SPECIAL TECHNIQUES-II

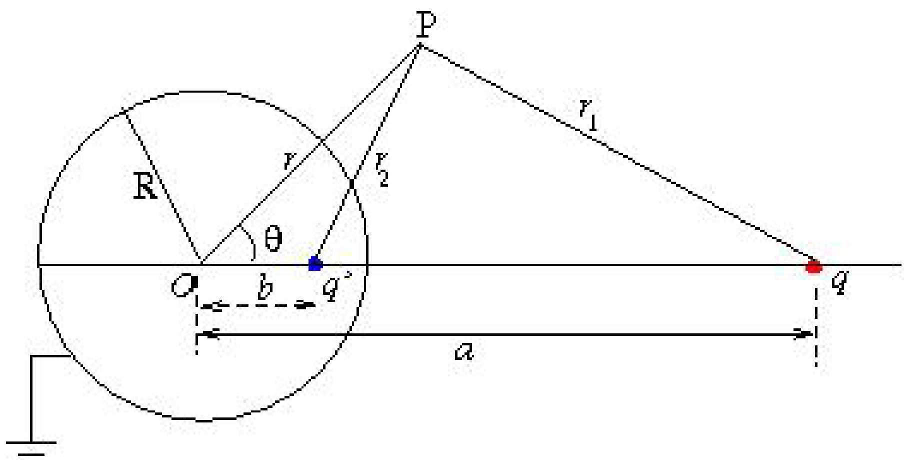
Lecture 18: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Method of Images for a spherical conductor

The method of images can be applied to the case of a charge in front of a grounded spherical conductor. The method is not as straightforward as the case of plane conductors but works equally well. Consider a charge q kept at a distance *a* from the centre of the grounded sphere. We wish to obtain an expression for the potential at a point P which is at the position (r, θ), the potential obviously will not depend on the azimuthal angle and hence that coordinate has been suppressed. Let the point P be at a distance r_1 from the location of the charge q.

The image charge is located at a distance b from the centre along the line joining the centre to the charge q. The line joining the charges and the centre is taken as the reference line with respect to which the angle θ is measured. Let P be at a distance *b* from the image charge. Let q' be the image charge.



The potential at P is given by $\varphi(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} + \frac{q'}{r_2}\right)$. Using the property of triangle, we can express the potential at P as,

$$\varphi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q'}{\sqrt{r^2+b^2-2br\cos\theta}}\right]$$

Since the potential vanishes at $r = R$ for all values of θ, the signs of q and q' must be opposite, and we must have,

$$q^2(R^2+b^2-2bR\cos\theta)=q'^2(\,R^2+a^2-2aR\cos\theta\,)$$

In order that this relation may be true for all values of θ, the coefficient of $\cos\theta$ from both sides of this equation must cancel,

$$2bRq^2=2aRq'^2$$

Since q and q' have opposite sign, this gives,

$$\boxed{q'=-\sqrt{\frac{b}{a}}\;q}$$

Substituting this in the θ independent terms above, we get,

$$R^2+b^2=\frac{b}{a}(R^2+a^2)$$

which gives $ab=R^2$. Thus $b=\frac{R^2}{a}, q'=-\frac{R}{a}q$.

It follows that if the object charge is outside the sphere, the image charge is inside the sphere. Using these, the potential at P is given by

$$\begin{aligned}\varphi(r,\theta)&=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ar\cos\theta}}+\frac{q'}{\sqrt{r^2+b^2-2br\cos\theta}}\right]\\&=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}-\frac{\frac{R}{a}}{\sqrt{r^2+\left(\frac{R^4}{a^2}\right)-2\left(\frac{R^2}{a}\right)r\cos\theta}}\right]\\&=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}-\frac{R}{\sqrt{r^2a^2+R^4-2R^2ar\cos\theta}}\right]\end{aligned}$$

The electric field can be obtained by computing gradient of the potential. It can be easily verified that the tangential component of the electric field $E_t=E_\theta=0$. The normal component is given by,

$$\begin{aligned}E_n&=E_r=-\frac{\partial\varphi}{\partial r}\\&=-\frac{q}{4\pi\epsilon_0}\left[\frac{-r+a\cos\theta}{(r^2+a^2-2ar\cos\theta\,)^{\frac{3}{2}}}+\frac{R(a^2r-R^2a\cos\theta)}{(r^2a^2+R^4-2R^2ar\cos\theta\,)^{\frac{3}{2}}}\right]\end{aligned}$$

The charge density on the surface of the sphere is $\epsilon_0E_n(r=R)$and is given by

$$\sigma(R,\theta)=-\frac{q}{4\pi}\left[\frac{-R+a\cos\theta}{(R^2+a^2-2aR\cos\theta\,)^{\frac{3}{2}}}+\frac{R(a^2R-R^2a\cos\theta)}{(R^2a^2+R^4-2R^3a\cos\theta\,)^{\frac{3}{2}}}\right]$$

$$= \frac{q}{4\pi R} \frac{R^2 - a^2}{(R^2 + a^2 - 2aR\cos\theta\ )^{\frac{3}{2}}}$$

The charge density is opposite to the sign of q since $R^2 - a^2 < 0$.

