MA2102: LINEAR ALGEBRA

Lecture 9: Quotient Spaces

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Indian Institute of Science Education & Research Kolkata

A *relation* on a set X is a subset $R \subset X \times X$. Given $(a, b) \in R$ we say a is related to b (by the relation R) and write $a \sim_R b$. If R is clear in the context, we simply write $a \sim b$.

- Reflextive : $x \sim x$ for all $x \in X$
- Symmetric : If $x \sim y$ then $y \sim x$
- Transitive : If $x \sim y$ and $y \sim z$ then $x \sim z$

Question *Does symmetry and transitivity imply reflexivity?*

Definition [Equivalence] A relation $\sim$ on a set X is called an equivalence relation if $\sim$ is reflexive, symmetric and transitive.

An equivalence relation $\sim$ divides X into equivalence classes, i.e.,

$$[x] := \{y \in X \mid y \sim x\}$$

is called the equivalence class of x.

Example 1 Consider the relation $\sim$ defined on $\mathbb{R}^2$ as follows. We say $(x_1, y_1) \sim (x_2, y_2)$ if and only if

$$(x_1, y_1) - (x_2, y_2) = (\lambda, \lambda)$$

for some $\lambda \in \mathbb{R}$. Show that this is an equivalence relation.

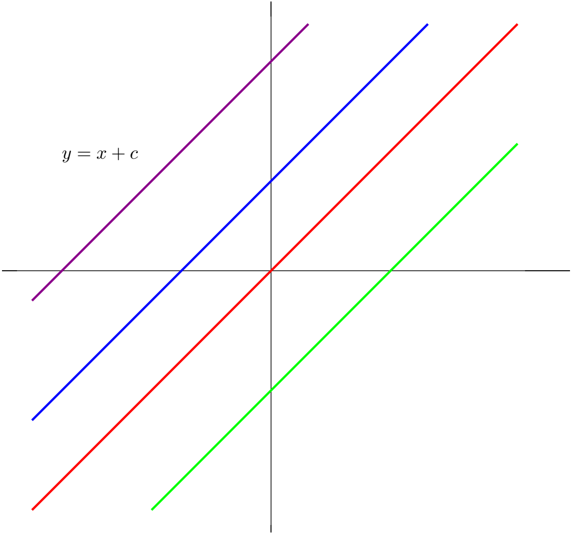
The equivalence classes are lines with slope 1 since

$$[(a, b)] = \{(x, y) \in \mathbb{R}^2 \,|\, (x - a, y - b) = (\lambda, \lambda) \text{ for some } \lambda \in \mathbb{R}\}$$

is precisely the line $y = x + (b - a)$. These lines are disjoint and cover the plane.

Observation The equivalence classes partition X. (exercise)

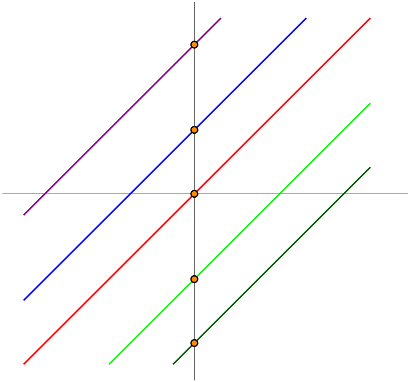
The set of equivalence classes will be denoted by $X/\sim$.



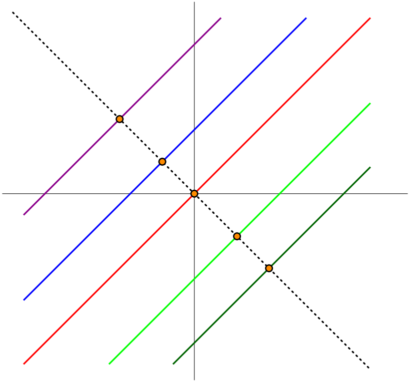
Question *Do these lines form a vector space?*

Method: We may *add* $y = x + c_1$ to $y = x + c_2$ to obtain $y = x + (c_1 + c_2)$ while we may scale $y = x + c_1$ by c to obtain $y = x + cc_1$.

Geometric meaning 1: We may choose one representative element $(0, c)$ for each equivalence class $y = x + c$. These points form the y-axis, which is a vector space.



Geometric meaning 2: We are choose one representative element $(-c/2, c/2)$ for each equivalence class $y = x + c$. These points form the line $y = -x$, which is a vector space.



Definition Given a vector subspace W of a vector space V (over a field F), consider the equivalence relation defined by $\sim_W$ as follows:

$$v_1 \sim v_2 \iff v_1 - v_2 \in W.$$

- $v \sim_W v$ as $v - v = \mathbf{0} \in W$
- if $v_1 \sim_W v_2$ then $v_1 - v_2 \in W$, whence $v_2 - v_1 \in W$
- if $v_1 \sim_W v_2$ and $v_2 - v_3 \in W$, then

$$v_1 - v_3 = (v_1 - v_2) + (v_2 - v_3) \in W.$$

Thus, $\sim_W$ is an equivalence relation and we denote the set of equivalence classes by V/W.

Show that the relation in example 1 is just $\sim_W$ for $V = \mathbb{R}^2$ and $W = \mathrm{span}((1,1))$.

Theorem *The set V/W is a vector space over F with the operations*

$$+ : V/W \times V/W \to V/W,\ [v_1]+[v_2] := [v_1+v_2]$$

$$\cdot : F \times V/W \to V/W,\ \lambda\cdot[v] := [\lambda v]$$

Proof.

It suffices to show that the operations are well-defined, i.e., if we choose $[v_1]=[u_1]$ and $[v_2]=[u_2]$, then $v_i - u_i \in W$, whence

$$(v_1+v_2)-(u_1+u_2) = v_1-u_1+v_2-u_2 \in W$$

implying that $[v_1]+[v_2]=[u_1]+[u_2]$. Also, if $[v]=[u]$, then $v-u\in W$ and

$$\lambda v - \lambda u = \lambda(v-u) \in W$$

implying that $\lambda\cdot[v]=\lambda\cdot[u]$. $\square$