Coefficient of Potential and Capacitance

Lecture 12: Electromagnetic Theory

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We know that inside a conductor there is no electric field and that the surface of a metal is an equipotential surface. Further, the tangential component of the electric field on the surface is zero leaving with only a normal component of the field on the surface. Our argument in arriving at these properties was largely intuitive and was based on our expectation that under electrostatic conditions, we expect the system of charges to stabilize and move around. Let us look at this problem from a mathematical stand point.

Consider surface of a conductor. Let us take the conductor surface to be in the xy plane. In the vacuum outside the conductor there are no free charges so that $\nabla \cdot \vec{E} = -\nabla^2\varphi = 0$. Further as we are concerned with only electrostatic phenomena, $\nabla \times \vec{E} = 0$ everywhere.

There are charges only on the surface of the conductor which would produce an electric field. The charges must be distributed in such a way that the fields they produce superpose so as to cancel the electric field inside the metal. So near the surface of the metal over a small a distance there is a change in the electric potential. Thus, if we take the outward normal to the surface of the conductor to be along the positive z direction, the z component of the electric field $E_z$must be large near the surface.

If the surface is homogeneous, $\frac{\partial E_z}{\partial x}$ or $\frac{\partial E_z}{\partial y}$ must be finite and continuous. Since the curl of the electric field is zero, this means, for instance,

$$\left(\nabla \times \vec{E}\right)_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0$$

and since $\frac{\partial E_z}{\partial y}$ is finite and continuous, so must be $\frac{\partial E_y}{\partial z}$, and by symmetry, $\frac{\partial E_x}{\partial z}$. Since the field, and its tangential component, are zero inside the surface, the tangential components ($E_y \text{or} E_z$) must be zero on the surface.

Total charge on the surface of the conductor is obtained by integrating the surface charge density, $\sigma = \epsilon_0 E_n = -\epsilon_0 \frac{\partial\varphi}{\partial n}$ which gives $Q = -\epsilon_0 \oint \frac{\partial\varphi}{\partial n} dS$.

Theorem : The potential function can be maximum only at the boundary of a region when the the electric field exists.

Proof : Suppose, the theorem is not true and the potential is maximum at a point P which does not lie on the boundary of a region of electric field. If this were so, we could enclose the point P by a small volume at every point of which the potential is less than that at P. Thus everywhere on this region

$\frac{\partial \varphi}{\partial n} < 0$. Thus the surface integral of this quantity $\oint \frac{\partial \varphi}{\partial n} dS < 0$. This contradicts the fact the according to Laplace's equation this integral should be zero as the surface integral is equal to the amount of charge enclosed by the surface and away from the boundary there are no charges.

Energy of a Charged Conductor in Electric Field

We will calculate the energy that is stored in the field produced by a charged conductor. In doing so we will subtract the energy that is stored within the volume of the conductor itself. Let V'denote all space around the conductor which excludes the conductor volume itself. The stored energy is given by

$$W = \frac{\epsilon_0}{2}\int_{V'} |E|^2 dv = -\frac{\epsilon_0}{2}\int_{V'} \vec{E} \cdot \nabla\varphi dv$$
$$= -\frac{\epsilon_0}{2}\int_{V'} \nabla \cdot (\varphi\vec{E}) dv + \frac{\epsilon_0}{2}\int_{V'} \varphi\nabla \cdot \vec{E} dv$$

where we have used the identity

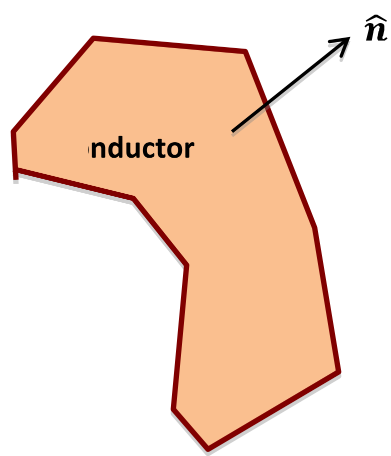
$$\nabla \cdot (\varphi\vec{E}) = \varphi\nabla \cdot \vec{E} + \vec{E} \cdot \nabla\varphi$$

The second integral on the right gives zero because outside the conductor there being no charge, $\nabla \cdot \vec{E} = 0$. The first integral can be converted to an integral over the surface using the divergence theorem. Since the volume excludes the conductor volume, the surface has two components, one at infinity, and the other at the surface of the conductor. The contribution to the surface integral from the surface at infinity is zero. We are then left with the integral over the surface of the conductor. Defining the normal direction on the conductor surface as being outward from the conductor surface, we observe that the boundary of the infinite region excluding the conductor volume has a normal direction which is opposite to the normal on the surface of the conductor.

