MA2102: LINEAR ALGEBRA

Lecture 32: Gram-Schmidt Process

13th November 2020

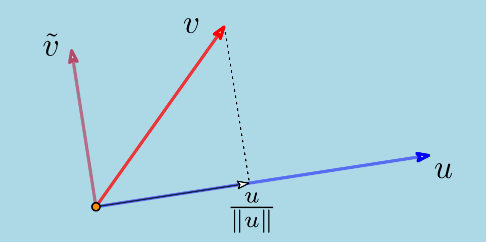
Indian Institute of Science Education & Research Kolkata

Given an *orthogonal basis* $\beta = \{v_1, \ldots, v_n\}$ of V, i.e., $\langle v_i, v_j \rangle = 0$ if $i \neq j$, we may form an orthonormal basis by normalizing. Consider

$$\tilde{\beta} = \left\{ \frac{v_1}{\|v_1\|}, \ldots, \frac{v_n}{\|v_n\|} \right\}.$$

Question *If β is a basis and not an orthogonal sequence, can we produce an orthonormal basis?*

Consider a basis $\{u, v\}$ of $\mathbb{R}^2$. We may subtract off the projection of v on the line spanned by u to obtain $\tilde{v}$ which is orthogonal to u.



Let $\hat{u} = \frac{u}{||u||}$ denote the normalized vector of u. The projection of v is given by $\langle v, \hat{u} \rangle \hat{u}$, whence

$$\tilde{v} = v - \frac{\langle v, u \rangle}{||u||^2} u.$$

More generally, if $\{w_1, \ldots, w_k\}$ is an orthogonal set and $w_i \neq 0$ for each i, then given any $v \in V$ we consider

$$\tilde{v} = v - \frac{\langle v, w_1 \rangle}{||w_1||^2} w_1 - \cdots - \frac{\langle v, w_k \rangle}{||w_k||^2} w_k.$$

Show that $\tilde{v}$ is orthogonal to each w_i. In particular, $\tilde{v}$ is orthogonal to any vector in $\operatorname{span}\{w_1, \ldots, w_k\}$. Note that

$$\operatorname{span}\{v, w_1, \ldots, w_k\} = \operatorname{span}\{\tilde{v}, w_1, \ldots, w_k\}.$$

More importantly, if $\{v, w_1, \ldots, w_k\}$ is linearly independent, then so is $\{\tilde{v}, w_1, \ldots, w_k\}$. If

$$c\tilde{v} + c_1 w_1 + \cdots + c_k w_k = 0,$$

then taking inner product with w_j we obtain $c_j = 0$ since $\langle \tilde{v}, w_j \rangle = 0$. Thus, $c\tilde{v} = 0$, implying that either $c = 0$ (we are done in that case) or $\tilde{v} = 0$. In such a case, $v \in \operatorname{span}\{w_1, \ldots, w_k\}$, contradicting that $\{v, w_1, \ldots, w_k\}$ is linearly independent. This procedure can be formalized to produce an orthonormal basis from any basis of V.

Gram-Schmidt Orthogonalization

Let $\beta = \{v_1, \ldots, v_m\}$ be a linearly independent set. Set $w_1 = v_1$,

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{||w_1||^2} w_1, \quad w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{||w_1||^2} w_1 - \frac{\langle v_3, w_2 \rangle}{||w_2||^2} w_2$$

and more generally

$$w_m = v_m - \frac{\langle v_m, w_1 \rangle}{||w_1||^2} w_1 - \cdots - \frac{\langle v_m, w_{m-1} \rangle}{||w_{m-1}||^2} w_{m-1}.$$

Then $w_i \neq \mathtt{0}$, $\{w_1, \ldots, w_m\}$ *is an orthogonal set and*

$$\operatorname{span}\{v_1, v_2, \ldots, v_k\} = \operatorname{span}\{w_1, w_2, \ldots, w_k\}$$

for $1 \leq k \leq m$.

Proof.

Let $P(j)$ be the statement that $w_1 \neq \mathtt{0}, \ldots, w_j \neq \mathtt{0}$ and $\{w_1, \ldots, w_j\}$ is an orthogonal set and

$$\operatorname{span}\{v_1, v_2, \ldots, v_j\} = \operatorname{span}\{w_1, w_2, \ldots, w_j\}.$$

It is evident that P(1) holds. Assume that $P(j-1)$ holds.

Induction hypothesis implies that

$$\operatorname{span}\{v_1, \dots, v_{j-1}, v_j\} = \operatorname{span}\{w_1, \dots, w_{j-1}, v_j\}.$$

It follows from the definition of w_j that the above is $\operatorname{span}\{w_1, \dots, w_{j-1}, w_j\}$. If $w_j = \mathtt{0}$, then $v_j \in \operatorname{span}\{w_1, \dots, w_{j-1}\}$. However, this implies $v_j \in \operatorname{span}\{v_1, \dots, v_{j-1}\}$, a contradiction to the linear independence of v_i's. Thus, $w_j \neq \mathtt{0}$ and $P(j)$ holds. $\square$

Corollary (1) Any finite dimensional inner product space V admits an orthonormal basis.

(2) If $\{e_1, \dots, e_k\}$ is an orthonormal set, then

$$||v||^2 \geq |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_k \rangle|^2.$$

The last inequality is called *Bessel's inequality*.

Proof.
(1) Choose a basis $\beta = \{v_1, \dots, v_n\}$ of V. By Gram-Schmidt algorithm, find $\{w_1, \dots, w_n\}$ which is an orthogonal set spanning V, whence it is an orthogonal basis. Normalize w_j's to $w_j/||w_j||$ to obtain an orthonormal basis.
(2) Extend $\{e_1, \dots, e_k\}$, a linearly independent set, to a basis $\beta = \{e_1, \dots, e_k, v_{k+1}, \dots, v_n\}$. Apply Gram-Schmidt algorithm (as indicated in (1) above) to obtain an orthonormal basis $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$. Now recall (cf. lecture 31) that

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n.$$

Taking length squared on both sides and using generalized Pythagoras Theorem, we get

$$||v||^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2 \geq |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_k \rangle|^2. \quad \square$$

Example Consider $P_3(\mathbb{R})$ equipped with the inner product arising from integration over $[0,1]$. Start with $\beta = \{1, x, x^2, x^3\}$. Set $w_1 = 1$ with $||1|| = 1$ and

$$w_2 = x - \frac{\langle x, 1\rangle}{||1||^2} 1 = x - \frac{1}{2}.$$

Note that (exercise) $||x - \frac{1}{2}||^2 = \frac{1}{12}$. Thus,

$$\begin{aligned} w_3 &= x^2 - \frac{\langle x^2, 1\rangle}{||1||^2} 1 - \frac{\langle x^2, x - \frac{1}{2}\rangle}{||x - \frac{1}{2}||^2}(x - \frac{1}{2}) \\ &= x^2 - x + \frac{1}{6} \end{aligned}$$

and satisfies $||w_3||^2 = \frac{1}{180}$ (exercise). We now look at

$$w_4 = x^3 - \frac{\langle x^3, 1\rangle}{||1||^2} 1 - \frac{\langle x^3, x - \frac{1}{2}\rangle}{||x - \frac{1}{2}||^2}(x - \frac{1}{2}) - \frac{\langle x^3, x^2 - x - \frac{1}{6}\rangle}{||x^2 - x - \frac{1}{6}||^2}(x^2 - x - \frac{1}{6}).$$

Show that $w_4 = x^3 + \frac{27}{2}x^2 - \frac{72}{5}x - \frac{41}{20}$ and compute $||w_4||^2$.