

## Elements of Vector Calculus : Line and Surface Integrals

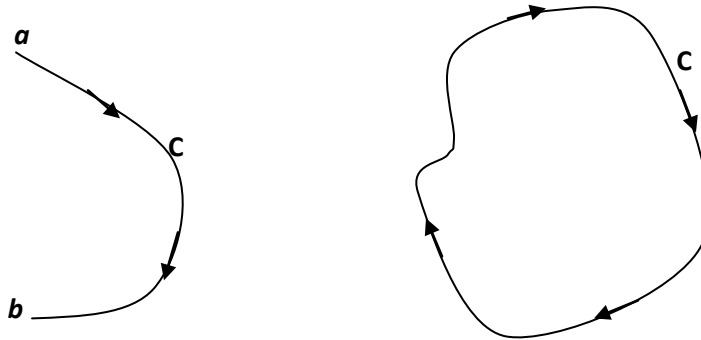
Lecture 2: Electromagnetic Theory

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In this lecture we will talk about special functions of vector fields and about operators which act on vector fields. We begin with the concept of a line integral.

We are familiar with normal integrals, which can be regarded as a sum. Let us consider a force acting on a particle in one dimension. The work done in moving the particle from a to b is given by the integral  $\int_a^b F(x)dx$ . The particle in this case can move parallel to the x-axis. How do we generalize this to higher dimension? Suppose, the force  $F$  acts on a particle taking it along a curve in two or three dimensions.

Work done is given by  $\int \vec{F} \cdot d\vec{l}$ , where the integral is taken along the path in which the particle moves, the path  $C$  can be either an open path or a closed path which terminates at its starting point.



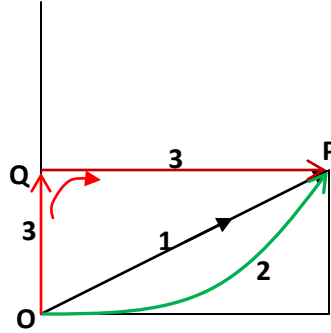
Unlike the case of normal integrals that we have learnt in school, the integral here does not depend only on the end-points but they may depend on the details of the path traversed. There are of course exceptions to this. For instance, the work done by gravitational force or electrostatic force depends only on the end points. Such forces are called “conservative forces”.

Along the curve one can always define a single parameter. For instance, taking time as the parameter,

we can express the work done along the curve as  $\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \frac{d\vec{r}}{dt} dt = \int_C \vec{F} \cdot \vec{v} dt$ , where  $\vec{v}$  is the velocity of the particle at time  $t$ .

### Example 1 :

We will illustrate calculation of the work by computing the work done by a force given by a two dimensional force  $\vec{F}(x, y) = (x^2 - y^2)\hat{i} + 2xy\hat{j}$  along three different paths connecting the origin  $O(0,0)$  to the point  $P(2,1)$ .



- (i) The path 1 is a straight line connecting O with P. The equation to the path is  $y=x/2$ .

$$\int \vec{F} \cdot d\vec{l} = \int F_x dx + \int F_y dy = \int_0^2 (x^2 - y^2) dx + \int_0^1 2xy dy$$

Substituting  $y=x/2$  in the first integral and  $x=2y$  in the second, we get,

$$\int \vec{F} \cdot d\vec{l} = \int_0^2 \frac{3x^2}{4} dx + \int_0^1 4y dy = 2 + \frac{4}{3} = \frac{10}{3}$$

- (ii) The second path is a parabola given by the equation  $y = \frac{x^2}{4}$ . The calculation proceeds exactly as in case (i) and we have,

$$\int \vec{F} \cdot d\vec{l} = \int_0^2 (x^2 - \frac{x^4}{16}) dx + \int_0^1 2 \cdot (2\sqrt{y}) y dy = \frac{34}{15} + \frac{8}{5} = \frac{58}{15}$$

- (iii) The third path is in two segments, OQ which connects  $(0,0)$  to  $(0,1)$  and then along QP connecting  $(0,1)$  to  $(2,1)$ . Note that in the first case,  $x=0$  so that  $\vec{F} = -y^2\hat{i}$ . Thus the integral reduces to  $\int F_x dx$ . However, along this path  $dx=0$ . Thus the work done along this path is zero.

