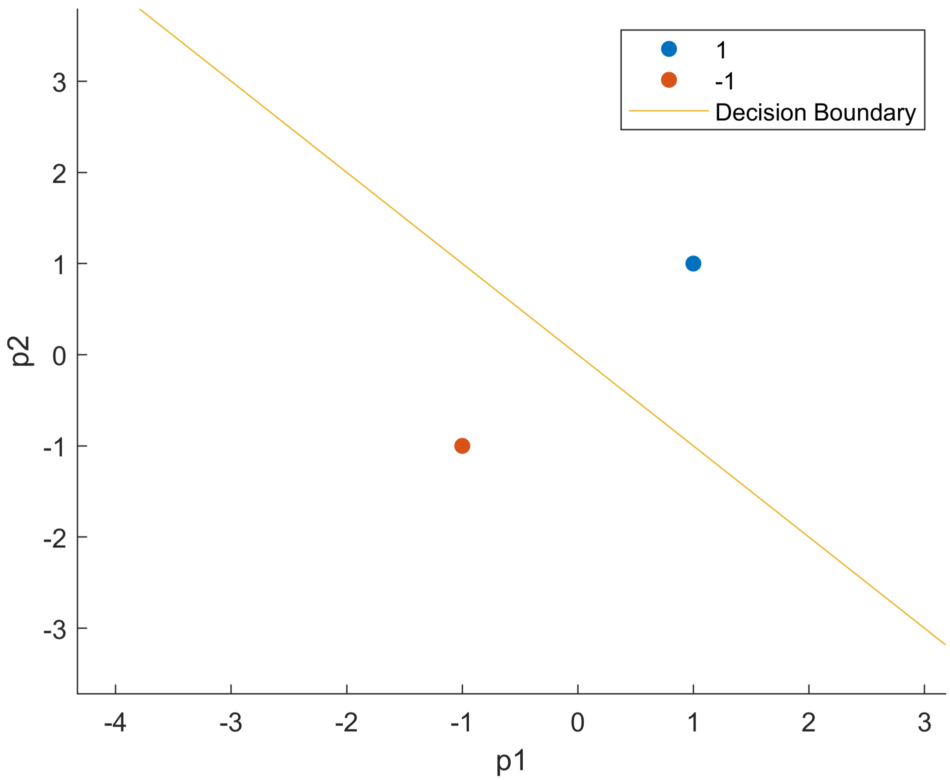

```
% sketching the trajectory on the contour plot
syms x1 x2 real
X = [x1; x2];
fx = 1 - 2*X'*[1; 1] + X'*[1 1; 1 1]*X;
fcontour(fx,[-0.5 2 -0.5 2])
hold on
plot(x_hist(1,:), x_hist(2,:));
hold on
scatter(x_hist(1,:), x_hist(2,:));
hold off
```



The decision boundary is calculated by setting the output of the network to 0:

$X^{*T} p = 0 \rightarrow [0.5, 0.5][p_1, p_2]^T = 0 \rightarrow p_1 = -p_2$

```
% sketching the decision boundary
scatter(patterns(1,1), patterns(2,1), 'filled')
hold on
scatter(patterns(1,2), patterns(2,2), 'filled')
hold on
syms p2 real
fplot(-p2)
legend('1', '-1', 'Decision Boundary')
xlabel('p1')
ylabel('p2')
```



As we can see, in minimizing the squared error between the patterns, the decision boundary seems to have passed from between the patterns in such a way that halves the distance between them, effectively putting as much distance as possible between both patterns.

v.

As we can see the algorithm converged to the same x^* ($[0.5\ 0.5]^T$ for both runs) which is expected because the convergence of the LMS algorithm is irrespective of the initial choice for the weights. Although the trajectory looks slightly different at first, the algorithm quickly converges to the same path after a few iterations.