MA2102: Linear Algebra

Lecture 23: Trace

14th October 2020

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The trace of square matrix is defined to be the sum of its diagonal entries, i.e.,

$$\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}$$

if $A \in M_n(F)$.

Definition [Trace] Let $T : V \to V$ be a linear map. Let V be of dimension n and let β be an ordered basis of V. The trace of T is defined to be the trace of $[T]_\beta$.

Question *Why is this notion of trace well-defined?*

To answer this, we note an useful property of trace (for matrices).

Lemma If $A, B, C \in M_n(F)$, then

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB).$$

Proof.

We first note that if $A, B \in M_n(F)$, then

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_{ij} = \operatorname{tr}(BA).$$

Now note that

$$\operatorname{tr}(ABC) = \operatorname{tr}((AB)C) = \operatorname{tr}(C(AB)) = \operatorname{tr}(CAB).$$

Iterating this once more, we obtain $\operatorname{tr}(CAB) = \operatorname{tr}(BCA)$. $\square$

We shall refer to this as the *cyclic* property of trace.

If $A_1, \ldots, A_k \in M_n(F)$, then show that

$$\operatorname{tr}(A_1 \cdots A_k) = \operatorname{tr}(A_j A_{j+1} \cdots A_{j-1}).$$

We observe that

$$\operatorname{tr}(QAQ^{-1}) = \operatorname{tr}(AQ^{-1}Q) = \operatorname{tr}(A)$$

for any invertible matrix $Q \in M_n(F)$. We refer to this property as *trace is conjugation invariant*.

If β, γ are two ordered bases of V, then we know that

$$[I_V]_\beta^\gamma [T]_\beta [I_V]_\gamma^\beta = [I_V]_\beta^\gamma [T]_\beta \big([I_V]_\beta^\gamma\big)^{-1} = [T]_\gamma.$$

Therefore,

$$\operatorname{tr}\big([T]_\gamma\big) = \operatorname{tr}\big([T]_\beta\big)$$

and trace of a linear map is well-defined. Let us discuss some examples via computations.

Examples (1) Consider the identity map $I_V : V \to V$. For any ordered basis β of V, $[I_V]_\beta = I_n$, where V has dimension n. Thus, trace of I_V is n.

(2) Consider the map

$$T : \mathbb{R}^n \to \mathbb{R}^n, \quad T(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n).$$

With respect to the standard basis β, the matrix $[T]_\beta$ is $D(\lambda_1, \ldots, \lambda_n)$, the diagonal matrix with entries $\lambda_1, \ldots, \lambda_n$. Thus, trace of T is $\lambda_1 + \cdots + \lambda_n$. If $\gamma = \{e_n, \ldots, e_1\}$ is another ordered basis, then

$$[T]_\gamma = D(\lambda_n, \ldots, \lambda_1)$$

and the trace of T is the same as earlier.

(3) Let $P : V \to V$ be a projection map, , i.e., $P^2 = P$.

Note that $v = v - P(v) + P(v)$ is a decomposition of v into $N(P)$ and $R(P)$. If $v \in N(P) \cap R(P)$, then $P(v) = \mathtt{0}_V$ as well as $v = P(w)$. Thus,

$$v = P(w) = P(P(w)) = P(v) = \mathtt{0}_V.$$

Therefore, $N(P) \cap R(P) = \{\mathtt{0}_V\}$ $V = R(P) \oplus N(P)$.

Let $k = \dim R(P)$ and $n - k = \dim N(P)$. If $V = W_1 \oplus W_2$, then show that the union of bases of W_1 and W_2 forms a basis of V. Form an ordered basis β of V from bases of $R(P)$ and $N(P)$, i.e.,

$$\beta = \{v_1, \ldots, v_k, u_1 \ldots, u_{n-k}\},$$

where v_i's form a basis of $R(P)$ and u_j's form a basis of $N(P)$. As $v_i = P(w_i)$ for some w_i,

$$P(v_i) = P^2(w_i) = P(w_i) = v_i.$$

The matrix associated with β is given by

$$[P]_\beta = \begin{pmatrix} I_k & 0_{(n-k)\times k} \\ 0_{k\times(n-k)} & 0_{n-k} \end{pmatrix}$$

whence the trace of P is k, the rank of P.

(4) Consider the vector space V of traceless 2×2 matrices. Let $A \in M_2(\mathbb{R})$ be an invertible matrix and consider the linear map

$$\mathrm{Ad}(A) : V \to V, \;\; B \mapsto ABA^{-1}.$$

This is well-defined as $\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B) = 0$. Consider the basis

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Let $\Delta = ad - bc$ denote the determinant of A. We can compute (exercise) the following:

$$\begin{aligned}
A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}A^{-1} &= \tfrac{1}{\Delta}\begin{pmatrix} ad+bc & -2ab \\ 2cd & -ad-bc \end{pmatrix} \\
A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}A^{-1} &= \tfrac{1}{\Delta}\begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \\
A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}A^{-1} &= \tfrac{1}{\Delta}\begin{pmatrix} bd & -b^2 \\ d^2 & -bd \end{pmatrix}
\end{aligned}$$

Thus, the matrix of $\mathrm{Ad}(A)$ with respect to β is

$$[\mathrm{Ad}(A)]_\beta = \tfrac{1}{\Delta}\begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix}.$$