Elements of Vector Calculus : Scalar Field & its Gradient

Lecture 1 : Electromagnetic Theory

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Introduction : In this set of approximately 40 lectures covering a course of one semester, I would take you through electrostatics , magnetostatics and electromagnetic phenomena, leading to both the differential and integral form of the Maxwell's equations. At the end of the course you would have an appreciation of what are the important phenomena and problems associated with electromagnetism. Later, I would discuss the electromagnetic waves and if time permits,I will have some discussion on antenna and radiation.

However, the course requires a good understanding of the subject of vector calculus. So in the initial lectures, we would spend some time in revising or providing an introduction to the essentials of vector calculus. It is not going to be rigorous the way a mathematician would like it to be but should adequately serve our purpose. In the first module of five lectures, we will discuss vector calculus and some of its basic applications. We will have discussions on the concept of a scalar field and a vector field, ordinary derivatives and gradient of a scalar function, line and surface integrals, divergence and curl of a vector field, Laplacian. We will enunciate two major theorems, viz., the divergence theorem and the Stoke's theorem.

 Concept of a Field :

By field, we basically mean something that is associated with a region of space. For instance, this room in which I am speaking can be considered to be a region in which a temperature field exists. Normally, we talk of the temperature of a room. However, this is in the sense of an average and does not provide detailed temperature profile inside the room. However, the temperature inside a room does vary from place to place. For instance, if you are in a kitchen, the temperature would be higher when you are close to stove and would be lower elsewhere. In principle, one can associate a temperature with every point inside the room. The field that we talked of here, viz. the temperature field is a scalar field because the field quantity "temperature" is a scalar.

The "field" is thus a region of space where with every point we can associate a scalar or a vector (it could be more generalized but for our purposes, these two will do). Coming to a vector field, as we know, a vector quantity has both magnitude and direction. Consider our room again. We can associate a gravitational field with it. Though we generally say that the acceleration due to gravity has a constant value inside the room, it is also meant in an average sense. In reality, its value and direction differs from place to place and a mass inside a room experiences a different force (both in magnitude and direction) depending on where in the room it is placed. If we talk of associating a force with every point in a certain region of space, we are talking about a vector field. In 2 dimensions, the force is a

function of positions x and y and in three dimensions it is a function of x, y and z. Other than gravitational field, examples of vector fields are electric field and magnetic field.

Pictorially, the scalar field being defined by a number associated with a point in space ,is usually represented using a fixed spatial structure called a grid. They are also represented by connecting all points having the same value of the scalar field by a contour (e.g. isothermals).

