Magnetized Material (contd.) and Electromagnetic Induction

Lecture 28: Electromagnetic Theory

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In the first half of this lecture we will continue with our discussion of magnetized material and in the second half we will introduce the phenomenon of electromagnetic induction.

A Uniformly Magnetized Sphere

In the last lecture we treated the problem of a uniformly magnetized sphere using the scalar potential. In this lecture we will revisit the same problem using the vector potential.

We had seen earlier that the magnetic vector potential due to a magnetic moment is given by the expression

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

where $\vec{r}$ is the position vector of the point of observation with respect to the position of the magnetic moment.

Using this we can write down the expression for the vector potential at a position $\vec{r}$ due to magnetic moments I n a magnetized material having a magnetization $\vec{M}(\vec{r'})$, where, as before, we have used the primed quantities to indicate the variable to be integrated

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{Vol} \frac{\vec{M}(\vec{r'}) \times (\vec{r} - \vec{r'})}{\left|\vec{r} - \vec{r'}\right|^3} d^3r'$$

We had, in the last lecture, converted this into two integrals, one over the volume and the other over the surface,

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi} \int_S \hat{n'} \times \left(\frac{\vec{M}(\vec{r'})}{\left|\vec{r} - \vec{r'}\right|} \right) dS' + \frac{\mu_0}{4\pi} \int \frac{1}{\left|\vec{r} - \vec{r'}\right|} \nabla' \times \vec{M}\left(\vec{r'}\right) d^3r'$$

Thus $\hat{n} \times \vec{M}$ takes the role of a surface current. We now identify, as we did in the electrostatic case, a bound volume current and a bound surface current, defined by
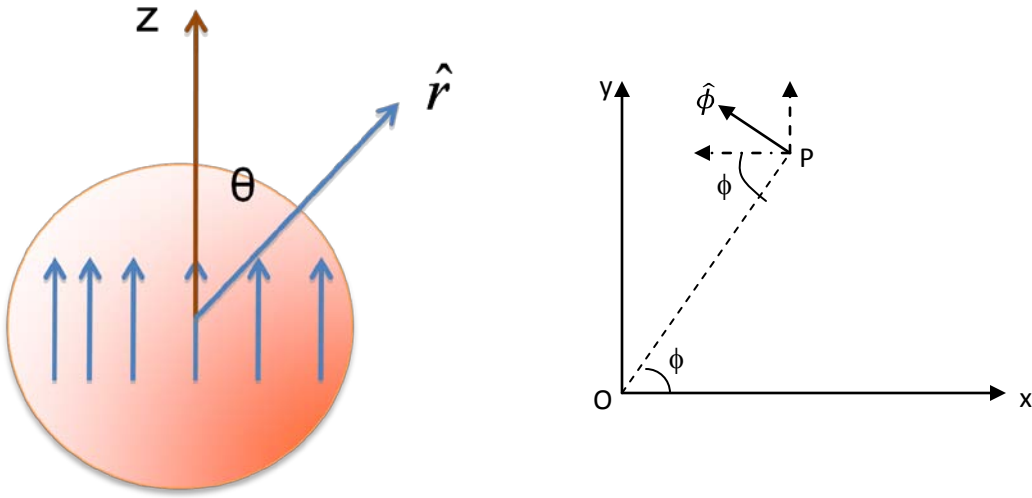
$$\vec{J}_M^v = \nabla \times \vec{M}$$

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$$\vec{J}_M^s = -\hat{n} \times \vec{M}$$

In the present case, since the magnetization is uniform, the bound volume current term is zero and we are left with only a surface term.

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi} \int_S \hat{n'} \times \left(\frac{\vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|} \right) dS'$$

Let us take the magnetization direction along the z axis,

In the figure above, the picture on the left shows the magnetized sphere with magnetization along the z direction. The normal to the surface of the sphere at every point is along the radial direction, so that the coordinate of an arbitrary point on the surface is (R, θ, ϕ). The angle ϕ is shown in the right hand figure, where the point P is the foot of the perpendicular from the point on the surface onto the equatorial plane, i.e. x-y plane. The angle that OP makes with the x axis is the azimuthal angle ϕ. Since the direction of the unit vector is always along the increasing value of the relevant coordinate, the unit vector $\hat{\phi}$ is as shown, and is given by,

$$\hat{\phi} = -\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}$$

Since the quantities which are integrated are primed, we will, in the following, use these relations with prime (the unit vectors of the Cartesian coordinates are always fixed). The surface current density is given by

$$\vec{J}_{sm}(\vec{r'}) = -\hat{n'} \times \vec{M}(\vec{r'}) = -M\sin\theta'\,\hat{r} \times \hat{k} = M\sin\theta'\,\widehat{\phi'}$$
$$= M\sin\theta'(-\sin\phi'\hat{\imath} + \cos\phi'\hat{\jmath})$$

