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QUANTUM FIELD THEORY (2020-02-07)

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A first version of this script has been translated into LaTeX by Andrea Federico Sanfilippo (TUM) in April 2019. He also prepared all the wonderful figures and diagrams.

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Der wackre Schwabe forcht' sich nit,
ging seines Weges Schritt vor Schritt.[1]

[1] Ludwig Uhland, often cited by Albert Einstein.

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Contents

1 Foreword

These notes are not original but mostly a combination of results found in books and lecture notes. My contribution is here foremost in the selection and the perspective provided. I have made liberal use of the following references:

The $\star$ indicates the technical level of the book. It is roughly proportional to the time needed per page for a full understanding. Note, that the Zee book in particular is written in a conversational tone, but covers more concepts than any of the other books.

- Martin Beneke - *Lecture notes on QFT* $\star\star\star$,
  I have benefited greatly from his notes which in turn mostly follow Weinberg's QFT I book, see below.
- My - *Relativity, Particles, Fields script* $\star$,
  which you should read to be prepared for what's coming.
- Peskin, Schroeder - *An Introduction to Quantum Field Theory* $\star\star$
- Schwartz - *Quantum Field Theory and the SM* $\star\star$
- Mandl, Shaw - *Quantum Field Theory* $\star$
- Zee - *Quantum Field Theory in a Nutshell* $\star$
- Srednicki - *Quantum Field Theory* $\star\star\star$
- Weinberg - *Quantum Field Theory I-III* $\star\star\star\star$
- Ramond - *Field Theory: a Modern Primer* $\star\star\star$
- Itzykson, Zuber - *Quantum Field Theory* $\star\star\star$
- Shifman, *Advanced topics in Quantum Field Theory* $\star\star\star$

and many more. I strongly suggest that you find a book (or books) you like from the ones above and study it as a complement to these lecture notes. My recommendation would be to use Schwartz for most of the lecture and Weinberg as additional reading, once you have grasped the main ideas of each section.

This course provides a introduction into the beautiful world of quantum field. It is the first real theory course in that it is the first course to tackle the theoretical framework that underlies all of nature. Let's go.

2
Bibliography

[1] Jun John Sakurai. *Modern Quantum Mechanics.*

[2] Michael E. Peskin, Daniel V. Schroeder. *An Introduction to Quantum Field Theory.*

[3] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model.*

[4] Andreas Weiler. *Relativity, Particles, Fields script.*

[5] Steven Weinberg. *Lectures on Quantum Mechanics.*

[6] Steven Weinberg. *The Quantum Theory of Fields, Volume I: Foundations.*

[7] Claude Itsykson, Jean-Bernard Zuber. *Quantum Field Theory.*

3
Introduction and Motivation

Quantum field theory (QFT) is the most important achievement of theoretical physics in the second half of the twentieth century. There are in essence three viewpoints to motivate its necessity which are compatible with each other.

3.1 Historical viewpoint

Every classical (physical) object should be rewritten in quantized form, e.g. the vector potential of the electromagnetic field should be translated to a field operator: $A^\mu(x) \to \hat{A}^\mu(x)$.

3.2 Wilsonian viewpoint

Two physical systems which look different at **short** distances may behave the same way at **large** distances, since short details become irrelevant. We can think of a theory in terms of an expansion:

$$\frac{l_{\text{short}}}{L} \ll 1\,, \tag{3.1}$$

where l_{short} is the microscopic distance scale of the theory and L is the length scale of an experiment. The QFTs then are interpreted as the universality classes of physical systems that one obtains by "zooming out" to long distances.

Examples:

a) Ising model:

$$H = -\sum_{\substack{i<j \\ \text{n.n.}}} s_i s_j\,. \tag{3.2}$$

It goes over into the Landau-Ginzburg theory

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda\phi^4 \right) \tag{3.3}$$

once we describe physics at distances $L \ll l_{\text{short}}$, where l_{short} is the lattice spacing in this case.

b) Fermi liquid of electrons in a metal: the temperature at which the scale of the lattice spacing becomes relevant is of the order of 10000 K, while the room temperature is roughly 300 K.

For details on this topic, see the beautiful lecture by Polchinski: hep-th/9210046

c) Standard Model of particle physics and gravity: we have the following hierarchy of scales:

$$\frac{M_{\text{Planck}}}{M_{\text{Weak}}} \gg 1\,. \tag{3.4}$$

d) Within the Standard Model we have several examples:

- Fermi-theory of β-decay
- Pions of Quantum Chromodynamics (QCD)
- Heavy quarks

In general, if there is a hierarchy between short and long distances, then one often can find a relatively simple and universal QFT describing the physical system.

Consequences:

- There is a true physical cut-off at short distances, the complete UV theory (i.e. including high energies/small distances) might not be a QFT.
- The UV and IR physics can be very different, e.g. the LHC has a hard time testing String Theory.
- The symmetries of the system are crucial and can be: preserved, broken, emergent, anomalous,...

  For instance, space-time symmetry (Poincaré invariance) determines what a theory can be (herein lies the difference between condensed matter and high energy physics).
- QFT allows us to describe both particle physics and statistical physics, since in both cases we are interested in (relatively) long-distance or macroscopic properties.

3.3 Weinbergian point of view

QFT is the only framework to obtain a

- Lorentz-invariant
- Quantum mechanical (unitary)
- Local

theory of scattering particles. Formally, we encapsulate a theory of scattering by the S-matrix:

$$S_{\alpha\beta} = \langle \alpha_{\text{out}} | S | \beta_{\text{in}} \rangle\,, \tag{3.5}$$

where $|\alpha_{\mathrm{out}}\rangle$ is the out-state in the distant future and $|\beta_{\mathrm{in}}\rangle$ is the in-state in the distant past, both describing well-separated particles.

Consequences:

- Particles ("atomic" states of the theory) are defined as irreducible representations (irreps) of the Poincaré group, so we define the Hilbert space of incoming/outgoing particles. Since the energies and momenta of well-separated particles **add** (no binding energy, etc.), we can use the creation/annihilation operators $a^\dagger_{\boldsymbol{p}}$, $a_{\boldsymbol{p}}$ of the quantized harmonic oscillator to describe each momentum $\boldsymbol{p}$ because the harmonic oscillator has evenly spaced energy levels.
- Locality constrains the form of the Hamiltonian (see cluster decomposition theorem)
- The S-matrix must be Poincaré invariant. It is defined as

$$S = S[\mathcal{H}(x)] = T\exp\left(-i\int_{-\infty}^{\infty} \mathrm{d}t\, V(t)\right), \tag{3.6}$$

$$V(t) = \int \mathrm{d}^3x\, \mathcal{H}(t, \boldsymbol{x}), \tag{3.7}$$

  with $\mathcal{H}(x)$ the Hamiltonian density. It requires the **causality condition**

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0, \quad (x-y)^2 < 0 \text{ spacelike separation.} \tag{3.8}$$

  If we write the theory in terms of a **local** Lagrangian, this condition can be shown to be satisfied.
- The symmetries constrain the asymptotic states of the S-matrix; one also uses gauge redundancies to describe certain massless particles in a manifestly local, Poincaré-invariant theory.
- One can show that only **massless** particles of spin ≤ 2 can couple such that there are long-range forces. Massless spin 1 particles must couple to a conserved current J_μ, massless spin 2 particles must couple to the energy-momentum tensor $T_{\mu\nu}$ which is also conserved, implying energy-momentum conservation.

  This is an intriguing result given the particles we have encountered in nature (spin 1: photon, spin 2: graviton, spin 3: $\varnothing$)

We will focus mostly on relativistic QFTs, the methods, however, will be general. Fundamental physics is very simple because quantum mechanics (QM) combined with special relativity (SR) is very powerful at constraining consistent theories.

4
Review of the Basic Paradigm

4.1 The action

A theory is defined by its action:

$$S = \int \mathrm{d}^d x \, \mathcal{L}(\phi_i, g_i)\,. \tag{4.1}$$

Here d denotes the number of dimension and usually we set $d = 1 + 3 = 4$; $\mathcal{L}$ is the Lagrangian density and is a function of the fields

$$\phi_i = \phi_i(t, \boldsymbol{x}) \tag{4.2}$$

of type i and the fundamental parameters

$$g_i = \alpha_{\mathrm{QED}}, m_e, \ldots \tag{4.3}$$

In a relativistic QFT, we need to decide which fields ϕ_i as irreps of the Poincaré group are present (i.e. scalars, fermions,...) and determine what the internal symmetries of the system are. The symmetries constrain the form of $\mathcal{L}$ and lead to selection rules.

4.2 Invariance under Lorentz transformations and space-time translations (Poincaré transformations)

An element of the Poincaré group $g = (\Lambda, a)$, with Λ a Lorentz-boost and spatial rotation and $a = a^\mu$ the translation four-vector, acts on a four-vector x^μ as

$$x^\mu \overset{g}{\mapsto} \Lambda^\mu{}_\nu x^\nu + a^\mu\,. \tag{4.4}$$

Quantum mechanical states also transform under the Poincaré group. The group element is mapped to a representation

$$(\Lambda, a) \mapsto U(\Lambda, a)\,, \tag{4.5}$$

which is an operator on Hilbert space:

$$|\psi'\rangle \equiv U(\Lambda, a)|\psi\rangle \quad \text{(state)}\,, \tag{4.6}$$

$$O' \equiv U(\Lambda, a) O U^{-1}(\Lambda, a) \quad \text{(operator)}\,. \tag{4.7}$$

If the physics is to be invariant under the transformation, that means that observables are unchanged:

$$|\langle\psi'|\psi'\rangle| = |\langle\psi|U^\dagger U|\psi\rangle| \overset{!}{=} |\langle\psi|\psi\rangle| \tag{4.8}$$

and $U(\Lambda, a)$ must be a unitary operator: $U^\dagger U = \mathbb{1}$.
A field operator is called a **scalar** operator if it transforms trivially under the Poncaré group:

$$U(\Lambda, a) O_s(x) U^{-1}(\Lambda, a) = O_s(\Lambda x + a)\,. \tag{4.9}$$

Further: The action S is invariant if $\mathcal{L}$ is a scalar operator. This is the case in relativistic QFT.
Compare to e.g. a vector operator:

Ex: Show that

$$\mathrm{d}^4 x \to \det(\Lambda)\mathrm{d}^4 x \quad \text{and} \quad |\det(\Lambda)| = 1$$

$$A_\mu(x) \to U(\Lambda, 0) A_\mu(x) U^{-1}(\Lambda, 0) = \Lambda_\mu{}^\nu A_\nu(\Lambda x)\,. \tag{4.10}$$

4.3 Lorentz group $SO(1,3)$

The structure of the Lorentz group is $SO(1,3)$ similar to $SO(4)$, which is the special orthogonal group acting on real 4-component vectors. $SO(4)$ has determinant one and preserves

$$v^T v = \tau^2 + x^2 + y^2 + z^2\,, \quad v = \begin{pmatrix} \tau \\ x \\ y \\ z \end{pmatrix}. \tag{4.11}$$

The Lorentz group is $SO(1,3)$: the group of linear transformations which preserve the quantity

$$\boxed{t^2 - x^2 - y^2 - z^2 = t^2 - \boldsymbol{x}^2} \tag{4.12}$$

$$= x_\mu x^\mu \equiv \eta_{\mu\nu} x^\mu x^\nu = \eta^{\mu\nu} x_\mu x_\nu\,, \tag{4.13}$$

$$x^\mu = \begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}, \quad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.14}$$

This means they satisfy

$$x_\mu x^\mu \to x'_\mu x'^\mu = x_\mu x^\mu \quad \text{or} \quad \Lambda^T \eta \Lambda = \eta\,. \tag{4.15}$$

Two examples of such a matrices are[1]

$$\Lambda_1(v) = \begin{pmatrix} \cosh(\beta_x) & \sinh(\beta_x) & 0 & 0 \\ \sinh(\beta_x) & \cosh(\beta_x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{Lorentz-boost,} \tag{4.16}$$

$$\Lambda_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\varphi) & -\sin(\varphi) & 0 \\ 0 & \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{spatial rotation.} \tag{4.17}$$

Instead of using the rapidity β, we can also write:

$$\boxed{x \to \frac{x + vt}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \quad t \to \frac{t + \frac{v}{c^2} x}{\sqrt{1 - \frac{v^2}{c^2}}}\,, \quad y \to y\,, \quad z \to z\,,} \tag{4.18}$$

[1] Recall the identity $\cosh^2\beta - \sinh^2\beta = 1$ to convince yourself.

so it follows that

Boost in x-direction with velocity v

$$\sinh(\beta_x) \overset{c=1}{=} \frac{v}{\sqrt{1-v^2}}\,. \tag{4.19}$$

These transformations give back the Galilei-transformations for $v \ll c$:

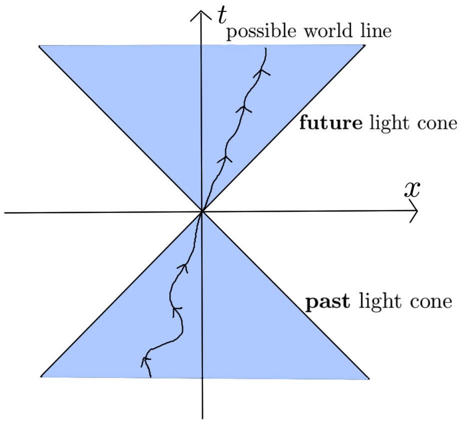
$$x \to x + vt\,, \quad t \to t\,. \tag{4.20}$$

The following statement will be important for the rest of the lecture:

> Two events with space-like separation cannot be causally connected to each other.

Why ist that?

$\Delta s^2 < 0$ space-like
$\Delta s^2 = 0$ light-like
$\Delta s^2 > 0$ time-like



1) We define two four-vectors x^μ, y^μ with space-like proper distance $\Delta s^2 = (x-y)^2 < 0$, for instance:

$$x^\mu = \begin{pmatrix} 0 \\ \boldsymbol{x} \end{pmatrix}\,, \quad y^\mu = \begin{pmatrix} 0 \\ \boldsymbol{y} \end{pmatrix}\,, \tag{4.21}$$

$$\Delta s^2 = \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu = 0 - (\boldsymbol{x}-\boldsymbol{y})^2 < 0\,. \tag{4.22}$$

They **cannot** be connected by a cause propagating at the speed of light. c is the maximal possible speed in special relativity. A light ray propagating in z-direction is

$$v^\mu_{\text{light}} = c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\,, \quad v_\mu v^\mu = c^2(1-1) = 0\,. \tag{4.23}$$

2) Two events with space-like separation have **no definite time-ordering**: we have

$$x^\mu = \begin{pmatrix} 0 \\ \boldsymbol{0} \end{pmatrix}\,, \quad y^\mu = \begin{pmatrix} 0 \\ \boldsymbol{y} \end{pmatrix}\,, \quad \Delta t = x^0 - y^0 = 0\,, \tag{4.24}$$

in one inertial frame. If we boost to a different one using Eq. (4.18) we get

$$x^\mu \overset{\Lambda_1(v)}{\to} \begin{pmatrix} 0 \\ \boldsymbol{0} \end{pmatrix}\,, \quad y^\mu \overset{\Lambda_1(v)}{\to} \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} v y_x \\ \boldsymbol{y} \end{pmatrix}\,, \quad \Delta t < 0 \tag{4.25}$$

and using yet another boost with $v \to -v$ we get

$$x^\mu \overset{\Lambda_1(-v)}{\to} \begin{pmatrix} 0 \\ \boldsymbol{0} \end{pmatrix}\,, \quad y^\mu \overset{\Lambda_1(-v)}{\to} \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} -v y_x \\ \boldsymbol{y} \end{pmatrix}\,, \quad \Delta t > 0\,. \tag{4.26}$$

Space-like separated events have no definite time ordering and are therefore **causally disconnected**.

4.4 Lagrange formalism and canonical quantization

The Lagrangian is defined as the integral over the Lagrangian density:

$$L = \int \mathrm{d}^3x\, \mathcal{L}(\phi_n(x), \nabla\phi_n(x), \dot{\phi}_n(x))\,, \tag{4.27}$$

where $x = x^\mu$, $\dot{\phi} = \frac{\partial \phi}{\partial t}$ and all the fields are evaluated at the same space-time point x: the Lagrangian is **local**. Non-local Lagrangians are usually incompatible with relativistic causality.
The action is automatically Lorentz-invariant if $\mathcal{L}$ is a Lorentz-scalar:

$$S = \int \mathrm{d}t\, L = \int \mathrm{d}^4x\, \mathcal{L}\,. \tag{4.28}$$

Mechanics:

$$S = \int \mathrm{d}t\; L(q, \dot{q}, t)$$
$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$
$$p_i = \frac{\partial L}{\partial \dot{q}}$$
$$H = \sum_i \dot{q}_i p_i - L$$

QM:

$$[q_i, p_j] = i\hbar\delta_{ij}$$
$$[q_i, q_j] = [p_i, p_j] = 0$$
$$\dot{q}_n = \frac{1}{i}[q_n, H]$$

Einstein summation convention is always implied unless stated otherwise.

4.4.1 Euler-Lagrange equations

We require that the action S be stationary, i.e. its variation with respect to the arguments of the Lagrangian density must vanish:

$$0 = \delta S = \delta\left(\int \mathrm{d}^4x\, \mathcal{L}(\phi_n, \partial_\mu\phi_n)\right)$$
$$= \int \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_n}\delta\phi_n + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\delta(\partial_\mu\phi_n) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial^\mu\phi_n)}\delta(\partial_\lambda\partial^\lambda\phi_n) + \ldots\right) \overset{*}{=}$$

We integrate by parts and assume that $\phi_n(x) \to 0$ fast enough for $|\boldsymbol{x}| \to \infty$:

$$\overset{*}{=} \int \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} + \partial_\mu\partial^\mu\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\partial^\lambda\phi_n)} + \ldots\right)\delta\phi_n\,. \tag{4.29}$$

Usually the Lagrangian is of the form

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{4.30}$$

and there are no $(\partial_\mu\partial^\mu\phi)^n$-terms (the reason why this is will become clear later) and we can truncate Eq. (4.29) after the second term.
We get the equations of motion (EOM):

$$\boxed{\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} - \frac{\partial\mathcal{L}}{\partial\phi_n} = 0\,.} \tag{4.31}$$

We define the canonically conjugate fields:

$$\pi_n(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_n(x))} \tag{4.32}$$

and Hamiltonian:

$$H = \int \mathrm{d}^3x\; \pi_n(x)\partial_0\phi(x) - \int \mathrm{d}^3x\; \mathcal{L} = \int \mathrm{d}^3x\; \mathcal{H}(x)\,, \tag{4.33}$$

with the **Hamiltonian density** $\mathcal{H}(x)$.
<u>Note:</u> The field $\phi_n(x)$ is obtained in the limit of infinitely many coordinates labelled by n and $\boldsymbol{x}$ (continuous index):

$$q_n(t) \to q_{n,\boldsymbol{x}}(t) \to \phi_{n,\boldsymbol{x}}(t) = \phi_n(t, \boldsymbol{x}) = \phi_n(x)\,, \tag{4.34}$$
$$\sum_n \to \sum_{n,\boldsymbol{x}} \to \sum_n \int \mathrm{d}^3x\,. \tag{4.35}$$

4.5 Canonical quantization

We impose equal-time commutation relations:

$$[\phi_n(t,\boldsymbol{x}),\pi_m(t,\boldsymbol{y})] = i\hbar\delta_{nm}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{4.36}$$
$$[\phi_n(t,\boldsymbol{x}),\phi_m(t,\boldsymbol{y})] = [\pi_n(t,\boldsymbol{x}),\pi_m(t,\boldsymbol{y})] = 0\,. \tag{4.37}$$

The time-evolution is given by the Heisenberg equation:

$$\partial_t\phi_n(t,\boldsymbol{x}) = \frac{1}{i}[\phi_n(t,\boldsymbol{x}),H] \tag{4.38}$$

and it has the formal solution

$$\phi_n(t,\boldsymbol{x}) = e^{iHt}\phi_n(0,\boldsymbol{x})e^{-iHt}\,, \tag{4.39}$$

since H does not depend explicitly on t.

Check:

$$\begin{aligned}\partial_t\phi_n(t,\boldsymbol{x}) &= iH\underbrace{e^{iHt}\phi_n(0,\boldsymbol{x})e^{-iHt}}_{=\phi_n(t,\boldsymbol{x})}+\\ &\quad+\underbrace{e^{iHt}\phi_n(0,\boldsymbol{x})e^{-iHt}}_{=\phi_n(t,\boldsymbol{x})}(-iH)\\ &= \frac{1}{i}(\phi_n(t,\boldsymbol{x})H - H\phi_n(t,\boldsymbol{x}))\\ &= \frac{1}{i}[\phi_n(t,\boldsymbol{x}),H] \quad \checkmark\,.\end{aligned}$$

Comments:

- Imposing the commutation relations at one time, e.g. $t = t_0$, preserves them at all times t:

$$[\phi_n(t,\boldsymbol{x}),\pi_m(t,\boldsymbol{y})] \tag{4.40}$$
$$= \left[e^{iH(t-t_0)}\phi_n(t_0,\boldsymbol{x})e^{-iH(t-t_0)}, e^{iH(t-t_0)}\pi_m(t_0,\boldsymbol{y})e^{-iH(t-t_0)}\right] \tag{4.41}$$
$$= e^{iH(t-t_0)}\underset{\substack{\uparrow\\ i\hbar\delta_{nm}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})}}{[\phi_n(t_0,\boldsymbol{x}),\pi_m(t_0,\boldsymbol{y})]}e^{-iH(t-t_0)} = i\hbar\delta_{nm}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{4.42}$$

- We usually work in the **Heisenberg picture** (unless otherwise stated):

> The time-dependence is in the operator ϕ, the states are time-independent.

  This choice is natural, since the equations of motion are for the field operator. The field operators are the central objects.

4.6 Noether theorem for fields

In our relativistic QFT we start with a Lagrangian and use it to canonically quantize the fields to arrive at a quantum Hamiltonian. Why should we bother with a Lagrangian at all then?

Because: Lagrangian densities are Lorentz-scalars while Hamiltonians are **not** ($\mathcal{H} = T_{00}$, 00-component of the energy-momentum tensor $T_{\mu\nu}$). It is hard to know wether a random Hamiltonian is Lorentz-invariant or not.[2]

The Noether theorem is important:

[2] In condensed matter physics one can directly start with a Hamiltonian since relativistic invariance is usually irrelevant.

- If there are continuous symmetries, then it follows that there are associated conservation laws.
- The conservation law involves more than just conserved charges: there are **conserved currents** $J_\mu(x)$ at every space-time point x!

A symmetry is a transformation of the fields

$$\phi'_n(x) = \phi_n(x) + \epsilon \underbrace{F_n(\phi_m(x), \partial_\mu \phi_m(x))}_{=\delta_F \phi_n(x)} + \mathcal{O}(\epsilon^2), \tag{4.43}$$

which leaves the **action** invariant:

$$S[\phi'_n(x)] = S[\phi_n(x)] \tag{4.44}$$

F_n is an infinitesimal transformation and it is **continuous**, since it can be continuously connected to $\mathbb{1}$ in the limit $\epsilon \to 0$

Compare to a discrete transformation, e.g. $\phi'(x) = -\phi(x)$).

Examples:

a) Translation:

$$\phi'_n(x) = \phi_n(x+a) = \phi_n(x) + a^\mu \partial_\mu \phi_n(x) + \ldots \tag{4.45}$$

$$a^\mu = \epsilon^\mu \implies F_n = \partial_\mu \phi_n(x) \tag{4.46}$$

b) Phase rotation:

$$\mathcal{L} = \partial_\mu \phi^* \phi^\mu \phi\,, \quad \phi'(x) = e^{i\alpha}\phi(x) = \phi(x) + i\alpha\phi(x) + \ldots \tag{4.47}$$

$$\alpha = \epsilon \implies F = i\phi \tag{4.48}$$

4.6.1 Noether theorem

> For every continuous symmetry of the action there exists a local current associated with that symmetry which is conserved when the equations of motion are satified.

The current is

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} F_n(x) - k^\mu(x) \tag{4.49}$$

with

$$\partial_\mu j^\mu(x)\Big|_{\text{EOM}} = 0 \quad \text{and charge} \quad Q \equiv \int \mathrm{d}^3x\, j^0(x)\,, \tag{4.50}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t}\Big|_{\text{EOM}} = 0 \quad \text{(charge conservation)}\,. \tag{4.51}$$

Derivation: By assumption S is invariant, $\mathcal{L}$ can therefore only change by a total derivative, which we will denote with $\partial_\mu k^\mu$. Varying the action by a symmetry transformation δ_F therefore gives

$$\delta_F S = \int \mathrm{d}^4x\, \delta_F \mathcal{L} = \int \mathrm{d}^4x \left(\mathcal{L}(\phi'_n, \partial_\mu \phi'_n) - \mathcal{L}(\phi_n, \partial_\mu \phi_n)\right) \tag{4.52}$$

$$\stackrel{!}{=} \int \mathrm{d}^4x\, \epsilon \partial_\mu k^\mu = 0\,, \tag{4.53}$$

where for the last equality we assume, as usual, that the fields vanish fast enough as $|\boldsymbol{x}| \to \infty$.
So we obtain

$$\delta_F \mathcal{L} = \epsilon \partial_\mu k^\mu \tag{4.54}$$

and in particular $k^\mu = 0$ if also $\mathcal{L}$ is invariant.

Furthermore, we know:

$$\delta_F \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_n} \underbrace{\delta_F \phi_n}_{\epsilon F_n} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta_F \underbrace{(\partial_\mu \phi)}_{\epsilon \partial_\mu F_n} \tag{4.55}$$

$$\overset{(*)}{=} \epsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} F_n \right) \tag{4.56}$$

(∗) Use EOM

$$\frac{\partial \mathcal{L}}{\partial \phi_n} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right)$$

to eliminate

$$\frac{\partial \mathcal{L}}{\partial \phi_n} .$$

Subtracting Eq. (4.54) from Eq. (4.56) gives:

$$\partial_\mu j^\mu(x) = 0 \quad \text{with} \quad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n(x))} F_n(x) - k^\mu(x) \tag{4.57}$$

and the corresponding conserved charge:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3 x \, \partial_0 j^0(x) = -\int \mathrm{d}^3 x \, \nabla \cdot \boldsymbol{j}(x) = 0 \, , \tag{4.58}$$

where we used that

$$\partial_\mu j^\mu = \partial_0 j^0 + \nabla \cdot \boldsymbol{j} = 0 \, . \tag{4.59}$$

5

Quantizing the Scalar Field

We will develop the formalism of QFT using the simplest representation of the Poincaré group: the scalar field. Later we will get to QFTs with more complicated Poincaré-representations, like spin $\frac{1}{2}$ fermions and spin 1 vector bosons. This section will be brief and is more of a short review of the canonical method before we move on to the path integral approach.

5.0.1 Dimensions

As usual we set

$$\hbar = 1\,, \quad c = 1\,, \tag{5.1}$$

except in cases where we want to show explicit dependence.
Definition:

$$[A] \equiv \text{mass dimension of } A \tag{5.2}$$

Then, using the following relations:

1) $E = mc^2$,
2) $E^2 = m^2c^4 + \boldsymbol{p}^2c^2$,
3) $\lambda = \frac{h}{p}$

and $c = 1$ we get

$$[m_{\text{el.}}] = 1 \text{ (by definition)}\,, \tag{5.3}$$

$$[H] = [\text{energy}] \overset{1)}{=} [m] = 1\,, \tag{5.4}$$

$$[\boldsymbol{p}] \overset{2)}{=} [\text{energy}] = 1\,, \tag{5.5}$$

$$[x] = [\text{length}] \overset{3)}{=} \left[\frac{1}{\text{momentum}}\right] = [\text{mass}^{-1}] = -1\,. \tag{5.6}$$

Furthermore:

$$\left[\mathrm{d}^3x\right] = -3\,, \quad [\partial_\mu] = \left[\frac{1}{x}\right] = +1\,, \tag{5.7}$$

$$[\mathcal{L}] = [\mathcal{H}] = +4\,, \quad [S] = [\hbar] = 0\,. \tag{5.8}$$

So

$$[\text{time}] = [\text{length}] = \left[\text{energy}^{-1}\right] = \left[\text{momentum}^{-1}\right]. \tag{5.9}$$

5.0.2 Free real scalar field

We want to derive the mass dimension of the field ϕ:

$$[\phi] = \alpha = ? \tag{5.10}$$

$$S = \frac{1}{2}\int \mathrm{d}^d x \left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) \tag{5.11}$$

$\overset{\text{EOM}}{\Longrightarrow} (\partial^2 + m^2)\phi(x) = 0$
The equations of motion are linear in ϕ.

We request that $[S] = [\hbar] \stackrel{!}{=} 0$ and therefore:

$$0 = [\mathrm{d}^d x][\partial_\mu^2][\phi^2] \tag{5.12}$$

$$0 = -d + 2 + 2\alpha \tag{5.13}$$

$$[\phi] = \frac{d-2}{2} \qquad \begin{cases} d = 3+1 & [\phi] = 1 \\ d = 1+1 & [\phi] = 0 \end{cases}. \tag{5.14}$$

Ex: Check $m^2\phi^2$!

5.0.3 Interaction terms

We write the Lagrangian as follows:

We usually call all terms which are $\sim \phi^n$ with $n \leq 3$ interaction terms, since they lead to non-linear equations of motion.

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{int}}\,, \tag{5.15}$$

$$\text{with} \quad \mathcal{L}_{\text{kin}} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{5.16}$$

$$\text{and} \quad \mathcal{L}_{\text{int}} = \lambda_0 + \lambda_1\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda^4}{4!}\phi^4 + \ldots\,, \tag{5.17}$$

Not all terms need to be kept

λ_0 : only relevant in presence of gravity, aka the cosmological constant

λ_1 : remove with the field redefinition $\phi = \phi' - \frac{\lambda_1}{m^2}$,

(we will set λ_1 to zero).

The dimensions of the remaining terms are as follows:

$$[m] = 1\,, \quad [\lambda_3] = 1\,, \quad [\lambda_4] = 0\,. \tag{5.18}$$

How about terms with more than two ∂_μ's, for instance

$$(\partial_\mu\partial^\mu\phi)(\partial_\lambda\partial^\lambda\phi) \tag{5.19}$$

Adding these terms to the Lagrangian is possible, but they lead to ghosts, except if they are used in an effective QFT. We will come back to this later.
How about terms like

$$\lambda_{12}\phi^{12} \qquad \text{or} \qquad \lambda_*(\partial_\mu\phi\partial^\mu\phi)\phi^3 \tag{5.20}$$

Both of them have couplings with negative mass dimension: $[\lambda_{12}] = -8$, $[\lambda_*] = -3$. We will ignore these for now and argue later why.

<u>For now:</u>

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \quad \text{with} \quad V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda^4}{4!}\phi^4\,. \tag{5.21}$$

The Euler-Lagrange equations take the following form:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \,, \tag{5.22}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \implies \partial_\mu \partial^\mu \phi + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0 \,, \tag{5.23}$$

$$(\partial^2 + m^2)\phi + \underbrace{\frac{\lambda_3}{2}\phi^2 + \frac{\lambda_4}{3!}\phi^3}_{=(*)} = 0 \,, \tag{5.24}$$

where the $(*)$-terms are non-linear and represent (self-)interactions of the field ϕ.

5.1 *Free real scalar field: creation and annihilation operators*

We can solve the free theory

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \implies (\partial^2 + m^2)\phi = 0 \,, \tag{5.25}$$

Note, if you have not seen this before, I recommend my Relativity, Particles, Field script or e.g. chapter 3 of Srednicki - Quantum Field Theory.

exactly with a Fourier transformation:

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} a_{\boldsymbol{p}} + e^{ipx} a^\dagger_{\boldsymbol{p}} \right) , \tag{5.26}$$

where

$$p \cdot x = E_{\boldsymbol{p}} t - \boldsymbol{p} \cdot \boldsymbol{x} \,, \quad E_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2} \,. \tag{5.27}$$

$\phi(x)$ is a real solution of the Klein-Gordon equation, since the second term in the expansion is the complex conjugate of the first. Note that in $p^\mu = (p^0, \boldsymbol{p})^T$, p^0 is fixed and we take $p^0 = E_{\boldsymbol{p}}$ which satisfies the KG equation.

We use the integration measure $\frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}$ instead of $\frac{\mathrm{d}^3 p}{(2\pi)^3}$ because it is Lorentz-invariant. We can express it as

$$\frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} = \frac{\mathrm{d}^4 p}{(2\pi)^4}(2\pi)\delta(p^2 - m^2)\theta(p^0) \tag{5.28}$$

On the right-hand side p^0 is integrated over and the value $p^0 = E_{\boldsymbol{p}} = +\sqrt{m^2 + \boldsymbol{p}^2}$ is picked by $\delta(p^2 - m^2)\theta(p^0)$.

Ex: Show this!

5.1.1 *Quantization*

We postulate that $\phi(x)$ and $\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi}$ satisfy canonical, equal-time commutation relations:

$$[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = i\hbar\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \,. \tag{5.29}$$

This is also often confusingly referred to as **second quantization**. However, the fact that there are discrete modes is a classical phenomenon (think of the eigenmodes of a string). The two steps are in fact (1) interpret these modes as having energy $E = \hbar\omega$ and (2) quantize each mode as a harmonic oscillator. In that sense we are only quantizing once.

and all others $[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = [\pi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = 0$. For this to hold, $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ must satisfy

$$[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p} - \boldsymbol{q}) \,, \tag{5.30}$$

$$[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = [a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = 0 \,. \tag{5.31}$$

We call $a_{\boldsymbol{p}}$ creation and $a^\dagger_{\boldsymbol{p}}$ annihilation operators. Why do we choose these names? On one hand, they act analogously to the

ladder operators of the quantized harmonic oscillator, and on the other hand there is another, more important reason which will become apparent later.
Let us evaluate the commutator using Eq. (5.30) and Eq. (5.31), we easily find

$$x^0 \neq y^0 : \quad [\phi(x), \phi(y)] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) . \tag{5.32}$$

Since it will be useful later, let us define this object.
Definition:

$$[\phi(x), \phi(y)] \equiv \Delta(x-y) \, . \tag{5.33}$$

Note: $\Delta(x-y)$ is Lorentz-invariant since $\frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}$ and $e^{ip(x-y)}$ are both Lorentz-invariant!

Evaluating $\Delta(x-y)$ at equal times $x^0 = y^0 = t$:

$$[\phi(t, \boldsymbol{x}), \phi(t, \boldsymbol{y})] = \Delta(x-y) \Big|_{x^0 = y^0 = t} \tag{5.34}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \right) \tag{5.35}$$

$$= 0 \quad \checkmark \, , \tag{5.36}$$

where we performed the substitution $\boldsymbol{p} \to -\boldsymbol{p}$ in the second term of Eq. (5.35).
From this we get

$$[\phi(t, \boldsymbol{x}), \pi(t, \boldsymbol{y})] = \left(\frac{\partial}{\partial y^0} \Delta(x-y) \right) \Big|_{y^0 = x^0} \tag{5.37}$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} iE_{\boldsymbol{p}} \left(e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \right) \tag{5.38}$$

$$= i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \quad \checkmark \tag{5.39}$$

and finally, using a similar argument as for $[\phi(x), \phi(y)]\Big|_{x^0=y^0} = 0$, we get

$$[\pi(x), \pi(y)] \Big|_{x^0 = y^0} = 0 \, . \tag{5.40}$$

We can now prove more generally that

$$[\phi(x), \phi(y)] = 0 \quad \text{if} \quad (x-y)^2 < 0 \quad \text{(space-like separation)} \tag{5.41}$$

holds.

Proof: Since $\Delta(x-y)$ is Lorentz-invariant, we can evaluate it in any convenient frame. For spacelike $(x-y)^2 < 0$ we can boost to a frame where

$$x^0 = y^0 = 0 \tag{5.42}$$

and so

$$(x-y)^2 = (x^0 - y^0)^2 - (\boldsymbol{x}-\boldsymbol{y})^2 = -(\boldsymbol{x}-\boldsymbol{y})^2 \, . \tag{5.43}$$

What happens at **light-like** distances is actually quite subtle. Naively, one would assume that a particle of mass m should not have a causal connection on the light cone (since it moves slower than the speed of light). However, the state $\phi(\boldsymbol{x})|0\rangle$ is an eigen-state of position (see RPF 5.4.1) and therefore contains modes of arbitrary momentum, in particular it contains states with $|\mathbf{k}| \gg m$ which propagate a light speed. Additionally, we know that the time derivative of a time-like commutator cannot vanish, since it has to reproduce the canonical commutator Eq. (5.29) between ϕ and π in the limit that $x \to y$.

The statement Eq. (5.41) then follows because in Eq. (5.36) we showed that the equal-time commutator vanishes.

For time-like distances $\Delta(x-y)$ does **not** vanish, e.g. for $z^\mu = x^\mu - y^\mu = (t,0,0,0)$, we get

$$\Delta(z) = [\phi(t,\boldsymbol{x}), \phi(0,\boldsymbol{x})] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} (-2i)\sin(E_{\boldsymbol{p}} t) \neq 0\,, \qquad (5.44)$$

which is non-vanishing.
We conclude that we have defined a theory of **causal quantum fields**. For now we are working with free fields but this property will continue to be true for **interacting fields**[1].

[1] i.e. including $\lambda_3\phi^3 + \ldots$ -terms.

The Hamilton operator of the free theory is

$$\begin{aligned}
H_0 &= \int \mathrm{d}^3 x \, \big(\underbrace{\pi(x)}_{=\dot\phi(x)}\dot\phi(x) - \mathcal{L}_0\big) = \int \mathrm{d}^3 x \, \frac{1}{2}\Big((\partial_t\phi)^2 + (\nabla\phi)^2 + m^2\phi^2\Big) \\
&= (\ldots \text{ See ex. } \ldots) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \frac{1}{2} E_{\boldsymbol{p}} (a_{\boldsymbol{p}} a_{\boldsymbol{p}}^\dagger + a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}) \\
&\underset{Eq.\ (5.30)}{=} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} E_{\boldsymbol{p}} (\underbrace{a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}}_{\equiv n_{\boldsymbol{p}}}) + \underbrace{\int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \frac{E_{\boldsymbol{p}}}{2} (2\pi)^3 \delta^{(3)}(0) 2E_{\boldsymbol{p}}}_{\text{Zero-point energy}}\,,
\end{aligned}$$

and we defined the **number operator** $n_{\boldsymbol{p}}$, which has as eigenvalue the number of particles in an n-particle state with momentum $\boldsymbol{p}$. This suggests an interpretation as the energy being stored in $n_{\boldsymbol{p}}$ quanta with momentum $\boldsymbol{p}$ and energy $E_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2}$.

5.2 A first encounter with Infinities in QFTs

We separate occurring infinities in QFT into IR-divergences and UV-divergences:

1) IR-divergences are caused by assuming infinitely large scales (or arbitrarily low energies).

2) UV-divergences are caused by extrapolating to infinitely small distances (or infinite energies).

We define the ground (or vacuum) state $|0\rangle$ through

$$a_{\boldsymbol{p}}|0\rangle \equiv 0 \quad \forall \boldsymbol{p}\,. \qquad (5.45)$$

Then we get for the ground state energy

$$\begin{aligned}
H_0|0\rangle = E_0|0\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} E_{\boldsymbol{p}} \Big(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + E_{\boldsymbol{p}} (2\pi)^3 \delta^{(3)}(0)\Big)|0\rangle \qquad (5.46) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} E_{\boldsymbol{p}} \cdot E_{\boldsymbol{p}} (2\pi)^3 \delta^{(3)}(0)|0\rangle \qquad (5.47) \\
&= \text{"}\infty^{2}\text{"}|0\rangle\,. \qquad (5.48)
\end{aligned}$$

Infinities are a common occurrence in QFT. They might signal that one is asking the wrong question, seeking the wrong observable or that a wrong procedure is being used.
We can take care of the IR-divergences by considering a finite volume V:

$$(2\pi)^3\delta^{(3)}(\boldsymbol{p}=0) = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}x\int_{-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}y\int_{-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}z\; e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\Big|_{\boldsymbol{p}=0} = V\,. \tag{5.49}$$

We now observe that we should be calculating energy densities instead of the total energy stored in (formally) infinite space-time:

$$\mathcal{E} = \frac{E}{V} = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2}E_{\boldsymbol{p}} = \frac{4\pi}{2(2\pi)^3}\int_0^\infty \mathrm{d}p\;\boldsymbol{p}^2\sqrt{m^2+\boldsymbol{p}^2}\longrightarrow\infty\,. \tag{5.50}$$

The result is still infinite, but now we only have one "power" of infinity instead of two. This is a UV-divergence which is caused by assuming that the QFT holds up to arbitrary energies (or up to the smallest distances): this is certainly too strong of an assumption. Let us therefore cut off the integral:

$$\mathcal{E}_0(\Lambda) = \frac{1}{4\pi^2}\int_0^\Lambda \mathrm{d}p\;\boldsymbol{p}^2\sqrt{m^2+\boldsymbol{p}^2} = a\Lambda^4+\ldots\,, \tag{5.51}$$

where a is some constant and we suppressed less divergent terms than Λ^4 after the last equality. We do not extrapolate beyond the scale Λ.

This is called **regularization**. The goal will be to find a regulator which respects the symmetries of the QFT. The one chosen above breaks Lorentz-invariance.

The next step is **renormaliziation**. We can renormalize the vacuum energy by adding a cosmological constant (cc):

$$\mathcal{L}_{\text{new}} = \mathcal{L} - V_0\,,\quad V_0 = \text{const.}\,,\quad [V_0] = +4\,. \tag{5.52}$$

In fact

$$\int \mathrm{d}^4x\,\sqrt{-g}V_0$$

is the correct form of the cc.

V_0 is a **bare** parameter which we choose to cancel the zero-point energy and give the **observed** value of the vacuum energy:

$$V_0(\Lambda) = -\mathcal{E}_0(\Lambda) + \chi_{\text{exp}}\,, \tag{5.53}$$

with $\chi_{\text{exp}} \approx (10^{-3}\text{ eV})^4$ the experimentally measured value. The resulting observable for the energy density of $|0\rangle$ is finite:

$$\frac{1}{V}H_0|0\rangle = (\mathcal{E}_0(\Lambda)+V_0(\Lambda))|0\rangle \tag{5.54}$$
$$= (\mathcal{E}_0(\Lambda)-\mathcal{E}_0(\Lambda)+\chi_{\text{exp}})|0\rangle \tag{5.55}$$
$$= \mathcal{E}_{\text{obs}}|0\rangle\,, \tag{5.56}$$

where $\mathcal{E}_{\text{obs}} = \chi_{\text{exp}}$ is clearly not a prediction but a parameter taken from experiments. Note that the **bare** parameter $V_0(\Lambda)$ is infinite in the limit $\Lambda\longrightarrow\infty$.
Much more on this later, see chapter 9.

5.3 Particle states and quanta of the scalar field

We now proceed to construct the Hilbert space of the particle states. We start from a **vacuum** $|0\rangle$:

$$\boxed{a_{\boldsymbol{p}}|0\rangle = 0 \quad \forall \boldsymbol{p}\,, \quad \langle 0|0\rangle = 1\,.} \tag{5.57}$$

We define a single particle state with momentum $\boldsymbol{p}$:

$$\boxed{|\boldsymbol{p}\rangle \equiv a^\dagger_{\boldsymbol{p}}|0\rangle\,,} \tag{5.58}$$

which leads to

$$\langle \boldsymbol{p}|\boldsymbol{k}\rangle = \langle 0|a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{k}}|0\rangle \tag{5.59}$$

$$\underset{Eq.\ (5.30)}{=} \overset{=1}{\langle 0|0\rangle} 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{k}) + \underset{=0}{\langle 0|a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{p}}|0\rangle} \tag{5.60}$$

$$= 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{k})\,. \tag{5.61}$$

We know that $|\boldsymbol{p}\rangle$ has energy $E_{\boldsymbol{p}} = \sqrt{m^2+\boldsymbol{p}^2}$. To proceed, recall the chapter about the Noether theorem and that 4-momentum is the Noether current of 4-translations:

$$j^\mu_\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} F_\alpha - k^\mu_\alpha \tag{5.62}$$

with

$$\phi(x+a) = \phi(x) + a^\alpha\partial_\alpha\phi(x) + \ldots\,, \quad F_\alpha = \partial_\alpha\phi \tag{5.63}$$

and

$$\mathcal{L}'(x) = \mathcal{L}(x+a) = \mathcal{L}(x) + a^\alpha\partial_\alpha\mathcal{L}(x) + \ldots\,, \tag{5.64}$$

$$\partial_\alpha\mathcal{L} = \partial_\mu(\delta^\mu_\alpha\mathcal{L}) \implies k^\mu_\alpha = \delta^\mu_\alpha\mathcal{L}\,, \tag{5.65}$$

so the conserved current reads

$$\boxed{T_\alpha{}^\mu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi - \delta^\mu_\alpha\mathcal{L}\,.} \tag{5.66}$$

This is the **energy-momentum tensor**. It contains four conserved charges (integrating over the components with $\mu = 0$):

$$P^\alpha = \int \mathrm{d}^3x\, T^{\alpha 0} = \int \mathrm{d}^3x \left(\underset{=\pi}{\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}}\partial^\alpha\phi - \eta^{\alpha 0}\mathcal{L}\right) \tag{5.67}$$

- **Energy** ($\alpha = 0$):

$$P^0 = \int \mathrm{d}^3x\,(\pi\dot{\phi} - \mathcal{L}) = H_0\,, \tag{5.68}$$

  time translation invariance implies energy conservation.

- **Momentum** ($\alpha = i$):

$$P^i = \int \mathrm{d}^3x\ \pi\partial^i\phi \quad \text{or} \quad \boldsymbol{P} = -\int \mathrm{d}^3x\ \pi\nabla\phi\,, \tag{5.69}$$

since $\partial^i = -\partial_i = -\frac{\partial}{\partial x^i} = -\nabla^i$.

Usually: $i = 1, 2, 3$

Let us express P^μ using the $a^\dagger$, a-expansion of $\phi(x)$:

$$\boldsymbol{P} = \int \mathrm{d}^3x\ \pi\nabla\phi = (\text{... See ex. ...}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\boldsymbol{p}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\,, \tag{5.70}$$

where we suppressed the contribution from the zero-point energy.
Let us apply this to a single-particle state:

$$\begin{aligned}
\boldsymbol{P}|\boldsymbol{p}\rangle &= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\boldsymbol{k}}}\boldsymbol{k}a^\dagger_{\boldsymbol{k}}a_{\boldsymbol{k}}a^\dagger_{\boldsymbol{p}}|0\rangle && (5.71)\\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\boldsymbol{k}}}\boldsymbol{k}a^\dagger_{\boldsymbol{k}}\Big(a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{k}} + 2E_{\boldsymbol{k}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{k})\Big)|0\rangle && (5.72)\\
&= 0 + \int \mathrm{d}^3k\ \delta^{(3)}(\boldsymbol{p}-\boldsymbol{k})\boldsymbol{k}a^\dagger_{\boldsymbol{k}}|0\rangle && (5.73)\\
&= \boldsymbol{p}a^\dagger_{\boldsymbol{p}}|0\rangle && (5.74)\\
&= \boldsymbol{p}|\boldsymbol{p}\rangle\,. && (5.75)
\end{aligned}$$

We see that $|\boldsymbol{p}\rangle$ is a momentum eigenstate with momentum $\boldsymbol{p}$. The same way one gets

$$H_0|\boldsymbol{p}\rangle = E_{\boldsymbol{p}}|\boldsymbol{p}\rangle, \quad E_{\boldsymbol{p}} = \sqrt{m^2+\boldsymbol{p}^2}\,. \tag{5.76}$$

Ex: Check it!

In conclusion: $|\boldsymbol{p}\rangle$ describes a relativistic particle with mass m, momentum $\boldsymbol{p}$ and energy $E_{\boldsymbol{p}} = \sqrt{m^2+\boldsymbol{p}^2}$.
We can find a conserved angular momentum

See ex.

$$J_i = Q_i = \epsilon_{ijk}\int \mathrm{d}^3x\ (x^jT^{0k} - x^kT^{0j}) \tag{5.77}$$

and let it act on a one-particle state with zero momentum:

$$J_i|\boldsymbol{p}=0\rangle = 0\,. \tag{5.78}$$

Hence the quanta of the scalar field are spin 0 particles!

5.3.1 Multiparticle states

From the vacuum $|0\rangle$ one can construct multiparticle states:

$$|0\rangle, \quad a^\dagger_{\boldsymbol{p}}|0\rangle\,, \quad a^\dagger_{\boldsymbol{p_1}}a^\dagger_{\boldsymbol{p_2}}|0\rangle\,, \quad \ldots \tag{5.80}$$

$$|\boldsymbol{p_1}\boldsymbol{p_2}...\boldsymbol{p_N}\rangle = \frac{1}{\mathcal{N}}\,a^\dagger_{\boldsymbol{p_1}}a^\dagger_{\boldsymbol{p_2}}...a^\dagger_{\boldsymbol{p_N}}|0\rangle\,. \tag{5.81}$$

With the normalization

$$\mathcal{N} = \prod_i \sqrt{n_i(\boldsymbol{p})!} \tag{5.79}$$

or the index i runs over all different values of the $\boldsymbol{p_i}$ in $\{\boldsymbol{p_1}, \boldsymbol{p_2}, ..., \boldsymbol{p_N}\}$ and $n_i(\boldsymbol{p})$ is the number of times $\boldsymbol{p}$ appears in $\{\boldsymbol{p_1}, \boldsymbol{p_2}, ..., \boldsymbol{p_N}\}$.
See ex.

The state in Eq. (5.81) has total momentum $\boldsymbol{p} = \boldsymbol{p_1} + \boldsymbol{p_2} + ... + \boldsymbol{p_N}$.
We can write Eq. (5.81) also as

$$|\boldsymbol{p_1}\boldsymbol{p_2}...\boldsymbol{p_N}\rangle = \frac{1}{\sqrt{N!}}\left(\prod_i \frac{1}{\sqrt{n_i(\boldsymbol{p})!}}\right)\sum_{\sigma\in S_N} P_\sigma\Big(|\boldsymbol{p_1}\rangle\otimes|\boldsymbol{p_2}\rangle\otimes...\otimes|\boldsymbol{p_N}\rangle\Big)\,, \tag{5.82}$$

where the sum runs over all permutations of the momenta in $\{\boldsymbol{p_1}, \boldsymbol{p_2}, ..., \boldsymbol{p_N}\}$.
We see in Eq. (5.81) that the state is automatically symmetric under the exchange of any two particles, hence the state is **bosonic**.

5.3.2 Fock space

The Fock space (FS) is the union of all N-particle Hilbert spaces, which are totally symmetric as discussed above, on which the field operators act:

$$F^S = \bigoplus_{N=0}^{\infty} H_N^S \,, \tag{5.83}$$

$$H_N^S = \underbrace{H \otimes H \otimes ... \otimes H}_{\text{symmetrized, } N \text{ factors}} \,, \tag{5.84}$$

$$H_0^S = \mathrm{span}(|0\rangle) \,, \tag{5.85}$$

where H is a single-particle Hilbert space and H_0^S is the 1-dimensional Hilbert space spanned by $|0\rangle$.
We see that QFT has resolved the wave-particle duality of early quantum theory. The fundamental entity is the quantum field.
We count the number of particles in a state using the number operator

$$N = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \,, \tag{5.86}$$

which satisfies

$$N|\boldsymbol{p_1 p_2 ... p_N}\rangle = n|\boldsymbol{p_1 p_2 ... p_N}\rangle \tag{5.87}$$

and it clearly commutes with the Hamiltonian of the free theory H_0:

$$[H_0, N] = 0 \,. \tag{5.88}$$

Therefore, in the free theory the particle number is conserved. This will **no longer** hold once we include interactions, e.g. $\mathcal{L}_0 \to \mathcal{L}_0 + \frac{\lambda_3}{3!}\phi^3 + \frac{\lambda_4}{4!}\phi^4 + ...$

5.3.3 Meaning of the state $\phi(x)|0\rangle$

5.4 The complex scalar field and charges

We now consider complex scalar fields, where ϕ in non-hermitian: $\phi \neq \phi^\dagger$. We could treat this using our formalism for real fields with

$$\phi(x) = \frac{1}{\sqrt{2}}\Big(\phi_1(x) + i\phi_2(x)\Big) \,, \quad \text{with } \phi_1, \phi_2 \text{ real fields} \,, \tag{5.89}$$

it is however more insightful to work directly with ϕ and $\phi^\dagger$ as **independent** variables.
The Hamiltonian density reads

$$\mathcal{H} = \dot{\phi}_1\pi_1 + \dot{\phi}_2\pi_2 - \mathcal{L}$$

$$\mathcal{H} = \pi\dot{\phi} + \pi^\dagger\dot{\phi}^\dagger - \mathcal{L} \tag{5.90}$$

ans we see that it is hermitan!
The Lagrangian of the complex scalar field is

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - V(\phi, \phi^\dagger) \,. \tag{5.91}$$

Note the absence of the factor $\frac{1}{2}$ for the canonical normalization. $V(\phi, \phi^\dagger)$ must contain only real combinations of ϕ, $\phi^\dagger$:

$$V(\phi, \phi^\dagger) = m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \tag{5.92}$$

$$+ [\lambda_4\phi^2 + \lambda_{31}\phi^3 + \lambda_{32}\phi^2\phi^\dagger + \lambda_{41}\phi^4 + \lambda_{42}\phi^3\phi^\dagger + \text{h.c.}] \tag{5.93}$$

$$m^2\,, \lambda \in \mathbb{R}\,, \quad \lambda_{21}\,, \lambda_{31}\,, \lambda_{41}\,, \lambda_{42} \in \mathbb{C}\,. \tag{5.94}$$

For a general $V(\phi, \phi^\dagger)$ the above Lagrangian describes a theory of two real spin 0 fields with different masses m_1, m_2, the corresponding fields are linear combinations of ϕ_1, ϕ_2.
For the complex scalar field we can consider a more interesting internal symmetry which finds regular applications: the **$U(1)$ symmetry** of global rephasings:

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x)\,, \quad \alpha \in \mathbb{R}\,. \tag{5.95}$$

This restricts the Lagrangian to exclude the terms in [...] in Eq. (5.93). The EOM then reads

$$\Box\phi + \frac{\mathrm{d}V}{\mathrm{d}\phi^\dagger} = (\Box + m^2)\phi + \frac{\lambda}{2}(\phi^\dagger\phi)\phi = 0 \tag{5.96}$$

and the Hamiltonian

$$H = \int \mathrm{d}^3x \left\{ \frac{\partial\phi^\dagger}{\partial t}\frac{\partial\phi}{\partial t} + (\nabla\phi^\dagger)(\nabla\phi) + m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \right\}, \tag{5.97}$$

where $\pi = \dot{\phi}^\dagger$.

5.4.1 Free theory and particle states of complex scalar field

The free EOM is

$$(\Box + m^2)\phi = 0 \tag{5.98}$$

and since ϕ does not have to be real/hermitian anymore we **try** the ansatz

This will turn out to be **wrong**!

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{-ipx} a_{\boldsymbol{p}}\,. \tag{5.99}$$

This solves the EOM if $E_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2}$ and it also satisfies the canonical commutation relations Eq. (5.29).

However, this ansatz is **not** correct, because it is **not consistent with causality**! We check the causality condition, e.g. between ϕ and $\phi^\dagger$ at space-like distances:

$$[\phi^\dagger(x), \phi(y)] = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{ip(x-y)} \neq 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{5.100}$$

This does not vanish for space-like separation, and therefore

$$[\mathcal{H}(x), \mathcal{H}(y)] \neq 0 \quad \text{for} \quad (x-y)^2 < 0\,. \tag{5.101}$$

We know how to construct causal QFTs if we also add a term like $+e^{ipx}a^\dagger_{\boldsymbol{p}}$ in Eq. (5.99). Since $\phi(x)$ is now **not** hermitian, we introduce a second set of annihilation and creation operators $b_{\boldsymbol{p}}$, $b_{\boldsymbol{p}}^{\dagger}$ which satisfy

$$[b_{\boldsymbol{p}}, b_{\boldsymbol{q}}^{\dagger}] = 2E_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}) \tag{5.102}$$

and all other commutators except $[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^{\dagger}]$ vanish.

The free EOM of the complex field have therefore the solution

Correct

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} a_{\boldsymbol{p}} + e^{ipx} b_{\boldsymbol{p}}^{\dagger} \right) . \tag{5.103}$$

We could have added $b_{\boldsymbol{p}}^{\dagger}$ with a coefficient λ, but we know from $[\phi(x), \phi(y)] = 0$, $(x-y)^2 < 0$ that $|\lambda| = 1$ must hold, and therefore $\lambda = e^{i\varphi}$. Such a phase can be absorbed by a redefinition of $b_{\boldsymbol{p}}^{\dagger}$ and the states.

For the complex field we have the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi^{\dagger} \tag{5.104}$$

and the canonical commutation relations

$$[\phi(x), \pi(y)]\Big|_{x^0=y^0} = i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{5.105}$$

while all other equal-time commutators vanish.

Interpretation: relativistic causality requires that the complex $U(1)$-invariant scalar field describes **two particles with equal mass** (since their energies have the same form, $E_{\boldsymbol{p}} = \sqrt{m^2 + \boldsymbol{p}^2}$) encapsulated in two different creation operators $a_{\boldsymbol{p}}^{\dagger}$, $b_{\boldsymbol{p}}^{\dagger}$. There are particles and anti-particles.

The free Hamiltonian is

$$H_0 = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} E_{\boldsymbol{p}} \left(a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} + b_{\boldsymbol{p}}^{\dagger} b_{\boldsymbol{p}} \right) , \tag{5.106}$$

where the zero-point energy was suppressed.

The internal $U(1)$-symmetry $\phi \to e^{-i\epsilon}\phi$ leads to a conserved current:

$$\delta\phi = -i\epsilon\phi = \epsilon F_{\phi} , \quad \delta\phi^{\dagger} = +i\epsilon\phi^{\dagger} = \epsilon F_{\phi^{\dagger}} , \tag{5.107}$$

Conserved current

$$j_{\mu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi)} F_{\phi} + \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi^{\dagger})} F_{\phi^{\dagger}} - \underset{\substack{\uparrow \\ =0}}{k_{\mu}} \tag{5.108}$$

$$= -i(\phi \partial_{\mu}\phi^{\dagger} - \phi^{\dagger}\partial_{\mu}\phi) \tag{5.109}$$

and k^{μ} vanishes since $\mathcal{L}$ is invariant. The associated conserved charge is

$$Q = \int \mathrm{d}^3x \, j^0(x) = -i \int \mathrm{d}^3x \, (\phi \partial_t \phi^{\dagger} - \phi^{\dagger}\partial_t\phi) \tag{5.110}$$

$$= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(a_{\boldsymbol{p}}^{\dagger} a_{\boldsymbol{p}} - b_{\boldsymbol{p}}^{\dagger} b_{\boldsymbol{p}} \right) = N_a - N_b , \tag{5.111}$$

with N_a, N_b the number operators for the a- and b-particles, respectively.

We define the particles as the a-particles, which have charge $+1$, and the antiparticles as the b-particles, which have charge -1; the number of the particles minus the number of the antiparticles determines the total charge Q.

Generally: relativistic causality requires that for any particle with non-zero charge there exists a corresponding antiparticle (with the same mass).
Since we know that here $\frac{d}{dt}Q = 0 = \partial_t Q$ and $\partial_t Q = i[H_0, Q]$ hold, we also know that

$$[H_0, Q] = 0\,. \tag{5.112}$$

This property will still be true once we move to interacting field theories, since [H,Q]=0 is a result of Noether's theorem.

6

Interacting Fields, the Interaction Picture, Perturbation Theory

6.1 Time dependence of interacting fields

Reminder: in the Heisenberg picture the operators are time dependent, while the states are time-independent.

$$\phi_H(x) = \phi_H(t, \boldsymbol{x}) = e^{iH_0 t}\phi_S(\boldsymbol{x})e^{-iH_0 t} \tag{6.1}$$

States:

$$|\psi(t)\rangle_S = e^{-iHt}|\psi\rangle_H ,$$

with $|\psi\rangle_H$ time-independent. They coincide at $t = 0$:

$$|\psi(0)\rangle_S = |\psi\rangle_H$$

$$_H\langle\psi|O_H(t)|\psi\rangle_H = {}_S\langle\psi(t)|O_S|\psi(t)\rangle_S$$

and we denote Heisenberg- and Schrödinger-operators with the subscripts H, S, respectively.
We have solved the real (free) theory including the time-dependence as

$$\phi_H(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(e^{-ipx}a_{\boldsymbol{p}} + e^{ipx}a^\dagger_{\boldsymbol{p}}\right) \tag{6.2}$$

where the time-dependence is in the exponentials and $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$ are time-independent.
Let us now step beyond the (boring) free theory and add interactions:

$$\mathcal{L} = \underset{\substack{\uparrow \\ \mathcal{L}_0 \text{ free/bilinear}}}{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2} - \underset{\substack{\uparrow \\ \mathcal{L}_{\text{int}} \text{ interaction/} \\ \text{trilinear and higher}}}{\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4} \tag{6.3}$$

and the EOM are

$$(\Box + m^2)\phi = -\frac{\lambda_3}{2}\phi^2 - \frac{\lambda_4}{3!}\phi^3 . \tag{6.4}$$

In general, there is no exact solution available to EOM of the above form, except for rare, special situations.
We could include the time-dependence of the terms $\lambda_3\phi^3$, $\lambda_4\phi^4$ in the time-dependence of the creation and annihilation operators:

$$\mathcal{L}_0 : \ a_{\boldsymbol{p}} , \ a^\dagger_{\boldsymbol{p}} \longrightarrow \mathcal{L}_0 + \mathcal{L}_{\text{int}} : \ a_{\boldsymbol{p}}(t) , \ a^\dagger_{\boldsymbol{p}}(t) , \tag{6.5}$$

but this is not useful, since this would only shift the problem to the unknown time-dependence of $a_{\boldsymbol{p}}(t)$, $a^\dagger_{\boldsymbol{p}}(t)$. Furthermore, there is no known general solution of

$$\phi(t, \boldsymbol{x}) = e^{iHt}\phi(0, \boldsymbol{x})e^{-iHt} , \quad H = H_0 + H_{\text{int}} , \tag{6.6}$$

with H_0 the free Hamiltonian and $H_{\rm int}$ the interaction Hamiltonian, or

$$\frac{\partial}{\partial t}\phi(t,\boldsymbol{x}) = i[H,\phi(t,\boldsymbol{x})]\,. \tag{6.7}$$

However, in the case of a **weak interaction**, we can solve the equation in a **perturbative** expansion.
It is convenient to work in the **interaction picture**, which is somewhat in-between the Heisenberg and Schrödinger pictures:

Interaction picture

For the H_0-time-dependence we use the Heisenberg picture, for the H_I-time-dependence the Schrödinger picture.

The exponential in 6.8 removes the H_0-time-dependence from the states.

$$|\psi(t)\rangle_I = e^{iH_0t}|\psi(t)\rangle_S \tag{6.8}$$
$$O_I(t) = e^{iH_0t}O_S e^{-iH_0t} \tag{6.9}$$

If λ_3, $\lambda_4 \ll 1$, then the most important time-dependence is generated by H_0.
The states in the Schrödinger picture satisfy the Schrödinger equation,

$$i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle_S = H_S|\psi(t)\rangle_S\,. \tag{6.10}$$

Using Eq. (6.8) in this equation we get

$$i\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-iH_0t}|\psi(t)\rangle_I\right) = (H_0+H_I)e^{-iH_0t}|\psi(t)\rangle_I\,, \tag{6.11}$$
$$\implies\quad i\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle_I = H_I|\psi(t)\rangle_I\,. \tag{6.12}$$

We now transition from the Heisenberg to the Interaction picture:

- For reference time (usually $t_0=0$) we fix

ϕ_H contains $H = H_0 + H_{\rm int}$-time-dependence

$$\phi_I(0,\boldsymbol{x}) = \phi_H(0,\boldsymbol{x})\,. \tag{6.13}$$

- At $t_0 = 0$ we split

$$H = H_0^H + H_{\rm int}^H = H_0^I + H_{\rm int}^I \tag{6.14}$$

  and at $t=0$ we also have $H_0^H = H_0^I$ because of Eq. (6.13).

- We define

$$\phi_I(t,\boldsymbol{x}) = e^{iH_0t}\phi_I(0,\boldsymbol{x})e^{-iH_0t}\,, \tag{6.15}$$

  with $H_0 = H_0^H(t=0)$, since H_0 is not time-independent in the Heisenberg picture even if H is (we could also use H_0^I since it is time-independent in the Interaction picture).

  Therefore we have

$$\frac{\partial}{\partial t}\phi_I(t,\boldsymbol{x}) = i[H_0^I,\phi_I(t,\boldsymbol{x})]\,. \tag{6.16}$$

- The time-evolution of the states in the interaction picture is given by

$$|\psi_I(t)\rangle = e^{iH_0t}\underbrace{e^{-iHt}|\psi\rangle_H}_{\substack{\text{time-dep.}\\ \text{Schrödinger state}}}\,, \tag{6.17}$$

where $|\psi\rangle_H$ is the time-independent Heisenberg state and e^{iH_0t} removes the H_0-time-evolution from the Schrödinger state.

We see that the two pictures are equivalent:

$$ {}_H\langle\psi|O_H(t)|\psi\rangle_H = {}_H\langle\psi|e^{iHt}\underbrace{O(0)}_{=O_I(0)}e^{-iHt}|\psi\rangle_H \tag{6.18}$$

$$= {}_H\langle\psi|e^{iHt}e^{-iH_0t}O_I(t)e^{iH_0t}e^{-iHt}|\psi\rangle_H \tag{6.19}$$

$$= {}_I\langle\psi(t)|O_I(t)|\psi(t)\rangle_I\,. \tag{6.20}$$

- We extract the Interaction picture interaction Hamiltonian H_I through

$$t=0: \quad H_{\text{int}}(\phi,\pi) = H^I_{\text{int}}(\phi_I,\pi_I) \tag{6.21}$$

and naturally we have

$$\pi_I = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_I)} \quad \text{or} \quad \dot\phi_I = i[H^I_0,\phi_I]\,. \tag{6.22}$$

$\dot\phi_I = \dot\phi_I(\phi_I,\pi_I)$ may be different to $\dot\phi = \dot\phi(\phi,\pi)$ if time derivatives are present in H_{int}.[1]

[1] which usually isn't the case since in most applications we have $H_{\text{int}} = V(\phi)$.

Important advantage: The interaction picture field operators have the same time-evolution as the free field operators, we can therefore express $H^I_{\text{int}}(\phi_I,\pi_I)$ in terms of free creation and annihilation operators. This is advantageous, since it is easy to calculate matrix elements in (free) Fock space.

Example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \mathcal{L}_{\text{int}}(\phi)\,, \quad \text{with e.g.} \quad \mathcal{L}_{\text{int}} = -\frac{\lambda_3}{3!}\phi^3\,. \tag{6.23}$$

We get

$$\mathcal{H} = \underbrace{\frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right)}_{=\mathcal{H}_0} \underbrace{-\mathcal{L}_{\text{int}}(\phi)}_{=+\mathcal{H}_{\text{int}}}\ . \tag{6.24}$$

Going over to the Interaction picture we have

$$\dot\phi_I(t,\boldsymbol{x}) = i[H^I_0(t),\phi_I(t,\boldsymbol{x})] = \int \mathrm{d}^3y\ i[\mathcal{H}^I_0(t,\boldsymbol{y}),\phi_I(t,\boldsymbol{x})] \tag{6.25}$$

$$= \pi_I(t,\boldsymbol{x})\,, \tag{6.26}$$

where for the last equality we used the canonical commutation relations and the commutator identity $[A^2,B] = A[A,B] + [A,B]A$. This means that the Legendre transformation is not affected and we have

$$\boxed{H^I_{\text{int}} = -\int \mathrm{d}^3x\ \mathcal{L}_{\text{int}}(\phi_I)\,.} \tag{6.27}$$

This holds almost always.

6.2 Perturbative expansion (Dyson's formula)

At a fixed time $t_0 = 0$ we can expand $\phi(x)$ as before:

$$\phi(0, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{i\boldsymbol{x}\cdot\boldsymbol{p}} a_{\boldsymbol{p}} + e^{-i\boldsymbol{x}\cdot\boldsymbol{p}} a^\dagger_{\boldsymbol{p}} \right) . \tag{6.28}$$

We now want to capture the full time dependence in a **perturbative** expansion.
In the Heisenberg picture we would have

$$\phi_H(t, \boldsymbol{x}) = e^{iH(t-t_0)} \phi_H(t_0, \boldsymbol{x}) e^{-iH(t-t_0)} , \tag{6.29}$$

for $H_{\text{int}} \equiv 0$, we have $H = H_0$ and

$$\phi_H(t, \boldsymbol{x}) \Big|_{H_{\text{int}}=0} = e^{iH_0(t-t_0)} \phi_H(t_0, \boldsymbol{x}) e^{-iH_0(t-t_0)} = \phi_I(t, \boldsymbol{x}) . \tag{6.30}$$

For perturbative H_{int} (e.g. $\lambda_3 \ll 1$) we get the most important part of the time evolution:

$$\phi_I(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} a_{\boldsymbol{p}} + e^{ipx} a^\dagger_{\boldsymbol{p}} \right) \Big|_{x^0 = t - t_0} . \tag{6.31}$$

We know further:

$$\begin{aligned} \phi_H(t, \boldsymbol{x}) &= e^{iH(t-t_0)} e^{-iH_0(t-t_0)} \phi_I(t, \boldsymbol{x}) e^{iH_0(t-t_0)} e^{-iH(t-t_0)} & (6.32) \\ &\equiv U^\dagger(t, t_0) \phi_I(t, \boldsymbol{x}) U(t, t_0) , & (6.33) \end{aligned}$$

with the unitary operator

$$\boxed{U(t, t_0) \equiv e^{iH_0(t-t_0)} e^{-iH(t-t_0)}} , \tag{6.34}$$

$U(t_0, t_0) = \mathbb{1}$
$U(t_2, t_1) = U(t_2, t_0) U^{-1}(t_1, t_0)$

which is the interaction picture propagator.
The states evolve as

$$|\psi(t)\rangle_I = U(t, t_0) |\psi(t_0)\rangle_I . \tag{6.35}$$

Then:

$$\boxed{i \frac{\partial}{\partial t} U(t, 0) = H^I_{\text{int}}(t) U(t, 0)} , \tag{6.36}$$

with

$$H^I_{\text{int}}(t) = e^{iH^I_0 t} H^I_{\text{int}}(0) e^{-iH^I_0 t} , \tag{6.37}$$

since

$$\begin{aligned} i \frac{\partial}{\partial t} U(t, 0) &= i \frac{\partial}{\partial t} \left(e^{iH_0 t} e^{-iHt} \right) \\ &= -H_0 e^{iH_0 t} e^{-iHt} + e^{iH_0 t} \underbrace{(H_0 + H_{\text{int}}(0))}_{=H(0)} \mathbb{1} e^{-iHt} \\ &= -H_0 e^{iH_0 t} e^{-iHt} + e^{iH_0 t} (H_0 + H_{\text{int}}(0)) e^{-iH_0 t} e^{iH_0 t} e^{-iHt} \\ &= -H_0 e^{iH_0 t} e^{-iHt} + \underbrace{e^{iH_0 t} e^{-iH_0 t}}_{=\mathbb{1}} H_0 e^{iH_0 t} e^{-iHt} + \\ &\quad + \underbrace{e^{iH_0 t} H_{\text{int}}(0) e^{-iH_0 t}}_{=H^I_{\text{int}}(t)} \underbrace{e^{iH_0 t} e^{-iHt}}_{=U(t,0)} \\ &= H^I_{\text{int}}(t) U(t, 0) . \quad \checkmark \end{aligned}$$

From now on we will abbreviate $U(t,0) = U(t)$.

Naively, we would expect the solution of Eq. (6.36) to be **Wrong!**

$$U(t) \sim \exp\left(-i \int_0^t \mathrm{d}t' \; H_I(t') \right), \tag{6.38}$$

but this ignores that the exponent is an operator with

$$[H_I(t), H_I(t')] \neq 0 \quad \text{for} \quad t \neq t' \,. \tag{6.39}$$

The iterative solution is the **Dyson series**:

$$U(t) = \mathbb{1} - i \int_{t_0=0}^{t} \mathrm{d}t_1' \; H^I_{\text{int}}(t_1') + \tag{6.40}$$

$$+ (-i)^2 \int_0^t \mathrm{d}t_1' \int_0^{t_1'} \mathrm{d}t_2' \; H^I_{\text{int}}(t_1') H^I_{\text{int}}(t_2') \tag{6.41}$$

$$+ ... + (-i)^n \int_0^t \mathrm{d}t_1' \int_0^{t_1'} \mathrm{d}t_2' ... \int_0^{t_{n-1}'} \mathrm{d}t_n' \; H^I_{\text{int}}(t_1') ... H^I_{\text{int}}(t_n') \tag{6.42}$$

$$+ ... \tag{6.43}$$

Note that the operator with the larger time is always on the left, we have $t_1' > t_2' > ... > t_n'$.
We define the **time-ordered product**:

$$T\{A(t_1)B(t_2)\} = \begin{cases} A(t_1)B(t_2) & t_1 > t_2 \\ B(t_2)A(t_1) & t_2 > t_1 \end{cases} \tag{6.44}$$

and we generalize this to n operators: the time-ordered product T orders operators from left to right by decreasing time argument.
We now observe that the operators in the Dyson series stand in time order and note that

$$T\{A(t_1)B(t_2)\} = T\{B(t_2)A(t_1)\} \,. \tag{6.45}$$

We simplify the expressions in the series by observing that

$$\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \; H^I_{\text{int}}(t_1) H^I_{\text{int}}(t_2) = \frac{1}{2} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \; T\{H^I_{\text{int}}(t_1) H^I_{\text{int}}(t_2)\} \,, \tag{6.46}$$

where all integrals have the same integration boundaries after the equality sign. This can be made clearer with graphic Fig. 6.1.
The right-hand side of Eq. (6.46) counts everything twice, since the integrand is symmetric under $t_1 \leftrightarrow t_2$:

$$\int_0^t \mathrm{d}t_1' \int_0^{t_1'} \mathrm{d}t_2' \; H^I_{\text{int}}(t_1') H^I_{\text{int}}(t_2') = \frac{1}{2} \int_0^t \mathrm{d}t_1' \mathrm{d}t_2' \begin{cases} H^I_{\text{int}}(t_1') H^I_{\text{int}}(t_2') & t_1' > t_2' \\ H^I_{\text{int}}(t_2') H^I_{\text{int}}(t_1') & t_2' > t_1' \end{cases} \tag{6.47}$$

$$= \frac{1}{2} \int_0^t \mathrm{d}t_1' \mathrm{d}t_2' \; T\{H^I_{\text{int}}(t_1') H^I_{\text{int}}(t_2')\} \tag{6.48}$$

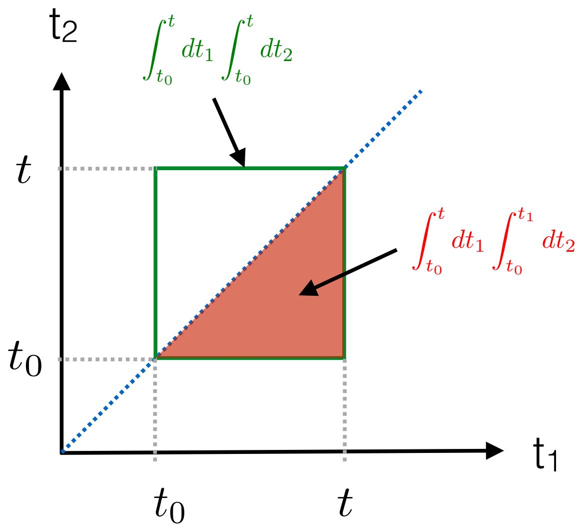


Figure 6.1: Illustration of the Dyson integration.

and for n integrations we have

$$\int_0^t \mathrm{d}t_1' ... \int_0^{t_{n-1}'} \mathrm{d}t_n' \; H_{\text{int}}^I(t_1')...H_{\text{int}}^I(t_n') = \frac{1}{n!} \int_0^t \mathrm{d}t_1'...\mathrm{d}t_n' \; T\{H_{\text{int}}^I(t_1')...H_{\text{int}}^I(t_n')\} \,. \tag{6.49}$$

Hence we can write:

$$\boxed{U(t,t_0) = T\Big\{\exp\Big(-i\int_{t_0}^t \mathrm{d}t' \; H_{\text{int}}^I(t')\Big)\Big\} ,} \tag{6.50}$$

with the time-ordered exponential $T\{\exp(...)\}$. This is **Dyson's formula**.

We can now perform calculations order by order in the interaction, deriving the time dependence of interacting/Heisenberg fields.

6.3 *A first encounter with the S-matrix*

Often in particle physics we prepare two well-separated particles: we define the initial state $|i\rangle$ as

$$|i\rangle : \quad t_i \to -\infty, \quad \text{measure initial momenta } \boldsymbol{p_1^i}, \boldsymbol{p_2^i} , \tag{6.51}$$

This is the simplified picture which we have employed in the RPF lecture. Note, that this is problematic if particles are self-interacting. We will discuss this in some detail once we get to the LSZ reduction.

then we let them scatter and measure the final state $|f\rangle$:

$$|f\rangle : \quad t_f \to +\infty, \quad \text{measure final momenta } \boldsymbol{p_1^f}, \boldsymbol{p_2^f} . \tag{6.52}$$

The probability amplitude for $|i\rangle$ to have evolved into $|f\rangle$ is

$$S_{fi} \equiv \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle f|U(t_f,t_i)|i\rangle , \tag{6.53}$$

with $U(t_f,t_i)$ as given in Eq. (6.50), and so

$$S_{fi} = \langle f|T\Big\{\exp\Big(-i\int_{-\infty}^{\infty} \mathrm{d}t' \; H_{\text{int}}^I(t')\Big)\Big\}|i\rangle \tag{6.54}$$

$$= \langle f|T\Big\{\exp\Big(-i\int \mathrm{d}^4x' \; \mathcal{H}_{\text{int}}^I(x)\Big)\Big\}|i\rangle . \tag{6.55}$$

Now we can express $|i\rangle$, $|f\rangle$ and $\mathcal{H}^I_{\text{int}}$ in terms of creation and annihilation operators.

Example:

This is a very crude example; more about this later when we rederive these expressions using the path integral formalism.

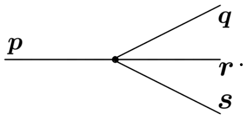
$$|i\rangle = |\boldsymbol{p^i_1}, \boldsymbol{p^i_2}\rangle, \quad |f\rangle = |\boldsymbol{p^f_1}, \boldsymbol{p^f_2}, \boldsymbol{p^f_3}, \boldsymbol{p^f_4}\rangle, \quad \mathcal{H}^I_{\text{int}} = \frac{\lambda}{4!}\phi^4_I. \tag{6.56}$$

We write out $\mathcal{H}^I_{\text{int}}$ in terms of $a_{\boldsymbol{p}}$, $a^\dagger_{\boldsymbol{p}}$:

$$\phi^4_I \supset (...)a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a^\dagger_{\boldsymbol{r}} a^\dagger_{\boldsymbol{s}} + (...)a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a^\dagger_{\boldsymbol{r}} a^\dagger_{\boldsymbol{s}} + (...) + ... \tag{6.57}$$

where the first term creates 4 particles with the corresponding momenta $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, the second destroys a particle with momentum $\boldsymbol{p}$ and creates three particles with momenta $\boldsymbol{q}$, $\boldsymbol{r}$, $\boldsymbol{s}$, and so on. The corresponding diagram to the second term is

$$a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a^\dagger_{\boldsymbol{r}} a^\dagger_{\boldsymbol{s}} =$$



. (6.58)

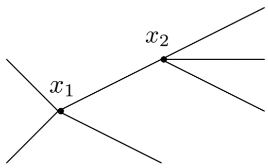
Then

$$S_{fi} \sim \langle 0| a_{\boldsymbol{p^f_1}} a_{\boldsymbol{p^f_2}} a_{\boldsymbol{p^f_3}} a_{\boldsymbol{p^f_4}} \Bigg[\mathbb{1} - i \int \mathrm{d}^4x \underbrace{\frac{\lambda}{4!}\phi^4_I(x)}_{=(1)} + \tag{6.59}$$

$$+ \frac{(-i)^2}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2\, T\left\{ \underbrace{\frac{\lambda}{4!}\phi^4_I(x_1)}_{=(2)} \underbrace{\frac{\lambda}{4!}\phi^4_I(x_2)}_{=(3)} \right\} + ... \Bigg] a^\dagger_{\boldsymbol{p_1}} a^\dagger_{\boldsymbol{p_2}} |0\rangle \tag{6.60}$$

and the only nonvanishing terms for (1) and (3) are the ones $\sim (a^\dagger)^3 a$ and for (2) the one $\sim (a^\dagger)^2 a^2$. Therefore we have

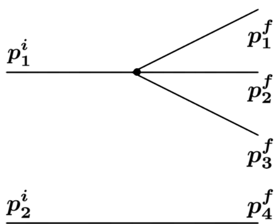
$$S_{fi} \sim \ 0 + \lambda \cdot 0 + \lambda^2 \cdot$$



$$+ (x_1 \leftrightarrow x_2) + ... \tag{6.61}$$

where the second term in Eq. (6.61) vanishes because all the momenta $\boldsymbol{p^i_1}$, $\boldsymbol{p^i_2}$, $\boldsymbol{p^f_1}$,...,$\boldsymbol{p^f_4}$ are different; since the relevant term is $\sim (a^\dagger)^3 a$, the scattering amplitude at $\mathcal{O}(\lambda)$ is described by diagrams like

$$\langle 0| a_{\boldsymbol{p^f_1}} a_{\boldsymbol{p^f_2}} a_{\boldsymbol{p^f_3}} a_{\boldsymbol{p^f_4}} \phi^4_I a^\dagger_{\boldsymbol{p^i_1}} a^\dagger_{\boldsymbol{p^i_2}} |0\rangle \ \sim$$



$$= 0 \tag{6.62}$$

since in, this example, $\boldsymbol{p^i_2} \neq \boldsymbol{p^f_4}$.

With this formalism and scattering theory, we can predict relativistic scattering as an expansion in λ, see e.g. [4].
We will however switch from the canonical to the **path integral** approach, since it is more powerful in the following applications:

Positives of the path integral approach

1. **Gauge theories** for QED, QCD, $SU(2)_L$ (spontaneous symmetry breaking) and gravity.
2. Allowing the discussion of **non-perturbative** effects (cases where there is an essential singularity for $\lambda \to 0$).
3. It is manifestly Lorentz-invariant, since it does not rely on the Hamiltonian for the time-evolution.

Negatives of the path integral approach

However: it is harder to show **unitarity** in the case of the path integral. We will rely on showing the equivalence to the canonical approach, where unitarity is manifest, if it matters.

7
Path Integral Representation of QFT

7.1 Motivating the path integral in quantum mechanics

We can define the path integral in single-particle quantum mechanics. The generalization to quantum fields is the straightforward:

$$\underset{\substack{\uparrow \\ n \text{ canonical degrees} \\ \text{of freedom (e.g. } n\text{=3,} \\ \text{particle in potential } V(\boldsymbol{x}))}}{q_n(t)} \quad \longrightarrow \quad \underset{\substack{\uparrow \\ \text{labels } m\text{ ,}\boldsymbol{x} \text{ degrees} \\ \text{of freedom, as before}}}{\phi_{n,\boldsymbol{x}}(t)} \tag{7.1}$$

The path integral was invented by Feynman in 1942, inspired by earlier work of Dirac in 1933.
Let us start by giving an intuitive description first: We have a double slit experiment

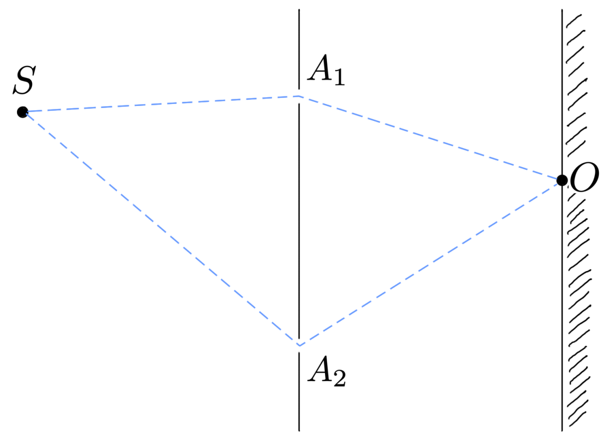


Figure 7.1: Double-slit experiment; two light rays starting at the source S through the slits A_1, A_2 and arriving at the observer O.

and the amplitude is given by

$$\mathcal{A}(\text{detected at } O) = \sum_i \mathcal{A}(S \to A_i \to O) \tag{7.2}$$

Now we add another screen, drill more holes in it and continue until we have infinitely many screens with infinitely many holes in them. The amplitude for a particle emitted at S to arrive at O in time T is then given by

$$\mathcal{A}(S \to O \text{ in } T) = \sum_{\text{paths}} \mathcal{A}(S \to O \text{ in } T \text{ along a particular path}) \tag{7.3}$$

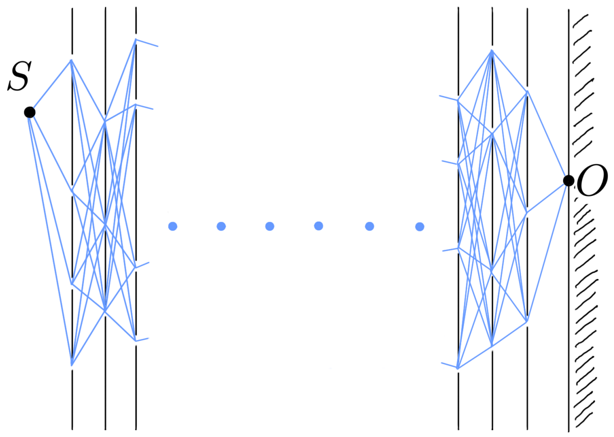


Figure 7.2: Inserting infinitely many screens with infinitely many holes between the source S and the observer O.

7.2 General path integral formula

7.2.1 Hamiltonian version of the path integral

QM

$Q_n(t)$, $P_n(t)$ are hermitian operators that satisfy the canonical commutation relations

$$[Q_n(t), P_m(t)] = i\delta_{nm}\,, \tag{7.4}$$
$$[Q_n(t), Q_m(t)] = [P_n(t), P_m(t)] = 0\,. \tag{7.5}$$

Q_n could be $\{X_1, X_2, X_3\}$ the position operator of a single particle or $\phi_{n,\boldsymbol{x}}$ the field value at every point in space. We start with the discrete index.

The eigenstates of Q_n are in the Schrödinger picture:

Because of Eq. (7.5) the Q_a commute with each other and can be diagonalized simultaneously.

$$Q_a|q\rangle = q_a|q\rangle\,, \tag{7.6}$$

where capital letters will always denote operators and the lowercase letters the eigenvalues.