We thus have

$$W = +\frac{\epsilon_0}{2}\oint \varphi\vec{E} \cdot d\vec{S}$$

the integral being over the surface of the conductor itself.



If there are multiple conductors, we can extend the same argument and get the total energy stored in the field as

$$W = \frac{\epsilon_0}{2}\sum_i \varphi_i \oint \vec{E}_i \cdot d\vec{S}_i$$

Where the potential has been taken out of the integral as the potential remains constant on each conductor surface.

The surface integral above, by Gauss's theorem is the amount of charge $Q_i$on the i −th conductor. We thus have,

$$W = \frac{1}{2}\sum_i \varphi_i Q_i$$

Since the field equations are linear, the potential and the charge are linear function of each other, the coefficient of proportionality can only depend on the shape of the conductor. Since, $Q = C\varphi$ for a single conductor, we have,

$$Q_i = \sum_j C_{i,j}\varphi_j$$
$$\varphi_i = \sum_j p_{i,j} Q_j$$

The coefficients $C_{i,j}$ are known as the **coefficients of capacitance** and $p_{i,j}$ are known as the **coefficients of potential.** In terms of these coefficients, the energy of a system of conductors can be written as

$$W = \frac{1}{2}\sum_{i,j} p_{ij} Q_i Q_j = \frac{1}{2}\sum_{i,j} C_{i,j}\varphi_i\varphi_j$$

It can be shown that these coefficients are symmetric under interchange of indices. To illustrate this, let us assume that we have a collection of conductors and we add a small charge ΔQ_k only to the k-th conductor, keeping the charges on all other conductors the same. Let us look at the amount by which the total energy of the system changes. We have,

$$\begin{aligned}
\Delta W &= \frac{\partial W}{\partial Q_k}\Delta Q_k \\
&= \frac{1}{2}\frac{\partial}{\partial Q_k}\left(\sum_{i,j} p_{ij} Q_i Q_j\right)\Delta Q_k \\
&= \frac{1}{2}\left(\sum_{i,j} p_{ij}\frac{\partial Q_i}{\partial Q_k} Q_j + \sum_{i,j} p_{i,j} Q_i \frac{\partial Q_j}{\partial Q_k}\right)\Delta Q_k \\
&= \frac{1}{2}\left(\sum_{i,j} p_{ij}\delta_{i,k} Q_j + \sum_{i,j} p_{i,j} Q_i \delta_{j,k}\right)\Delta Q_k \\
&= \frac{1}{2}\left(\sum_j p_{k,j}\, Q_j + \sum_i p_{i,k} Q_i\right)\Delta Q_k
\end{aligned}$$

$$= \frac{1}{2} \sum_j (p_{k,j} + p_{j,k}) Q_j \Delta Q_k \qquad (1)$$

Where, in deriving the last step from its preceding step, we have changed the dummy summation index from I to j.

However, we can get an equivalent expression for the change in the energy from the expression $W = \frac{1}{2} \sum_i \varphi_i Q_i$. It follows that if the charge on the k-th conductor is increased by ΔQ_k, the energy would change by

$$\Delta W = \varphi_k \Delta Q_k = \sum_j p_{k,j} Q_j \Delta Q_k \qquad (2)$$

(Remember that the factor of ½ in the expression for energy came in order to avoid double counting of interactions between pairs. Since we are only interested in change in potential energy due to an additional charge being given to the k-th conductor, the change in the potential energy is simply given by the product of the additional charge with the potential at the site of the k-th conductor.)

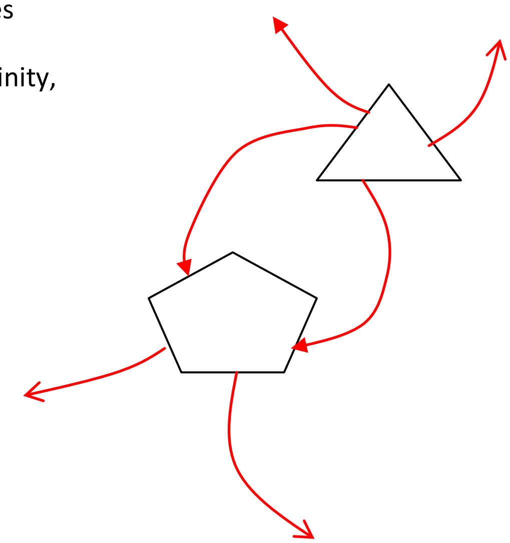
Comparing (1) and (2), we get

$$p_{k,j} = p_{j,k}$$

We can also show that the diagonal term $p_{i,i}$ is the largest coefficient. Further, all coefficients are positive.

