# Wave Guides

Lecture 36: Electromagnetic Theory

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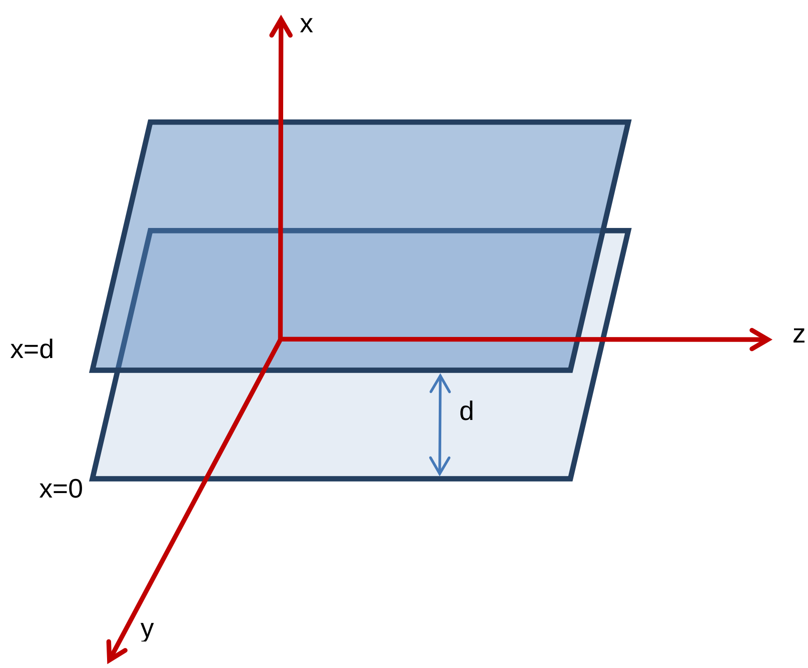
We have seen that electromagnetic waves carry both energy and momentum. It should therefore be possible to transmit energy contained in the wave from one place to another. For low frequencies (typically, less than 1 MHz), this is done by parallel transmission lines or coaxial cables. However, for higher frequencies, such as microwave frequencies, we need special conduits such a hollow metal tubes or optical fibers.

We will first explain the basic concepts of guiding waves by taking the simple case of a pair of parallel, infinite metal plates with a separation d between them. The wave is made to propagate in the hollow region between the plates, which we take to be empty space. We have seen that the electric field is primarily confined to the surface, penetrating a small "skin depth" which becomes smaller with increasing frequency. As a result we take the electric field to vanish at the surface of the guide.




Let us rewrite the Maxwell's equations for the case where there are free charges or currents.

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = -i\omega\mu\vec{H}$$

$$\nabla \times \vec{H} = \vec{J} + \epsilon \frac{\partial \vec{E}}{\partial t} = (\sigma + i\omega\mu)\vec{E}$$

where we have time variation to be $e^{i\omega t}$.

Taking the curl of the these equations and substituting from the other equations, we get

$$\nabla^2 \vec{E} = i\omega\mu(\sigma + i\omega\epsilon)\vec{E}$$
$$\nabla^2 \vec{H} = i\omega\mu(\sigma + i\omega\epsilon)\vec{H}$$

For propagation between the plannes, $\sigma = 0$, so that we have,

$$\nabla^2 \vec{E} = -\omega^2 \epsilon\mu \vec{E}$$
$$\nabla^2 \vec{H} = -\omega^2 \epsilon\mu \vec{H} \qquad (1)$$

We will return to thiese equations a little while later.

Consider the original curl equations and write them in component form (with $\sigma = 0$)

$$\nabla \times \vec{H} = i\omega\mu\vec{E}$$

gives,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\omega\mu E_x$$
$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i\omega\mu E_y$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\mu E_z$$

Parallely, the derivatives of electric field components satisf,

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega\mu H_x$$
$$\frac{\partial E_x}{\partial z} - \frac{\partial E}{\partial x} = -i\omega\mu H_y$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z$$

The solutions of these set of equations can be classified into three distinct groups. The direction of propagation being along the z direction, we term this direction as the longitudinal direction and a direction perpendicular to it (i.e. x and y directions) as the transverse direction. The distinct solutions are grouped as

