Angular Momentum, Electromagnetic Waves

Lecture33: Electromagnetic Theory

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As before, we keep in view the four Maxwell's equations for all our discussions.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \Leftrightarrow \nabla \cdot \vec{D} = \rho_{free}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

In the last lecture, we have seen that the electromagnetic field carries both energy and momentum and any discussion on conservation of these two quantities must keep this into account. In this lecture, we will first talk about the possibility of angular momentum being associated with the electromagnetic field and in the second half, introduce the electromagnetic waves.

Angular Momentum

Electromagnetic field, in addition to storing energy and momentum, also has angular momentum and this can be exchanged with the charged particles of the system.

We have seen that the momentum density of the electromagnetic field is given by $\frac{\vec{S}}{c^2} = \frac{1}{c^2}\vec{E} \times \vec{H}$. Based on this, we define the angular momentum density of the electromagnetic field about an arbitrary origin as

$$\vec{l}_{em} = \frac{1}{c^2}\vec{r} \times \left(\vec{E} \times \vec{H}\right)$$

so that the angular momentum of the field is given by

$$\vec{L}_{em} = \frac{1}{c^2}\int \vec{r} \times \left(\vec{E} \times \vec{H}\right) dV$$

Let us start with a collection of charges and currents. The expression for the force density (i.e. the rate of change of momentum density) for the particles is given by the Lorentz force expression

$$\frac{d}{dt}\vec{l}_{mec\,h} = \vec{r} \times \left(\rho\vec{E} + \vec{J} \times \vec{B}\right)$$

We will do some aklgebraic manipulation on the right hand side by using Maxell's equations, We take the medium to be the free space.

First, we substitute $\epsilon_0(\nabla \cdot \vec{E})$ for ρ and replace for the current density, $\vec{J} = \frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. With this we have,

$$\frac{d}{dt}\vec{l}_{mec\,h} = \vec{r} \times \left[\epsilon_0 \vec{E}(\nabla \cdot \vec{E}) + \left(\frac{1}{\mu_0}\nabla \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \times \vec{B}\right]$$

We next add some terms to makes this expression look symmetrical in electric and magnetic fields. To match the first term we add a term $\frac{1}{\mu_0}\vec{B}(\nabla \cdot \vec{B})$, which is identically zero. To make the second term symmetric, we add and subract a term $\epsilon_0\left(\nabla \times \vec{E}\right) \times \vec{E}$. With these we get,

$$\frac{d}{dt}\vec{l}_{mec\,h} = \epsilon_0 \vec{r} \times [\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E})] + \frac{1}{\mu_0}\vec{r} \times [\vec{B}(\nabla \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{B})] - \epsilon_0 \vec{r} \times \left(\frac{\partial \vec{E}}{\partial t} \times \vec{B}\right) + \epsilon_0 \vec{r} \times \vec{E} \times (\nabla \times \vec{E})$$

where we have interchanged the order of cross product in a couple of places by changing sign.

The last two terms on right can be simplified as follows. Using $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we can write,

$$\begin{aligned} -\epsilon_0 \vec{r} \times \left(\frac{\partial \vec{E}}{\partial t} \times \vec{B}\right) + \epsilon_0 \vec{r} \times \vec{E} \times (\nabla \times \vec{E}) &= -\epsilon_0 \vec{r} \times \left(\frac{\partial \vec{E}}{\partial t} \times \vec{B}\right) - \epsilon_0 \vec{r} \times \vec{E} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) \\ &= -\epsilon_0 \vec{r} \times \frac{d(\vec{E} \times \vec{B})}{dt} = -\epsilon_0 \mu_0 \vec{r} \times \frac{d\vec{S}}{dt} \\ &= -\frac{d}{dt}\vec{l}_{em} \end{aligned}$$

Thus we have,

$$\frac{d}{dt}(\vec{l}_{mec\,h} + \vec{l}_{em}) = \epsilon_0 \vec{r} \times [\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E})] + \frac{1}{\mu_0}\vec{r} \times [\vec{B}(\nabla \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{B})]$$

