

Solutions to Laplace's Equations

Lecture 14: Electromagnetic Theory

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We have seen that in the region of space where there are no sources of charge, Laplace's equation gives the potential. In this lecture, we will discuss solutions of Laplace's equation subject to some boundary conditions.

Formal Solution in One Dimension

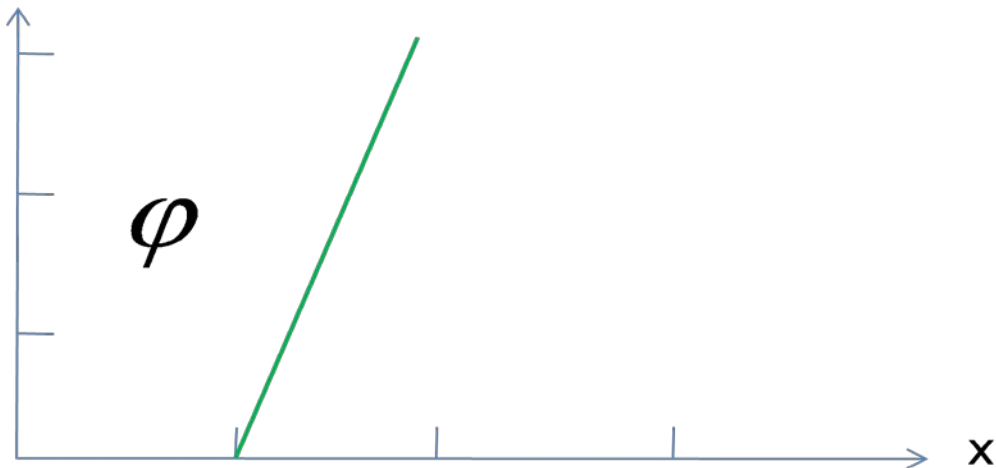
The solution of Laplace's equation in one dimension gives a linear potential,

$$\frac{d^2\varphi}{dx^2} = 0$$

has the solution $\varphi(x) = mx + c$, where m and c are constants. The solution is featureless because it is a monotonically increasing or a decreasing function of x . Further, since the potential at x is the average of the potentials at $x+a$ and $x-a$, it has no minimum or maximum.

$$\varphi(x) = \frac{1}{2}(\varphi(x+a) + \varphi(x-a))$$

This feature, as we will subsequently see, is common to the solutions in higher dimensions as well.



Formal Solution in Two Dimensions

In two dimensions, the Laplace's equation is

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

Since the derivatives are partial, we must have

$$\frac{\partial^2 \varphi}{\partial x^2} = -\frac{\partial^2 \varphi}{\partial y^2} = a$$

where a is a constant. Thus the solution to the equation is

$$\varphi(x, y) = \frac{a}{4}(x^2 - y^2) \quad (1)$$

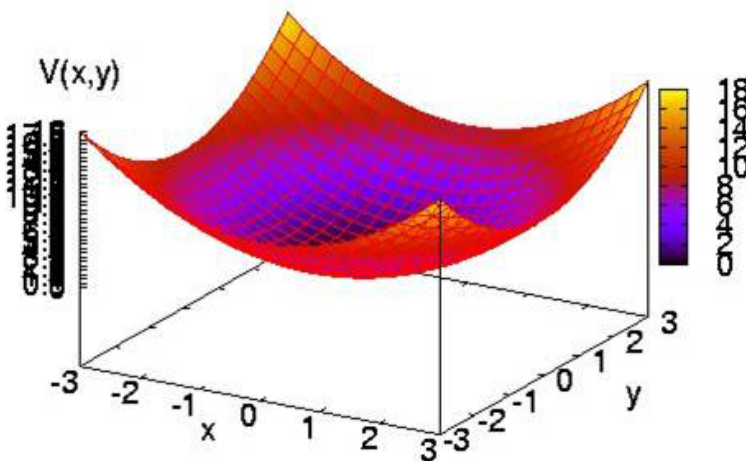
Before looking at this solution let us look at a closely resembling Poisson's equation in 2 dimensions,