The eigenstates are orthonormal:

$$\langle q'|q\rangle = \prod_a \delta(q'_a - q_a) \equiv \delta(q - q') \tag{7.7}$$

and the corresponding completeness relation reads

$$\mathbb{1} = \int \left(\prod_a \mathrm{d}q_a\right) |q\rangle\langle q|\,. \tag{7.8}$$

We also have analogous relations for P_a:

$$P_a|p\rangle = p_a|p\rangle\,, \tag{7.9}$$

$$\langle p'|p\rangle = \prod_a (2\pi)\delta(p'_a - p_a) \equiv (2\pi)^N \delta(p'-p)\,, \tag{7.10}$$

$$\mathbb{1} = \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right)|p\rangle\langle p|\,. \tag{7.11}$$

The scalar product of $|q\rangle$ with $|p\rangle$ is

Where N is the number of coordinates.

$$\langle q|p\rangle = \prod_a \exp(ip_a q_a) \equiv e^{ipq}\,. \tag{7.12}$$

To see that the normalization is correct, we use Eq. (7.11) and multiply from the left with $\langle q|$ and from the right with $|q'\rangle$:

$$\langle q| \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right)|p\rangle\langle p|q'\rangle = \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right) e^{ip(q-q')} = \delta(q-q') \quad \checkmark\,.$$

and similarly

$$\langle p| \int \left(\prod_a \mathrm{d}q_a\right)|q\rangle\langle q|p'\rangle = \int \left(\prod_a \mathrm{d}q_a\right) e^{iq(p-p')} = (2\pi)^N \delta(p-p') \quad \checkmark\,.$$

The commutation relations Eq. (7.4), Eq. (7.5) imply Eq. (7.12):

$$p_m\langle q|p\rangle = \langle q|P_m|p\rangle \underset{\uparrow\atop \text{Eq. (7.4), Eq. (7.5)}}{=} \frac{1}{i}\frac{\partial}{\partial q_m}\langle q|p\rangle\,, \tag{7.13}$$

$$\implies \quad \langle q|p\rangle = \prod_n \exp(iq_n p_n)\,. \tag{7.14}$$

Now we transition from the Schrödinger picture to the Heisenberg picture:

$$Q_a^H(t) \equiv e^{iHt} Q_a^S e^{-iHt}\,, \tag{7.15}$$

$$P_A^H(t) \equiv e^{iHt} P_a^S e^{-iHt}\,. \tag{7.16}$$

The corresponding eigenstates are

$$Q_a^H(t)|q;t\rangle_H = q_a|q;t\rangle_H\,, \tag{7.17}$$

$$P_a^H(t)|p;t\rangle_H = p_a|p,t\rangle_H\,, \tag{7.18}$$

where $|q;t\rangle_H$, $|p,t\rangle_H$ are given by

The time dependence is given by e^{iHt} and **not** by e^{-iHt} since $|q;t\rangle_H$ is the eigenstate of $Q_a^H(t)$ and **not** the result of letting $|q\rangle$ evolve in time.

$$|q;t\rangle_H = e^{iHt}|q\rangle_S\,, \tag{7.19}$$

$$|p,t\rangle_H = e^{iHt}|p\rangle_S\,. \tag{7.20}$$

Check:

$$Q_a^H(t)|q;t\rangle_H = e^{iHt}Q_a^S e^{-iHt}e^{iHt}|q\rangle_S = e^{iHt}Q_a^S|q\rangle_S \tag{7.21}$$

$$= q_a e^{iHt}|q\rangle_S = q_a|q;t\rangle_H \quad \checkmark\,. \tag{7.22}$$

For the states $|q;t\rangle_H$ and $|p;t\rangle_H$ the same orthonormality and completeness relations hold, Eq. (7.7)-Eq. (7.12). For the following we will drop the subscripts S and H and work in the Heisenberg picture.

We want to know: if after a measurement at time t the system is in a definite state $|q;t\rangle$, what is then the probability for it to be in the state $|q';t'\rangle$ at time t'?

The central dynamical problem is therefore to calculate $\langle q';t'|q;t\rangle$. This is easy to do for an infinitesimal displacement $t' = \tau + d\tau$, $t = \tau$:

$$\langle q';\tau + d\tau|q;\tau\rangle = \langle q',\tau|e^{-iHd\tau}|q;\tau\rangle \tag{7.23}$$

The classical Hamiltonian is given as a function $H(p,q)$, which we can also write as $H(Q(t),P(t))$ in the Heisenberg picture of operators. We will use **Weyl-ordering**:

Weyl ordering seems like only one of many choices to write expressions like $p^n q^m$. There is of course a $\mathcal{O}(\hbar)$ ambiguity in transitioning from a classical to a quantum mechanical system. Classical dynamics is contained in the QM system as a limit in $\hbar \to 0$, but we cannot predict quantum dynamics from classical behaviour. So we make a choice here.

All the Q's stand to the left and all the P's to the right.

We can achieve this by using the commutation relations; this defines $H(Q(t),P(t))$. The Weyl-ordering allows us to identify in Eq. (7.23)

$$\langle q',\tau|e^{-iH(Q(t),P(t))d\tau}|q;\tau\rangle = \langle q',\tau|e^{-iH(q'_n,P(t))d\tau}|q;\tau\rangle\,. \tag{7.24}$$

To deal with $P(t)$ we use 7.11 in the Heisenberg picture:

$$\langle q';\tau + d\tau|q;\tau\rangle = \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right)\langle q';\tau|e^{-iHd\tau}|p;\tau\rangle\underbrace{\langle p;\tau|q;\tau\rangle}_{=e^{-ipq}} \tag{7.25}$$

$$= \int \left(\prod_a \frac{\mathrm{d}p_a}{2\pi}\right)\exp\Big(-i\underbrace{H(q'_n,p_n)}_{\text{c-number!}}d\tau + i(q'-q)p\Big)\,, \tag{7.26}$$

with each p_a-integration from $-\infty$ to $+\infty$.

<u>**Note:**</u> we could only replace $H(Q(t),P(t)) \to H(q'_n,p_n)$ in Eq. (7.25) because with $d\tau$ infinitesimal $\exp(-iHd\tau) \approx \mathbb{1} - iHd\tau$ is linear in (the Weyl-ordered) H.
We now split $[t_i,t_f]$ into $N+1$ intervals:

$$t_i = t_0 < t_1 < ... < t_N < t_{N+1} \equiv t_f\,,\quad d\tau = \frac{t_f - t_i}{N+1}\,. \tag{7.27}$$

We insert complete sets of position eigenstates and use the above result for the infinitesimal time interval. To simplify the notation we will use

$$\int \prod_n \mathrm{d}q'_n \equiv \int \mathrm{d}q'\,,\qquad \int \prod_n \frac{\mathrm{d}p_n}{2\pi} \equiv \int \mathrm{d}\tilde{p}'\,. \tag{7.28}$$

Therefore

$$\langle q_f;t_f|q_i;t_i\rangle = \int \mathrm{d}q^1\mathrm{d}q^2...\mathrm{d}q^N\langle q_f;t_f|q^N;t_N\rangle\langle q^N;t_N|q^{N-1};t_{N-1}\rangle...\langle q^1;t_1|q_i;t_i\rangle \tag{7.29}$$

$$= \int \prod_{k=1}^{N}\mathrm{d}q^k\prod_{l=0}^{N}\mathrm{d}\tilde{p}^l\exp\left(i\sum_{m=1}^{N+1}(q^m - q^{m-1})p^{m-1} - H(q^m,p^{m-1})d\tau\right) \tag{7.30}$$

and we now fearlessly take the $N \to \infty$ limit:

$$N \longrightarrow \infty : \begin{cases} q_n^k & \longrightarrow \quad q_n(t) \\ |q^k; t_k\rangle & \\ p_n^k & \longrightarrow \quad p_n(t) \\ |p^k; t_k\rangle & \\ q_n^k - q_n^{k-1} & \longrightarrow \quad \dot{q}_n(t)dt \\ \sum_{m=1}^{N+1} dt & \longrightarrow \quad \int_{t_i}^{t_f} \mathrm{d}t \end{cases} \tag{7.31}$$

This implies for the integration measure:

$$\int \prod_{k=1}^{N} \prod_{a} \mathrm{d}q_a^k \prod_{l=0}^{N} \prod_{b} \frac{\mathrm{d}p_b^l}{2\pi} \longrightarrow \int_{\substack{q(t_i)\equiv t_i \\ q(t_f)\equiv q_f}} \prod_t \prod_a \mathrm{d}q_a(t) \prod_t \prod_b \frac{\mathrm{d}p_b(t)}{2\pi} \tag{7.32}$$

$$\equiv \int_{q_i}^{q_f} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)] \tag{7.33}$$

"Path integral"

This defines an integral **over all functions** $q_n(t)$, $p_n(t)$ = "paths" from $q_n(t_i)$ to $q_n(t_f)$.
Hence we get for the Hamilton path integral:

$$\langle q_f; t_f | q_i; t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)] e^{i\int_{t_i}^{t_f} \mathrm{d}t \, \left(\sum_n \dot{q}_n p_n - H(q_n(t), p_n(t))\right)},$$

i.e. the quantum transition amplitude = integration over infinite paths $\cdot\, e^{i(\dots)}$; note that no operators are left over.
The measure $\int \mathcal{D}q\mathcal{D}p$ is defined using the limit $N \to \infty$ from the discrete version.

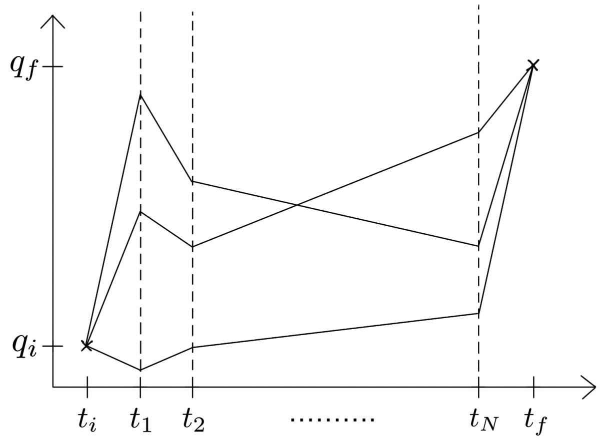


Figure 7.3: Some of the paths contributing to the path integral.

At every t_k we integrate over all q_k.
Interpretation: we integrate over all paths/trajectories from $q(t_i) = q_i$ to $q(t_f) = q_f$. Each path is weighted with a complex phase of $\exp\left(i\int_{t_i}^{t_f} \mathrm{d}t \, \left(\sum_n \dot{q}_n p_n - H(q_n(t), p_n(t))\right)\right)$. For p_k one integrates over all **unconstrained** boundary values p_i, p_f and also over all p_k.

Note that q_k, p_k do not satisfy the classical equations of motion, e.g. they do **not** satisfy the Hamilton equations

$$\dot{q}_k \neq \frac{\partial H}{\partial p_k} \quad \text{and} \quad \dot{p}_k \neq -\frac{\partial H}{\partial q_k}\,! \tag{7.34}$$

7.2.2 Generalization to fields

The Hamiltonian H is now given by

$$H = \int \mathrm{d}^3x\ \mathcal{H}(\phi(x), \Pi(x))\,, \tag{7.35}$$

Here we consider just one scalar field $\phi(x)$ and its conjugate momentum $\Pi(x)$, but in general $\mathcal{H}$ depends on many different fields.

where for this section we write the conjugate momentum as $\Pi(x)$ to avoid confusion later. We substitute

$$\sum_n \longrightarrow \int \mathrm{d}^3x\,, \quad |q;t\rangle \longrightarrow |\varphi;t\rangle\,. \tag{7.36}$$

$|\varphi;t\rangle$ is the eigenvector satisfying the eigenvalue equation

$$\phi(t,\boldsymbol{x})|\varphi;t\rangle = \varphi(\boldsymbol{x})|\varphi;t\rangle\,, \tag{7.37}$$

analogously to Eq. (7.17), and a similar eigenvalue equation holds for $\Pi(x)$, with eigenvalue $\pi(\boldsymbol{x})$. Here $\varphi\,,\pi : \mathbb{R}^3 \to \mathbb{C}$ are **c-functions**!
The Hamilton **QFT path integral** is then

$$\langle\varphi_f;t_f|\varphi_i;t_i\rangle = \int_{\substack{\varphi(t_i,\boldsymbol{x})=\varphi_i(\boldsymbol{x})\\ \varphi(t_f,\boldsymbol{x})=\varphi_f(\boldsymbol{x})}} \mathcal{D}[\varphi(x)]\mathcal{D}[\pi(x)]e^{i\int_{t_i}^{t_f}\mathrm{d}t\int \mathrm{d}^3x\,\left(\pi(x)\dot{\varphi}(x)-\mathcal{H}(\varphi,\pi)\right)}\,,$$

i.e. the quantum transition amplitude = sum over all classical field configurations satisfying the boundary values φ_i, φ_f and weight specified by the integrand.

Note that we have introduced a time index for φ and π to account for the fact that the eigenvalues can be different at each time t.

7.3 Path integral for Green functions

Our goal will be to calculate scattering amplitudes of particle states and Green functions, not of field configurations.
As a step towards this, we generalize our path integral formula to:

$$\langle q_f;t_f|O_a(t_a)O_b(t_b)\cdot ...|q_i;t_i\rangle\,, \tag{7.38}$$

where the O_i can be any operator $O(t) = O(P_n(t), Q_n(t))$ with the convention that all Q's are positioned on the **right** of all the P's (note that this is the opposite convention to the one used for the Hamiltonian in the path integral).
We use the same strategy a s before and divide $[t_i, t_f]$ into intervals $[t_k, t_k + d\tau]$ and insert complete sets of momentum eigenstates $|p, t_k\rangle$.

<u>**Note:**</u> this can only be done if the operators $O_a(t_a)$, $O_b(t_b)$,... are **time-ordered**, i.e. $t_f > t_a > t_b > ... > t_i$. If this is not the case, we must put them in order first.. This means, we are computing

$$\langle q_f;t_f|T\{O_a(t_a)O_b(t_b)\cdot ...\}|q_i;t_i\rangle\,. \tag{7.39}$$

Example: if $t_a \in [t, t+d\tau]$, we get terms like

$$\langle q'; t+d\tau | O_a(P_n(t_a), Q_n(t_a)) | q; t\rangle = \tag{7.40}$$

$$= \int \left(\prod_n \frac{\mathrm{d}p_n}{2\pi}\right) \langle q'| e^{-iH(Q_n,P_n)d\tau} |p\rangle\langle p| O_a(P_n(t_a), Q_n(t_a)) |q\rangle \tag{7.41}$$

$$= \int \left(\prod_n \frac{\mathrm{d}p_n}{2\pi}\right) \exp\left(-iH(q'_n, p_n)d\tau + i\sum_n (q'_n - q_n)p_n\right) \cdot O_a(p_n, q_n)\,, \tag{7.42}$$

where $O_a(p_n, q_n)$ is a **c-function** of p_n, q_n after the last equality, no longer an operator.
We follow the same steps as before to get

$$\begin{aligned}\langle q_f; t_f | T\{O_a(t_a)O_b(t_b)\cdot ...\} | q_i; t_i\rangle &= \\ = \int \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)] \Big[& e^{i\int_{t_i}^{t_f} \mathrm{d}t \, \left(\sum_n \dot{q}_n p_n - H(q_n(t), p_n(t))\right)} \cdot \\ & \cdot O_a(p_n(t_a), q_n(t_a)) O_b(p_n(t_b), q_n(t_b)) \cdot ... \Big]\end{aligned} \tag{7.43}$$

We now want to discuss objects appearing in the S-matrix:

$$\langle\Omega| T\{O_a(t_a)O_b(t_b)\cdot ...\} |\Omega\rangle\,, \tag{7.44}$$

with $|\Omega\rangle$ the vacuum state of the full (interacting) theory. The O_i's are now some functions of the field operators $\phi(x_i)$, $\partial_\mu\phi(x_i)$, etc.
We will derive the Hamilton path integral formula for Green functions:

$$\langle\Omega| T\{O_a(t_a)O_b(t_b)...\} |\Omega\rangle =$$

$$= |N|^2 \int \mathcal{D}[\varphi]\mathcal{D}[\pi] \exp\left(i\int \mathrm{d}^4x \left(\pi\dot{\varphi} - \mathcal{H} + \frac{i\varepsilon}{2}\varphi^2\right)\right) O_a(x_a)O_b(x_b)...\,, \tag{7.45}$$

where we have suppressed the x-dependences in the exponent.
The $i\varepsilon$ is the one you may have seen in the Feynman propagator:

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{p^2 - m^2 + i\varepsilon}\,. \tag{7.46}$$

We start from Eq. (7.43), switch to the field notation and insert a complete set of coordinate eigenstates at t_i and t_f:

$$\begin{aligned}\langle\Omega| T\{O_a(t_a)O_b(t_b)\cdot ...\} |\Omega\rangle &= \\ \overset{(*)}{=} \lim_{\substack{t_i\to-\infty \\ t_f\to+\infty}} \int \prod_{\boldsymbol{x}} \mathrm{d}\varphi_i(\boldsymbol{x}) \prod_{\boldsymbol{x}} & \mathrm{d}\varphi_f(\boldsymbol{x}) \langle\Omega|\varphi_f; t_f\rangle\langle\varphi_f; t_f| \cdot \\ & \cdot T\{O_aO_b\cdot ...\} |\varphi_i; t_i\rangle\langle\varphi_i; t_i|\Omega\rangle\,.\end{aligned} \tag{7.47}$$

This is an integral over **all** initial and final field configurations, so that $\mathcal{D}[\varphi_n(\boldsymbol{x})]$ is now **unconstrained**, as is $\mathcal{D}[\pi_n(\boldsymbol{x})]$ (i.e. there are no boundary conditions).

$$\begin{aligned}\overset{(*)}{=} \int \mathcal{D}[\varphi]\mathcal{D}[\pi] & \langle\Omega|\varphi; t_f = +\infty\rangle\langle\varphi; t_i = -\infty|\Omega\rangle \cdot \\ & \cdot \exp\left(i\int \mathrm{d}^4x \left(\pi(x)\dot{\varphi}(x) - \mathcal{H}(\phi, \pi)\right)\right) O_a(x_a)O_b(x_b)...\end{aligned} \tag{7.48}$$

We need to evaluate $\langle\varphi;\mp\infty|\Omega\rangle$. Since we can think of $\varphi(\boldsymbol{x})$ as coordinates of the system, we can interpret the above bracket as the wavefunction of the vacuum in coordinate representation. As an analogy, compare to $\psi(\boldsymbol{x}) = \langle\boldsymbol{x}|\psi\rangle$ in quantum mechanics.
We now assume that for $t \longrightarrow \mp\infty$ the field behaves as a **free** field:

$$\phi(x) \overset{t\to-\infty}{\longrightarrow} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(e^{-ipx}a_{\boldsymbol{p},\mathrm{in}} + e^{ipx}a^\dagger_{\boldsymbol{p},\mathrm{in}}\right), \tag{7.49}$$

$$\phi(x) \overset{t\to+\infty}{\longrightarrow} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(e^{-ipx}a_{\boldsymbol{p},\mathrm{out}} + e^{ipx}a^\dagger_{\boldsymbol{p},\mathrm{out}}\right), \tag{7.50}$$

where $a_{\boldsymbol{p},\mathrm{in/out}}$, $a^\dagger_{\boldsymbol{p},\mathrm{in/out}}$ satisfy the standard free field commutation relations. Similarly, we take $\Pi(x) = \dot{\phi}(x)$ as free for $t_{i/f} \longrightarrow \mp\infty$. We will discuss these asymptotic states in more detail later when we develop scattering theory.
As in the exercise sheet[1], we can express the creation and annihilation operators as

[1] sheet1, ex. 1

$$a_{\boldsymbol{p}} = e^{iE_{\boldsymbol{p}}t}\Big(E_{\boldsymbol{p}}\tilde{\phi}_{\boldsymbol{p}}(t) + i\tilde{\Pi}_{\boldsymbol{p}}(t)\Big), \quad \text{with} \quad \tilde{\phi}_{\boldsymbol{p}}(t) = \int \mathrm{d}^3 x\, e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\phi(x)\,. \tag{7.51}$$

This allows us to write

$$a_{\boldsymbol{p},\mathrm{in/out}} = \lim_{t\to\mp\infty}\int \mathrm{d}^3 x\, e^{ipx}E_{\boldsymbol{p}}\left(\phi(x) + \frac{i}{E_{\boldsymbol{p}}}\Pi(x)\right) \tag{7.52}$$

Since $a_{\boldsymbol{p},\mathrm{in/out}}|\Omega\rangle = 0$ holds, since we assume that for $t \to \pm\infty$ matrix elements may be calculated as if there where no interactions. We can write:

$$\begin{aligned} 0 &= \langle\varphi;\mp\infty|a_{\boldsymbol{p},\mathrm{in/out}}|\Omega\rangle && (7.53)\\ &= \lim_{t\to\mp\infty}\int \mathrm{d}^3 x\, e^{ipx}E_{\boldsymbol{p}}\langle\varphi;\mp\infty|\phi(x) + \frac{i}{E_{\boldsymbol{p}}}\Pi(x)|\Omega\rangle && (7.54)\\ &= \lim_{t\to\mp\infty}\int \mathrm{d}^3 x\, e^{ipx}E_{\boldsymbol{p}}\left(\varphi(\boldsymbol{x}) + \frac{1}{E_{\boldsymbol{p}}}\frac{\delta}{\delta\varphi(\boldsymbol{x})}\right)\langle\varphi;\mp\infty|\Omega\rangle\,, && (7.55)\end{aligned}$$

where in the last step we have used the analogue of the position space representation of the momentum operator, since:

$$\mathrm{QM:} \qquad \langle q;t|P_n(t) = -i\frac{\partial}{\partial q_n}\langle q;t|$$

The QFT generalisation replaces a derivative with a functional derivative

$$\begin{array}{llll} \mathrm{QM}\to\mathrm{QFT}: & P_n(x) \longrightarrow \Pi(x) & & \\ \text{derivative:} & \dfrac{\partial}{\partial q_n} \longrightarrow \dfrac{\delta}{\delta\varphi(\boldsymbol{x})} & \text{functional derivative} \\ \text{wave-function:} & \psi(q_n,t) \longrightarrow \psi[\varphi(t,\boldsymbol{x})] & \text{wave-functional} \\ & = \langle q;t|\psi\rangle \quad\ = \langle\varphi;t|\psi\rangle & \end{array}$$

Therefore:

$$\begin{aligned} \langle\varphi;t|\Pi(t,\boldsymbol{x})|\psi\rangle &= -i\frac{\delta}{\delta\varphi(\boldsymbol{x})}\langle\varphi;t|\psi\rangle && (7.56)\\ &= -i\frac{\delta}{\delta\varphi(\boldsymbol{x})}\psi[\varphi(t,\boldsymbol{x})] && (7.57)\end{aligned}$$

From Eq. (7.55) we see that $\langle\varphi;\mp\infty|\Omega\rangle$ satisfies

$$\int \mathrm{d}^3x\ e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\left(\frac{\delta}{\delta\varphi(\boldsymbol{x})}+E_{\boldsymbol{p}}\varphi(\boldsymbol{x})\right)\langle\varphi;\mp\infty|\Omega\rangle = 0\,. \tag{7.58}$$

This is a first order **functional** differential equation. We can guess the solution by comparing to an ordinary differential equation of the same form:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+ax\right)f(x) = 0 \tag{7.59}$$

$$\Longrightarrow \quad f(x) = \text{const.}\cdot e^{-\frac{1}{2}ax^2} \tag{7.60}$$

We can take the ansatz

$$\boxed{\langle\varphi;\mp\infty|\Omega\rangle = N\exp\left(-\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y\ K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{x})\varphi(\boldsymbol{y})\right)} \tag{7.61}$$

Plugging this into Eq. (7.58):

$$\int \mathrm{d}^3x\ e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\left(-\int \mathrm{d}^3y\ K(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y})+E_{\boldsymbol{p}}\varphi(\boldsymbol{x})\right)N\exp(...) \stackrel{!}{=} 0\,, \tag{7.62}$$

which is satisfied for all $\varphi(\boldsymbol{y})$ if

$$\int \mathrm{d}^3x\ e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}K(\boldsymbol{x},\boldsymbol{y}) = E_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{y}}\,. \tag{7.63}$$

By inverting the Fourier transform we get

$$\boxed{K(\boldsymbol{x},\boldsymbol{y}) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}E_{\boldsymbol{p}}} \tag{7.64}$$

We will not need the explicit form but K can be written for $r\neq 0$ as

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{m}{2\pi^2 r}\partial_r\left(\frac{1}{r}K_{-1}(mr)\right)\,,$$

with $r=|\boldsymbol{x}-\boldsymbol{y}|$ and $K_n(x)$ the Hankel functions. For $m\longrightarrow 0$ we get the limit

$$K = -\frac{1}{\pi^2 r^4}\,.$$

We therefore get

$$\begin{aligned}&\langle\Omega|\varphi;+\infty\rangle\langle\varphi;-\infty|\Omega\rangle =\\ &= |N|^2\exp\left(-\frac{1}{2}\int \mathrm{d}^3x\mathrm{d}^3y\ K(\boldsymbol{x},\boldsymbol{y})\Big[\varphi(\infty,\boldsymbol{x})\varphi(\infty,\boldsymbol{y})+\varphi(-\infty,\boldsymbol{x})\varphi(-\infty,\boldsymbol{y})\Big]\right)\,.\end{aligned} \tag{7.65}$$

<u>**Details:**</u> We can rewrite the exponent as

$$f(\infty)+f(-\infty) = \lim_{\varepsilon\to 0^+}\varepsilon\int_{-\infty}^{\infty}\mathrm{d}t\ f(t)e^{-\varepsilon|t|} \tag{7.66}$$

which works for any resonably smooth function $f(t)$. Here we use $f(t)=\varphi(t,\boldsymbol{x})\varphi(t,\boldsymbol{y})$.

Back to Eq. (7.65):

$$\langle\Omega|\varphi;+\infty\rangle\langle\varphi;-\infty|\Omega\rangle = \tag{7.67}$$

$$= |N|^2\exp\left(\frac{1}{2}\varepsilon\int \mathrm{d}^3x\mathrm{d}^3y\int_{-\infty}^{\infty}\mathrm{d}t\ e^{-\varepsilon|t|}K(\boldsymbol{x},\boldsymbol{y})\varphi(t,\boldsymbol{x})\varphi(t,\boldsymbol{y})\right) \tag{7.68}$$

$$= |N|^2\exp\left(-\frac{1}{2}\varepsilon\int \mathrm{d}^4x\ \varphi(t,\boldsymbol{x})^2\right).] \tag{7.69}$$

In the last step we have used that

$$\varepsilon K(\boldsymbol{x},\boldsymbol{y}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\varepsilon E_{\boldsymbol{p}} \tag{7.70}$$

$$\approx \int \frac{\mathrm{d}^3p}{(2\pi)^3}\tilde{\varepsilon} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \tilde{\varepsilon}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{7.71}$$

where we have introduces another infinitesimal $\tilde{\varepsilon} > 0$. The meaning of this will become clearer later.

We finally derive the Hamilton path integral for Green functions:

$$\langle\Omega|T\{O_a(t_a)O_b(t_b)...\}|\Omega\rangle = \tag{7.72}$$

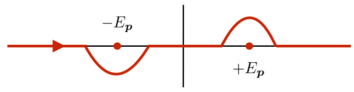
$$= |N|^2 \int \mathcal{D}[\varphi]\mathcal{D}[\pi] \exp\left(i\int \mathrm{d}^4x\, \left(\pi\dot{\varphi} - \mathcal{H} + i\frac{\varepsilon}{2}\varphi^2\right)\right)\cdot$$

$$\cdot\, O_a(x_a)O_b(x_b)... \tag{7.73}$$

Anticipating the result:

$$\frac{i}{p^2-m^2+i\varepsilon} \quad \text{vs.} \quad \frac{i}{p^2-m^2+i\varepsilon E_{\boldsymbol{p}}}$$

and we can replace $\varepsilon E_{\boldsymbol{p}} \to \varepsilon$. The contour integration for the Feynman-propagator is the following:



The normalization is chosen such that $\langle\Omega|\Omega\rangle = 1$:

$$|N|^{-2} = \int \mathcal{D}[\varphi]\mathcal{D}[\pi] \exp\left(i\int \mathrm{d}^4x\, \left(\pi(x)\dot{\phi}(x) - \mathcal{H}(x) + \frac{i\varepsilon}{2}\varphi^2\right)\right) \tag{7.74}$$

Since $\mathcal{H}(x) \supset \frac{m^2}{2}\varphi(x)^2$, we will not write $\frac{i\varepsilon}{2}\varphi(x)^2$ out explicitly but assume that $\frac{m^2}{2}$ should be interpreted as

$$\frac{m^2}{2} \longrightarrow \frac{m^2}{2} - \frac{i\varepsilon}{2}\,. \tag{7.75}$$

7.4 *Lagrangian version of the path integral*

The integrand in the exponent of the path integral (PI) formula looks like the Lagrangian associated to $\mathcal{H}$ ($\mathcal{L} = \pi\dot{\varphi} - \mathcal{H}$). This is somewhat misleading since π is an integration variable and **not** given by $\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}$. However, for an important class of theories, the path integral over π can be done by just replacing them with the values dictated by the canonical formalism.

The conditions for this are:

- $\mathcal{H}(x)$ is at most quadratic in $\pi(x)$,
- the $O_i(\pi(x),\varphi(x))$ do **not** depend on $\pi(x)$.

7.4.1 *The Gaussian path integral*

We start with finite-dimensional Gaussian integrals:

$$\text{1-dim.:} \quad \int_{-\infty}^{\infty} \mathrm{d}x\, \exp\left(-\frac{a}{2}x^2\right) = \sqrt{\frac{2\pi}{a}} \tag{7.76}$$

$$n\text{-dim.:} \quad \int_{-\infty}^{\infty} \mathrm{d}x_1...\mathrm{d}x_n\, \exp\left(-\frac{1}{2}x_iA_{ij}x_j\right) \overset{(*)}{=} \sqrt{\frac{2\pi}{a_1}}\sqrt{\frac{2\pi}{a_2}}\cdot ... \cdot \sqrt{\frac{2\pi}{a_n}}$$

$$= \frac{(2\pi)^{\frac{n}{2}}}{\sqrt{\det(A)}}$$

($*$) Diagonalize A, use Eq. (7.76).

with the symmetric coefficient matrix $A_{ij} = A_{ji}$.
Gaussian integral with a "source":

$$\int_{-\infty}^{\infty} \mathrm{d}x\ \exp\left(-\frac{a}{2}x^2 - bx\right) \stackrel{(*)}{=} \int_{-\infty}^{\infty} \mathrm{d}x\ \exp\left[-\frac{a}{2}\left(x+\frac{b}{a}\right)^2 + \frac{b^2}{2a}\right] \tag{7.77}$$

$$= \sqrt{\frac{2\pi}{a}}\exp\left(\frac{b^2}{2a}\right) \tag{7.78}$$

(∗) Complete the square and shift integration variable:

$$x \longrightarrow x + \frac{b}{a}\,.$$

and in N dimensions we have

$$\int_{-\infty}^{\infty} \mathrm{d}x_1 ... \mathrm{d}x_N\ \exp\Big(\underbrace{-\frac{1}{2}x_k A_{kl} x_l - B_k x_k - C}_{:=Q(x)}\Big) \tag{7.79}$$

$$= \left[\det\left(\frac{A}{2\pi}\right)\right]^{-\frac{1}{2}} \exp\left(\frac{1}{2}B_m [A^{-1}]_{mn} B_n - C\right) \tag{7.80}$$

$$= \left(\prod_{k=1}^{N}\sqrt{\frac{2\pi}{a_k}}\right) e^{-Q(\bar{x})}\,, \tag{7.81}$$

where

$$\bar{x} = -A^{-1}B\,, \quad \left.\frac{\partial Q}{\partial x_k}\right|_{x=\bar{x}} = 0 \tag{7.82}$$

is the stationary point of the quadratic form $Q(x)$. The a_k are the eigenvalues of the symmetric matrix A, and we count a degenerate eigenvalue several times in Eq. (7.81). We also need to assume that $\det(A) \neq 0$, in order for the inverse A^{-1} to exist.

Generalize to functional integral:

$$x_k \longrightarrow x(t)\,, \quad \int \prod_{k=1}^{N} \mathrm{d}x_k \longrightarrow \int \mathcal{D}[x(t)]\,, \tag{7.83}$$

$$A_{kl} \longrightarrow A(t,t')\,, \quad B_k \longrightarrow B(t)\,. \tag{7.84}$$

$$\int \mathcal{D}[x(t)] e^{-\frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\ A(t,t')x(t)x(t') - \int \mathrm{d}t\ B(t)x(t) - C}$$

$$= \left[\det\left(\frac{A}{2\pi}\right)\right]^{-\frac{1}{2}} e^{\frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\ B(t)A^{-1}(t,t')B(t') - C} \tag{7.85}$$

and we can again rewrite the integrand in the exponent in Eq. (7.85) as

$$A(t,t')\bar{x}(t)x(t')\,, \quad \text{with} \quad \bar{x}(t) = -\int \mathrm{d}t'\ A^{-1}(t,t')B(t')\,. \tag{7.86}$$

How do we calculate $A^{-1}(t,t')$ or $\det\big(A(t,t')\big)$?
The eigenvalue equation to the eigenvector x in the discrete case goes over into

$$\sum_l A_{kl}x_l = \lambda x_k \ \longrightarrow\ \int \mathrm{d}t'\ A(t,t')x(t') = \lambda x(t) \tag{7.87}$$

where $A(t,t')$ is an integral kernel and $x(t)$ is now an eigenfunction. This allows us to define

$$\mathrm{tr}(A) = \sum_n \lambda_n \quad \text{and} \quad \det(A) = \prod_n \lambda_n \,. \tag{7.88}$$

These are usually infinite sums/products. They are defined only when they are convergent.

<u>**Inverse:**</u>

$$\sum_l [A^{-1}]_{kl}[A]_{lk'} = \delta_{kk'} \longrightarrow \int \mathrm{d}t \; A^{-1}(t_1,t)A(t,t_2) = \delta(t_1-t_2)\,. \tag{7.89}$$

Note that this applies to any $A(t,t')$, which can also be differential operators, e.g.

$$A(t,t') = \delta(t-t')\frac{\mathrm{d}^n}{\mathrm{d}t^n}\,, \tag{7.90}$$

where $\delta(t-t')$ means that the operator is "diagonal".

We now apply this to the PI and we focus on the $\mathcal{D}[\pi]$-integration:

$$\int \mathcal{D}[\pi] \exp\left(i \int \mathrm{d}^4x \; \pi(x)\dot{\varphi}(x) - \mathcal{H}(x)\right) \overset{(*)}{=} \tag{7.91}$$

where $\mathcal{H}(x)$ is a quadratic form in π by assumption. We rewrite the quadratic part as

$$-\int \mathrm{d}^4x \mathrm{d}^4y \; i\pi(x)A[\varphi(x)]\pi(y) \quad \text{with} \quad A[\varphi(x)] \propto \delta^4(x-y) \tag{7.92}$$

and so

$$\overset{(*)}{=} \left(\det\left(2\pi i A[\varphi(x)]\right)\right)^{-\frac{1}{2}} \exp\left(i\int \mathrm{d}^4x \left(\bar{\pi}\dot{\varphi} - \mathcal{H}(\varphi,\bar{\pi})\right)\right) \tag{7.93}$$

where the factor 2π appears in the determinant because

$$\int \frac{\mathrm{d}p_i}{2\pi} \exp\left(-\frac{a}{2}p_i^2\right) = \frac{1}{2\pi}\sqrt{\frac{2\pi}{a}} = \sqrt{\frac{1}{2\pi a}}\,. \tag{7.94}$$

As above, $\bar{\pi}(x)$ is the stationary point of the quadratic form:

$$\frac{\delta}{\delta\pi(x)}\left(\int \mathrm{d}^4y \left(\pi(y)\dot{\varphi}(y) - \mathcal{H}(\varphi(y),\pi(y))\right)\right) = \dot{\varphi}(x) - \frac{\partial\mathcal{H}}{\partial\pi(x)} \overset{!}{=} 0 \tag{7.95}$$

and we get $\bar{\pi}(x)$ by solving

$$\dot{\varphi}(x) = \left.\frac{\partial\mathcal{H}}{\partial\pi(x)}\right|_{\pi(x)=\bar{\pi}(x)}\,. \tag{7.96}$$

This is one of the Hamilton equations! This implies

$$\bar{\pi}(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}(x)}\,. \tag{7.97}$$

Hence we find that in this case $\bar{\pi}(x)$ **is** the canonically conjugate momentum! We can express it in terms of φ and $\dot{\varphi}$ and conclude that the Gaussian integration "performs" a Legendre transformation to

$$\stackrel{(*)}{=} \left(\det\left(2\pi i A[\varphi(x)]\right)\right)^{-\frac{1}{2}} \exp\left(i\int d^4x\, \mathcal{L}(\varphi(x), \partial_\mu\varphi(x))\right). \quad (7.98)$$

We have found the Lagrange version of the PI for Green functions, which is only valid if $\mathcal{H}$ is quadratic in π:

$$\begin{aligned}&\langle\Omega|T\{O_a(t_a)O_b(t_b)...\}|\Omega\rangle = \\ &\quad = |N|^2\int\mathcal{D}[\varphi]\left(\det\left(2\pi i A[\varphi(x)]\right)\right)^{-\frac{1}{2}}\cdot \\ &\quad\cdot\exp\left(i\int d^4x\,\mathcal{L}(\varphi(x),\partial_\mu\varphi(x))\right)O_a(x_a)O_b(x_b)...\end{aligned} \quad (7.99)$$

The $O_i(x_i)$ can only depend on $\varphi(x)$, not on $\pi(x)$.

with

$$|N|^{-2} = \int\mathcal{D}[\varphi]\left(\det\left(2\pi i A[\varphi(x)]\right)\right)^{-\frac{1}{2}}\exp\left(i\int d^4x\,\mathcal{L}(\varphi(x),\partial_\mu\varphi(x))\right)$$

Note:

- We have left out the factor $2\pi i$ in det(...) since it cancels between the numerator and $|N|^2$.
- If $A[\varphi(x)]$ does not depend on $\varphi(x)$, $\det(A[\varphi(x)])$ can be pulled out before the PI and also cancels. If $A[\varphi(x)]$ depends on $\varphi(x)$ then we can use

  $$\det(A) = e^{\mathrm{tr}(\ln(A))}, \quad \text{since} \quad \ln(\det(A)) = \ln\left(\prod_i a_i\right) = \mathrm{tr}(\ln(A)),$$

  to absorb it into a redefinition of $\mathcal{L}$. In the following this will not be relevant, hence we will drop the $\left(\det(A[\varphi(x)])\right)$ from the PI.
- **Example:**

  $$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2\right) + \mathcal{L}_{\text{int}}(\varphi) \quad (7.100)$$

  $$\Longrightarrow \quad \mathcal{H} = \frac{1}{2}\left(\pi^2 + (\nabla\varphi)^2 + m^2\varphi^2\right) - \mathcal{L}_{\text{int}}(\varphi) \quad (7.101)$$

  $$\Longrightarrow A[\varphi(x)] = \delta^{(4)}(x-y) \quad (7.102)$$

  This is the most important case!

Counterexample: Non-linear sigma model:

$$\mathcal{L} = \frac{1}{2(1+\varphi^a\varphi^a)}\partial_\mu\varphi_a\partial^\mu\varphi_a + ...$$

$$\Longrightarrow \quad \mathcal{H} = \frac{1}{2}(1+\varphi_a^2)\pi^2 + ...$$

$$\Longrightarrow \mathcal{D}[\varphi(x)](\det(A))^{-\frac{1}{2}} = \mathcal{D}[\varphi(x)]\left(\det(1+\varphi_a^2)\right)^{-1},$$

which gives additional terms in $\mathcal{L}$:

$$\mathcal{L} \to \mathcal{L} + i\delta^{(4)}(0)\ln(1+\varphi_a^2)$$

This form of the path integral shows that quantum amplitudes involve **all** particle trajectories weighted by a phase factor $e^{iS[q(t)]}$:

$$\langle q_f; t_f|q_i; t_i\rangle \sim \int\mathcal{D}[q(t)]\exp\left(i\underbrace{\int_{t_i}^{t_f} dt\, L(q(t),\dot{q}(t))}_{(1)}\right) \quad (7.103)$$

where (1) is the **action** $S[q(t)]$. The double slit experiment is the simplest realization of this fact.

7.5 *The classical limit and ordering ambiguities*

We can directly see from the above formula how the motion of **classical** trajectories emerges, as the unique trajectory for which $\delta S = 0$.
Restoring $\hbar$ we have

$$\langle q_f; t_f | q_i; t_i \rangle \sim \int \mathcal{D}[q(t)] e^{\frac{i}{\hbar} S[q(t)]} \,. \tag{7.104}$$

The classical limit is given as $\hbar \longrightarrow 0$ (recall $[Q,P] = i\hbar$) and the integral is dominated by the stationary point

$$\delta S = 0 \,. \tag{7.105}$$

Otherwise the integrand has infinitely many quickly varying contributions which average to zero.
For ordinary integrals, this is captured in the **method of stationary phase** (related to the method of steepest descent).

We see that the general $\hbar$-dependence is

$$\langle q_f; t_f | q_i; t_i \rangle = \hbar^0 F^0 + \hbar^1 F_1 + \hbar^2 F_2 + \ldots \tag{7.106}$$

Another way of seeing this is by taking $\hbar \to 0$ in the imaginary direction; then the solution is dominated by

$$S[q_0] \,, \quad \text{with} \quad \delta S\Big|_{q_0} = 0 \,, \quad \tilde{\hbar} \equiv -i\hbar$$

since the weight is now $\exp\left(-\frac{S[q]}{\hbar}\right)$.

In the step where we go from the classical to the quantum mechanical case, we require the $\hbar \longrightarrow 0$ limit to match, and since QM is the more fundamental theory, we should not expect the classical theory to tell us about the F_is, $i \geq 1$. The ordering ambiguity (which is $\mathcal{O}(\hbar)$) tells us that we need to perform measurements to figure out what the fundamental theory is and therefore what the correct ordering of P and Q is (if it matters at all).

7.6 *Perturbative solution of Green functions*

We have two options to evaluate the path integrals:

1. Discretize space and time and evaluate them numerically. This is called lattice field theory and was invented by Wilson. A subtlety: we need to analytically continue our expressions to Euclidean space-time.

2. Find a perturbative expansion for the interactions:

$$\langle \Omega | T\{O_a(t_a) O_b(t_b) \ldots\} | \Omega \rangle = \\ = |N|^2 \int \mathcal{D}[\varphi] e^{i \int \mathrm{d}^4x \, \mathcal{L}_0} \sum_{N=0}^{\infty} \frac{1}{N!} \left(i \int \mathrm{d}^4x \, \mathcal{L}_{\text{int}} \right)^N O_a(x_a) O_b(x_b) \ldots \tag{7.107}$$

   This can be done since it is a Gaussian integral with a polynomial prefactor. An analogy is

$$\int_{-\infty}^{\infty} \mathrm{d}q \; e^{-\frac{1}{2} m^2 q^2 - \frac{\lambda}{4!} q^4} = \int_{-\infty}^{\infty} \mathrm{d}q \; e^{-\frac{1}{2} m^2 q^2} \left[1 - \frac{\lambda}{4!} q^4 + \frac{1}{2} \left(\frac{\lambda}{4!} \right)^2 q^8 + \ldots \right] , \tag{7.108}$$

which we can easily evaluate if we know the Gaussian result.

This works if the interaction is "weak enough" such that we can truncate the sum at a finite order.

Here we will focus on the second option.
A trick to calculate the series in form of the Gaussian is to use a generating functional. In our previous analogy:

$$Z[J] = \int_{-\infty}^{\infty} \mathrm{d}q \ \exp\Big(- \frac{1}{2} m^2 q^2 - \frac{\lambda}{4!} q^4 + Jq \Big) \tag{7.109}$$

is the generating functional of the interacting theory, with interaction $-\frac{\lambda}{4!} q^4$ and

$$Z_0[J] = \int_{-\infty}^{\infty} \mathrm{d}q \ \exp\Big(- \frac{1}{2} m^2 q^2 + Jq \Big) \tag{7.110}$$

is the generating functional of the free theory.
We want to calculate terms in a power series, e.g.

$$\int_{-\infty}^{\infty} \mathrm{d}q \ e^{-\frac{1}{2}m^2 q^2} \cdot q^{4n} = \Big(\frac{\mathrm{d}}{\mathrm{d}J} \Big)^{4n} \int_{-\infty}^{\infty} \mathrm{d}q \ e^{-\frac{1}{2}m^2 q^2 + Jq} \Big|_{J=0} , \tag{7.111}$$

which we can use to write

$$Z[J] = e^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^4} \int_{-\infty}^{\infty} \mathrm{d}q \ e^{-\frac{1}{2}m^2 q^2 + Jq} = e^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^4} Z_0[J] \tag{7.112}$$

$$= \sqrt{\frac{2\pi}{m^2}} e^{-\frac{\lambda}{4!}\left(\frac{\mathrm{d}}{\mathrm{d}J}\right)^4} e^{\frac{J^2}{2m^2}} . \tag{7.113}$$

We will use a similar trick in field theory.
We can write correlators in the interacting theory ($Z[J]$) as a sum over "correlators" in the free theory ($Z_0[J]$).
We can express the $Z[J]$-expansion using Feynman "diagrams" where each vertex gets a λ and the vertices are connected by propagators associated with factors $\frac{1}{m^2}$:

$$Z[J] = \exp \Bigg[\underbrace{- \frac{\lambda}{4!} \Big(\frac{\mathrm{d}}{\mathrm{d}J} \Big)^4}_{\text{"vertices"}} \Bigg] \underbrace{Z_0[J]}_{=\exp\left(\frac{J^2}{2m^2}\right) \text{ "propagators"}} \tag{7.114}$$

7.6.1 *Green function of the free theory*

The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2} \Big(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \Big) . \tag{7.115}$$

We define

$$Z_0[J] \equiv |N|^2 \int \mathcal{D}[\varphi] \exp \Big(i \int \mathrm{d}^4 x \ \Big(\mathcal{L}_0 + J(x) \varphi(x) \Big) \Big) . \tag{7.116}$$

This is the **generating functional** for the free Green functions, since we can get the n-point functions

$$G_0(x_1, ..., x_n) \equiv \langle \Omega | T\{ \phi(x_1) ... \phi(x_n) \} | \Omega \rangle \tag{7.117}$$

$$= \frac{1}{i} \frac{\delta}{\delta J(x_1)} \cdot ... \cdot \frac{1}{i} \frac{\delta}{\delta J(x_n)} Z_0[J] \Big|_{J=0} \tag{7.118}$$

from it.
As discussed e.g. in the RPF script, using the rules for functional differentiation we get

$$\frac{1}{i}\frac{\delta}{\delta J(y)}\exp\left(i\int \mathrm{d}^4x\ J(x)\varphi(x)\right)=\varphi(y)\exp\left(i\int \mathrm{d}^4x\ J(x)\varphi(x)\right).$$

We can evaluate the PI that defines $Z_0[J]$, since it is a Gaussian, using Eq. (7.85):

$$\begin{aligned}
Z_0[J] &= |N|^2\int \mathcal{D}[\varphi]e^{-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ \varphi(x)D(x,y)\varphi(y)-\int \mathrm{d}^4x\ (-iJ(x))\varphi(x)} \\
&\overset{\substack{Eq.\ (7.85)\\ \downarrow}}{=} \underbrace{|N|^2\sqrt{\det\left(\frac{D(x,y)}{2\pi}\right)}}_{\substack{\uparrow\\ =Z_0[J=0]=1\\ \text{definition of } |N|^2!}}e^{\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ (-iJ(x))D^{-1}(x,y)(-iJ(y))}
\end{aligned}$$

with

$$D(x,y)=i\delta^{(4)}(x-y)(\partial_\mu\partial^\mu+m^2)\,. \tag{7.119}$$

Therefore, we get the **generating functional of the free theory**:

$$\boxed{Z_0[J]=\exp\left(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ J(x)\Delta_F(x-y)J(y)\right),} \tag{7.120}$$

where

$$\Delta_F(x-y)\equiv D^{-1}(x,y)\,, \tag{7.121}$$

which depends only on $x-y$ since $D(x,y)$ does.
Using the definition of the functional inverse, Eq. (7.89), we compute $\Delta_F(x-y)$:

$$\int \mathrm{d}^4z\ i\delta^{(4)}(x-z)(\partial_\mu\partial^\mu_{(z)}+m^2)D^{-1}(z,y)\overset{!}{=}\delta^{(4)}(x-y) \tag{7.122}$$

$$\Longrightarrow\quad (\Box_x+m^2)i\Delta_F(x-y)=\delta^{(4)}(x-y)\,. \tag{7.123}$$

We Fourier-transform to solve:

$$\int \mathrm{d}^4x\ e^{ip(x-y)}(\Box_x+m^2)i\Delta_F(x-y)=1 \tag{7.124}$$

and by integrating by parts we let $\Box_x$ act on $e^{ip(x-y)}$, which is equivalent to setting $\Box_x\to(-ip^\mu)(-ip_\mu)=-p^2$, and obtain

$$\int \mathrm{d}^4x\ e^{ip(x-y)}\Delta_F(x-y)=\frac{i}{p^2-m^2}\,, \tag{7.125}$$

or

$$\boxed{\Delta_F(x-y)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}e^{-ip(x-y)}\frac{i}{p^2-m^2+i\varepsilon}\,.} \tag{7.126}$$

This is the **Feynman-propagator**.
We have made the $i\varepsilon$ explicit ($m^2\to m^2-i\varepsilon$, Eq. (7.75)), otherwise the d^4p-integral would not be well-defined due to the singularity at

$p^2 = m^2$. The "$i\varepsilon$" gives the prescription on how the integral must be taken around the singularities:

$$p^2 - m^2 = (p^0)^2 - \boldsymbol{p}^2 - m^2 = 0 \tag{7.127}$$

$$\implies \quad p^0 = \pm E_{\boldsymbol{p}} = \pm\sqrt{m^2 + \boldsymbol{p}^2} \tag{7.128}$$

In the complex p^0-plane:

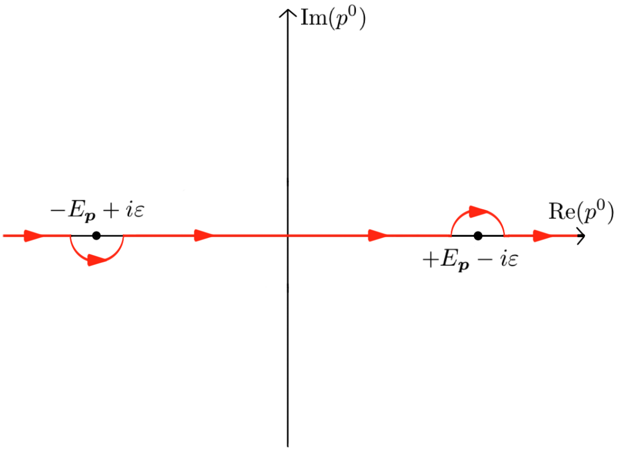


Figure 7.4: Integration contour for the evaluation of the defining integral of the Feynman-propagator.

We find that the Feynman-propagator is the inverse of the quadratic part of the Lagrangian in the $i\varepsilon$-prescription.
Had we been more careful and had we kept the $E_{\boldsymbol{p}}$ in $K(x, y)$ we would have obtained[2]:

$$\frac{i}{p^2 - m^2 + i\varepsilon E_{\boldsymbol{p}}} = \frac{i}{p^2 - m^2 + i\varepsilon}\,, \quad \text{since} \quad \varepsilon E_{\boldsymbol{p}} \sim \varepsilon\,. \tag{7.129}$$

[2] see Weinberg QFT1, chapter 9.4

Further, we see that the $\varepsilon e^{-\varepsilon|t|}$-factor in the more careful derivation (we dropped it after Eq. (7.69)) would give a correction of higher order in ε.

7.6.2 Free n-point function

We consider first the case $n = 1$:

$$G_0(z) = \frac{1}{i}\frac{\delta}{\delta J(z)} Z_0[J]\bigg|_{J=0} \tag{7.130}$$

$$= \frac{1}{i}\bigg(-\frac{1}{2}\int \mathrm{d}^4y\ \Delta_F(z-y)J(y) - \tag{7.131}$$

$$-\frac{1}{2}\int \mathrm{d}^4x\ J(x)\Delta_F(x-z)\bigg) Z_0[J]\bigg|_{J=0} \tag{7.132}$$

$$= 0\,. \tag{7.133}$$

The free n-point function vanishes for any odd n, since there is always a factor J left over which makes it vanish for $J = 0$.
We can see this also from

$$\int_{-\infty}^{\infty} \mathrm{d}q\ q^{2n+1} e^{-\frac{m^2}{2}q^2}, \quad n = 1, 2, 3, \dots \tag{7.134}$$

since q^{2n+1} is odd under $q \to -q$ and the rest is even. Analogously

$$\int \mathcal{D}[\varphi] e^{iS[\varphi]} \tag{7.135}$$

is even under $\varphi \to -\varphi$ and so an integral like

$$\int \mathcal{D}[\varphi]\varphi^{2n+1}e^{iS[\varphi]} \tag{7.136}$$

vanishes.
Now we consider the case $n = 2$:

We will now use a "summation convention" for integrals:

$$A_z B_z \equiv \int \mathrm{d}^4 z\ A(z)B(z)\,,$$
$$\Delta^F_{x,y} \equiv \Delta_F(x-y)\,.$$

$$\begin{aligned}
&\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = &(7.137)\\
&= \frac{1}{i^2}\frac{\delta}{\delta J(x)}\frac{\delta}{\delta J(y)}e^{-\frac{1}{2}\int \mathrm{d}^4z_1\mathrm{d}^4z_2\ J(z_1)\Delta_F(z_1-z_2)J(z_2)}\Big|_{J=0} &(7.138)\\
&= -\frac{\delta}{\delta J(x)}\Big(-\frac{1}{2}\Delta^F_{y,z_2}J_{z_2} - \frac{1}{2}J_{z_1}\Delta^F_{z_1,y}\Big)e^{-\frac{1}{2}J_{z_1}\Delta^F_{z_1,z_2}J_{z_2}}\Big|_{J=0} &(7.139)\\
&= \frac{1}{2}\Big(\Delta_F(y-x) + \Delta_F(x-y) + \mathcal{O}(J)\Big)\Big|_{J=0} &(7.140)\\
&= \Delta_F(x-y)\,, &(7.141)
\end{aligned}$$

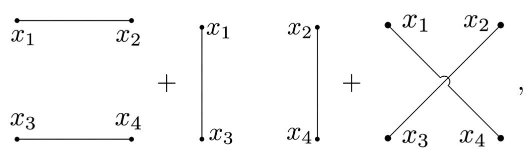
where the terms $\mathcal{O}(J)$ come from differentiating $e^{-\frac{1}{2}J_{z_1}\Delta^F_{z_1,z_2}J_{z_2}}$ and for the last equality we used the symmetry of Δ_F, $\Delta_F(x-y) = \Delta_F(y-x)$.
Therefore we find that the free two-point function coincides with the Feynman-propagator:

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \Delta_F(x-y)\,. \tag{7.142}$$

Now consider the case $n = 4$:

$\phi_i \equiv \phi(x_i)$, analogously for J_i.

$$\begin{aligned}
&\langle\Omega|T\{\phi_1\phi_2\phi_3\phi_4\}|\Omega\rangle = \frac{1}{i^4}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}e^{-\frac{1}{2}J_x\Delta^F_{x,y}J_y}\Big|_{J=0} &(7.143)\\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\Big(-J_x\Delta^F_{x,4}e^{-\frac{1}{2}J_y\Delta^F_{y,z}J_z}\Big)\Big|_{J=0} &(7.144)\\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\Big(\Big[-\Delta^F_{3,4} + J_x\Delta^F_{x,4}J_y\Delta^F_{y,3}\Big]e^{-\frac{1}{2}J_y\Delta^F_{y,z}J_z}\Big)\Big|_{J=0} &(7.145)\\
&= \frac{\delta}{\delta J_1}\Big(\Big[\Delta^F_{3,4}J_z\Delta^F_{z,2} + \Delta^F_{2,4}J_y\Delta^F_{y,3} + J_x\Delta^F_{x,4}\Delta^F_{2,3} + &(7.146)\\
&\quad + J_x\Delta^F_{x,4}J_y\Delta^F_{y,3}J_z\Delta^F_{z,2}\Big]e^{-\frac{1}{2}J_y\Delta^F_{y,z}J_z}\Big)\Big|_{J=0} &(7.147)\\
&= \Delta^F_{3,4}\Delta^F_{1,2} + \Delta^F_{2,4}\Delta^F_{1,3} + \Delta^F_{1,4}\Delta^F_{2,3} &(7.148)\\
&= \sum \text{"over all pairwise and complete contractions"} &(7.149)\\
&= \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 &(7.150)
\end{aligned}$$

$=$  , (7.151)

where in general we have:

$$\langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle = \begin{cases} 0, & n \text{ odd} \\ \sum_{\substack{\text{pairwise}\\ \text{contractions}}} \Delta^F_{i_1,i_2}\Delta^F_{i_3,i_4}\cdot\ldots\cdot\Delta^F_{i_{n-1},i_n}\,, & n \text{ even}\end{cases} \tag{7.152}$$

for n-point functions of the free theory.

7.7 Free complex scalar field: propagator

As before, a $U(1)$-invariant non-hermitian scalar field has the free Lagrangian:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \,, \tag{7.153}$$

where we also imply the substitution $m^2 \longrightarrow m^2 - i\varepsilon$ here.
The generating functional reads

$$Z_0[J, J^*] = |N|^2 \int \mathcal{D}[\varphi]\mathcal{D}[\varphi^*] \exp\left(i \int \mathrm{d}^4x \left(\mathcal{L}_0 + J\varphi + J^*\varphi^*\right)\right) \tag{7.154}$$

and φ, φ^* are treated as independent variables.
We can use the previous calculation with $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\phi_i^* = \phi_i$ hermitian/real, to derive

$$Z_0[J, J^*] = \exp\left(- \int \mathrm{d}^4x \mathrm{d}^4y \; J(x)\Delta_F(x-y)J^*(y)\right). \tag{7.155}$$

Note the absence of the factor $\frac{1}{2}$ compared to the real scalar field case; $\Delta_F(x-y)$ is the same propagator as for the real field.
Green functions are generated by ϕ, $\phi^\dagger$ via

$$\varphi(x) \longleftrightarrow \frac{1}{i}\frac{\delta}{\delta J(x)}, \quad \varphi^*(x) \longleftrightarrow \frac{1}{i}\frac{\delta}{\delta J^*(x)} \tag{7.156}$$

and

$$\wick{\c\phi(x)\c\phi(y)} = \langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = 0 \tag{7.157}$$

$$\wick{\c\phi^\dagger(x)\c\phi^\dagger(y)} = 0 \tag{7.158}$$

$$\wick{\c\phi(x)\c\phi^\dagger(y)} = \langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \Delta_F(x-y) \tag{7.159}$$

$$\wick{\c\phi^\dagger(x)\c\phi(y)} = \Delta_F(x-y) \tag{7.160}$$

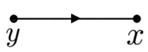
It is simple to see that Eq. (7.157), Eq. (7.158) hold by plugging in the expansion in creation and annihilation operators:

$$\phi \sim a_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}} \,.$$

But we need a combination $aa^\dagger$ or $bb^\dagger$ acting on $|\Omega\rangle$ to get a non-vanishing contribution.

Using the Feynman diagram notation:

$$\wick{\c\phi(x)\c\phi^\dagger(y)} = \tag{7.161}$$



so the arrow indicates ϕ at x and $\phi^\dagger$ at y.
As before, the n-point function of the free theory is given by all complete, non-zero pairwise contractions between ϕ and $\phi^\dagger$, i.e. there have to be the same number of ϕ and $\phi^\dagger$ for the n-point function not to vanish.

8

Interacting Theories, Perturbative Expansion of Green Functions and Derivation of the Feynman Rules

8.1 Generating functional of an interacting theory

We divide the total Lagrangian again into

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}\,, \quad \text{with e.g.} \quad \mathcal{L}_{\text{int}} = -\frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 \tag{8.1}$$

and the generating functional is

$$Z[J] = \frac{\int \mathcal{D}[\varphi] \exp\Big(i \int \mathrm{d}^4x\; (\mathcal{L} + J\varphi)\Big)}{\int \mathcal{D}[\varphi] \exp\Big(i \int \mathrm{d}^4x\; \mathcal{L}\Big)}\,. \tag{8.2}$$

As discussed, the Green functions are defined as

$$G(x_1, ...x_n) = \langle\Omega|T\{\phi(x_1)...\phi(x_n)\}|\Omega\rangle \tag{8.3}$$

$$= \prod_{i=1}^{n} \frac{1}{i}\frac{\delta}{\delta J(x_i)} Z[J]\bigg|_{J=0}\,. \tag{8.4}$$

The important difference is that even though the expression looks the same, the $\phi(x_i)$ now are **not** free fields anymore.
We expand the interaction

$$\exp\Big(i \int \mathrm{d}^4x\; \mathcal{L}_{\text{int}}\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(i \int \mathcal{L}_{\text{int}}\Big)^n\,, \tag{8.5}$$

assuming it to be weak. As in the case of the simple 1-dimensional integral we can "pull out" the interaction as a functional derivative of the free correlation function as follows.
With the generating functional of the free theory

$$Z_0[J] = \exp\Big(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\; J(x)\Delta_F(x-y)J(y)\Big) \tag{8.6}$$

we can calculate for the generating functional of the full theory:

Use

$$\varphi(x) \longleftrightarrow \frac{1}{i}\frac{\delta}{\delta J(x)}$$

$$Z[J] = |N|^2 \int \mathcal{D}[\varphi] e^{i\int \mathrm{d}^4x\, (\mathcal{L}_0 + J\varphi)} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \mathrm{d}^4 z_1 ... \mathrm{d}^4 z_n \cdot \tag{8.7}$$

$$\cdot\, \mathcal{L}_{\text{int}}(\varphi(z_1)) \cdot ... \cdot \mathcal{L}_{\text{int}}(\varphi(z_n)) \tag{8.8}$$

$$= |N|^2 \underbrace{\sum_{n=0}^{\infty} \frac{1}{n!} \left[i \int \mathrm{d}^4x\, \mathcal{L}_{\text{int}} \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right]^n}_{= \exp \left[i \int \mathrm{d}^4x\, \mathcal{L}_{\text{int}} \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \right]} \underbrace{\int \mathcal{D}[\varphi] e^{i\int \mathrm{d}^4x\, (\mathcal{L}_0 + J\varphi)}}_{= Z_0[J]} . \tag{8.9}$$

So we find

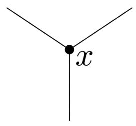
$$\boxed{Z[J] = |N|^2 \exp \left(i \int \mathrm{d}^4x\, \mathcal{L}_{\text{int}} \left[\frac{1}{i} \frac{\delta}{\delta J(x)} \right] \right) Z_0[J] .} \tag{8.10}$$

This is the **central identity of quantum field theory**, which we can also write as

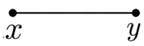
$$Z[J] = |N|^2 \underbrace{e^{i\int \mathrm{d}^4x\, \mathcal{L}_{\text{int}} \left[\frac{1}{i} \frac{\delta}{\delta J(x)} \right]}}_{\text{vertices}} \cdot \underbrace{e^{-\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y\, J(x) \Delta_F(x-y) J(y)}}_{\text{propagators}} \tag{8.11}$$

e.g.

$$\mathcal{L}_{\text{int}} = -\frac{\lambda_3}{3!} \phi(x)^3 \quad \Longrightarrow \quad \mathcal{L}_{\text{int}} = \tag{8.12}$$



$$\Delta_F(x - y) = \tag{8.13}$$



Now we just need to compute the results ($\widehat{=}$ Feynman diagrams). Since $\mathcal{L}_{\text{int}}$ is small, the sum can be truncated at the order of perturbation expansion required. From the discussion of the free theory we know that

$$\int \mathcal{D}[\varphi] e^{i\int \mathrm{d}^4x\, \mathcal{L}_0} \cdot (\text{polynomial in } \varphi) = \begin{array}{c}\text{sum over all non-vanishing} \\ \text{pairwise contractions}\end{array} \tag{8.14}$$

This is simply a general result for Gaussian integrals with polynomial prefactors (see Exercise).

Therefore we can write

$$\int \mathcal{D}[\varphi] e^{i\int \mathrm{d}^4x\, \mathcal{L}_0} \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \int \mathrm{d}^4x\, \mathcal{L}_{\text{int}} \right)^n \varphi(x_1) \cdot ... \cdot \varphi(x_n)$$

$$= \sum_{n=0}^{\infty} \int \mathrm{d}^4 z_1 ... \mathrm{d}^4 z_n \cdot \left[\begin{array}{c} \text{sum over all pairwise} \\ \text{contractions of fields in} \\ \mathcal{L}_{\text{int}}(\varphi(z_1)) \cdot ... \cdot \mathcal{L}_{\text{int}}(\varphi(z_n)) \varphi(x_1) \cdot ... \cdot \varphi(x_n) \end{array} \right]$$

We will truncate the sum at a finite order in n.

8.1.1 Example: 2-point function in ϕ^3-theory

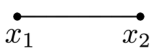
We assume the following interaction

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!} \phi^3 \tag{8.15}$$

and assume that g is small.
We want to now calculate the two point function in the full theory, so we want express

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)\} | \Omega \rangle \tag{8.16}$$

in terms of a truncated, perturbative expansion in powers of g.
The contributions in terms of powers of g are

$\mathcal{O}(g^0)$: 

$= \Delta_F(x_1 - x_2)$

$\mathcal{O}(g^1)$: $\quad 0 \quad$ (no non-vanishing contraction of 5 fields $\phi(x_1)\phi(x_2)\phi(z)^3$, need an even number)

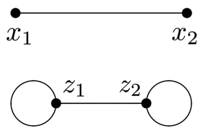
$\mathcal{O}(g^2)$: $\quad$ Pairwise contractions of $\phi(x_1)\phi(x_2)\phi(z_1)^3\phi(z_2)^3$:

...

We will only expand up to terms of order g^2 here.

Diagrams:

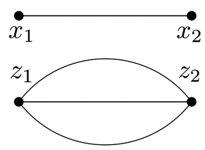
1)



$$\begin{aligned}&\frac{1}{2} \cdot 3 \cdot 3 \cdot \left(-\frac{ig}{3!} \right)^2 \Delta_F(x_1 - x_2) \cdot \\ &\cdot \int \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \; \Delta_F(z_1 - z_2) \Delta_F(0)^2 \,,\end{aligned} \tag{8.17}$$

where the factor $\frac{1}{2}$ is the $\frac{1}{n!}$ in the exponential and the factor 9 accounts for the fact that there are 9 contractions which lead to the same expression/diagram, namely 3 legs starting from z_1 that can be contracted with the 3 legs starting from z_2.

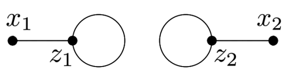
2)



$$\begin{aligned}&\frac{1}{2} \cdot 3 \cdot 2 \cdot \left(-\frac{ig}{3!} \right)^2 \Delta_F(x_1 - x_2) \cdot \\ &\cdot \int \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \Delta_F(z_1 - z_2)^3\end{aligned} \tag{8.18}$$

The factor $3 \cdot 2$ accounts for the fact that there are 3! possibilities to connect the three legs starting from z_1 to the three legs starting from z_2.

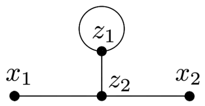
3)



$$\begin{aligned}&\frac{1}{2} \cdot 2 \cdot 3 \cdot 3 \cdot \left(-\frac{ig}{3!} \right)^2 \int \mathrm{d}^4 z_1 \mathrm{d}^4 z_2 \cdot \\ &\cdot \Delta_F(x - z_1) \Delta_F(x_2 - z_2) \Delta_F(0)^2 \\ &= G^{(1)}(x_1) G^{(1)}(x_2) \,,\end{aligned} \tag{8.19}$$

where the factor 2 accounts for the fact that interchanging z_1 and z_2 gives the same diagram. The expression coincides with the product of two one-point functions $G^{(1)}(x_i)$ at $\mathcal{O}(g^1)$.

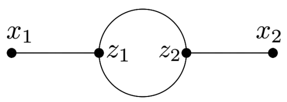
4)



$$\frac{1}{2}\cdot 2\cdot 3\cdot 3\cdot 2\cdot\left(-\frac{ig}{3!}\right)^2\int \mathrm{d}^4z_1\mathrm{d}^4z_2\cdot$$
$$\cdot\,\Delta_F(x_1-z_1)\Delta_F(0)\Delta_F(z_1-x_2)\cdot$$
$$\cdot\,\Delta_F(z_1-z_2)\,. \tag{8.20}$$

Here the first factor 2 again accounts for the interchangeability of z_1 and z_2, then there are 3 possibilities for x_1 to contract to z_1 and two to contract x_2 to z_1, and finally there are three possibilities to contract z_1 to z_2.

5)



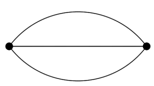
$$\frac{1}{2}\cdot 2\cdot 3\cdot 3\cdot 2\left(-\frac{ig}{3!}\right)^2\int \mathrm{d}^4z_1\mathrm{d}^4z_2\cdot$$
$$\cdot\,\Delta_F(x_1-z_1)\Delta_F(z_2-x_2)\cdot$$
$$\cdot\,\Delta_F(z_1-z_2)^2\,. \tag{8.21}$$

The first factor 2 accounts for the interchangeability of z_1 and z_2, the two factors 3 represent the 9 possible contractions of x_1 to z_1 and x_2 to z_2 and the last factor 2 accounts for the two possible contractions of z_1 to z_2.

Diagrams 1) and 2) are **disconnected** (not connected to $\phi(x_1)$ and $\phi(x_2)$) and belong to the class of diagrams known as **vacuum polarization subdiagrams**. They do not contribute to the two-point function. But why is that?

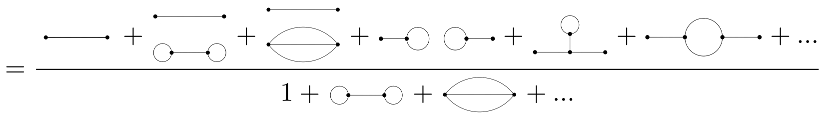
We also need to include the denominator of Eq. (8.2):

$$\int \mathcal{D}[\varphi]e^{i\int \mathrm{d}^4x\,(\mathcal{L}_0+\mathcal{L}_{\rm int})} = 1 +$$

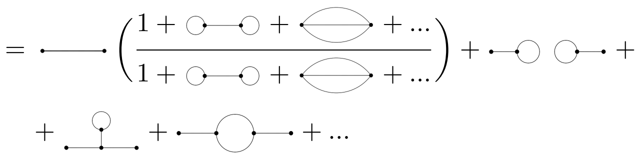
$$+$$

$$+\,\mathcal{O}(g^3) \tag{8.22}$$

Dividing by this contribution gives

$$\frac{\int \mathcal{D}[\varphi] \exp\left(i \int \mathrm{d}^4x \; (\mathcal{L} + J\varphi)\right)}{\int \mathcal{D}[\varphi] \exp\left(i \int \mathrm{d}^4x \; \mathcal{L}\right)} = \tag{8.23}$$



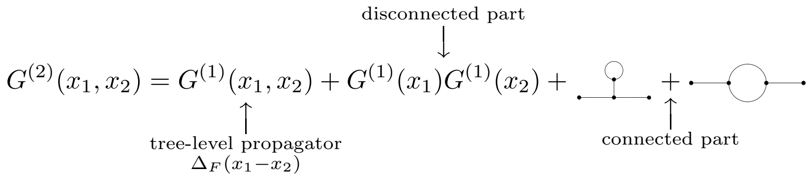
(8.24)



(8.25)

(8.26)

In conclusion, we find for the two-point function to $\mathcal{O}(g^2)$:



(8.27)

A diagram consists of:

- products of propagators,
- integrations over internal vertex points (here: z_1, z_2),
- vertex factors (here: $-\frac{ig}{3!}$),
- a symmetry factor for the number of contractions.

We consider two diagrams as **identical** if they differ only by a relabeling (here $z_1 \longleftrightarrow z_2$) of internal vertex points.
We can now summarize the Feynman rules in position space. We will be brief since most of the time we are interested in momentum space calculations.

8.2 Feynman rules for position space Green functions

Consider the Lagrangian

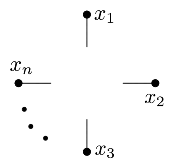
$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \quad \text{with} \quad \mathcal{L}_{\text{int}} = g_{l_1 \dots l_k} \phi_1 \dots \phi_k \tag{8.28}$$

The perturbative expansion of $\langle \Omega | T\{\phi_{l_1}(x_1) \cdot \dots \cdot \phi_{l_n}(x_n)\} | \Omega \rangle$ is computed using the following set of rules:

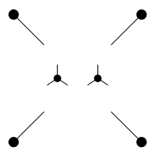
1. We determine all non-vanishing contractions $\phi_l(x)\phi_{l'}(y)$ (Feynman propagators) **and** vertex factors $ig_{l_1 \dots l_k}$.
2. Start with (**external**) points x_i for each position at which the fields in the n-point function are evaluated. Draw a line from each point.
3. Calculating the $\mathcal{O}(g^m)$ contribution: Draw m **internal** vertex points z_1,...,z_m for every $\mathcal{L}_{\text{int}}(\phi)$.
4. A line can either connect to an existing line, resulting in a Feynman propagator connecting the two lines, or it can split due to an interaction.
5. Leave out disconnected vacuum polarization subdiagrams.
6. Compute the symmetry factor by counting contractions and integrate over $\int d^4 z_i$ at each internal point z_i. (This last point will mostly not be very important, as it is an artifact of real scalar field theory.)

1.

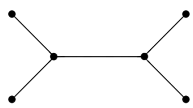
2.



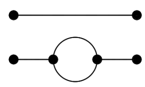
3.



for $m = 2$.
4.



or



also for $m = 2$.

8.3 Green functions in momentum space

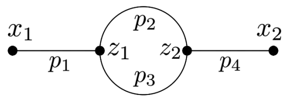
We define

$$\begin{aligned} &\int d^4x_1 \dots d^4x_n \; e^{ip_1x_1 + \dots ip_nx_n} G(x_1, \dots, x_n) \\ &\equiv (2\pi)^{(4)} \delta^4(p_1 + \dots + p_n) \widetilde{G}(p_1, \dots, p_n) \end{aligned} \tag{8.29}$$

Note: the independent variables in $\widetilde{G}$ are p_i^μ, where p_i^0 is free and **not** on-shell $\Rightarrow p_i^0 \neq \sqrt{m^2 + \boldsymbol{p_i^2}}$.

We will find the momentum space Green functions much more useful, especially for scattering problems. The meaning of the overall δ-function will become clear after we've looked at an example.

Example:



$$\overset{(*)}{=} \frac{1}{2} \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot \left(- \frac{ig}{3!} \right)^2 \int \mathrm{d}^4 z_1 \int \mathrm{d}^4 z_2 \Delta^F_{x,z_1} \Delta^F_{z_2,x_2} (\Delta^F_{z_1,z_2})^2 \tag{8.30}$$

We now write every propagator in momentum space (setting $m = 0$ for simplicity):

$$\overset{(*)}{=} -\frac{g^2}{2} \int \mathrm{d}^4 z_1 \int \mathrm{d}^4 z_2 \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_3}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_4}{(2\pi)^4} \cdot$$
$$\cdot\, e^{-ip_1(x_1-z_1)} e^{-ip_2(z_2-x_2)} e^{-ip_3(z_1-z_2)} e^{-ip_4(z_1-z_2)} \cdot$$
$$\cdot \frac{i}{p_1^2 + i\varepsilon} \frac{i}{p_2^2 + i\varepsilon} \frac{i}{p_3^2 + i\varepsilon} \frac{i}{p_4^2 + i\varepsilon}$$

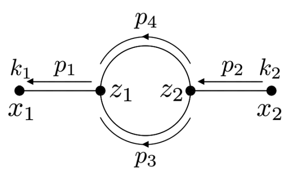
We see that if we perform the z_1 and z_2 integrals, we get the factors

$$\underset{\substack{\uparrow \\ z_1}}{\delta^{(4)}(p_1 - p_3 - p_4)} \quad \text{and} \quad \underset{\substack{\uparrow \\ z_2}}{\delta^{(4)}(-p_2 + p_3 + p_4)} \tag{8.31}$$

corresponding to the 4-momentum being conserved at each vertex z_1 and z_2.

Let us now Fourier-transform according to Eq. (8.29):

$$\xrightarrow{\text{FT}} -\frac{g^2}{2} \int \mathrm{d}^4 x_1 \int \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 p_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_3}{(2\pi)^4} \int \frac{\mathrm{d}^4 p_4}{(2\pi)^4} \cdot$$
$$\cdot\, (2\pi)^4 \delta^{(4)}(p_1 - p_3 - p_4)(2\pi)^4 \delta^{(4)}(-p_2 + p_3 + p_4) \cdot$$
$$\cdot\, e^{-ip_1 x_1} e^{+ip_2 x_2} e^{ik_1 x_1} e^{ik_2 x_2} \frac{i}{p_1^2 + i\varepsilon} \frac{i}{p_2^2 + i\varepsilon} \frac{i}{p_3^2 + i\varepsilon} \frac{i}{p_4^2 + i\varepsilon} \overset{(*)}{=}$$

$=$ 

$$p_1 - p_3 - p_4 = 0$$
$$-p_2 + p_3 + p_4 = 0$$
$$p_1 = +k_1$$
$$p_2 = -k_2$$

Therefore we have

$$k_1 - p_3 - p_4 = 0$$
$$k_2 + p_3 + p_4 = 0$$
$$\implies k_1 + k_2 = 0$$

The arrows indicate whether a momentum is **incoming** at z_i. We use the following convention:

- For **incoming** momenta use $e^{-ip_i x_i}$.
- For **outgoing** momenta use $e^{ip_i x_i}$.

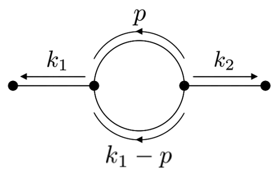
Finally, integrating over x_1 and x_2 gives us more δ-factors, $\delta^{(4)}(p_1 - k_1)\delta^{(4)}(p_2 + k_2)\big((2\pi)^4\big)^2$ and so we obtain

$$\overset{(*)}{=} -\frac{g^2}{2} \frac{i}{k_1^2 + i\varepsilon} \frac{i}{k_2^2 + i\varepsilon} (2\pi)^4 \delta^{(4)}(k_1 + k_2) \int \frac{\mathrm{d}^4 p_3}{(2\pi)^4} \frac{i}{p_3^2 + i\varepsilon} \cdot \tag{8.32}$$
$$\cdot \frac{i}{(k_1 - p_3)^2 + i\varepsilon} \tag{8.33}$$

and renaming $p_3 = p$ we finally get

$$\widetilde{G}(k_1, k_2) = -\frac{g^2}{2} \frac{i}{k_1^2 + i\varepsilon} \frac{i}{k_2^2 + i\varepsilon} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p^2 + i\varepsilon)((k_1 - p)^2 + i\varepsilon)} \,. \tag{8.34}$$

It made sense to remove the overall δ-distribution from the definition since it will always show up. Why?
The reason is that imposing 4-momentum conservation at each vertex through the overall δ of the following diagram



is the same as requesting that the momentum-space Green functions be translation-invariant:

$$G(x_1, x_2, ..., x_n) = G(x_1 + a, x_2 + a, ..., x_n + a) \quad \forall a \,. \tag{8.35}$$

<u>**Check:**</u>

$$\int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 ... \mathrm{d}^4 x_n \; e^{ip_1 x_1 + ... + ip_n x_n} G(x_1, x_2, ..., x_n) \,, \tag{8.36}$$

use $a = -x_1$ and the translation-invariance of G:

$$= \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 ... \mathrm{d}^4 x_n \; \exp\left(i \sum_{j=1}^{n} p_j x_j \right) G(0, x_2 - x_1, ..., x_n - x_1) \,. \tag{8.37}$$

Now we introduce new integration variables: $y_2 = x_2 - x_1, \ldots, \; y_k = x_k - x_1$,

$$\begin{aligned} &= \int \mathrm{d}^4 x_1 e^{ip_1 x_1 + ip_2 x_1 + ... + ip_n x_1} \int \mathrm{d}^4 y_2 ... \mathrm{d}^4 y_n e^{ip_2 y_2 + ... + ip_n y_n} \cdot \\ &\quad \cdot G(0, y_2, ..., y_n) \\ &= (2\pi)^4 \delta^{(4)}(p_1 + p_2 + ... + p_n) \int \mathrm{d}^4 y_2 ... \mathrm{d}^4 y_n e^{ip_2 y_2 + ... + ip_n y_n} \cdot \\ &\quad \cdot G(0, y_2, ..., y_n) \quad \checkmark \end{aligned}$$

<u>**Important:**</u> how many p_i integrations are left over?
Eliminating as many internal p_i as possible using δ-functions, we find one **overall** δ-function and a number of p_i-integrations.

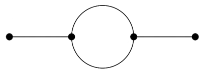
$$\underset{\substack{\uparrow \\ \text{number} \\ \text{of loops}}}{L} = \overset{\substack{\text{number of} \\ \text{propagators/lines} \\ (\text{each for one } \int \mathrm{d}^4 p_i) \\ \downarrow}}{P} - \underset{\substack{\uparrow \\ \text{number of} \\ \text{internal and} \\ \text{external vertex} \\ \text{points} \\ (\text{each for one } \delta) \\ \int \mathrm{d}^4 z_i \text{ or } \int \mathrm{d}^4 x_i}}{V} + \overset{\substack{\text{number of} \\ \text{connected components} \\ (\text{overall } \delta\text{'s are useless and} \\ \text{for each disconnected} \\ \text{component there is one } \delta\text{-fct.}) \\ \downarrow}}{n} \tag{8.38}$$

Examples:

1)

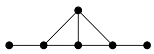
$$L = 1 - 2 + 1 = 0 \tag{8.39}$$

2)



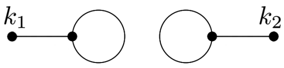
$$L = 4 - 4 + 1 = 1 \tag{8.40}$$

3)



$$L = 7 - 6 + 1 = 2 \tag{8.41}$$

4)



$$L = 4 - 4 + 2 = 2 \tag{8.42}$$

with

$$(2\pi)^8 \delta^{(4)}(k_1)\delta^{(4)}(k_2) \frac{1}{k_1^2 + i\varepsilon} \frac{1}{k_2^2 + i\varepsilon} \left(- \frac{ig}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{i}{p^2 + i\varepsilon} \right)^2 . \tag{8.43}$$

8.4 Feynman rules in momentum space

We label external momenta as p_i and internal momenta as k_i:

1. Draw external points for each field in $\langle \Omega | T\{\phi_1 ... \phi_n\} | \Omega \rangle$ and add vertices up to the given order of expansion (the same as in position space).
2. Each external line gets a factor $\widetilde{\Delta}_F(p_i) = \frac{i}{p_i^2 - m^2 + i\varepsilon}$ and each internal line is assigned a $\widetilde{\Delta}_F(k_i)$.
3. Momentum is conserved at each vertex.
4. Integrate over all undetermined 4-momenta.
5. Apply symmetry factors.
6. Sum over all possible diagrams.

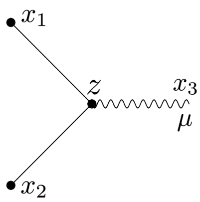
Example:

$$\mathcal{L} = \mathcal{L}_0 + g\phi\partial_\mu \phi A^\mu \tag{8.44}$$

What is the vertex factor in momentum space? Naively it should correspond to some of the p_μ entering the vertex. But which one? Choose a Green function with the same fields as the vertex:

position $\longrightarrow$ momentum
$\partial_\mu \longrightarrow ik_\mu$

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)A_\mu(x_3)\} | \Omega \rangle = \quad + \mathcal{O}(g^2) \tag{8.45}$$



$$= \sum_{\text{all contractions}} ig \int \mathrm{d}^4 z \; \phi(x_1)\phi(x_2)A_\mu(x_3)\phi(z)\partial_\rho\phi(z)A^\rho(z) \tag{8.46}$$

$$= ig \int \mathrm{d}^4 z \left[\Delta_F^{\mu\rho}(x_3 - z) \Big(\Delta_F(x_1 - z)\partial_\rho^z \Delta_F(x_2 - z) + (x_1 \longleftrightarrow x_2) \Big) \right] . \tag{8.47}$$

With

$$\overbrace{A_\mu(x)A_\nu(x)} = \Delta^F_{\mu\nu}(x-y)$$

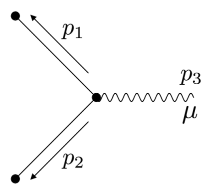
and all others vanishing. We will explore this later when we go beyond scalar particles.

We Fourier-transform:

Using the convention of all momenta outgoing e^{+ipx}.

$$\begin{aligned}
&\xrightarrow{\text{FT}} (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\widetilde{G}(p_1,p_2,p_3)\\
&\quad= \int \mathrm{d}^4z\mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{d}^4x_3\ \exp\Big(i(p_1x_1+p_2x_2+p_3x_3)\Big)\cdot\Big[\text{Eq. (8.47)}\Big]\\
&\quad= \int \mathrm{d}^4z \int \prod_{j=1}^{3}\left[\frac{\mathrm{d}^4k_j}{(2\pi)^4}\mathrm{d}^4x_j \exp\Big(ip_jx_j\Big)\right]\Big\{e^{-ik_3(x_3-z)}\widetilde{\Delta}_F^{\mu\rho}(k_3)\cdot\\
&\qquad\cdot\Big[e^{-ik_1(x_1-z)}\widetilde{\Delta}_F(k_1)ig\partial_\rho^z\Big(e^{-ik_2(x_2-z)}\widetilde{\Delta}_F(k_2)\Big)+(x_1\leftrightarrow x_2)\Big]\Big\}\\
&\quad\overset{p_i=k_i}{=} (2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\widetilde{\Delta}_F(p_1)\widetilde{\Delta}_F(p_2)\widetilde{\Delta}_F^{\mu\rho}(p_3)\cdot(-g)(p_1+p_2)_\rho\,.
\end{aligned}$$

We obtained the desired vertex factor:



$$\leftrightarrow \quad -g(p_1+p_2)_\mu\,.$$

It is symmetric under the exchange $p_1 \leftrightarrow p_2$, as it should be for identical bosons. Note, that we need to keep the conventions for the momenta in mind.

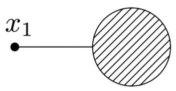
8.5 *Composite operator Green functions*

Instead of $\langle\Omega|T\{\phi_1(x_1)\phi_2(x_2)...\}|\Omega\rangle$, we often also want to calculate Green functions of operators, e.g.

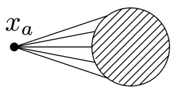
$$O_a = \phi(x)^2 \quad\text{or}\quad \phi(x_1)^3\phi(x_2) \quad\text{etc.} \tag{8.48}$$

These are both local, composite operators.
We can just perform a straightforward extension of the Feynman rules:

$$\phi(x_1) \qquad \mapsto \qquad O_a(x_1) \tag{8.49}$$



$\mapsto$



(8.50)

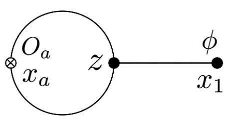
and we sum again over all contractions. Here x_1 and x_a are external points, O_a is an operator containing n fields and the striped circle symbolizes the rest of the diagram.

Example:

$$\mathcal{L}_{\text{int}} = -\frac{g}{3!}\phi^3 \quad\text{and}\quad O_a(x_a) = \phi(x_a)^2 \tag{8.51}$$

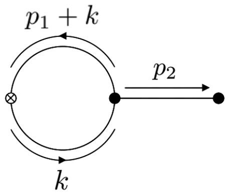
$$\langle\Omega|T\{O_a(x_a)\phi(x_1)\}|\Omega\rangle = \langle\Omega|T\{\phi(x_a)^2\phi(x_1)\}|\Omega\rangle \tag{8.52}$$

$=$



$+\,\mathcal{O}(g^2)$ (8.53)

and in momentum space:



$$, \quad \text{with external momenta } p_1\,, p_2\,. \tag{8.54}$$

8.6 The T^* product and comparison to the canonical formalism

We have so far required $O_a(x_a)$ in the path integral derivation to depend only on $\phi(x)$ and not on $\pi(x)$ which is a strong restriction, since we cannot deal with terms $\sim \pi \sim \dot{\phi}$ or any four-derivative $\partial_\mu \phi$.

We define a $\boldsymbol{T^*}$ time ordering:

$$\langle \Omega | T^* \{\partial_0 \phi(x) ...\} | \Omega \rangle = \partial_0^{(x)} \langle \Omega | T\{\phi(x)...\} | \Omega \rangle \tag{8.55}$$
$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon t} \Big(\langle \Omega | T\{\phi(x+\epsilon t)...\} | \Omega \rangle - \langle \Omega | T\{\phi(x)...\} | \Omega \rangle \Big)\,, \tag{8.56}$$

where T is the usual time ordering defined by the path integral.

