Electromagnetic Waves

Lecture 35: Electromagnetic Theory

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Propagation of Electromagnetic Waves in a Conducting Medium

We will consider a plane electromagnetic wave travelling in a linear dielectric medium such as air along the z direction and being incident at a conducting interface. The medium will be taken to be a linear medium. So that one can describe the electrodynamics using only the E and H vectors. We wish to investigate the propagation of the wave in the conducting medium.

As the medium is linear and the propagation takes place in the infinite medium, the vectors $\vec{E}, \vec{H}$ and $\vec{k}$ are still mutually perpendicular. We take the electric field along the x direction, the magnetic field along the y- dirrection and the propagation to take place in the z direction. Further, we will take the conductivity to be finite and the conductor to obey Ohm's law, $\vec{J} = \sigma\vec{E}$. Consider the pair of curl equations of Maxwell.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu\frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon\frac{\partial \vec{E}}{\partial t} = \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t}$$

Let us take $\vec{E}, \vec{H}$ and $\vec{k}$ to be respectively in x, y and z direction. We the nhave,

$$\left(\nabla \times \vec{E}\right)_y = \frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t}$$

i.e.,

$$\frac{\partial E_x}{\partial z} + \mu\frac{\partial H_y}{\partial t} = 0 \qquad (1)$$

and

$$\left(\nabla \times \vec{H}\right)_x = -\frac{\partial H_y}{\partial z} = \sigma E_x + \epsilon\frac{\partial E_x}{\partial t}$$

i.e.

$$\frac{\partial H_y}{\partial z} + \sigma E_x + \epsilon\frac{\partial E_x}{\partial t} = 0 \quad (2)$$

We take the time variation to be harmonic ($\sim e^{i\omega t}$) so that the time derivative is equivalent to a multiplication by $i\omega$. The pair of equations (1) and (2) can then be written as

$$\frac{\partial E_x}{\partial z} + i\mu\omega \mathrm{H_y} = 0$$
$$\frac{\partial H_y}{\partial z} + \sigma E_x + i\omega\epsilon E_x = 0$$

We can solve this pair of coupled equations by taking a derivative of either of the equations with respect to z and substituting the other into it,

$$\frac{\partial^2 E_x}{\partial z^2} + i\mu\omega \frac{\partial \mathrm{H_y}}{\partial z} = \frac{\partial^2 E_x}{\partial z^2} - i\mu\omega(\sigma + i\omega\epsilon)E_x = 0$$

Define, a complex constant γ through

$$\gamma^2 = i\mu\omega(\sigma + i\omega\epsilon)$$

in terms of which we have,

$$\frac{\partial^2 E_x}{\partial z^2} - \gamma^2 E_x = 0 \quad (3)$$

In an identical fashion, we get

$$\frac{\partial^2 H_y}{\partial z^2} - \gamma^2 H_y = 0 \quad (4)$$

Solutions of (3) and (4) are well known and are expressed in terms of hyperbolic functions,

$$E_x = A\cosh(\gamma z) + B\sinh(\gamma z)$$
$$H_y = C\cosh(\gamma z) + D\sinh(\gamma z)$$

where A, B, C and D are constants to be determined. If the values of the electric field at z=0 is E_0 and that of the magnetic field at z=0 is H_0, we have $A = E_0$ and $C = H_0$.

In order to determine the constants B and D, let us return back to the original first order equations (1) and (2)

$$\frac{\partial E_x}{\partial z} + i\mu\omega \mathrm{H_y} = 0$$
$$\frac{\partial H_y}{\partial z} + \sigma E_x + i\omega\epsilon E_x = 0$$

Substituting the solutions for E and H

$$\gamma E_0 \sinh(\gamma z) + B\gamma\cosh(\gamma z) + i\omega\mu(\mathrm{H_0}\cosh(\gamma z) + D\sinh(\gamma z)) = 0$$

This equation must remain valid for all values of z, which is possible if the coefficients of $sinh$ and cosh terms are separately equated to zero,

$$E_0\gamma + i\omega\mu D = 0$$
$$B\gamma + i\omega\mu H_0 = 0$$

The former gives,

$$
\begin{aligned}
D &= -\frac{\gamma}{i\omega\mu} E_0 \\
&= -\frac{\sqrt{i\mu\omega(\sigma + i\omega\epsilon)}}{i\omega\mu} \\
&= -\sqrt{\frac{\sigma + i\omega\epsilon}{i\omega\mu}} E_0 \\
&= -\frac{E_0}{\eta}
\end{aligned}
$$

where

$$\eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}$$

Likewise, we get,

$$B = -\eta H_0$$

Substituting these, our solutions for the E and H become,

$$
\begin{aligned}
E_x &= E_0 \cosh(\gamma z) - \eta H_0 \sinh(\gamma z) \\
H_y &= H_0 \cosh(\gamma z) - \frac{E_0}{\eta} \sinh(\gamma z)
\end{aligned}
$$