1. Solutions for which $E_z = 0$ , i.e. the non-zero electric field is transverse to the direction of propagation. This is called the **"Transverse Electric " or TE mode**. In this case the longitudinal component of magnetic field is non-vanishing ($H_z \neq 0$). The solution, therefore, is also referred to as H-mode.
2. Solutions for which $H_z = 0$ , i.e. the non-zero magnetic field is transverse to the direction of propagation. This is called the "**Transverse Magnetic " or TM mode**. In this case the longitudinal component of electric field is non-vanishing ($E_z \neq 0$). The solution, therefore, is also referred to as E-mode.
3. In some situations, it is possible to have the longitudinal component of both electric and magnetic field to be simultaneously zero, like the case of propagation of electromagnetic wave in free space. This special solution is called **"Transverse Electric and Magnetic Mode" or TEM mode.**

It is of course possible for a solution not to belong to any of these distinct categories in which case it would be a **"mixed mode"** solution.

Since the wave propagates along the z direction, the z dependence of the field is specified,

$$\vec{E}(x,y,z) = \vec{E}(x,y)e^{i\omega t - \gamma z}$$

where, the complex factor $\gamma = \alpha + i\beta$. Thus $\partial/\partial z$ is equivalent to multiplication by $-\gamma$. Further, since the plates are of infinite extent in y direction, there is no field variation in this direction so that we can replace the derivative $\partial/\partial y$ by zero. Using these, we can rewrite the equations above as

$$\begin{aligned} \gamma H_y &= i\omega\mu E_x \\ -\gamma H_x - \frac{\partial H_z}{\partial x} &= i\omega\mu E_y \\ \frac{\partial H_y}{\partial x} &= i\omega\mu E_z (2) \end{aligned}$$

and

$$\begin{aligned} \gamma E_y &= -i\omega\mu H_x \\ -\gamma E_x - \frac{\partial E_z}{\partial x} &= -i\omega\mu H_y \\ \frac{\partial E_y}{\partial x} &= -i\omega\mu H_z (3) \end{aligned}$$

The wave equation (1) takes the form,

$$\left(\frac{\partial^2}{\partial x^2}+\gamma^2\right)\vec{E} = -\omega^2\epsilon\mu\vec{E}$$

$$\left(\frac{\partial^2}{\partial x^2}+\gamma^2\right)\vec{H} = -\omega^2\epsilon\mu\vec{H} \qquad (4)$$

We will discuss in detail the TE solution and leave the TM solution as an exercise.

**TE- Mode :**

In this case $E_z = 0$. From Eqn. (2) , we get $H_y = \text{constant}$, which we can choose to be zero. This in turn implies, from Eqn. (3), $E_x = 0$. We will first solve for $E_y$ using Eqn. (4).

$$\left(\frac{\partial^2}{\partial x^2}+\gamma^2\right)E_y = -\omega^2\epsilon\mu E_y$$

Define $k^2 = \gamma^2 + \omega^2\epsilon\mu$. We have,

$$\frac{\partial^2}{\partial x^2}E_y + k^2 E_y = 0$$

the solution of which are well known to be

$$E_y = A\sin kx + B\cos kx$$

(complete solution will be obtained by multiplying this with $e^{-\gamma z}e^{i\omega t}$. )
We now insert the boundary condition, $E_y = 0$ on both the plates, i.e. at $x = 0$ and at $x = d$. The former gives $B = 0$, so that $E_y = A\sin kx$. The latter condition restricts the values that *k* can take to $\frac{n\pi}{d}$, where $n = 1,2, \ldots$ .( $n$ cannot take the value zero because that would make the field identically zero. )
Thus we have,

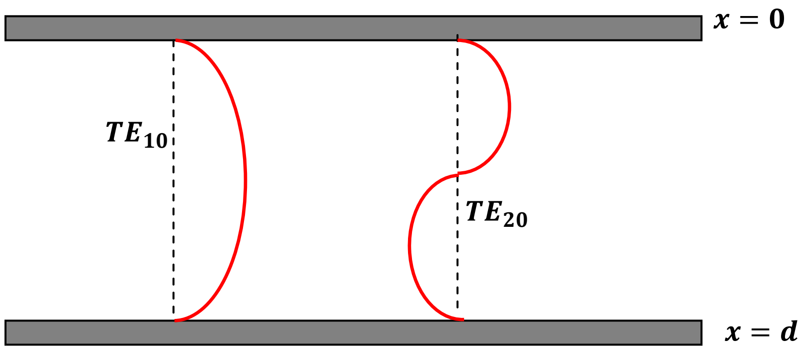
$$E_y = E_{y0}\sin\left(\frac{n\pi}{d}x\right)e^{-\gamma z}e^{i\omega t} \, (5)$$

where $A = E_{y0}$ is the maximum value of the field.
We can now use this expression in Eqn. (3) to obtain the magnetic field components