$$\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial y^2} = a$$

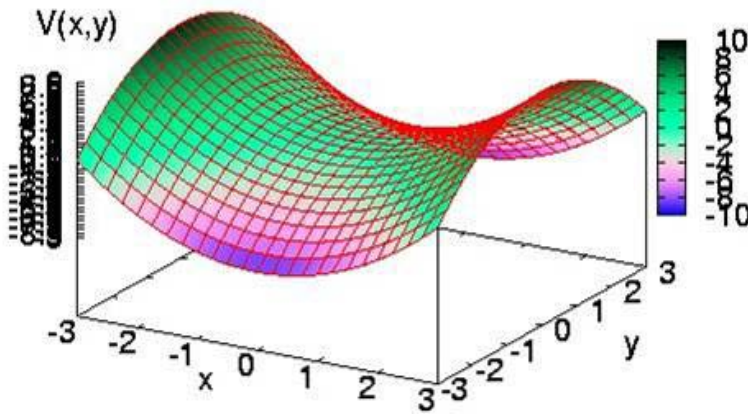
which has the solution,

$$\varphi'(x, y) = \frac{a}{4}(x^2 + y^2) \quad (2)$$

The solution looks like cup having a minimum at $x = 0, y = 0$.



On the other hand, if you look at the solution (1) of the Laplace's equation, the graph of the potential as a function of x and y looks like the following :



Note that the function does not have a minimum or a maximum. The point $x=0, y=0$ is a “saddle point” where the function rises in one direction and decreases in another direction, much like a saddle on a horse where the seat slopes downwards along the back but rises along its length both forward and backward.

[Mathematically, in two dimensions, if a function f partial derivatives $f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yy} = \frac{\partial^2 f}{\partial y^2}$ such that $f_{xx}f_{yy} - f_{xy}^2 < 0$, the function has a saddle point. If the expression has a positive value, the function has a maximum or a minimum depending on whether f_{xx} and f_{yy} are both negative or both positive respectively. For eqn. (1) at the point $(0,0)$ the expression is $-a^2/4$ at $(0,0)$]

This is true in any dimension because for a minimum the second partial derivative with respect to each variable must be negative so that their sum cannot add to zero as is required by Laplace’s equation.

Earnshaw’s Theorem

A consequence of solutions Laplace’s equation not having a minimum is that a charge cannot be held in equilibrium by electrostatic forces alone (i.e., if we are to keep a charge in equilibrium, we need forces other than just electrostatic forces; or for that matter, magnetostatic or static gravitational forces). This is known as the Earnshaw’s theorem.

The proof is elementary. In a region of space containing no charge, Laplace’s equation is valid for the potential ϕ . If a charge is to be kept in this potential, its potential energy $q\phi$ also satisfies the Laplace’s equation. Since the solutions of Laplace’s equation do not have minima, the charge cannot be in static equilibrium.

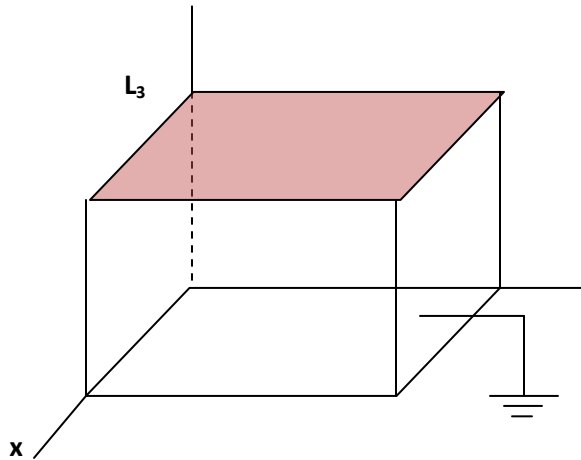
In the remaining part of this lecture we will discuss the solutions of Laplace's equation in three dimensions in different coordinate systems.