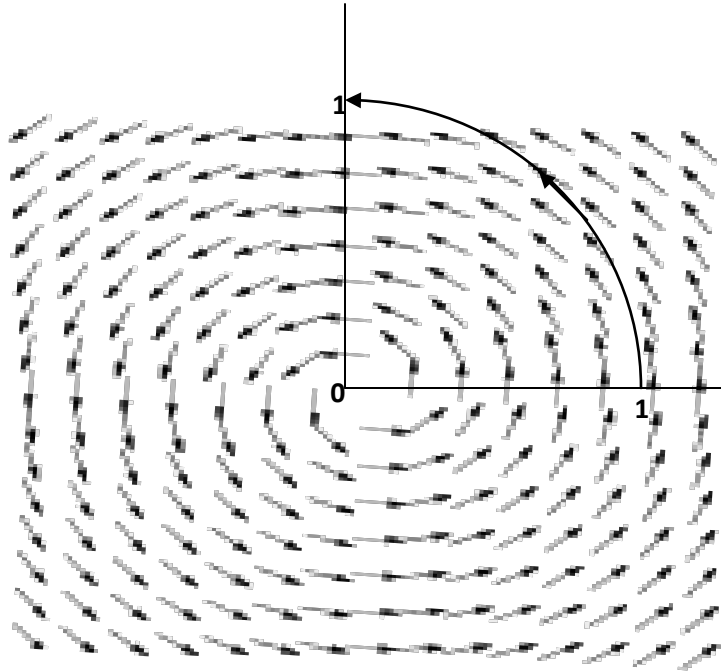
From Q to P,  $dy$  is zero, and  $y=1$ , thus for this path  $\vec{F} = (x^2 - 1)\hat{i}$ ,  $\int \vec{F} \cdot d\vec{l} = \int_0^2 F_x dx = \int_0^2 (x^2 - 1) dx = \frac{2}{3}$

We can see from the above example that the **integral is path dependent**.

### Example 2 :

As a second example, consider a force  $\vec{F} = -y\hat{i} + x\hat{j}$  along the first quadrant of a circle  $x^2 + y^2 = 1$  taken in anti-clockwise fashion.

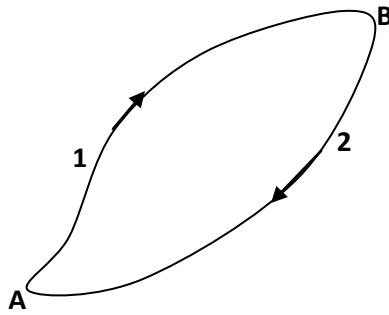
Look at the sketch of the path. In this quadrant  $x$  and  $y$  are both positive. You could sketch the vector field and see that vectors at each point is "somewhat" directed upward, implying that the direction of the tangent to the path and the force makes an acute angle. Thus we expect  $\vec{F} \cdot d\vec{l}$  to be positive.



Since the particle moves on a circular path of unit radius , we can parameterize the position by  $x = \cos \theta, y = \sin \theta$ , so that  $dx = -\sin \theta d\theta, dy = \cos \theta d\theta$ . Thus

$$\begin{aligned} \int_c \vec{F} \cdot d\vec{l} &= \int_c (-ydx + xdy) \\ &= \int_0^{\frac{\pi}{2}} (\sin^2 \theta + \cos^2 \theta) d\theta = \frac{\pi}{2} \end{aligned}$$

In general, the line integral depends on the path of integration and not just on its end points. In cases where the value of the integral depends on the path of integration, the force is called “non-conservative”. If  $\int \vec{F} \cdot d\vec{l}$  depends only on its end points, the force is said to be “conservative”. In this case, it can be further seen that  $\oint \vec{F} \cdot d\vec{l} = 0$ , where the circle symbol over the integration sign means an integral over a closed contour. This becomes obvious if we look at the following. Suppose, we are computing the line integral over a closed path which goes from A to B along a path 1 but return from B to the initial point A through path 2.



