# Radiation

Lecture 39: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

So far we have discussed propagation of electromagnetic waves both in free space and in waveguides without worrying about how they are produced. In the following two lectures we will discuss the production of electromagnetic waves. Once produced they carry their energy and momentum and propagate in free space, may be to be received by an antenna. An antenna is a physical device which transmits or receives electromagnetic waves.

What causes a source to emit radiation? We have seen that charges at rest creates electrostatic field. A steady current, on the other hand is a source of magnetic field. Neither of these lead to radiation. However, it is known that accelerating charges emit radiation.

To discuss this in some detail, we fall back on our workhorse, viz., the Maxwell's equations. Let us recall the potential formulation of the electromagnetic field that we had discussed earlier.

Using the expression for Faraday's law,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We rewrite this in terms of vector potential $\vec{A}$,

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

This enablesus to express the quantity within the bracket as gradient of a scalar function V,

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Using Lorentz gauge condition,

$$\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

the Maxwell's equations can be expressed as a pair of homogeneous wave equations,

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \quad (1)$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} \qquad (2)$$

Since the structure of both equations is same, we can solve either of them and write the solution to the remaining equation by similarity. We will solve the equation for the vector potential. The method of solution depends on defining a "**Green's function**".

Let us define a time Fourier transform through the equation,

$$\vec{A}(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{A}(\vec{x},\omega)\, e^{i\omega t}\, d\omega$$

the inverse of which is

$$\vec{A}(\vec{x},\omega) = \int_{-\infty}^{\infty} \vec{A}(\vec{x},t)\, e^{-i\omega t}\, dt$$

Substituting in Eqn. (2)

$$\begin{aligned}\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{A}(\vec{x},\omega)\, e^{i\omega t}\, d\omega \\ &= -\mu_0 \vec{J}(\vec{x},t) \\ &= -\frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} J(\vec{x},\omega)\, e^{i\omega t}\, d\omega\end{aligned}$$

Taking inverse Fourier transform for both sides, we get

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{A}(\vec{x},\omega) = -\mu_0 J(\vec{x},\omega)$$

Defining $k = \frac{\omega}{c}$, we get,

$$(\nabla^2 + k^2)\vec{A}(\vec{x},\omega) = -\mu_0 J(\vec{x},\omega)$$

To obtain the solution of this equation, we define a subsidiary equation for a function known as the Green's function

$$(\nabla^2 + k^2)G(\vec{x} - \vec{x'}) = -4\pi\delta^3(\vec{x} - \vec{x'})$$

Thus the Green's function satisfies a very similar equation in which the inhomogeneous term of the original equation has been replaced by a delta function. In terms of the Green's function the formal solution of the equation for the vector potential can be written as a convolution,

$$\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int G\left(\vec{x} - \vec{x'}\right)\vec{J}\left(\vec{x'},\omega\right)d^3x'$$

To see how it comes about,

$$(\nabla^2 + k^2)\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int (\nabla^2 + k^2)G(\vec{x} - \overrightarrow{x'})\vec{J}(\overrightarrow{x'},\omega)d^3x'$$
$$= \frac{\mu_0}{4\pi}\int -4\pi\delta^3(\vec{x} - \overrightarrow{x'})\vec{J}(\overrightarrow{x'},\omega)d^3x' = -\mu_0 J(\vec{x},\omega)$$

Remember that in the above, the primed coordinate $\overrightarrow{x'}$ is for the source and the unprimed quantity refers to the point where the field is being evaluated. The solution must be spherically symmetric about the source point, i.e. depend on $R = \left|\vec{x} - \overrightarrow{x'}\right|$. With this substitution, the equation for the Green's function becomes,

$$(\nabla^2 + k^2)G(R) = -4\pi\delta^3(R)$$

First consider $R \neq 0$, for which the equation becomes

$$(\nabla^2 + k^2)G(R) = 0$$

Expressing the Laplacian in spherical coordinates, as there is no dependence on angles,

$$\frac{1}{R}\frac{d^2}{dR^2}(RG) + k^2G = 0$$

the solution for which is seen to be

$$RG \sim e^{\pm ikR}$$

so that the Green's function, excepting at the origin has the solution as a linear combination of an outgoing wave and an incoming wave.

$$G = A\frac{e^{ikR}}{R} + B\,\frac{e^{-ikR}}{R}$$

What about the solution at the origin?

