

Poisson's and Laplace's Equations

Lecture 13: Electromagnetic Theory

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We will spend some time in looking at the mathematical foundations of electrostatics. For a charge distribution defined by a charge density ρ , the electric field in the region is given by $\nabla \cdot \vec{E} = \rho/\epsilon_0$, which gives, for the potential ϕ , the equation which is known as the **Poisson's equation**,

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

In particular, in a region of space where there are no sources, we have

$$\nabla^2 \phi = 0$$

Which is called the **Laplace's equation**. In addition, under static conditions, the equation

$$\nabla \times \vec{E} = 0$$

is valid everywhere.

Solutions of Laplace's equation are known as **Harmonic functions**.

The expressions for the Laplacian operator in Cartesian, spherical and cylindrical coordinates are given by the following expressions :

Cartesian

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2}$$

Cylindrical (ρ, θ, z)

$$\nabla^2 \phi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

(It may be noted that we have used the symbol ϕ for the potential and a closely similar symbol ϕ for the azimuthal angle of the spherical coordinate system, which should not cause confusion).

The formal solution of Poisson's equation has been known to us from our derivation of the form of potential using Coulomb's law. It is easy to check that

the expression

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{All\ Space} \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^3r'$$

Where we have used the primed variable \vec{r}' to indicate the integrated variable and \vec{r} as the position coordinate of the point where the potential is calculated. We will operate both sides of the above equation with ∇^2 . Since ∇^2 acts on the variable \vec{r} , we can take it inside the integral on the right and make it operate on the function $\frac{1}{|\vec{r} - \vec{r}'|}$, and get,

$$\nabla^2 \varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{All\ Space} \rho(\vec{r}') \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) d^3r'$$

Recall that $\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi\delta^3(\vec{r} - \vec{r}')$, so that the integral on the right is easily computed using the property of the delta function to be $-\frac{\rho}{\epsilon_0}$.

Uniqueness Theorem :

There may be many solutions to Poisson's and Laplace's equations. However, the solutions that interest us in Physics are those which satisfy the given boundary conditions. Other than the assumed existence of the solutions subject to given boundary conditions, one theorem that comes to our help is what is known as the Uniqueness Theorem. The theorem basically states that corresponding to various possible solutions of Laplace's or Poisson's equation, the solution that satisfies the given boundary condition is unique, i.e., no two different solutions can satisfy the given equation with specified boundary conditions.

The great advantage of the theorem lies in the fact that if we obtain a solution by some technique or even by intuition, we need not look any further, the solution that we have at hand is the only possible solution.

What are these boundary conditions? Consider, for instance, a situation in which we are required to find the solution of either of these equations in a region of volume V bounded by a surface S . Within this region, we have, for instance, a set of conductors with surfaces S_1, S_2, \dots on which we know the values of the potential. In addition, we are given the potential function on the surface S itself. This is an example of boundary condition, known as the **Dirichlet boundary condition**.

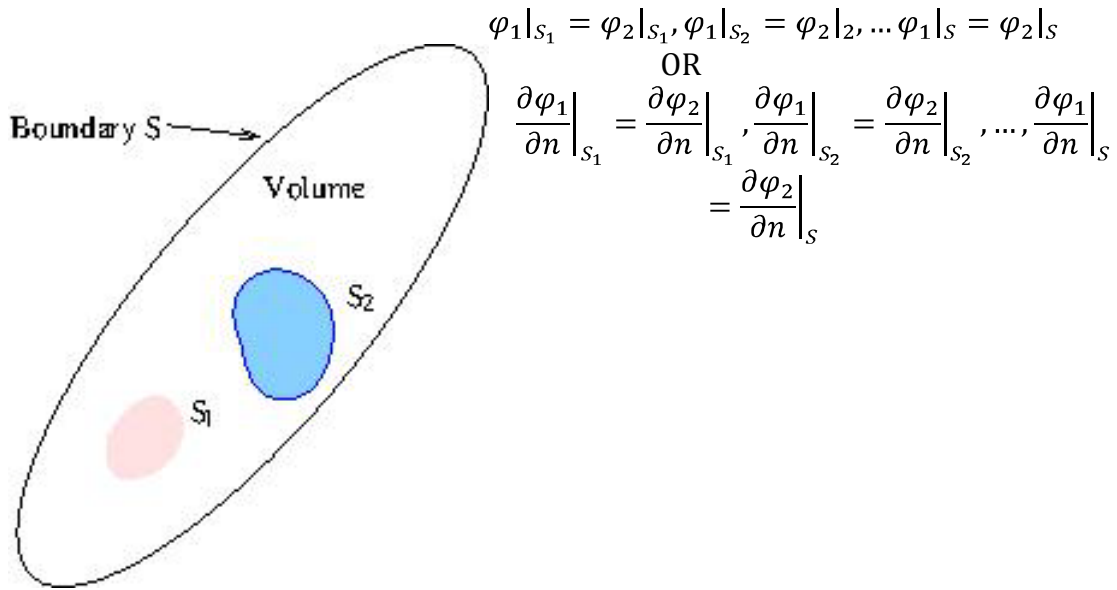
It is also possible that instead of specifying the values of potential on the conductors, we are given the electric fields on the surface of the conductors, which as we know, are directed normal to the surface of the conductors. This would be yet another example of boundary conditions. Boundary conditions where the normal derivatives of the function to be derived are specified on some surface or surfaces are known as the **Neumann boundary condition**.