Since the integral depends only on the end points we can write for the path 1 :  $\int_A^B \vec{F} \cdot d\vec{l} = \phi(B) - \phi(A)$ , where  $\frac{\partial \phi}{\partial z}$  is the value of the indefinite integral. In the returning path 2 : we have  $\int_B^A \vec{F} \cdot d\vec{l} = \phi(A) - \phi(B)$ . Thus over the closed contour, we have,  $\oint \vec{F} \cdot d\vec{l} = 0$ .

We will spend some time on conservative field. This is because, we will be dealing with electrostatic field which is a conservative field.

Suppose, we have a force field which can be expressed as a gradient of some scalar function, i.e.  $\vec{F} = \nabla \phi$ . Note that not all forces can be written this way. But in those cases, where it can be so written, we have,

$$\begin{aligned} \int_A^B \vec{F} \cdot d\vec{l} &= \int_A^B \left( \hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z} \right) (\hat{i} dx + \hat{j} dy + \hat{k} dz) \\ &= \int_A^B \left( \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = \int_A^B d\phi = \phi(B) - \phi(A) \end{aligned}$$

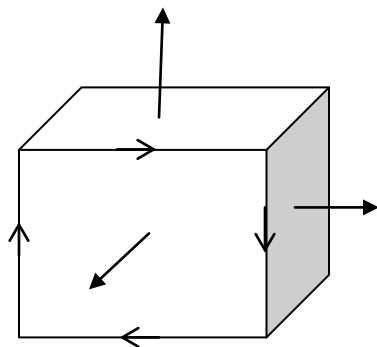
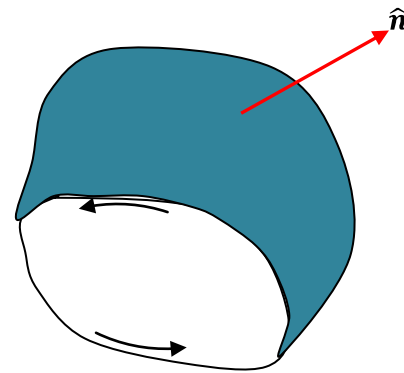
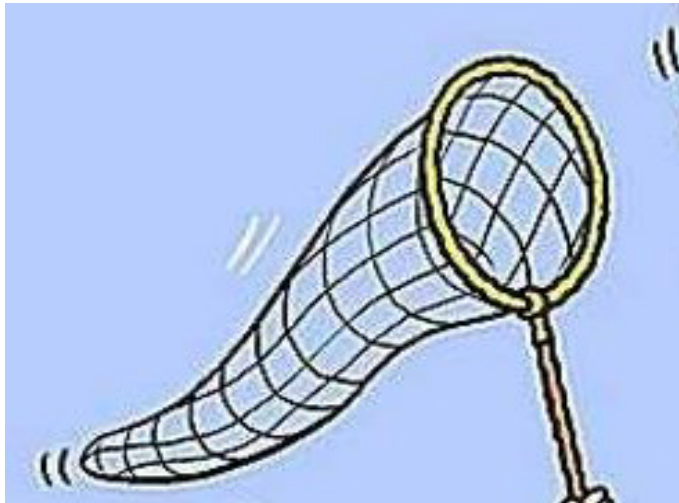
So if a force field is expressible as a gradient of a scalar function, then the line integral would depend only on end points and not depend on path. Such a field is therefore conservative. We make a statement here that a conservative field can be expressed as a gradient of a scalar field. Generally, in Physics, we define the force as negative gradient of a scalar function  $\phi$ , i.e.,  $\vec{F} = -\nabla \phi$ . We call such a function as “potential”. (A mathematician would call  $\phi$  to be a potential, whether we have the negative sign or not; in Physics, the negative gradient of potential is the force).

