Qtext :

Answer the following questions with justifications.

a) i) Using only the techniques of linear algebra studied in the course so far, compute the maximum of the expression $2x_1^2 + 3x_2^2 - 2x_1x_2$, subject to $x_1^2 + x_2^2 = 1$. Give sufficient justifications. (3)

ii) Let L be a lower triangular $n \times n$ matrix and $\boldsymbol{A} = LL^T$. Assume the eigenvector matrix of $\boldsymbol{A}$ can be written as $\boldsymbol{S}$, and its eigenvalue matrix is $\boldsymbol{\Lambda}$. Can you write the SVD of $\boldsymbol{A}$ in terms of $\boldsymbol{S}$ and $\boldsymbol{\Lambda}$? Give justification for your answer. (2)

b) i) A particle moves in space with respect to time in the following fashion $[x_1(t), x_2(t), x_3(t)] = [r\sin(\omega t), r\cos(\omega t), kt]$, where r, ω and k are constants. Let D denote the Euclidean distance of the particle from the origin. Calculate the gradient $\frac{dD}{d\boldsymbol{x}}$, and then calculate $\frac{dD}{dt}$ using the chain rule. (2)

ii) Using Taylor's series find a linear approximation $g(x, y)$ to the function $f(x, y) = \alpha x^2 + \beta y^2$ around the point (x_0, y_0). What is the condition needed on α and β to ensure that $f(x, y) \geq g(x, y)$ for all x, y? Give justification for your answer. (3)

Qtext :

Answer the following questions with justifications.

a) Let $g(u) = \begin{pmatrix} \log(u) \\ \exp(u^2) \\ \sin(u) \end{pmatrix}$, $f(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$ where $u \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3$. Compute $\frac{\partial g}{\partial u}, \frac{\partial f}{\partial \mathbf{x}}$ and hence $\frac{\partial (f \circ g)}{\partial u}$ (2.5)

b) Assuming the result that the geometric multiplicity is same as the algebraic multiplicity of each each eigenvalue for real symmetric matrices, prove that number of nonzero eigenvalues is equal to the rank of real symmetric matrix $\mathbf{A}$. Then if $\mathbf{A} = \mathbf{b}\mathbf{b}^T$ where $\mathbf{b} = [1, y, z]^T$, find the rank and all eigenvalues. (3)

c) A student doing some data analysis arrived at a real matrix $\mathbf{A}$ and he got (2.5)

$$(\mathbf{A}^3)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\delta \end{pmatrix}$$

where $\delta > 0$. The professor rejected his analysis saying there is no real $\mathbf{A}$ with $(\mathbf{A}^3)^2$ equal to the above given matrix. Prove or disprove the claim made by the professor with proper justification.

d) If there are two input $\mathbf{x} = [x_1, x_2]^T$ which determine the output $f = [f_1, f_2]^T$ where the output is given by the formula given below.

$$\begin{aligned} f_1 &= x_1^2 + \alpha x_2 + 2x_1 \cos(x_2) \\ f_2 &= \beta x_1 + x_2^2 + 2x_2 \cos(x_1) \end{aligned}$$

i) Compute $\nabla_{\mathbf{x}} f(0,0) = \frac{df}{d\mathbf{x}}(0,0)$. (1)

ii) Find the condition on α and β for $\nabla_{\mathbf{x}} f(0,0)$ to be positive definite showing the details of the computation. (1)

Answer the following questions with justifications. (3)

a) Assume that you are given a matrix $A \in \mathbb{R}^{n \times n}$ where $A = [a_{ij}]$ $(1 \leq i\,,\, j \leq n)$ having the property $\sum_{j=1}^{n} a_{ij} = 0 \quad \forall\, i = 1, 2, \ldots, n.$

i) Can zero be one of the eigenvalues of this matrix ? Give reasoning for your answer.

ii) Derive the determinant of this matrix.

iii) Derive at least one eigenvector-eigenvalue pair of this matrix .

b) Derive a lower triangular matrix L so that $A = LL^T$ for the matrix A given below. Is such a decomposition always possible? Justify. Clear derivation of entries of L must be provided. (4)

$$A = \begin{bmatrix} 42 & 32 & 37 \\ 32 & 34 & 31 \\ 37 & 31 & 35 \end{bmatrix}$$

c) A professor teaching linear algebra wanted to test students understanding of eigenvalues and eigenvectors. For some real number β, he constructed a 3×3 matrix C given by (3)

$$C = \begin{bmatrix} 2 & \beta & 0 \\ \beta & 2 & \beta \\ 0 & \beta & 2 \end{bmatrix}$$

i) If it is given that C has three non-zero eigenvalues, then can you compute all the possible values of β?

ii) If it is given that C has 3 positive eigenvalues, then can you compute