MA2102: Linear Algebra

Lecture 21: Invertibility

7th October 2020

Indian Institute of Science Education & Research Kolkata

Definition [Invertible Maps] A linear map $T : V \to W$ is called invertible if there exists $S : W \to V$ such that $S \circ T = I_V$ and $T \circ S = I_W$.

The existence of S with the properties imply that T is a bijection. Thus, T is a linear isomorphism and S is unique. Conversely, if T is a linear isomorphism, then the set theoretic inverse $S : W \to V$ of T is a linear map satisfying the required properties. Thus, linear isomorphisms and invertible linear maps are synonymous.

Remark If V is finite dimensional and $T : V \to W$ is invertible, then T is a linear isomorphism, whence $\dim V = \dim W$.

Definition [Invertible Matrix] A matrix $A \in M_{m\times n}(\mathbb{R})$ is called invertible if there exists a matrix $B \in M_{n\times m}(\mathbb{R})$ such that

$$AB = I_m, \qquad BA = I_n.$$

Observation If $A \in M_{m\times n}(\mathbb{R})$ is invertible, then $m = n$.

Proof.

Let B be the (why?) inverse of A. Then A, B define linear maps

$$L_A : \mathbb{R}^n \to \mathbb{R}^m, \;\; L_B : \mathbb{R}^m \to \mathbb{R}^n$$

via the matrices acting on vectors. With respect to the standard bases β_k of $\mathbb{R}^k$ we have (exercise)

$$[L_A]_\beta^\gamma = A, \;\; [L_B]_\gamma^\beta = B.$$

Moreover, we have

$$L_A \circ L_B = I_{\mathbb{R}^m}, \;\; L_B \circ L_A = I_{\mathbb{R}^n}.$$

Thus, m must equal n. $\square$

The following properties hold for invertible linear maps. If $T : V \to W$ and $S : U \to V$ are invertible linear maps, then

- $(T \circ S)^{-1} = S^{-1} \circ T^{-1}$
  Follows by composing both sides with $T \circ S$.
- $(T^{-1})^{-1} = T$
  Follows by composing both sides with T^{-1}.

The following result connects the two notions of invertibility.

Theorem

Let V and W be finite dimensional vector spaces with ordered bases β and γ respectively. If $T : V \to W$ is a linear map, then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Moreover, we have

$$[T^{-1}]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}.$$

Proof.

If T is invertible, then $\dim V = \dim W = n$ by our earlier remark, whence $[T]_\beta^\gamma \in M_n(\mathbb{R})$. As $T \circ T^{-1} = I_W$ and $T^{-1} \circ T = I_V$, we get

$$\begin{array}{rcccccl} I_n & = & [I_V]_\beta & = & [T^{-1} \circ T]_\beta & = & [T^{-1}]_\gamma^\beta [T]_\beta^\gamma \\ I_n & = & [I_W]_\gamma & = & [T \circ T^{-1}]_\gamma & = & [T]_\beta^\gamma [T^{-1}]_\gamma^\beta \end{array}$$

For the converse, assume that $A = [T]_\beta^\gamma$ is invertible with B its inverse. By our earlier observation, $m = n$. It suffices to show that T is injective. If $v \in N(T)$, then $T(v) = \mathtt{0}_W$. As

$$A[v]^\beta = [T]_\beta^\gamma [v]^\beta = [T(v)]^\gamma = \mathtt{0}$$

multiplying by B on the left, we conclude that $[v]^\beta = \mathtt{0}$. This implies that $v = \mathtt{0}_V$ and that T is injective. $\square$

Example We specialize to the case of $T : V \to V$, where V is finite dimensional. Let β, γ be two (ordered) bases of V. We had seen that

$$[I_V]_\beta^\gamma [T]_\beta [I_V]_\gamma^\beta = [T]_\gamma.$$

Let $Q = [I_V]_\beta^\gamma$ denote the change of basis (from β to γ) matrix. Then we may rewrite the above identity as

$$Q[T]_\beta Q^{-1} = [T]_\gamma.$$

Remark Note that $[T]_\beta$ is invertible if and only if $[T]_\gamma$ is invertible as both are equivalent to T being invertible.

In order to define meaningful invariants of linear maps $T : V \to V$ we need to define scalar quantities associated to matrices which are unchanged under *conjugation*.

In subsequent lectures we will explore trace, rank and determinant as potential invariants. Let us focus on conjugation.

Definition [Similarity] A matrix $A \in M_n(\mathbb{R})$ is said to be similar to $B \in M_n(\mathbb{R})$ if there exists an invertible matrix $Q \in M_n(\mathbb{R})$ such that

$$QAQ^{-1} = B.$$

Observe that if we define $A \sim B$ by the relation of similarity, then $\sim$ is an equivalence relation.

- [reflexive] $I_n A I_n^{-1} = A$
- [symmetric] If $QAQ^{-1} = B$, then $Q^{-1}BQ = A$
- [transitive] If $QAQ^{-1} = B$ and $PBP^{-1} = C$ then

$$(PQ)A(PQ)^{-1} = PQAQ^{-1}P^{-1} = PBP^{-1} = C.$$

For $M_1(\mathbb{R}) = \mathbb{R}$, the relation of similarity is not interesting - every real number is its own equivalence class as multiplication is commutative. When $n = 2$, multiplication in $M_2(\mathbb{R})$ is not commutative. We will see later that

$$\operatorname{trace}(PAP^{-1}) = \operatorname{trace}(A) \quad \text{and} \quad \det(PAP^{-1}) = \det(A).$$

Question *Do the trace and determinant determine the similarity class of $A \in M_2(\mathbb{R})$?*

The matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

are traceless and have zero determinant. However, they are not similar (exercise).