Consider two conductors k and j. Suppose we bring in a unit positive charge from infinity to the k-th conductor, keeping all other conductors uncharged. By definition of the potential for a single conductor $\varphi_k = p_{k,k} Q_k$. Thus if $Q_k = 1, p_{k,k}$ is equal to the potential of the k-th conductor. Let us now bring the j-th conductor into the picture.

Since k-th conductor has a net positive charge, field lines Arising out of this conductor will either terminate at infinity, or terminate on the j-th conductor from which the field lines will go to infinity. Since Q_k is the only charge, the potential on the j-th conductor is $\varphi_j = p_{j,k} Q_k$. The potential on the k-th conductor is still $\varphi_k = p_{k,k} Q_k$ since $Q_j = 0$. Thus, we have, $\varphi_k > \varphi_j > 0$, which gives, $p_{k,k} > p_{j,k} > 0$.



In Lecture 5 we proves Green's theorem according to which, given to scalar fields φ and ψ in a volume V bounded by a surface S,

$$\int_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi)\, dV = \oint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot \hat{n} dS$$

Let φ a potential field which arises from a volume charge distribution ρ in V and a surface charge distribution σ on S. Likewise, let ψbe a potential which arises from ρ' in V and σ' on S. Using Maxwell's equation, we have, $\nabla^2 \varphi = -\frac{\rho}{\epsilon_0}$ and $\nabla^2 \psi = -\frac{\rho'}{\epsilon_0}$. Let the electric field corresponding to φ and ψ be $\vec{E}$ and $\vec{E}'$ respectively. We then have,

$$\begin{aligned} -\frac{1}{\epsilon_0}\int_V (\varphi\rho' - \psi\rho) dV &= -\oint_S \left(\varphi \overrightarrow{E'} - \psi \vec{E}\right) \cdot \hat{n} dS \\ = \frac{1}{\epsilon_0}\oint_S (\varphi\sigma' - \psi\sigma) dS \end{aligned}$$

Thus

$$\int_V \varphi\rho' dV + \oint \varphi\sigma' dS = \int_V \psi\rho dV + \oint \psi\sigma dS$$

This is known as **Green's Reciprocity Theorem**.

Consider the volume to be bounded by conducting surfaces. Since the potential is constant on the conducting surfaces, we can write,

$$\int_V \varphi\rho' dV + \sum \varphi_i Q_i' = \int_V \psi\rho dV + \sum \psi_i Q_i$$

where the sum is taken over the surface of the conductors.

Example :

A point charge q is located at a distance r from a spherical conductor of radius R (<r). What is the potential on the surface of the sphere?

Since the only charge is the point charge at a distance r, we take volume charge density to be zero. We can consider the system as consisting of two conductors with the point charge being considered as a conductor with an infinitesimally small radius. If we put the charge $Q_s = q$ on the surface of the sphere, the potential that it produces at the location of the charge at a distance r, is $\varphi = \frac{q}{4\pi\epsilon_0 r}$. If we use the reciprocity theorem, denoting the potential on the sphere when the charge is at the distance r from by ψ ,

$$q.\frac{q}{4\pi\epsilon_0 r} = q\psi$$

which gives $\psi = \frac{q}{4\pi\epsilon_0 r}$. (Note that this also follows from the definition of the coefficient of potential,

Let the point charge be conductor 1 and the sphere as conductor 2.

$$\varphi_1 = p_{11}q_1 + p_{12}q_2$$
$$\varphi_2 = p_{21}q_1 + p_{22}q_2$$

Let $q_1 = 0, q_2 = q$, I.e. let the sphere be given a charge q. The potential at conductor 1 due to the charge q being given to the sphere is $\varphi_1 = 0 + p_{12}q$. However, the potential at the conductor 1 due to the charge q on the sphere is $\varphi_1 = \frac{q}{4\pi\epsilon_0 r}$. Thus $p_{12} = \frac{1}{4\pi\epsilon_0 r}$. By symmetry of the coefficient of potential, $p_{21} = p_{12} = \frac{1}{4\pi\epsilon_0 r}$. The potential on the sphere when the charge on the point conductor is then,

$$\varphi_2 = p_{21}q_1 + p_{22}q_2$$

We have here $q_1 = q, q_2 = 0$ so that $\varphi_2 = \frac{q}{4\pi\epsilon_0 r}$.

