

Conservation of Energy and Momentum

Lecture 31: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Coulomb Gauge and the Potential formulation of Maxwell's equations

The Maxwell's equations in their final form are

$$\begin{aligned}\nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \Leftrightarrow \nabla \cdot \vec{D} = \rho_{free} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{H} &= \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}\end{aligned}$$

We had, in the last lecture, made a reformulation of these equations in terms of scalar and vector potentials. This gave us two “coupled” equations for four quantities, i.e. 3 components of the vector potential and one component of scalar potential. We had seen that these equations get decoupled in Lorentz gauge.

We had a lot of discussion on Coulomb gauge in which the divergence of the vector potential is zero. Is this gauge any good to be used now?

Recall the pair of equations,

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0} \quad (1)$$

$$\left(\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \right) = \nabla \left(\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right) - \mu_0 \vec{J} \quad (2)$$

Note that first equation gets decoupled in Coulomb gauge and becomes a Poisson's equation for the scalar potential with the formal solution,

$$V(\vec{x}, t) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|}$$

The second equation is not straightforward and requires a lot of mathematical manipulation before the decoupling can be seen.

To avoid repeating the same expression unnecessarily, we will concentrate on the right hand side of eqn. (2). In Coulomb gauge the term with the divergence drops out and the right hand side of (2) becomes,

$$\frac{1}{c^2} \nabla \left(\frac{\partial V}{\partial t} \right) - \mu_0 \vec{J}$$

We insert the formal solution of the scalar potential into this equation,

$$\frac{1}{c^2} \frac{1}{4\pi\epsilon_0} \nabla \left(\int d^3x' \frac{\partial \rho(\vec{x}', t)}{\partial t} \frac{1}{|\vec{x} - \vec{x}'|} \right) - \mu_0 \vec{J}$$

Now the term $\frac{\partial \rho(\vec{x}', t)}{\partial t}$ is a current and we replace it by $-\nabla' \cdot \vec{J}(\vec{x}', t)$ using the equation of continuity and the term becomes,

$$\frac{1}{c^2} \frac{1}{4\pi\epsilon_0} \nabla \left(\int d^3x' \left(-\nabla' \cdot \vec{J}(\vec{x}', t) \right) \frac{1}{|\vec{x} - \vec{x}'|} \right) - \mu_0 \vec{J} \quad (I)$$

We will do some further simplification to (I). But first let us recall that we have learnt that a vector is completely determined when its divergence and curl are specified. This allows us to write the current density vector inside the integral as a sum of one part \vec{J}_t whose divergence is zero and another part \vec{J}_l whose curl is zero,

$$\vec{J} = \vec{J}_t + \vec{J}_l$$

Using the identity,

$$\nabla \times (\nabla \times \vec{J}) = \nabla(\nabla \cdot \vec{J}) - \nabla^2 \vec{J}$$

we can write,

$$\begin{aligned} \nabla^2 \vec{J}_t &= -\nabla \times (\nabla \times \vec{J}_t) \equiv -\nabla \times (\nabla \times \vec{J}) \\ \nabla^2 \vec{J}_l &= \nabla(\nabla \cdot \vec{J}_l) \equiv \nabla(\nabla \cdot \vec{J}) \end{aligned}$$

where we have used $\nabla \cdot \vec{J}_t = 0$ and $\nabla \times \vec{J}_l = 0$ in arriving at both these relations. We will shortly return to using these relations.