### Surface Integral :

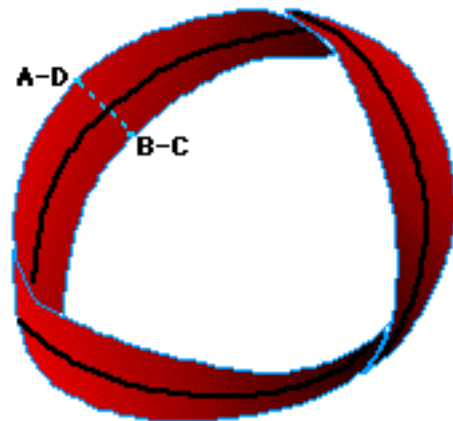
We define surface integral as an integral of a vector function over a surface. It is given by  $\int_S \vec{F} \cdot \hat{n} dS$ , where  $\hat{n}$  is the “outward” normal to the surface over which the integral is taken. We will shortly define the meaning of the phrase “outward”. The quantity  $\int_S \vec{F} \cdot \hat{n} dS$  is also known as the flux of the vector field through the surface S.

Let us consider an open surface. An open surface is bounded by a curve, for instance, a cup or a butterfly net. In both cases, there is a rim which is the bounding curve of the surface. The direction of the outward normal is defined by the “right handed rule”—if the bounding curve of a surface is traversed in the direction of rotation of a right handed screw, the direction in which the head of the screw moves is the direction of the outward normal. For being able to define a surface integral, we need what are known as “two sided surfaces”. Most of the common surfaces that we meet are two sided, they have an inside surface and an outside surface, separated by an edge or a rim. Take for example, the butterfly net shown here. The circular rim separates the

inside surface from the outside surface. One cannot go from a point on the outside surface to the inside surface or vice versa without crossing the edge.



**Defining an outward normal**



**A Möbius strip**

There are one-sided surfaces too for which one cannot define a surface integral. One such example is a Möbius Strip. A Möbius strip is easily constructed by taking a strip of paper and sticking the short edges after giving a half-turn so that the edge on one side is glued to the other side. In such a case one can seamlessly traverse the entire surface without having to cross an edge. Note that in this case the normal at a point cannot be uniquely defined.

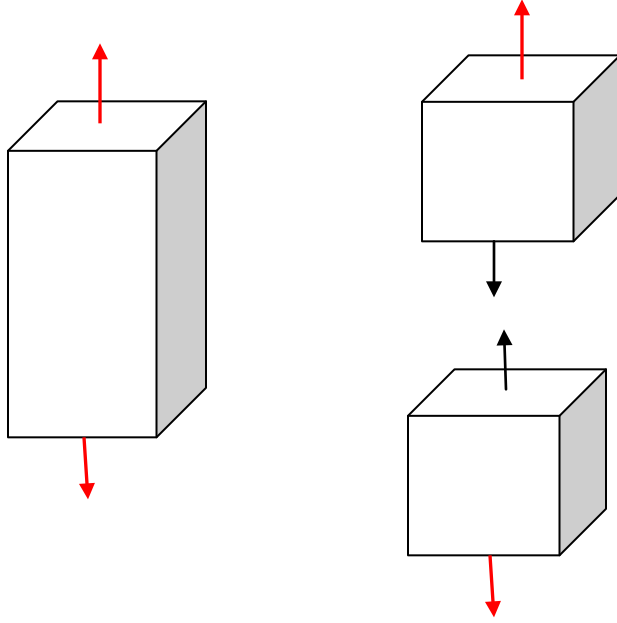
### **Divergence of a Vector :**

We define divergence of a vector field at a point as the limit of surface integral to the volume enclosed by such a surface as the volume enclosed goes to zero. Thus

$$\text{div } \vec{F} = \lim_{\Delta V \rightarrow 0} \frac{\int_S \vec{F} \cdot \hat{n} dS}{\Delta V}$$

This is a point relationship, i.e. a relationship defined at every point in the region where a vector field exists. We imagine an infinitely small volume around the point and compute the surface integral of the vector field over the surface defining this volume.

