MA2102: LINEAR ALGEBRA

Lecture 16: Matrix Reloaded

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Let us continue with examples of matrices associated to linear maps.

Examples (4) Consider the linear map

$$D : P_3(\mathbb{R}) \to P_2(\mathbb{R}), \;\; p \mapsto p'$$

with ordered bases $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. The matrix $[D]_\beta^\gamma$ is given by

$$[D]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

We may change the codomain to $P_3(\mathbb{R})$ and consider the same map, relabelled as

$$\mathscr{D} : P_3(\mathbb{R}) \to P_3(\mathbb{R}), \;\; p \mapsto p'.$$

The matrix with respect to β is given by

$$[\mathscr{D}]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The determinant is zero and $\mathscr{D}$ is not an isomorphism as it is not injective. The trace of the matrix is zero.

Consider the ordered basis

$$\gamma = \{1 + x + x^2 + x^3, x + x^2 + x^3, x^2 + x^3, x^3\}.$$

It can be seen that (exercise)

$$[\mathscr{D}]_\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 3 \\ -3 & -3 & -3 & -3 \end{pmatrix}.$$

The matrix has determinant zero and trace zero as well.

Remark It is not a coincidence that the determinant and trace of $[\mathscr{D}]_\beta$ and $[\mathscr{D}]_\gamma$ are equal. We shall see prove this later.

(5) Consider the integration map

$$\mathscr{S} : P_n(\mathbb{R}) \to P_{n+1}(\mathbb{R}), \ \ p(x) \mapsto \int_0^x p(t)\,dt.$$

With ordered bases $\beta = \{1, x, \ldots, x^n\}$ and $\gamma = \{1, x, \ldots, x^{n+1}\}$, the matrix is given by

$$[\mathscr{I}]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{n+1} \end{pmatrix}.$$

(6) Consider the transpose map

$$T : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \;\; A \mapsto A^t.$$

Consider two ordered basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

If any element of β or γ is symmetric, then T fixes it. Then

$$[T]_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad [T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The determinant of both matrices are -1 and the trace of both matrices are 2.

(7) Consider the counter-clockwise rotation by θ, i.e.,

$$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2, \quad T_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

With respect to the standard ordered basis $\beta = \{(1,0),(0,1)\}$, we have

$$[T_\theta]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

With respect to $\gamma = \{(1,1),(1,-1)\}$ we see that

$$T_\theta(1,1) = (\cos\theta - \sin\theta, \sin\theta + \cos\theta) = \cos\theta(1,1) - \sin\theta(1,-1)$$

$$T_\theta(1,-1) = (\cos\theta + \sin\theta, \sin\theta - \cos\theta) = \sin\theta(1,1) + \cos\theta(1,-1).$$

Thus, we conclude that

$$[T_\theta]_\gamma = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Show that $[T_\theta]_\beta$ and $[T_\theta]_\gamma$ satisfy $AA^t = I_2$. In fact, both these matrices have determinant $\cos^\theta + \sin^2\theta = 1$ and trace $2\cos\theta$. Moreover,

$$\begin{aligned} ||T_\theta(x,y)||^2 &= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \\ &= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta \\ &\quad + x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta \\ &= x^2 + y^2 \\ &= ||(x,y)||^2. \end{aligned}$$

Such a linear map is called a linear isometry.

Question *Is there an ordered basis β such that $[T_\theta]_\beta$ is diagonal?*

We note that T_0 and T_π are the identity and negative of the identity maps respectively. These are diagonal in the standard basis of $\mathbb{R}^2$. So, we ask this question for $\theta \neq 0, \pi$. If possible, let $\beta = \{v_1, v_2\}$ be such a basis for T_θ. If $[T_\theta]_\beta = D(\lambda_1, \lambda_2)$, then

$$T_\theta(v_1) = \lambda_1 v_1, \;\; T_\theta(v_2) = \lambda_2 v_2.$$

As T_θ is an isometry, it follows that $\lambda_1, \lambda_2 \in \{\pm 1\}$. Thus, either T_θ maps v_1 to itself or rotates it by π, which cannot happen as $\theta \neq 0, \pi$. Thus, $[T_\theta]_\beta$ is not diagonal for any β if $\theta \neq 0, \pi$.

Remark The word "*matrix*" is late Latin for "*womb*".