

Wave Guides and Resonating Cavities

Lecture38: Electromagnetic Theory

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Resonating Cavity (contd.)

We have shown in the last lecture that the electric field components for $TE_{l,m,n}$ are given by

$$\begin{aligned}E_x &= E_{x0} \cos(k_x x) \sin(k_y y) \sin(k_z z) \\E_y &= E_{y0} \sin(k_x x) \cos(k_y y) \sin(k_z z) \\H_x &= \frac{1}{-i\omega\mu} \left(-\frac{\partial E_y}{\partial z} \right) = \frac{k_z}{i\omega\mu} E_{y0} \sin(k_x x) \cos(k_y y) \cos(k_z z) \\H_y &= \frac{1}{-i\omega\mu} \left(\frac{\partial E_x}{\partial z} \right) = -\frac{k_z}{i\omega\mu} E_{x0} \cos(k_x x) \sin(k_y y) \cos(k_z z) \\H_z &= \frac{1}{-i\omega\mu} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -\frac{1}{i\omega\mu} (E_{y0} k_x - E_{x0} k_y) \cos(k_x x) \cos(k_y y) \sin(k_z z)\end{aligned}$$

Let us consider $TE_{1,0,1}$ i.e., $l = n = 1, m = 0$. This implies that $k_y = 0$, leading to $E_x = 0$ and $H_y = 0$. Thus the only non-zero components are

$$\begin{aligned}E_y &= E_{y0} \sin(k_x x) \sin(k_z z) \\H_x &= \frac{k_z}{i\omega\mu} E_{y0} \sin(k_x x) \cos(k_z z) \\H_z &= -\frac{1}{i\omega\mu} E_{y0} k_x \cos(k_x x) \sin(k_z z) \equiv H_{z0} \cos(k_x x) \sin(k_z z)\end{aligned}$$

where

$$H_{z0} = \frac{ik_x}{\omega\mu} E_{y0} = \frac{i\pi}{\omega\mu a} E_{y0}$$

In terms of H_{z0} , the fields are rewritten as,

$$\begin{aligned}H_z &= H_{z0} \cos(k_x x) \sin(k_z z) \\H_x &= -\frac{k_z}{k_x} H_{z0} \sin(k_x x) \cos(k_z z) = -\frac{a}{d} H_{z0} \sin(k_x x) \cos(k_z z) \\E_y &= E_{y0} \sin(k_x x) \sin(k_z z) = \frac{\omega\mu a}{i\pi} H_{z0} \sin(k_x x) \sin(k_z z)\end{aligned}$$

Q- Factor of a Resonator

The Q-factor, a short form for Quality factor of a resonator is defined as the ratio of the amount of energy stored in the cavity and the amount of energy lost per cycle through the walls of the cavity. In the following we will calculate the Q factor for the $TE_{1,0,1}$ mode of the cavity.

The deterioration of the cavity is because of two factors, viz. the finite conductivity of the walls of the cavity and an imperfect dielectric in the space between. We will consider the space between the walls to be vacuum so that it is only the finite conductivity of the walls that we need to worry about. The conductivity, though not infinite, is nevertheless large so that the skin depth is small. The tangential component of the magnetic field will be assumed to be confined to a depth equal to the skin depth which leads to a surface current J_s .

Let us first calculate the stored energy. The average energy stored is given by

$$\langle W \rangle = \frac{\epsilon}{2} \int |E|^2 dV = \frac{\epsilon}{2} \int |E_y|^2 dV$$

since the only non-zero component of the electric field is along the y direction. Substituting the expression for E_y , we get

$$\langle W \rangle = \frac{\epsilon}{2} \left(\frac{\omega \mu a}{\pi} \right)^2 |H_{z0}|^2 \int_0^a dx \sin^2(k_x x) \int_0^d dz \sin^2(k_z z) \int_0^b dy$$

The first two integrals respectively give $\frac{a}{2}$ and $\frac{d}{2}$ while the last one gives b . We thus have,

$$\langle W \rangle = \frac{\epsilon}{2} \left(\frac{\omega \mu a}{\pi} \right)^2 |H_{z0}|^2 \frac{abd}{4} = \frac{\epsilon \mu^2 \omega^2 a^3 bd}{8\pi^2} |H_{z0}|^2 \quad (1)$$

