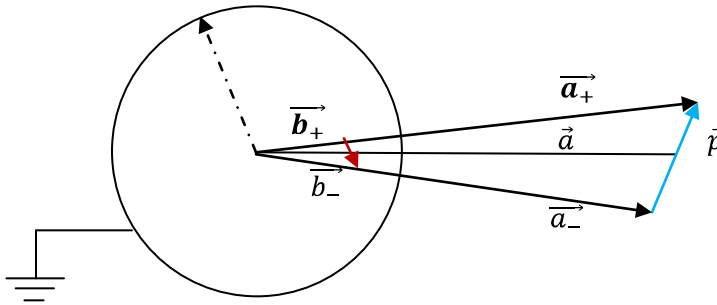# SPECIAL TECHNIQUES-II

Lecture 19: Electromagnetic Theory

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Method of Images for a spherical conductor

Example :A dipole near aconducting sphere

The dipole is placed at the position $\vec{a}$. It has a dipole moment $\vec{p} = q(\overrightarrow{a_+} - \overrightarrow{a_-})$. The corresponding image charges are located $\overrightarrow{b_+}$and$\overrightarrow{b_-}$ (the direction of the dipole moment being from the negative charge to the positive charge, the image dipole moment vector is directed as shown in the figure sine the image charge has opposite sign with respect to real charge.) The image charges are as follows :

Corresponding to the charge $+q$ at $\overrightarrow{a_+}$ the image charge is

$$-q' = -\frac{R}{a_+}q \ \ at \ \ \overrightarrow{b_+} = \frac{R^2}{a_+}\hat{a}_+$$

And, corresponding to the charge $-q$ at $\overrightarrow{a_-}$ the image charge is

$$+q" = \frac{R}{a_-}q \ \ at \ \ \overrightarrow{b_-} = \frac{R^2}{a_-}\hat{a}_-$$

The image approximates a dipole with dipole moment

$$\vec{p'} = \left(q\frac{R^3}{a_-^3}\vec{a}_- - q\frac{R^3}{a_+^3}\vec{a}_+ \right)$$

where,

$$\vec{a}_\pm = \vec{a} \pm \frac{(\vec{a}_+ - \vec{a}_-)}{2}$$

We have,

$$\frac{1}{a_\pm^3} = \frac{1}{\left[\left| \vec{a} \pm \frac{(\vec{a}_+ - \vec{a}_-)}{2} \right|^2 \right]^{\frac{3}{2}}}$$

$$= \frac{1}{\left[a^2 + \frac{|\vec{a}_+ - \vec{a}_-|^2}{4} \pm \vec{a} \cdot (\vec{a}_+ - \vec{a}_-) \right]^{\frac{3}{2}}}$$

$$= \frac{1}{a^3}\left(1 \mp \frac{3}{2}\frac{\vec{a} \cdot (\vec{a}_+ - \vec{a}_-)}{a^2} \right)$$

where $|\vec{a}_+ - \vec{a}_-| \ll a$.

The dipole moment of the image is thus calculated as follows :

$$\vec{p'} = \left(q\frac{R^3}{a_-^3}\vec{a}_- - q\frac{R^3}{a_+^3}\vec{a}_+ \right)$$

$$= \frac{R^3}{a^3} q \left[\left(1 + \frac{3}{2}\frac{\vec{a} \cdot (\vec{a}_+ - \vec{a}_-)}{a^2} \right)\vec{a}_- - \left(1 - \frac{3}{2}\frac{\vec{a} \cdot (\vec{a}_+ - \vec{a}_-)}{a^2} \right)\vec{a}_+ \right]$$

We can rearrange these to write,

$$\vec{p'} = \frac{R^3}{a^3}\left[q(\vec{a}_+ - \vec{a}_-) + \frac{3}{2}\frac{\vec{a} \cdot q(\vec{a}_+ - \vec{a}_-)}{a^2}(\vec{a}_+ - \vec{a}_-) \right]$$

$$= \frac{R^3}{a^3}\left(-\vec{p} + \frac{3}{2}\frac{\vec{a} \cdot \vec{p}}{a^2}(\vec{a}_+ - \vec{a}_-) \right)$$

The image charges, being of unequal magnitude, there is some excess charge within the sphere. The excess charge is given by

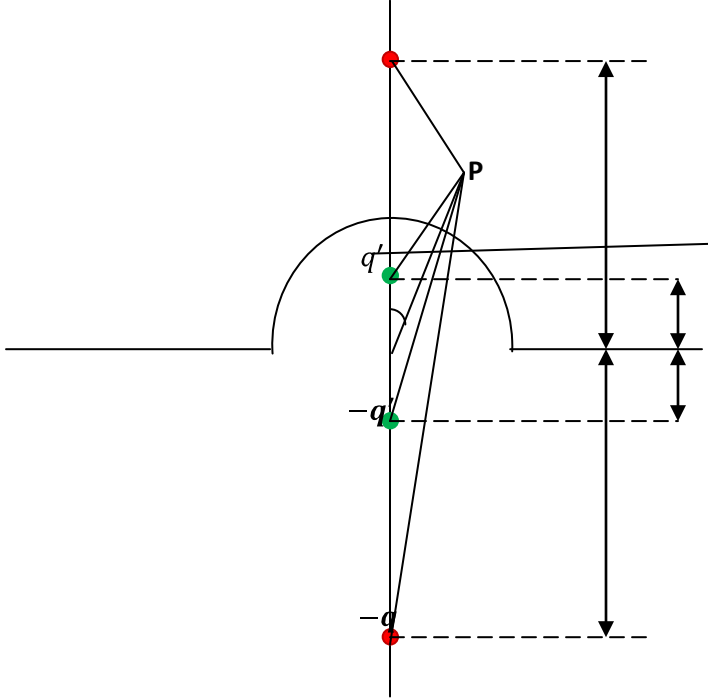$$q_{excess} = Rq\left(\frac{1}{a_-} - \frac{1}{a_+} \right) \approx Rq\frac{a_+ - a_-}{a^2}$$

$$a_\pm = \sqrt{a^2 + \frac{d^2}{4} \mp ad\cos\theta} \approx a\left(1 \mp \frac{d}{2a}\cos\theta \right)$$

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$$a_+ - a_- \approx -d \cos \theta$$

so that,

$$q_{excess} = -\frac{Rqd \cos \theta}{a^2} = +R\frac{\vec{p} \cdot \vec{a}}{a^3}$$

Example : A charge q in front of an infinite grounded conducting plane with a hemispherical boss of radius R directly in front of it.