We recall that the elements of the stress tensor was defined as

$$\vec{T}_{\alpha,\beta} = \epsilon_0 \left[E_\alpha E_\beta - \frac{1}{2}|E|^2 \delta_{\alpha,\beta}\right] + \frac{1}{\mu_0}\left[B_\alpha B_\beta - \frac{1}{2}|B|^2 \delta_{\alpha,\beta}\right]$$

It can be shown (see Tutorial assignment 1) that the right hand side simplifies to $\vec{r} \times (\nabla \cdot \overleftrightarrow{T})$, so that

$$\boxed{\frac{d}{dt}(\vec{l}_{mec\,h} + \vec{l}_{em}) = \vec{r} \times (\nabla \cdot \overleftrightarrow{T})}$$

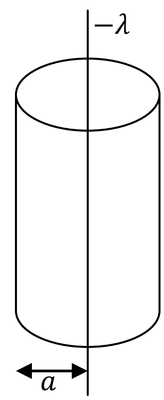
On integrating over the volume one gets a statement of conservation of angular momentum.

$$\frac{d}{dt}\left(\vec{L}_{mec\,h} + \int_{vol} \vec{l}_{em}\, d^3r\right) = \int \vec{r} \times (\overleftrightarrow{T}) \cdot d\vec{S}$$

which sates that the total rate of change of momentum (of particles and field) is equal to the flux of the torque through the surface.

Example : The Feynman Paradox – a variant of Feynman disk

We discuss a variant of the famous Feynman disk problem, which is left as an exercise. There is an infinite line charge of charge density $-\lambda$ is surrounded by an insulating cylindrical surface of radius a having *a* surface charge density $\sigma = +\frac{\lambda}{2\pi a}$, so that the net charge of the system is zero. The cylinder can freely rotate about the z axis, which coincides with the line charge. Because of Gauss's law, the electric field exists only within the cylinder. The system is immersed in a uniform magnetic field $\vec{B} = B_0\hat{z}$ along the z axis. The system is initially at rest. If the magnetic field is now reduced to zero, the cylinder will be found to rotate.



The explanation of rotation lies in conservation of angular momentum. As the system was initially at rest, the initial mechanical angular momentum is zero. The field angular momentum can be calculated as follows.
The electric field is given by
$\vec{E} = -\frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$. The magnetic field is circumferential and is given by $\vec{B} = B_0\hat{z}$.
Thus, the initial field angular momentum per unit length is given by, (taking the angular momentum about the axis)

$$\vec{L} = \int \left(\vec{r} \times \frac{\vec{S}}{c^2}\right) 2\pi r\, dr$$
$$= -\frac{\lambda B_0}{2\pi} 2\pi \int_0^a [\hat{r} \times (\hat{r} \times \hat{z})] r dr = \frac{\lambda B_0 a^2}{2}\hat{z}$$

where we have used $\mu_0\epsilon_0 = \frac{1}{c^2}$.
This is the net angular momentum of the field plus the mechanical system because the latter is at rest. If the magnetic field is reduced to zero, because of a changing flux through any surface parallel to the xy plane, there is an azimuthal current generated. This is caused by the rotating cylinder which rotates with an angular speed ω, giving rise to an angular momentum $I\omega$, where I is the moment of inertia per unit length of the cylinder about the z axis. If the time period of rotation is T, the current per unit length is given by $J_\phi = \frac{Q}{T} = 2\pi a\sigma\frac{\omega}{2\pi} = a\sigma\omega = \frac{\lambda\omega}{2\pi}$. Since the current is circumferential, the magnetic field (like in a solenoid) is along the z direction and is given by $\vec{B}_{final} = \mu_0 J\hat{z} = \frac{\mu_0\lambda\omega}{2\pi}\hat{z}$. The final angular momentum is thus given by

$$\frac{\lambda B_{final}\, a^2}{2}\hat{z} = \frac{\mu_0 \lambda^2 a^2}{4\pi}\omega$$