It is interesting to note that unlike in the case of a conducting plane, the magnitude of the image charge is not equal to that of the object charge but has a reduced value,

$$Q_{ind} = 2\pi\int_0^{\pi} \sigma(R,\theta)R^2 \sin\theta d\theta = \frac{q}{2}\int_{-1}^{+1} \frac{R(R^2 - a^2)}{(R^2 + a^2 - 2aR\mu)^{3/2}} \frac{d\mu}{(-2aR)}$$
$$= \frac{qR(a^2 - R^2)}{4aR}\left(R^2 + a^2 - 2aR\mu\right)^{1/2}\Big|_{-1}^{1} = -q\frac{R}{a}$$

$$Q_{ind} = 2\pi \int_0^{2\pi} \sigma(R,\theta)R^2 \sin\theta d\theta = \frac{q}{2}\int_{-1}^{+1} \frac{R(R^2 - a^2)}{(R^2 + a^2 - 2aR\mu)} \frac{d\mu}{(-2aR)}$$
$$= \frac{qR(a^2 - R^2)}{4aR}(R^2 + a^2 - 2aR\mu)|_{-1}^{+1} = -q\frac{R}{a}$$

Thus the potential can be written as

$$\boxed{\varphi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r} - \vec{a}|} - \frac{\left(\frac{R}{a}\right)q}{\left|\vec{r} - \frac{R^2}{a^2}\vec{a}\right|}\right]}$$

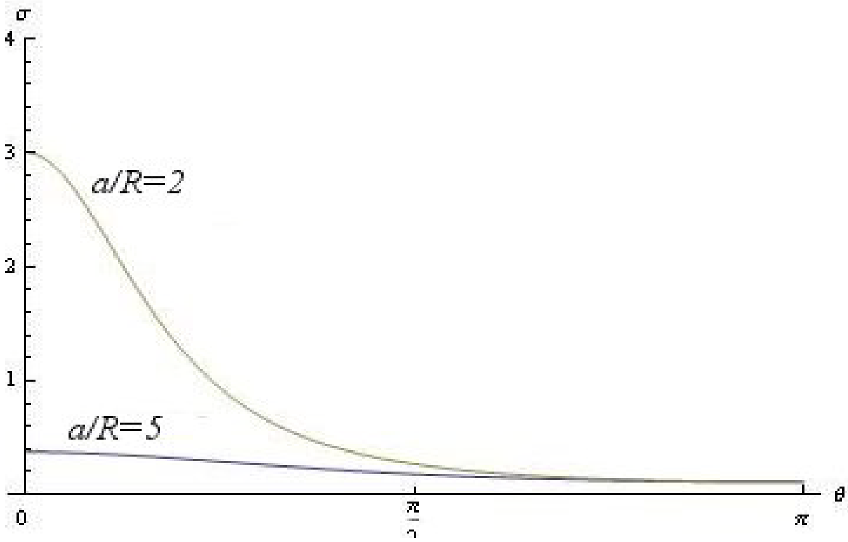
The following figure shows the variation of charge density on the surface as a function of the angle θ. As expected, when the charge comes closer to the sphere, the charge density peaks around θ=0.

Since the distance between the charge and its image is $a - b$, the force exerted on the charge by the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{|a - b|^2} = \frac{1}{4\pi\epsilon_0}\frac{q^2 Ra}{|a^2 - R^2|^2}$$

For $a \gg R$, the force is proportional to the inverse cube of the distance of the charge from the centre. For $a \approx R$, let α be the distance of q from the surface of the sphere, $a = R + \alpha$; $\alpha \ll R$,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2 R(R + \alpha)}{|2R\alpha + \alpha^2|^2} \sim \frac{1}{\alpha^2}$$



Consider the limit of $R \to \infty$, (one has to be careful here as the origin would now have shifted to extreme left; instead, measure distance from the surface of the sphere. The charge q is located at a distance $d = a - R$ from the surface,

$$q' = -q\frac{R}{a} = -q\frac{R}{R+d} \to -q \text{ , as } R \to \infty$$

The distance of the image charge, $b = \frac{R^2}{a} = \frac{R^2}{d+R}$ so that the distance of the image from the surface $b' = R - b = R - \frac{R^2}{d+R} = \frac{Rd}{d+R} \to d$ as $R \to \infty$. Thus we recover the case of a charge in front of an infinite plane.

A Few Special Cases

1. If the sphere, instead of being grounded, is kept at a constant potential φ_0, we can solve the problem by putting an additional charge $4\pi\epsilon_0\varphi_0 R$ distributed uniformly over its surface.
2. If the sphere was insulated, conducting and has a charge Q, we put a charge $Q - q'$ over the surface after disconnecting from the ground. The potential at any point then is the sum of the potential due to the image problem and that due to a charge $Q - q'$ at the centre.

$$\varphi(r,\theta) = 1/4\pi\epsilon_0\left[\frac{q}{|\vec{r}-\vec{a}|} - \frac{\left(\frac{R}{a}\right)q}{\left|\vec{r}-\frac{R^2}{a^2}\vec{a}\right|} + \frac{Q+\left(\frac{R}{a}\right)q}{r}\right]$$

Method of Inversion

The problem of finding the image charge for a sphere indicates that there is a symmetry associated with finding the potential. The potential expression has a symmetry about the centre of the sphere, which can be termed as the "centre of inversion" and the potential expression is symmetric if we let $r \to r' = R^2/r$. It may be observed that if $\varphi(r,\theta,\phi)$ is the potential due to a collection of charges $\{q_i\}$ located at $\{(a_i,\theta_i,\phi_i)\}$,then the potential $\varphi'(r,\theta,\phi)$ due to the charges $q_i' = \frac{R}{a_i} q_i$ located at the inversion points $\left\{\frac{R^2}{a_i},\theta,\phi\right\}$ are related by the relation

$$\varphi'(r,\theta,\phi) = \frac{R}{r}\varphi(\frac{R^2}{r},\theta,\phi)$$

Consider a charge q located at $(a, \theta = 0)$. The potential at $P(r,\theta,\phi)$ due to this charge is given by

$$\varphi(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{a^2 + r^2 - 2ar\cos\theta}}$$

The potential at $P(r,\theta,\phi)$ due to charges Rq/a located at $b = R^2/a$ is given by

$$\varphi'(r,\theta,\phi) = \frac{1}{4\pi\epsilon_0}\frac{(\frac{R}{a})q}{\sqrt{\frac{R^4}{a^2} + r^2 - 2\frac{R^2}{a}r\cos\theta}}$$

