MA2102: LINEAR ALGEBRA

Lecture 30: Inner Products

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We may equip a vector space with extra structure in order to introduce geometry. For instance, we may talk about *length* of a vector and *angles* between vectors. We introduce length and angles indirectly by means of something which resembles the classical dot product.

Important All vector spaces are over $\mathbb{R}$ or $\mathbb{C}$.

Definition [Inner Product] An inner product space is a vector space V equipped with a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

which satisfies

(i) [linearity in first variable] $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$
$\langle cv, w \rangle = c \langle v, w \rangle$

(ii) [conjugate symmetry] $\langle v, w \rangle = \overline{\langle w, v \rangle}$

(iii) [positivity] $\langle v, v \rangle > 0$ if $v \neq 0$.

Remark A few observations are in order.

- By (i) $\langle 0, v\rangle = \langle 0+0, v\rangle = \langle 0, v\rangle + \langle 0, v\rangle$, whence $\langle 0, v\rangle = 0$.
- If $F = \mathbb{R}$, then (ii) becomes $\langle v, w\rangle = \langle w, v\rangle$ (symmetry).
- If $F = \mathbb{C}$, then (ii) implies that $\langle v, v\rangle$ is real. Thus, condition (iii) is meaningful.
- Note that

$$\langle v, cw\rangle = \overline{\langle cw, v\rangle} = \bar{c}\overline{\langle w, v\rangle} = \bar{c}\,\langle v, w\rangle\,.$$

We also have

$$\begin{aligned}\langle v, w_1 + w_2\rangle &= \overline{\langle w_1 + w_2, v\rangle} \\ &= \overline{\langle w_1, v\rangle} + \overline{\langle w_2, v\rangle} \\ &= \langle v, w_1\rangle + \langle v, w_2\rangle\,.\end{aligned}$$

Thus, $\langle\cdot,\cdot\rangle$ is linear in the second variable if $F = \mathbb{R}$ and *conjugate linear* in the second variable if $F = \mathbb{C}$.

Definition [Length] The length of a vector $v \in (V, \langle \cdot, \cdot \rangle)$ is defined as

$$\|v\| := \langle v, v \rangle^{\frac{1}{2}}.$$

By positivity, $\|v\| = 0$ if and only if $v = 0$.

Observation If $\langle v_1, v \rangle = \langle v_2, v \rangle$ for all $v \in V$, then $v_1 = v_2$.

Proof.

The hypothesis implies that

$$0 = \langle v_1, v \rangle - \langle v_2, v \rangle = \langle v_1 - v_2, v \rangle$$

for all $v \in V$. Substituting $v = v_1 - v_2$, we obtain $\|v_1 - v_2\| = 0$, whence $v_1 = v_2$. □

Show that $\| \cdot \|^2$ satisfies the parallelogram law, i.e.,

$$\|v - w\|^2 + \|v + w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Examples (1) The *standard* inner product on $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle_{\text{std}}$ and given by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle_{\text{std}} = x_1 y_1 + \cdots + x_n y_n.$$

This is the classical dot product of vectors in $\mathbb{R}^n$.

(2) The *standard* inner product on $\mathbb{C}^n$ is given by

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle_{\text{std}} = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}.$$

For $\mathbb{C}$, we have $\langle z, w \rangle = z\overline{w}$. Note that conjugation in the second coordinates is essential as otherwise positivity would fail.

(3) Let $V = C[0,1]$ denote the vector space of all continuous real-valued functions. Define

$$\langle f, g \rangle := \int_0^1 f(t) g(t) \, dt.$$

The inner product is linear in both variables as well as symmetric. To verify positivity, let $f \in C[0,1]$. The function f^2 is non-negative and integrable, whence

$$\langle f,f\rangle = \int_0^1 f(t)^2 dt \ \geq \ 0.$$

Use continuity to show that if $\langle f,f\rangle = 0$, then f is the zero function. It follows that the length squared of x^k is $\frac{1}{2k+1}$.

(4) Let V be the vector space of continuous complex-valued functions defined on $[0,2\pi]$. Note that V is complex vector space. Show that

$$\langle f,g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}\,dt$$

defines an inner product.

(5) Consider the inner product defined on $M_n(\mathbb{C})$ as follows

$$\langle A,B\rangle := \operatorname{trace}(B^*A).$$

Linearity in first variable and conjugate linearity in second variable is immediate. Since $B^* = \overline{B}^t$, we have

$$\overline{\langle A,B\rangle} = \overline{\operatorname{trace}(B^*A)} = \operatorname{trace}(\overline{B^*A}) = \operatorname{trace}(B^t\overline{A}).$$

On the other hand, we have

$$\langle B,A\rangle = \operatorname{trace}(A^*B) = \operatorname{trace}((A^*B)^t) = \operatorname{trace}(B^t\overline{A}).$$

Show that $\langle A,A\rangle = \sum_{i,j}|a_{ij}|^2$. In particular, $||I_n||^2 = n$.

This inner product reduces to

$$\langle A,B\rangle := \operatorname{trace}(B^tA).$$

for $A,B \in M_n(\mathbb{R})$.

Remark Given an inner product $\langle \cdot, \cdot \rangle$ on V, a positive scalar multiple $c \langle \cdot, \cdot \rangle$ (for $c > 0$) is also an inner product. Restrictions of inner products on V to any subspace W also define inner products on W.

(6) Let us consider the restriction of the inner product in example (3) to the subspace $P_n(\mathbb{R})$ of $C[0,1]$. As $\langle f, g \rangle$ is obtained integrating fg over 0 to 1, we see that

$$\left\langle x^k, x^l \right\rangle = \int_0^1 t^k t^l dt = \frac{1}{k+l+1}.$$

Thus, elements of the standard basis $\{1, x, \ldots, x^n\}$ are not mutually perpendicular.

Prove the Cauchy-Schwarz inequality, i.e., $|| \langle v, w \rangle || \leq ||v|| \cdot ||w||$. We may use this to prove the triangle inequality: $||v + w|| \leq ||v|| + ||w||$.