Let us take a small sphere of radius $R_0$ around the origin. Since the argument of the delta function is included within the sphere, we have,

$$\int_{R_0}(\nabla^2 + k^2)G(R)d^3R = -4\pi\int_{R_0}\delta^3(R)d^3R = -4\pi$$

Since $G(R)$ is singular only at the origin, we can calculate the integral over it within the sphere, by replacing it by the expression obtained for it except at the origin,

$$\int k^2G(R)d^3R = \int k^2\left(\frac{A}{R} + \frac{B}{R}\right)d^3R = 4\pi k^2(A+B)\int_0^{R_0}\frac{1}{R}R^2dR = 4\pi(A+B)k^2\frac{R_0^2}{2} \to 0$$

as $R_0 \to 0$.

Thus,

$$\int_{R_0} \nabla^2 G(R) d^3R = -4\pi$$

As $R_0 \to 0$, $G = \frac{A+B}{R}$, so that

$$\nabla^2 G(R) = (A+B)\nabla^2 \frac{1}{R} = -4\pi(A+B)\delta^3(R)$$

Substituting this into the preceding integral, we get

$$A + B = 1$$

Thus we have,

$$G(\vec{x} - \vec{x'}) = \begin{cases} \dfrac{e^{ik\left|\vec{x}-\vec{x'}\right|}}{\left|\vec{x}-\vec{x'}\right|} & \text{outgoing wave} \\ \dfrac{e^{-ik\left|\vec{x}-\vec{x'}\right|}}{\left|\vec{x}-\vec{x'}\right|} & \text{incoming wave} \end{cases}$$

We will take the outgoing solution and proceed further.

$$\vec{A}(\vec{x}, \omega) = \frac{\mu_0}{4\pi} \int G(\vec{x} - \vec{x'}) \vec{J}(\vec{x'}, \omega) d^3x'$$
$$= \frac{\mu_0}{4\pi} \int \frac{e^{ik\left|\vec{x}-\vec{x'}\right|}}{\left|\vec{x}-\vec{x'}\right|} \int_{-\infty}^{\infty} \vec{J}(\vec{x'}, t') e^{i\omega t'} dt' d^3x'$$

Let us take the inverse Fourier transform of the above to get $\vec{A}(\vec{x}, t)$. It may, however, be remembered that $k = \frac{\omega}{c}$ has $\omega$ dependence,

$$\vec{A}(\vec{x}, t) = \frac{1}{2\pi} \frac{\mu_0}{4\pi} \int \frac{d^3x'}{\left|\vec{x}-\vec{x'}\right|} \int_{-\infty}^{\infty} dt' J(\vec{x'}, t') \int_{-\infty}^{\infty} e^{i\omega t'} e^{\frac{i\omega}{c}\left|\vec{x}-\vec{x'}\right|} e^{-i\omega t} d\omega$$
$$= \frac{1}{2\pi} \frac{\mu_0}{4\pi} \int \frac{d^3x'}{\left|\vec{x}-\vec{x'}\right|} \int_{-\infty}^{\infty} dt' J(\vec{x'}, t') 2\pi\delta\left(t' - t + \frac{\omega}{c}\left|\vec{x}-\vec{x'}\right|\right)$$
$$= \frac{\mu_0}{4\pi} \int \frac{d^3x'}{\left|\vec{x}-\vec{x'}\right|} J\left(\vec{x'}, t - \frac{\omega}{c}\left|\vec{x}-\vec{x'}\right|\right)$$

Parallely, one can write, for the scalar potential

$$V(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{d^3x'}{\left|\vec{x}-\vec{x'}\right|} \rho\left(\vec{x'}, t - \frac{\omega}{c}\left|\vec{x}-\vec{x'}\right|\right)$$

Note that the potential at time t at the position where it is being calculated is given in terms of current in the source at an "earlier time" $t - \frac{\omega}{c}\left|\vec{x} - \overrightarrow{x'}\right|$. The potential is influenced by the wave which came from the source at a time which is earlier by the time taken by the electromagnetic wave to travel this distance. This is known as the "retarded" potential. The other solution is the incoming wave solution which would give to an "advanced" solution.

