# MA2102: Linear Algebra

## Lecture 8: Replacement Theorem

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Let us recall the statement of the main result.

**Replacement Theorem**

*Let $V$ be a vector space that is spanned by a set $S$ of size $n$. Let $L$ be a linearly independent set of size $m$. Then*

*(i) $m \leq n$ and*

*(ii) there exists $T \subseteq S$ of size $n-m$ such that $T \cup L$ spans $V$.*

We often use the phrase "*$L$ can be extended to a basis and $S$ can be reduced to a basis*".

Proof.

Let $L = \{u_1, \ldots, u_m\}$ and $S = \{s_1, \ldots, s_n\}$ be as given. Since $u_m \in V = \mathrm{span}(S)$, we have

$$u_m = a_1 s_1 + \cdots + a_n s_n$$

for some scalars $a_i$.

If all $a_i$'s are zero, then $u_m = 0$, which contradicts linear independence of $L$. Thus, some $a_i$ must be non-zero; by rearranging indices, assume that $a_n \neq 0$. Therefore,

$$s_n = -\frac{1}{a_n}u_m - \frac{a_1}{a_n}s_1 - \cdots - \frac{a_{n-1}}{a_n}s_{n-1}$$

and if $\mathscr{S}_1 = \{s_1, \ldots, s_{n-1}, u_m\}$, then $s_n \in \operatorname{span}(\mathscr{S}_1) = V$.

Now $u_{m-1} \in V = \operatorname{span}(\mathscr{S}_1)$ implies that

$$u_{m-1} = a_1 s_1 + \cdots a_{n-1}s_{n-1} + b_m u_m{}^{1}.$$

If all the $a_j$'s are zero, then $u_{m-1} = b_m u_m$ and $\{u_{m-1}, u_m\}$ is linearly dependent, a contradiction. Thus, by rearranging the indices for $s_j$'s, we may assume that $a_{n-1} \neq 0$.

[1]Note that these scalars $a_i$'s are not the same as the earlier $a_i$'s.

Therefore,

$$s_{n-1} = -\frac{1}{a_{n-1}}u_{m-1} - \frac{b_m}{a_{n-1}}u_m - \frac{a_1}{a_{n-1}}s_1 - \cdots - \frac{a_{n-2}}{a_{n-1}}s_{n-2}$$

and if $\mathscr{S}_2 = \{s_1, \ldots, s_{n-2}, u_{m-1}, u_m\}$, then $s_{n-1} \in \operatorname{span}(\mathscr{S}_2) = V$.

Iterate this process to obtain

$$\mathscr{S}_m = \{s_1, \ldots, s_{n-m}, u_1, \ldots, u_m\} = \{s_1, \ldots, s_{n-m}\} \cup \{u_1, \ldots, u_m\}$$

which spans $V$. This proves both (i) and (ii). $\square$

**Remark** The set $T$ is not unique. This is evident from the choices in the proof as well as by the following example. Let $V = P_3(\mathbb{R})$ with $S = \{1-x, x-x^2, 2x, 3-4x\}$ and $L = \{1+x^2, x+x^2\}$. Then adjoining either $2x$ or $x-x^2$ or $3-4x$ to $L$ would span $V$. (exercise)

**Corollary C** Let $V$ be a finite dimensional vector space and $W$ be a (vector) subspace. Then

(i) $\dim_F(W) \leq \dim_F(V)$

(ii) If $\dim_F(W) = \dim_F(V)$ then $W = V$.

Proof.

Let $n = \dim_F(V)$ be the dimension of $V$ and let $\beta$ be a basis. Any finite subset $L$ of $W$ of size more than $n$ is linearly dependent because if $L$ was linearly independent, then by (i) of Replacement Theorem, $|L| \leq n = |\beta|$, a contradiction.

We now construct a basis of $W$ as follows. If $W = \{0\}$, then $\varnothing$ is a basis and $\dim_F(W) = 0$. Otherwise, choose a non-zero $u_1 \in W$. Iterate this process to choose $u_1, u_2, \ldots, u_k \in W$ such that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent. By the initial observation, this process has to terminate, i.e., there exists $k_0$ such that $\beta_0 = \{u_1, u_2, \ldots, u_{k_0}\} \subset W$ is linearly independent and no extension of this in $W$ is linearly independent.

Claim: $\beta_0$ is a basis of $W$. If $u \in W - \mathrm{span}(\beta_0)$, then $\{u, u_1, u_2, \ldots, u_{k_0}\}$ is linearly dependent, i.e.,

$$cu + c_1 u_1 + \cdots + c_{k_0} u_{k_0} = 0$$

and not all of $c, c_1, \ldots, c_{k_0}$ can be zero. Note that $c \neq 0$, whence $u \in \mathrm{span}(\beta_0)$, i.e., $W = \mathrm{span}(\beta_0)$.

Part (i) follows from applying Replacement Theorem to $S = \beta$ and $L = \beta_0$. For part (ii), if $|\beta| = |\beta_0|$, then $\beta_0$ is also a basis of $V$ (cf. Corollary B (b)). $\square$

Recall the sum $A + B$ of two subsets of a vector space $V$.

**Definition** [Sum] Let $W_1, W_2$ be vector subspaces of a vector space $V$. The set $W_1 + W_2$ is called the sum of $W_1$ and $W_2$.

The set $W_1 + W_2$ is called a direct sum if $W_1 \cap W_2 = \{\mathbf{0}\}$. We denote a direct sum by $W_1 \oplus W_2$.

Show that $W_1 + W_2$ is a subspace.

**Examples** (1) If $W_1$ is the $x$-axis and $W_2$ is the $y$-axis in $V = \mathbb{R}^3$, then $W_1 + W_2$ is a direct sum and equals the $xy$-plane.

(2) If $W_1$ is the $xy$-plane and $W_2$ is the $yz$-plane in $V = \mathbb{R}^3$, then $W_1 + W_2$ is *not* a direct sum as $W_1 \cap W_2$ is the $y$-axis.

(3) If $W_1$ is the space of symmetric matrices and $W_2$ is the set of skew-symmetric matrices in $V = M_n(\mathbb{R})$, then $W_1 + W_2$ is a direct sum as $W_1 \cap W_2 = \{\mathbf{0}\}$ (exercise).

We shall see later that for a finite dimensional vector space $V$,

$$\dim_F(W_1 \oplus W_2) = \dim_F(W_1) + \dim_F(W_2).$$

**Question** *What is the correct generalization of the dimension formula for $W_1 + W_2$?*

**Observation** The subspace $W_1 + W_2$ is a direct sum if and only if every $v \in W_1 + W_2$ can be expressed uniquely as $v = w_1 + w_2$ for $w_i \in W_i$.

Proof.

Let $v \in W_1 \cap W_2$. Any element of $W_1 + W_2$ can be expressed as

$$w_1 + w_2 = (w_1 - v) + (w_2 + v)$$

and this is non-unique if and only if $v = \mathbf{0}$. $\square$