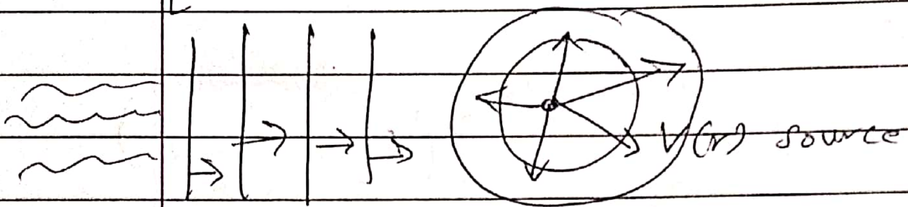
Quantum Scattering

$$H = \frac{p^2}{2m} + V(r).$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi = E\psi.$$

Consider positive energy $E = \frac{\hbar^2 k^2}{2m}$.

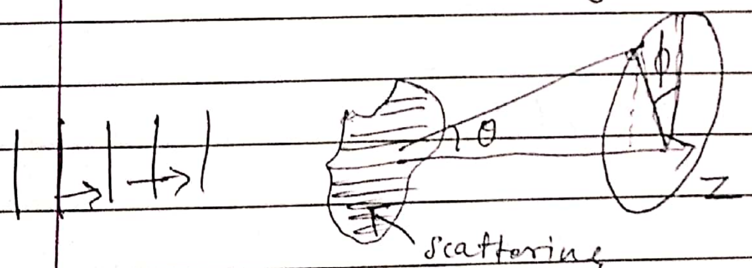
$$\left[-\frac{\hbar^2}{2m}(\nabla^2 + k^2) + V(r)\right]\psi = 0.$$



incoming can be a plane wave. $\phi(r) = e^{ikz}$.
and the second wave $\psi_{scat}(r) \approx \frac{e^{ikr}}{r}$
to die out.

$$\psi_s(r) = f_k(\theta, \phi) \cdot \frac{e^{ikr}}{r}.$$

for angle.



total $\psi(r) = \psi_s(r) + \phi(r) \approx e^{ikz} + f_k(\theta,\phi)\frac{e^{ikr}}{r}$.

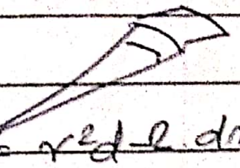
$d\sigma$ differential cross section $= \left(\frac{\#\text{ particles per unit time through } d\Omega}{\#\text{ incident flux}}\right)$

incident flux $= \vec{J} = \frac{\hbar}{m}\, im(\phi(r)\nabla\phi^*)$ $\quad \phi = e^{ikz}$.

$$= \frac{\hbar k}{m}\hat{z}$$

Now # of particle.

$dn =$ # particle in volume



$dV = r^2 d\Omega\, dr$

$$dn = |\psi(r)|^2 d^3r$$
$$= \left|f_k(\theta,\phi)\frac{e^{ikr}}{r}\right|^2 \cdot r^2 d\Omega\, dr = |f_k(\theta,\phi)|^2 dr\, d\Omega$$

$dt = \frac{dr}{v}$ $\quad v$: classical velocity

$$\frac{dn}{dt} = |f_k(\theta,\phi)|^2 \frac{d\Omega\, dr}{dr} v = \frac{\hbar k}{m} |f_k(\theta,\phi)|^2 d\Omega$$

$d\sigma = |f_k(\theta,\phi)|^2 d\Omega \Rightarrow$ Differential cross section

$$\frac{d\sigma}{d\Omega} = |f_k(\theta,\phi)|^2$$

Free particle & spherical. $\left(Y_{lm}\frac{u}{r} = \psi\right)$.

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right) u_{El}(r) = \frac{\hbar^2 k^2}{2m} u_{El}.$$

$$\Rightarrow u_{El}(r) = \underbrace{A_l \cdot r \cdot j_l(kr)}_{\text{sph. Bessel}} + \underbrace{B_l \cdot r\, \eta_l(kr)}_{\text{sph. Neum}}$$

$$r j_l(kr) \underset{r\to\infty}{\sim} \sin\left(kr - \frac{l\pi}{2}\right), \quad r\eta_l(kr) \sim -\cos\left(kr - \frac{\pi l}{2}\right)$$

$$e^{ikz} = e^{ikr\cos\theta} = \sum a_l P_l(\cos\theta)\, j_l(kr)$$
$$= \sqrt{4\pi} \sum_{l=0}^{\infty} \sqrt{2l+1}\, (i)^l\, Y_{l,0}(\theta)\, j_l(kr)$$

as $j_l(kr) \longrightarrow \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right) = \frac{1}{2ik}\left(\frac{e^{ik(r-\frac{l\pi}{2})} - e^{-ik(\dots)}}{r}\right)$

