Elements of Vector Calculus :Laplacian

Lecture 5: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Till now we have talked about operators such as gradient, divergence and curl which act on scalar or vector fields. These operators are all first order differential operators. Gradient operator, acting on a scalar field, gives a vector field. Divergence, on the other hand, acts on a vector field giving a scalar field. The curl operator, acting on a vector field, gives another vector field. The Cartesian expressions for these operators are as follows : The operator ∇ has the form

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

In terms of this, the three operators are written as

$$\nabla f = \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\nabla \times \vec{F} = \hat{\imath}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

where we have denoted a scalar field by f and a vector field by $\vec{F}$.

We will define another useful operator, known as the Laplacian operator, which is a second order differential operator acting on a scalar field (and with some conventional usage on a vector field). This operator is denoted by ∇^2, which is actually a $\nabla \cdot \nabla$ operator, i.e. it is a divergence of a gradient operator. Since the gradient operates on a scalar field giving rise to a vector, the divergence operator can act on this finally resulting on a scalar field. Thus

$$\nabla^2 f = \nabla \cdot (\nabla f) = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, denoted by ∇^2 is known as the **Laplacian.**

A class of functions, known as "**Harmonic Functions**" satisfy what is known as the **Laplacian equation**,

$$\boldsymbol{\nabla^2 f = 0}$$

In electromagnetic theory, in particular, one often finds ∇^2 operator acting on a vector field. As has been explained above, a Laplacian can only act on a scalar field. However, often we have equations where the Laplacian operator acts on components of a vector field, which are of course scalars. Thus $\nabla^2\vec{F}$ is used as a short hand notation, which actually means $\nabla^2\vec{F} = \widehat{u_1}\nabla^2 F_1 + \widehat{u_2}\nabla^2 F_2 + \widehat{u_3}\nabla^2 F_3$, where $\widehat{u_1}, \widehat{u_2}, \widehat{u_3}$ are the unit vectors along three orthogonal directions in the chosen coordinate system and F_1, F_2, F_3 are the components of the vector field $\vec{F}$ in these directions. Thus, in Cartesian coordinates, we have $\nabla^2\vec{F} = \hat{\imath}\nabla^2 F_x + \hat{\jmath}\nabla^2 F_y + \hat{k}\nabla^2 F_z$.

In electrodynamics, several operator identities using the operator $\nabla$is frequently used. Here is a list of them. They are not proved here but you are strongly advised to prove some of them.

1. $\nabla \times (\nabla f) = 0$. This is obvious because ∇f represents a conservative field, whose curl is zero.
2. $\nabla \cdot (\nabla \times \vec{F}) = 0$. We have seen that $\nabla \times \vec{F}$ represents a solenoidal field, which is divergenceless. We will see later that the magnetic field B is an example of a solenoidal field.
3. $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$. This operator identity is very similar to the vector triple product. (for ordinary vectors, we have, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$).
4. $\nabla(fg) = g\nabla f + f\nabla g$, where both f and are scalar fields.
5. $\nabla \cdot (f\vec{F}) = f\nabla \cdot \vec{F} + \vec{F} \cdot \nabla f$
6. $\nabla \times (f\vec{F}) = f(\nabla \times \vec{F}) + \nabla f \times \vec{F}$

Green's Identities :

We will now derive two important identities which go by the name Green's identities.

Let the vector field $\vec{A}$ be single valued and continuously differentiable inside a volume V bounded by a surface S. By divergence theorem, we have

$$\int_V (\vec{\nabla} \cdot \vec{A}) dV = \int_S \vec{A} \cdot \hat{n} dS$$

If we choose $\vec{A}$ to be given by $\vec{A} = \varphi \nabla \psi$, where φ and ψ are two scalar fields, then we get, using the relation (5) above,

$$\boxed{\int_V [\varphi \nabla^2 \psi + (\nabla\varphi \cdot \nabla\psi)] dV = \oint_S \varphi (\nabla\psi \cdot \hat{n}) dS} \qquad (1)$$

This is known as Green's first identity. By interchanging φ and ψ, we get,

$$\int_V [\psi \nabla^2 \varphi + (\nabla\varphi \cdot \nabla\psi)] dV = \oint_S \psi (\nabla\varphi \cdot \hat{n}) dS \qquad (2)$$

If we subtract (2) from (1) we g

$$\boxed{\int_V (\varphi \nabla^2 \psi - \psi \nabla^2 \varphi)\, dV = \oint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot \hat{n} dS} \qquad (3)$$

Equation (3) is Green's second identity and is also known as the **Green's Theorem**.