Thus

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S \frac{M \sin\theta'(-\sin\phi'\hat{\imath} + \cos\phi'\hat{\jmath})}{|\vec{r} - \vec{r'}|} dS'$$

We will show later that the integral with the first term in the numerator gives zero. Let us calculate the second term. As before, we will expand the denominator in spherical harmonics. The numerator can be written as follows :

$$\sin\theta' \cos\phi' = -\sqrt{\frac{2\pi}{3}} \left(Y_{1,1}(\theta',\phi') - Y_{1,-1}(\theta',\phi')\right)$$

$$\sin\theta' \sin\phi' = \sqrt{\frac{2\pi}{3}} \left(Y_{1,1}(\theta',\phi') + Y_{1,-1}(\theta',\phi')\right)$$

$$\frac{1}{|\vec{r} - \vec{r'}|} = 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}^*(\theta',\phi') Y_{l,m}(\theta,\phi)$$

The surface integral is essentially an integral over the solid angle. The second term of the numerators would then give,

$$\vec{A}(\vec{r}) = -\sqrt{\frac{2\pi}{3}} \frac{\mu_0 M}{4\pi} \hat{\jmath} \int_S 4\pi \sum_{l,m} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{l,m}^*(\theta',\phi') \left(Y_{1,1}(\theta',\phi')\right.$$
$$\left. - Y_{1,-1}(\theta',\phi')\right) Y_{l,m}(\theta,\phi) R^2 d\Omega'$$

By orthogonality of spherical harmonics, $l = 1, m = \pm 1$ and we get

$$\vec{A}(\vec{r}) = -\sqrt{\frac{2\pi}{3}} \frac{\mu_0 M}{3} \frac{r_<}{r_>^2} R^2 \left[Y_{1,1}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\right]\hat{\jmath}$$

We can define the point of observation in the x-z plane because the sphere, other than for the z direction which defines magnetization direction, has symmetry in the xy plane so that we have liberty of defining the x axis. With this choice, $\phi = 0$ and we are left with,

$$\vec{A}(\vec{r}) = -\sqrt{\frac{2\pi}{3}} \frac{\mu_0 M}{3} \frac{r_<}{r_>^2} R^2 \left[Y_{1,1}(\theta,0) - Y_{1,-1}(\theta,0)\right]\hat{\jmath}$$

We now substitute the expressions for spherical harmonics

$$\vec{A}(\vec{r}) = \frac{\mu_0 M}{3} \frac{r_<}{r_>^2} R^2 2 \sin\theta$$

This also explains why the first term in the numerator gives zero for in that case the two spherical harmonics are to be added which gives zero (for , $\phi = 0$).
Inside the sphere, $r_> = R, r_< = r$ so that the vector potential inside the sphere is,

$$\vec{A}(r < R) = \frac{2\mu_0 M}{3} r \sin\theta = \frac{2}{3}\mu_0 M x \hat{\jmath}$$

Outside the sphere, $r_> = r, r_< = R$ so that the vector potential inside the sphere is,

$$\vec{A}(r < R) = \frac{2\mu_0 M}{3} \frac{R^3}{r^2} \sin\theta \, \hat{j}$$

Thus inside the sphere, the vector potential is in the y direction but its magnitude is proportional to x. We had seen that this is the vector potential one would get for a constant magnetic field in the z direction., because

$$\overrightarrow{B_{in}} = \nabla \times \vec{A} = \frac{\partial A_y}{\partial x} \hat{k} = \frac{2}{3} \mu_0 M \hat{k}$$

Suppose now the magnetized sphere were put in a uniform magnetic field $\overrightarrow{B_0}$, since the field inside is uniform, the net field within the sphere will be given by

$$\vec{B} = \frac{2}{3} \mu_0 \overrightarrow{M} + \vec{B}_0$$

The H field is given by

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \overrightarrow{M} = \frac{\vec{B}_0}{\mu_0} - \frac{\overrightarrow{M}}{3}$$

If the material is paramagnetic, we can write $\vec{B} = \mu \vec{H}$ so that we get,

$$\frac{2}{3} \mu_0 \overrightarrow{M} + \vec{B}_0 = \mu \left(\frac{\vec{B}_0}{\mu_0} - \frac{\overrightarrow{M}}{3} \right)$$

which gives

$$\overrightarrow{M} = \frac{3}{\mu_0} \frac{\mu - \mu_0}{\mu + 2\mu_0} \vec{B}_0$$

This gives the net magnetic field inside the material to be

$$\vec{B} = \frac{3\mu}{\mu + 2\mu_0} \vec{B}_0$$

You could compare this expression with the corresponding relation for the electric field inside a uniform dielectric in the presence of an external electric field.