Example 2 :

Two conductors having capacitances C_1 and C_2 are placed apart at a distance r from each other. Calculate the capacitance of the pair.

Let us put a charge q on conductor 1 keeping the conductor 2 uncharged. The potential of the conductor 1 is then $\varphi_1 = \frac{q}{C_1}$ and the potential on the conductor 2 due to the charge q at a distance r is $\varphi_2 = \frac{q}{4\pi\epsilon_0 r}$. Since $\varphi_1 = \frac{q}{C_1}$ when $q_1 = q, q_2 = 0$ and is $\frac{q}{4\pi\epsilon_0 r}$ when$q_1 = 0, q_2 = q$, from the relation $\varphi_1 = p_{11}q_1 + p_{12}q_2$, we have $p_{11} = \frac{1}{C_1}, p_{12} = \frac{1}{4\pi\epsilon_0 r}$. Likewise, $p_{22} = \frac{1}{C_2}$. Thus we have,

$$\varphi_1 = \frac{1}{C_1}q_1 + \frac{1}{4\pi\epsilon_0 r}q_2$$
$$\varphi_2 = \frac{1}{4\pi\epsilon_0 r}q_1 + \frac{1}{C_2}q_2$$

If we look at it as a matrix equation, we have $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}\frac{1}{C_1} & \frac{1}{4\pi\epsilon_0 r}\\ \frac{1}{4\pi\epsilon_0 r} & \frac{1}{C_2}\end{pmatrix}\begin{pmatrix}q_1\\ q_2\end{pmatrix}$. We can invert this equation to express the charges in terms of the potential on the two conductors,

$$\begin{pmatrix}q_1\\ q_2\end{pmatrix} = \begin{pmatrix}\frac{1}{C_1} & \frac{1}{4\pi\epsilon_0 r}\\ \frac{1}{4\pi\epsilon_0 r} & \frac{1}{C_2}\end{pmatrix}^{-1}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

The inverse of the matrix is easily found to be

$$\frac{\begin{pmatrix} \frac{1}{C_2} & -\frac{1}{4\pi\epsilon_0 r} \\ -\frac{1}{4\pi\epsilon_0 r} & \frac{1}{C_1} \end{pmatrix}}{\frac{1}{C_1C_2} - \left(\frac{1}{4\pi\epsilon_0 r}\right)^2}$$

Thus the coefficients of capacitances are given by,

$$C_{11} = \frac{1}{C_2}\frac{1}{\frac{1}{C_1C_2} - \left(\frac{1}{4\pi\epsilon_0 r}\right)^2}$$
$$C_{22} = \frac{1}{C_1}\frac{1}{\frac{1}{C_1C_2} - \left(\frac{1}{4\pi\epsilon_0 r}\right)^2}$$
$$C_{12} = -\frac{1}{4\pi\epsilon_0 r}\frac{1}{\frac{1}{C_1C_2} - \left(\frac{1}{4\pi\epsilon_0 r}\right)^2} = C_{21}$$

Capacitors are devices which are used to store charges. Consider a collection of a system of conductors of which a pair hold equal and opposite charge $\pm Q$, the others being neutral. Using the coefficients of potential, we can write the potential of these two conductors as

$$\varphi_1 = p_{11}Q - p_{12}Q + \varphi_{others}$$
$$\varphi_2 = p_{12}Q - p_{22}Q + \varphi_{others}$$

whereφ_{others} is the potential of the uncharged conductors. The difference in potential of the two conductors is

$$\Delta\varphi = (p_{11} + p_{22} - 2p_{12})Q$$

The capacitance is thus given by

$$C = \frac{Q}{\Delta\varphi} = (p_{11} + p_{22} - 2p_{12})^{-1}$$

Force on a system of conductors

Let us consider what happens when we displace a system of charges by$d\vec{r}$. If the force acting on the charges is $\vec{F} \cdot d\vec{r}$. This work must be done by the charges at the cost of their energy,

$$\vec{F} \cdot d\vec{r} = -dW$$

We can then write $\vec{F} = -\nabla W$.