Since a vector field has a magnitude and direction, it is a little more complicated to represent it graphically. Let us consider a two dimensional vector field $\vec{F}(x,y) = y\hat{\imath} - 2x\hat{\jmath}$ as an example. We can use a graph paper with conventional x and y axes. How does one represent the vector field? We take some unit to represent a unit length of the vector field. In the figure below we have taken one fifth the unit spacing along x or y axes.
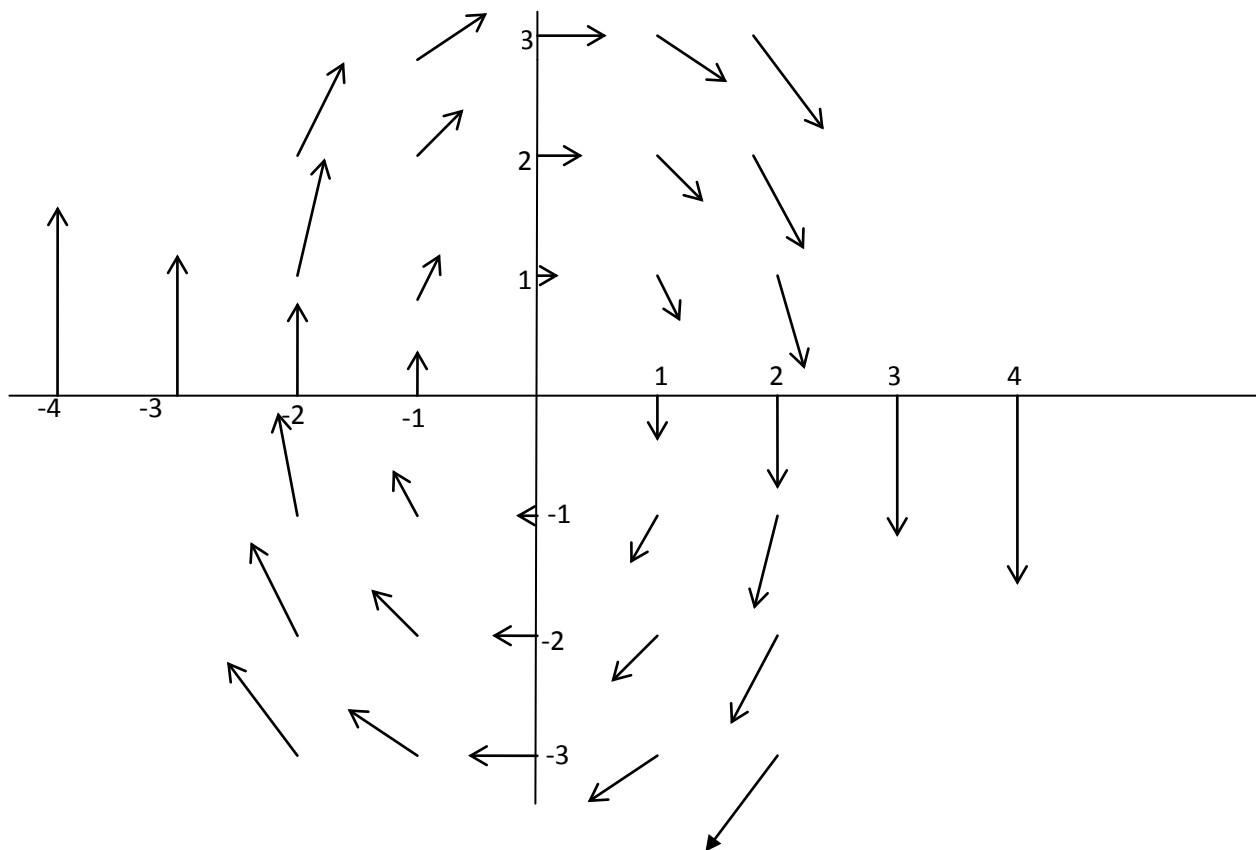

Figure 1: Graphical representation of a vector field $\vec{F} = y\hat{\imath} - 2x\hat{\jmath}$

you can use a computer package like Mathematica . The output from such a package for the vector field above is shown below,
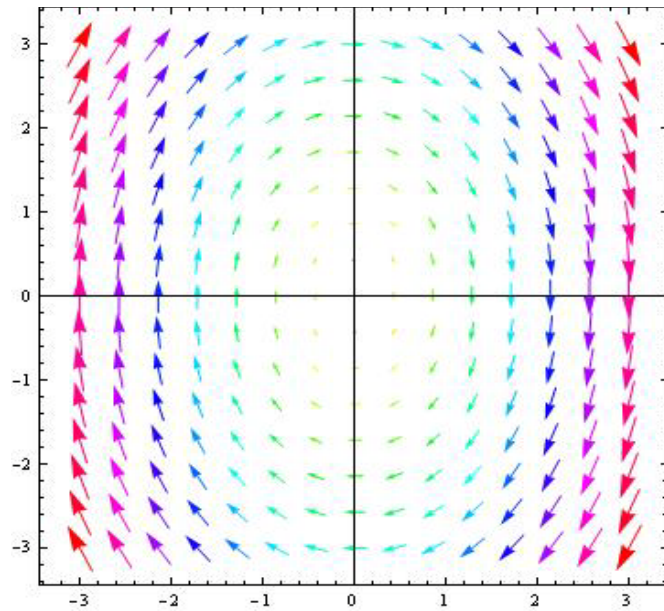
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Figure 2 : Vector field $\vec{F} = y\hat{i} - 2x\hat{j}$ represented in Mathematica