Ferromagnetism

We will leave the discussion on magnetostatics with a short comment on material known as ferromagnets.

We have seen that in a paramagnet, the magnetization is proportional to the applied magnetic field and the magnetization would go to zero when such applied field is withdrawn. Consider a ferromagnet. When an external magnetic field is applied, the magnetization rises and as the field increases further, the rise, which is initially linear, becomes nonlinear. The cause of magnetization is alignment of the internal magnetic moments in the direction of the applied field. As the strength of the field increases, at some value of the applied field, all the magnetic moments are aligned and any further increase in the applied field has no effect on the magnetization because once all the magnetic moments are aligned, there is nothing to be done any further. This results in a "**saturation magnetization**".
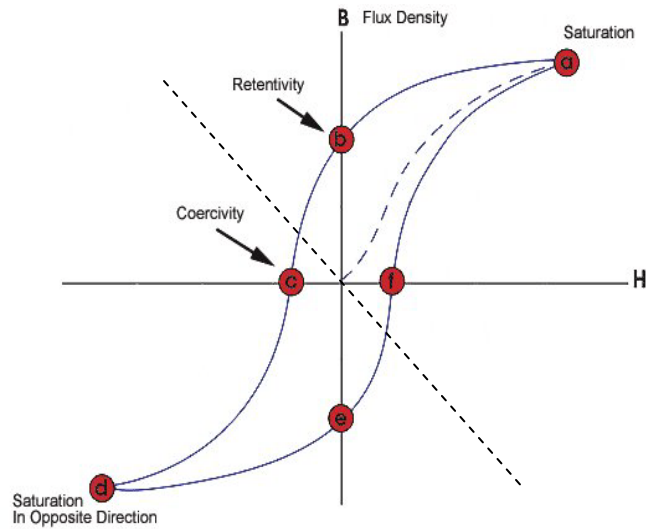
Suppose, now, we reduce the magnetic field gradually. The magnetization would decrease but interestingly, it does not take the path along which it increased when the magnetic field was increasing. It takes a different path and even when the applied field has become zero, there is some remnant magnetization left. This is called "hysteresis" , which is to say that a system has memory of the fact that it had been subjected to a magnetic field in the past. The magnetization value when the applied field is zero is known as the "**retentivity**".

If we now want, the magnetization to become zero, it would be necessary to apply a field in the reverse direction. The strength of the reverse field at which the magnetization becomes zero is called "coercive field". The hysteresis loop has a symmetry about the positive and the negative direction of the applied field, as shown in the figure.

Returning back to the case of a uniformly magnetized sphere, we had seen that

$$\vec{B} = \frac{2}{3}\mu_0\overrightarrow{M} + \vec{B}_0$$

$$\vec{H} = \frac{\vec{B}_0}{\mu_0} - \frac{\overrightarrow{M}}{3}$$

Note that H field acts to demagnetize the sphere.
From which, we get,

$$\vec{B} = 3\vec{B}_0 - 2\mu_0\vec{H}$$

i.e. , the B-H curve, in this case has a slope of -2. If one draws a straight line of slope -2, it will intersect the BH curve in the second quadrant from which the value of $\vec{B}$ and $\mu_0\vec{H}$ inside the sphere can

be found. In the absence of external magnetic field, the permeability is $\mu = -2\mu_0$. One can also calculate the magnetization from the known values of $\vec{B}$ and $\mu_0 \vec{H}$ from the relations given above.

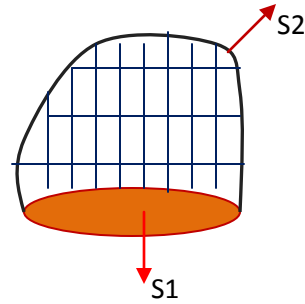**Time Varying Magnetic Field**

Flux of magnetic field has been defined as

$\int \vec{B} \cdot d\vec{S}$ over any surface, the direction of

$d\vec{S}$ is along the outward normal. Consider,

For instance a volume V bounded by two

Surfaces, a disk and a surface shaped

Like a fisherman's net. Since the surface

Integral over any closed surface is zero, it follows that the flux of the magnetic field through the surface S1 is the same as the flux through S2.