$$\Rightarrow \boxed{e^{ikz} = \frac{\sqrt{4\pi}}{\cancel{k}} \sum \sqrt{2l+1}\, i^l\, Y_{l0}(\theta)\, \frac{1}{2i}\left(\frac{e^{i(kr-\frac{l\pi}{2})} - e^{-i(kr-\frac{l\pi}{2})}}{r}\right) \text{ for } r \gg a}$$

range

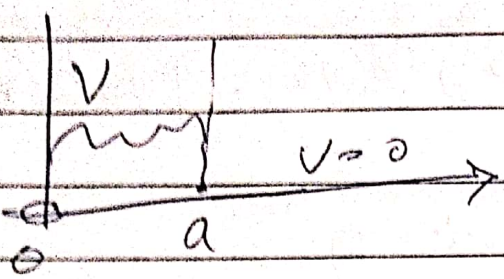
plane wave in sph. cor

1-D scattering

$V=0 \Rightarrow \phi = \sin(kx) = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$

$\psi(x) = \frac{1}{2i}\left(\text{outgoing} - e^{-ikx}\right)$

$x > a$



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outgoing $= e^{ikx+2i\delta_k}$

δ_k is phase shift

$\psi(x) = \underbrace{\phi(x) + \psi_s(x)}$.

same incoming but phase shifted outgoing

$\Rightarrow \frac{1}{2i}\left(e^{ikx+2i\delta_x} - \cancel{e^{-ikx}}\right) = \psi_s + \frac{1}{2i}\left(e^{ikx}\right)$

$\Rightarrow \psi_s(x) = \frac{1}{2i} e^{ikx}\left(e^{2i\delta_k} - 1\right)$

in 3-D

$$\psi(\vec{r}) = \frac{\sqrt{4\pi}}{k}\sum_{l=0}^{\infty}\sqrt{2l+1}\,(i)^l\, Y_{l0}\,\frac{1}{2ir}\Big(\underset{\text{outgoing}}{e^{i(kr-\frac{l\pi}{2})}} - \underset{\text{incoming}}{e^{-i(kr-\frac{l\pi}{2})}}\Big)$$

$$+ \; f_k(\theta)\frac{e^{ikr}}{r}$$

$$= \frac{\sqrt{4\pi}}{k}\sum_{l=0}^{\infty}\sqrt{2l+1}\,(i)^l\, Y_{l0}\,\frac{1}{2ir}\Big(\underset{\text{~~total~~ outgoing}}{e^{i(kr-\frac{l\pi}{2}+2\delta_l)}} - \underset{\text{will remain}}{e^{-i(kr-\frac{l\pi}{2})}}\Big)$$

$f_k(\theta,\phi) = f_k(\theta)$ as the incoming is along z-axis with no ϕ dep.

$$\Rightarrow \boxed{f_k(\theta) = \frac{\sqrt{4\pi}}{k}\sum_{l=0}^{\infty}\sqrt{2l+1}\; Y_{l,0}(\theta)\, e^{i\delta_l}\sin(\delta_l)}$$

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$$\sigma = \int |f_k(\theta)|^2 \, d\Omega = \frac{4\pi}{k^2} \sum_{l,l'} \sqrt{(2l+1)(2l'+1)}\, e^{-i\delta_l} \sin(\delta_l) . e^{i\delta_{l'}} \sin(\delta_{l'}) \int d\Omega \, Y_{l0}^*(\theta) Y_{l'0}(\theta)$$

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2(\delta_l)$$

$f_k(\theta \neq 0)$ is forward scattering like for $\theta = 0$.

$$Y_{l,0}(\theta = 0) = \sqrt{\frac{2l+1}{4\pi}}$$

$$f_k(\theta = 0) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) . e^{i\delta_l} \sin(\delta_l)$$

$$im(f_k(\theta = 0)) = \frac{1}{k} \frac{k^2}{4\pi} \sigma$$

$$\Rightarrow \boxed{\sigma = \frac{4\pi}{k} im(f(0))} \leftarrow \text{Optical theorem}$$

* The whole scattering is whatever was incident minus the shadow of the object.

$$\psi(x) = (A_l j_l(kr) + B_l \eta_l(kr)) Y_{lm}(\theta)$$

$$\rightarrow \left(\frac{A_l}{kr} \sin(kr - \frac{l\pi}{2}) - \frac{B_l}{kr} \cos(kr - \frac{l\pi}{2}) \right) Y_{l0}(\theta)$$

$$\tan(\delta_l) = -\frac{B_l}{A_l} \leftarrow \text{let}$$

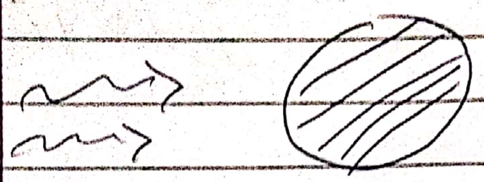
$$\Rightarrow \psi \sim \frac{A}{kr \cos(\delta_l)} \left(\sin(kr - \frac{l\pi}{2}) \cos(\delta_l) + \cos(kr - \frac{l\pi}{2}) \sin(\delta_l) \right) Y_{l0}(\theta)$$

