MA2102: Linear Algebra

Lecture 17: Dual Spaces

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Recall the following result (cf. lecture 15).

Proposition Let V and W be vector spaces (over $\mathbb{R}$) of dimension n and m over respectively. The space $\mathscr{L}(V, W)$ is isomorphic to $M_{m\times n}(\mathbb{R})$.

A linear isomorphism was constructed by first choosing ordered bases β and γ of V and W respectively. Then we verified that the map

$$\Phi : \mathscr{L}(V, W) \to M_{m\times n}(\mathbb{R}), \;\; \Phi(T) = [T]_\beta^\gamma.$$

was an isomorphism.

There are two extreme cases: either $V = \mathbb{R}$ or $W = \mathbb{R}$. In the first case, any linear map from $\mathbb{R}$ to W is determined uniquely by its value at 1, i.e., the map

$$\mathrm{ev} : \mathscr{L}(\mathbb{R}, W) \to W, \;\; T \mapsto T(1)$$

is a linear isomorphism (exercise).

We also have the *coordinate* map

$$C : W \to \mathbb{R}^n, \;\; w \mapsto [w]^\gamma.$$

We had observed (cf. lecture 14) that this was an isomorphism. The composition gives us a linear isomorphism

$$C \circ \mathrm{ev} : \mathscr{L}(\mathbb{R}, W) \to \mathbb{R}^n, \;\; T \mapsto [T(1)]^\gamma.$$

Notice that $C \circ \mathrm{ev}$ is the linear isomorphism Φ corresponding to $\beta = \{1\}$ and γ.

Definition [Dual Spaces] Let V be a vector space over $\mathbb{R}$. The dual of V is defined to be $\mathscr{L}(V, \mathbb{R})$. It is denoted by V^*.

Remark The dual is defined similarly for a vector space V over F, by setting $V^* = \mathscr{L}(V, F)$.

Note that if V is finite dimensional, then $\dim V^* = \dim V$. Dual spaces appear frequently in physics.

- General Relativity : vectors and covectors
- Quantum Mechanics : $\langle\psi|$ bras and $|\psi\rangle$ kets
- Solid State Physics : crystalline and reciprocal lattice

Examples (1) [Dual of $\mathbb{R}^n$] Let $T \in (\mathbb{R}^n)^*$, i.e., $T : \mathbb{R}^n \to \mathbb{R}$ is a linear map. If $T(\mathbf{e}_i) = a_i$, then

$$\begin{aligned} T(v) &= T(v_1 e_1 + \cdots + v_n e_n) \\ &= a_1 v_1 + \cdots + a_n v_n \\ &= \langle (a_1, \ldots, a_n), (v_1, \ldots, v_n) \rangle \,. \end{aligned}$$

Thus, T may be identified with the vector $\mathbf{a} = (a_1, \ldots, a_n)$. This defines a map from the dual of $\mathbb{R}^n$ to $\mathbb{R}^n$.

Consider the linear maps

$$T_i : \mathbb{R}^n \to \mathbb{R}, \ \ (v_1, \dots, v_n) \mapsto v_i.$$

Note that

$$T(v) = a_1 v_1 + \cdots + a_n v_n = a_1 T_1(v) + \cdots + a_n T_n(v).$$

Therefore, any $T \in (\mathbb{R}^n)^*$ can be expressed as a linear combination of T_j's. As $(\mathbb{R}^n)^*$ has dimension n, the set $\{T_1, \dots, T_n\}$ forms a basis. Elements of $(\mathbb{R}^n)^*$ are called *covectors* or *linear functionals*.

(2) [Dual Basis] Choose an ordered basis $\beta = \{v_1, \dots, v_n\}$ of V. Define linear maps

$$v_i^* : V \to \mathbb{R}, \ \ a_1 v_1 + \cdots + a_n v_n \mapsto a_i.$$

Claim *The set $\{v_1^*, \cdots, v_n^*\}$ forms a basis of V^*.*

If $v = a_1 v_1 + \cdots + a_n v_n$ and $T \in V^*$, then

$$T(v) = a_1 T(v_1) + \cdots + a_n T(v_n) = T(v_1) v_1^*(v) + \cdots + T(v_n) v_n^*(v).$$

This implies that we have an equality of linear maps

$$T = T(v_1) v_1^* + \cdots + T(v_n) v_n^*.$$

Thus, $\beta^* = \{v_1^*, \ldots, v_n^*\}$ span V^*. As $\dim V^* = n$, β is a basis.

Definition [Dual Basis] The set β^* is called the dual basis of β.

Note that we have

$$v_i^*(v_j) = \delta_{ij}.$$

Consider the linear map $T_\beta : V \to V^*$

$$T_\beta(a_1 v_1 + \cdots + a_n v_n) = a_1 v_1^* + \cdots + a_n v_n^*.$$

Show that $T_\beta : V \to V^*$ is a linear isomorphism. This implies that there are plenty of linear isomorphisms between V and V^*.

Remark There is no canonical isomorphism between V and V^*. It seemed so in the case of $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ due to the presence of the vector dot product.

Consider $\beta = \{(1,0),(0,1)\}$ and $\gamma = \{(1,0),(1,1)\}$ as two ordered bases of $\mathbb{R}^2$. As

$$T_\beta(x,y) = xT_\beta(\mathbf{e}_1) + yT_\beta(\mathbf{e}_2) = x\mathbf{e}_1^* + y\mathbf{e}_2^*,$$

the map T_β is given by

$$T_\beta(x,y) : \mathbb{R}^2 \to \mathbb{R}, \ \ (a,b) \mapsto ax + by.$$

Note that $(a,b) = (a-b)(1,0) + b(1,1)$ in terms of γ.

Note that

$$
\begin{aligned}
T_\gamma(x,y) &= T_\gamma((x-y)(1,0)+y(1,1)) \\
&= (x-y)T_\gamma(1,0)+yT_\gamma(1,1) \\
&= (x-y)(1,0)^*+y(1,1)^*. \\
\Rightarrow\ T_\gamma(x,y)(a,b) &= T_\gamma(x,y)\big((a-b)(1,0)+b(1,1)\big) \\
&= (a-b)(x-y)+by.
\end{aligned}
$$

Thus, the map T_γ is given by

$$T_\gamma : \mathbb{R}^2 \to (\mathbb{R}^2)^*,\ T_\gamma(x,y):(a,b)\mapsto a(x-y)+b(2y-x).$$

It follows that T_β and T_γ are quite different maps!

Question *Given $T : V \to W$, is there a linear map $T^* : W^* \to V^*$?*