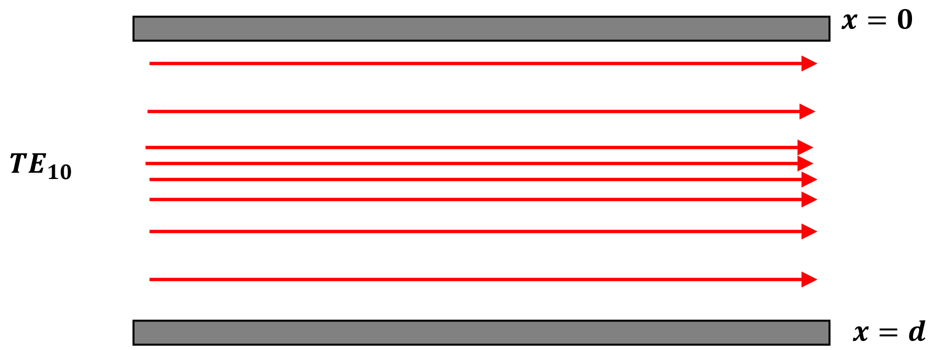
$$H_x = -\frac{\gamma}{i\omega\mu}E_y = -E_{y0}\frac{\gamma}{i\omega\mu}\sin\left(\frac{n\pi}{d}x\right)e^{-\gamma z}e^{i\omega t} \, (6)$$

$$H_z = -\frac{1}{i\omega\mu}\frac{\partial E_y}{\partial x} = -E_{y0}\frac{n\pi}{i\omega\mu d}\cos\left(\frac{n\pi}{d}x\right)e^{-\gamma z}e^{i\omega t} \qquad (7)$$

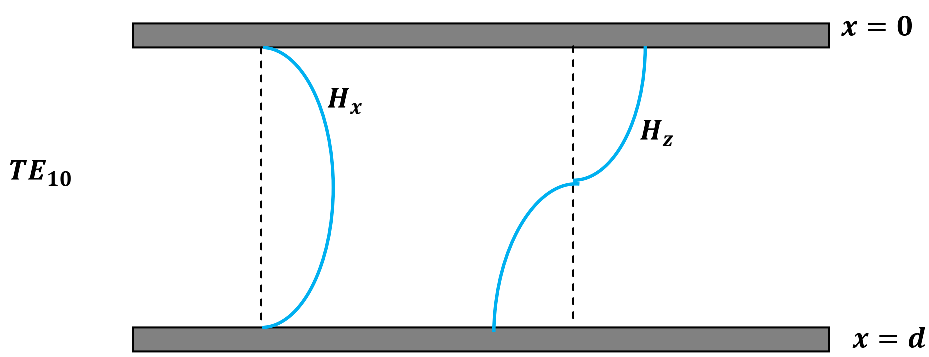
Modes are named by specifying the value that $n$ takes. As we have seen the lowest mode is $n = 1$. This is termed as **TE$_{10}$** mode, the meaning of the second index will be clear when we discuss rectangular waveguides but for the present case it remains zero for all values of *n*. The following figure gives the electric field profile for n=1 and n=2 for a fixed z.

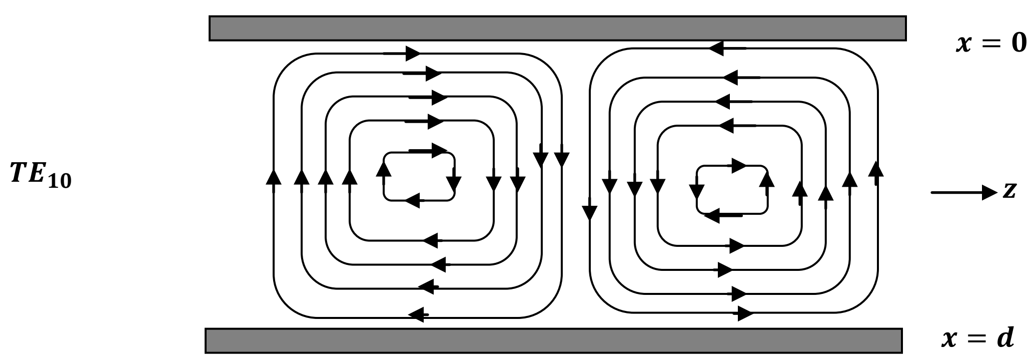



If we look along the direction of propagation, for the $TE_{10}$mode, thefield lines crowdat the centre of the guide, where the field strength is strongest.