In addition to these two types of boundary conditions, there are other possible boundary conditions. For instance in **Cauchy boundary condition**, the value of both the potential and its normal derivative are specified. There could be **mixed**

boundary conditions in which different types of boundary conditions may be specified on different parts of the boundary.

Uniqueness theorem applies to those cases where there is only one type of boundary condition, viz. either Dirichlet or Neumann boundary condition.

To prove the uniqueness theorem, let us assume that contrary to the assertion made in the theorem, there exist two solutions φ_1 and φ_2 of either Poisson's or Laplace's equation which satisfy the same set of boundary conditions on surfaces S_1, S_2, \dots and the boundary S . The conditions, as stated above, may be either of Dirichlet type or Neumann type :



Let us define a new function $\varphi = \varphi_1 - \varphi_2$. In view of the fact that φ_1 and φ_2 satisfy the same boundary conditions, the boundary condition satisfied by φ are $\varphi|_{S_1} = \varphi|_{S_2} = \dots = \varphi|_S = 0$ (Dirichlet) OR $\frac{\partial \varphi}{\partial n}\bigg|_{S_1} = \frac{\partial \varphi}{\partial n}\bigg|_{S_2} = \dots = \frac{\partial \varphi}{\partial n}\bigg|_S = 0$ (Neumann).

Further, whether φ_1 and φ_2 satisfy Poisson's or Laplace's equation, their difference $\varphi = \varphi_1 - \varphi_2$ satisfies Laplace's equation.

In order to prove the Uniqueness theorem we will use Green's First Identity, derived in Module 1, which states that for two arbitrary scalar fields ϕ and ψ , the following identity holds,

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3r = \oint_S \phi \frac{\partial \psi}{\partial n} dS$$

where S is the boundary defining the volume V . We choose $\phi = \psi = \varphi$, to get,

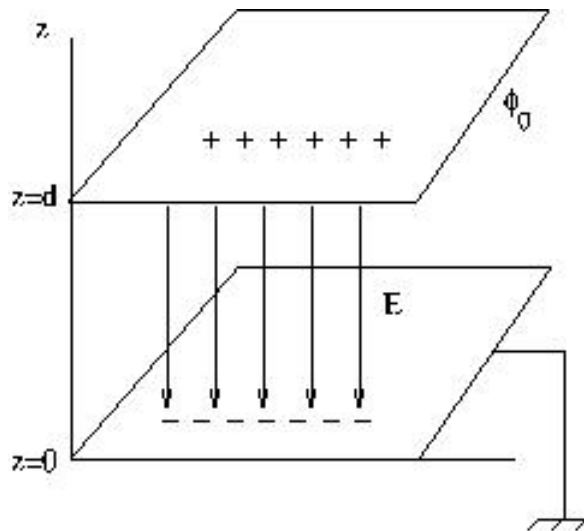
$$\int_V (\varphi \nabla^2 \varphi + |\nabla \varphi|^2) d^3r = \oint_S \varphi \frac{\partial \varphi}{\partial n} dS$$

Since on the surface, either the Dirichlet or the Neumann boundary condition is valid, the right hand side of the above is zero everywhere. (Note that the surface consists of $S + S_1 + S_2 + \dots$, with the direction of normal being outward on S and inward on the conductor surfaces enclosed).

Since φ satisfies Laplace's equation, we are then left with $\int_V |\nabla \varphi|^2 d^3r = 0$. The integrand, being a square of a field is positive everywhere in the volume and its integral can be zero only if the integral itself is identically zero. Thus we have, $\nabla \varphi = 0$, which leads to $\varphi = \text{constant}$ everywhere on the volume. If Dirichlet boundary condition is satisfied, then $\varphi = 0$ on the surface and therefore, it is zero everywhere in the volume giving, $\varphi_1 = \varphi_2$. If, on the other hand, Neumann boundary condition is satisfied, we must have $\frac{\partial \varphi}{\partial n} = 0$, i.e. $\varphi = \varphi_1 - \varphi_2 = \text{constant}$ everywhere. Since the constant can be chosen arbitrarily, we take the constant to be zero and get $\varphi_1 = \varphi_2$.