Electromotive Force

We had learnt earlier that electrostatic field is conservative so that $\oint \vec{E} \cdot d\vec{l} = 0$. If we assume something like the Ohm's law to be valid, this tells you that a purely irrotational field cannot drive a current through a circuit because the integral denotes the work done in taking a unit charge through the circuit. As the current is E/R, the rate at which the energy is dissipated is $\vec{J} \cdot \vec{E}$ and this cannot obviously be provided by a purely electrostatic field.

The agency responsible for providing current in a circuit is known as the "electromotive force" which is an unfortunate terminology which has remained for historic reasons. The correct dimension is that of electromagnetic work per unit charge.

The source which provides this current has got to be non-conservative, in a circuit connected to a battery, it is usually the chemical energy which gets converted to electrical energy and it is the battery which provides the non-conservative force. Suppose we assume that such a field exists, we can write the total electric field as a sum of two parts, one part $\vec{E}$ which is conservative and the other $\vec{E'}$ which is non-conservative. The current density is proportional to the sum of both these.

If we take a line integral of the total electric field, $\vec{E}_t = \vec{E} + \vec{E'}$, the line integral of the conservative part in the part of the circuit outside the battery is precisely canceled by the contribution within the battery. The electromotive force is then the line integral of the non-conservative part within the battery, i.e. $\mathcal{E} = \int \vec{E'} \cdot d\vec{l}$. However, in the outside circuit, the non-conservative field is zero and so it is immaterial if we add the integral over the outside circuit as well, making $\oint \vec{E'} \cdot d\vec{l}$. But now, since the closed loop

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integral of the conservative part of the field is anyway zero, we may as well define the electromotive force as the line integral of the total electric field itself, both conservative and non-conservative. We define the electromotive force as

$$\mathcal{E} = \oint \overrightarrow{E_t} \cdot d\vec{l}$$

Where the subscript t is unnecessary and will be dropped later.

Open circuit voltage:

In the case there is no current, and if $\sigma \neq 0$, this implies that the sum of the conservative and non-conservative fields must be zero because

$$\vec{J} = \sigma(\vec{E} + \overrightarrow{E'}) = 0$$

If we now take the line integral of the electric field between two points at the terminating points of the non-conservative region, e.g., between the terminals of a battery, we would have,

$$\varphi_2 - \varphi_1 = -\int_1^2 \vec{E} \cdot d\vec{l} = \int_1^2 \overrightarrow{E'} \cdot d\vec{l} = \oint \overrightarrow{E'} \cdot d\vec{l} = \mathcal{E}$$

i.e. the open circuit voltage between two points is equal to the total electromotive force in the circuit.

Faraday's Law

Michael Faraday enunciated the law of electromagnetic induction from experimental observations. He found that whenever there is a relative motion between a circuit and a source of magnetic field (such as a current or a permanent magnet), or if there is no relative motion but if the magnetic field is changing with time, there an emf induced in the circuit which is proportional to the rate of change of the magnetic flux through the circuit,

$$\mathcal{E} \propto \frac{d\Phi_B}{dt}$$

The law is usually written as

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

The minus sign here is not so much an algebraic quantity but a reminder of another law, known as "Lenz's law" which states that if the circuit can support a current, the direction of the current would be such that the magnetic field generated by the induced current would tend to oppose the cause of the emf itself, viz., if the flux was increasing, the direction will be so as to decrease it and vice versa.

Motional emf

Explanation of Faraday's law is fairly straightforward for the case where there is a relative motion between a circuit and the source of the magnetic field, the corresponding emf is known as the "motional emf". In such a case, the charges in the circuit will be seen to be moving by an observer who is stationary in the laboratory. As the charges are moving, they will be subjected to sidewise Lorentz force which will result in stretching of the circuit.