The image charges along with their magnitudes and positions are shown in the figure. Here, $b = \frac{a^2}{R}, q' = q\frac{R}{a}$.The pairs $\pm q$ and $\pm q'$ make the potential on the plane zero, while the pairs q, q' and $-q, -q'$ make the potential on the hemisphere vanish. By uniqueness theorem, these four are the appropriate charges to satisfy the required boundary conditions.

If we take an arbitrary point $P(r, \theta)$, the potential at that point due to the four charge system is given by

$$\varphi(r, \theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} - \frac{q}{\sqrt{r^2 + a^2 + 2ra \cos \theta}} - \frac{\frac{qR}{a}}{\sqrt{r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a} \cos \theta}} \right.$$
$$\left. + \frac{\frac{qR}{a}}{\sqrt{r^2 + \frac{R^4}{a^2} + \frac{2rR^2}{a} \cos \theta}} \right]$$

The electric field is obtained by taking the gradient of the above. As we are only interested in computing the surface charge density, we will only compute the radial component of above and evaluate it at r=R,

$$
\begin{aligned}
E_r &= \frac{1}{4\pi\epsilon_0}\left[\frac{q(r-a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{\frac{3}{2}}} - \frac{q(r+a\cos\theta)}{(r^2+a^2+2ra\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qR}{a}\left(r-\frac{R^2}{a}\cos\theta\right)}{\left(r^2+\frac{R^4}{a^2}-\frac{2rR^2}{a}\cos\theta\right)^{\frac{3}{2}}}\right.\\
&\qquad\left. + \frac{\frac{qR}{a}\left(r+\frac{R^2}{a}\cos\theta\right)}{\left(r^2+\frac{R^4}{a^2}+\frac{2rR^2}{a}\cos\theta\right)^{\frac{3}{2}}}\right]\\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q(r-a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{\frac{3}{2}}} - \frac{q(r+a\cos\theta)}{(r^2+a^2+2ra\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qR}{a}\frac{R^2}{a^2}\left(r\frac{a^2}{R^2}-a\cos\theta\right)}{\frac{R^3}{a^3}\left(r^2\frac{a^2}{R^2}+R^2-2ra\cos\theta\right)^{\frac{3}{2}}}\right.\\
&\qquad\left. + \frac{\frac{qR}{a}\frac{R^2}{a^2}\left(r\frac{a^2}{R^2}+a\cos\theta\right)}{\frac{R^3}{a^3}\left(r^2\frac{a^2}{R^2}+R^2-2ra\cos\theta\right)^{\frac{3}{2}}}\right]
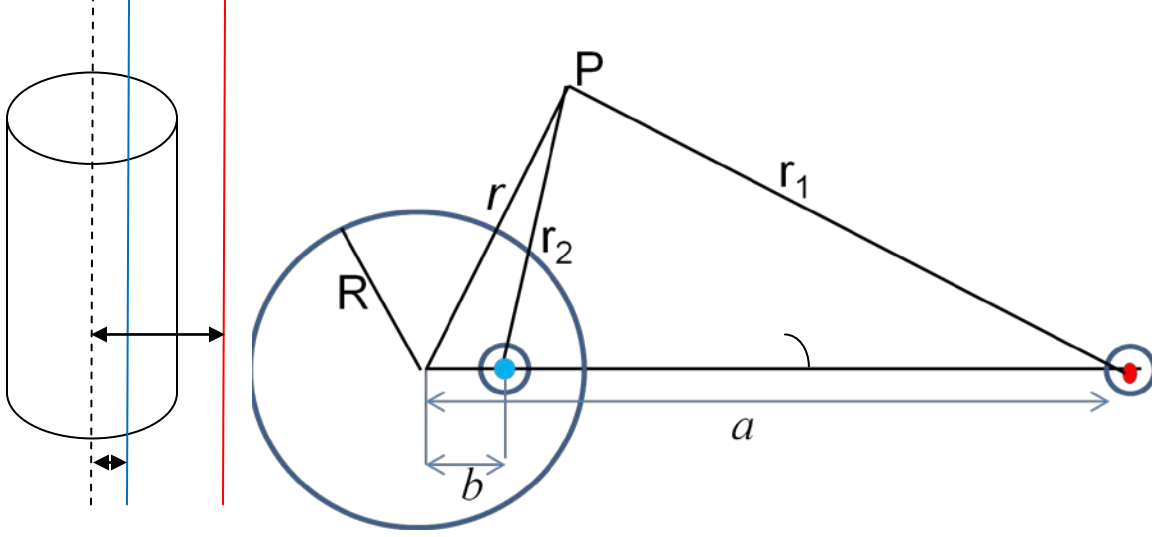\end{aligned}
$$

Substituting r=R and combining the first term with the third and the second with the fourth, we get,

$$
\begin{aligned}
\sigma &= \epsilon_0 E_r|_{r=R}\\
&= \frac{qR}{4\pi}\left(1-\frac{a^2}{R^2}\right)\left[\frac{1}{(R^2+a^2-2Ra\cos\theta)^{\frac{3}{2}}} - \frac{1}{(R^2+a^2+2Ra\cos\theta)^{\frac{3}{2}}}\right]
\end{aligned}
$$

The charge density can be integrated over the hemisphere to give the total charge on the hemispherical boss,

$$
\begin{aligned}
Q_{boss} &= \frac{qR}{4\pi}\left(1-\frac{a^2}{R^2}\right)\left[\int_0^{\frac{\pi}{2}}\frac{2\pi R^2\sin\theta\, d\theta}{(R^2+a^2-2Ra\cos\theta)^{\frac{3}{2}}} - \int_0^{\frac{\pi}{2}}\frac{2\pi R^2\sin\theta\, d\theta}{(R^2+a^2+2Ra\cos\theta)^{\frac{3}{2}}}\right]\\
&= -\frac{qR}{2}\left(1-\frac{a^2}{R^2}\right)\left[\int_0^1\frac{R^2\, d\mu}{(R^2+a^2-2Ra\mu)^{\frac{3}{2}}} - \int_0^1\frac{R^2\, d\mu}{(R^2+a^2+2Ra\mu)^{\frac{3}{2}}}\right]\\
&= -q + \frac{q(a^2-R^2)}{a\sqrt{a^2+R^2}}
\end{aligned}
$$

