# MA2102: LINEAR ALGEBRA

## Lecture 4: Span

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Indian Institute of Science Education & Research Kolkata

**Example 1** There are several interesting subspaces of $M_{n\times n}(\mathbb{R})$, denoted by $M_n(\mathbb{R})$.

(a) Symmetric matrices: Consider the subset

$$\mathrm{Sym}_n := \{A \in M_n(\mathbb{R}) \mid A = A^T\}.$$

If $A, B \in \mathrm{Sym}_n$, then

$$(A+B)^T = A^T + B^T = A + B$$

and $(cA)^T = cA^T = cA$, it follows that $\mathrm{Sym}_n$ is a vector space. Note that this *looks like* $\mathbb{R}^{1+2+\cdots+n}$.

(b) Traceless matrices: Consider the subset

$$W_n = \{A \in M_n(\mathbb{R}) \mid \operatorname{trace} A := a_{11} + \cdots + a_{nn} = 0\}.$$

Since $\operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B$ and $\operatorname{trace}(cA) = c \operatorname{trace} A$, we conclude that $W_n$ is a subspace. Note that this *looks like* $\mathbb{R}^{n^2-1}$.

(c) Diagonal matrices: Consider the subset

$$D_n = \{A \in M_n(\mathbb{R}) \,|\, a_{ij} = 0 \text{ if } i \neq j\}.$$

It can be verified that $D_n$ is a subspace as it closed under addition and scaling. Note that this *looks like* $\mathbb{R}^n$.

(d) Scalar matrices: Consider the subset

$$S_n = \{\lambda I_n \in M_n(\mathbb{R}) \,|\, \lambda \in \mathbb{R}\}.$$

It can be verified that $S_n$ is a subspace as it closed under addition and scaling. Note that this *looks like* $\mathbb{R}$. Show that $S_n$ is a field.

**Example 2** There are several interesting subspaces of $P(\mathbb{R})$, the vector space of all polynomials.

(a) Polynomials of degree at most $n$: We had seen earlier that $P_n(\mathbb{R})$ is a subspace.

(b) Even polynomials: Consider the subset

$$E := \{a_0 + a_2x^2 + \cdots + a_{2k}x^{2k} \,|\, a_i \in \mathbb{R}, k \in \mathbb{N} \cup \{0\}\}.$$

As this is closed under addition and scaling, it is a subspace.

(c) Truncated polynomials: Fix $k \in \mathbb{N}$ and consider the subset

$$T_k := \{a_{k+1}x^{k+1} + \cdots + a_m x^m \,|\, a_i \in \mathbb{R}, m \geq k+1\}.$$

For instance, $T_1$ consists of all polynomials with no constant and linear term. Show that $T_k$ is a subspace.

In all the examples, the subspaces can be *generated* by smaller sets.

- In example 1(a), for $n = 2$, any symmetric matrix can be written as

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- In example 1(c), for $n = 3$, any diagonal matrix can be written as

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- In example 2(a), for $n = 5$, any poynomial of degree at most 5 is naturally a combination of $1, x, x^2, x^3, x^4$ and $x^5$.

Consider two vectors $\mathbf{v} = (1, 0, -2)$ and $\mathbf{w} = (0, 1, 3)$ in $\mathbb{R}^3$. Show that $W_0 = \{c\mathbf{v} + d\mathbf{w} \mid c, d \in \mathbb{R}\}$ is a subspace.

Let $W$ be a subspace containing $\mathbf{v}, \mathbf{w}$. Any vector of the form $c\mathbf{v} + d\mathbf{w}$ lies in $W$. Thus, $W_0 \subseteq W$, implying that $W_0$ is the smallest subspace containing $\mathbf{v}$ and $\mathbf{w}$.

What does $W_0$ look like? It is not the zero vector space as $(0, 1, 3) \in W_0$. If $W_0$ was a line, then any two non-zero vectors in $W_0$ would be scalar multiple of each other. However, there exists no $c \in \mathbb{R}$ such that $c(0, 1, 3) = (1, 0, -2)$. Is $W_0$ all of $\mathbb{R}^3$? Show that $\mathbf{v} \times \mathbf{w}$ is not in $W_0$. Thus, $W_0$ is smaller than $\mathbb{R}^3$ and has to be a plane (from geometric considerations).