Comment: the $\langle \Omega | T\{\partial_0 \phi(x)...\} | \Omega \rangle$ **exists**, we only did not know how to use the PI to calculate it.

The T^* definition is different from T. Recall the definition:

$$T\{A(x)B(y)\} = \theta(x^0 - y^0) A(x) B(y) + \theta(y^0 - x^0) B(y) A(x)$$

and with the choice $A(x) = \partial_0 \phi(x)$:

$$T\{\partial_0^{(x)} \phi B(y)\} = \theta(x^0 - y^0) \partial_0^{(x)} \phi(x) B(y) + \theta(y^0 - x^0) B(y) \partial_0^{(x)} \phi(x)\,.$$

Compare this to T^*:

$$\begin{aligned} T^*\{\partial_0 \phi(x) B(y)\} = \partial_0^{(x)} T\{\phi(x) B(y)\} &= T\{\partial_0^{(x)} \phi(x) B(y)\} + \\ &+ \delta(x^0 - y^0) \phi(x) B(y) - \delta(y^0 - x^0) B(y) \phi(x) \\ &= T\{\partial_0^{(x)} \phi(x) B(y)\} + [\phi(x), B(y)] \delta(x^0 - y^0)\,. \end{aligned}$$

Example:

$$\begin{aligned} \langle \Omega | T\{\partial_\mu \phi(x) \partial_\nu \phi(y)\} | \Omega \rangle &= \partial_\mu^{(x)} \langle \Omega | T\{\phi(x) \partial_\nu \phi(y)\} | \Omega \rangle - \\ &\quad - \langle \Omega | [\phi(x), \partial_\nu \phi(y)] | \Omega \rangle \delta(x^0 - y^0) \\ &= \partial_\mu^{(x)} \langle \Omega | T\{\phi(x) \partial_\nu \phi(y)\} | \Omega \rangle - i \delta_\nu^0 \delta^{(4)}(x-y) \\ &= \underbrace{\partial_\mu^{(x)} \partial_\nu^{(y)} \langle \Omega | T\{\phi(x) \phi(y)\} | \Omega \rangle}_{= \langle \Omega | T^* \{\partial_\mu \phi(x) \partial_\nu \phi(y)\} | \Omega \rangle} - i \delta_\mu^0 \delta_\nu^0 \delta^{(4)}(x-y) \\ &= |N|^2 \int \mathcal{D}[\varphi] e^{i \int \mathrm{d}^4 x \mathcal{L}} \partial_\mu \phi(x) \partial_\nu \phi(y) - i \delta_\mu^0 \delta_\nu^0 \delta^{(4)}(x-y)\,, \end{aligned}$$

$i\delta_\mu^0 \delta_\nu^0 \delta^{(4)}(x-y)$ is not manifestly covariant, since we single out time derivatives.

where in the second equality we used the equal-time canonical commutation relations Eq. (4.36) to simplify $[\phi(x), \partial_\nu \phi(y)]$, since $\pi = \partial_0 \phi$ in this case, and in the last equality we used the definition of the path integral.

- The PI always uses the T^* product, since T is not always defined in the Lagrangian version of the PI.
- If $O_a(x_a)$ does not contain $\pi(x)$ (which mostly means time derivatives or Lorentz-covariant derivatives ∂_μ), then $T^* = T$.
- $\boxed{\text{In the following we will always mean } T^* \text{ when we write } T!}$

8.6.1 *Comparison of the canonical operator formalism and path integrals*

We define the theory using a Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \tag{8.57}$$

$$\langle \Omega | T^* \{ O_a(x_a) \cdot \ldots \cdot O_n(x_n) \} | \Omega \rangle$$

$$= \frac{\int \mathcal{D}[\varphi] \exp\left(i \int \mathrm{d}^4 x \, \mathcal{L} \right) O_a(x_a) \ldots}{\int \mathcal{D}[\varphi] \exp\left(i \int \mathrm{d}^4 x \, \mathcal{L} \right)}$$

Feynman path integral
Calculate perturbatively by using the polynomial Gaussian integrals.

in the interaction picture

$$H = H_0^I + H_{\text{int}}^I$$

$$\langle \Omega | T \{ O_a(x_a) \cdot \ldots \cdot O_n(x_n) \} | \Omega \rangle$$

$$= \frac{\langle 0 | T \{ O_a^I(x_a) \ldots O_n^I(x_n) e^{-i \int_{-\infty}^{\infty} \mathrm{d}t \, H_{\text{int}}^I} \} | 0 \rangle}{\langle 0 | T \{ e^{-i \int_{-\infty}^{\infty} \mathrm{d}t \, H_{\text{int}}^I} \} | 0 \rangle}$$

with $|0\rangle$ the free Fock space vacuum, see e.g. the RPF script.

Gell-Mann-Low formula
Evaluate fields in the interaction picture, develop techniques to calculate a, $a^\dagger$ expressions in the interaction picture.

As mentioned in the introduction: the path integral is manifestly Lorentz-covariant since $\mathcal{L}_{\text{int}}$ and T^* are, whereas H_{int}^I and T are not (both yield the same result as long as $O_a(x_a)$ does not contain $\pi(x)$).

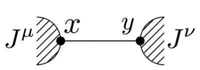
What happens if $\mathcal{L}_{\text{int}}$ (but not $O_a(x_a)$) contains derivatives?

Example:

$$\mathcal{L}_{\text{int}} = -g J^\mu \partial_\mu \phi \tag{8.58}$$

where J^μ is a current that depends on other fields, e.g. $J^\mu = \overline{\Psi} \gamma^\mu \Psi$. We want to evaluate $\langle \Omega | T \{ J_\mu(x) J_\nu(y) \} | \Omega \rangle$ to $\mathcal{O}(g^2)$!

A) Path integral:

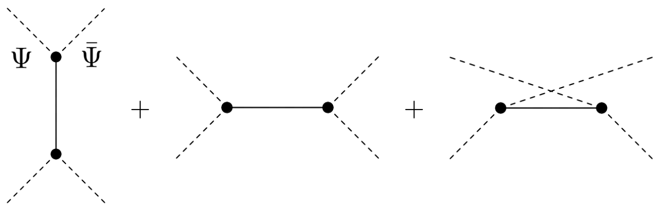


$$= |N|^2 \int \mathcal{D}[\varphi] e^{i \int \mathrm{d}^4 x \, \mathcal{L}} \partial_\mu \phi(x) \partial_\nu \phi(y) \tag{8.59}$$

$$= \langle \Omega | T^* \{ \partial_\mu \phi(x) \partial_\nu \phi(y) \} | \Omega \rangle \,, \tag{8.60}$$

which is Lorentz-covariant!

If we wrote $J^\mu = \overline{\Psi}\gamma^\mu\Psi$, this would be



(8.61)

B) Operator formalism:

$$\mathcal{H}^I_{\text{int}} = gJ^\mu\partial_\mu\phi_I + g^2(J^0)^2 \tag{8.62}$$

where the second term comes from conjugate momentum w.r.t. full Hamiltonian

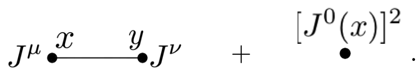
$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \underset{\substack{\uparrow \\ \text{from } \frac{1}{2}(\partial_\mu\phi)^2}}{\dot{\phi}} - \overset{\substack{\text{from } \mathcal{L}_{\text{int}} \\ \downarrow}}{J^0 g}\,, \tag{8.63}$$

but:

$$\pi_I = \dot{\phi}_I \tag{8.64}$$

is the conjugate momentum with respect to H_0.

Calculating this with the Gell-Mann-Low formula, we would have



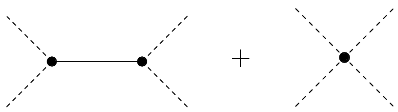
(8.65)

The first diagram is the part involving ϕ, corresponding to $\langle\Omega|T\{\partial_\mu\phi(x)\partial_\nu\phi(y)\}|\Omega\rangle$, which also contains the non-covariant term $-i\delta^0_\mu\delta^0_\nu\delta^4(x-y)$.

The second diagram is the local term contribution for $x = y$ and $\mu = \nu = 0$; the corresponding expression for $(J^0)^2$ is $+i\delta^0_\mu\delta^0_\nu\delta^4(x-y)$. Therefore, we see that the non-covariant contributions cancel and the result is the same as in A)! If we simply ignore the non-covariant terms in the operator formalism, then the Feynman rules are the same as in the path integral formalism.

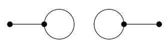
See zuber [7], chapter 6.1.4 for a proof.

If we wrote $J^\mu = \overline{\Psi}\gamma^\mu\Psi$, the above diagrams would correspond to

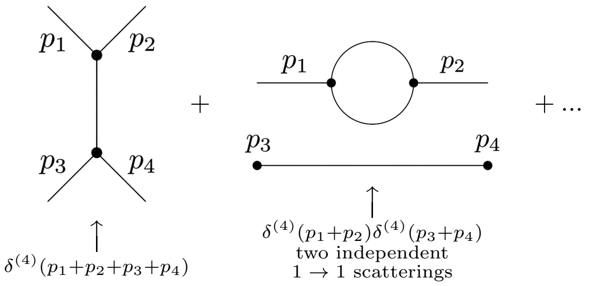


(8.66)

8.7 *Connected and one-particle-irreducible (1PI) Green functions*

Disconnected diagrams like  are not relevant for scattering processes, as evident from the δ-function appearing in each disconnected contribution.

They correspond to several independent scattering processes of fewer particles, e.g. in ϕ^3-theory for $2 \longrightarrow 2$ scattering:



(8.67)

We **want** the generating functional of **connected** Green functions. We know that

$$
\begin{aligned}
G(x) =& \; G^C(x)\,, && (8.68)\\
G(x_1,x_2) =& \; G^C(x_1,x_2) + G^C(x_1)G^C(x_2)\,, && (8.69)\\
& \text{(this defines } G^C(x_1,x_2))\,, && (8.70)\\
G(x_1,x_2,x_3) =& \; G^C(x_1,x_2,x_3) + G^C(x_1,x_2)G^C(x_3) && (8.71)\\
& + G^C(x_1,x_3)G^C(x_2) + G^C(x_2,x_3)G^C(x_1) && (8.72)\\
& + G^C(x_1)G^C(x_2)G^C(x_3)\,, && (8.73)\\
& \text{(this defines } G^C(x_1,x_2,x_3))\,, && (8.74)
\end{aligned}
$$

C means "connected".

and so on. This allows us to recursively define connected n-point functions:

$$G(x_1,...,x_n) = \sum_{\substack{\text{all partitions } I_\alpha \\ \text{of } \{1,...,n\} \\ \text{i.e. } \bigcup_\alpha I_\alpha = I}} \prod_\alpha G^C(x_{I_\alpha})\,. \tag{8.75}$$

We define:

$$\boxed{Z[J] \equiv e^{iW[J]}\,,} \tag{8.76}$$

This is called a "cumulant expansion".

or equivalently

$$W[J] = -i\ln(Z[J])\,. \tag{8.77}$$

This defines $W[J]$ and we claim that $iW[J]$ generates the **connected Green functions**:

$$\boxed{G^C(x_1,...,x_n) \equiv \frac{1}{i}\frac{\delta}{\delta J(x_1)} \cdot ... \cdot \frac{1}{i}\frac{\delta}{\delta J(x_n)} iW[J]\,.} \tag{8.78}$$

<u>Proof:</u>
<u>$n = 1$</u>:

$$G(x) = \frac{1}{i}\frac{\delta}{\delta J(x)} Z[J]\Big|_{J=0} = \left(\frac{1}{i}\frac{\delta(iW)}{\delta J(x)}\right) e^{iW[J]}\Big|_{J=0} \tag{8.79}$$

$$\underset{\substack{\uparrow \\ 1=Z[0]=e^{iW[0]}}}{=} \frac{1}{i}\frac{\delta}{\delta J(x)} W[J]\Big|_{J=0} = G^C(x)\,. \tag{8.80}$$

This is clearly ok.

$n = 2$: We need to show that we get the combinatoric structure of of Eq. (8.69):

$$G(x_1, x_2) = \left(\frac{1}{i} \frac{\delta}{\delta J(x_1)} \frac{1}{i} \frac{\delta}{\delta J(x_2)} (iW[J]) + \frac{1}{i} \frac{\delta(iW)}{\delta J(x_1)} \frac{1}{i} \frac{\delta(iW)}{\delta J(x_2)} \right) e^{iW[J]} \Big|_{J=0} \tag{8.81}$$

$$= \left(\underset{\substack{\uparrow \\ G^C(x_1,x_2)}}{\frac{1}{i} \frac{\delta}{\delta J(x_1)} \frac{1}{i} \frac{\delta}{\delta J(x_2)}} (iW[J]) \Big|_{J=0} \right) + G^C(x_1) G^C(x_2) \quad \checkmark . \tag{8.82}$$

$n = 3$: It is evident that we get the structure in Eq. (8.71)-Eq. (8.73) by taking the $\frac{1}{i} \frac{\delta}{\delta J(x_3)}$ derivative of the first line of the case $n = 2$ expression, and so on.

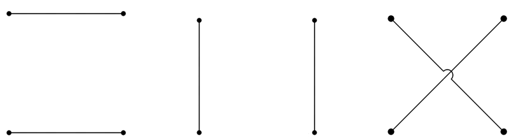
For the **free** scalar field we have

$$Z_0[J] = e^{-\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \; J(x) \Delta_F(x-y) J(y)} \tag{8.83}$$

and so:

$$iW_0[J] = \ln(Z_0[J]) = -\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \; J(x) \Delta_F(x-y) J(y) \, . \tag{8.84}$$

This is interesting! Only the 2-point function is non vanishing, while all the other connected n-point functions vanish in the free theory. This is clear if we look at the diagrams in the free theory, e.g. for the 4-point function:



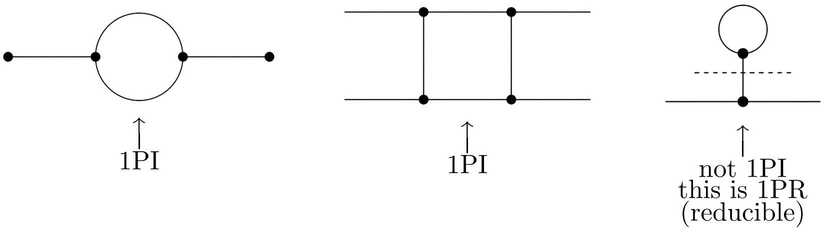
. (8.85)

These are all disconnected diagrams (there are no internal interaction vertices). Connected Green functions are important for **scattering**.

Another important subclass of diagrams are called **one-particle-irreducible** (1PI).
Definition:

> A 1PI diagram does not fall apart into disconnected diagrams by cutting an **internal** line.

Examples:



1PI diagrams are important for renormalization and we will discuss the 1PI generating functional in QFT2 in the context of spontaneous symmetry breaking.

8.8 *The path integral master formula*

We derived classical conservation laws for continuous transformations that leave the action invariant. Consider an infinitesimal transformation:

$$\phi_n(x) \longrightarrow \phi'_n(x) = \phi_n(x) + \epsilon F_n[\phi_n, x]\,, \tag{8.86}$$

where this transformation is not yet restricted to being a symmetry. This means for the generating functional:

$$Z[J_n] = |N|^2 \int \mathcal{D}[\varphi_n] \exp\left(iS[\varphi_n] + i\int \mathrm{d}^4x\ J_n(x)\varphi_n(x)\right) \tag{8.87}$$

$$\overset{\substack{\text{relabel integration}\\ \text{variable } \varphi\to\varphi'\\ \downarrow}}{=} |N|^2 \int \mathcal{D}[\varphi'_n] \exp\left(iS[\varphi'_n] + i\int \mathrm{d}^4x\ J_n(x)\varphi'_n(x)\right) \tag{8.88}$$

$$= |N|^2 \int \mathcal{D}[\varphi_n] \left|\det\left[\frac{\delta\varphi'_n(y)}{\delta\varphi_n(x)}\right]\right| \exp\Bigg\{ iS[\varphi_n] + \tag{8.89}$$

$$+\, i\int \mathrm{d}^4x\ J_n(x)\varphi_n(x) + \int \mathrm{d}^4x \left(i\frac{\delta S[\varphi'_n]}{\delta\varphi_n(x)}\delta\varphi_n(x) + \right. \tag{8.90}$$

$$\left. +\, iJ_n(x)\delta\varphi_n(x)\right) + \underset{\substack{\uparrow\\ \mathcal{O}(\epsilon^2)}}{\dots}\Bigg\}\,. \tag{8.91}$$

We now rewrite the Jacobian using the identity $\ln(\det(A)) = \mathrm{tr}(\ln(A))$:

$$\left|\det\left[\frac{\delta\varphi'_n(y)}{\delta\varphi_n(x)}\right]\right| \underset{\substack{\uparrow\\ \text{must be positive}\\ \text{since } \det(\dots)=1\\ \text{for } \epsilon\to 0}}{=} \exp\left[\mathrm{tr}\left(\ln\left(1 + \epsilon\frac{\delta F_n[\varphi_n(y), y]}{\delta\varphi_n(x)}\right)\right)\right] \tag{8.92}$$

$$\underset{\substack{\uparrow\\ \ln(1+x)\approx x\\ e^x\approx 1+x\\ \text{for } x\ll 1}}{=} 1 + \epsilon\cdot\mathrm{tr}\left(\frac{\delta F_{n'}[\varphi_n(y), y]}{\delta\varphi_n(x)}\right) \tag{8.93}$$

$$= 1 + \epsilon\int \mathrm{d}^4x \sum_n \left(\frac{\delta F_n[\varphi_n(x), x]}{\delta\varphi_n(x)}\right) + \epsilon^2\cdot\dots\,, \tag{8.94}$$

where the trace in Eq. (8.93) is taken over n, n' and x, y, which translates to the sum and integral in the next line, setting $x = y$ and $n = n'$.

We find for $Z[J_n]$:

$$Z[J_n] = \int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4x\ J_n(x)\varphi_n(x)} \cdot \left[1 + \int \mathrm{d}^4x \left\{ \sum_n \frac{\delta F_n[\varphi_n(x),x]}{\delta\varphi_n(x)} + i\left(\frac{\delta S[\varphi_n]}{\delta\varphi_n(x)} J_n(x)\right) F_n[\varphi_n(x),x] \right\}\right]$$
$$\stackrel{!}{=} |N|^2 \int \mathcal{D}[\varphi_n] \exp\left(iS[\varphi_n] + i\int \mathrm{d}^4x\ J_n(x)\varphi_n(x)\right),$$

where we require the last equality to hold since we just relabelled the integration variables. Subtracting the two equations from each other leads to

$$0 = \int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4x\ J_n(x)\varphi_n(x)} \cdot \\ \cdot \int \mathrm{d}^4x \left\{ \sum_n \frac{\delta F_n[\varphi_n(x),x]}{\delta\varphi_n(x)} + i\left(\frac{\delta S[\varphi_n]}{\delta\varphi_n(x)} + J_n(x)\right) F_n[\varphi_n(x),x] \right\}. \tag{8.95}$$

We can now use this to generate identities between Green functions by taking derivatives with respect to J.
We found the **master formula**:

$$0 = \int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4x J_n\varphi_n} \cdot \int \mathrm{d}^4x \Bigg\{ \\ \sum_n \frac{\delta F_n[\varphi_n(x),x]}{\delta\varphi_n(x)} + i\left(\frac{\delta S[\varphi_n]}{\delta\varphi_n(x)} + J_n(x)\right) F_n[\varphi_n(x),x] \Bigg\}. \tag{8.96}$$

for an infinitesimal change in the fields

$$\phi_n(x) \longrightarrow \phi'_n(x) = \phi_n(x) + \epsilon F_n[\phi'_n(x),x]. \tag{8.97}$$

8.8.1 *Equation of motion for Green functions*

Recall the classical EOM:

$$\frac{\delta S[\phi_n]}{\delta\phi_n(x)} = \frac{\partial\mathcal{L}}{\partial\phi_n(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n(x))} = 0. \tag{8.98}$$

Does this mean that for operators $\phi(x) \longrightarrow \hat{\phi}(x)$ a similar equation holds? $\frac{\delta S}{\delta\hat{\phi}} = 0$ as an operator equation. ✓

Example:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right) - \frac{\lambda}{4!}\phi^4 \tag{8.99}$$

$$\frac{\delta S}{\delta\phi} = 0 \iff (\Box + m^2)\phi + \frac{\lambda}{3!}\phi^3 = 0 \tag{8.100}$$

We will show that a similar equation holds for time ordered products of multiple operators and the equations of motion:

$$\langle\Omega|T\left\{\frac{\delta S}{\delta\phi_n}\phi_{n_1}(x_1)\phi_{n_2}(x_2)\cdot\ldots\cdot\phi_{n_N}(x_N)\right\}|\Omega\rangle = ? \tag{8.101}$$

Proof using the master formula:
We use the master formula Eq. (8.96) and $\phi'_n(x) = \phi_n(x) + \epsilon_n(x)$:

- $F_n[\phi'_n(x), x] = \epsilon_n(x)$.
- The term coming from the path integral measure $\sum_n \frac{\delta F_n}{\delta\varphi_n}$ vanishes because F_n is independent of ϕ_n.
- This has to hold for arbitrary $\epsilon_n(x) \Longrightarrow$ the **integrand** vanishes: to see this, just pick $\epsilon_n(x) = \delta^{(4)}(x-y) \longrightarrow \{...\} = 0$.

Our master equation becomes the **Schwinger-Dyson-equation**:

$$\boxed{0 = \int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4y\ J_n(y)\varphi_n(y)} \left(\frac{\delta S[\varphi_n]}{\delta\varphi_n(x)} + J_n(x)\right).} \tag{8.102}$$

Eq. (8.102) also be derived from

$$\int \mathcal{D}[\varphi] \frac{\delta}{\delta\varphi(x)}(...) = 0$$

the total functional derivative, with $(...) = \exp\left(iS[\varphi] + i\int \mathrm{d}^4y\ J(y)\varphi(y)\right)$. Then Eq. (8.102) follows.

Comment: we can write Eq. (8.102) as

$$\boxed{\left(S'\left[\frac{1}{i}\frac{\delta}{\delta J_n(x)}\right] + J_n(x)\right) Z[J] = 0\,,} \tag{8.103}$$

where $S'[\phi_n] = \frac{\delta S}{\delta\phi_n}$. This is an alternative form of the Schwinger-Dyson equation.

We see that by setting $J_n \equiv 0$ in Eq. (8.102) that

$$\langle\Omega|T\frac{\delta S}{\delta\phi_n(x)}|\Omega\rangle = 0 \tag{8.104}$$

i.e. our quantum fields satisfy the equations of motion.
We can use Eq. (8.102) to generate the expression

$$\langle\Omega|T\left\{\frac{\delta S[\varphi_n]}{\delta\phi_n(x)}\phi_1\cdot ...\cdot\phi_N\right\}|\Omega\rangle \tag{8.105}$$

by taking $\frac{1}{i}\frac{\delta}{\delta J_{n_1}(x_1)}\cdot ...\cdot\frac{1}{i}\frac{\delta}{\delta J_{n_N}(x_N)}$ derivatives and then setting $J(x_i) = 0$.

$$\langle\Omega|T\left\{\frac{\delta S[\varphi_n]}{\delta\phi_n(x)}\phi_1\cdot ...\cdot\phi_N\right\}|\Omega\rangle =$$
$$\sum_{i=1}^{N} i\delta^{(4)}(x-x_i)\delta_{nn_i}\langle\Omega|T\{\phi_{n_1}(x_1)\cdot ...\cdot \underset{\substack{\uparrow\\ \text{omit this field}}}{\cancel{\phi_{n_i}(x_i)}}\cdot ...\cdot\phi_{n_N}(x_N)\}|\Omega\rangle \tag{8.106}$$

The RHS comes from $\frac{1}{i}\frac{\delta}{\delta J(x)}$ applied to the second term in Eq. (8.102), since

$$\frac{1}{i}\frac{\delta J_n(x)}{\delta J_{n_i}(x_i)} = -i\delta_{n,n_i}\delta^{(4)}(x-x_i)\,.$$

This allows us to establish **exact** relations between Green functions.

Example: Again in ϕ^4-theory:

$$\langle\Omega|T\{[-(\Box+m^2)\phi(x) - \frac{\lambda}{3!}\phi(x)^3]\phi(y)\}|\Omega\rangle = i\delta^{(4)}(x-y)\,,$$
$$(\Box+m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle + \frac{\lambda}{3!}\langle\Omega|T\{\phi(x)^3\phi(y)\}|\Omega\rangle = -i\delta^{(4)}(x-y)\,.$$

We are allowed to pull out the derivative because this is a T^* product, since we are working within the path integral formalism.

This holds to **all orders** in perturbation theory! We find a relationship between the full two-point function and $\langle\Omega|T\{\phi(x)^3\phi(y)\}|\Omega\rangle$ (see ex.). For $\mathcal{L}=\mathcal{L}_0$ or $\lambda=0$, this contains

$$(\Box+m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle=-i\delta^{(4)}(x-y)\,, \tag{8.107}$$

which is nothing but the Green function equation for the propagator.

8.9 Internal symmetries and Ward-identities

Symmetries are particularly interesting transformations and lead to strong constraints on Green functions using our master formula Eq. (8.96). They correspond to the quantum field theoretic realization of Noether's theorem.

We will now consider internal symmetries like the $U(1)$-rotation of the complex scalar field $\phi\to e^{i\alpha}\phi$, or $\phi'=\phi+i\alpha\phi$. We are therefore interested in the following:

$$\phi_n'(x)=\phi_n(x)+\epsilon\theta^a \underset{\substack{\text{only implicit } x\text{-dependence}\\ \text{through } \phi(x)}}{\underset{\uparrow}{F_n^a[\phi_{n'}]}}\,. \tag{8.108}$$

More specifically: we are considering internal symmetries of the form

$$\phi_n'(x)=D_{nn'}(\theta^a)\phi_{n'}(x)\,, \tag{8.109}$$

so no translations or Lorentz-transformations which change the space-time point, e.g. only rotations in field space.

The parameters θ^a, $a=1,...,\dim(G)$ parametrize the transformations of the $\dim(G)$-dimensional symmetry group G.

<u>**Example:**</u> 2 complex scalar fields in the fundamental $SU(2)$-representation: $G=SU(2)$,

$$\begin{pmatrix}\phi_1'(x)\\ \phi_2'(x)\end{pmatrix}=\exp\left(i\epsilon\theta^a \overset{\text{generators}}{\overset{\downarrow}{T^a}}\right)\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \tag{8.110}$$

$$=\exp\left(i\epsilon\theta^a \overset{\text{Pauli-matrices}}{\overset{\downarrow}{\frac{\sigma^a}{2}}}\right)\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} \tag{8.111}$$

$$\approx(\mathbb{1}_{2\times2}+i\epsilon\theta^a T^a)_{nn'}\phi_{n'}(x)\,,\quad n'=1,2\,. \tag{8.112}$$

We find as the infinitesimal transformation:

$$\epsilon\theta^a F_n^a[\phi_{n'}]=\epsilon\theta^a iT^a_{nn'}\phi_{n'}(x)\,. \tag{8.113}$$

Assume for now that θ^a is independent of x. This is called a **global** symmetry (vs. x-dependent **local** symmetries).

If the transformation is a symmetry, then:

$$S[\phi_n']=S[\phi_n]\quad\text{(the action is invariant)}\,, \tag{8.114}$$

or

$$0 = \frac{\partial}{\partial\theta^a} S[\phi_n + \epsilon\theta^a F_n^a[\phi_{n'}]]\,, \tag{8.115}$$

$$0 \overset{\text{chain rule}}{\downarrow=} \int \mathrm{d}^4x\, \frac{\delta S[\phi_n']}{\delta\phi_n(x)} \frac{\delta\phi_n(x)}{\delta\theta^a}\,, \tag{8.116}$$

$$0 = \int \mathrm{d}^4x\, \frac{\delta S[\phi_n']}{\delta\phi_n(x)} \epsilon F_n^a[\phi_{n'}]\,. \tag{8.117}$$

Plugging this into our master-formula Eq. (8.96), the $\frac{\delta S}{\delta\varphi_n}F_n$-term vanishes!

Important: We have to distinguish two cases.

1. The path integral measure is **not** invariant, $\left|\frac{\delta\varphi_n'(y)}{\delta\varphi_n(x)}\right| \neq 1$, i.e. $\sum_n \frac{\delta F_n}{\delta\varphi_n} \neq 0$ in the master formula. This is an **anomalous** symmetry. The **classical** action is invariant, but the measure is not and so $Z[J]$ is not invariant either, which implies that the **quantum** correlations functions do **not** respect the symmetry.

   This has deep implications and connects to the topology of gauge groups, non-perturbative phenomena and the chiral spectrum of gauge theories (see QFT 2 for this beautiful subject).

2. The path integral measure **is** invariant. The master formula becomes

   $$\int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4y\; J_n(x)\varphi_n(x)} \int \mathrm{d}^4x\; J_n(x) F_n^a[\varphi_{n'}(x)] = 0\,.$$

   From the above, we derive the Ward-Takahashi-identities by taking $\frac{1}{i}\frac{\delta}{\delta J_{n_1}(x_1)} \cdot \ldots \cdot \frac{1}{i}\frac{\delta}{\delta J_{n_N}(x_N)}$ derivatives (and setting $J(x_i) = 0$):

$$\begin{aligned}\sum_{k=1}^{m} \langle\Omega| T\{\phi_{n_1}(x_1)\cdot\ldots\cdot\phi_{n_{k-1}}(x_{k-1}) F_{n_k}^a[\phi_l(x_k)]\cdot \\ \cdot\phi_{n_{k+1}}(x_{k+1})\cdot\ldots\cdot\phi_{n_m}(x_m)\}|\Omega\rangle = 0\,,\end{aligned} \tag{8.118}$$

Ward-Takahashi-identities

e.g.

$$\frac{1}{i}\frac{\delta}{\delta J_1(x_1)}\frac{1}{i}\frac{\delta}{\delta J_2(x_2)}\Bigg(\int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4y\; J_n(y)\varphi_n(y)}\cdot \tag{8.119}$$

$$\cdot\int \mathrm{d}^4z\; J_n(z) F_n^a[\varphi_m(z)]\Bigg)\Bigg|_{J_i(x)=0} \tag{8.120}$$

$$= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\Bigg[\int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4y\; J_n(y)\varphi_n(y)}\Bigg(\varphi_2(x_2)\cdot \tag{8.121}$$

$$\cdot\int \mathrm{d}^4z\; J_n(z) F_n^a(z) + \frac{1}{i}F_2^a[\varphi_m(x_2)]\Bigg)\Bigg]\Bigg|_{J_i(x)=0} \tag{8.122}$$

$$= \int \mathcal{D}[\varphi_n] e^{iS[\varphi_n]+i\int \mathrm{d}^4y\; J_n(y)\varphi_n(y)}\Bigg(\varphi_1(x_1)\varphi_2(x_2)\cdot \tag{8.123}$$

$$\cdot\int \mathrm{d}^4z\; J_n(z) F_n^a(z) + \frac{1}{i}\varphi_2(x_2) F_1^a[\varphi_m(x_1)] + \tag{8.124}$$

$$+\frac{1}{i}\varphi_1(x_1) F_2^a[\varphi_m(x_2)]\Bigg)\Bigg|_{J_i(x)=0}\,, \tag{8.125}$$

so we get

$$\langle\Omega|T\{\varphi_1(x_1)F_2^a[\varphi_m(x_2)]\}|\Omega\rangle + \langle\Omega|T\{F_1^a[\varphi_m(x_1)]\varphi_2(x_2)\}|\Omega\rangle = 0\,. \tag{8.126}$$

Example: Use $U(1)$-symmetric scalar field:

$$\delta\phi = i\theta\phi\,, \quad \delta\phi^\dagger = -i\theta\phi^\dagger\,, \tag{8.127}$$

$$F_n[\phi] = \begin{cases} i\phi\,, & \text{if } n = \phi \\ -i\phi^\dagger\,, & \text{if } n = \phi^\dagger \end{cases}\,. \tag{8.128}$$

The Ward-Takahashi-identities then read:

1)

$$0 = \langle\Omega|T\{i\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle + \langle\Omega|T\{\phi(x_1)(-i\phi^\dagger(x_2))\}|\Omega\rangle \tag{8.129}$$
$$0 = \langle\Omega|T\{\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle - \langle\Omega|T\{\phi(x_1)\phi^\dagger(x_2)\}|\Omega\rangle \tag{8.130}$$

This one is **trivial**.

2) Taking the derivatives $\frac{\delta}{\delta J_\phi(x_1)}\frac{\delta}{\delta J_\phi(x_2)}\frac{\delta}{\delta J_{\phi^\dagger}(x_3)}$ we get

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle + \langle\Omega|T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle \tag{8.131}$$
$$- \langle\Omega|T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle = 0\,, \tag{8.132}$$
$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi^\dagger(x_3)\}|\Omega\rangle = 0\,. \tag{8.133}$$

We generalize: any n-point function with unequal number of ϕ and $\phi^\dagger$ vanishes. This implies: in scattering processes, the charge carried by the ϕ-particle is conserved, which is consistent with the Noether theorem. Even if the symmetry is anomalous (path integral measure transforms non-trivially) we can determine anomalous Ward-identities by including the $\sum_n \frac{\delta F_n}{\delta\varphi_n}$-term.

8.9.1 Local transformations and Noether current

We derive stronger constraints on Green functions by using insertions of Noether currents in correlation functions.
To derive this we assume now that $\theta^a = \theta^a(x)$ holds (local transformation). In general, the action will no longer be invariant.

Claim:

$$\int \mathrm{d}^4x\, \frac{\delta S}{\delta\phi_n(x)} F_n^a[\phi_n(x)]\theta^a(x) = -\int \mathrm{d}^4x\, \left(\partial^\mu j_\mu^a(x)\right)\theta^a(x)\,, \tag{8.134}$$

with

$$j_\mu^a(x) = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_n(x))} F_n^a[\phi_{n'}(x)] - k_\mu^a \tag{8.135}$$

the Noether current corresponding to the transformation parametrized by θ^a:

$$\phi_n(x) \longrightarrow \phi_n'(x) = \phi_n(x) + \epsilon F_n^a[\phi_{n'}(x), x] \tag{8.136}$$

Proof: We take the LHS of the claim:

$$\int \mathrm{d}^4x \, \frac{\delta S}{\delta\phi_n(x)} F_n^a[\phi_{n'}(x)]\theta^a(x) = \int \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_n} - \partial^\mu \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_n)} \right) \cdot \tag{8.137}$$

$$\cdot F_n^a[\phi_{n'}(x)]\theta^a(x) \tag{8.138}$$

$$= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_n} F_n + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_n)} \partial^\mu F_n - \partial^\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_n)} F_n \right) \right] \theta^a(x) \tag{8.139}$$

$$= \int \mathrm{d}^4x \, \partial^\mu \left(k_\mu^a - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_n)} F_n^a[\phi_{n'}(x)] \right) \theta^a(x) = \text{RHS} \quad \checkmark \, , \tag{8.140}$$

where for the last equality we used that

$$\frac{1}{\epsilon} \frac{\partial\mathcal{L}\big(\phi_n + \epsilon\theta^a F_n^a, \partial_\mu(\phi_n + \epsilon\theta^a F_n^a)\big)}{\partial\theta^a} = \tag{8.141}$$

$$= \frac{1}{\epsilon} \Big(\mathcal{L}(\phi_n', \partial_\mu\phi_n') - \mathcal{L}(\phi_n, \partial_\mu\phi_n) \Big) = \frac{1}{\epsilon} \frac{\delta\mathcal{L}}{\delta\theta^a} \overset{\substack{\text{definition of} \\ k_\mu^a\text{, see Eq. (4.54)} \\ \downarrow}}{=} \partial^\mu k_\mu^a \, . \tag{8.142}$$

If the field satisfies the EOM, $\frac{\delta S}{\delta\phi_n} = 0$, then we directly get $\partial_\mu j_a^\mu = 0$, since $\theta^a(x)$ is an arbitrary function. This is therefore an alternative derivation of Noether's theorem.
We use this result in our master formula Eq. (8.96), assuming that the global symmetry is not anomalous, i.e. $\sum_n \frac{\delta F_n}{\delta\varphi_n} = 0$:

$$0 = \int \mathcal{D}[\varphi_n] e^{iS[\varphi_n] + i\int \mathrm{d}^4y \, J_n(y)\varphi_n(y)} \Big(-\partial^\mu j_\mu^a[\varphi_n(x)] + J_n(x) F_n[\varphi_{n'}(x)] \Big) \, , \tag{8.143}$$

where we have again used that $\theta^a(x)$ is an arbitrary function.
Taking m $\frac{1}{i}\frac{\delta}{\delta J_l(x)}$ derivatives gives the **Ward-identity**:

$$\begin{aligned} &\partial_\mu^{(x)} \langle\Omega| T\{ j_a^\mu[\phi_n(x)] \phi_{n_1}(x_1) \cdot \ldots \cdot \phi_{n_m}(x_m) \} |\Omega\rangle = \\ &= (-i) \sum_{k=1}^m \delta^{(4)}(x - x_k) \langle\Omega| T\{ \phi_{n_1}(x_1) \cdot \ldots \cdot \phi_{n_{k-1}}(x_{k-1}) \cdot \\ &\qquad \cdot F_{n_k}^a[\phi_l(x_k)] \phi_{n_{k+1}}(x_{k+1}) \cdot \ldots \cdot \phi_{n_m}(x_m) \} |\Omega\rangle \, . \end{aligned}$$

QFT Noether theorem

Analogously to the equations of motion $\frac{\delta S}{\delta\phi_n} = 0$, the conservation of the current holds, up to **contact terms** on the RHS for arbitrary insertion into the Green function.
Contact terms: δ-function terms for which x coincides with one of the x_k, dependent on the infinitesimal transformation $F_{n_k}^a[\phi_l(x_k)]$.

Reminder: T in the above expression actually means T^*, otherwise we wouldn't have been allowed to move the derivative outside of the $T\{...\}$ product.

Examples:

1)

$$\partial_\mu \langle\Omega| T\{ j^\mu[\phi_n(x)] \} |\Omega\rangle = 0 \tag{8.144}$$

with j^μ the $U(1)$-symmetry current, as in Eq. (8.127) and Eq. (8.128).

2)

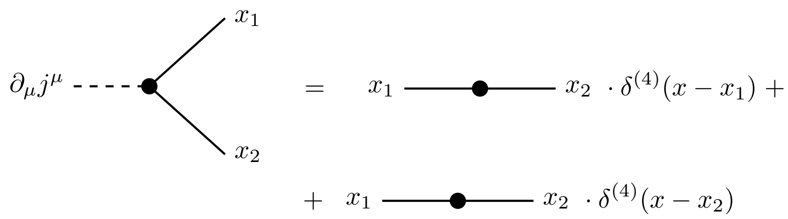
$$\partial_\mu \langle \Omega | T\{ j^\mu(x) \phi(x_1) \phi^\dagger(x_2) \} | \Omega \rangle = (-i) \delta^{(4)}(x - x_2) \cdot \tag{8.145}$$

$$\cdot \langle \Omega | T\{ \phi(x_1) (-i\phi^\dagger(x_2)) \} | \Omega \rangle - i\delta^{(4)}(x - x_1) \langle \Omega | T\{ i\phi(x_1) \phi^\dagger(x_2) \} | \Omega \rangle \, , \tag{8.146}$$

$$\partial_\mu \langle \Omega | T\{ j^\mu(x) \phi(x_1) \phi^\dagger(x_2) \} | \Omega \rangle = \delta^{(4)}(x - x_1) \cdot \tag{8.147}$$

$$\cdot \langle \Omega | T\{ \phi(x_1) \phi^\dagger(x_2) \} | \Omega \rangle - \delta^{(4)}(x - x_2) \langle \Omega | T\{ \phi(x_1) \phi^\dagger(x_2) \} | \Omega \rangle \, , \tag{8.148}$$

again with the $U(1)$ current as in Eq. (8.127) and Eq. (8.128). Diagrammatically, this corresponds to



9 Scattering Theory

9.1 Motivation

We can learn a lot about a microphysical system by performing measurements at macroscopic distances compared to the size of the micro-system and the range of the relevant interactions, and at large time differences compared to the duration of the interaction.

Example:

$$(\Box + m^2)\phi(x) = \delta^{(3)}(\boldsymbol{x}) \quad \text{static source}\,, \tag{9.1}$$

$$\phi(\boldsymbol{x}) = \frac{1}{-\nabla^2 + m^2}\delta^{(3)}(\boldsymbol{x})\,, \tag{9.2}$$

$$\xrightarrow{\text{FT}} \quad \phi(\boldsymbol{x}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\boldsymbol{p}^2 + m^2} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \underset{\substack{\uparrow \\ \text{see [4], chapter 6.7.1}}}{=} \frac{1}{4\pi r} e^{-mr}\,, \quad r \equiv |\boldsymbol{x}|\,. \tag{9.3}$$

The interaction is exponentially small beyond $r \gg \frac{1}{m}$.

We can probe micro-distances by using high-energies/momenta, e.g. at LHC: 13 trillion electronvolts of energy
In this measurement we do not need to know the time-dependence in detail, it is enough to get the relation between

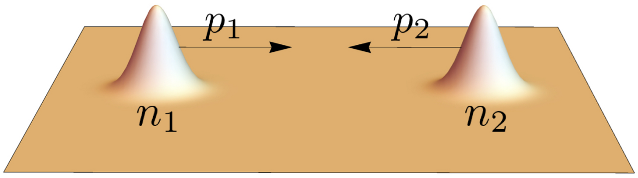
$$\begin{array}{ccccc} \text{initial state} & & \star & & \text{final state} \\ t \to -\infty & \longrightarrow & & \longrightarrow & t \to +\infty \\ \text{(formally)} & & \text{interaction} & & \end{array}$$

Scattering matrix.

Initial and final states should correspond to well-separated particles and therefore behave like in a non-interacting system.
We assume that the interactions have finite range, i.e. "the interaction strength decreases faster than $\frac{1}{r}$ for $r \longrightarrow \infty$". As an example for such an interaction, see the Klein-Gordon example above: $\phi \sim \frac{1}{r}e^{-mr}$.

Preparation of states for scattering:

- **<u>Initial state:</u>** $t \longrightarrow -\infty$, $|\psi_i\rangle$

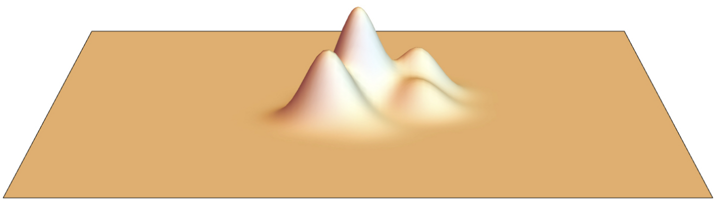


  The states are localized in momentum space around p_i, yet they are so broad that (p_i, n_i) do **not** overlap in space! E.g.

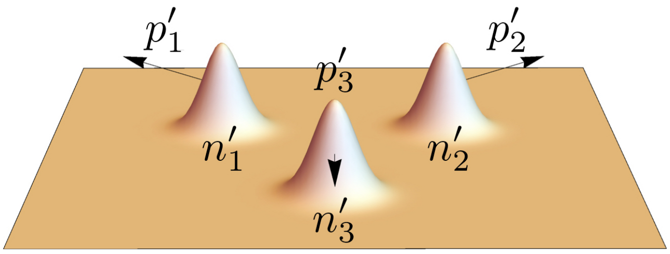
$$|\psi_i\rangle = \int \frac{\mathrm{d}^3 p_1}{(2\pi)^3 2E_{\boldsymbol{p_1}}} \int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E_{\boldsymbol{p_2}}} f(p_1) f(p_2) |p_1, p_2, \mathrm{in}\rangle \,, \tag{9.4}$$

  with $|p_i\rangle$ momentum eigenstates.

- **<u>Interaction:</u>** Wave packets overlap at some intermediate time (when exactly does not matter).



- **<u>Final state:</u>** $t \longrightarrow +\infty$, $|\psi_f\rangle$



  The wave packets are well-separated in space again, the momenta are detected together with the other properties of the particles.

Repeating the experiment many times (LHC: 600 000 000 $\frac{\text{collisions}}{\text{second}}$!) we can determine

$$|\langle\psi_f|\psi_i\rangle|^2 \,. \tag{9.5}$$

We **want** to compute this probability.

To properly define the asymptotic states in the interacting theory, we need to define **particle states of the interacting theory**.

We can derive P^μ from the full Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ (as a conserved spatial integral $\int \mathrm{d}^3x\ T^{0\mu}$). We require $H = P^0$ to be bounded from below (to avoid spontaneous decay into states with ever lower energies). We set without loss of generality the energy of the lowest energy state as $E_0 = 0$.
The vacuum is Poincaré invariant:

$$U(\Lambda, a)|\Omega\rangle = |\Omega\rangle \tag{9.6}$$

which implies

$$P^\mu|\Omega\rangle = 0 \tag{9.7}$$

since

$$U(0,a)|\Omega\rangle = e^{iP^\mu a_\mu}|\Omega\rangle \overset{!}{=} |\Omega\rangle \tag{9.8}$$

since P^μ generates spacetime-translations.

9.2 One-particle states

We now assume that the eigenvalues of $P_\mu P^\mu = P^2 \geq 0$. One -particle states are part of the **discrete** spectrum of P^2 (with eigenvalues M^2).
We write the basis states of P^2 as

Note: Trivial in field theory:

$$p^2 = m^2$$

with m the mass in the free Lagrangian $\mathcal{L}_0$.

$$|p, \underset{\substack{\uparrow \\ \text{other properties}}}{n}\rangle\,, \tag{9.9}$$

which are labelled by $\boldsymbol{p}$. The corresponding eigenvalue of $H = P^0$ is then $p^0 = \sqrt{M^2 + \boldsymbol{p}^2}$.
We can use $\boldsymbol{p}$ as a label since it commutes with $\boldsymbol{P}^2$ ($\boldsymbol{P}^2$ is the Casimir operator, more about this later).
Important:

> One-particle state of the free theory **do not coincide** with the one-particle states of the interacting theory.

Example:

$$\mathcal{L} = \frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\Big) + \mathcal{L}_{\text{int}}(\phi)\,, \tag{9.10}$$

$$\mathcal{L}_{\text{int}} \equiv 0 \quad \longrightarrow \quad P^2 \text{ has one discrete eigenvalue: } p^2 = m^2\,, \tag{9.11}$$

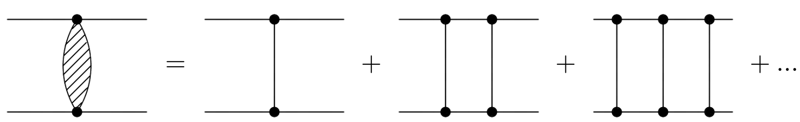
$$\mathcal{L}_{\text{int}} \neq 0 \quad \longrightarrow \quad \text{Shifted eigenvale: } M^2 = M^2(m^2, \lambda) \tag{9.12}$$

$$\approx m^2 + \mathcal{O}(\lambda)\,. \tag{9.13}$$

Or: we get new eigenvalues in case of bound states:

$$M^2 \approx (2m)^2 + \mathcal{O}(\lambda)\,, \tag{9.14}$$

which lead to further **one-particle states**. A bound state is represented by the diagrams



What does the spectrum look like? Consider $(H, |\boldsymbol{p}|)$-hyperboloids:

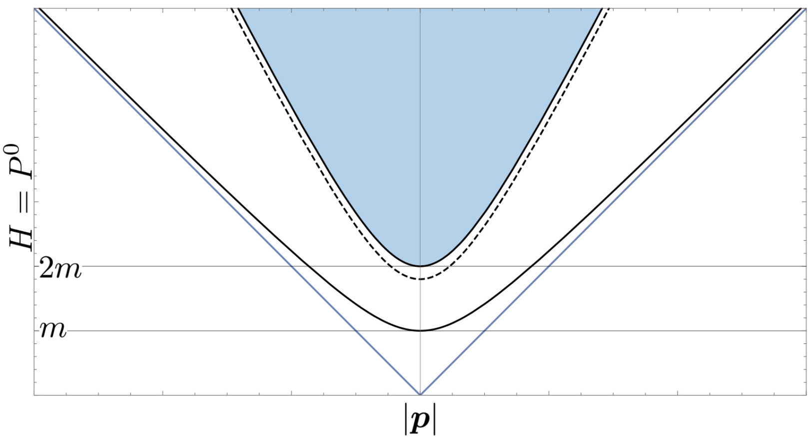


Figure 9.1: Energy spectrum of one- and multiparticle states. The solid black lines represent one- and two-particle states, while the dashed black line represents a bound state of two particles. The shaded blue region stands for the continuum of multiparticle states, while the vacuum is located at the origin of the plot.

For a typical theory the states consist of one ore more particles of mass m. There is a hyperboloid of one-particle states and a continuum of hyperboloids of two-particles, three-particle states and so on.
There may also be one or more bound-state hyperboloids below the threshold for the creation of two free particles.

9.3 Many particles and scattering states

From the single particle states we can create the Fock space of many-particle states (tensor product - as for free particles):

$$|\phi_\alpha\rangle = |p_1, n_1; p_2, n_2; ...\rangle\,. \tag{9.15}$$

Define new $\widetilde{H}_0$, $\widetilde{H}_{\text{int}}$ as a different partition of

$$H = \widetilde{H}_0 + \widetilde{H}_{\text{int}}\,, \tag{9.16}$$

where $\widetilde{H}_0$ contains free-field terms of single-particle states of the full Hamiltonian H (including e.g. bound states, if they exist). The $|\phi_\alpha\rangle$ are eigenstates of $\widetilde{H}_0$ with the eigenvalues

$$\sum_i p_i^0 = \sum_i \sqrt{M_i^2 + \boldsymbol{p_i}^2}\,. \tag{9.17}$$

The spectrum of the new "free" theory contained in $\widetilde{H}_0$ contains single-particle states (including bound states) and a **continuum** of multiparticle states. Why is there a continuum in the first place? Consider as an example two particles $(M, \boldsymbol{p_1})$, $(M, \boldsymbol{p_2})$:

$$P^2|\phi_\alpha\rangle = \left[\left(\sqrt{M^2+\boldsymbol{p_1}^2} + \sqrt{M^2+\boldsymbol{p_2}^2}\right)^2 - (\boldsymbol{p_1}+\boldsymbol{p_2})^2\right]|\phi_\alpha\rangle\,, \tag{9.18}$$

Note: For one-particle states P^2 is

$$P^2|\phi_1\rangle = (\sqrt{m^2+\boldsymbol{p_1}^2}^2 - \boldsymbol{p_1}^2)|\phi_1\rangle = m^2|\phi_1\rangle\,,$$

as expected.

so the mass is a continuous function of $\boldsymbol{p_1}^2$ and $\boldsymbol{p_2}^2$ and its value is therefore part of a continuum.

Since by assumption $\widetilde{H}_{\text{int}}$ has finite range, we expect that the full H has eigenstates $|\psi_\alpha\rangle$ with the same eigenvalue $E_\alpha = \sum_\alpha p_\alpha^0$ as $|\phi_\alpha\rangle$

for $\widetilde{H}_0$ since we can construct spatially separated wave-packets of localized particles.
We conclude:

$$H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle\,, \tag{9.19}$$

with $|\psi_\alpha\rangle$ a state of non-interacting, spatially separated particles with $\alpha = p_1, n_1; p_2, n_2; \ldots$.
Define in- and out-states in the Heisenberg picture:

Recall: States in the Heisenberg picture do not depend on time.

$$\underset{t\to-\infty}{\underset{\uparrow}{|\psi_\alpha^{\text{in}}\rangle}} \quad \text{and} \quad \underset{t\to+\infty}{\underset{\uparrow}{|\psi_\alpha^{\text{out}}\rangle}} \tag{9.20}$$

and $|\psi_\alpha^{\text{in/out}}\rangle$ are an ensemble of well-separated particles.
Definition of the S(cattering)-matrix:

$$S_{\beta\alpha} = \langle\psi_\beta^{\text{out}}|\psi_\alpha^{\text{in}}\rangle \tag{9.21}$$

which is the "transition amplitude for a system that can be described as a set of free particles α at $t \longrightarrow -\infty$ to a system of free particles β at $t \longrightarrow +\infty$".
Note that the S-matrix is usually defined using **momentum eigenstates** (which are maximally localized). Instead of this idealization, we can use **wave-packets** with mean momenta as α and β to make sure that states are also localized in space. Almost all of the time this can be ignored in practice, and it only makes our calculations more cumbersome.

*9.4 Lippmann-Schwinger equation**

*Not discussed in the lecture

Let us first present a simplified ansatz and then be a bit more rigorous.
When conducting scattering experiments, we are interested in solving for what happens at late times in terms of an initial state at early times:
Say we start off at early times with a state

$$\widetilde{H}_0|\phi_S\rangle = E|\phi_S\rangle\,. \tag{9.22}$$

If energies are continuous, there will be an eigenstate of the full theory with the **same** eigenvalue:

$$(\widetilde{H}_0 + \widetilde{H}_{\text{int}})|\psi_S\rangle = E|\psi_S\rangle\,. \tag{9.23}$$

We can formally write:

$$\boxed{|\psi_S\rangle = |\phi_S\rangle + \frac{1}{E - \widetilde{H}_0}\widetilde{H}_{\text{int}}|\psi_S\rangle\,,} \tag{9.24}$$

Lippmann-Schwinger equation

which we can verify by multiplying both sides by $(E - \widetilde{H}_0)$:

$$(E - \widetilde{H}_0)|\psi_S\rangle = \overset{=0}{\overset{\downarrow}{(E - \widetilde{H}_0)|\phi_S\rangle}} + \widetilde{H}_{\text{int}}|\psi_S\rangle \tag{9.25}$$

$$E|\psi_S\rangle = (\widetilde{H}_0 + \widetilde{H}_{\text{int}})|\psi_S\rangle \quad \checkmark \tag{9.26}$$

We need to be a bit more careful here, since the inverse of $(E - \widetilde{H}_0)$ is not well-defined. E is an eigenvalue of $\widetilde{H}_0$ and so $\det(E - \widetilde{H}_0) = 0$ and $\frac{1}{E-\widetilde{H}_0}$ is singular.
Let's be more careful: in the Schrödinger picture we have

$$\left(i\frac{\partial}{\partial t} - \widetilde{H}_0^S\right)|\psi_S(t)\rangle = \widetilde{H}_{\text{int}}^S|\psi_S(t)\rangle\,. \tag{9.27}$$

For $t \longrightarrow -\infty$ $|\psi_S(t)\rangle$ should be solving the free Schrödinger equation:

$$|\psi_S(t)\rangle \xrightarrow{t\to-\infty} |\phi_S(t)\rangle \quad \text{with} \quad \left(i\frac{\partial}{\partial t} - \widetilde{H}_0^S\right)|\phi_S(t)\rangle = 0\,. \tag{9.28}$$

We now take the Green function of the Schrödinger equation:

$$\left(i\frac{\partial}{\partial t} - \widetilde{H}_0^S\right)D_\pm(t,t') = \delta(t-t') \tag{9.29}$$

with

$$D_\pm(t,t') = 0\begin{cases} t' > t, & \text{for } D_+ \\ t' < t, & \text{for } D_- \end{cases} \tag{9.30}$$

e.g.

$$i\frac{\partial}{\partial t}D_-(t,t')$$
$$= i\left(-\frac{1}{i}\right)\left((-i\widetilde{H}_0^S) - \delta(t-t')\right)e^{-i\widetilde{H}_0^S(t-t')}$$
$$= \widetilde{H}_0^S D_- + \delta(t-t') \quad \checkmark$$

where we used that for $t = t'$ $e^{-i\widetilde{H}_0^S(0)} = 1$.

and

$$D_\pm(t,t') = \pm\frac{1}{i}e^{-i\widetilde{H}_0^S(t-t')}\cdot\begin{cases}\theta(t-t'), & \text{retarded } (+) \\ \theta(t'-t), & \text{advanced } (-)\end{cases}. \tag{9.31}$$

Therefore:

$$\boxed{|\psi_S(t)\rangle = |\phi_S(t)\rangle + \int_{-\infty}^{\infty}\mathrm{d}t'\; D_+(t,t')\widetilde{H}_{\text{int}}^S|\psi_S(t')\rangle\,.} \tag{9.32}$$

We can deduce that $|\psi_S(t)\rangle$ solves Eq. (9.27).
As before, we consider wave-packets around $\alpha = p_1, n_1; p_2, n_2$ for $t \longrightarrow -\infty$. We prefer to work with momentum eigenstates (which are spatially delocalized, e.g. in QM $\langle \boldsymbol{x}|\boldsymbol{p}\rangle \sim e^{i\boldsymbol{p}\cdot\boldsymbol{x}}$). We compensate for the non-localization by switching off the interaction. We replace

$$\widetilde{H}_{\text{int}}^S \longrightarrow \widetilde{H}_{\text{int}}^S e^{\epsilon t}\,, \quad \epsilon > 0 \text{ infinitesimal.} \tag{9.33}$$

with

$$\underset{\substack{\uparrow \\ \text{Heisenberg}}}{|\phi_S(t)\rangle = |\phi_\alpha\rangle e^{-iE_\alpha t}}\,, \quad \underset{\substack{\uparrow \\ \text{Heisenberg,} \\ \text{equal at } \boldsymbol{t=0}}}{|\psi_S(t)\rangle = |\psi_\alpha\rangle e^{-iE_\alpha t}}\,. \tag{9.34}$$

Evaluating Eq. (9.32) at $t = 0$ gives:

$$|\psi_\alpha^{\text{in}}\rangle = |\phi_\alpha\rangle + \frac{1}{i}\int_{-\infty}^{\overset{\theta(t-t')}{\downarrow}\; 0}\mathrm{d}t\;\underbrace{e^{+i\widetilde{H}_0^S t'}\,\widetilde{H}_{\text{int}}^S e^{+\epsilon t'}\,e^{-iE_\alpha t'}}_{\exp\left(-it'(E_\alpha - \widetilde{H}_0^S + i\epsilon)\right)\widetilde{H}_{\text{int}}^S}|\psi_\alpha^{in}\rangle\,, \tag{9.35}$$

$$\boxed{|\psi_\alpha^{\text{in}}\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - \widetilde{H}_0 + i\epsilon}\widetilde{H}_{\text{int}}|\psi_\alpha^{in}\rangle\,.} \tag{9.36}$$

Central equation of scattering theory in quantum theory (including QFT).

which is the Lippmann-Schwinger equation. At $t = 0$ we have $\widetilde{H}_0 = \widetilde{H}_0^S$ and $\widetilde{H}_{\rm int} = \widetilde{H}_{\rm int}^S$. An analogous consideration can be done for $|\psi_\alpha^{out}\rangle$ with $t \to \infty$ by replacing $+i\epsilon \longrightarrow -i\epsilon$.

The Lippmann-Schwinger equation implies

$$|\psi_\alpha^{\rm in}\rangle = |\psi_\alpha^{\rm out}\rangle = |\phi_\alpha\rangle \quad \text{for} \quad \widetilde{H}_{\rm int} = 0 \tag{9.37}$$

$$H|\psi_\alpha^{\rm in}\rangle = E_\alpha|\psi_\alpha^{\rm in}\rangle \tag{9.38}$$

(see discussion at the beginning and operate with $(H_0 - E)$ on the Lippmann-Schwinger equation).
Note that $|\phi_\alpha\rangle$ are plane wave states in position space, but the $|\psi_\alpha^{\rm in/out}\rangle$ are not, because plane waves are not eigenstates of the full $H = \widetilde{H}_0 + \widetilde{H}_{\rm int}$. However, the $|\psi_\alpha^{\rm in/out}\rangle$ are stationary scattering states of H.

To understand the formalism better, consider a quantum-mechanical system with a (non relativistic) single-particle Hamiltonian

$$H = \underset{\underset{\widetilde{H}_0}{\uparrow}}{\frac{\boldsymbol{P}^2}{2m}} + \underset{\underset{\widetilde{H}_{\rm int}}{\uparrow}}{V(\boldsymbol{X})}\,, \quad |\phi_\alpha\rangle \longleftrightarrow e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\,, \tag{9.39}$$

$|\phi_\alpha\rangle$ is an incoming plane wave with $E_\alpha = \frac{\boldsymbol{p}^2}{2m}$.
The full Hamiltonian has two degenerate (i.e. with the same energy) scattering eigenstates to the same E_α:

$$|\psi_\alpha^{\rm in/out}\rangle \longleftrightarrow \underset{\underset{\text{plane wave}}{\uparrow}}{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}} + \frac{\overset{\overset{\text{in- or outgoing spherical wave}}{\downarrow}}{e^{\pm i|\boldsymbol{p}||\boldsymbol{x}|}}}{|\boldsymbol{x}|} f_\pm(|\boldsymbol{p}|, \underset{\underset{\text{scattering angle}}{\uparrow}}{\cos\theta})\,. \tag{9.40}$$

For more details, see e.g. [1], chapter 7. One could use the Lippmann-Schwinger equation to find an iterative solution to the scattering problem:

$$|\psi_\alpha^{\rm in}\rangle = |\phi_\alpha\rangle + \frac{1}{E_\alpha - \widetilde{H}_0 + i\epsilon}\widetilde{H}_{\rm int}|\psi_\alpha^{\rm in}\rangle\,. \tag{9.41}$$

$|\psi_\alpha^{\rm in}\rangle$ appears on both sides of the equation and we can iterate:

$$|\psi_\alpha^{\rm in}\rangle = |\phi_\alpha\rangle + \underset{\underset{(*)}{\uparrow}}{\frac{1}{E_\alpha - \widetilde{H}_0 + i\epsilon}}\widetilde{H}_{\rm int}|\phi_\alpha\rangle + \tag{9.42}$$

$$+ \frac{1}{E_\alpha - \widetilde{H}_0 + i\epsilon}\widetilde{H}_{\rm int}\frac{1}{E_\alpha - \widetilde{H}_0 + i\epsilon}\widetilde{H}_{\rm int}|\phi_\alpha\rangle + \ldots \tag{9.43}$$

The truncated sum after $(*)$ is known as the first-order Born-approximation, see e.g. [1], chapter 7 or [5], chapter 7 and 8.
The Lippmann-Schwinger equation implies (see ex.)

$$e^{-iHt}\int \mathrm{d}x\, f(x)|\psi_\alpha^{in/out}\rangle \quad \xrightarrow{t\to\mp\infty} \quad e^{-i\widetilde{H}_0 t}\int \mathrm{d}x\, f(x)|\phi_\alpha\rangle\,. \tag{9.44}$$

An intuitive explanation for this relation is that for $t \longrightarrow \mp\infty$ the Heisenberg state $|\psi_\alpha^{\text{in/out}}\rangle$ looks like a set of free particles with the same momenta.
We can write this as

$$|\psi_\alpha^{\text{in/out}}\rangle = \Omega^{\text{in/out}}|\phi_\alpha\rangle\,, \tag{9.45}$$

with

$$\Omega^{\text{in/out}} = \lim_{t\to\mp\infty} e^{iHt}e^{-i\widetilde{H}_0 t}\,. \tag{9.46}$$

These are called the Møller operators. We define the scattering matrix as

$$S_{\beta\alpha} = \langle\psi_\beta^{\text{out}}|\psi_\alpha^{\text{in}}\rangle \equiv \langle\phi_\beta|\underset{\substack{\uparrow\\ \text{scattering operator}}}{S}|\phi_\alpha\rangle\,, \tag{9.47}$$

so

$$S = \left(\Omega^{\text{out}}\right)^\dagger\Omega^{\text{in}} = \lim_{t_2\to\infty}\lim_{t_1\to-\infty} e^{i\widetilde{H}_0 t_2}e^{-iH(t_2-t_1)}e^{-i\widetilde{H}_0 t_1}\,. \tag{9.48}$$

Comments:

1. S is unitary:

$$S^\dagger S = SS^\dagger = \mathbb{1}\,, \tag{9.49}$$

   since $\Omega^{in/out}$ are.

2. Poincaré-invariance requires that given $|\phi'_\alpha\rangle = U(\Lambda,a)|\phi_\alpha\rangle$ we get

$$\langle\phi'_\alpha|S|\phi'_\beta\rangle \stackrel{!}{=} \langle\phi_\alpha|S|\phi_\beta\rangle\,, \tag{9.50}$$

   and therefore

$$U^\dagger SU \stackrel{!}{=} S \quad \text{or} \quad [S,U]=0\,. \tag{9.51}$$

   That this holds is not obvious here, since H and $\widetilde{H}_0$ do not commute with Lorentz-transformations (they are the 0-components of the four-vector P^μ). One can show that S is Lorentz-invariant if derived from a Lorentz-invariant Lagrangian.

3. The equality

$$S|\psi_\alpha^{\text{out}}\rangle = |\psi_\alpha^{\text{in}}\rangle \tag{9.52}$$

   holds.

   Proof:

$$\begin{aligned}
\langle\psi_\beta^{\text{out}}|S|\psi_\alpha^{\text{out}}\rangle &= \langle\phi_\beta|\left(\Omega^{\text{out}}\right)^\dagger S\Omega^{\text{out}}|\phi_\alpha\rangle & (9.53)\\
&= \lim_{t\to+\infty}|\phi_\beta\rangle e^{i\widetilde{H}_0 t}(e^{-iHt}\underset{\substack{\uparrow\\ =S\text{, since } S \text{ is time}\\ \text{translation invariant}\\ \left(\text{this is } U(0,a^0=t,a^i=0)\right)\,.}}{S}e^{iHt})e^{-i\widetilde{H}_0 t}|\phi_\alpha\rangle & (9.54)\\
&= \lim_{t\to+\infty}\underset{\substack{\uparrow\\ =1 \text{ since } E_\beta=E_\alpha\\ \text{(energy conservation)}}}{e^{it(E_\beta-E_\alpha)}}\langle\phi_\beta|S|\phi_\alpha\rangle & (9.55)\\
&= \langle\psi_\beta|S|\psi_\alpha\rangle = \langle\psi_\alpha^{\text{out}}|\psi_\alpha^{\text{in}}\rangle\,. & (9.56)
\end{aligned}$$

Remark: We assume that all $|\phi_\alpha\rangle$ (or all $|\psi_\alpha^{\text{in}}\rangle$ or all $|\psi_\alpha^{\text{out}}\rangle$) form a **basis** of Hilbert space ("asymptotic completeness") of the theory. Since the $|\phi_\alpha\rangle$ are orthonormal, it follows that $|\psi_\alpha^{\text{in/out}}\rangle$ are orthonormal since $\Omega^{\text{in/out}}$ are unitary.

The spaces of the $|\psi_\alpha^{\text{in}}\rangle$ and $|\psi_\alpha^{\text{out}}\rangle$ are isomorphic Fock spaces!

$$|\Omega\rangle_{in} = |\Omega\rangle_{out} = |\Omega\rangle$$

because of Poincaré-invariance!

From now on we will use the shorthand

$$\langle\phi_\beta|\phi_\alpha\rangle \equiv \delta(\beta-\alpha) \underset{\text{meaning}}{\underset{\uparrow}{=}} (2\pi)^3 2E_{\boldsymbol{p}_\alpha}\delta^{(3)}(\boldsymbol{p}_\beta - \boldsymbol{p}_\alpha) \tag{9.57}$$

if written out with the proper normalization.

9.5 Asymptotic fields

The Fock space of the $|\psi_\alpha^{\text{in/out}}\rangle$-states contains approximately non-interacting multiparticle states. The corresponding creation and annihilation operators are

$$a^{\dagger\,\text{in}}_{\boldsymbol{p},n}\,, \quad a^{\text{in}}_{\boldsymbol{p},n} \quad \text{and} \quad a^{\dagger\,\text{out}}_{\boldsymbol{p},n}\,, \quad a^{\text{out}}_{\boldsymbol{p},n}\,. \tag{9.58}$$

The single-particle states are

$$a^{\dagger\,\text{in/out}}_{\boldsymbol{p},n}|\Omega\rangle\,. \tag{9.59}$$

We can now define the asymptotic in- and out-fields for each particle (including bound states):

$$\phi_n^{\text{in/out}}(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(\underset{p^0=E_{\boldsymbol{p}}}{\underset{\uparrow}{e^{-ipx}}}a^{\text{in/out}}_{\boldsymbol{p},n} + e^{ipx}\underset{\substack{\text{or } b^\dagger \text{ if the}\\ \text{field is complex}}}{\underset{\uparrow}{a^{\dagger\,\text{in/out}}_{\boldsymbol{p},n}}}\right) \tag{9.60}$$

The time evolution of the states is according to $\widetilde{H}_0$.

Note: Asymptotic fields can be different than those in H_0 (in general no relation between the two types of fields exists).

9.5.1 Weakly-coupled QFT: asymptotic fields

The spectrum of P^2 is a small deformation of the spectrum of the free theory (given by H_0), the properties of the two are **continuously connected** to each other (they transform continuously into each other as the interaction strengths goes to zero).
In the free theory:

$$\phi_n^{\text{in}}(x) = \phi_n^{\text{out}}(x) \tag{9.61}$$

and they are the same as the fields appearing in $\mathcal{L}_0$.
Hence: We substitute one $\phi_n^{\text{in}}(x)$ or $\phi_n^{\text{out}}(x)$ for each ϕ_n in $\mathcal{L}$ (with the same properties as the free field: real or complex, its spin etc.).
The fields can differ only by a **constant** factor at early/late times:

$$\underset{\substack{\text{interacting}\\ \text{field of } \mathcal{L}}}{\underset{\uparrow}{\langle\psi_1|\phi_n|\psi_2\rangle}} \xrightarrow{t=x^0\to\mp\infty} \sqrt{Z_n}\,\langle\psi_1|\underset{\substack{\text{free field for}\\ \text{particle with mass}\\ M=m+\mathcal{O}(\lambda)\neq m}}{\underset{\uparrow}{\phi_n^{\text{in/out}}}}|\psi_2\rangle \tag{9.62}$$

and $\phi_n^{\text{in/out}}(x)$ must be the same field type as $\phi_n(x)$ since field characteristics (spin, charge,...) cannot be continuously deformed as $\lambda \longrightarrow 0$.
The asymptotic field obey the Klein-Gordon equation as well:

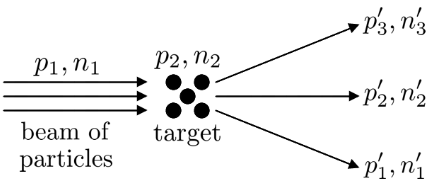
$$(\Box + M^2)\phi_n^{\text{in/out}}(x) = 0\,. \tag{9.63}$$

Note that Eq. (9.62) holds in the weak sense. What does **not** hold is that operator products in the interacting theory (e.g. powers of $\phi_n(x)$) approach corresponding products of $\phi_n^{\text{in/out}}(x)$.

9.6 The scattering cross-section

We will relate S-matrix elements to scattering cross-sections which are directly measurable in collider experiments. We will also derive expressions for decay rates, which are also directly experimentally accessible.

1) **Cross-sections:**



$$\overset{\alpha}{\begin{matrix} p_1, n_1 \\ p_2, n_2 \end{matrix}} \quad \longrightarrow \quad \overset{\beta}{\begin{matrix} p_1', n_1',\ p_2', n_2' \\ p_3',\ n_3',\ldots \end{matrix}}$$

Example: Imagine being Ernest Rutherford and you want to know the size of a gold nucleus. You collide α-particles with a gold foil and measure how many α's are scattered and from this you can determine the cross-section area $\sigma = \pi r^2$ (this is the classical result, the quantum mechanical result is different).
Assume at first that there is just **one** nucleus:

$$\sigma = \frac{\text{number of particles scattered}}{\text{time} \cdot \underbrace{\text{number density in beam} \cdot \text{velocity of particles in the beam}}_{\equiv \Phi \text{ incoming flux}}} \tag{9.64}$$

$$= \frac{1}{T} \cdot \frac{1}{\Phi} \cdot N \quad (\text{unit: meter}^2)\,. \tag{9.65}$$

We can also measure differential quantities, e.g. $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \mathrel{\hat{=}} \text{"number of particles scattered into a certain solid angle } \mathrm{d}\Omega\text{"} \tag{9.66}$$

and we can also define a transition rate from the state α to the state β:

$$\begin{aligned} R_{\beta\alpha} &= (\text{transition rate})_{\beta\alpha} && (9.67)\\ &= (\text{incoming flux}) \cdot (\text{number of target particles}) \cdot \sigma_{\beta\alpha}\,. && (9.68) \end{aligned}$$

More, generally, we define

$$\mathrm{d}\sigma_{\beta\alpha} \equiv \frac{\text{Number of final states } \beta \text{ per time and target particles}}{\text{Number of incoming particles per time and area}}\,. \tag{9.69}$$

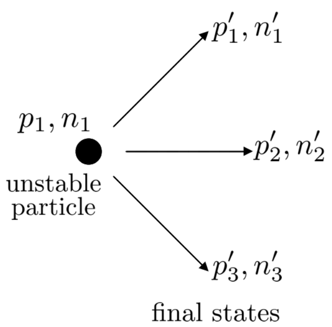
This is a quantum mechanical probability for scattering:

$$P = \frac{(\mathrm{N_{scattered}})_\beta}{(\mathrm{N_{incoming}})_\alpha} \quad \Longrightarrow \quad \mathrm{d}\sigma_{\beta\alpha} = \frac{1}{T}\frac{1}{\Phi_\alpha}\mathrm{d}P_\beta\,. \tag{9.70}$$

The differential number of scattering events (as measured e.g. in a collider) is given by

$$\mathrm{d}N_{\beta\alpha} = \underset{\substack{\uparrow \\ \text{Luminosity} \\ \text{(units: 1/cross-section)}}}{L} \cdot \mathrm{d}\sigma_{\beta\alpha}\,. \tag{9.71}$$

2) **Decay rates:**



$$\mathrm{d}\Gamma_{\beta\alpha} = \frac{1}{T}\mathrm{d}P_{\beta\alpha} \tag{9.72}$$

$$\mathrm{d}\Gamma_{\beta\alpha} = \begin{matrix}\text{"Probability that a one-particle state } \alpha\ (p_1, n_1) \\ \text{turns into a multiparticle state } \beta \text{ over the time } T\text{"}\end{matrix} \tag{9.73}$$

The life-time of a state α is given by

$$\tau_\alpha = \frac{1}{\Gamma_{\text{total}}}\,, \tag{9.74}$$

with the total decay rate

$$\Gamma_{\text{total}} = \int \mathrm{d}\beta\ \mathrm{d}\Gamma_{\beta\alpha}\,. \tag{9.75}$$

It seems impossible to use the S-matrix for this, since an incoming particle cannot be an asymptotic state at $t = -\infty$ if it is to decay. However, this is not a problem, since we assume interactions only happen over a finite time T (formally we switch off the interactions responsible for the decay at $t \longrightarrow \mp\infty$!). The decay rate is therefore calculated as a $1 \longrightarrow m$ scattering process.

We want: $d\sigma_{\beta\alpha}$ and $d\Gamma_{\beta\alpha}$ in terms of $S_{\beta\alpha}$! We first rewrite $S_{\beta\alpha}$:

$$S_{\beta\alpha} = \langle \phi_\beta | S | \phi_\alpha \rangle \equiv \underset{\substack{\uparrow \\ \text{contribution from} \\ \text{no scattering}}}{\delta(\beta - \alpha)} + i(2\pi)^4 \underset{\substack{\uparrow \\ (*)}}{\delta^{(4)}(p_\beta - p_\alpha)} T_{\beta\alpha} \,, \tag{9.76}$$

Definition of the T-matrix: $T_{\beta\alpha}$. $|\phi_\alpha\rangle = |\boldsymbol{p}_1 \boldsymbol{p}_2 \dots\rangle$ is a set of in (or out) states.

where the δ-function $(*)$ can always be extracted, since $[S, P_\mu] = 0$ holds and therefore the total four-momentum P_μ^{total} is conserved:

$$p_\beta = p_\alpha \,, \quad p_\beta = p'_1 + p'_2 + \dots \,, \quad p_\alpha = p_1 + p_2 + \dots \tag{9.77}$$

Note: In non-relativistic quantum mechanics S does not commute with $\boldsymbol{P}$ if H depends on $\boldsymbol{X}$ via $V(\boldsymbol{X})$, so it's not relativistically invariant. In this case only the energy E is conserved and one writes $i(2\pi)\delta(E_\beta - E_\alpha)$ only.

For the derivation of the cross-section formula we temporarily put the system in a **finite box** of volume V. This changes our normalization:

Instead of the canonical normalization

$$(2\pi)^3 2E_{\boldsymbol{p}} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'}) \,.$$

$$\langle p', n' | p, n \rangle_V = N_p \delta_{nn'} \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'}) \quad \text{and} \quad N_p = \frac{(2\pi)^3}{V} \,. \tag{9.78}$$

Let us check the normalization

$$\langle p, n | p, n \rangle_V = N_p \delta^{(3)}(0) = \frac{N_p}{(2\pi)^3} \underbrace{\int d^3x \, \underbrace{e^{i(\boldsymbol{p}-\boldsymbol{p})\cdot \boldsymbol{x}}}_{=1}}_{=V} = 1 \quad \checkmark \,. \tag{9.79}$$

Check normalization.

Particle states are normalized and dimensionless and therefore $S^V_{\beta\alpha}$ is also dimensionless, as it should be for a probability amplitude.
We now compute the **transition rate** ($\hat{=}$ transition probability per unit time): we take $\alpha \longrightarrow \beta$ with $\beta \neq \alpha$ (otherwise no scattering would have happened).

$$R_{\beta\alpha} = \frac{|\langle \psi^{\text{out}}_\beta | \psi^{\text{in}}_\alpha \rangle|^2}{T} \underset{\substack{\uparrow \\ t_i \to -\infty \\ t_f \to +\infty}}{=} \lim_{T \to \infty} \frac{1}{T} \left| i(2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) T^V_{\beta\alpha} \right|^2 \tag{9.80}$$

$$\overset{Eq.\ (9.84)}{=} V(2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) \prod_{i=1}^{2} \prod_{j=1}^{n'} \left(\frac{N_i}{(2\pi)^3 2E_{\boldsymbol{p_i}}} \right) \left(\frac{N_j}{(2\pi)^3 2E_{\boldsymbol{p_j}}} \right) |T_{\beta\alpha}|^2 \,, \tag{9.81}$$

where compared to the usual normalization $T^V_{\beta\alpha}$ is

$$T^V_{\beta\alpha} = \prod_{i=1}^{2} \prod_{j=1}^{n'} \sqrt{\frac{N_i}{(2\pi)^3 2E_{\boldsymbol{p_i}}}} \sqrt{\frac{N_j}{(2\pi)^3 2E_{\boldsymbol{p_j}}}} T_{\beta\alpha} \tag{9.82}$$

and we have used that

Recall a similar argument for $\delta(\dots)^2$ in the derivation of Fermi's Golden Rule in QM.

$$\left|\delta^{(4)}(p_\beta - p_\alpha)\right|^2 \overset{\substack{\text{Only allowed if you}\\ \text{are a physicist!}}}{=} \frac{1}{(2\pi)^4}\underbrace{\int \mathrm{d}^4x\, e^{i(p_\beta - p_\alpha)x}}_{=V\cdot T}\underbrace{\delta^{(4)}(p_\beta - p_\alpha)}_{=1} \tag{9.83}$$

$$= \frac{V\cdot T}{(2\pi)^4}\delta^{(4)}(p_\beta - p_\alpha)\,. \tag{9.84}$$

Since momentum is continuous, we can only calculate the transition rate into final states with momenta $\{\boldsymbol{p'_j}\}$ in a small volume $\mathrm{d}^3p'_j$ around $\boldsymbol{p'_j}$.

We multiply Eq. (9.81) by $\prod_{j=1}^{n'}\frac{\mathrm{d}^3p'_j}{N_j}$, where the division by N_j is fixed by the completeness relation for fields. We obtain

$$\mathrm{d}R_{\beta\alpha} = \prod_{i=1}^{2}\frac{N_i}{(2\pi)^3 2E_{\boldsymbol{p_i}}}\prod_{j=1}^{n'}\frac{\mathrm{d}^3p'_j}{(2\pi)^3 2E_{\boldsymbol{p_j}}}V(2\pi)^4\delta^{(4)}(p_\beta - p_\alpha)|T_{\beta\alpha}|^2\,. \tag{9.85}$$

In the conventional normalization:

$$\langle\phi_\beta|S|\phi_\alpha\rangle \sim \delta^{(4)}(p_\beta - p_\alpha)T_{\beta\alpha}$$

with

$$|\phi_\beta\rangle = a^\dagger_{\boldsymbol{p'_1}}a^\dagger_{\boldsymbol{p'_2}}\cdot\ldots\cdot a^\dagger_{\boldsymbol{p'_{n'}}}|\Omega\rangle$$

$$|\phi_\alpha\rangle = a^\dagger_{\boldsymbol{p_1}}a^\dagger_{\boldsymbol{p_2}}|\Omega\rangle$$

$$[a_{\boldsymbol{p}}] = [a^\dagger_{\boldsymbol{p}}] = -1$$

and therefore, using the first relation, we get for the mass dimension of $T_{\beta\alpha}$

$$-n' - 2 = -4 + [T_{\beta\alpha}]$$

$$\implies [T_{\beta\alpha}] = -(2+n') + 4 \quad \checkmark$$

We check the mass dimensions (here we write the mass dimensions of the single factors in the same order as they are above):

$$[\mathrm{d}R_{\beta\alpha}] = 2\cdot(-1+3) + n'\cdot(-1+3) - 3 + (-4) + 2\cdot(-(2+n')+4) = +1 \quad \checkmark$$

since the rate should have dimension (second^{-1}).

The transition rate has to be **divided** by:

1. The flux of incoming particles $\left(\frac{\text{number}}{\text{time}}\cdot\frac{1}{\text{area}}\right)$:

$$\underbrace{\frac{|\langle p_1 n_1|p_1 n_1\rangle|}{V}}_{\text{particle density}}\cdot\overbrace{|\boldsymbol{v_1} - \boldsymbol{v_2}|}^{\text{relative velocity}} = \frac{N_1}{(2\pi)^3}|\boldsymbol{v_1} - \boldsymbol{v_2}|\,, \tag{9.86}$$

$$V = \frac{(2\pi)^3}{N_1}\,. \tag{9.87}$$

2. The number of target particles:

$$|\langle p_2 n_2|p_2 n_2\rangle| = \frac{N_2 V}{(2\pi)^3} = 1 \tag{9.88}$$

So we get

$$\mathrm{d}\sigma_{\beta\alpha} = \frac{1}{4E_{\boldsymbol{p_1}}E_{\boldsymbol{p_2}}|\boldsymbol{v_1} - \boldsymbol{v_2}|}\prod_{j=1}^{n'}\frac{\mathrm{d}^3p_j}{(2\pi)^3 2E_{\boldsymbol{p_j}}}(2\pi)^4\delta^{(4)}(p_\beta - p_\alpha)|T_{\beta\alpha}|^2\,, \tag{9.89}$$

$$\frac{1}{4E_{\boldsymbol{p_1}}E_{\boldsymbol{p_2}}|\boldsymbol{v_1} - \boldsymbol{v_2}|} = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}\,, \tag{9.90}$$

where the last equality holds in a frame where the particles 1 and 2 collide head-on: $\boldsymbol{p_1}||\boldsymbol{p_2}$, $\boldsymbol{p_1}\cdot\boldsymbol{p_2} < 0$, $\boldsymbol{v_i} \equiv \frac{\boldsymbol{p_i}}{E_i}$.

see exercise for the details

We can write this as

$$\mathrm{d}\sigma_{\beta\alpha} \equiv \frac{1}{(2E_{\boldsymbol{p_1}})(2E_{\boldsymbol{p_2}})|\boldsymbol{v_1}-\boldsymbol{v_2}|}|T_{\beta\alpha}|^2 \underset{\substack{\uparrow \\ \text{Lorentz-invariant phase space}}}{\mathrm{d}\Pi_{\mathrm{LIPS}}} ,$$

$$\text{with} \quad \mathrm{d}\Pi_{\mathrm{LIPS}} \equiv \prod_{\substack{\text{final states} \\ j=1}}^{n'} \frac{\mathrm{d}^3 p_j}{(2\pi)^3 2E_{\boldsymbol{p_j}}} (2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) \,. \tag{9.91}$$

Ex: Show Lorentz-invariance of the measure!

All the factors V, T have dropped out and we can trivially take the limit $V \longrightarrow \infty$, $T \longrightarrow \infty$.

Remarks:

1. We have assumed all particles different in **final** state β. If some are identical: $n' = n_1' + n_2' + ...$, with n_i' the number of identical particles of type $2'$, one has to multiply $\mathrm{d}\Pi_{\mathrm{LIPS}}$ by

$$\frac{1}{S!} = \frac{1}{n_1'!n_2'! \cdot ...} \,, \tag{9.92}$$

   since we must count states β that are indistinguishable (i.e. they differ only **exchange of identical** particles) in the momentum integral **only once**.

2. General particle states are defined by momentum and **spin**:

$$|p_i, s_i, n_i\rangle \,, \quad s = -J, -J+1, ..., J-1, J \quad \text{for a spin-}J\text{ particle.} \tag{9.93}$$

   In Eq. (9.91) we assume all spins are measured to take definite values. If we do **not** measure the spin orientation in the **final** state, we need to sum over all possible final spin orientations:

$$\sum_{s_1', s_2', ...} (...) = \text{"}\textbf{final state spin sum}\text{"} \,. \tag{9.94}$$

   If the spin of the **initial** state is not prepared, and often it is random, then we must take

$$\mathrm{d}\sigma_{\beta\alpha} \longrightarrow \frac{1}{(2J_1+1)(2J_2+1)} \sum_{s_1, s_2} \mathrm{d}\sigma_{\beta\alpha} \tag{9.95}$$

   the "**initial spin average**".

If we prepare the initial spins with some probability P_α, then we take

$$\mathrm{d}\sigma_{\beta\alpha} \longrightarrow \int\!\!\!\!\!\!\sum P_\alpha(\mathrm{spin})\mathrm{d}\sigma_{\beta\alpha} \,.$$

3. If we are not interested in a fully differential cross-section (in fact: any measurement will imply finite $\prod_{j=1}^{n'} \mathrm{d}^3 p_j$ integrations), we can integrate over regions of momentum space of the final states (also called "**phase space**"). The **total** scattering cross-section is defined as

$$\sigma_{\beta\alpha} = \int \mathrm{d}\sigma_{\beta\alpha} \,, \tag{9.96}$$

   where one integrates over the full phase space.

9.7 Differential decay rate and width

We defined above the quantity

$$\mathrm{d}\Gamma_{\beta\alpha} = \frac{1}{T}\mathrm{d}P_{\beta\alpha} \tag{9.97}$$

as the probability that a one-particle state (p_1, n_1) will turn into a multiparticle state in the time T.
The same steps as above lead to the formula

$$\boxed{\mathrm{d}\Gamma_{\beta\alpha} = \frac{1}{2E_1}\mathrm{d}\Pi_{\mathrm{LIPS}}|T_{\beta\alpha}|^2\,,} \tag{9.98}$$

where in the rest frame of the decaying particle we have $2E_1 = 2m_1$.
This formula is easy to remember: just drop all the factors from $\mathrm{d}\sigma_{\beta\alpha}$ that do not make sense for a single decaying particle ($|\boldsymbol{v_1} - \boldsymbol{v_2}|$ and $\frac{1}{2E_2}$).
After a time t, the probability that the particle (p_1, n_1) has not decayed yet is

$$P = e^{-\Gamma t}\,. \tag{9.99}$$

Γ is also called the **width** of the particle, it is the inverse of the **life-time**

$$\tau = \frac{1}{\Gamma}\,. \tag{9.100}$$

Consider the uncertainty principle: if a particle exists for a time τ, any measurement of its energy (or mass in its rest frame) must be uncertain by $\sim \frac{1}{\tau} = \Gamma$. Therefore a series of measurements of mass will have a characteristic spread of order Γ!

