Harmonic oscillator :—

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Quantisation in energy basis.

$$\hat{H}|\psi\rangle = i\hbar \frac{d}{dt}|\psi\rangle$$

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$$

$$\langle H\rangle = \frac{1}{2m}\langle\psi|p^\dagger p|\psi\rangle + \frac{1}{2}m\omega^2\langle\psi|x^\dagger x|\psi\rangle \geq 0.$$

Set up of eigenvalue equation :—

$$\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\right)|E\rangle = E|E\rangle$$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2\right)\psi = E\psi$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2x^2\right)\psi = 0$$

$$x = \sqrt{\frac{\hbar}{m\omega}}\,y \,, \quad \varepsilon = \frac{E}{\hbar\omega}$$

$$\Rightarrow \frac{d^2\psi}{dy^2} + (2\varepsilon - y^2)\psi = 0$$

* for $y \to \infty$ limit

$$\psi'' - y^2\psi = 0 \quad \Rightarrow \psi = Ay^m e^{\pm y^2/2}$$

* for $y \to 0$ limit

$$\psi'' + 2\varepsilon\psi = 0$$

$$\Rightarrow \psi = A\cos(\sqrt{2\varepsilon}\,y) + B\sin(\sqrt{2\varepsilon}\,y)$$

$$\therefore \psi = u(y).e^{\pm y^2/2}$$

$\Rightarrow u'' - 2yu' + (2\varepsilon - 1)u = 0$

Apply power series solution,

$$u(y) = \sum c_n y^n$$

$$\Rightarrow \sum c_n\left[n(n-1)y^{n-2} - 2ny^n + (2\varepsilon - 1)y^n\right] = 0$$

$$\Rightarrow \sum y^n\left[c_{n+2}(n+2)(n+1) + c_n(2\varepsilon - 1 - 2n)\right] = 0$$

$$\Rightarrow c_{n+2} = \frac{c_n(2n + 1 - 2\varepsilon)}{(n+2)(n+1)}, \quad \frac{c_{n+2}}{c_n} \to \frac{2}{n}$$

$$u(y) \xrightarrow[y \to \infty]{} y^m e^{y^2/2}$$

if $\varepsilon_n = n + \frac{1}{2}$, c_{n+2} vanishes.

$\Rightarrow$ for odd 'n', even c_n vanish
for even 'n', odd c_n vanish

$$\Rightarrow E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

These polynomials are Hermite,

$H_0(y) = 1$, $H_1(y) = 2y$, $H_2(y) = -2(1 - 2y^2)$.

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar\, 2^{2n}(n!)^2}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\left[\sqrt{\frac{m\omega}{2\hbar}}\, x\right]$$

7.3.6

$$V(x) = \frac{1}{2} m\omega^2 x^2 \qquad x > 0$$

$$\infty \qquad x \leq 0$$

$\psi = A \sin\sqrt{2\varepsilon}\, y$ as $y \to 0$.

as $\psi(0) = 0$.

$\Rightarrow \psi = u(y) e^{-y^2/2}$

$u(y) = 0 \quad y = 0.$

$\Rightarrow C_0 = 0$

$\Rightarrow$ all C_n for even $n = 0$

only odd 'n' allowed.

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad n\text{-odd}.$$

$$\Rightarrow \frac{3}{2}\hbar\omega,\ \frac{7}{2},\ \frac{11}{2},\ \frac{15}{2} \ldots$$

7.3.7 :-

$$|0\rangle = \int dx\, |x\rangle\langle x|0\rangle$$

$$\langle p|0\rangle = \int dx\, \langle p|x\rangle\langle x|0\rangle$$

$$= \int dx \frac{1}{\sqrt{2\pi\hbar}} e^{-i\frac{px}{\hbar}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(\frac{-m\omega x^2}{2\hbar}\right)$$

$$= \frac{1}{\cancel{\sqrt{2\pi\hbar}}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot \left(\sqrt{\frac{\cancel{2\pi\hbar}}{m\omega}}\right) e^{-\frac{p^2}{2\hbar m\omega}}$$

$$= \left(\frac{1}{m\pi\hbar\omega}\right)^{1/4} e^{-\frac{p^2}{2\hbar m\omega}}$$