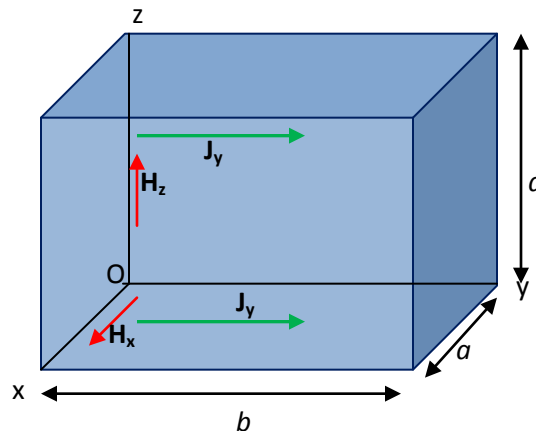
We will now calculate the loss through the walls of the wave guide.

We know that the surface current causes a discontinuity in the magnetic field. As the magnetic field decrease fast within the skin depth region, we will assume the field to be zero in this region and calculate the surface current from the relation

$$\vec{J}_s = \hat{n} \times \vec{H}_t$$

where \hat{n} is the unit normal pointing into the resonator.

For the rectangular parallelepiped we have three pairs of symmetric surfaces, contribution from each pair being the same. For instance we have a front face at $x = a$ and a back face at $x = 0$.



For both the front face the normal direction is along $-\hat{i}$ and for the back face it is along $+\hat{i}$ since it is inward into the cavity. The surface current density is thus given by

$$\vec{J}_s = (\mp \hat{i}) \times (H_y \hat{j} + H_z \hat{k}) = \pm H_z \hat{j}$$

since $H_y = 0$ in this mode. We are, however interested in $|\vec{J}_s|^2$ so that contribution from the two walls add up and we get the loss from these two sides as

$$\begin{aligned} \text{Loss}_1 &= 2 \times \frac{1}{2} R_s \int |J_s|^2 dy dz \\ &= R_s |H_{z0}|^2 \int_0^b dy \int_0^d \sin^2(k_z z) dz \\ &= \frac{1}{2} R_s |H_{z0}|^2 b d \end{aligned}$$

we can in a similar way calculate the contribution from the left ($y=0$) and the right ($y=b$) faces for which $\hat{n} = \pm \hat{j}$,

$$\vec{J}_s = (\pm \hat{j}) \times (H_x \hat{i} + H_z \hat{k}) = \mp H_x \hat{k} \pm H_z \hat{i}$$

so that

$$\begin{aligned} |J_s|^2 &= |H_x|^2 + |H_z|^2 \\ &= |H_{z0}|^2 \left[\left(\frac{a}{d}\right)^2 \sin^2(k_x x) \cos^2(k_z z) + \cos^2(k_x x) \sin^2(k_z z) \right] \end{aligned}$$

Substituting this and integrating, the loss from the two side walls becomes

$$\text{Loss}_2 = R_s |H_{z0}|^2 \left[\left(\frac{a}{d}\right)^2 \frac{ad}{4} + \frac{ad}{4} \right]$$

In a similar way the loss from the top and the bottom faces can be found to be

$$\text{Loss}_3 = R_s |H_{z0}|^2 \left[\left(\frac{a}{d}\right)^2 \frac{ab}{4} \right]$$

Adding, all the losses, the total loss becomes

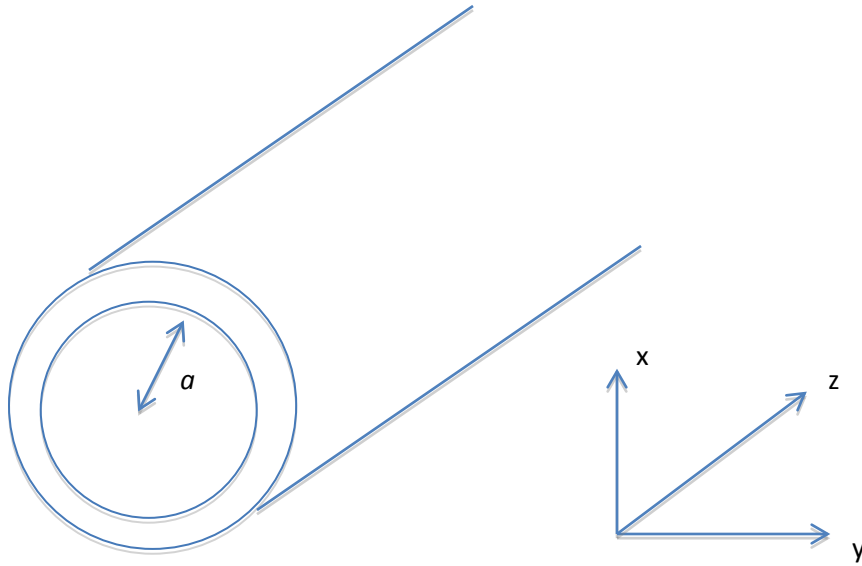
$$\text{Loss} = \frac{R_s |H_{z0}|^2}{4d^2} [d^3(2b+a) + a^3(2b+d)] \quad (2)$$