$$\sim \frac{A}{kr} \frac{1}{\cos\delta_l} \sin(kr - \frac{l\pi}{2} + \delta_l)$$

$$e^{i(kr + \frac{l\pi}{2} + 2\delta_l)} - e^{-i(kr \mp \frac{l\pi}{2})}$$

for away

* Example :-



$$V = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

$$\Psi = \sum_{l=0} \left(A_l J_l(kr) + B_l \eta_l(kr) \right) P_l(\cos\theta)$$

$$\Rightarrow \psi(a) = 0$$

$$\Rightarrow A_l j_l(ka) + B_l \eta_l(ka) = 0.$$

$$\Rightarrow \tan(\delta_l) = \frac{J_l(ka)}{\eta_l(ka)} = \frac{B_l}{A_l}$$

$$\sin^2(\delta_l) = \frac{J_l^2(ka)}{J_l^2(ka) + \eta_l^2(ka)}$$

$$\Rightarrow \sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2(\delta_l) = \boxed{\frac{4\pi}{k^2} \sum (2l+1) \frac{J_l^2}{J_l^2 + \eta_l^2}}$$

$ka << 1 \rightarrow$ small energy

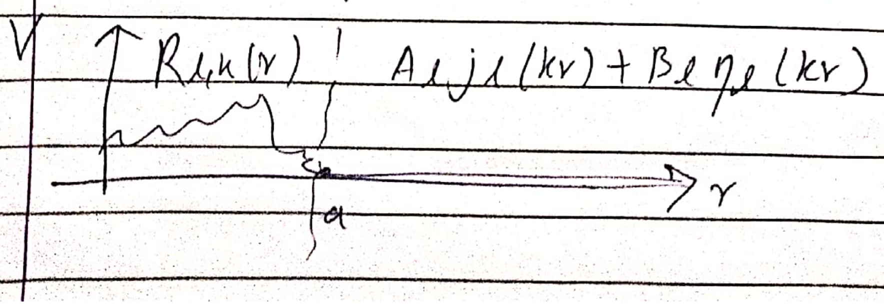
$$\sigma \approx 4\pi a^2$$

* General phase shift computation :- spher $V(r)$

$$E = \frac{\hbar^2 k^2}{2m}, \quad \left(\frac{-\hbar^2}{2m} \frac{d^2}{dr^2} + V(r) + \frac{\hbar^2 l(l+1)}{2m \cdot r^2} \right) u_{E,l,k}(r) = \frac{\hbar k^2}{2m} u_{l,k}(r).$$

$$R_{l,k}(r) = \frac{u_{l,k}(r)}{r}.$$

$$V(r) = \begin{cases} V(r) & r < a \\ 0 & r > a \end{cases}$$



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Continuity at $r=a$ $\Rightarrow$ and differentiability.

$$R_{l,k}(a) = A_l j_l(ka) + B_l \eta_l(ka)$$

$$\Rightarrow a R_{l,k}'(a) = ka\left(A_l j_l'(ka) + B_l \eta_l'(ka)\right)$$

$$\frac{R'(a)}{R(a)} = ka \frac{J_l'(ka) - \tan\delta_l \,\eta_l'(ka)}{J_l(ka) - \tan\delta_l \,\eta_l(ka)}$$

$$\Rightarrow \boxed{\tan(\delta_l) = \frac{J_l'(ka) - \frac{R_l'(a)}{k R_l(a)} J_l(ka)}{\eta_l'(ka) - \frac{R_l'(a)}{k R_l(a)} \eta_l(ka)}}$$

Once you solve for $R_{l,k}$, you done.

Now let's find these R's.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = E\psi \qquad E = \frac{\hbar k^2}{2m},$$

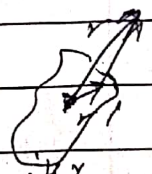
$$\Rightarrow (\nabla^2 + k^2)\psi(\bar{r}) = V(r)\psi(r)$$

$$\Rightarrow \psi(\bar{r}) = \psi_0(\bar{r}) + \int d^3\vec{r}\,' \, G(r-r') \cdot U(r')\psi(r')$$

$$(\nabla^2 + k^2)\psi_0 = 0$$

$$G = -\frac{1}{4\pi} \frac{e^{ik|r-r'|}}{|r-r'|} \qquad |r-r'| \approx r, \text{ -- denom } \cancel{\text{numerator}}$$

$$|r-r'| \approx r - n \cdot r'$$



$$\psi(\vec{r}) = e^{ikz} + \left(-\frac{1}{4\pi}\int d^3\vec{r}\, e^{-ikn\cdot r} V(r') \cdot \psi(r')\right)\frac{e^{ikr}}{r}$$

$$\Rightarrow \boxed{f_k(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\, e^{-ikn\cdot r'} V(r')\psi(r')}$$

