Solutions to Laplace's Equations- II

Lecture 15: Electromagnetic Theory

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Laplace's Equation in Spherical Coordinates :

In spherical coordinates the equation can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}=0(1)$$

Note that we are using the notation ϕ to denote the azimuthal angle and φ to denote the potential function.

As with the rectangular coordinates, we will attempt a separation of variable, writing,

$$\varphi(r,\theta,\phi)=R(r)P(\theta)F(\phi)$$

Inserting this into the Laplace's equation and dividing throughout by $\mathrm{r}^2\sin^2\theta\, R(r)P(\theta)F(\phi)$, we get,

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{\sin\theta}{P}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right)=-\frac{1}{F}\frac{\partial^2 F}{\partial\phi^2} \qquad (2)$$

The left hand side depends only on (r,θ) while the right hand side on ϕ alone. Thus each of the terms must be equated to a constant, which we take as m^2. Writing the right hand side as

$$-\frac{1}{F}\frac{d^2F}{d\phi^2}=m^2$$

Note that since the only dependence is on ϕ, we need not write the partial derivative and have replaced it by ordinary derivative. The solution of this equation is $F(\phi)=\ Ae^{\pm im\phi}$, where A is a constant. Note that the potential function, and hence, F is single valued. Thus if we increase the azimuthal angle by 2π, we must have the same value for F, so that,

$$e^{\pm im(\phi+2\pi)}=e^{\pm im\phi}$$

This requires m to be an integer. This allows us to restrict the domain of $\phi : [0,2\pi]$.

We now rewrite (2) as,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{P\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P}{\partial\theta}\right) + \frac{m^2}{\sin^2\theta} \qquad (3)$$

Using identical argument as above, because the left hand side is a function of r alone while the right hand side is a function of θ alone, we must equate each side of (3) to a constant. For reasons that will become clear later, we write this constant as $l(l+1)$, which is quite general as we have not said what l is. Thus we have,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

We can simplify this equation by making a variable transformation, $\mu = \cos\theta,\ \ d\mu = -\sin\theta d\theta$, using which we get,

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + \left[l(l+1) - \frac{m^2}{1-\mu^2}\right]P = 0$$

The domain of θ being $[0:\pi]$, the range of μ is $[-1:+1]$. We will not attempt to solve this equation as it turns out that the equation is a rather well known equation in the theory of differential equations and the solutions are known to be polynomials in μ.

They are known as "**Associated Legendre Polynomials**" and are denoted by $P_{lm}(\mu)\text{or}P_{lm}(\cos\theta)$. We will point out the nature of the solutions.

It turns out that unless l happens to be an integer, the solutions of the above equation will diverge for $\mu \to \pm 1$. Thus, physically meaningful solutions exist for integral values of l only. Let us look at some special cases of the solutions. For a given l, m takes integral values from – l to $+l$, i.e. $m = -l, -l + 1, -l + 2, \dots, l-1, l$.

A particularly simple class of solution occur when the system has azimuthal symmetry, i.e., the system looks the same in the xy plane no matter from which angle ϕ we look at it. This implies that our solutions must be ϕ independent, i.e. *m=0*.

In such a case, the equation for the associated Legendre polynomial takes the form,

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + l(l+1)P = 0$$

The solutions of this equation are known as ordinary "**Legendre Polynomials**" and are denoted by $P_l(\mu)\text{or}P_l(\cos\theta)$. Let us look at some of the lower order Legendre polynomials.

Take $= 0$: The equation becomes,

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) = 0$$

It is trivial to check that the solution is a constant. We take the constant to be 1 for normalization purpose. Thus $P_0(\cos\theta) = 1$

Take $l = 1$: The equation is

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + 2P = 0$$

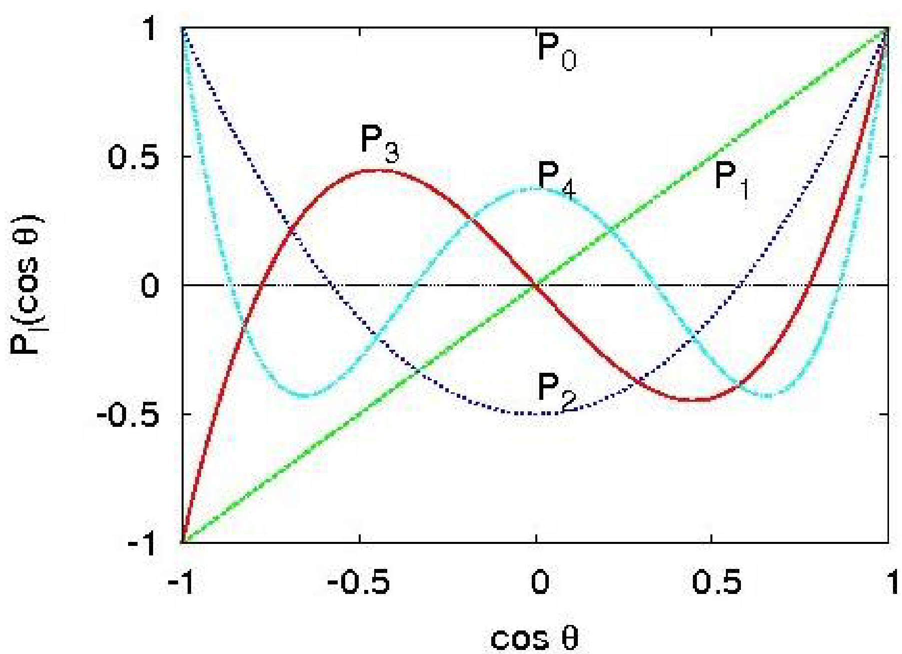
It is straightforward to check that the solution is $P_1(\mu) = \mu = \cos\theta$