Example 1 : Parallel Plate Capacitor :

Consider a parallel plate capacitor with a plate separation of d between the plates. The lower plate is grounded while the upper plate is maintained at a constant potential φ_0 . Between the two plates there are no sources and hence Laplace's equation is valid in this region. Since the plates are assumed infinite in the x, y directions, the only variation is with respect to the z direction and we have,



$$\nabla^2 \varphi \equiv \frac{\partial^2 \varphi}{\partial z^2} = 0$$

The solution of this equation is straightforward, and we get,

$$\varphi(z) = Az + B$$

where A and B are constants.

Substituting the boundary conditions at $z=0$ and at $z=d$,

$$\varphi(z=0) = 0 = B$$

$$\varphi(z=d) = \varphi_0 = Ad$$

which gives, $A = \frac{\varphi_0}{d}$, $B = 0$

Substituting these, we get

$$\varphi(z) = \frac{\varphi_0}{d} z$$

The electric field in the region is given by $\vec{E} = -\nabla \varphi = -\hat{k} \frac{\partial \varphi}{\partial z} = -\frac{\varphi_0}{d} \hat{k}$. This shows that the electric field between the plates is constant and is directed from the upper plate toward the lower plate. The upper plate gets positively charged and the lower plate is negatively charged. We can find out the charge densities on the plates by taking the normal component of the electric field. For instance on the upper plate, the direction of normal being in the negative z direction, $\hat{n} = -\hat{k}$, we get,

$$\sigma = \epsilon_0 E_n = + \frac{\varphi_0}{d} \epsilon_0$$

Likewise, the normal to the lower plate being in the positive z direction, the charge density on that plate is equal and opposite. If we multiply the charge density by the area of the plate A, we get (area is taken to be large so that the edge effects are neglected),

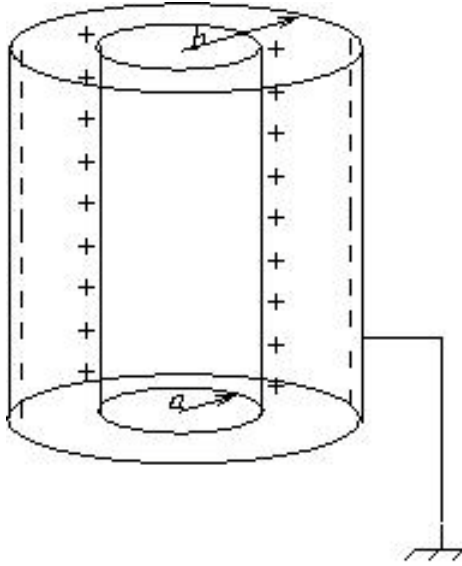
$$Q = A\sigma = A \frac{\varphi_0}{d} \epsilon_0$$

to be the charge on the upper plate. The lower plate has equal but opposite charge. Dividing the amount of charge on the positive plate by the potential difference φ_0 between the plates, we get the capacitance of the parallel plate capacitor to be $C = \frac{\epsilon_0 A}{d}$.

Example 2 : Coaxial Cable :

Consider a coaxial cable of inner radius a and outer radius b. The outer conductor is grounded while the inner conductor is maintained at a constant potential φ_0 . Taking the z axis of the cylindrical coordinate system along the axis of the cylinders, we can write down the Laplace's equation in the space between the cylinders as follows :

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \varphi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$



Here, ρ is the radial distance from the axis and θ is the polar angle. We assume the cylinders to be of infinite extend in the z direction. This implies that there is azimuthal symmetry and no variation with respect to z as well. The solution can only depend on the radial distance ρ . Giving,

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \varphi}{\partial \rho} \right) = 0$$

The solution of the above equation is easy to obtain, and we get,

$$\varphi(\rho) = A \ln \rho + B$$

where A and B are two constants which must be determined from the boundary conditions.