The Q factor is given by ω times the quotient of [1] with (2)

$$\begin{aligned} Q &= \omega \frac{\frac{\epsilon \mu^2 \omega^2 a^3 b d}{8\pi^2}}{\frac{R_s}{4d^2} [d^3(2b+a) + a^3(2b+d)]} \\ &= \omega^3 \frac{\epsilon \mu^2 a^3 b d^3 \sigma \delta}{2\pi^2 [d^3(2b+a) + a^3(2b+d)]} \end{aligned}$$

where we have substituted $R_s = \frac{1}{\sigma\delta}$. thus the quality factor can be determined from a knowledge of the dimension of the cavity, the operating frequency and the nature of the material of the walls.

Circular Wave Guides



The waveguide that we consider are those with circular cross section. The direction of propagation is still the z axis so that the geometry is cylindrical. This geometry is of great practical value as optical fibers used in communication have this geometry. However, optical fibres are dielectric wave guides whereas what we are going to discuss are essentially hollow metal tube of circular cross section.

We take the cylindrical coordinates to be (ρ, ϕ, z) and the radius of cross section to be a . We will first express the two curl equations in cylindrical.

As there are no real current, we have,

$$(\nabla \times \vec{H})_\rho = \frac{1}{\rho} \frac{\partial}{\partial \phi} H_z - \frac{\partial H_\phi}{\partial z} = i\omega\epsilon E_\rho$$

$$(\nabla \times \vec{H})_\phi = \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = i\omega\epsilon E_\phi$$

$$(\nabla \times \vec{H})_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) - \frac{1}{\rho} \frac{\partial}{\partial \phi} H_z = i\omega\epsilon E_z$$

As before, we will replace $\frac{\partial}{\partial z} \rightarrow -\gamma$ so that these equations become

$$\begin{aligned} \frac{1}{\rho} \frac{\partial}{\partial \phi} H_z + \gamma H_\phi &= i\omega\epsilon E_\rho \\ -\gamma H_\rho - \frac{\partial H_z}{\partial \rho} &= i\omega\epsilon E_\phi \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho H_\phi) - \frac{1}{\rho} \frac{\partial}{\partial \phi} H_z &= i\omega\epsilon E_z \end{aligned}$$

The other set of the curl equations are obtained from Faraday's law and can be easily written down from the above set by replacing ϵ with $-\mu$ and interchanging E and H.

We get,

$$\begin{aligned} \frac{1}{\rho} \frac{\partial}{\partial \phi} E_z + \gamma E_\phi &= -i\omega\mu H_\rho \\ -\gamma H_\rho - \frac{\partial E_z}{\partial \rho} &= -i\omega\mu H_\phi \\ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\phi) - \frac{1}{\rho} \frac{\partial}{\partial \phi} E_z &= -i\omega\mu H_z \end{aligned}$$

As in the case of rectangular wave guides, we can still classify the modes as TE or TM. Consider one pair of the above equations,

$$\begin{aligned} i\omega\epsilon E_\rho &= \frac{1}{\rho} \frac{\partial}{\partial \phi} H_z + \gamma H_\phi \\ i\omega\mu H_\phi &= \gamma H_\rho + \frac{\partial E_z}{\partial \rho} \end{aligned}$$

we can eliminate H_ϕ from these two equations, and express E_ρ as

$$(\gamma^2 + \epsilon\mu\omega^2)E_\rho = -\frac{i\omega\mu}{\rho} \frac{\partial}{\partial \phi} H_z - \gamma \frac{\partial E_z}{\partial \rho}$$

Define, $\gamma^2 + \epsilon\mu\omega^2 = k^2$ to rewrite this equation as

$$k^2 E_\rho = -\frac{i\omega\mu}{\rho} \frac{\partial}{\partial \phi} H_z - \gamma \frac{\partial E_z}{\partial \rho}$$

