MA2102: LINEAR ALGEBRA

Lecture 28: Eigenspaces

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A natural generalization of the notion of eigenvectors is the concept of eigenspaces.

Definition [Eigenspace] Let $T: V \to V$ be a linear map. If λ is an eigenvalue, then its eigenspace is defined as

$$E_\lambda := \{v \in V \mid T(v) = \lambda v\}.$$

Note that $E_\lambda \neq \emptyset$ and it is a (vector) subspace. If T is diagonalizable, with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then there exists a basis

$$\beta = \{v_1, \ldots, v_n\}$$

consisting of eigenvectors. Such a basis is called an eigenbasis. We may rearrange the v_j's such that the first r_1 vectors are eigenvectors for λ_1, the next r_2 vectors are eigenvectors for λ_2 and so on all the way up to the last r_k vectors are eigenvectors for λ_k.

It follows that

$$r_1 + \cdots + r_k = n.$$

Consider the span S of $\{v_1, \ldots, v_{r_1}\}$. We know that it is contained in E_{λ_1}. If $v \in E_{\lambda_1}$, then write v in terms of v_j's and look at

$$\begin{aligned}
T(v) &= T(c_1 v_1 + \cdots + c_n v_n) \\
&= \lambda_1 (c_1 v_1 + \cdots + c_{r_1} v_{r_1}) + \lambda_2 (c_{r_1+1} v_{r_1+1} + \cdots + c_{r_1+r_2} v_{r_1+r_2}) \\
&\quad + \cdots + \lambda_k (c_{r_1+\cdots+r_{k-1}+1} v_{r_1+\cdots+r_{k-1}+1} + \cdots + c_{r_1+\cdots+r_k} v_{r_1+\cdots+r_k}) \\
\lambda_1 v &= \lambda_1 (c_1 v_1 + \cdots + c_{r_1} v_{r_1}) + \lambda_1 (c_{r_1+1} v_{r_1+1} + \cdots + c_{r_1+r_2} v_{r_1+r_2}) \\
&\quad + \cdots + \lambda_1 (c_{r_1+\cdots+r_{k-1}+1} v_{r_1+\cdots+r_{k-1}+1} + \cdots + c_{r_1+\cdots+r_k} v_{r_1+\cdots+r_k})
\end{aligned}$$

It follows from $T(v) = \lambda_1 v$ that $c_j = 0$ for $j > r_1$, whence $v \in S$. This establishes the equality $E_{\lambda_1} = \mathrm{span}_F\{v_1, \ldots, v_{r_1}\}$.

Show that $E_{\lambda_j} = \text{span}_F\{v_{r_1+\cdots+r_{j-1}+1}, \ldots, v_{r_1+\cdots+r_j}\}$.

If $v \in E_{\lambda_i} \cap E_{\lambda_j}$ for $i \neq j$, then

$$\lambda_i v = T(v) = \lambda_j v,$$

implying that $(\lambda_i - \lambda_j)v = \mathbf{0}$. This implies that $v = \mathbf{0}$,

$$E_{\lambda_i} \cap E_{\lambda_j} = \{\mathbf{0}\}$$

and $E_{\lambda_i} + E_{\lambda_j}$ is a direct sum. More generally, we claim that

$$E_{\lambda_1} + \cdots + E_{\lambda_k} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}.$$

This may be proven by induction, starting with $E_{\lambda_1} + E_{\lambda_2} = E_{\lambda_1} \oplus E_{\lambda_2}$.

Lemma If $v_1, \ldots, v_k$ are eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, \ldots, v_k\}$ is linearly independent.

Proof.
Let j be the maximum index such that $\{v_1, \ldots, v_j\}$ is linearly independent. If $j = k$ we are done; otherwise

$$v_{j+1} = c_1 v_1 + \cdots + c_j v_j.$$

Applying T to both sides we obtain

$$\lambda_{j+1}(c_1 v_1 + \cdots + c_j v_j) = \lambda_{j+1} v_{j+1} = c_1 \lambda_1 v_1 + \cdots + c_j \lambda_j v_j.$$

This leads us to

$$c_1(\lambda_{j+1} - \lambda_1) v_1 + \cdots + c_j(\lambda_{j+1} - \lambda_j) v_1 = 0$$

implying that $c_1 = \cdots = c_j = 0$. Hence, $v_{j+1} = 0$, a contradiction. $\square$

Show that V is the direct sum of E_{λ_i}'s, when T is diagonalizable, i.e., every v can be written uniquely as $v = v_1 + \cdots + v_k$, where $v_j \in E_{\lambda_j}$.

Example (1) Let $T : V \to V$ admit two eigenvalues λ, μ and $\dim_F V = 2$. Note that $\dim_F E_\lambda \geq 1$ as well as $\dim_F E_\mu \geq 1$. As $E_\lambda \oplus E_\mu$ has dimension at least 2, we must have $V = E_\lambda \oplus E_\mu$. Thus, both the eigenspaces have dimension 1. Choose a basis v and w of E_λ and E_μ respectively. Then $\beta = \{v, w\}$ is an eigenbasis of V and T is diagonal with respect to β.

(2) Let $T : R^2 \to \mathbb{R}^2$ be given by

$$T(x, y) = (x + 2y, 3x + 2y).$$

The matrix with respect to the standard basis is

$$A = [T]_\beta = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$$

The characteristic polynomial is $p_A(x) = (x - 4)(x + 1)$ (exercise).

By the previous example, we know that $\mathbb{R}^2 = E_4 \oplus E_{-1}$. If (x,y) is an eigenvector with eigenvalue 4, then

$$(x+2y, 3x+2y) = (4x, 4y).$$

Solving we obtain $(x,y) = c(2,3)$. Show that any element of E_{-1} is of the form $c(1,-1)$. Thus, T is diagonalizable with $\{(2,3),(1,-1)\}$ as an eigenbasis.

(3) Let $P : V \to V$ be a projection map. If $P(v) = \lambda v$ and $v \neq 0$, then applying P we obtain $\lambda v = \lambda^2 v$. This implies that $\lambda \in \{0,1\}$ and eigenvalues of P can be either 0 or 1. We have seen earlier that E_0 is the null space of P, E_1 is the range of P and $V = E_0 \oplus E_1$.

(4) Consider the map $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$, $p(x) \mapsto xp'(x)$. As $T(x^k) = kx^k$, for $k = 0, \ldots, n$, the standard basis $\beta = \{1, x, x^2, \ldots, x^n\}$ is an eigenbasis.

There are $n+1$ distinct eigenvalues and

$$P_n(\mathbb{R}) = E_0 \oplus \cdots \oplus E_n$$

(5) Consider the *shear* matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and the associated linear map

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \;\; T(x, y) = (x + y, y).$$

The only eigenvalue is 1 and

$$E_1 = \{(0, y) \in \mathbb{R}^2 \,|\, y \in \mathbb{R}\}.$$

Thus, A or T is not diagonalizable. However, A has the same trace, rank, determinant, eigenvalues and characteristic polynomial as I_2.