Take $l = 2$: The equation is

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dP}{d\mu}\right) + 6P = 0$$

It can be verified that the solution is $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1) = \frac{1}{2}(3\cos^2\theta - 1)$

The solutions of a few lower order polynomials are shown below.



It can be verified that the Legendre polynomials of different orders are orthogonal,

$$\int_{-1}^{1} d\mu P_m(\mu) P_n(mu) = \frac{2\delta_{m,n}}{(2m+1)}$$

We are now left with only the radial equation,

$$\frac{d}{dr}\left(r^2\frac{dR}{\partial r}\right) = l(l+1)R$$
$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = l(l+1)r$$

A simple inspection tells us that the solutions are power series in r. Taking the solution to be of the form $R \sim r^n$, we get, on substituting into the radial equation,

$$n(n-1)r^n + 2nr^n - l(l+1)r^n = 0$$

Equating the coefficient of r^n to zero, we get,

$$n(n-1) + 2n - l(l+1) = 0$$
$$\left(n+\frac{1}{2}\right)^2 = \left(l+\frac{1}{2}\right)^2$$

which gives the value of $n = l$ or $-(l+1)$.

Thus, the function $R(r)$ has the form $A\,r^l + \frac{B}{r^{l+1}}$. Substituting the solutions obtained for Rand P, we get the complete solution for the potential for the azimuthally symmetric case (m=0) to be,

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) \qquad (4)$$

Example 1: A sphere of radius R has a potential $\varphi(R,\theta) = \varphi_0\cos^2\theta$ on its surface. Determine the potential outside the sphere.

Since we are only interested in solutions outside the sphere, in the solution (4), the term r^l for$l > 0$ cannot exist as it would make the potential diverge at infinity. Setting $A_0 = 0$, we get the solution to be

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

We can determine the constants B_l by looking at the surface potential For this purpose, we have to re-express the given potential in terms of Legendre polynomials.

$$\begin{aligned}\varphi(R,\theta) &= \varphi_0\cos^2\theta\\ &= \frac{\varphi_0}{3}(3\cos^2\theta - 1 + 1)\\ &= \varphi_0\left(\frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta)\right)\end{aligned}$$

Comparing this with the expression

$$\varphi(R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)$$

We conclude that $l = 0, 2$ and the corresponding coefficients are given by

$$B_0 = \frac{\varphi_0}{3} R$$
$$B_2 = \frac{2\varphi_0}{3} R^3$$

Thus the potential outside the sphere is given by

$$\varphi(r,\theta) = \frac{\varphi_0}{3}\left(\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right)$$

Complete Solution in Spherical Polar (without azimuthal symmetry)

If we do not have azimuthal symmetry, we get the complete solution by taking the product of R, P and F. We can write the general solution as

$$\varphi(r,\theta,\phi) = \sum_{l,m}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_{lm}(\cos\theta)\left(C_m e^{im\phi} + D_m e^{-im\phi}\right)$$
$$= \sum_{l,m}\left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) Y_{lm}(\theta,\phi)$$

where the constants have been appropriately redefined. The functions $Y_{lm}(\theta,\phi)$ introduced above are known as “**Spherical Harmonics**”.These are essentially products of associated Legendre polynomials introduced earlier and functions $e^{im\phi}$ which form a complete set for expansion of an arbitrary function on the surface of a sphere. The normalized spherical harmonics are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) e^{im\phi}$$

The functions are normalized as follows :

$$\int_0^{2\pi} d\phi \int_0^{\pi} Y_{lm}(\theta,\phi) Y^*_{l'm'}(\theta,\phi) \sin\theta d\theta d\phi = \delta_{l,l'}\delta_{m,m'}$$

For a given l, the spherical harmonics are polynomials of degree l in $\sin\theta$ and $\cos\theta$.