Example : Line charge near a conducting cylinder

The potential at an arbitrary point $P(r, \theta)$ is given by

$$\varphi(r, \theta) = \frac{\lambda}{2\pi\epsilon_0} \ln r_1 - \frac{\lambda'}{2\pi\epsilon_0} \ln r_2$$

$$= -\frac{1}{4\pi\epsilon_0} (\lambda \ln(a^2 + r^2 - 2ar\cos\theta) + \lambda' \ln(b^2 + r^2 - 2br\cos\theta))$$

Equating the tangential component of electric field on the surface to zero, we have,

$$E_t = -\frac{\partial\varphi}{\partial\theta}\bigg|_{r=R} = \frac{1}{4\pi\epsilon_0} \left[\frac{\lambda(2aR\sin\theta)}{a^2 + R^2 - 2aR\cos\theta} + \frac{\lambda'(2bR\sin\theta)}{b^2 + R^2 - 2bR\cos\theta} \right] = 0$$

Taking one of the terms to the other side and cross multiplying,

$$\lambda(2aR\sin\theta) \ (b^2 + R^2 - 2bR\cos\theta) = -\lambda'(2bR\sin\theta)(a^2 + R^2 - 2ar\cos\theta)$$

As this is valid for all values of θ, we have, on equating coefficient of $\sin\theta$ and that of $\sin 2\theta$ to zero,

$$2aR\lambda(b^2 + R^2) = -2bR\lambda'(a^2 + R^2)$$
$$-2\lambda abR = 2\lambda'abR$$

The second relation gives, $\lambda' = -\lambda$ and substituting this in the first relation, we get,

$$a(b^2 + R^2) = b(a^2 + R^2)$$
$$b = \frac{R^2}{a}$$

CONFORMAL MAPPING

A technique which is very useful, particularly in handling two dimensional problems is the method of Conformal mapping which uses complex variable.

We are familiar with the scalar potential corresponding to the electrostatic field. If the electric field is in two dimensions, say, in the x,y plane, we can think of something like a "vector potential" $\vec{A}(0,0,A)$ which has only one component in the third dimension, i.e. along the z direction. We had defined, $\vec{E} = -\nabla\varphi$. If we define vector potential by the relation $\vec{E} = \nabla \times \vec{A}$, then we have, the components of the electric field are given by,

$$E_x = -\frac{\partial \varphi}{\partial x}, \qquad E_y = -\frac{\partial \varphi}{\partial y}$$

In terms of the "vector potential" which has only one (z) component,

$$E_x = \left(\nabla \times \vec{A}\right)_x = \frac{\partial A}{\partial y}$$
$$E_y = \left(\nabla \times \vec{A}\right)_y = -\frac{\partial A}{\partial x}$$

Thus, we have,

$$\frac{\partial \varphi}{\partial x} = -\frac{\partial A}{\partial y}$$
$$\frac{\partial \varphi}{\partial y} = \frac{\partial A}{\partial x}$$

Those of you who are familiar with the theory of complex variables may recognize these equations, albeit with different notations.

Consider functions of a complex variable $z = x + iy$. The function $z \to f(z) = w$ is an image of z plane onto the complex w plane. We write the real and the imaginary parts of w as $u + iv$. Consider a simple map given by $f(z) = z^2 = x^2 - y^2 + 2ixy$. This implies $u = x^2 - y^2$, $v = 2xy$.

To see how points in the z plane transform to those in the w plane, consider for instance, three concentric quartercircles in the first quadrant of the z plane, i.e. the x y plane. We have the radii of the circles $a = 1,2,3$. We know that if we use polar coordinates, the coordinates are given by $x = R\cos\theta, y = R\sin\theta$ with $0 \le \theta \le \pi/2$.

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The three quarter circles are mapped on to three concentric semi-circles of radii 1, 4 and 9 by the mapping $f(z) = z^2$. This is because,

$$u = x^2 - y^2 = R^2(\cos^2\theta - \sin^2\theta) = R^2\cos 2\theta$$
$$v = 2xy = 2R^2\cos\theta\sin\theta = R^2\sin 2\theta$$

so that the radius of the circles $u^2 + v^2 = R^4$ are R^2 and the range of polar angle is doubled. In the figure we have drawn two straight lines which gets mapped onto straight lines because if the equation to the straight line in the z plane is $y = mx$, the image in the u-v plane is given by