Consider the following.



On the left is a rectangular parallelepiped whose outward normals from the top and the bottom faces are shown. If we imagine the parallelepiped to be sliced into two parts, as shown to the right, while there is no change in the outward normals of the top face of the upper half and the bottom face of the lower half. However at the interface where the section is made, the normals are oppositely directed and the contributions to the surface integral from the two surfaces cancel. The same would be true for the other three pairs of faces of the parallelepiped. Thus, if we have a macroscopic volume defined by a bounding surface, we could split the volume into a large number of small volumes and the surface integrals of all volumes which are inside cancel out leaving us with contribution from the outside surface only. Thus the flux over a closed surface can be written as a sum over the surfaces of elemental volumes which make up the total volume.

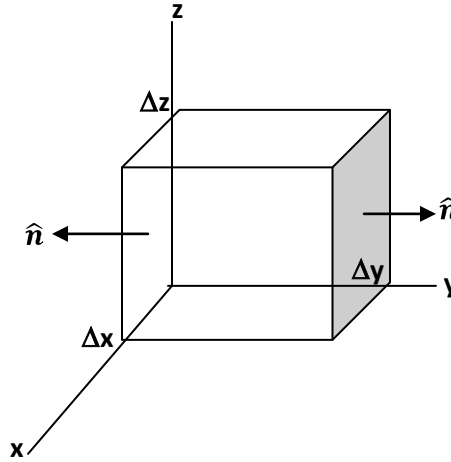
$$\int_S \vec{F} \cdot \hat{n} dS = \sum \int_{\Delta S} \vec{F} \cdot \hat{n} dS = \sum \lim_{\Delta V \rightarrow 0} \left( \frac{1}{\Delta V} \int_{\Delta S} \vec{F} \cdot \hat{n} dS \right) \Delta V$$

The quantity in the bracket was defined as the divergence of  $\vec{F}$  at the location of the elemental volume. Thus the sum is nothing but an integral of the divergence over the volume of the system. This gives us the **Divergence Theorem** :

$$\boxed{\int_S \vec{F} \cdot \hat{n} dS = \int_V \text{div } \vec{F} dV}$$

### Expression for Divergence in Cartesian Coordinates :

Since divergence is meaningful in the limit of the volume going to zero, we can calculate it by taking an infinitesimally small parallelepiped of dimensions  $\Delta x \times \Delta y \times \Delta z$  which is oriented along the axes of the coordinate system.



Consider the flux from two opposite faces shown in the figure. Note that the left face is at some value of  $y$  and the normal is directed along  $-\hat{j}$  direction. Thus the flux from this face is  $-F_y(y)\Delta x\Delta z$ . The opposite face is at  $y+\Delta y$  and the normal to this face is along  $+\hat{j}$  direction. The flux from the face is  $+F_y(y+\Delta y)\Delta x\Delta z$ . The net outward flux from these two faces can be obtained by subtracting the former expression from the latter. Retaining only first order term in a Taylor series expansion of  $F_y$ , we have

$$F_y(y+\Delta y) - F_y(y) = \frac{\partial F_y}{\partial y} \Delta y$$

Thus the net flux from the two faces is  $\frac{\partial F_y}{\partial y} \Delta x \Delta y \Delta z = \frac{\partial F_y}{\partial y} \Delta V$

By symmetry, one can get expressions for the contributions from the other two pairs of faces. The net outward flux from the six faces is thus,

$$\vec{F} \cdot \hat{n} dS = \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV$$

Summing over, we have

$$\int_S \vec{F} \cdot \hat{n} dS = \int_V \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) dV$$

However, the left hand side of this equation is identically equal to  $\int_V \text{div } \vec{F} dV$ . Since the volume we have taken is arbitrary, we have

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

This is the Cartesian expression for the divergence of a vector field  $\vec{F}$ .