Born-approximation

$$\psi(r) = e^{ikr} + \int d^3r' \, G(r-r') \, U(r') \, \psi(r')$$

iteratively replace $r \to r'$, $r' \to r''$, $r'' \to r'''$

$$\psi(r') = e^{ikr'} + \int d^3r'' \, G(r'-r'') \, V(r'') \, \psi(r'')$$

$$\Rightarrow \quad \psi(r) = e^{ik_i r} + \int d^3r' \, G(r-r') \, V(r') e^{ik_i r'} + \int d^3r' \, G(r-r') \, V(r') \int d^3r'' \, G(r'-r'') \, V(r'') e^{ik_i r''}$$

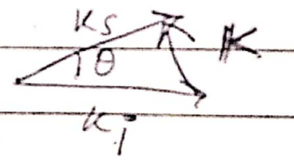
$$+ \int d^3r' \, G(r-r') \, V(r') \int d^3r'' \, G(r'-r'') \, V(r'') \int d^3r''' \, G(r''-r''') \, V(r''') \psi(r''')$$

$$\psi = \underbrace{\underbrace{e^{ik_i r} + \int GVe^{ik_i r}}_{\text{first}} + \int GV \int GVe^{ik_i r}}_{\text{second approximation}} + \iint \cdots + \cdots$$

If we use First-Born approximation,

$$\psi^{Born}(r) = e^{ik_i r} - \frac{1}{4\pi}\left(\int d^3r' \, e^{-i(k_s-k_i)r'} \, V(r')\right)\frac{e^{ikr'}}{r}$$

let $\mathbf{K} = k_s - k_i$



$$|\mathbf{K}| = 2k\sin\frac{\theta}{2}$$

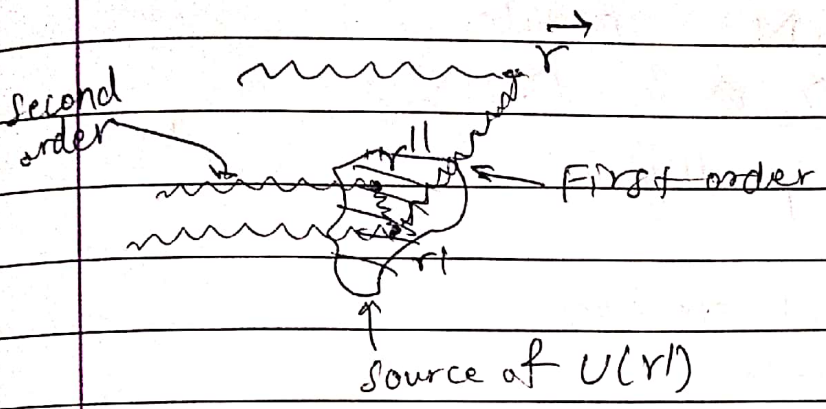
$$f_k(\theta) = -\frac{m}{2\pi\hbar^2}\int_0^\infty 2\pi r^2 dr \int_{-1}^{1} d(\cos\theta) \, e^{-ikr\cos\theta} \, V(r)$$

$\hat{k} = \hat{z}$

$$\boxed{f_k^{Born}(\theta) = -\frac{2m}{\mathbf{K}\hbar^2}\int_0^\infty dr \, r \, V(r)\sin(\mathbf{K}r)}$$

$$\mathbf{K} = 2k\sin\left(\frac{\theta}{2}\right)$$

Zeroth order will be the incident plane directly hitting target at $\vec{r}$.



* Example Yukawa-potential :— just coulomb with μ as mass of photon.

$$V(r) = V_0 \frac{e^{-\mu r}}{r}, \quad V_0 = ZZ'e^2.$$

$$\Rightarrow f_k(\theta) = -\frac{2m}{k\hbar^2}\int_0^\infty dr \, . \, r \, . \, V_0 \frac{e^{-\mu r}}{r} \sin(kr)$$

$$= -\frac{2mV_0}{\hbar^2(k^2+\mu^2)}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\hbar^2}\right)^2 . \frac{1}{(2k^2(1-\cos\theta)+\mu^2)^2}$$

$$\mu \to 0, \quad d\Omega = d(\cos\theta)\, d\phi.$$

$$\Rightarrow \sigma = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{4\pi}{4k^2\mu^2+\mu^4} \longrightarrow \infty.$$

due to long range force.

$$\frac{d\sigma}{d\Omega} = \frac{(2m)^2(ZZ'e^2)^2}{16(\hbar k)^4 . (\sin^4(\frac{\theta}{2}))}$$