The wave is propagating in the z direction. Let us evaluate the fields when the wave has reached $z = l$,

$$
\begin{aligned}
E_x &= E_0 \cosh(\gamma l) - \eta H_0 \sinh(\gamma l) \\
H_y &= H_0 \cosh(\gamma l) - \frac{E_0}{\eta} \sinh(\gamma l)
\end{aligned}
$$

If l is large, we can approximate

$$\sinh(\gamma l) \approx \cosh(\gamma l) = \frac{e^{\gamma l}}{2}$$

we then have,

$$
\begin{aligned}
E_x &= (E_0 - \eta H_0) \frac{e^{\gamma l}}{2} \\
H_y &= (H_0 - \frac{E_0}{\eta}) \frac{e^{\gamma l}}{2}
\end{aligned}
$$

The ratio of the magnitudes of the electric field to magnetic field is defined as the "characteristic impedance" of the wave

$$\left|\frac{E_x}{H_y}\right| = \eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}$$

Suppose we have lossless medium, σ=0, i.e. for a perfect conductor, the characteristic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

If the medium is vacuum, $\mu = \mu_0 = 4\pi \times\ 10^{-7}\ \mathrm{N/A^2}$ and $\epsilon = \epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/N - m^2}$ gives $\eta \approx 377\ \Omega$. The characteristic impedance, as the name suggests, has the dimension of resistance.

In this case, $\gamma = i\sqrt{\mu\epsilon}$.

Let us look at the full three dimensional version of the propagation in a conductor. Once again, we start with the two curl equations,

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \sigma\vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}$$

Take a curl of both sides of the first equation,

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2\vec{E} = -\mu \frac{\partial(\nabla \times \vec{H})}{\partial t}$$

As there are no charges or currents, we ignore the divergence term and substitute for the curl of H from the second equation,

$$\nabla^2\vec{E} = \mu \frac{\partial}{\partial t}\left(\sigma\vec{E} + \epsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$= \sigma\mu \frac{\partial \vec{E}}{\partial t} + \sigma\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

We take the propagating solutions to be

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

so that the above equation becomes,

$$k^2\vec{E} = (i\omega\mu\sigma + \omega^2\mu\epsilon)\vec{E}$$

so that we have, the complex propagation constant to be given by

$$k^2 = i\omega\mu\sigma + \omega^2\mu\epsilon$$

so that

$$k = \sqrt{\omega\mu(\omega\epsilon + i\sigma)}$$

k is complex and its real and imaginary parts can be separated by standard algebra, we have

$$k = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\left(1+\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}\right)^{1/2} + i\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)^{1/2}\right]$$

Thus the propagation vector β and the attenuation factor α are given by

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)}$$

The ration $\frac{\sigma}{\omega\epsilon}$ determines whether a material is a good conductor or otherwise. Consider a good conductor for which $\sigma \gg \omega\epsilon$. For this case, we have,

$$\beta = \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$$

The speed of electromagnetic wave is given by

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\sigma\mu}}$$

The electric field amplitude diminishes with distance as $e^{-\alpha z}$. The distance to which the field penetrates before its amplitude diminishes be a factor e^{-1} is known as the "**skin depth**" , which is given by

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}}$$

The wave does not penetrate much inside a conductor. Consider electromagnetic wave of frequency 1 MHz for copper which has a conductivity of approximately $6 \times 10^7 \Omega^{-1}\mathrm{m}^{-1}$. Substituting these values, one gets the skin depth in Cu to be about 0.067 mm. For comparison, the skin depth in sea water which is conducting because of salinity, is about 25 cm while that for fresh water is nearly 7m. Because of small skin depth in conductors, any current that arises in the metal because of the electromagnetic wave is confined within a thin layer of the surface.