Uniqueness Theorem :

An important result for the vector fields is that in a region of volume V defined by a closed surface S, a vector field is uniquely specified by

1. its divergence
2. its curl and
3. its normal component at all points on the surface S.

This can be shown by use of the Green's theorem stated above. Consider two vector fields $\vec{A}$ and $\vec{B}$ which are specified inside such a volume and let us assume that they have identical divergences, curl as also identical values at all points on the defining surface. The uniqueness theorem implies that the two vector fields are identical.

To see this let us define a third vector $\vec{C} = \vec{A} - \vec{B}$. By the properties that we have assumed for $\vec{A}$ and $\vec{B}$ it follows that curl of $\vec{C}$ is zero. Thus $\vec{C}$ is a conservative field and hence can be expressed as a gradient of some potential. Let $\vec{C} = -\nabla\varphi$. The equality of the divergences of $\vec{A}$ and $\vec{B}$ implies that the divergence of $\vec{C}$ is zero, which, in turn, implies that $\nabla^2\varphi = 0$, i.e. the scalar potential φ satisfies the Laplace's equation at every point inside the volume V.

The third property, viz., the identity of the normal components of $\vec{A}$ and $\vec{B}$ at every point on S implies that the normal component of $\vec{C}$ is zero. Thus $\vec{C} \cdot \hat{n} = \nabla\varphi \cdot \hat{n} = 0$. Let us use Green's first identity , given by eqn. (1) above taking $\varphi = \psi$, we get

$$\int_V [\varphi\nabla^2\varphi + |\nabla\varphi|^2] dV = \oint_S \varphi(\nabla\varphi \cdot \hat{n}) dS \equiv 0$$

the last relation follows because $\nabla\varphi \cdot \hat{n} = 0$ everywhere on the surface. Thus we have,

$$\int_V [\varphi\nabla^2\varphi + |\nabla\varphi|^2] dV = 0$$

Since $\nabla^2\varphi = 0$, everywhere on the surface, it gives $\int_V |\nabla\varphi|^2 dV = 0$. Since an integral of a function which is positive everywhere in the volume of integration cannot be zero unless the integrand identically vanishes, we get $\nabla\varphi = \vec{C} = 0$, proving the theorem.

Dirac Delta Function

Before we conclude our discussion of the mathematical preliminaries, we would introduce you to a very unusual type of function which is known as Dirac's δ function. In a strict sense, it is not a function and mathematicians would like to call it as "generalized function" or a "distribution". The defining properties of a delta function are as follows :

$$\delta(x) = 0, \quad \text{if } x \neq 0,$$

$$\int_{-\varepsilon}^{\varepsilon} f(x)\delta(x)dx = f(0), \quad \text{if } f(x) \text{ is continuous at x=0}$$

In the last relation, it is to be noted that the limits of integration includes the point where the argument of the delta function vanishes. Note that the function is *not* defined at the point x=0. It easily follows that if we take a test function f(x) , we have