Consider the circuit shown in which an element $d\vec{l}$ is instantaneously moving along the direction shown, stretching and deforming the circuit. Let us take the magnetic field into the plane of the paper. As moving charge in the element $d\vec{l}$ will be subject to Lorentz force $q\vec{v} \times \vec{B}$, which leads to an emf given by
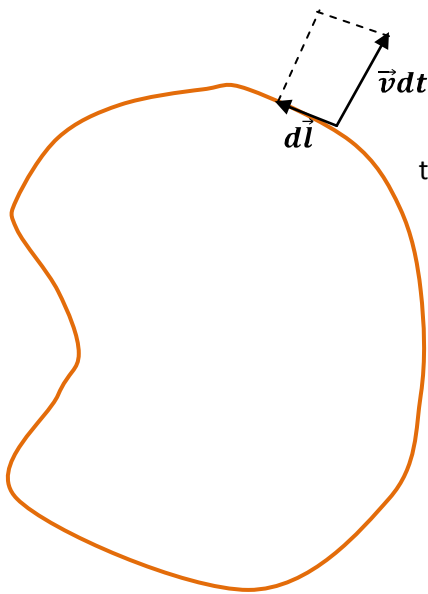
$$\mathcal{E} = \oint \frac{\vec{F} \cdot d\vec{l}}{q} = \oint \left(\vec{v} \times \vec{B} \right) \cdot d\vec{l}$$

Using the cyclicity of the scalar triple product,

$$\mathcal{E} = \oint \vec{B} \cdot \left(d\vec{l} \times \vec{v} \right)$$

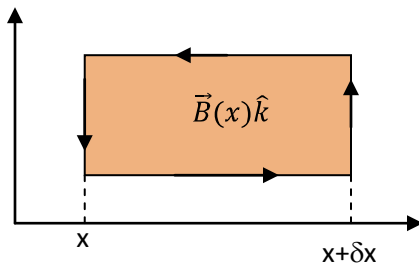$d\vec{l} \times \vec{v}$ is the rate at which the area is increasing,

so that

$$\mathcal{E} = -\oint \vec{B} \cdot \frac{d\vec{S}}{dt} = -\frac{d\Phi_B}{dt}$$

The minus sign follows from the usual direction of the area in terms of outward normal.

Consider the case where the loop not being stretched but is moving bodily. For simplicity, we consider a rectangular loop moving along its length in an inhomogeneous magnetic field.

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Let the instantaneous position of the loop be as shown in the figure. The force on a charge which has a velocity in the positive x direction (because of the direction in which the loop is moving) is along the negative y direction. The magnitude of such force on the charges in the top and the bottom are the same and since the line integral is taken along opposite directions, the contribution to the emf from these two sides would cancel, leaving us with the contribution from the left and the right edges, which gives for the line integral

$$\oint (\vec{v} \times \vec{B}) \cdot d\vec{l} = -\left(\vec{v} \times \vec{B}(x)\right) \cdot (-\hat{j})\delta y + \left(\vec{v} \times \vec{B}(x + \delta x)\right) \cdot (-\hat{j})\delta y$$

where δy is the width of the rectangle. Assuming δx to be small, we expand the magnetic field strength in a Taylor series,

$$B(x + \delta x) = B(x) + \frac{\partial B}{\partial x}\delta x$$

which gives,

$$\mathcal{E} = -\left(\vec{v} \times \frac{\partial B}{\partial x}\hat{k}\right) \cdot (-\hat{j})\delta x \, \delta y$$

Since $\vec{v} = \frac{\partial x}{\partial t}$, this gives,

$$\mathcal{E} = -\left(v \times \frac{\partial B}{\partial x}\right)\delta x \, \delta y = -\frac{\partial B}{\partial t}\delta S = -\frac{d\Phi_B}{dt}$$

It is important to realize that in the above example, we have shown that a motional emf is developed because the charges experience a force. However, there is no way a charge can tell whether the force arises due to its state of motion with respect to the source of a magnetic field so that it experiences a changing flux or it is just because the magnetic field itself is changing with time. In either case the effect would be the same. This is what was borne out by the experiments of Faraday.

Thus for whatever reason, there is a change in the magnetic flux in a circuit, it leads to an induced emf, which is the same as an induced electric field. This electric field is not conservative and we have,

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{S}$$

Using Stoke's theorem, we can write the line integral as a surface integral of a curl,

$$\int_S (\nabla \times \vec{E}) \cdot d\vec{S} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S}$$

The time derivative can be taken inside the integral and we can equate the integrands, as the relation above is true for arbitrary surface,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Note that in the absence of time varying field, the electric field is conservative. This is the modified form of Maxwell's equation including time varying magnetic field.


