MA2102: LINEAR ALGEBRA

Lecture 24: Rank

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Let $T : V \to W$ be a linear map between finite dimensional vector spaces over a field F. The rank of T is the dimension of $R(T)$, the range of T. Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis of V. The range is spanned by $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$, i.e.,

$$\operatorname{range}(T) = \dim_F\big(\operatorname{span}\{T(v_1), \ldots, T(v_n)\}\big).$$

By Replacement Theorem, a subset of $T(\beta)$ is a basis of $R(T)$. By rearranging indices, we may assume that $\{T(v_1), \ldots, T(v_k)\}$ is a basis of $R(T)$.

Choose an ordered basis γ of W and consider $[T]_\beta^\gamma$. The j^{th} column $\mathbf{c}_j$ of this matrix is the coordinate vector of $T(v_j)$ with respect to the basis γ.

Claim: *The first k columns* $\mathbf{c}_1, \ldots, \mathbf{c}_k$ *are linearly independent in* F^m.

If $\lambda_1 \mathbf{c}_1 + \cdots + \lambda_k \mathbf{c}_k = \mathtt{0} \in F^m$, then

$$\begin{aligned} \mathtt{0}_{F^m} &= \lambda_1 [T(v_1)]^\gamma + \cdots + \lambda_k [T(v_k)]^\gamma \\ &= [\lambda_1 T(v_1)]^\gamma + \cdots + [\lambda_k T(v_k)]^\gamma \\ &= [\lambda_1 T(v_1) + \cdots + \lambda_k T(v_k)]^\gamma . \end{aligned}$$

In particular, $\lambda_1 T(v_1) + \cdots + \lambda_k T(v_k) = \mathtt{0}_W$, which implies that $\lambda_i = \mathtt{0}$. Show that any column $\mathbf{c}_j$ is in the span of $\mathbf{c}_1, \ldots, \mathbf{c}_k$. Thus, we see that the column rank of $[T]_\beta^\gamma$ is k. As row rank of a matrix and column rank are equal, we have

$$\operatorname{rank}(T) = \text{ column rank of } [T]_\beta^\gamma = \text{ row rank of } [T]_\beta^\gamma .$$

As $[T]_\beta^\gamma$ is the transpose of $[T^*]_{\gamma^*}^{\beta^*}$, we conclude that

$$\operatorname{rank}(T) = \text{ row rank of } [T]_\beta^\gamma = \text{ column rank of } [T^*]_{\gamma^*}^{\beta^*} = \operatorname{rank}(T^*).$$

It follows from Rank-Nullity Theorem that

$$\text{nullity}(T) = n - \text{rank}(T) = n - \text{rank}(T^*).$$

Show that T is injective if and only if T^* is surjective.

We shall see a general approach that explains why the rank of various matrices associated with a linear map are equal.

Observation: *Let $P \in M_m(\mathbb{R}), Q \in M_n(\mathbb{R})$ and $A \in M_{m\times n}(\mathbb{R})$. If P is injective and Q is surjective, then rank of PAQ equals rank of A.*

This follows from a sequence of equalities:

$$\begin{aligned}
\text{column rank of } PAQ &= \dim_{\mathbb{R}}\{PAQw \mid w \in \mathbb{R}^n\} \\
&= \dim_{\mathbb{R}}\{AQw \mid w \in \mathbb{R}^n\} \quad \text{(injectivity of } P) \\
&= \dim_{\mathbb{R}}\{Aw' \mid w' \in \mathbb{R}^n\} \quad \text{(surjectivity of } Q) \\
&= \text{column rank of } A.
\end{aligned}$$

Remark The condition on P being injective is equivalent to P being invertible. Similarly, Q being surjective is equivalent to Q being invertible.

Corollary Let $T : V \to W$ be a linear map between finite dimensional vector spaces. If β, β' are ordered bases of V and γ, γ' are ordered bases of W, then

$$\text{rank of } [T]_{\beta}^{\gamma} = \text{rank of } [T]_{\beta'}^{\gamma'}.$$

Proof.

We know that

$$[I_W]_{\gamma}^{\gamma'}[T]_{\beta}^{\gamma}[I_V]_{\beta'}^{\beta} = [T]_{\beta'}^{\gamma'}.$$

As the change of basis matrices are invertible, the conclusion follows from the observation. □

The observation implies that rank is a conjugation invariant.

Example [Polynomials] Consider $P_n(\mathbb{R})$ and the map

$$T : P_n(R) \to P_n(\mathbb{R}), \quad T(p(x)) = p(x+1).$$

Show that T is a linear map. For instance, if $n = 2$ then

$$T(x^2 + 2x - 3) = (x+1)^2 + 2(x+1) - 3 = x^2 + 4x.$$

The matrix of T with respect to $\beta = \{1, x, \ldots, x^n\}$ is given by

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & \cdots & n \\ 0 & 0 & 1 & \cdots & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Show that T has full rank.

We may also consider the *binomial basis*, i.e.,

$$\gamma = \left\{ \binom{x}{0}, \binom{x}{1}, \ldots, \binom{x}{n} \right\}.$$

The notation is reminiscent of binomial coefficients, i.e.,

$$\binom{x}{k} := \frac{x(x-1)\cdots(x-(k-1))}{k!}.$$

This agrees with ${}^nC_k$ (when we put $x = n$) and is actually a polynomial of degree k. Show that γ is a basis of $P_n(\mathbb{R})$.

We claim that

$$T\left(\binom{x}{k}\right) = \binom{x+1}{k} = \binom{x}{k} + \binom{x}{k-1}.$$

Recall the binomial identity

$$^{n+1}C_k =^n C_k +^n C_{k-1}.$$

Consider the polynomial

$$P(x) = \binom{x+1}{k} - \binom{x}{k} - \binom{x}{k-1}$$

of degree k. Any integer $n \geq k$ is a root of $P(x)$. Since P admits infinitely many roots, it must be identically zero. The matrix of T with respect to γ is given by

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$