Intermediate summary:

- We know how to compute Green functions.
- We know how to calculate scattering cross-sections and decay widths in terms of single-particle $\widetilde{H}_0$ states and T-matrix elements.

We now need a relationship between the T-matrix and Green functions and we need to find out how to extract single-particle states from Green functions.

9.8 Field strength renormalization, two-point function and spectral representation

Consider a complex scalar field (just to change things up a bit, you can go back to the real field by replacing $\phi = \phi^\dagger$ everywhere). It transforms under the Poincaré-group as

$$U(\Lambda, a)\phi(x)U^{-1}(\Lambda, a) = \phi(\Lambda x + a)\,, \tag{9.101}$$

and as usual we assume that the Lagrangian $\mathcal{L}$ is Poincaré-invariant.

We build a basis of Hilbert space using the eigenstates of the four-momentum operator P_μ

$$\mathbb{1} = \sum\!\!\!\!\!\!\!\int_\lambda |\lambda\rangle\langle\lambda| \quad \text{(completeness)}\,, \tag{9.102}$$

with the states $|\lambda\rangle$ characterized by the total momentum $\boldsymbol{P}$ (because of translational invariance):

$$|\lambda\rangle \mapsto |\boldsymbol{P}\underset{\substack{\uparrow\\ \text{whatever is needed}\\ \text{to specify } \lambda\\ \text{uniquely}}}{\lambda}\rangle \equiv |\lambda_{\boldsymbol{p}}\rangle\,. \tag{9.103}$$

Assuming that we use relativistic normalization for $|\lambda_{\boldsymbol{p}}\rangle$ (as for single-particle states) we can write:

$$\boxed{\mathbb{1} = |\Omega\rangle\langle\Omega| + \sum\!\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}(\lambda)}} |\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}|\,,} \tag{9.104}$$

where we separated out the vacuum state and the second term contains single and multi-particle states. Let's explore these a bit more.

As usual, we have the relations

$$\langle\lambda'_{\boldsymbol{q}}|\lambda_{\boldsymbol{p}}\rangle = 2E_{\boldsymbol{p}(\lambda)}(2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\,, \tag{9.105}$$

$$E_{\boldsymbol{p}(\lambda)} = \sqrt{m_\lambda^2 + \boldsymbol{p}(\lambda)^2}\,, \quad \text{i.e.} \quad \boldsymbol{P}^2|\lambda_{\boldsymbol{p}}\rangle = m_\lambda^2|\lambda_{\boldsymbol{p}}\rangle \tag{9.106}$$

and in particular:

$$H|\underset{\substack{\uparrow\\ \text{total } \boldsymbol{p}=0}}{\lambda_0}\rangle = m_\lambda|\lambda_0\rangle\,. \tag{9.107}$$

If $|\lambda_0\rangle$ refers to a **single-particle state**, then

$$m_\lambda = \text{"mass of particle"} = M\,. \tag{9.108}$$

If $|\lambda_0\rangle$ is a **two-particle state**, then

$$m_\lambda^2 = \Big(\sqrt{M^2+\boldsymbol{p_1}^2} + \sqrt{M^2+\boldsymbol{p_2}^2}\Big)^2 - \Big(\underset{\substack{\uparrow\\ =\boldsymbol{p}=0\\ \text{since } \lambda_0}}{\boldsymbol{p_1}+\boldsymbol{p_2}}\Big)^2 = 4\Big(M^2+\boldsymbol{p_1}^2\Big)\,, \tag{9.109}$$

where for the last equality we used that $\boldsymbol{p_1} = -\boldsymbol{p_2}$. Therefore here λ parametrizes a continuum of asymptotic two-particle in-states.

Now consider:

$$\langle\Omega|\phi(x)\mathbb{1}\phi^\dagger(y)|\Omega\rangle \overset{(*)}{=} \langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi^\dagger(y)|\Omega\rangle + \tag{9.110}$$

$$+ \sum\!\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}(\lambda)}} \langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle\langle\lambda_{\boldsymbol{p}}|\phi^\dagger(y)|\Omega\rangle\,. \tag{9.111}$$

We manipulate the matrix element:

With the translation

$$U(1,x) = e^{iPx}\,,$$

$$P_\mu|\lambda_{\boldsymbol{p}}\rangle = p_\mu|\lambda_{\boldsymbol{p}}\rangle\,, \quad P_\mu|\Omega\rangle = 0\,.$$

$$\begin{aligned}
\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle &= \langle\Omega|U(\mathbb{1},x)\phi(0)U^{-1}(\mathbb{1},x)|\lambda_{\boldsymbol{p}}\rangle && (9.112)\\
&= \langle\Omega|e^{iPx}\phi(0)e^{-iPx}|\lambda_{\boldsymbol{p}}\rangle && (9.113)\\
&= e^{-ipx}\langle\Omega|\phi(0)|\lambda_{\boldsymbol{p}}\rangle && (9.114)\\
&\underset{\substack{\uparrow\\ \text{choose boost } U(\Lambda_{\boldsymbol{p}},0) \text{ such}\\ \text{that } \boldsymbol{p} \text{ goes to } 0,\ \phi(0) \text{ does}\\ \text{not change under boosts}}}{=} e^{-ipx}\langle\Omega|U^{-1}(\Lambda_{\boldsymbol{p}},0)\phi(0)U(\Lambda_{\boldsymbol{p}},0)|\lambda_{\boldsymbol{p}}\rangle && (9.115)\\
&= e^{-ipx}\langle\Omega|\phi(0)|\lambda_0\rangle && (9.116)
\end{aligned}$$

and define

$$\boxed{Z(\lambda) \equiv \langle\Omega|\phi(0)|\lambda_0\rangle\,.} \tag{9.117}$$

Similarly we can apply this to the first term, substituting $|\lambda_{\boldsymbol{p}}\rangle \longrightarrow |\Omega\rangle$:

$$\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle = v\,. \tag{9.118}$$

We find that the vacuum expectation value has to be constant, v =const., by translation invariance.
We can redefine the field, $\phi'(x) \equiv \phi(x) - v$, such that

$$\langle\Omega|\phi'(x)|\Omega\rangle = 0\,. \tag{9.119}$$

In the following, we will assume that we are working with $\phi'(x)$.
Then:

$$\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle \overset{(*)}{=} \sum\!\!\!\!\!\!\int_\lambda \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}(\lambda)}} e^{-ip(x-y)}|Z(\lambda)|^2\,, \tag{9.120}$$

$$\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle = \int_0^\infty \mathrm{d}M^2\,\rho(M^2)\underbrace{\int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}(\lambda)}} e^{-ip(x-y)}}_{\equiv\triangle}\,, \tag{9.121}$$

where we defined the **spectral function**

$$\rho(M^2) \equiv \sum\!\!\!\!\!\!\int_\lambda \delta(M^2 - m_\lambda^2)|Z(\lambda)|^2\,, \tag{9.122}$$

which is **positive**, $\rho(M^2) \geq 0$, as we can read off from its definition.
The integral $\triangle$ can be written as

$$\Delta(x-y, M^2) = \langle\Omega|\phi_0(x)\phi_0^\dagger(y)|\Omega\rangle\,, \tag{9.123}$$

where $\phi_0(x)$ is a free field satisfying

$$(\Box + M^2)\phi_0(x) = 0\,, \tag{9.124}$$

therefore with mass M and energy $p^0 = \sqrt{M^2 + \boldsymbol{p}^2}$.
Instead of $\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle$ we can also consider

$$\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \theta(x^0-y^0)\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle+\theta(y^0-x^0)\langle\Omega|\phi^\dagger(y)\phi(x)|\Omega\rangle\,. \tag{9.125}$$

Going through the same steps as above, we get the same result with the substitution

$$\begin{matrix}\Delta(x-y, M^2) \\ = \langle\Omega|\phi_0(x)\phi_0^\dagger(y)|\Omega\rangle\end{matrix} \quad\longrightarrow\quad \begin{matrix}\Delta_F(x-y, M^2) \\ = \langle\Omega|T\{\phi_0(x)\phi_0^\dagger(y)\}|\Omega\rangle\end{matrix} \tag{9.126}$$

We find the **spectral representation of the exact two-point function in the interacting theory**:

$$\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \int_0^\infty \mathrm{d}M^2\, \rho(M^2)\Delta_F(x-y,M^2)\,,$$
$$\langle\Omega|\phi(x)\phi^\dagger(y)|\Omega\rangle = \int_0^\infty \mathrm{d}M^2\, \rho(M^2)\Delta(x-y,M^2)\,.$$

(9.127)

(9.128)

Note that the spectral function is normalized:

$$\int_0^\infty \mathrm{d}M^2\, \rho(M^2) = 1\,. \tag{9.129}$$

Why? We show it using the canonical commutation relations:

$$i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) = \langle\Omega|[\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})]|\Omega\rangle \tag{9.130}$$
$$\overset{(*)}{=} \int_0^\infty \mathrm{d}M^2\, \rho(M^2)\langle\Omega|[\phi_0(t,\boldsymbol{x}),\pi_0(t,\boldsymbol{y})]|\Omega\rangle \tag{9.131}$$
$$= i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\int_0^\infty \mathrm{d}M^2\, \rho(M^2)\,, \tag{9.132}$$

where of the equality $(*)$ we assumed that π and π_0 have the same functional form in ϕ, $\dot{\phi}$ or ϕ_0, $\dot{\phi}_0$. This should hold for $t \longrightarrow \pm\infty$ with interactions switched off.
Therefore, by comparing sides, we see

$$\int_0^\infty \mathrm{d}M^2\, \rho(M^2) \overset{!}{=} 1 \quad \checkmark\,. \tag{9.133}$$

One-particle states show interesting implications.

Assume now, for simplicity, that there is only **one single-particle state**.[1] Then for this single particle state $|\lambda_0\rangle$, we define

$$\langle\Omega|\phi(0)|\lambda_0\rangle = \sqrt{Z} \tag{9.134}$$

and there is no dependence on additional parameters λ on the RHS, since there is just one state. Note, do not confuse the single particle overlap ($\sqrt{Z}$) here with the general definition in Eq. (9.117) ($Z(\lambda)$).
For the free field we have

$$\langle\Omega|\phi_0(0)|\lambda_0\rangle = 1\,, \tag{9.135}$$

therefore $Z_{\text{free}} = 1$. Hence we can write

$$\rho(M^2) = Z\cdot\delta(M^2-m^2) + \begin{pmatrix}\text{terms from}\\ \text{multiparticle}\\ \text{states}\end{pmatrix}\,. \tag{9.136}$$

Since $\rho(M^2) \geq 0$, we find $0 \leq Z \leq 1$ and $Z = 1$ if and only if $\phi(x)$ is free, since then also

$$\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle = 0 \tag{9.137}$$

for multiparticle states $|\lambda_{\boldsymbol{p}}\rangle$ must hold.

[1] this is not true in general, as for instance complex scalar fields ϕ have particle and anti-particle states, there might also be bound states, and many more instances.

Important: We found

$$\langle\Omega|\phi(x)|\lambda_{\boldsymbol{p}}\rangle = \sqrt{Z}e^{-ipx} \tag{9.138}$$

here. This is exactly what follows from Eq. (9.62):

$$\phi(x) \xrightarrow{t\to\mp\infty} \sqrt{Z}\phi^{\text{in/out}}(x) \tag{9.139}$$

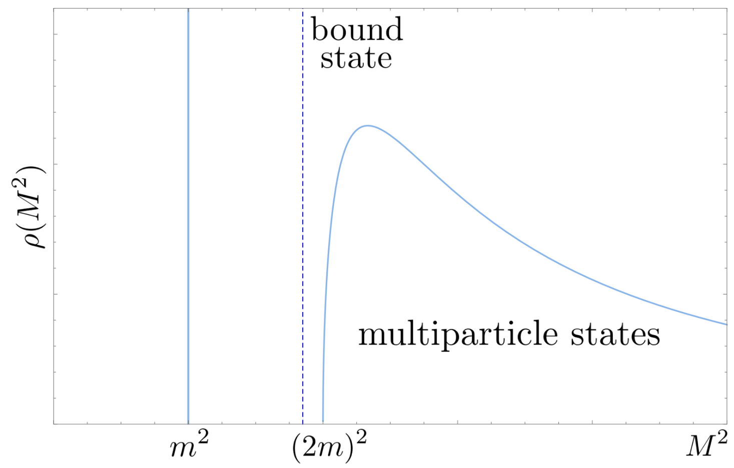
for the asymptotic free $\phi^{in/out}(x)$ field.

$\sqrt{Z}$ is called the **field strength renormalization constant**. It quantifies how much the field overlap with the single-particle state is changed by the interaction (recall: $\sqrt{Z_{\text{free}}} = 1$).
We can inspect the Fourier-transform of the two-point function:

$$\int \mathrm{d}^4(x-y)\, e^{ip(x-y)}\langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle \tag{9.140}$$

$$= \int_0^\infty \mathrm{d}M^2\, \rho(M^2) \underbrace{\int \mathrm{d}^4(x-y)\, e^{ip(x-y)}\Delta_F(x-y, M^2)}_{=\frac{i}{p^2-m^2+i\varepsilon}} \tag{9.141}$$

$$= \frac{iZ}{p^2-m^2+i\varepsilon} + \int_{m^2_{\text{bound}}}^\infty \mathrm{d}M^2\, \underset{\substack{\uparrow\\ \text{excluding the}\\ \text{one-particle state}}}{\rho(M^2)} \frac{i}{p^2-M^2+i\varepsilon} \tag{9.142}$$



and in the complex p^2-plane:
We have established the connection between Green functions and particles: particles can be found from the poles of the two-pint function.

- The **location** of the pole is the **mass** (physical mass or pole mass).
- The **residue** gives the field strength renormalization constant.

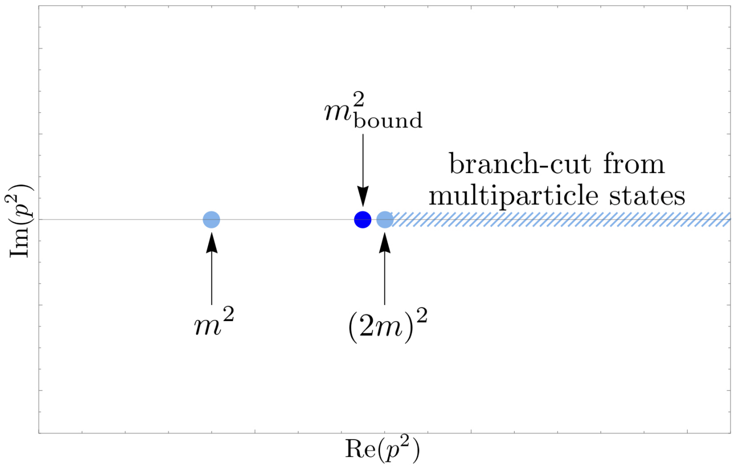
$\rho(M^2)$ is smooth,

$$\int \mathrm{d}x\, \frac{1}{x-a} \approx \ln(x-a)\,,$$

therefore the branch-cut is like the one of the complex logarithm $\ln(x-a)$ with $x = p^2$ and $a = (2m)^2$.

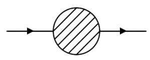
Example:

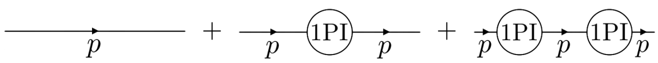
$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m_0^2\phi^\dagger\phi + \mathcal{L}_{\text{int}}\,, \tag{9.143}$$



where m_0 is not the physical mass, the parameter in the Lagrangian would be the mass in the **free theory**.

$$\int \mathrm{d}^4(x-y)\; e^{ip(x-y)} \langle\Omega|T\{\phi(x)\phi^\dagger(y)\}|\Omega\rangle = \tag{9.144}$$





$$= \quad + \quad + \quad + \ldots \tag{9.145}$$

$$= \frac{i}{p^2 - m_0^2 + i\varepsilon} + \frac{i}{p^2 - m_0^2 + i\varepsilon}\Big(\underbrace{-i\Pi(p^2, m_0^2)}_{=\text{(1PI)}}\Big)\frac{i}{p^2 - m_0^2 + i\varepsilon} + \ldots \tag{9.146}$$

$$= \frac{i}{p^2 - m_0^2 - \Pi(p^2, m_0^2) + i\varepsilon}\,. \tag{9.147}$$

Geometric series:

$$\frac{1}{a-b} = \frac{1}{a}\frac{1}{1-\frac{b}{a}} = \frac{1}{a}\Big(1 + \frac{b}{a} + \frac{b^2}{a^2} + \ldots\Big)$$
$$= \frac{1}{a} + \frac{1}{a}b\frac{1}{a} + \frac{1}{a}b\frac{1}{a}b\frac{1}{a} + \ldots$$

The mass of the particle is therefore given implicitly by

$$\boxed{p^2 - m_0^2 - \Pi(p^2, m_0^2)\Big|_{p^2=m^2} = 0\,.} \tag{9.148}$$

If $\Pi(p^2, m_0^2)$ is computed in perturbation theory, i.e.

$$\Pi(p^2, m_0^2) = \lambda\Pi^{(1)} + \lambda^2\Pi^{(2)} + \ldots\,, \tag{9.149}$$

we can solve this iteratively:

$$m^2 = m_0^2 + \delta m^2\,, \quad \delta m^2 = \lambda\delta m^2_{(1)} + \ldots \tag{9.150}$$

δm^2 is the **mass renormalization**, i.e. the contribution to the particle mass coming from interactions. An analogy to this effect in quantum mechanics is the shift of atomic energy levels coming from interactions, e.g. spin-orbit coupling, spin-spin coupling, and more. The field strength renormalization follows by expanding Eq. (9.148) around the pole location m^2:

$$p^2 - m_0^2 - \Pi(p^2, m_0^2) = \underbrace{\big(m^2 - m_0^2 - \Pi(m^2, m_0^2)\big)}_{\substack{=0 \text{ by definition} \\ \text{of } m^2}} + \tag{9.151}$$

$$+ \left(1 - \frac{\partial \Pi}{\partial p^2}\right)\bigg|_{p^2=m^2} (p^2 - m^2) + \ldots \tag{9.152}$$

and therefore

$$\frac{i}{p^2 - m_0^2 - \Pi(p^2, m_0^2) + i\varepsilon} \xrightarrow{p^2 \to m^2} \frac{i}{p^2 - m^2 + i\varepsilon} \cdot \frac{1}{1 - \frac{\partial \Pi}{\partial p^2}}\bigg|_{p^2=m^2} + \tag{9.153}$$

$$+ \begin{pmatrix}\text{non-singular} \\ \text{terms}\end{pmatrix} . \tag{9.154}$$

The field strength renormalization is therefore given by

$$\boxed{Z^{-1} = 1 - \frac{\partial \Pi}{\partial p^2}\bigg|_{p^2=m^2} .} \tag{9.155}$$

9.8.1 Constraints on $\Box^2$ terms from the spectral decomposition

The spectral representation gives powerful exact constraints. Suppose we were to add a term $\sim \Box^2$ to $\mathcal{L}_{\text{int}}$:

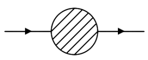
$$\mathcal{L} = -\phi^\dagger \Big(\Box + \underset{\substack{\uparrow \\ \text{const.}}}{c}\frac{\Box^2}{\Lambda^2} + m^2\Big)\phi + \mathcal{L}_{\text{int}}(\phi^\dagger, \phi) . \tag{9.156}$$

This would lead to the free propagator

$$\underset{p}{\longrightarrow} = \frac{i}{p^2 - m^2 - c\frac{p^4}{\Lambda^2}} \overset{p^2 \to \infty}{\longrightarrow} -\frac{i}{c}\frac{\Lambda^2}{p^4} , \tag{9.157}$$

which decays faster, compared to $\frac{i}{p^2 - m^2 + i\varepsilon} \sim \frac{1}{p^2}$ for $p^2 \longrightarrow \infty$.
Now consider

$$\Pi_{2\text{pt}}(p^2) = \int \mathrm{d}^4(x-y)\; e^{ip(x-y)} \langle \Omega | T\{\phi(x)\phi^\dagger(y)\} | \Omega \rangle \tag{9.158}$$

$=$ 

$$= \int \mathrm{d}M^2\; \rho(M^2) \frac{i}{p^2 - M^2 + i\varepsilon} , \tag{9.159}$$

where $\rho(M^2)$ includes the one-particle state. Let us now analytically continue this:

$$p^0 \longrightarrow ip^0 , \quad p^2 = -\underset{\substack{\uparrow \\ \text{euclidean}}}{p_E^2} = -\Big(\big(p^0\big)^2 + \big(p^1\big)^2 + \big(p^2\big)^2 + \big(p^3\big)^2\Big) . \tag{9.160}$$

We now want to look at $\Pi_{2\text{pt}}(-p_E^2)$ to show that $\Pi_{2\text{pt}} \sim \frac{1}{p^4}$ for $p^2 \longrightarrow \infty$ is forbidden by unitarity.
Consider

$$\Big|\Pi_{2\text{pt}}(-p_E^2)\Big| = \left| \int_0^\infty \mathrm{d}M^2\; \frac{\rho(M^2)}{p_E^2 + M^2} \right| \geq \left| \int_0^{M_0^2} \mathrm{d}M^2\; \frac{\rho(M^2)}{p_E^2 + M^2} \right| \tag{9.161}$$

for any M_0^2, where we have used that $\rho(M^2) \geq 0$. From this:

$$\lim_{p_E^2 \to \infty} p_E^2 \left| \Pi_{2\text{pt}}(-p_E^2) \right| \geq \lim_{p_E^2 \to \infty} p_E^2 \left| \int_0^{M_0^2} \mathrm{d}M^2 \, \frac{\rho(M^2)}{2p_E^2} \right| = \frac{A}{2}\,, \quad (9.162)$$

since $p_E^2 > M_0^2$ in the limit $p_E^2 \longrightarrow \infty$ and therefore $p_E^2 + M_0^2 < 2p_E^2$. In the last equality we have introduced the finite positive number

$$A = \int_0^{M_0^2} \mathrm{d}M^2 \, \rho(M^2)\,. \quad (9.163)$$

A propagator $\frac{1}{p^2 - m^2 - c\frac{p^4}{\Lambda^2}}$ would violate this bound!

A propagator cannot decrease faster than $\dfrac{1}{p^2}$ at large p^2.

(9.164)

Comment: this analysis of $\langle \Omega | T\{\phi(x)\phi^\dagger(y)\} | \Omega \rangle$ (two-point function) is also valid if $\phi(x)$ is replaced by some other composite operator $O(x) = F[\phi(x)]$, which might have different Lorentz-properties as the fundamental fields in $\mathcal{L}[\phi(x)]$
This is important e.g. for bound states:

- QCD at low energies ($E < 1$ GeV): There are **no** particle states in the spectrum corresponding to quarks and leptons.

- Hadron bound states correspond to the poles of the two-point function $\langle \Omega | T\{O(x)O(y)\} | \Omega \rangle$ with the non-vanishing overlap $\langle \Omega | O(0) | \text{hadron} \rangle = \sqrt{Z_O}$.

- In QED the free theory contains e^-, e^+, γ, but with interactions there are positronium bound states (e^+e^-). These bound states do not appear in electron two-point functions:

$$\langle \Omega | (\text{electron field}) | \text{positronium} \rangle = 0\,, \quad (9.165)$$

  since the quantum numbers (positronium vs. electron) do not match. **But** composite operator two-point functions can exhibit positronium poles.

E.g. naive quark content
- π^+ ($u\bar{d}$) 140 MeV
- K^+ ($u\bar{s}$) 494 MeV
- B^0 ($d\bar{b}$) 5279 MeV
- p (uud) 938 MeV

etc.

9.9 LSZ reduction

The main result in this section will be the derivation of the Lehmann-Symanzik-Zimmermann reduction formula, which relates S-matrix elements for asymptotic momentum eigenstates

$$\langle k_1, ..., k_m; \text{out} | p_1, ..., p_n; \text{in} \rangle \quad (9.166)$$

to an expression involving the Green functions.
We will derive that

LSZ-reduction

$$\begin{aligned}
&\langle k_1, ..., k_m; \text{out}|p_1, ..., p_n; \text{in}\rangle = \\
&= \frac{1}{\sqrt{Z}^{n+m}} \prod_{i=1}^{m} \left(i \int \mathrm{d}^4 x_i \; e^{ik_i x_i} (\Box_{x_i} + m^2) \right) \cdot \\
&\quad \cdot \langle \Omega | T\{\phi(x_1) \cdot ... \cdot \phi(x_m) \phi^\dagger(y_1) \cdot ... \cdot \phi^\dagger(y_n)\} |\Omega\rangle \cdot \\
&\quad \cdot \prod_{j=1}^{n} \left(i \int \mathrm{d}^4 y_j \; (\underbrace{\overleftarrow{\Box}_{y_j}}_{\text{acts to the left}} + m^2) e^{-ip_j y_j} \right).
\end{aligned} \tag{9.167}$$

The time-ordered correlation function $\langle\Omega|T\{\phi(x_1)\cdot ... \cdot \phi(x_m)\phi^\dagger(y_1)\cdot ... \cdot \phi^\dagger(y_n)\}|\Omega\rangle$ can be very complicated and contains much more information than the S-matrix elements.
The factors $(\Box + m^2) \to (-p^2 + m^2)$ project onto the S-matrix. $(-p^2 + m^2)$ vanishes for asymptotic states: these factors will remove all terms from $\langle\Omega|T\{\phi(x_1)\cdot ... \cdot \phi(x_m)\phi^\dagger(y_1)\cdot ... \cdot \phi^\dagger(y_n)\}|\Omega\rangle$ except for those with poles of the form $\frac{1}{p^2-m^2}$, i.e. propagators of on-shell particles.
The LSZ-formula tells us that the S-matrix projects out one-particle asymptotic states from time-ordered products of fields.
We assume again that $\phi(x)$ is a complex scalar field with non-vanishing overlap with the particles in the spectrum. Similarly to the analysis of the two-point function, we can generalize the above statement:

$$\begin{aligned}
\phi(x) \longrightarrow& \sqrt{Z}\phi^{\text{in/out}}(x) \\
&= \sqrt{Z} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} a_{\boldsymbol{p}}^{\text{in/out}} + e^{ipx} b_{\boldsymbol{p}}^{\dagger\,\text{in/out}} \right).
\end{aligned} \tag{9.168}$$

We want $\langle k_1, ..., k_m; \text{out}|p_1, ..., p_n; \text{in}\rangle$. We first express a, $a^\dagger$ and b, $b^\dagger$ in terms of asymptotic fields:

$$i\partial_0 \phi^{\text{in/out}}(x) = \frac{1}{2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(e^{-ipx} a_{\boldsymbol{p}}^{\text{in/out}} - e^{ipx} b_{\boldsymbol{p}}^{\dagger\,\text{in/out}} \right) \tag{9.169}$$

We solve Eq. (9.168) and Eq. (9.169):

$$a^{\text{in/out}} = \int \mathrm{d}^3 x \; e^{ipx} \left(E_{\boldsymbol{p}} \phi^{\text{in/out}}(x) + i\partial_0 \phi^{\text{in/out}}(x) \right) \tag{9.170}$$

$$= i \int \mathrm{d}^3 x \; e^{ipx} \overleftrightarrow{\partial_0} \phi^{\text{in/out}}(x) \tag{9.171}$$

and similarly

$$b^{\dagger\,\text{in/out}} = -i \int \mathrm{d}^3 x \; e^{-ipx} \overleftrightarrow{\partial_0} \phi^{\text{in/out}}(x), \tag{9.172}$$

with

$$\overleftrightarrow{\partial_0} = \underbrace{\overrightarrow{\partial_0}}_{\substack{\text{acts on}\\ \text{the right}}} - \underbrace{\overleftarrow{\partial_0}}_{\substack{\text{acts on}\\ \text{the left}}} \tag{9.173}$$

Now consider $O(y_1, ..., y_n)$, which is a product of $\psi(y_1), ..., \psi(y_n)$, where ψ can be either ϕ or $\phi^\dagger$:

$$a_{\boldsymbol{p}}^{\text{out}} T\{O(y_1, ..., y_n)\} - T\{O(y_1, ..., y_n)\} a_{\boldsymbol{p}}^{\text{in}} = \tag{9.174}$$

$$\overset{(*)}{=} \frac{1}{\sqrt{Z}} \int_{-\infty}^{\infty} \mathrm{d}x^0 \underset{\substack{\uparrow \\ \text{this picks intergrand} \\ \text{at } t=\pm\infty}}{\partial_0} \Big[\underbrace{\int \mathrm{d}^3x\, e^{ipx} T\{(p^0\phi(x) + i\partial_0\phi(x))}_{=a^{\text{in/out}}} O(y_1, ..., y_n)\} \Big], \tag{9.175}$$

where we have used that:

- $$\phi(x) \overset{x^0 \to \mp\infty}{\longrightarrow} \sqrt{Z}\phi^{\text{in/out}}(x)$$
- For $x^0 \longrightarrow \infty$ $x^0 > y_i^0$ holds for all i and so
$$\phi(x) T\{O(y_1, ..., y_n)\} = T\{\phi(x) O(y_1, ..., y_n)\}$$
- Similarly for $x^0 \longrightarrow -\infty$:
$$T\{O(y_1, ..., y_n)\}\phi(x) = T\{O(y_1, ..., y_n)\phi(x)\}$$
- Express the difference of boundary terms with
$$\int_{-\infty}^{\infty} \mathrm{d}x^0\, \partial_0 F(x) = F(x^0 = \infty, \boldsymbol{x}) - F(x^0 = -\infty, \boldsymbol{x}) \tag{9.176}$$

and get acting with the time derivative on the term in the brackets

$$\overset{(*)}{=} \frac{1}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{ipx} i \underbrace{(p^0 - i\partial_0)(p^0 + i\partial_0)}_{=(p^0)^2 + (\partial_0)^2} T\{\phi(x) O(y_1, ..., y_n)\} \tag{9.177}$$

where

$$\begin{aligned} e^{ipx} i (p^0)^2 &= i e^{ipx} (\boldsymbol{p}^2 + m^2) = i e^{ipx} (-\overset{\substack{\text{acts to the left} \\ \downarrow}}{\nabla_x^2} + m^2) \\ &\overset{\text{IBP}}{=} i e^{ipx} (-\nabla_x^2 + m^2) \end{aligned}$$

and

$$(\partial_0)^2 - \nabla_x^2 = \Box_x$$

so finally:

$$\overset{(*)}{=} \frac{i}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{ipx} (\Box_x + m^2) T\{\phi(x) O(y_1, ..., y_n)\}\,. \tag{9.178}$$

Similarly:

$$b_{\boldsymbol{p}}^{\dagger\,\text{out}} T\{O(y_1, ..., y_n)\} - T\{O(y_1, ..., y_n)\} b_{\boldsymbol{p}}^{\dagger\,\text{in}} = \tag{9.179}$$

$$= \frac{(-i)}{\sqrt{Z}} \int \mathrm{d}^4x\, e^{-ipx} (\Box_x + m^2) T\{\phi(x) O(y_1, ..., y_n)\} \tag{9.180}$$

Now replace $O \to O^\dagger$ and take the adjoint of the above equations. We get same the results with the substitutions

$$a \longrightarrow a^\dagger\,, \quad b^\dagger \longrightarrow b\,, \tag{9.181}$$

$$\pm i e^{\pm ipx} \longrightarrow \mp i e^{\mp ipx}\,, \quad \phi(x) \longrightarrow \phi^\dagger(x)\,. \tag{9.182}$$

We finally apply this to:

$$\langle k_1, ..., k_m; \text{out}|p_1, ..., p_n; \text{in}\rangle = \tag{9.183}$$
$$= \langle\Omega|(a^{\text{out}}_{\boldsymbol{k_m}} \cdot ... \cdot a^{\text{out}}_{\boldsymbol{k_1}} a^{\dagger\, in}_{\boldsymbol{p_1}} \cdot ... \cdot a^{\dagger\,\text{in}}_{\boldsymbol{p_n}})|\Omega\rangle \tag{9.184}$$
$$= \frac{1}{\sqrt{Z}}\left(i\int \mathrm{d}^4x_1\, e^{ik_1x_1}(\Box_{x_1}+m^2)\right)\langle\Omega|(a^{\text{out}}_{\boldsymbol{k_m}} \cdot ... \cdot a^{\text{out}}_{\boldsymbol{k_2}}\phi(x_1)\cdot \tag{9.185}$$
$$\cdot\, a^{\dagger\,\text{in}}_{\boldsymbol{p_1}} \cdot ... \cdot a^{\dagger\,\text{in}}_{\boldsymbol{p_n}})|\Omega\rangle + \langle\Omega|(a^{\text{out}}_{\boldsymbol{k_m}} \cdot ... \cdot a^{\text{in}}_{\boldsymbol{k_1}} a^{\dagger\,\text{in}}_{\boldsymbol{p_1}} \cdot ... \cdot a^{\dagger\,\text{in}}_{\boldsymbol{p_n}})|\Omega\rangle \tag{9.186}$$

where for the last equality we used Eq. (9.178) for $a^{\text{out}}_{\boldsymbol{k_1}}$ choosing $O = a^{\text{out}}_{\boldsymbol{k_m}} \cdot ... \cdot a^{\text{out}}_{\boldsymbol{k_2}} a^{\dagger\,\text{in}}_{\boldsymbol{p_1}} \cdot ... \cdot a^{\dagger\,\text{in}}_{\boldsymbol{p_n}}$. The result is time-ordered, since $a^{\text{out}}_{\boldsymbol{p}} = a_{\boldsymbol{p}}(t=\infty)$ and $a^{\text{in}}_{\boldsymbol{q}} = a_{\boldsymbol{q}}(t=-\infty)$.

In a true scattering process, we assume that $k_i \neq p_j\ \forall i,j$, otherwise some of the particles would just have continued on their paths without interaction; this would correspond to disconnected subdiagrams. The second term of the above result can now be shown to vanish by commuting $a^{\text{in}}_{\boldsymbol{k_1}}$ to the right:

$$\langle\Omega|(a^{\text{out}}_{\boldsymbol{k_m}} \cdot ... \cdot a^{\text{out}}_{\boldsymbol{k_2}} a^{in}_{\boldsymbol{k_1}} a^{\dagger\,\text{in}}_{\boldsymbol{p_1}} \cdot ... \cdot a^{\dagger\,\text{in}}_{\boldsymbol{p_n}})|\Omega\rangle \tag{9.187}$$
$$= \langle\Omega|(a^{\text{out}}_{\boldsymbol{k_m}} \cdot ... \cdot a^{\text{out}}_{\boldsymbol{k_2}} a^{\dagger\,\text{in}}_{\boldsymbol{p_1}} \cdot ... \cdot a^{\dagger\,\text{in}}_{\boldsymbol{p_n}})\big(\underset{\overset{\uparrow}{=0}}{a^{\text{in}}_{\boldsymbol{k_1}}|\Omega\rangle}\big) = 0 \tag{9.188}$$

Note that this is not possible for $a^{\text{out}}_{k_i}$, since there are no simple commutation relations between in- and out-operators, e.g.

$$[a^{\text{out}}_{k_1}, a^{\dagger\,\text{in}}_{p_1}] =? \tag{9.189}$$

Now we **repeat** this process for the a's for $k_2, ..., k_m$, then for the $a^\dagger$'s for $p_1, ..., p_n$ until all a's and $a^\dagger$'s are converted into ϕ's and $\phi^\dagger$'s.

The result is

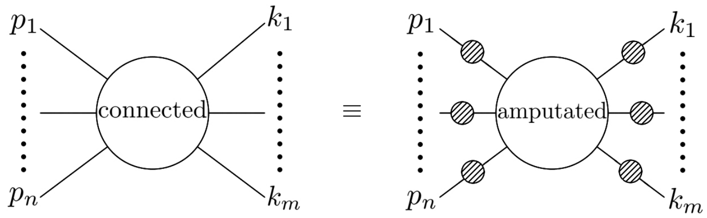
$$
\begin{aligned}
&\langle k_1, ..., k_m; \text{out} | p_1, ..., p_n; \text{in} \rangle = \\
&= \frac{1}{\sqrt{Z}^{n+m}} \prod_{i=1}^{m} \left(i \int \mathrm{d}^4 x_i \; e^{i k_i x_i} (\Box_{x_i} + m^2) \right) \cdot \\
&\quad \cdot \langle \Omega | T\{\phi(x_1) \cdot ... \cdot \phi(x_m) \phi^\dagger(y_1) \cdot ... \cdot \phi^\dagger(y_n)\} | \Omega \rangle \cdot \\
&\quad \cdot \prod_{j=1}^{n} \left(i \int \mathrm{d}^4 y_j \; (\overleftarrow{\Box}_{y_j} + m^2) e^{-i p_j y_j} \right)
\end{aligned} \tag{9.190}
$$

$$
\begin{aligned}
&= \frac{(-i)^{n+m}}{\sqrt{Z}^{n+m}} \prod_{i=1}^{m} \left(k_i^2 - m^2 \right) \prod_{j=1}^{n} \left(p_j^2 - m^2 \right) \cdot \\
&\quad \cdot \int \mathrm{d}^4 x_1 .... \mathrm{d}^4 x_m \mathrm{d}^4 y_1 ... \mathrm{d}^4 y_n \exp \left(i \sum_{i=1}^{m} k_i x_i - i \sum_{j=1}^{n} p_j y_j \right) \cdot \\
&\quad \cdot \langle \Omega | T\{\phi(x_1) \cdot ... \cdot \phi(x_m) \phi^\dagger(y_1) \cdot ... \cdot \phi^\dagger(y_n)\} | \Omega \rangle \, .
\end{aligned} \tag{9.191}
$$

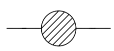
This is the **LSZ reduction formula** which relates S-matrix elements to momentum space Green functions.
The k_i, p_j satisfy the on-shell condition $k_1^2 = ... = k_m^2 = p_1^2 = ... = p_n^2 = m^2$, so it appears that the overall factor gives zero. However, the Green function has **poles** at these points!

9.9.1 Amputated Green function

We define the amputated Green function as



,

where the p_i are the incoming momenta in $e^{-ip_i x_i}$ and the k_j are the outgoing momenta in $e^{-ik_j x_j}$ and  is the exact two-point function.

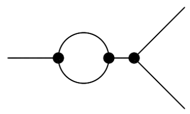
Written in terms of formulas, this corresponds to

$$\widetilde{G}^C(q_1, q_2, ..., q_k) \equiv \Big[\prod_{l=1}^{k} \underbrace{\widetilde{G}(q_l)}_{\substack{\text{2-pt function depends only}\\ \text{on one momentum}}}\Big] \widetilde{G}^{\text{amp}}(q_1, ..., q_k)\,, \tag{9.192}$$

which defines $\widetilde{G}^{\text{amp}}$ for $k > 2$.

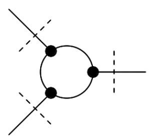
Example:

1)



is a connected but **not** amputated three-point function.

2)



without the external propagators is an amputated three-point function.

From our discussion of the spectral representation of the two-point function:

$$\widetilde{G}(k) \xrightarrow{k^0 \to E_{\boldsymbol{k}} = \sqrt{m^2+\boldsymbol{k}^2}} \frac{iZ}{k^2 - m^2} + \begin{pmatrix}\text{non-singular}\\ \text{terms}\end{pmatrix}\,, \tag{9.193}$$

with m the physical mass and **not** the free Lagrangian parameter m_0.
Using this in the LSZ formula Eq. (9.191) together with Eq. (9.192) where expressions of the above form appear yields

$$\frac{(-i)}{\sqrt{Z}}(k_i^2 - m^2)\,\widetilde{G}(k_i) \xrightarrow{k_i^0 \to E_{\boldsymbol{k}}} \sqrt{Z} \tag{9.194}$$

and therefore:

Recall the definition of the momentum space Green function:

$$\int \mathrm{d}^4x ... \mathrm{d}^4x_n e^{ip_1x_1 + ... + ip_nx_n} G(x_1, ..., x_n)$$
$$\equiv (2\pi)^4 \delta^{(4)}(p_1 + ... + p_n) \widetilde{G}(p_1, ..., p_n)\,.$$

Recall the definition of the T-matrix:

$$\langle k_1, ..., k_m; \text{out}|p_1, ..., p_n; \text{in}\rangle = (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{m} k_i - \sum_{j=1}^{n} p_j\right) \cdot$$

$$\cdot \left(\sqrt{Z}\right)^{n+m} \widetilde{G}^{\text{amp}}(k_1, ..., k_m; \underbrace{-p_1, ..., -p_n}_{\substack{\text{because } p_j \text{ were} \\ \text{defined as ingoing}}})\Big|_{\substack{\text{on-shell:} \\ k_i^0 = \sqrt{m^2 + \boldsymbol{k}_i^2} \\ p_j^0 = \sqrt{m^2 + \boldsymbol{p}_j^2}}}$$

$$= i(2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{m} k_i - \sum_{j=1}^{n} p_j\right) T_{k_1,...,k_m;p_1,...,p_n} \,. \tag{9.195}$$

$$\tag{9.196}$$

Comments:

- We only need **connected** Green functions, the disconnected parts correspond to special momentum configurations which are a product of **unrelated** subscatterings.
- Rule for computing Green functions:

outgoing	ϕ	$\phi^\dagger$
incoming	$\phi^\dagger$	ϕ
	particle	**anti-particle**

  where the ϕ's and $\phi^\dagger$'s in the particle and anti-particle case differ only by the sign of their momentum in $\widetilde{G}^{\text{amp}}$: $p \longleftrightarrow -p$.
- Different fields ϕ_1, ϕ_2 will have **different** $\sqrt{Z_1}$, $\sqrt{Z_2}$ factors.
- We will discuss fields with **spin** later which will carry extra factors on the external lines (corresponding to the polarizations of the particles).

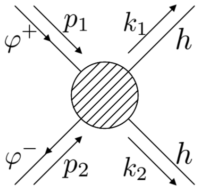
9.10 Example computation of a scattering cross-section

Consider a theory with a complex scalar field ϕ and a real scalar field H defined by the following Lagrangian:

$$\mathcal{L} = \underbrace{\frac{1}{2}\left(\partial_\mu H \partial^\mu H - m_H^2 H^2\right) + \partial_\mu \phi^\dagger \partial^\mu \phi - m_\phi^2 \phi^\dagger \phi}_{=\mathcal{L}_0} + \underbrace{\lambda \phi^\dagger \phi H + \frac{g_3}{3!} H^3 - \frac{g_4}{2} \phi^\dagger \phi H^2}_{=\mathcal{L}_{\text{int}}} \,. \tag{9.197}$$

Additional interactions $\sim H^4, \sim (\phi^\dagger \phi)^2$ etc. are not needed here. We assume weak couplings: $\lambda, g_3, g_4 \ll 1$ and $g_4 \sim \lambda^2 \sim g_3^2$. The theory contains a real scalar particle h and a charged scalar φ^+ and its antiparticle φ^-.

Goal: Compute the scattering cross-section of the reaction $\varphi^+\varphi^- \longrightarrow hh$ at lowest order in perturbation theory.



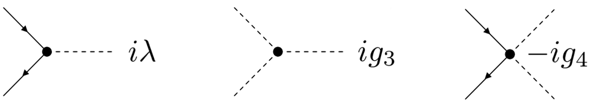
We need to look at

$$\langle\Omega|T\{H(x_1)H(x_2)\phi(x_3)\phi^\dagger(x_4)\}|\Omega\rangle\,, \tag{9.198}$$

which we evaluate on-shell, amputated and ignore disconnected subdiagrams.

Feynman rules:

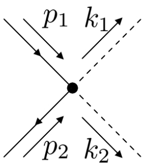
(propagators)



(vertices)

Diagrams:

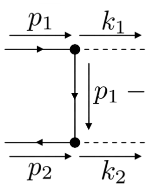
1)



$$-\frac{ig_4}{2}\underbrace{(HH\phi\phi^\dagger)}_{\substack{\text{from}\\ \langle\Omega|T\{HH\phi\phi^\dagger\}|\Omega\rangle}}\quad\underbrace{(\phi^\dagger\phi HH)}_{\text{from } i\mathcal{L}_{\text{int}}}\cdot 2 = -ig_4 \tag{9.199}$$

and the factor 2 takes into account that there are two identical contractions between the H fields. The result is the same as the Feynman rule for the vertex.

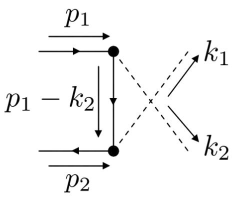
2)



$$2\cdot\frac{1}{2!}(i\lambda)^2 HH\phi^\dagger\phi\quad\phi^\dagger\phi H\quad\phi^\dagger\phi H\,, \tag{9.200}$$

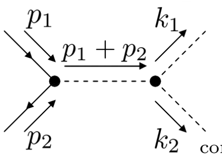
where again the factor 2 takes into account that there are two identical contractions, between the ϕ's and $\phi^\dagger$'s this time, after integrating over z_1 and z_2. Here the other contraction between the H's does **not** lead to the same diagram, which is obvious from the momentum flow, see 3).

3)



This is the resulting diagram from the other H contraction above.

4)



$$\underset{\substack{\uparrow \\ \text{identical} \\ \text{contractions of } H}}{3!} \cdot \overset{\substack{\text{two interactions} \\ \downarrow}}{\frac{1}{2!}} \cdot 2(i\lambda)\frac{g_3}{3!} HH\phi^\dagger\phi \quad \phi^\dagger\phi H \quad HHH \tag{9.201}$$

and the last factor 2 comes from $\mathcal{L}^2_{\text{int}} \supset 2\lambda\phi^\dagger\phi H \cdot \frac{g_3}{3!}H^3$.

We therefore find the following scattering matrix element:

$$T = (-i)\underset{\substack{\uparrow \\ 1+\mathcal{O}(\lambda, g_3, g_4) \\ \text{at lowest order, we} \\ \text{can set } Z_H = Z_\phi = 1}}{\left(\sqrt{Z_H}\right)^2\left(\sqrt{Z_\phi}\right)^2} \widetilde{G}^{\text{amp}}(k_1, k_2; -p_1, -p_2)\Big|_{\substack{\text{on-shell} \\ k_1+k_2=p_1+p_2 \\ \uparrow \\ \text{at lowest order we can} \\ \text{neglect the mass shifts} \\ \delta m^2_\phi,\, \delta m^2_H \text{ since} \\ \text{they are } \mathcal{O}(\lambda, g_3, g_4)}} \tag{9.202}$$

$i(2\pi)^4\delta^{(4)}(p_f - p_i)T$
$= (2\pi)^4\delta^{(4)}(p_f - p_i)\sqrt{Z}^{n+m}\widetilde{G}(k_1, ...; p_1, ...)$

$$= (-i)\Bigg(\underset{\substack{\uparrow \\ 1.}}{-ig_4} - \lambda^2\left[\underset{\substack{\uparrow \\ 2.}}{\frac{i}{(p_1 - k_1)^2 - m^2_\phi}} + \underset{\substack{\uparrow \\ 3.}}{\frac{i}{(p_1 - k_2)^2 - m^2_\phi}}\right] -$$

$$- \lambda g_3 \underset{\substack{\uparrow \\ 4.}}{\frac{i}{(p_1 + p_2)^2 - m^2_H}}\Bigg)\,. \tag{9.203}$$

We see that the amputated Green function is much simpler: no external leg propagators have to be included.
This is a $2 \longrightarrow 2$-scattering: we can make use of the Mandelstam variables

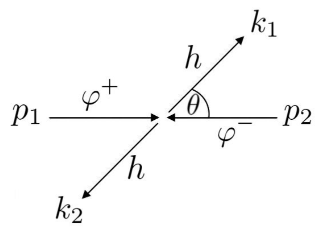
$$s = (p_1 + p_2)^2\,, \quad t = (p_1 - k_1)^2\,, \quad u = (p_1 - k_2)^2\,. \tag{9.204}$$

Using these we rewrite

$$iT = (-i)\left[g_4 + \lambda^2\left(\frac{1}{t - m^2_\varphi} + \frac{1}{u - m^2_\varphi}\right) + \frac{\lambda g_3}{s - m^2_H}\right]. \tag{9.205}$$

Scattering cross sections: We want to compute $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ in the center-of-mass (CMS) system:

$$\text{CMS:} \quad \boldsymbol{p_1} + \boldsymbol{p_2} = 0 = \boldsymbol{k_1} + \boldsymbol{k_2} \tag{9.206}$$



We choose:

$$p_1 = \begin{pmatrix} E \\ 0 \\ 0 \\ |\boldsymbol{p}| \end{pmatrix}, \quad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -|\boldsymbol{p}| \end{pmatrix}, \tag{9.207}$$

$$k_1 = \begin{pmatrix} E \\ |\boldsymbol{k}|\sin\theta \\ 0 \\ |\boldsymbol{k}|\cos\theta \end{pmatrix}, \quad k_2 = \begin{pmatrix} E \\ -|\boldsymbol{k}|\sin\theta \\ 0 \\ -|\boldsymbol{k}|\cos\theta \end{pmatrix}, \tag{9.208}$$

$$|\boldsymbol{p}| = \sqrt{E^2 - m_p^2}, \quad |\boldsymbol{k}| = \sqrt{E^2 - m_k^2}. \tag{9.209}$$

Note that we can choose the xz-plane for scattering, since momentum conservation implies that initial and final state momenta lie in a plane (use the cylindrical symmetry of the initial state to rotate the system into the xz-plane).
Using the Mandelstam variables, Eq. (9.204), we write

$$s = (p_1 + p_2)^2 = 4E^2, \tag{9.210}$$

Energy of $\varphi^{\pm}$-particles in the collision beams. At LHC: $s = 13$ TeV!

$$\hat{t} = (p_1 - k_1)^2 - m_\varphi^2 = -2E^2 + m_H^2 + 2\sqrt{E^2 - m_\varphi^2}\sqrt{E^2 - m_H^2}\cos\theta, \tag{9.211}$$

$$\hat{u} = (p_1 - k_2)^2 - m_\varphi^2 = -2E^2 + m_H^2 - 2\sqrt{E^2 - m_\varphi^2}\sqrt{E^2 - m_H^2}\cos\theta, \tag{9.212}$$

See Ex!

where in the last equation we already used that $\cos(\pi + \theta) = -\cos\theta$, resulting from the exchange $k_1 \longleftrightarrow k_2$.
Note that

$$\begin{aligned} \hat{t} \leq &-2E^2 + m_H^2 + 2\sqrt{E^2 - m_\varphi^2}\sqrt{E^2 - m_H^2} \\ &= -\left(\sqrt{E^2 - m_\varphi^2} - \sqrt{E^2 - m_H^2}\right)^2 - m_\varphi^2 \leq -m_\varphi^2. \end{aligned} \tag{9.213}$$

This means that $t, u \leq 0$ are negative! We can now compute the differential cross-section Eq. (9.91):

$$\begin{aligned} \mathrm{d}\sigma = &\overset{\substack{\text{Final state contains} \\ \text{two identical particles} \\ \varphi^+\varphi^- \to hh \\ \downarrow}}{\frac{1}{2}} \frac{\mathrm{d}^3 k_1}{(2\pi)^3 2E_{\boldsymbol{k_1}}} \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_{\boldsymbol{k_2}}} \left(\frac{1}{\underbrace{}_{}\overset{\substack{=(2E_{\boldsymbol{p_1}})(2E_{\boldsymbol{p_2}})|\boldsymbol{v_1}-\boldsymbol{v_2}|\,,\ (\text{see Ex}) \\ \downarrow}}{4\sqrt{(p_1 \cdot p_2)^2 - m_\varphi^4}}} \right) \cdot \\ &\cdot (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2)|T|^2. \end{aligned} \tag{9.214}$$

Using:

$$\mathrm{d}^3k_1 = \mathrm{d}|\boldsymbol{k_1}|\underset{\substack{\uparrow \\ \text{spherical coordinates}}}{\mathrm{d}\Omega}|\boldsymbol{k_1}|^2\,, \quad \mathrm{d}|\boldsymbol{k_1}| = \mathrm{d}E_{\boldsymbol{k_1}}\frac{E_{\boldsymbol{k_1}}}{|\boldsymbol{k_1}|}\,, \tag{9.215}$$

$$E_{\boldsymbol{k_1}} = \sqrt{m_H^2 + \boldsymbol{k_1}^2} \overset{-\boldsymbol{k_1}=\boldsymbol{k_2}}{=} \sqrt{m_H^2 + \boldsymbol{k_2}^2} = E_{\boldsymbol{k_2}}\,, \tag{9.216}$$

the $\delta^{(3)}(...)$ to remove d^3k_2 and

$$\delta(E_{\boldsymbol{k_1}}+E_{\boldsymbol{k_2}}-E_{\boldsymbol{p_1}}-E_{\boldsymbol{p_2}}) = \delta(2E_{\boldsymbol{k_1}}-2E_{\boldsymbol{p_1}}) = \frac{1}{2}\delta(E_{\boldsymbol{k_1}}-E_{\boldsymbol{p_1}}) \tag{9.217}$$

to remove $\mathrm{d}E_{\boldsymbol{k_1}}$, we get the simplified expression

$$\mathrm{d}\sigma = \frac{1}{2}\frac{1}{4\pi^2}\frac{E_{\boldsymbol{k_1}}|\boldsymbol{k_1}|^2}{|\boldsymbol{k_1}|}\mathrm{d}\Omega\frac{1}{4E_{\boldsymbol{k_1}}E_{\boldsymbol{k_2}}}\frac{1}{4\sqrt{(2E^2-m_\varphi^2)^2-m_\varphi^4}}\frac{1}{2}|T|^2 \tag{9.218}$$

and finally, using

$$(2E^2-m_\varphi^2)^2 - m_\varphi^4 = 4E^2(E^2-m_\varphi^2)\,, \tag{9.219}$$

we obtain

$$\mathrm{d}\sigma = \frac{1}{512\pi^2E^2}\sqrt{\frac{E^2-m_H^2}{E^2-m_\varphi^2}}|T|^2\mathrm{d}\Omega\,. \tag{9.220}$$

Since $|T|^2$ depends on $\cos\theta$ only, we can integrate over the azimuthal angle and write

$$\mathrm{d}\Omega = \mathrm{d}\varphi\mathrm{d}\cos\theta = 2\pi\mathrm{d}\cos\theta \tag{9.221}$$

and therefore get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = 2\pi\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(E_1,\cos\theta)\,, \tag{9.222}$$

$$\sigma_{\mathrm{tot}} = \int_{-1}^{1}\mathrm{d}\cos\theta\,\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} \qquad \text{(total cross-section)}\,. \tag{9.223}$$

Interesting limits:

$$iT = (-i)\left[g_4 + \lambda^2\left(\frac{1}{t-m_\varphi^2}+\frac{1}{u-m_\varphi^2}\right)+\frac{\lambda g_3}{s-m_H^2}\right] \tag{9.224}$$

1) **High energy** and no forward/backward-scattering: $|\cos\theta|$ is not near to 1 and $E \gg m_\varphi, m_H$. Since then $s,t,u \sim E^2$ and

$$iT = -ig_4 + \mathcal{O}\left(\frac{1}{E^2}\right)\,, \tag{9.225}$$

the cross-section is **isotropic** and dominated by the four-point interaction.

2) High-energy and **forward/backward-scattering**: $|\cos\theta| \approx 1$, so

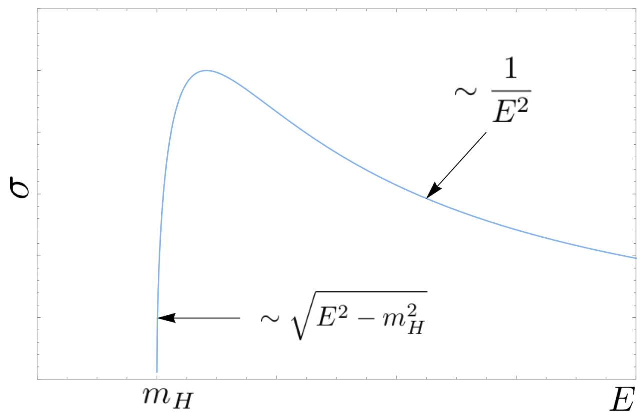
$$\hat{t} \approx -2E^2(1-\cos\theta)\,, \quad \hat{u} \approx -2E^2(1+\cos\theta) \tag{9.226}$$

and

$$\begin{aligned} -T &\approx g_4 - \frac{\lambda^2}{2E^2}\left(\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}\right)+\frac{\lambda g_3}{4E^2} \\ &\approx g_4 - \frac{\lambda^2}{E^2}\frac{1}{1-\cos^2\theta} = g_4 - \frac{\lambda^2}{E^2}\frac{1}{\sin^2\theta}\,. \end{aligned} \tag{9.227}$$

Therefore there is a forward- and backward-enhancement from the t- and u-channels, since the mediator is "massless" (massless compared to $E \gg m_\varphi$).

3) Energy dependence of the **total** $\varphi^+\varphi^- \longrightarrow hh$ **cross-section**:



The cross-section shows a threshold at $E \approx m_H$, i.e. $s = 4m_H^2$, since at lower energies no h-pair can be produced and therefore the studied scattering process cannot happen. Further, with Eq. (9.211) and Eq. (9.212), we find $\hat{t} = \hat{u} = -m_H^2$. At energies near the **threshold** $E \approx m_H$ the T-matrix element behaves like

$$iT \approx (-i)\left(g_4 - \frac{2\lambda^2}{m_H^2} + \frac{\lambda g_3}{3m_H^2}\right) = \text{const.} \tag{9.228}$$

and the total cross-section has the energy dependence

$$\sigma \sim \sqrt{E^2 - m_H^2}\,, \tag{9.229}$$

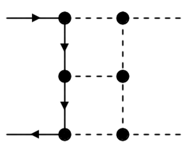
whereas at **high energies** $E \gg m_H, m_\phi$ it behaves like

$$\sigma = \frac{4\pi}{512\pi^2} \cdot \frac{g_4^2}{E^2} \sim \frac{1}{E^2}\,. \tag{9.230}$$

This has been a computation to lowest order.

Outlook: QFT calculations become very involved

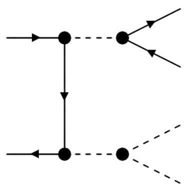
1. At higher order in perturbation theory, e.g. a representative diagram (there are of course more)



the two-loop correction from diagrams like the one above,

$$T \sim \int \mathrm{d}^4 q_1 \mathrm{d}^4 q_2 \,(\ldots)\,, \tag{9.231}$$

2. When there are many particles in the final state, e.g. the $2 \longrightarrow 4$ scattering process



where

$$\mathrm{d}\Pi_{\mathrm{LIPS}} = \prod_{i=1}^{4} \frac{\mathrm{d}^3 k_i}{(2\pi)^3 2E_{\boldsymbol{k_i}}} (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{4} k_i - p_1 - p_2 \right). \qquad (9.232)$$

This integral is hard to evaluate even numerically due to singularities at the end points (think of the points $|\cos\theta| = 1$ in the $2 \longrightarrow 2$ scattering process when $m_\varphi = 0$) and the "curse of dimensionality": $\mathrm{d}\Pi_{\mathrm{LIPS}}(2 \to 4)$ is $4 \cdot 3 - 4 = 8$ -dimensional, and therefore to get just 10 points in each dimension we need 10^8 interpolation points!

10
Renormalization

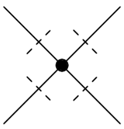
10.1 A simplified example

Before we get into any physical calculations, let us give an overview of how things are going to work out.
Consider

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{\lambda}{4!}\phi^4 \,. \tag{10.1}$$

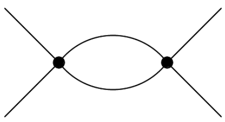
At tree level, the T-matrix element corresponding to the four-point function is

$$iT_1 = \quad = -i\lambda \,. \tag{10.2}$$

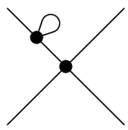


The leading-order correction to this is

$$iT_2 = \quad + \ldots = (-i\lambda)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}(\ldots) \tag{10.3}$$

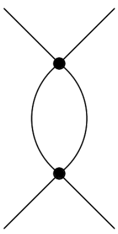


Note that we do not need e.g.

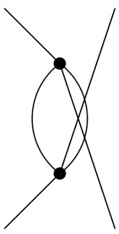


since this loop is part of the amputated two-point function and contributes to Z and m.

and $\left(\sqrt{Z}\right)^4 = 1 + \mathcal{O}(\lambda^2)$, so we can just set $\sqrt{Z} = 1$ at one-loop order. We will ignore the

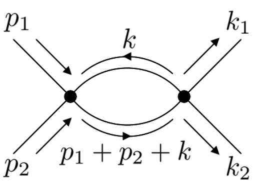


t-channel ,



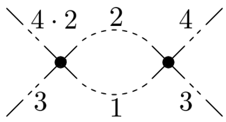
u-channel (10.4)

contributions to keep things simpler for now (see the full calculation in a few pages).



$$= \underset{\substack{\uparrow \\ (*)}}{\frac{1}{2}}(-i\lambda)^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2} \tag{10.5}$$

with $p \equiv p_1 + p_2$. The symmetry factor $(*)$ results from



$$\frac{1}{2!}\left(-\frac{i\lambda}{4!}\right)^2 \cdot \underset{\substack{\uparrow \\ \text{connect to} \\ \text{external legs}}}{(4\cdot 3)} \cdot 4\cdot 3\cdot \overset{\substack{z_1\leftrightarrow z_2 \\ \downarrow}}{2}\cdot \underset{\substack{\uparrow \\ \text{connect} \\ \text{internal}}}{2} = \frac{(4!)^2}{2!(4!)^2}(-i\lambda)^2 = \frac{1}{2}(-i\lambda)^2 \quad \checkmark\,. \tag{10.6}$$

1) iT_2 is Lorentz-invariant, it can only depend on $s = p^2$.

2) iT_2 is dimensionless.

3) For large k:

$$\int \mathrm{d}^4k \frac{1}{k^2}\frac{1}{(p-k)^2} \overset{k\gg p}{\sim} \int_C^\Lambda \mathrm{d}k\, k^3 \frac{1}{k^4} \approx \ln(\Lambda)\,. \tag{10.7}$$

From 1)-3) we conclude that

$$iT_2 \sim \ln\left(\frac{s}{\Lambda^2}\right). \tag{10.8}$$

A quick way to see this is:

$$\frac{\partial}{\partial s} iT_2(s) = \frac{p^\mu}{2s}\frac{\partial}{\partial p^\mu} iT_2 = -\frac{\lambda^2}{2s}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\cdot\frac{(p-k)\cdot p}{(p-k)^4} \tag{10.9}$$

$$\frac{\partial}{\partial s} T_2 = -\frac{\lambda^2}{32\pi^2}\cdot\frac{1}{s}\,, \tag{10.10}$$

Note that both Eq. (10.9) and Eq. (10.10) are finite! We will do many of these integrals soon.

since[1]

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\cdot\frac{(p-k)\cdot p}{(p-k)^4} = \frac{i}{16\pi^2}\,, \tag{10.11}$$

[1] We will calculate it explicitly in Sec. 10.5.3

which means:

$$T_2(s) = -\frac{\lambda^2}{32\pi^2}\ln(s) + C\,, \tag{10.12}$$

where C is an integration constant. Since iT_2 is divergent, C will have to be infinite. We write this constant as

More on this later. This chapter is on the "mechanisms" that will be used, the algebra will come later.

$$C = \frac{\lambda^2}{32\pi^2}\ln(\Lambda^2)\,, \tag{10.13}$$

such that

$$T_2(s) = -\frac{\lambda^2}{32\pi^2}\ln\left(\frac{s}{\Lambda^2}\right). \tag{10.14}$$

The s-, t-, and u-diagrams together would have given

$$T_2 = -\frac{\lambda^2}{32\pi^2}\left[\ln\left(\frac{s}{\Lambda^2}\right)+\ln\left(\frac{t}{\Lambda^2}\right)+\ln\left(\frac{u}{\Lambda^2}\right)\right],$$

which we have ignored here for simplicity.

The total matrix element reads

$$T(s) = -\lambda - \frac{\lambda^2}{32\pi^2}\ln\left(\frac{s}{\Lambda^2}\right), \tag{10.15}$$

which is divergent! However, this is not yet an observable, since we have not determined the λ that enters the expression.

Note: Differences are finite!

$$T(s_1) - T(s_2) = \frac{\lambda^2}{32\pi^2} \ln\left(\frac{s_2}{s_1}\right). \tag{10.16}$$

Shouldn't we expect $T(s)$ to be finite? After all

$$\sigma \sim |T|^2 \tag{10.17}$$

is a physical cross-section!
The key to the solution of this predicament is $\boldsymbol{\lambda}$, which we need to be more careful about. It characterizes the **strength** of the $\boldsymbol{\phi\phi \longrightarrow \phi\phi}$ **interaction**.

How can we measure it? By measuring the $\phi\phi \longrightarrow \phi\phi$ cross-section. **But** $\phi\phi \longrightarrow \phi\phi$ contains the $\lambda(...) + \lambda^2(...) + ...$ corrections as given above, and it is therefore impossible to extract λ from this.

Let us therefore **define** a **renormalized coupling**, λ_R, as the value of the total matrix element at the scale $s = s_0$:

$$\boxed{\lambda_R \equiv -T(s_0) = \lambda + \frac{\lambda^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + ...} \tag{10.18}$$

Since λ_R is observable and therefore must be finite, λ must be infinite to cancel the infinity from $\ln(\Lambda^2)$.

We can now solve for λ_R in perturbation theory:

$$\lambda = \lambda_R + a\lambda_R^2 + ... \tag{10.19}$$

We substitute this into Eq. (10.18):

$$\begin{aligned}
\lambda_R &= \left(\lambda_R + a\lambda_R^2 + ...\right) + \frac{(\lambda_R + a\lambda_R^2 + ...)^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + ... && (10.20)\\
&= \lambda_R + a\lambda_R^2 + \frac{\lambda_R^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + \mathcal{O}\left(\lambda_R^3\right). && (10.21)
\end{aligned}$$

Therefore

$$a = -\frac{1}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right). \tag{10.22}$$

and:

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + ... \tag{10.23}$$

This should be though of as an expansion in λ_R, even though the second term can be larger than the first as $\Lambda \longrightarrow \infty$.
Now we measure the cross-section at a different $s \neq s_0$. The T-matrix element is now

$$\begin{aligned}
T(s) &= -\lambda - \frac{\lambda^2}{32\pi^2} \ln\left(\frac{s}{\Lambda^2}\right) + ... && (10.24)\\
&= -\left[\lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln\left(\frac{s_0}{\Lambda^2}\right) + ...\right] - \frac{\lambda_R^2}{32\pi^2} \ln\left(\frac{s}{\Lambda^2}\right) + ... && (10.25)\\
&= -\lambda_R - \frac{\lambda_R^2}{32\pi^2} \ln\left(\frac{s}{s_0}\right) + ..., && (10.26)
\end{aligned}$$

which is **finite** order by order, and we therefore get a physical prediction.
The $\phi\phi \longrightarrow \phi\phi$ cross-section at the scale s differs from the one at the scale s_0 by logarithmic terms.

Note:

- λ_R is observable and therefore determines the exact cross-section at s_0.
- Logarithms are characteristic of loop effects, tree-level graphs give only rational polynomials in momenta and couplings for iT.
- This procedure is called **on-shell perturbation theory** (or physical perturbation theory).

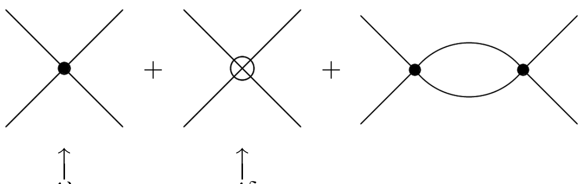
10.2 Counterterms

A simpler way to get the same result is to add **counterterms** to the Lagrangian, which are other interactions but infinite:

$$\mathcal{L} \longrightarrow \mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4\,, \tag{10.27}$$

where the counterterm is infinite ($\sim \ln(\Lambda^2)$) but formally of order λ_R^2.
The amplitude is:

$iT =$



(10.28)

$-i\lambda_R$ $\quad -i\delta_\lambda$

$$T(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{s}{\Lambda^2}\right) + \mathcal{O}\big(\lambda_R^4\big) \tag{10.29}$$

We adjust δ_λ such that the exact cross-section at s_0 comes out correctly:

$$\delta_\lambda = -\frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{s_0}{\Lambda^2}\right) \tag{10.30}$$

an infinite constant which cancels the third term in Eq. (10.29) for $s = s_0$.
Then

$$T(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{s}{s_0}\right), \tag{10.31}$$

which is finite and $T(s_0) = -\lambda_R$.
The procedure is called **renormalized perturbation theory**, i.e. using counterterms and expanding in the physical λ_R. It is equivalent to the on-shell perturbation theory but in practice the renormalized perturbation theory with counterterms is often much easier.
Note that this was a toy example and we have ignored two diagrams. This was just to show the general structure of the renormalization

procedure. In the following we will be back to doing actual calculations.

The Lagrangian before renormalization is often called the **bare Lagrangian**:

$$\mathcal{L} = \frac{1}{2}\Big(\partial_\mu \phi_0 \partial^\mu \phi_0 - m_0^2 \phi_0^2\Big) - \frac{\lambda_0}{4!}\phi_0^4 \,, \tag{10.32}$$

where ϕ_0 is the bare field (not to be confused with the free field) and λ_0, m_0 are the bare (unrenormalized) parameters.

10.3 Self-energy calculation

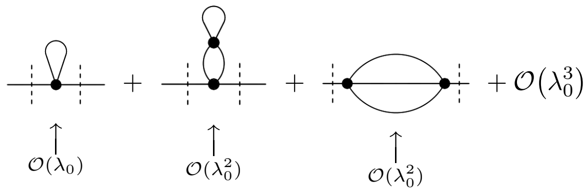
The two point function reads:

$$\int \mathrm{d}^4(x-y)\; e^{ip(x-y)} \langle \Omega | T\{\phi_0(x)\phi_0(y)\}|\Omega\rangle = \frac{i}{p^2 - m_0^2 - \Pi(p^2, m_0^2) + i\varepsilon}\,, \tag{10.33}$$

where, again, ϕ_0 is a solution of the **full** theory, **not** the free one, the subscript 0 here signals the bare quantities.
As before, we have

$$-i\Pi(p^2, m_0^2) = \tag{10.34}$$

$$= \tag{10.35}$$



$$= \underbrace{\frac{1}{2}}_{\text{symmetry factor}}(-i\lambda_0)\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{i}{k^2 - m_0^2 + i\varepsilon} + \mathcal{O}(\lambda_0^2)\,. \tag{10.36}$$

We perform a **Wick rotation** to determine this integral (we could also look for residues):

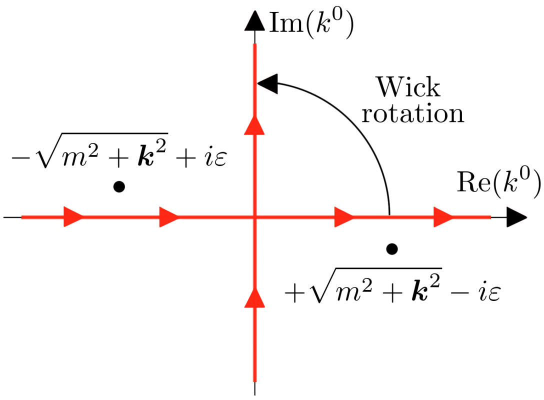


Figure 10.1: Sketch of the Wick rotation. There are no poles obstructing the rotation, so we can perform it.

where the poles are at:

$$0 = k^2 - m^2 + i\varepsilon \quad \Longrightarrow \quad E_\pm = \pm\sqrt{m^2 + \boldsymbol{k}^2 - i\varepsilon} = \pm\sqrt{m^2 + \boldsymbol{k}^2} \mp i\varepsilon\,. \tag{10.37}$$

Going from the Minkowskian over to the Euclidean metric

$$\eta^E_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10.38}$$

we have

$$k^0_E = -ik^0\,, \quad \boldsymbol{k_E} = \boldsymbol{k}\,, \tag{10.39}$$

$$k^2 = (k^0)^2 - \boldsymbol{k}^2 = -(k^0_E)^2 - \boldsymbol{k}^2_{\boldsymbol{E}} = -k^2_E \tag{10.40}$$

and therefore

$$\int \mathrm{d}^4k\; f(k^2) = \int_{-\infty}^{\infty} \mathrm{d}k^0 \int \mathrm{d}^3k\; f(k^2) = \int_{-i\infty}^{i\infty} \mathrm{d}k^0 \int \mathrm{d}^3k\; f(k^2) \tag{10.41}$$

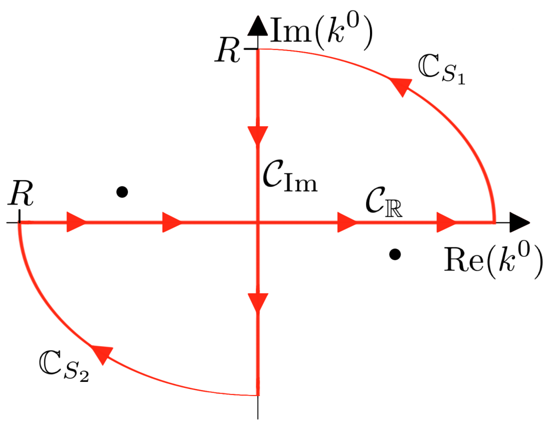
$$= i\int_{-\infty}^{\infty} \mathrm{d}k^0_E \int \mathrm{d}^3k\; f(k^2) = i\int_{-\infty}^{\infty} \mathrm{d}^4k_E\; f(-k^2_E)\,. \tag{10.42}$$

Thus

$$-i\Pi(p^2, m_0^2) = -i\frac{\lambda_0}{2}\cdot i\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{i}{-k^2_E - m_0^2 + i\varepsilon} \overset{(*)}{=} \tag{10.43}$$

and we can now drop the $i\varepsilon$ in the denominator since we are never close to the pole.

The Wick rotation works because we can analytically continue the integral into the complex plane using the following contour:



There are no poles inside the contour, so by the residue theorem

$$\oint \mathrm{d}^4k\; f(k^2) = 0\,, \tag{10.44}$$

or, equivalently, by dividing the contour into four pieces,

$$0 = \underbrace{\int_{\mathbb{C}_{S_1}} \mathrm{d}^4k\, f(k^2) + \int_{\mathbb{C}_{S_2}} \mathrm{d}^4k\, f(k^2)}_{=0 \text{ as } R\to\infty} + \int_{\mathcal{C}_{\mathbb{R}}} \mathrm{d}^4k\, f(k^2) + \int_{\mathcal{C}_{\mathrm{Im}}} \mathrm{d}^4k\, f(k^2)\,. \tag{10.45}$$

Therefore

$$\int_{\mathcal{C}_{\mathbb{R}}} \mathrm{d}^4k\, f(k^2) = -\int_{\mathcal{C}_{\mathrm{Im}}} \mathrm{d}^4k\, f(k^2) = \int_{-\mathcal{C}_{\mathrm{Im}}} \mathrm{d}^4k\, f(k^2) \tag{10.46}$$

and finally

$$\int_{-\mathcal{C}_{\mathrm{Im}}} \mathrm{d}^4k\, f(k^2) = i\int \mathrm{d}^4k_E\, f(-k_E^2)\,. \tag{10.47}$$

Therefore the procedure is justified.

Carrying on with the computation:

$$\overset{(*)}{=} -\frac{i\lambda_0}{2}\int \frac{\mathrm{d}^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m_0^2} = -\frac{i\lambda_0}{32\pi^2}\int_0^\infty \mathrm{d}k_E^2\,\frac{k_E^2}{k_E^2+m_0^2} \longrightarrow \infty\,. \tag{10.48}$$

$$\int \mathrm{d}^4k_E = \int_0^\infty \mathrm{d}k_E^2\,\frac{1}{2}k_E^2\int \mathrm{d}\Omega^{(4)}$$

where

$$\int \mathrm{d}\Omega^{(4)} = 2\pi^2$$

is the surface area of the unit sphere in four dimensions, and therefore

$$\int \mathrm{d}^4k_E = \pi^2\int_0^\infty \mathrm{d}k_E^2\, k_E^2\,.$$

We regularize this by applying a momentum cut-off, $|k_E| < \Lambda$:

$$-i\Pi(p^2, m_0^2) = -\frac{i\lambda_0}{32\pi^2}\left[\Lambda^2 - m_0^2\ln\left(\frac{\Lambda^2+m_0^2}{m_0^2}\right)\right] \tag{10.49}$$

$$= -\frac{i\lambda_0}{32\pi^2}\left[\Lambda^2 - m_0^2\ln\left(\frac{\Lambda^2}{m_0^2}\right) + \mathcal{O}\left(\frac{m_0^4}{\Lambda^2}\right)\right] \tag{10.50}$$

with $\Lambda \gg m_0$.
The physical mass and field strength renormalization are given by, see Eq. (9.155),

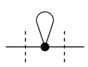
$$m^2 - m_0^2 - \Pi(p^2 = m^2, m_0^2) = 0 \tag{10.51}$$

$$Z = \left.\frac{1}{1-\frac{\partial\Pi}{\partial p^2}}\right|_{p^2=m^2} \tag{10.52}$$

and plugging in the above result for $\Pi(p^2, m_0^2)$ we obtain using

$$m^2 = m_0^2 + \frac{\lambda_0}{32\pi^2}\left[\Lambda^2 - m_0^2\ln\left(\frac{\Lambda^2}{m_0^2}\right) + \ldots\right] \tag{10.53}$$

$$Z = 1 + \mathcal{O}\big(\lambda_0^2\big) \tag{10.54}$$

The fact that $Z = 1$ at one-loop order is peculiar to ϕ^4-theory, since the diagram  does not depend on p^2, and therefore $\frac{\partial\Pi}{\partial p^2} = 0$ at the one loop. p^2 enters the expressions starting at two loops.

10.4 Renormalization paradigm

Where do the divergences come from? They are due to momentum modes k which are much larger than the masses and external momenta. Why do we need to integrate k up to infinity? Because interactions are **local** (large k, small x) and particles are treated as point-like. We can get an intuitive understanding by considering a local 3-point vertex,

$$S \supset \int d^4x\, \phi(x)^3$$
$$= \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\tilde{\phi}(k_1)\tilde{\phi}(k_2)\tilde{\phi}(k_3)(2\pi)^4\delta^{(4)}(k_1+k_2+k_3)$$

where we see in Fourier space, that a low-momentum excitation $k_1 \approx 0$ will be coupled without any suppression to high-momentum modes k_2 and $k_3 \approx -k_2$. Local couplings require the existence of infinitely many degrees of freedom at very large momenta, with 'hard interactions'[2] and therefore ultraviolet divergences are never far. Deeper physical consequences of quantum field theory arise from this circumstance like the running of couplings.

In Fourier space

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx}\tilde{\phi}(k)$$

[2] coupling low to high momentum modes unsuppressed

We have not measured these infinitely energetic momentum modes. The observations in the lab go up to a finite $E < E_{\text{max}}$, or until a minimal scale $x_{\text{min}} \sim \frac{1}{E_{\text{max}}}$, whereas from the theoretical side we have a theory defined by $\mathcal{L}(\phi_0, m_0, \lambda_0)$, which is local, Lorentz-invariant to all energies and we extrapolate our theory up to $E \longrightarrow \infty$. But at the very least, with increasing energies, new particles which are **not** included in our theory will appear or our theory could have to be replaced by something entirely different (maybe string theory).

This observation motivates the **modern view** of QFTs: a QFT should be regarded as an effective low-energy theory which is valid up to some scale Λ. This can be condensed into the motto "*ignorance is no shame*".

But then how can we predict anything?
Observables measured at $E \ll \Lambda$ cannot depend in an essential way on physics at much higher energies (or smaller distances) than E, otherwise one could resolve arbitrarily small structures using finite energy, in contradiction to the uncertainty principle.
The renormalization procedure looks as follows:

$$\mathcal{L}(\phi_0, m_0, \lambda_0) \xrightarrow{\substack{\text{replace bare parameters } m_0,\lambda_0 \\ \text{by } m_R(m_0,\lambda_0), \lambda_R(m_0,\lambda_0) \\ \text{(physical renormalization parameters)}}} f(m,\lambda) \quad (10.55)$$

were the observables m, λ should be independent of the used regularization procedure.

1) The definition of $m_R(m_0, \lambda_0)$ and $\lambda_R(m_0, \lambda_0)$ depends on the used regularization (see the computation of λ_R in Eq. (10.18)).

2) Observables should be independent of the regularization (see the independence of $T_2(s)$ on Λ in Eq. (10.26)).

λ_0, m_0 are auxiliary parameters and almost any regularisation procedure can be chosen, we choose the one which makes computations simplest. A regularisation (say cutting off momenta, $|k| < \Lambda$), is a UV deformation of the theory, affecting processes at arbitrarily high energies about which we are ignorant anyway.

10.5 Regularization methods: cut-off and dimensional regularization

A general integral like

$$A(a,\Delta) \equiv \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{(k^2-\Delta+i\varepsilon)^a} = (-1)^a \cdot i \int \frac{\mathrm{d}^4k_E}{(2\pi)^4} \frac{1}{(k_E^2+\Delta)^a} \tag{10.56}$$

with a cut-off regularization

$$\int \mathrm{d}^4k_E \quad \longrightarrow \quad \pi^2 \int_0^{\Lambda^2} \mathrm{d}k_E^2 \; k_E^2 \quad \text{(like before)}\,, \tag{10.57}$$

$$A(a,\Delta) = i(-1)^a \frac{\pi^2}{(2\pi)^4} \int_0^{\Lambda^2} \mathrm{d}k_E^2 \; \frac{k_E^2}{(k_E^2+\Delta)^a}\,. \tag{10.58}$$

This integral is finite for $a > 2$, since

$$A(a,\Delta) \sim \int \mathrm{d}k_E^2 \frac{k_E^2}{(k_E^2)^a}\,. \tag{10.59}$$

We have already discussed the case $a = 1$, in which it is quadratically divergent.
For $a = 2$:

$$A(2,\Delta) = \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2+\Delta}{\Delta}\right) - \frac{\Lambda^2}{\Delta+\Lambda^2}\right] \tag{10.60}$$

$$= \frac{i}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{\Delta}\right) - 1 + \mathcal{O}\left(\frac{\Delta}{\Lambda^2}\right)\right]. \tag{10.61}$$

This is logarithmically divergent.

10.5.1 Dimensional regularization

Also called dim.reg.

Assume that space-time and momentum space are d-dimensional, k_E^μ is then a d-dimensional vector. Then we have

$$\int \frac{\mathrm{d}^dk}{(2\pi)^d} = \int \frac{\mathrm{d}\Omega_d}{(2\pi)^d} \int_0^\infty \mathrm{d}k^2 \; \frac{1}{2}\left(k^2\right)^{\frac{d}{2}-1} \tag{10.62}$$

$$= \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{1}{\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \mathrm{d}k_E^2 \; \left(k^2\right)^{\frac{d}{2}-1}\,, \tag{10.63}$$

where we have used that the surface area of a d-dimensional unit sphere is

$$\int \mathrm{d}\Omega_d = \frac{(\pi)^{\frac{d}{2}}}{\frac{1}{2}\Gamma\left(\frac{d}{2}\right)}\,. \tag{10.64}$$

The surface of the d-dimensional unit sphere can be calculated in the following way:

$$\begin{aligned}(\sqrt{\pi})^d &= \left(\int_{-\infty}^\infty \mathrm{d}x \; e^{-x^2}\right)^d \\ &= \int \mathrm{d}^dx \; \exp\left(-\sum_{i=1}^d x_i^2\right) \\ &= \int_0^\infty \mathrm{d}x \; x^{d-1}e^{-x^2} \int \mathrm{d}\Omega_d \\ &= \frac{1}{2}\Gamma\left(\frac{d}{2}\right)\int \mathrm{d}\Omega_d\,,\end{aligned}$$

where we used the definition of the Euler Gamma function

$$\Gamma(z) \equiv \int_0^\infty \mathrm{d}y \; y^{z-1}e^{-y}\,,$$

with the choice $y = x^2$ and $z = \frac{d}{2}$. Therefore we obtain the desired result

$$\int \mathrm{d}\Omega_d = \frac{(\pi)^{\frac{d}{2}}}{\frac{1}{2}\Gamma\left(\frac{d}{2}\right)}\,.$$

We transition from 4 to d dimensions by

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \longrightarrow \mu^{4-d} \int \frac{\mathrm{d}^dk}{(2\pi)^d}\,, \tag{10.65}$$

where we have introduced an arbitrary constant μ ($[\mu] = 1$) to keep the mass dimension of the integral the same.
Therefore we have

$$A(a,\Delta) = \frac{i}{(4\pi)^{\frac{d}{2}}} \cdot \frac{(-1)^a}{\Gamma\left(\frac{d}{2}\right)} \mu^{4-d} \int_0^\infty \mathrm{d}l \frac{l^{\frac{d}{2}-1}}{(l+\Delta)^a} \overset{(*)}{=} \tag{10.66}$$

with $l \equiv k^2$. Note that $A(a,\Delta)$ is convergent when $\frac{d}{2} < a$:

$$\frac{l^{\frac{d}{2}-1}}{(l+\Delta)^a} \sim l^{\frac{d}{2}-a-1} \quad \text{for} \quad l \gg \Delta\,. \tag{10.67}$$

<u>**Idea:**</u> Compute the integral for $\frac{d}{2} < a$ and then **analytically continue** it (in complex d-space) to $d = 4$.

$$\overset{(*)}{=} \frac{i}{(4\pi)^{\frac{d}{2}}} \frac{(-1)^a}{\Gamma\big(\frac{d}{2}\big)} \Delta^{\frac{d}{2}-a} \mu^{4-d} \int_0^1 \mathrm{d}x\, x^{a-1-\frac{d}{2}} \bar{x}^{\frac{d}{2}-1}\,, \tag{10.68}$$

where we have introduced

$$x = \frac{\Delta}{l+\Delta}\,, \qquad \bar{x} = 1-x \tag{10.69}$$

and used

$$l = \Delta\frac{1-x}{x}\,, \quad \mathrm{d}l = -\Delta\frac{1}{x^2}\mathrm{d}x\,, \tag{10.70}$$

$$l(x=0) = \infty\,, \quad l(x=1) = 0\,, \tag{10.71}$$

which means for the integrand

$$\begin{aligned} \mathrm{d}l\frac{l^{\frac{d}{2}-1}}{(l+\Delta)^a} &= -\mathrm{d}x\Delta\frac{1}{x^2} \cdot \Delta^{\frac{d}{2}-1}\bar{x}^{\frac{d}{2}-1}x^{1-\frac{d}{2}} \cdot \Delta^{-a}x^a \\ &= -\mathrm{d}x\Delta^{\frac{d}{2}-a}x^{a-1-\frac{d}{2}}\bar{x}^{\frac{d}{2}-1} \end{aligned} \tag{10.72}$$

Now we can use the identity

$$\int_0^1 \mathrm{d}x\, x^{\alpha-1}\bar{x}^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,, \tag{10.73}$$

choosing $\alpha = a - \frac{d}{2}$ and $\beta = \frac{d}{2}$, we finally obtain

$$\boxed{A(a,\Delta) = \frac{i}{(4\pi)^2} \frac{\Gamma\big(a-\frac{d}{2}\big)}{\Gamma(a)} \left(\frac{\Delta}{4\pi\mu^2}\right)^{\frac{d}{2}-2} (-\Delta)^{2-a}\,.} \tag{10.74}$$

Any loop integral, after some massaging, can be brought into this form.

For the following we will need some properties of the **Gamma-function** which is an extension of the factorial to complex numbers. It is defined as

$$\Gamma(z) = \int_0^\infty \mathrm{d}t\, t^{z-1} e^{-t} \; : \tag{10.75}$$

1) $\Gamma(x+1) = x\Gamma(x)$

2) $\Gamma(n+1) = n!$ with $n \in \mathbb{N}$

3) $\Gamma(x)$ is analytic with simple poles at $x = 0, -1, -2, \ldots$

4)
$$\Gamma(1+\epsilon) = \exp\left(-\epsilon\gamma_E + \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_n \epsilon^n \right) \tag{10.76}$$

with the Euler-Mascheroni-constant $\gamma_E = 0.5772\ldots$ and the Riemann Zeta-function

$$\zeta_k = \sum_{m=1}^{\infty} \frac{1}{m^k} \,. \tag{10.77}$$

Using property 1., we obtain

$$\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)\,, \tag{10.78}$$

$$\Longrightarrow \Gamma(\epsilon) = \frac{1}{\epsilon}\Gamma(1+\epsilon) \overset{4.}{=} \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\,. \tag{10.79}$$

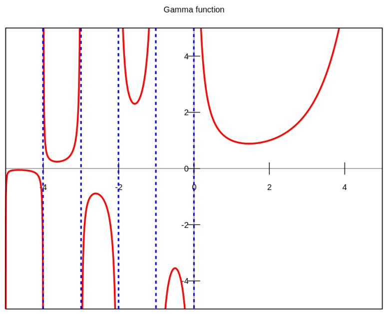


Figure 10.2: The gamma function along part of the real axis.