Suppose, instead, we have a system of conductors which are kept at some fixed potential $\{\varphi_i\}$. The energy of the system is

$$W = \frac{1}{2}\sum_i \varphi_i q_i$$

where $q_i = -\epsilon_0 \oint_S \frac{\partial \varphi_i}{\partial n_i} dS$ is the charge on the surface of the i-th conductor. It may be note that the charges q_i are unknown and must be obtained by solving Poisson's equation.

Suppose an electric force displaces the system of conductors by $d\vec{r}$. The work done by the force is $\vec{F}\cdot d\vec{r}$. The problem at this stage is that we neither know the force nor know the work done. However, if we are armed with a knowledge of the total energy of the system, we can gain this information. The work can be done in two ways .

1. By a decrease in the stored electrostatic energy W^{es} of the system : In this process the charges do not flow into or out of the system. The force is given by $F_x = -\left.\frac{\partial W^{es}}{\partial x}\right|_Q$ where $Q = \sum_i q_i$ with similar expressions for other components of the force.
2. By an increase in energy : If the potentials on the conductors are to be kept constant, work is done by drawing energy from the battery. This would result in an increase in the energy of the system and the force would be given by $F_x = \left.\frac{\partial W^{battery}}{\partial x}\right|_\varphi$.

Example 3 : Force on a capacitor plate :

Consider the force exerted on one of the plates of a parallel plate capacitor. Imagine one of the plates as fixed and the other free to move. Assume that the plates are charged to $\pm Q$ and the battery disconnected. In this situation, the charge on the plates is held constant. Now, the electrostatic energy is $\frac{Q^2}{2C}$. The capacitance C depends on the instantaneous distance x of the movable plate from the fixed plate,

$$C = \frac{A\epsilon_0}{x}$$

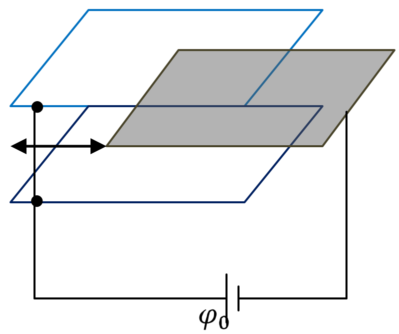
Thus

$$W^{es} = Q^2 x/2A\epsilon_0$$

The force acting on the plate is $F_x = -\frac{dW}{dx} = -\frac{Q^2}{2A\epsilon_0}$, the negative sign shows that the force tends to reduce the distance between the plates.

Example 4 :

A parallel plate capacitor of width w and spacing d (<< w) between the plates is held fixed with *both the plates* being kept at a potential φ_0 with respect to the ground. A grounded conducting sheet of metal is partially inserted midway between the plates so that the left edge is at a distance x inside the capacitor plates. Calculate the force with which the sheet is being drawn inside.



Since both plates of the capacitor are at the same potential, the electric field between the plates is zero. However, when the grounded sheet is inserted, it creates an electric field in the region where the sheet is there. Since the spacing between the sheets and either plate is d/2, the electric field in the region is $|E| = V/(d/2) = \frac{2V}{d}$. Since the electric field is constant, the electrostatic energy

$$W^{es} = \frac{\epsilon_0}{2}\int |E|^2 dv = \frac{2\epsilon_0 V^2}{d} wx$$

The integration is over the volume $(= wxd)$ where the electric field exists. Thus

$$F_x = +\frac{2\epsilon_0 V^2}{d} w$$

The positive sign indicates that the force tends to increase x i.e., draw the sheet inside.

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Tutorial Assignment

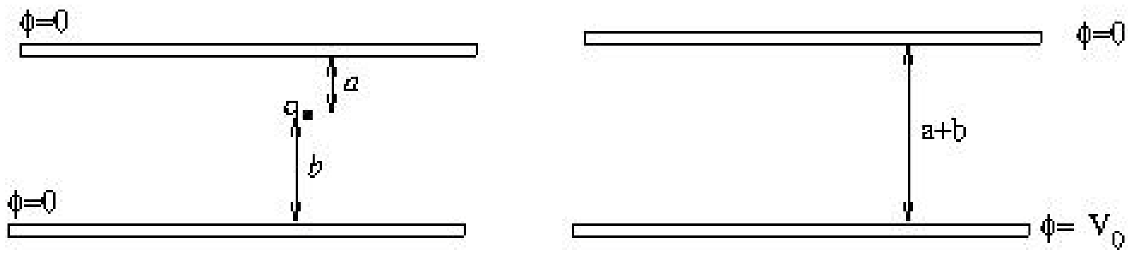
(In solving these problems use the concepts developed in this lecture)