. Thus we must have,

$$I\omega + \frac{\mu_0 \lambda^2 a^2}{4\pi}\omega = \frac{\lambda B_0 a^2}{2}$$

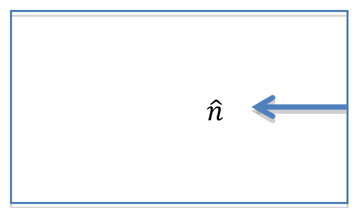
which allows us to determine ω

$$\omega = \frac{\lambda B_0 a^2}{2}\frac{1}{I + \frac{\mu_0 \lambda^2 a^2}{4\pi}}$$

Example : Radiation Pressure

An experimentally verifiable example of the fact that the electromagnetic field stores momentum is provided by the pressure exerted by radiation confined inside a cavity. Consider an enclosure in the shape of a rectangular parallelepiped and let us consider the right most wall of the enclosure. The normal direction to the wall is to the let and the fieldexerts a force on this wall along the x direction.

The force exerted on an area dS of the wall is given by $d\vec{F} = \overleftrightarrow{T} \cdot d\vec{S}$



Since the force is exerted in x direction we only need

$$T_{xx} = \epsilon_0 \left(E_x^2 - \frac{1}{2}|E|^2\right) + \frac{1}{\mu_0}\left(B_x^2 - \frac{1}{2}|B|^2\right)$$

If the radiation is isotropic, we can write,

$E_x^2 = \frac{1}{3}|E|^2$ and $B_x^2 = \frac{1}{3}|B|^2$ so that,

$$dF_x = \frac{1}{6}\left(\epsilon_0 E^2 + \frac{B^2}{2\mu_0}\right) = \frac{1}{3}u$$

where u is the energy density. This, incidentally, was the starting point for proving Stefan Boltzmann's law.

Plane Wave solutions to Maxwell's Equations

In the following we obtain a special solution to the Maxwell's equations for a linear , isotropic medium which is free of sources of charges and currents.

We have then the following equations to solve:

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t}$$

Taking the curl of the third equation, and substituting the first equation therein, we have,

$$\nabla \times \left(\nabla \times \vec{E}\right) = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\nabla^2 \vec{E} = -\frac{\partial\left(\nabla \times \vec{B}\right)}{\partial t}$$

Substituting last equation in this, we get,

$$\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

In a similar way, we get an identical looking equation for the magnetic field,

$$\nabla^2 \vec{B} = \mu\epsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

These represent wave equations with the velocity of the wave being $1/\sqrt{\mu\epsilon}$.

It may be noted that, in a general curvilinear coordinate system, we do not have, $\left(\nabla^2 \vec{E}\right)_x = \nabla^2 \vec{E}_x$. However, it would be true in a Cartesian coordinates where the unit vectors are fixed.

Let us look at “plane wave” solutions to these equations. A plane wave is one for which the surfaces of constant phases, viz. wavefronts, are planes. Time harmonic solutions are of the form $\sin(\vec{k} \cdot \vec{r} - \omega t)$ or $\cos(\vec{k} \cdot \vec{r} - \omega t)$. However, mathematically it turns out to be simple to consider an exponential form and take, at the end of calculations, the real or the imaginary part.

We take the solutions to be of the form,

$$\vec{E} = \vec{E}_0 e^{i\left(\vec{k} \cdot \vec{r} - \omega t\right)}$$
$$\vec{B} = \vec{B}_0 e^{i\left(\vec{k} \cdot \vec{r} - \omega t\right)}$$

Note that the surfaces of constant phase are given by

$$\vec{k} \cdot \vec{r} - \omega t = \text{constant}$$

At any time t, since $\omega t = \text{constant}$, we have, the surface given by

$$\xi = \vec{k} \cdot \vec{r} = \text{constant}$$

Represent the wave-fronts. This is obviously an equation to a plane.$\vec{k}$ is known as the **"propagation vector"**. As time increases these wave fronts move forward, I.e the distance from source increases. One could also look at the backward moving waves which would be of the form $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}+\omega t)}$.

