MA2102: Linear Algebra

Lecture 27: Diagonalization

3rd November 2020

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Suppose $T : V \to V$ is a linear map. In order to compute trace and determinants of T^k, it definitely helps if there exist a basis β such that $[T]_\beta$ is a diagonal matrix. In that case,

$$\operatorname{trace}(T^k) = \operatorname{trace}\big([T^k]_\beta\big) = \operatorname{trace}\big(([T]_\beta)^k\big),$$

which is easy to compute. Similarly, determinant of T becomes the product of the diagonal entries of $[T]_\beta$. As determinant is multiplicative,

$$\det(T^k) = \det\big([T^k]_\beta\big) = \det([T]_\beta)^k.$$

Definition [Diagonalizable] A matrix $A \in M_n(F)$ is called diagonalizable if A is similar to a diagonal matrix, i.e., there exists an invertible matrix $C \in M_n(F)$ such that CAC^{-1} is diagonal.

A linear map $T : V \to V$ is called diagonalizable if $[T]_\beta$ is diagonal for some basis β.

Remark If γ is another basis of V, then $[T]_\gamma$, being similar to $[T]_\beta$, is diagonalizable as a matrix. Conversely, suppose $[T]_\gamma$ is diagonalizable for some basis γ, then $Q[T]_\gamma Q^{-1}$ is diagonal for some matrix Q. Use Q to define a basis β such that $[T]_\beta$ is diagonal. Thus, $T : V \to V$ is diagonalizable if $[T]_\gamma$ is diagonalizable for any basis γ.

If such a basis β exists, then write $\beta = \{v_1, \ldots, v_n\}$ and

$$[T]_\beta = D(\lambda_1, \ldots, \lambda_n).$$

In other words, $v_j \neq \mathbf{0}$ and $T(v_j) = \lambda_j v_j$.

Definition [Eigenvector] A non-zero vector $v \in V$ is called an eigenvector of $T : V \to V$ if $T(v) = \lambda v$ for some scalar $\lambda \in F$. The scalar λ is called the eigenvalue of the eigenvector v.

Definition [Eigenvector] A non-zero vector $v \in F^n$ is called an eigenvector of $A \in M_n(F)$ if $Av = \lambda v$ for some $\lambda \in F$. The scalar λ is called the eigenvalue of A corresponding to the eigenvector v.

Remark Eigenvalues are also called proper values, latent values, spectral values or characteristic values. 'Eigen' is a German word meaning *own*.

Geometrically, if v is an eigenvector of T, then consider the 1-dimensional subspace L spanned by v. It follows that $T(L) \subseteq L$, where equality is achieved if and only if the eigenvalue if non-zero.

Example (1) Any non-zero vector v in the null space of $T : V \to V$ is an eigenvector with eigenvalue $\mathtt{0}$.

(2) Any non-zero vector v is an eigenvector of $I_V : V \to V$ with eigenvalue 1.

Theorem [Characterization of eigenvalues]
The following conditions are equivalent:

(a) λ *is an eigenvalue for* T*;*

(b) $T-\lambda I_V$ *is not invertible;*

(c) $\det(T-\lambda I_V)=0$.

Proof.

(a) $\Rightarrow$ (b): There exists $v\neq 0$ such that $T(v)=\lambda v$, whence $v\in N(T-\lambda I_V)$. Thus, $T-\lambda I_V$ is not invertible.

(b) $\Rightarrow$ (c): If $T-\lambda I_V$ is not invertible as a linear map, then $[T-\lambda I_V]_\beta$ is not invertible as a matrix, whence $\det([T]_\beta-\lambda I_n)=0$.

(c) $\Rightarrow$ (a): The assumption implies that $T-\lambda I_V$ is not invertible, i.e., it is not of full rank. Thus, there exists a non-zero v such that $(T-\lambda I_V)(v)=0$. □

Observe that λ is an eigenvalue of T if and only is λ is an eigenvalue of $[T]_\beta$ for any ordered basis β.

Example (1) If $A \in M_n(F)$ is upper triangular, then λ is an eigenvalue if and only if $\lambda = a_{jj}$ for some $j \in \{1, \ldots, n\}$. Note that $\det(A)$ is the product of the eigenvalues.

(2) If $v \in V$ satisfies $T(v) = \lambda v$, then $T^k(v) = \lambda^k v$. Thus, eigenvectors of T are always eigenvectors of T^k for $k \in \mathbb{N}$. More generally, let $p(x) = a_0 + a_1 x + \cdots + a_k x^k$ be a polynomial of degree k. It follows that v is an eigenvector of $p(T)$ since

$$p(T)(v) = (a_0 I_V + a_1 T + \cdots + a_k T^k)(v) = (a_0 + a_1 \lambda + \cdots + a_k \lambda^k)v.$$

(3) Let $T : V \to V$ be linear and V be finite dimensional. Then T is invertible if and only if 0 is not an eigenvalue of T.

(4) Recall that $[T^*]_{\beta^*} = [T]_\beta^t$ for $T : V \to V$ and any basis β of V. Thus,

$$\det\big([T^*]_{\beta^*} - \lambda I_n\big) = \det\big([T]_\beta^t - \lambda I_n\big) = \det\big([T]_\beta - \lambda I_n\big).$$

Therefore, eigenvalues of T and T^* coincide.

Note that not all linear maps admit eigenvalues. For instance, let R_θ denotes the counter-clockwise rotation on the plane by an angle θ, i.e.,

$$R_\theta : \mathbb{R}^2 \to \mathbb{R}^2, \ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let $\theta \neq 0, \pi$, i.e., R_θ is not $\pm I_2$. If (a, b) is an eigenvector with *real* eigenvalue λ, then

$$a\cos\theta - b\sin\theta = \lambda a, \quad a\sin\theta + b\cos\theta = \lambda b.$$

Squaring both equations and adding, we obtain $a^2+b^2=\lambda^2(a^2+b^2)$, whence $\lambda=\pm 1$ as $a^2+b^2\neq 0$. Show that $\lambda=\pm 1$ is not possible.

Consider the complexified map

$$R_\theta:\mathbb{C}^2\to\mathbb{C}^2,\ \begin{pmatrix} x \\ y \end{pmatrix}\mapsto\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

By condition (c) of the Theorem, we solve for

$$0=(\cos\theta-\lambda)(\cos\theta-\lambda)+\sin^2\theta=\lambda^2-2\lambda\cos\theta+1.$$

This indeed has two complex roots although it has no real roots.

Remark The map R_θ is an isometry and the eigenvalues (real or complex) of any isometry must have absolute value 1. From geometry, we see that if v is an eigenvector, then only rotation by 0 or π maps v to v or $-v$ respectively.