MA2102: LINEAR ALGEBRA

Lecture 7: Dimension

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Let us recall the notion of a basis of a vector space.

Definition [Basis] A basis β of a vector space V is a subset such that β is linearly independent and spans V.

Proposition Let V be a vector space that admits a finite subset S that spans V. Then there exists a subset $\beta \subseteq S$ which is a basis for V.

Proof.

If $S = \varnothing$ or $\{\mathtt{0}\}$, then $\mathrm{span}(S) = \{\mathtt{0}\}$. We may choose $\beta = \varnothing \subseteq S$ to be our basis. If $S \neq \varnothing$, then choose a non-zero $u_1 \in S$. By choice, $\{u_1\}$ is linearly independent. If possible, choose $u_2 \in S$ such that $\{u_1, u_2\}$ is linearly independent. If no such choice is possible, then any for $v \in S - \{u_1\}$, the set $\{u_1, v\}$ is linearly dependent, i.e., $cu_1 + dv = \mathtt{0}$ for some scalars c, d, at least one of these being non-zero. As $d \neq \mathtt{0}$, $v \in \mathrm{span}(\{u_1\})$, implying that $V = \mathrm{span}(S) = \mathrm{span}(\{u_1\})$.

Iterate this process to choose $u_1, u_2, \dots, u_k \in S$ such that $\beta = \{u_1, \dots, u_k\}$ is linearly independent and one of the two mutually cases happen:

(a) $S = \beta$ and we are done;

(b) for any $v \in S - \beta$, the set $\beta \cup \{v\}$ is linearly dependent.

As S is finite, such a selection can always be made. Now focus on case (b): for any such v we have a non-trivial linear combination

$$c_1 u_1 + \cdots + c_k u_k + dv = 0.$$

If $d = 0$, then that contradicts the linear independence of u_j's. Thus, $d \neq 0$ and

$$v = -\frac{c_1}{d} u_1 - \cdots - \frac{c_k}{d} u_k \in \operatorname{span}(\beta).$$

Hence, $S \subseteq \operatorname{span}(\beta)$.

Now, taking spans on both sides we obtain

$$V = \mathrm{span}(S) \subseteq \mathrm{span}(\mathrm{span}(\beta)) = \mathrm{span}(\beta) \subseteq V.$$

This means that $\mathrm{span}(\beta) = \mathrm{span}(S) = V$. As β is linearly independent and spans V, we have found a finite basis of V. $\square$

Prove the proposition by eliminating vectors from S one at a time, using an earlier observation from lecture 6.

Remark The above proof will not work if S is not finite. However, there are vector spaces which do not have finite basis. One may consider $P(\mathbb{R})$ or the set of real sequences, for instance.

Definition Let V be a vector space. We say V is infinite dimensional if it has no finite basis.

Replacement Theorem
Let V be a vector space that is spanned by a set S of size n. Let L be a linearly independent set of size m. Then
(i) $m \leq n$ and
(ii) there exists $T \subseteq S$ of size $n - m$ such that $T \cup L$ spans V.

Remark It follows from the proof of the previous proposition and (ii) that L can be extended to a basis of V. We also know that S can be reduced to a basis of V. We shall see that any two basis have the same size for a vector space as above. Thus, part (i) $m \leq n$ is often rephrased as *L can be extended to a basis and S can be reduced to a basis*.

Corollary A Let V be a vector space with a finite basis. Then any two bases have the same size.

This common integer is called the dimension of the vector space V over the field F. We denote this number by $\dim_F(V)$.

Proof.

Let β be a basis of V of size n. If β' is another basis of size m, then

$$m \leq n\ (S = \beta, L = \beta') \text{ and } n \leq m\ (S = \beta', L = \beta)$$

due to (i). If β' is infinite, then choose $L = \{u_1, \ldots, u_{n+1}\} \subset \beta'$. As L is linearly independent, applying (i) to L and $S = \beta$ we obtain $n + 1 \leq n$, a contradiction. $\square$

Corollary B Let V be a vector space of dimension n.

(a) Any spanning set S of V contains at least n elements. If S is of size n, then S is a basis.

(b) Any linearly independent set consisting of n vectors is a basis.

(c) Any linearly independent set can be extended to a basis.

Example (1) The dimension of $\mathbb{R}^n$ over $\mathbb{R}$ is n.
(2) The dimension of $M_n(F)$ over F is n^2.
(3) The dimension of $P_n(\mathbb{R})$ over $\mathbb{R}$ is $n+1$.
(4) The dimension of $n \times n$ real traceless matrices W_n is $n^2 - 1$.
(5) The dimension of $n \times n$ real symmetric matrices Sym_n is $1 + 2 + \cdots + n$.
(6) The dimension of $\mathbb{C}$ over $\mathbb{C}$ is 1, i.e., $\dim_{\mathbb{C}}(\mathbb{C}) = 1$. The dimension of $\mathbb{C}$ over $\mathbb{R}$ is 2, i.e., $\dim_{\mathbb{R}}(\mathbb{C}) = 2$.

Question *What is the dimension of the subspace of real skew-symmetric $n \times n$ matrices?*

Proof.
(a) If S is linearly independent, then S is a basis. By Corollary A, $|S| = n$.

If S is linearly dependent, then by proposition choose $T \subset S$ such that T is a basis. By Corollary A, $|T| = n$, whence $|S| \geq |T| = n$. If $|S| = n$, then $T = S$ is a basis.

(b) Let L be a linearly independent set of size n. Let β be a basis of size n. Apply (ii) of Theorem to $S = \beta$ and $L = L$ to obtain T, of size $0 = n - n$, such that $T \cup L = L$ spans V.

(c) Let L be a linearly independent set of size m and β be a basis of size n. By (ii) of Theorem, there exists $T \subset \beta$ of size $n - m$ such that $T \cup L$ spans V. Thus, $|T \cup L| \leq n$ while using (a) of this corollary, we get $|T \cup L| \geq n$. Therefore, $T \cup L$ is a spanning set of size n and by (a) of this corollary, it is a basis. $\square$