MA2102: LINEAR ALGEBRA

Lecture 10: Linear Maps

8th September 2020

Indian Institute of Science Education & Research Kolkata

We want to study *appropriate* notion of maps (or transformation) between vector spaces.

Definition [Linear Map] A map $T : V \to W$ between vector spaces (over a field F) is called a linear transformation if

(i) $T(v_1 + v_2) = T(v_1) + T(v_2)$ for any $v_1, v_2 \in V$;

(ii) $T(\lambda v) = \lambda T(v)$ for any $\lambda \in F$ and $v \in V$

Properties of linear maps A few observations are in order.

- $T(\mathtt{0}_V) = \mathtt{0}_W$

Note that

$$T(\mathtt{0}_V) = T(\mathtt{0}_V + \mathtt{0}_V) = T(\mathtt{0}_V) + T(\mathtt{0}_V)$$

implies that $T(\mathtt{0}_V) = \mathtt{0}_W$.

We may also conclude the same by putting $\lambda = \mathtt{0}, v = \mathtt{0}_V$ *in (ii).*

- $T(c_1v_1+\cdots+c_kv_k)=c_1T(v_1)+\cdots+c_kT(v_k)$

Note that the equation is true for $k=1$ by (ii). For $k=2$

$$T(c_1v_1+c_2v_2)=T(c_1v_1)+T(c_2v_2)=c_1T(v_1)+c_2T(v_2).$$

More generally,

$$\begin{aligned} T(c_1v_1+\cdots+c_kv_k) &= T(c_1v_1)+T(c_2v_2+\cdots+c_kv_k) \\ &= c_1T(v_1)+T(c_2v_2+\cdots+c_kv_k). \end{aligned}$$

We iterate this to obtain our identity.

Remark This property implies (i) (set $k=2$ and $c_1=c_2=1$) and (ii) (set $k=1$). We may define a linear map as a map $T:V\to W$ satisfying this property.

Example We shall discuss several instances of linear maps.

(1) Scaling: Let $T:\mathbb{R}\to\mathbb{R}$, $x\mapsto 3x$. Since

$$T(x+y)=3(x+y)=3x+3y=T(x)+T(y)$$

and $T(cx)=3cx=cT(x)$, T is a linear map. Show that any linear map $T:\mathbb{R}\to\mathbb{R}$ is of the form $T(x)=cx$.

(2) Dilation: Let $T:V\to V$ be $T(v)=2v$. This is a linear map *but* may be zero if $2=1+1$ is zero in F. If 2 is invertible in the field F, then T is injective. If $T(v)=T(w)$, then $2(v-w)=\mathtt{0}$, which implies that $v=w$.

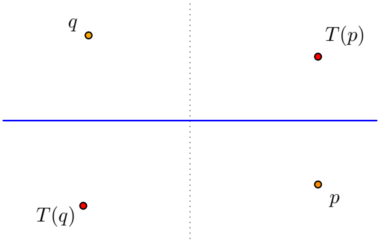
(3) Identity: Let $I:V\to V$, $v\mapsto v$ be the identity map.

(4) Trivial: Let $\mathbb{O}:V\to W$, $v\mapsto \mathtt{0}_W$. If $W=V$ then the trivial map is dilation by zero.

(5) Matrices: Any $A \in M_{m\times n}(\mathbb{R})$ defines a map $L_A : \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

(6) Reflections: Consider the reflection T of $\mathbb{R}^2$ across the x-axis.



$T(x,y) = (x,-y)$

Compute the reflection S across the line $y = x$ and show that S, T are linear maps.

Remark Reflection across lines that do not pass through the origin, are not linear maps as $(0,0)$ is not mapped to $(0,0)$.

(7) Rotations: Consider the linear map

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \;\; T(x,y) = (-y,x).$$

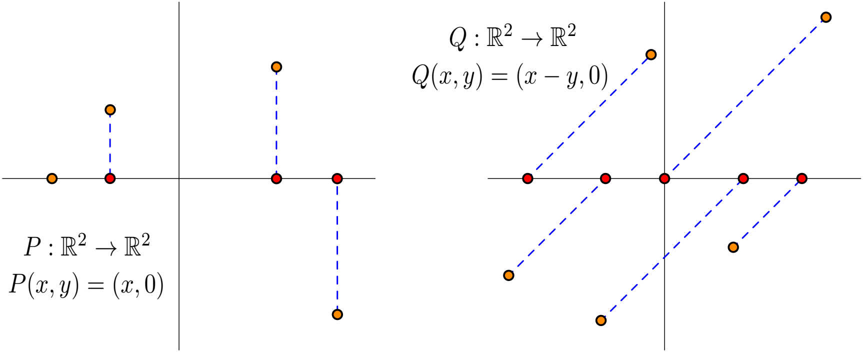
Note that $(x,y) \cdot T(x,y) = 0$ and $||(x,y)|| = ||T(x,y)||$. This implies that T is a rotation counter-clockwise by $\pi/2$. More generally, consider the counter-clockwise rotation by angle θ

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2, \;\; \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show that T_θ is a linear map. Moreover, show that $T_{\theta_1+\theta_2} = T_{\theta_1} \circ T_{\theta_2}$.

In complex co-ordinates, $T_\theta : \mathbb{C} \to \mathbb{C}$ is multiplication by $e^{i\theta}$.

(8) Projections: Consider the linear maps given below:



Show that P and Q are linear maps which satisfy $A^2 = A$.

Other standard example include

$$P : \mathbb{R}^3 \to \mathbb{R}^3,\ P(x, y, z) = (x, y, 0).$$

(9) Inclusions: Consider the linear maps below:

$$\begin{aligned} T_1 : \mathbb{R} \to \mathbb{R}^2, &\quad x \mapsto (x, 0) \\ T_2 : \mathbb{R} \to \mathbb{R}^2, &\quad x \mapsto (x, x) \\ T_3 : \mathbb{R} \to \mathbb{R}^2, &\quad x \mapsto (x, mx). \end{aligned}$$

We are identifying $\mathbb{R}$ with a line in $\mathbb{R}^2$. We may also consider

$$T : \mathbb{R}^2 \to \mathbb{R}^3,\ T(x, y) = (x, y, ax + by).$$

It is an injective linear map whose image is the plane $z = ax + by$.