MA2102: LINEAR ALGEBRA

Lecture 33: Orthogonal Projection

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Let W be a subspace of an inner product space V. The inner product $\langle \cdot, \cdot \rangle$ when restricted to W induces an inner product on W.

- choose a basis $\gamma = \{w_1, \ldots, w_k\}$ of W
- apply Gram-Schmidt to obtain an orthogonal basis of W

$$\tilde{\gamma} = \{v_1, \ldots, v_k\}$$

- $\tilde{\gamma}$ is linearly independent in V
- extend $\tilde{\gamma}$ to a basis $\beta = \{v_1, \ldots, v_k, u_{k+1}, \ldots, u_n\}$ of V
- apply Gram-Schmidt to β to obtain an orthogonal basis

$$\tilde{\beta} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}.$$

Example Let $V = \mathbb{R}^4$ with the standard inner product. Let

$$W := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2020x_1 = x_2 + x_3 + x_4\}$$

be a 3-dimensional subspace.

Show that $\gamma = \{(1, 2020, 0, 0), (1, 0, 2020, 0), (1, 0, 0, 2020)\}$ is a basis of W. We label the vectors in γ as w_1, w_2, w_3, in order. Note the following:

$$||w_j||^2 = 1 + 2020^2 = c^2, \;\; \langle w_i, w_j \rangle = 1 \text{ if } i \neq j.$$

Applying Gram-Schmidt algorithm, we get $v_1 = w_1$ and

$$\begin{aligned}
v_2 &= w_2 - \tfrac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1 = w_2 - \tfrac{1}{c^2} w_1 \\
||v_2||^2 &= \langle w_2 - \tfrac{1}{c^2} w_1, w_2 - \tfrac{1}{c^2} w_1 \rangle = c^2 - \tfrac{1}{c^2} \\
v_3 &= w_3 - \tfrac{\langle w_3, v_1 \rangle}{||v_1||^2} v_1 - \tfrac{\langle w_3, v_2 \rangle}{||v_2||^2} v_2 \\
&= w_3 - \tfrac{1}{c^2} w_1 - \tfrac{1 - \frac{1}{c^2}}{c^2 - \frac{1}{c^2}} (w_2 - \tfrac{1}{c^2} w_1) \\
&= w_3 - \tfrac{1}{c^2 + 1} w_2 - \tfrac{1}{c^2 + 1} w_1.
\end{aligned}$$

To extend $\{v_1, v_2, v_3\}$ to a basis of $\mathbb{R}^4$, it suffices to find u_4 such that $u_4 \notin W$. We may set $u_4 = (1,1,1,1)$ and $\beta = \{v_1, v_2, v_3, u_4\}$. Applying Gram-Schmidt to β amounts to modifying u_4:

$$v_4 = u_4 - \frac{\langle u_4, v_1\rangle}{||v_1||^2} v_1 - \frac{\langle u_4, v_2\rangle}{||v_2||^2} v_2 - \frac{\langle u_4, v_3\rangle}{||v_3||^2} v_3.$$

Show that

$$\langle u_4, v_1\rangle = 2021, \ \langle u_4, v_2\rangle = \frac{c^2-1}{c^2} 2021, \ \langle u_4, v_3\rangle = \frac{c^2-1}{c^2+1} 2021.$$

It is left as an exercise to normalize the orthogonal basis $\tilde{\beta} = \{v_1, v_2, v_3, v_4\}$. Note that the span of v_4 is a line which is orthogonal to W.

Definition [Orthogonal complement] Let W be a subspace of an inner product space V. The orthogonal complement $W^{\perp}$ of W in V is defined as

$$W^{\perp} := \{v \in V \,|\, \langle v, w\rangle = 0 \text{ for all } w \in W\}.$$

Note that $W \cap W^{\perp} = \{\mathbf{0}\}$ due to positivity (exercise). In particular, $\dim W^{\perp} \leq n-k$.

Proposition The dimension of $W^{\perp}$ is $n-k$, if W and V have dimension k and n respectively.

