# MA2102: Linear Algebra

## Lecture 14: Ordered Basis

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**Definition** [Ordered Basis] Let $V$ be a finite dimensional vector space. An ordered basis of $V$ is a basis $\beta$ of $V$, endowed with a specific order.

For instance, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1\}$ are both bases of $\mathbb{R}^3$ but different as ordered bases.

**Proposition** Any two vector spaces of the same finite dimension are isomorphic.

Proof.
Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_n\}$ be ordered basis of $V$ and $W$ respectively. Define a linear map $T : V \to W$ by first declaring $T(v_i) = w_i$. Since we want $T$ to be linear, define

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n.$$

Any $v \in V$ can be written as a *unique* linear combination of $v_i$'s. Thus,

$$T(v) = T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

and $T$ is defined on all of $V$. Since any $w \in W$ is of the form $c_1 w_1 + \cdots + c_n w_n$, it is the image of $c_1 v_1 + \cdots + c_n v_n$ under $T$, whence $T$ is surjective. Show that $T$ is linear. We may now invoke the Rank-Nullity Theorem to conclude that $T$ is injective. Thus, $T$ is a linear isomorphism. □

**Corollary** Any vector space $V$ of dimension $n$ over $\mathbb{R}$ is isomorphic to $\mathbb{R}^n$.

If we choose the standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ and an ordered basis $\beta = \{v_1, \ldots, v_n\}$ of $V$, then a linear isomorphism is given by

$$T : \mathbb{R}^n \to V, \;\; T(c_1, \ldots, c_n) = c_1 v_1 + \cdots + c_n v_n.$$

**Example** [Rank-Nullity Theorem revisited] Let $T : V \to W$ be a linear map. Since $T$ maps $V$ to $R(T)$, the range of $T$, we may consider the surjective linear map $T : V \to R(T)$. Define a new map

$$\mathscr{T} : V/N(T) \to R(T), \ \ [v] \mapsto T(v).$$

- [well-defined] Assume that $[v_1] = [v_2]$

$$\begin{aligned} [v_1] = [v_2] &\Leftrightarrow v_1 - v_2 \in N(T) \\ &\Leftrightarrow T(v_1 - v_2) = \mathtt{0}_W \\ &\Leftrightarrow T(v_1) - T(v_2) = \mathtt{0}_W \\ &\Leftrightarrow \mathscr{T}[v_1] - \mathscr{T}[v_2] = \mathtt{0}_W. \end{aligned}$$

- [injective] Follows from the equivalences above.
- [surjective] By construction.

- [linearity] $\mathscr{T}(c[v]) = \mathscr{T}([cv]) = T(cv) = cT(v) = c\mathscr{T}([v])$

$$\mathscr{T}([v_1]+[v_2]) = \mathscr{T}([v_1+v_2]) = T(v_1+v_2) = \mathscr{T}([v_1]) + \mathscr{T}([v_2])$$

Thus, $\mathscr{T} : V/N(T) \to R(T)$ is a linear isomorphism.

**Remark** This isomorphism is also known as the 1st Isomorphism Theorem in other contexts (like Group Theory).

If $V$ is finite dimensional, then $R(T)$, being the span of $T(v_j)$'s where $\{v_1, \ldots, v_n\}$ is a basis of $V$, is also finite dimensional. As $N(T)$ is a subspace of $V$, it is also finite dimensional. By part (iii) of Proposition (cf. lecture 13), we know that $R(T)$ and $V/N(T)$ have the same dimension. Show that $\dim_F(V/N(T)) = \dim_F V - \dim_F N(T)$. Thus,

$$\dim_F V - \text{nullity}(T) = \dim_F V - \dim_F N(T) = \dim_F R(T) = \text{rank}(T).$$

The quotient space can be finite dimensional even when $V$ and $W$ are not.

**Example** Let $V = P(\mathbb{R})$, the space of all polynomials and let

$$W = \{p(x) \in P(\mathbb{R}) \mid p(x) = a_1 x + a_2 x^2 + \cdots + a_k x^k\}.$$

Show that $W$ is an infinite dimensional subspace. Consider the quotient space $V/W$. Since a polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ is equivalent to the constant polynomial $a_0$, we expect $V/W$ to be of dimension 1. Define

$$T : V/W \to \mathbb{R}, \quad T([p(x)]) = p(0).$$

This is a well-defined and surjective map. Show that $T$ is a linear isomorphism.

**Definition** [Coordinate Vector] Let $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis of a vector space $V$. Any $v \in V$ can be expressed uniquely as $v = a_1 v_1 + \cdots + a_n v_n$. Then

$$[v]^\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

is called the coordinate vector of $v$ relative to $\beta$.

**Examples** (1) Let $V = \mathbb{R}^3$ and $v = (1, 2, 3)$. Consider the ordered bases

$$\begin{aligned} \beta &= \{(1,0,0),(0,1,0),(0,0,1)\} \\ \beta' &= \{(1,0,0),(0,0,1),(0,1,0)\} \\ \beta'' &= \{(1,2,3),(1,0,0),(0,1,0)\}. \end{aligned}$$

The coordinate vectors are given by

$$[v]^{\beta} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ [v]^{\beta'} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \ [v]^{\beta''} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(2) Let $V = P_2(\mathbb{R})$ and $v = 1 + x + 2x^2$. Consider the ordered bases[1]

$$\beta = \{1, x, x^2\}, \ \beta' = \{(1, x^2, 1 + x\}, \ \beta'' = \{(1 + x, 2x^2, 5\}.$$

The coordinate vectors are given by

$$[v]^{\beta} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ [v]^{\beta'} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \ [v]^{\beta''} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

[1] Show that these are bases.