Solution in Cartesian Coordinates :

The Laplace's equation in Cartesian coordinates is given by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (3)$$

We will look into the solution of this equation in a region defined by a parallelepiped of dimensions $L_1 \times L_2 \times L_3$. In principle, one could choose boundary conditions such as prescribing the potential on each of the six surfaces of the parallelepiped. However, we consider a situation in which five walls of the parallelepiped are conductors which are joined together so that each is maintained at the same potential, which we take to be zero. The sixth plate of the parallelepiped at $z = L_3$ is made with a different sheet of metal and is kept at a potential which is a known function $f(x, y)$ of position on the plate.



Summarizing the boundary conditions, we have,

$$\begin{aligned} \varphi(0, y, z) &= \varphi(x, 0, z) = \varphi(x, y, 0) = 0 \\ \varphi(L_1, y, z) &= \varphi(x, L_2, z) = 0 \\ \varphi(x, y, L_3) &= f(x, y) \end{aligned}$$

We will use a technique known as separation of variables to solve the equation. We assume that the solution $\varphi(x, y, z)$ can be written as a product of three independent functions, X, Y and Z, the first one depending on the variable x alone, the second on variable y alone and the third on the variable z alone. It is not that such a trick will always work, but in cases where it does, the uniqueness theorem would guarantee us that that would be the only solution. Keeping this in view, let us attempt such a solution:

$$\varphi(x, y, z) = X(x)Y(y)Z(z)$$

Substituting this in eqn. (3) and dividing both sides by the product $X(x)Y(y)Z(z)$, we get,

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0$$

In this equation there are three terms, the first depending on x alone, the second on y alone and the third on z alone. Since x,y,z are arbitrary, the equation can be satisfied only if each of the terms is a constant and the three constants are such that they add up to zero.

Let us write

$$\begin{aligned} \frac{1}{X} \frac{\partial^2 X}{\partial x^2} &= -k_1^2 \\ \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} &= -k_2^2 \\ \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} &= k_3^2 \end{aligned}$$

Where k_1, k_2 and k_3 are constants, which satisfy $k_1^2 + k_2^2 = k_3^2$.

The boundary conditions on five faces of the parallelepiped where $\varphi = 0$, can be written as

$$\begin{aligned} X(0) &= X(L_1) = 0 \\ Y(0) &= Y(L_2) = 0 \\ Z(0) &= 0 \end{aligned}$$

However, on the sixth face we must have $Z(L_3) = \text{constant}$ because the potential on this face depends only on x,y whereas Z can only depend on z.

Consider the first equation, viz., $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -k_1^2$. The solution of this equation is well known to be linear combination of sine and cosine functions. Keeping the boundary conditions $X(0) = X(L_1) = 0$ in mind, we can write the solution to be given by

$$X(x) = A \sin k_1 x$$

where $k_1 = \frac{m\pi}{L_1}$, where m is an integer not equal to zero (as it would make the function identically vanish) and A is a constant .

In a very similar way the solution for Y is written as

$$Y(y) = B \sin k_2 y$$

With B being a constant and $k_2 = \frac{n\pi}{L_2}$. Once again n is a non-zero integer.

The solution for Z is a little more complicated because of the constraint $k_1^2 + k_2^2 = k_3^2$. Since k_1^2 and k_2^2 are positive integers, k_3^2 is also positive.

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = k_3^2$$

Then has solutions which are hyperbolic functions,

$$Z(z) = C \sinh(k_3 z)$$

with $k_3 = \sqrt{\left(\frac{m\pi}{L_1}\right)^2 + \left(\frac{n\pi}{L_2}\right)^2} = k_{mn}$. Thus, we can write the expression for the potential as

$$\varphi(x, y, z) = \sum_{m,n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi}{L_1} x\right) \sin\left(\frac{n\pi}{L_2} y\right) \sinh(k_{mn} z)$$

The solution above satisfy the first five boundary conditions. Let us see the effect of the last boundary condition.

$$\varphi(x, y, z = L_3) = f(x, y) = \sum_{m,n=1}^{\infty} C_{mn} \sin\left(\frac{m\pi}{L_1} x\right) \sin\left(\frac{n\pi}{L_2} y\right) \sinh(k_{mn} L_3)$$

