MA2102: Linear Algebra

Lecture 15: Matrix Representations

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Recall the definition of an ordered basis. For instance,

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

forms an ordered basis of $M_2(\mathbb{R})$ (exercise). Since any

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$$

can be expressed as

$$\tfrac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tfrac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \tfrac{b-c}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \tfrac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

the coordinate vector of A is given by $[A]^{\beta} = (\frac{a+d}{2}\ \frac{a-d}{2}\ \frac{b-c}{2}\ \frac{b+c}{2})^t$.

Let $T : V \to W$ be a linear map. Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$ be ordered bases for V and W respectively. We express

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

for unique scalars a_{ij}. We may form a $m \times n$ matrix $A = ((a_{ij}))$.

Question *Given $v \in V$, what is the relationship between the coordinate vectors $[v]^{\beta}$ and $[Tv]^{\gamma}$?*

If $v = x_1 v_1 + \cdots + x_n v_n$ then

$$\begin{aligned} T(v) &= x_1 T(v_1) + \cdots + x_n T(v_n) \\ &= \sum_{i=1}^{m} x_1 a_{i1} w_i + \cdots + \sum_{i=1}^{m} x_n a_{in} w_i \\ &= \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} x_j \Big) w_i. \end{aligned}$$

Therefore, we see that

$$[v]^\beta = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \Rightarrow \quad [Tv]^\gamma = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

In other words, $[Tv]^\gamma = A[v]^\beta$.

Definition [Matrix Representation] The $m \times n$ matrix A associated to $T : V \to W$ is called the matrix representation of T with respect to β and γ. We write $A = [T]_\beta^\gamma$ and thus $[Tv]^\gamma = [T]_\beta^\gamma [v]^\beta$.

Convention *If $\beta = \gamma$, then we denote A by $[T]_\beta$.*

Show that A, β and γ determines a unique linear map $T : V \to W$ such that $A = [T]_\beta^\gamma$.

Examples (1) Let $\mathbb{1} : V \to V$ be the identity map and let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis of V. Then

$$[\mathbb{1}]_\beta := [\mathbb{1}]_\beta^\beta = I_n$$

is the $n \times n$ identity matrix.

(2) Consider the dilation map

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (\lambda x, \lambda y).$$

Let $\beta = \{(1, 0), (0, 1)\}$, $\gamma = \{(0, 1), (1, 0)\}$ and $\eta = \{(1, 1), (1, -1)\}$ be three bases for $\mathbb{R}^2$. It follows that

$$[T]_* = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \; [T]_\beta^\gamma = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \; [T]_\beta^\eta = \begin{pmatrix} \lambda/2 & \lambda/2 \\ \lambda/2 & -\lambda/2 \end{pmatrix}.$$

(3) Let V be a finite dimensional vector space. Consider a linear map $T : V \to V$, i.e., $T \in \mathscr{L}(V)$. Choose a basis $\{v_1, \ldots, v_k\}$ of the null space $N(T)$. By Replacement Theorem, extend this to an ordered basis $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ of V. Set $w_j = T(v_j)$ for $j \geq k+1$.

Claim: The set $L = \{w_{k+1}, \ldots, w_n\}$ is linearly independent.

Method 1: The set L spans $R(T)$, which has dimension $n-k$. Thus, L is a basis of $R(T)$.

Method 2: If $c_{k+1}w_{k+1} + \cdots + c_n w_n = \mathtt{0}_W$, then $T(c_{k+1}v_{k+1} + \cdots + c_n v_n) = \mathtt{0}_W$. Thus, $c_{k+1}v_{k+1} + \cdots + c_n v_n \in N(T)$, i.e.,

$$c_{k+1}v_{k+1} + \cdots + c_n v_n = a_1 v_1 + \cdots + a_k v_k.$$

As β is a basis, all a_i's and c_j's must be zero.

Method 3: Consider the linear isomorphism (cf. lecture 14)

$$\mathscr{T} : V/N(T) \to R(T), \;\; [v] \mapsto T(v).$$

The set $\{[v_{k+1}], \ldots, [v_n]\}$ is a basis of $V/N(T)$. Thus, (cf. lecture 13) the image of a basis under a linear isomorphism is a basis of $R(T)$.

Extend L to an ordered basis $\gamma = \{w_1, \ldots, w_k, w_{k+1}, \ldots, w_n\}$ of V. As

$$T(v_1) = \cdots = T(v_k) = 0_W$$
$$T(v_j) = w_j, \qquad j = k+1, \ldots, n$$

we conclude that

$$[T]_\beta^\gamma = \begin{pmatrix} 0_{k\times k} & 0_{(n-k)\times k} \\ 0_{(n-k)\times k} & I_{n-k} \end{pmatrix}.$$

Thus, there exist bases with respect to which the matrix representation is diagonal.

Proposition Let V and W be vector spaces (over $\mathbb{R}$) of dimension n and m over respectively. The space $\mathscr{L}(V,W)$ is isomorphic to $M_{m\times n}(\mathbb{R})$.

Proof.

Let β and γ be ordered bases for V and W respectively. Consider

$$\Phi : \mathscr{L}(V,W) \to M_{m\times n}(\mathbb{R}), \ \ \Phi(T) = [T]_\beta^\gamma.$$

Note that

$$\Phi(S+T) = [S+T]_\beta^\gamma = [S]_\beta^\gamma + [T]_\beta^\gamma = \Phi(S) + \Phi(T)$$

$$\Phi(\lambda T) = [\lambda T]_\beta^\gamma = \lambda [T]_\beta^\gamma = \lambda \Phi(T)$$

implies that Φ is linear. Since any matrix A determines $T : V \to W$ such that $[T]_\beta^\gamma = A$ (cf. exercise at the beginning), Φ is surjective. If $\Phi(T) = 0_{m\times n}$, then T is the trivial map, whence Φ is injective. $\square$