Back to our integral, Eq. (10.74):

$$A(a,\Delta) = \frac{i}{(4\pi)^2} \frac{\Gamma\left(a - \frac{d}{2}\right)}{\Gamma(a)} \left(\frac{\Delta}{4\pi\mu^2} \right)^{\frac{d}{2}-2} (-\Delta)^{2-a} \tag{10.80}$$

is singular for $d = 4$ at $a = 1, 2$, because, choosing $d = 4 - 2\epsilon$ in the Gamma function $\Gamma\left(a - \frac{d}{2}\right)$, leads to

$$\Gamma(a-2+\epsilon) = \begin{cases} a = 1: & \Gamma(-1+\epsilon) \overset{1.}{=} \frac{\Gamma(\epsilon)}{-1+\epsilon} \approx -\Gamma(\epsilon) + \ldots \\ a = 2: & \Gamma(\epsilon) \end{cases} \tag{10.81}$$

and we have already seen that $\Gamma(\epsilon)$ diverges like $\frac{1}{\epsilon}$ for $\epsilon \longrightarrow 0$, i.e. for $d \longrightarrow 4$.

The original integral diverges for **all $a \leq 2$**, but dimensional regularization and analytic continuation in d gives finite values except for $a = 1, 2$. To take care of the divergent choices $a = 1, 2$, we expand A around $d = 4 - 2\epsilon$ with $\epsilon \ll 1$:

$$A(a=1,\Delta) = \frac{i}{(4\pi)^2} (-\Delta) \underset{\substack{\uparrow \\ =-\Gamma(\epsilon)}}{\frac{\Gamma(-1+\epsilon)}{\Gamma(1)}} \left(\frac{\Delta}{4\pi\mu^2} \right)^{-\epsilon} \tag{10.82}$$

and we now use

$$\Gamma(-1+\epsilon) = \frac{\Gamma(\epsilon)}{-1+\epsilon} = -(1+\epsilon+\ldots)\Gamma(\epsilon) = -\left(\frac{1}{\epsilon} - \gamma_E + 1 + \mathcal{O}(\epsilon) \right),$$

$$x^\epsilon = e^{\ln(x^\epsilon)} = e^{\epsilon \ln(x)} = 1 + \epsilon \ln(x) + \ldots$$

and therefore

$$A(1,\Delta) = \frac{i}{(4\pi)^2}\Delta\left(\frac{1}{\epsilon} - \gamma_E + 1 + \mathcal{O}(\epsilon)\right)\left(1 - \epsilon \ln\left(\frac{\Delta}{4\pi\mu^2}\right) + ...\right) \tag{10.83}$$

$$= \frac{i}{(4\pi)^2}\Delta\left(\frac{1}{\epsilon} - \ln\left(\frac{\Delta}{4\pi e^{-\gamma_E}\mu^2}\right) + 1 + \mathcal{O}(\epsilon)\right) \tag{10.84}$$

and, similarly, for $a = 2$

$$A(2,\Delta) = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \ln\left(\frac{\Delta}{4\pi e^{-\gamma_E}\mu^2}\right) + \mathcal{O}(\epsilon)\right). \tag{10.85}$$

We defined a new scale $\widetilde{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$, since this combination shows up frequently.

Comparing to the cut-off regularization, we find the correspondence

$$\frac{1}{\epsilon} - \ln\left(\frac{\Delta}{\widetilde{\mu}^2}\right) \longleftrightarrow \ln\left(\frac{\Lambda^2}{\Delta}\right). \tag{10.86}$$

However, the constant terms differ and there is no explicit quadratic divergence in dimensional regularization.

10.5.2 Feynman parameters

If we have to calculate integrals with more than one internal propagator, we can use the following trick to combine them into one numerator

$$\begin{aligned}&\frac{1}{P_1^{a_1}P_2^{a_2}\cdot ... \cdot P_n^{a_n}}\\ &= \frac{\Gamma(a_1+a_2+...+a_n)}{\Gamma(a_1)\Gamma(a_2)\cdot ... \cdot\Gamma(a_n)}\int_0^1 \mathrm{d}x_1...\mathrm{d}x_n\ \delta(1-x_2-x_2-...-x_n)\cdot\\ &\cdot \frac{x_1^{a_1-1}x_2^{a_2-1}\cdot ... \cdot x_n^{a_n-1}}{(x_1P_1+x_2P_2+...+x_nP_n)^{a_1+a_2+...+a_n}},\end{aligned}$$

where e.g.

$$P_1^{a_1} = (p^2 - m^2 + i\varepsilon)^{a_1} \tag{10.87}$$

and the x_i are called **Feynman parameters**.

Applications:

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\ \frac{1}{(xA+(1-x)B)^2}, \tag{10.88}$$

$$\frac{1}{A^aB^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \mathrm{d}x\ \frac{x^{a-1}\bar{x}^{b-1}}{(xA+\bar{x}B)^{a+b}}, \tag{10.89}$$

$$\frac{1}{A^aB^bC^c} = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^1 \mathrm{d}x\mathrm{d}y\ \frac{(xy)^{a-1}(x\bar{y})^{b-1}(\bar{x})^{c-1}}{(xyA+x\bar{y}B+\bar{x}C)^{a+b+c}}, \tag{10.90}$$

with $\bar{x} \equiv 1-x$ and $\bar{y} \equiv 1-y$.

10.5.3 *Calculating the integral from the introduction*

With these new tools at our disposal, we can now easily calculate the integral Eq. (10.11)

$$I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2} \cdot \frac{(p-k)\cdot p}{(p-k)^4} = \frac{i}{16\pi^2}\,, \tag{10.91}$$

from the introduction. First, we use Feynman parametrization Eq. (10.89) and complete the square in the denominator with $\tilde{k} = k - xp$

$$I = \frac{\Gamma(3)}{\Gamma(1)\Gamma(2)} \int_0^1 dx \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{x(p-k)\cdot p}{(x(p-k)^2 + \bar{x}k^2)^3} \tag{10.92}$$

$$= 2\int_0^1 dx \int \frac{\mathrm{d}^4 \tilde{k}}{(2\pi)^4} \frac{x(p^2 - p\cdot\tilde{k} - xp^2)}{(\tilde{k}^2 + x\bar{x}p^2)^3} \tag{10.93}$$

$$= 2x\bar{x}p^2 \int_0^1 dx \int \frac{\mathrm{d}^4 \tilde{k}}{(2\pi)^4} \frac{1}{(\tilde{k}^2 + x\bar{x}p^2)^3} \tag{10.94}$$

where in the last step we have used that $\int \tilde{k}_\mu f(\tilde{k}^2) = 0$ due to anti-symmetry of the integrand. Now we can plug it into our master formula Eq. (10.74) with $a = 3$ and $\Delta = -x\bar{x}p^2$,

$$I = i(4\pi)^{-2+\epsilon}\,\Gamma(1+\epsilon) \int_0^1 dx \underbrace{\left(\frac{p^2\bar{x}x}{\mu^2}\right)^{-\epsilon}}_{1-\epsilon\ln(\ldots)}$$

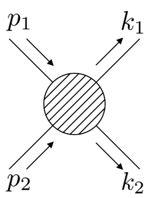
The integral is finite, we can directly take $\epsilon \to 0$ and we can perform the trivial x integration to derive

$$I = \frac{i}{16\pi^2} + \mathcal{O}(\epsilon)$$

Voilà!

10.6 *Revisiting $\phi\phi \longrightarrow \phi\phi$ scattering at one loop*

$$\langle k_1, k_2; \mathrm{out}|p_1, p_2; \mathrm{in}\rangle\Big|_{\mathrm{connected}} =$$



$$\Big|_{\substack{\text{connected,}\\ \text{amputated,}\\ \text{on-shell}}} \tag{10.95}$$

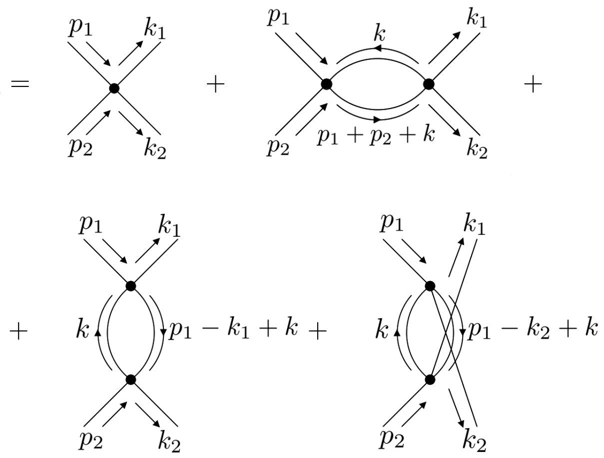
$$= (2\pi)^4\delta^{(4)}(k_1 + k_2 - p_1 - p_2)\underbrace{\widetilde{G}^{(4)}_{\mathrm{amp}}}_{=iT_{\alpha\beta}}(k_1, k_2, -p_1, -p_2)\Big|_{k_i^2 = p_i^2 = m^2} \tag{10.96}$$

and at one loop

$$\sqrt{Z} = 1 + \mathcal{O}(\lambda^2)\,. \tag{10.97}$$

The amputated Green function is

$\widetilde{G}^{(4)}_{\text{amp}} =$



(10.98)

$$
\begin{aligned}
&= -i\lambda_0 + (-i\lambda_0)^2 \cdot \frac{1}{2}\left[\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2 - m_0^2 + i\varepsilon}\cdot\right.\\
&\cdot \frac{i}{(p_1+p_2+k)^2 - m_0^2 + i\varepsilon} + (p_1+p_2 \longleftrightarrow p_1 - k_1) + \qquad (10.99)\\
&\left. + (p_1+p_2 \longleftrightarrow p_1 - k_2)\right].
\end{aligned}
$$

In dim. reg. the dimensions of the fields and parameters are d-dependent:

$$[\mathcal{L}] = d\,, \quad [\partial_\mu] \overset{\text{by def.}}{=} 1\,, \quad [m_0] = 1\,, \tag{10.100}$$

$$[\partial_\mu\phi_0\partial^\mu\phi_0] = d \quad \Longrightarrow \quad [\phi_0] = \frac{d}{2} - 1\,, \tag{10.101}$$

$$[\lambda_0\phi_0^4] = d \quad \Longrightarrow \quad [\lambda_0] = d - 4\left(\frac{d}{2} - 1\right) = 4 - d = 2\epsilon\,. \tag{10.102}$$

Therefore the bare coupling λ_0 is no longer dimensionless! We must substitute

$$\lambda_0 \longrightarrow \lambda_0\widetilde{\mu}^{-2\epsilon} \quad \text{such that} \quad [\lambda_0\widetilde{\mu}^{-2\epsilon}] = 0\,. \tag{10.103}$$

We now evaluate the integral in Eq. (10.99) in dim. reg. using Feynman parameters:

$$\widetilde{\mu}^{4-d}\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2 - m_0^2 + i\varepsilon)((p+k)^2 - m_0^2 + i\varepsilon)} \tag{10.104}$$

$$= \widetilde{\mu}^{4-d}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(xk^2 + \bar{x}(p+k)^2 - m_0^2 + i\varepsilon)^2} \overset{(*)}{=} \tag{10.105}$$

with $p \equiv p_1 + p_2$. We rewrite the denominator as

$$k^2 + 2\bar{x}k\cdot p + \bar{x}p^2 - m_0^2 + i\varepsilon = (k + \bar{x}p)^2 + x\bar{x}p^2 - m_0^2 + i\varepsilon \tag{10.106}$$

and change the integration variable to $k' \equiv k + \bar{x}p$:

$$\overset{(*)}{=} \widetilde{\mu}^{4-d} \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \frac{1}{(k'^2 - [m_0^2 - x\bar{x}p^2 - i\varepsilon])^2} \tag{10.107}$$

$$\overset{A(2,\Delta)}{=} \frac{i}{(4\pi)^2} \left[\frac{1}{\epsilon} - \int_0^1 \mathrm{d}x \ln\left(\frac{m_0^2 - x\bar{x}p^2 - i\varepsilon}{\widetilde{\mu}^2} \right) + \mathcal{O}(\epsilon) \right] \tag{10.108}$$

$$= \frac{i}{(4\pi)^2} \Bigg[\underbrace{\frac{1}{\epsilon} - \ln\left(\frac{m_0^2}{\widetilde{\mu}^2} \right)}_{\substack{=\ln\left(\frac{\Lambda^2}{m_0^2}\right)-1 \\ \text{in cut-off reg.}}} - \underbrace{\int_0^1 \mathrm{d}x \, \ln\left(1 - x\bar{x}\frac{p^2}{m_0^2} - i\varepsilon \right)}_{\equiv A(p^2)} + \mathcal{O}(\epsilon) \Bigg] . \tag{10.109}$$

Notice the distinction between the dimensional regularisation ϵ and the ε in the propagators.
The result is different to the previously studied $\phi\phi \longrightarrow \phi\phi$ scattering, since there we set the bare mass $m_0 = 0$ for simplicity, while here we do not. Hence, the amputated Green function is

$$\widetilde{G}^{(4)}_{\text{amp}} = -i\lambda_0 \left\{ 1 - \frac{\lambda_0 \overset{\substack{\text{to get the correct} \\ \text{mass dimension} \\ \text{of the coupling} \\ \downarrow}}{\widetilde{\mu}^{-2\epsilon}}}{32\pi^2} \left[\frac{3}{\epsilon} - 3\ln\left(\frac{m_0^2}{\widetilde{\mu}^2} \right) - A(s) - A(t) - A(u) \right] + \mathcal{O}(\lambda_0^2) \right\} \tag{10.110}$$

and it is divergent for $\epsilon \longrightarrow 0$.

The renormalized coupling defines the physical interaction strength through the value of the $2 \longrightarrow 2$ scattering amplitude at a fixed s, t and u (e.g. $s = 4m^2$, $t = u = 0$):

$$\left(\sqrt{Z}\right)^4 \widetilde{G}^{(4)}_{\text{amp}} \Big|_{\substack{s=4m^2 \\ t=u=0}} \equiv -i\lambda_R \widetilde{\mu}^{+2\epsilon} , \tag{10.111}$$

where the factor $\widetilde{\mu}^{2\epsilon}$ is there to allow us to set $[\lambda_R] = 0$, i.e. it is only relevant in dimensional regularization.
Therefore, by plugging Eq. (10.110) into Eq. (10.111) we obtain

$$\lambda_R \widetilde{\mu}^{2\epsilon} = \lambda_0 \left\{ 1 - \frac{\lambda_0 \widetilde{\mu}^{-2\epsilon}}{32\pi^2} \left[\frac{3}{\epsilon} - 3\ln\left(\frac{m_0^2}{\widetilde{\mu}^2} \right) - A\left(4m^2\right) \right] + \mathcal{O}\left(\lambda_0^2\right) \right\} .$$

Now we express the scattering amplitude in terms of λ_R, as was done in the introduction of this chapter, and obtain

$$\left(\sqrt{Z}\right)^4 \widetilde{G}^{(4)}_{\text{amp}} = -i\lambda_R \widetilde{\mu}^{2\epsilon} \left[1 - \frac{\lambda_R}{32\pi^2} \Big(A(s) + A(t) + A(u) - A(4m^2) \Big) + \mathcal{O}\left(\lambda_R^2\right) \right] .$$

The divergence vanishes and the limit $\epsilon \longrightarrow 0$ is finite.

We can now predict $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ including quantum corrections, i.e. loop diagrams and the final result is independent of the regularisation.

This worked because the UV divergence $\frac{1}{\epsilon}$ was independent of the kinematics. This is the case because it is due to loop momenta $k \gg m_0, p$ which we can set to 0 as $k \longrightarrow \infty$.

The definition of λ_R is not unique, but we can relate different λ_R to each other, e.g.

$$\left(\sqrt{Z}\right)^4 \widetilde{G}^{(4)}_{\text{amp}} \Big|_{s=t=u=-4v^2} = -i\lambda'_R \widetilde{\mu}^{2\epsilon} , \tag{10.112}$$

with v being some scale, then the analogous calculation as above leads to

$$(\sqrt{Z})^4 \widetilde{G}^{(4)}_{\text{amp}} = -i\lambda'_R\left[1-\frac{\lambda'_R}{32\pi^2}\Big(A(s)+A(t)+A(u)-3A(-4v^2)\Big)+\mathcal{O}\big({\lambda'_R}^2\big)\right]. \tag{10.113}$$

But since

$$\lambda'_R = \lambda_R\left[1+\frac{\lambda_R}{32\pi^2}\Big(3A\big(-4v^2\big)-A(4m^2)\Big)+\mathcal{O}\big(\lambda_R^2\big)\right], \tag{10.114}$$

the numerical prediction for $\widetilde{G}^{(4)}_{\text{amp}}$ is the **same** if we relate λ_R and λ'_R to each other according to Eq. (10.114) consistently.

In actual calculations we use renormalized perturbation theory, which never uses explicit bare parameters. It is usually simpler for bookkeeping, especially at higher orders.

1) Write

$$\phi_0 = \sqrt{Z}\phi_R\,, \quad m_0^2 = m_R^2+\delta m^2\,, \quad \lambda_0 = Z_\lambda\lambda_R\widetilde{\mu}^{2\epsilon}\,. \tag{10.115}$$

2) Since $Z-1$, $Z_\lambda - 1$, δm^2 are at least $\mathcal{O}(\lambda)$, we can write the Lagrangian in terms of renormalized fields and parameters (we will drop the R from now on, i.e. $\lambda_R \longrightarrow \lambda$ etc.):

$$\mathcal{L} = \frac{1}{2}\Big(\partial_\mu\phi_0\partial^\mu\phi_0 - m_0^2\phi_0^2\Big) - \frac{\lambda_0}{4!}\phi_0^4 \tag{10.116}$$

$$= \frac{1}{2}Z\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z\big(m^2+\delta m^2\big)\phi^2 - Z_\lambda Z^2\widetilde{\mu}^{2\epsilon}\frac{\lambda}{4!}\phi^4 \tag{10.117}$$

$$= \underbrace{\frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\Big) - \frac{\lambda}{4!}\phi^4}_{\equiv\mathcal{L}_r} + \tag{10.118}$$

$$+\underbrace{\frac{1}{2}(Z-1)\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\Big((Z-1)m^2+Z\delta m^2\Big)\phi^2 - \big(Z_\lambda Z^2-1\big)\frac{\lambda\widetilde{\mu}^{2\epsilon}}{4!}\phi^4}_{\equiv\mathcal{L}_{c.t.}}$$

$$= \mathcal{L}_r + \mathcal{L}_{c.t}\,, \tag{10.119}$$

where $\mathcal{L}_r$ has the same functional form as $\mathcal{L}$ and $\mathcal{L}_{c.t}$ contains the counterterms, which we treat as "new interactions".

3) Compute the Green function using $\mathcal{L}_r$ and $\mathcal{L}_{c.t}$. The Green functions are directly the renormalized ones (only $\phi = \phi_R$ appears) and e.g. the Feynman propagator is

$$\frac{i}{p^2-m^2+i\varepsilon} \tag{10.120}$$

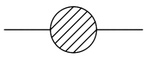
(and not with m_0 the bare mass). The result is directly in terms of m, λ.

The **counterterms**

$$\delta_Z \equiv Z-1\,, \quad \delta_m \equiv (Z-1)m^2+Z\delta m^2\,, \quad \delta_\lambda \equiv Z_\lambda Z^2 - 1\,, \tag{10.121}$$

will have to be determined oder by order in λ using 3 renormalization conditions.

The above used **renormalization scheme** corresponds to

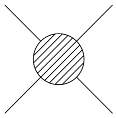

$$= \frac{i}{p^2 - m^2 - \Pi(p^2, m^2) + i\varepsilon}$$

exact, renormalized two-point function

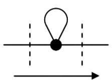
$$\xrightarrow{p^2 \to m^2} \frac{i}{p^2 - m^2 + i\varepsilon} \quad \textbf{to all orders in } \boldsymbol{\lambda}! \tag{10.122}$$

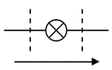
$$\Pi(p^2 = m^2, m^2) = 0 \quad \text{and} \quad \left.\frac{\partial \Pi}{\partial p^2}\right|_{p^2 = m^2} = 0\,. \tag{10.123}$$

We also have


$$\Bigg|_{\substack{\text{amputated,} \\ s=4m^2, t=u=0}} = -i\lambda\widetilde{\mu}^{2\epsilon} \quad \textbf{to all orders in } \boldsymbol{\lambda}! \tag{10.124}$$

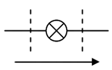
Explicitly:

$$-i\Pi(p^2, m^2) =$$ 

$+$ 

$+ \ldots$ (10.125)

The Feynman rule for the counterterm above is



$$= -i\big(\delta_Z p^2 + \delta_m\big) \tag{10.126}$$

and we already calculated the expression corresponding to the first diagram in Eq. (10.50). Therefore

$$-i\Pi(p^2, m^2) = -\frac{i\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^2}\right)\right] - i\big(\delta_Z p^2 + \delta_m\big) \tag{10.127}$$

and, setting $p^2 = m^2$,

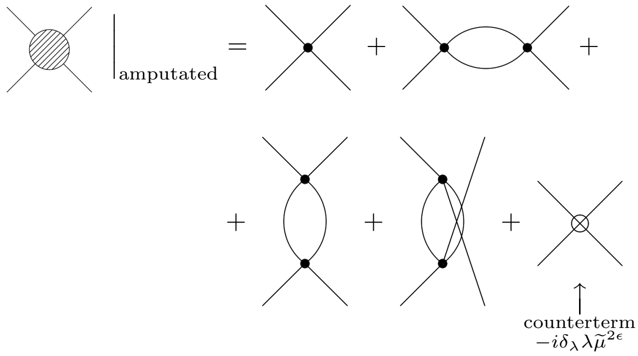
$$\delta_Z p^2 + \delta_m = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^2}\right)\right]. \tag{10.128}$$

By comparing coefficients we read off

$$\delta_Z = 0 \tag{10.129}$$

$$\Longrightarrow \delta_m \underset{\text{up to } \mathcal{O}(\lambda)}{=} \delta m^2 = -\frac{\lambda}{32\pi^2}\left[\Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) + \mathcal{O}\left(\frac{m^4}{\Lambda^2}\right)\right], \tag{10.130}$$

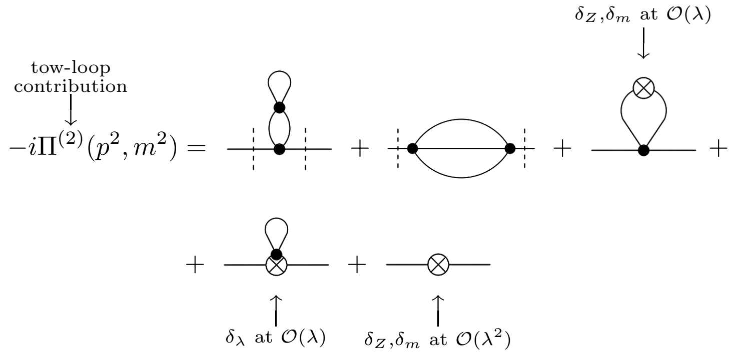
in agreement with Eq. (10.54).



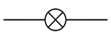
(10.131)

$$= -i\lambda \widetilde{\mu}^{2\epsilon} \tag{10.132}$$

determines δ_λ at $\mathcal{O}(\lambda)$.
To next order in perturbation theory:



(10.133)

If the scheme is consistent, the condition Eq. (10.123) determines δ_m, δ_Z at $\mathcal{O}(\lambda^2)$ from the two-loop contribution $\Pi^{(2)}$. The left-over divergence  must have the correct form $(ap^2 + b)$, otherwise $\Pi^{(2)}$ could not be made finite using the renormalization of m, λ and ϕ.

10.7 Renormalization schemes

We so far defined m, Z as the location and residue of the pole of the exact two-point function. Let us call these m_{phys}, Z_{OS} (on-shell).
We can choose other **renormalization schemes**; this amounts to a **reparametrization** of the theory (there is no change in the physical observables). For the bare field we write

$$\phi_0 = \sqrt{Z_{\text{OS}}}\phi_{\text{OS}} = \sqrt{Z}\phi \tag{10.134}$$

and for the bare mass

$$m_0^2 = m_{\text{phys}}^2 + \delta m_{\text{phys}}^2 = m^2 + \delta m^2 \,, \tag{10.135}$$

where the quantities without a subscript belong to a different scheme.
We relate the quantities in the differnt schemes through

$$m_{\text{phys}} = m_{\text{phys}}(m, \lambda) \,, \quad Z_{\text{OS}} = Z_{\text{OS}}(m, \lambda) \tag{10.136}$$

and all the quantities are free of divergences, similarly to the relationship between λ and λ'.

Comments:

- Irrespective of the convention for the renormalized mass: $\widetilde{G}_{\text{amp}}$ is computed for on-shell momenta $p_i^2 = m_{\text{phys}}^2$.

- The factor Z in the LSZ reduction formula is the residue Z_{OS} of the bare two-point function. In a different scheme (e.g. ϕ vs. ϕ_{OS}) we must multiply $\widetilde{G}_{\text{amp}}$ by $\left(\sqrt{\frac{Z_{\text{OS}}}{Z}}\right)^n$. Note that

$$\frac{Z_{\text{OS}}}{Z} = 1 + \mathcal{O}(\lambda) \tag{10.137}$$

  and it is free of divergences.

10.7.1 The $\overline{\text{MS}}$-scheme

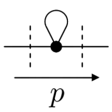
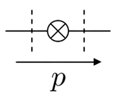
$\overline{\text{MS}}$="m-s-bar", i.e. minimal subtraction.

This is also very useful if the particles cannot appear as external states: the light quarks (u, d, s) appear only as mesons or hadrons! They never appear on-shell. See also: the renormalon problem of perturbation theory https://en.wikipedia.org/wiki/Renormalon

The $\overline{\text{MS}}$-scheme is the most widely used renormalization scheme in high-order calculations. The scheme itself is simple: the counterterms δ_m, δ_λ contain only divergences $\frac{1}{\epsilon}$, $\frac{1}{\epsilon^2}$,...

The parameters $\overline{m}$, $\overline{\lambda}$, where the overline signifies that they are in the $\overline{\text{MS}}$-scheme, have no direct physical interpretation, but we can still relate observables to them in an unambiguous, divergence-free way.

The $\overline{\text{MS}}$ mass is calculated at the one-loop order again by

$-i\Pi(p^2, m^2) =$  $+$  $+ \ldots$ (10.138)

$$= \underset{\sim A(1,\Delta)}{\underset{\uparrow}{-}} \frac{i\lambda}{32\pi^2}(-m^2)\left[\frac{1}{\epsilon} - \ln\left(\frac{m^2}{\widetilde{\mu}^2}\right) + 1\right] - i(\delta_Z p^2 + \delta_m)\,. \tag{10.139}$$

In the on-shell scheme, we have

$$\delta_m = \frac{\lambda}{32\pi^2} m^2 \left[\frac{1}{\epsilon} - \ln\left(\frac{m^2}{\widetilde{\mu}^2}\right) + 1\right], \tag{10.140}$$

such that $\Pi\big|_{p^2=m^2} = 0$, while in the $\overline{\text{MS}}$-scheme we have

$$\delta_m = \frac{\lambda}{32\pi^2} m^2 \cdot \frac{1}{\epsilon}\,. \tag{10.141}$$

The bare mass is

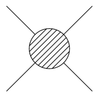
$$m_0^2 = m_{\text{phys}}^2 + \delta m_{\text{phys}}^2 = m_{\text{phys}}^2 \left\{1 + \frac{\lambda}{32\pi^2}\left[\frac{1}{\epsilon} - \ln\left(\frac{m_{\text{phys}}^2}{\widetilde{\mu}^2}\right) + 1\right] + \mathcal{O}(\lambda^2)\right\} \tag{10.142}$$

$$= \overline{m}^2(\widetilde{\mu}) + \delta\overline{m}^2(\widetilde{\mu}) = \overline{m}^2(\widetilde{\mu})\left[1 + \frac{\lambda}{32\pi^2} \cdot \frac{1}{\epsilon} + \mathcal{O}(\lambda^2)\right] \tag{10.143}$$

and therefore we can establish the relation

$$\underset{\text{independent of }\widetilde{\mu}}{\underset{\uparrow}{m^2_{\text{phys}}}} = \overset{\substack{\overline{\text{MS}}\text{-mass depends on }\widetilde{\mu}\text{ in a way,}\\ \text{such that the LHS is independent of }\widetilde{\mu}}}{\overset{\downarrow}{\overline{m}^2(\widetilde{\mu})}}\left\{1 + \frac{\lambda}{32\pi^2}\left[\ln\left(\frac{\overline{m}^2(\widetilde{\mu})}{\widetilde{\mu}^2}\right) + 1\right] + \mathcal{O}(\lambda^2)\right\}. \tag{10.144}$$

The same holds for the coupling:



$$\Big|_{\text{amputated}} = -i\lambda\widetilde{\mu}^{2\epsilon}\Big\{1 - \frac{\lambda}{32\pi^2}\left[\frac{3}{\epsilon} - 3\ln\left(\frac{m^2}{\widetilde{\mu}^2}\right) - \right. \tag{10.145}$$
$$\left. - A(s) - A(t) - A(u)\right] + \delta_\lambda\Big\}\,.$$

In the on-shell scheme we obtain

$$\delta_\lambda^{\text{OS}} = \frac{\lambda}{32\pi^2}\left[\frac{3}{\epsilon} - 3\ln\left(\frac{m^2}{\widetilde{\mu}^2}\right) - A(4m^2)\right], \tag{10.146}$$

while in the $\overline{\text{MS}}$-scheme we have

$$\delta_\lambda^{\overline{\text{MS}}} = \frac{\lambda}{32\pi^2}\cdot\frac{3}{\epsilon}\,. \tag{10.147}$$

Therefore we relate

$$\lambda = \left(\frac{\sqrt{Z_{\overline{\lambda}}}}{\sqrt{Z_\lambda}}\right)^4 \overline{\lambda}(\widetilde{\mu}) = (1 + \delta_{\overline{\lambda}} - \delta_\lambda + ...)\overline{\lambda}(\widetilde{\mu}) \tag{10.148}$$

and no δ_Z appears here since it vanishes at one loop.

All couplings, masses, etc. depend on the **scale** $\widetilde{\mu}$ in the $\overline{\text{MS}}$-scheme, where $\widetilde{\mu}$ is the arbitrary scale used to define the d-dimensional integration (in particular, $\widetilde{\mu}$ does **not** go to infinity!). Similarly, there is a scale dependence in the OS-scheme, e.g. $\lambda = \lambda\big(s_0 = 4m^2\big)$, since renormalized objects depend on the renormalization point.

Since from now on only $\widetilde{\mu}$ will appear in formulas, and no longer just μ, we replace $\widetilde{\mu} \longrightarrow \mu$ and keep in mind what is actually meant.

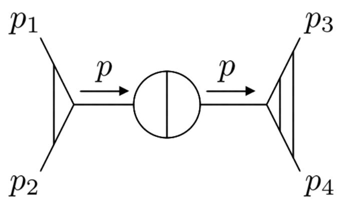
10.8 The systematics of renormalization

Can we make all possible Green functions finite by renormalizing just the fields, masses and couplings? This question is non-trivial, since a we have just a finite number of parameters and the structure of the divergences must match the terms we can add to the Lagrangian. This investigation will lead us to a better understanding of the nature of quantum corrections and is a path to the important concept of effective field theories.

10.8.1 Degree of divergence

This discussion will be general (i.e. we will have no specific Lagrangian in mind). The theory we consider may contain many different fields, which we will label by the subscript f.

For the following it is sufficient to consider 1PI diagrams. Why is that? 1PI subdiagrams in a general diagram are connected by lines that do not contain any loop momenta, e.g.



where $p = p_1 + p_2$. We therefore need only to consider 1PI self-energy and 1PI vertex graphs. In the following we will denote a 1PI Feynman diagram by γ.
Assuming that all loop moment go to infinity at the same rate, we define:

$$
\begin{aligned}
D(\gamma) &= \max_{\text{all terms}} \left\{ \begin{matrix}\text{powers of loop} \\ \text{momenta in numerator} \\ \text{including integration measure}\end{matrix} - \begin{matrix}\text{powers of loop} \\ \text{momenta in} \\ \text{denominator}\end{matrix} \right\} \\
&= \text{"}\textbf{superficial degree of divergence}\text{"}\,.
\end{aligned}
$$

Example:

$$
\int \underset{\substack{\uparrow \\ 4}}{\frac{\mathrm{d}^4 k_1}{(2\pi)^4}} \underset{\substack{\uparrow \\ 4}}{\frac{\mathrm{d}^4 k_1}{(2\pi)^4}} \underset{\substack{\uparrow \\ -2}}{\frac{1}{k_1^2}} \underset{\substack{\uparrow \\ -2}}{\frac{1}{(k_1 - k_2)^2}} \left(- \underset{\substack{\uparrow \\ 0}}{2\eta_{\mu\nu}} + \underset{\substack{\uparrow \\ 2}}{\frac{k_{1\mu} k_{2\nu}}{m^2}} \right) \underset{\substack{\uparrow \\ -4}}{\frac{1}{((k_1 + p)^2 - M^2)^2}}\,, \tag{10.149}
$$

$$
\Longrightarrow \quad D(\gamma) = \max\{0, 2\} = 2\,. \tag{10.150}
$$

This expression is therefore called "quadratically divergent".

An expression with $D(\gamma) = 0$ is called logarithmically divergent, e.g.

$$
\int_C^\Lambda \frac{\mathrm{d}x}{x} = \ln(\Lambda) + ... \tag{10.151}
$$

and if $D(\gamma) \geq 0$ it is divergent.

10.8.2 Computation of the superficial degree of divergence

Propagators are of the form

$$
\frac{1}{(k^2)^{1-s_f}}\,, \quad s_f \geq 0 \tag{10.152}
$$

for the field type f and $k^2 \gg m^2$ (for the scalar field: $s_f = 0$).

$$
\underset{\substack{\uparrow \\ \text{1PI diagram} \\ \gamma}}{D(\gamma)} = \sum_f \overset{\substack{\text{number of internal} \\ \text{lines (propagators)} \\ \downarrow}}{I_f}(2s_f - 2) + \sum_i \underset{\substack{\uparrow \\ \text{number of} \\ \text{vertices of} \\ \text{type } i}}{v_i} \overset{\substack{\text{number of derivatives} \\ \text{on fields in vertex} \\ \text{of type } i \\ \downarrow}}{a_i} \quad + \quad \underset{\substack{\uparrow \\ \text{space-time} \\ \text{dimensions}}}{d} \cdot \overset{\substack{\text{number} \\ \text{of loops} \\ \downarrow}}{L}\,. \tag{10.153}
$$

We have already derived a formula for the number of loops in a connected diagram in Eq. (8.38), which we now reformulate:

$$L = \underbrace{\sum_f I_f}_{\substack{\text{each line} \\ \left(\int \mathrm{d}^4 k\right)}} - \overbrace{\sum_i v_i}^{\substack{\text{4-momentum} \\ \text{conservation} \\ \delta^{(4)}(\ldots)}} + \underbrace{1}_{\substack{\text{overall trivial} \\ \delta\text{-function}}} . \tag{10.154}$$

Further, we can include the number of external legs of type f E_f:

$$E_f + 2I_f = \sum_i v_i n_{if} \,, \tag{10.155}$$

where every external (internal) line contributes one (two) end(s) which have to be connected to an internal vertex of type i, hence the different factor for E_f and I_f. n_{if} is the number of fields of type f present in the vertex of type i. The vertices supply $\sum_i v_i n_{if}$ available ends.

Using all this, we eliminate the number of loops L and the number of internal lines I_f to obtain

Recall:

1. s_f: power of the propagator
$$\frac{1}{(k^2)^{1-s_f}}$$
2. a_i: number of derivatives in the vertex v_i
3. n_{if}: number of fields of type f in the vertex v_i

$$D(\gamma) = d - \sum_f E_f \left(\frac{d}{2} - 1 + s_f\right) - \sum_i v_i \Delta_i \,, \tag{10.156}$$

$$\Delta_i \equiv d - a_i - \sum_f n_{if} \left(\frac{d}{2} - 1 + s_f\right) . \tag{10.157}$$

Δ_i is a characteristic quantity for each vertex of type i.

Case $\Delta_i \geq 0$:
For all vertices i in the theory:

$$D(\gamma) \leq 4 - \sum_f E_f (1 + s_f) \,. \tag{10.158}$$

The degree of divergence **decreases** when the number of **external** legs **increases**. Only a **finite** number of 1PI functions can be divergent. An example from scalar field theory is

$$\mathcal{L} = -\frac{1}{2}\phi_0(\Box + m^2)\phi_0 - \frac{\lambda_3}{3!}\phi_0^3 - \frac{\lambda_4}{4!}\phi_0^4 \,. \tag{10.159}$$

$$\begin{gathered} s_f = 0\,, \quad d = 4\,, \quad a_3 = a_4 = 0\,, \\ \Delta_{\lambda_3} = 4 - 0 - 3 \cdot (2-1) = 1\,, \\ \Delta_{\lambda_4} = 4 - 0 - 4 \cdot (2-1) = 0\,, \\ [\lambda_3] = 1\,, \quad [\lambda_4] = 0\,. \end{gathered}$$

with Δ_i the mass dimension of the coupling.

This suggests that the divergences can be removed by counterterms corresponding to field, mass and coupling (λ_3 and λ_4) renormalization.

		Green function	maximal value of $D(\gamma)$
δm^2, Z	$\longleftarrow$	$\phi\phi$	2
Z_{λ_3}	$\longleftarrow$	$\phi\phi\phi$	1
Z_{λ_4}	$\longleftarrow$	$\phi\phi\phi\phi$	0

Note, sometimes one needs **fewer** counterterms. For instance,

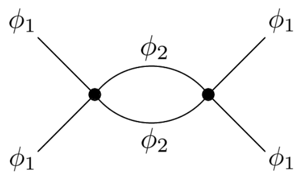
$$\mathcal{L} = -\frac{1}{2}\phi_0(\Box + m^2)\phi_0 - \frac{\lambda_4}{4!}\phi_0^4 \tag{10.160}$$

has a $\phi \longrightarrow -\phi$ symmetry, which forbids Green functions with an odd number of fields (in particular terms $\sim \phi^3$!).

The converse also occurs: one could need **more** counterterms than the ones appearing in the original Lagrangian. Consider the following example

$$\mathcal{L} = -\frac{1}{2}\sum_{i=1}^{2}\Big(\phi_i(\Box + m^2)\phi_i\Big) - \frac{\lambda}{4}\phi_1^2\phi_2^2\,. \tag{10.161}$$

We can have $\phi_1\phi_1 \longrightarrow \phi_1\phi_1$ scattering, e.g.



To cancel the divergence occurring in this diagram, we need the counterterm $\frac{\tilde{\lambda}}{4!}\delta_{\lambda_1}\phi_1^4$, but we can only get it if the term $\frac{\tilde{\lambda}}{4!}\phi_1^4$ is present in the Lagrangian from the beginning! Similarly for terms $\sim \phi_2^4$. Hence we **must** add these to $\mathcal{L}$ and determine the values from $\phi_1\phi_1 \longrightarrow \phi_1\phi_1$, $\phi_2\phi_2 \longrightarrow \phi_2\phi_2$, $\phi_1\phi_1 \longrightarrow \phi_2\phi_2$ scattering independently. The ϕ_1^4- and ϕ_2^4-terms are **not** forbidden by any symmetry of $\mathcal{L}$ in Eq. (10.161).

Important conclusion:

A theory can be rendered finite by a reparametrization of a finite number of couplings if:

1. $\Delta_i \geq 0$ for all vertices.
2. $\mathcal{L}$ contains all vertices compatible with the symmetries of the (regularized) theory.

Note that the regularization may break the symmetries of the non-regularized theory, which would then no longer be symmetries of $\mathcal{L}$.

These theories are called **renormalizable**.

An interaction is called **renormalizable** if $\Delta_i = 0$ and **non-renormalizable** if $\Delta_i < 0$. For super-renormalizable theories the degree of divergence $D(\gamma)$ decreases for increasing number of vertices:

$$D(\gamma) = ... - \sum_i v_i\Delta_i\,. \tag{10.162}$$

Ultimatively $D(\gamma) < 0$ if v_i is large enough.

Hence: if all $\Delta_i > 0$, the loop diagrams will be **finite** after a certain order. Therefore there is only a finite number of divergent subdiagrams and the theory is **super-renormalizable**.

In $d = 4$ dimensions, the only super-renormalizable theory is

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda}{3!}\phi^3 \,. \tag{10.163}$$

However: this theory possesses no stable ground state.

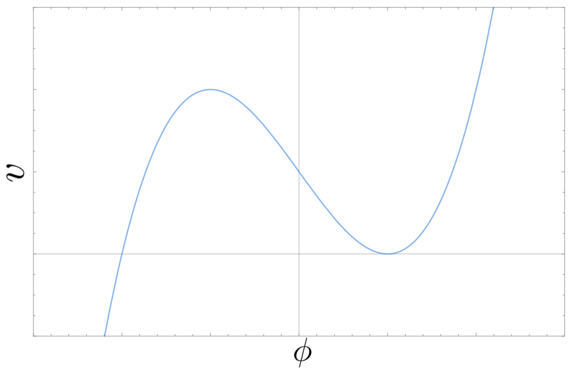


Figure 10.3: Plot of the potential generated by the field ϕ defined by Eq. (10.163).

10.9 Relevant, marginal and irrelevant

Consider the example scalar field theory in 4D with arbitrary powers of interaction terms

$$\int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda_3\,\phi^3 - \lambda_4\,\phi^4 - \lambda_5\,\phi^5 + \dots\right)$$

Study theory at long distances in **scaling limit**

$$x^\mu \to l x^\mu, \quad l \to \infty, \quad d^4x \to d^4x\, l^4 \tag{10.164}$$

Consider the kinetic terms

$$\int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \to \int d^4x \frac{1}{2} l^2 \partial_\mu\phi\partial^\mu\phi \tag{10.165}$$

which change. We want to keep canonical kinetic terms, since it will allow us to compare the interaction terms to the kinetic terms of a free field[3]. We therefore define a field

[3] we are including the mass among interactions here.

$$\phi'(x) \equiv l^{-1}\phi(lx)$$

to find

$$\int d^4x \left(\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m^2 l^2 \phi'^2 - \lambda_3\, l\phi'^3 - \lambda_4\, l^0\phi'^4 - \lambda_5\, l^{-1}\phi'^5 + \dots\right) \tag{10.166}$$

In the low-energy or **long-distance** limit, $l \to \infty$. We see that the kinetic term stays the same, but **relevant** operators grow in importance

$$\int d^4x \left(-\frac{1}{2}m^2 l^2 \phi'^2 - \lambda_3\, l\phi'^3\right)$$

whereas **marginal** operators stay constant

$$\int d^4x \left(-\lambda_4 \, l^0 \phi'^4\right)$$

and **irrelevant** operators shrink in importance

$$\int d^4x \left(-\lambda_5 \, l^{-1} \phi'^5 - \lambda_6 \, l^{-2} \phi'^6 + \ldots\right)$$

Since $[\lambda_5] = -1$, we can define a dimensionless coupling $\tilde{\lambda}_5$ and absorb the scale into M. The operator becomes

$$-\frac{\tilde{\lambda}_5}{M} \, l^{-1} \phi'^5 + \ldots$$

We see that in the limit that the defining scale of the operator is much higher and $1/l \ll M$, the effective coupling is very small

$$\lambda_5^{\text{eff}} = \frac{\tilde{\lambda}_5}{lM} \ll 1$$

Historically, one imposed renormalizability to preserve predictivity. If only marginal and relevant operators are present $\Delta_i \geq 0$ and counterterms to absorb infinities are also only from marginal and relevant set of operators.

We see however that even if we include arbitrary powers of higher-dimensional operators, their contribution to physical processes is suppressed by powers of $(lM)^n$.

10.10 Non-renormalizable and effective quantum field theories

Example:

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda_4}{4!}\phi^4 - \frac{\lambda_6}{6!}\phi^6 \,, \tag{10.167}$$

$$\Delta_{\phi^4} = 0 \,, \quad \Delta_{\phi^6} = -2 \,. \tag{10.168}$$

This corresponds to one $\Delta_i < 0$, then

$$D(\gamma) = d - \sum_f E_f \left(\frac{d}{2} - 1 + s_f\right) - \underset{\substack{\uparrow \\ (*)}}{\sum_i v_i \Delta_i} \,, \tag{10.169}$$

where $(*)$ contains a *negative* Δ_i, hence **any** Green function with any number of external legs E_f becomes divergent if only the number v_i becomes large.
Therefore we would need to add an infinite number of terms with more and more fields (i.e. E_f) to $\mathcal{L}_{\text{int}}$ (constrained by the symmetries of the Lagrangian, as usual) to absorb infinities. Hence, we would need to perform infinitely many measurements to fix these couplings.

Are non-renormalizable theories unpredictive? This was the prevailing attidude in the early days of QFT (60's and 70's). However, the modern view is very different!
In our example we had

$$\mathcal{L}_{\text{int}} = -\frac{\lambda_4}{4!}\phi^4 - \frac{\lambda_6}{6!}\phi^6\,, \tag{10.170}$$
$$\text{with}\quad [\phi]=1\,, \qquad \text{coupling}\quad [\lambda_4]=0 \quad\text{and}\quad [\lambda_6]=-2\,.$$

We rewrite λ_6 using a dimensionless coupling $\hat{\lambda}_6$:

$$\lambda_6 = \frac{\hat{\lambda}_6}{M^2}\,, \tag{10.171}$$

with some scale M such that $\hat{\lambda}_6 = \mathcal{O}(1)$.

This holds in general: since $[\mathcal{L}] = d$, the dimension of a coupling g_i is

$$d = [g_i] + \underbrace{a_i}_{\substack{\text{number of}\\ \text{derivatives}}} + \sum_f n_{if}\overbrace{\left(\frac{d}{2} - 1 + s_f\right)}^{\substack{=[\psi_f]\\ \text{dimensions of } \psi_f}}\,, \tag{10.172}$$

where the last observation can be made clear by considering the dimensions of the kinetic term of the Lagrangian:

$$d = 2[\psi_f] + 2(1-s_f) \quad\Longrightarrow\quad [\psi_f] = \frac{d}{2} - 1 + s_f\,, \tag{10.173}$$

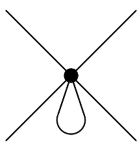
with $1 - s_f$ the power of the squared momentum in the propagator. Hence, by comparing Eq. (13.67) to Eq. (13.68), we read off

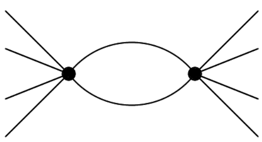
$$[g_i] = \Delta_i\,! \tag{10.174}$$

We therefore rewrite g_i as

$$g_i = \frac{\hat{g}_i}{M^{-\Delta_i}}\,. \tag{10.175}$$

In Eq. (10.167) at one loop the contributions to $2 \longrightarrow 2$ and $4 \longrightarrow 4$ scattering are

1)  $\sim \dfrac{\hat{\lambda}_6}{M^2}\,,$

2)  $\sim \dfrac{\hat{\lambda}_6^2}{M^4}\,.$

Both loops are divergent and 1) contributes to δ_{λ_4} while 2) requires the term $-\frac{\lambda_8}{8!}\phi^8$ to be added to the Lagrangian. Now we can continue this ad infinitum by looking at the analogous loop diagram to

2) generated by the $-\frac{\lambda_8}{8!}\phi^8$ term, which would give a divergent $6 \to 6$ process, requiring a $-\frac{\lambda_{12}}{12!}\phi^{12}$ term to be added, and so on.

Consider a **scattering process with momenta**[4] $< \boldsymbol{E}$. An non-renormalizable or irrelevant interaction with $\Delta_i < 0$ contributes

[4] say in the CMS frame

$$|T_{\beta\alpha}| \supset \hat{g}_i \frac{E^D}{M^{|\Delta_i|}}\,, \tag{10.176}$$

where $D - |\Delta_i|$ is the assumed dimension of $|T_{\beta\alpha}|$. If the energy E is much smaller than the dimension of the scale M, $E \ll M$, then the contributions of the non-renormalizable interactions will be suppressed, the more **the higher the dimension of the operator**. In physics we can only ever make predictions with finite accuracy; as long as $E \ll M$ only a **finite number** of non-renormalizable interactions is ever relevant for a prediction at a given accuracy. Therefore non-renormalizable QFTs are as predictive as renormalizable theories (as long as $E \ll M$). If one needs to increase the accuracy, then one might need to include more higher dimensional operators, but the required number will always be finite.

Summary:

- **renormalizable QFTs:** the interaction operators have dimension ≤ 4, such that $\Delta_i \geq 0$. The theory can in principle be used to calculate processes at arbitrarily high energy scales E (up to so-called Landau poles, which we will discuss soon).
- **non-renormalizable QFTs:** the interaction operators have dimensions > 4, such that $\Delta_i < 0$. For finite E ($\ll M$) only a finite number of these operators are relevant. Therefore they are predictive QFTs for phenomena below a certain scale $E < M$. These theories are also called **effective field theories** (EFTs).

In 4 dimensions.

Since gravity is non-renormalizable and automatically includes irrelevant terms, we ultimatively think of the Standard Model (SM) with gravity as an EFT. Additionally, there are many phenomena (e.g. dark matter) which the SM cannot explain, which also points to the fact of it being an EFT.

Also: how can we know that a theory is truly renormalizable? We can only ever perform experiments at finite E, we will therefore never know wether a renormalizable theory is in fact only the leading term of a non-renormalizable theory for $E \ll M$.

10.10.1 An effective theory example

Effective theories are useful even if we have a description in terms of a renormalizable theory.Let us illustrate this with the following example:

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi + \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - M^2\varphi^2\right) - \frac{k_3}{3!}\varphi^3 - g\varphi\phi^\dagger\phi\,, \tag{10.177}$$

φ has mass M, ϕ has mass m and $M \gg m$.

with $k_3\,, g \sim \mathcal{O}(M)$ (this theory is even super-renormalizable). We now assume that

$$m^2 \ll M^2 \tag{10.178}$$

and that the external momenta fulfill

$$p_i \cdot p_j \ll M^2 \quad \forall i, j\,, \quad p_i \sim E_i\,. \tag{10.179}$$

Then we only have to consider external lines of ϕ, $\phi^\dagger$, since there is not enough energy to produce on-shell φ-particles with $M \gg E$, since Eq. (10.179) implies

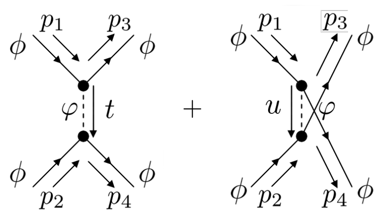
$$p_i^2 \ll M^2\,. \tag{10.180}$$

Since we cannot produce heavy φ-particles (by assumption), we can write an EFT containing only light ϕ fields, which reproduces scattering amplitudes of the full theory to arbitrary accuracy!
Our goal is therefore to find a $\mathcal{L}_{\text{eff}}$ such that

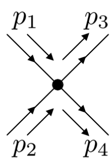
$$T_{\beta\alpha}\Big|_{\text{full }\mathcal{L}(\phi,\varphi)} = T_{\beta\alpha}\Big|_{\mathcal{L}_{\text{eff}}(\phi)} + \mathcal{O}\left[\left(\frac{E}{M}\right)^a\right]. \tag{10.181}$$

depending on observed accuracy

1) Consider $\phi\phi \longrightarrow \phi\phi$ scattering:



$$= (-ig)^2\left(\frac{i}{t - M^2 + i\varepsilon} + \frac{i}{u - M^2 + ive}\right) = (+i)\frac{2g^2}{M^2} + \mathcal{O}\left(g^2\frac{E^2}{M^2}\right) \tag{10.182}$$

$=$  $+\,\mathcal{O}\left(g^2\dfrac{E^2}{M^2}\right),$

where $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ as usual.

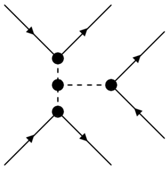
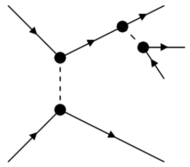
We can reproduce this effective interaction of light ϕ's with

$$\mathcal{L}_{\text{eff}}^{(4)} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda_4}{4}(\phi^\dagger\phi)^2 \tag{10.183}$$

$$\text{and} \quad \lambda_4 = -\frac{2g^2}{M^2} \quad \text{(dimensionless)}. \tag{10.184}$$

2) What about $\phi\phi \longrightarrow \phi\phi\phi^\dagger\phi$ scattering?

In the full theory the relevant diagrams are:


$+$

$+$ (3 more).

1)

This is not completely reproduced by $\mathcal{L}_{\text{eff}}^{(4)}$ which for instance does not contain 1), only e.g.

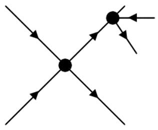
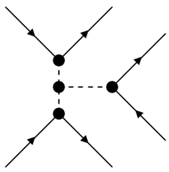

$+$ (3 more).

Diagram 1) leads to the effective interaction



$$\longrightarrow (-ig)^3(-ik_3)\cdot\frac{(-i)^3}{M^6} \tag{10.185}$$

$$\Longrightarrow \quad \frac{g^3k_3}{M^6}(\phi^\dagger\phi)^3\,. \tag{10.186}$$

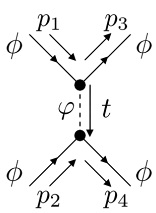
We need new terms:

$$\mathcal{L}_{\text{eff}}^{(6)} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda_4}{4}(\phi^\dagger\phi)^2 - \frac{\lambda_6}{8M^2}(\phi^\dagger\phi)^3\,, \tag{10.187}$$

$$\text{with} \quad \lambda_6 \sim \frac{g^3k_3}{M^4} \quad \text{dimensionless.} \tag{10.188}$$

We have added terms $\frac{(\phi^\dagger\phi)^2}{M^2}$, $\frac{(\phi^\dagger\phi)^3}{M^4}$.

At this order in $1/M$, we need to include the momentum expansion of the propagator, e.g.



$$\approx \frac{g^2}{M^2}\left(1+\frac{(p_1-p_3)^2}{M^2}+\frac{(p_1-p_3)^4}{M^4}+...\right). \tag{10.189}$$

At order $\frac{1}{M^4}$ we will get momentum-dependent vertices, corresponding to

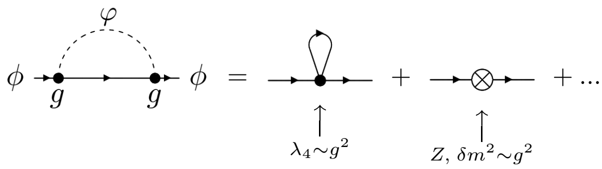
$$\sim \frac{1}{M^2}\phi^\dagger(\partial_\mu\phi)(\partial^\mu\phi^\dagger)\phi$$

and similar. We can consistently follow this process to arbitrary order in $\frac{E}{M}$ to get closer to the UV theory if necessary.

3) **What about loop corrections?**

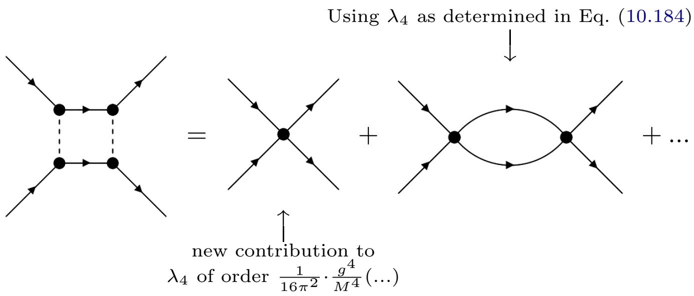
Consider:

a)



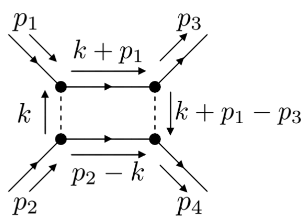
This determines mass and field renormalization due to additional φ-terms.

b)



We call this successive calculation of corrections to couplings in EFT **matching**.

It is important that all the generated interactions are **local**, which means that all the momentum dependence entering corrections in EFTs is **polynomial** in the external momenta (and **not** $\ln\left(\frac{p^2}{m^2}\right)$). We will show an example and sketch the argument.

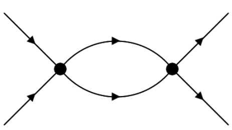


where the loop momentum k is integrated over.

1) **Low-energy modes** $\boldsymbol{k} \sim \mathcal{O}(p_i) \ll M$**:** We replace

$$\frac{1}{k^2 - M^2 + i\varepsilon}, \quad \frac{1}{(k + p_1 - p_2)^2 - M^2 + i\varepsilon} \tag{10.190}$$

by $-\frac{1}{M^2}$. This gives

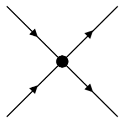


which is a complicated (usually non-polynomial function) of p_i, see Eq. (10.110).

2) **High-energy modes $\boldsymbol{k} \sim \mathcal{O}(M) \gg p_i, m$:** We can treat the p_i as small parameters and expand in p_i:

$$\int \mathrm{d}^4 k \frac{1}{k^4(k^2 - M^2)^2} \cdot (1 + \text{polynomial in } p_i) \ , \tag{10.191}$$

which is



a **local** vertex which might contain derivatives (depending on order in $1/M$).

The construction therefore always works out, since non-local **IR** or low-energy contributions 1. are contained in $\boldsymbol{\mathcal{L}}_{\text{eff}}$; the diagram in 1. is in the EFT. The remainder, 2., or **UV** contributions, are **local**.

Comments: EFTs allow two perspectives:

1) If we know the full theory (also called UV completion), the EFT is still useful because it is simpler: it contains fewer fields and only the IR dynamics. We also need to fix the λ_4, λ_6,... only once and they are independent of the process.

2) If we have not probed energy scales $E \sim M^2$ yet, we do not know the UV completion. All the accessible phenomena can be described by a non-renormalizable theory of ϕ -interactions. The determined couplings λ_4, λ_6,... then constrain the form of the UV completion.

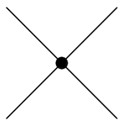
The renormalization of composite operators is left to the reader as self-study. This topic can be found in [3], page 430 and [2], chapter 18.2.

11 The Renormalization Group

We observe that regularization introduces new scales: Λ in cut-off regularization and μ (or $\widetilde{\mu}$) in dimensional regularization (recall that μ is **not** a cut-off, the analogue to Λ in dim. reg. would be $\frac{1}{\epsilon}$).

Which is not a group since it is not invertible. The renormalization group can be described as "a clever idea that lets you get something for nothing".

Even after renormalization a scale dependence usually remains (even after taking the limit $\Lambda \to \infty$ or $\epsilon \to 0$), like



$$\Bigg|_{\substack{s=s_0\\ t=u=0}} = -i\lambda_R\,, \tag{11.1}$$

see Eq. (10.28), or the dim. reg. scale μ in the $\overline{\mathrm{MS}}$-scheme:

$$\lambda = \overline{\lambda}(\mu)\,. \tag{11.2}$$

If $\mathcal{L}$ was scale-invariant classically (i.e. it contained no dimensionful parameters, e.g. masses), scale-invariance is usually broken in the regularized theory (even after $\Lambda \longrightarrow \infty$ or $\epsilon \longrightarrow 0$). From this, it follows that dilatation symmetry is anomalous (usually).

Let us consider the high-energy scattering $\phi\phi \longrightarrow \phi\phi$, in particular the 1-loop matrix element Eq. (10.26):

$$T(s) = -\lambda_R - 3\frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{s}{s_0}\right) + ... \tag{11.3}$$

The perturbative expansion breaks down in the high-energy limit $s \longrightarrow \infty$ even for small λ_R when

$$3\frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{s}{s_0}\right) \sim \mathcal{O}(1)\,. \tag{11.4}$$

That is a strange: the result diverges for $s \longrightarrow \infty$ (or $s_0 \longrightarrow 0$). This is a general fact: if the renormalization scale (here: s_0) is far away from the "characteristic" scale of the process (here: s), then perturbation theory is not well-behaved.

11.1 Computation of the scale-dependence

Let us use the fact that the **bare** coupling is independent of μ:

Recall that we defined in Sec. 10.7.1

$$Z_\lambda^{\overline{\mathrm{MS}}} = Z_\lambda Z^2$$

in the $\overline{\mathrm{MS}}$-scheme.

$$\mathcal{L} = -\frac{\lambda_0}{4!}\phi_0^4 + ... = -Z_\lambda^{\overline{\mathrm{MS}}}\mu^{2\epsilon}\lambda_R(\mu)\frac{\phi_R^4}{4!} \tag{11.5}$$

$$\text{or}\quad \lambda_R(\mu) = \mu^{-2\epsilon}\left(Z_\lambda^{\overline{\mathrm{MS}}}\right)^{-1}\lambda_0\,, \tag{11.6}$$

and since λ_0 is independent of μ we have

$$\frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_0 = 0 \tag{11.7}$$

and therefore, by plugging in the redefinition of λ_0,

We now drop the subscript R:

$$\lambda_R(\mu) \equiv \lambda(\mu)$$

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda(\mu) = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\left[\mu^{-2\epsilon}\left(Z_\lambda^{\overline{\mathrm{MS}}}\right)^{-1}\right]\lambda_0 \tag{11.8}$$

$$= -\epsilon\mu^{-2\epsilon}\left(Z_\lambda^{\overline{\mathrm{MS}}}\right)^{-1}\lambda_0 - \mu^{-2\epsilon}\frac{1}{\left(Z_\lambda^{\overline{\mathrm{MS}}}\right)^2}\left(\mu^2\frac{\mathrm{d}Z_\lambda^{\overline{\mathrm{MS}}}}{\mathrm{d}\mu^2}\right)\lambda_0 \tag{11.9}$$

and plugging the definition of $\lambda(\mu)$ in again, we finally obtain

$$\mu^{2\epsilon} = e^{\epsilon\ln(\mu^2)}$$
$$\frac{\mathrm{d}\mu^{2\epsilon}}{\mathrm{d}\ln(\mu^2)} = \epsilon\mu^{2\epsilon}$$
$$\frac{\mathrm{d}}{\mathrm{d}\ln(\mu^2)} = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}$$

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda(\mu) = -\epsilon\lambda(\mu) - \frac{1}{\left(Z_\lambda^{\overline{\mathrm{MS}}}\right)}\left(\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}Z_\lambda^{\overline{\mathrm{MS}}}\right)\lambda(\mu) \tag{11.10}$$

$$\equiv\beta(\lambda) \quad (\text{finite as } \epsilon \longrightarrow 0) \tag{11.11}$$

where we know that $\beta(\lambda)$ is finite since the renormalized coupling is finite. This is the **renormalization group equation** for $\lambda(\mu)$.

We can solve it by separating variables:

For the separation of variables to work, we need that $Z^{\overline{\mathrm{MS}}}$ does not explicitly depend on μ. See Eq. (10.147), where this is clearly satisfied. It would not be true in the *OS* scheme, see Eq. (10.146).

$$\frac{\mathrm{d}\mu^2}{\mu^2} = \frac{\mathrm{d}\lambda}{\beta(\lambda)} \quad\Longrightarrow\quad \ln\left(\frac{\mu_2^2}{\mu_1^2}\right) = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)}\frac{\mathrm{d}\lambda'}{\beta(\lambda')}\,. \tag{11.12}$$

Let's consider the ϕ^4 -example:

$$Z_\lambda^{\overline{\mathrm{MS}}} = 1 + \frac{3\lambda}{32\pi^2}\cdot\frac{1}{\epsilon} + ... \tag{11.13}$$

and we now drop the superscript $\overline{\mathrm{MS}}$: $Z_\lambda^{\overline{\mathrm{MS}}} = Z_\lambda$. The beta-function is then

$$\beta(\lambda) = \frac{\mathrm{d}\lambda}{\mathrm{d}\ln(\mu^2)} \overset{(*)}{=} -\epsilon\lambda(\mu) - Z_\lambda^{-1}\frac{\mathrm{d}Z_\lambda}{\mathrm{d}\ln(\mu^2)}\cdot\lambda(\mu)$$
$$= -\epsilon\lambda(\mu) - \left(1+\frac{3\lambda}{32\pi^2}\cdot\frac{1}{\epsilon}+...\right)^{-1}\frac{\mathrm{d}\lambda}{\mathrm{d}\ln(\mu^2)}\cdot\frac{3}{32\pi^2}\cdot\frac{1}{\epsilon}\cdot\lambda(\mu)$$
$$\overset{\text{plug in }(*)}{=} -\epsilon\lambda(\mu) - \left(1-\frac{3\lambda}{32\pi^2}\cdot\frac{1}{\epsilon}+...\right)\big(-\epsilon\lambda(\mu)-..\big)\frac{3}{32\pi^2}\cdot\frac{1}{\epsilon}\cdot\lambda(\mu)$$

where we iteratively solve to obtain the expression

$$\beta_\epsilon(\lambda) = -\epsilon\lambda(\mu) + \frac{3}{32\pi^2}\cdot\lambda(\mu)^2 + \mathcal{O}\big(\lambda(\mu)^3\big)\,. \tag{11.14}$$

Taking the limit $\epsilon \longrightarrow 0$, we see that the result is finite

$$\beta_{\epsilon\to0}(\lambda) = \frac{3}{32\pi^2}\lambda^2 + ... = \beta_0\lambda^2 + ... \tag{11.15}$$

and plugging this result in Eq. (11.12) we get

$$\ln\left(\frac{\mu_2^2}{\mu_1^2}\right) = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)}\mathrm{d}\lambda'\left[\frac{1}{\beta_0\lambda'^2} + \mathcal{O}\left(\frac{1}{\lambda}\right)\right] \tag{11.16}$$

$$= \frac{1}{\beta_0}\left(\frac{1}{\lambda(\mu_1)} - \frac{1}{\lambda(\mu_2)}\right) + \mathcal{O}\big(\ln(\lambda)\big) \tag{11.17}$$

and finally, solving for $\lambda(\mu_2)$,

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \beta_0 \lambda(\mu_1) \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \,. \tag{11.18}$$

Therefore, given $\lambda(\mu_1)$ we can compute $\lambda(\mu_2)$.
We can perform an analogous computation for the masses:

$$m_0 = Z_m m(\mu) \,, \quad \frac{\mathrm{d}}{\mathrm{d}\mu^2} m_0 = 0 \tag{11.19}$$

and going through the same steps as above we get

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} m(\mu) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \left(Z_m^{-1} \right) m_0 \tag{11.20}$$

$$= \gamma_m(\lambda) m(\mu) \,, \tag{11.21}$$

with

$$\gamma_m(\lambda) \equiv \left(-\frac{1}{Z_m} \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} Z_m \right) . \tag{11.22}$$

This is the **renormalization group equation (RGE) for the running mass $m(\mu)$**.
We solve this, again, by separating variables and using $\frac{\mathrm{d}\mu^2}{\mu^2} = \frac{\mathrm{d}\lambda}{\beta(\lambda)}$

$$\frac{\mathrm{d}m}{m(\mu)} \overset{(*)}{=} \frac{\mathrm{d}\mu^2}{\mu^2} \gamma_m(\lambda) = \mathrm{d}\lambda \frac{\gamma_m(\lambda)}{\beta(\lambda)} \tag{11.23}$$

which gives, when integrated,

$$\ln\left(\frac{m(\mu_2)}{m(\mu_1)}\right) = \int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \mathrm{d}\lambda \frac{\gamma_m(\lambda)}{\beta(\lambda)} \,, \tag{11.24}$$

or

$$m(\mu_2) = m(\mu_1) \exp\left(\int_{\lambda(\mu_1)}^{\lambda(\mu_2)} \mathrm{d}\lambda \frac{\gamma_m(\lambda)}{\beta(\lambda)} \right) . \tag{11.25}$$

The equality $(*)$ assumes that $\gamma_m(\lambda) \equiv \left(-\frac{1}{Z_m} \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} Z_m \right)$ does not depend on m. This is not true in general but it is satisfied in the $\overline{\mathrm{MS}}$ (and MS) schemes, since all poles $\frac{1}{\epsilon}$ arise from momentum regions that are much larger thant p, k, m and the renormalization condition does not contain m.

11.2 *Overview of the possible scale dependence*

A) **Trivial infrared (IR) fixed point.** This is realized in ϕ^4 - theory. The form of $\lambda(\mu)$ is

$$\lambda(\mu_2) = \frac{\lambda(\mu_1)}{1 - \beta_0 \lambda(\mu_1) \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \tag{11.26}$$

and $\beta(\lambda) > 0$ for small λ:

Choosing $\mu_1 \sim \mathcal{O}(m)$ we can compute $\lambda(\mu_2)$ for $\mu_2 \sim \mathcal{O}(\sqrt{s})$. Here we only need $\lambda(\mu)$ to be small for $\mu \in [\mu_1, \mu_2]$, but $\lambda(\mu) \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)$ does **not** have to be small, as is required in perturbation theory!

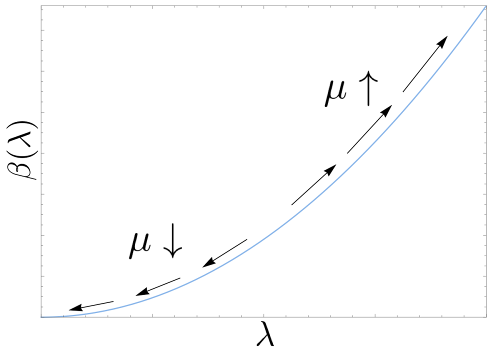


Figure 11.1: Plot of $\beta(\lambda)$ with a trivial IR fixed point. The arrows indicate in which direction β shifts when the scale μ is raised or lowered.

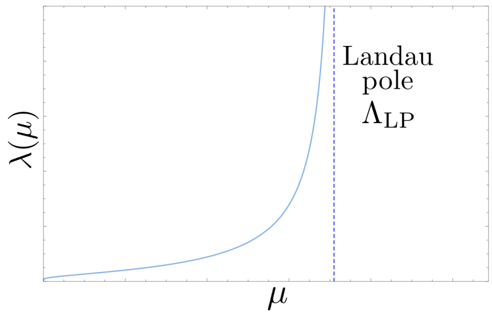


Figure 11.2: Plot of $\lambda(\mu)$ as given in Eq. (11.26). The function exhibits a Landau pole at a high scale.

Since $\beta_0 > 0$, $\lambda(\mu)$ **increases** with μ and diverges at around[1]

$$\Lambda_{\text{LP}} \approx \mu \exp\left(\frac{1}{2\beta_0\lambda(\mu)}\right), \tag{11.27}$$

which is the **Landau pole** scale.

We say that there is a trivial IR fixed point if for $\mu \longrightarrow 0$ also $\lambda(\mu) \longrightarrow 0$ and $\beta(\lambda \longrightarrow 0) \longrightarrow 0$ holds, which also means that λ stays fixed (at 0).

[1] Perturbation theory becomes unreliable near the pole, since the coupling blows up.x

Comments:

- Since $\lambda(\mu)$ becomes large near the Landau pole Λ_{LP}, we do not really know what happens in its vicinity using perturbation theory only. We conclude that the interaction becomes non-perturbative and that Λ_{LP} is especially large if $\lambda(\mu)$ is small at low μ.
- We define a fixed point as the point where $\beta(\lambda_*) = 0$. Hence here we have a trivial IR fixed point ($\mu \longrightarrow 0$). At a fixed point (FP) the running stops. Trivial FP's are the ones with vanishing couplings, since the theory effectively becomes free.
- The use of running couplings gives us something for free! We get an automatic **resummation** of certain large terms **to all orders**. If we choose $q^2 = s = -t = -u$, then the T-matrix element for $\phi\phi \longrightarrow \phi\phi$ scattering in ϕ^4 -theory reads

$$iT_{\phi\phi\longrightarrow\phi\phi} = -i\lambda(q)\left[1 + \frac{\lambda(q)}{32\pi^2}\cdot\begin{pmatrix}\text{coefficients of}\\ \mathcal{O}(1),\text{no large logs}\end{pmatrix}\right]. \tag{11.28}$$

  Inserting

$$\lambda(q) = \frac{\lambda(s_0)}{1 - \frac{3}{32\pi^2}\lambda(s_0)\ln\left(\frac{q^2}{s_0}\right)} \tag{11.29}$$

one recovers the expression in Eq. (10.31),

$$iT_{\phi\phi\longrightarrow\phi\phi} = -i\lambda_R + i\frac{\lambda_R^2}{32\pi^2}\ln\left(\frac{s}{s_0}\right). \tag{11.30}$$

B) **Trivial ultraviolet (UV) fixed point.** Assuming an expansion

$$\beta(\lambda) = \beta_0\lambda^2 + \dots, \quad \textbf{now} \quad \beta_0 < 0\,, \tag{11.31}$$

then

$$\lambda(\mu_{\text{high}}) = \frac{\lambda(\mu_{\text{low}})}{1 + (-\beta_0)\lambda(\mu_{\text{low}})\ln\left(\frac{\mu_{\text{high}}^2}{\mu_{\text{low}}^2}\right)}. \tag{11.32}$$

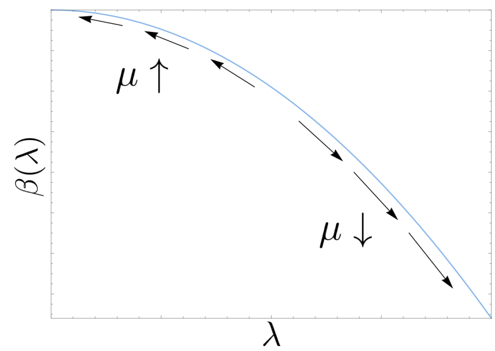


Figure 11.3: Plot of $\beta(\lambda)$ with a trivial UV fixed point. The arrows indicate in which direction β shifts when the scale μ is raised or lowered. Note that β is negative.

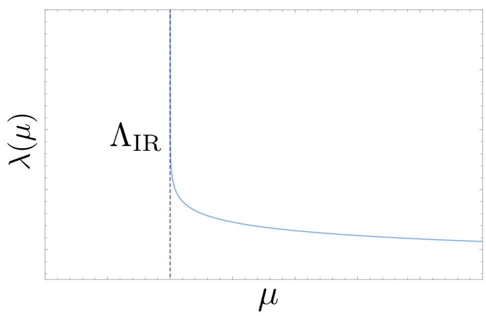


Figure 11.4: Plot of $\lambda(\mu)$ as given in Eq. (11.32). The function exhibits an IR pole at a low scale.

The IR pole is roughly at

$$\Lambda_{\text{IR}} = \mu_{\text{high}}\exp\left(-\frac{1}{|\beta_0|\mu_{\text{high}}}\right). \tag{11.33}$$

The natural emergence of a typical energy scale, here Λ_{IR}, from a theory which was formerly scale-invariant is called **dimensional transmutation**.

Take the example of a scalar field theory with vanishing mass and only a $\lambda_4\phi^4$ interaction. We can see in Eq. (10.166), that this theory would be scale invariant (no l dependence!). A non-vanishing β function, however, would introduce an explicit scale.

The coupling strength here decreases as μ increases and the effective interaction strength becomes weaker for higher energy scattering (i.e. there is a trivial UV fixed point). On the other hand, as μ decreases $\lambda(\mu)$ becomes larger and "infinite" at some scale, here Λ_{IR} (which, again, is not accessible through perturbation theory).

This occurs in nature for QCD (quantum chromodynamics): we measure

$$\alpha_s^{\overline{\text{MS}}}(M_Z) \equiv \frac{g_s^{\overline{\text{MS}}}}{4\pi} \approx 0.118\,, \tag{11.34}$$

with M_Z the mass of the Z^0-boson, and

- $\alpha_s^{\overline{\mathrm{MS}}}(\mu) \longrightarrow 0$ for $\mu \longrightarrow \infty$, we say that $\alpha_s^{\overline{\mathrm{MS}}}$ is "asymptotically free".
- $\alpha_s^{\overline{\mathrm{MS}}}(\mu) \longrightarrow \infty$ for $\mu \approx \Lambda_{\mathrm{IR}} \approx 200$ MeV.

The beta-function in QCD is given to first order by

$$\beta_{\mathrm{QCD}}(g_s) = -\left(11 - \frac{2n_f}{3}\right)\frac{g_s^2}{16\pi^2}\,, \tag{11.35}$$

There are three quark generations:

$$\binom{u}{d}\,,\quad \binom{c}{s}\,,\quad \binom{t}{b}\,.$$

with the number of quark flavors $n_f = 6$.

This makes plausible that QCD forms neutral bound states at low energies (p, n, π, K,...) out of quarks and gluons.

C) **Non-trivial fixed points.** This happens when $\beta(\lambda_*) = 0$ for $\lambda_* \neq 0$. We distinguish two cases:

1) **UV-stable fixed points.**

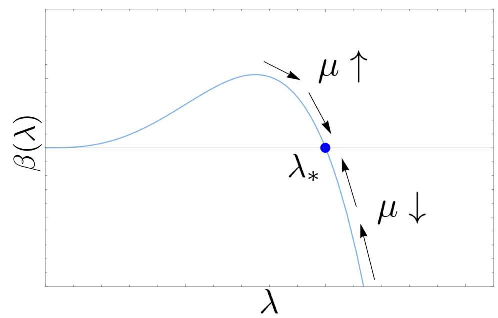


Figure 11.5: Plot of $\beta(\lambda)$ with a UV-stable fixed point at λ_*.

Independently of wether we start within the domain of attraction, $\lambda \longrightarrow \lambda_*$ for $\mu \longrightarrow \infty$:

- For $\lambda < \lambda_*$ $\beta(\lambda) > 0$ and λ **increases** with μ until λ_* is reached.
- For $\lambda > \lambda_*$ $\beta(\lambda) < 0$ and λ **decreases** with μ until λ_* is reached again.

For $\mu \longrightarrow 0$ the behavior depends if one starts with $\lambda > \lambda_*$ or $\lambda < \lambda_*$.

2) **IR stable fixed point.** For increasing μ λ is repelled away from λ_*, where $\beta(\lambda_*) = 0$, while it is attracted to it with decreasing μ.

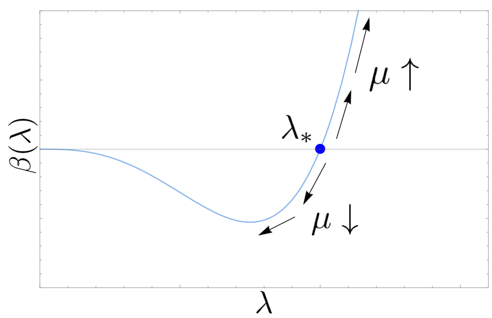


Figure 11.6: Plot of $\beta(\lambda)$ with a IR-stable fixed point at λ_*.

3) **Exceptional case.** $\beta(\lambda) > 0 \; \forall \lambda$ but it increases very slowly such that

$$\int_{\lambda_0}^{\infty} \frac{\mathrm{d}\lambda}{\beta(\lambda)} = \infty = \ln\left(\frac{\mu_{\text{high}}^2}{\mu_{\text{low}}^2}\right), \tag{11.36}$$

e.g. $\beta(\lambda) \sim c\lambda^{-k}$, with $0 < k < 1$, for $\lambda \longrightarrow \infty$. In this case $\lambda_* = \infty$ is a UV fixed point.

11.3 Renormalization group flow

We can generalize the above to theories with several coupling constants λ_i (see Ex!). We start with

$$[\lambda_i] = \Delta_i = 0 \quad \text{(dimensionless)}. \tag{11.37}$$

Then we obtain

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_i = \beta_i(\lambda_j) = \beta_i^{jk}\lambda_j\lambda_k + \dots, \tag{11.38}$$

which is a coupled system of differential equations, a so-called "dynamical system" (in mathematics terminology).
The fixed points are now defined as the solutions λ_i^* for which

$$\beta_i(\lambda_j^*) = 0 \quad \forall i\,. \tag{11.39}$$

Fixed points can be attractive or repulsive, depending on the eigenvalues near the fixed point:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_i = \underbrace{\left[\frac{\partial\beta_i}{\partial\lambda_k}\bigg|_{\lambda_k=\lambda_k^*}\right]}_{\substack{\equiv A_{ik} \\ \text{constant matrix}}}(\lambda_k - \lambda_k^*) + \dots \tag{11.40}$$

The fixed point is attractive in direction of negative eigenvalues of the matrix A_{ik}, and it is repulsive in the direction of positive eigenvalues of A_{ik}.

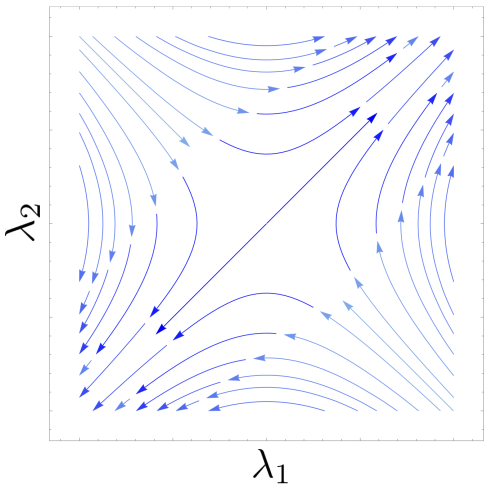


Figure 11.7: Exemplary plot for the RGE-flow. The arrows indicate in which direction the fixed points are attractive or repulsive.

We now include **dimensionful** coupling constants and consider the

RGE flow.

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi + \sum_i \lambda_i O_i \tag{11.41}$$

$$= \frac{1}{2}\Big(\partial_\mu\phi\partial^\mu\phi - m(\mu)^2\phi^2\Big) - \frac{\lambda_4(\mu)}{4!}\phi^4 - \frac{\lambda_6(\mu)}{6!}\phi^6 - ... + \mathcal{L}_{c.t.} \tag{11.42}$$

and as before $[\lambda_i] = \Delta_i$. We rewrite the dimensionful coupling constants using dimensionless ones:

$$\hat\lambda_2 = \frac{m^2}{\mu^2}\,, \quad \hat\lambda_4 = \lambda_4\,, \quad \hat\lambda_6 = \mu^2\lambda_6\,, \tag{11.43}$$

where we have made the μ-dependence implicit. Then the dynamical system becomes:

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\hat\lambda_i = \beta_i(\hat\lambda_j) \tag{11.44}$$

$$= -\frac{\Delta_i}{2}\hat\lambda_i + \beta_i^{jk}\hat\lambda_j\hat\lambda_k + ...\,, \tag{11.45}$$

since e.g.

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\hat\lambda_2 = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\left(\frac{m^2(\mu)}{\mu^2}\right) \tag{11.46}$$

$$= -1\cdot\left(\frac{m^2(\mu)}{\mu^2}\right) + \underbrace{\frac{\mathrm{d}}{\mathrm{d}\mu^2}\Big(m^2(\mu)\Big)}_{\text{dimensionless}} \tag{11.47}$$

$$= -\hat\lambda_2 + \beta^{jk}_{\hat\lambda_2}\hat\lambda_j\hat\lambda_k\,. \tag{11.48}$$

The μ-dependence for small $\hat\lambda_i$ is determined by the trivial factor $(\mu^2)^{\frac{\Delta_i}{2}}$.

Now we assume that the theory is valid at a high scale Λ, which is to be much larger than the energy used in the experiments we perform. We also assume that all $\hat\lambda_i \sim \mathcal{O}(1)$ and that they allow for perturbative expansions. Then it holds:

$$\frac{\mathrm{d}\hat\lambda_i}{\hat\lambda_i} = -\frac{\Delta_i}{2}\frac{\mathrm{d}\mu^2}{\mu^2}\,, \tag{11.49}$$

which, when integrated, gives

$$\ln(\hat\lambda_i) = -\frac{\Delta_i}{2}\ln(\mu^2) + \text{const.}\,, \tag{11.50}$$

or, equivalently,

$$\hat\lambda_i(\mu) = \hat\lambda_i(\Lambda)\left(\frac{\mu}{\Lambda}\right)^{-\Delta_i} + \text{corrections}\,. \tag{11.51}$$

- **Non-renormalizable couplings** ($\Delta_i < 0$) become unimportant for $\mu \sim E \ll \Lambda$:

  $$\hat\lambda_i(\mu) \sim \left(\frac{\mu}{\Lambda}\right)^{|\Delta_i|} \ll 1\,. \tag{11.52}$$

  This explains why low-energy phenomena are described approximately by renormalizable QFTs! Irrelevant

  Operators O_i with $[O_i] > 4$ (and therefore $\Delta_i < 0$) are called **irrelevant**.

- Renormalizable couplings and operators with $[O_i] = 4$ are called **marginal**. Wether they grow or not with decreasing μ is determined by the next term $\beta_i^{jk}\hat{\lambda}_j\hat{\lambda}_k$ and not by their canonical mass dimension. The marginal running is, if it is perturbative, very slow. Marginal

- Super-renormalizable couplings **grow** for $\mu \ll \Lambda$. In particular, the mass term belongs to this category:

$$m^2(\mu) = \mu^2\hat{\lambda}_2(\mu) = \Lambda^2\hat{\lambda}_2(\Lambda) = \mathcal{O}(\Lambda^2)\,! \tag{11.53}$$

Relevant

  A mass term $m^2(\mu) \ll \Lambda^2$ requires a fine-tuning of $\hat{\lambda}_2(\Lambda)$ to very small values.

  These operators with $[O_i] < 4$ are called **relevant**.

11.4 Wilsonian RGE

Before μ was the renormalization scale, and $\lambda_i(\mu)$ was the renormalized coupling after taking the cut-off to ∞ (i.e. $\epsilon \longrightarrow 0$ or $\Lambda \longrightarrow \infty$). This is called **continuum** RGE.
There is another useful picture (introduced by Wilson in the 1960's), which is convenient for systems where the cut-off is physical (e.g. lattice size, atomic spacing,...). An example is a system near a second order phase transition:

$$\Lambda \sim \frac{1}{a} \gg m \sim \frac{1}{\xi}\,, \tag{11.54}$$

where a is approximately the atomic spacing and ξ quantifies the correlation length or thermal fluctuation of the system. We can describe the system using a microscopic system with the following Lagrangian:

$$\mathcal{L}[\phi_i, \lambda_i(\Lambda), \underset{\substack{\uparrow \\ \text{physical cut-off}}}{\Lambda}] \supset \sum_i \overset{\substack{\text{non-renormalizable interactions} \\ \downarrow}}{\lambda_i(\Lambda)} O_i\,. \tag{11.55}$$

If we are only interested in the long-range (low E) physics, we can "integrate out" the high momentum modes by lowering the cut-off to Λ'. One then defines $\lambda_i(\Lambda')$ such that the long-range (IR) physics is unchanged:

$$\mathcal{L}_{\text{eff}}[\phi_i, \lambda_i(\Lambda'), \Lambda'] = \mathcal{L}[\phi_i, \lambda_i(\Lambda), \Lambda] + \mathcal{O}\left(\frac{1}{\Lambda'}\right). \tag{11.56}$$

One then finds that the $\lambda_i(\Lambda')$ for $\Lambda' \ll \Lambda$ are independent of the **initial values** of **irrelevant** operators.
Therefore second order phase transitions are characterized by a universal behavior and are determined by the IR fixed point of the flow of renormalizable couplings (irrespective of the microphysics!).
For $m \ll \Lambda$ fine-tuning is needed , which is provided by the experimental setup (e.g. tuning the temperature to the critical value).
Indeed, in the Landau-Ginzburg theory the term

$$-\frac{1}{2}c(T - T_c)\phi^2\,, \tag{11.57}$$

with T_c the critical temperature of the system, plays the role of the "mass term" of the scalar field.