Some of the lower order Spherical harmonics are listed below.

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_{2,2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{2i\phi}$$

$$Y_{2,1} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi}$$

$$Y_{2,0} = \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1)$$

We have not listed the negative m values as they are related to the corresponding positive m values by the property

$$Y_{l,-m}(\theta,\phi) = (-1)^m Y_{l,m}(\theta,-\phi)$$

Example 2 : A sphere of radius R has a surface charge density given by $\sigma = \sigma_0 \sin 2\theta \sin\phi$. Determine the potential both inside and outside the sphere.

Solution : Surface charge density implies a discontinuity in the normal component of the electric field

$$\left.\frac{\partial\varphi}{\partial r}\right|_{R^-} - \left.\frac{\partial\varphi}{\partial r}\right|_{R^+} = \frac{\sigma_0 \sin 2\theta \sin\phi}{\epsilon_0}$$

where$R_\pm = R \pm \delta$.

We need to express the right hand side in terms of spherical harmonics. Using the table of spherical harmonics given above, we can see that

$$\sigma_0 \sin 2\theta \sin\phi = 2\sigma_0 \sin\theta\cos\theta \frac{\left(e^{i\phi} - e^{-i\phi}\right)}{2i}$$

$$= i\sigma_0 \sqrt{\frac{8\pi}{15}}\, (\mathrm{Y}_{2,1} + \mathrm{Y}_{2,-1})$$

Consider the general expression for the potential given earlier,

$$\varphi(r,\theta,\phi)=\sum_{l,m}\left(A_{lm}r^l+\frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

We need to take the derivative of this expression just inside and just outside the surface. Inside the surface, the origin being included, $B_{lm}=0$ and outside the surface, the potential should vanish at infinity, requiring $A_{lm}=0$. Thus, we have

Inside :$\varphi(r,\theta,\phi)=\sum_{l,m}A_{lm}r^lY_{lm}(\theta,\phi)$

Outside :$\varphi(r,\theta,\phi)=\sum_{l,m}\frac{B_{lm}}{r^{l+1}}Y_{lm}(\theta,\phi)$

Taking derivatives with respect to r and substituting r=R,

$$\frac{\sigma}{\epsilon_0}=\left(\sum_{l,m}A_{lm}Y_{lm}(\theta,\phi)\,lR^{l-1}+\frac{B_{lm}(l+1)}{R^{l+2}}Y_{lm}(\theta,\phi)\right)$$
$$\equiv i\frac{\sigma_0}{\epsilon_0}\sqrt{\frac{8\pi}{15}}\,(\mathrm{Y}_{2,1}+\mathrm{Y}_{2,-1})$$

Comparing, we notice that only $l=2$ terms are required in the sum. Comparing, we get,

$$2RA_{2,\pm1}+\frac{3}{R^4}B_{2,\pm1}=i\frac{\sigma_0}{\epsilon_0}\sqrt{\frac{8\pi}{15}}$$

We get another connection between the coefficients by using the continuity of the tangential component of the electric fields inside and outside, given by derivatives with respect to and ϕ . Since the angle part is identical in the expressions for the potential inside and outside the sphere, we have,

$A_{lm}R^l=\frac{B_{lm}}{R^{l+1}}$, which gives $B_{2,\pm1}=R^5A_{2,\pm1}$.

These two equations allow us to solve for $A_{2,\pm1}$ and $B_{2,\pm1}$ and we get,

$$A_{2,\pm1}=\frac{i\sigma_0}{5R\epsilon_0}\sqrt{\frac{8\pi}{5}}$$
$$B_{2,\pm1}=\frac{i\sigma_0R^4}{5\epsilon_0}\sqrt{\frac{8\pi}{5}}$$

Thus, the potential in this case is given by, Inside :$\varphi(r,\theta,\phi) = \frac{i\sigma_0}{5R\epsilon_0}\sqrt{\frac{8\pi}{5}}r^2\left(\mathrm{Y}_{2,1} + \mathrm{Y}_{2,-1}\right) = \frac{\sigma_0 r^2}{5R\epsilon_0}\sin 2\theta \sin\phi$

Outside :$\varphi(r,\theta,\phi) = \frac{i\sigma_0 R^4}{5\epsilon_0}\sqrt{\frac{8\pi}{5}}\frac{1}{r^3}\left(\mathrm{Y}_{2,1} + \mathrm{Y}_{2,-1}\right) = \frac{\sigma_0 R^4}{5r^3\epsilon_0}\sin 2\theta \sin\phi$

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Tutorial Assignment

1. A sphere of radius R, centered at the origin, has a potential on its surface given by $\varphi = \varphi_0\cos^3\theta$. Find the potential outside the sphere.
2. A spherical shell of radius R has a charge density $\sigma = \sigma_0\cos\theta$ glued on its surface. There are no charges either inside or outside. Find the potential both inside and outside the sphere.