Magnetostatics IV

Lecture 27: Electromagnetic Theory

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Tutorial Assignment

1. A square loop of side a lies in the xy plane, as shown. A magnetic field exists in the region directed along the z direction and varies with time and space as $\vec{B} = B_0 t^2 x \hat{k}$, where B_0 is appropriately dimensioned. Calculate the emf developed in the loop. If the x component of the induced electric field is zero, obtain an expression for the electric field induced and show that the line integral of the electric field correctly gives the emf calculated.
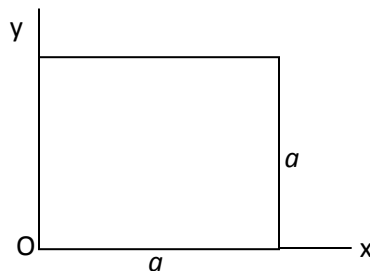
2. A flip coil is a coil that is flipped (turned by 180^0) rapidly in a static magnetic field. What is the charge transported through the coil when it is so flipped? Assume that the resistance of the coil is R and that Ohm's law remains valid.

3. Faraday's disk generator is a metal disk of radius R rotated about its axis with a constant angular speed w in a constant magnetic field B directed parallel to the axis of rotation. Find the emf developed between the axis and the rim. Verify your result by direct application of Faraday's law and by calculating motional emf.

4. A massless conducting rod of resistance R lies on two perfectly conducting horizontal rails separated by a distance d on a table, at one end the rods are connected by a conducting wire. . A vertically upward magnetic field B exists in the region. If the rod is connected by a pulley

arrangement which supports a mass m which falls down as the rod moves outward on the rails. Calculate the terminal velocity of the mass.
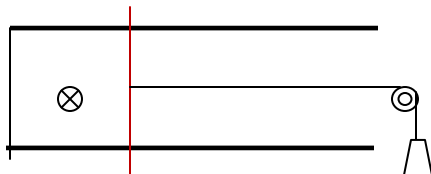
5. A conducting loop of radius a , mass m and resistance R falls freely in such a way that its plane remains parallel to the xy plane. An inhomogeneous magnetic field $\vec{B} = B_0 \, (1 + kz)\hat{z}$ exists in the region, where z is the height through which the loop falls. Calculate the terminal velocity of the loop.

Solutions to Tutorial Assignments

1. Emf is given by Faraday's law, $\mathcal{E} = -\frac{d}{dt} \int \vec{B} \cdot d\hat{S} = -\frac{d}{dt} (B_0 t^2) \int_0^a xa \, dx = -B_0 t \, a^3$
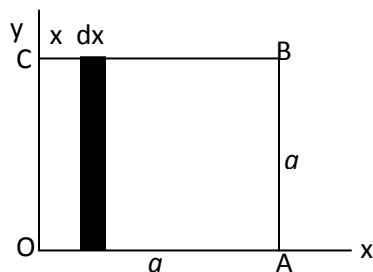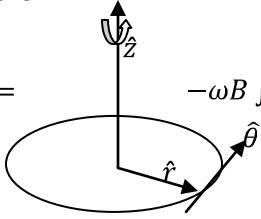

Calculate the curl of the electric field $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -2B_0 tx\hat{k}$. Expanding the curl and equating its z component with the rhs, and setting $E_x = 0$, we get

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} = -2B_0 tx$$

On integrating, we get, $\vec{E} = -B_0 tx^2 \hat{j}$. Consider the loop OABC in the counterclockwise direction. The contribution to the line integral $\oint \vec{E} \cdot d\vec{l}$ from the sides OA and BC are zero because the line is along the x direction while the field is along y. Contribution from CO is also zero because on this line x=0 implies E=0. The only contribution is from AB (x=a), in which the line integral is

$$\int_0^a E_y dy = -B_0 ta^2 \int_0^a dy = -B_0 \, ta^3$$

2. The current through the loop is given by $\frac{\mathcal{E}}{R} = \frac{1}{R}\frac{d}{dt}\int \vec{B}\cdot d\vec{S}$. Thus the charge transported is $I\Delta t = \frac{1}{R}(\phi_f - \Phi_i)$, which depends on the difference between the initial and the final flux. If the coil has an area A and has N turns, assuming that the magnetic field is perpendicular to the plane of the coil in the initial and final configurations where the normal makes angle of 0^0 and 180^0, the charge is $\delta Q = \frac{2NAB}{R}$.