Reflection and Transmission from interface of a conductor

Consider an electromagnetic wave to be incident normally at the interface between a dielectric and a conductor. As before, we take the media to be linear and assume no charge or current densities to exist anywhere. We then have a continuity of the electric and the magnetic fields at the interface so that

$$E_I + E_R = E_T$$
$$H_I + H_R = H_T$$

The relationships between the magnetic field and the electric field are given by

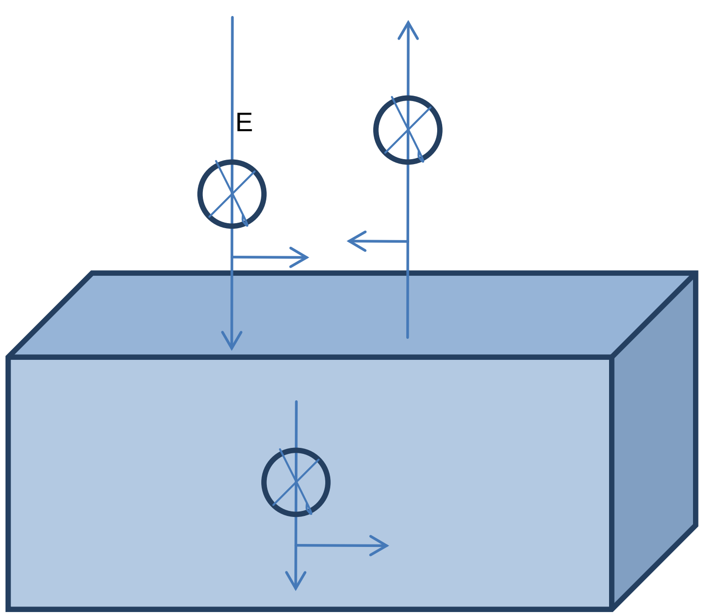
$$H_I = \frac{E_I}{\eta_1}$$
$$H_R = -\frac{E_R}{\eta_1}$$
$$H_T = \frac{E_T}{\eta_2}$$

the minus sign in the second relation comes because of the propagation direction having been reversed on reflection.

Solving these, we get,

$$\frac{E_R}{E_I} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$
$$\frac{E_T}{E_I} = \frac{E_R}{E_I} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

The magnetic field expressions are given by interchanging η_1 and η_2 in the above expressions.



Let us look at consequence of this. Consider a good conductor such as copper. We can see that η_2 is a small complex number. For instance, taking the wave frequency to be 1 MHz and substituting conductivity of Cu to be $6 \times 10^6 \Omega^{-1} m^{-1}$, we can calculate η_2 to be approximately $(1+i) \times 2.57 \times 10^{-4} \Omega$ whereas the vacuum impedance $\eta_1 = 377\ \Omega$. This implies

$$\frac{E_R}{E_I} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \approx -1$$

which shows that a good metal is also a good reflector. On the other hand, if we calculate the transmission coefficient we find it to be substantially reduced, being only about $(1+i) \times 1^{-6}$.

For the transmitted magnetic field, the ratio $\frac{H_T}{H_I}$ is approximately +2. Though E is reflected with a change of phase, the magnetic field is reversed in direction but does not undergo a phase change. The continuity of the magnetic field then requires that the transmitted field be twice as large.

Surface Impedance

As we have seen, the electric field is confined to a small depth at the conductor interface known as the skin depth. We define surface impedance as the ratio of the parallel component of electric field that gives rise to a current at the conductor surface,

$$Z_s = \frac{E_\parallel}{K_s}$$

where K_s is the surface current density.

Assuming that the current flows over the skin depth, one can write, for the current density, (assuming no reflection from the back of this depth)

$$J = J_0 e^{-\gamma z}$$

Since the current density has been taken to decay exponentially, we can extend the integration to infinity and get

$$K_s = \int J_0 e^{-\gamma z} dz = \frac{J_0}{\gamma}$$

The current density at the surface can be written as $\sigma E_\parallel$. For a good conductor, we have,

$$\gamma = \sqrt{i\omega\mu\sigma + \omega^2 \mu\epsilon} \approx (1+i)\sqrt{\omega\mu\sigma/2}$$

Thus

$$Z_s = \frac{E_\parallel}{K_s} = \frac{\gamma}{\sigma} = \sqrt{\frac{\omega\mu}{2\sigma}}(1+i) = \frac{1}{\sigma\delta}(1+i) \equiv R_s + iX_s$$

where the surface resistance R_s and surface reactance X_s are given by

$$R_s = \frac{1}{\sigma\delta} = X_s$$

The current profile at the interface is as shown.

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Tutorial Assignment

1. A 2 GHz electromagnetic propagates in a non-magnetic medium having a relative permittivity of 20 and a conductivity of 3.85 S/m. Determine if the material is a good conductor or otherwise. Calculate the phase velocity of the wave, the propagation and attenuation constants, the skin depth and the intrinsic impedance.