Thus we have succeeded in expressing E_ρ in terms of derivatives of the z component of E and H. In a similar way we can express all the four components.

$$k^2 E_\rho = -\gamma \frac{\partial E_z}{\partial \rho} - \frac{i\omega\mu}{\rho} \frac{\partial}{\partial \phi} H_z \quad (1)$$

$$k^2 E_\phi = -\frac{\gamma}{\rho} \frac{\partial}{\partial \phi} E_z + i\mu\omega \frac{\partial H_z}{\partial \rho} \quad (2)$$

$$k^2 H_\rho = \frac{i\omega\epsilon}{\rho} \frac{\partial}{\partial \phi} E_z - \gamma \frac{\partial H_z}{\partial \rho} \quad (3)$$

$$k^2 H_\phi = -i\mu\epsilon \frac{\partial E_z}{\partial \rho} - \frac{\gamma}{\rho} \frac{\partial}{\partial \phi} H_z \quad (4)$$

$$k^2 = \gamma^2 + \epsilon\mu\omega^2$$

We will now obtain solutions for E_z and H_z using Helmholtz equation. In the following we will discuss the TE modes for which $E_z = 0$. We need to then find H_z using

$$\nabla^2 H_z = -\omega^2 \mu \epsilon H_z$$

Writing the Laplacian in cylindrical coordinates,

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} H_z \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} H_z + \frac{\partial^2}{\partial z^2} H_z = -\omega^2 \mu \epsilon H_z$$

We use technique of separation of variables by defining,

$$H_z(\rho, \phi, z) = R(\rho)F(\phi)Z(z)$$

Substituting this into the Helmholtz equation and dividing throughout by $R(\rho)F(\phi)Z(z)$, we get,

$$\frac{1}{R} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} R \right) + \frac{1}{F} \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} F + \omega^2 \mu \epsilon = -\frac{1}{Z} \frac{\partial^2}{\partial z^2} Z$$

The left hand side of this equation is a function of (ρ, ϕ) while the right hand side is a function of z alone. Thus we can equate each side to a constant. Anticipating propagation along the z direction, we equate the right hand side to $-\gamma^2$ so that, we have,

$$\begin{aligned} \frac{\partial^2}{\partial z^2} Z &= -\gamma^2 Z \\ \frac{1}{R} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} R \right) + \frac{1}{F} \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} F + (\omega^2 \mu \epsilon + \gamma^2) &= 0 \end{aligned}$$

The last equation needs to be separated into a term involving ρ alone and another depending on ϕ alone. This is done by multiplying the equation by ρ^2 ,

$$\frac{1}{R} \rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} R \right) + (\omega^2 \mu \epsilon + \gamma^2) \rho^2 = -\frac{1}{F} \frac{\partial^2}{\partial \phi^2} F$$

Once again, we equate each of the terms to a constant n^2 . We thus have the following pair of equations

$$\begin{aligned} \frac{\partial^2}{\partial \phi^2} F + n^2 F &= 0 \\ \rho \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} R \right) + k_\rho^2 \rho^2 R - n^2 R &= 0 \end{aligned}$$

where $k_\rho^2 = \omega^2 \mu \epsilon + \gamma^2$. The former equation has the solution

