MA2102: Linear Algebra

Lecture 20: Change of Coordinates

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It is natural to compare the matrix associted to $T \circ S$ and the matrices associated to T and S.

Proposition Let U, V, W and X be finite dimensional vector spaces with ordered bases α, β, γ and δ respectively.

(1) If $S : V \to W$ and $T : W \to X$ are linear maps, then

$$[T \circ S]_{\beta}^{\delta} = [T]_{\gamma}^{\delta}[S]_{\beta}^{\gamma}.$$

(2) If $R : U \to V$ is a linear map, then

$$\left([T]_{\gamma}^{\delta}[S]_{\beta}^{\gamma}\right)[R]_{\alpha}^{\beta} = [T]_{\gamma}^{\delta}\left([S]_{\beta}^{\gamma}[R]_{\alpha}^{\beta}\right).$$

Remark Part (1) is proved assuming matrix multiplication. Part (2) proves associativity of matrix multiplication, since a fixed pair of bases and a matrix gives rise to a unique linear map.

Proof.
Let $\beta = \{v_1, \ldots, v_l\}$, $\gamma = \{w_1, \ldots, w_m\}$ and $\delta = \{x_1, \ldots, x_n\}$. Then

$$\begin{aligned}
(T \circ S)(v_j) &= T(S(v_j)) = T\Big(\sum_{i=1}^{m} s_{ij} w_i\Big) \\
&= \sum_{i=1}^{m} s_{ij} T(w_i) = \sum_{i=1}^{m} s_{ij} \Big(\sum_{k=1}^{n} t_{ki} x_k\Big) \\
&= \sum_{k=1}^{n} \Big(\sum_{i=1}^{m} t_{ki} s_{ij}\Big) x_k.
\end{aligned}$$

Thus, the $(k,j)^{\text{th}}$ entry of $[T \circ S]_\beta^\gamma$ equals the $(k,j)^{\text{th}}$ entry of $[T]_\gamma^\delta [S]_\beta^\gamma$.

For part (2), note that

$$(T \circ S) \circ R = T \circ (S \circ R).$$

Applying (1) to both sides we obtain

$$[T\circ S]^{\delta}_{\beta}[R]^{\beta}_{\alpha}=[T]^{\delta}_{\gamma}[S\circ R]^{\gamma}_{\alpha}$$

which upon applying (1) again, yields

$$\big([T]^{\delta}_{\gamma}[S]^{\gamma}_{\beta}\big)[R]^{\beta}_{\alpha}=[T]^{\delta}_{\gamma}\big([S]^{\gamma}_{\beta}[R]^{\beta}_{\alpha}\big).$$

This completes the proof. $\square$

Remark Part (2) can be proved directly as a consequence of associativity of matrix multiplication. As for (1), one could enforce it by declaring that this is what we want. This would then define matrix multiplication to be what it is.

Before we discuss an application and some examples, note that the identity map $I_V : V \to V$, when written with respect to an ordered basis β gives us

$$[I_V]_{\beta} := [I_V]^{\beta}_{\beta} = I_n, \quad n = \dim V.$$

Corollary Let $T : V \to V$ be a linear map and let β, γ be two ordered bases of V of size n. Then the following hold:

$$
\begin{aligned}
[I_V]_\beta^\gamma [I_V]_\gamma^\beta &= [I_V]_\gamma^\gamma = I_n \\
[I_V]_\gamma^\beta [T]_\gamma^\gamma &= [T]_\gamma^\beta \\
[T]_\gamma^\gamma [I_V]_\beta^\gamma &= [T]_\beta^\gamma.
\end{aligned}
$$

Proof.

Let $I_V : V \to V$ be the identity map. The first claim follows from $I_V \circ I_V = I_V$ and $[I_V]_\beta = I_n$ while the last two claims follow from $T \circ I_V = T = I_V \circ T$. □

Note that

$$[I_V]_\gamma^\beta [T]_\gamma [I_V]_\beta^\gamma = [T]_\gamma^\beta [I_V]_\beta^\gamma = [T \circ I_V]_\beta^\beta = [T]_\beta.$$

Observe that

$$[I_V]^\gamma_\beta [I_V]^\beta_\gamma \; = \; I_n \; = \; [I_V]^\beta_\gamma [I_V]^\gamma_\beta.$$

Thus, these are square matrices which are inverses of each other.

Definition [Change of Basis] The matrix $[I_V]^\gamma_\beta$ associated to the identity map $I_V : V \to V$ and two ordered bases β, γ of V is called the *change of coordinate* matrix (or change of basis) matrix. It expresses the basis vectors of β in terms of γ.

Recall the linear isomorphisms (cf. lecture 14)

$$\phi_\beta : \mathbb{R}^n \to V, \;\; (c_1, \ldots, c_n) \mapsto c_1 v_1 + \cdots + c_n v_n$$

where $\beta = \{v_1, \ldots, v_n\}$. Consider the map

$$T := (\phi_\gamma)^{-1} \circ \phi_\beta : \mathbb{R}^n \to \mathbb{R}^n.$$

Let us write down the matrix of T with respect to the standard basis. The j^{th} column is given by $(\phi_\gamma)^{-1} \circ \phi_\beta(e_j) = (\phi_\gamma)^{-1}(v_j)$. If $\gamma = \{v_1', \ldots, v_n'\}$ and the entries of the change of basis matrix are a_{ij}, then

$$
\begin{aligned}
(\phi_\gamma)^{-1}(v_j) &= (\phi_\gamma)^{-1}(a_{1j}v_1' + \cdots + a_{nj}v_n') \\
&= a_{1j}(\phi_\gamma)^{-1}(v_1') + \cdots + a_{nj}(\phi_\gamma)^{-1}(v_n') \\
&= a_{1j}e_1 + \cdots + a_{nj}e_n.
\end{aligned}
$$

Thus, the j^{th} column is precisely the j^{th} column of $[I_V]_\beta^\gamma$. This is the reason for the nomenclature.

Example Let $\beta = \{e_1, e_2\}, \gamma = \{2e_1, 3e_2\}, \eta = \{e_2, e_1\}$ be three ordered bases of $\mathbb{R}^2$. What are the six change of basis matrices?

Convince yourself (exercise) that

$$[I_{\mathbb{R}^2}]_\beta^\gamma = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}, \ [I_{\mathbb{R}^2}]_\gamma^\eta = \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}, \ [I_{\mathbb{R}^2}]_\beta^\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the last matrix can also be obtained via matrix multiplication, i.e., $[I_{\mathbb{R}^2}]_\beta^\gamma [I_{\mathbb{R}^2}]_\gamma^\eta = [I_{\mathbb{R}^2}]_\beta^\eta$. Similarly, we see that

$$[I_{\mathbb{R}^2}]_\gamma^\beta = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \ [I_{\mathbb{R}^2}]_\eta^\gamma = \begin{pmatrix} 0 & 1/3 \\ 1/2 & 0 \end{pmatrix}, \ [I_{\mathbb{R}^2}]_\beta^\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Compute the change of basis matrix $[I_{\mathbb{R}^2}]_\eta^\delta$, where $\delta = \{e_1 + e_2, e_1 - e_2\}$.