Let us get back to the relation (I). Note that the gradient outside is taken with respect to the point of observation. We can take it inside the integration and it will act only on $\frac{1}{|\vec{x}-\vec{x}'|}$. However, since this depends only on the difference of \vec{x} and \vec{x}' , we can replace ∇ by ∇' by incorporating a minus sign. With this (I) becomes

$$\frac{1}{c^2} \frac{1}{4\pi\epsilon_0} \left(\int d^3x' (\nabla' \cdot \vec{J}(\vec{x}', t)) \nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) \right) - \mu_0 \vec{J}$$

We will simplify this further by using chain rule differentiation,

$$(\nabla' \cdot \vec{J}(\vec{x}', t)) \nabla' \left(\frac{1}{|\vec{x}-\vec{x}'|} \right) = \nabla' \left(\frac{\nabla' \cdot \vec{J}(\vec{x}', t)}{|\vec{x}-\vec{x}'|} \right) - \frac{1}{|\vec{x}-\vec{x}'|} \nabla' (\nabla' \cdot \vec{J}(\vec{x}', t))$$

Substituting this into the preceding term, we have two integrals, one of which is a volume integral of a gradient. This term can be converted to a surface integral like the way we do for the divergence theorem and take to surface to infinity to make this term zero. This implies the remaining integral is over all space and we get

$$\begin{aligned} & -\frac{1}{c^2} \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\nabla' (\nabla' \cdot \vec{J}(\vec{x}', t))}{|\vec{x}-\vec{x}'|} - \mu_0 \vec{J} \\ & = -\frac{\mu_0}{4\pi} \int d^3x' \frac{\nabla' (\nabla' \cdot \vec{J}_l(\vec{x}', t))}{|\vec{x}-\vec{x}'|} - \mu_0 \vec{J} \end{aligned}$$

in obtaining this step, we have used $c^2 = 1/\mu_0\epsilon_0$ and used the fact that $\nabla' \cdot \vec{J}(\vec{x}', t)$ can be written as $\nabla' \cdot \vec{J}_l(\vec{x}', t)$ because the transverse part has zero divergence. Further, we can use the fact that \vec{J}_l is irrotational to write,

$$\nabla' (\nabla' \cdot \vec{J}_l(\vec{x}', t)) = \nabla'^2 \vec{J}_l(\vec{x}', t) + \nabla' \times (\nabla' \times \vec{J}_l(\vec{x}', t)) \equiv \nabla'^2 \vec{J}_l(\vec{x}', t)$$

With this (I) takes the form

$$-\frac{\mu_0}{4\pi} \int d^3x' \frac{\nabla'^2 \vec{J}_l(\vec{x}', t)}{|\vec{x}-\vec{x}'|} - \mu_0 \vec{J}$$

At this stage, we will use the Green's identity for fields T and U according to which

$$\int_V d^3x (T \nabla^2 U - U \nabla^2 T) = \int_S d\vec{S} \cdot (T \nabla U - U \nabla T)$$

As our fields vanish at infinity, we have,

$$\int_V d^3x T \nabla^2 U = \int d^3x U \nabla^2 T$$

Using this, we get,

$$\begin{aligned} \left(\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} \right) &= -\frac{\mu_0}{4\pi} \int d^3x' \vec{j}_l(\vec{x}', t) \nabla'^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) - \mu_0 \vec{j} \\ &= -\frac{\mu_0}{4\pi} \int d^3x' \vec{j}_l(\vec{x}', t) \left(-\delta^3(\vec{x} - \vec{x}') \right) - \mu_0 \vec{j} \\ &= \mu_0 \vec{j}_l(\vec{x}, t) - \mu_0 \vec{j} = -\mu_0 \vec{j}_t \end{aligned}$$

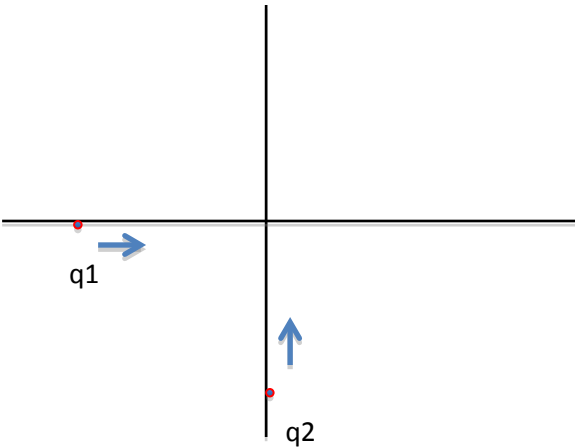
Thus the original equation for the vector potential has now been completely decoupled from the scalar part and we have, instead, an inhomogeneous wave equation,