**Example : Localized oscillation of charge or current**

Let us consider a localized source of charge or current, given by

$$\vec{J}\left(\overrightarrow{x'}, t'\right) = \vec{J}\left(\overrightarrow{x'}\right)e^{-i\omega t'}$$

$$\rho\left(\overrightarrow{x'}, t'\right) = \rho\left(\overrightarrow{x'}\right)e^{-i\omega t'}$$

$$\vec{A}(\vec{x}, t) = \frac{\mu_0}{4\pi}\int \frac{d^3x'}{\left|\vec{x} - \overrightarrow{x'}\right|} J\left(\overrightarrow{x'}, t - \frac{\omega}{c}\left|\vec{x} - \overrightarrow{x'}\right|\right)$$

$$= \frac{\mu_0}{4\pi}\int \frac{d^3x'}{\left|\vec{x} - \overrightarrow{x'}\right|}\vec{J}\left(\overrightarrow{x'}\right)\exp\left(-i\omega\left(t - \frac{\omega}{c}\left|\vec{x} - \overrightarrow{x'}\right|\right)\right)$$

$$= \frac{\mu_0}{4\pi}e^{-i\omega t}\int \frac{e^{ik\left|\vec{x} - \overrightarrow{x'}\right|}}{\left|\vec{x} - \overrightarrow{x'}\right|}\vec{J}\left(\overrightarrow{x'}\right)d^3x'$$

Thus we have,

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{e^{ik\left|\vec{x} - \overrightarrow{x'}\right|}}{\left|\vec{x} - \overrightarrow{x'}\right|}\vec{J}\left(\overrightarrow{x'}\right)d^3x'$$

and, similarly,

$$V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{e^{ik\left|\vec{x} - \overrightarrow{x'}\right|}}{\left|\vec{x} - \overrightarrow{x'}\right|}\rho\left(\overrightarrow{x'}\right)d^3x'$$

where$k = \frac{\omega}{c}$.

There are three regions of interest, assuming that the source is confined to a small region of space with typical dimension d. The distance at which the field is being measured is R= $\left|\vec{x} - \overrightarrow{x'}\right|$ and the wavelength of the radiation is $\lambda$.

1. Near field for which $d \ll R \ll \lambda$
2. Intermediate zone for which $d \ll R \approx \lambda$
3. Radiation zone or Far field with $d \ll \lambda \ll R$

Our primary interest will be in the radiation zone, which we will discuss in detail in the next lecture. We will have very little to say about the intermediate zone and its analysis is very complicated.

Consider the near field for which $d \ll R \ll \lambda$. In this region $kR = \frac{2\pi R}{\lambda} \ll 1$ as a result of which we can replace $e^{ik\left|\vec{x}-\vec{x'}\right|} \approx 1$. The expression for the vector potential becomes

$$A(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{1}{\left|\vec{x}-\vec{x'}\right|}\vec{J}\left(\vec{x'}\right)d^3x'$$

which is the familiar expression for the vector potential in magnetostatics. The only difference from the magnetostatic case is the time variation of $\vec{A}(\vec{x},t)$ which is simply to multiply the expressions obtained in the static case by $e^{-i\omega t}$. Thus we can say that the situation is "quasi-static" and we can borrow all our results from magnetostatics to discuss this region.