1. Consider two infinite grounded conductors separated by a distance of d. A point charge q is situated at a distance of *a* from the top conductor and *b* from the bottom conductor. What is the net charge induced on each of the conductors? How would the answer change if there is only the top conductor?

2. There is a cavity of irregular shape inside a spherical conductor of radius R. A charge q is placed inside the cavity being insulated from the walls of the cavity by an insulating prop. What is the potential at a distance r > R from the centre of the sphere?
3. Consider two infinite cylindrical conductors of radius a , their axes being parallel and separated from each other by a distance $r > 2a$. They carry equal and opposite charge of charge density $\pm\lambda$ per unit length. Obtain an expression for the capacitance of the pair. What is the force exerted on one of the plates by the other if the potential difference between the plates is maintained at V?
4. Two concentric spherical conducting shells of radii R_1 andR_2 carry chargeQ_1s and Q_2 respectively. Calculate the capacitance of the pair. What is the electrostatic energy stored in the pair?

Solutions to Tutorial Assignment

1. The original problem is shown in the figure to the left. Consider a second problem where the top plate is kept at a potential V_0 and the bottom plate is grounded. Both these situation have the same volume of space and are bounded by conducting surfaces.



In order to apply Green's reciprocity theorem, we observe that that the charge and potential distribution of the two cases are as follows :

If the induced charges on the top plate on the top is Q_{top} that induced on the bottom plate isQ_{bottom},

the volume charge densit $\rho = q\delta(z-b)$y, where the perpendicular distance z is measured from the bottom plate. The surface charges on the left are Q_{top} and Q_{bottom}. $\varphi = 0$ for both the plates.

The charge density for the second case is zero both in volume and on the surfaces. The potentials are given by $\psi = 0$ for top plate, $\psi = V_0$ for the bottom plate. The electric field being uniform between the plates, the potential at a distance z from the bottom plate is $\frac{V_0(a+b-z)}{a+b}$. Thus the potential at a distance b from the bottom is $\frac{V_0 a}{a+b}$. Using,

$$\int_V \varphi\rho' dV + \oint \varphi\sigma' dS = \int_V \psi\rho dV + \oint \psi\sigma dS$$

expression on the left is zero (since $\rho' = \sigma' = 0$) while that on the right is

$$\frac{V_0 a}{a+b}q + V_0 Q_{bottom}$$

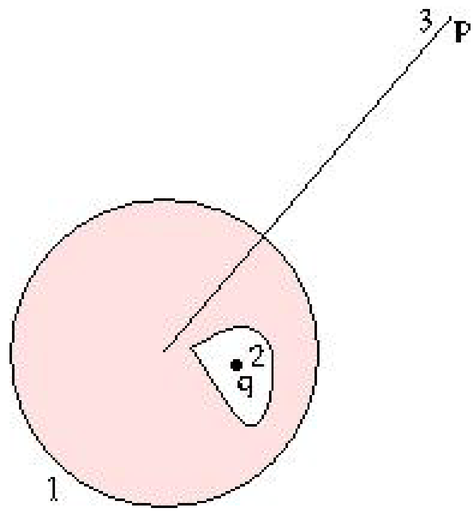
Equating this to zero we get

$Q_{bottom} = -\frac{a}{a+b}q$. By symmetry, one can write $Q_{top} = -\frac{b}{a+b}q$.

If only the top plate remains, we can simulate the configuration by removing the bottom plate to infinity and get taking the limit $b \to \infty$, $Q_{top} = -q$. (We will see later that this is the solution obtained by method of images.)