$\varphi(\rho = b) = 0$ gives $A \ln b + B = 0$, so that $B = -A \ln b$. The boundary condition at $\rho = a$ gives $\varphi_0 = A \ln a - A \ln b$, so that

$$A = \frac{\varphi_0}{\ln\left(\frac{a}{b}\right)}$$

Substituting these in the expression for the potential, we get the potential in the region between the cylinders to be given by

$$\varphi(\rho) = \varphi_0 \frac{\ln(\frac{\rho}{b})}{\ln(\frac{a}{b})}$$

The electric field in the region is given by the negative gradient of the potential which is simply the derivative with respect to ρ ,

$$\begin{aligned}\vec{E} &= -\nabla\varphi = -\hat{\rho} \frac{\partial}{\partial\rho} \varphi(\rho) \\ &= -\hat{\rho} \frac{\varphi_0}{\ln(\frac{a}{b})} \frac{1}{\rho}\end{aligned}$$

Since $\ln(\frac{a}{b}) < 0$, the direction of the electric field is outward from the inner conductor, which gets positively charged. The charge density of the inner conductor is given by

$$\sigma = \epsilon_0 E_n = +\varphi_0 \epsilon_0 \frac{1}{a \ln(\frac{b}{a})}$$

where we have taken care of the minus sign by inverting the argument of log. Note that the expression for the charge density on the outer plate will not be identical because of the fact that the radii of the two cylinders are different. The total charge per unit length on the inner conductor is given by

$$Q = \varphi_0 \epsilon_0 \frac{2\pi a}{a \ln(\frac{b}{a})} = \frac{2\pi \varphi_0 \epsilon_0}{\ln(\frac{b}{a})}$$

The magnitude of the charge per unit length on the outer conductor can be seen to be the same. The capacitance per unit length is thus given by

$$C = \frac{2\pi\epsilon_0}{\ln(\frac{b}{a})}$$

Example 3 : Spherical Capacitor:

Consider a spherical capacitor with the inner conductor having a radius a and the outer conductor a radius b . The outer conductor is grounded and the inner conductor is maintained at a constant potential φ_0 . Because of spherical symmetry, the potential can only depend on the radial distance r . The radial part of Laplace's equation, which is valid in the space between the conductors is given by

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\varphi}{dr} \right) = 0$$

This is a differential equation in a single variable, and the solution can be easily obtained as

$$\varphi(r) = -\frac{A}{r} + B$$

where A and B are constants.

Inserting the boundary conditions,

$$\varphi(r = b) = 0$$

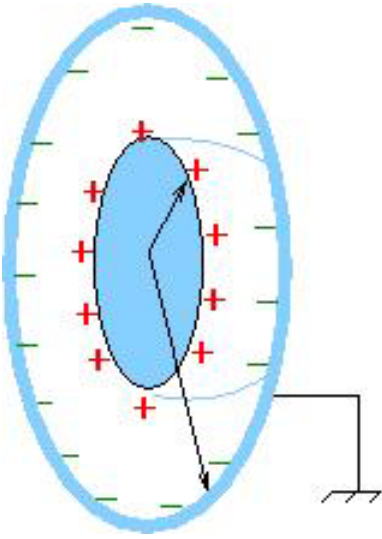
gives $B = \frac{A}{b}$. The other boundary condition $\varphi(r = a) = \varphi_0$ gives

$$\varphi_0 = -\frac{A}{a} + \frac{A}{b}$$

$$A = -\frac{\varphi_0 ab}{b - a}$$

Substituting these into the solution, we have,

$$\varphi(r) = \frac{\varphi_0 ab}{b - a} \left(\frac{1}{r} - \frac{1}{b} \right)$$



As before, we find the charge density on the spheres by taking the normal component of the electric field. The electric field is given by,

$$\begin{aligned} \vec{E} &= -\frac{d\varphi}{dr} \hat{r} \\ &= \frac{\varphi_0 ab}{b - a} \frac{1}{r^2} \hat{r} \end{aligned}$$

The charge density on the inner plate is given by

$$\sigma_{in} = \epsilon_0 \frac{\varphi_0 ab}{b - a} \frac{1}{a^2}$$

The total charge on the inner sphere is

$$Q = \frac{4\pi\epsilon_0\varphi_0 ab}{b - a}$$

It can be checked that the outer sphere has equal and opposite negative charge.

The capacitance is given by

$$C = \frac{4\pi\epsilon_0 ab}{b - a}$$

The capacitance of a single spherical conductor is obtained by taking the outer sphere to infinity, i.e., $b \rightarrow \infty$, which gives the capacitance for a single conductor to be $4\pi\epsilon_0 a$. This result is not surprising because for a single conductor containing charge Q , the potential is $\varphi = \frac{Q}{4\pi\epsilon_0 a}$, so that the capacitance, which is given by $\frac{Q}{\varphi} = 4\pi\epsilon_0 a$.