$$\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}_t$$

Electromagnetic Momentum

We have seen that the electromagnetic field carries energy. A natural question arises as to whether it carries momentum as well. The answer is affirmative and we will illustrate this by a simple procedure. A more rigorous derivation requires use of the theory of relativity.

Suppose, we have two charged particles, q_1 and q_2 , the former moving along the x axis while the latter moves along the y axis.



The force on q_1 just as it passes by the origin is purely electrical as it lies along the direction of motion of q_2 , and is given by

$$\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{d^2}$$

where d is the distance between the charges at that instant. This is also the magnitude of the electric force on q_2 due to q_1 .

However, if we look at the force on q_2 due to q_1 , there is also a magnetic force. This is because the moving charge q_1 creates a magnetic field in the z direction which exerts a force on q_2 . This is certainly anomalous and is in apparent violation of the third law. We say it is an apparent violation because the third law is essentially a statement of conservation of momentum and it is the total momentum of the system that needs to be conserved. In this case, in addition to the two charges, there exists the electromagnetic field and if the field itself carries momentum, there is no violation. This is intuitive but the fact that electromagnetic field carries momentum is a fact.

Let us look at the force exerted on a system of sources (charges and currents) and electromagnetic field. Let \vec{P}_{mech} represent the moment associated with the sources. The force exerted on the system of sources is then given by Lorentz force equation

$$\begin{aligned}\vec{F} &= \frac{d\vec{P}_{mech}}{dt} = \int \rho(\vec{E} + \vec{v} \times \vec{B})d^3x \\ &= \int (\rho\vec{E} + \vec{j} \times \vec{B})d^3x\end{aligned}$$

Once again, for simplicity, we will consider linear electric and linear magnetic material. We will use Maxwell's equations to cast these equations in terms of field variables. We replace ρ using Gauss's law and the current density from Ampere's law,

$$\rho = \epsilon_0 \nabla \cdot \vec{E}$$

and

$$\vec{j} = \frac{1}{\mu_0} \left[\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right]$$

Substituting these in the force equation, we get,

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$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} = \int \epsilon_0 \vec{E} \nabla \cdot \vec{E} d^3x + \int \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} d^3x - \epsilon_0 \int \frac{\partial \vec{E}}{\partial t} \times \vec{B} d^3x$$

Let us simplify some of the terms,

$$\begin{aligned}\frac{\partial \vec{E}}{\partial t} \times \vec{B} &= \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) - \vec{E} \times \frac{\partial \vec{B}}{\partial t} \\ &= \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) + \vec{E} \times (\nabla \times \vec{E})\end{aligned}$$

In the last expression we have used Faraday's law. Thus we have,

$$\begin{aligned}\frac{d\vec{P}_{mech}}{dt} &= \int \epsilon_0 \vec{E} \nabla \cdot \vec{E} d^3x - \int \frac{1}{\mu_0} \vec{B} \times (\nabla \times \vec{B}) d^3x - \epsilon_0 \int \vec{E} \times (\nabla \times \vec{E}) d^3x \\ &\quad - \epsilon_0 \int \frac{\partial}{\partial t} (\vec{E} \times \vec{B}) d^3x\end{aligned}$$

In the second term we have changed the order of cross product and hence a minus sign.