# Radiation

Lecture 39: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

**Tutorial Assignment**

1. Consider a thought experiment in which a charge q appears abruptly at the origin at time $t = t_1$ and disappears equally abruptly at time $t = t_2$. Find the retarded potentials for the problem. Determine the corresponding fields.
2. A long straight wire lying along the z axis carries a current given by

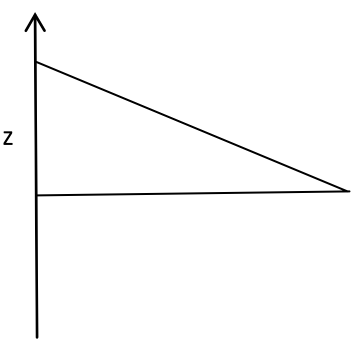
$$I(t) = \begin{cases} 0 & (t \le 0) \\ I_0 t & (t > 0) \end{cases}$$

Determine the retarded potentials and the electric and the magnetic fields.

**Solutions to Tutorial Assignments**

1. Since the charge is static only retarded scalar potential is generated. For time $t < t_1 + \frac{x}{c}$ the information that a charge has appeared at the origin has not reached the position x. Thus the potential is zero for such time. Likewise, though the charge has disappeared at time $t_2$, this information would not reach the position x till a later time $t_2 + \frac{x}{c}$. Thus the potential is $V(\vec{x}) = \frac{1}{4\pi\epsilon_0}\frac{q}{x}$ for $t_1 < t - \frac{x}{c} < t_2$ and zero at other times. The corresponding field is $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$ in the same time interval and is zero at other times.
2. The wire being charge neutral, the scalar potential is zero. The vector potential, because of cylindrical symmetry depends only on the distance *s* of the point of observation P from the wire, and is given by

$$\vec{A}(s,t) = \frac{\mu_0}{4\pi}\hat{z}\int \frac{I(t')}{r}dz$$




Consider an element dz at a distance z form the origin (see figure). Information about the current element would reach the point P at a time $\frac{r}{c} = \frac{\sqrt{z^2+s^2}}{c}$ later than the time the current was actually established. Thus the current, as seen by the point P is given by

$$I(t') = \begin{cases} 0 & (t' \le 0) \\ I_0 t' & (t' > 0) \end{cases}$$

where $t' = t - \frac{r}{c} = t - \frac{\sqrt{z^2+s^2}}{c}$. Now let us look at the range of integration. For an element dz located at a distance z from the origin to influence the field at the point P, its distance from P should be less than ct. Thus, $\sqrt{z^2+s^2} < ct$ . Hence the potential is given by

$$\vec{A}(s,t) = 2\frac{\mu_0}{4\pi}I_0\hat{z}\int_0^{\sqrt{c^2t^2-s^2}} \frac{t - \frac{\sqrt{z^2+s^2}}{c}}{\sqrt{z^2+s^2}}dz$$

(the factor of 2 in front is because only the upper half of the wire is considered in the integration). The integral is straightforward and gives,

$$\vec{A}(s,t) = \frac{\mu_0}{2\pi} I_0 \hat{z}\left[t \ln\left(\frac{ct + \sqrt{c^2t^2 - s^2}}{s}\right) - \frac{\sqrt{c^2t^2 - s^2}}{c}\right]$$

The electric field is given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{2\pi} I_0 \hat{z} \ln\left(\frac{ct + \sqrt{c^2t^2 - s^2}}{s}\right)$$

and the magnetic field is given by

$$\vec{B} = \nabla \times \vec{A} = \frac{\partial A_z}{\partial s}\hat{\varphi} = \frac{\mu_0}{2\pi s c} I_0 \sqrt{c^2t^2 - s^2}\,\hat{\varphi}$$

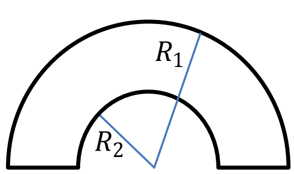
# Radiation

Lecture 39: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

**Self Assessment Questions**

1. An long straight wire lies along the z direction. At time $t = 0$, a constant current $I_0$ is switched on. Calculate the vector potential and the electric and the magnetic fields.
2. A conducting wire is bent into a loop of two semicircles, one of radius $R_1$ and the other $R_2, (< R_1)$connected by straight segments. The loop carries a current $I(t) = I_0 t$, which is directed anticlockwise on the shorter semicircle and clockwise on the other one.. Calculate the vector potential at the centre of the loop. Calculate the electric field at the centre. Can you calculate the magnetic field from the expression of the vector potential?

