# MA2102: LINEAR ALGEBRA

## Lecture 25: Determinant

27th October 2020

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The determinant of $2 \times 2$ matrices may be thought of as a function

$$\det : M_2(F) \to F, \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc.$$

We shall often denote determinant of $A$ by $|A|$. Note that determinant is not linear as

$$0 = \det(I_2 - I_2) \neq \det(I_2) + \det(-I_2) = 2.$$

It also does not scale linearly, i.e., $\det(\lambda A) = \lambda^2 \det(A)$. We also know that $A \in M_2(F)$ is invertible if and only if $\det(A) = ad - bc \neq 0$, with the inverse given by

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Some key properties of determinant (for $2 \times 2$ matrices):

- $\det(I_2) = 1$

  *This may be referred to as normalization.*

- $\det(A) = \det(A^t)$

  *This is an algebraic property; a symmetry of the determinant.*

- $\det(AB) = \det(A)\det(B)$

  - *This is a computation, to be interpreted as a homomorphism later.*
  - *It implies that* $\det(C^{-1}) = \det(C)^{-1}$.
  - *It implies that* $\det(CAC^{-1}) = \det(A)$.

- $\det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = -\det\begin{pmatrix} R_2 \\ R_1 \end{pmatrix}$, $\det\begin{pmatrix} C_1 & C_2 \end{pmatrix} = -\det\begin{pmatrix} C_2 & C_1 \end{pmatrix}$

  - *Interchanging rows or columns changes the sign.*
  - *Determinant of a matrix with identical rows is zero.*

- $\det\begin{pmatrix} R_1+\lambda R \\ R_2 \end{pmatrix} = \det\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \lambda \det\begin{pmatrix} R \\ R_2 \end{pmatrix}$

  - $(a+\lambda a')d-(b+\lambda b')c=(ad-bc)+\lambda(a'd-b'c)$

  - *This property holds if we fix row 1 and add to row 2.*

  - *Determinant is a linear function of each row when the other row is fixed. Same considerations with columns.*

Recall that the adjoint of $A$, denoted by $\text{adj}(A)$, is defined by

$$\text{adj}(A)=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Note that the determinant of $\text{adj}(A)$ and $A$ are the same. This will, however, not be true for $n\times n$ matrices for $n\geq 3$. Moreover, we see that

$$\text{adj}(A)A=\det(A)I_2.$$

There is a characterization of determinants for $2 \times 2$ matrices.

**Proposition** Let $\delta : M_2(F) \to F$ be a function such that

(i) it is a linear function of each row when the other row is fixed;

(ii) if two rows of $A$ are identical, then $\delta(A) = 0$;

(iii) $\delta(I_2) = 1$.

Then $\delta(A) = \det(A)$ for any $A \in M_2(F)$.

Proof.

Note that

$$\delta\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = a\delta\begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} = ad\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = ad.$$

If $A$ has rank 0, then $A$ is the zero matrix and by (ii), $\delta(A) = 0 = \det(A)$. If $A$ has rank 1, then row rank is 1. Thus, one of the rows of $A$ is a multiple of the other row. By (i) and (ii), $\delta(A) = 0 = \det(A)$.

If $A$ has rank 2, then $A$ can be transformed to a diagonal matrix or anti-diagonal matrix via elementary row operations of the form

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \mapsto \begin{pmatrix} R_1 + \lambda R_2 \\ R_2 \end{pmatrix} \text{ and/or } \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} \mapsto \begin{pmatrix} R_1 \\ R_2 + \lambda R_1 \end{pmatrix}$$

It follows from (i) and (ii) that $\delta$ remains unchanged under this transformation. For instance, if $a \neq 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 = R_2 - \frac{c}{a} R_1} \begin{pmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{pmatrix} \xrightarrow{\delta} a(d - \frac{bc}{a})\delta \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

The last quantity is $ad - bc = \det(A)$. The case when $a = 0$ and necessarily $c \neq 0$ can be computed similarly (exercise). $\square$

**Definition** [Determinant] For $A \in M_n(F)$ the minor $\tilde{A}_{ij}$ of $A$ assocaited to the $i^{\text{th}}$ and $j^{\text{th}}$ column is the $(n-1)\times(n-1)$ matrix obtained by deleting the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $A$.

The determinant of $A$ is defined by picking a row $i$ and setting

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}\det(\tilde{A}_{ij}).$$

The definition of determinant is recursive, depending on determinants of smaller matrices. We are implicitly using the normalization that the determinant of a $1 \times 1$ matrix is the entry itself.

**Remark** The definition is referred to as the cofactor expansion along row $i$. Determinant can be defined using a cofactor expansion along any row or column. This is an amazing which we will prove in the special lecture.

We may also expand along column $j$, i.e.,

$$\det(A) = \sum_{i=1}^{n} a_{ij}(-1)^{i+j}\det(\tilde{A}_{ij}).$$

Show that this property is equivalent to $\det(A) = \det(A^t)$.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \text{ (1st row)} = -cb + da \text{ (2nd row)}.$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{cases} +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +g\begin{vmatrix} b & c \\ e & f \end{vmatrix} - h\begin{vmatrix} a & c \\ d & f \end{vmatrix} + i\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{cases}$$

All of these equal $aei - afh - bdi + bfg + cdh - cge$.

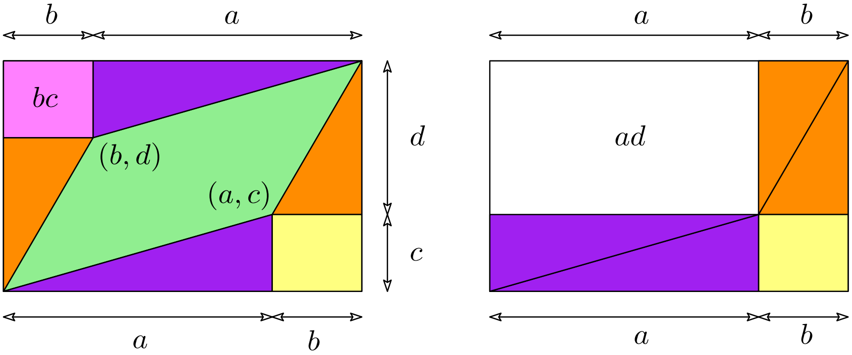


Figure: Signed area of the parallelogram is the determinant