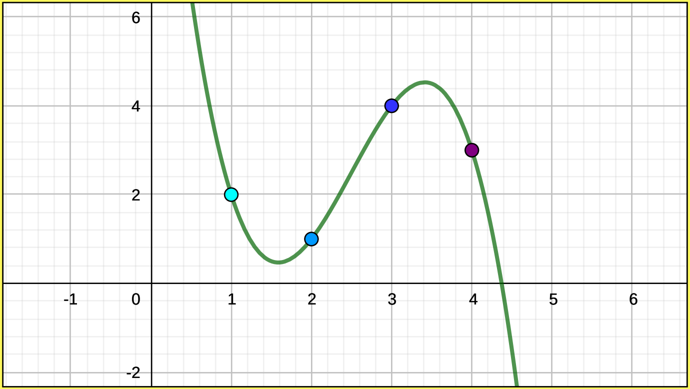

Lagrange Interpolation

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- Fit (or interpolate) a curve through a given set of points.
- Polynomials $p(x)$ passing through $(x_1, y_1), \ldots, (x_k, y_k)$.

Aim: Search for $p(x)$ with the lowest degree such that $p(x_i) = y_i$[1]

$k = 1$: $p(x) = y_1$ (degree 0)

graph of $p(x)$ is a horizontal line

$k = 2$: $(x, p(x))$ defines a line through (x_1, y_1) and (x_2, y_2)

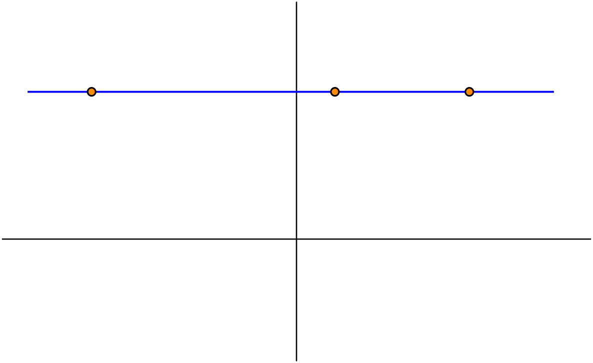
$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad \Rightarrow \quad y = y_1 \frac{x - x_2}{x_1 - x_2} + y_2 \frac{x - x_1}{x_2 - x_1}$$

Set $p(x) = y$ (degree at most 1)

[1] It is assumed that the x_i's are distinct.

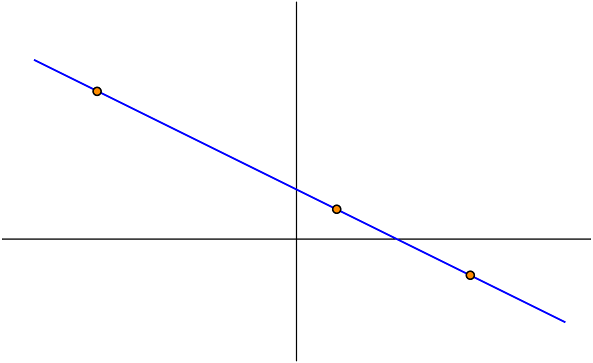
$k = 3$: If y_i's are equal, then $p(x) = y_1$ (degree 0)

graph of $p(x)$ is a horizontal line



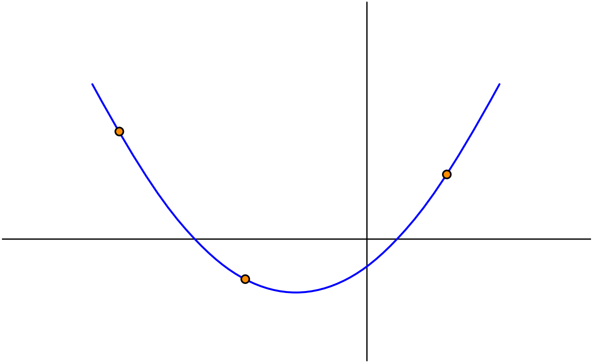
$k = 3$: If (x_i, y_i)'s are collinear, then $p(x)$ is of degree 1

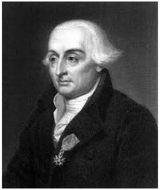
graph of $p(x)$ is a line



$k = 3$: In general $p(x)$ is of degree 2

graph of $p(x)$ is a parabola





Setup of the problem: Given $a_0 < a_1 < \cdots < a_n$ and $b_0 \ldots, b_n$, find

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

such that $p(a_i) = b_i$.