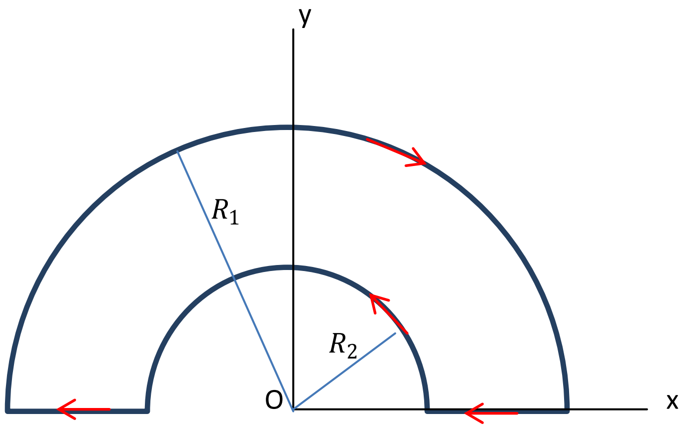


**Solutions to Self Assessment Questions**

1. The problem is very similar to Problem 2 of the Tutorial. The vector potential is given by

$$\vec{A}(s,t) = \frac{\mu_0}{2\pi} I_0 \hat{z} \ln\left(\frac{ct + \sqrt{c^2t^2 - s^2}}{s}\right)$$

2.




Since the current is switched on at t=0, the (retarded) vector potential at the centre O is given by

$$\vec{A} = \frac{\mu_0 I_o}{4\pi} \int \frac{\left(t - \frac{r}{c}\right)}{r} d\vec{l} = \frac{\mu_0 I_o}{4\pi} t \int \frac{1}{r} d\vec{l} - \frac{\mu_0 I_o}{4\pi c} \int d\vec{l}$$

Since $\oint d\vec{l} = 0$, the second integral vanishes. We are left with $\vec{A} = \frac{\mu_0 I_o}{4\pi} t \int \frac{1}{r} d\vec{l}$. Note that the integral is a line integral over the loop, which is directed differently at different points of the loop. Let us take the loop to be in the x-y plane with directions as shown in the figure. Recall that the direction of $d\vec{l}$ is along the direction of the current. The contribution from the two straight sections give $-\frac{\mu_0 I_o}{4\pi} t \ln \frac{R_2}{R_1} \hat{\imath}$, the negative sign is due to the fact that the current in both the sections are directed along the negative x direction.

We now have to perform the integration over the two semicircles. Consider the integration over the bigger semicircle. We can resolve $d\vec{l} = R_1 d\theta(\cos\theta \hat{\imath} + \sin\theta \hat{\jmath})$. The denominator of the integral also has $R_1$ so that the integral becomes independent of $R_1$ (In a similar way, the integral of the smaller semicircle will be independent of $R_2$). The y-component of the integral would vanish by symmetrically placed elements on either side of the y-axis, leaving us with $\frac{\mu_0 I_0 t}{4\pi}\hat{\imath}$ as the value of the integral. The integral over the smaller semicircle is done in a similar way but its direction would be opposite because of the direction of the current. Thus the contribution from the two semicircles would cancel and the vector potential at the centre would be given by $-\frac{\mu_0 I_o}{4\pi} t \ln\frac{R_2}{R_1}\hat{\imath}$ . The electric field is trivial and is given by

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{\mu_0}{4\pi} I_0 \ln\frac{R_1}{R_2}\hat{\imath}$$

Since the vector potential has been calculated only at a single point, it is not possible to calculate its curl and hence the magnetic field cannot be determined by the above calculation.