11.5 RGE-flow of Green functions

Collecting our results, for now we obtained several RGEs:

$$\beta_i(\lambda_j) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\lambda_i \,, \tag{11.58}$$

$$\gamma_m(\lambda_j) = \frac{\mu^2}{m}\frac{\partial m}{\partial \mu^2} \quad \text{(mass anomalous dimension)}\,, \tag{11.59}$$

$$\gamma_\phi(\lambda_j) = \frac{\mu^2}{Z_\phi}\frac{\partial Z_\phi}{\partial \mu^2} \quad \text{(field anomalous dimension)}\,. \tag{11.60}$$

We even have one for composite operators:

$$\gamma_{mn}(\lambda_i) = \left(Z^{-1}\right)_{mk}\mu^2\frac{\partial Z_{kn}}{\partial \mu^2}\,. \tag{11.61}$$

We will now derive the scale-dependence of n-point functions:

$$\widetilde{G}^{(n)}(p_i; m, \lambda)\,, \quad \text{mass dimension: } d_G \tag{11.62}$$

is the renormalized n-point function, expressed in terms of the renormalized quantities m, λ, ϕ. Now we rescale all the momenta by a common factor:

$$p_i \longrightarrow a p_i \tag{11.63}$$

$$\frac{p_i}{p_j} \longrightarrow \frac{p_i}{p_j}\,, \quad \frac{p_i}{m} \longrightarrow \frac{a p_i}{m} = \frac{p_i}{\left(\frac{m}{a}\right)}\,. \tag{11.64}$$

Hence:

$$\widetilde{G}^{(n)}(a p_i; m, \lambda) = a^{d_G}\widetilde{G}^{(n)}\left(p_i; \frac{m}{a}, \lambda\right), \tag{11.65}$$

but this is **wrong**, since $\widetilde{G}^{(n)}$ depends on the renormalization scale μ (or the cut-off Λ if we are working with the regularized Green function rather than the renormalized one).
The correct statement is:

$$\widetilde{G}^{(n)}(a p_i; m, \lambda, \mu) = a^{d_G}\widetilde{G}^{(n)}\left(p_i; \frac{m}{a}, \lambda, \overset{!!!}{\overset{\downarrow}{\frac{\mu}{a}}}\right). \tag{11.66}$$

If we take a to be large, the RHS of Eq. (11.66) will contain large logarithms and render perturbation theory useless, since

$$\ln\left(\frac{p_i}{\frac{\mu}{a}}\right) \approx \ln(a) \tag{11.67}$$

for $p_i \sim \mathcal{O}(\mu)$. Our goal is therefore to find an RGE for the RHS of Eq. (11.66) to allow us to change $\frac{\mu}{a}$ back to μ.

The bare Green function is independent of μ:

Bare Green Function

$$\langle\Omega|T\{\phi_0(x_1)\dots\phi_0(x_n)\}|\Omega\rangle \tag{11.68}$$

with the bare field $\phi_0(x) = \sqrt{Z_\phi}\phi(x)$ in terms of the renormalized field we find the expression in Eq. (11.69).

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\Big(\underbrace{\sqrt{Z_\phi}^{\,n}\,\widetilde{G}^{(n)}(p_i, m(\mu), \lambda(\mu); \mu)}_{=\widetilde{G}^{0,(n)}(p_1,\dots,p_n)}\Big) = 0 \tag{11.69}$$

and therefore, by the chain rule,

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\mu^2\frac{\partial\lambda}{\partial\mu^2}\frac{\partial}{\partial\lambda}+\mu^2\frac{\partial m}{\partial\mu^2}\frac{\partial}{\partial m}+\frac{n}{2}\frac{\mu^2}{Z_\phi}\frac{\partial Z_\phi}{\partial\mu^2}\right)\widetilde{G}^{(n)}(p_i,m(\mu),\lambda(\mu);\mu)=0$$

and, plugging in the definitions from Eq. (11.58)-Eq. (11.60) we finally get

$$0=\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta(\lambda)\frac{\partial}{\partial\lambda}+\gamma_m(\lambda)m\frac{\partial}{\partial m}+\frac{n}{2}\gamma_\phi(\lambda)\right)\widetilde{G}^{(n)}\,. \tag{11.70}$$

Callan - Symanzik - Gell-Mann - Low formula

We can simplify this complicated partial differential equation. If the $m(\mu)$, $\lambda(\mu)$ are chosen such that they solve the RGEs Eq. (11.58)-Eq. (11.60) with the initial conditions $\lambda(\mu_0)=\lambda$, $m(\mu_0)=m$, then

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\frac{n}{2}\gamma_\phi(\lambda(\mu))\right)F(\mu)=0 \tag{11.71}$$

and therefore

$$F(\mu)=F(\mu_0)\exp\left(-\frac{n}{2}\int_{\mu_0^2}^{\mu^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\gamma_\phi(\lambda(\mu'))\right), \tag{11.72}$$

or:

$$\widetilde{G}^{(n)}(p_i,m(\mu),\lambda(\mu);\mu)=\widetilde{G}^{(n)}(p_i,m(\mu_0),\lambda(\mu_0);\mu_0)\exp\left(-\frac{n}{2}\int_{\mu_0^2}^{\mu^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\gamma_\phi(\lambda(\mu'))\right).$$

We can use this in the rescaling equation Eq. (11.66) with $p_i\longrightarrow ap_i$:

$$G^{(n)}(ap_i,m(\mu_0),\lambda(\mu_0);\mu_0)\cdot\exp\left(-\frac{n}{2}\int_{\mu_0^2}^{\mu^2}\frac{\mathrm{d}\mu'^2}{\mu'^2}\gamma_\phi(\lambda(\mu'))\right)=a^{d_G}\widetilde{G}^{(n)}\left(p_i,\frac{m(\mu)}{a},\lambda(\mu),\frac{\mu}{a}\right)$$

Now we choose $\mu=a\mu_0$ and rename $\mu_0\to\mu$:

$$\widetilde{G}^{(n)}(ap_i,m(\mu),\lambda(\mu);\mu)=a^{d_G}\underbrace{\exp\left(\int_1^a\frac{\mathrm{d}a'}{a'}n\gamma_\phi(\lambda(a'\mu))\right)}_{(*)}\widetilde{G}^{(n)}\left(p_i,\frac{m(a\mu)}{a},\lambda(a\mu);\mu\right). \tag{11.73}$$

Compared to the naive (wrong) result of Eq. (11.65), there is an additional factor $(*)$ which modifies the naive dimensional scaling (a^{d_G}), which involves the **anomalous dimension of the field**, and a change in the coupling constants, which takes into account the scale dependence of $m(a\mu)$ and $\lambda(a\mu)$.

Note: For amputated Green functions:

$$\mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\left(\sqrt{Z_\phi}^{-n}\widetilde{G}^{(n)}_{\text{amp}}\right)=0 \tag{11.74}$$

and hence we must substitute $n\gamma_\phi\longrightarrow -n\gamma_\phi$ in Eq. (11.73).

11.5.1 Applications and examples

Consider the two-point function of a scalar field $\widetilde{G}^{(2)}(p^2,m(\mu),\lambda(\mu);\mu)$. The mass dimension of it is $d_G=-2$.

Now we take $p \longrightarrow p_0$ and $a = \frac{p}{p_0}$:

$$\widetilde{G}^{(2)}(p^2, m(\mu), \lambda(\mu); \mu) = \frac{p_0^2}{p^2} \exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} 2\gamma_\phi\big(\lambda(a'\mu)\big)\right) \widetilde{G}^{(2)}\left(p^2, \frac{m(a\mu)}{a}, \lambda(a\mu); \mu\right).$$

We distinguish two cases:

1) **Small coupling:**

$$\gamma_\phi(\lambda) = \gamma_0 \lambda + \ldots, \quad \beta(\lambda) = \beta_0 \lambda^2 + \ldots \tag{11.75}$$

and therefore

$$\exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} 2\gamma_\phi\big(\lambda(a'\mu)\big)\right) = \exp\left(\int_{\mu^2}^{a\mu^2} \frac{\mathrm{d}\mu'^2}{\mu'^2} \gamma_\phi\big(\lambda(\mu')\big)\right) \tag{11.76}$$

$$= \exp\left(\int_{\lambda(\mu)}^{\lambda(a\mu)} \mathrm{d}\lambda' \frac{\gamma_\phi(\lambda')}{\beta(\lambda')}\right) \tag{11.77}$$

$$\approx \exp\left(\int_{\lambda(\mu)}^{\lambda(a\mu)} \mathrm{d}\lambda' \frac{\gamma_0}{\beta_0 \lambda'}\right) \tag{11.78}$$

$$= \exp\left[\frac{\gamma_0}{\beta_0} \ln\left(\frac{\lambda(a\mu)}{\lambda(\mu)}\right)\right] \tag{11.79}$$

$$= \left(\frac{\lambda(a\mu)}{\lambda(\mu)}\right)^{\frac{\gamma_0}{\beta_0}}. \tag{11.80}$$

Therefore

$$\widetilde{G}^{(2)}(p^2, m(\mu), \lambda(\mu); \mu) \overset{p^2 \to \infty}{\sim} \frac{1}{p^2} \underset{\substack{\uparrow \\ (*)}}{\lambda(p)^{\frac{\gamma_0}{\beta_0}}} \cdot \left[1 + \mathcal{O}\left(\lambda, \frac{m^2}{p^2}\right)\right], \tag{11.81}$$

where $(*)$ is a deviation from the naive scaling behavior $\frac{1}{p^2}$ resulting from dimensional analysis (which would indicate that $p^2 \longrightarrow \infty$ dominates over m^2).

2) **Non-trivial fixed-point:** for $a \longrightarrow \infty$ $\lambda \longrightarrow \lambda_*$, see Fig. 11.5. We can approximate

$$\exp\left(\int_1^a \frac{\mathrm{d}a'}{a'} 2\gamma_\phi\big(\lambda(a'\mu)\big)\right) = \exp\left(2\gamma_\phi(\lambda_*) \int_1^a \frac{\mathrm{d}a'}{a'}\right) \tag{11.82}$$

$$= \left(\frac{p^2}{p_0^2}\right)^{\gamma_\phi(\lambda_*)} \tag{11.83}$$

and therefore

$$\widetilde{G}^{(2)}(p^2, m(\mu), \lambda(\mu); \mu) \sim \underset{\substack{\uparrow \\ \text{compared to small-coupling} \\ \text{logarithmic modification}}}{\frac{1}{(p^2)^{1-\gamma_\phi(\lambda_*)}}} \cdot \left(1 + \mathcal{O}\left(\lambda, \frac{m^2}{p^2}\right)\right). \tag{11.84}$$

Here the power law changes and the field dimension changes from 1 to $1 - \gamma_\phi(\lambda_*)$ near the fixed point.

This shows that near the fixed point λ_* quantum fields behave differently than their canonical mass dimension, with important implications for renormalization theory.

We based our analysis on perturbation theory; if a theory has a non-trivial UV fixed point (and the question about the existence of it cannot be answered in perturbation theory), then, even if it looks non-renormalizable by perturbative power counting (e.g. the propagators are $\sim \frac{1}{p^2} \xrightarrow{p^2 \to 0} \infty$) may in fact be renormalizable! In this case the anomalous dimensions near the fixed point are crucial, which are however difficult non-perturbative issues which can be solved only in a few cases.

E.g. gravity! http://www.scholarpedia.org/article/Asymptotic_Safety_in_quantum_gravity

But: Recall the constraint from Eq. (9.164) from the spectral decomposition, stating that a two-point function can go to 0 maximally as fast as $\frac{1}{p^2}$ for $p^2 \longrightarrow \infty$.

12
Poincaré Group, Spin and Relativistic Particles

Until now we have been discussing the general features of QFT. Now we will explore what the Lagrangian for the real world could possibly be.
There is a deep connection between spin and Lorentz-invariance that is obscure in non-relativistic quantum mechanics.

12.1 Symmetries and their representations

We will use the following notation: G will stand for a group and g for an element of that group. We define the **representation** of a group element through the mapping

$$g \mapsto U(g) \tag{12.1}$$

and U is unitary if it leaves the scalar product invariant:

$$\langle\psi|\varphi\rangle \mapsto \langle\psi|U^\dagger(g)U(g)|\varphi\rangle = \langle\psi|\varphi\rangle\,. \tag{12.2}$$

If G is a Lie group, we can write g as

$$g = e^{-i\epsilon^A t^A}\,, \tag{12.3}$$

where ϵ^A are coordinates on the group manifold and t^A are the **generators** of the Lie group G. This induces the following form for the representation:

$$U(g) = e^{-i\epsilon^A T^A}\,, \tag{12.4}$$

with T^A the representation of the generators t^A.

The generators of the Lie group span the Lie algebra associated to the group and obey

$$[T^A, T^B] = i\underbrace{f^{ABC}}_{\text{structure constants}} T^C + i\overbrace{Z^{AB}}^{\text{central charge}}\mathbb{1}\,. \tag{12.5}$$

Lie groups are **continuous groups**[1], parametrized by real parameters ϵ^A, which connect any group element continuously to the identity $\mathbb{1}$.

[1] as opposed to discrete groups, where the elements are separated – this makes Lie groups differentiable manifolds.

Lie-group example: the group $SU(2)$, or the special unitary group in 2 dimensions.

$$U^\dagger U = \mathbb{1}, \qquad \text{and} \qquad \det U = 1 \tag{12.6}$$

The generators obey

$$[t^i, t^j] = i\epsilon^{ijk} t^k$$

with ϵ^{ijk} the Levi-Civita tensor. In the fundamental representation, the group acts on complex 2-vectors:

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto U(g) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} .$$

U is a complex 2×2 matrix, which we can write as

$$U = e^{-i\theta^i T^i} .$$

The generators are hermitian $U^\dagger U = \mathbb{1} \to 0 = T^i - (T^i)^\dagger$ and traceless $0 = \ln \det U = \operatorname{Tr} \ln U \to \operatorname{Tr} T^i = 0$, after using the generators and expanding. Therefore we have three generators which we parametrize with

$$T^i = \frac{\sigma^i}{2}$$

with σ^i the Pauli-matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The $SU(2)$ group elements can also be parametrized using

$$g = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} , \quad \alpha, \beta \in \mathbb{C}$$

which is unitary and satisfies $\det U = 1$, if the parameters fulfil the condition

$$|\alpha|^2 + |\beta|^2 = 1 . \quad (*)$$

By writing

$$\alpha = a + ib , \quad \beta = c + id ,$$

the condition $(*)$ becomes

$$a^2 + b^2 + c^2 + d^2 = 1$$

and we see explicitly that the group manifold of $SU(2)$ is the unit 3-sphere S^3.

12.2 Unitary representations of the Poincaré group

The group of space-time translations and Lorentz transformations is called **Poincaré group**, or $ISO(1,3)$, i.e. the group of isometries of Minkowski space.

If we perform a boost or a rotation only the spins or momenta change, the other quantum numbers stay the same. Therefore, particles can be loosely defined as a *set of states that mix only among themselves under Poincaré transformations.*

States transform as

$$|\psi\rangle \mapsto U|\psi\rangle\,, \tag{12.7}$$

with U a representation of the Poincaré group. We take a basis of states $\{|\psi_i\rangle\}$ for $|\psi\rangle$ and write the above as

$$|\psi_i\rangle \mapsto U_{ij}|\psi_j\rangle\,, \tag{12.8}$$

with i, j discrete or continuous indices. If there is **no** subset of states $\{|\psi_i'\rangle\}$ of $\{|\psi_i\rangle\}$ which transform only among themselves under U_{ij}, then we call the representation **irreducible**.

We want **unitary** representations, since we want matrix elements

$$M = \langle\psi_1|\psi_2\rangle \tag{12.9}$$

to be Poincaré-**invariant**: if M is to be Poincaré-invariant and $|\psi_1\rangle$, $|\psi_2\rangle$ transform covariantly, then we request

$$M = \langle\psi_1|U^\dagger U|\psi_2\rangle \stackrel{!}{=} \langle\psi_1|\psi_2\rangle \tag{12.10}$$

and therefore $U^\dagger U \stackrel{!}{=} \mathbb{1}$ for M to be invariant, which is the definition of unitarity.

The unitary representations are only a small subset of all the representations of the Poincaré group. For example, we will see that the four-vector representation A_μ is **not** unitary, but the unitary ones are the only ones for which we can compute Poincaré-invariant matrix elements, and we therefore need to find them.

Particles transform under irreducible unitary representations of the Poincaré group.

12.3 Unitarity vs Lorentz-invariance

There is a conflict between unitarity and Lorentz-invariance. Recall the treatment of spin in non-relativistic quantum mechanics: a general state can be written as

$$|\psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle\,, \tag{12.11}$$

with norm

$$\langle\psi|\psi\rangle = |c_1|^2 + |c_2|^2 > 0\,. \tag{12.12}$$

since the basis states are orthonormal.
This norm is invariant under rotations. If we choose e.g. the 2-axis as the rotation axis, the representation of the rotation is

$$e^{i\theta\frac{\sigma^2}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{12.13}$$

and

$$|\uparrow\rangle \mapsto \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle\,, \tag{12.14}$$

$$|\downarrow\rangle \mapsto -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle\,. \tag{12.15}$$

Now we do the same for a basis of four states $|V_\mu\rangle$ which transforms as a four-vector under Lorentz-transformations. We can choose an arbitrary linear combination:

$$|\psi\rangle = c_0|V_0\rangle + c_1|V_1\rangle + c_2|V_2\rangle + c_3|V_3\rangle\,, \tag{12.16}$$

with norm

$$\langle\psi|\psi\rangle = |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2\,. \tag{12.17}$$

This is the norm for any basis and it is always positive.
However: The norm is **not** Lorentz-invariant!

Four-vector example:

1) Start with

$$|\psi\rangle = |V_0\rangle\,, \quad \langle\psi|\psi\rangle = 1\,. \tag{12.18}$$

2) Boost in x-direction:

$$|\psi'\rangle = \cosh\beta|V_0\rangle + \sinh\beta|V_1\rangle\,. \tag{12.19}$$

The norm is now

$$\langle\psi'|\psi'\rangle = \cosh^2\beta + \sinh^2\beta \neq 1 \neq \langle\psi|\psi\rangle\,. \tag{12.20}$$

Thus the probability of finding a state depends on the reference frame, which is a strange result.
The boost matrix is clearly not unitary:

$$\Lambda = \begin{pmatrix} \cosh\beta & \sinh\beta & 0 & 0 \\ \sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \quad \Lambda^\dagger \neq \Lambda^{-1}\,. \tag{12.21}$$

Could we modify the norm to be (N'-norm)

$$\langle\psi|\psi\rangle_{N'} = |c_0|^2 - |c_1|^2 - |c_2|^2 - |c_3|^2\,? \tag{12.22}$$

This norm would be Lorentz-invariant! However, it would **not** be positive definite, which is not a problem by itself. The probability would still be greater or equal to 0, since

$$P = |\langle\psi|\psi\rangle_{N'}|^2 \geq 0\,. \tag{12.23}$$

However, probabilities will no longer be less or equal to 1. Take the example from above:

1)

$$|\psi\rangle = |V_0\rangle\,, \quad \langle\psi|\psi\rangle_{N'} = |c_0|^2 = 1^2 = 1\,. \tag{12.24}$$

2) Boost in x-direction:

$$|\psi'\rangle = \cosh\beta|V_0\rangle + \sinh\beta|V_1\rangle\,. \tag{12.25}$$

The boosted "new norm" is then

$$\langle\psi'|\psi'\rangle_{N'} = |\cosh\beta|^2 - |\sinh\beta|^2 = 1\,. \tag{12.26}$$

3) **Problem:** The probability of finding $|\psi'\rangle$ in the state $|\psi\rangle = |V_0\rangle$ is

$$|\langle V_0|\psi'\rangle|^2 = \cosh^2\beta\,. \tag{12.27}$$

Since for $\beta \neq 0$ $\cosh\beta > 1$, this cannot define a probability!

Lorentz-transformations can now **mix positive and negative norm** states, and therefore there is no way to interpret this in terms of probabilities. A probability interpretation with $0 \leq P \leq 1$ requires only positive (or only negative) norm states.

Summary: The physics of quantum mechanics requires a probability interpretation, and therefore the use of the metric δ^μ_ν, while Lorentz-invariance requires the Minkowski-metric $\eta_{\mu\nu}$. It appears that the two exclude each other.

12.4 Poincaré group and particles in relativistic QFT

Lorentz-transformations (Λ) and space-time translations (a) form a group:

$$\begin{aligned} x''^\mu &= {\Lambda_2}^\mu{}_\nu x'^\nu + a_1^\mu = {\Lambda_2}^\mu{}_\nu(\Lambda_1 x + a_1)^\nu + a_2^\mu &(12.28)\\ &= (\Lambda_2\Lambda_1)^\mu{}_\nu x^\nu + (\Lambda_2 a_1 + a_2)^\mu\,. &(12.29)\end{aligned}$$

We denote elements of the Poincaré group as (Λ, a):

- $\Lambda = \mathbb{1}$: space-time translations.
- $a = 0$: Lorentz group.

Lorentz-transformations leave the product $x^\mu y_\mu$ invariant:

$$x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\nu \stackrel{!}{=} \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha y^\beta \tag{12.30}$$

and this is equivalent to requesting that

$$\eta_{\alpha\beta} = \Lambda^\mu{}_\alpha\eta_{\mu\nu}\Lambda^\nu{}_\beta \tag{12.31}$$

or

$$\boxed{\eta = \Lambda^T\eta\Lambda\,.} \tag{12.32}$$

From Eq. (12.32) follows

$$\big(\det(\Lambda)\big)^2 = 1 \quad\Longrightarrow\quad \det(\Lambda) = \pm 1 \tag{12.33}$$

and

$$\eta_{\mu\nu}\Lambda^\mu{}_0\Lambda^\nu{}_0 = 1 \quad\Longrightarrow\quad \left(\Lambda^0{}_0\right)^2 = 1 + \sum_{i=1}^{3}\left(\Lambda^i{}_0\right)^2, \tag{12.34}$$

which means that

$$\Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1\,. \tag{12.35}$$

We therefore distinguish two subgroups:

1) $\det(\Lambda) = 1$

2) $\Lambda^0{}_0 \geq 1$

and the transformations which have both of the above properties are called proper ($\det(\Lambda) = 1$), orthochronous ($\Lambda^0{}_0 \geq 1$) Lorentz-transformations.
The Lorentz group has four connected components:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\det(\Lambda) = 1\,,\ \Lambda^0{}_0 \geq 1 \\ \text{othrocronous proper } \mathcal{L}_+^{\uparrow}\end{array}} \arrow[r, "\mathcal{P} = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & -\mathbb{1}_{3\times 3}\end{pmatrix}", "\begin{array}{c}\text{parity}\\ \text{space inversion}\end{array}"'] \arrow[d, "\mathcal{T} = \begin{pmatrix} -1 & \mathbf{0}^T \\ \mathbf{0} & \mathbb{1}_{3\times 3}\end{pmatrix}", "\text{time reversal}"'] & \boxed{\begin{array}{c}\det(\Lambda) = -1\,,\ \Lambda^0{}_0 \geq 1 \\ \mathcal{L}_-^{\uparrow}\end{array}} \arrow[d, "\mathcal{T}"'] \\
\boxed{\begin{array}{c}\det(\Lambda) = -1\,,\ \Lambda^0{}_0 \leq -1 \\ \mathcal{L}_+^{\downarrow}\end{array}} \arrow[r, "\mathcal{P}"] & \boxed{\begin{array}{c}\det(\Lambda) = 1\,,\ \Lambda^0{}_0 \leq -1 \\ \mathcal{L}_-^{\downarrow}\end{array}}
\end{tikzcd}$$

We can decompose the Lorentz group $\mathcal{L}$ in the following way:

$$\mathcal{L} = \mathcal{L}_+^{\uparrow} \otimes \mathcal{P} \otimes \mathcal{T}\,. \tag{12.36}$$

We only need to require orthochronous, proper Lorentz-invariance for the theory to be relativistically invariant. The weak interactions in the Standard Model do **not** respect $\mathcal{P}$ and $\mathcal{T}$.

12.4.1 Poincaré-algebra

We want to find the commutator of the generators of the transformations $U(\Lambda, a)$, with

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = U(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)\,. \tag{12.37}$$

We write infinitesimal Lorentz-transformations and space-time translations as

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + \dots\,, \quad a^\mu = \epsilon^\mu + \dots\,, \tag{12.38}$$

with

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad \text{antisymmetric}, \tag{12.39}$$

note that the upper index has been lowered. It is antisymmetric because from the defining property of Lorentz-transformations

Eq. (12.32) follows:

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\mu\nu}\left(\delta^{\mu}_{\alpha} + \omega^{\mu}{}_{\alpha}\right)\left(\delta^{\nu}_{\beta} + \omega^{\nu}{}_{\beta}\right) \tag{12.40}$$

$$= \eta_{\alpha\beta} + \underbrace{\left(\omega_{\beta\alpha} + \omega_{\alpha\beta}\right)}_{\stackrel{!}{=}0} + \ldots \tag{12.41}$$

For a given representation $U(\Lambda, a)$ we can write

$$U(1+\omega, \epsilon) = \mathbb{1} - \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_{\mu}P^{\mu} + \ldots, \tag{12.42}$$

where the $J^{\mu\nu} = -J^{\nu\mu}$ are the six generators of Lorentz-transformations and the four P^{μ} are components of four-momentum which generate space-time translations.

$U^{-1}(\Lambda, a) = U\left(\Lambda^{-1}, -\Lambda^{-1}a\right)$ from

$U(\Lambda, a)U^{-1}(\Lambda, a) = \mathbb{1}$.

Now we consider

$$\underset{\text{in } \mathcal{L}_{+}^{\uparrow}!}{\underbrace{U(\Lambda, a)U(1+\omega, \epsilon)U^{-1}(\Lambda, a)}} \tag{12.43}$$

and expand:

$$U(\Lambda, a)J^{\mu\nu}U^{-1}(\Lambda, a) = \Lambda_{\rho}{}^{\mu}\Lambda_{\sigma}{}^{\nu}\left(J^{\rho\sigma} - a^{\rho}P^{\sigma} + a^{\sigma}P^{\rho}\right), \tag{12.44}$$

$$U(\Lambda, a)P^{\mu}U^{-1}(\Lambda, a) = \Lambda_{\rho}{}^{\mu}P^{\rho}. \tag{12.45}$$

Therefore $J^{\mu\nu}$ transforms as a tensor under Lorentz transformations, while P^{μ} transforms as a vector. These relations will be derived in the exercise!

Now we take (Λ, a) as infinitesimal and expand the above to linear order. This gives us

Poincaré-algebra

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(\eta^{\mu\sigma}J^{\nu\rho} + \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho}\right) \tag{12.46}$$

$$\left[P^{\mu}, J^{\nu\rho}\right] = i\left(\eta^{\mu\nu}P^{\rho} - \eta^{\mu\rho}P^{\nu}\right) \tag{12.47}$$

$$\left[P^{\mu}, P^{\nu}\right] = 0 \tag{12.48}$$

The generators of spatial rotations are

$$\boldsymbol{J} = \left(J^1, J^2, J^3\right) = \left(J^{23}, J^{31}, J^{12}\right), \tag{12.49}$$

which satisfy

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k, \tag{12.50}$$

while the generators of boosts are

$$\boldsymbol{K} = \left(K^1, K^2, K^3\right) = \left(J^{01}, J^{02}, J^{03}\right). \tag{12.51}$$

Two boosts give a rotation,

$$\left[K^i, K^j\right] = -i\epsilon^{ijk}J^k, \tag{12.52}$$

and a boost rotates like a three-vector:

$$\left[J^i, K^j\right] = i\epsilon^{ijk}K^k. \tag{12.53}$$

For energy, momentum and angular momentum we have the following relations:

$$\left[P^0, J^i\right] = 0\,, \quad \left[P^0, P^\mu\right] = 0\,, \tag{12.54}$$

which imply angular momentum and four-momentum conservation, and

$$\left[P^0, K^i\right] = -iP^i\,, \quad \left[P^i, J^j\right] = i\epsilon^{ijk}P^k\,, \quad \left[P^i, K^j\right] = -i\delta^{ij}P^0\,. \tag{12.55}$$

This algebra is independent of any particular representation $U(\Lambda, a)$. There are representatives which are not unitary. As you might know from the rotation group, $\mathcal{L}_+^\uparrow$ has projective representations, i.e. representations up to a sign.

12.4.2 Classifying particle states

If $|\psi\rangle$ is a one-particle state, then the transformed state $U(\Lambda, a)|\psi\rangle$ should be, too (the notion of particle should be frame-independent). It should also not be divisible into separate entities. We therefore **define**:

$$\left\{U(\Lambda, a)|\psi\rangle \middle| \forall(\Lambda, a) \in ISO(1,3)\right\} = \begin{array}{c}\text{minimal invariant subspace}\\ \text{on which a representation}\\ U(\Lambda, a)\text{ is irreducible}\end{array}.$$

Therefore a relativistic particle is defined by an irreducible representation (rep) of the Poincaré group.

We characterize irreducible, unitary reps by invariant operators. Invariant operators are Lorentz scalars (but not all Lorentz scalars are invariants). We construct from

$$P^\mu\,, \quad J^{\mu\nu}\,, \quad W_\mu \equiv \frac{1}{2}\underset{\underset{\epsilon_{0123}=-1}{\uparrow}}{\epsilon_{\mu\nu\rho\sigma}}J^{\nu\rho}P^\sigma\,, \tag{12.56}$$

with W_μ the Pauli-Lubanski vector, the two Poincaré-invariants

$$P^2 = P_\mu P^\mu\,, \quad W^2 = -W_\mu W^\mu\,, \tag{12.57}$$

which commute with all the generators P^μ, $J^{\mu\nu}$. Other possible combinations vanish

$$P^\mu W_\mu = 0\,, \quad P^\mu P^\nu J_{\mu\nu} = 0\,, \quad W^\mu W^\nu J_{\mu\nu} = 0 \tag{12.58}$$

or as $J_{\mu\nu}J^{\mu\nu}$, which, however, are **not** invariant.[2]

[2] See exercise: $[J_{\mu\nu}J^{\mu\nu}, P_\kappa] \neq 0$

We will understand the physical significance of the two invariants soon: P^2 will lead to the mass and W^2 to the spin of a particle.

12.5 One-particle basis states

A relativistic particle is characterized by two invariant quantities, the eigenvalues of P^2 (mass) and W^2 (spin). This defines **mass** and **spin**.[3]

[3] We follow Weinberg I here.

For a given eigenvalue of P^2, W^2, we label the basis states with a maximal set of operators,each commuting with P^2 and W^2 (and itself). We use:

$$P^\mu\,,\ \mu = 1,2,3 \quad \text{and} \quad W^3\,. \tag{12.59}$$

We write the basis states of a representation with mass m and spin s as

$$\underset{\substack{\uparrow \\ \text{eigenvalues of } \boldsymbol{P}}}{|\boldsymbol{p}}, \overset{\substack{\text{eigenvalues of } W^3 \\ \downarrow}}{s}\rangle\,. \tag{12.60}$$

- We do not need P^0 to characterize the state because it is determined from the P^2-eigenvalue through the on-shell condition

$$\left(P^0\right)^2 = m^2 + \boldsymbol{p}^2\,. \tag{12.61}$$

- The $\boldsymbol{p}$ spectrum is continuous, therefore the rep to which the basis states belong must be infinite dimensional (all unitary reps of the Poincaré group are infinite-dimensional).
- We still need to determine the possible values of s and the eigenvalues of W^2.

We normalize the one-particle states as follows:

$$\langle \boldsymbol{p}, s|\boldsymbol{p'}, s'\rangle = (2\pi)^3 2p^0 \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'})\underset{\substack{\uparrow \\ \text{new!}}}{\delta_{ss'}}\,. \tag{12.62}$$

$p^0 = \sqrt{m^2 + \boldsymbol{p}^2}$

We can use all of the previously developed objects (Fock space, multiparticle states,...) as for the scalar field. What is **new** here is the spin degree of freedom.

The eigenvalues of P^2,

$$P^2|\boldsymbol{p}, s\rangle = m^2|\boldsymbol{p}, s\rangle\,, \tag{12.63}$$

can be:

a) $m^2 > 0$

b) $m^2 = 0$

c) $m^2 < 0$ tachyons (faster than light)

d) $m^2 = 0$ **and** $\boldsymbol{p} = 0$ (vacuum)

12.5.1 Massive particle states ($m^2 > 0$)

The energy of a particle is given as usual by

$$p^0 = \pm\sqrt{m^2 + \boldsymbol{p}^2} \tag{12.64}$$

and we assume $p^0 > 0$, which is preserved under $\mathcal{L}_+^\uparrow$. Our goal is to find the values of s for a given irreducible representation (irrep) and give it physical meaning. What we are after is how a state with spin transforms under a Lorentz transformation

$$U(\Lambda)|\boldsymbol{p}, s\rangle = ?$$

It cannot be just $U(\Lambda)|\boldsymbol{p}, s\rangle = |\Lambda \boldsymbol{p}, s\rangle$ since e.g. the spin will change under a general rotation or pick up a phase.
Note however:

$$P^\mu\big(U(\Lambda)|\boldsymbol{p}, s\rangle\big) \underset{\substack{\uparrow \\ \text{uses} \\ U^{-1}P^\mu U = \Lambda^\mu{}_\nu P^\nu}}{=} \big(\Lambda p\big)^\mu \big(U(\Lambda)|\boldsymbol{p}, s\rangle\big) \tag{12.65}$$

We see, that $U(\Lambda)|\boldsymbol{p}, s\rangle$ is a linear combination of $|\boldsymbol{\Lambda p}, s'\rangle$.

For $m^2 > 0$ and a given p^μ we can find Lorentz-transformations $L(p)$ such that

$$p^\mu = L^\mu{}_\nu(p) k^\nu \tag{12.66}$$

with $k^\mu = m(1, 0, 0, 0)^T$. $L^{-1}(p)$ is then the boost into the rest frame of p.
We **define** the states of momentum p as

$$|\boldsymbol{p}, s\rangle \equiv U\big(L(p)\big)|\boldsymbol{k}, s\rangle\,. \tag{12.67}$$

There is a freedom in defining the s labels for a single particle state at rest since there is no preferred direction in this frame (spatial rotations do not change the frame). In a way, we can also see this as a redundancy which we remove by fixing it with Eq. (12.67). So maybe a better way at looking at this equation this way: $|\boldsymbol{k}, s\rangle \equiv U^{-1}\big(L(p)\big)|\boldsymbol{p}, s\rangle$, it fixes s by using a definition which makes sense once we have a direction $\boldsymbol{p}$ to compare s to.

We now see how the s labels are related for different momenta. Important, we have the *same* s on both sides. Again, we define s for general momentum states Eq. (12.67).

The interpretation of $|\boldsymbol{k}, s\rangle$ is the rest frame state:

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} k^\sigma = \frac{m}{2}\epsilon_{\mu\nu\rho 0} J^{\nu\rho} \tag{12.68}$$

$$W_0 = 0\,, \quad W_i = mJ^i \tag{12.69}$$

$$W^2 = -W_\mu W^\mu = W_i W_i = m^2 \boldsymbol{J}^2 \tag{12.70}$$

E.g.

$$W_1 = \frac{m}{2}\epsilon_{1\nu\rho 0} J^{\nu\rho} = \frac{m}{2}\big(J^{23} - J^{32}\big) = mJ^1$$

So we see that invariance of W^2 is related to angular momentum, which for $\boldsymbol{p} = 0$ is the intrinsic spin of the particle in its rest frame. The label s is therefore the eigenvalue of $W_3 \sim J^3$ (the spin orientation) which can take the values

$$\underbrace{-j, -j+1, ..., j-1, j}_{2j+1 \text{ numbers}} \quad \text{for spin } j\,, \tag{12.71}$$

A massive particle has intrinsic spin $j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ with $2j + 1$ degrees of freedom.

where j can be an integer of a half-integer.

The relation to the irreps of $SO(3)$ is not accidental: $SO(3)$ is the group which leaves $k^\mu = m(1, 0, 0, 0)^T$ invariant. This is called the **little group** whose representations induce reps of the full Poincaré group.

We find for translations

$$U(\mathbb{1}, a)|\boldsymbol{p}, s\rangle = e^{ia^\mu P_\mu}|\boldsymbol{p}, s\rangle = e^{ia^\mu p_\mu}|\boldsymbol{p}, s\rangle \tag{12.72}$$

for general boosts

$$U(\Lambda,0)|\boldsymbol{p},s\rangle = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}|\boldsymbol{p},s\rangle = \sum_{s'=-j}^{j} D^{(j)}_{ss'}(w)|\boldsymbol{\Lambda p},s'\rangle \tag{12.73}$$

with

$$w \equiv L^{-1}(\Lambda p)\Lambda L(p)\,. \tag{12.74}$$

w is a spatial rotation which depends on the given $\boldsymbol{p}$ and Λ and $D^{(j)}_{s's}(w)$ is the **Wigner function**[4] for w. We derive this relation with

[4] see also math lectures.

$$\begin{aligned}
U(\Lambda,0)|\boldsymbol{p},s\rangle &= U(\Lambda,0)U(L(p),0)|\boldsymbol{k},s\rangle && (12.75)\\
&= U(\Lambda L(p),0)|\boldsymbol{k},s\rangle && (12.76)\\
&= U(L(\Lambda p)\underbrace{L^{-1}(\Lambda p)\Lambda L(p)}_{=w(\Lambda,p)},0)|\boldsymbol{k},s\rangle && (12.77)
\end{aligned}$$

$w(\Lambda,p)$ is a pure rotation that leaves k^μ invariant:[5]

[5] i.e. it only changes the s label and not the $\boldsymbol{k}$ label!

$$\begin{aligned}
L(p)k &\equiv p\\
wk &= L^{-1}(\Lambda p)\Lambda L(p)k = L^{-1}(\Lambda p)L(\Lambda p)k\\
&= k \quad \checkmark
\end{aligned}$$

and is therefore an element of $SO(3)$.. To finally derive Eq. (12.73), we use the fact that the transformation with $U(L(\Lambda p),0)$ is defined in Eq. (12.67).

Note, w operates on a vector as follows:

$$w = \underset{\substack{\uparrow\\ \text{back to } k}}{L^{-1}(\Lambda p)} \quad\cdot\quad \overset{\substack{\text{rotate/boost}\\ \downarrow}}{\Lambda} \quad\cdot\quad \underset{\substack{\uparrow\\ \text{boost } k\to p}}{L(p)} \tag{12.78}$$

$$k \longrightarrow p \longrightarrow \Lambda p \longrightarrow k$$

With

$$U(L(p))|\boldsymbol{k},s\rangle = |\boldsymbol{p},s\rangle$$

we get:

$$\begin{aligned}
U(\Lambda,0)|\boldsymbol{p},s\rangle &= U(L(\Lambda p),0)U(\overset{\substack{\text{pure rotation}\\ \downarrow}}{w},0)|\boldsymbol{k},s\rangle\\
&= U(L(\Lambda p),0)\sum_{s'} D^{(j)}_{ss'}(w)|\boldsymbol{k},s'\rangle\\
&= \sum_{s'} D^{(j)}_{ss'}(w)|\Lambda\boldsymbol{p},s'\rangle \quad \checkmark\,.
\end{aligned}$$

You can convince yourself that if $\Lambda^\mu{}_\nu$ is an arbitrary (pure) rotation $\mathcal{R}$ (no boost), the Wigner rotation w is the same as $\mathcal{R}$ for all p^μ.[6]

[6] See Weinberg I, chapter 2.5.

Thus the states of a moving massive particle have the same transformation under rotation as in non-relativistic quantum mechanics. This is good news: the whole apparatus of spherical harmonics and Clebsch-Gordan coefficients can be carried over from non-relativistic quantum mechanics to relativistic QFT.

Is this a unitary rep? A general Poincaré-transformation is, as we derived,

$$U(\Lambda,a)|\boldsymbol{p},s\rangle = e^{ia^\mu(\Lambda p)_\mu}\sum_{s'} D^{(j)}_{ss'}(w)|\boldsymbol{\Lambda p},s'\rangle \tag{12.79}$$

and therefore

$$\langle\boldsymbol{p},s|U^\dagger(\Lambda,a)U(\Lambda,a)|\boldsymbol{p'},s'\rangle \overset{(1)}{=} \langle\boldsymbol{\Lambda p},s|\boldsymbol{\Lambda p'},s'\rangle \overset{(2)}{=} \langle\boldsymbol{p},s|\boldsymbol{p'},s'\rangle\,, \tag{12.80}$$

where the equality (1) holds because

$$e^{ia^\mu(\Lambda p)_\mu} e^{-ia^\mu(\Lambda p)_\mu} = 1 \tag{12.81}$$

and the $D^{(j)}_{s's}(w)$ are orthonormal functions, i.e.

$$\sum_{s''} D^{(j)}_{s''s}{}^*(w) D^{(j)}_{s''s'}(w) = \delta_{ss'}\,, \tag{12.82}$$

and the equality (2) holds because the normalization we chose in Eq. (12.62),

$$\langle \boldsymbol{p}, s|\boldsymbol{p'}, s'\rangle = (2\pi)^3 2E_{\boldsymbol{p}}\, \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p'})\delta_{ss'}\,, \tag{12.83}$$

is Lorentz-invariant!

12.5.2 Massless particle states $(m = 0)$

Since there is no rest frame for particles with $m = 0$ or $p^2 = 0$, we cannot simply use the limiting case of $m \neq 0$! Massless particles always move at light speed, $p^0 = |\boldsymbol{p}|$.
We choose the reference momentum $k^\mu = n(1, 0, 0, 1)^T$, with n an arbitrary real number, and we then obtain a general momentum as before via

$$p^\mu = L(p)^\mu{}_\nu k^\nu\,. \tag{12.84}$$

As above

$$\begin{aligned} U(\Lambda, 0)|\boldsymbol{p}, s\rangle &= U(\Lambda, 0)U(L(p), 0)|\boldsymbol{k}, s\rangle = U(\Lambda L(p), 0)|\boldsymbol{k}, s\rangle && (12.85)\\ &= U(L(\Lambda p), 0)U(w, 0)|\boldsymbol{k}, s\rangle\,, && (12.86)\end{aligned}$$

since

$$\Lambda L(p) = L(\Lambda p)\, \underbrace{L^{-1}(\Lambda p)\Lambda L(p)}_{=w(p)} = L(\Lambda p)w(p)\,, \tag{12.87}$$

but now $w(p)$ leaves the *massless* reference momentum $k^\mu = n(1, 0, 0, 1)^T$ invariant, because

$$w(p)k = L^{-1}(\Lambda p)\Lambda L(p)k \overset{Eq.\ (12.84)}{\downarrow}{=} L^{-1}(\Lambda p)\Lambda p = k \quad \checkmark \tag{12.88}$$

and the last equality holds because

$$\Lambda p = L(\Lambda p)k \quad \Longrightarrow \quad k = L^{-1}(\Lambda p)\Lambda p\,. \tag{12.89}$$

From $w^\mu{}_\nu k^\nu = k^\mu$ or $\Lambda^\mu{}_\nu k^\nu = k^\mu$ with $\Lambda = w$ we get, using $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + ...$,

$$\omega^\mu{}_\nu k^\nu \overset{!}{=} 0 \tag{12.90}$$

$$\to \quad \omega_{\mu\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \\ \alpha & 0 & \theta & -\alpha \\ \beta & -\theta & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix}. \tag{12.91}$$

Let us derive this form. Starting from a general antisymmetric matrix,

$$\omega'_{\mu\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & C \\ \alpha & 0 & \theta & D \\ \beta & -\theta & 0 & E \\ -C & -D & -E & 0 \end{pmatrix}, \tag{12.92}$$

we use equation Eq. (12.90) to constrain the matrix elements. 4 equations give us 3 conditions, e.g. the first line of Eq. (12.90) reads

$$(0, -\alpha, -\beta, C) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0 \quad \Longrightarrow \quad C = 0 \tag{12.93}$$

and similarly from the other lines we obtain

$$D = -\alpha\,, \quad E = -\beta\,. \tag{12.94}$$

Therefore we get for the representation of the Wigner rotation

$$U(\mathbb{1} + \omega, 0) = \mathbb{1} - i\Big(\theta J^3 + \alpha\underbrace{\left(K^1 + J^2\right)}_{\equiv A} + \beta\underbrace{\left(K^2 + J^1\right)}_{\equiv B}\Big) + \ldots \tag{12.95}$$

J^3 is the generator of rotations about the z-axis and A and B are combinations of rotations and boosts in the x and y directions. They satisfy the **$m = 0$ little group algebra**

$$[A, B] = 0\,, \quad [J^3, A] = iB\,, \quad [J^3, B] = -iA\,. \tag{12.96}$$

A, B and J^3 generate $ISO(2)$, the isometries of the 2-dimensional euclidean plane, consisting of rotations (generated by J^3) and translations (generated by A and B). We can interpret the algebra like this since the generators of translations commute (compare to $[P_x, P_y] = 0$) and the generator of rotations satisfies an analogous relation to $[J^3, P_x] = iP_y$, as expected.

We found therefore a set of three eigenvalues for s, namely the eigenvalues (α, β, σ) of (A, B, J_3). The eigenvalue spectrum of A and B is continuous (think of 2D momentum), but to date no particles with "continuous spin" are known (this is an active field of research in theoretical physics). We therefore must require $\alpha = \beta = 0$.
If $\alpha = \beta = 0$ then $s = \sigma$ is determined by J_3:

Fixing a different value would be impossible: since A and B do not commute with J^3, then the rep would still be infinite-dimensional, because if α and β are eigenvalues, then

$$\alpha\cos\theta + \beta\sin\theta = \alpha'$$
$$-\alpha\sin\theta + \beta\cos\theta = \beta'$$

would also be eigenvalues.

$$J_3|\boldsymbol{k}, \sigma\rangle = \sigma|\boldsymbol{k}, \sigma\rangle\,, \tag{12.97}$$
$$U(w(\theta), 0)|\boldsymbol{k}, \sigma\rangle = e^{-i\theta\sigma}|\boldsymbol{k}, \sigma\rangle\,, \tag{12.98}$$
$$U(\Lambda, 0)|\boldsymbol{p}, \sigma\rangle = e^{-i\theta(\Lambda,p)\sigma}|\Lambda\boldsymbol{p}, \sigma\rangle\,, \tag{12.99}$$

where $\theta(\Lambda, p)$ is determined for given Λ, p by determining w. We see that the Wigner rotation matrix $D_{ss'}$ in Eq. (12.73) is replaced by a phase.

For a 2π-rotation $U(\Lambda, 0)$ must be ± 1 [7], and therefore σ can only be an **integer** or a **half-integer**. Note that in this case the representation space is **one-dimensional**, as opposed to $2j+1$-dimensional for a massive particle.

[7] see exercise

The eigenvalue σ is called **helicity**, $|\sigma|$ is the spin. It is the helicity because

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} k^\sigma = \frac{1}{2}\left(\epsilon_{\mu\nu\rho 0} J^{\nu\rho} + \epsilon_{\mu\nu\rho 3} J^{\nu\rho}\right) n \tag{12.100}$$

$$= (-J^3, A, B, J^3)^T n\,, \tag{12.101}$$

where in the second equality we have used that $k^\mu = n(1,0,0,1)^T$. Therefore:

$$W_\mu |\boldsymbol{k}, \sigma\rangle = -k_\mu J^3 |\boldsymbol{k}, \sigma\rangle = k_\mu J_3 |\boldsymbol{k}, \sigma\rangle = \sigma k_\mu |\boldsymbol{k}, \sigma\rangle\,, \tag{12.102}$$

i.e. $W_\mu = \sigma k_\mu$, so σ represents the spin component along the direction of motion.

Note: unlike s, the **helicity $\boldsymbol{\sigma}$ is Lorentz-invariant.**

Why does the photon have two polarization (i.e. helicity) states? The reason is the **$\mathcal{CPT}$ theorem**:

> Any local, relativistic QFT is invariant
> under simultaneous $\mathcal{CPT}$ transformations.

Therefore:

$$\text{Particle with helicity } \sigma \xrightarrow{\mathcal{CPT}} \text{Antiparticle with helicity } -\sigma$$

$\mathcal{CPT}$ requires a reducible Lorentz-rep!

Since the photon is its own antiparticle, the local QFT of photons must contain both helicities: σ and $-\sigma$.

12.5.3 Transformation of creation and annihilation operators

The transformation laws for single particle states uniquely determine the transformation properties of creation and annihilation operators under Lorentz transformations. Consider the creation operator $a_s^\dagger(\boldsymbol{p})$ for a massive state of momentum $\boldsymbol{p}$ and spin s, see Eq. (5.58)

$$|\boldsymbol{p}, s\rangle \equiv a_s^\dagger(\boldsymbol{p})|0\rangle\,,$$

Applying a Lorentz transformation ($a_\mu = 0$), we have

$$\begin{aligned} U(\Lambda)|\boldsymbol{p}, s\rangle &= \left(U(\Lambda) a_s^\dagger(\boldsymbol{p}) U^{-1}(\Lambda)\right) U(\Lambda)|0\rangle \\ &= \left(U(\Lambda) a_s^\dagger(\boldsymbol{p}) U^{-1}(\Lambda)\right)|0\rangle \end{aligned}$$

where we have used that the vacuum $|0\rangle$ is Lorentz invariant. To satisfy Eq. (12.73), we must have

$$U(\Lambda)\, a_s^\dagger(\boldsymbol{p})\, U^{-1}(\Lambda) = \sum_{s'=-j}^{j} D^{(j)}_{ss'}\left(w(\Lambda, p)\right)\, a_{s'}^\dagger(\boldsymbol{\Lambda p}) \tag{12.103}$$

Taking the hermitian conjugate of this equation and using the unitarity of $U(\Lambda)$ (i.e. $U(\Lambda)^\dagger = U(\Lambda)^{-1}$), we get the transformation law for the annihilation operator as well

$$U(\Lambda)\, a_s(\boldsymbol{p})\, U^{-1}(\Lambda) = \sum_{s'=-j}^{j} D^{(j)\,*}_{ss'}\,(w(\Lambda, p))\; a_{s'}(\boldsymbol{\Lambda p}) \tag{12.104}$$

which simplifies for mass-less fields with $D^{(\sigma)}_{ss'} = e^{\pm i\sigma\theta(\Lambda,p)}$ and helicity $h = \pm\sigma$ to

$$U(\Lambda)\, a_s(\boldsymbol{p})\, U^{-1}(\Lambda) = \sum_{\sigma=\pm h} e^{\mp i\sigma\theta(\Lambda,p)}\, a_\sigma(\boldsymbol{\Lambda p}) \tag{12.105}$$

We will get to the transformation properties of field operators in a little bit.

12.5.4 Parity and time-reversal

For rotations and boosts we have

$$\mathcal{P}\boldsymbol{J}\mathcal{P}^{-1} = +\boldsymbol{J}\,, \quad \mathcal{P}\boldsymbol{K}\mathcal{P}^{-1} = -\boldsymbol{K}\,,$$
$$\mathcal{T}\boldsymbol{J}\mathcal{T}^{-1} = -\boldsymbol{J}\,, \quad \mathcal{T}\boldsymbol{K}\mathcal{T}^{-1} = +\boldsymbol{K}\,.$$

For momentum:

$$\mathcal{P}\boldsymbol{P}\mathcal{P}^{-1} = -\boldsymbol{P}\,, \quad \mathcal{T}\boldsymbol{P}\mathcal{T}^{-1} = -\boldsymbol{P}\,,$$

while for energy:

$$\mathcal{P}H\mathcal{P}^{-1} = \mathcal{T}H\mathcal{T}^{-1} = H\,.$$

$\mathcal{P}$ and $\mathcal{T}$ must be represented as unitary or anti-unitary operators. For four-vectors:

$$\mathcal{P} = \mathcal{P}^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{(parity)}\,. \tag{12.106}$$

Using Eq. (12.44) and Eq. (12.45) and leaving the i's for a moment

$$U(\Lambda,0) i J^{\mu\nu} U^{-1}(\Lambda,0) = i\,\Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu J^{\rho\sigma}\,, \tag{12.107}$$
$$U(\Lambda,0) i P^{\mu} U^{-1}(\Lambda,0) = i\,\Lambda_\rho{}^\mu P^{\rho}\,, \tag{12.108}$$

and setting $\Lambda = \mathcal{P}$ we get for a **parity** transformation $\mathcal{P}$,

$$U_{\mathcal{P}} i J^{\mu\nu} U_{\mathcal{P}}^{-1} = i\,\mathcal{P}_\rho{}^\mu \mathcal{P}_\sigma{}^\nu J^{\rho\sigma}\,, \tag{12.109}$$
$$U_{\mathcal{P}} i P^{\mu} U_{\mathcal{P}}^{-1} = i\,\mathcal{P}_\rho{}^\mu P^{\rho} = i\begin{pmatrix} P^0 \\ -\boldsymbol{P} \end{pmatrix}\,. \tag{12.110}$$

The same relations hold for **time-reversal** $\mathcal{T}$, just substituting

$$\mathcal{P}^\mu{}_\nu \longrightarrow \mathcal{T}^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \quad U_{\mathcal{P}} \longrightarrow U_{\mathcal{T}}\,. \tag{12.111}$$

However, $U_{\mathcal{T}}$ must be anti-unitary[8] $\hat{U}_{\mathcal{T}}$. Definition of an *anti-unitary operator*:

$$\hat{U}_{\mathcal{T}} i \hat{U}_{\mathcal{T}}^{-1} = -i\,,$$

or

$$\hat{U}_{\mathcal{T}} i = -i\hat{U}_{\mathcal{T}}\,.$$

where we use the 'hat' to indicate anti-unitary operators. Now, for the zero-component of the four momentum, the Hamiltonian H, we

[8] see also the math lecture

get

$$U_{\mathcal{P}} i H U_{\mathcal{P}}^{-1} = iH\,, \tag{12.112}$$
$$\hat{U}_{\mathcal{T}} i H \hat{U}_{\mathcal{T}}^{-1} = -iH\,. \tag{12.113}$$

Why? Suppose that $U_{\mathcal{T}}$ was unitary; then we could just cancel the i's and obtain[9]

$$U_{\mathcal{T}} H = -H U_{\mathcal{T}}\,. \tag{12.114}$$

[9] Which we can't since the i's in do not commute with the $\hat{U}_{\mathcal{T}}$'s.

Taking an eigenstate $|\psi\rangle$ of H with energy $E > 0$, then

$$H(U_{\mathcal{T}}|\psi\rangle) = -U_{\mathcal{T}} H|\psi\rangle = -E(U_{\mathcal{T}}|\psi\rangle)\,, \tag{12.115}$$

with the disastrous conclusion that for any state $|\psi\rangle$ with energy E there is another state $U_{\mathcal{T}}|\psi\rangle$ with energy $-E$. Since the spectrum of H is not bounded from above, it would then be also not bounded from below. This is not a sensible theory.

Similarly if $U_{\mathcal{P}}$ was anti-unitary, i.e.

$$\hat{U}_{\mathcal{P}} i H \hat{U}_{\mathcal{P}}^{-1} = iH\,, \quad -i\hat{U}_{\mathcal{P}} H \hat{U}_{\mathcal{P}}^{-1} = iH\,,$$

or

$$\hat{U}_{\mathcal{P}} H = -H\hat{U}_{\mathcal{P}}$$

and therefore

$$H(\hat{U}_{\mathcal{P}}|\psi\rangle) = -\hat{U}_{\mathcal{P}}(H|\psi\rangle) = -E(\hat{U}_{\mathcal{P}}|\psi\rangle)\,,$$

which leads to again to the conclusion that the spectrum of H is not bounded from below.

12.5.5 Transforming one-particle states

For **massive particles** we have

$$U_{\mathcal{P}}|\boldsymbol{p}, s\rangle = \underset{\substack{\uparrow \\ \text{intrinsic parity of the particle} \\ \text{(independent of } \boldsymbol{p},\, s\text{)}}}{\eta}|\mathcal{P}\boldsymbol{p}, \overset{\substack{\text{rotating spin is the same as} \\ \text{watching it in a mirror} \\ \downarrow}}{s}\rangle \tag{12.116}$$

with the intrinsic parity $|\eta| = 1$.

$$\mathcal{P}p^\mu = \begin{pmatrix} \sqrt{m^2 + \boldsymbol{p}^2} \\ -\boldsymbol{p} \end{pmatrix}\,. \tag{12.117}$$

We can always choose $|\boldsymbol{k}, s\rangle$ to be an eigenstate of $U_{\mathcal{T}}$ and then

$$U_{\mathcal{T}}|\boldsymbol{p}, s\rangle = (-1)^{j-s}|-\boldsymbol{p}, -s\rangle\,, \tag{12.118}$$

see [6], chapter 2.6.

e.g. a scalar $\phi(x)$ and a pseudo-scalar $a(x)$ transform as

$$\phi(x) \to +\phi(\mathcal{P}x)$$
$$a(x) \to -a(\mathcal{P}x)$$

with $\mathcal{P}x_\mu = (x_0, -x_i)$.

For **massless particles** holds

$$U_{\mathcal{P}}|\boldsymbol{p}, \sigma\rangle = \eta e^{\mp i\pi\sigma}|\mathcal{P}\boldsymbol{p}, \underset{\substack{\uparrow \\ \text{helicity, \textbf{not} spin}}}{-\sigma}\rangle\,, \tag{12.119}$$

where the upper/lower sign comes in if the y-component of $\boldsymbol{p}$ is positive or negative. This complication is due to $\mathcal{P}$ changing the reference momentum $k^\mu = n(1,0,0,1)^T$ and we therefore consider the operation

Again $|\eta| = 1$.

$$U(R_2)\mathcal{P}\,, \tag{12.120}$$

where R_2 is a 180°-rotation about the y-axis, which takes k into $\mathcal{P}k$ and $U(R_2)\mathcal{P}$ therefore leaves k invariant and these operations therefore define the little group.

Why does the helicity change sign?

$$\text{helicity} \sim \underset{\substack{\uparrow \\ \text{no sign change under } \mathcal{P}}}{\text{spin}} \cdot \overset{\substack{\text{sign change under } \mathcal{P} \\ \downarrow}}{\text{momentum}} \sim \text{sign change.}$$

Time reversal acts on massless states as

Also here $|\xi| = 1$.

$$U_{\mathcal{T}}|\boldsymbol{p}, \sigma\rangle = \xi e^{\pm i\pi\sigma}|\mathcal{T}\boldsymbol{p}, \sigma\rangle\,, \tag{12.121}$$

see [6], chapter 2.6.

12.6 Internal symmetries and charge conjugation

The three-momentum and spin often do **not** yet uniquely define a particle state, e.g. quark states $|\boldsymbol{p}, s\rangle$ are spin $\frac{1}{2}$ particles which are three-fold degenerate (3 colors), or gluons are spin 1 particles which are eight-fold degenerate: the invariant property is color, which is associated to an **internal** symmetry group.

Let $\mathcal{S}$ be the complete symmetry group. $\mathcal{S}$ must contain the Poincaré group $ISO(1,3)$:

$$ISO(1,3) \subset \mathcal{S}\,. \tag{12.122}$$

Particles are characterized by **invariant** operators of $\mathcal{S}$, one-particle states furnish irreducible representations of $\mathcal{S}$. An important result is the **Coleman-Mandula theorem**, which states that the Poincaré group and any other internal symmetry group G cannot be combined in any other but the trivial way:

$$\mathcal{S} = ISO(1,3) \otimes G\,, \tag{12.123}$$

which implies that for generators of the space-time $A_{ISO(1,3)}$ and the internal group B_G;

$$[A_{ISO(1,3)}, B_G] = 0\,. \tag{12.124}$$

In other words: *internal symmetry generators are Lorentz-scalars.*

Exception: Supersymmetry. Space-time generators carry Lorentz-indices:

$$\{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu\,,$$

the θs are generators with Lorentz-indices.

Otherwise the S-matrix would be overconstrained and trivial, since conserved currents would have multiple Lorentz-indices: say

$$\overset{\substack{\text{internal symmetry} \\ \downarrow}}{\underset{\substack{\uparrow \\ \text{Lorentz-index}}}{t^a_\nu}}\,, \tag{12.125}$$

and there would be a conserved current

$$\partial_\mu \left(j^\mu\right)^a_\nu = 0\,. \tag{12.126}$$

The Ward-identities would then imply selection rules would which involve momenta, e.g. $2 \longrightarrow 2$ scattering which is described by two continous kinematic variables (e.g. $s, \cos\theta$), would only be constrained to specific k^1_μ, k^2_ν which would be non-analytic.

The internal symmetry group G acts on particle states as

$$U(g)|\boldsymbol{p}, s; n\rangle\,, \tag{12.127}$$

with n a set of labels for the eigenvalues of the maximal set of commuting generators from G. So far we have observed only finite-dimensional representations of internal symmetries in Nature, with finite discrete eigenvalues:

$$\underbrace{U(g)}_{\substack{\uparrow\\ \text{internal symmetry rep;}\\ \text{unitary operator on}\\ \text{hilbert space}}}|\boldsymbol{p}, s; n\rangle = \sum_{n'} \overbrace{D_{nn'}(g)}^{\substack{\text{matrix of numbers}\\ \text{depending on } g\\ \downarrow}}\underbrace{|\boldsymbol{p}, s; n'\rangle}_{\substack{\uparrow\\ \boldsymbol{p}\text{, } s \text{ independent, they do}\\ \textbf{not} \text{ change under } U(g)}}\,. \tag{12.128}$$

Example: Two complex scalar fields which transform under $SU(2)$:

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}\,, \tag{12.129}$$

$$\phi_n \mapsto U(g)_{nn'}\phi_{n'} = \underbrace{\exp(i\theta^a\sigma^a)_{nn'}}_{=D_{nn'}(g)}\phi_{n'}\,. \tag{12.130}$$

12.6.1 Anti-particles

A particle transforms as Eq. (12.128) under G, anti-particles are states

$$|\boldsymbol{p}, s; \bar{n}\rangle \tag{12.131}$$

which:

a) have the same mass as $|\boldsymbol{p}, s; n\rangle$.

b) have the same spin j (if $m > 0$) and opposite helicity σ (if $m = 0$).

c) transform under all internal symmetries with the complex conjugated representation:

$$U(g)|\boldsymbol{p}, s; \bar{n}\rangle = \sum_{\bar{n}'} D_{\bar{n}\bar{n}'}{}^*(g)|\boldsymbol{p}, s; \bar{n}'\rangle \quad \forall g \in G\,. \tag{12.132}$$

If an anti-particle state exists, then **charge conjugation** $\mathcal{C}$ acts as

$$U_{\mathcal{C}}|\boldsymbol{p}, s; n\rangle = \xi_{\mathcal{C}}|\boldsymbol{p}, s; \bar{n}\rangle\,, \quad |\xi_{\mathcal{C}}|^2 = 1\,, \tag{12.133}$$

$$U_{\mathcal{C}}|\text{particle}\rangle = |\text{anti-particle}\rangle\,. \tag{12.134}$$

Now we have all ingredients to define relativistic fields.

12.7 Relativistic fields

We write field operators as

$$\phi_{\alpha,n}(x)\,, \tag{12.135}$$

with α the Lorentz-index and n the internal symmetry-index. The field operators transform under a transformation U on Hilbert space as

$$\phi'(x) = U\,\phi(x)\,U^{-1}\,. \tag{12.136}$$

For a relativistic field and a Poincaré-transformation $(\Lambda, a) \in ISO(1,3)$ we define

$$U(\Lambda,a)\phi_{\alpha,n}(x)U^{-1}(\Lambda,a) \equiv \sum_{\alpha'} D_{\alpha\alpha'}\underset{\substack{\uparrow\\ \text{matrix of c-numbers}}}{\left(\Lambda^{-1}\right)}\phi_{\alpha',n}(\Lambda x + a)\,, \tag{12.137}$$

whereas under internal symmetries it transforms as

$$U(g)\,\phi_{\alpha,n}(x)\,U^{-1}(g) = \sum_{n'} D_{nn'}(g)\phi_{\alpha,n'}(x)\,. \tag{12.138}$$

Both α and n are from a finite set.
Local fields with definite Lorentz-transformation properties allow us to easily write down Lorentz-invariant theories which satisfy relativistic causality. We just need to make sure that the Lagrangian is a Lorentz-scalar and that field operators commute for space-like separation.
We know that the $D(\Lambda)$ form a rep of the Lorentz group. Two Lorentz-transformations Λ_1^{-1}, Λ_2^{-1} can be combined as

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2) \tag{12.139}$$

and the D are finite-dimensional matrices. We therefore have found a finite-dimensional matrix representation of $\mathcal{L}_+^\uparrow$ and contrarily to $U(\Lambda,a)$, the $D(\Lambda)$ *need not be unitary.*[10]

[10] Difference between fields and particle states!

The Lorentz group is generated by

$$J^{\mu\nu} = \begin{pmatrix} 0 & K^1 & K^2 & K^3 \\ -K^1 & 0 & J^3 & -J^2 \\ -K^2 & -J^3 & 0 & J^1 \\ -K^3 & J^2 & -J^1 & 0 \end{pmatrix}\,, \tag{12.140}$$

$$\boldsymbol{J} = \left(J^{23}, J^{31}, J^{12}\right)\,, \quad \boldsymbol{K} = \left(J^{01}, J^{02}, J^{03}\right)\,, \tag{12.141}$$

with J^i the generators of rotations and K^i the generators of boosts, as before. Now we will complexify the generators:

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K})\,, \quad \boldsymbol{B} = \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K})\,, \tag{12.142}$$

which leads to a useful separation:

$$\left[A^i, A^j\right] = i\epsilon^{ijk}A^k\,, \quad \left[B^i, B^j\right] = i\epsilon^{ijk}B^k\,, \quad \left[A^i, B^j\right] = 0\,. \tag{12.143}$$

This is the angular momentum or $SU(2)$ algebra for A and B. We have therefore found the following isomorphism:

$$\begin{matrix}\text{Complexified Lie-algebra}\\ \text{of } SO(1,3)\end{matrix} \quad \cong \quad \begin{matrix}\text{Lie-algebra of}\\ SU(2)_A \otimes SU(2)_B\end{matrix}$$

Hence the irreps of the Lorentz group can be labelled by the labels (A, B) of the irreps of the two $SU(2)$s, where A and B can be integers or half-integers. The dimension of the chosen irrep is then $(2A+1)\cdot(2B+1)$.

Recall how the boost and rotation generators transformed under parity:

$$\mathcal{P}\boldsymbol{J}\mathcal{P} = \boldsymbol{J}\,, \quad \mathcal{P}\boldsymbol{K}\mathcal{P} = -\boldsymbol{K}\,. \tag{12.144}$$

Therefore

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K}) \quad \overset{\mathcal{P}}{\longleftrightarrow} \quad \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K}) = \boldsymbol{B}\,, \tag{12.145}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ trasform into each other under parity and a general representation (A, B) fulfills

$$(A, B) \quad \overset{\mathcal{P}}{\longleftrightarrow} \quad (B, A)\,. \tag{12.146}$$

Only reps with $A = B$ are invariant under parity:

$$(A, A) \quad \overset{\mathcal{P}}{\longrightarrow} \quad (B, B) = (A, A) \quad \checkmark\,. \tag{12.147}$$

The trivial representation is given by the **scalar field**:

$$(A, B) = (0, 0)\,, \quad D(\Lambda) = \mathbb{1}\,, \quad A^i = 0\,,\ B^i = 0\,, \tag{12.148}$$

$$\Longrightarrow \quad U(\Lambda, a)\phi(x)U^{-1}(\Lambda, a) = \phi(\Lambda x + a)\,. \tag{12.149}$$

We start with the free field operators for a general Lorentz-rep. The free field operators create and annihilate particles at x^μ, which imposes non-trivial constraints:

general representation

$$\phi^{(+)}_{\alpha,n}(x) \equiv \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} u_{\alpha,n}(x, \boldsymbol{p}, s) a(\boldsymbol{p}, s, n)\,, \tag{12.150}$$

Destruction

$$\phi^{(-)}_{\alpha,n}(x) \equiv \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} v_{\alpha,n}(x, \boldsymbol{p}, s) a^\dagger(\boldsymbol{p}, s, n)\,, \tag{12.151}$$

Creation

where $u_{\alpha,n}$, $v_{\alpha,n}$ are coefficients which have yet to be determined.

We focus on the Poincaré properties for now and drop the n-index.

- $\phi^{(\pm)}$ transform in the matrix representation of $\mathcal{L}^\uparrow_+$.
- The a, $a^\dagger$ inherit the transformation properties from the unitary reps of $ISO(1,3)$, which operates on **single particle** states created/destroyed by $a^\dagger/a$, see Sec. 12.5.3.

Therefore:

$$|\boldsymbol{p}, s\rangle = a^\dagger(\boldsymbol{p}, s)|\Omega\rangle \tag{12.152}$$

Case $\underline{\boldsymbol{m > 0}}$: see Eq. (12.103) and Eq. (12.104)

$$U(\Lambda, b)a^\dagger(\boldsymbol{p}, s)U^{-1}(\Lambda, b) = e^{ib^\mu(\Lambda p)_\mu} \sum_{s'} D^{(j)}_{ss'}\big(w(\Lambda, p)\big)a^\dagger(\Lambda p, s')\,,$$

$$U(\Lambda, b)a(\boldsymbol{p}, s)U^{-1}(\Lambda, b) = e^{-ib^\mu(\Lambda p)_\mu} \sum_{s'} D^{(j)*}_{ss'}\big(w(\Lambda, p)\big)a(\Lambda p, s')\,,$$

because we had found for one-particle states in Eq. (12.72), Eq. (12.73):

$$U(\mathbb{1}, b)|\boldsymbol{p}, s\rangle = e^{ib^\mu p_\mu}|\boldsymbol{p}, s\rangle\,, \tag{12.153}$$

$$U(\Lambda, 0)|\boldsymbol{p}, s\rangle = \sum_{s'} D^{(j)}_{ss'}(w)|\boldsymbol{\Lambda p}, s'\rangle\,. \tag{12.154}$$

Case $\boldsymbol{m=0}$:

$$
\begin{aligned}
U(\Lambda,b)a^{\dagger}(\boldsymbol{p},\sigma)U^{-1}(\Lambda,b) &= e^{ib^{\mu}(\Lambda p)_{\mu}}e^{-i\theta(\Lambda p)\sigma}a^{\dagger}(\Lambda p,\sigma)\,,\\
U(\Lambda,b)a(\boldsymbol{p},\sigma)U^{-1}(\Lambda,b) &= e^{-ib^{\mu}(\Lambda p)_{\mu}}e^{i\theta(\Lambda p)\sigma}a(\Lambda p,\sigma)\,.
\end{aligned}
$$

Similar relations hold for time-reversal, parity, charge conjugation and the internal symmetries.

We can derive the x-dependence of u_α, v_α from the translation symmetry, generated by P^μ. Take the destruction field in Eq. (12.150):

$$
\begin{aligned}
\phi_\alpha^{(+)}(x+b) &= U(\mathbb{1},b)\phi_\alpha^{(+)}(x)U^{-1}(\mathbb{1},b)\\
&= \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}u_\alpha(x;\boldsymbol{p},s)U(\mathbb{1},b)a(\boldsymbol{p},s)U^{-1}(\mathbb{1},b)\\
&= \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}u_\alpha(x;\boldsymbol{p},s)e^{-ib\cdot p}a(\boldsymbol{p},s)\\
&\overset{!}{=}\sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}u_\alpha(x+b;\boldsymbol{p},s)a(\boldsymbol{p},s)\,,
\end{aligned}
\tag{12.155}
$$

from which see that

$$
u_\alpha(x+b;\boldsymbol{p},s) = e^{-ibp}u_\alpha(x;\boldsymbol{p},s)\,. \tag{12.156}
$$

So, setting $b=x$ and $x=0$ in the LHS of Eq. (12.156):

$$
u_\alpha(x;\boldsymbol{p},s) = e^{-ipx}u_\alpha(0;\boldsymbol{p},s) \equiv e^{-ipx}u_\alpha(\boldsymbol{p},s) \tag{12.157}
$$

where we dropped the 0 in the last step since u_α is now independent of x.
Similarly for $\phi^{(-)}$:

$$
\phi_\alpha^{(-)} \sim v_\alpha(x;\boldsymbol{p},s)a^{\dagger}(\boldsymbol{p},s)\,, \tag{12.158}
$$

$$
v_\alpha(x;\boldsymbol{p},s) = e^{ipx}v_\alpha(\boldsymbol{p},s)\,. \tag{12.159}
$$

The x-dependence of the expansion is therefore fully fixed:

$$
\phi_\alpha^{(+)}(x) \equiv \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}e^{-ipx}u_\alpha(\boldsymbol{p},s)a(\boldsymbol{p},s,n)\,, \tag{12.160}
$$

$$
\phi_\alpha^{(-)}(x) \equiv \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}e^{ipx}v_\alpha(\boldsymbol{p},s)a^{\dagger}(\boldsymbol{p},s,n)\,. \tag{12.161}
$$

We will now Lorentz-boost and rotate to determine the behaviour of $u_\alpha(\boldsymbol{p},s)$ and $v_\alpha(\boldsymbol{p},s)$. We will find that knowing the transformation behavior under the **little group** is enough.

Let us consider a particle with mass $m>0$: we use

$$
U(\Lambda,0)\phi_\alpha^{(\pm)}(x)U^{-1}(\Lambda,0) = \sum_{\alpha'}D_{\alpha\alpha'}\left(\Lambda^{-1}\right)\phi_{\alpha'}^{(\pm)}(\Lambda x) \tag{12.162}
$$

LHS, we have

$$U(\Lambda,0)\phi_\alpha^{(+)}(x)U^{-1}(\Lambda,0)=\sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}e^{-ipx}u_\alpha(\boldsymbol{p},s)\cdot\sum_{s'}D_{ss'}^{(j)*}(w)a(\boldsymbol{\Lambda p},s')\,, \tag{12.163}$$

and on the RHS

$$\begin{aligned}\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})\phi_{\alpha'}^{(+)}(\Lambda x)&=\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})\sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\cdot e^{-ip\cdot\Lambda x}u_{\alpha'}(\boldsymbol{p},s)a(\boldsymbol{p},s)\\&=\sum_{\alpha'}D_{\alpha\alpha'}(\Lambda^{-1})\sum_{s'}\int\frac{\mathrm{d}^3\Lambda p}{(2\pi)^3 2E_{\boldsymbol{\Lambda p}}}\cdot e^{-i\Lambda p\cdot\Lambda x}u_{\alpha'}(\boldsymbol{\Lambda p},s')a(\boldsymbol{\Lambda p},s')\,,\end{aligned} \tag{12.164}$$

where we performed the substitution $\boldsymbol{p}\longrightarrow\boldsymbol{\Lambda p}$ and renamed $s\longrightarrow s'$ in the last equality. Equating the RHS of Eq. (12.163) and Eq. (12.164) we get, by cancelling $\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}a(\Lambda p,s')e^{-ipx}$,

$$\sum_s u_\alpha(\boldsymbol{p},s)\underset{\substack{\uparrow\\\text{unitary little}\\\text{group transformation}}}{D_{ss'}^{(j)*}(w)}=\sum_{\alpha'}\underset{\substack{\uparrow\\\text{non-unitary}\\\text{Lorentz-rep}}}{D_{\alpha\alpha'}(\Lambda^{-1})}u_{\alpha'}(\boldsymbol{\Lambda p},s')\,. \tag{12.165}$$

We multiply this from the left by $D(\Lambda)$ and from the right with $D_{s't}^{(j)}(w)$, using

$$D(\Lambda)D(\Lambda^{-1})=\mathbb{1} \tag{12.166}$$

and the unitarity of $D_{ss'}^{(j)}$:

$$D_{\beta\alpha}(\Lambda)u_\alpha(\boldsymbol{p},s)\underbrace{D_{ss'}^{(j)*}(w)D_{s't}^{(j)}(w)}_{\substack{=\delta_{st},\text{ unitarity of}\\\text{the little group,}\\(D^{(j)})^\dagger D^{(j)}=\mathbb{1}}}=\underbrace{D_{\beta\alpha}(\Lambda)D_{\alpha\alpha'}(\Lambda^{-1})}_{\substack{=\delta_{\beta\alpha'},\text{ since}\\D(\Lambda^{-1})=D^{-1}(\Lambda)}}u_{\alpha'}(\boldsymbol{\Lambda p},s')D_{s't}^{(j)}(w)\,. \tag{12.167}$$

Einstein summation convention is implied.

We can perform analogous computations for $v_\alpha(\boldsymbol{p},s)$ and get

$$D_{\beta\alpha}(\Lambda)u_\alpha(\boldsymbol{p},s)=u_\beta(\boldsymbol{\Lambda p},s')D_{s's}^{(j)}\big(w(\Lambda,p)\big)\,, \tag{12.168}$$

$$D_{\beta\alpha}(\Lambda)v_\alpha(\boldsymbol{p},s)=v_\beta(\boldsymbol{\Lambda p},s')D_{s's}^{(j)*}\big(w(\Lambda,p)\big)\,, \tag{12.169}$$

or in words

Lorentz-transformation of Lorentz-index α	=	Little group transformation of spin index s

where the LHS of the equations is summed over α and the RHS is summed over s'.

The result for **massless** particles is simpler: we just have to substitute

$$D_{s's}^{(j)}\longrightarrow e^{-i\theta\sigma} \tag{12.170}$$

in the above equation Eq. (12.168) and Eq. (12.169) and there is no sum over s'.

Now we can simplify the above using the **reference momentum**

Recall the definition of $L(p)$:

$$L(p)k=p\,,$$

and therefore

$$L(k)=\mathbb{1}\,.$$

$p \longrightarrow k$ with

$$m^2 > 0: \quad k = (m, 0, 0, 0)^T \tag{12.171}$$

$$m^2 = 0: \quad k = n(1, 0, 0, 1)^T \tag{12.172}$$

and use $\Lambda p \longrightarrow p$ with $\Lambda = L(p)$ (a standard boost). The Wigner rotation

$$w(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p) \tag{12.173}$$

becomes

$$w(\Lambda, k) = L^{-1}(\Lambda k)\Lambda \underbrace{L(k)}_{=\mathbb{1}} = L^{-1}([\underbrace{L(p)k}_{=p}])\overbrace{\Lambda}^{=L(p)}\mathbb{1} = L^{-1}(p)L(p) = \mathbb{1} \tag{12.174}$$

and hence for the reference momentum:

$$D^{(j)}_{s's}\big(w(L(p), k)\big) = D^{(j)}_{s's}(\mathbb{1}) = \delta_{s's} \tag{12.175}$$

and for $m^2 = 0$

$$e^{-i\theta(\Lambda, k)} = 1 \quad \Longrightarrow \quad \theta(\Lambda, p) = 0\,. \tag{12.176}$$

Using Eq. (12.168) and Eq. (12.169) for the reference momentum:

$$u_\beta(\boldsymbol{p}, s')\underbrace{D^{(j)}_{s's}\big(w(L(p), k)\big)}_{=\delta_{s's}} = D_{\beta\alpha}(L(p))u_\alpha(\boldsymbol{k}, s) \tag{12.177}$$

and

$$u_\beta(\boldsymbol{p}, s) = \sum_\alpha D_{\beta\alpha}\big(L(p)\big)u_\alpha(\boldsymbol{k}, s)\,, \tag{12.178}$$

$$v_\beta(\boldsymbol{p}, s) = \sum_\alpha D_{\beta\alpha}\big(L(p)\big)v_\alpha(\boldsymbol{k}, s)\,, \tag{12.179}$$

$\Lambda = L(p)$
$p \longrightarrow k$
$\Lambda p \longrightarrow p$

with the usual modifications for massless particles.

Conclusion: We only need to determine the $u_\alpha(\boldsymbol{k}, s)$, $v_\alpha(\boldsymbol{k}, s)$ for the reference momentum! We can get all the other polarizations/spins $u_\alpha(\boldsymbol{p}, s)$, $v_\alpha(\boldsymbol{p}, s)$ by simple Lorentz-transformations of Eq. (12.178) and Eq. (12.179).

We now use a transformation out of the little group $\big(m^2 > 0 : SO(3),\ m^2 = 0 : ISO(2)\big)$, which we call R. Then Eq. (12.168) and Eq. (12.169) give for $\Lambda \longrightarrow R$, $p \longrightarrow k$:

The elements of the little group are transformations that leave k invariant:

$$Rk = k$$

$$D_{\beta\alpha}(R)u_\alpha(\boldsymbol{k}, s) = u_\beta(\boldsymbol{k}, s')D^{(j)}_{s's}(R)\,, \tag{12.180}$$

$$D_{\beta\alpha}(R)v_\alpha(\boldsymbol{k}, s) = v_\beta(\boldsymbol{k}, s')D^{(j)*}_{s's}(R)\,, \tag{12.181}$$

since $\Lambda k = Rk = k$.

Example: scalar field. The scalar field has $D(\Lambda) = \mathbb{1}$ (trivial Lorentz-rep), so

$$u_\alpha(\boldsymbol{k}, s) = \sum_{s'} u_\alpha(\boldsymbol{k}, s')D^{(j)}_{s's}(R) \quad \forall R \tag{12.182}$$

and therefore $\boldsymbol{j} = \mathbf{0}$ (or $\sigma = 0$): the scalar field can only describe particles with spin or helicity 0.
We can find a simple solution to Eq. (12.180), Eq. (12.181), namely $u_\alpha(\boldsymbol{k}, s) = v_\alpha(\boldsymbol{k}, s) \equiv 1$; this is unique, since we can choose the phase and normalization freely. This gives

$$\phi^{(+)}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{-ipx} a_{\boldsymbol{p}}\,, \tag{12.183}$$

$$\phi^{(-)}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} e^{ipx} a^\dagger_{\boldsymbol{p}}\,. \tag{12.184}$$

Recall from the initial chapter on the free scalar field:

- Relativistic causality requires that $\phi_\alpha(x)$ contains annihilation **and** creation operators. For the real field, this means

$$\phi = \phi^{(+)} + \phi^{(-)}\,. \tag{12.185}$$

- A complex field requires an anti-particle. If the particle transforms under the complex rep of the internal symmetry group G:

$$U(g)a(\boldsymbol{p}, s, n)U^{-1}(g) = \sum_{n'} D^*_{nn'}(g)a(\boldsymbol{p}, s, n')\,, \tag{12.186}$$

but

$$U(g)a^\dagger(\boldsymbol{p}, s, n)U^{-1}(g) = \sum_{n'} D_{nn'}(g)a^\dagger(\boldsymbol{p}, s, n')\,. \tag{12.187}$$

Therefore, we cannot combine a and $a^\dagger$ in the same field since

$$\phi \sim a + a^\dagger \xrightarrow{G} D^*(g)a + D(g)a^\dagger\,, \tag{12.188}$$

which is **not** a covariant transformation: we need just D^* or just D. ϕ **must** contain the anti-particle creation operator $b^\dagger$, which transforms under the complex conjugated rep by definition:

$$U(g)b^\dagger(\boldsymbol{p}, s, n)U^{-1}(g) = \sum_{n'} D^*_{nn'}(g)b^\dagger(\boldsymbol{p}, s, n')\,. \tag{12.189}$$

Now we have

$$\phi \sim a + b^\dagger \xrightarrow{G} \phi' = D^*(g)a + D^*(g)b^\dagger = D^*(g)\phi \quad \checkmark\,. \tag{12.190}$$

We find: relativistic causality **requires** anti-particles whenever particles transform under the **complex** rep of the internal symmetry group (note: there are also **real** reps, where $D^*(g) = D(g)$).

12.8 Classification and summary

We sum up our results concerning Lorentz representations:

1) The complexified Lie-algebra of the Lorentz group $SO(1,3)$ is locally isomorphic to the Lie-algebra of the product group

$SU(2)_A \otimes SU(2)_B$. Therefore we can classify the reps of the Lorentz group using the eigenvalues (A, B) of the two generators

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{J} + i\boldsymbol{K}) \in su(2)_A\,, \quad \boldsymbol{B} = \frac{1}{2}(\boldsymbol{J} - i\boldsymbol{K}) \in su(2)_B\,, \tag{12.191}$$

where $su(2)_i$ is the Lie-algebra of the Lie group $SU(2)_i$.

2) The eigenvalues of $\boldsymbol{A}$, $\boldsymbol{B}$ are

$$A, B = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\,, \tag{12.192}$$

which can be obtained through the two Casimir operators $\boldsymbol{A}^2$, $\boldsymbol{B}^2$ satisfying

$$[\boldsymbol{A}^2, A_i] = [\boldsymbol{A}^2, B_i] = 0\,, \tag{12.193}$$
$$[\boldsymbol{B}^2, A_i] = [\boldsymbol{B}^2, B_i] = 0\,, \tag{12.194}$$

which have the eigenvalues $A(A+1)$, $B(B+1)$. Therefore we obtain the following Lorentz reps:

		A			
		0	$\frac{1}{2}$	1	...
	0	(0,0)	$\left(0, \frac{1}{2}\right)$	(0,1)	...
B	$\frac{1}{2}$	$\left(\frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\left(\frac{1}{2}, 1\right)$	...
	1	(1,0)	$\left(1, \frac{1}{2}\right)$	(1,1)	...
	...	...	...	...	...

and the dimension of each rep is $(2A+1)(2B+1)$.

3) These reps transform in the $m^2 > 0$ example, with $SO(3)$ as the little group , as:

a) (0,0): **1** of $SO(3)$, scalar, dim=1.

b) $\left(\frac{1}{2}, 0\right)$, $\left(0, \frac{1}{2}\right)$: **2** of $SO(3)$, Weyl-spinor (right/left-handed), dim=2.

c) $\left(\frac{1}{2}, \frac{1}{2}\right)$: $\mathbf{1} \oplus \mathbf{3}$ of $SO(3)$, vector, dim=4.

d) $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ is the Dirac spinor, dim=4.

We will discover the subtleties in the following.

The maximal spin in the rep (A, B) is $|\boldsymbol{A} + \boldsymbol{B}| = J_{\max}$:

	(0,0)	$\left(\frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$	...
$J_{\max}$	0	$\frac{1}{2}$	1	...

13
Spin $\frac{1}{2}$ Particles and Spinor Fields

Now we will step beyond the trivial Lorentz rep (i.e. the scalar) and construct a quantum field theory of spin $\frac{1}{2}$ fields.

13.1 Spinor representations $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$

Matrix representations are labelled by "spins" (A, B). The smallest nontrivial reps are $\left(\frac{1}{2},0\right)$ or $\left(0,\frac{1}{2}\right)$, with dimension $2\cdot\frac{1}{2}+1=2$.

$\underline{\left(\frac{1}{2},0\right):}$ The generators read

$$\boldsymbol{A}=\frac{\boldsymbol{\sigma}}{2}\,,\quad \boldsymbol{B}=0\,, \tag{13.1}$$

with

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\,,\quad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\,,\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\,, \tag{13.2}$$

the Pauli-matrices. We know the representation of the $J^{\mu\nu}$:

$$J^{0i}=K^i=-i\big(A^i-B^i\big)=-i\frac{\sigma^i}{2}\,, \tag{13.3}$$

$$J^{ij}=\epsilon^{ijk}\big(A^k+B^k\big)=\frac{1}{2}\epsilon^{ijk}\sigma^k\,. \tag{13.4}$$

We define

$$\sigma^\mu\equiv(\mathbb{1},\sigma^i)^T\,,\quad \sigma^{\mu\nu}\equiv\frac{i}{4}(\sigma^\mu\bar\sigma^\nu-\sigma^\nu\bar\sigma^\mu)\,, \tag{13.5}$$

$$\bar\sigma^\mu\equiv(\mathbb{1},-\sigma^i)^T\,,\quad \bar\sigma^{\mu\nu}\equiv\frac{i}{4}(\bar\sigma^\mu\sigma^\nu-\bar\sigma^\nu\sigma^\mu)\,. \tag{13.6}$$

The representation of $J^{\mu\nu}$ on $\left(\frac{1}{2},0\right)$ is therefore

$$\Big[J^{\mu\nu}\Big]_{(\frac{1}{2},0)}\equiv\bar\sigma^{\mu\nu}\,, \tag{13.7}$$

see check on the margin, with

$$\Lambda\mapsto D_{(\frac{1}{2},0)}(\Lambda)\equiv D_R(\Lambda)=\exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar\sigma^{\mu\nu}\right)\,. \tag{13.8}$$

$D_R(\Lambda)$ operates on two-component objects $\psi_{R,\alpha}$, $\alpha=1,2$, which we call **right-handed spinors**.