$$\int_{a-\epsilon}^{a+\epsilon} f(x)\delta(x-a)dx = f(a)$$

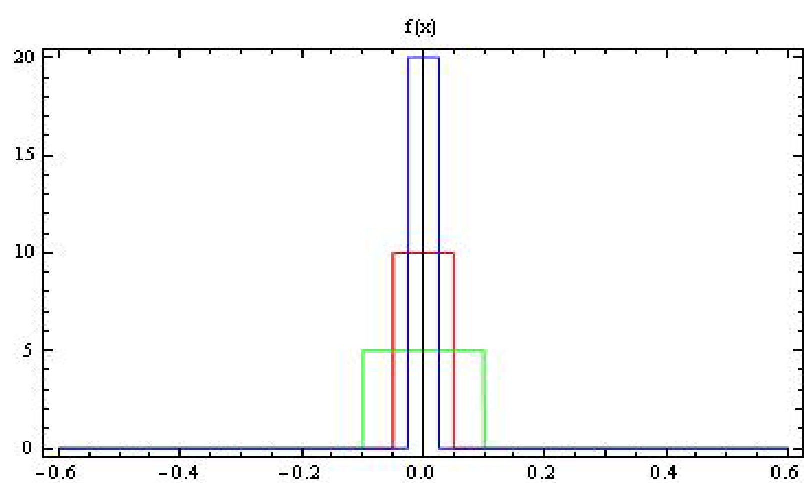
We can easily see why it is not a function in a strict sense. The function is zero at every point other than at one point where it is not defined. We know that a Riemann integral is defined in terms of the area enclosed by the function with the x axis between the limits. If we have a function which is zero everywhere excepting at a point, no matter what be the value of the function at such a point, the width of a point being zero, the area enclosed is zero. As a matter of fact, a standard theorem in Riemann integration states that if a function is zero everywhere excepting at a discrete set of points, the Riemann integral of such a function is zero.

How do we then understand such a function? The best way to look at a delta function is as a limit of a sequence of functions. We give a few such examples.

1. Consider a sequence of functions defined as follows :

$$\delta_n(x) = \begin{cases} 0 \quad \text{if} \quad x < -\dfrac{1}{2n} \\ n \;\text{ if } -\dfrac{1}{2n} \le x \le \dfrac{1}{2n} \\ 0 \;\text{ if } x > \dfrac{1}{2n} \end{cases}$$

For a fixed n, it represents a rectangle of height n, spread from $-\frac{1}{2n}$ to $+\frac{1}{2n}$. The figure below sketches a few such rectangles.



As n becomes very large, the width of the rectangle decreases but height increases in such a proportion that the area remains fixed at the value 1. As $n \to \infty$, the width becomes zero but the area is still finite. This is the picture of the delta function that we talked about.

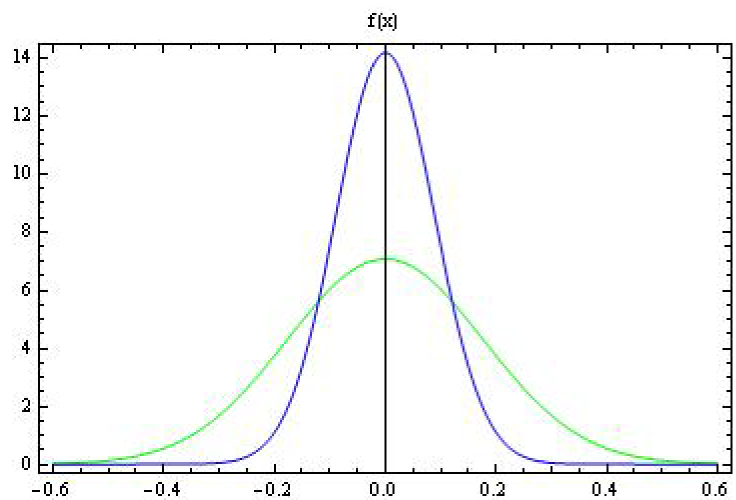
2. As second example, consider a sequence of Gaussian functions given by