As time progresses, the surfaces of constant phase satisfy the equation

$$|k|\zeta - \omega t = \text{constant}$$

where $\zeta = \hat{k} \cdot \vec{r}$. Thus the angular frequency ω ιs given by

$$\omega = k\frac{d\zeta}{dt}$$

$\frac{d\zeta}{dt} = \frac{\omega}{k} = v$ is known as the "phase velocity".

Substituting our solutions into the two divergence equations, we get,

$$\vec{k} \cdot \vec{E} = 0$$
$$\vec{k} \cdot \vec{B} = 0$$

Thus the direction of both electric and magnetic fields are perpendicular to the propagation vector. If we restrict ourselves to non-conducting media, we would, in addition, have, from the curl equation,

$$\nabla \times \vec{B} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

$$i\vec{k} \times \vec{B} = -i\mu\epsilon\omega\vec{E}$$

This also shows that the electric field is perpendicular to the magnetic field vector. Thus the electric field, the magnetic field and the direction of propagation form a right handed triad. If the propagation vector is along the z direction, the electric and magnetic field vectors will be in the xy plane being perpendicular each other.

Note that using the exponential form has the advantage that the action of operator ∇ is equivalent to replacing it by $i\vec{k}$ and the time derivative is equivalent to a multiplication by $-i\omega$.

We can take the cross product of the equation $i\vec{k} \times \vec{B} = -i\mu\epsilon\omega\vec{E}$ with $\vec{k}$ to get,

$$\vec{k} \times (\vec{k} \times \vec{B}) = -\omega\mu\epsilon\vec{k} \times \vec{E}$$

expanding the scalar triple product,

$$\vec{k}(\vec{k} \cdot \vec{B}) - k^2\vec{B} = -\omega\mu\epsilon\vec{k} \times \vec{E}$$

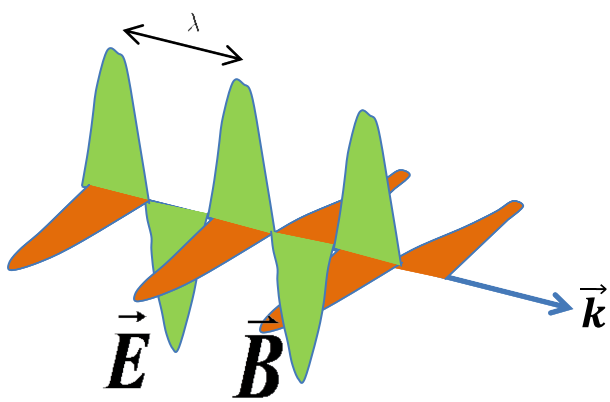
substituting $\vec{k} \cdot \vec{B} = 0$, we get,

$$\vec{B} = \frac{\omega\mu\epsilon}{k^2}\vec{k} \times \vec{E} = \frac{\omega\mu\epsilon}{k}\hat{k} \times \vec{E}$$

=

The ratio of the strength of the magnetic field to that of the electric field is given by $\frac{\omega\mu\epsilon}{k} = \frac{1}{v}$

Where we have used the expressions for the velocity of the wave, $v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$. In free space this velocity is the same as the speed of light which explains why the magnetic field associate with the electromagnetic field is difficult to observe.



In the above figure the electric field variation is shown by the green curve and that of magnetic field by the orange curve.

We have seen that electromagnetic field stores energy and momentum. A propagating electromagnetic wave carries energy and momentum in its field. The energy density of the electromagnetic wave is given by

$$u = \frac{1}{2}\epsilon|E|^2 + \frac{|B|^2}{2\mu}$$
$$= \frac{1}{2}\epsilon\left(|E|^2 + \frac{|B|^2}{\mu\epsilon}\right) = \epsilon|E|^2$$

where we have used the relation $|E| = \frac{|B|}{\sqrt{\mu\epsilon}}$.