Harmonic oscillator using ladder operators :-

Define ladder operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right); \quad \hat{a}^{\dagger} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$$

$$[a, a^{\dagger}] = 1$$

The Hamiltonian can be rewritten as,

$$\hat{H} = \left(a^{\dagger}a + \frac{1}{2}\right)\hbar\omega.$$

$$\hat{N} = a^{\dagger}a, \quad \hat{N}|n\rangle = n|n\rangle$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle, \quad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

For constructing their matrix,

$$a^{\dagger} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad a = \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\Rightarrow |n\rangle = \frac{(a^{\dagger})^{n}}{\sqrt{n!}}|0\rangle$$

Coherent States :-

They happen to be the eigenstates of the annihilation operator ($\hat{a}$).

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha \in \mathbb{C}$$

as $\hat{a}$ is non-hermitian.

let $|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\Rightarrow \sqrt{n+1}\, c_{n+1} = \alpha c_n$$

① $c_1 = \alpha c_0$, $c_2 = \dfrac{\alpha^2}{\sqrt{2!}} c_0$

$$\ldots \quad c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$|\alpha\rangle = c_0 \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

which when normalized.

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}\hbar\omega\left(n+\frac{1}{2}\right)t} |n\rangle$$

$$|\alpha\rangle = \underbrace{D(\alpha)}_{\text{displacement}} |0\rangle \qquad D(\alpha) = e^{(\alpha a^\dagger - \alpha^* a)}$$

Squeezed states :—

$\overset{\omega_1}{\vdash\!\!\sim\!\sim\!\circ}\, m_1$

$$H_1 = \frac{p^2}{2m_1} + \frac{1}{2} m_1 \omega_1 x^2$$

$$\Delta x = \sqrt{\frac{\hbar}{2m_1\omega_1}}, \quad \Delta p = \sqrt{\frac{\hbar m_1 \omega_1}{2}}$$

for ground state;

Hamiltonian changes,

$$H_2 = \frac{p^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 x^2$$

$$\Delta x = \sqrt{\frac{m_2\omega_2}{m_1\omega_1}} \sqrt{\frac{\hbar}{2m_2\omega_2}}, \quad \Delta p = \sqrt{\frac{m_1\omega_1}{m_2\omega_2}} \sqrt{\frac{\hbar m_2 \omega_2}{2}}$$

$$e^{-\gamma} = \sqrt{\frac{m_2\omega_2}{m_1\omega_1}}$$

$$a_1 + a_1^\dagger = e^{\gamma}(a_2 + a_2^\dagger)$$
$$a_1 - a_1^\dagger = e^{-\gamma}(a_2 - a_2^\dagger)$$

$$\Rightarrow a_1 = a_2 \cosh(\gamma) + a_2^\dagger \sinh(\gamma)$$
$$a_1^\dagger = a_2^\dagger \cosh(\gamma) + a_2 \sinh(\gamma)$$

Bogoliubov transformation

$$a_1 |0\rangle_1 = 0$$
$$(a_2 \cosh\gamma + a_2^\dagger \sinh\gamma)|0_1\rangle = 0.$$

Expect a solution of $|0_1\rangle = c_0|0\rangle_2 + c_2 a_2^\dagger a_2^\dagger |0\rangle_2 + c_4 a_2^\dagger a_2^\dagger a_2^\dagger a_2^\dagger |0\rangle \ldots$

$$\rightarrow |0\rangle_1 = N(\gamma) e^{\left(-\frac{1}{2} f(\gamma) a_2^\dagger a_2^\dagger\right)} |0\rangle_2$$

