MA2102: LINEAR ALGEBRA

Lecture 35: Self-adjoint Operators

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Recall the adjoint T^* of a linear map $T : V \to W$. Note that

$$\begin{aligned}\langle v, (S^* + T^*)(w)\rangle_V &= \langle v, S^*(w)\rangle_V + \langle v, T^*(w)\rangle_V \\ &= \langle S(v), w\rangle_W + \langle T(v), w\rangle_W \\ &= \langle (S+T)(v), w\rangle_W,\end{aligned}$$

implying that $(S+T)^* = S^* + T^*$. Similarly,

$$\langle v, \bar{c}T^*(w)\rangle_V = c\langle v, T^*(w)\rangle_V = c\langle T(v), w\rangle_W = \langle (cT)(v), w\rangle_W$$

implies that $(cT)^* = \bar{c}T^*$. Show that $(T^*)^* = T$.

Remark Consider the adjoint map

$$\mathrm{adj} : \mathscr{L}(V, V) \to \mathscr{L}(V, V), \quad T \mapsto T^*.$$

We know that

$$(S+T)^* = S^* + T^*, \; (cT)^* = \bar{c}T^*, \; (ST)^* = T^*S^*, \; (T^*)^* = T.$$

Thus, adj is similar to complex conjugation.

Definition [Self-adjoint] A linear map $T : V \to V$ is called self-adjoint if $T = T^*$.

Equivalent Definition 2 A linear map $T : V \to V$ is called self-adjoint if for any $v, w \in V$

$$\langle T(v), w \rangle = \langle v, T(w) \rangle.$$

Justification We have seen that T^* is uniquely determined by the implicit defining criteria

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle. \tag{1}$$

Putting $T^* = T$ gives definition 2. The converse is clear.

Equivalent Definition 3 A linear map $T : V \to V$ is called self-adjoint if for any orthonormal basis β we have $[T]_\beta$ is *Hermitian*, i.e., $[T]_\beta^* = [T]_\beta$.

Justification Note that $T = T^*$ if and only if $[T]_\beta = [T^*]_\beta$. We had proved that (cf. lecture 34) that $[T^*]_\beta = [T]_\beta^*$. Thus, $T = T^*$ if and only if $[T]_\beta = [T]_\beta^*$.

Remark • The orthonormality of β in definition 3 is necessary, as otherwise definition 3 would be independent of the inner product.

• For self-adjoint maps $T : V \to V$ of real vector spaces, the matrix $[T]_\beta$ is symmetric for any orthonormal basis β. The converse also holds.

Examples (1) The scaling map $\lambda I_V : V \to V$ is self-adjoint if and only if $\lambda = \overline{\lambda}$.

(2) Let $P : V \to V$ be an orthogonal projection to the subspace W. Then (cf. homework 10) we have $\langle P(v), w\rangle = \langle v, P(w)\rangle$, whence P is self-adjoint.

(3) Fix non-zero vectors $v_1, v_2 \in V$ and define

$$T : V \to V, \;\; T(v) := \langle v, v_1 \rangle v_2.$$

Note that

$$\langle T(v), w \rangle = \langle \langle v, v_1 \rangle v_2, w \rangle = \langle v, v_1 \rangle \langle v_2, w \rangle = \langle v, \overline{\langle v_2, w \rangle} v_1 \rangle.$$

Thus, $T^*(w) = \langle w, v_2 \rangle v_1$ and T is self-adjoint if and only if $v_1 = cv_2, c \in \mathbb{R}$ (exercise).

(4) Let $p \in M_n(\mathbb{C})$ be an invertible matrix. Consider the linear map

$$T : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \;\; T(A) = PAP^{-1}.$$

Observe that

$$\langle A, T^*(B) \rangle = \langle T(A), B \rangle = \mathrm{trace}(B^* PAP^{-1}) = \mathrm{trace}(P^{-1} B^* PA).$$

Thus, $T^*(B) = P^* B (P^*)^{-1}$ and T is self-adjoint if and only if

$$PAP^{-1} = P^*A(P^*)^{-1} \text{ for any } A \in M_n(\mathbb{C}).$$

In other words, $P^{-1}P^*$ commutes with any matrix. Therefore, $P^{-1}P^* = \lambda I_n$, whence $P^* = \lambda P$. Show that $|\lambda| = 1$.

If T is a self-adjoint map, then let $w \in N(T)$. Note that for any $v \in V$

$$0 = \langle v, T(w) \rangle = \langle T(v), w \rangle.$$

Therefore, $w \in R(T)^{\perp}$ and $N(T) \subset R(T)^{\perp}$. By Rank-Nullity Theorem, we conclude that

$$V = N(T) \oplus R(T)$$

is an orthogonal direct sum decomposition.

Remark For non-self-adjoint maps T, we can show that

$$N(T) = R(T^*)^{\perp}, \; R(T) = N(T^*)^{\perp}.$$

Let $T : V \to V$ be a self-adjoint operator on a complex vector space V. Recall that $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if λ is a root of $\det(T - xI_n) = 0$. Fundamental Theorem of Algebra implies that all polynomials with complex coefficients have complex roots. If $\lambda \in \mathbb{C}$ is an eigenvalue and $v \neq 0$ satisfies $T(v) = \lambda v$, then

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle T(v), v\rangle = \langle v, T(v)\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle.$$

As $\langle v, v\rangle > 0$, we conclude that $\lambda \in \mathbb{R}$.

Now consider two different eigenspaces, i.e., if $\lambda_1 \neq \lambda_2$ are eigenvalues of T with v_i an eigenvector associated to λ_i, $i = 1, 2$, then

$$\lambda_1\langle v_2, v_2\rangle = \langle T(v_1), v_2\rangle = \langle v_1, T(v_2)\rangle = \langle v_1, \lambda_2 v_2\rangle = \lambda_2\langle v_1, v_2\rangle.$$

As $\lambda_1 \neq \lambda_2$, we conclude that v_1 and v_2 are orthogonal to each other.

Observation A self-adjoint operator has only real eigenvalues. Moreover, the eigenspaces are orthogonal to each other.

In fact, more is true (stated *without* proof).

Theorem
Let V be a finite dimensional real vector space. If $T : V \to V$ is a linear map, then T is self-adjoint if and only if there exists an orthonormal eigenbasis.

Remark

- The orthonormality condition is essential. For instance, if $Q(x, y) = (x - y, 0)$ (cf. remark in lecture 33), then $[Q]_\beta = D(1, 0)$ with $\beta = \{(1, 0), (1, 1)\}$. The basis β is an eigenbasis but nor orthonormal. We see that

$$\langle Q(1,1), (1,0)\rangle = 0 \neq 1 = \langle (1,1), (1,0)\rangle = \langle (1,1), Q(1,0)\rangle.$$

- It is called *spectral theorem* for real self-adjoint maps.