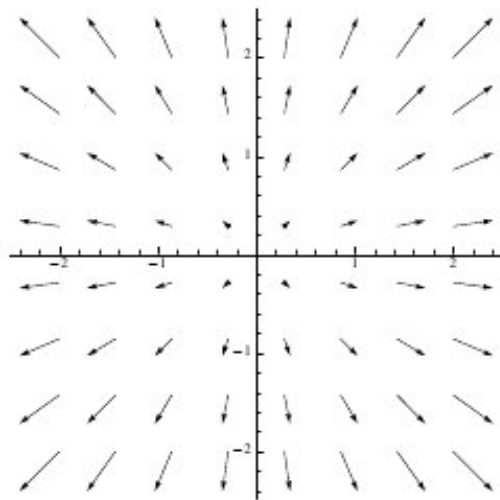
We recall that the gradient operator  $\vec{\nabla}$  was given by

$$\vec{\nabla} = \hat{i}dx + \hat{j}dy + \hat{k}dz$$

Using this, one can write

$$\text{div } \vec{F} = \vec{\nabla} \cdot \vec{F}$$

Physically, divergence, as the name suggests, shows how much does the field at a point diverges from its value at that point to the neighbouring region. In the figure below, the vector field  $\vec{F} = \hat{i}x + \hat{j}y$ , which has a positive divergence has been plotted in the x-y plane in the range  $x: [-2, +2]$   $y: [-2, +2]$ . It can be seen that the fields spread outward from the origin. A field with negative divergence would instead converge.





## Elements of Vector Calculus : Line and Surface Integrals

Lecture 2: Electromagnetic Theory

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### Tutorial :

1. Calculate the line integral of the vector field  $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$  along the following two paths joining the origin to the point P(1,1,1). (a) Along a straight line joining the origin to P, (ii) along a path parameterized by  $x = t, y = t^2, z = t^3$ .
2. Calculate the line integral of  $\vec{F} = (x^2 + y^2)\hat{i} + xy\hat{j}$  over a quarter circle in the upper half plane along the path connecting (3,0) to (0,3). What would be the result if the path was taken along the same circular path but in the reverse direction?
3. Calculate the line integral of the scalar function  $F(x, y) = xy^3$  over the right half of the semi-circle  $x^2 + y^2 = 4$  along the counterclockwise direction from (0,-2) to (0,+2).
4. A two dimensional force field is given by  $(2xy^2 + 3x^2)\hat{i} + (2x^2y + 4y^3)\hat{j}$ . Find a potential function for this force field.
5. Calculate the flux of a constant vector field  $V\hat{k}$ , through the curved surface of a hemisphere of radius R whose base is in the x-y plane.
6. Calculate the divergence of the position vector  $\vec{r}$ .
7. Calculate the divergence of  $\vec{F} = -z\hat{i} + xz\hat{j} + zy\hat{k}$ .
8. Calculate the divergence of  $\vec{F} = xz\hat{i} - e^z\hat{j} + \cos z\hat{k}$ .