2. Let us identify three conductors, viz., the sphere, which is conductor 1, the point conductor inside the irregular cavity the conductor 2 and a third point conductor 3 at the location P. From the definition of coefficients of potential we have,

$$\varphi_1 = p_{11}q_1 + p_{12}q_2 + p_{13}q_3$$
$$\varphi_2 = p_{21}q_1 + p_{22}q_2 + p_{23}q_3$$
$$\varphi_3 = p_{31}q_1 + p_{32}q_2 + p_{33}q_3$$



These coefficients depend only on geometry and can therefore be determined by specifying charge distributions. Suppose, we put $q_1 = q_2 = 0, q_3 = q$, we get

$\varphi_1 = p_{13}q, \varphi_2 = p_{23}q$. However, in the absence of a charge in the cavity, these two potentials must be the same. Hence$p_{13} = p_{23}$.

Next, let us put a charge q on the sphere leaving others uncharged, i.e., $q_1 = q, q_2 = q_3 = 0$. in this case, since the charge given to a conductor would only be symmetrically distributed on its outer surface, we would have $\varphi_3 = \frac{q}{4\pi\epsilon_0 r}$. We then have, $\varphi_1 = p_{11}q, \varphi_2 = p_{21}q, \varphi_3 = p_{31}q = \frac{q}{4\pi\epsilon_0 r}$ which gives, $p_{31} = \frac{1}{4\pi\epsilon_0 r}$. Thus we have,

$$p_{31} = p_{13} = p_{23} = p_{32} = \frac{1}{4\pi\epsilon_0 r}$$

Let us substitute these into the equation for φ_3 when $q_1 = q_3 = 0, q_2 = q$

$$\varphi_3 = p_{31}q_1 + p_{31}q_2 + p_{33}q_3 = p_{31}q = \frac{q}{4\pi\epsilon_0 r}$$

3. The potential due to a cylindrical conductor of radius of cross section a carrying a linear charge density λ at a distance ρ (>a) from the axis of the cylinder is$\frac{\lambda}{2\pi\epsilon_0}\ln\frac{\rho_0}{\rho}$, where ρ_0 is an arbitrary distance at which the potential is set to be zero (remember that for a cylindrical conductor we cannot take the reference point at infinite distance from the conductor). Using this it is straightforward to show that the capacitance of the pair is given by $C = \pi\epsilon_0/\ln\left(\frac{r-a}{a}\right)$ If the a constant potential difference V is maintained between the conductors, the electrostatic energy is given by $\frac{1}{2}CV^2 = \frac{\pi\epsilon_0}{2\ln\left(\frac{r-a}{a}\right)}V^2$ so that the force is given by

$$\frac{dW}{dr} = \frac{\pi\epsilon_0}{2\left(\ln\left(\frac{r-a}{a}\right)\right)^2}\frac{V^2}{r}$$

4. When there is a charge Q_1 on the inner conductor and Q_2 on the outer conductor, the potential on the outer conductor is given by$\varphi_2 = \frac{Q_1+Q_2}{4\pi\epsilon_0 R_2}$. The field between the spheres bein$\frac{Q_1}{4\pi\epsilon_0 r^2}$g, one can easily calculate the potential on the inner sphere to be given b$\varphi_1 = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1}{R_1} + \frac{Q_2}{R_2}\right)$y. Thus the potential matrix is given by

$$\frac{1}{4\pi\epsilon_0}\begin{pmatrix} \frac{1}{R_1} & \frac{1}{R_2} \\ \frac{1}{R_2} & \frac{1}{R_2} \end{pmatrix}$$

The capacitance matrix is given by

$$C = 4\pi\epsilon_0 \frac{R_1 R_2^2}{R_2 - R_1}\begin{pmatrix} \frac{1}{R_2} & -\frac{1}{R_2} \\ -\frac{1}{R_2} & \frac{1}{R_1} \end{pmatrix}$$

The electrostatic energy is given by

$$W = \frac{1}{2}\sum_{i,j} C_{ij}\varphi_1\varphi_j = \frac{1}{2}\sum_{i,j} p_{ij}Q_iQ_j$$

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Self Assessment Quiz

1. Calculate the capacitance of two conductors carrying equal and opposite charges in terms of the coefficient of potential.
2. Three concentric shells of radii $R_1 < R_2 < R_3$ respectively carry charges $+Q, +Q$ and $-2Q$. Calculate the energy of the system.
3. Two concentric spherical shells of radii $R_1 < R_2$ are both grounded and a point charge q is placed at a distance r between them, being insulated from both the surfaces. Calculate the energy of the system charges.
4. Calculate the work done in doubling the separation between the plates of a parallel plate capacitor from d to 2d, if the potential difference between the plates remains φ.