$$u = x^2 - y^2 = (1 - m^2)x^2$$
$$v = 2xy = 2mx^2$$

so that $v = \frac{2m}{1-m^2}u$, i.e. straight lines get mapped onto straight lines. The shaded region in the figure to the left maps to the shaded region in the figure to the right.
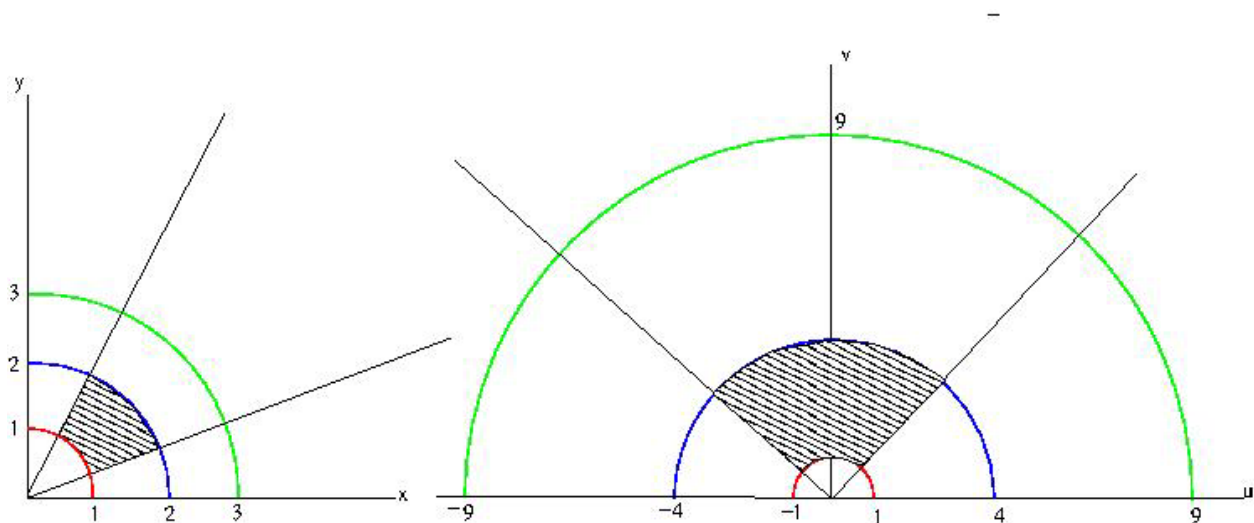
We define loosely a "well behaved function" as one where the mapping takes neighbouring points in the z plane to neighbouring points in the w-plane. In complex variable theory, we define an "analytic function" as a function which is differentiable in the complex plane. There are some subtle points of difference in the differentiability of a function of a real variable and that of a complex variable.

For a function of a real variable x, the derivative of a function at a point x is given by

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A natural extension of this definition to the case of complex variable would be to define the derivative at a point z as,

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

The important difference between the former case and the present case is the way Δz approaches zero. In case of the real variable x, there was just two ways of approaching the point x, from its left or from its right. The only requirement of differentiability was that the left and the right limits must exist and be the same. (This is what made $|x|$ not differentiable at $x = 0$). In case of a complex variable, however, there are infinite ways in which the function can approach the point z. Some of these ways are shown in the figure by arrows.
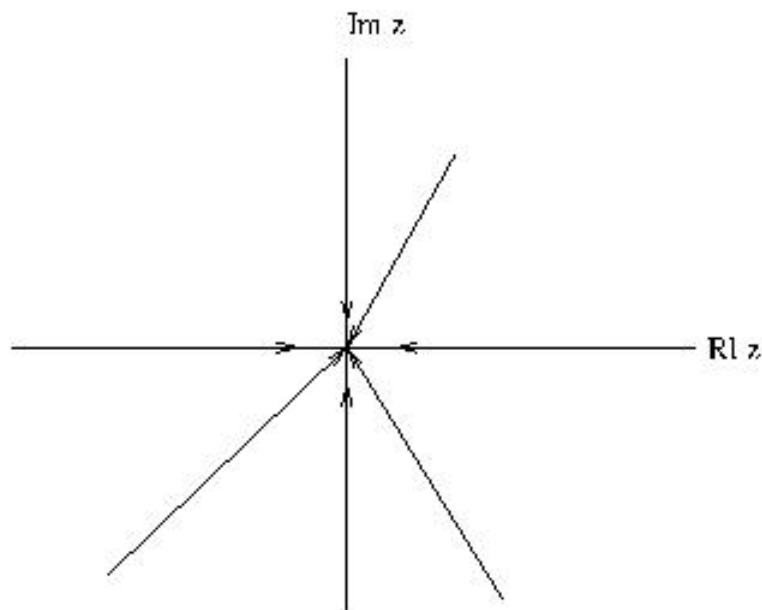
The derivative is defined if the ratio above has a unique value irrespective of the path along which the limit is approached.

If, for instance, the function approaches the limit along the real axis, we would have,

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{[u(x + \Delta x, y) - u(x, y)] + i[v(x + \Delta x, y) - v(x, y)]}{\Delta x}$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

On the other hand, if the limit is approached along the positive imaginary axis, we would have,

$$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{[u(x, y + \Delta y) - u(x, y)] + i[v(x, y + \Delta y) - v(x, y)]}{i\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Since these two ways of approaching the limit must give the same answer, we have, on equating real and imaginary parts of these two expressions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These two relations are known as the **Cauchy-Riemann conditions** and are taken as the test of analyticity of a function at a point in the complex plane.

Since we are in two dimensions, by taking partial derivative of the former equation with respect to x and of the latter with respect to y and adding them, we get

$$\nabla^2 u = 0$$

and, likewise,

$$\nabla^2 v = 0$$

Thus the real and imaginary part of w satisfies Laplace's equation in two dimension. This is the connection of what we are discussing with the electrostatic potential problem.

We had seen that the pair of potentials (φ, A) satisfies,

$$\frac{\partial \varphi}{\partial x} = -\frac{\partial A}{\partial y}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial A}{\partial x}$$

Thus we can identify $(\varphi, -A)$ with the pair (u, v) of a complex variable and define a "complex potential"

$$w = \varphi - iA$$

We then have,

$$\frac{\partial w}{\partial x} = \frac{\partial \varphi}{\partial x} - i\frac{\partial A}{\partial x} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y} = -E_x + iE_y$$

Let us look at the lines of force. Remember that the direction of the lines of force is given by

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-\partial A/\partial x}{\partial A/\partial y}$$

Cross multiplying we have,

$$\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = dA = 0$$

Thus the imaginary part of the complex potential is constant along the lines of force. The constancy of the real part φ corresponds to equipotential curves in two dimensions.

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = -E_x dx - E_y dy = -\vec{E} \cdot \vec{dl}$$

If the mapping is such that it preserves the angles between curves in both magnitude and sense, it is known as conformal mapping. (In the following figure, two curves on the left are mapped by some conformal transformation to the curves on the right).
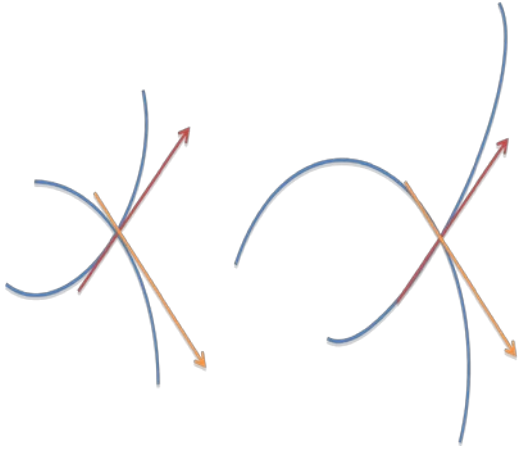

It is known from the theory of complex variables that mapping by analytic functions are conformal, except at the "critical points" where the derivative of the function vanishes. Thus the mapping $f(z) = z^2$ is conformal, excepting at the origin.