### Solution to Tutorial Problems :

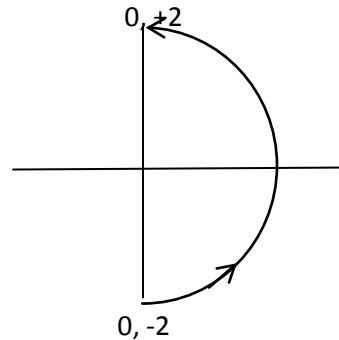
- (a) Along the straight line path, we have,  $x=y=z$ , so that  $\int \vec{F} \cdot d\vec{l} = \int (F_x dx + F_y dy + F_z dz)$  can be expressed as  $\int_0^1 4x^2 dx - \int_0^1 y^2 dy + \int_0^1 z^2 dz = \frac{4}{3}$ .

(b) Substituting the parameterized form, we have,  $dx = dt, dy = 2t dt, dz = 3t^2 dt$ . The line integral is  $\int_0^1 4t^4 dt - \int_0^1 2t^5 dt + \int_0^1 3t^7 dt = \frac{101}{120}$
- The path integral is most conveniently done by parameterizing  $x = 3 \cos \theta, y = 3 \sin \theta$ , so that  $dx = -3 \sin \theta, dy = 3 \cos \theta$ . The integral is then given by  $-27 \int_0^{\frac{\pi}{2}} \sin \theta d\theta + 27 \int_0^{\frac{\pi}{2}} \cos^2 \theta \sin \theta d\theta = -18$ . If the curve is traversed in the opposite direction, the line integral would become +18.
- Since the path is along a circle of radius 2, we can parameterize by  $x = 2 \cos \theta, y = 2 \sin \theta$

$$dl = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = 2d\theta$$

The line integral is

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2^5 \cos \theta \sin^3 \theta d\theta \\ &= 32 \frac{\sin^4 \theta}{4} \Big|_{-\pi/2}^{\pi/2} = 16 \end{aligned}$$

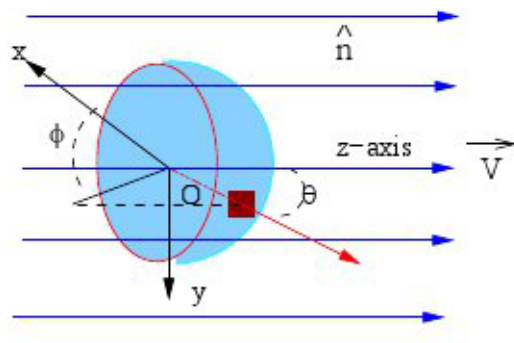


- Let  $\vec{F} = \nabla \phi$ , we then have

$$\begin{aligned} F_x &= \frac{\partial \phi}{\partial x} = 2xy^2 + 3x^2 \Rightarrow \phi = x^2 y^2 + x^3 + C_1(y) \\ F_y &= \frac{\partial \phi}{\partial y} = 2x^2 y + 4y^3 \Rightarrow \phi = x^2 y^2 + y^3 + C_2(x) \end{aligned}$$

The potential function is therefore given by  $\phi = x^2 y^2 + x^3 + y^3 + C$ , where C is an arbitrary constant.

- In this case it is convenient to use the spherical polar coordinates, the surface element is  $dS = R^2 \sin \theta d\theta d\phi$  and it is along the radial direction, which makes an angle  $\theta$  with the z-axis. Thus  $\int_S \vec{F} \cdot d\vec{S} = |V| R^2 \int_0^{2\pi} d\phi \int_0^{\frac{\pi}{2}} \sin \theta \cos \theta d\theta = |V| R^2 \pi$



6.  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ . Thus  $\text{div } \vec{r} = 1 + 1 + 1 = 3$ . (This is an important relation which we use frequently.)
7.  $\nabla \cdot \vec{F} = 0 + 0 + y = y$ .
8.  $\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + z = z + 0 - \sin z = z - \sin z$

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### Self Assessment Quiz

1. Find the line integral of the vector field  $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$  along a path  $y = \sqrt{x}$  from (0,0) to (1,1).