$$\delta_\sigma(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

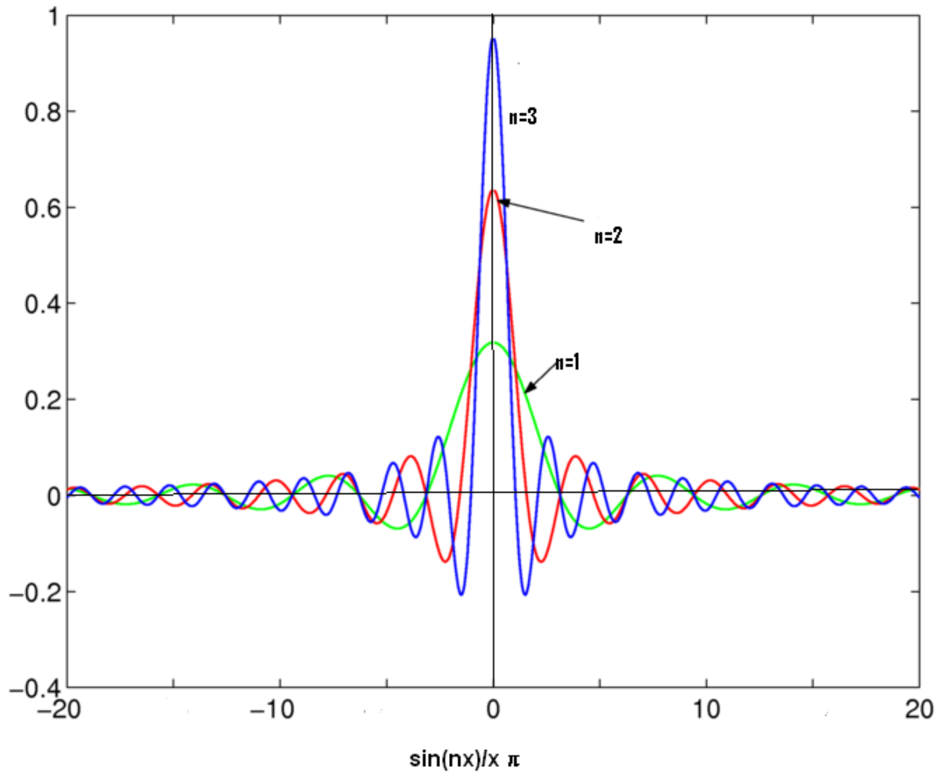
These functions are defined such that their integral is normalized to 1,

$$\int_{-\infty}^{+\infty} \delta_\sigma(x)dx = 1$$

for any value of σ. For instance, if we take σ to be a sequence of decreasing fractions 1,1/2, …,1/100, .. the functions represented by the sequence are $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \frac{2}{\sqrt{2\pi}} e^{-x^2/8}, \cdots \frac{100}{\sqrt{2\pi}} e^{-\frac{x^2}{20000}}, \cdots$ which are fast decreasing functions and the limit of these when σ approaches zero is zero provided $x \neq 0$.Note, however, the value of the function at x=0 increases with decreasing σ though the integral remains fixed at its value of 1.



3. A third sequence that we may look into is a sequence of "sinc" functions which are commonly met with in the theory of diffraction $\delta_\epsilon = \frac{\sin(x/\epsilon)}{\pi x}$. Note that as ϵ becomes smaller and smaller, the width of the central pattern decreases and the function gets peaked about the origin. Further, it can be shown that $\int_{-\infty}^{+\infty} \frac{\sin x}{\pi x} = 1$. Thus the function $\delta_\epsilon(x)$ is a representation of the delta function in the limit of $\epsilon \to 0$.



This gives us a very well known integral representation of the delta function

$$\int_{-\infty}^{+\infty} e^{ikx}dx = 2\pi\delta(x)$$

One can prove this as follows:

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{ikx}dx &= \lim_{n\to\infty}\int_{-n}^{+n} e^{ikx}dx \\ &= \lim_{n\to\infty}\frac{e^{ikx}}{ix} = \lim_{n\to\infty}\frac{e^{ixn}-e^{-ixn}}{ix} \\ &= \lim_{n\to\infty}\frac{2\sin xn}{x} = 2\pi\delta(x)\end{aligned}$$

where we have set $n = 1/\epsilon$ to arrive at the last result. It may be noted that the integrals above are convergent in their usual sense because sin (nx) does not have a well defined value for $n \to \pm\infty$)