The corresponding magnetic fields for $TE_{10}$ are shown below:



**Cutoff Frequency**

Is transmission in this manner always possible? We have,

$$\gamma = \sqrt{k^2 - \omega^2 \epsilon\mu} = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \omega^2 \epsilon\mu}$$

Propagating solution implies that $\gamma = i\beta$. Thus we requires,

$$\omega > \omega_c = \frac{1}{\sqrt{\mu\epsilon}} \frac{n\pi}{d}$$

If the frequency is less than this, the wave attenuates. The phase velocity for the propagating solution is given by

$$v_\phi = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2 \epsilon\mu - \left(\frac{n\pi}{d}\right)^2}}$$

As frequency decreases and approaches the critical value, it becomes infinite. For very large frequencies, the velocity in vacuum approaches that of light $1/\sqrt{\mu\epsilon}$.

**TM Mode**

We will not work out the TM mode algebra. In this case $H_z = 0$. The non-zero field components are

$$\begin{aligned} H_y &= H_y^0 \cos\left(\frac{m\pi}{d}x\right) e^{i\omega t - i\beta z} \\ E_x &= H_y^0 \frac{\beta}{\omega\epsilon} \cos\left(\frac{m\pi}{d}x\right) e^{i\omega t - i\beta z} \\ E_z &= H_y^0 \frac{im\pi}{\omega\epsilon d} \cos\left(\frac{m\pi}{d}x\right) e^{i\omega t - i\beta z} \end{aligned}$$

**TEM Mode**

Note that in TM case , unlike in the case of TE modes, we can have m=0 here because the solutions are in terms of cosine functions. In this case we have,

$$\begin{aligned} H_y &= H_y^0 \\ E_x &= H_y^0 \frac{\beta}{\omega\epsilon} \\ E_z &= H_y^0 \frac{im\pi}{\omega\epsilon d} \end{aligned}$$

which gives the ratio $\frac{E_x}{H_y} = \frac{\beta}{\omega\epsilon} = \sqrt{\frac{\mu}{\epsilon}}$, which is the intrinsic impedance we have seen to characterize propagation of wave in a uniform medium.

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**Tutorial Assignment**

1. For a guided wave between two infinite conducting planes separated by a distance of 0.25 m, find the cutoff frequency for the $TM_{20}$ mode. If the operating frequency is 3 GHz, find the phase velocity of the wave.
2. A $TE_{10}$ mode propagates between two parallel planes separated by a distance of 0.25 m. The planes are lossy and have a conductivity of $5 \times 10^7$ S/m. If the maximum electric field strength between the planes is 1000 V/m, determine the power loss per square meter on each plate when the operating frequency is 2 GHz.

**Solutions to Tutorial Assignments**

1. The critical angular frequency $\omega_c$ is given by

$$\omega_c = \frac{1}{\sqrt{\mu_0\epsilon_0}}\frac{m\pi}{a} = \frac{cm\pi}{a}$$

so that the cutoff frequency is given by

$$f_c = \frac{cm}{2a} = 3 \times 10^8 \times \frac{2}{2 \times 0.25} = 1.2 \text{ GHz}$$

If the operating frequency is 3 GHz, the propagation constant is given by

$$\beta = \sqrt{\omega^2 \mu_0 \epsilon_0 - \left(\frac{m\pi}{a}\right)^2} = \sqrt{\frac{4\pi^2 f^2}{c^2} - \frac{4\pi^2}{a^2}} = 2\pi \sqrt{\frac{f^2}{c^2} - \frac{1}{a^2}} = 57.59$$

.

The phase velocity is given by

$$\frac{\omega}{\beta} = \frac{2\pi\nu}{\beta} = 3.27 \times 10^8 \text{ m/s}$$

2. The amplitude of linear current density on the plates is equal to the tangential component of the magnetic field on the planes,

$$\left|J_{sy}\right| = \left|H_z\right|_{x=0,x=a} = \frac{n\pi}{\omega\mu d} E_{y0}$$

Power loss per unit length of conductor

$$\frac{1}{2} J_{sy}^2 R_s = \frac{1}{2}\left(\frac{n\pi}{\omega\mu_0 d} E_{y0}\right)^2 \sqrt{\omega\mu_m/2\sigma_m}$$

