MA2102: LINEAR ALGEBRA

Lecture 12: Rank-Nullity Theorem

11th September 2020

Indian Institute of Science Education & Research Kolkata

IISER KOLKATA

Recall that for a linear map $T : V \to W$, we observed:

- *T is injective (one-to-one) if and only if $N(T) = \{0_V\}$*
- *T is surjective (onto) if and only if $R(T) = W$*

Let us revisit the previous examples through this table.

Eg	$N(T)$	$R(T)$	Nullity	Rank	Domain
1	$\{0\}$	$\mathbb{R}$	0	1	$\mathbb{R}$
2	$\{0\}$ (if $2 \neq 0$ in F)	V	0	$\dim_F(V)$	V
2	V (if $2 = 0$ in F)	$\{0\}$	$\dim_F(V)$	0	V
3	$\{0_V\}$	V	0	$\dim_F(V)$	V
4	V	$\{0_W\}$	$\dim_F(V)$	0	V

Eg	$N(T)$	$R(T)$	Nullity	Rank	Domain
5	kernel of A	column space	nullity(A)	rank of A	$\mathbb{R}^n$
6	$\{(0,0)\}$	$\mathbb{R}^2$	0	2	$\mathbb{R}^2$
7	$\{(0,0)\}$	$\mathbb{R}^2$	0	2	$\mathbb{R}^2$
8	y-axis	y-axis	1	1	$\mathbb{R}^2$
	line $y=x$	x-axis	1	1	$\mathbb{R}^2$
	z-axis	xy-plane	1	2	$\mathbb{R}^3$
9	$\{(0,0)\}$	a line	0	1	$\mathbb{R}$
	$\{(0,0)\}$	a plane	0	2	$\mathbb{R}^2$
10	constant polynomials	$P_{n-1}(\mathbb{R})$	1	n	$P_n(\mathbb{R})$
13	W	V/W	$\dim(W)$	$\dim(V) - \dim(W)$	V

All the examples point towards the following result.

Rank-Nullity Theorem
Let V be a finite dimensional vector space. If $T : V \to W$ is a linear map, then

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(V).$$

It is also referred to as the *Dimension Theorem*.

Corollary Let $T : V \to W$ be a linear map and suppose that

$$\dim(V) = \dim(W) < \infty.$$

Then the following are equivalent:
(i) T is one-to-one;
(ii) T is onto.

Proof.

We establish the following sequence of equivalences:

$$\begin{aligned}
T \text{ is one-to-one} \quad &\Leftrightarrow \quad N(T) = \{0_V\} \ \text{[observation]} \\
&\Leftrightarrow \quad \text{nullity}(T) = 0 \ \text{[definition]} \\
&\Leftrightarrow \quad \text{rank}(T) = \dim(V) \ \text{[Theorem]} \\
&\Leftrightarrow \quad \text{rank}(T) = \dim(W) \ \text{[hypothesis]} \\
&\Leftrightarrow \quad \text{range}(T) = W \ \text{[corollary C]} \\
&\Leftrightarrow \quad T \text{ is onto} \ \text{[definition]}
\end{aligned}$$

This completes the proof. □

Question *Let* $T : V \to V$ *be an injective map and* $\dim(V) = n$*. Is* $T^{-1} : V \to V$ *a linear map?*

Note that T^{-1} is a bijection.

If $T^{-1}(w_i) = v_i$ for $i = 1, 2$, then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$$

implies that $T^{-1}(w_1 + w_2) = v_1 + v_2$. Similarly,

$$T(cv_1) = cT(v_1) = cw_1$$

implies that $T^{-1}(cw_1) = cT^{-1}(w_1)$. Thus, T^{-1} is a linear map.

Proof of Theorem.

Let $k = \dim(N(T)) \leq \dim(V) = n$. Choose a basis $\{v_1, \ldots, v_k\}$ of $N(T)$. By Replacement Theorem, extend this linearly independent set to a basis

$$\beta = \{v_1 \ldots, v_k, v_{k+1}, \ldots, v_n\}$$

of V.

Note that $\{T(v_1), \ldots, T(v_n)\}$ spans $R(T)$. However,

$$T(v_1) = T(v_2) = \cdots = T(v_k) = \mathtt{0}_W.$$

We will show that $\dim R(T) = n - k$.
Claim: The set $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis of $R(T)$.
It suffices to prove linear independence. Let

$$b_{k+1} T(v_{k+1}) + \cdots + b_n T(v_n) = \mathtt{0}_W.$$

By linearity of T, $b_{k+1} v_{k+1} + \cdots + b_n v_n \in N(T)$. Thus, there exists scalars $a_1, \ldots, a_k$ such that

$$a_1 v_1 + \cdots + a_k v_k = b_{k+1} v_{k+1} + \cdots + b_n v_n.$$

As β is linearly independent, all the a_i's and b_j's are zero. $\square$

Example Consider the map $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by

$$T(p) = 2p' + 3\int_0^x p(t)\,dt.$$

Note that

$$T(1) = 3x,\ \ T(x) = 2 + \tfrac{3}{2}x^2,\ \ T(x^2) = 4x + x^3.$$

It can be shown (exercise) that $\beta = \{T(1), T(x), T(x^2)\}$ is linearly independent. Since $R(T) = \mathrm{span}(\beta)$, rank of T is 3. By Rank-Nullity Theorem, nullity is 0, whence $N(T) = \{0\}$.