MA2102: LINEAR ALGEBRA

Lecture 13: Linear Isomorphism

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We study the space of all linear maps together. Given vector spaces V and W (over a field F), consider

$$\mathscr{L}(V,W) := \{T : V \to W \mid T \text{ is linear}\}.$$

When $W = V$, we simplify $\mathscr{L}(V,V)$ to $\mathscr{L}(V)$, the set of linear self-maps of V. Given $S, T : V \to W$, set

$$(S+T)(v) := S(v) + T(v)$$

Claim: $S + T$ is a linear map
Check that

$$(S+T)(cv) = S(cv) + T(cv) = cS(v) + cT(v) = c(S+T)(v)$$

Given two vectors $v_1, v_2 \in V$

$$\begin{aligned}(S+T)(v_1+v_2) &= S(v_1+v_2)+T(v_1+v_2)\\ &= S(v_1)+S(v_2)+T(v_1)+T(v_2)\\ &= (S+T)(v_1)+(S+T)(v_2)\end{aligned}$$

This defines a map

$$+ : \mathscr{L}(V,W)\times\mathscr{L}(V,W)\to\mathscr{L}(V,W),\ \ (S,T)\mapsto S+T$$

This is commutative and associative (as addition of functions behave likewise). The zero map $\mathbb{O} : V \to W$ is the additive identity in $\mathscr{L}(V,W)$.

To define the scaling, consider for any scalar c and $T \in \mathscr{L}(V,W)$ the map

$$(cT)(v) := cT(v)$$

Show that cT is a linear map. This defines a map

$$\cdot : F \times \mathscr{L}(V,W) \to \mathscr{L}(V,W), \ \ (c,T) \mapsto cT$$

Thus, $1T = T$ and it can be seen that $(-1)T$ is the inverse of T. All other axioms can now be verified easily.

Observation *The set $\mathscr{L}(V,W)$ is a vector space.*

Examples (1) Consider $\mathscr{L}(\mathbb{R},\mathbb{R}^n)$ and let $T:\mathbb{R}\to\mathbb{R}^n$ be a linear map. Since $\{1\}$ is a basis of $\mathbb{R}$, T is determined by $T(1)$ ($T(c) = cT(1)$). It seems intuitively clear that the map

$$\mathrm{ev} : \mathscr{L}(\mathbb{R},\mathbb{R}^n) \to \mathbb{R}^n, \ \ T \mapsto T(1)$$

is a bijection. It is onto (given $\mathbf{v} \in \mathbb{R}^n$ set $T(c) = c\mathbf{v}$).

It is one-to-one (if $T(1) = S(1)$, then $T(c) = cT(1) = cS(1) = S(c)$). Moreover,

$$\mathrm{ev}(S+T) = (S+T)(1) = S(1) + T(1) = \mathrm{ev}(S) + \mathrm{ev}(T)$$

$$\mathrm{ev}(cT) = (cT)(1) = cT(1) = c\,\mathrm{ev}(T)$$

implies that ev is actually a linear bijection. It follows from lecture 12 that ev^{-1} is a linear map (exercise).

(2) We shall see later that $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ can be identified (by a linear bijection) to $M_{m\times n}(\mathbb{R})$.

Definition [Linear Isomorphism] A linear map $T : V \to W$ is called a linear isomorphism if T is one-to-one and onto.

We say V and W are isomorphic (as vector spaces) if there exists a linear isomorphism $T : V \to W$.

Note that T being one-to-one and onto guarantees that $T^{-1} : W \to V$ is a well-defined linear map (cf. lecture 12). Thus, an alternative definition of isomorphism can be the existence of a linear map $S : W \to V$ such that $T \circ S = \mathrm{id}_W$ and $S \circ T = \mathrm{id}_V$. This implies that T is a bijection and $S = T^{-1}$.

Remark The notion of isomorphism is an equivalence relation.

- The identity map $\mathrm{id} : V \to V$ is a linear isomorphism.
- If $S : V \to W$ is a linear isomorphism, then $S^{-1} : W \to V$ is also an isomorphism.
- If $S : V \to W$ and $T : U \to V$ are linear isomorphisms, then $S \circ T : U \to W$ is also an isomorphism.

We may now make precise what we had meant by "looks like". It means "is isomorphic to".

Proposition *Let $T : V \to W$ be a linear map.*
(i) *Let $\{w_1, \dots, w_n\} \subset W$ be a linearly independent set and let T be surjective. If $v_i \in V$ such that $T(v_i) = w_i$, then $\{v_1, \dots, v_n\}$ is linearly independent.*
(ii) *Let $\{v_1, \dots, v_n\} \subset V$ be a linearly independent set. If T is injective, then $\{T(v_1), \dots, T(v_n)\}$ is linearly independent.*
(iii) *Let T be a linear isomorphism. If $\beta = \{v_1, \dots, v_n\}$ is a basis for V, then $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.*

Proof.
(ii) If $c_1 T(v_1) + \cdots + c_n T(v_n) = 0_W$, then by linearity of T, $T(c_1 v_1 + \cdots + c_n v_n) = 0_W$. Thus, $c_1 v_1 + \cdots + c_n v_n = 0_V$ due to injectivity of T. As v_j's are linearly independent, c_j's are all zero.

(i) If $c_1 v_1 + \cdots + c_n v_n = \mathtt{0}_V$, then by linearity of T,

$$\mathtt{0}_W = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T(v_1) + \cdots + c_n T(v_n) = c_1 w_1 + \cdots + c_n w_n.$$

As w_j's are linear independent, c_j's are zero.

(iii) Since β spans V and T is onto, $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ spans W. As β is linearly independent, by (ii) $T(\beta)$ is linearly independent. This proves that $T(\beta)$ is a basis. $\square$

Remark The proposition works for infinite linearly independent sets as well as for infinite dimensional vector spaces. For instance, in (i) we may choose any linearly independent set S in V to prove that $T(S)$ is linearly independent. Similarly, in (iii), the image of a basis is a basis, even if V is not finite dimensional.