MA2102: Linear Algebra

Lecture 22: Similar Matrices

9th October 2020

Indian Institute of Science Education & Research Kolkata

Recall that $A \in M_n(\mathbb{R})$ is *similar* to $B \in M_n(\mathbb{R})$ if there exists an invertible matrix $Q \in M_n(\mathbb{R})$ such that $B = QAQ^{-1}$. Note that given two similar matrices A and B, Q is not unique as

$$(\lambda Q)A(\lambda Q)^{-1} = B$$

for any non-zero scalar λ. Similarity is an equivalence relation.

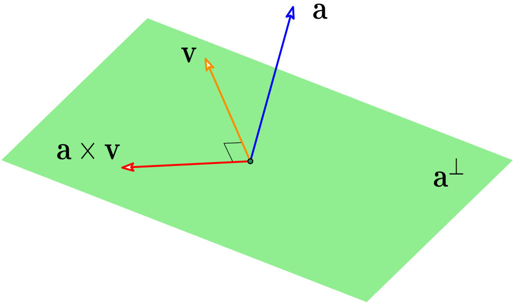
Examples (1) [Change of basis] If $T : V \to V$ is a linear map and β, γ are two ordered bases of V, then $[T]_\beta$ and $[T]_\gamma$ are similar via the change of basis matrix.

(2) [Cross product] Choose a unit vector $\mathbf{a} \in \mathbb{R}^3$. Consider the map

$$X_\mathbf{a} : \mathbb{R}^3 \to \mathbb{R}^3, \ \ \mathbf{v} \mapsto \mathbf{a} \times \mathbf{v}.$$

Show that this is a linear map.

It maps **a** to zero and maps $\mathbf{a}^\perp$ to itself.



The map $X_{\mathbf{a}} : \mathbf{a}^\perp \to \mathbf{a}^\perp$ is a rotation by an angle of $\pi/2$.

Remark The notion of clockwise or counterclockwise does not make sense as $\mathbf{a}^\perp$ can be identified with $\mathbb{R}^2$ in many ways.

On a formal note, we know that

$$\mathbf{a} \cdot X_{\mathbf{a}}(\mathbf{v}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{v}) = \mathbf{v}(\mathbf{a} \times \mathbf{a}) = 0.$$

Thus, $R(X_{\mathbf{a}}) \subset \mathbf{a}^{\perp}$. Recall that if θ is the angle between $\mathbf{a}$ and $\mathbf{v}$, then

$$||\mathbf{a} \times \mathbf{v}|| = ||\mathbf{a}||||\mathbf{v}|| \sin\theta = ||\mathbf{v}|| \sin\theta.$$

If $\mathbf{v} \in \mathbf{a}^{\perp}$, then $||X_{\mathbf{a}}(\mathbf{v})|| = ||\mathbf{v}||$. Moreover, the vector triple product identity implies that

$$X_{\mathbf{a}}^2(\mathbf{v}) = \mathbf{a} \times (\mathbf{a} \times \mathbf{v}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{v}) - \mathbf{v}(\mathbf{a} \cdot \mathbf{a}) = -\mathbf{v}.$$

Thus, $R(X_{\mathbf{a}}) = \mathbf{a}^{\perp}$. We also know that $N(X_{\mathbf{a}}) = \mathbb{R}\mathbf{a}$. If $\mathbf{v} \in \mathbb{R}^3$, then

$$\mathbf{v} = (\mathbf{v} \cdot \mathbf{a})\mathbf{a} + \big(\mathbf{v} - (\mathbf{v} \cdot \mathbf{a})\mathbf{a}\big) \in N(X_{\mathbf{a}}) \oplus R(X_{\mathbf{a}})$$

implies that $\mathbb{R}^3 = N(X_{\mathbf{a}}) \oplus R(X_{\mathbf{a}})$.

