MA2102: LINEAR ALGEBRA

Lecture 26: Determinant: Properties

28th October 2020

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We shall think of $A \in M_n(F)$ as

$$A = (\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n),$$

an ordered collection of n (column) vectors. As the determinant can be computed by a cofactor expansion along any row or column, if $A = (\mathbf{v}_1\ \cdots\ \mathbf{v}_{j-1}\ \mathbf{v}_j + \mathbf{w}_j\ \mathbf{v}_{j+1}\ \cdots\ \mathbf{v}_n)$, then by a cofactor expansion along the j^{th} column,

$$\begin{aligned} \det(\mathbf{v}_1\ \cdots\ \mathbf{v}_{j-1}\ \mathbf{v}_j + \mathbf{w}_j\ \mathbf{v}_{j+1}\ \cdots\ \mathbf{v}_n) &= \det(\mathbf{v}_1\ \cdots\ \mathbf{v}_{j-1}\ \mathbf{v}_j\ \mathbf{v}_{j+1}\ \cdots\ \mathbf{v}_n) \\ &\quad + \det(\mathbf{v}_1\ \cdots\ \mathbf{v}_{j-1}\ \mathbf{w}_j\ \mathbf{v}_{j+1}\ \cdots\ \mathbf{v}_n). \end{aligned}$$

Similarly, we have

$$\det(\mathbf{v}_1\ \cdots\ \mathbf{v}_{j-1}\ \lambda\mathbf{v}_j\ \mathbf{v}_{j+1}\ \cdots\ \mathbf{v}_n) = \lambda \det(\mathbf{v}_1\ \cdots\ \mathbf{v}_{j-1}\ \mathbf{v}_j\ \mathbf{v}_{j+1}\ \cdots\ \mathbf{v}_n).$$

Hence, determinant is a linear function of any column when the other columns are fixed. The same proeprty holds for rows.

- If $A \in M_n(F)$ has one row or column zero, then $\det(A) = 0$.
  - use cofactor expansion along the zero column or row
- If A has two identical rows or columns, then $\det(A) = 0$.
  - refer to homework 8 (exercise)
- If B is obtained from A by interchanging two distinct columns (or rows), then $\det(B) = -\det(A)$.

$$\begin{aligned}
0 &= \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ (\mathbf{v}_i + \mathbf{v}_j) \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ (\mathbf{v}_i + \mathbf{v}_j) \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) \\
&= \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ (\mathbf{v}_i + \mathbf{v}_j) \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_i \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) \\
&\quad + \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ (\mathbf{v}_i + \mathbf{v}_j) \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_j \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) \\
&= \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ \mathbf{v}_i \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_i \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) \\
&\quad + \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ \mathbf{v}_i \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_j \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) \\
&\quad + \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ \mathbf{v}_j \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_i \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) \\
&\quad + \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{i-1} \ \mathbf{v}_j \ \mathbf{v}_{i+1} \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_j \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n).
\end{aligned}$$

- Determinant is unchanged by adding a mutiple of another column (resp. row) to a column (resp. row).

$$\det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_j + \lambda \mathbf{v}_i \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n) = \det(\mathbf{v}_1 \ \cdots \ \mathbf{v}_{j-1} \ \mathbf{v}_j \ \mathbf{v}_{j+1} \ \cdots \ \mathbf{v}_n)$$

This naturally leads to the following result.

Lemma If E is an elementary matrix, then

$$\det(EA) = \det(E)\det(A) = \det(AE).$$

Proof.
Recall the three types of elementary matrices:
Row(column) switching: The matrix $T_{i,j} \in M_n(F)$ is obtained by interchanging the i^{th} and j^{th} row of I_n. Note that

$$T_{i,j}^{-1} = T_{i,j}, \qquad \det(T_{i,j}) = -1.$$

The matrix $T_{i,j}A$ is the matrix produced by exchanging row i and row j of A. Similarly, $AT_{i,j}$ is obtained by interchanging columns i and j of A. Thus,

$$\det(T_{i,j}A) = \det(AT_{i,j}) = -\det(A) = \det(T_{i,j})\det(A).$$

Row(column) scaling: The matrix $D_i(\lambda) \in M_n(F)$ is the diagonal matrix with 1's on the diagonal except at the $(i,i)^{\text{th}}$ place, where it is a non-zero scalar λ. Note that

$$D_i(\lambda)^{-1} = D_i(1/\lambda), \qquad \det(D_i(\lambda)) = \lambda.$$

The matrix $D_i(\lambda)A$ is the matrix produced by scaling row i of A by λ. Similarly, the matrix $AD_i(\lambda)$ is the matrix produced by scaling column i of A by λ. Thus,

$$\det(D_i(\lambda)A) = \det(AD_i(\lambda)) = \lambda\det(A) = \det(D_i(\lambda))\det(A).$$

Row(column) addition: The matrix $L_{i,j}(\mu) \in M_n(F)$ is the identity matrix with μ in the $(j,i)^{\text{th}}$ entry, i.e., it is obtained from I_n by replacing the i^{th} column $\mathbf{e}_i$ with $\mathbf{e}_i + \mu\mathbf{e}_j$. Note that

$$L_{i,j}(\mu)^{-1} = L_{i,j}(-\mu), \qquad \det(L_{i,j}(\mu)) = 1.$$

The matrix $L_{i,j}(\mu)A$ is the matrix produced by adding μ times row i to row j. Similarly, the matrix $AL_{i,j}(\mu)$ is the matrix produced by adding μ times column j to column i. Thus,

$$\det(L_{i,j}(\mu)A) = \det(AL_{i,j}(\mu)) = \det(A) = \det(L_{i,j}(\mu))\det(A).$$

This completes the proof. $\square$

Theorem

A matrix $A \in M_n(F)$ is invertible if and only if $\det(A)$ *is non-zero.*

Proof.

Any $A \in M_n(F)$ of rank r can be transformed into block identity matrix, i.e.,

$$E_1 \cdots E_k A E'_1 \cdots E'_l = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix},$$

for appropriate choices of elementary matrices E_i's and E'_j's. The determinant of the block identity matrix is 1 if and only if $r = n$ which is equivalent to A having rank n. This is equivalent to A being invertible. If $r = n$, then using the lemma repeatedly, we obtain

$$\det(E_1) \cdots \det(E_k) \det(A) \det(E'_1) \cdots \det(E'_l) = 1.$$

As elementary matrices have non-zero determinants, we conclude that $\det(A) \neq 0$ if and only if A is invertible. $\square$

We may deduce from the Lemma and the Theorem that

$$\det(AB) = \det(A)\det(B)$$

for any $A, B \in M_n(F)$. This will be proved in the special lecture.

Remark The restriction of determinant to $GL_n(F)$, the set of invertible matrices, defines a *homomorphism*.

It follows that if A is invertible, then

$$\det(A^{-1}) = \det(A)^{-1}.$$

We also note that

$$\det(CAC^{-1}) = \det(A).$$

Definition [Determinant] Let $T : V \to V$ be a linear map and V be finite dimensional. The determinant of T is defined to be the determinant of $[T]_\beta$ for any choice of ordered basis β.

If β, γ are two ordered bases, then $[T]_\beta$ and $[T]_\gamma$ are conjugate to each other, via the change of basis matrix. As determinant is conjugation invariant, we have

$$\det([T]_\beta) = \det([T]_\gamma).$$

Thus, the notion of determinant of a linear map $T : V \to V$ is well-defined.