$$F = A \cos(n\phi) + B \sin(n\phi)$$

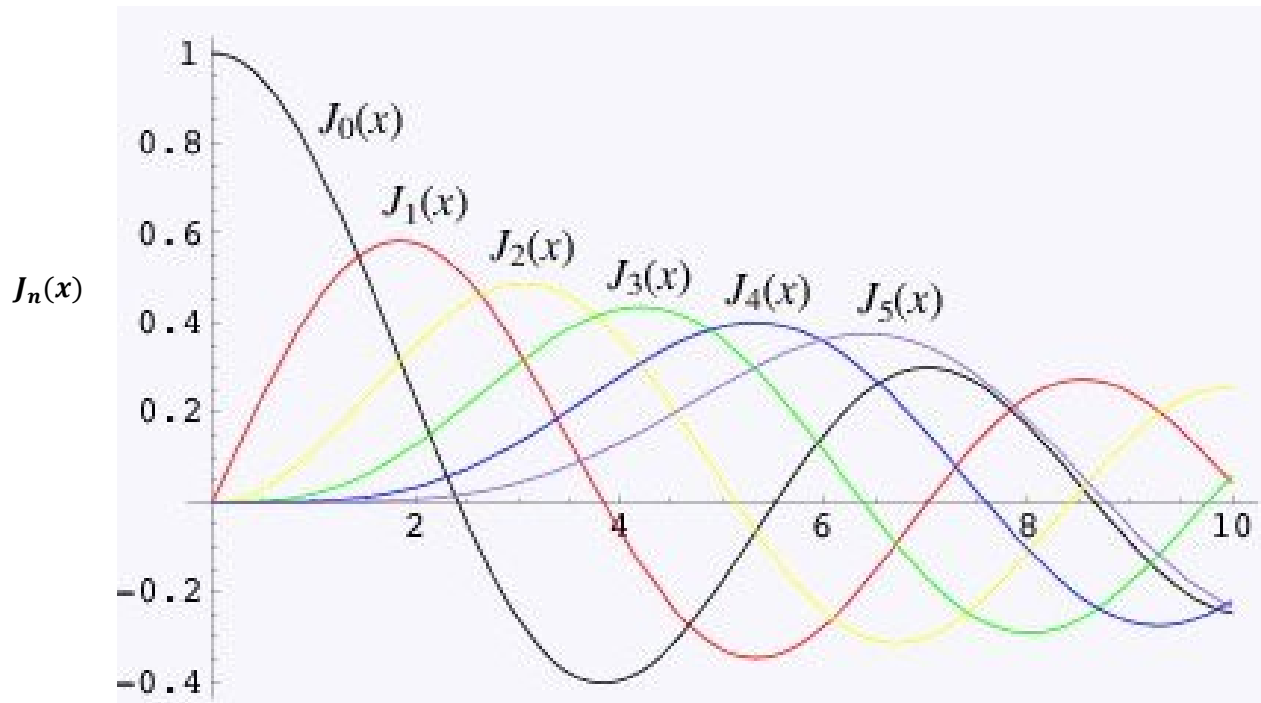
Singlevaluedness of F requires that if ϕ changes by 2π , the solution must remain the same. This, in turn, requires that n is an integer. We are now left with only the last equation which can be written in an expanded form,

$$\rho^2 \frac{\partial^2}{\partial \rho^2} R + \rho \frac{\partial}{\partial \rho} R + k_\rho^2 \rho^2 R - n^2 R = 0$$

Defining a new variable $x = k_\rho \rho$ (not to be confused with the Cartesian variable x), the equation can be written as

$$\frac{d^2}{dx^2} y + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2} \right) y = 0$$

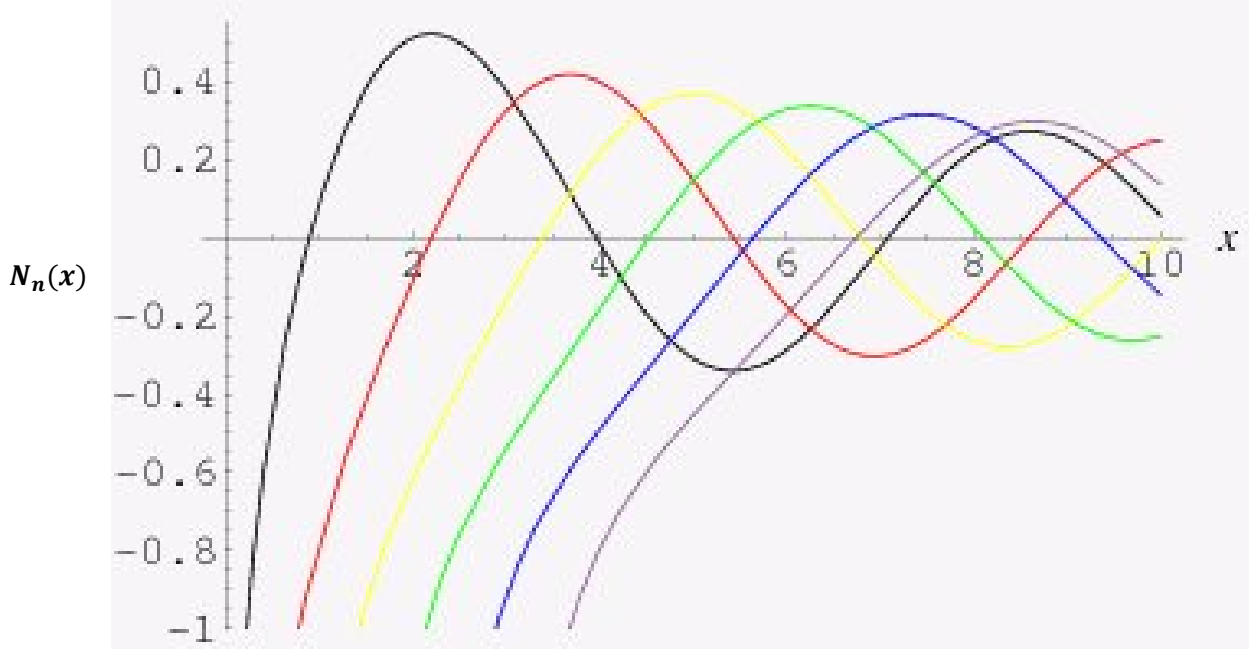
where, we have written y in place of R . This equation is the well known Bessel equation, the solutions of which are linear combinations of Bessel functions of first kind $J_n(x)$ and that of the second kind $N_n(x)$. The Bessel functions of second kind are also known as Neumann function. The following graph shows the variation of these functions with distance x .