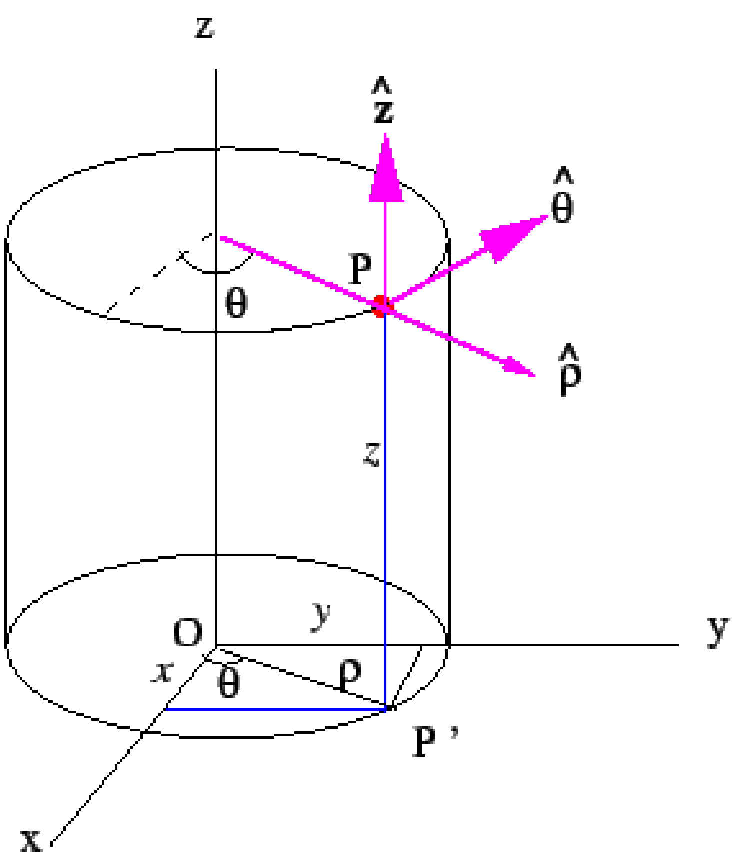
4. An yet another representation is as a sequence of Lorentzian functions $\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}$. It can be seen that $\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\epsilon}{x^2+\epsilon^2}dx = 1$.

Some properties of δ- functions which we state without proof are as follows:

1. $\delta(-x) = \delta(x)$
2. $x\delta(x) = 0$
3. $\delta(ax) = \frac{1}{|a|}\delta(x), \quad a \neq 0$
4. $f(x)\delta(x-a) = f(a)\delta(x-a)$
5. $\int_{-\infty}^{\infty}\delta(x-y)\delta(y-z)dy = \delta(x-z)$

We end this lecture by summarizing properties of cylindrical and spherical coordinate systems that we have been using (and will be using) in these lectures.

Cylindrical Coordinate system

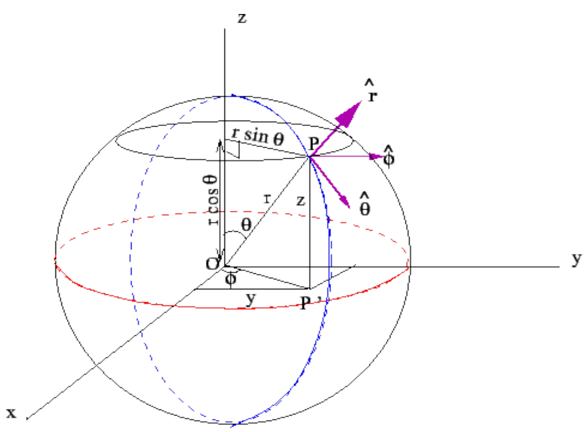


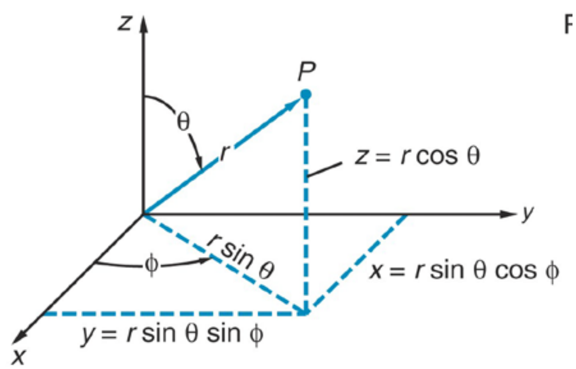
The unit vectors of the cylindrical coordinates are shown above. It may be noted that the z axis is identical to that of the Cartesian coordinates. The variables ρ and θ are similar to the two dimensional polar coordinates with their relationship with the Cartesian coordinates being given by $x = \rho\cos\theta, y = \rho\sin\theta$. An arbitrary line element in this system is $d\vec{l} = \hat{\rho}d\rho + \hat{\theta}\,\rho d\theta + \hat{k}dz$.