Choose $\mathbf{b}$ of length 1 such that $\mathbf{a}\cdot\mathbf{b}=0$. Show that $\gamma=\{\mathbf{a},\mathbf{b},\mathbf{a}\times\mathbf{b}\}$ is a basis of $\mathbb{R}^3$. If $\beta=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$, then

$$[X_{\mathbf{a}}]_\gamma=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

On the other hand, if $\mathbf{a}=(x_1,x_2,x_3)$, then

$$\begin{aligned} X_{\mathbf{a}}(\mathbf{e}_1) &= (0,x_3,-x_2)=0\mathbf{e}_1+x_3\mathbf{e}_2-x_2\mathbf{e}_3 \\ X_{\mathbf{a}}(\mathbf{e}_2) &= (-x_3,0,x_2)=-x_3\mathbf{e}_1+0\mathbf{e}_2+x_2\mathbf{e}_3 \\ X_{\mathbf{a}}(\mathbf{e}_3) &= (x_2,-x_1,0)=x_2\mathbf{e}_1-x_1\mathbf{e}_2+0\mathbf{e}_3 \end{aligned}$$

$$[X_{\mathbf{a}}]_\beta=\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

Note that both matrices have trace zero and determinant zero. The change of basis matrix is given by

$$Q = [I_{\mathbb{R}^3}]_\gamma^\beta = \begin{pmatrix} | & | & | \\ \mathbf{a} & \mathbf{b} & \mathbf{a}\times\mathbf{b} \\ | & | & | \end{pmatrix}$$

Show that $Q[X_{\mathbf{a}}]_\gamma Q^{-1} = [X_{\mathbf{a}}]_\beta$.

To compute the map $X_{\mathbf{a}}^2$ and associated matrices, we check that

$$[X_{\mathbf{a}}^2]_\gamma = [X_{\mathbf{a}}]_\gamma [X_{\mathbf{a}}]_\gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This is expected, however, as $X_{\mathbf{a}}^2$ on $\mathbf{a}^\perp$ is rotation by π or acts as $-I_{\mathbf{a}^\perp}$.

The matrix $[X_{\mathbf{a}}^2]_\beta$ can be computed in one of two ways:

- $[X_{\mathbf{a}}^2]_\beta = [X_{\mathbf{a}}]_\beta [X_{\mathbf{a}}]_\beta$
- $[X_{\mathbf{a}}^2]_\beta = Q[X_{\mathbf{a}}^2]_\gamma Q^{-1}$

A computation shows that

$$[X_{\mathbf{a}}^2]_\beta = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -x_2^2 - x_3^2 & x_1x_2 & 0x_1x_3 \\ x_1x_2 & -x_1^2 - x_3^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_1^2 - x_2^2 \end{pmatrix}$$

The trace is -2 as $||\mathbf{a}||^2 = 1$. This equals the trace of $[X_{\mathbf{a}}^2]_\gamma$.

(3) [2×2 matrices] Let $Q \in M_2(\mathbb{R})$ be an invertible matrix, i.e., there exists another matrix $P \in M_2(\mathbb{R})$ such that $PQ = QP = I_2$. We can explicitly compute P from Q.

If the matrix Q is given by

$$Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

then invertibility of Q implies that $\alpha\delta - \beta\gamma \neq 0$. The matrix

$$P = \frac{1}{\alpha\delta-\beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

is the inverse to Q. If $A \in M_2(\mathbb{R})$, then QAQ^{-1} is given by

$$\frac{1}{\alpha\delta-\beta\gamma} \begin{pmatrix} a\alpha\delta - b\alpha\gamma + c\beta\delta - d\beta\gamma & -a\alpha\beta + b\alpha^2 - c\beta^2 + d\alpha\beta \\ a\gamma\delta - b\gamma^2 + c\delta^2 - d\gamma\delta & -a\beta\gamma + b\alpha\gamma - c\beta\delta + d\alpha\delta \end{pmatrix}.$$

This implies that

$$\operatorname{trace}(A) = \operatorname{trace}(QAQ^{-1}) = a + d.$$