It is seen that the Neumann function diverges at the origin and hence is not an acceptable solution. The asymptotic behavior of these functions are

$$J_n(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right)$$

$$N_n(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4}\right)$$



The complete solution is thus given by

$$E_z(\rho, \phi, z) = J_n(k_\rho \rho)(A_n \cos(n\phi) + B_n \sin(n\phi))e^{-\gamma z}$$

$$H_z(\rho, \phi, z) = J_n(k_\rho \rho)(C_n \cos(n\phi) + D_n \sin(n\phi))e^{-\gamma z}$$

For TE modes,

$$E_z = 0, \left. \frac{\partial H_z}{\partial n} \right|_{surface} = 0$$

For TM modes,

$$H_z = 0, E_z|_{surface} = 0$$

Let us look at the TE mode in a little more detail. The tangential component of the electric field must be zero at the metallic boundary. Thus $E_\phi = 0$ at $\rho = a$. Substituting the expression for E_ϕ (with $E_z = 0$) we require, from Eqn. (2)

$$E_\phi = \frac{i\mu\omega}{k_\rho^2} \frac{\partial H_z}{\partial \rho} = \frac{i\mu\omega}{k_\rho^2} J'_n(k_\rho \rho)(A_n \cos(n\phi) + B_n \sin(n\phi))e^{-\gamma z}$$

where J'_n is the derivative of Bessel function with respect to ρ . This expression vanishes at $\rho = a$ provided

$$J'_n(k_\rho a) = 0$$

which provides a restriction on propagation of TE mode. The zeros of the Bessel function J_n are listed in the following table, (m-th zero of J_n is p_{nm})

n	m=1	m=2	m=3
0	2.4048	5.5201	8.6537
1	3.8317	7.0156	10.1735
2	5.1356	8.4172	11.6198

The zeros of the derivative of the Bessel function are listed in the following table (m-th zero of J_n is p_{nm}'),

n	m=1	m=2	m=3
0	3.8317	7.0156	10.1735
1	1.8412	5.3314	8.5363
2	3.0542	6.7061	9.9695

For propagation to take place, $\gamma = \sqrt{k_p^2 - \omega^2 \mu \epsilon}$ must be imaginary. This is possible if $\omega > \omega_c$, where

$$\omega_c = \frac{1}{\sqrt{\mu \epsilon}} k_p = \frac{1}{\sqrt{\mu \epsilon}} \frac{p_{nm}'}{a}$$

where p_{nm}' gives the m-th zero of the derivative of Bessel function. The corresponding mode is classified as TE_{nm} . Similarly, one can show that the cutoff frequency for the TM modes are given by

$$\omega_c = \frac{1}{\sqrt{\mu \epsilon}} \frac{p_{nm}}{a}$$

From the tables given above, it is clear that the lowest mode is TE_{11} followed by TM_{01} .