Another important quantity is the volume element. This is obtained by multiplying a surface elent in the ρ–θ plane with dz. And is given by $dV = \rho d\theta d\rho dz$

Spherical Coordinate system

For systems showing spherical symmetry, it is often convenient to use a spherical polar coordinates. The variables (r, θ, φ) and the associated unit vectors are shown in the figure.





To define the coordinate system, choose the origin to be at the centre of a sphere with the Cartesian axis defined, as shown. An arbitrary point P is on the surface of a sphere of radius r so that OP=r, the radially outward direction OP is taken to be the direction of the vector $\hat{r}$. The other two coordinates are fixed as follows. The angle which OP makes with the z axis is the polar angle q and the direction perpendicular to OP along the direction of increasing θ is the direction of the unit vector $\hat{\theta}$. Angle q varies from 0 to π. The azimuthal angle φ is fixed as follows. We drop a perpendicular PP' from P onto the xy plane. The foot of the perpendicular P' lies in the xy plane. If we join OP', the angle that OP' makes with the x-axis is the angle φ. Note that as the point P moves on the surface of the sphere keeping the angle θ fixed, i.e. as it describes a cone, the angle φ increases from a value 0 to 2π. The relationship between the spherical polar and the Cartesian coordinates is as follows:

The volume element in the spherical polar is visualized as follows :

$$x = r\sin\theta\cos\varphi$$

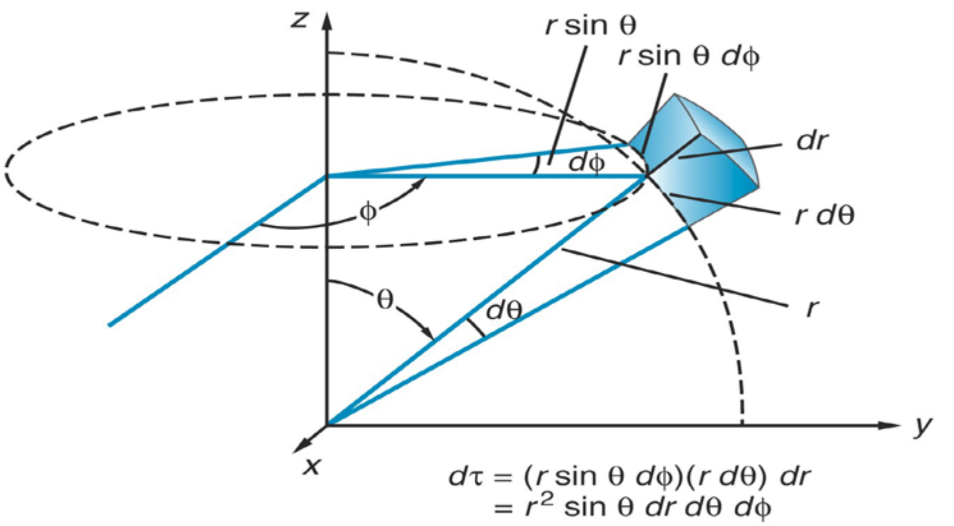
$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \cos^{-1}\frac{z}{r}$$

$$\varphi = \tan^{-1}\frac{y}{x}$$



The expression for the divergence and the Laplacian in the spherical coordinates are given by

$$\nabla\cdot\vec{V}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(V_\varphi)$$

$$\nabla^2V=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}(V)$$

Delta function in three dimensions :

In three dimensions, the delta function is a straightforward generalization of that in one dimension

$$\delta^3(\vec{r})=\delta(x)\delta(y)\delta(z)$$

With the property

$$\int f(\vec{r})\delta^3(\vec{r}-\vec{a})d^3r=f(\vec{a})$$

,

Where the region of integration includes the position $\vec{a}$.

Example : In electrostatics we often use the delta function. In dealing with a point charge located at the origin, one comes up with a situation where one requires the Laplacian of the Coulomb potential, i.e. of 1/r. In the following we obtain $\nabla^2\left(\frac{1}{r}\right)$.