In electrostatics we deal with force field due to charges. In Fig. 3 we show the force on a unit positive charge due to two equal and opposite charges . The vector field has been plotted at close enough points so that the field lines appear continuous. To find the force on a positive charge at a point, we need to draw a tangent to the field lines at that point. These are known as "lines of force" in electrostatics". The arrow on the lines show the direction in which the charge moves.
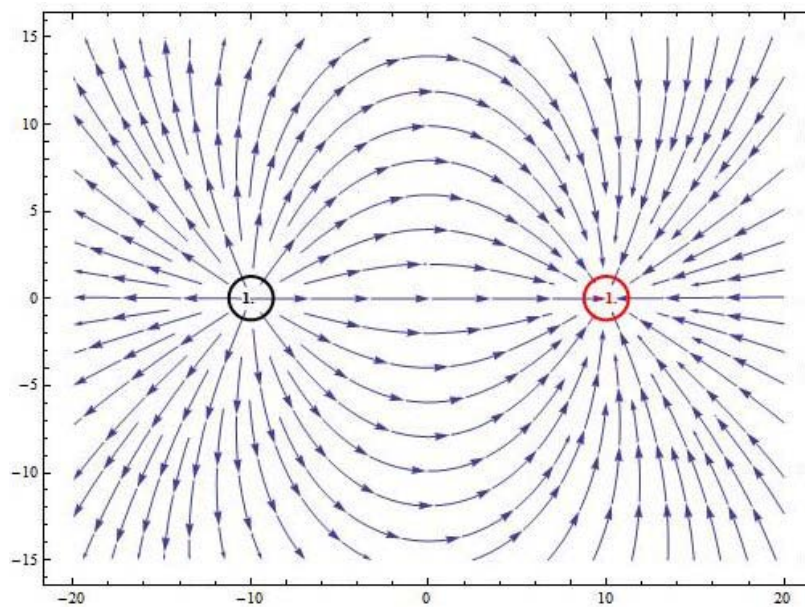

Figure 3 : Lines of force due to two opposite charges.

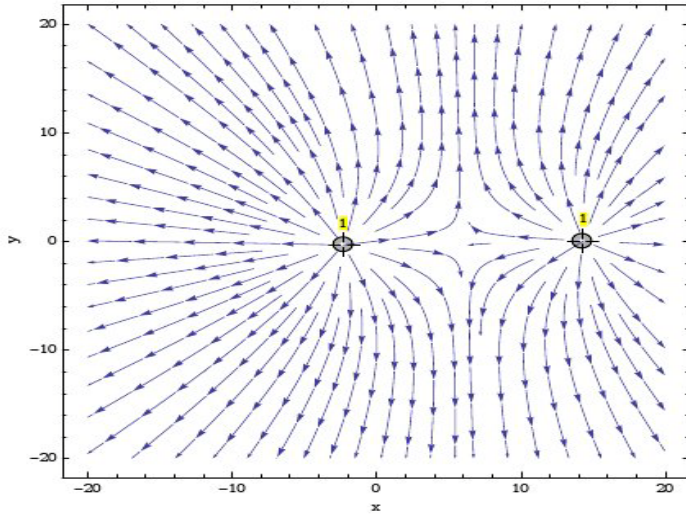Figure 4 shows the force field due to two similar charges, a positive charge is repelled by both the charges.

Figure 4 : Force field due to two similar charges.

Directional Derivatives :

Let me first remind you of the definition of ordinary derivative of a function f(x) of a single variable x. We define it by the relationship $\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$. This means that the value of the function f at the point $x + \Delta x$ is its value at the point x plus the derivative of the function time the increment in the value of x. If we are given a function in one dimension, the derivative at a point is the slope of the function at that point. If the slope is positive the value of the function increases from its value at a neighbouring point, it decreases if the slope is negative.

$$f(x + \Delta x) = f(x) + \frac{df}{dx}\Delta x$$

What happens in higher dimensions? We are familiar with the concept of "partial derivative". Suppose we have a function of x and y. The partial derivative with respect to x means that when the differentiation is done with respect to the variable x, we treat the variable y as a constant. Similarly, in taking partial derivative with respect to y, the value of x is kept fixed.

