MA2102: Linear Algebra

Lecture 5: Linear Dependence

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Indian Institute of Science Education & Research Kolkata

- $\mathrm{span}(S_1 \cup S_2) = \mathrm{span}(S_1) + \mathrm{span}(S_2)$

Before proving this, recall that

$$T_1 + T_2 := \{v + w \mid v \in T_1, w \in T_2\}$$

for any two subsets T_1, T_2 *of a vector space* V. *For example, if* $A = \mathbb{Z}$ *and* $B = (0,1)$ *inside* $\mathbb{R}$, *then* $A + B = \mathbb{R} - \mathbb{Z}$. *Similarly, if* $C = [0,1)$, *then* $A + C = \mathbb{R}$.

As $S_i \subset S_1 \cup S_2$, *we have* $\mathrm{span}(S_i) \subseteq \mathrm{span}(S_1 \cup S_2)$. *As* $\mathrm{span}(S_1 \cup S_2)$ *is a vector space, if* $v \in \mathrm{span}(S_1)$ *and* $w \in \mathrm{span}(S_2)$, *then* $v + w \in \mathrm{span}(S_1 \cup S_2)$. *Thus, we have the* $\supseteq$ *inclusion.*

Conversely, if $v = c_1 v_1 + \cdots + c_k v_k + d_1 w_1 + \cdots + d_l w_l$ *with* $v_i \in S_1$ *and* $w_j \in S_2$, *then* $v \in \mathrm{span}(S_1) + \mathrm{span}(S_2)$.

Let V_1, V_2 be subspaces of V. Show that $V_1 \cap V_2$ is a subspace.

In fact, $V_1 + V_2$ is the smallest subspace containing V_1 and V_2.

- $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$

As $S_1 \cap S_2 \subset S_i$, *we have* $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

Let V_1, V_2 be subspaces of V. Show that $V_1 + V_2$ is a subspace.

Remark The converse is not true. Let v be a non-zero vector. Let $S_1 = \{v\}$ and $S_2 = \{-v\}$ with $\text{span}(S_1) = \text{span}(S_2)$. Then

$$\{0\} = \text{span}(\varnothing) = \text{span}(S_1 \cap S_2) \not\supseteq \text{span}(S_1).$$

Illustrative Examples

(1) Consider the set S

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

We observe that $M_2(\mathbb{R}) = \text{span}(S)$. In fact, there is no unique way of representing $A \in M_2(\mathbb{R})$ as a linear combination of elements of S. For example,

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $M_2(\mathbb{R})$ is the span of

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}.$$

Convince yourself that $M_2(\mathbb{R})$ cannot be the span of a set of size three.

(2) Consider $\operatorname{span}(\{1-x, 1+x, x^2-x\})$ in $P(\mathbb{R})$, the space of all polynomials. Note that if $p(x)$ is in the span, then $\deg(p) \leq 2$. On the other hand,

$$1 = \tfrac{1}{2}(1-x) + \tfrac{1}{2}(1+x)$$

$$x = \tfrac{1}{2}(1+x) - \tfrac{1}{2}(1-x)$$

$$x^2 = 1(x^2-x) + \tfrac{1}{2}(1+x) - \tfrac{1}{2}(1-x)$$

implies that any polynomial of degree at most 2 is in the span. Thus, the span is $P_2(\mathbb{R})$.

Show that if $a(1-x) + b(1+x) + c(x^2-x) = 0$, then $a = b = c = 0$.

Convince yourself that $P_2(\mathbb{R})$ cannot be the span of a set of size two.

(3) Let $\mathbf{u}_1 = (2,-1,4)$, $\mathbf{u}_2 = (1,-1,3)$, $\mathbf{u}_3 = (1,1,-1)$ and $\mathbf{u}_4 = (1,-2,-1)$. If $\mathbf{u}_1$ is a linear combination of the other $\mathbf{u}_i$'s, then

$$S := \operatorname{span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}) = \operatorname{span}(\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}).$$

Observe that

$$\mathbf{u}_3 = 2\mathbf{u}_1 - 3\mathbf{u}_2 = 2\mathbf{u}_1 - 3\mathbf{u}_2 + 0\mathbf{u}_4.$$

This implies that $S = \operatorname{span}(\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4\})$. The above relation can be rewritten as

$$2\mathbf{u}_1 - 3\mathbf{u}_2 - 1\mathbf{u}_3 + 0\mathbf{u}_4 = 0,$$

i.e., there is a *non-trivial* linear combination of $\mathbf{u}_i$'s which is zero. Show that no non-trivial combination of $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_4$ is zero.
In fact, from basic matrix theory, we can see that $S = \mathbb{R}^3$.

Definition [Linear Dependence] A subset S of a vector space (over a field F) is called linearly dependent if there exists distinct vectors $v_1, \ldots, v_k$ in S and scalars $c_1, \ldots, c_k \in F$, not all zero, such that

$$c_1 v_1 + \cdots + c_k v_k = \mathtt{0}.$$

A set S is called linearly independent if it is not lineraly dependent.

Remarks (1) Note that if $v \in S$, then $2v + (-2)v = \mathtt{0}$, although true, does not imply anything about S. However, if $-v \in S$, then $2v + 2(-v) = \mathtt{0}$ implies that S is linearly dependent.

(2) Any set containing the zero vector is always linearly dependent.

A linear combination $c_1 v_1 + \cdots + c_k v_k = \mathtt{0}$ is called non-trivial if some c_i's are non-zero.

Example (1) Consider the set S consisting of

$$\mathbf{u}_1 = (1,0),\ \mathbf{u}_2 = (0,1),\ \mathbf{u}_3 = (1,1),\ \mathbf{u}_4 = (-2,-2).$$

Observe that the following relations hold

$$\begin{aligned} \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 &= (0,0) \\ 2\mathbf{u}_1 + 2\mathbf{u}_2 + \mathbf{u}_3 + 2\mathbf{u}_4 &= (0,0) \\ \mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3 &= (0,0) \\ 2\mathbf{u}_3 + \mathbf{u}_4 &= (0,0) \end{aligned}$$

Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$, $\{\mathbf{u}_2, \mathbf{u}_3\}$, $\{\mathbf{u}_1, \mathbf{u}_3\}$ are linearly independent and all these sets span $\mathbb{R}^2$ while $\{\mathbf{u}_3, \mathbf{u}_4\}$ is linearly dependent.

Question *Can you think of a linearly indepedent set of size* 3 *in* $\mathbb{R}^2$*?*

Example (1) Consider the subsets

$$S_1 = \{1-x, 1+x, x\}, \quad S_2 = \{1-x, 1+x, e\}$$

of $P(\mathbb{R})$, the set of polynomials. Note that e stands for the constant polynomial that takes the value e. Since

$$\begin{aligned} -1(1-x)+1(1+x)-2(x) &= 0 \\ 1(1-x)+1(1+x)-\tfrac{2}{e}(e) &= 0 \end{aligned}$$

the sets S_1, S_2 are linearly dependent. However, any subset of size two in S_1 or S_2 is linearly independent (exercise).

Question *If S is linearly dependent, can we find $v \in S$ such that* $\text{span}(S) = \text{span}(S - \{v\})$*?*