Since the function does not have angular dependence we get, by direct differentiation, for $r\neq 0$,

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right)$$
$$= \frac{1}{r^2}\frac{d}{dr}(-1) = 0$$

This is obviously not valid for r=0. To find out what happens at r=0, consider, integrating $\nabla^2\left(\frac{1}{r}\right)$ over a sphere of radius R about the origin. R can have any arbitrary value. We have using, the divergence theorem,

$$\int_V \nabla^2\left(\frac{1}{r}\right) d^3r = \int_S \nabla\left(\frac{1}{r}\right) d^2r$$

where the surface integral is over the surface of the sphere. Since, r=R on the surface of the sphere (and is non-zero), $\nabla\left(\frac{1}{r}\right) = -\frac{1}{R^2}$ on the surface. We then have

$$\int_V \nabla^2\left(\frac{1}{r}\right) d^3r = \int_S \nabla\left(\frac{1}{r}\right) d^2r = -\frac{1}{R^2}\int_S d^2r = -\frac{1}{R^2}.4\pi R^2 = -4\pi$$

This implies that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$.

Elements of Vector Calculus :Laplacian

Lecture 5: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Tutorial Problems :

1. Calculate the Laplacian of $f = 5x^4 + 3y^3 + 2z^2$
2. Obtain an expression for the Laplacian operator in the cylindrical coordinates.
3. Obtain the Laplacian of $r^{-n} = (x^2 + y^2 + z^2)^{-n/2}, n \neq -1$
4. Show that $\nabla^2(uv) = \nabla^2 u + \nabla^2 v + 2(\nabla u \cdot \nabla v)$
5. Find the Laplacian of the vector field $\vec{F} = x^2 y\hat{\imath} + \ln x\,\hat{\jmath} + \sin z\,\hat{k}$

Solutions to Tutorial Problems :

1. $\nabla^2 f = \frac{\partial^2}{\partial x^2}(5x^4) + \frac{\partial^2}{\partial y^2}(3y^3) + \frac{\partial^2}{\partial z^2}(2z^2) = 60x^2 + 18y + 5$

2. In cylindrical coordinates, the z-axis is the same as in Cartesian. Thus we need to only express $\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$ in polar coordinates and add $\frac{\partial^2 f}{\partial z^2}$ to the result. We have $x = \rho\cos\theta, y = \rho\sin\theta$. We can thus write,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$$

Since $\rho = (x^2 + y^2)^{\frac{1}{2}}, \frac{\partial \rho}{\partial x} = \frac{x}{\rho} = \cos\theta,$ and $\tan\theta = \frac{y}{x},\ \sec^2\theta\frac{\partial\theta}{\partial x} = -\frac{y}{x^2}, \Rightarrow$

$$\frac{\partial\theta}{\partial x} = -\cos^2\theta.\frac{y}{x^2} = -\frac{\sin\theta}{\rho}$$

In a similar way, $\frac{\partial \rho}{\partial y} = \sin\theta, \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{\rho}$

Thus $\frac{\partial f}{\partial x} = \cos\theta\frac{\partial f}{\partial \rho} - \frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial\rho}\left(\cos\theta\frac{\partial f}{\partial\rho}\right)\frac{\partial\rho}{\partial x} + \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial\rho}\right)\frac{\partial\theta}{\partial x} - \frac{\partial}{\partial\rho}\left(\frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}\right)\frac{\partial\rho}{\partial x} - \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}\right)\frac{\partial\theta}{\partial x} \\
&= \frac{\partial}{\partial\rho}\left(\cos\theta\frac{\partial f}{\partial\rho}\right)\cos\theta - \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial f}{\partial\rho}\right)\frac{\sin\theta}{\rho} - \frac{\partial}{\partial\rho}\left(\frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}\right)\cos\theta + \frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\rho}\frac{\partial f}{\partial\theta}\right)\frac{\sin\theta}{\rho} \\
&= \cos^2\theta\frac{\partial^2 f}{\partial\rho^2} - \frac{\cos\theta\sin\theta}{\rho}\frac{\partial^2 f}{\partial\theta\partial\rho} + \frac{\sin^2\theta}{\rho}\frac{\partial f}{\partial\rho} + \frac{\cos\theta\sin\theta}{\rho^2}\frac{\partial f}{\partial\theta} - \frac{\cos\theta\sin\theta}{\rho}\frac{\partial^2 f}{\partial\theta\partial\rho} \\
&\qquad - \frac{\cos\theta\sin\theta}{\rho^2}\frac{\partial f}{\partial\theta} + \frac{\sin^2\theta}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} \\
&= \cos^2\theta\frac{\partial^2 f}{\partial\rho^2} + \frac{\sin^2\theta}{\rho}\frac{\partial f}{\partial\rho} + \frac{\sin^2\theta}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} - \frac{2\cos\theta\sin\theta}{\rho}\frac{\partial^2 f}{\partial\theta\partial\rho}
\end{aligned}$$

