MA2102: LINEAR ALGEBRA

Lecture 18: More Duals

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Let us discuss some examples of dual vectors.

Examples (1) $[V = \mathbb{R}^n]$ If $\mathbf{v} \in \mathbb{R}^n$, then define

$$L_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}^n, \;\; \mathbf{w} \mapsto \mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i.$$

It follows from the properties of dot product that $L_{\mathbf{v}}$ is a linear map.

(2) $[V = M_n(\mathbb{R})]$ Consider the trace map

$$\mathrm{tr} : M_n(\mathbb{R}) \to \mathbb{R}, \;\; A \mapsto \mathrm{trace}(A).$$

Since it is a linear map, $\mathrm{tr} \in (M_n(\mathbb{R}))^*$.

(3) $[V = C([0, 2\pi], \mathbb{R})]$ Let us clarify that V is the set of continuous real valued function defined on $[0, 2\pi]$. Show that V is a vector space.

Consider the maps

$$S_n : V \to \mathbb{R}, \qquad f(x) \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt$$

$$C_n : V \to \mathbb{R}, \qquad f(x) \mapsto \frac{1}{2\pi}\int_0^{2\pi} f(t)\cos(nt)\,dt.$$

The key analytical point is that the integrals make sense. Once this is ensured,

$$\begin{aligned} S_n(f+g) &= \frac{1}{2\pi}\int_0^{2\pi}(f(t)+g(t))\sin(nt)\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\sin(nt)\,dt + \frac{1}{2\pi}\int_0^{2\pi} g(t)\sin(nt)\,dt \\ &= S_n(f)+S_n(g). \end{aligned}$$

Show that $S_n(cf) = cS_n(f)$.

In a similar manner we may prove that C_n is a linear map.

Remark For a given $f \in V$, these numbers $\{C_n(f), S_n(f)\}_{n\in\mathbb{Z}}$ are called the *Fourier coefficients* of f. Although V is infinite dimensional, and so is V^*, these dual vectors $\{C_n, S_n\}_{n\in\mathbb{Z}}$ form a “basis” in some appropriate sense.

(4) $[\, V = P_1(\mathbb{R})]$ Consider the maps

$$f_1 : P_1(\mathbb{R}) \to \mathbb{R}, \qquad p(x) \mapsto \int_0^1 p(t)\,dt$$

$$f_2 : P_1(\mathbb{R}) \to \mathbb{R}, \qquad p(x) \mapsto \int_0^2 p(t)\,dt.$$

These maps are linear, whence $f_1, f_2 \in V^*$. Note that

$$f_1(1) = 1, \quad f_1(x) = \tfrac{1}{2}, \quad f_2(1) = 2, \quad f_2(x) = 2.$$

Claim: *The set $\{f_1, f_2\}$ forms a basis of $P_1(\mathbb{R})^*$.*

It suffices to show that this set is linearly independent as $P_1(\mathbb{R})^*$ has dimension 2. Suppose that $a_1 f_1 + a_2 f_2 = \mathtt{0}$, i.e., the left hand side is the zero linear functional. Thus,

$$\mathtt{0} = (a_1 f_1 + a_2 f_2)(1) = a_1 f_1(1) + a_2 f_2(1) = a_1 + 2a_2.$$

$$\mathtt{0} = (a_1 f_1 + a_2 f_2)(x) = a_1 f_1(x) + a_2 f_2(x) = \frac{a_1}{2} + 2a_2.$$

The only solution is $a_1 = a_2 = \mathtt{0}$.

Question *Is there a basis $\{p_1, p_2\} \subset P_1(\mathbb{R})$ such that $\{f_1, f_2\}$ is the dual basis?*

We are seeking $p_1 = a + bx, p_2 = c + dx$ such that $p_1^* = f_1$ and $p_2^* = f_2$.

In other words, we want

$$f_1(a+bx)=1,\ f_2(a+bx)=0,\ f_1(c+dx)=0,\ f_2(c+dx)=1.$$

These equalities translate to

$$a+b/2=1,\ 2a+2b=0,\ c+d/2=0,\ 2c+2d=1.$$

Show that the solution is given by $a=2, b=-2, c=-1/2, d=1$. Thus, $\{f_1, f_2\}$ is the dual basis of $\{2-2x, -\frac{1}{2}+x\}$.

Analogous to dual spaces, we have the following notion.

Definition [Dual Map] Let $T: V \to W$ be a linear map. The map T^*, defined by

$$T^*: W^* \to V^*,\ L \mapsto L \circ T$$

is called the dual of T.

By definition, if $L : W \to \mathbb{R}$, then

$$T^*(L)(v) := (L \circ T)(v) = L(T(v)).$$

Moreover, $T^*(L)$ is a linear map, i.e., T^* is well defined. Now

$$T^*(L_1 + L_2) = (L_1 + L_2) \circ T = L_1 \circ T + L_2 \circ T$$

$$T^*(\lambda L) = (\lambda L) \circ T = \lambda(L \circ T) = \lambda T^*(L)$$

imply that T^* is a linear map.

Question *How are the rank and nullity of T and T^* related?*

In fact, we may choose ordered bases β and γ of V and W respectively. Consider the dual bases β^* and γ^*.

Question *What is the relationship between $[T]_\beta^\gamma$ and $[T^*]_{\gamma^*}^{\beta^*}$?*

We may talk about iterated duals. In particular, let

$$V^{**} = (V^*)^* = \mathscr{L}(\mathscr{L}(V, \mathbb{R}), \mathbb{R})$$

be the *double dual* of V.

	V	V^*	V^{**}
Elements	*vector*	*dual vector* or *covector*	*dual of dual vector* (?)
Notation	v	$T : V \to \mathbb{R}$	$L : V^* \to \mathbb{R}$
Example	v	v^*	$\mathrm{ev}_v : V^* \to \mathbb{R}$ $\mathrm{ev}_v(T) = T(v)$

Proposition The evaluation map

$$\Phi : V \to V^{**}, \;\; v \mapsto \mathrm{ev}_v$$

is a linear isomorphism if V is finite dimensional.