Thus,

$$\begin{aligned}\varphi\left(\frac{R^2}{r},\theta,\phi\right) &= \frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{a^2 + \frac{R^4}{r^2} - 2\frac{R^2}{r}a\cos\theta}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{\left(\frac{a}{r}\right)\sqrt{r^2 + \frac{R^4}{a^2} - 2\frac{R^2}{a}r\cos\theta}} \\ &= \frac{R}{r}\varphi'(r,\theta,\phi)\end{aligned}$$

Example : Conducting sphere in a uniform electric field

To produce the field we put a charge $-Q$ at $z=+a$ and a charge $+Q$ at $z=-a$. The field at an arbitrary point P is given by

$$\vec{E}(\vec{r})=\frac{Q}{4\pi\epsilon_0}\left[\frac{\vec{r}+a\hat{k}}{\left|\vec{r}+a\hat{k}\right|^3}-\frac{\vec{r}-a\hat{k}}{\left|\vec{r}-a\hat{k}\right|^3}\right]$$

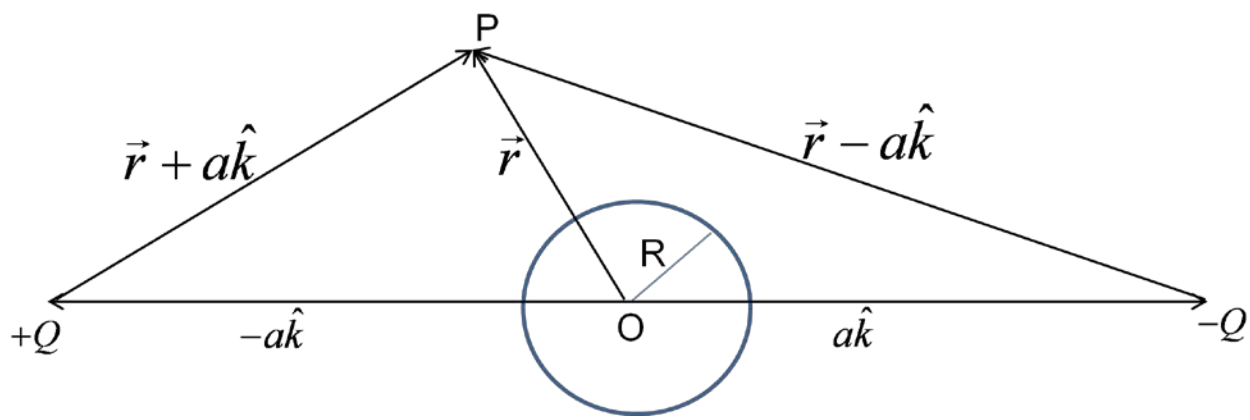
We have, if the charges are far away from the sphere, $a\gg r,R$

$$\begin{aligned}\frac{1}{\left|\vec{r}\pm a\hat{k}\right|^3}&=\frac{1}{\left(a^2+r^2\pm 2a\vec{r}\cdot\hat{k}\right)^{\frac{3}{2}}}\\&=\frac{1}{(a^2+r^2\pm 2az)^{\frac{3}{2}}}\\&=\frac{1}{a^3}\left(1\pm\frac{2z}{a}\right)^{\frac{3}{2}}\approx\frac{1}{a^3}\end{aligned}$$

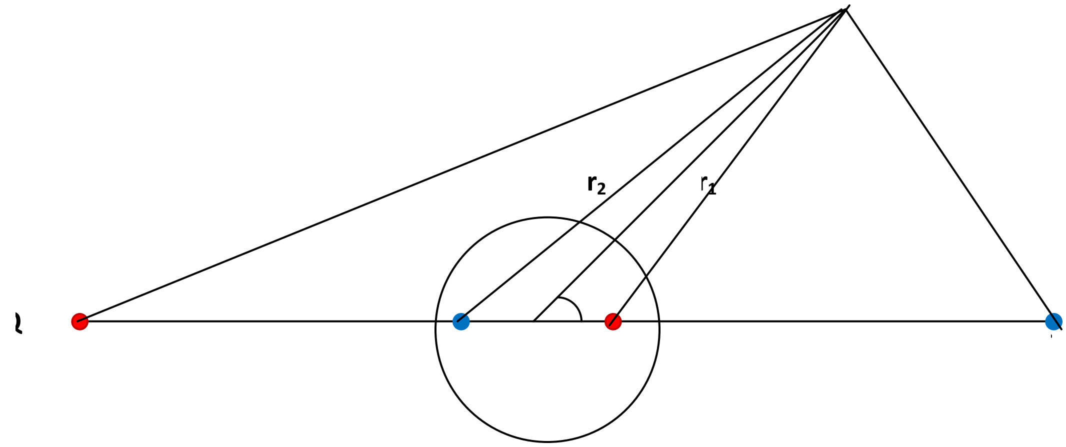
so that

$$\vec{E}(\vec{r})=\frac{Q}{4\pi\epsilon_0}\frac{2a\hat{k}}{a^3}=\frac{Q}{2\pi\epsilon_0 a^2}\hat{k}=E_0\hat{k}$$

so that the situation represents a constant electric field in the z direction with $E_0=\frac{Q}{2\pi\epsilon_0 a^2}$.



The potential due to the pair of charges is $-E_0\cos\theta$.