- Calculate the line integral of the vector field  $\vec{F} = 2x\hat{i} + 3y\hat{j} + 4z\hat{k}$  along the following two paths joining the origin to the point P(1,1,1). (a) Along a straight line joining the origin to P, (ii) along a path parameterized by  $x = t, y = t^2, z = t^3$ .
- From the result of Problem 2, can you conclude that the force is conservative? If so, determine a potential function for this vector field.
- A potential function is given by  $\phi(x, y, z) = \frac{x^3}{a^3} - \frac{2xyz}{b^3} + \frac{y^2z}{c^3}$ , where  $a, b$  and  $c$  are constants. Find the force field.
- A conservative force field is given by  $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ . Calculate the work done by the force in taking a particle from the origin to the point (1,1,2).
- A force field is given by  $\vec{F} = -\frac{\vec{r}}{r^3}$ , where  $r$  is the distance of the point from the origin. Calculate the divergence at a point other than the origin.

### Solutions to Self Assessment Quiz

- Substituting  $y = \sqrt{x}, x = y^2$ , the line integral can be expressed as  $\int_0^1 (x^2 + x)dx - 2 \int_0^1 y^3 dy = \frac{1}{3}$ .
- (i) For the straight line path  $x=y=z$ ,  $\int \vec{F} \cdot d\vec{l} = \int (F_x dx + F_y dy + F_z dz) = \int_0^1 2x dx + \int_0^1 3y dy + \int_0^1 4z dz = 9/2$ . (ii) For the second path  $dx = dt, dy = 2t dt, dz = 3t^2 dt$ , so that  $\int \vec{F} \cdot d\vec{l} = \int_0^1 (2t + 6t^3 + 12t^5) dt = 9/2$ .
- Just from the fact that line integrals along two different paths give the same result, one cannot conclude that the force is conservative. However, in this particular case, the vector field happens to be conservative. Let the potential function be  $\phi, \vec{F} = \nabla\phi$ . Equating components of the force, we get,

$$F_x = \frac{\partial\phi}{\partial x} = 2x \Rightarrow \phi = x^2 + C_1(y, z)$$

$$F_y = \frac{\partial\phi}{\partial y} = 3y \Rightarrow \phi = \frac{3y^2}{2} + C_2(x, z)$$

$$F_z = \frac{\partial\phi}{\partial z} = 4z \Rightarrow \phi = 2z^2 + C_3(x, y)$$

Clearly, the function is given by  $\phi = x^2 + \frac{3y^2}{2} + 2z^2 + C$ , where  $C$  is an arbitrary constant, which can be taken to be zero. The line integral can therefore be written as

$$\int \nabla\phi \cdot d\vec{l} = \int_0^P d\phi = \phi|_{0,0,0}^{1,1,1} = 1 + \frac{3}{2} + 2 = \frac{9}{2}$$

As expected.

- The components of the force are obtained as follows :  $F_x = \frac{\partial\phi}{\partial x} = \frac{3x^2}{a^3} - \frac{2yz}{b^3}$ ,  $F_y = \frac{\partial\phi}{\partial y} = -\frac{2xz}{b^3} + \frac{2yz}{c^3}$ ,  $F_z = \frac{\partial\phi}{\partial z} = -\frac{2xy}{b^3} + \frac{y^2}{c^3}$ .
- Since the force is conservative it can be expressed as a gradient of a scalar potential. Writing  $\vec{F} = \nabla\phi$ , We can show that the potential is given by  $\phi = x^2y + xz^3$ . The work done is  $\phi(1,1,2) - \phi(0,0,0) = 9 - 0 = 9$  units.

6. Given  $\vec{F} = -\frac{\vec{r}}{r^3}$ , where  $r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ . Using chain rule (since the function depends only on the distance  $r$ ),  $\nabla \cdot \vec{F} = -\frac{1}{r^3} \nabla \cdot \vec{r} - \nabla \left( \frac{1}{r^3} \right) \cdot \vec{r} = -\frac{3}{r^3} + \frac{3}{r^4} r = 0$ .