Example : Conformal mapping $f(z) = z^2$

As was explained earlier, the circle of radius R are mapped onto circles of radius R² in this map. Let us look at how the lines y= constant or x= constant map.

Let $y = c$.We then have $u = x^2 - y^2 = x^2 - c^2$ and $v = 2xy = 2xc$. We thus have, $u = \frac{v^2}{4c^2} - c^2$, i.e. $v^2 = 4c^2(c^2 + u)$, which are parabola indicated by blue curves in the figure. These represent lines of force in the case of complex potential formulation.

Similarly when x= constant=c, we have, $v^2 = 4c^2(c^2 - u)$ which are oppositely directed parabola shown by the red curves. These would represent equipotentials.
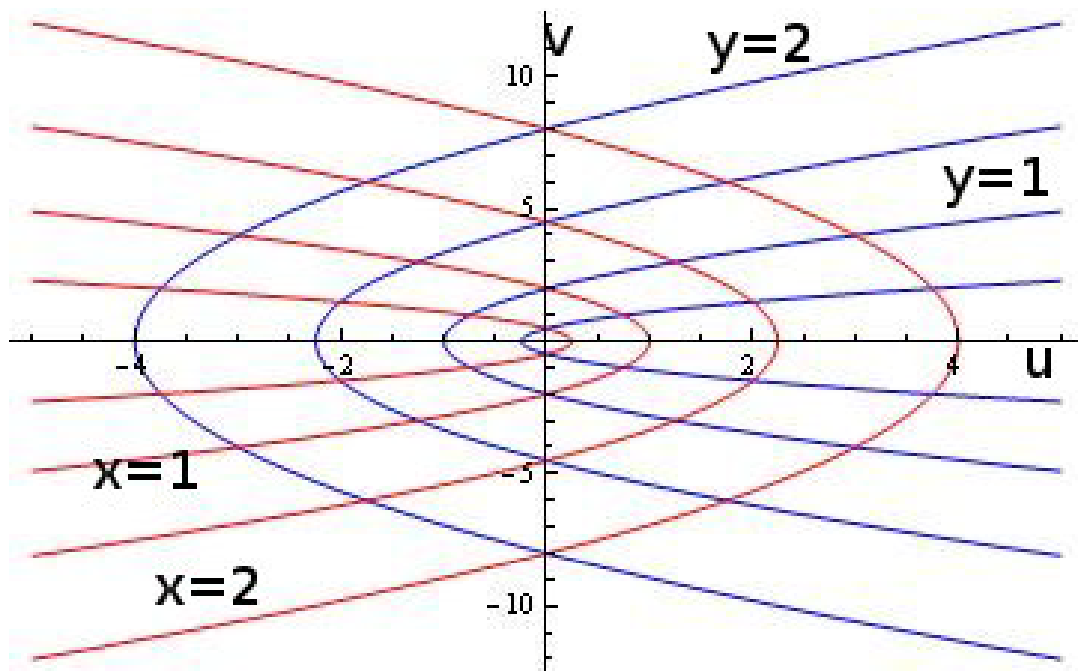
In applying the technique of conformal mapping to solving electrostatics, we map the given problem to a coordinate system where it is amenable to an easier solution. After obtaining the solution we map it back to the original coordinate system.

Example : Find the electric field in the region between two infinite conducting cylinders of inner radius a and outer radius b. The potential on the inner cylinder is kept fixed at φ_1 and the outer cylinder at φ_2.

As the cylinders are infinite, the potential or the electric field have no z dependence and the variation is only in the two dimensional (r, θ) plane, where the cross sections are circles. We can do a conformal mapping . Since polar equation to a circle is $x = r \cos\theta, y = r \sin\theta$, in the complex z plane, the equation is $z = r\, e^{i\theta}$. We will map this onto the w-plane by a conformal transformation. An obvious candidate for this mapping is the log function $w = \ln z$ which would take the circles $r = a$ and $r = b$ to straight lines on which u is constant, as

$$w = \ln z = \ln r e^{i\theta} = \ln r + i\theta = u + iv$$

so that the circles get mapped to lines $u = \ln r$. Thus the two circles become mapped to two straight lines $u = \ln a$ and $u = \ln b$. As we need the expression for the potential between the two circles, we need to solve for u in the region between the two lines.
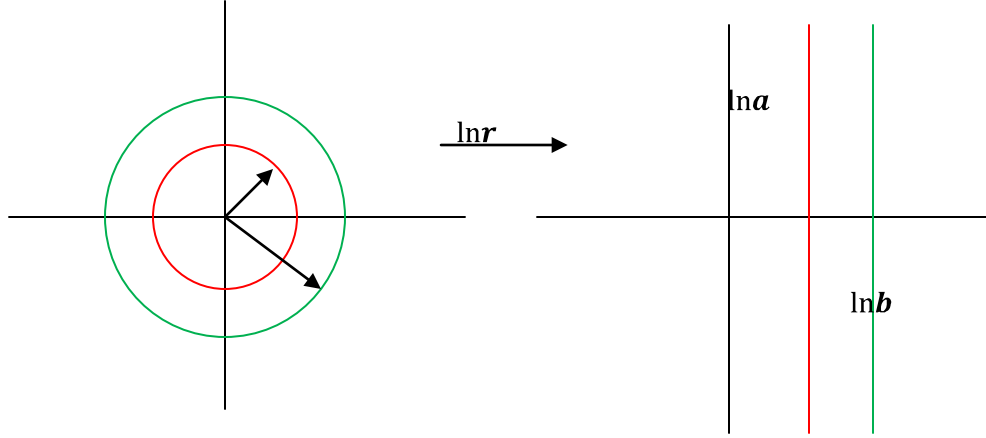
Since the potential in the original problem was a function of r alone, in the transformed space the potential between the lines would be function of u alone. Since the potential satisfies Laplace's equation, we have $\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial u^2} = 0$ so that the potential has the form $\varphi = C_1 u + C_2$. To fix the constants, use the boundary conditions $\varphi = \varphi_1$ when $u = \ln a$ and $\varphi = \varphi_2$ when $u = \ln b$. Substituting these the equation to the potential is found to be

$$\varphi = \frac{\varphi_1 - \varphi_2}{\ln\left(\frac{a}{b}\right)} u + \frac{\varphi_2 \ln a - \varphi_1 \ln b}{\ln\left(\frac{a}{b}\right)}$$