The potential due to the two image charges (shown in the figure, inside the sphere)

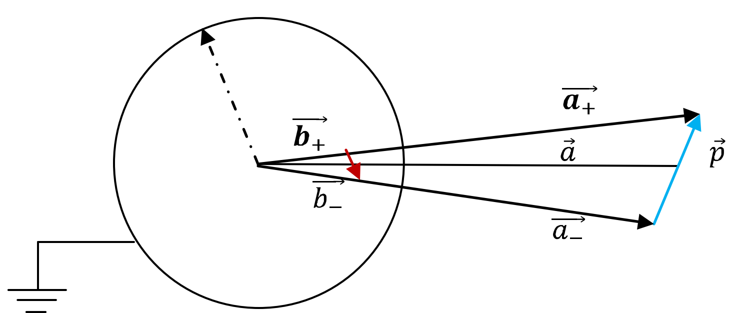
$$\begin{aligned}\varphi_{image} &= \frac{1}{4\pi\epsilon_0}\frac{QR}{a}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\\ &= \frac{1}{4\pi\epsilon_0}\frac{QR}{a}\left[\frac{1}{\left(\frac{R^4}{a^2}+r^2-2r\frac{R^2}{a}\cos\theta\right)^{\frac{1}{2}}}-\frac{1}{\left(\frac{R^4}{a^2}+r^2+2r\frac{R^2}{a}\cos\theta\right)^{\frac{1}{2}}}\right]\\ &= \frac{1}{4\pi\epsilon_0}\frac{QR}{a}\left[\frac{1}{r}\left(1+\frac{R^2}{ar}\cos\theta\right)-\frac{1}{r}\left(1-\frac{R^2}{ar}\cos\theta\right)\right]\\ &= \frac{1}{2\pi\epsilon_0}\frac{QR^3}{a^2r^2}\cos\theta = \frac{E_0R^3}{r^2}\cos\theta\end{aligned}$$

Thus the net potential is given by the sum of the potentials due to the charges $\pm Q$ and the potential due to the image charges,

$$\varphi = E_0\cos\theta\left(\frac{R^3}{r^2}-r\right)$$

as was obtained by earlier treatment.

Example :A dipole near aconducting sphere



The dipole is placed at the position $\vec{a}$. It has a dipole moment $\vec{p} = q(\overrightarrow{a_+} - \overrightarrow{a_-})$. The corresponding image charges are located $\overrightarrow{b_+}$and$\overrightarrow{b_-}$ (the direction of the dipole moment being from the negative charge to the positive charge, the image dipole moment vector is directed as shown in the figure sine the image charge has opposite sign with respect to real charge.) The image charges are as follows :

Corresponding to the charge $+q$ at $\overrightarrow{a_+}$ the image charge is

$$-q' = -\frac{R}{a_+}q \quad at \quad \overrightarrow{b_+} = \frac{R^2}{a_+}\hat{a}_+$$

And, corresponding to the charge $-q$ at $\overrightarrow{a_-}$ the image charge is

$$+q" = \frac{R}{a_-}q \quad at \quad \overrightarrow{b_-} = \frac{R^2}{a_-}\hat{a}_-$$

The image approximates a dipole with dipole moment

$$\overrightarrow{p'} = \left(q\frac{R^3}{a_-^3}\vec{a}_- - q\frac{R^3}{a_+^3}\vec{a}_+\right)$$

where,

$$\vec{a}_\pm = \vec{a} \pm \frac{(\vec{a}_+ - \vec{a}_-)}{2}$$

We have,

$$\begin{aligned}\frac{1}{a_\pm^3} &= \frac{1}{\left[\left|\vec{a} \pm \frac{(\vec{a}_+ - \vec{a}_-)}{2}\right|^2\right]^{\frac{3}{2}}} \\ &= \frac{1}{\left[a^2 + \frac{|\vec{a}_+ - \vec{a}_-|^2}{4} \pm \vec{a}\cdot(\vec{a}_+ - \vec{a}_-)\right]^{\frac{3}{2}}} \\ &= \frac{1}{a^3}\left(1 \mp \frac{3}{2}\frac{\vec{a}\cdot(\vec{a}_+ - \vec{a}_-)}{a^2}\right)\end{aligned}$$

where$|\vec{a}_+ - \vec{a}_-| \ll a$.

The dipole moment of the image is thus calculated as follows :

$$\begin{aligned}\overrightarrow{p'} &= \left(q\frac{R^3}{a_-^3}\vec{a}_- - q\frac{R^3}{a_+^3}\vec{a}_+\right) \\ &= \frac{R^3}{a^3}q\left[\left(1 + \frac{3}{2}\frac{\vec{a}\cdot(\vec{a}_+ - \vec{a}_-)}{a^2}\right)\vec{a}_- - \left(1 - \frac{3}{2}\frac{\vec{a}\cdot(\vec{a}_+ - \vec{a}_-)}{a^2}\right)\vec{a}_+\right]\end{aligned}$$

We can rearrange these to write,

$$\overrightarrow{p'} = \frac{R^3}{a^3}\left[q(\vec{a}_+ - \vec{a}_-) + \frac{3}{2}\frac{\vec{a}\cdot q(\vec{a}_+ - \vec{a}_-)}{a^2}(\vec{a}_+ - \vec{a}_-)\right]$$
$$= \frac{R^3}{a^3}\left(-\vec{p} + \frac{3}{2}\frac{\vec{a}\cdot\vec{p}}{a^2}(\vec{a}_+ - \vec{a}_-)\right)$$

The image charges, being of unequal magnitude, there is some excess charge within the sphere. The excess charge is given by