The Pauli matrices satisfy

$$[\sigma^i,\sigma^j]=2i\epsilon^{ijk}\sigma^k\,,$$

therefore

$$\begin{aligned}J^{ij}&=\frac{i}{4}(\bar\sigma^i\sigma^j-\bar\sigma^j\sigma^i)=-\frac{i}{4}2i\epsilon^{ijk}\sigma^k\\&=\frac{1}{2}\epsilon^{ijk}\sigma^k\quad\checkmark\end{aligned}$$

and

$$\begin{aligned}J^{0i}&=\frac{i}{4}(\bar\sigma^0\sigma^i-\bar\sigma^i\sigma^0)=\frac{i}{4}(\mathbb{1}\cdot\sigma^i+\mathbb{1}\sigma^i)\\&=\frac{i}{2}\sigma^i=-J^{i0}\quad\checkmark\,.\end{aligned}$$

$\underline{\left(0, \frac{1}{2}\right)}$: The generators are now

$$\boldsymbol{A} = 0\,, \quad \boldsymbol{B} = \frac{\boldsymbol{\sigma}}{2} \tag{13.9}$$

and from this follows through the same steps as above:

$$\left[J^{\mu\nu}\right]_{(0,\frac{1}{2})} \equiv \sigma^{\mu\nu}\,, \quad J^{i0} = i\frac{\sigma^i}{2}\,, \quad J^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k \tag{13.10}$$

and

$$D_L(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right). \tag{13.11}$$

This defines **left-handed spinors**:

$$\psi_L \xrightarrow{\Lambda} D_L(\Lambda)\psi_L\,. \tag{13.12}$$

Comments:

- $D_L(\Lambda)$ and $D_R(\Lambda)$ are **not** unitary matrices! This is expected, since $\Lambda^\mu{}_\nu$ is not unitary either. We see it from the fact that even if $\boldsymbol{A}$ and $\boldsymbol{B}$ are hermitian (and $e^{i\theta^a A^a}$ and $e^{i\theta^a B^a}$ would be unitary), the boosts $\boldsymbol{K} = -i(\boldsymbol{A} - \boldsymbol{B})$ are not hermitian. The matrices $D_{L/R}(\Lambda) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\overset{(-)\mu\nu}{\sigma}\right)$ generate arbitrary complex 2×2 matrices M with $\det(M) = 1$. The constraint $\det(M) = 1$ follows since

$$\det\left[\exp\left(\frac{i}{2}\omega_{\mu\nu}\overset{(-)\mu\nu}{\sigma}\right)\right] = \exp\left(\frac{i}{2}\omega_{\mu\nu}\underbrace{\mathrm{tr}\left[\overset{(-)\mu\nu}{\sigma}\right]}_{=0}\right) = 1\,. \tag{13.13}$$

  These matrices form $Sl(2,\mathbb{C})$ (which is six-dimensional). The spinors ψ_L and ψ_R are therefore $Sl(2,\mathbb{C})$-spinors and are different from $SU(2)$-spinors from non-relativistic quantum mechanics on which the $j = \frac{1}{2}$ rep of $SU(2)_j$ operates (which is of course unitary).

- The Λ in the $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$ reps under a spatial rotation of 2π have the property

$$D_{L/R}(\underset{\substack{\uparrow\\ \text{rotation axis}}}{\boldsymbol{n}}, \overset{\substack{\text{angle}\\ \downarrow}}{2\pi}) = -1\,. \tag{13.14}$$

  Just as $SU(2)$ is the universal covering group of $O(3)$, $Sl(2,\mathbb{C})$ is the universal covering group of $SO(1,3)$.

- The two reps for ψ_L and ψ_R are **not** equivalent, i.e. no matrix S exists, such that

$$D_{(0,\frac{1}{2})}(\Lambda) = S D_{(\frac{1}{2},0)}(\Lambda) S^{-1} \quad \forall\Lambda\,. \tag{13.15}$$

  However, the following holds:

  1) $D_L^\dagger(\Lambda) = D_R^{-1}(\Lambda)\,,$ (13.16)

  2) $D_L(\Lambda) = \epsilon D_R^*(\Lambda)\epsilon^{-1}\,, \quad \text{with} \quad \epsilon \equiv -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,.$ (13.17)

Why do the $D_{L/R}(\Lambda)$ generate general matrices in $Sl(2,\mathbb{C})$? The generator matrices of $Sl(2,\mathbb{C})$ must be traceless:

$$\det[\exp(i\theta^a\chi^a)] = \exp[i\theta^a \underbrace{\mathrm{tr}(\chi^a)}_{=0}] = 1\,.$$

A basis of traceless $\mathbb{R}^{2\times 2}$ matrices is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3\,, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where the last two matrices are linear combinations of σ^1 and σ^2. Choosing arbitrary complex coefficients we can write $M \in Sl(2,\mathbb{C})$ as

$$M = \exp[i(\boldsymbol{\theta} + i\boldsymbol{\eta})\cdot\boldsymbol{\sigma}]\,,$$

which has the form of $D_{L/R}(\Lambda)$, q.e.d.

1) We see this because $\sigma^{\mu\nu\dagger} = \bar{\sigma}^{\mu\nu}$ holds. We check it:

$$\begin{aligned}\sigma^{\mu\nu\dagger} &= \frac{i}{4}\Big(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\Big)^\dagger = -\frac{i}{4}\Big((\bar{\sigma}^\nu)^\dagger(\sigma^\mu)^\dagger - (\bar{\sigma}^\mu)^\dagger(\sigma^\nu)^\dagger\Big)\\ &= -\frac{i}{4}\big(\bar{\sigma}^\nu\sigma^\mu - \bar{\sigma}^\mu\sigma^\nu\big) = -\bar{\sigma}^{\nu\mu} = \bar{\sigma}^{\mu\nu}\,,\end{aligned} \tag{13.18}$$

and so:

$$\begin{aligned}D_L^\dagger(\Lambda) &= \exp\left(\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu\dagger}\right) = \exp\left(\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)\\ &= D_R^{-1}(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{-1}.\end{aligned} \tag{13.19}$$

2) See exercise.

An important implication of this is that given a left-handed Weyl-spinor ψ_L, then

$$\big(\psi_L\big)^{\mathcal{C}} = \epsilon\psi_L^* \tag{13.20}$$

transforms as a right-handed Weyl-spinor!

$$\epsilon\psi_L^* \xrightarrow{\Lambda} \epsilon\big(D_L(\Lambda)\psi_L\big)^* = \epsilon D_L^*(\Lambda)\epsilon^{-1}\epsilon\psi_L^* \stackrel{2)}{=} D_R(\Lambda)\big(\epsilon\psi_L^*\big) \quad \checkmark \tag{13.21}$$

Two other important relations are the following: σ^μ and $\bar{\sigma}^\mu$ "transform" like four-vectors:

$$D_L^{-1}(\Lambda)\sigma^\mu D_R(\Lambda) = \Lambda^\mu{}_\nu\sigma^\nu \tag{13.22}$$

$$D_R^{-1}(\Lambda)\bar{\sigma}^\mu D_L(\Lambda) = \Lambda^\mu{}_\nu\bar{\sigma}^\nu \tag{13.23}$$

13.2 Dirac spinors

Historically, electrons where first discovered as matter particles of quantum electrodynamics (QED), which is a parity-invariant theory: the two-dimensional reps ψ_L and ψ_R are not parity-invariant since they change into each other under parity transformations:

$$\left(\frac{1}{2},0\right) \xrightarrow{\mathcal{P}} \left(0,\frac{1}{2}\right). \tag{13.24}$$

However, the four dimensional rep given by the direct sum $\big(\frac{1}{2},0\big) \oplus \big(0,\frac{1}{2}\big)$ is $\mathcal{P}$-invariant. Lorentz-transformations in this rep are given by

$$\Lambda \mapsto \begin{pmatrix} D_L(\Lambda) & 0 \\ 0 & D_R(\Lambda)\end{pmatrix} = D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda)\,. \tag{13.25}$$

The above rep is reducible for $\mathcal{L}_+^\uparrow$, but not for $\mathcal{P}\otimes\mathcal{L}_+^\uparrow$. It operates on four-component **Dirac spinors**:

$$\Psi_\alpha\,, \quad \alpha = 1,2,3,4\,. \tag{13.26}$$

We define:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0\end{pmatrix} \quad \text{(Weyl representation of the } \gamma^\mu)\,, \tag{13.27}$$

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix}\sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu}\end{pmatrix}\,, \tag{13.28}$$

such that

$$D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right), \tag{13.29}$$

which operates on

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} . \tag{13.30}$$

The γ matrices have the following properties:

- The γ^μ satisfy the following anti-commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}_{4\times 4} \tag{13.31}$$

  Let's check this:

$$\{\gamma^\mu, \gamma^\nu\} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \tag{13.32}$$

$$= \begin{pmatrix} \sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu + \bar\sigma^\nu\sigma^\mu \end{pmatrix}, \tag{13.33}$$

  with, as before,

$$\sigma^\mu = (\mathbb{1}, \sigma^i)^T, \quad \bar\sigma^\mu = (\mathbb{1}, -\sigma^i)^T . \tag{13.34}$$

  We now check the different components of Eq. (13.33):

$$\mu = \nu = 0: \quad 2\begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & \mathbb{1}_{2\times 2} \end{pmatrix} = 2\mathbb{1}_{4\times 4}, \tag{13.35}$$

$$\mu = i\,, \nu = 0: \quad \begin{pmatrix} \sigma^i - \sigma^i & 0 \\ 0 & -\sigma^i + \sigma^i \end{pmatrix} = 0\,, \tag{13.36}$$

$$\mu = i\,, \nu = i: \quad \begin{pmatrix} -2(\sigma^i)^2 & 0 \\ 0 & -2(\sigma^i)^2 \end{pmatrix} = -2\mathbb{1}_{4\times 4} \quad \checkmark . \tag{13.37}$$

  Eq. (13.31) defines a Clifford algebra and one can deduce all the properties of the γ^μ just from their anti-commutation relation. We will from now on omit the unit matrix in Eq. (13.31).

- An important connection to the vector rep of the Lorentz group is

$$D^{-1}_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda)\gamma^\mu D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu . \tag{13.38}$$

- We can use different bases for the γ^μ, unitary transformations

$$\gamma^\mu \longrightarrow U^\dagger\gamma^\mu U\,, \quad U^\dagger U = UU^\dagger = \mathbb{1}\,, \tag{13.39}$$

  do **not** change their properties, e.g. in the Dirac basis the γ^μ have the following form:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} . \tag{13.40}$$

This form of the γ^μ was introduced in the lecture "Quantum Mechanics 2".

13.3 Field operator for spinors and Lagrangians

Consider a left-handed spinor, belonging to $\left(0, \frac{1}{2}\right)$. The associated field operator has to satisfy

$$U(\Lambda, a)\psi_\alpha^{(\pm)}(x)U^{-1}(\Lambda, a) = \sum_{\alpha'} \underbrace{D_{(0,\frac{1}{2})}(\Lambda)_{\alpha\alpha'}}_{=D_L(\Lambda)_{\alpha\alpha'}} \psi_{\alpha'}^{(\pm)}(\Lambda x + a)\,, \tag{13.41}$$

i.e. it has to transform as like a left-handed Weyl-spinor.

This leads to conditions on the coefficients of a, $a^\dagger$, which we denoted as $u_\alpha(\boldsymbol{p}, s)$ and $v_\alpha(\boldsymbol{p}, s)$, see Eq. (12.180), Eq. (12.181). We already showed that it is enough to know the action of a little group transformation R on u_α and v_α in the refererence momentum k, e.g.

$$u_\alpha(\boldsymbol{k}, s')D^{(j)}_{s's}(R) = D_L(R)_{\alpha\alpha'}u_{\alpha'}(\boldsymbol{k}, s)\,, \tag{13.42}$$

and we can then boost to general momenta p using

$$u_\alpha(\boldsymbol{p}, s) = \sum_{\alpha'} D_L\big(L(p)\big)_{\alpha\alpha'}u_{\alpha'}(\boldsymbol{k}, s)\,. \tag{13.43}$$

The same holds for v_α.

For a given j we sum over $2j+1 = s$ values, therefore:

$$u \equiv \underset{\substack{\uparrow\\ 2}}{u_\alpha}(\boldsymbol{k}, \underset{\substack{\uparrow\\ 2j+1}}{s})\,. \tag{13.44}$$

u is a $2\cdot(2j+1)$-dimensional matrix.

A **general result** from representation theory is that if D_1 (here $D^{(j)}$) and D_2 (here D_L) are irreps, then Eq. (13.42), which we can write as

$$\underset{\substack{\uparrow\\ 2\cdot(2j+1)}}{u}\overset{\substack{(2j+1)(2j+1)\\ \downarrow}}{D^{(j)}} = \underset{\substack{\uparrow\\ 2\cdot 2}}{D_L}\overset{\substack{2\cdot(2j+1)\\ \downarrow}}{u}\,, \tag{13.45}$$

can only be satisfied if D_1 and D_2 have the **same** dimensions. So only the choice $j = \frac{1}{2}$ is possible, therefore free 2-spinors create and destroy particles with spin $\frac{1}{2}$.

We can write for pure rotations R

$$D^{(\frac{1}{2})}_{s's}(R) = \left[\exp\left(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right]_{s's}, \quad D_L(R)_{\alpha\alpha'} = \left[\exp\left(-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\right]_{\alpha\alpha'}, \tag{13.46}$$

therefore Eq. (12.180) and Eq. (12.181) infinitesimally become

$$u\sigma^i = \sigma^i u\,, \tag{13.47}$$
$$v(-\sigma^{i*}) = \sigma^i v\,, \tag{13.48}$$

(recall that v_α transforms with the complex conjugated rep $D^{(\frac{1}{2})^*}$), and therefore we obtain

$$[u, \sigma^i] = 0\,, \tag{13.49}$$
$$[v\epsilon^{-1}, \sigma^i] = 0\,. \tag{13.50}$$

The relation

$$\epsilon\sigma^{i*}\epsilon^{-1} = -\sigma^i$$

holds and therefore

$$v(-\sigma^{i*})\epsilon^{-1} = v\epsilon^{-1}\epsilon(-\sigma^{i*})\epsilon^{-1} = v\epsilon^{-1}\sigma^i\,,$$

which, by Eq. (13.48), means that

$$v\epsilon^{-1}\sigma^i = \sigma^i v\epsilon^{-1}\,.$$

We found that $\boldsymbol{u}$ **and** $\boldsymbol{v\epsilon^{-1}}$ **commute with all** $\boldsymbol{\sigma^i}$. This means that u and $v\epsilon^{-1}$ must be proportional to the Casimir operator $\boldsymbol{\sigma}^2 \sim \mathbb{1}_{2\times 2}$, which is the only invariant operator under σ^i, i.e.

$$u_\alpha(\boldsymbol{k}, s) = C_u \delta_{\alpha s}\,, \tag{13.51}$$

$$v_\alpha(\boldsymbol{k}, s) = -C_v \epsilon_{\alpha s} \tag{13.52}$$

and there is no $\boldsymbol{k}$-dependence since it is fixed to be the reference momentum. Written as 2-spinors, this is

$$u(\boldsymbol{k}, 1/2) = \begin{pmatrix} C_u \\ 0 \end{pmatrix}\,, \quad u(\boldsymbol{k}, -1/2) = \begin{pmatrix} 0 \\ C_u \end{pmatrix} \tag{13.53}$$

and

$$v(\boldsymbol{k}, 1/2) = \begin{pmatrix} 0 \\ C_v \end{pmatrix}\,, \quad v(\boldsymbol{k}, -1/2) = \begin{pmatrix} -C_v \\ 0 \end{pmatrix}\,. \tag{13.54}$$

To get the general $u(\boldsymbol{p}, s)$, $v(\boldsymbol{p}, s)$ we need to find a rep of the standard boost. For $\boldsymbol{m > 0}$:

$$D_{R/L}\big(L(p)\big) = \frac{p^0 + m \pm \boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p^0+m)}}\,, \quad p^0 = \sqrt{m^2 + \boldsymbol{p}^2}\,. \tag{13.55}$$

<u>**Derivation:**</u> we boost from the reference momentum $k^\mu = (m, \boldsymbol{0})^T$ to a general momentum $p^\mu = (p^0, \boldsymbol{p})^T$. We first boost in the x-direction using

$$\Lambda = \begin{pmatrix} \cosh(\omega) & \sinh(\omega) & 0 & 0 \\ \sinh(\omega) & \cosh(\omega) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13.56}$$

and let it act on k^μ:

$$\Lambda \begin{pmatrix} m \\ \boldsymbol{0} \end{pmatrix} = \begin{pmatrix} m\cosh(\omega) \\ m\sinh(\omega) \\ 0 \\ 0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} p^0 \\ |\boldsymbol{p}| \\ 0 \\ 0 \end{pmatrix} = e^{i\omega_{01}K^1} \begin{pmatrix} m \\ \boldsymbol{0} \end{pmatrix}\,. \tag{13.57}$$

Therefore it follows that

$$\cosh(\omega) = \frac{p^0}{m} = \gamma\,, \quad \sinh(\omega) = \frac{|\boldsymbol{p}|}{m} = \sqrt{\gamma^2 - 1}\,. \tag{13.58}$$

x was arbitrary, we take now the same value of ω for a boost in an arbitrary direction $\boldsymbol{n} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|}$:

$$D_{R/L}\big(L(p)\big) = \exp\Big(i\omega\boldsymbol{n}\cdot\boldsymbol{K}\Big)\,. \tag{13.59}$$

We know the representation of $\boldsymbol{K}$:

$$\boldsymbol{K} = \mp\frac{i\boldsymbol{\sigma}}{2} \quad \text{for} \quad \begin{cases} \left(\frac{1}{2}, 0\right) \\ \left(0, \frac{1}{2}\right) \end{cases}\,. \tag{13.60}$$

So, using the identities

$$\sinh\left(\frac{x}{2}\right) = \frac{\sinh(x)}{\sqrt{2\cosh(x)+1}}\,, \quad \cosh\left(\frac{x}{2}\right) = \sqrt{\frac{\cosh(x)+1}{2}} \tag{13.61}$$

we obtain:

$$\begin{aligned} D_{R/L}\big(L(p)\big) &= \cosh\left(\frac{\omega}{2}\right)\mathbb{1} \pm \boldsymbol{n}\cdot\boldsymbol{\sigma}\sinh\left(\frac{\omega}{2}\right) \\ &\overset{\substack{Eq.\ (13.58),\\ Eq.\ (13.61)}}{\downarrow}{=} \sqrt{\frac{\frac{p^0}{m}+1}{2}}\mathbb{1} \pm \boldsymbol{n}\cdot\boldsymbol{\sigma}\frac{\frac{|\boldsymbol{p}|}{m}}{\sqrt{2\frac{p^0}{m}+1}} \\ &= \frac{p^0+m\pm\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p^0+m)}} \quad \checkmark\,. \end{aligned} \tag{13.62}$$

13.4 Neutral spinor fields (Majorana spinor) and anti-commuting fields

The neutral spinor is the analogue to the real scalar field and the fields satisfy

$$\Big[\psi_\alpha^{(-)}(x), \psi_\beta^{(+)}(y)\Big]_{\underset{\substack{\uparrow\\ \text{causality}}}{\mp}} \neq 0 \quad \text{for} \quad (x-y)^2 < 0\,, \tag{13.63}$$

with

$$[A,B]_\mp = AB \mp BA \tag{13.64}$$

$$\text{or} \quad [A,B]_- = [A,B]\,, \qquad [A,B]_+ = \{A,B\}\,. \tag{13.65}$$

We must use a superposition of $\psi_\alpha^{(+)}$, $\psi_\alpha^{(-)}$ containing annihilation/creation operators to define a causal field

$$\psi_\alpha(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\Big(e^{-ipx}u_\alpha(\boldsymbol{p},s)a(\boldsymbol{p},s) + \lambda e^{ipx}v_\alpha(\boldsymbol{p},s)a^\dagger(\boldsymbol{p},s)\Big)\,. \tag{13.66}$$

Computing the (anti-)commutators:

$$[\psi_\alpha(x),\psi_\beta(y)]_\mp = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\Big(e^{-ip(x-y)}\lambda u_\alpha(\boldsymbol{p},s)v_\beta(\boldsymbol{p},s) \mp e^{ip(x-y)}\lambda v_\alpha(\boldsymbol{p},s)u_\beta(\boldsymbol{p},s)\Big)\,, \tag{13.67}$$

$$[\psi_\alpha(x),\psi_\beta^\dagger(y)]_\mp = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\Big(e^{-ip(x-y)}u_\alpha(\boldsymbol{p},s)u^*_\beta(\boldsymbol{p},s) \mp e^{ip(x-y)}|\lambda|^2 v_\alpha(\boldsymbol{p},s)v^*_\beta(\boldsymbol{p},s)\Big)\,, \tag{13.68}$$

where we assumed standard commutation relations (wrong!) for the creation/annihilation operators when performing the above computations:

$$[a(\boldsymbol{p},s), a^\dagger(\boldsymbol{p}',s')]_\mp = 2E_{\boldsymbol{p}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{p}')\delta_{ss'}\,. \tag{13.69}$$

We will make a choice + or − later.

To simplify the expressions in Eq. (13.67), Eq. (13.68), we compute the four spin sums appearing in them.

The following relations hold:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \sigma^i\sigma^j p_i p_j = \frac{1}{2}\{\sigma^i,\sigma^j\}p_i p_j = \boldsymbol{p}^2\,,$$

hermiticity

$$\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}\,,$$

and

$$\begin{aligned}
D_L\big(L(p)\big)D_L^\dagger\big(L(p)\big) &= \left(\frac{p^0+m-\boldsymbol{\sigma}\cdot\boldsymbol{p}}{\sqrt{2m(p^0+m)}}\right)^2 \\
&= \frac{(p^0+m)^2-2(p^0+m)\boldsymbol{\sigma}\cdot\boldsymbol{p}+\boldsymbol{p}^2}{2m(p^0+m)} \\
&= \frac{1}{m}\left(-\boldsymbol{\sigma}\cdot\boldsymbol{p}+\frac{{p^0}^2+2p^0m+\overset{={p^0}^2}{\overbrace{(m^2+\boldsymbol{p}^2)}}}{2(p^0+m)}\right) \\
&= \frac{1}{m}\left(-\boldsymbol{\sigma}\cdot\boldsymbol{p}+\frac{2p^0(p^0+m)}{2(p^0+m)}\right) \\
&= \frac{1}{m}(p^0-\boldsymbol{\sigma}\cdot\boldsymbol{p}) \\
&= \frac{1}{m}\sigma^\mu p_\mu \quad \checkmark\,.
\end{aligned}$$

The spin-sums are therefore:

$$\sum_s u_\alpha(\boldsymbol{p},s)u_\beta^*(\boldsymbol{p},s) = D_L\big(L(p)\big)_{\alpha\alpha'}D_L^*\big(L(p)\big)_{\beta\beta'}\sum_s \overset{u_\alpha^s=C_u\delta_{\alpha s}}{\overbrace{u_{\alpha'}(\boldsymbol{k},s)}}u_{\beta'}^*(\boldsymbol{k},s) \tag{13.70}$$

$$= |C_u|^2\Big(D_L\big(L(p)\big)D_L^\dagger\big(L(p)\big)\Big)_{\alpha\beta} \tag{13.71}$$

$$= \frac{|C_u|^2}{m}(p^0-\boldsymbol{\sigma}\cdot\boldsymbol{p})_{\alpha\beta} \tag{13.72}$$

$$= \frac{|C_u|^2}{m}\sigma^\mu_{\alpha\beta}p_\mu\,. \tag{13.73}$$

Similarly:

$$\sum_s v_\alpha(\boldsymbol{p},s)v_\beta^*(\boldsymbol{p},s) = \frac{|C_v|^2}{m}\sigma^\mu_{\alpha\beta}p_\mu\,, \tag{13.74}$$

$$\sum_s u_\alpha(\boldsymbol{p},s)v_\beta(\boldsymbol{p},s) = -C_uC_v\epsilon_{\alpha\beta}\,, \tag{13.75}$$

$$\sum_s v_\alpha(\boldsymbol{p},s)u_\beta(\boldsymbol{p},s) = C_uC_v\epsilon_{\alpha\beta}\,. \tag{13.76}$$

Finally:[1]

$$[\psi_\alpha(x),\psi_\beta(y)]_\mp = -C_uC_v\lambda\epsilon_{\alpha\beta}\Big(\Delta_+(x-y)\pm\Delta_+(y-x)\Big)\,, \tag{13.77}$$

$$[\psi_\alpha(x),\psi_\beta^\dagger(y)]_\mp = \sigma^\mu_{\alpha\beta}\frac{|C_u|^2}{m}\left(\partial_\mu^{(x)}\Delta_+(x-y)\mp|\lambda|^2\left|\frac{C_v}{C_u}\right|^2\partial_\mu^{(y)}\Delta_+(y-x)\right)\,, \tag{13.78}$$

[1] Compare to the signs of the scalar field Sec. 5.1.1

with

$$\Delta_+(x-y) = \int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}e^{-ipx}\,. \tag{13.79}$$

We recall from the discussion of the free scalar field that $\Delta_+(x)\neq 0$ for $x^2<0$ (space-like distance), so the two terms on the RHS's of Eq. (13.77) and Eq. (13.78) have to cancel each other for these commutators to vanish and the theory to be causal. We further use $\Delta_+(-x) = \Delta_+(x)$, which means that $\partial_\mu^{(x)}\Delta_+(x)$ is odd under $x \longrightarrow -x$. Therefore it follows:

> The commutators $[\cdot,\cdot]_\mp$ vanish for space-like separation only if one chooses the **anti-commutation** relations and $|\lambda|^2\left|\frac{C_v}{C_u}\right|^2 = 1$. Relativistic causality requires that **spin** $\frac{1}{2}$ fields are **fermionic**.

This is a special case of the **spin-statistics theorem**, which follows from Lorentz-invariance and causality:

$$\begin{aligned}\text{integer spin} &= \text{bosons},\\ \text{half-integer spin} &= \text{fermions}.\end{aligned}$$

We choose

$$\lambda = 1\,, \quad C_u = \sqrt{m}\,, \quad C_v = \sqrt{m}e^{i\delta}\,, \tag{13.80}$$

without loss of generality. The corresponding field is a Majorana field. Its particle excitations are neutral spin $\frac{1}{2}$ particles, which are their own anti-particles.

The above field expansion satisfies the Majorana equation:

$$i\bar{\sigma}^\mu\partial_\mu\psi - m\big(\psi^T\epsilon\big)^\dagger = 0\,, \tag{13.81}$$

or, in components,

$$i\bar{\sigma}^\mu_{\alpha\beta}\partial_\mu\psi_\beta + m\epsilon_{\alpha\beta}\psi^*_\beta = 0\,. \tag{13.82}$$

Comments:

- For a single Majorana field we can rephase the mass $m = |m|e^{i\delta}$ to be real $\psi \longrightarrow e^{i\frac{\delta}{2}}\psi$, which sets $\delta = 0$.
- For a right-handed spinor field we must replace $\bar{\sigma}^\mu$ by σ^μ in Eq. (13.82).
- The field equation is a first-order differential equation. It implies the Klein-Gordon equation for each component ψ_α:

  $$(\Box + m^2)\psi_\alpha = 0\,. \tag{13.83}$$

  To show this we use that

  $$\sigma^\nu\bar{\sigma}^\mu + \sigma^\mu\bar{\sigma}^\nu = 2\eta^{\mu\nu} \tag{13.84}$$

  and

  $$(\bar{\sigma}^\mu)^* = -\epsilon\sigma^\mu\epsilon\,. \tag{13.85}$$

  Multiplying Eq. (13.81) from the left with $i\sigma^\mu\partial_\mu$ gives

  $$\underbrace{-\sigma^\nu\partial_\nu\bar{\sigma}^\mu\partial_\mu\psi}_{(1)} + \underbrace{mi\sigma^\mu\partial_\mu\big(\epsilon\psi^*\big)}_{(2)} = 0\,. \tag{13.86}$$

We rewrite (1):

$$-\sigma^\nu\partial_\nu\bar{\sigma}^\mu\partial_\mu\psi = -\frac{1}{2}(\sigma^\nu\bar{\sigma}^\mu + \sigma^\mu\bar{\sigma}^\nu)\partial_\mu\partial_\nu\psi \overset{\substack{Eq.\ (13.84)\\ \downarrow}}{=} -\Box\psi\,. \tag{13.87}$$

(2) looks like the first term in Eq. (13.81), but with the additional $\epsilon_{\alpha\beta}$. To investigate it further, we take the complex conjugate of Eq. (13.81),

$$-i(\bar{\sigma}^\mu)^*\partial_\mu\psi^* + m\epsilon\psi = 0\,. \tag{13.88}$$

and use Eq. (13.85):

$$i\epsilon\sigma^\mu\epsilon\partial_\mu\psi^* + m\epsilon\psi = 0\,. \tag{13.89}$$

Multiplying the above with ϵ from the left and using that $\epsilon^2 = -\mathbb{1}_{2\times 2}$, we finally get

$$i\sigma^\mu\partial_\mu(\epsilon\psi^*) = -m\psi\,. \tag{13.90}$$

Therefore:

$$-\Box\psi_\alpha + m(-m\psi_\alpha) = 0\,, \tag{13.91}$$

$$\iff \quad (\Box + m^2)\psi_\alpha = 0 \quad \checkmark\,. \tag{13.92}$$

13.5 Field Lagrangian and equations of motion

We do not need to go through the particle representation to find the EOM, we can also take the field representation and investigate the terms in the Lagrangian.
The Lagrangian should be a Lorentz scalar. The term

$$\psi_L^\dagger\bar{\sigma}^\mu\psi_L \tag{13.93}$$

transforms as

$$\psi_L^\dagger\bar{\sigma}^\mu\psi_L \longrightarrow \Lambda^\mu{}_\nu\psi_L^\dagger\bar{\sigma}^\mu\psi_L \tag{13.94}$$

and

$$i\partial_\mu \longrightarrow \Lambda_\mu{}^\nu i\partial_\nu\,, \tag{13.95}$$

therefore

$$\psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L\,, \quad \psi_R^\dagger i\sigma^\mu\partial_\mu\psi_R\,, \tag{13.96}$$

are Lorentz scalars (these will be the kinetic terms), as well as

$$\psi_L^T\epsilon\psi_L\,, \quad \psi_R^T\epsilon\psi_R\,, \tag{13.97}$$

which will be the Majorana mass terms. We can check that $\psi_L^T\epsilon\psi_L$ is a scalar using

$$D_L^\dagger(\Lambda) = D_R^{-1}(\Lambda) \tag{13.98}$$

and

$$D_L(\Lambda) = \epsilon D_R^*(\Lambda)\epsilon^{-1} = -\epsilon D_R^*(\Lambda)\epsilon\ : \tag{13.99}$$

under Lorentz transformations

$$\psi_L^T\epsilon\psi_L \xrightarrow{\Lambda} \psi_L^T D_L^T(\Lambda)\epsilon D_L(\Lambda)\psi_L = \psi_L^T\epsilon\psi_L \quad \checkmark\,, \tag{13.100}$$

with

$$\begin{aligned} D_L^T(\Lambda)\epsilon D_L(\Lambda) &= -D_L^T(\Lambda)\epsilon D_L(\Lambda)\epsilon\cdot\epsilon \overset{\overset{Eq.\ (13.99)}{\downarrow}}{=} D_L^T(\Lambda)D_R^*(\Lambda)\epsilon \\ &\underset{\underset{Eq.\ (13.98)}{\uparrow}}{=} D_R^{*\,-1}(\Lambda)D_R^*(\Lambda)\epsilon = \epsilon \quad \checkmark\,. \end{aligned} \tag{13.101}$$

Therefore, the free Majorana Lagrangian is

$$\boxed{\mathcal{L} = \psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L - \frac{m}{2}\Big[\psi_L^T\epsilon\psi_L - \psi_L^\dagger\epsilon\psi_L^*\Big]} \tag{13.102}$$

Comments:

- The canonical mass dimension of ψ_L is $[\psi_L] = \frac{3}{2}$, since

$$\Big[\psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L\Big] = 4 \quad\longleftrightarrow\quad 2[\psi_L] = 3\,. \tag{13.103}$$

- The Lagrangian is real, such that the action is real:

$$\Big[\psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L\Big]^* = \Big[\psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L\Big]^\dagger \tag{13.104}$$

$$= (-i)\partial_\mu\psi_L^\dagger(\bar{\sigma}^\mu)^\dagger\psi_L \tag{13.105}$$

$$\overset{\text{IBP}}{=} i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L + (\text{total derivative})\,. \tag{13.106}$$

$(\bar{\sigma}^\mu)^\dagger = \bar{\sigma}^\mu\,,\quad \sigma^{\mu\,\dagger} = \sigma^\mu$

- The ψ_α are anti-commuting objects (more later) since operators will anti-commute and otherwise

$$\psi_L^T\epsilon\psi_L = \underset{\underset{\text{antisymm. in }\alpha\leftrightarrow\beta}{\uparrow}}{\epsilon_{\alpha\beta}}\psi_{L,\alpha}\psi_{L,\beta} \tag{13.107}$$

would vanish if $\psi_{L,\alpha}\psi_{L,\beta}$ wasn't antisymmetric in $\alpha \longleftrightarrow \beta$ as well.

- We get the EOM through the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\psi_{L,\alpha}} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_{L,\alpha})} = 0\,. \tag{13.108}$$

It is simpler to use the variation w.r.t. $\psi_L^\dagger$:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_{L,\alpha}^\dagger)} = 0\,, \tag{13.109}$$

$$\frac{\partial\mathcal{L}}{\partial\psi_{L,\alpha}^\dagger} = \underset{\underset{(*)}{\uparrow}}{-}i\bar{\sigma}^\mu_{\alpha\beta}\partial_\mu\psi_{L,\beta} + \frac{m}{2}\Big[-\epsilon_{\alpha\beta}\psi_{L,\beta}^\dagger + \psi_{L,\gamma}^\dagger\epsilon_{\gamma\alpha}\Big]\,. \tag{13.110}$$

The derivative in Eq. (13.110) is a right-derivative (we will define it later), and the minus sign $(*)$ comes from the anti-commutation of $\frac{\partial}{\partial\psi_{L,\alpha}^\dagger}$ and $\psi_{L,\beta}$. Therefore,

$$i\bar{\sigma}^\mu_{\alpha\beta}\partial_\mu\psi_{L,\beta} + m\epsilon_{\alpha\beta}\psi_\beta^\dagger = 0 \tag{13.111}$$

follows.

The canonically conjugate momentum is

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_{L,\alpha})} = \psi^\dagger_{L,\beta} i \underbrace{\bar{\sigma}^0_{\beta\alpha}}_{=\delta_{\alpha\beta}} = i\psi^\dagger_\alpha \,. \tag{13.112}$$

No derivative appears, since the EOM is a first-order differential equation. The conjugate momentum is simply the hermitian conjugate of ψ_L multiplied by i.

Instead of going through the discussion of the particle reps and the little group, we can proceed as in the chapter on scalar field theory at the beginning. We would get

$$H_0 = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} E_{\boldsymbol{p}} a^\dagger(\boldsymbol{p}, s) a(\boldsymbol{p}, s) \tag{13.113}$$

if we imposed anti-commutation rules and used $u(\boldsymbol{p}, s)$, $v(\boldsymbol{p}, s)$ as we previously derived.
We now impose **equal-time <u>anti</u>-commutation relations**:

$$\{\pi_\alpha(t, \boldsymbol{x}), \psi_\beta(t, \boldsymbol{y})\} = i\delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y})\,, \tag{13.114}$$
$$\{\psi_\alpha(t, \boldsymbol{x}), \psi_\beta(t, \boldsymbol{y})\} = 0\,, \tag{13.115}$$
$$\{\pi_\alpha(t, \boldsymbol{x}), \pi_\beta(t, \boldsymbol{y})\} = 0\,. \tag{13.116}$$

We can check that this holds:

$$\{\psi_\alpha(t, \boldsymbol{x}), \psi_\beta(t, \boldsymbol{y})\} \overset{Eq.\ (13.77)}{=} -C_u C_v \lambda \epsilon_{\alpha\beta} \underbrace{\Big(\Delta_+(x-y) - \Delta_+(y-x)\Big)}_{=0 \text{ for space-like } (x-y)^2<0}$$

and

$$\begin{aligned}\{\psi_\alpha(t, \boldsymbol{x}), i\psi^\dagger_\beta(t, \boldsymbol{y})\} &= i \cdot i\sigma^\mu_{\alpha\beta} \underbrace{\frac{|C_u|^2}{m}}_{=1} \Big(\partial^{(x)}_\mu \Delta_+(x-y) + \partial^{(y)}_\mu \Delta_+(x-y)\Big)\Big|_{x^0=y^0} \\ &= -\sigma^\mu_{\alpha\beta} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} (-ip_\mu)\Big(e^{-ip(x-y)} + e^{ip(x-y)}\Big)\Big|_{x^0=y^0} .\end{aligned}$$

The $\boldsymbol{p}$-terms cancel after substituting $\boldsymbol{p} \longrightarrow -\boldsymbol{p}$ in the second term.
The only thing that remains is

$$\{\psi_\alpha(t, \boldsymbol{x}), i\psi^\dagger_\beta(t, \boldsymbol{y})\} = i\underbrace{\sigma^0_{\alpha\beta}}_{=\delta_{\alpha\beta}} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} 2E_{\boldsymbol{p}} \underbrace{e^{-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})}}_{x^0=y^0} \tag{13.117}$$
$$= i\delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \quad \checkmark\,. \tag{13.118}$$

This was the free field theory. Can we include interactions?

13.5.1 Interactions

Can we include self-interactions like ϕ^3 or ϕ^4 in the scalar field theory? Not at low E, since

$$\frac{\widetilde{g}}{\Lambda^2}\Big(\psi^T_L \epsilon \psi_L\Big)\Big(\psi^T_L \epsilon \psi_L\Big), \quad \text{with} \quad [\psi_L] = \frac{3}{2}\,, \tag{13.119}$$

Figure 13.1: Heisenberg proposed a non-linear spinor theory as his 'weltformel' (at the 1958 Planck Centenary in West Berlin). Ex: show why this makes no sense. See also "Heisenberg's 1958 Weltformel and the Roots of Post-Empirical Physics", Alexander S. Blum, which you can download at the UB.

is an irrelevant operator (not important at low energy), since it has mass dimension 6.

If there is a scalar field,

$$\mathcal{L}_{\text{int}} = -g\phi\psi_L^T\epsilon\psi_L + \text{h.c.} \tag{13.120}$$

is the only possible term. If ϕ is complex we can perform a $U(1)$ transformation $\phi \longrightarrow e^{-i\alpha}\phi$ and $\psi_L \longrightarrow e^{i\frac{\alpha}{2}}\psi_L$, but for this to be a symmetry of the theory the Majorana mass term would have to vanish, $m \longrightarrow 0$, since it is not $U(1)$-invariant:

$$\frac{m}{2}\left[\psi_L^T\epsilon\psi_L - \psi_L^\dagger\epsilon\psi_L^*\right] \quad \overset{U(1)}{\longrightarrow} \quad \frac{m}{2}\left[e^{i\alpha}\psi_L^T\epsilon\psi_L - e^{-i\alpha}\psi_L^\dagger\epsilon\psi_L^*\right]. \tag{13.121}$$

In the presence of a vector field A_μ (more on this later),

$$A_\mu \longrightarrow {\Lambda_\mu}^\nu A_\nu\,, \tag{13.122}$$

then a possible marginal interaction is

$$\mathcal{L}_{\text{int}} = g\psi_L^\dagger i\bar{\sigma}^\mu A_\mu\psi_L\,. \tag{13.123}$$

13.6 Dirac fermions or charged and massive spinors

We first make the naive (and wrong) attempt:

$$\underset{\substack{\uparrow \\ L \text{ or } R \text{ two-spinor}}}{\psi_\alpha} \sim u_\alpha(\boldsymbol{p},s)a(\boldsymbol{p},s) + v_\alpha(\boldsymbol{p},s)\underset{\substack{\uparrow \\ \text{anti-particle creation operator}}}{b^\dagger}(\boldsymbol{p},s)\,. \tag{13.124}$$

This would be fine for **internal** symmetries, since anti-particles transform under the complex conjugated rep of G by definition:

$$a \longrightarrow D(g)a\,, \quad b \longrightarrow D^\dagger(g)b\,, \quad b^\dagger \longrightarrow D(g)b^\dagger \tag{13.125}$$

$$\text{and} \quad \psi \longrightarrow D(g)a + D(g)b^\dagger = D(g)\psi \quad \checkmark\,. \tag{13.126}$$

This does not work, however, because of Lorentz invariance: the field equation would contain

$$\psi_\alpha^{\mathcal{C}} \sim u_\alpha(\boldsymbol{p},s)b(\boldsymbol{p},s) + v_\alpha(\boldsymbol{p},s)a^\dagger(\boldsymbol{p},s)\,, \tag{13.127}$$

which we cannot obtain from $\psi^\dagger$!
<u>**Recall:**</u> When we changed from the real scalar field ϕ to the complex one, the degrees of freedom went from 1 to 2. We therefore treated ϕ and $\phi^\dagger$ as independent canonical variables.
That is **not possible here**. We would like to go from the real ψ_α with 2 d.o.f. to the complex ψ_α with 4 d.o.f, but we cannot treat ψ and $\psi^\dagger$ as independent canonical variables, since

$$\pi_\alpha = i\psi_\alpha^\dagger \tag{13.128}$$

is the **conjugate momentum**.

The four-component field is constructed from two Majorana-fields: the spinor belonging to the rep $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ is the **Dirac-spinor**:

$$\Psi_\alpha(x)=\sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\Big(e^{-ipx}u_\alpha(\boldsymbol{p},s)a(\boldsymbol{p},s)+e^{ipx}v_\alpha(\boldsymbol{p},s)b^\dagger(\boldsymbol{p},s)\Big)\,, \tag{13.129}$$

with $\alpha=1,2,3,4$. It transforms under the Lorentz group as

$$\Psi\xrightarrow{\Lambda}\begin{pmatrix}D_L(\Lambda)&0\\0&D_R(\Lambda)\end{pmatrix}\Psi\,. \tag{13.130}$$

Writing it as

$$\Psi=\begin{pmatrix}\psi_L\\\psi_R\end{pmatrix} \tag{13.131}$$

we can use the γ^μ to simplify the expressions.

Summary:

1) The field expansions for spin $\frac{1}{2}$ fermionic and spin $\frac{1}{2}$ antifermionic fields are given in terms of annihilation/creation operators a, $a^\dagger$ and b, $b^\dagger$ with anti-commutation relations between them, where the only non-vanishing relations are

$$\{a(\boldsymbol{p},s),a^\dagger(\boldsymbol{p'},s')\}=2E_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'})\delta_{ss'}\,, \tag{13.132}$$
$$\{b(\boldsymbol{p},s),b^\dagger(\boldsymbol{p'},s')\}=2E_{\boldsymbol{p}}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'})\delta_{ss'}\,. \tag{13.133}$$

2) The spin polarization coefficients have four entries:

$$u(\boldsymbol{p},s)=\frac{\not{p}+m}{\sqrt{2m(E_{\boldsymbol{p}}+m)}}u(\boldsymbol{k},0)\,,\quad k^\mu=\begin{pmatrix}m\\\boldsymbol{0}\end{pmatrix}\,, \tag{13.134}$$
$$v(\boldsymbol{p},s)=-\frac{\not{p}-m}{\sqrt{2m(E_{\boldsymbol{p}}+m)}}v(\boldsymbol{k},s)\,, \tag{13.135}$$

where we have introduced the Feynman-slash notation

$$\not{a}\equiv a_\mu\gamma^\mu\,. \tag{13.136}$$

Furthermore

$$u(\boldsymbol{k},1/2)=\sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}\,,\quad u(\boldsymbol{k},-1/2)=\sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}\,, \tag{13.137}$$

$$v(\boldsymbol{k},1/2)=\sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\,,\quad v(\boldsymbol{k},-1/2)=\sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}\,. \tag{13.138}$$

We define the useful "Dirac-adjoint"

$$\overline{\Psi}_\alpha\equiv\left(\Psi^\dagger\gamma^0\right)_\alpha=\Psi^\dagger_\beta\gamma^0_{\beta\alpha}\,, \tag{13.139}$$

which we can use to set

$$\bar{u}(\boldsymbol{p}, s) = u^\dagger(\boldsymbol{p}, s)\gamma^0 \,, \tag{13.140}$$
$$\bar{v}(\boldsymbol{p}, s) = v^\dagger(\boldsymbol{p}, s)\gamma^0 \,. \tag{13.141}$$

We have the two spin sums

$$\sum_{s=\pm\frac{1}{2}} u_\alpha(\boldsymbol{p}, s)\bar{u}_\beta(\boldsymbol{p}, s) = (\not{p} + m)_{\alpha\beta} \,, \tag{13.142}$$
$$\sum_{s=\pm\frac{1}{2}} v_\alpha(\boldsymbol{p}, s)\bar{v}_\beta(\boldsymbol{p}, s) = (\not{p} - m)_{\alpha\beta} \,. \tag{13.143}$$

3) The anti-commutator of the field operators is

$$\{\Psi_\alpha(x), \overline{\Psi}_\beta(y)\} = (i\not{\partial}^{(x)} + m)_{\alpha\beta}\Big(\Delta_+(x-y) - \Delta_+(y-x)\Big) \,. \tag{13.144}$$

4) Parity, charge-conjugation and time-reversal act on the fields as follows:

$$U_{\mathcal{P}}\Psi(x)U_{\mathcal{P}}^{-1} = \eta_{\mathcal{P}}^*\gamma^0\Psi(t, -\boldsymbol{x}) \,, \tag{13.145}$$
$$U_{\mathcal{T}}\Psi(x)U_{\mathcal{T}}^{-1} = \xi_{\mathcal{T}}^*\gamma^1\gamma^3\Psi(-t, \boldsymbol{x}) \,, \tag{13.146}$$
$$U_{\mathcal{C}}\Psi(x)U_{\mathcal{C}}^{-1} = -\xi_{\mathcal{C}}^* i\gamma^2\Psi^*(x) \,. \tag{13.147}$$

$$\overline{\eta}_{\mathcal{P}} \equiv -\eta_{\mathcal{P}}^* \,,$$
$$\bar{\xi}_{\mathcal{T}} \equiv \xi_{\mathcal{T}}^* \,,$$
$$\bar{\xi}_{\mathcal{C}} \equiv \xi_{\mathcal{C}}^* \,,$$

where the overline symbolizes that the quantity is associated to an anti-particle. In particular, an anti-particle has the opposite parity of the corresponding particle.

These relations can be derived from the properties of a, $a^\dagger$, b and $b^\dagger$ under $U_i[...]U_i^{-1}$ using the relations given in section Sec. 12.6.1, Sec. 12.5.5, and Sec. 12.5.4.

5) The free field equation for Dirac spinors is the **Dirac equation**

$$(i\not{\partial} - m)\Psi(x) = 0 \,, \quad \not{\partial} = \gamma^\mu\partial_\mu \,. \tag{13.148}$$

We can derive it from the expansion of $\Psi \sim a + b^\dagger$ with

$$(\not{p} - m)u(\boldsymbol{p}, s) = \frac{(\not{p} - m)(\not{p} + m)}{\sqrt{2m(E_{\boldsymbol{p}} + m)}}u(\boldsymbol{k}, s) = 0 \,, \tag{13.149}$$

since

$$\not{p}\not{p} = p_\mu p_\nu\gamma^\mu\gamma^\nu = p_\mu p_\nu \underbrace{\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}}_{=\eta^{\mu\nu}} = p^2 = m^2 \,, \tag{13.150}$$

and similarly

$$(\not{p} + m)v(\boldsymbol{p}, s) = 0 \,. \tag{13.151}$$

After Fourier-transforming $\Psi(x)$, Eq. (13.148) becomes

$$(\not{p} - m)\widetilde{\Psi}(p) = 0 \tag{13.152}$$

We can see that this implies 4 real d.o.f.. Naively, Ψ_α, $\alpha = 1, 2, 3, 4$ is complex and it contains 8 real parameters. We take

$$p^\mu = \begin{pmatrix} m \\ \boldsymbol{0} \end{pmatrix} \,, \quad \not{p} = m\gamma^0 = \begin{pmatrix} 0 & m\mathbb{1}_{2\times2} \\ m\mathbb{1}_{2\times2} & 0 \end{pmatrix} \,, \tag{13.153}$$

so the EOM Eq. (13.152) becomes

$$m \begin{pmatrix} -\mathbb{1}_{2\times 2} & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & -\mathbb{1}_{2\times 2} \end{pmatrix} \widetilde{\Psi}(p) = 0 \tag{13.154}$$

$$\Longleftrightarrow \quad \widetilde{\Psi}_1 - \widetilde{\Psi}_3 = 0 \qquad \widetilde{\Psi}_2 - \widetilde{\Psi}_4 = 0\,. \tag{13.155}$$

and we see that the two equations for the complex spinors reduce the number of independent *real* degrees of freedom of a massive spinor on-shell from eight to just four.

With

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \quad \text{and} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{13.156}$$

we get from Eq. (13.148)

$$i\bar{\sigma}^\mu \partial_\mu \psi_L - m\psi_R = 0\,, \tag{13.157}$$

$$i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0\,. \tag{13.158}$$

The mass term couples ψ_L and ψ_R. For $m = 0$ ψ_L and ψ_R decouple.

If we require

$$\psi_R \equiv -(\epsilon\psi_L)^* \quad \text{"reality" condition for spinors,} \tag{13.159}$$

which is consistent with the Lorentz-transformation properties, then ψ_L and ψ_R are not independent. ψ_L and ψ_R satisfy the Majorana equation for **neutral** spinors. Indeed:

$$U_{\mathcal{C}}\Psi(x)U_{\mathcal{C}}^{-1} = -\xi_{\mathcal{C}}^* i\gamma^2 \Psi^*(x) \tag{13.160}$$

$$= -\xi_{\mathcal{C}}^* \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} \tag{13.161}$$

$$\underset{\epsilon=-i\sigma^2}{\overset{\uparrow}{=}} \xi_{\mathcal{C}}^* \begin{pmatrix} \epsilon\psi_R^* \\ -\epsilon\psi_L^* \end{pmatrix} = \xi_{\mathcal{C}}\Psi(x) \quad \checkmark\,, \tag{13.162}$$

where we used Eq. (13.159) and $\epsilon^2 = -\mathbb{1}_{2\times 2}$:

$$\epsilon\psi_R^* = \psi_L\,, \quad -\epsilon\psi_L^* = \psi_R\,. \tag{13.163}$$

The field is therefore self-conjugated and the particle is the same as the anti-particle, so the field is neutral. For four-spinors, the reality condition Eq. (13.159) translates to

$$-i\gamma^2\Psi^* = \Psi \tag{13.164}$$

and it is the analogue to the **reality condition** $\phi^\dagger = \phi$ for the scalar field.

6) The free Lagrangian is

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\Psi\,. \tag{13.165}$$

Is it Lorentz-invariant? We check the mass term:

$$\overline{\Psi}\Psi = \Psi^\dagger \gamma^0 \Psi = \Psi^\dagger \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix} \Psi \tag{13.166}$$

$$= \psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \tag{13.167}$$

and since

$$D_L^{-1}(\Lambda) = D_R^\dagger(\Lambda)\,, \tag{13.168}$$

see Eq. (13.16), this is Lorentz-invariant; similarly, we can show that $\overline{\Psi} i \partial\!\!\!/ \Psi$ is invariant.

Ex: show this!

The possible (marginal) interaction terms are:

$$-g\phi\overline{\Psi}\Psi + \text{h.c.}\,, \tag{13.169}$$

$$-g\overline{\Psi}\gamma^\mu A_\mu \Psi + \text{h.c.}\,. \tag{13.170}$$

13.7 *Massless spin $\frac{1}{2}$ particles*

There seems to be a smooth $m \longrightarrow 0$ limit. If we take the Majorana field ,

$$\psi_\alpha(x) \sim \sum_{s=\pm\frac{1}{2}} u(\boldsymbol{p}, s) a(\boldsymbol{p}, s) + v(\boldsymbol{p}, s) a^\dagger(\boldsymbol{p}, s)\,, \tag{13.171}$$

in the limit $m \longrightarrow 0$ the spin polarization coefficients become

$$u(\boldsymbol{p}, s) = \frac{\sigma^\mu p_\mu}{\sqrt{2E_{\boldsymbol{p}}}} \xi_s\,, \quad v(\boldsymbol{p}, s) = \frac{\sigma^\mu p_\mu}{\sqrt{2E_{\boldsymbol{p}}}} \epsilon \xi_s e^{i\delta}\,, \tag{13.172}$$

$$\text{with} \quad \xi_s = \begin{cases} (1,0)^T\,, & s = +\frac{1}{2} \\ (0,1)^T\,, & s = -\frac{1}{2} \end{cases}\,. \tag{13.173}$$

So ψ_α seems to create/destroy spin $\frac{1}{2}$ particles (similarly for the particles and antiparticles in the case of the Dirac field). The field equations are indeed the same for $m \longrightarrow 0$, but the **interpretation** is not correct.
We need Eq. (12.168), Eq. (12.169) with $p^\mu = k^\mu = n(1,0,0,1)^T$ the reference momentum:

$$u(\boldsymbol{k}, \sigma) e^{-i\theta(w)\sigma} = \overset{\substack{\text{in little group} \\ \downarrow}}{D(w)} u(\boldsymbol{k}, \sigma)\,, \tag{13.174}$$

$$v(\boldsymbol{k}, \underset{\substack{\uparrow \\ \text{helicity}}}{\sigma}) e^{i\theta(w)\sigma} = D(w) v(\boldsymbol{k}, \sigma)\,, \tag{13.175}$$

and $\theta(w)$ the angle of rotation around the z-axis generated by J^3. In the $\left(\frac{1}{2}, 0\right)$ or $\left(0, \frac{1}{2}\right)$ rep we have

$$D(w) = \mathbb{1} - i\big[\theta J^3 + \alpha\big(J^2 + K^1\big) + \beta\big(K^2 - J^1\big)\big] + \ldots \tag{13.176}$$

and we know

$$J^3 = \frac{\sigma^3}{2}\,, \quad J^2 + K^1 = \frac{\pm i\sigma^1 + \sigma^2}{2}\,, \quad K^2 - J^1 = \frac{\pm i\sigma^2 - \sigma^1}{2}\,, \tag{13.177}$$

with the $+$ sign for $(0,\frac{1}{2})$ and the $-$ sign for $(\frac{1}{2},0)$. Therefore:

$$D(w) = \mathbb{1} - i\theta\frac{\sigma^3}{2} - i(\pm i\alpha - \beta)\frac{\sigma^1 \mp i\sigma^2}{2} + \ldots\,, \tag{13.178}$$

from which we know:

1) Since we must set $\boldsymbol{A} = \boldsymbol{B} = 0$:

$$(\sigma^1 \mp i\sigma^2)u(\boldsymbol{k},\sigma) = 0\,, \tag{13.179}$$
$$(\sigma^1 \mp i\sigma^2)v(\boldsymbol{k},\sigma) = 0\,, \tag{13.180}$$

with

$$\sigma^1 - i\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\,, \quad \sigma^1 + i\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\,. \tag{13.181}$$

2) The J^3 eigenvalue σ is

$$\frac{\sigma^3}{2}u(\boldsymbol{k},\sigma) = \sigma u(\boldsymbol{k},\sigma)\,, \tag{13.182}$$
$$\frac{\sigma^3}{2}v(\boldsymbol{k},\sigma) = -\sigma v(\boldsymbol{k},\sigma)\,. \tag{13.183}$$

Now we have to find solutions to the above equations.

For a **left-handed** (massless) spinor we take the upper sign everywhere and from 1) becomes

$$(\sigma^1 - i\sigma^2)u(\boldsymbol{k},s) = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}u(\boldsymbol{k},s) = 0 \tag{13.184}$$

and therefore

$$u(\boldsymbol{k},s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_{-\frac{1}{2}}\,, \tag{13.185}$$

From 2) we get the helicity $\sigma = -\frac{1}{2}$, and so we write

$$u(\boldsymbol{k},-1/2)\,. \tag{13.186}$$

The same way we get

$$v(\boldsymbol{k},\sigma) = \begin{pmatrix} 0 \\ * \end{pmatrix}\,, \quad \text{helicity} \quad \sigma = +\frac{1}{2}\,. \tag{13.187}$$

We choose

$$v(\boldsymbol{k},\sigma=1/2) = \epsilon\xi_{\frac{1}{2}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{13.188}$$

The field expansion is therefore

$$\psi_L(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(e^{-ipx}\underbrace{u(\boldsymbol{p},-1/2)\,a(\boldsymbol{p},-1/2)}_{\text{particle with helicity }\sigma=-\frac{1}{2}} + e^{ipx}\underbrace{v(\boldsymbol{p},1/2)\,a^\dagger(\boldsymbol{p},1/2)}_{\text{particle with helicity }\sigma=\frac{1}{2}}\right)\,. \tag{13.189}$$

We therefore found that a left-handed, massless spinor annihilates particles with helicity $-\frac{1}{2}$ and creates anti-particles with helicity $\frac{1}{2}$. If the particle is neutral, i.e. particle and anti-particle are the same, then the field creates and annihilates the same particle in two different helicity states.If the field is complex, then the anti-particle must have the opposite helicity as the particle. Massless two-component spinors are **Weyl-spinors**.

For a **right-handed spinor** 1) becomes

$$(\sigma^1 + i\sigma^2)u(\boldsymbol{k}, s) = 2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u(\boldsymbol{k}, s) \tag{13.190}$$

and therefore

$$u(\boldsymbol{k}, s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{13.191}$$

From 2) we obtain the helicity $\sigma = \frac{1}{2}$. Analogously we get

$$v(\boldsymbol{k}, s) = \begin{pmatrix} * \\ 0 \end{pmatrix} , \quad \sigma = -\frac{1}{2} . \tag{13.192}$$

A right-handed spinor field **creates anti-particles** with helicity $-\frac{1}{2}$ and annihilates particles with helicity $\frac{1}{2}$.

- We always get both helicities $\pm\frac{1}{2}$, even though Lorentz-invariance does not require it.
- This discussion is relevant for the (almost massless) neutrinos in the Standard Model: ν are described by left-handed spinors with $\sigma = -\frac{1}{2}$ and $\bar{\nu}$ are described by right-handed spinors with $\sigma = \frac{1}{2}$.
- Weinberg was able to show the following general theorem: if $\psi(x)$ is in the (A, B) rep, then ψ destroys particles with helicity $\sigma = A - B$ and creates anti-particles with $\sigma = B - A$.

14 Path Integral for Fermion Fields

We now want to apply the QFT results of the scalar field to the spinor fields and transfer the formalism of the LSZ reduction, Feynman diagrams and Green functions.
Our coordinates in the functional integration are now **anti-commuting**:

$$\{\Psi_\alpha(t,\boldsymbol{x}), i\Psi^\dagger_\beta(t,\boldsymbol{y})\} = i\delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{14.1}$$

where the Ψ_α play the role of the coordinates Q_n and the $i\Psi^\dagger_\alpha$ play the role of the conjugate momenta P_n. We need to rederive the path integral formula using the **anti-**commutation relations

$$\{Q_n, P_m\} = i\delta_{nm}\,, \tag{14.2}$$

$$\{Q_n, Q_m\} = \{P_n, P_m\} = 0 \tag{14.3}$$

as the starting point (we have used the discrete notation as in section 6.1).
There is an immediate problem: for a position eigenstate $|q\rangle$:

$$Q_n|q\rangle = q_n|q\rangle \quad \forall n\,. \tag{14.4}$$

Then

$$0 = \{Q_n, Q_m\}|q\rangle = (q_n q_m + q_m q_n)|q\rangle\,, \tag{14.5}$$

from which we read off that

$$q_n q_m = -q_m q_n \quad \forall n, m\,. \tag{14.6}$$

But this cannot hold for (non-vanishing) c-numbers (ordinary $\mathbb{C}$-numbers).

14.1 Grassmann variables

Grassmann variables are anti-commuting "numbers", **not** operators, for which anti-commutation is no problem. For a single Grassmann number ξ

$$\xi^2 = 0\,, \tag{14.7}$$

holds, since it anti-commutes with itself.
We now define functions, integration, derivatives, etc.

14.1.1 Functions

A general function has the form

$$f(\xi) = f_0 + f_1\xi\,, \quad f_0, f_1 \in \mathbb{C}\,, \tag{14.8}$$

and no higher powers of ξ can appear since $\xi^2 = 0$.

14.1.2 Differentiation

We have the choice between two options:

$$f(\xi + \mathrm{d}\xi) = \begin{cases} f(\xi) + \mathrm{d}\xi f'(\xi) + \dots\,, & \text{left-derivative} \\ f(\xi) + f'(\xi)\mathrm{d}\xi + \dots\,, & \text{right-derivative} \end{cases}\,. \tag{14.9}$$

We choose the **right-derivative** in this lecture.

Example: with η, ξ Grassmann variables and $a, b, c, d \in \mathbb{C}$ we define:

$$g(\xi, \eta) = a + b\xi + c\eta + d\xi\eta\,. \tag{14.10}$$

The right-derivative of g with respect to ξ is defined through

$$g(\xi + \mathrm{d}\xi, \eta) = g(\xi, \eta) + g'(\xi, \eta)\mathrm{d}\xi + \dots\,, \tag{14.11}$$

$$\mathrm{d}\xi \cdot \xi = -\xi \mathrm{d}\xi$$
$$\mathrm{d}\xi\eta = -\eta\mathrm{d}\xi$$

where here $'$ denotes right-derivation with respect to ξ. Plugging in the definition:

$$a + b(\xi + \mathrm{d}\xi) + c\eta + d(\xi + \mathrm{d}\xi)\eta = g(\xi, \eta) + g'(\xi, \eta)\mathrm{d}\xi + \dots \tag{14.12}$$

and by comparing coefficients we read off

$$\frac{\mathrm{d}}{\mathrm{d}\xi} g(\xi, \eta) = g'(\xi, \eta) = b - d\eta\,. \tag{14.13}$$

For the left-derivative we would have $g(\xi + \mathrm{d}\xi, \eta) = g(\xi, \eta) + \mathrm{d}\xi g'(\xi, \eta) + \dots$ and therefore, by the same steps,

$$g'(\xi, \eta)\Big|_{\text{left}} = b + d\eta\,.$$

The right-derivative is consistent with

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\alpha)} = i\psi_\alpha^\dagger\,, \tag{14.14}$$

with

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu \partial_\mu \psi\,. \tag{14.15}$$

The product rule is

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big(f(\xi)g(\xi)\Big) = f(\xi)g'(\xi) + (-1)^{|g|} f'(\xi)g(\xi)\,. \tag{14.16}$$

$$\text{with} \quad |g| = \begin{cases} 0\,, & \text{if } g(\xi) \text{ is a c-valued function} \\ 1\,, & \text{if } g(\xi) \text{ is a Grassmann-valued function} \end{cases}\,. \tag{14.17}$$

14.1.3 Integration

We want the Grassmann integration to have the following properties:

1) linearity,

2) integration over total derivatives vanishes.

From property 2) we get

$$0 = \int \mathrm{d}\xi \, \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) \overset{Eq.\ (14.8)}{=} f_1 \int \mathrm{d}\xi \cdot 1 \tag{14.18}$$

and therefore we find

$$\int \mathrm{d}\xi \cdot 1 = 0 \,. \tag{14.19}$$

This guarantees that integration behaves as for ordinary integrals:

$$\int_{-\infty}^{\infty} \mathrm{d}x \; f(x) = \int_{-\infty}^{\infty} \mathrm{d}x \; f(x+a) \;:$$

$$\int \mathrm{d}\xi \; f(\xi) = \int \mathrm{d}\xi \; f(\xi + \eta) \,.$$

Property 1) gives us

$$\int \mathrm{d}\xi \; f(\xi) = f_1 \int \mathrm{d}\xi \; \xi \equiv f_1 \,, \tag{14.20}$$

where we defined

$$\int \mathrm{d}\xi \; \xi = 1 \,. \tag{14.21}$$

14.1.4 Variable transformations

We define

$$\xi' = a\xi \,, \quad a \in \mathbb{C} \,, \tag{14.22}$$

and therefore

$$1 = \int \mathrm{d}\xi \; \xi = \frac{1}{a} \int \mathrm{d}\xi \; \xi' = \frac{1}{a} \int \mathrm{d}\xi' \underbrace{\left|\left| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right|\right|}_{\uparrow\ \text{Jacobian}} \xi' \,, \tag{14.23}$$

$$\implies \quad \left|\left| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right|\right| = a \,. \tag{14.24}$$

Notice that for ordinary integrals we would get $\left|\left| \frac{\mathrm{d}x}{\mathrm{d}x'} \right|\right| = \frac{1}{a}$!

14.1.5 δ-function

The δ-function is simply

$$\delta(\xi) = \xi \,. \tag{14.25}$$

Check:

$$\begin{aligned}
\int \mathrm{d}\xi \; \delta(\xi - \xi_0) f(\xi) &= \int \mathrm{d}\xi \; (\xi - \xi_0)(f_0 + f_1\xi) && (14.26)\\
&= \int \mathrm{d}\xi \; (f_0\xi - \xi_0 f_1 \xi) && (14.27)\\
&= f_0 + f_1 \underbrace{\int \mathrm{d}\xi \; \xi}_{=1} \cdot \xi_0 && (14.28)\\
&= f_0 + f_1\xi_0 = f(\xi_0) \quad \checkmark \,. && (14.29)
\end{aligned}$$

Furthermore:

$$\int \mathrm{d}\xi \; e^{\xi\xi_0} = \int \mathrm{d}\xi \; (1 + \xi\xi_0) = \xi_0 = \delta(\xi_0) \,. \tag{14.30}$$

We generalize to multiple, independent Grassmann variables. Let ξ_i, $i = 1, ..., N$ be a Grassmann variable, i.e.

$$0 = \{\xi_i, \xi_j\} = \{\mathrm{d}\xi_i, \xi_j\} = \{\mathrm{d}\xi_i, \mathrm{d}\xi_j\} \,. \tag{14.31}$$

A general function is then

$$f(\xi) = f_0 + f_1^i \xi_i + f_2^{ij} \xi_i \xi_j + \dots + f_N \underset{\substack{\uparrow \\ \text{define using descending index}}}{\xi_N \xi_{N-1} \cdot \dots \cdot \xi_1} . \tag{14.32}$$

<u>**Important:**</u> We must be careful about the signs from the permutations when taking derivatives $\frac{\mathrm{d}}{\mathrm{d}\xi_i} f(\xi)$ from the right.

We integrate over multiple Grassmann variables as follows:

$$\int \mathrm{d}^N \xi \; f(\xi) \equiv \int \mathrm{d}\xi_1 \mathrm{d}\xi_2 \cdot \dots \cdot \mathrm{d}\xi_N \; f(\xi) = f_N \,, \tag{14.33}$$

$$\int \mathrm{d}\xi_i \; \xi_i = 1 \quad \forall i \,, \tag{14.34}$$

where Eq. (14.33) holds because f_N is the coefficient of the only term of $f(\xi)$ where all Grassmann variables appear and therefore it is the only one where one or more of the above integrals does not vanish, since

$$\int \mathrm{d}\xi_i \cdot 1 = 0 \quad \forall i \,. \tag{14.35}$$

We can perform a linear variable transfomation

$$\xi_i' = A_{ij} \xi_j \tag{14.36}$$

and the above integral becomes formally

$$\int \mathrm{d}^N \xi \; f(A\xi) = \int \mathrm{d}^N \xi' \left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\| f(\xi') = f_N \left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\| \tag{14.37}$$

and we now investigate what $\left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\|$ is:

$$\int \mathrm{d}^N \xi \; f_N (A\xi)_N \cdot \dots \cdot (A\xi)_1 \tag{14.38}$$

$$= \int \mathrm{d}^N \xi \; f_N A_{N i_N} \cdot \dots \cdot A_{1 i_1} \xi_{i_N} \cdot \dots \cdot \xi_{i_1} \tag{14.39}$$

$$= f_N \int \mathrm{d}^N \xi \left(\sum_{\sigma \in S_N} \mathrm{sign}(\sigma) A_{1\sigma(i_1)} \cdot \dots \cdot A_{N\sigma(i_N)} \right) \xi_N \cdot \dots \cdot \xi_1 \tag{14.40}$$

$$= f_N \int \mathrm{d}^N \xi \; \det(A) \xi_N \cdot \dots \cdot \xi_1 = f_N \det(A) \,, \tag{14.41}$$

where the σ are permutations of the N indices and the sign comes from the anticommutativity of the ξ_i. We found

See https://en.wikipedia.org/wiki/Determinant

$$\boxed{\left\| \frac{\mathrm{d}\xi}{\mathrm{d}\xi'} \right\| = \det(A) \,,} \tag{14.42}$$

opposed to the usual Jacobian $\frac{1}{\det(A)}$ for ordinary integrals.

14.1.6 δ-function and translation invariance

Analogously to before we have

$$\delta^{(N)}(\xi) = \xi_N \cdot \dots \cdot \xi_1 \tag{14.43}$$

and the integrals are translationally invariant:

$$\begin{aligned}\int \mathrm{d}^N\xi\ f(\xi+\eta) &= f_N \int \mathrm{d}^N\xi\ (\xi_N+\eta_N)\cdot\ldots\cdot(\xi_1+\eta_1) \\ &= \int \mathrm{d}^N\xi\ f_N\xi_N\cdot\ldots\cdot\xi_1 = \int \mathrm{d}^N\xi\ f(\xi)\,.\end{aligned} \tag{14.44}$$

We can also introduce **complex** Grassmann variables:

$$\int \mathrm{d}\xi\ \xi = \int \mathrm{d}\xi^*\ \xi^* = 1\,, \tag{14.45}$$

where we now treat ξ and ξ^* as two independent "real" Grassmann variables. Further

$$\int \mathrm{d}\xi \mathrm{d}\xi^*\ \xi^*\xi = 1 \tag{14.46}$$

holds. We now define:

$$\mathrm{d}^N\xi\mathrm{d}^N\xi^* \equiv \mathrm{d}\xi_1\mathrm{d}\xi_1^*\cdot\ldots\cdot\mathrm{d}\xi_N\mathrm{d}\xi_N^*\,, \tag{14.47}$$

notice the special ordering convention. With this, we have

$$\begin{aligned}\int \mathrm{d}^N\xi\mathrm{d}^N\xi^*\ e^{\xi^\dagger A\xi} &\overset{\substack{\xi'=A\xi\\ \downarrow}}{=} \int \mathrm{d}^N\xi'\mathrm{d}^N\xi^* \left|\left|\frac{\mathrm{d}\xi}{\mathrm{d}\xi'}\right|\right| e^{\xi^\dagger\xi'} \\ &= \det(A)\int \mathrm{d}^N\xi'\mathrm{d}^N\xi^*\frac{1}{N!}(\xi^\dagger\xi')^N \\ &= \det(A)\int \mathrm{d}^N\xi'\mathrm{d}^N\xi^*\frac{1}{N!}(\xi_1^*\xi_1'+\ldots+\xi_N^*\xi_N')^N \\ &\overset{(*)}{=} \det(A)\int \mathrm{d}^N\xi'\mathrm{d}^N\xi^*\ \xi_N^*\xi_N'\cdot\ldots\cdot\xi_1^*\xi_1' \\ &= \det(A)\,,\end{aligned}$$

where after the equality $(*)$ the factor $\frac{1}{N!}$ is gone since there are exactly $N!$ terms in $(\xi^\dagger\xi')^N$ where all $\xi_i^*\xi_i'$ are different. Further, *pairs* of Grassmann variables, like $\xi_1^*\xi_1'$ commute.

Finally, we are now ready to derive the **path integral formula**

14.2 Path integral formula

To simplify the notation we will use the discrete notation for Ψ_n, $\Psi_n^\dagger$:

$$\{\Psi_n, \Psi_m^\dagger\} = \delta_{nm} \,. \tag{14.48}$$

The index n is equivalent to $(\alpha, \boldsymbol{x})$, where α is the spinor index, at $t = 0$. The Heisenberg equation determines the time dependence.
We now define the eigenstates of the fermionic canonical variables: the ground state $|0\rangle$ is defined by

$$\Psi_N|0\rangle = 0 \quad \text{and so} \quad \langle 0|\Psi_n^\dagger = 0 \,. \tag{14.49}$$

This state exists always: we can take any non-zero state $|\psi\rangle$ and define

$$|0\rangle \equiv \prod_n \Psi_n|\psi\rangle \tag{14.50}$$

and now $\Psi_m|\psi\rangle = 0 \; \forall n$ since with Eq. (14.48), we have $\Psi_m^2 = 0$. We normalize

$$\langle 0|0\rangle = 1 \,. \tag{14.51}$$

We now define a "coherent fermion state"

$$|\xi\rangle \equiv \exp\left(- \sum_n \xi_n \Psi_n^\dagger \right) |0\rangle \,, \tag{14.52}$$

which is an eigenstate for **all** Ψ_n:

$$\Psi_n|\xi\rangle = \xi_n|\xi\rangle \quad \forall n \,. \tag{14.53}$$

Proof:

$$(\Psi_n - \xi_n)|\xi\rangle = (\Psi_n - \xi_n) \exp\left(- \sum_m \xi_m \Psi_m^\dagger \right) |0\rangle \tag{14.54}$$

$$= (\Psi_n - \xi_n) e^{-\xi_n \Psi_n^\dagger} \exp\left(- \sum_{m \neq n} \xi_m \Psi_m^\dagger \right) |0\rangle \tag{14.55}$$

$$= (\Psi_n - \xi_n)(1 - \xi_n \Psi_n^\dagger) \exp\left(- \sum_{m \neq n} \xi_m \Psi_m^\dagger \right) |0\rangle \,. \tag{14.56}$$

We simplify:

$$\begin{aligned}(\Psi_n - \xi_n)(1 - \xi_n \Psi_n^\dagger) &= \Psi_n - \xi_n - \Psi_n \xi_n \Psi_n^\dagger = \Psi_n - \xi_n + \xi_n \Psi_n \Psi_n^\dagger \\ &= \Psi_n - \xi_n \Psi_n^\dagger \Psi_n \,,\end{aligned} \tag{14.57}$$

where we used that

$$\{\Psi_n, \Psi_n^\dagger\} = 1 \,. \tag{14.58}$$

So finally:

$$(\Psi_n - \xi_n)|\xi\rangle = (\Psi_n - \xi_n \Psi_n^\dagger \Psi_n) \exp\left(- \sum_{m \neq n} \xi_m \Psi_m^\dagger \right) |0\rangle = 0 \,, \tag{14.59}$$

where we anti-commuted the Ψ_ns through the exponential until they act on $|0\rangle$ and vanish. Since all the Grassmann variables in the

exponential satisfy $m \neq n$, the anti-commutation relations with Ψ_n are trivial.
Similarly:

$$\langle\xi| = \langle 0| \exp\left(- \sum_n \Psi_n \xi_n^* \right) \tag{14.60}$$

and here

$$\langle\xi_n|\Psi_n^\dagger = \langle\xi_n|\xi_n^* \,. \tag{14.61}$$

For complex Grassmann variables η, ξ we define the **complex conjugated** product as

$$(\xi\eta)^* = \eta^*\xi^* \,. \tag{14.62}$$

We also need the **completeness relation** for the derivation of the path integral formula:

$$\mathbb{1} = \int \prod_n \mathrm{d}\xi_n^* \mathrm{d}\xi_n \; |\xi\rangle \exp\left(- \sum_n \xi_n \xi_n^* \right) \langle\xi| \,. \tag{14.63}$$

Proof: We first show that the scalar product is

$$\langle\xi|\eta\rangle = \exp\left(\sum_n \xi_n^* \eta_n \right), \tag{14.64}$$

since:

$$\begin{aligned} \langle\xi|\eta\rangle &= \langle 0| e^{-\sum_n \psi_n \xi_n^*} e^{-\sum_n \eta_n \psi_n^*} |0\rangle & (14.65)\\ &= \langle 0| e^{-\sum_{n\neq 1} \psi_n \xi_n^*} e^{-\psi_1 \xi_1^*} e^{-\eta_1 \psi_1^*} e^{-\sum_{n\neq 1} \eta_n \psi_n^*} |0\rangle & (14.66) \end{aligned}$$

and

$$\begin{aligned} e^{-\psi_1\xi_1^*} e^{-\eta_1\psi_1^*} &= (1-\psi_1\xi_1^*)(1-\eta_1\psi_1^*) = 1 - \psi_1\xi_1^* - \eta_1\psi_1^* + \xi_1^*\eta_1 \overset{\text{use } \{\psi_n,\psi_n^*\}=1}{\overbrace{(\psi_1\psi_1^*)}} \\ &= 1 + \xi_1^*\eta_1 + \begin{pmatrix} \text{contributions that vanish} \\ \text{when we anti-commute } \psi_1^* \text{ to} \\ \text{the left and } \psi_1 \text{ to the right} \end{pmatrix} \\ &= e^{\xi_1^*\eta_1} \,. \end{aligned}$$

So:

$$\begin{aligned} \langle\xi|\eta\rangle &= e^{\xi_1^*\eta_1} \langle 0| e^{-\sum_{n\neq 1} \psi_n \xi_n^*} e^{-\sum_{n\neq 1} \eta_n \psi_n^*} |0\rangle & (14.67)\\ &= e^{\xi_1^*\eta_1} e^{\xi_2^*\eta_2} \cdot \ldots \cdot e^{\xi_N^*\eta_N} & (14.68)\\ &= \exp\left(\sum_n \xi_n^* \eta_n \right). & (14.69) \end{aligned}$$

Then we can show that

$$\left[\int \prod_n \mathrm{d}\xi_n^* \mathrm{d}\xi_n \; |\xi\rangle \exp\left(- \sum_n \xi_n \xi_n^* \right) \langle\xi| \right] |\eta\rangle = |\eta\rangle \quad \forall |\eta\rangle \tag{14.70}$$

Use

$$\langle\xi|\eta\rangle = \exp\left(\sum_n \xi_n^* \eta_n \right)$$

to show it.

using similar algebra as before (see Ex.).

Now we can derive the path integral formula. We start with the transition matrix element of the eigenstates of the canonical coordinates at times t_i and t_f:

$$\langle\xi_f; t_f|\xi_i; t_i\rangle = \langle\xi_f| e^{-iH(t_f - t_i)} |\xi_i\rangle \,, \tag{14.71}$$

where

$$H = H[\Psi_n^\dagger, \Psi_m] \tag{14.72}$$

with the convention that all the $\Psi_n^\dagger$ are positioned to the left of the Ψ_n.

As before, we divide the time interval $t_f - t_i$ in Eq. (14.71) into small intervals $\Delta t \ll 1$, insert several complete sets of states and evaluate the resulting terms of the form

$$\langle \xi^k | e^{iH\Delta t} | \xi^{k-1} \rangle \approx e^{-iH(\xi^{k^*}, \xi^{k-1})\Delta t} \exp\left(\sum_n \xi_n^{k^*} \xi_n^{k-1} \right), \tag{14.73}$$

where the last exponential comes from the scalar product

$$\langle \xi^k | \xi^{k-1} \rangle = \exp\left(\sum_n \xi_n^{k^*} \xi_n^{k-1} \right). \tag{14.74}$$

Therefore, the transition matrix element is

$$\langle \xi_f; t_f | \xi_i; t_i \rangle = \exp\left(\sum_n \xi_n^*(t_f) \xi_n(t_f) \right) \int \mathcal{D}[\xi_n^*(t), \xi_n(t)] \cdot \exp\left[i \int_{t_i}^{t_f} \mathrm{d}t \left(i \sum_n \xi_n^*(t) \dot{\xi}_n(t) - H\big(\xi_n^*(t), \xi_n(t)\big) \right) \right],$$

with the boundary conditions

$$\xi(t_i) = \xi_i\,, \quad \xi^*(t_f) = {\xi_f}^*\,. \tag{14.75}$$

The $\xi^*(t_i)$ and $\xi(t_f)$ are integrated over (i.e. they are not fixed).

14.3 Green functions of fermionic fields

We now transition from the discrete to the continuous field operators:

$$\Psi_n \longrightarrow \Psi_\alpha(\boldsymbol{x})\,, \quad \sum_n \longrightarrow \sum_\alpha \int \mathrm{d}^3x\,, \tag{14.76}$$

where we used the aforementioned equivalence $n \hat{=} (\alpha, \boldsymbol{x})$. We will use the summation convention for contractions and omit $\sum_\alpha$ in the following, unless explicitly stated otherwise.

As before (see section 6.2) we start with

$$\langle \psi_f; t_f | T\{O_a(x_a) O_b(x_b) \cdot ...\} | \psi_i; t_i \rangle \tag{14.77}$$

We define a T^*-ordering as:

$$T\{O_a(x_a)O_b(x_b)\} = \theta(x_a^0 - x_b^0) O_a(x_a) O_b(x_b) \pm \theta(x_b^0 - x_a^0) O_b(x_b) O_a(x_a)\,, \tag{14.78}$$

Recall the T^* product:
$\langle \Omega | T\{\partial_t \phi(x) \cdot ...\} | \Omega \rangle = \partial_0 \langle \Omega | T\{\phi(x) \cdot ...\} | \Omega \rangle$,
there are no contact terms.

where we use the upper sign if O_a and O_b are **bosonic** and the lower one if O_a and O_b are **fermionic**.

Then (also as before), we multiply by $\langle \Omega | \psi_f; t_f \rangle$ from the left and by $\langle \psi_i; t_i | \Omega \rangle$ from the right and integrate over ψ_i and ψ_f. This again gives us the $+i\varepsilon$-prescription:

$$\begin{aligned} &\langle \Omega | T\{O_a(x_a) O_b(x_b) \cdot ...\} | \Omega \rangle = \\ &= |N|^2 \int \mathcal{D}[\psi_\alpha(x), \psi_\alpha^*(x)] \exp\left(i \int \mathrm{d}^4x \left(\psi_\alpha^* i\partial_0 \psi_\alpha - \mathcal{H}(\psi^*, \psi) + i\varepsilon \psi_\alpha^* \psi_\alpha \right) \right) \cdot \\ &\qquad \cdot O_a(x_a) O_b(x_b) \cdot ... \end{aligned} \tag{14.79}$$

This looks like the Hamiltonian path integral version for the scalar field, we integrate over the Grassmann variables ψ and ψ^*, which correspond to q and p.

Recall the scalar Hamiltonian path integral:

$$\langle q_f, t_f | q_i, t_i \rangle = = \int \mathcal{D}[q]\mathcal{D}[p] \exp\Big(i \int \mathrm{d}t \sum_n \dot{q}_n p_n - H(q,p)\Big) .$$

The Hamiltonian version had independent q and p. Only after integrating over p did we get

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial \pi}$$

on page 54.

__**Difference:**__ the above is **also** the Lagrangian version! The reason is that for spinor fields

$$P_n \longleftrightarrow i\Psi_n^\dagger \quad \text{and} \quad \dot{Q}_n \longleftrightarrow \partial_0 \Psi_n \tag{14.80}$$

are independent, since the field EOM is a first order differential equation. Therefore

$$\underbrace{\psi^\dagger i \partial_0 \psi}_{\hat{=} p_n \dot{q}_n} - \mathcal{H} \tag{14.81}$$

is the Lagrangian $\mathcal{L}$ and we do not need to integrate over $p_n \longleftrightarrow \psi_n^\dagger$ to establish the relation. So:

$$\langle \Omega | T\{O_a(x_a) O_b(x_b) \cdot ...\} | \Omega \rangle = |N|^2 \int \mathcal{D}[\psi, \psi^*] \exp\left(i \int \mathrm{d}^4x \, \mathcal{L}(\psi, \psi^*)\right) \cdot \\ \cdot O_a(x_a) O_b(x_b) \cdot ... ,$$

exactly as in Eq. (7.99) for scalar field theory.

14.4 Feynman rules for spinor fields

We start with the Dirac field. We use the change of variables $\Psi^\dagger \longrightarrow \Psi^\dagger \gamma^0 = \overline{\Psi}$ for convenience.
The generating functional is

$$Z[\eta, \overline{\eta}] \equiv |N|^2 \int \mathcal{D}[\psi, \overline{\psi}] \exp\left(i \int \mathrm{d}^4x \Big(\mathcal{L} + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\Big)\right) . \tag{14.82}$$

The functional derivatives acting on it give

$$\frac{1}{i}\frac{\delta}{\delta \eta(x)} \longleftrightarrow \overline{\psi}(x) , \qquad \underset{\substack{\uparrow \\ \text{right-derivative and anti-commutation}}}{-}\frac{1}{i}\frac{\delta}{\delta \overline{\eta}(x)} \longleftrightarrow \psi(x) . \tag{14.83}$$

The $\eta(x)$, $\overline{\eta}(x)$ are Grassmann-valued fields (not operators).

14.4.1 Free theory

$$Z_0[\eta, \overline{\eta}] = |N_0|^2 \int \mathcal{D}[\psi, \overline{\psi}] \exp\left(i \int \mathrm{d}^4x \Big(\overline{\psi}(i\partial\!\!\!/ - m + i\varepsilon)\psi + \\ + \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\Big)\right) . \tag{14.84}$$

We perform a change of variables

$$\psi(x) = \psi'(x) - \frac{1}{i\partial\!\!\!/ - m + i\varepsilon}\eta(x) , \tag{14.85}$$

which gives

$$Z_0[\eta,\overline{\eta}] = |N_0|^2 \int \mathcal{D}[\psi',\overline{\psi'}] \exp\Big(i \int \mathrm{d}^4x \Big(\overline{\psi'}(i\gamma^\mu\partial_\mu - m + i\varepsilon)\psi' - \\ - \overline{\eta}(i\not{\partial} - m + i\varepsilon)^{-1}\eta\Big)\Big) \tag{14.86}$$

$$\underset{\substack{\uparrow \\ \text{Gaussian Grassmann integral} \\ \int \mathrm{d}^N\xi \mathrm{d}^N\xi^* \exp(\xi^\dagger A\xi) = \det(A)}}{=} |N_0|^2 \det\Big(i(i\not{\partial} - m + i\varepsilon)\Big) \exp\Big(-i\int \mathrm{d}^4x\, \overline{\eta}\frac{1}{i\not{\partial} - m + i\varepsilon}\eta\Big)\,. \tag{14.87}$$

The factor

$$|N_0|^2 \det\Big(i(i\not{\partial} - m + i\varepsilon)\Big) \tag{14.88}$$

is independent of η and $\overline{\eta}$ and is equal to 1 since

$$Z_0[0,0] = 1 \tag{14.89}$$

must hold. Therefore:

$$Z_0[\eta,\overline{\eta}] = \exp\Big(-\int \mathrm{d}^4x \mathrm{d}^4y\, \overline{\eta}_\alpha(x)\Delta_F^{\alpha\beta}(x-y)\eta_\beta(y)\Big)\,, \tag{14.90}$$

where

$$\Big(\frac{i}{i\not{\partial} - m + i\varepsilon}\Big)^{\alpha\beta} \eta_\beta(x) = \int \mathrm{d}^4y\, \Delta_F^{\alpha\beta}(x-y)\eta_\beta(y)\,. \tag{14.91}$$

Δ_F is the Feynman-propagator for the Dirac spinor field:

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)}\widetilde{\Delta}_F(p)\,. \tag{14.92}$$

Inserting this into

$$\Big(\frac{i\not{\partial}_x - m + i\varepsilon}{i}\Big)\Delta_F(x-y) = \delta^{(4)}(x-y) \tag{14.93}$$

gives:

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \underbrace{(-i)\big(\not{p} - m + i\varepsilon\big)\widetilde{\Delta}_F(p)}_{\overset{!}{=}1} = \delta^{(4)}(x-y)\,, \tag{14.94}$$

and so

$$\boxed{\widetilde{\Delta}_F(p) \equiv \frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon} = \frac{i}{\not{p} - m + i\varepsilon}\,,} \tag{14.95}$$

because

$$\frac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}(-i)\big(\not{p} - m\big) = \frac{\not{p}\not{p} - m^2}{p^2 - m^2} = 1 \tag{14.96}$$

satisfies Eq. (14.94).
The contraction is therefore

$$\Psi(x)\overline{\Psi}(y) \equiv \Delta_F(x-y) = \tag{14.97}$$

,

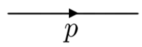
or, in momentum space,

$$\overline{\Psi} \sim a^\dagger + b$$

creates a fermion and destroys an anti-fermion,

$$\Psi \sim a + b^\dagger$$

destroys a fermion and creates an anti-fermion.