We now revert back to the original coordinate system by realizing that $u = \ln r$. Thus the potential is given by,

$$\varphi = \frac{\varphi_1 - \varphi_2}{\ln\left(\frac{a}{b}\right)} \ln r + \frac{\varphi_2 \ln a - \varphi_1 \ln b}{\ln\left(\frac{a}{b}\right)}$$

Tutorial Assignment

1. Consider the conformal mapping $w = f(z) = z + \frac{1}{z}$. At which point(s) the mapping is not conformal? How are circles in z plane mapped onto the w-plane?

2. Verify that for a complex function $w = f(z) = z^3 = u + iv$, u is a harmonic function.

3. A wedge shaped region is bounded by two semi-infinite conducting planes intersecting along the z axis, the angle of the wedge being $\frac{\pi}{3}$. The plane at $\theta = 0$ is maintained at zero potential while the one at $\theta = \frac{\pi}{3}$ is at a constant potential φ_0. Find an expression for the potential inside the wedge. [Hint : Consider conformal transformation in two steps, first $z :\to w = z^3$ and then $w :\to f = \frac{1}{\pi}\ln w$.
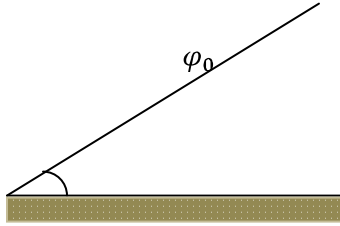
4. **(Hard Problem)** A long cylinder is split into two halves by a thin insulation. One half of the cylinder is grounded while the other half is maintained at a constant potential φ_0. Take the length of the cylinder along the z direction. Obtain an expression for the potential inside the cylinder.

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Solutions to Tutorial Assignment

1. $\frac{df}{dz} = 1 - \frac{1}{z^2}$ which vanishes at $z = \pm 1$. Hence the function is analytic everywhere except at these points and the mapping is not conformal at these points.

   Use the polar representation $z = r\, e^{i\theta}$.
   $$f(z) = re^{i\theta} + \frac{1}{r}e^{-i\theta} = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$
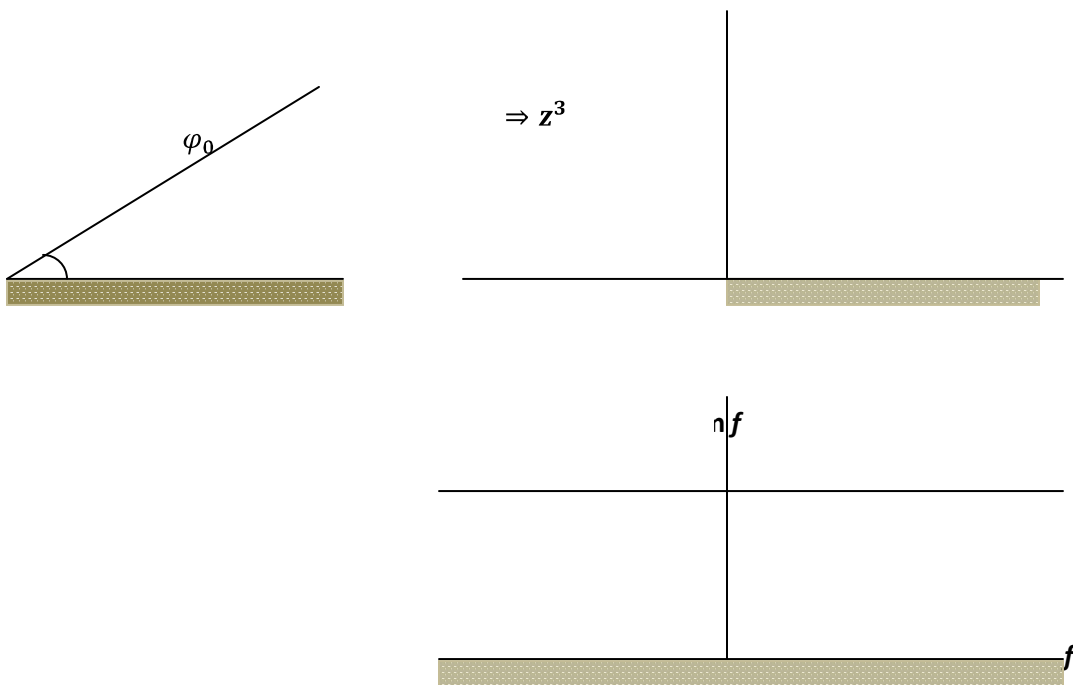 $$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

   so that $u = \left(r + \frac{1}{r}\right)\cos\theta;\ \ v = \left(r - \frac{1}{r}\right)\sin\theta$. The equation satisfied by u and v are

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2} + \frac{v}{\left(r-\frac{1}{r}\right)^2} = 1$$

Circles in z plane are of constant r. Thus these are mapped on to ellipses, except near the origin where the above equation is not satisfied.

2. $f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2 - y^3)$, so that $u = x^3 - 3xy^2$. It can be easily checked that $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$.

3.