3. In cylindrical coordinates, $\vec{v} = \omega R\hat{\theta}$, $\vec{B} = B\hat{k}$. Since we are interested in calculating the motional emf between axis and the rim, we take a line connecting the centre of the disk and the circumference so that $d\vec{l} = \hat{r}dr$.

$$\mathcal{E} = -\int (\vec{v}\times\vec{B})\cdot d\vec{l} = -\int (\omega r\hat{\theta}\times B\hat{k})\cdot\hat{r}dr$$

$$= -\omega B\int_0^R rdr = -\frac{\omega BR^2}{2}$$

One can directly apply Faraday's law by considering a radial vector and calculate the area that it sweeps in time dt.

As it sweeps a sector of the circle of angle ωdt, the area is $dA = \frac{R^2 d\theta}{2}$ so that $-\frac{d\Phi}{dt} = -\frac{BR^2}{2}\frac{d\theta}{dt} = -\frac{\omega BR^2}{2}$.

4. As the rod moves to right by a distance x, it sweeps an area Lx so that the rate of change of flux is $-BL\frac{dx}{dt} = -BLv$. The current in the circuit is $\frac{BLv}{R}$. The force experienced by the rod is $ILB = \frac{B^2 L^2 v}{R}$. Terminal velocity is attained when this force equals mg, i.e. $v_t = \frac{mgR}{B^2 L^2}$.

5. When the loop has fallen through a distance z, the flux through the loop is $\Phi = B_0(1 + kz)\pi a^2$ so that the emf induced in the loop is $-\frac{d\Phi}{dt} = -B_0 k \frac{dz}{dt}\pi a^2 = B_0 k v \pi a^2$, where v is the instantaneous downward velocity of the loop. The current induced in the loop is $I = B_0 k v \frac{\pi a^2}{R}$ in the counterclockwise direction, as viewed from above so that the field due to the induced current tends to decrease the flux. The rate at which energy is dissipated is $I^2 R = B^2 k^2 v^2 \pi^2 a^4 R$. The potential energy decreases by mgz so that the rate of change of change of potential energy is mgv. (Remember that once terminal velocity is attained there is no change in the kinetic energy so that the loss in potential energy is at the expense of Joule heat alone.) Equating these, we get, $v_t = \frac{mgR}{B^2 k^2 \pi^2 a^4}$.
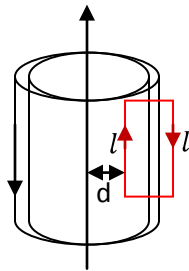
Self Assessment Questions

1. A conducting bar of resistance R slides frictionlessly with a constant speed upward over a pair of resistanceless rails inclined at an angle 2q under the action of some external force. A uniform magnetic field of strength B subsists in the region perpendicular to the plane of the rails as shown. Find the current induced in the circuit both by direct use of Faraday's law and by explicit use of Lorentz force for calculation of motional emf.
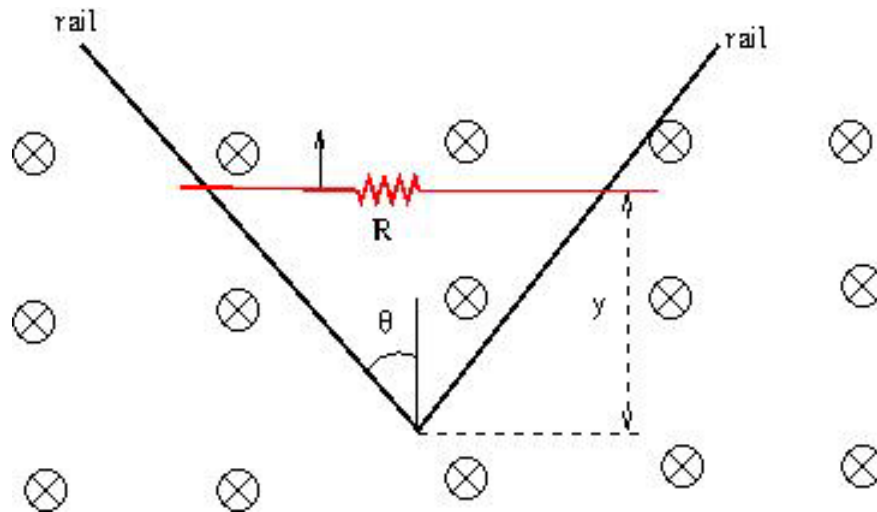
2. A rectangular conducting loop of width w , length L and resistance R falls under gravity keeping its plane always vertical . An inhomogeneous magnetic field $\vec{B} = -\frac{B_0 y}{y_0} \hat{k}$ points into the page (xy) plane). Find the terminal velocity of the loop.
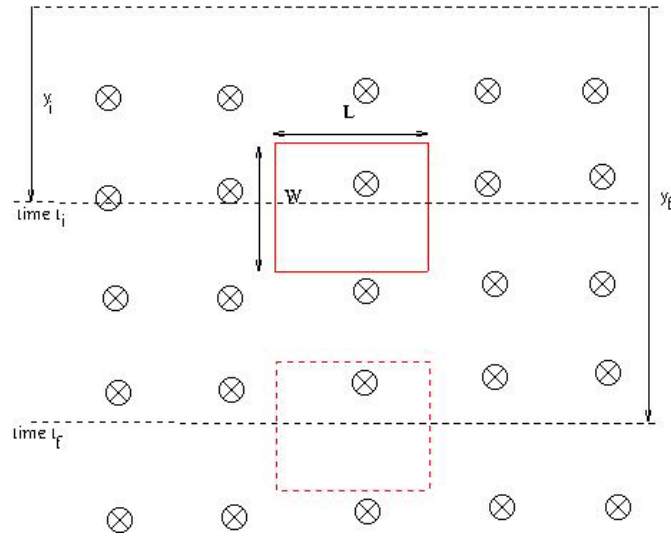
3. A circular loop of radius R rotates with an angular speed ω in a uniform magnetic field directed along the x axis. The loop rotates about an axis parallel to the z axis. Calculate the emf induced in the loop.