What if both x and y are to be allowed to vary simultaneously? The problem is that there are many ways the two variables can change simultaneously. Same is true for a function of three or more variables. The concept of derivative is thus to be generalized.

Suppose ϕ is a scalar function of the variables x, y and z. Starting from a point $P_0(x_0,y_0,z_0)$ if we move along an arbitrary direction by a length Δs, the value of the function at the destination $P(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ will be given by its value at the initial point plus the derivative of the function

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computed along the direction in which we moved times the length Δs. Such a derivative is called the directional derivative. Since $\overrightarrow{\Delta s} = \Delta x \hat{\imath} + \Delta y\, \hat{\jmath} + \Delta z \hat{k}$, we could go from the point P_0 to the point P by going by a distance Δx along the x direction, keeping y and z constant, then going by an amount Δy along the y direction and finally by Δz along the z direction and arrive at the point P.

This is graphically shown in two dimensions :
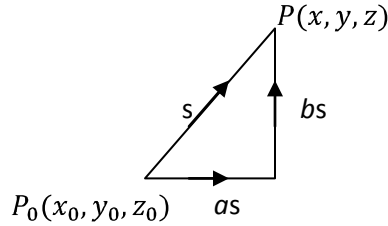
Using the definition of partial derivatives, we have,

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} \qquad (1)$$

where $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ and $\frac{\partial \phi}{\partial z}$ respectively represent the partial derivatives of ϕ with respect to x, y and z respectively. Equation (1) gives the directional derivative of the scalar function ϕ along the direction $\overrightarrow{ds}$.

Example :

We will illustrate the concept of directional derivative by calculating the directional derivative of the scalar function $\phi(x,y) = x^2 + y^2$ along three different directions : along (i) $\hat{\imath} + 2\hat{\jmath}$ (ii) $-2\hat{\imath} + 2\hat{\jmath}$ and (ii) $\hat{\imath} + \alpha\hat{\jmath}$ at the point (1,2).

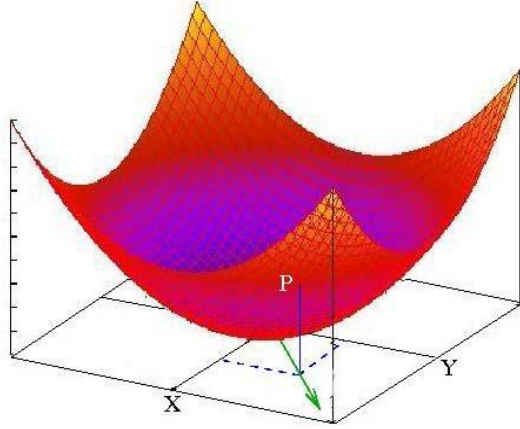(i) The figure below shows the function $\phi(x,y) = x^2 + y^2$ plotted along the z axis. It is a cup like structure.

Figure 5 : A plot of the function x²+y²

Since the function is in two dimensions, we have $\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds}$.

The partial derivatives are given by $\frac{\partial\phi}{\partial x} = 2x, \frac{\partial\phi}{\partial y} = 2y$ so that $\frac{d\phi}{ds} = 2x\frac{dx}{ds} + 2y\frac{dy}{ds}$. In order to calculate $\frac{ds}{dx}$ and $\frac{ds}{dy}$, we observe that along the given direction $\hat{\imath} + 2\hat{\jmath}$, the coordinates x and y are related by $y = 2x$ so that $\frac{dy}{dx} = 2$. We have $ds = \sqrt{(dx)^2 + (dy)^2}$ which gives $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{5}$ and $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{\frac{5}{2}}$. Plugging these into the expression for directional derivative, we get, $2y = 2x\sqrt{5} + 2y\frac{\sqrt{5}}{2}$. At the point (1,2), the directional derivative is $\frac{d\phi}{ds} = 2\sqrt{5}$

(ii)   The calculation is very similar to (i). The answer is zero.