Will have to find $N(\gamma)$, $f(\gamma)$ -

$$\left(\cosh\gamma\left[a_2^{\dagger}, -\frac{1}{2}f(\gamma)\, a_2^{\dagger}a_2^{\dagger}\right] + a_2^{\dagger}\sinh(\gamma)\right) e^{-\frac{1}{2}f(\gamma)\, a_2^{\dagger}a_2^{\dagger}} |0_2\rangle$$

$$[A, e^{B}] = [A,B]\, e^{B}$$

$$\Rightarrow f(\gamma) = \tanh(\gamma)$$

$$|0\rangle_1 = N(\gamma)\, e^{\left(-\frac{1}{2}\tanh(\gamma)\, a_2^{\dagger}a_2^{\dagger}\right)} |0\rangle_2$$

$$N(\gamma) = {}_2\langle 0|0\rangle_1 = \int_{-\infty}^{\infty} dx\, (\psi_0^2(x))^* \psi_0^1(x)$$

$$= \left[\frac{m_1\omega_1\, m_2\omega_2}{(\pi\hbar)^2}\right]^{1/4} \int_{-\infty}^{\infty} dx\, e^{\left(-\left(\frac{m_1\omega_1 + m_2\omega_2}{2\hbar}\right)x^2\right)}$$

$$= \frac{1}{\sqrt{\cosh(\gamma)}}$$

$$\Rightarrow |0\gamma\rangle = \frac{1}{\sqrt{\cosh\gamma}}\, e^{\left(-\frac{1}{2}\tanh\gamma\, a^{\dagger}a^{\dagger}\right)} |0\rangle$$

$$|0\rangle_\gamma = e^{-\frac{\gamma}{2}\left(a^{\dagger}a^{\dagger} - aa\right)} |0\rangle$$

WTF IS SQUEEZE ?

EXPLAIN "FUTURE ADWAIT"!

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Cavity Quantum Electrodynamics

PHOTON STATES :-

$$E = \frac{1}{2}\int d^3x\, \varepsilon_0 \left[\vec{E}^2(r,t) + c^2 \vec{B}^2(r,t) \right]$$

~~$E_x(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}} \sin(kz)$~~ $V =$ volume.

"ω" $k = \frac{\omega}{c}$

$$E_x(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}}\, \omega\, q(t) \sin(kz).$$

$$i B_y(z,t) = \sqrt{\frac{2}{\varepsilon_0 V}}\, p(t) \cos(kz)$$

$$E = \frac{1}{2}\left[p^2(t) + \omega^2 q^2(t) \right]$$

$$H = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2 \right)$$

Heisenberg's equations of motion,

$$i\hbar \frac{\partial}{\partial t}(\hat{A}) = [A, H]$$

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}\,(a + a^{\dagger})$$

$$p = -i\sqrt{\frac{\hbar\omega}{2}}\,(a - a^{\dagger})$$

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2} \right)$$

$$\Rightarrow \hat{q}(t) = \sqrt{\frac{\hbar}{2\omega}}\left(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^{\dagger} \right)$$

(from Heisenberg's)

$$\hat{E}_x(z,t) = \varepsilon_0\left(e^{-i\omega t}\hat{a} + e^{i\omega t} a^{\dagger} \right)\sin(kz)$$

--- Electric field operator

Energy eigenstate of $\hat{H}$ is the photon state.

$$\langle E_x \rangle_{|n\rangle} = \langle n | \hat{E}_x | n \rangle$$

$$= \mathcal{E}_0 \left(e^{-i\omega t} \langle n | \hat{a} | n \rangle + e^{i\omega t} \langle n | a^\dagger | n \rangle \right) \sin(kz)$$

$$= 0$$

$$\langle E_x \rangle_{|\alpha\rangle} = \langle \alpha | E_x | \alpha \rangle$$

$$= \mathcal{E}_0 \left(e^{-i\omega t} \langle \alpha | a | \alpha \rangle + e^{i\omega t} \langle \alpha | a^\dagger | \alpha \rangle \right)$$

$$= \mathcal{E}_0 \left(e^{-wit} \alpha + \alpha^* e^{i\omega t} \right) \sin(kz)$$

$$= 2 \mathcal{E}_0 \, \mathrm{Re}\left(\alpha e^{-i\omega t} \right) \sin(kz)$$

$$\alpha = |\alpha| e^{i\theta}$$

$$\Longrightarrow \quad 2\mathcal{E}_0 |\alpha| \cos(\omega t - \theta) \sin(kz).$$

$$\langle H \rangle = \hbar\omega \left(\langle N \rangle + \tfrac{1}{2} \right) = \hbar\omega \left(|\alpha|^2 + \tfrac{1}{2} \right)$$