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Tutorial Assignment

1. If $\varphi(r, \theta, \phi)$ satisfies Laplace's equation inside a sphere, show that the average of $\varphi(r, \theta, \phi)$ over the surface is equal to the value of the potential at the origin.
2. Consider two concentric spherical shells of radii R_1 and R_2 . A charge Q_1 resides on the inner sphere and Q_2 on the outer sphere. The potential is set to be zero at infinite distance. Find the potential at the centre of the sphere.

Solutions to tutorial assignment

1. The average of the potential over the surface is
 $\langle \varphi \rangle = \frac{1}{4\pi R^2} \int \varphi(R, \theta, \phi) R^2 \sin \theta d\theta d\phi$. Since the angles are averaged out, the right hand side can only depend on R . Thus we have

$$\frac{d\langle \varphi(R, \theta, \phi) \rangle}{dr} = \frac{1}{4\pi} \int \frac{\partial \varphi(R, \theta, \phi)}{\partial R} \sin \theta d\theta d\phi$$

$$\frac{\partial \varphi(R, \theta, \phi)}{\partial R} = \nabla \varphi \cdot \hat{r}$$

Since the outward normal on the sphere is along \hat{r} , we have $d\vec{S} = \sin \theta d\theta d\phi \hat{r}$ and we can write,

$$\frac{d\langle \varphi(R, \theta, \phi) \rangle}{dr} = \frac{1}{4\pi} \int \nabla \varphi \cdot d\vec{S}$$

The right hand side can be converted to a volume integral using the divergence theorem,

$$\frac{d\langle \varphi(R, \theta, \phi) \rangle}{dr} = \frac{1}{4\pi} \int \nabla^2 \varphi d^3r = 0$$

since $\varphi(r, \theta, \phi)$ satisfies Laplace's equation. Thus the average is independent of the radius of the sphere and we can continuously decrease the size of the sphere so that it reduces to a point at the origin.

2. The potential on the surface of the inner sphere can be easily shown to be $\frac{Q_2}{R_2} + \frac{Q_1}{R_1}$. This must also be the potential at the origin by uniqueness theorem as derived in Problem 3.

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Self Assessment Quiz

1. If φ is a solution of Laplace's equation, show that one or more derivatives of φ with respect to rectangular coordinates also satisfy Laplace's equation.
2. Using the property of solutions to Laplace's equation show that there cannot be an electrostatic field inside a hollow conductor unless there is a charge in that region.
3. A conductor has a hollow scooped out inside. The conductor carries a charge and the potential of the conductor is φ_0 with respect to its zero at infinity. Show that the potential inside the hollow is also φ_0 .
4. Consider two point conductors separated by a distance of 10m. One of the conductors is maintained at a potential of 100 V while the other maintained at 200 V. Determine the potential function and the corresponding electric field..

Solutions to Self Assessment Quiz

$$\begin{aligned} 1. \quad \nabla^2 \frac{\partial \varphi}{\partial x} &= \nabla \cdot \left(\hat{i} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y \partial x} \hat{j} + \frac{\partial^2 \varphi}{\partial z \partial x} \hat{k} \right) = \frac{\partial^3 \varphi}{\partial x^3} + \frac{\partial^3 \varphi}{\partial^2 y \partial x} + \frac{\partial^3 \varphi}{\partial^2 z \partial x} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial^3 \varphi}{\partial x^2} + \frac{\partial^3 \varphi}{\partial y^2} + \frac{\partial^3 \varphi}{\partial z^2} \right) = 0 \end{aligned}$$

2. The inner surface of the conductor which encloses the hollow is an equipotential. Let the potential be φ_0 . If there are no charges in the

hollow, the potential there must satisfy Laplace's equation. We know that solutions to Laplace's equations have no local minima or maxima. Thus the potential inside the hollow must be the same as that on its boundary, i.e., the potential has a constant value φ_0 . Thus the electric field inside the hollow is zero.

3. Conductor is an equipotential. Thus the inner surface of the hollow is also at a potential φ_0 . The potential inside the hollow must have the same value of the potential because (i) being a constant, it satisfies Laplace's equation and (ii) it satisfies Dirichlet boundary condition on the surface. By uniqueness theorem, this is the only possible solution.
4. Let the first conductor be located at the origin and the second at $x=10$. On the entire x axis other than at these two points, Laplace's equation is applicable, which has the solution $V(x) = Ax + B$. Inserting boundary conditions we have, $A = 10, B = 100$ so that the potential is $V(x) = 10x + 100$ Volts. The electric field is $\vec{E} = -\frac{dV}{dx}\hat{i} = -10\hat{i}$ V/m.