(iii)   Following the method outlined in (i) above, the directional derivative at the point (1,2) can be shown to be given by $\frac{d\phi}{ds} = \frac{2+4\alpha}{\sqrt{1+\alpha^2}}$. The directional derivative has a maximum when $\alpha = 2$. Thus the directional derivative at (1,2) has a maximum in the direction of $\hat{\imath} + 2\hat{\jmath}$. It may be noted that this is the radial direction at that point.

Suppose the direction cosines of the direction that we move is (a,b,c), the unit vector in this direction represented by $\hat{u}$ is given by $\hat{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, with $a^2 + b^2 + c^2 = 1$. We have,

$x = x_0 + as, y = y_0 + bs, z = z_0 + cs$, which gives $\frac{dx}{ds} = a, \frac{dy}{ds} = b, \frac{dz}{ds} = c$

$P(x, y, z),$

$\hat{u}\,s$ $bs\hat{\jmath}$

$P_0(x_0, y_0, z_0)$ $as\hat{\imath}$
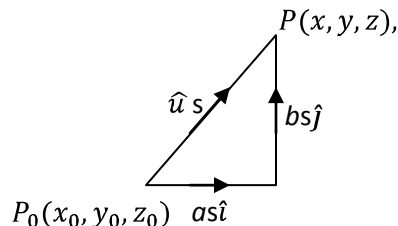
Which results in the directional derivative along $\hat{u}$ is given by

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$

$$= a\frac{\partial\phi}{\partial x} + b\frac{\partial\phi}{\partial y} + c\frac{\partial\phi}{\partial z}$$

$$\equiv \vec{\nabla\phi} \cdot \hat{u} = |\nabla\phi|.\,1.\cos\theta$$

Where the "gradient operator" $\vec{\nabla}$ is given by

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial x} + \hat{k}\frac{\partial}{\partial z}$$

where θ is the angle between the gradient and he direction in which the directional derivative is taken. Thus

(1) the magnitude of the gradient at a point is the maximum possible magnitude of the directional derivative at that point, and

(2) the direction of the gradient is that direction in which the directional derivative takes maximum value.

What does this physically mean? Suppose you are on a hill , not quite at the summit. If you want to come down to the base, there are many directions that you can take. Of all such possible directions, the fastest will be one which is steepest, i.e. with maximum slope.
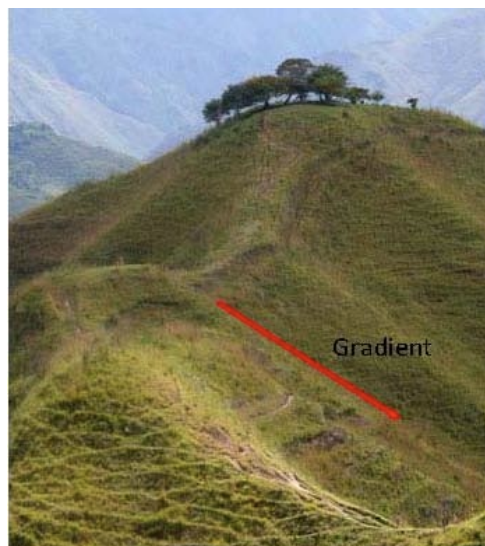

Figure 6 : Showing the direction of gradient on a hill

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Since the rate of change in the value of the function is maximum along the gradient, it follows that such a direction is perpendicular to a surface on which the function is constant. Such a surface is called a "Level Surface". Returning to the function $\phi(x, y) = x^2 + y^2$, level surface (rather a level curve in
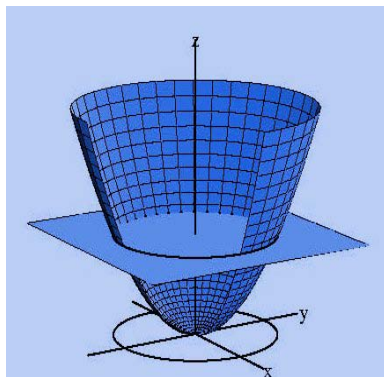

Figure 7 : Level surface

this case) is the intersection of the plane z= constant with the surface $z(x, y) = x^2 + y^2$, which are family of circles. In Physics, the corresponding surface would be an equipotential surface and the direction of the gradient would correspond to the direction of the electric field.