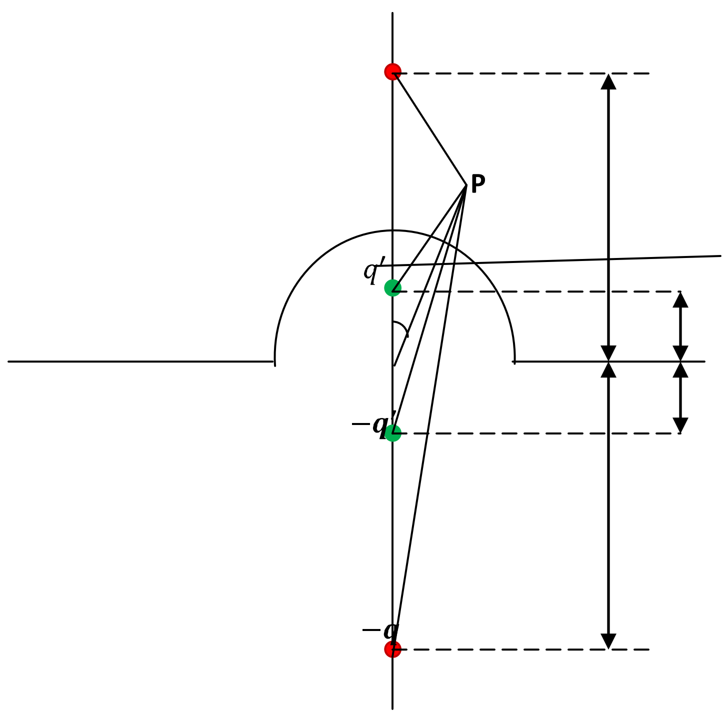
$$q_{excess} = Rq\left(\frac{1}{a_-} - \frac{1}{a_+}\right) \approx Rq\frac{a_+ - a_-}{a^2}$$

$$a_\pm = \sqrt{a^2 + \frac{d^2}{4} \mp ad\cos\theta} \approx a\left(1 \mp \frac{d}{2a}\cos\theta\right)$$
$$a_+ - a_- \approx -d\cos\theta$$

so that,

$$q_{excess} = -\frac{Rqd\cos\theta}{a^2} = +R\frac{\vec{p}\cdot\vec{a}}{a^3}$$

Example : A charge q in front of an infinite grounded conducting plane with a hemispherical boss of radius R directly in front of it.



The image charges along with their magnitudes and positions are shown in the figure. Here, $b = \frac{a^2}{R}, q' = q\frac{R}{a}$.The pairs$\pm q$ and $\pm q'$ make the potential on the plane zero, while the pairs q, q' and $-q, -q'$ make the potential on the hemisphere vanish. By uniqueness theorem, these four are the appropriate charges to satisfy the required boundary conditions.

If we take an arbitrary point $P(r, \theta)$, the potential at that point due to the four charge system is given by

$$\varphi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{q}{\sqrt{r^2 + a^2 + 2ra\cos\theta}} - \frac{\frac{qR}{a}}{\sqrt{r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a}\cos\theta}} + \frac{\frac{qR}{a}}{\sqrt{r^2 + \frac{R^4}{a^2} + \frac{2rR^2}{a}\cos\theta}}\right]$$

The electric field is obtained by taking the gradient of the above. As we are only interested in computing the surface charge density, we will only compute the radial component of above and evaluate it at r=R,

$$E_r = \frac{1}{4\pi\epsilon_0}\left[\frac{q(r - a\cos\theta)}{(r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} - \frac{q(r + a\cos\theta)}{(r^2 + a^2 + 2ra\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qR}{a}\left(r - \frac{R^2}{a}\cos\theta\right)}{\left(r^2 + \frac{R^4}{a^2} - \frac{2rR^2}{a}\cos\theta\right)^{\frac{3}{2}}} + \frac{\frac{qR}{a}\left(r + \frac{R^2}{a}\cos\theta\right)}{\left(r^2 + \frac{R^4}{a^2} + \frac{2rR^2}{a}\cos\theta\right)^{\frac{3}{2}}}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q(r - a\cos\theta)}{(r^2 + a^2 - 2ra\cos\theta)^{\frac{3}{2}}} - \frac{q(r + a\cos\theta)}{(r^2 + a^2 + 2ra\cos\theta)^{\frac{3}{2}}} - \frac{\frac{qR}{a}\frac{R^2}{a^2}\left(r\frac{a^2}{R^2} - a\cos\theta\right)}{\frac{R^3}{a^3}\left(r^2\frac{a^2}{R^2} + R^2 - 2ra\cos\theta\right)^{\frac{3}{2}}} + \frac{\frac{qR}{a}\frac{R^2}{a^2}\left(r\frac{a^2}{R^2} + a\cos\theta\right)}{\frac{R^3}{a^3}\left(r^2\frac{a^2}{R^2} + R^2 - 2ra\cos\theta\right)^{\frac{3}{2}}}\right]$$

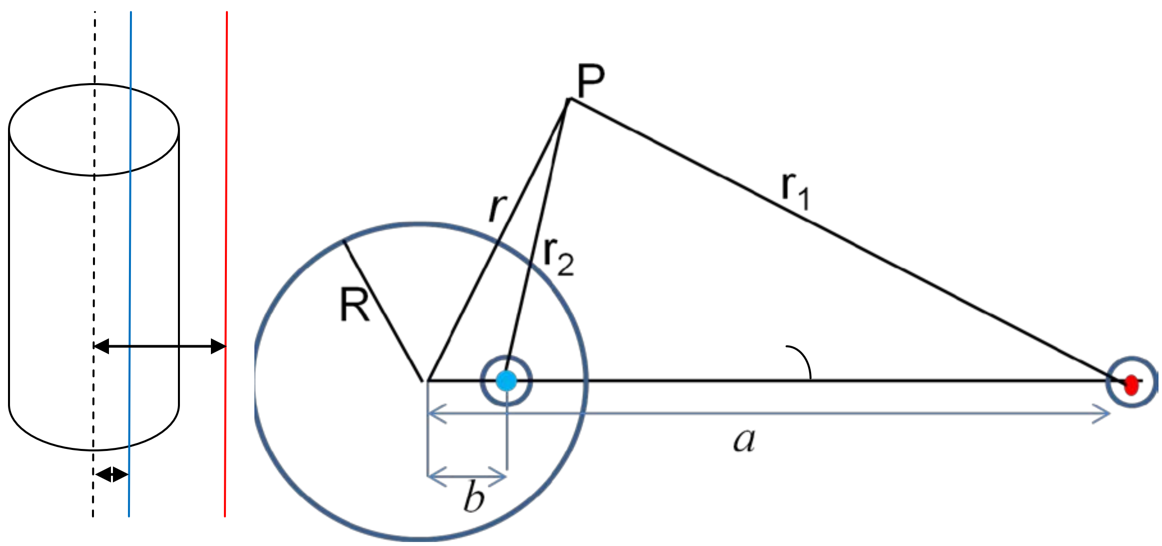
Substituting r=R and combining the first term with the third and the second with the fourth, we get,