The last term on the right can be seen to be $\epsilon_0 \mu_0 \int \frac{\partial}{\partial t} (\vec{E} \times \vec{H}) d^3x = \frac{1}{c^2} \int \frac{\partial}{\partial t} \vec{S} d^3x$, where $\vec{S} = \vec{E} \times \vec{H}$ is the Poynting vector. Since the expression has the dimension of force, $\frac{S}{c^2}$ has the dimension of momentum density and we will identify this term as the rate of change of momentum associated with the electromagnetic field (radiation field) and take it to the left hand side to be added to the rate of change of momentum of the sources P_{rad} ,

$$\frac{d\vec{P}_{mech}}{dt} + \frac{dP_{rad}}{dt} = \int \epsilon_0 \vec{E} \nabla \cdot \vec{E} d^3x - \int \frac{1}{\mu_0} \vec{B} \times (\nabla \times \vec{B}) d^3x - \epsilon_0 \int \vec{E} \times (\nabla \times \vec{E}) d^3x$$

The expression on the right looks asymmetric in the electric and magnetic quantities which can be rectified easily. Let us look at the electric field terms represented by the first and the third terms on the right.

The integrand is $\vec{E} \nabla \cdot \vec{E} - \vec{E} \times (\nabla \times \vec{E})$, other than for a factor of ϵ_0 . In order to simplify this to desired form, we will calculate the Cartesian components of this. Let us find what its x component is and then we will add up the three components.

$$\begin{aligned} [\vec{E} \nabla \cdot \vec{E} - \vec{E} \times (\nabla \times \vec{E})]_x &= E_x \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) - E_y (\nabla \times \vec{E})_z + E_z (\nabla \times \vec{E})_y \\ &= E_x \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) - E_y \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) + E_z \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \\ &= \frac{1}{2} \frac{\partial}{\partial x} (E_x^2 - E_y^2 - E_z^2) + \frac{\partial}{\partial y} (E_x E_y) + \frac{\partial}{\partial z} (E_x E_z) \\ &= \frac{\partial}{\partial x} (E_x^2) - \frac{1}{2} \frac{\partial}{\partial x} E^2 + \frac{\partial}{\partial y} (E_x E_y) + \frac{\partial}{\partial z} (E_x E_z) \end{aligned}$$

From symmetry, one can write the y and z components,

$$\begin{aligned} [\vec{E} \nabla \cdot \vec{E} - \vec{E} \times (\nabla \times \vec{E})]_y &= \frac{\partial}{\partial y} (E_y^2) - \frac{1}{2} \frac{\partial}{\partial y} E^2 + \frac{\partial}{\partial z} (E_y E_z) + \frac{\partial}{\partial x} (E_y E_x) \\ [\vec{E} \nabla \cdot \vec{E} - \vec{E} \times (\nabla \times \vec{E})]_z &= \frac{\partial}{\partial z} (E_z^2) - \frac{1}{2} \frac{\partial}{\partial z} E^2 + \frac{\partial}{\partial x} (E_z E_x) + \frac{\partial}{\partial y} (E_z E_y) \end{aligned}$$

Let us add the three components. The second term of each of the expressions add to give us

$$-\frac{1}{2} \left(\hat{i} \frac{\partial}{\partial x} E^2 + \hat{j} \frac{\partial}{\partial y} E^2 + \hat{k} \frac{\partial}{\partial z} E^2 \right) = -\frac{1}{2} \nabla E^2$$

The remaining terms look messy and we will return to them shortly.

Let us consider the magnetic field terms. In order to make it symmetrical with the magnetic field, we need to add a term $\frac{1}{\mu_0} \vec{B} \nabla \cdot \vec{B}$ to the triple vector product term. This is simply adding

zero because $\nabla \cdot \vec{B} = 0$. However, this will make the electric and the magnetic fields at par and we would get a contribution of $-\frac{1}{2}\nabla B^2$ similar to the case of the electric field. This would give us a term

$$-\nabla \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) = -\nabla u$$

where u is the energy density of electromagnetic field. Thus this term (i.e. the gradient of the energy density) represents the momentum density term.

We will return to a discussion of the remaining terms in the next lecture.