4. A long coaxial cable consists of a thin wire surrounded by a concentric shell of radius R. Current goes up through the wire and returns through the shell. If the current varies with time at a constant rate $\frac{dI}{dt}$, calculate the emf developed and the induced electric field in the rectangular loop of side l shown in the figure.

Solutions to Self Assessment Questions

1. From geometry, we have, $\tan\theta = w/y$ where 2w is the instantaneous length of the bar enclosed by the circuit and y the distance from the junction of the rails. The area swept by the bar is

$$A = wy = y^2\tan\theta$$

The emf, as given by Faraday's law is

$$\mathcal{E} = -B\frac{dA}{dt} = -2By\tan\theta\frac{dy}{dt} = -2Byv\tan\theta = -2Bv^2t\tan\theta$$

assuming that the bar started sliding at t=0 from the junction. The current is obtained by dividing this expression by R.

Since only the bar is moving, the motional emf is only along the sliding bar and is given by $\mathcal{E} = \int vBdl = vB\,(2w) = vB\,2y\tan\theta$, which gives the same result as above.

2. At a time t, when the centre of the loop is at the position y, the flux threading the loop is given by $\int \vec{B} \cdot d\vec{S} = \frac{LB_0}{y_0} \int_{y-\frac{w}{2}}^{y+\frac{w}{2}} y\,dy = \frac{LB_0 yw}{y_0}$. The emf is given by $\frac{LB_0 w}{y_0}\frac{dy}{dt} = \frac{LwvB_0}{y_0}$. The current direction is counterclockwise as per Lenz's law.

   Let us look at the problem from Lorentz force point of view. The force on the current is ILB. The force on the two sides cancel, being equal and opposite. The vertical forces are $mg - ILB_0(y_{top} - y_{bot}) = mg - ILB_0 w$. The terminal velocity occurs when the net force is zero, so that the current in the loop should be $\frac{mg}{LB_0 w}$, which requires the terminal velocity to be
$$v_t = \frac{mgy_0 R}{(LwB_0)^2}$$

3. If θ is the instantaneous angle between the normal to the loop and the direction of the magnetic field,
$$\mathcal{E} = -\frac{d}{dt}\int B\cos\theta\,dS = -BA\frac{d}{dt}(\cos\omega t) = B\omega\,(\sin\omega t)\,\pi R^2$$

4. We assume that the instantaneous field is given by the corresponding steady state expression, I.e. the magnetic field at a distance r for a long straight wire is given by $\frac{\mu_0 I(t)}{2\pi r}$. Thus the flux through a strip at a distance r from the wire between the wire and the she shell is $B(t)ldr$ so that the total flux is (since the field outside is zero),
$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_d^R \frac{\mu_0 I(t)}{2\pi r}ldr = -\frac{\mu_0 l}{2\pi}\frac{dI(t)}{dt}\ln\frac{R}{d}$$
Since the electric field can only depend on the distance,
$$El = -\frac{\mu_0 l}{2\pi}\frac{dI(t)}{dt}\ln\frac{R}{d}$$
which gives,
$$E = -\frac{\mu_0}{2\pi}\frac{dI(t)}{dt}\ln\frac{R}{d}$$