$$\sigma = \epsilon_0 E_r|_{r=R}$$
$$= \frac{qR}{4\pi}\left(1 - \frac{a^2}{R^2}\right)\left[\frac{1}{(R^2 + a^2 - 2Ra\cos\theta)^{\frac{3}{2}}} - \frac{1}{(R^2 + a^2 + 2Ra\cos\theta)^{\frac{3}{2}}}\right]$$

The charge density can be integrated over the hemisphere to give the total charge on the hemispherical boss,

$$Q_{boss} = \frac{qR}{4\pi}\left(1-\frac{a^2}{R^2}\right)\left[\int_0^{\frac{\pi}{2}} \frac{2\pi R^2 \sin\theta d\theta}{(R^2+a^2-2Ra\cos\theta)^{\frac{3}{2}}} - \int_0^{\frac{\pi}{2}} \frac{2\pi R^2 \sin\theta d\theta}{(R^2+a^2+2Ra\cos\theta)^{\frac{3}{2}}}\right]$$
$$= -\frac{qR}{2}\left(1-\frac{a^2}{R^2}\right)\left[\int_0^{1} \frac{R^2 d\mu}{(R^2+a^2-2Ra\mu)^{\frac{3}{2}}} - \int_0^{1} \frac{R^2 d\mu}{(R^2+a^2+2Ra\mu)^{\frac{3}{2}}}\right]$$
$$= -q + \frac{q(a^2-R^2)}{a\sqrt{a^2+R^2}}$$

Example : Line charge near a conducting cylinder



The potential at an arbitrary point $P(r,\theta)$ is given by

$$\varphi(r,\theta) = \frac{\lambda}{2\pi\epsilon_0}\ln r_1 - \frac{\lambda'}{2\pi\epsilon_0}\ln r_2$$
$$= -\frac{1}{4\pi\epsilon_0}(\lambda\ln(a^2+r^2-2ar\cos\theta) + \lambda'\ln(b^2+r^2-2br\cos\theta))$$

Equating the tangential component of electric field on the surface to zero, we have,

$$E_t = -\left.\frac{\partial\varphi}{\partial\theta}\right|_{r=R} = \frac{1}{4\pi\epsilon_0}\left[\frac{\lambda(2aR\sin\theta)}{a^2+R^2-2aR\cos\theta} + \frac{\lambda'(2bR\sin\theta)}{b^2+R^2-2bR\cos\theta}\right] = 0$$

Taking one of the terms to the other side and cross multiplying,

$$\lambda(2aR\sin\theta)\ \ (b^2+R^2-2bR\cos\theta) = -\lambda'(2bR\sin\theta)(a^2+R^2-2ar\cos\theta)$$

As this is valid for all values of θ, we have, on equating coefficient of $\sin\theta$ and that of $\sin 2\theta$ to zero,

$$2aR\lambda(b^2+R^2) = -2bR\lambda'(a^2+R^2)$$
$$-2\lambda abR = 2\lambda' abR$$

The second relation gives, $\lambda' = -\lambda$ and substituting this in the first relation, we get,

$$a(b^2+R^2) = b(a^2+R^2)$$
$$b = \frac{R^2}{a}$$

SPECIAL TECHNIQUES-II

Lecture 18: Electromagnetic Theory

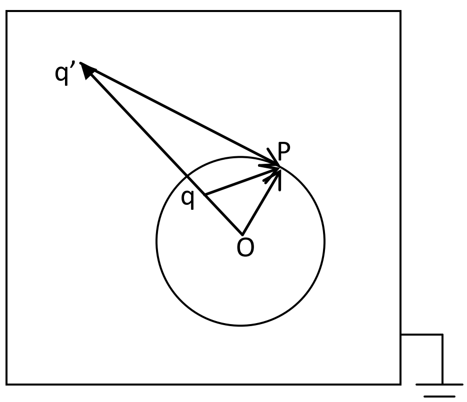
Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Tutorial Assignment

1. A charge q is placed in front of a conducting sphere of radius R which is maintained at a constant potential φ_0. Obtain an expression for the potential at points outside the sphere.
2. A spherical cavity of radius R is scooped out of a block of conductor which is maintained at zero potential. A point charge is placed inside the cavity at a distance r_0 from its centre. Obtain the image charges, an expression for the potential inside the cavity as also the induced charge on the cavity wall.
3. A conducting sphere of radius R, containing a charge Q, is kept at a height h above a grounded, infinite plane. Calculate the amount and location of the image charges.
4. Two identical spheres of radius R each, contain charge Q and $-Q$. The spheres are separated by a distance d. Locate the image charges.