The field pattern of the waveguide are shown in the following (source – Web)

Wave Type	TM_{01}	TM_{02}	TM_{11}	TE_{01}	TE_{11}
Field distributions in cross-sectional plane, at plane of maximum transverse fields					
Field distributions along guide					
Field components present	E_z, E_r, H_ϕ	E_z, E_r, H_ϕ	$E_z, E_r, E_\phi, H_r, H_\phi$	H_z, H_r, E_ϕ	$H_z, H_r, H_\phi, E_r, E_\phi$

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Tutorial Assignment

1. TE_{10} mode propagates in an air filled rectangular waveguide of dimension $1\text{cm} \times 0.5\text{cm}$. The frequency of the propagating wave is 3×10^{12} Hz. It is desired to construct a cavity out of this waveguide by closing the third dimension so that the cavity resonates in TE_{102} mode. (a) What should be the length of the cavity? (b) Write down the electric field for the resonating mode.
2. For a cubical cavity, what is the degeneracy of the lowest resonant frequency, i.e. how many different modes correspond to the lowest possible frequency?
3. A circular (cylindrical) waveguide of radius 4 cm is filled with a material of dielectric constant 2.25. The guide is operated at a frequency of 2 GHz. For the dominant TE mode, determine the cutoff frequency, the guide wavelength and the bandwidth for a single mode operation (assuming only TE modes).
4. If the cylindrical cavity of circular cross section is closed at both ends by perfectly conducting discs, it becomes a cavity resonator. Find the resonant frequencies of a cavity resonator of length d and radius a .

Solutions to Tutorial Assignments

1. For $l = 1, m = 0, n = 2$, the resonant frequency is given by

$$\omega = 2\pi\nu = c \left[\left(\frac{\pi}{a} \right)^2 + \left(\frac{2\pi}{d} \right)^2 \right]^{1/2}$$

Substituting $a = 0.01$, and the value of the frequency, we get, $d = 1.15$ cm. With these values, we have $k_x = \frac{\pi}{a} = 100\pi$, $k_y = 0$, $k_z = 100\sqrt{3}\pi$ (m^{-1}). The non-vanishing field components are

$$E_y = E_{y0} \sin(k_x x) \sin(k_z z) \cos(\omega t)$$

The other two non-vanishing field components can be calculated by using Maxwell's equations,

$$\frac{\partial E_y}{\partial z} = \mu \frac{\partial H_x}{\partial t}, \quad \frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t}$$

$$H_x = \left(\frac{k_z}{\omega\mu} \right) E_{y0} \sin(k_x x) \cos(k_z z) \sin(\omega t)$$

$$H_z = \left(\frac{k_x}{\omega\mu} \right) E_{y0} \cos(k_x x) \sin(k_z z) \sin(\omega t)$$

2. Since all the three dimensions are the same, we would have the same frequency for (100), (010) and (001) modes. Counting TE and TM, there are 6 modes corresponding to the lowest frequency.
3. For TE_{mn} mode the cutoff frequency is given by

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}} \frac{p'_{nm}}{a}$$

For the dominant mode $m=n=1$, $p'_{nm} = 1.8412$. Substituting $\frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\sqrt{2.25}} = 2 \times 10^8$ m/s, and $a=0.04$ m, we get $\omega_c = 9.206 \times 10^9$ rad/s corresponding to 1.47 GHz. To find the guide wavelength, note that the propagation vector

$$\beta = \frac{2\pi}{\lambda} = 2\pi\sqrt{\mu\epsilon}\sqrt{v^2 - v_c^2} = \frac{2\pi}{2 \times 10^8} \sqrt{4 \times 10^{18} - (1.47 \times 10^9)^2}. \text{ Solving, we get } \lambda = 0.14 \text{ m.}$$

To find the bandwidth of single mode TE operation note that since the cutoff frequency for the next higher mode TE_{21} is $3.0542/1.8412=1.66$ times the cutoff frequency for TE_{11} mode, the bandwidth is fixed.

4. If the cavity resonator is closed at both ends, there would be standing waves formed along the z direction. Since the tangential component of the electric field is continuous (and therefore vanishes near the surface of the perfect conductor,) we must have $\frac{\partial E_z}{\partial z} = 0$ at $z=0$ and $z=d$. Taking the origin at the centre of one of the disks, for TM mode, E_z is given by

$$E_z = E_0 J_n(k_\rho \rho) (A \sin n\phi + B \cos n\phi) \cos \frac{l\pi z}{d}$$

Since $k_\rho^2 + k_z^2 = \omega^2 \mu\epsilon$, the resonant frequencies for the TM modes are

$$\omega = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{1.8412}{a} \right)^2 + \left(\frac{\pi}{d} \right)^2}$$

Note that unlike the dominant frequency of the TM mode which is fixed by the dimension a , the TE_{111} frequency can be tuned by adjusting the value of the length of the resonator. If d is made sufficiently large, the frequency can be made to be lower than that of the TM mode.