2. An electromagnetic wave with its electric field parallel to the plane of incidence is incident from vacuum onto the surface of a perfect conductor at an angle of incidence θ. Obtain an expression for the total electric and the magnetic field.

Solutions to Tutorial Assignments

1. One can see that $\omega\epsilon = 2\pi \times (2\times 10^{9}) \times (20 \times 8.85 \times 10^{-12}) = 2.22$ S/m. The ratio of conductivity σ to $\omega\epsilon$ is 1.73 which says it is neither a good metal nor a good dielectric. The propagation constant β and the attenuation constant α are given by,

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}+1\right)} = 229.5\ \text{rad/m}$$

$$\alpha = \omega\sqrt{\frac{\mu\epsilon}{2}}\sqrt{\left(\sqrt{1+\frac{\sigma^2}{\omega^2\epsilon^2}}-1\right)} = 132.52\ \frac{\text{Np}}{\text{m}}$$

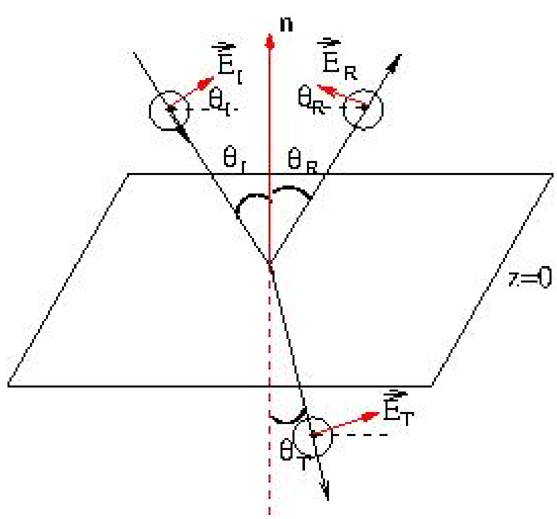
The intrinsic impedance is

$$\eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}} = 50.8 + 29.9\, i\ \ \Omega$$

The phase velocity is $\frac{\omega}{\beta} = 5.4 \times 10^{7}\ \text{m/s}$.

The skin depth is $\delta = \frac{1}{\alpha} = 7.5\ \text{mm}$

2. The case of p polarization is shown.



Let the incident plane be y-z plane. Let us look at the magnetic field. We have, since both the incident and the reflected fields are in the same medium,

$$\frac{E_i}{H_i} = \frac{E_r}{H_r} = \eta$$

Let us write the incident magnetic field as

$$\vec{H_i} = H_i e^{-i\beta(y\sin\theta - z\cos\theta)}\hat{x}$$

The reflected magnetic field is given by

$$\vec{H_r} = H_r e^{-i\beta(y\sin\theta + z\cos\theta)}\hat{x}$$

Since the tangential components of the electric field is continuous, we have,

$$E_i \cos\theta = E_r \cos\theta_r$$

As $\theta_i = \theta_r$, we have $E_i = E_r$, and consequently, $H_i = H_r$. Thus the total magnetic field can be written as

$$\vec{H_t} = \vec{H_i} + \vec{H_r} = H_i e^{-i\beta y\sin\theta} 2\cos(\beta z\cos\theta)\hat{x}$$
$$= 2\frac{E_i}{\eta} e^{-i\beta y\sin\theta} \cos(\beta z\cos\theta)\hat{x}$$

The electric field has both y and z components,

$$E_{iy} = E_i \cos\theta e^{-i\beta(y\sin\theta - z\cos\theta)}$$
$$E_{iz} = E_i \sin\theta e^{-i\beta(y\sin\theta - z\cos\theta)}$$

The reflected electric field also has both components,

$$E_{ry} = -E_i \cos\theta e^{-i\beta(y\sin\theta + z\cos\theta)}$$
$$E_{iz} = E_i \sin\theta e^{-i\beta(y\sin\theta + z\cos\theta)}$$

Adding these two the total electric field, has the following components,

$$E_{ty} = E_i \cos\theta e^{-i\beta y\sin\theta}\left(e^{i\beta z\cos\theta} - e^{-i\beta z\cos\theta}\right)$$
$$= 2iE_i \cos\theta e^{-i\beta y\sin\theta} \sin(\beta z\cos\theta)$$
$$E_{iz} = E_i \sin\theta e^{-i\beta y\sin\theta}\left(e^{i\beta z\cos\theta} + e^{-i\beta z\cos\theta}\right)$$
$$= 2E_i \sin\theta e^{-i\beta y\sin\theta} \cos(\beta z\cos\theta)$$

3.