Solution to Tutorial Assignment

1. We have seen that when the sphere is maintained at zero potential, we can introduce an image charge $q' = -\frac{qR}{a}$ at a distance $\frac{R^2}{a}$ from the centre of the sphere. If we want the sphere to be maintained at a constant potential, we imagine an additional image charge q"at the centre of the sphere which keeps the potential constant at $\frac{q"}{4\pi\epsilon_0 R}$. Equating this to φ_0, we take this charge at the centre to be equal to $q" = 4\pi\varphi_0\epsilon_0 R$. The potential at an arbitrary point located at a distance r from the centre is thus given by
$$\varphi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r}-\vec{a}|} - \frac{\left(\frac{R}{a}\right)q}{\left|\vec{r}-\frac{R^2}{a^2}\vec{a}\right|} + \frac{4\pi\varphi_0\epsilon_0 R}{r}\right]$$
2. Let us put an image charge q' at a distance $r_0{}'$ along the line joining the centre of the sphere to the position of the charge q which is taken along the direction $\hat{n}_0$. Take a point within the cavity along a direction $\hat{n}$. The potential at such an arbitrary point is given by



$$\varphi(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|r\hat{n} - r_0\hat{n}_0|} + \frac{q'}{|r\hat{n} - r'_0\hat{n}_0|}\right]$$
If the point is chosen on the surface of the cavity, this must give zero value for the potential. Following identical method as shown in the lecture, the image charge is found to be $q' = -q\frac{R}{r_0}$ and the location of the image charge is $\frac{R^2}{r_0}$. The surface charge density is found to be given by
$$\frac{q}{4\pi R}\frac{r_0^2 - R^2}{(R^2 + r_0^2 - 2r_0 R\cos\theta\ )^{\frac{3}{2}}}$$

3. Imagine the charge Q to be at the centre of the sphere. The sphere is at a constant potential. In order to keep the plane at zero potential an image charge $Q_1' = -Q$ is there at a distance h below

the plane, i.e. at a distance 2h from the centre of the sphere. This would disturb the potential on the sphere which has to compensated by an image charge $Q_1 = +Q\,\frac{R}{2h}$ at a distance $b_1 = \frac{R^2}{2h}$ from the centre of the sphere. As this image charge is located at a distance $h - b_1$ from the plane, the image charge due to this is $Q_2' = -Q_1$ at a distance $h - b_1$ below the plane, i.e. at a distance $2h - b_1$ from the centre of the sphere. The image in the sphere is a charge $Q_2 = Q_1\,\frac{R}{2h-b_1}$ at a distance $b_2 = \frac{R^2}{2h-b_1}$ from the centre of the sphere. Thus we see that there would be infinite number of images $Q_1', Q_2', Q_3', \cdots$ below the plane and $Q_1, Q_2, Q_3, \ldots$ inside the sphere. There seems to be a pattern for the magnitude and the location of these images. Inside the sphere,$Q_1 = Q\,\frac{R}{2h}$

at $b_1 = \frac{R^2}{2h}, Q_2 = Q_1\,\frac{R}{2h-b_1}$ at $b_2 = \frac{R^2}{2h-b_1},$ etc. If we define $Q_0 = Q, b_0 = 0$, it seems to suggest $Q_n = Q_{n-1}\,\frac{R}{2h-b_{n-1}}$ at $b_n = \frac{R^2}{2h-b_{n-1}}$. This assertion can be easily proved as follows. If true, the nth image is at a distance $h - b_n$ from the plane. The image of this in the plane is at a distance $2h - b_n$ from the centre of the sphere and has a charge $-Q_n$. The image in the sphere is $+Q_n\,\frac{R}{2h-b_n}$ at a distance $\frac{R^2}{2h-b_n}$ from the centre of the sphere. These are, respectively, the charge Q_{n+1} and the distance b_{n+1}.

Thus we have

$$Q_n = Q_{n-1}\frac{R}{2h - b_{n-1}} = Q_{n-1}\frac{b_n}{R}$$
$$Q_{n+1} = Q_n\frac{R}{2h - b_n}$$

Rearranging,

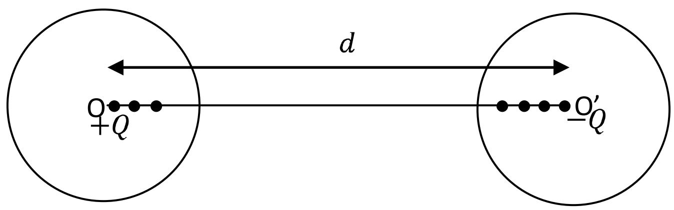
$$\frac{1}{Q_{n+1}} + \frac{1}{Q_{n-1}} = \frac{1}{Q_n}\left[\frac{2h - b_n}{R} + \frac{b_n}{R}\right]$$
$$= \frac{2h}{R}\frac{1}{Q_n}$$

This is a difference equation which can be solved by assuming a solution of the form, $\frac{1}{Q_n} = A\lambda^n$, which gives $\lambda^{n+1} - \frac{2h}{R}\lambda^n + \lambda^{n-1} = 0$, which has the non-zero solution $\lambda = \frac{h}{R} \pm \frac{\sqrt{h^2-R^2}}{R}$. It can be seen that in order that the series converges for all distances $h > R$, the positive square root is to be taken. The total charge in the sphere is

$$\sum_{n=0}^{\infty} Q_n = Q_0 \sum_{n=0}^{\infty} (\frac{1}{\lambda})^n = \frac{Q_0}{1 - \frac{1}{\lambda}} = \frac{Q_0}{1 - \frac{h}{r} - \frac{\sqrt{h^2-R^2}}{R}}$$

4. Consider the image of charge +Q on the other sphere. It is a charge $Q_1' =- Q\frac{R}{d}$ at a distance $b_1' = \frac{R^2}{d}$ from the centre of right hand sphere O′. The distance of this from O is $d' = d - \frac{R^2}{d} = \frac{d^2-R^2}{d}$ from O.