$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}\,. \tag{14.98}$$

The arrow in the above diagrams denotes fermion flow: it goes from $\overline{\Psi}$ to Ψ and we use the convention that momentum flows in the same direction.
The other contractions vanish:

$$\Psi(\overbracket{x)\Psi(}y) = 0 = \overline{\Psi}(\overbracket{x)\overline{\Psi}(}y) \tag{14.99}$$

in analogy to the scalar field.
We generate n-point functions of the free theory through

$$\begin{aligned}&\langle\Omega|T\{\Psi(x_1)\cdot \ldots \cdot \overline{\Psi}(x_k)\cdot \ldots \cdot \Psi(x_l)\cdot \ldots\}|\Omega\rangle = \\ &= \ldots \cdot \frac{-1}{i}\frac{\delta}{\delta\overline{\eta}(x_l)}\cdot \ldots \cdot \frac{1}{i}\frac{\delta}{\delta\eta(x_k)}\cdot \ldots \cdot \frac{-1}{i}\frac{\delta}{\delta\overline{\eta}(x_1)} Z_0[\eta,\overline{\eta}]\bigg|_{\eta=\overline{\eta}=0}\,.\end{aligned} \tag{14.100}$$

Notice the order of the functional derivatives! Recall the generating functional of the free theory:

Compare to the generating functional of the free complex scalar field:

$$Z_0[J,J^*] = \exp\left(-\int \mathrm{d}^4x\mathrm{d}^4y\; J^*(x)\Delta_F(x-y)J(y)\right).$$

$$Z_0[\eta,\overline{\eta}] = \exp\left(-\int \mathrm{d}^4x\mathrm{d}^4y\;\overline{\eta}(x)\Delta_F(x-y)\eta(y)\right). \tag{14.101}$$

We can use the same argumentation as in the scalar field case to show that the general n-point function is given by the sum over all non-vanishing pairwise contractions.

Important: we must keep track of the signs from the exchange of fields:

$$\langle\Omega|T\{\Psi(x_1)\overline{\Psi}(x_2)\}|\Omega\rangle = -\langle\Omega|T\{\overline{\Psi}(x_2)\Psi(x_1)\}|\Omega\rangle\,, \tag{14.102}$$

which we can see from the Grassmann path integral.We can understand the rule using an example.

Example: free four-point function (to simplify the notation we use again the "summation convention" for integrals that we already introduced in section 6.5.2):

$$\langle\Omega|T\{\Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4\}|\Omega\rangle = \left(-\frac{1}{i}\right)^2\left(\frac{1}{i}\right)^2\frac{\delta}{\delta\eta_4}\frac{\delta}{\delta\overline{\eta}_3}\frac{\delta}{\delta\eta_2}\frac{\delta}{\delta\overline{\eta}_1}e^{-\overline{\eta}_x\Delta^F_{xy}\eta_y}\bigg|_{\eta=\overline{\eta}=0} \tag{14.103}$$

$$= \frac{\delta}{\delta\eta_4}\frac{\delta}{\delta\overline{\eta}_3}\frac{\delta}{\delta\eta_2}\Bigg(\underset{\substack{\uparrow\\ \text{from right-derivative}}}{-}\left(-\Delta^F_{1y}\eta_y\right)e^{-\overline{\eta}_x\Delta^F_{xy}\eta_y}\Bigg)\bigg|_{\eta=\overline{\eta}=0} \tag{14.104}$$

$$= \frac{\delta}{\delta\eta_4}\frac{\delta}{\delta\overline{\eta}_3}\left(\left(\Delta^F_{12}-\Delta^F_{1y}\eta_y\overline{\eta}_x\Delta^F_{x2}\right)e^{-\overline{\eta}_{z_1}\Delta^F_{z_1z_2}\eta_{z_2}}\right)\bigg|_{\eta=\overline{\eta}=0} \tag{14.105}$$

$$= \frac{\delta}{\delta\eta_4}\left[\Delta^F_{12}\Delta^F_{3x}\eta_x-\Delta^F_{1y}\eta_y\Delta^F_{32}+\left(\text{vanishing terms for }\frac{\delta}{\delta\eta}(\ldots)\bigg|_0\right)\right] \tag{14.106}$$

$$= \Delta^F_{12}\Delta^F_{34}\overset{\substack{\text{different sign compared}\\ \text{to the scalar field!}\\ \downarrow}}{-}\Delta^F_{14}\Delta^F_{32} \tag{14.107}$$

$$= \Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4+\Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4 = \Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4+\Psi_1\overline{\Psi}_4\overline{\Psi}_2\Psi_3 \tag{14.108}$$

$$= \Psi_1\overline{\Psi}_2\Psi_3\overline{\Psi}_4-\Psi_1\overline{\Psi}_4\Psi_3\overline{\Psi}_2\,. \tag{14.109}$$

The rule is: draw all contractions and multiply each by $(-1)^\sigma$, where σ is the number of commutations you have to do to bring fields into the correct $\Psi\overline{\Psi}$ order.

14.4.2 Interactions

As before, we use the expansion

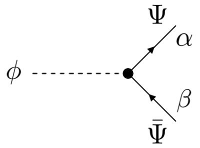
$$\exp\left(i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\right) = \sum_{m=0}^{\infty}\frac{1}{m!}\left(i\int \mathrm{d}^4x\,\mathcal{L}_{\text{int}}\right)^m \tag{14.110}$$

and truncate the series after a finite m. This gives a Gaussian path integral with polynomial prefactor and to compute it, as before, we sum over all non-vanishing pairwise contractions and keep track of the signs resulting from the permutation of the fields.

Example:

$$\mathcal{L} = \mathcal{L}_0^{\text{free}}+g\overline{\Psi}\Psi\phi+e\overline{\Psi}\gamma^\mu\Psi A_\mu \tag{14.111}$$

The vertices of this theory are

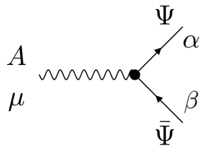


$$ig\delta_{\alpha\beta}\,, \tag{14.112}$$

since

$$\overline{\Psi}\Psi = \overline{\Psi}_\alpha\Psi_\beta\delta_{\alpha\beta}\,, \tag{14.113}$$

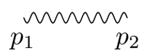
and

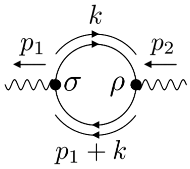


$$ie\gamma^{\mu}_{\alpha\beta}\,. \tag{14.114}$$

Consider the two-point function

$$\int \mathrm{d}^4x\mathrm{d}^4y\; e^{ip_1x+ip_2y}\langle\Omega|T\{A_\mu(x)A_\nu(y)\}|\Omega\rangle\Big|_{\text{connected}} \tag{14.115}$$

$$=$$ 

$$+$$ 

$$+\,\mathcal{O}\big(e^4, e^2g^2\big) \tag{14.116}$$

$$\begin{aligned}&= (2\pi)^4\delta^{(4)}(p_1+p_2)\bigg(\widetilde{\Delta}_F^{\mu\nu}(p_1)+\widetilde{\Delta}_F^{\mu\sigma}(p_1)\widetilde{\Delta}_F^{\rho\nu}(p_2)\cdot\\ &\cdot\,(ie)^2\underset{\substack{\uparrow\\(*)}}{(-1)}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\gamma^{\rho}_{\alpha\beta}i(\not{k}+m)_{\beta\gamma}\gamma^{\sigma}_{\gamma\delta}i(\not{p}_1+\not{k}+m)_{\delta\alpha}}{(k^2-m^2+i\varepsilon)\big((p_1+k)^2-m^2+i\varepsilon\big)}\bigg)\,.\end{aligned} \tag{14.117}$$

The minus sign $(*)$ comes from the exchange of the fields in the contraction

$$A_\mu(x)A_\nu(y)\big(\overline{\Psi}\gamma^\sigma A_\sigma\Psi\big)(z_1)\big(\overline{\Psi}\gamma^\rho A_\rho\Psi\big)(z_2)\,, \tag{14.118}$$

since it takes an odd number of exchanges to write

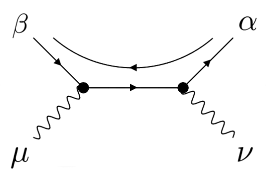
$$\Psi\overline{\Psi}(z_2-z_1)\Psi\overline{\Psi}(z_1-z_2)\,. \tag{14.119}$$

We **generalize** to the **Feynman rules for spinors**:

- Rule of fermion loops: there is a (-1) associated to each closed fermion loop.

- Spinor indices are contracted by starting at the **end** of the fermion line, then working in the **opposite** direction of the arrow. For closed fermion lines we can start anywhere and get a **trace**.

  <u>**Example:**</u>



$$\gamma^{\mu}_{\beta\delta}i(\not{p}+m)_{\delta\gamma}\gamma^{\nu}_{\gamma\alpha} \tag{14.120}$$

14.5 Feynman rules for spinors

We will now derive the LSZ reduction formula for spinors to express

$$\langle k_1, s_1, ..., k_m, s_m; \text{out}|p_1, s_1', ..., p_n, \underset{\substack{\uparrow \\ \text{spin label}}}{s_n'}; \text{in}\rangle \tag{14.121}$$

in terms of Green functions.
We define asymptotic in/out-fields:

$$\Psi_\alpha(x) \longrightarrow \sqrt{Z}\Psi_\alpha^{\text{in/out}}(x)\,, \tag{14.122}$$

with the complex field

$$\begin{aligned}\Psi_\alpha^{\text{in/out}}(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \Big(& e^{-ipx} u_\alpha(\boldsymbol{p}, s) a^{\text{in/out}}(\boldsymbol{p}, s) + \\ & + e^{ipx} v_\alpha(\boldsymbol{p}, s) b^{\dagger\,\text{in/out}}(\boldsymbol{p}, s)\Big)\end{aligned} \tag{14.123}$$

and $b^\dagger \longrightarrow a^\dagger$ if the field is real.

Main result: the S-matrix is given by the on-shell, amputated Green function multiplied by the polarization functions $u(\boldsymbol{p}, s)$ or $v(\boldsymbol{p}, s)$. Consider the scattering of charged spin $\frac{1}{2}$ particles:

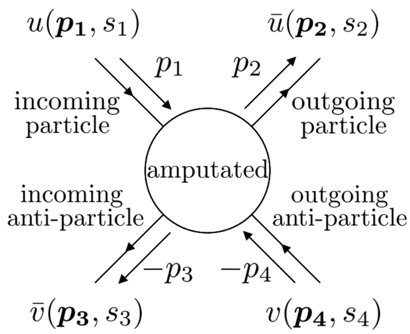
$$S = \mathbb{1} + iT \tag{14.124}$$

and

$$\begin{aligned} iT_{k_1,s_1,...;p_1,s_1',...} = \left(\sqrt{Z}\right)^{n+m} \prod_{i=1}^{m} \bar{u}_{\alpha_i}(\boldsymbol{k_i}, s_i) \prod_{j=1}^{n} u_{\beta_j}(\boldsymbol{p_j}, s_j) \cdot \\ \cdot \widetilde{G}^{\text{amp}}(k_1, ..., k_m; -p_1, ..., -p_n)_{\alpha_1,...,\alpha_m;\beta_1,...,\beta_n}\,. \end{aligned} \tag{14.125}$$

The Green functions now carry the spinor indices α_i, β_i from the matrix rep of the fields. The external particles carry the spin labels s_i and the polarization spinors u, v carry mixed labels (α, s) to convert the amputated Green function into the T-matrix element of the scattering process.

General rule:



The orientation of the anti-particle line is reversed because

$$\Psi \sim e^{-ipx} u_\alpha(\boldsymbol{p}, s) a(\boldsymbol{p}, s) + \ldots, \quad \overline{\Psi} \sim e^{-ipx} \bar{v}_\alpha(\boldsymbol{p}, s) b(\boldsymbol{p}, s) + \ldots \tag{14.126}$$

and the line goes from $\overline{\Psi}$ to Ψ.

- The same holds for Weyl-2-spinors:

$$\bar{u},\ \bar{v} \longrightarrow u^*,\ v^* \tag{14.127}$$

  and the rest stays the same.

- If particle and anti-particle are the same, then $u = v^*$ or some other reality condition relates u to v^*.

14.5.1 Example: Compton scattering of a scalar particle on an electron:

We assume a canonical Lagrangian with the following fields

$$\begin{aligned} \phi \quad & \text{real scalar field} \\ \Psi \quad & \text{Dirac field of the electron} \\ \mathcal{L}_{\text{int}} &= -g\overline{\Psi}\Psi\phi \end{aligned} \tag{14.128}$$

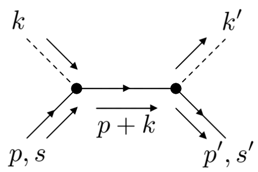
The usual situation is that the incoming electrons have unpolarized spins and the final state spins are not measured. We therefore must take the **spin average** of the squared matrix element as follows:

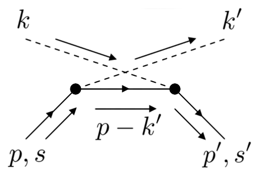
$$\overline{|T|^2} = \underset{\text{average over initial}}{\frac{1}{2}} \overset{\text{sum over final}}{\sum_{s,s'=\pm\frac{1}{2}}} |T|^2 . \tag{14.129}$$

To compute the average we use the spin sums we already derived:

$$\sum_{s=\pm\frac{1}{2}} u(\boldsymbol{p}, s)\bar{u}(\boldsymbol{p}, s) = \not{p} + \underset{\text{electron mass}}{m} . \tag{14.130}$$

The relevant tree-level Feynman diagrams are

$iT =$ 

$+$ 

(14.131)

$$= \bar{u}(\boldsymbol{p'}, s')\left[ig\frac{i(\not{p} + \not{k} + m)}{(p+k)^2 - m^2 + i\varepsilon} ig + ig\frac{i(\not{p} - \not{k}' + m)}{(p-k')^2 - m^2 + i\varepsilon} ig\right] u(\boldsymbol{p}, s) \tag{14.132}$$

$$\equiv (ig)^2 i\bar{u}_\alpha(\boldsymbol{p'}, s') M_{\alpha\beta} u_\beta(\boldsymbol{p}, s) . \tag{14.133}$$

For the scattering cross-section we need $|T|^2 = T^* \cdot T$.
We use:

$$(\gamma^0)^\dagger = \gamma^0$$
$$(\gamma^0)^2 = \mathbb{1}$$
$$\gamma^0 (\gamma^\mu)^\dagger \gamma^0 = \gamma^\mu$$

$$\big(\bar{u}(\boldsymbol{p'},s')\gamma^{\mu_1}\cdot\ldots\cdot\gamma^{\mu_n}u(\boldsymbol{p},s)\big)^* \tag{14.134}$$
$$= (u^\dagger(\boldsymbol{p'},s')\gamma^0\gamma^{\mu_1}\cdot\ldots\cdot\gamma^{\mu_n}u(\boldsymbol{p},s))^\dagger \tag{14.135}$$
$$= u^\dagger(\boldsymbol{p},s)(\gamma^{\mu_n})^\dagger\cdot\ldots\cdot(\gamma^{\mu_1})^\dagger\gamma^0 u(\boldsymbol{p'},s') \tag{14.136}$$
$$= u^\dagger(\boldsymbol{p},s)\gamma^0\,\underbrace{\gamma^0(\gamma^{\mu_n})^\dagger\gamma^0}_{=\gamma^{\mu_n}}\cdot\ldots\cdot\gamma^0(\gamma^{\mu_1})^\dagger\gamma^0 u(\boldsymbol{p'},s') \tag{14.137}$$
$$= \bar{u}(\boldsymbol{p},s)\gamma^{\mu_n}\cdot\ldots\cdot\gamma^{\mu_1}u(\boldsymbol{p'},s') \tag{14.138}$$

(the order of the γ's is exchanged).
Therefore:

Here: $\overline{M} = M\Big|_{i\varepsilon\longrightarrow -i\varepsilon}$

$$|T|^2 = g^4\Big[\bar{u}(\boldsymbol{p'},s')Mu(\boldsymbol{p},s)\Big]\Big[\bar{u}(\boldsymbol{p},s)\overline{M}u(\boldsymbol{p'},s')\Big] \tag{14.139}$$
$$= g^4\big[u(\boldsymbol{p},s)\bar{u}(\boldsymbol{p},s)\big]_{\alpha\beta}\overline{M}_{\beta\gamma}\big[u(\boldsymbol{p'},s')\bar{u}(\boldsymbol{p'},s')\big]_{\gamma\delta}M_{\delta\alpha}\,. \tag{14.140}$$

Now we use Eq. (14.130) and so

$$\overline{|T|^2} = \frac{1}{2}g^4\mathrm{tr}\Big((\not{p}+m)\overline{M}(\not{p}'+m)M\Big) \tag{14.141}$$
$$= \frac{1}{2}g^4\mathrm{tr}\Bigg((\not{p}+m)\left[\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}+\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\right]\cdot \tag{14.142}$$
$$\cdot(\not{p}'+m)\left[\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}+\frac{\not{p}-\not{k}'+m}{(p-k')^2-m^2}\right]\Bigg)\,. \tag{14.143}$$

We assume that ϕ is massless:

$$k^2 = k'^2 = 0 \tag{14.144}$$
$$\text{and}\quad (p+k)^2-m^2 = \underbrace{p^2}_{=m^2}+2p\cdot k+\underbrace{k^2}_{=0}-m^2 = 2p\cdot k\,. \tag{14.145}$$

Therefore

$$(\not{p}+m)(\not{p}+\not{k}+m) = \underbrace{(\not{p}+m)(\not{p}-m}_{=p^2-m^2=0}+\not{k}+2m) = (\not{p}+m)(\not{k}+2m) \tag{14.146}$$

and

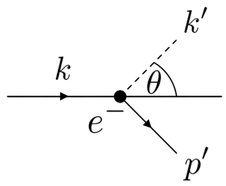
$$|T|^2 = \frac{1}{2}g^4\mathrm{tr}\Bigg((\not{p}+m)\left[\frac{\not{k}+2m}{2p\cdot k}+\frac{-\not{k}'+2m}{-2p\cdot k}\right](\not{p}'+m)\cdot \cdot\left[\frac{\not{k}+2m}{2p\cdot k}+\frac{-\not{k}'+2m}{-2p\cdot k}\right]\Bigg)\,. \tag{14.147}$$

Now we have to compute four Dirac-traces (this computation can also be done in Mathematica):

We can ignore traces of odd numbers of γ matrices, since they vanish (see Ex.).

$$\mathrm{tr}[(\not{p}+m)(\not{k}+2m)(\not{p}'+m)(-\not{k}'+2m)] = \tag{14.148}$$
$$= 4m^2\mathrm{tr}(\mathbb{1}_{4\times4})+2m^2\underbrace{\mathrm{tr}(\not{p}\not{k})}_{=p\cdot k}+4m^2\underbrace{\mathrm{tr}(\not{p}\not{p}')}_{=p\cdot p'}+$$
$$+2m^2\mathrm{tr}(\not{p}(-\not{k}'))+2m^2\mathrm{tr}(\not{k}\not{p}')+m^2\mathrm{tr}(\not{k}(-\not{k}'))+ \tag{14.149}$$
$$+2m^2\mathrm{tr}(\not{p}'(-\not{k}'))+\mathrm{tr}(\not{p}\not{k}\not{p}'(-\not{k}'))$$
$$= 16m^4+8m^2\left(p\cdot k+2p\cdot p'-p\cdot k'+k\cdot p'-\frac{1}{2}k\cdot k'-p'\cdot k'\right)-$$
$$-4[(p\cdot k)(p'\cdot k')-(p\cdot p')(k\cdot k')+(p\cdot k')(k\cdot p')] \tag{14.150}$$

We can now either introduce Mandelstam variables (see Ex.), or choose a particular reference frame; Compton scattering is usually expressed in the rest frame of the incoming electron:



$$p = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \,, \quad k = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix} \,, \quad p' = \begin{pmatrix} m + E - E' \\ -E' \sin\theta \\ 0 \\ E - E' \cos\theta \end{pmatrix} \,, \quad k' = \begin{pmatrix} E' \\ E' \sin\theta \\ 0 \\ E' \cos\theta \end{pmatrix} \,. \tag{14.151}$$

Following the same discussion as before for the kinematics and the evaluation of the phase space integral (see the exercise for details) we finally obtain the formula

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{g^4}{32\pi m^2} \left(\frac{E'}{E} \right)^2 \left(\frac{E}{E'} + \frac{E'}{E} + 2\cos(2\theta) \right) . \tag{14.152}$$

15 Massive Vector Fields

We will discuss fields which transform according to the vector/fundamental rep of the Lorentz group:

$$\Lambda \mapsto D(\Lambda)^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\nu}\,. \tag{15.1}$$

In the (A, B) classification, this is equivalent to the irrep $\left(\frac{1}{2}, \frac{1}{2}\right)$ and we will find that these fields correspond to particles with

$$\boldsymbol{m > 0} \text{ and spin } 1 \quad \text{and} \quad \boldsymbol{m = 0} \text{ and helicity } 1\,.$$

We can compare this rep to $\left(\frac{1}{2}, 0\right)$ and $\left(0, \frac{1}{2}\right)$, since

$$\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right) \tag{15.2}$$

and therefore we can represent this as

$$\underset{\substack{\uparrow \\ \text{Lorentz} \\ \text{four-vector}}}{V^{\mu}} \quad \mapsto \quad \underset{\substack{\uparrow \\ \text{hermitian} \\ 2\times 2 \text{ matrices}}}{V} \equiv \sigma^{\mu} V_{\mu}\,. \tag{15.3}$$

Recall also the relation

$$D_L^{-1}(\Lambda)\sigma^{\mu} D_R(\Lambda) = \Lambda^{\mu}{}_{\nu}\sigma^{\nu}$$

which encapsulates this algebraically.

The degrees of freedom stay the same, since on the RHS we have complex 2×2-matrices which have 8 free parameters and the hermicity $A^{\dagger} = A$ gives us 4 constraining equations, so the total degrees of freedom are $8 - 4 = 4$, the same as on the LHS.

The matrix V transforms under the Lorentz group as

$$\Lambda^{\mu}{}_{\nu} V^{\nu} \mapsto \underset{\substack{\uparrow \\ D_L}}{D_{(0,\frac{1}{2})}}(\Lambda) V \underset{\substack{\uparrow \\ D_R^{-1}}}{D^{-1}_{(\frac{1}{2},0)}}(\Lambda) \tag{15.4}$$

and this **defines** the action of the $\left(\frac{1}{2}, \frac{1}{2}\right)$ rep on hermitian matrices. Using

$$D_R^{\dagger}(\Lambda)\sigma^{\mu} D_R(\Lambda) = \Lambda^{\mu}{}_{\nu}\sigma^{\nu}\,, \tag{15.5}$$

$$D_R^{\dagger}(\Lambda) = D_L^{-1}(\Lambda) = D_L\left(\Lambda^{-1}\right), \quad D_R(\Lambda) = D_R^{-1}\left(\Lambda^{-1}\right) \tag{15.6}$$

(see Sec. 13), we know

$$V \longrightarrow D_L(\Lambda)\sigma^{\mu} D_R^{-1}(\Lambda) V_{\mu} = \left(\Lambda^{-1}\right)^{\mu}{}_{\nu}\sigma^{\nu} V_{\mu}\,, \tag{15.7}$$

therefore we read off

$$V'_\mu = \Lambda_\mu{}^\nu V_\nu \quad \text{or} \quad V'^\mu = \Lambda^\mu{}_\nu V^\nu \,. \tag{15.8}$$

So V^μ transforms as a four-vector if V transforms under the rep $\left(\frac{1}{2}, \frac{1}{2}\right)$.

15.1 Massive vector fields

We will encounter a general problem: a Lorentz tensor field

$$A^{\mu_1,\dots,\mu_n} \tag{15.9}$$

has 4^n components, but a massive particle has only $2j+1$ spin degrees of freedom. Usually an irreducible rep of the Lorentz group is equivalent to a reducible rep of the little group (here $SO(3)$). For the vector field:

$$\underset{\substack{\uparrow \\ \text{four-vector } V^\mu \\ \text{under Lorentz}}}{4} \quad = \quad \overset{\substack{V^0 \text{ is a scalar under} \\ \text{spatial rotations } (SO(3)) \\ \downarrow}}{1} \oplus \underset{\substack{\uparrow \\ V^i \text{ is a three-vector} \\ \text{under little group} \\ \text{rotations}}}{3} \,. \tag{15.10}$$

A massive graviton would be described by a symmetric rank 2-tensor:

$$(4\otimes 4)\Big|_{\text{symmetric}} = \begin{pmatrix} 1 & 5 & 6 & 7 \\ & 2 & 8 & 9 \\ & & 3 & 10 \\ & & & 4 \end{pmatrix}$$

$$\begin{aligned}(1\oplus 3)(1\oplus 3) &= 1 \oplus 3 \oplus (3\otimes 3)\Big|_{\text{symmetric}} \\ &\hat{=} h_{00} \oplus h_{i0} \oplus \overline{h_{ij}} \oplus \mathrm{tr}(h_{ij}) \\ &\hat{=} 1 \oplus 3 \oplus 5 \oplus 1 \,,\end{aligned}$$

with $\overline{h_{ij}}$ traceless.

We expect an extra condition on the vector field which eliminates the spin 0 degree of freedom if we want it to describe spin 1 particles only.

15.1.1 Lagrangian

The most general Lorentz-invariant quadratic tensors are

$$A_\mu A^\mu \,, \quad \partial_\mu A^\nu \partial^\mu A_\nu \,, \quad \partial_\mu A^\nu \partial_\nu A^\mu \,, \quad \partial_\mu A^\mu \partial_\nu A^\nu \,. \tag{15.11}$$

Defining

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15.12}$$

we define the most general Lagrangian:

$$\mathcal{L} = -\frac{a}{4} F_{\mu\nu} F^{\mu\nu} + \frac{b}{2}\big(\partial_\mu A^\nu\big)\big(\partial_\mu A^\nu\big) + \frac{c}{2}\big(\partial_\mu A^\mu\big)\big(\partial_\nu A^\nu\big) + \frac{m^2}{2} A_\mu A^\mu \,. \tag{15.13}$$

The conjugate momenta are

$$\pi_\rho = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\rho)} = -\frac{a}{2} F^{\mu\nu}\Big(\delta^0_\mu \eta_{\rho\nu} - \delta^0_\nu \eta_{\rho\mu}\Big) + \frac{b}{2}\Big(\delta^0_\mu \eta_{\rho\nu} \partial^\nu A^\mu + \eta^{0\nu}\delta^\mu_\rho \partial_\mu A_\nu\Big) + c\delta^0_\mu \delta^\mu_\rho \partial_\nu A^\nu \tag{15.14}$$

$$= aF_\rho{}^0 + b\partial_\rho A^0 + c\delta^0_\rho \partial_\mu A^\mu \,. \tag{15.15}$$

If for a specific $\rho = \mu$ the conjugate momentum π_μ vanishes, then this A_μ is **not** a generalized coordinate. Therefore we can get rid of the $SO(3)$ scalar d.o.f. A_0 if

$$\pi^0 = 0 = a\underset{\substack{\uparrow \\ =0}}{F_0{}^0} + b\partial_0 A^0 + c\big(\partial_0 A^0 - \partial_i A^i\big) \tag{15.16}$$

$$= (b+c)\partial_0 A^0 - c\partial_i A^i \,. \tag{15.17}$$

Therefore we must choose

$$b = c = 0 \tag{15.18}$$

and the Lagrangian becomes

$[A^\mu] = 1$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu \,, \tag{15.19}$$

where we have chosen $a = 1$, which leads to the canonical form of the Hamiltonian H. We can generalize this for a complex A^μ like in the case of the scalar field.
The second term in the Lagrangian is like the Maxwell mass term.
The sign is the correct one, since

$$\frac{m^2}{2} A_\mu A^\mu \longrightarrow -\frac{m^2}{2} \boldsymbol{A} \cdot \boldsymbol{A} \tag{15.20}$$

for the canonical variables A^i. A^0 appears, but we solve for A^0 using

$$\pi^0 \big(A^0, \boldsymbol{A}\big) = 0 \,. \tag{15.21}$$

15.1.2 Equations of motion

We can extend the Lagrangian by the following term:

$$\mathcal{L} = \mathcal{L}_0 - J_\mu A^\mu \,, \tag{15.22}$$

with J_μ some combination of other fields (there can be also other couplings). The conjugate momenta are

$$\pi^0 = 0 \,, \tag{15.23}$$

$$\pi_i = -\pi^i = -F^{i0} \,, \tag{15.24}$$

where the latter is the "electric field", which is the conjugate momentum to A^i. The Euler-Lagrange equation gives us

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = m^2 A^\mu - J^\mu \,, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} \,, \tag{15.25}$$

$$\boxed{\partial_\nu F^{\nu\mu} + m^2 A^\mu = J^\mu \,.} \tag{15.26}$$

This is the **Maxwell-Proca equation**.

In the free theory ($J^\mu \equiv 0$) we can let ∂_μ act on Eq. (15.26), which gives

$$\underbrace{\partial_\mu \partial_\nu F^{\nu\mu}}_{=0} + m^2 \partial_\mu A^\mu = 0 \tag{15.27}$$

and therefore

$$\boxed{\partial_\mu A^\mu = 0 \,.} \tag{15.28}$$

This is a Lorentz-invariant condition which projects out the scalar degree of freedom A^0. From this follows also

$$\partial_\nu F^{\nu\mu} = \partial_\nu \big(\partial^\nu A^\mu - \partial^\mu A^\nu\big) \overset{\text{Eq. (15.28)}}{=} \Box A^\mu \tag{15.29}$$

and plugging this into Eq. (15.26) with $J^\mu = 0$ we obtain

$$\boxed{(\Box + m^2)A^\mu = 0\,.} \tag{15.30}$$

As in the spinor case, the Klein-Gordon equation holds for every component of A^μ.

The free field can be expanded as follows:

$$A^\mu(x) = \sum_{\lambda=0,\pm1} \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\boldsymbol{p}}} \Big(e^{-ipx}\epsilon^\mu(\boldsymbol{p},\lambda)a(\boldsymbol{p},\lambda) + e^{ipx}\epsilon^{*\mu}(\boldsymbol{p},\lambda)a^\dagger(\boldsymbol{p},\lambda)\Big)\,. \tag{15.31}$$

The **polarization vectors** $\epsilon^\mu(\boldsymbol{p},\lambda)$ correspond to the $u_\alpha(\boldsymbol{p},s)$ in the general expansion. We sum over three λ values since for $j = 1$ there are three J^3 eigenvalues $s = -1, 0, 1$ for the spin of the particles.

The polarization vectors must satisfy

$$\epsilon^\mu(\boldsymbol{p},\lambda)p_\mu = 0 \tag{15.32}$$

and

$$\boxed{\sum_{\lambda=0,\pm1} \epsilon^\mu(\boldsymbol{p},\lambda)\epsilon^{*\nu}(\boldsymbol{p},\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}\,.} \tag{15.33}$$

In the reference frame $p^\mu = k^\mu = (m, \mathbf{0})^T$, Eq. (15.33) becomes

$$\sum_{\lambda=0,\pm1} \epsilon^i(\boldsymbol{k},\lambda)\epsilon^{*j}(\boldsymbol{k},\lambda) = \delta^{ij}\,, \tag{15.34}$$

which is consistent with

$$\epsilon^0(\boldsymbol{k},\lambda) = 0\,. \tag{15.35}$$

We now check that using

$$\epsilon^\mu(\boldsymbol{p},\lambda) = L(p)^\mu{}_\nu \epsilon^\nu(\boldsymbol{k},\lambda)\,, \tag{15.36}$$

with $L(p)k = p$, we get Eq. (15.33) back.

We choose

$$\epsilon^\mu(\boldsymbol{k},\lambda=0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\,, \quad \epsilon^\mu(\boldsymbol{k},\lambda=\pm1) = \mp\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix} \tag{15.37}$$

Polarization vectors for massive spin 1 particles.

and

$$p^\mu = \begin{pmatrix}p^0\\0\\0\\p\end{pmatrix}\,, \quad p^0 = \sqrt{m^2+p^2}\,. \tag{15.38}$$

Therefore

$$L(p)k = p\,, \quad k^\mu = \begin{pmatrix}m\\\mathbf{0}\end{pmatrix}\,, \tag{15.39}$$

or

$$L(p) = \begin{pmatrix} \cosh(\beta) & 0 & 0 & \sinh(\beta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\beta) & 0 & 0 & \cosh(\beta) \end{pmatrix}, \tag{15.40}$$

with

$$\cosh(\beta) = \frac{p^0}{m}, \quad \sinh(\beta) = \frac{p}{m}, \tag{15.41}$$

as before. Since $p^\mu = (p^0, 0, 0, p)^T$,

$$\epsilon^\mu(\boldsymbol{p}, \lambda = \pm 1) = \epsilon^\mu(\boldsymbol{k}, \lambda = \pm 1) \tag{15.42}$$

and so

$$\sum_{\lambda=\pm 1} \epsilon^\mu(\boldsymbol{p}, \lambda)\epsilon^{*\nu}(\boldsymbol{p}, \lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15.43}$$

For $\lambda = 0$ we get

$$\epsilon^\mu(\boldsymbol{p}, \lambda = 0) = \begin{pmatrix} \frac{p}{m} \\ 0 \\ 0 \\ \frac{1}{m}\sqrt{m^2 + p^2} \end{pmatrix} \tag{15.44}$$

and therefore

$$\epsilon^\mu(\boldsymbol{p}, \lambda = 0)\epsilon^{*\nu}(\boldsymbol{p}, \lambda = 0) = \begin{pmatrix} \frac{p^2}{m^2} & 0 & 0 & \frac{p}{m^2}\sqrt{m^2 + p^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p}{m^2}\sqrt{m^2 + p^2} & 0 & 0 & \frac{p^2 + m^2}{m^2} \end{pmatrix}.$$

We now compare the sum of these results componentwise to Eq. (15.33):

$$\mu, \nu = 1, 2: \quad p_1 = p_2 = 0, \sum_{\lambda=0,\pm 1} \epsilon^\mu(\boldsymbol{p}, \lambda)\epsilon^{*\nu}(\boldsymbol{p}, \lambda) = -1 \quad \checkmark, \tag{15.45}$$

$$\mu = \nu = 0: \quad -1 + \frac{p^0 p^0}{m^2} = \frac{-m^2 + (p^2 + m^2)}{m^2} = \frac{p^2}{m^2} \quad \checkmark, \tag{15.46}$$

$$\mu = \nu = 3: \quad 1 + \frac{p^2}{m^2} = \frac{p^2 + m^2}{m^2} \quad \checkmark, \tag{15.47}$$

$$\mu = 3,\ \nu = 0: \quad 0 + \frac{p}{m^2}\sqrt{m^2 + p^2} \quad \checkmark. \tag{15.48}$$

Do these satisfy the compatibility equation between spin 1 particles and field transformations of Lorentz-reps? The equation to be satisfied is

$$\sum_{s'} u_\alpha(\boldsymbol{k}, s') D^{(j)}_{s's}(R) = \sum_{\alpha'} D_{\alpha\alpha'}(R) u_{\alpha'}(\boldsymbol{k}, s) \tag{15.49}$$

and plugging in the $\epsilon^\mu(\boldsymbol{k}, \lambda)$ we get

$$\sum_{\lambda'=0,\pm 1} \epsilon^i(\boldsymbol{k}, \lambda') D^{(1)}_{\lambda'\lambda}(R) = \sum_j R^{ij} \epsilon^j(\boldsymbol{k}, \lambda). \tag{15.50}$$

Here, the Wigner function is

$$D^{(1)}_{\lambda'\lambda}(R) = \exp\Big(- i\theta n^k \big[J^k\big]^{(1)}_{\lambda'\lambda}\Big) \tag{15.51}$$

and the rotation matrix is

$$R^{ij} = \delta^{ij} - \epsilon^{kij} n^k \theta + ... \tag{15.52}$$

Infinitesimally, we can write this as

$$\sum_{\lambda'=0,\pm1} \epsilon^i(\boldsymbol{k},\lambda')\big[J^k\big]^{(1)}_{\lambda'\lambda} = -i\epsilon^{kij}\epsilon^k(\boldsymbol{k},\lambda) \tag{15.53}$$

and

$$\big[J^k\big]^{(1)}_{\lambda'\lambda} \equiv \langle j=1,\lambda'|J^k|j=1,\lambda\rangle \tag{15.54}$$

are the matrix elements of the angular momentum operator J^k in the $|jm\rangle$ basis. With the representation

$$\big[J^3\big]^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{15.55}$$

one can see that Eq. (15.50) works out. Therefore we can conclude that A^μ creates and destroys particles with spin 1.

Are the particles bosons or fermions? We write

$$A_\mu(x) = \underset{\substack{\uparrow \\ \text{annihilation}}}{A^{(+)}_\mu(x)} + \lambda \underset{\substack{\uparrow \\ \text{creation}}}{A^{(-)}_\mu(x)} \tag{15.56}$$

and compute the commutator $[\cdot,\cdot]_-$ and anti-commutator $[\cdot,\cdot]_+$ using canonical commutation or anti-commutation relations for a and $a^\dagger$, respectively. For spacelike separations $(x-y)^2 < 0$ we get

$$\big[A^\mu(x), A^\nu(y)\big]_\mp = \lambda(1\mp 1)\Big(\eta^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{m^2}\Big)\Delta_+(x-y)\,, \tag{15.57}$$

$$\big[A^\mu(x), A^{\dagger\nu}(y)\big]_\mp = \big(1\mp|\lambda|^2\big)(-1)\Big(\eta^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{m^2}\Big)\Delta_+(x-y) \tag{15.58}$$

and both vanish for $|\lambda| = 1$ and commutators! We see, that spin 1 particles are **bosons**.

15.2 Path integral for spin 1 fields

The Hamilton version follows using exactly the same steps as the derivation for the neutral scalar:

$$\langle\Omega|T\{O_a(x_a)O_b(x_b)...\}|\Omega\rangle = |N|^2 \int \mathcal{D}[\boldsymbol{A},\boldsymbol{\pi}] \exp\Big(i\int \mathrm{d}^4x\, \Big(\boldsymbol{\pi}\cdot\dot{\boldsymbol{A}} - \mathcal{H}(\boldsymbol{\pi},\boldsymbol{A}) + i\varepsilon...\Big)\Big) O_a(x_a)O_b(x_b)...$$

Since A^0 is not a canonical coordinate, the path integral is only taken over $\boldsymbol{A}$ and $\boldsymbol{\pi}$.

We eliminate A^0 using the conjugate momentum and the EOM for A_0 Eq. (15.26):

$\partial_i F^{i0} + m^2 A^0 = J^0$

$$E^i = \pi^i = F^{i0} = \partial^i A^0 - \partial^0 A^i \,,$$
$$A^0 = -\frac{1}{m^2}\Big(\nabla\cdot\boldsymbol{E} - J^0\Big)\,,$$
$$\text{and} \quad \partial_0 \boldsymbol{A} = -\boldsymbol{E} - \nabla A^0 = -\boldsymbol{E} + \frac{1}{m^2}\nabla\Big(\nabla\cdot\boldsymbol{E} - J^0\Big)\,.$$

We now compute $\mathcal{H}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{\pi} = \boldsymbol{E}$:

$$\begin{aligned}\mathcal{H} &= \pi_\mu \dot{A}^\mu - \mathcal{L} = -\boldsymbol{\pi}\cdot\partial_0\boldsymbol{A} - \mathcal{L}\\ &= \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\left(\nabla\times\boldsymbol{A}\right)^2 + \frac{1}{(2m)^2}\left(\nabla\cdot\boldsymbol{E}\right)^2 + \frac{1}{2}m^2\boldsymbol{A}^2 - &&\mathcal{H}_0\\ &- \boldsymbol{J}\cdot\boldsymbol{A} - \frac{1}{m^2}J^0\nabla\cdot\boldsymbol{E} + \frac{1}{2m^2}\left(J^0\right)^2. &&\mathcal{H}_{\text{int}}\end{aligned}$$

This looks very non-covariant. We can use the following trick: we can add

$$\text{const.} = \int \mathcal{D}[A^0]\exp\left(i\frac{m^2}{2}\int \mathrm{d}^4x\,\left(A^0 - \frac{1}{m^2}\left(\nabla\cdot\boldsymbol{\pi} - J^0\right)\right)^2\right) \quad (15.59)$$

to the integrand of the path integral. It has no effect on the result of the path integral, since *const.* is independent of $\boldsymbol{\pi}$. To see this, we make the substitution

$$A^0 = A'^0 + \frac{1}{m^2}\Big(\nabla\cdot\boldsymbol{\pi} - J^0\Big)\,, \quad (15.60)$$

which has unit Jacobian and the constant then cancels with $|N|^2$.

The path integral measure therefore becomes $\mathcal{D}[\boldsymbol{A},\boldsymbol{\pi},A^0]$ and the integrand contains the new Hamiltonian density

$$\mathcal{H}' = \frac{1}{2}\boldsymbol{\pi}^2 + (\nabla\times\boldsymbol{A})^2 - \frac{1}{2}m^2 A_\mu A^\mu + J_\mu A^\mu + A^0\nabla\cdot\boldsymbol{\pi}\,. \quad (15.61)$$

Now we integrate over $\boldsymbol{\pi}$ (completing the square as in the scalar case) to get the Lagrangian version of the path integral:

$$\langle\Omega|T\{O_a(x_a)O_b(x_b)\cdot\ldots\}|\Omega\rangle = |N|^2\int \mathcal{D}[A^\mu]e^{i\int \mathrm{d}^4x\,\mathcal{L}}O_a(x_a)O_b(x_b)\cdot\ldots\,, \quad (15.62)$$

where $\mathcal{L}$ is manifestly covariant:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu - J_\mu A^\mu\,. \quad (15.63)$$

Note: A^0 is now an independent field configuration (i.e. it is not determined by $\boldsymbol{A}$).

15.2.1 Propagator

The generating functional is

$$Z[J] = \frac{\int\mathcal{D}[A^\mu]\ \exp\Big(i\int \mathrm{d}^4x(\mathcal{L} + J_\mu A^\mu)\Big)}{\int\mathcal{D}[A^\mu]\exp\Big(\int \mathrm{d}^4x\ \mathcal{L}\Big)}\,. \quad (15.64)$$

In the **free theory** the result is

$$Z_0[J] = \exp\left(-\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ J_\mu(x)\Delta_F^{\mu\nu}(x-y)J_\nu(y)\right), \quad (15.65)$$

with

$$\Big(-\eta_{\mu\nu}\big(\Box_x+m^2+i\varepsilon\big)+\partial_\mu\partial_\nu\Big)i\Delta_F^{\nu\rho}(x-y)=\delta_\mu^\rho\delta^{(4)}(x-y)\,, \tag{15.66}$$

because

$$\mathcal{L}_0=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\big(m^2-i\varepsilon\big)A_\mu A^\mu \tag{15.67}$$

$$=-\frac{1}{2}A^\mu\Big(-\eta_{\mu\nu}\big(\Box_x+m^2+i\varepsilon\big)+\partial_\mu\partial_\nu\Big)A^\nu\,. \tag{15.68}$$

Fourier-transforming Eq. (15.65), it becomes

$$\Big(\eta_{\mu\nu}\big(p^2-m^2+i\varepsilon\big)-p_\mu p_\nu\Big)i\widetilde{\Delta}_F^{\nu\rho}(p)=\delta_\mu^\rho\,. \tag{15.69}$$

There are only two Lorentz structures which can fulfill this equation and therefore we make the ansatz

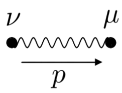
$$\widetilde{\Delta}_F^{\nu\rho}(p)\equiv A\big(p^2\big)\eta^{\nu\rho}+B\big(p^2\big)p^\nu p^\rho\,, \tag{15.70}$$

which leads to the free Feynman propagator

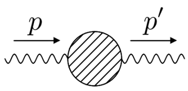
$$\boxed{\widetilde{\Delta}_F^{\mu\nu}(p)=\frac{i}{p^2-m^2+i\varepsilon}\left(-\eta^{\mu\nu}+\frac{p^\mu p^\nu}{m^2}\right).} \tag{15.71}$$

The Feynman rules of the free theory are the follwing:

$$\overbracket{A^\mu(x)A^\nu(y)}=\Delta_F^{\mu\nu}(x-y) \tag{15.72}$$



$$\widetilde{\Delta}_F^{\mu\nu}(p)\quad\text{(no arrows since } A_\mu^*=A_\mu\text{)} \tag{15.73}$$

$\epsilon^\mu(\boldsymbol{p},\lambda)$  $\epsilon^{*\mu}(\boldsymbol{p},\lambda)$ (15.74)

The polarization vector $\epsilon^\mu(\boldsymbol{p},\lambda)$ is associated with the incoming particle, while the vector $\epsilon^{*\mu}(\boldsymbol{p},\lambda)$ is associated with the outgoing one.

15.2.2 Interactions

The marginal interaction built out of only A^μ fields are

$$A_\mu A^\mu\partial_\nu A^\nu\,,\quad A_\mu A_\nu\partial^\mu A^\nu\,,\quad (A_\mu A^\mu)^2 \tag{15.75}$$

and there a re no relevant ones, whereas the marginal interactions involving spinor or scalar fields are

$$\psi_L^\dagger\bar\sigma^\mu\psi_L A_\mu\,,\quad \psi_R^\dagger\sigma^\mu\psi_R A_\mu\,,\quad \overline{\Psi}\gamma^\mu\Psi A_\mu\,, \tag{15.76}$$

$$\phi\partial^\mu\phi A_\mu\,,\quad \phi^2A_\mu A^\mu\,,\quad \phi A_\mu A^\mu\,. \tag{15.77}$$

<u>However:</u> a theory with these interactions in **not** renormalizable! Our previous discussion of marginal/relevant interaction, which led to renormalizable theories was based on the quantity

$$\Delta_i=4-a_i-\sum_f n_{if}[\psi_f]\,, \tag{15.78}$$

with $[\psi_f] = 1 + s_f$ the mass dimension of the field of type f, where $s_f = 0$ for ϕ and A^μ and $s_f = \frac{1}{2}$ for $\psi_{L/R}$ and Ψ.
Now: the propagator is

$$\widetilde{\Delta}_F^{\mu\nu}(p) = \frac{i}{p^2 - m^2 + i\varepsilon}\Bigg(- \eta^{\mu\nu} + \underset{\substack{\uparrow \\ \text{dangerous}}}{\frac{p^\mu p^\nu}{m^2}}\Bigg) \tag{15.79}$$

and

$$\widetilde{\Delta}_F^{\mu\nu} \sim \frac{1}{(p^2)^{1-s_f}} \underset{\substack{\uparrow \\ Eq.\ (15.79)}}{\sim} \text{const.} \quad \implies \quad s_f = 1\,. \tag{15.80}$$

Therefore one can show using the interactions we listed above

$$\Delta_i = d - a_i - \sum_f n_{if}(1 + s_f) \tag{15.81}$$

$$\implies \quad \Delta_i < 0 \quad \textbf{always.} \tag{15.82}$$

Massive vector fields have no renormalizable interactions.

Examples:

$$A_\mu A^\mu \partial_\nu A^\nu : \quad a_i = 1\,,\ n_{iA} = 3\,, \tag{15.83}$$

$$\implies \quad \Delta_i = 4 - 1 - 3 \cdot 2 = -3\,, \tag{15.84}$$

or

$$\phi^2 A_\mu A^\mu : \quad a_i = 0\,,\ n_{i\phi} = 2\,,\ n_{iA} = 2\,, \tag{15.85}$$

$$\implies \quad \Delta_i = 4 - 0 - 2 \cdot 1 - 2 \cdot 2 = -2\,. \tag{15.86}$$

This can also be understood as a failure to satisfy perturbative unitarity. The theory is still unitary (after all, we have a hermitian Hamiltonian), but the tree-level result grows with energy (recall the propagator) and we will eventually need to add loop diagrams as important as tree diagrams to recover perturbativity.

As an EFT, the theory is valid for $E \ll m$, otherwise

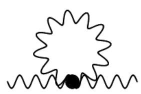
$$\frac{p^\mu p^\nu}{m^2} \sim \frac{E^2}{m^2} \gg 1\,. \tag{15.87}$$

However, for $E \ll m$, massive spin 1 fields should be integrated out. Can we even write down a sensible EFT? Yes, for weak enough interactions:

$$\mathcal{L}_{\text{int}} \supset g^2 A^4 \tag{15.88}$$

each interaction gives

$$\underset{\substack{\uparrow \\ \text{1 loop}}}{\frac{g^2}{16\pi^2}} \cdot \frac{E^2}{m^2}\,, \tag{15.89}$$



$$\sim \frac{g^2}{16\pi^2} \cdot \frac{E^2}{m^2}$$

and so $E \ll \frac{4\pi m}{g}$ is good for an EFT.
This is the reason we knew that the LHC would discover something, since the W and Z bosons ($m_W \approx 80$ GeV, $m_Z \approx 90$ GeV with $g \approx 1$) are massive spin 1 particles, which require a UV completion. We found that this UV completion looks very much like the Higgs scalar field and massless vector fields of the Standard Model.[1]

[1] In QFT2, you will learn about the Higgs mechanism, which allows us to give masses to vector bosons while preserving perturbative unitarity/renormalizability.

16 Massless vector particles and gauge redundancy

Recall, that given the transformation properties of the polarization four-vectors, the field operator $A^\mu(x)$ transform as a four-vector under a generic Lorentz transformation Λ, see Eq. (12.162)

$$U(\Lambda,0)A^\mu(x)U^{-1}(\Lambda,0) = \sum_{\alpha'} D^{\alpha\alpha'}\left(\Lambda^{-1}\right)A^{\alpha'}(\Lambda x) \tag{16.1}$$

$$= (\Lambda^{-1})^\mu{}_\nu\, A^\nu(\Lambda x) \tag{16.2}$$

In summary, when we apply a Lorentz transformation $U(\Lambda)$ to the field-operator, the only objects that transform non-trivially are the creation and annihilation operators, which inherit the transformation properties of the single-particle states they create or destroy. Since the polarization four-vectors, when transformed covariantly according to the Lorentz index they carry, behave exactly in the same way as single particle states when acted upon by $U(\Lambda)$ (that is: p changes to Λp, and the spin label transforms with the D matrix), then we can reabsorb the transformation of the creation and annihilation operators into the polarization four-vectors via a Lorentz-covariant transformation of the latter. We are then left with a Lorentz matrix acting on the field-operator, with the usual Lorentz contraction of indices.

The fact that the field operator transforms covariantly under Lorentz transformations, ensures that in order to construct a Lorentz invariant theory where our spin-1 particles interact with others and among themselves, it is enough to write down an interaction Lagrangian where $A^\mu(x)$ always appears in Lorentz invariant contractions, like $A^\mu(x)A_\mu(x)$ etc. That is, it is enough that the interaction Lagrangian looks Lorentz-invariant. This sounds trivial, but as we will see in the following, for massless particles this is not the case.

This line of reasoning leads us all the way to the Einstein-Hilbert action, diffeomorphism redundancy and the universality of gravity.[1]

[1] It couples universally to the energy-momentum of particles, i.e. no coupling constant.

16.1 Massless particles and Lorentz-transformations

Recall that for massless particles, the reference momentum is $k^\mu = (k,0,0,k)^T$ and the little group transformations can be parametrized using $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + ...$

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & -\alpha & -\beta & 0 \\ \alpha & 0 & \theta & -\alpha \\ \beta & -\theta & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix} . \tag{16.3}$$

or

$$U(\mathbb{1}+\omega,0) = \mathbb{1} - i\Big(\theta J^3 + \alpha \underbrace{(K^1+J^2)}_{\equiv A} + \beta \underbrace{(K^2+J^1)}_{\equiv B}\Big) + ...$$

we chose the A and B eigenvalues $\alpha = \beta = 0$ and the J_3 eigenvalue to be the helicity (here ± 1). The polarization four-vectors for single particle states with k must be eigenvectors of J_3 with eigenvalues ± 1. We choose for the reference momentum

You can easily check that for a z rotation with J_3, the polarization vectors indeed transform like eigenvectors with eigenvalue $\sigma = \pm 1$. A J_3 rotation $R_z(\theta)^\mu{}_\rho$ with angle θ gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c \pm is \\ -is \pm c \\ 0 \end{pmatrix} = e^{\pm i 1\cdot\theta} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} .$$

with $c = \cos\theta$ and $s = \sin\theta$, which is as we want

$$e^{iJ_3\theta}|k,\sigma\rangle = e^{i\sigma\theta}|k,\sigma\rangle$$

$$e^\mu_{\pm 1}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} . \tag{16.4}$$

and for general p

$$e^\mu_{\pm 1}(p) = L(p)^\mu{}_\nu \, e^\nu_{\pm 1}(k)$$

We can express $L(p)$ as a boost in z to the energy $|\boldsymbol{p}|$ followed by a general rotation to point in to $\boldsymbol{p}$. The boost in z-direction will not affect the polarization vectors in Eq. (16.4) and the rotation will just mix the spatial components. We see therefore that the time component is always zero

$$e^0_{\pm 1}(p) = 0 \tag{16.5}$$

This is problematic, because this can not be a Lorentz invariant statement. Furthermore, for the reference momentum $k = n(1,0,0,1)^T$ and Eq. (16.4), we clearly have

$$k_\mu e^\mu_{\pm 1}(k) = 0 \qquad \Leftrightarrow \qquad p_\mu e^\mu_{\pm 1}(p) = 0 \tag{16.6}$$

where the second equality holds because of Lorentz-invariance. We define the free massless spin 1 (or photon field) using the expansion

$$A^\mu(x) = \sum_{\sigma=\pm 1} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \Big(e^{-ipx} \epsilon^\mu(\boldsymbol{p},\sigma) a(\boldsymbol{p},\sigma) + e^{ipx} \epsilon^{*\mu}(\boldsymbol{p},\sigma) a^\dagger(\boldsymbol{p},\sigma) \Big) . \tag{16.7}$$

Clearly, thanks to Eq. (16.5), we have to have

$$A^0(x) \equiv 0$$

which is clearly at odds with A^μ transforming covariantly under Lorentz. We find that the field operator for a *massless spin 1 field* A^μ *cannot transform like a Lorentz 4-vector under Lorentz boosts!*[2] Due to Eq. (16.6), we also have

[2] Otherwise $A^0 = 0$ would not be true!

$$\nabla \cdot \mathbf{A}(x) \equiv 0$$

What went wrong?

16.1.1 *Lorentz-transformation of massless spin1 fields*

The problem lies in the A and B generators of the Little Group which we have discarded so far by choosing $\alpha = \beta = 0$ eigenvalues for our representation. If we however transform the polarization vectors in these directions using Eq. (16.3), we get for the Little group element $W = \Lambda(\alpha, \beta, \theta = 0)$

$$\Lambda(\alpha, \beta, \theta = 0)^\mu{}_\nu \, e^\nu_{\pm 1}(k) = e^\nu_{\pm 1}(k) + (\alpha \pm i\beta)\frac{1}{\sqrt{2}k} k^\mu$$

which then generalizes to an arbitrary Lorentz-transformation using Eq. (12.75) as before

$$\begin{aligned}\Lambda^\mu{}_\nu \, e^\nu_{\pm 1}(p) &= (L(\Lambda p) \cdot W(\Lambda, p))^\mu{}_\nu \, e^\nu_{\pm 1}(k) \\ &= L(\Lambda p) e^{\pm i\theta(\Lambda, p)} \left(e^\nu_{\pm 1}(k) + (\alpha \pm i\beta)\frac{1}{\sqrt{2}k} k^\mu \right) \\ &= e^{\pm i\theta(\Lambda, p)} \left(e^\nu_{\pm 1}(\Lambda p) + (\alpha \pm i\beta)\frac{1}{\sqrt{2}k} (\Lambda p)^\mu \right)\end{aligned}$$

we solve for $e^\nu_{\pm 1}(p)$ to get

$$e^{\mp i\theta(\Lambda, p)} e^\nu_{\pm 1}(p) = (\Lambda^{-1})^\mu{}_\nu \, e^\nu_{\pm 1}(\Lambda p) + (\alpha \pm i\beta)\frac{1}{\sqrt{2}k} p^\mu \qquad (16.8)$$

Let us now investigate the implications for the free feel operator. Using the expansion above and the transformation properties of the creation and annihilation operators Eq. (12.105), we get with $c_\pm = \frac{1}{\sqrt{2}k}(\alpha \pm i\beta)$

$$U(\Lambda) \, a_s(\boldsymbol{p}) \, U^{-1}(\Lambda) = \sum_{\sigma = \pm h} e^{\mp i\sigma\theta(\Lambda, p)} \, a_\sigma(\boldsymbol{\Lambda p})$$

$$\begin{aligned}U(\Lambda) \, A_\mu(x) \, U^{-1}(\Lambda) A^\mu(x) &= \sum_{\sigma = \pm 1} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} e^{\mp i\sigma\theta(\Lambda, p)} \epsilon^\mu(\boldsymbol{p}, \sigma) a(\Lambda \boldsymbol{p}, \sigma) + \text{h.c.} \right), \\ &= \sum_{\sigma = \pm 1} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} ((\Lambda^{-1})^\mu{}_\nu \epsilon^\mu(\Lambda \boldsymbol{p}, \sigma) + c\, p^\mu) a(\Lambda \boldsymbol{p}, \sigma) + \text{h.c.} \right), \\ &= (\Lambda^{-1})^\mu{}_\nu \sum_{\sigma = \pm 1} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ip(\Lambda x)} \epsilon^\mu(\boldsymbol{p}, \sigma) a(\boldsymbol{p}, \sigma) + \text{h.c.} \right) + \partial_\mu \lambda_\Lambda(x), \\ &= (\Lambda^{-1})^\mu{}_\nu \, A_\mu(\Lambda x) + \partial_\mu \lambda_\Lambda(x).\end{aligned}$$

where we have used Eq. (16.8) and that $p \cdot x = (\Lambda p) \cdot (\Lambda x)$ and that the integration measure is Lorentz invariant $p \to \Lambda p$. This is very interesting. If you compare this to a covariant field operator, see Eq. (16.1), we end up with an additional **non-covariant** piece

$$U(\Lambda) \, A_\mu(x) \, U^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu \, A_\mu(\Lambda x) + \partial_\mu \lambda_\Lambda(x). \qquad (16.9)$$

which is the divergence of some Λ dependent scalar function $\lambda(x)$.

We conclude, that we cannot just rely on Lorentz-invariance looking Lagrangians to build a **truly** Lorentz-invariant theory. A term like $A_\mu A^\mu$ which would be Lorentz-invariant for a massive vector, would not be Lorentz-invariant for a *massless vector field*, due to the additional term in Eq. (16.9). The form of the additional term provides us with a clue to solve the problem. If besides looking Lorentz-invariant, the Lagrangian is additionally invariant under the replacement

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x). \tag{16.10}$$

for a generic function $\lambda(x)$, then the inhomogeneous piece in the Lorentz transformation of A_μ is not catastrophic and the Lagrangian is *truly* Lorentz invariant. We will discuss this in more detail later, but for instance, we can take

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{16.11}$$

which is not affected by the replacement Eq. (16.10)

$$\begin{aligned} F_{\mu\nu} \longrightarrow & \partial_\mu(A_\nu + \partial_\nu \lambda) - \partial_\nu(A_\mu + \partial_\mu \lambda) && (16.12)\\ &= F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\lambda(x) && (16.13)\\ &= F_{\mu\nu} && (16.14) \end{aligned}$$

which means for the field operator

$$U(\Lambda) F_{\mu\nu} U^{-1}(\Lambda) \longrightarrow (\Lambda^{-1})^\mu{}_\rho (\Lambda^{-1})^\nu{}_\sigma \, F^{\rho\sigma}(\Lambda x) \tag{16.15}$$

which shows that it is a true Lorentz-tensor!

If we use $F_{\mu\nu}$ as a building block and contact it with itself, we have found a truly Lorentz-invariant term $F_{\mu\nu}F^{\mu\nu}$, which we will use as the kinetic term.

We can couple A_μ to true Lorentz-vectors if they satisfy a conservation equation

$$\partial_\mu J^\mu(x) = 0 \tag{16.16}$$

because then with

$$\mathcal{L}_{int} = A_\mu(x) J^\mu(x) \tag{16.17}$$

we get with Eq. (16.10), that

$$\begin{aligned} \mathcal{L}_{int} \to & \mathcal{L}_{int} + \partial_\mu \lambda(x) J^\mu(x) \\ &= \mathcal{L}_{int} - \lambda(x) \partial_\mu J^\mu(x) \\ &= \mathcal{L}_{int} \end{aligned}$$

where we have integrated by parts and used the conservation equation.

16.2 Einstein-Hilbert from Gauge redundancy

The graviton is a massless spin 2 particle. Let us see what we can learn from the discussion above about gravity. The field tensor is a symmetry 2-tensor

$$h_{\mu\nu}(x) = h_{\nu\mu}(x) \tag{16.18}$$

The polarization tensors have to be **helicity 2** eigenstates. These can be constructed from our massless spin 1 helicity vectors as follows,

$$e^{\mu\nu}_{\pm 2}(k) = e^{\mu}_{\pm 1}(k) e^{\nu}_{\pm 1}(k)$$

Let us check the helicity property

$$R_z(\theta)^{\mu}{}_{\rho} R_z(\theta)^{\nu}{}_{\sigma} e^{\rho\sigma}_{\pm 2}(k) = e^{\pm 2i\theta} e^{\mu\nu}_{\pm 2}(k)$$

where we have used the fact that the helicity 2 tensor is a trivial product of the spin 1 vectors, which are eigenvectors of the two rotations with eigenvalues $e^{\pm i\theta}$. The polarization tensor takes the form

Related to the usual gravitational wave polarization ... TODO

$$e^{\mu\nu}_{\pm 2}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm i & 0 \\ 0 & \pm i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A general polarization tensor is defined as

$$e^{\mu\nu}_{\pm 2}(k) = L(p)^{\mu}{}_{\rho}\, L(p)^{\nu}{}_{\sigma}\, e^{\rho\sigma}_{\pm 2}(k)$$

The free graviton field operator is

$$h^{\mu\nu}(x) = \sum_{\sigma=\pm 2} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} \epsilon^{\mu\nu}(\boldsymbol{p},\sigma) a(\boldsymbol{p},\sigma) + \text{h.c.} \right). \quad (16.19)$$

In complete analogy to the argument for the photon in Eq. (16.8), we find that the polarization tensors satisfy

$$e^{\mp 2i\theta(\Lambda,p)} e^{\nu}_{\pm 2}(p) = (\Lambda^{-1})^{\mu}{}_{\rho} (\Lambda^{-1})^{\nu}{}_{\sigma}\, e^{\rho\sigma}_{\pm 2}(\Lambda p) + p^{\mu} v^{\nu}_{\pm} + v^{\mu}_{\pm} p^{\nu} \quad (16.20)$$

with a α, β dependent vector

$$v^{\mu}_{\pm} = (\Lambda^{-1})^{\mu}{}_{\rho} (\alpha \pm i\beta) \frac{1}{\sqrt{2}k} e^{\rho}_{\pm 1}(\Lambda p)$$

This directly implies for the Lorentz transformations of the graviton field operator, that

$$U(\Lambda)\, h^{\mu\nu}(x)\, U^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\rho} (\Lambda^{-1})^{\nu}{}_{\sigma}\, h^{\rho\sigma}(\Lambda x) + \partial^{\mu} \xi^{\nu}_{\Lambda}(x) + \partial^{\nu} \xi^{\mu}_{\Lambda}(x). \quad (16.21)$$

which is like **not Lorentz covariant**, like the photon field operator.

Let us construct a truly Lorentz covariant tensor, like in the case of the field strength Eq. (16.11) above. We require invariance under Lorentz and the under additional

$$h^{\mu\nu}(x) \to h^{\mu\nu}(x) + + \partial^{\mu} \xi^{\nu}(x) + \partial^{\nu} \xi^{\mu}(x) \quad (16.22)$$

The simplest tensor is the following

$$R_{\mu\nu\rho\sigma} \equiv \frac{1}{2} \left(\partial_{\mu} \partial_{\rho} h_{\nu\sigma} - \partial_{\nu} \partial_{\rho} h_{\mu\sigma} - \partial_{\mu} \partial_{\sigma} h_{\nu\sigma} + \partial_{\nu} \partial_{\sigma} h_{\mu\sigma} \right)$$

which is the **linearized Riemann tensor**. All the non-covariant terms cancel and we can use $R_{\mu\nu\rho\sigma}$ to build truly Lorentz-invariant interactions and kinetic terms for the graviton.

It is quite remarkable, that we derived this just using the consistency of QM with Lorentz invariance.

Just like for the photon, we can couple $h_{\mu\nu}$ to a **conserved, symmetric** tensor $T^{\mu\nu}$. There is exactly one such conserved tensor in QFT, the **energy-momentum tensor**. The conservation implies

$$\partial_\mu T^{\mu\nu} = 0$$

The interaction Lagrangian is

$$\mathcal{L}_{int} = h_{\mu\nu} T^{\mu\nu}$$

which is invariant under the gauge transformation Eq. (16.21), because

$$\begin{aligned}\mathcal{L}_{int} &\to \mathcal{L}_{int} + 2(\partial_\mu \xi_\nu) T^{\mu\nu}(x) \\ &= \mathcal{L}_{int} - 2\xi_\nu (\partial_\mu T^{\mu\nu}(x)) \\ &= \mathcal{L}_{int}\end{aligned}$$

and is therefore Lorentz-invariant.

You can see the implications for massless spin 3 fields: there is no conserved 3 tensor in QFT (recall the Coleman-Mandula theorem) and we therefore cannot couple this field like photons or gravitons directly. One can show that this means that there are no long-range forces associated with massless spin 3 fields, which could explain why we haven't see any such fields.

16.3 Comments on gauge invariance

Of course invariance under the formal replacements like Eq. (16.10) and Eq. (16.22) is what usually deserves the name of **gauge invariance**. However from the constructive viewpoint we have taken here, it is clear that gauge invariance is not a symmetry on which we have any choice: it is forced upon us by combining quantum mechanics and Lorentz-invariance. In particular, if we want to describe massless particles through local field operators like $A\mu(x)$ and $h_{\mu\nu}(x)$, then the theory is truly Lorentz invariant only if it is gauge-invariant—because these field operators do not transform covariantly under Lorentz transformations.
However **gauge-invariance is not a symmetry in any physical sense**: recall that the non-covariant piece in the transformation of $A\mu(x)$ and $h_{\mu\nu}(x)$ stems from the special little-group transformations under which the particles' states are invariant. This means that a gauge transformation like Eq. (16.10) and Eq. (16.22) is not associated with any transformation in the particles' Hilbert space. Therefore, unlike ordinary symmetries, gauge-invariance does not yield any non-trivial relation between different observables like e.g. different scattering amplitudes—because it acts trivially on the particles' states!
Gauge invariance in a sense is a **redundancy** in our description of massless particles: we have a class of transformations under which the field operators transform, but the states don't. It had better be that the different field configurations that we span by applying such transformations are physically all equivalent—that they describe the same state of the system—which we achieve by demanding that the Lagrangian be invariant under those transformations.

16.4 Massless vectors as a limit of massive vectors

The Lagrangian admits a smooth limit for $m \to 0$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \tag{16.23}$$

However, in other ways the limit $m \longrightarrow 0$ is subtle and **not** smooth. We can already anticipate the issue by remembering that the *particle states* of a massive spin-1 vector has three degrees of freedom ($0, \pm 1$ spin) whereas a massless vector has only two degrees of freedom (± 1 helicity)

a) The Hamiltonian is singular:

$$\mathcal{H} \supset \frac{1}{m^2} \left(\nabla \cdot \boldsymbol{E} - J^0 \right)^2 . \tag{16.24}$$

Taking the limit $m \longrightarrow 0$ is only possible if we enforce

$$\nabla \cdot E = J^0 \,, \tag{16.25}$$

which is the Gauss law. From the EOM

$$\partial_\nu F^{\nu\mu} + m^2 A^\mu = J^\mu \tag{16.26}$$

we get for $m^2 \longrightarrow 0$, by letting ∂_μ act on this,

$$\partial_\mu \partial_\nu F^{\nu\mu} = \partial_\mu J^\mu \quad \Longrightarrow \quad \partial_\mu J^\mu = 0 \,. \tag{16.27}$$

A massless vector field A^μ couples to a **conserved** current!

b) The polarization sum,

$$\sum_{\lambda = 0, \pm 1} \epsilon^\mu(\boldsymbol{p}, \lambda) \epsilon^{*\nu}(\boldsymbol{p}, \lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \,, \tag{16.28}$$

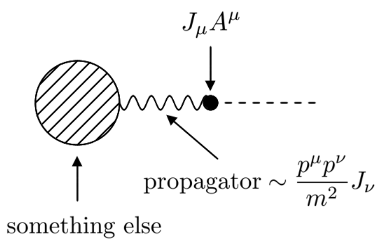
is singular. This arises from the longitudinal polarization

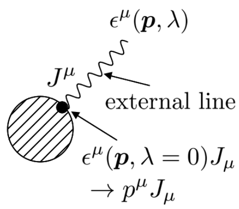
$$p^\mu = \begin{pmatrix} E_{\boldsymbol{p}} \\ 0 \\ 0 \\ |\boldsymbol{p}| \end{pmatrix} , \quad \epsilon^\mu(\boldsymbol{p}, \lambda = 0) = \begin{pmatrix} \frac{|\boldsymbol{p}|}{m} \\ 0 \\ 0 \\ \frac{E_{\boldsymbol{p}}}{m} \end{pmatrix} , \tag{16.29}$$

and high energies $|\boldsymbol{p}|$, $E_{\boldsymbol{p}} \gg m$:

$$\epsilon^\mu(\boldsymbol{p}, \lambda = 0) = \frac{p^\mu}{m} + \mathcal{O}\left(\frac{m}{E_{\boldsymbol{p}}} \right) . \tag{16.30}$$

These singular terms contribute to the matrix element:





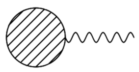
However, the singular polarization is without effect, if we enforce

$$p_\mu \underset{\substack{\uparrow \\ \text{Fourier transform}}}{\widetilde{J}^\mu} = 0\,, \tag{16.31}$$

which again leads us to the current being conserved,

$$\partial_\mu J^\mu = 0\,. \tag{16.32}$$

We can express the decoupling of the singular ($\lambda = 0$) term by assuming that it never coupled to the current in the first place, which requires



$$\epsilon^\mu(\boldsymbol{p}, \lambda = 0) = 0\,, \tag{16.33}$$

M_μ

or

$$p_\mu M^\mu = 0\,, \tag{16.34}$$

or the invariance of $\epsilon^\mu M_\mu$ under the shift

$$\epsilon^\mu(\boldsymbol{p}, \lambda) \longrightarrow \epsilon^\mu(\boldsymbol{p}, \lambda) + \alpha(p) p^\mu\,. \tag{16.35}$$

In the field expansion, this means

$$\begin{aligned} A^\mu(x) \longrightarrow & A^\mu(x) + \sum_{\lambda=0,\pm1} \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}} \left(e^{-ipx} p^\mu \alpha(p) a(\boldsymbol{p}, \lambda) + \text{h.c.} \right) \\ &= A^\mu(x) + \underset{\substack{\uparrow \\ \text{from } p^\mu}}{\partial^\mu} \overset{\substack{\text{scalar field} \\ \downarrow}}{\alpha(x)}\,. \end{aligned} \tag{16.36}$$

So the Lagrangian should be invariant under

$$A^\mu(x) \longrightarrow A^\mu(x) + \partial^\mu \alpha(x) \tag{16.37}$$

for any $\alpha(x)$. This is a **local** identification of the same field state. This is not a symmetry, since it doesn't correspond to different states whose transformation leaves $\mathcal{L}$ invariant. Rather $A^\mu(x)$ and $A^\mu(x) + \partial^\mu \alpha(x)$ are the same physical state (they are in the same equivalence class). This is consistent with $\partial_\mu J^\mu(x) = 0$, since

$$\int \mathrm{d}^4 x\; J_\mu(x) A^\mu(x) \longrightarrow \int \mathrm{d}^4 x\; J_\mu(x) \Big(A^\mu(x) + \partial^\mu \alpha(x) \Big) \tag{16.38}$$

$$\overset{\text{IBP}}{=} \int \mathrm{d}^4 x\; J_\mu(x) A^\mu(x) - \alpha(x) \partial_\mu J^\mu(x) \tag{16.39}$$

$$\underset{\substack{\uparrow \\ \partial_\mu J^\mu(x)=0}}{=} \int \mathrm{d}^4 x\; J_\mu(x) A^\mu(x) \quad \checkmark\,. \tag{16.40}$$

c) Massless "spin 1" particles have helicity $\sigma = \pm 1$, which corresponds to **two** degrees of freedom (vs. massive spin 1: $2j + 1 = 3$ d.o.f.). Again: the $\lambda = 0$ polarization must be irrelevant.

<u>**Tangential discussion:**</u> Why don't we describe photons as massless

$$(1,0) \oplus (0,1) \tag{16.41}$$

fields? This contains just the two helicity states $\sigma = \pm 1$ from the start.

We can understand the $(1,0)$ rep as

$$\underset{\psi_R}{\underset{\uparrow}{\left(\frac{1}{2},0\right)}} \otimes \underset{\xi_R}{\underset{\uparrow}{\left(\frac{1}{2},0\right)}} = \overset{\text{anti-symmetric}}{\overset{\downarrow}{(0,0)}} \oplus \underset{\text{symmetric}}{\underset{\uparrow}{(1,0)}}\,, \tag{16.42}$$

using angular momentum addition again:

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1\,. \tag{16.43}$$

An example for a Lorentz-singlet would be

$$\epsilon_{\alpha\beta}\psi_L^\alpha \xi_L^\beta\,, \quad \alpha,\beta = 1,2\,. \tag{16.44}$$

The symmetric combination transforms as

$$F'_{\alpha'\beta'} = D_R^{\alpha'\alpha} D_R^{\beta'\beta} F_{\alpha\beta}\,. \tag{16.45}$$

$$\bar\sigma^{\mu\nu} \sim \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\,,$$
$$\bar\sigma^\mu\sigma^\nu \longrightarrow D_L^\dagger \bar\sigma^\mu \underbrace{D_L D_R^\dagger}_{=\mathbb{1}} \sigma^\nu D_R\,,$$
therefore
$$\bar\sigma^{\mu\nu} \longrightarrow D_L^\dagger \bar\sigma^{\mu\nu} D_R\,.$$
Using this, we see
$$\epsilon\bar\sigma^{\mu\nu} \longrightarrow \epsilon D_L^\dagger \bar\sigma^{\mu\nu} D_R \underset{\epsilon D_L^\dagger = D_R^T\epsilon}{\underset{\uparrow}{=}} D_R^T \bar\sigma^{\mu\nu} D_R \quad \checkmark\,.$$

We define the vector rep as

$$F^{\mu\nu} \equiv \frac{i}{4}\left(\epsilon\bar\sigma^{\mu\nu}\right)_{\alpha\beta} F_{\alpha\beta}\,, \tag{16.46}$$

analogously to the $\left(\frac{1}{2},\frac{1}{2}\right)$ case, where we had

$$A = A_\mu\sigma^\mu\,, \quad A \longrightarrow D_L A D_R^{-1}\,. \tag{16.47}$$

$F^{\mu\nu}$ transforms as a rank-2 Lorentz-tensor and is **self-dual**:

$$\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = F^{\mu\nu}\,. \tag{16.48}$$

What if we used it for the $\sigma = 1$-photon?

$$F^{\mu\nu} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_{\boldsymbol{p}}}\left(e^{-ipx}\epsilon^{\mu\nu}(\boldsymbol{p},\sigma=1)a(\boldsymbol{p},\sigma=1) + \text{h.c.}\right), \tag{16.49}$$

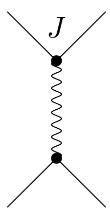
where Eq. (15.49) implies:

$$\epsilon^{\mu\nu}(\boldsymbol{p},\pm1) \sim p^\mu\epsilon^\nu(\boldsymbol{p},\pm1) - p^\nu\epsilon^\mu(\boldsymbol{p},\pm1)\,. \tag{16.50}$$

The field $F^{\mu\nu}$ would have to couple to a current:

$$\mathcal{L}_{\text{int}} = -J_{\mu\nu}F^{\mu\nu} \tag{16.51}$$

Particle exchange leads to:



$$\sim \underset{\text{propagator}}{\underset{\uparrow}{\frac{p^\mu p^\nu}{p^2}}} \quad \text{vs} \quad \frac{1}{p^2} \quad \text{of photon.} \tag{16.52}$$

The Fourier transform

$$\int \frac{\mathrm{d}^3 p}{(2\pi)^3} e^{i\boldsymbol{p}\cdot\boldsymbol{r}} \tag{16.53}$$

of it implies that the static potential ($t = 0$) falls of as $\frac{1}{r^3}$ (and not as $V \sim \frac{1}{r}$ as it should be for the Coulomb interaction): Therefore, this would prevent long-range interactions (i.e. $\sim \frac{1}{r}$), which we observe in Nature.

In my QFT2 class, we will understand that spin 2 fields have to couple to the energy-momentum tensor $T_{\mu\nu}$ (with universal coupling for all fields) and that **massless** particles with higher spins (> 2) **cannot** mediate long-range forces, which clearly explain why we haven't seen any in Nature. There are **massive** higher spin fields, e.g. in some resonances of QCD.

16.4.1 Gauge-invariant Lagrangians

The Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_\mu A^\mu \tag{16.54}$$

is gauge-invariant:

$$\begin{aligned} F_{\mu\nu} \longrightarrow & \partial_\mu (A_\nu + \partial_\nu \alpha) - \partial_\nu (A_\mu + \partial_\mu \alpha) && (16.55)\\ &= F_{\mu\nu} + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\alpha(x) && (16.56)\\ &= F_{\mu\nu} && (16.57)\end{aligned}$$

and if $\partial_\mu J^\mu(x) = 0$ holds, then

$$\mathcal{L} \longrightarrow \mathcal{L} \quad \checkmark\,. \tag{16.58}$$

16.4.2 Gauge-invariant derivative

The gauge field A^μ couples to conserved currents. These are the results of internal symmetries, e.g. a $U(1)$-symmetric charged scalar has the **global** symmetry

$$\Psi \longrightarrow e^{i\alpha} \Psi \tag{16.59}$$

$$\overline{\Psi} i \partial\!\!\!/ \Psi$$

is invariant under

$$\Psi \longrightarrow e^{i\alpha} \Psi\,.$$

How can we render interactions invariant?

$$\overline{\Psi} \gamma^\mu A_\mu \Psi \longrightarrow \overline{\Psi} \gamma^\mu A_\mu \Psi + \overline{\Psi} \gamma^\mu \Psi \partial_\mu \alpha(x) \tag{16.60}$$

can be made invariant if we substitute $\alpha \longrightarrow \alpha(x)$ in Eq. (16.59),

$$\Psi(x) \longrightarrow e^{i\alpha(x)} \Psi(x) \tag{16.61}$$

turning it into a **local** symmetry, and if we replace

$$\partial_\mu \longrightarrow D_\mu \quad \text{with} \quad D_\mu \Psi = \big(\partial_\mu - i A_\mu(x)\big)\Psi\,. \tag{16.62}$$

Now we have:

$$\begin{aligned} D_\mu \Psi \longrightarrow D'_\mu \Psi' &= \Big(\partial_\mu - i\big(A_\mu(x) + \partial_\mu \alpha(x)\big)\Big) e^{i\alpha(x)} \Psi && (16.63)\\ &= e^{i\alpha(x)} \Big(\partial_\mu + i\big(\partial_\mu \alpha(x)\big) - i A_\mu(x) - i\big(\partial_\mu \alpha(x)\big)\Big)\Psi\,, && (16.64)\end{aligned}$$

which means that it transforms as

$$D_\mu\Psi(x) \longrightarrow e^{i\alpha(x)} D_\mu\Psi(x)\,. \tag{16.65}$$

Therefore the Lagrangian is invariant, since it is invariant under global $U(1)$-symmetries, e.g.

$$\overline{\Psi}\not{D}\Psi \longrightarrow \overline{\Psi}e^{-i\alpha(x)}e^{i\alpha(x)}\not{D}\Psi = \overline{\Psi}\not{D}\Psi\,. \tag{16.66}$$

$\overline{\Psi} \longrightarrow (\Psi e^{i\alpha})^\dagger\gamma^0 = \overline{\Psi}e^{-i\alpha}$

Similarly for the scalar field ϕ:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\gamma^\mu D_\mu - m_\Psi)\Psi + (D_\mu\phi)^\dagger D^\mu\phi - \\ &- m_\phi^2\phi^\dagger\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2\end{aligned} \tag{16.67}$$

We can also introduce a **gauge coupling** g by redefining

$$A^\mu \longrightarrow gA^\mu\,, \quad D_\mu = \partial_\mu - igA_\mu\,, \tag{16.68}$$

as well as the transformation

$$A^\mu \longrightarrow A^\mu + \frac{1}{g}\partial^\mu\alpha(x) \tag{16.69}$$

and

$$\mathcal{L}_{\text{int}} = \underbrace{g\overline{\Psi}\gamma^\mu\Psi}_{=J^\mu} A_\mu\,, \tag{16.70}$$

where J^μ is, as expected, the Noether current of the global (i.e. $\alpha =$const.) transformations:

$$\Psi(x) \longrightarrow \Psi'(x) = e^{i\alpha}\Psi(x) \approx \Psi(x) + \underbrace{i\alpha\Psi(x)}_{=\alpha F(x)} + \ldots\,, \tag{16.71}$$

$$\Longrightarrow \quad j^\mu \underset{k^\mu\equiv 0}{\overset{\uparrow}{=}} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi(x))}F(x) = \overline{\Psi}i\gamma^\mu i\Psi = -\overline{\Psi}\gamma^\mu\Psi\,, \tag{16.72}$$

therefore we know that J^μ is conserved.
This differential operator is **not** invertible, since eigenvectors of the form $A_\nu(x) \sim \partial_\nu\lambda(x)$ have the eigenvalue zero.
Fixing initial data, we would not be able to determine A_μ uniquely, since both A_μ and $A_\mu + \partial_\mu\lambda(x)$ would both satisfy the EOM. However, if we are willing to **identify** A_μ and $A_\mu + \partial_\mu\lambda(x)$, this is not a problem!
Alternatively: we can choose a fixed gauge and perform a gauge transformation such that the equation is satisfied. Some possibilities are:

1) **Lorentz gauge:**

$$\partial_\mu A^\mu = 0\,. \tag{16.73}$$

It is manifestly Lorentz-invariant and leaves the free choice of those $\lambda(x)$ which satisfy$\partial_\mu\partial^\mu\lambda(x) = 0$.

2) **Coulomb gauge:**

$$\nabla\cdot\boldsymbol{A} = 0\,. \tag{16.74}$$

It is not manifestly Lorentz-invariant, but it allows the easiest counting of the degrees of freedom:

$$\underset{\underset{4}{\uparrow}}{A^{\mu}}\,, \quad \underset{\underset{-1}{\uparrow}}{A^0 = 0}\,, \quad \underset{\underset{-1}{\uparrow}}{\nabla \cdot \boldsymbol{A} = 0} \quad \Longrightarrow \quad 2 \text{ d.o.f.} \tag{16.75}$$

Gauge invariance for scalar fields requires the term

$$D_\mu \phi^\dagger D^\mu \phi \supset \phi^\dagger \phi A_\mu A^\mu \,. \tag{16.76}$$

16.4.3 Gauge fixing and degrees of freedom

We saw that for massive spin 1 particles the condition $\partial_\mu A^\mu = 0$ reduced the degrees of freedom from 4 to 3, whereas in the massless case we had only 2 degrees of freedom.
As before, A^0 is not a canonical coordinate and therefore $\pi^0 = 0$, but now we can solve the following:

$$\partial_\mu F^{\mu\nu} = 0\,, \quad \text{take } \nu = 0\,, \tag{16.77}$$

$$\Longrightarrow \quad \nabla^2 A^0 + \nabla \cdot \frac{\partial \boldsymbol{A}}{\partial t} = 0\,. \tag{16.78}$$

If we now fix $\boldsymbol{A}$, $\partial_t \boldsymbol{A}$ at some time t_0, then A^0 is fully fixed:

1)
$$A^{=}(\boldsymbol{x}) = \int \mathrm{d}^3 x' \frac{\nabla_{x'}}{4\pi|\boldsymbol{x} - \boldsymbol{x'}|} \cdot \frac{\partial \boldsymbol{A}(\boldsymbol{x'})}{\partial t}\,, \tag{16.79}$$
which one can check using the Green function method for $\nabla^2\Phi$, see [4], chapter 4.9.5. A^0 is not dynamical.

2) With a gauge transformation we can remove the additional component:
$$A^\mu \longrightarrow A^\mu + \partial^\mu \alpha(x)\,. \tag{16.80}$$

Note that the gauge redundancy is also visible in Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = 0 \quad \Longleftrightarrow \quad \big(\eta_{\mu\nu}(\partial_\lambda \partial^\lambda) - \partial_\mu \partial_\nu\big) A^\nu(x) = 0 \tag{16.81}$$

16.5 Quantum Electrodynamics (QED)

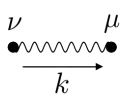
QED contains a massive spin $\frac{1}{2}$ electron and a massless spin1 photon. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \underset{\underset{\text{gauge fixing, see QFT 2}}{\uparrow}}{\frac{1}{2\xi}} \big(\partial_\mu A^\mu\big)^2 + \overline{\Psi}\big(i\gamma^\mu D_\mu - m\big)\Psi\,, \tag{16.82}$$

with

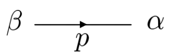
$$D_\mu = \partial_\mu - ieA_\mu\,, \tag{16.83}$$

e the electric charge of the electron and m the mass of the electron. The Feynman rules of QED are the following: the photon propagator is



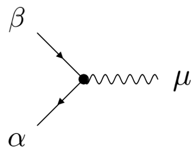
$$\frac{i}{k^2 + i\varepsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2} \right), \tag{16.84}$$

where we call the gauge with $\xi = 1$ Feynman gauge. The electron propagator is



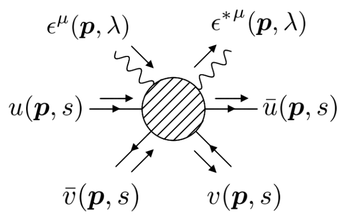
$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon} \tag{16.85}$$

and the electron-photon vertex factor is



$$+ i\gamma^{\mu}_{\alpha\beta} \,. \tag{16.86}$$

For the polarization vectors the rule is the following:



where all the lines on the left are incoming and the ones on the right are outgoing.

We will discuss an application of QED in QFT2 before moving on to gauge redundancy on steroids (aka Yang-Mills and gravity).

17
Outlook

This chapter should be familiar to you if you have read the RPF notes. It is even more applicable this time.

If you found this interesting you should continue and attend the QFT2 course in the next semester. You have yet to encounter many of the truly awesome and deep insights that quantum field theory contains. QFT is the language in which the laws of Nature are written. As Sidney Coleman said: *"Not only God knows, I know, and by the end of the semester, you will know."*

An anecdote about Coleman: Steve Weinberg was giving the gauge seminar one Wednesday and was, as I recall nearly finished or just finishing his talk. Sidney had not arrived. So-Young Pi asked Steve a question and Steve replied (as I recall) “That’s a good question. I don’t know the answer; I haven’t thought about it.” At that exact instant Sidney enters the room, hears Steve’s reply, heads for the coffee pot, and says “I know the answer; what’s the question?” He was told the question, and he answered it. (Correctly, of course.) – as told by Ken Lane.