The constants C_{mn} can be derived from above equation because $f(x,y)$ is a known function. For this we use the orthogonal property of the trigonometric functions,

$$\int_0^L dx \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{m'\pi}{L} x\right) = \frac{L}{2} \delta_{m,m'}$$

This gives,

$$C_{mn} = \frac{4}{L_1 L_2 \sinh(k_{mn} L_3)} \int_0^{L_1} dx \int_0^{L_2} dy f(x, y) \sin\left(\frac{m\pi}{L_1} x\right) \sin\left(\frac{n\pi}{L_2} y\right)$$

Take for instance, $f(x, y) = \text{constant} = f_0$

The integral above is then product of two integrals,

$$\begin{aligned} \int_0^L dx \sin\left(\frac{m\pi}{L} x\right) &= \frac{L}{m\pi} \cos\left(\frac{m\pi}{L} x\right) \Big|_0^L = \frac{L}{m\pi} [(-1)^m - 1] \\ &= \begin{cases} -2 \frac{L}{m\pi} & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even} \end{cases} \end{aligned}$$

Thus C_{mn} is non-zero only when both m and n are odd and has the value

$$C_{mn} = f_0 \frac{16}{mn\pi^2} \frac{1}{\sinh(k_{mn} L_3)}$$

In the next lecture we will obtain solutions of Laplace's equation in spherical coordinates.

Solutions to Laplace's Equations

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Tutorial Assignment

1. Find an expression for the potential in the region between two infinite parallel planes $z = 0$ and $z = 1$, the potential on the planes being given by the following :
$$\varphi(z = 0) = 5 \cos 4x \cos 3y$$
$$\varphi(z = 1) = 6 (\cosh 5) \cos 4x \cos 3y$$
2. Obtain a solution to Laplace's equation in two dimensions in Cartesian coordinates assuming that the principle of variable separation holds.
3. Find the solution of two dimensional Laplace's equation inside a rectangular region bounded by $x = 0, y = 0, x = a$ and $y = b$. The potential has value zero on the first three boundaries and takes value $\varphi(x, y = b) = \varphi_0 x/a$ on the fourth side.

Solution to Tutorial Assignment

1. We know that the general solution is a product of linear combination of either sine, cosine or hyperbolic sine and hyperbolic cosine functions with a relationship between the arguments. . In this case, since on both plates the dependence on x and y is given as $\cos 4x \cos 3y$, the general solution is

$$\varphi(x, y, z) = A \cos 4x \cos 3y \cosh 5z + B \cos 4x \cos 3y \sinh 5z$$

Where the argument of the hyperbolic function is obtained from the relation $k_3^2 = k_1^2 + k_2^2 = 16 + 9 = 25$. Substituting the boundary condition $\varphi(z = 0) = 5 \cos 4x \cos 3y$, we get

$$5 \cos 4x \cos 3y = A \cos 4x \cos 3y \Rightarrow A = 5$$

and $\varphi(z = 1) = 6 (\cosh 5) \cos 4x \cos 3y$, we get

$$6 \cosh 5 \cos 4x \cos 3y = 5 (\cosh 5) \cos 4x \cos 3y + B \cos 4x \cos 3y (\sinh 5)$$

which gives $B = \coth 5$. Thus

$$\varphi(x, y, z) = 5 \cos 4x \cos 3y \cosh 5z + (\coth 5) \cos 4x \cos 3y \sinh 5z$$

2. We start with $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$. Writing $\varphi(x, y) = X(x)Y(y)$, we get, on substituting and dividing throughout by XY ,

$$\frac{1}{X} \frac{d^2 X}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = k^2$$

The solution is then a product of $X(x) = A_1 \sin kx + B_1 \cos kx$ and $Y(y) = C_1 \sinh ky + D_1 \cosh ky$. We can reduce the number of constants from 5 to 4 by redefining the constants. For instance, if $A_1 C_1 \neq 0$, we define the constants as $A_1 C_1 = A$, $\frac{B_1}{A_1} = B$, $\frac{D_1}{C_1} = C$ and write,

$$\varphi(x, y) = A (\sin kx + B \cos kx) (\sinh ky + C \cosh ky)$$

The constants A, B, C and k are to be determined from boundary conditions.