φ_0

$\Rightarrow z^3$

$\Downarrow \frac{1}{\pi} \ln w$

$n|f$

f

The sequence of transformation is shown in the diagram. The location of points is to point out the sequence in which points are mapped and are not to scale. In the first transformation, writing $z = re^{\frac{i\pi}{3}}$ for the coordinates of the inclined plane z^3 takes it to $r^3 e^{i\pi} = -r^3$, i.e. it is mapped to the negative real axis. The horizontal section remains on the positive real axis, though the distances are scaled as cube of the original distance. (In general if the angle that the incline makes with the horizontal is π/n, the mapping is by z^n.

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In the second transformation, we use $f = \frac{1}{\pi}\ln w$ so that the original incline which now is along

$w = -r^3 = r^3 e^{i\pi}$ is mapped to $f = \frac{1}{\pi}\ln(r^3 e^{i\pi}) = \frac{1}{\pi}\ln r^3 + i$, so that the real axis is scaled but now Im $f = 1$. Thus the original inclined plane is now parallel to the original horizontal plane. Clearly, the variation in the potential can only be along the imaginary f direction. Denote the real and the imaginary parts of f as f_1 and f_2 respectively. Since the potential satisfied Laplace's equation, and, taking into account the fact that the only variation is with respect to f_2, we have,

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial f_2^2} = 0$$

Which has the solution $\varphi = af_2 + b$, where a and b are constants to be determined from boundary conditions. We have, $\varphi = 0$ when $f_2 = 0$ (lower plate), which gives $b = 0$. Since $\varphi = \varphi_0$ when $f_2 = 1$ (upper plate), we get,

$$\varphi(f_1, f_2) = \varphi_0 f_2$$

We have to now retrace our path and carry out a sequence of two inverse transformation to go from f plane to w plane and back to z plane.

$$\varphi(u, v) = \varphi_0 \mathrm{Im}\left(\frac{1}{\pi}\ln w\right)$$
$$\varphi(x, y) = \frac{\varphi_0}{\pi}\mathrm{Im}(\ln z^3) = \frac{\varphi_0}{\pi}\mathrm{Im}\,(3\ln z)$$

Now, $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$ so that its imaginary part is θ. Thus the potential is given by $\varphi = \frac{3\varphi_0\theta}{\pi} = 3\frac{\varphi_0}{\pi}\tan^{-1}\frac{y}{x}$.

4. let the length of the cylinder be perpendicular to the plane of the paper. A cross section in the (ρ, θ) plane is shown in the figure. The radius of the circle is taken to be unity. The upper half (I.e. $0 \le \theta \le \pi$) is at a potential φ_0 while the lower half ($\pi \le \theta \le 2\pi$) is grounded.
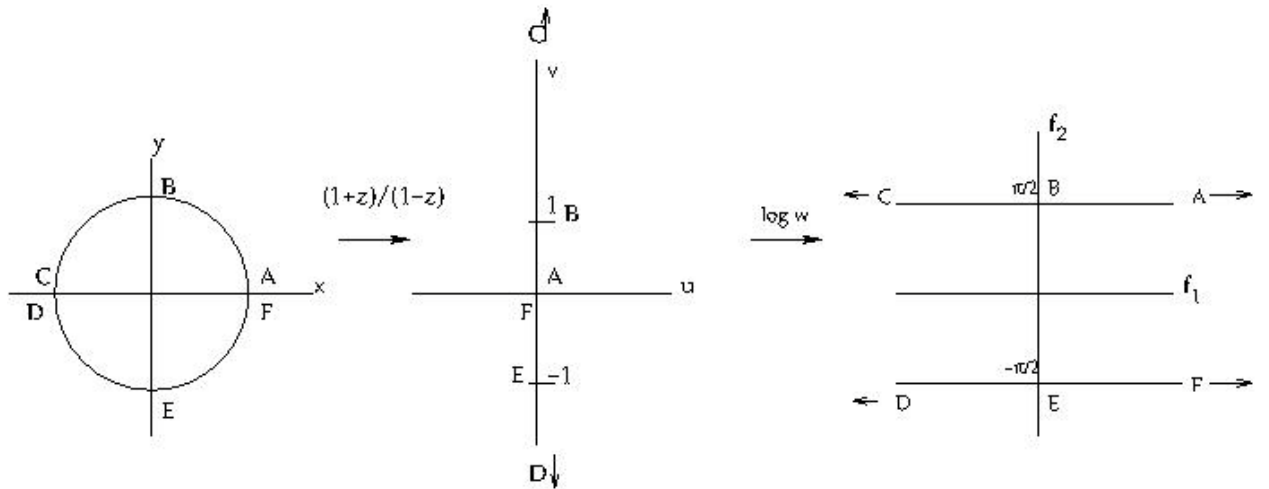
The successive transformation which convert the original problem to the problem of a parallel plate capacitor is shown in the figure. The first transformation takes all the points on the circle to the imaginary axis of the z plane,

$$w = \frac{1+z}{1-z} = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{2\sin\frac{\theta}{2}}{-2i\cos\frac{\theta}{2}} = i\cot\frac{\theta}{2} = iv$$

The following table shows the location of corresponding points before and after the transformation.

Point	A	B	C	D	E	F
θ	0+	$\pi/2$	π	π+	$3\pi/2$	2π
cot (θ/2)	∞	1	0+	0-	-1	$-\infty$

Thus the upper half of the cylinder is mapped to the positive imaginary axis and the lower half to the negative imaginary axis. We can write

$$w = \begin{cases} i\left|\cot\dfrac{\theta}{2}\right| & \text{for upper half} \\ -i\left|\cot\dfrac{\theta}{2}\right| & \text{for lower half} \end{cases}$$

Which can be written as $e^{\pm i\frac{\pi}{2}}\left|\cot\dfrac{\theta}{2}\right|$, the upper sign for the upper half and the lower sign for the lower half. The next mapping takes w to f where

$$f = \ln w = \pm\frac{i\pi}{2} + \ln\left|\cot\frac{\theta}{2}\right|$$

Thus the upper half becomes a line parallel to the real axis at $f_2 = \frac{\pi}{2}$ and the lower half a line at $f_2 = -\frac{\pi}{2}$.