This plane $W_0$ is called the subspace *spanned* by $\mathbf{v}, \mathbf{w}$. We often write $W_0 = \text{span}(\{\mathbf{v}, \mathbf{w}\})$.

**Definition** [Span] Given a subset $S$ of a vector space $V$, the span of $S$ is defined to be

$$\mathrm{span}(S) := \{v \in V \mid v = c_1 v_1 + \cdots + c_k v_k \text{ for some } v_i \in S \text{ and } c_i \in \mathbb{R}\}$$

A vector is in the span of $S$ if it can be expressed as a finite linear combination of vectors in $S$.

- If $v \in \mathrm{span}(S)$, then $v = c_1 v_1 + \cdots + c_k v_k$, whence

$$cv = (cc_1)v_1 + \cdots + (cc_k)v_k \in \mathrm{span}(S).$$

- If $v, w \in \mathrm{span}(S)$, then $v = c_1 v_1 + \cdots + c_k v_k$ and $w = d_1 w_1 + \cdots + d_l w_l$. Thus,

$$v + w = c_1 v_1 + \cdots + c_k v_k + d_1 w_1 + \cdots + d_l w_l \in \mathrm{span}(S).$$

**Examples** We shall discuss a few instances of span.

**(1)** $\operatorname{span}(\varnothing) = \{0\}$ (convention)

**(2)** $\operatorname{span}(\{v\}) = \{\lambda v \mid \lambda \in \mathbb{R}\}$ is the "line" passing through $v$. It is an actual line if $v \neq 0$ and the zero vector space if $v = 0$.

**(3)** $\operatorname{span}\Big(\underbrace{(1,0,0,\ldots,0)}_{e_1}, \underbrace{(0,1,0,\ldots,0)}_{e_2}, \cdots, \underbrace{(0,0,0,\ldots,1)}_{e_n}\Big) = \mathbb{R}^n$ since $(x_1, x_2, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n$.

**(4)** $\operatorname{span}\left(\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}\right)$ is the set of traceless $2 \times 2$ matrices.

**(5)** $\operatorname{span}\left(\{1, x, x^2\}\right) = P_2(\mathbb{R})$ is the set of polynomials of degree at most 2.

**Proposition** *If $S$ be a subset of a vector space $V$, then* span($S$) *is a vector subspace containing $S$. Conversely, if $W$ is a subspace containing $S$, then $W$ contains* span($S$).

Proof.

We had seen that span($S$) is a subspace. Conversely, if $W$ contains $S$ and $v_1, \ldots, v_k \in S$, then

$$c_1 v_1 + \cdots + c_k v_k \in W$$

for $c_i \in \mathbb{R}$. $\square$

We shall be concerned with finding smallest sets $S$ which span a given subspace $W$ of $V$. For example, $\{(1,0),(0,1)\}$ span $\mathbb{R}^2$ while $\{(1,2),(0,3)\}$ also span $\mathbb{R}^2$ (exercise). The set $\{(1,1),(2,0),(1,-1)\}$ also span $\mathbb{R}^2$. It seems believable that any set of size 1 cannot span $\mathbb{R}^2$.

**Properties of span** A few relevant observations are in order.

- $\mathrm{span}(\varnothing) = \{\mathbf{0}\}$
- $\mathrm{span}(\mathrm{span}(S)) = \mathrm{span}(S)$

*Since a set is always contained in its span, the inclusion $\supseteq$ follows. On the other hand*

$$a_1(c_{11}v_{11} + \cdots + c_{1k}v_{1k_1}) + \cdots + a_l(c_{l1}v_{l1} + \cdots + c_{lk_l}v_{lk_l})$$

*belongs to* $\mathrm{span}(S)$.

Show that $\mathrm{span}(W) = W$ if and only if $W$ is a subspace.

- $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2))$ if $S_1 \subseteq S_2$

**Remark** The converse is not true. Both $S_1 = \{(1,0),(0,1)\}$ and $S_2 = \{(1,2),(0,3)\}$ span $\mathbb{R}^2$ but they are disjoint.