Solutions to Self Assessment Quiz

1. Let $Q_1 = -Q_2 = Q$. We have,

$$\varphi_1 = p_{11}Q_1 + p_{12}Q_2 = p_{11}Q - p_{12}Q$$
$$\varphi_2 = p_{21}Q_1 + p_{22}Q_2 = p_{21}Q - p_{22}Q$$

Use, $p_{12} = p_{21}$ to get, $\Delta\varphi = \varphi_1 - \varphi_2 = Q(p_{11} + p_{22} - 2p_{12})$. The capacitance

$$C = \frac{Q}{\Delta\varphi} = \frac{1}{p_{11} + p_{22} - 2p_{12}}$$

2. The potential on the outermost conductor is $\varphi_3 = \frac{Q_1+Q_2+Q_3}{4\pi\epsilon_0 R_3}$. Potential on the central conductor is

$$\varphi_2 = \frac{Q_1+Q_2+Q_3}{4\pi\epsilon_0 R_3} - \frac{1}{4\pi\epsilon_0}\int_{R_3}^{R_2}\frac{Q_1+Q_2}{r^2}dr = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1+Q_2}{R_2}+\frac{Q_3}{R_3}\right)$$

The potential on the innermost conductor is

$$\varphi_1 = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1+Q_2}{R_2}+\frac{Q_3}{R_3}\right) - \frac{1}{4\pi\epsilon_0}\int_{R_2}^{R_1}\frac{Q_1}{r^2}dr = \frac{Q_1}{R_1}+\frac{Q_2}{R_2}+\frac{Q_3}{R_3}$$

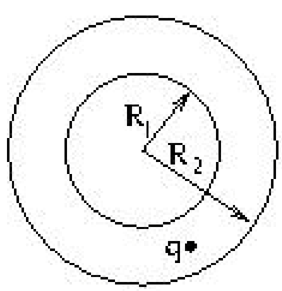
The matrix for the coefficient of potential is

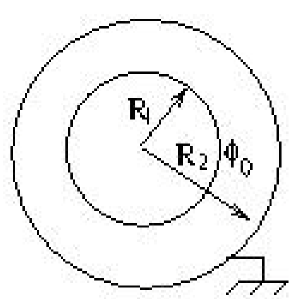
$$\frac{1}{4\pi\epsilon_0}\begin{pmatrix} \frac{1}{R_1} & \frac{1}{R_2} & \frac{1}{R_3} \\ \frac{1}{R_2} & \frac{1}{R_2} & \frac{1}{R_3} \\ \frac{1}{R_3} & \frac{1}{R_3} & \frac{1}{R_3} \end{pmatrix}$$

The energy is given by, on substituting $Q_1 = Q_2 = Q, Q_3 = -2Q$

$$W = \frac{1}{2}\sum_{i,j} p_{ij}Q_iQ_j = \frac{Q^2}{16\pi^2\epsilon_0^2}\left(\frac{1}{R_1}+\frac{3}{R_2}-\frac{4}{R_3}\right)$$

3. We use Green's reciprocal theorem by constructing an artificial problem (figure to right) where the outer sphere is grounded but the inner sphere is maintained at a constant potential φ_0 without any charge being between them.





In the second case, the potential in the region between the spheres satisfies Laplace's equation in variable r of spherical coordinates (as it can only depend on r) which has the solution $\varphi_0 = A + \frac{B}{r}$, where A and B are constants. Substituting boundary conditions, the potential at r is given by

$$\varphi(r) = \varphi_0\frac{a(b-r)}{r(b-a)}$$

Since both spheres in given configuration is grounded, their potential is zero. Let Q_a and Q_b be charges induced on the inner sphere and outer sphere respectively. We then have,

$$\varphi_0 Q_a + \varphi(r) q = 0$$

which can be solved to give

$$Q_a = -aq \frac{b-r}{r(b-a)}$$

The charge induced on the outer sphere may be found by changing the boundary condition on the right to grounded inner sphere and the outer sphere being at a constant potential. It can also be inferred by realizing that the total induced charge should be $-q$.

4. The work done is calculated by the difference between initial and final energies. Since the potential difference between the plates is maintained at φ, the energy is given by the formula $\frac{C\varphi^2}{2}$. If the initial separation is d, the capacitance is $\frac{A\epsilon_0}{d}$ and it becomes $\frac{A\epsilon_0}{2d}$ when the separation is doubled. Work done is then $\frac{A\epsilon_0}{4d}\varphi^2$