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Self Assessment Questions

1. For electromagnetic wave propagation inside a good conductor, show that the electric and the magnetic fields are out of phase by 45^0.
2. A 2 kHz electromagnetic propagates in a non-magnetic medium having a relative permittivity of 20 and a conductivity of 3.85 S/m. Determine if the material is a good conductor or otherwise. Calculate the phase velocity of the wave, the propagation and attenuation constants, the skin depth and the intrinsic impedance.
3. An electromagnetic wave with its electric field perpendicular to the plane of incidence is incident from vacuum onto the surface of a perfect conductor at an angle of incidence θ. Obtain an expression for the total electric and the magnetic field.

Solutions to Self Assessment Questions

1. From the text, we see that the ratio of electric field to magnetic field is given by

$$\frac{E_x}{H_y} = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}}$$

For a good conductor, we can approximate this by $\sqrt{\frac{i\omega\mu}{\sigma}} = e^{\frac{i\pi}{4}}\sqrt{\frac{\omega\mu}{\sigma}}$.

2. One can see that $\omega\epsilon = 2\pi \times (2 \times 10^3) \times (20 \times 8.85 \times 10^{-12}) = 2.22 \times 10^{-6}$ S/m. The ratio of conductivity σ to $\omega\epsilon$ is 1.73×10^6 which says it is r a good metal. The propagation constant β and the attenuation constant α are given by,

$$\beta = \sqrt{\frac{\omega\mu\sigma}{2}} = 0.176 \text{ rad/m}$$
$$\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = 0.176 \text{ Np/m}$$

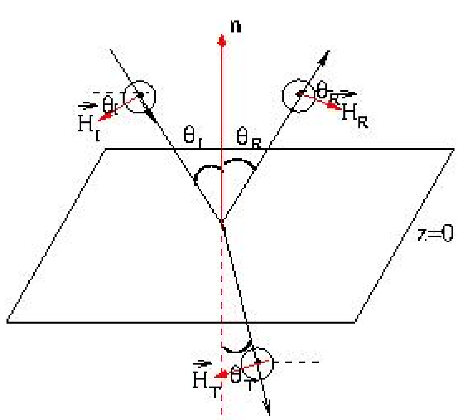
The intrinsic impedance is

$$\eta = \sqrt{\frac{i\omega\mu}{\sigma + i\omega\epsilon}} = 0.045(1 + i)\,\Omega$$

The phase velocity is $\frac{\omega}{\beta} = 7.14 \times 10^4$ m/s.

The skin depth is $\delta = \frac{1}{\alpha} = 5.68$ m.

3. The direction of electric and magnetic field for s polarization is as shown below.



Let the incident plane be y-z plane. The incident electric field is taken along the x direction and is given by

$$\vec{E_i} = E_i e^{-i\beta(y\sin\theta - z\cos\theta)}\hat{x}$$
$$\vec{E_r} = -E_i e^{-i\beta(y\sin\theta + z\cos\theta)}\hat{x}$$

The minus sign comes because at z=0, for any y, the tangential component of the electric field must be zero.

The total electric field is along the x direction and is given by the sum of the above,

$$\vec{E_t} = 2iE_i e^{-i\beta y\sin\theta}\sin(\beta z\cos\theta)\,\hat{x}$$

which is a travelling wave in the y direction but a standing wave in the z direction. Since the wave propagates in vacuum, we have,

$$\frac{E_i}{H_i} = \frac{E_r}{H_r} = \eta$$

The magnetic field has both y and z components.

$$H_{ty} = H_{iy} + H_{ry} = -H_i \cos\theta \ e^{-i\beta(y \sin\theta - z\cos\theta)} - H_i \cos\theta e^{-i\beta(y \sin\theta + z\cos\theta)}$$
$$= -2H_i \cos\theta e^{-i\beta y \sin\theta} \cos(\beta z \cos\theta)$$
$$= -2\frac{E_i}{\eta} \cos\theta e^{-i\beta y \sin\theta} \cos(\beta z \cos\theta)$$
$$H_{tz} = H_{iz} + H_{rz} = -H_i \sin\theta \ e^{-i\beta(y \sin\theta - z\cos\theta)} + H_i \sin\theta \ e^{-i\beta(y \sin\theta + z\cos\theta)}$$
$$= 2iH_i \cos\theta e^{-i\beta y \sin\theta} \sin(\beta z \cos\theta)$$
$$= 2\frac{E_i}{\eta} \cos\theta e^{-i\beta y \sin\theta} \sin(\beta z \cos\theta)$$