We take $\mu_m = \mu_0$. The loss for $TE_{10}$ is (in J/m)

$$\frac{1}{2\sqrt{2\sigma}} \frac{\pi^2}{(2\pi\nu)^{1.5}\mu_0^{1.5} d} E_{y0}^2 = \frac{1}{80\pi}$$

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**Self Assessment Questions**

1. For a guided wave between two infinite conducting planes separated by a distance of 0.2m. If the operating frequency is 3.3 GHz, find the number of distinct modes that can travel in the guide.
2. A $TE_{20}$ mode is propagating along the z direction between two parallel conducting planes separated by 0.2m along the x- direction. Find the cutoff frequency. Determine $\lambda_x$. If the operating frequency is 2.5 GHz, determine $\lambda_z$ and the non-vanishing components of the electric and the magnetic field. If the operating frequency is 1.2 GHz, calculate the distance over which the strength of the fields reduce to 1/e of their value.
3. Calculate the rate at which energy is transmitted in a parallel plane waveguide operating in $TE_{10}$ mode.

**Solutions to Self Assessment Questions**

1. The propagation constant is given by

$$\beta = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n\pi}{d}\right)^2} = \sqrt{\left(\frac{2\pi \times 3.3 \times 10^9}{3 \times 10^8}\right)^2 - (5n\pi)^2}$$
$$= \pi\sqrt{(22)^2 - (5n)^2}$$

For propagation to take place $22 > 5n$, so that n<5. This implies 4 TE modes, 4 TM modes and one TEM mode, giving a total of 9 modes.

2. The cutoff frequency is given by $\nu_c = c\,\frac{n}{2a} = 3 \times 10^8 \times \frac{1}{0.2} = 1.5$ GHz. The wavelength in the x direction for n=2 is $\lambda_x = \frac{2d}{n} = d = 0.2$ m. If the operating frequency is $2.5 \times 10^9$Hz., the propagation vector is given by (using n=2)

$$\beta = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{n\pi}{d}\right)^2} = 2\pi\sqrt{\left(\frac{2.5\times 10^9}{3\times 10^8}\right)^2 - 25}$$

$$= \frac{2\pi}{3}\sqrt{625-225} = \frac{40\pi}{3}$$

The fields are as follows :

$$E_y = E_{y0}\sin(10\pi x)e^{-\frac{40\pi i z}{3}}e^{i\omega t}$$

$$H_x = -E_{y0}\frac{\beta}{\omega\mu}\sin(10\pi x)e^{-\frac{40\pi i z}{3}}e^{i\omega t}$$

$$H_z = -E_{y0}\frac{2\pi}{i\omega\mu d}\cos(10\pi x)e^{-\frac{40\pi i z}{3}}e^{i\omega t}$$

To determine $\lambda_z$, we use the fact that it is equal to the distance over which the phase of the propagating wave changes by $2\pi$. Thus

$$\frac{40\pi\lambda_z}{3} = 2\pi$$

,

so that $\lambda_z = 3/20$m.

If the operating frequency is 1.2 GHz, the wave attenuates, and we have,

$$\alpha = \sqrt{\left(\frac{n\pi}{d}\right)^2 - \frac{(2\pi\nu)^2}{c^2}} = 6\pi$$

so that the attenuation distance is $\frac{1}{6\pi}$ m.

3. The fields are given by

$$E_y = E_{y0}\sin\left(\frac{\pi}{d}x\right)e^{-i\beta z}$$

$$H_x = -E_{y0}\frac{\beta}{\omega\mu}\sin\left(\frac{\pi}{d}x\right)e^{-i\beta z}$$

$$H_z = -E_{y0}\frac{\pi}{i\omega\mu d}\cos\left(\frac{\pi}{d}x\right)e^{-i\beta z}$$

Power transmitted per unit area is

$$\frac{1}{2}Re\left(\vec{E}\times\vec{H}^*\right) = -\frac{1}{2}\left(E_y H_x^*\right)$$

$$= \frac{1}{2}\frac{\beta}{\omega\mu}\left|E_{y0}\right|^2\sin^2\left(\frac{\pi x}{d}\right)$$

Power transmitted in z direction through an area of unit width,

$$P = \frac{1}{2}\frac{\beta}{\omega\mu}\left|E_{y0}\right|^2\int_0^d \sin^2\left(\frac{\pi x}{d}\right)dx$$

$$= \frac{1}{4}\frac{\beta d}{\omega\mu}\left|E_{y0}\right|^2$$