Solutions to Tutorial Assignment

1. The problem has azimuthal symmetry. Since we are interested in potential outside the sphere, we put all $A_l = 0$ and have, $\varphi(r,\theta) = \frac{B_l}{r^{l+1}}P_l(\cos\theta)$. On the surface, the potential can be written as $\varphi(R,\theta) = \varphi_0\cos^3\theta = \frac{\varphi_0}{5}(2P_3 + 3P_1)$. Comparing this with the general expression for the potential, we have, $B_1 = \frac{3\varphi_0 R^2}{5}, B_3 = \frac{2\varphi_0 R^4}{5}, B_l = 0\ \forall l \neq 1,3$. Thus the potential function is given by
$$\varphi(r) = \frac{3\varphi_0 R^2}{5}\frac{1}{r^2}\cos\theta + \frac{2\varphi_0 R^4}{5}\frac{1}{r^4}P_3(\cos\theta)$$

2. General expressions for potential inside and outside are given by, $\varphi_{out}(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$, $\varphi_{in}(r,\theta) = \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$. The surface charge density is given by the discontinuity of normal component of the potential at r=R, ie.,
$$\sigma = \epsilon_0\left(\sum_{l=0}^{\infty}A_l l R^{l-1}P_l(\cos\theta) + \sum_{l=0}^{\infty}\frac{B_l(l+1)}{R^{l+2}}P_l(\cos\theta)\right) = \sigma_0\cos\sigma_0 P_1(\cos\theta)$$

The potential must be continuous across the surface. Since the Legendre polynomials are orthogonal, we have,

$$\frac{B_l}{R^{l+1}} P_l(\cos\theta) = A_l R^l P_l(\cos\theta)$$

Thus $B_l = A_l R^{2l+1}$. The charge density expression contains only $P_l(\cos\theta)$ term on the right. Clearly, only the $l = 1$ term should be considered in the expressions and all other coefficients must add up so as to give zero. We have,

$$A_1 + \frac{2B_1}{R^3} = \frac{\sigma_0}{\epsilon_0}$$

From the continuity of the potential, we had, $B_1 = R^3 A_1$. Thus we have, $A_1 = \frac{\sigma_0}{3\epsilon_0}$, $B_1 = R^3 \frac{\sigma_0}{3\epsilon_0}$. Thus the potential is given by

$$\varphi(r < R, \theta) = \frac{\sigma_0}{3\epsilon_0} r \cos\theta$$

$$\varphi(r > R, \theta) = \frac{\sigma_0 R^3}{3\epsilon_0} \frac{1}{r^2} \cos\theta$$

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Self Assessment Quiz

1. The surface of a sphere of radius R has a potential $\varphi = \varphi_0 \cos(3\theta)$. If there are no charges outside the sphere, obtain an expression for the potential outside.
2. A sphere of radius R has a surface potential given by $\varphi(\theta, \phi) = \varphi_0 \sin\theta \sin\phi$. Obtain an expression for the potential inside the sphere.

Solutions to Self Assessment Quiz

1. One has to first express $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ in terms of Legendre polynomials. It can be checked that $\cos(3\theta) = \frac{8P_3}{5} - \frac{3P_0}{5}$. The general expression for the potential outside the sphere is $\varphi_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$. Comparing this with the boundary condition at r=R, $\frac{B_3}{R^4} = \frac{8\varphi_0}{5}, \frac{B_1}{R^2} = -\frac{3}{5}\varphi_0$; all other coefficients are zero. Thus, $\varphi_{out}(r,\theta) = -\frac{3}{5}\varphi_0 \frac{R^2}{r^2} P_1(\cos\theta) + \frac{8}{5}\varphi_0 \frac{R^4}{r^4} P_3(\cos\theta)$.
2. The potential inside has the form $\varphi(r,\theta,\phi) = \sum_{l,m} A_{lm} r^l Y_{lm}(\theta,\phi)$. At r=R, the potential can be expressed in terms of spherical harmonics as

$$\begin{aligned}\varphi(\theta,\phi) &= \varphi_0 \sin\theta \sin\phi \\ &= \frac{1}{2i}\varphi_0 \sin\theta\left(e^{i\phi} - e^{-i\phi}\right) \\ &= i\sqrt{\frac{2\pi}{3}}\varphi_0 (Y_{1,1} + Y_{1,-1})\end{aligned}$$

Thus only $l = 1, m = \pm 1$ terms are there in the expression for the potential. We have,

$$\varphi(r,\theta,\phi) = i\sqrt{\frac{2\pi}{3}}\varphi_0 \frac{r}{R}\left(Y_{1,1} + Y_{1,-1}\right) = \varphi_0 \frac{r}{a}\sin\theta\sin\phi$$