The Poynting vector is given by

$$\vec{S} = \vec{E} \times \vec{H} = \frac{\vec{E} \times \vec{B}}{\mu}$$

Recall that the difference between H and B is due to bound currents which cannot transport energy. Replacing the real form of the electromagnetic field, we get, taking the electric field along the x direction, the magnetic field along the y direction and the propagation vector along the z direction,

$$\vec{S} = c\epsilon_0 E^2 \cos^2(kz - \omega t)\hat{k}$$

where we have assumed the electromagnetic wave to be travelling in free space. The intensity of the wave is defined to be the time average of the Poynting vector,

$$I = \frac{1}{2} c\epsilon_0 E^2 \hat{k}$$

where the factor of ½ comes from the average of the $\cos^2(kz - \omega t)$ over a period.

In general, given the propagation direction, the electric and the magnetic fields are contained in a perpendicular to it. The fields can point in arbitrary direction in the plane as long as they are perpendicular to each other. If the direction of the electric field is random with time, the wave will be known as an "unpolarized wave". Specifying the direction of the electric field (or equivalently of the orthogonal magnetic field) is known as a statement on the statement of polarization of the wave.

Consider the expression for the electric field at a point at a given time. Assuming that it lies in the xy plane, we can write the following expression for the electric field

$$\vec{E}(z,t) = Re\left[\left(E_{0x}\hat{\imath} + E_{0y}\hat{\jmath}\right)e^{i(kz-\omega t)}\right]$$

In general, the quantities inside the bracket are complex. However, we can specify some specific relations between them.

1. Suppose E_{0x} and E_{0y} are in phase, i.e., suppose,
$$E_{0x} = |E_{0x}|e^{i\phi}$$
$$E_{0y} = |E_{0y}|e^{i\phi}$$
then we can express the field as
$$\vec{E}(z,t) = \left(|E_{0x}|\hat{\imath} + |E_{0y}|\hat{\jmath}\right)\cos(kz - \omega t + \phi)$$
The magnitude of the electric vector changes from 0 to $\sqrt{|E_{0x}|^2 + |E_{0y}|^2}$ but its direction remains constant as the wave propagates along the z direction. Such a wave is called "**linearly polarized**" wave. The wave is also called plane "**plane polarized**" as at a given time the electric vectors at various locations are contained in a plane.

2. Instead of the phase between the x and y components of the electric field being the same, suppose the two components maintain a constant phase difference as the wave moves along, we have

$$E_{0x} = |E_{0x}|$$
$$E_{0y} = |E_{0y}|e^{i\phi}$$

If the phase difference happens to be $\frac{\pi}{2}$, we can write the expression for the electric field as

$$\vec{E}(z,t) = Re\left[\left(E_{0x}\hat{\imath} + E_{0y}\hat{\jmath}\right)e^{i(kz-\omega t)}\right]$$
$$= |E_{0x}|\cos(kz-\omega t)\hat{\imath} + |E_{0y}|\sin(kz-\omega t)\hat{\jmath}$$

Consider the time variation of electric field at a particular point in space, say at z=0. As t increases from 0 to $\frac{\pi}{2\omega}$, E_x decreases from its value $|E_{0x}|$ to zero while E_y increases from zero to $|E_{0y}|$ the electric vector rotating counterclockwise describing an ellipse. This is known as "**elliptic polarization**". (In the general case of an arbitrary but constant phase difference, the state of polarization is elliptic with axes making an angle with x and y axes. Alternatively, a state of arbitrary polarization can be expressed as a linear combination of circular or linear polarizations.)

3. A special case of elliptic polarization is "circular polarization" where the amplitudes of the two components are the same, viz., $|E_{0x}| = |E_{0y}|$, when the electric vector at a point describes a circle with time.