In the present case $\phi(x, y) = x^2 + y^2$

$\nabla\phi = 2x\hat{\imath} + 2y\hat{\jmath} = 2\hat{r}$, which, as expected, is in the radial direction which is normal to the level curve, which is a circle.

Gradient of a Scalar Field is a Vector Field and its direction is normal to the level surface.

Formal Proof : Consider a level curve which is parameterized by a variable t, which varies from point to point on the curve. Example of such a parameter for the circle is angle θ, so that $x = R\cos\theta$, $y = R\sin\theta$, where R is the radius (which is fixed) and q is the polar angle $0 \leq \theta \leq 2\pi$.

The position vector of a point on the curve is given by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. Let the level curve be given by $\phi(x(t), y(t), z(t)) = $ constant. The tangent to the curve is $\vec{r}'(t) = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + \frac{dz}{dt}\hat{k}$. Obviously, on the level curve $\frac{d\phi}{dt} = 0$. But,

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$
$$= \overrightarrow{\nabla\phi} \cdot \vec{r}'(t)$$
$$= 0$$

Which shows that the gradient is normal to the level curve.

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Tutorial :

1. Find the directional derivative of the function $f(x, y) = 3x^2 y$ at a point (-2,1) along the direction $4\hat{\imath} + 3\hat{\jmath}$.
2. Find the directional derivative of the function $f(x, y, z) = 3x^3 + 2xy^2 + xyz$ along the direction $2\,\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$ at a point (1,1,1)
3. Consider a function $f(x, y, z) = x^2 + y^2 - z^2$. What is the direction in which maximum change of the function takes place at the point (2,2,1)?
4. Find the equation to the tangent plane to the surface $x^2 + y^2 - z^2 = 7$ at the point (2,2,1).

   Solutions :

   1. The unit vector along $4\hat{\imath} + 3\hat{\jmath}$ is $\hat{u} = \frac{4\hat{\imath}+3\hat{\jmath}}{5}$. Gradient of the given function is $6xy\hat{\imath} + 3x^2\hat{\jmath}$. Thus the directional derivative at (x,y) is $\left(\frac{24}{5}\right)xy + \left(\frac{9}{5}\right)x^2$. At (-2,1) its value is -12/5.

   2. The unit vector along $2\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$ is $\hat{u} = \frac{\hat{\imath}+2\hat{\jmath}+2\hat{k}}{5}$. Gradient of the given function is $(9x^2 + 2y^2 + yz)\hat{\imath} + (4xy + xz)\,\hat{\jmath} + xy\hat{k}$. At (1,1,1) the gradient is $12\hat{\imath} + 5\hat{\jmath} + \hat{k}$ Thus the directional derivative at (1,1,1) is 8.

   3. Maximum change takes place in the direction of the gradient. In this case the unit vector along the gradient is $\frac{4\hat{\imath}+4\hat{\jmath}-2\hat{k}}{6}$.

   4. The level surface is $x^2 + y^2 - z^2 = 7$. Normal to the surface is in the direction of gradient which is $4\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$. The equation to the tangent plane is given by $f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$, where f_x, f_y and f_z are the partial derivative of the function f(x,y,z)= constant at the point (x_0, y_0, z_0). In this case $f_x = 2x_0 = 4, f_y = 2y_0 = 4, f_z = -2z_0 = -2$. Thus the equation is $4(x - 2) + 4(y - 2) - 2(z - 1) = 0$.

Self Assessment Quiz

1. Find the directional derivative of the function $f(x, y) = x^2 y$ at the point (2,3) along the direction $3\hat{\imath} - 4\hat{\jmath}$.

2. In the above problem, what is the directional directive along the direction $4\hat{\imath} + 3\hat{\jmath}$ which is perpendicular to the direction $3\hat{\imath} - 4\hat{\jmath}$?