First Proof: The equations $p(a_i) = b_i$ can be written as

$$c_0 + c_1 a_i + c_2 a_i^2 + \cdots + c_n a_i^n = b_i.$$

Rewrite these as

$$\begin{pmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & & \ddots & & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}$$

The $(n+1)\times(n+1)$ matrix[2] has non-zero determinant as the a_j's are distinct. There are unique scalars c_j's satisfying our need. $\square$

Consider the evaluation map

$$\mathrm{ev}: P_n(\mathbb{R}) \to \mathbb{R}^{n+1}, \quad q(x) \mapsto (q(a_0), q(a_1), \ldots, q(a_n)).$$

This is a linear map between vector spaces of dimension $n+1$.

By Rank-Nullity Theorem, ev is one-to-one if and only if it is onto.

Second Proof (constructive): We show that ev is onto. This ensures that there is a unique polynomial of degree at most n for our purposes. Suppose we have $p_j(x) \in P_n(\mathbb{R})$ such that

$$p_j(a_i) = \delta_{ij}.$$

[2]It is often called a Vandermonde matrix.

The polynomial

$$p(x) = b_0 p_0(x) + \cdots + b_n p_n(x)$$

is the required polynomial. To construct p_j, consider

$$p_j(x) = \frac{(x - a_0)\cdots(x - a_{j-1})(x - a_{j+1})\cdots(x - a_n)}{(a_j - a_0)\cdots(a_j - a_{j-1})(a_j - a_{j+1})\cdots(a_j - a_n)}$$

Each p_j has degree n. Thus, degree of p is at most n. $\square$

The p_j's form a basis of $P_n(\mathbb{R})$.

The cover graph is that of the cubic satisfying

$$p(1) = 2,\ p(2) = 1,\ p(3) = 4,\ p(4) = 3.$$

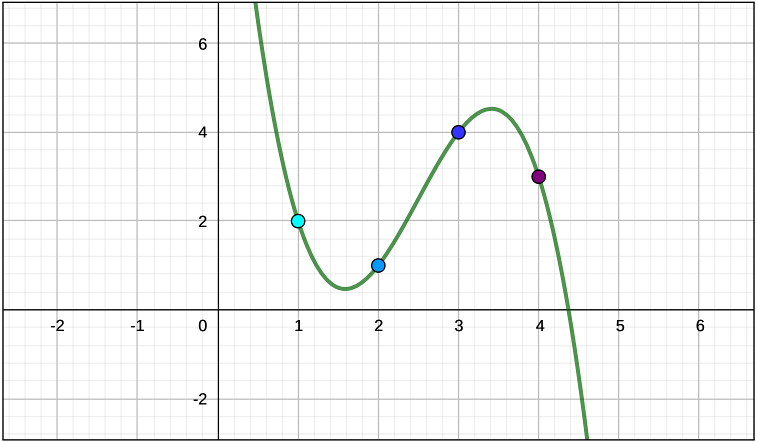


Figure: The interpolating cubic $p(x) = -\frac{4}{3}x^3 + 10x^2 - \frac{65}{3}x + 15$

Third Proof (existential): We show that ev is one-to-one. This ensures that there is a unique polynomial of degree at most n for our purposes. If $\mathrm{ev}(p) = \mathrm{ev}(q)$, then $p - q$ has $n+1$ distinct roots $(a_0, \ldots, a_n)$ while its degree is at most n. This is possible only when $p = q$. $\square$