3. We have seen that the general solution is a product of linear combination of sine and cosine in one of the variables and hyperbolic sine and cosine in the other variable. Since the potential is to be zero for $x = a$ for all values of y, this is possible only if we choose the solution for the variable x to be the trigonometric function and for y to be hyperbolic function. This is because a sine function can become zero at values of its argument other than zero, a hyperbolic function cannot. Further, only sine and sinh functions need to be considered so that at $x=0$ and at $y=0$, the function vanishes. Thus we write

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \sin kx \sinh ky$$

In this form, each term of the series automatically satisfies the boundary conditions at $x=0$ and at $y=0$. Since the potential is to become zero for all values of y for $x=a$, this is possible if $\sin ka = 0$, i.e. if $k = n\pi/a$. Thus the potential function becomes

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right)$$

We have to satisfy the only remaining boundary condition, viz., for $y = b$, $\varphi(x, y) = \varphi_0 x$. At $y = b$, we have,

$$\varphi(x, b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}b\right) = \varphi_0 x$$

To determine the coefficients A_n , we multiply both sides of the expression by $\frac{2}{a} \sin\left(\frac{m\pi}{a}x\right)$ and integrate over x from 0 to a

$$\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a}b\right) \frac{2}{a} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx = \frac{2}{a^2} \varphi_0 \int_0^a x \sin\left(\frac{m\pi x}{a}\right) dx$$

The integral on the left gives $\frac{a}{2} \delta_{m,n}$ while that on the right gives $-\frac{a^2}{m\pi} (-1)^m$. Thus,

$$A_m = \frac{2\varphi_0}{m\pi} (-1)^m \frac{1}{\sinh\left(\frac{n\pi b}{a}\right)}$$

This completely determines the potential.

Solutions to Laplace's Equations

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Self Assessment Quiz

1. Does the function $f(x, y) = \ln(x^2 + y^2)$ satisfy Laplace's equation?
2. Find an expression for the potential in the region between two infinite parallel planes $z = 0$ and $z = \frac{\pi}{2}$ on which the potentials are respectively given by e^{3x+4y} and 0.
3. Find an expression for the potential in the region between two infinite parallel planes, the potential on which are given by the following :

$$\varphi(z = 0) = 0$$

$$\varphi(z = 2/13) = 3 \sin 12x \cos 5y$$

Solution to self assessment Quiz

1. Yes, a direct calculation shows

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} = -\frac{\partial^2 f}{\partial y^2}$$

2. In this case also we have solutions which are written as products of separate variables. The solution can be written as (since X and Y are hyperbolic functions, the function Z is a linear combination of sine and cosine),

$$\varphi(x, y, z) = e^{3x+4y}(A \sin 5z + B \cos 5z)$$

The constants A and B can be determined by the boundary conditions. For $z=0$, $\varphi(x, y, 0) = e^{3x+4y}$ gives $B=1$. For $z = \frac{\pi}{2}$, $\varphi = 0$ gives $A=0$. Thus

$$\varphi(x, y, z) = e^{3x+4y} \cos 5z$$

3. The z dependence of the potential is a linear combination of hyperbolic sine and cosine function with argument $\sqrt{12^2 + 5^2}z = 13z$. Thus the general expression is

$$\varphi(x, y, z) = A \sin 12x \cos 5y \cosh 13z + B \sin 12x \cos 5y \sinh 13z$$

For $z=0$, the potential being zero, the cosh term vanishes, $A=0$. $\varphi(z = 2/13) = 3 \sin 12x \cos 5y$ gives $B \sinh 2 = 3$ so that

$$\varphi(x, y, z) = \frac{3}{\sinh 2} \sin 12x \cos 5y \sinh 13z$$