Tutorial Assignment

1. Prove the relation

$$\epsilon_0 \vec{r} \times \left[\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E})\right] + \frac{1}{\mu_0} \vec{r} \times \left[\vec{B}(\nabla \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{B})\right] = \vec{r} \times (\nabla \cdot \overleftrightarrow{T})$$

2. Two concentric shells of radii a and b carry charges $\pm q$. At the centre of the shells a dipole of magnetic moment $m\hat{k}$ is located. Find the angular momentum in the electromagnetic field for this system.

Solutions to Tutorial Assignments

1. We will simplify only the electrical field term, the magnetic field term follows identically. We have,

$$\begin{aligned}
&\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E}) \\
&\qquad = (\hat{\imath}E_x + \hat{\jmath}E_y + \hat{k}E_z)\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) - (\hat{\imath}E_x + \hat{\jmath}E_y + \hat{k}E_z) \\
&\qquad \times \left[\hat{\imath}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\right]
\end{aligned}$$

Let us consider the x component of both sides,

$$\begin{aligned}
\left[\vec{E}(\nabla \cdot \vec{E}) - \vec{E} \times (\nabla \times \vec{E})\right]_x &= E_x\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) - E_y\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + E_z\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x}(E_x^2 - E_y^2 - E_z^2) + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z) \\
&= \frac{\partial}{\partial x}E_x^2 - \frac{\partial}{\partial x}|E|^2 + \frac{\partial}{\partial y}(E_x E_y) + \frac{\partial}{\partial z}(E_x E_z)
\end{aligned}$$

The y and z components follow by symmetry. The x component of the divergence of the stress tensor can be seen to be the same, as,

$$
\begin{aligned}
\left(\nabla\cdot\overleftrightarrow{T}\right)_x &= \frac{\partial T_{xx}}{\partial x}+\frac{\partial T_{xy}}{\partial y}+\frac{\partial T_{xz}}{\partial z} \\
&= \frac{\partial}{\partial x}\left(E_x^2-\frac{E^2}{2}\right)+\frac{\partial}{\partial y}\left(E_xE_y\right)+\frac{\partial}{\partial z}\left(E_xE_z\right)
\end{aligned}
$$

2. In a spherical coordinate system, the magnetic field due to the dipole is given by $\vec{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[2\cos\theta\hat{r}+\sin\theta\hat{\theta}\right]$. The electric field is confined only within the shells and is given by $\vec{E}=\frac{q}{4\pi\epsilon_0 r^2}\hat{r}$. The Poynting vector is given by

$$\vec{S}=\frac{\vec{E}\times\vec{B}}{\mu_0}=\frac{qm}{16\pi^2\epsilon_0 r^5}\sin\theta\hat{\phi}$$

The angular momentum density about the centre of the shells is then given by

$$\vec{r}\times\frac{\vec{S}}{c^2}=-\frac{qm}{16\pi^2\epsilon_0 r^4}\sin\theta\hat{\theta}$$

The total angular momentum of the electromagnetic field can be obtained by integrating this over the volume. However, since $\hat{\theta}$ is not a constant unit vector, we first convert this to Cartesian and write the total angular momentum as

$$\vec{L}=-\frac{qm}{16\pi^2\epsilon_0}\int\frac{1}{r^4}\sin\theta\left[\cos\theta\cos\phi\,\hat{\imath}+\cos\theta\sin\phi\hat{\jmath}-\sin\theta\hat{k}\right]d^3r$$

The first two terms, when integrated gives zero because the integral over ϕvanishes. We are left with

$$
\begin{aligned}
\vec{L} &= \frac{qm}{16\pi^2\epsilon_0}\hat{k}\int\frac{1}{r^4}\sin^2\theta d^3r \\
&= \frac{qm}{16\pi^2\epsilon_0}\hat{k}\;2\pi\int_a^b\frac{1}{r^2}dr\;\int_0^\pi\sin^3\theta d\theta \\
&= \frac{qm}{8\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)\times\frac{4}{3}\hat{k}=\frac{qm}{6\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)\hat{k}
\end{aligned}
$$