Proof.

Choose a basis $\gamma = \{w_1, \ldots, w_k\}$ of W and extend this to a basis $\beta = \{w_1, \ldots, w_k, u_{k+1}, \ldots, u_n\}$ of V. Apply Gram-Schmidt to obtain an orthonormal basis $\tilde{\beta} = \{v_1, \ldots, v_n\}$. By construction,

$$\operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\{w_1, \ldots, w_k\} = W$$

and each of $v_{k+1}, \ldots, v_n$ lie in $W^{\perp}$. Thus, $\dim W^{\perp} \geq n-k$, whence $\{v_{k+1}, \ldots, v_n\}$ is an orthonormal basis of $W^{\perp}$. $\square$

It follows that we have a direct sum decomposition $V = W \oplus W^{\perp}$.

Examples (1) In the previous example, $W^{\perp}$ is the line spanned by v_4.

(2) The orthogonal complement of $P_1(\mathbb{R})$ in $P_2(\mathbb{R})$ with respect to $\langle p,q\rangle = \int_0^1 p(t)q(t)dt$ must be a line. We had seen (cf. lecture 32) that $\{1, x-\frac{1}{2}\}$ is an orthogonal basis of $P_1(\mathbb{R})$ with $x^2-x-\frac{1}{6}$ being orthogonal to $P_1(\mathbb{R})$. Thus,

$$P_1(\mathbb{R})^{\perp} = \{c(x^2-x-\tfrac{1}{6}) \,|\, c \in \mathbb{R}\}.$$

(3) Let Sym_n be the subspace of symmetric matrices in $V = M_n(\mathbb{R})$, equipped with $\langle A,B\rangle = \mathrm{trace}(B^tA)$. Let A be skew-symmetric, i.e., $A^t = -A$. By cyclicity and symmetry of trace and elements of Sym_n being symmetric,

$$\mathrm{trace}(B^tA) = \mathrm{trace}(AB^t) = \mathrm{trace}(AB) = \mathrm{trace}(B^tA^t) = -\mathrm{trace}(B^tA).$$

It follows that $A \in \mathrm{Sym}_n^{\perp}$.

More generally, the subspace Skew_n of skew-symmetric matrices is contained in $\mathrm{Sym}_n^\perp$. However,

$$\dim \mathrm{Sym}_n^\perp = n^2 - \tfrac{n^2+n}{2} = \tfrac{n^2-n}{2}$$

equals the dimension of Skew_n. Thus, Skew_n is the orthogonal complement of Sym_n. Give a geometric proof of this fact by interpreting the inner product on $M_n(\mathbb{R})$ as an appropriate inner product on $\mathbb{R}^{n^2}$.

Definition [Orthogonal Projection] Let W be a subspace of an inner product space V. Using the decomposition $V = W \oplus W^\perp$, every $v \in V$ can be written uniquely as $w + w'$, where $w \in W$ and $w' \in W^\perp$. The map defined by

$$P : V \to V, \;\; P(v) = w$$

is called the orthogonal projection of V onto W.

Show that P is a linear map which is a projection. Moreover, null space of P is $W^{\perp}$ and range of P is W. It follows that there is a unique orthogonal projection associated to a subspace.

Remark Not all projections are orthogonal. For instance, consider the (skew) projection

$$Q:\mathbb{R}^2\to\mathbb{R}^2,\ \ Q(x,y)=(x-y,0).$$

If Q is an orthogonal projection, then nullity of Q must be the y-axis, the orthogonal complement of x-axis. However, this fails to hold.

We revisit example (3) and note that

$$A=\tfrac{1}{2}(A+A^t)+\tfrac{1}{2}(A-A^t)\in \mathrm{Sym}_n\oplus \mathrm{Skew}_n.$$

Thus,

$$P:M_n(\mathbb{R})\to M_n(\mathbb{R}),\ \ P(A)=\tfrac{1}{2}(A+A^t)$$

is the orthogonal projection associated to Sym_n.