3. Sketch the g vector field $y\hat{\imath} + x\hat{\jmath}$ in the region $x \in [-2,2], y \in [-2,2]$.

4. Find a normal to the surface $x = y^2 + z^2$ at the point (2,1,1)

5. Evaluate grad r where r is the distance from origin.

6. Find the tangent plane and a normal line to the surface $-x^3 + 3xyz + z^2 = 9$ at the point (1,1,2)

Solutions :

1. $\nabla f = 2xy\hat{\imath} + x^2\hat{\jmath} = 12\hat{\imath} + 4\hat{\jmath}$. The unit vector in the given direction is $\hat{u} = \frac{3\hat{\imath}-4\hat{\jmath}}{5}$. Thus $\frac{df}{ds} = 4$.

2. The given direction being perpendicular to the direction of the gradient, is along the level surface. The directional derivative is zero along a level surface.

3. The sketch is as under
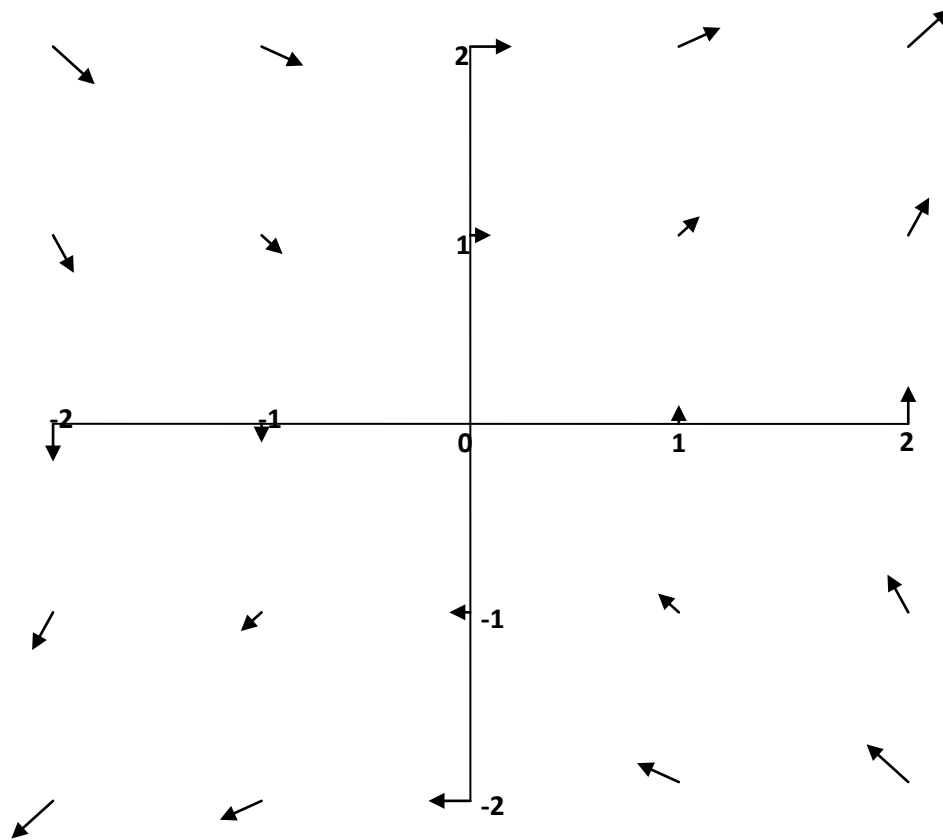
4. Define $f(x,y,z) = y^2 + z^2 - x$. The gradient is $\nabla f = -\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$. Unit normal to the level surface $f = 0$ is $= \frac{-\hat{\imath}+2\hat{\jmath}+2\hat{k}}{3}$.

5. $r = (x^2 + y^2 + z^2)^{1/2}$, $\nabla r = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \hat{r}$

6. At the point (1,1,2) $\nabla f = 3\hat{\imath} + 6\hat{\jmath} + 7\hat{k}$. The tangent plane is 3(x-1)+6(y-1)+7(z-2)=0. The unit normal, which is along the gradient is parameterized by x=1+3t, y=1+6t, z=2+7t.