One can similarly calculate $\frac{\partial^2 f}{\partial y^2}$ and show that it is given by

$$\frac{\partial^2 f}{\partial y^2} = \sin^2\theta\frac{\partial^2 f}{\partial\rho^2} + \frac{\cos^2\theta}{\rho}\frac{\partial f}{\partial\rho} + \frac{\cos^2\theta}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{2\cos\theta\sin\theta}{\rho}\frac{\partial^2 f}{\partial\theta\partial\rho}$$

Adding these terms and further adding the term $\frac{\partial^2 f}{\partial z^2}$, we get

$$\begin{aligned}
\nabla^2 f &= \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2} \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}
\end{aligned}$$

3. In Cartesian coordinates , $\frac{\partial r^n}{\partial x} = \frac{\partial r^{-n}}{\partial r}\frac{\partial r}{\partial x} = -nr^{-n-1} \times \frac{x}{r} = -nxr^{-n-2}$, where we have used $\frac{dr}{dx} = \frac{x}{r}$. The second derivative with respect to x, $\frac{\partial^2 r^n}{\partial x^2} = -n\frac{\partial}{\partial x}(xr^{-n-2}) = -nr^{-n-2} + n(n+2)r^{-n-3}\frac{x^2}{r}$. On

adding the second differentiation with respect to y and z (which can be written by symmetry), we get

$$\nabla^2 r^n = -3nr^{-n-2} + n(n+2)r^{-n-4}(x^2+y^2+z^2)$$
$$= (-3n + n^2 + 2n)r^{-n-2} = n(n-1r^{-n-2})$$

Where we have used $x^2+y^2+z^2 = r^2$.

4. $\nabla^2(uv) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(uv)$. Consider $\frac{\partial^2}{\partial x^2}(uv) = \frac{\partial}{\partial x}\left[u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}\right] = u\frac{\partial^2 v}{\partial x^2}+v\frac{\partial^2 u}{\partial x^2}+2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$.
   Adding three components result follows.
5. The Laplacian of a vector field is

$$\hat{\imath}\nabla^2 F_x + \hat{\jmath}\nabla^2 F_y + \hat{k}\nabla^2 F_z$$
$$= \hat{\imath}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(x^2 y)$$
$$+\hat{\jmath}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\ln x) + \left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)(\sin z)$$
$$= 2y\hat{\imath} - \frac{1}{x^2}\hat{\jmath} - \sin z\hat{k}$$

Elements of Vector Calculus :Laplacian

Lecture 5: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Self Assessment Quiz

1. Evaluate $\nabla^2(x^2-y)(x+z)$.
2. Find the Laplacian of $e^x + \log y + \cos z$
3. Determine $\int_{-1}^{2}[\sin x\,\delta(x+2) - \cos x\,\delta(x)]\,dx$
4. Evaluate $\int_{-3}^{2}(x^3-2x^2+3x+1)\delta(x+2)dx$

Self Assessment Quiz

1. $\nabla(x^2 - y) = 2x\hat{\imath} - \hat{\jmath},\ \nabla(x+z) = \hat{\imath} + \hat{k}, \nabla^2(x^2 - y) = 2, \nabla^2(x+z) = 0$. Using $\nabla^2(uv) = \nabla^2 u + \nabla^2 v + 2(\nabla u \cdot \nabla v)$, we get the result to be 6x+2z.
2. Answer : $e^x - \frac{1}{y^2} - \cos z$
3. Only the delta function at x=0 contributes because the argument of the first delta function $x = -2$ is not in the limits of integration. The result is $-\cos 0 = -1$
4. In this case $x = -2$ is inside the limit of integration. Thus the result is $(-2)^3 - 2(-2)^2 + 3(-2) + 1 = -21$.