MA2102: LINEAR ALGEBRA

Lecture 2: Vector Space

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We will study sets V (think of $\mathbb{R}^3$ or $\mathbb{R}^n$) with a binary operation called *addition*, defined over a set F (think of $\mathbb{R}$), having *addition* and *multiplication*, such that the eight axioms are satisfied.

Expected properties of F The set of scalars is called a field.

- addition in F is commutative and associative;
- multiplication in F is commutative and associative;
- there is an additive identity, i.e., $0 + \lambda = \lambda$;
- additive inverses exist;
- there is a multiplicative identity, i.e., $1 \cdot \lambda = \lambda$;
- multiplicative inverses exist for non-zero elements;
- distributive law holds.

Remark We usually assume that $1 \neq 0$ in the definition. Thus, a field always has at least two elements.

Examples $\mathbb{R}$ real numbers, $\mathbb{Q}$ rational numbers, $\mathbb{C}$ complex numbers.

Question *What about the set* $\{0,1\}$*?*

Definition [Vector Space] A vector space V, over a field F, is a set with a binary operation

$$+ : V \times V \to V \qquad (\textit{addition})$$

and an operation

$$\cdot : F \times V \to V \qquad (\textit{scaling})$$

satisfying the following axioms:

Axiom I $v + w = w + v$ for any $v, w \in V$ (commutative)

Axiom II $u + (v + w) = (u + v) + w$ for any $u, v, w \in V$ (associative)

Axiom III there exists $\mathtt{0} \in V$ such that $\mathtt{0} + v = v$ for any $v \in V$ (identity)

Axiom IV given $u \in V$, there exists $v \in V$ such that $u + v = \mathtt{0}$ (inverse)

Axiom V there exists $1 \in F$ such that $1 \cdot v = v$ for any $v \in V$
Axiom VI $(ab) \cdot v = a \cdot (b \cdot v)$ for any $v \in V$ and $a, b \in F$
Axiom VII $a \cdot (v + w) = a \cdot v + a \cdot w$ for $v, w \in V$ and $a \in F$
Axiom VIII $(a + b) \cdot v = a \cdot v + b \cdot v$ for $v \in V$ and $a, b \in F$

Examples (1) The set $V = \{\theta\}$ is a vector space over $\mathbb{R}$ (in fact, over any field F). It is called the zero vector space.

(2) The sets $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ and, more generally, $\mathbb{R}^n$ are vector spaces over $\mathbb{R}$.

Definition [Field] A field F is a set with two binary operations

$$+ : F \times F \to F, \quad \cdot : F \times F \to F$$

such that

Axiom I $+$ and $\cdot$ are commutative and associative

Axiom II $+$ has (additive) identity and inverses

Axiom III $\cdot$ has (multiplicative) identity and inverses for non-zero elements

Axiom IV $\cdot$ should distribute over $+$.

Examples (1) $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$.

(2) The set $\{0, 1\}$ is a field.

(3) Show that any field $F \subset \mathbb{R}$ must contain $\mathbb{Q}$.

(4) Consider the set

$$\mathbb{Q}[\sqrt{2}] := \{p + q\sqrt{2} \mid p, q \in \mathbb{Q}\}.$$

Axioms I, II and IV are easily verified. If $p + q\sqrt{2}$ is non-zero, then at least p or q is non-zero.

Consider the number

$$\alpha := \frac{p}{p^2-2q^2} - \frac{q}{p^2-2q^2}\sqrt{2}.$$

Show that $\alpha \neq 0$ and it is the multiplicative inverse of $p+q\sqrt{2}$.
Note that p^2-2q^2 can never be zero unless p, q are both zero.

Properties of a field:

- (cancellation law) If $a+b=a+c$, then $b=c$.

Let a' be an additive inverse of a, i.e., $a+a'=0$. By commutativity, $a'+a=0$. Add a' to both sides to obtain

$$a'+(a+b)=a'+(a+c).$$

By associativity and $a'+a=0$, this means $0+b=0+c$, whence $b=c$.

- The additive identity is unique.

Let 0 and $0'$ be two additive identity. Then $0 = 0 + 0' = 0'$.

- Additive inverses are unique.

Consequence of cancellaton law.

- Multiplicative identity is unique.

Similar proof as uniqueness of 0.

- (cancellation law II) If $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.

Let a' be a multiplicative inverse of a, i.e., $a \cdot a' = 1$. By commutativity, $a' \cdot a = 1$. Multiply a' to both sides to obtain

$$a' \cdot (a \cdot b) = a' \cdot (a \cdot c).$$

By associativity and $a' \cdot a = 1$, this means $1 \cdot b = 1 \cdot c$, whence $b = c$.

- Multiplicative inverses are unique.

Consequence of cancellation law II.

Verify all the properties listed above.

Properties of a vector space:

- (cancellation law) If $u+v=u+w$, then $v=w$.

Same proof as in the case of a field.

- Additive identity is unique.

Same proof as in the case of a field.

- Given $v \in V$, there exists a unique $w \in V$ such that $v+w=\mathtt{0}$.

Consequence of cancellation law.

- $\mathtt{0} \cdot v=\mathtt{0}$ for any $v \in V$.

Since $\mathtt{0} \cdot v=(\mathtt{0}+\mathtt{0}) \cdot v=\mathtt{0} \cdot v+\mathtt{0} \cdot v$, *use cancellation law.*

Matrices as vectors: Consider the set $M_{m\times n}(\mathbb{R})$ of $m \times n$ matrices with real entries.

- Addition of matrices is defined via entrywise addition.
- The zero matrix $0_{m\times n}$ is the additive identity.
- $(cA)_{ij} := cA_{ij}$ defines scaling for matrices.

Show that $M_{m\times n}(\mathbb{R})$ is a vector space over $\mathbb{R}$. Note that $M_{n\times 1}(\mathbb{R})$ *looks like* $\mathbb{R}^n$, while $M_{3\times 2}(\mathbb{R})$ *looks like* $\mathbb{R}^6$. For $M_{m\times n}(\mathbb{R})$ to be closed under matrix multiplication, we need $m = n$.

Question *Is $M_{n\times n}(\mathbb{R})$ a field?*

Answer Apart from the case $n = 1$, it is not. Justify this by looking at non-zero matrices which are not invertible.

Remark The matrices $M_{m\times n}(\mathbb{Q})$ and $M_{m\times n}(\mathbb{C})$ are vector spaces over $\mathbb{Q}$ and $\mathbb{C}$ respectively.