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Self Assessment Questions

1. A metal sphere of radius R has a charge Q and is uniformly magnetized with a magnetization $\vec{M}$. Calculate the angular momentum of the field about the centre of the sphere.
2. In Problem 1, if the magnetization of the sphere is gradually and uniformly reduced to zero (probably by heating the sphere through the Curie temperature), calculate the torque exerted by the induced electric field on the sphere and show that the angular momentum is conserved in the process.

Solutions to Self Assessment Questions

1. The charge being uniformly distributed over its surface, the electric field exists only outside the sphere and is given by $\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$. The magnetic field inside is constant and is given outside the sphere by $\vec{B} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right]$. One can calculate the Poynting vector and angular momentum density closely following Problem 2 of the tutorial. The total angular momentum is obtained by integrating over all space outside the sphere. Following this we get

$$\vec{L} = \frac{qm\mu_0}{16\pi^2}\hat{k}\int\frac{1}{r^4}\sin^2\theta d^3r\,\hat{k}$$
$$= \frac{qm}{16\pi^2}\mu_0 2\pi\int_R^\infty\frac{1}{r^2}dr\ \int_0^\pi\sin^3\theta d\theta\,\hat{k}$$
$$= \frac{qm\mu_0}{8\pi}\frac{1}{R}\frac{4}{3}\hat{k}$$

Substituting $m = \frac{4\pi R^3}{3}M$, we get

$$\vec{L} = \frac{2qM\mu_0}{9}R^2\hat{k}$$

2. When the sphere is slowly demagnetized from M to zero, there is a changing flux which induces an electric field. The induced electric field is circumferential and can be calculated from Faraday's law. The magnetic field due to the magnetized sphere inside the sphere is constant and is given by $\frac{2}{3}\mu_0 M$. Consider a circumferential loop on the surface between polar angles θ and $\theta + d\theta$. The radius of the circular loop is $R\sin\theta$ and the area of the loop is $\pi(R\sin\theta)^{\ 2}$. The changing flux through this area is $\frac{2}{3}\mu_0\frac{dM}{dt}\pi(R\sin\theta)^{\ 2}$ and this changing flux is equal to the emf induced in the loop of radius $R\sin\theta$, so that we have, the electric field magnitude to be given by

$$E\ (2\pi R\sin\theta) = \frac{2}{3}\mu_0\frac{dM}{dt}\pi(R\sin\theta)^{\ 2}$$

which gives,

$$\vec{E} = \frac{1}{3}\mu_0 R\sin\theta\frac{dM}{dt}\hat{\phi}$$

If we consider an area element $dS = 2\pi R^2\sin\theta d\theta$ of the surface, the charge on the surface $dQ = \sigma 2\pi R^2\sin\theta d\theta = \frac{Q}{2}\sin\theta d\theta$ experiences a force

$$dF = \frac{1}{6}\mu_0 Rsin\,\theta\frac{dM}{dt}\frac{Q}{R}\sin\theta d\theta = \frac{1}{6}\mu_0 QR\ \frac{dM}{dt}\sin^2\theta d\theta$$

The torque about the z axis is obtained by multiplying this with the distance $R\sin\theta$ of this element about the z axis.

$$d\tau = \ \frac{Q}{6}R^2\mu_0\frac{dM}{dt}\sin^3\theta d\theta$$

Total torque is obtained by integrating over the angle

$$\tau = \frac{Q}{6}R^2\mu_0\frac{dM}{dt}\int_0^\pi\sin^3\theta d\theta = \frac{2Q}{9}R^2\mu_0\frac{dM}{dt}$$

The change in angular momentum is $\int\tau dt = \frac{2Q}{9}R^2\mu_0 M$, as was obtained in Problem 1.