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Self Assessment Questions

1. Consider TE_{011} mode in a cubical cavity of side a . Write the expressions for the electric and magnetic fields and show that they are orthogonal.
2. For the above problem determine the total energy stored in the electric and magnetic fields.
3. A cubical cavity of side 3 cm has copper walls (conductivity $\sigma = 6 \times 10^7$ S/m). Calculate the Q factor of the cavity for TE_{101} mode.
4. Consider a cylindrical cavity resonator. Show that the dominant frequency is TE mode of the resonator.

Solutions to Self Assessment Questions

1. By definition of TE mode, $E_z=0$. Further, for 011 modes, $k_x = 0, k_y = k_z = \frac{2\pi}{a}$. This gives $E_y = 0$. We have,

$$E_x = Rl E_0 \sin(k_y y) \sin(k_z z) e^{i\omega t} = E_0 \sin(k_y y) \sin(k_z z) \cos(\omega t)$$

$$H_y = Rl \left(\frac{1}{-i\omega\mu} \frac{\partial E_x}{\partial z} \right) = \frac{E_0 k_z}{\omega\mu} \sin(k_y y) \cos(k_z z) \sin(\omega t)$$

$$H_x = 0$$

$$H_z = Rl \left(\frac{1}{i\omega\mu} \frac{\partial E_x}{\partial y} \right) = \frac{E_0 k_y}{\omega\mu} \cos(k_y y) \sin(k_z z) \sin(\omega t)$$

2. Total energy in the electric field is obtained by

$$\begin{aligned} & \frac{1}{2} \epsilon E_0^2 \cos^2(\omega t) \int_0^a dx \int_0^a \sin^2(k_y y) dy \int_0^a \sin^2(k_z z) dz \\ &= \frac{1}{8} \epsilon_0 E_0^2 a^3 \cos^2(\omega t) \end{aligned}$$

One can similarly find the magnetic field energy from $\frac{\mu}{2} \int H^2 dV$. This gives,

$$\frac{1}{8} \frac{(k_y^2 + k_z^2)}{\mu\omega^2} E_0^2 a^3 \sin^2(\omega t)$$

Using the relation $k_y^2 + k_z^2 = \omega^2/c^2$, this expression can be written as $\frac{1}{8} \epsilon_0 E_0^2 a^3 \sin^2(\omega t)$. So that the total energy in the field is $\frac{1}{8} \epsilon_0 E_0^2 a^3$.

3. The resonant frequency of the cavity for TE₁₀₁ mode is given by $\omega = \pi c \sqrt{2/a^2} = 4.44 \times 10^{10}$ rad/s. Using the expression given in the text, the Q factor can be written as follows for the case of $a = b = c$.

$$Q = \frac{\omega^3 \epsilon \mu^2 \sigma \delta a^3}{24\pi^2}$$

Equivalent expressions for Q factor can be written using $R_s = \frac{1}{\sigma\delta}$ and the preceding expression for ω as

$$Q = \frac{\omega \mu a}{6R_s}$$

Where $R_s = \sqrt{\frac{\omega\mu}{\sigma}} = 3 \times 10^{-2}$. Substituting these values we get $Q \approx 9300$.

4. For the TE mode of the cavity, B_z must vanish at the surface of the end faces. The z-component of the magnetic field is, therefore, given by

$$B_z = B_0 J_n(k_\rho \rho) (A \sin n\varphi + B \cos n\varphi) \sin \frac{l\pi z}{d}$$

The resonant frequencies are given by

$$\omega = \frac{1}{\sqrt{\mu\epsilon}} \sqrt{\left(\frac{p'_{mn}}{a}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$

where p'_{mn} are the zeros of the derivatives of Bessel functions. Note that unlike in the case of TM modes, in this case $l \neq 0$ because that would make the fields vanish. For TM modes, the dominant frequency is TM_{010} , for which the frequency is $\omega = \frac{c}{a} p_{01} = 2.4048 \frac{c}{a}$. The dominant mode for TE is TE_{111} for which the frequency is given by

$$\omega = c \sqrt{\left(\frac{p'_{mn}}{a}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$