Clearly the potential can only depend on f_2. As the potential satisfies Laplace's equation in only one variable f_2, the solution can be written as $\varphi = Af_2 + B$, where A and B are constants to be determined from the boundary conditions : $\varphi = 0$ for $f_2 = \frac{\pi}{2}$ and $\varphi = \varphi_0$ for $f_2 = -\frac{\pi}{2}$, which gives,

$$\varphi = \frac{\varphi_0}{\pi}f_2 + \frac{\varphi_0}{2}$$

The next task is to express the potential in terms of variables in the w and z plane. We have,

$$f_2 = \text{Im}(\ln w) = \text{Im}\ln\left(\frac{1+z}{1-z}\right)$$
$$= \text{Im}\ln\left[\frac{(1+x)+iy}{(1-x)-iy}\right]$$

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$$= \operatorname{Im} \ln \left[\frac{(1 - x^2 - y^2) + 2iy}{(1 - x)^2 + y^2} \right]$$

$$= \tan^{-1} \frac{2y}{1 - x^2 - y^2}$$

where the last step is obtained by writing the imaginary part of the log function by expressing the numerator in a polar representation. Thus the potential is given by

$$\varphi = \frac{\varphi_0}{\pi} \tan^{-1} \frac{2y}{1 - x^2 - y^2} + \frac{\varphi_0}{2}$$

SPECIAL TECHNIQUES-II

Lecture 19: Electromagnetic Theory

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Self assessment Quiz

1. Consider a conformal mapping given by $w = \sin z$. How are lines parallel to x and y axes map?

2. A semi-circular region is bounded by the x-axis $(-1 \leq x \leq +1)$ and a circular arc of radius 1 about the centre. The x – axis is grounded while the circular arc is maintained at a constant potential φ_0. Consider a conformal transformation $w = i \frac{1-z}{1+z}$ to convert the region into a planar region, solve the Laplace's equation and obtain an expression for the potential in the semi-circular region.

3. Two semi-infinite conductors make right angles with each other. The horizontally placed conductor is grounded while the vertical one is at a potential φ_0, there being a thin insulation. Find an expression for the potential in the region between the two planes.

Solutions to Self assessment Quiz

1. $u + iv = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$ so that $u = \sin x \cosh y, \; v = \cos x \sinh y$. For $x = a$, we have $\cosh y = \dfrac{u}{\sin a}$, $\sinh y = \dfrac{v}{\cos a}$ so that, $\dfrac{u^2}{\sin^2 a} - \dfrac{v^2}{\cos^2 a} = 1$, which are hyperbola with foci at $(\pm 1, 0)$. For *y=b,* we have $\sin x = \dfrac{u}{\cosh b}$, $\cos x \dfrac{v}{\sinh b}$ so that the equation of trajectory is $\dfrac{u^2}{\cosh^2 b} + \dfrac{v^2}{\sinh^2 b} = 1$, which represent ellipses with foci at $(\pm 1, 0)$.

2. Consider a mapping $w = i\dfrac{1-z}{1+z}$. For the semi-circle of unit radius, $z = e^{i\theta}$, so that,

$$w = i\frac{1-z}{1+z} = i\frac{1-e^{i\theta}}{1+e^{i\theta}} = i\frac{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}} = (i)\frac{-2i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} = u$$

with $0 \leq \theta \leq \pi$. The semi-circle is mapped on to the real axis since v=0. For $\theta = 0, u = \tan 0 = 0$ and for $\theta = \pi$, $u = \tan\frac{\pi}{2} = \infty$. Thus the semicircular part is mapped along the positive real axis from 0 to infinity.

For the line $-1 \leq x \leq +1$, $w = i\dfrac{1-x}{1+x}$, is purely imaginary and has a value 0 for x=1 and is $i\infty$ for $x = -1$. Thus the real line is mapped along the imaginary axis from the origin to infinity.
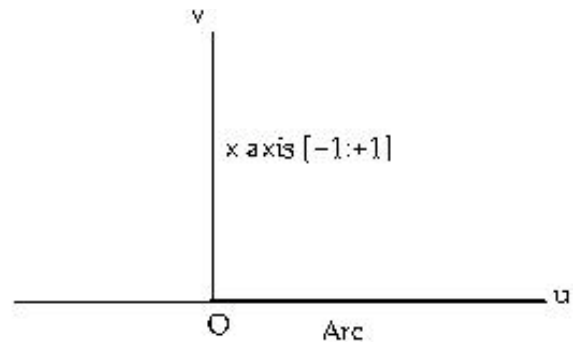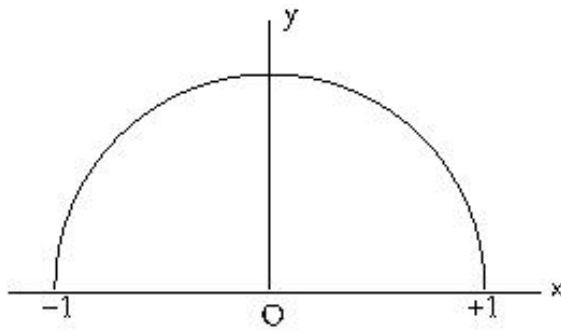
For the region bounded by the lines $u : [0 : \infty], v : [0, \infty]$ the transformation is given by

$$w = i\frac{1-z}{1+z} = i\frac{(1-x) - iy}{(1+x) + iy} = i\frac{(1-x^2-y^2) - 2iy}{(1+x)^2 + y^2}$$

which gives,

$$u = \frac{2y}{(1+x)^2 + y^2}, \; v = \frac{(1-x^2-y^2)}{(1+x)^2 + y^2}$$

Since φ satisfies two dimensional Laplace's equation, $\dfrac{\partial^2 \varphi}{\partial u^2} + \dfrac{\partial^2 \varphi}{\partial v^2} = 0$, subject to boundary condition $\varphi(u = 0) = \varphi_0, \varphi(v = 0) = 0$ the solution is given by $\varphi(u, v) = \dfrac{2\varphi_0}{\pi}\tan^{-1}\dfrac{u}{v}$. The potential as a function of (x, y) is obtained by substituting for u and v the expressions above.

3. The solution of this problem proceeds identically with that for the tutorial problem 3 with the change that the first conformal transformation is $w = z^2$. The expression for potential is $\varphi = \frac{2\varphi_0}{\pi} \tan^{-1} \frac{y}{x}$.