   Similarly, image of $-Q$ in the sphere to the left is $Q_1 = +Q\frac{R}{d}$ at $b_1 = \frac{R^2}{d}$. The image of Q_1' in the sphere to the left is $Q_2 = +\left(Q\frac{R}{d}\right)\frac{Rd}{d^2-R^2} = \frac{Q_1 R}{d-\frac{R^2}{d}} = Q_1\frac{R}{d-b_1}$ at $b_2 = \frac{R^2}{d'} = \frac{R^2 d}{d^2-R^2} = \frac{R^2}{d-b_1}$.



It can be seen that there are infinite number of images with the images on the left being

$Q_n = Q_{n-1}\frac{R}{d-b_{n-1}}$ at $b_n = \frac{R^2}{d-b_{n-1}}$.

SPECIAL TECHNIQUES-II

Lecture 18: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

Self Assessment Quiz

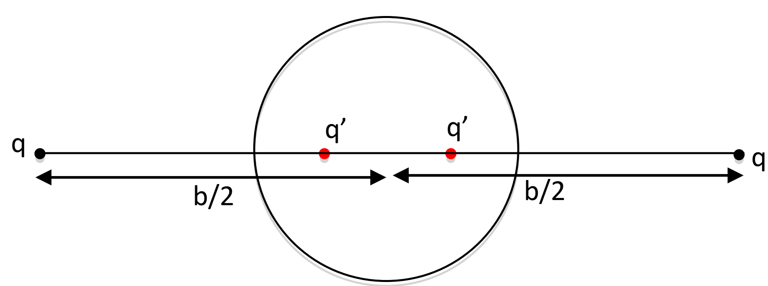
1. Consider a point charge in front of an insulated, charged conducting sphere of radius R. If the sphere is to contain a charge Q, what is the potential outside the sphere?
2. A conducting sphere of radius R has a charge Q. A charge q is located at a distance 3R from the centre of the sphere. Calculate the potential at a distance R/2 from the centre of the sphere along the line joining the centre with the charge q.
3. Two point charges q each are at a distance d from each other. What should be the minimum radius R of a grounded sphere that can be put with its centre at the mid point of the line joining the two charges so that the mutual repulsion between the point charges is compensated? Assume d >> R.
4. Two spheres, each of radius R contain identical charge Q. The spheres are separated by a negligible distance. Locate all image charges.

Solutions to Self Assessment Quiz

1. Initially, assume that the sphere is grounded. If the charge q is at the position $a\hat{n}$ with respect to the centre of the sphere, an image charge $q' = -q\frac{R}{a}$ located at $\frac{R^2}{a}\hat{n}$ satisfies the required boundary conditions. In order that the sphere has a total charge Q, we now disconnect it from the ground and add $Q + \frac{qR}{a}$ amount of charge to the sphere, which will be uniformly distributed over the surface. The potential at $\vec{r}$ is then given by,

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\vec{r} - \hat{n}a|} - \frac{q(\frac{R}{a})}{\left|\vec{r} - \frac{R^2}{a}\hat{n}\right|} + \frac{Q + \frac{qR}{d}}{r}\right]$$

2. The image of q in the sphere is a charge $-q\frac{R}{3R} = -\frac{q}{3}$ at $\frac{R^2}{3R} = \frac{R}{3}$ from the centre. Since the sphere is a conductor all parts of it are equipotential. As the overall charge is Q, it can be looked upon as an image charge $-\frac{q}{3}$ at R/3 which cancels the potential due to the charge q outside and a charge at the centre equal to $Q + \frac{q}{3}$. The potential of the sphere due to this is $\frac{1}{4\pi\epsilon_0}\frac{Q+q/3}{R}$.
3. Each of the point charge gives an image charge within the sphere with charge $-q\ \frac{R}{b/2} = -q\frac{2R}{b}$ at a distance $\frac{R^2}{b/2} = \frac{2R^2}{b}$ from the centre towards the charge.



These two image charges, having opposite sign to that of the original charges, attract each of them. The distance of these image charges from either of the charges is $\frac{b}{2} \pm \frac{2R^2}{b} = \frac{b^2 \pm 4R^2}{2b}$. The net attraction due to these charges must cancel the repulsive force $\frac{q^2}{4\pi\epsilon_0 b^2}$. Thus, we have,

$$\frac{q^2}{b^2} = q^2 \frac{2R}{b}\left[\frac{4b^2}{(b^2+4R^2)^2} + \frac{4b^2}{(b^2-4R^2)^2}\right]$$

Simplifying, we get,

$$b^8 + 256\, R^8 - 32\, R^4 = 16\, Rb^7 + 256\, R^5 b^3$$

Since b>> R, as a first approximation we can neglect all terms other than the first term on either sides of this equation, and get, $b^8 = 16Rb^7$ which gives $R = \frac{b}{16}$.

4. The problem is very similar to Problem 4 of the tutorial assignment with the difference that both the charges being +Q, the image charges alternate in sign. Taking d=2R, we have, for image on the left sphere,

$$Q_1 = -Q\frac{R}{d} = -\frac{Q}{2}\,\mathrm{at} b_1 = \frac{R^2}{2R} = \frac{R}{2}$$
$$Q_2 = +\frac{Q}{2}\frac{R}{2R - \frac{R}{2}} = +\frac{Q}{3}\,\mathrm{at} b_2 = \frac{R^2}{2R - \frac{R}{2}} = \frac{2}{3}R$$
$$Q_3 = -\frac{Q}{3}\frac{R}{2R - \frac{2}{3}R} = -\frac{Q}{4}\,\mathrm{at} b_3 = \frac{R^2}{2R - \frac{2R}{3}} = \frac{3R}{4}$$

etc. The total induced charge is $-Q\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots\right) = -Q(1 - \ln 2)$.