MA2102: Linear Algebra

Lecture 29: Characteristic Polynomial

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Motivated by the observation that λ is an eigenvalue of T if and only if $\det(T-\lambda I_V)=0$, we consider the polynomial

$$p_T(x)=\det(T-xI_V).$$

If V has dimension n over F, then choose a basis β of V. If $[T]_\beta$ has entries a_{ij}, then

$$[T-xI_V]_\beta=[T]_\beta-xI_n=\begin{pmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{pmatrix}.$$

Thus, $p_T(x)$ is a polynomial of degree n. The roots of this polynomial (in F) are the eigenvalues of T.

Definition For $A \in M_n(F)$ the polynomial

$$p_A(x) = \det(A - xI_n)$$

is called the characteristic polynomial of A.

For a linear map $T : V \to V$ the polynomial $p_T(x) = \det(T - xI_V)$ is called the characteristic polynomial of T.

Example (1) For $A \in M_2(F)$ given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the characteristic polynomial is given by

$$p_A(x) = (a-x)(d-x) - bc = x^2 - (a+d)x + (ad-bc).$$

Thus, $p_A(x)$ has $-$trace(A) and $\det(A)$ as coefficients.

(2) For $A = D(\lambda_1, \ldots, \lambda_n) \in M_n(F)$, we have

$$p_A(x) = (x - \lambda_1) \cdots (x - \lambda_n).$$

Prove an analogous result for upper triangular matrices.

(3) Let $T : V \to V$ be a nilpotent operator, i.e., $T^k = 0$ for some positive integer k. If $T(v) = \lambda v$ for a non-zero vector v, then

$$0 = T^k(v) = \lambda^k v$$

which implies that $\lambda^k = 0$. Thus, zero is the only eigenvalue of T. To prove this using characteristic polynomial, suppose $\lambda \neq 0$ is an eigenvalue, i.e., $\det(T - \lambda I_V) = 0$. This means

$$0 = \det([T]_\beta - \lambda I_n) = \det(-\lambda(\lambda^{-1}[T]_\beta + I_n)) = (-\lambda)^n \det(\lambda^{-1}[T]_\beta + I_n).$$

Note that $([T]_\beta)^k = 0_k$ and $I_n + \lambda^{-1}[T]_\beta$ is invertible (exercise).

Thus, $(-\lambda)^n \det(\lambda^{-1}[T]_\beta + I_n)$ is non-zero, a contradiction. We see that zero is the only possible eigenvalue. Note that there is a smallest integer l such that $T^l = 0$ but T^{l-1} is not the zero map. Therefore, there exists $v \in V$ such that $T^{l-1}(v) \neq 0$ but $T^l(v) = 0$. In particular, $T^{l-1}(v) \neq 0$ is a null vector for T. As T is not injective, $\det(T) = 0$.

Given $A \in M_n(F)$, consider the set

$$S = \{I_n, A, A^2, \ldots, A^{n^2}\}.$$

There are two mutually exclusive and exhaustive cases:

Case 1: Either two elements in S are the same, i.e., $A^k = A^l$ for some $0 \leq k < l \leq n^2$. Then A satisfies the equation $x^k = x^l$.

Case 2: S is of size $n^2 + 1$. As $M_n(F)$ has dimension n^2, the set S is linearly dependent.

There exists $a_i \in F$ such that

$$a_0 I_n + a_1 A + \cdots + a_{n^2} A^{n^2} = 0_n.$$

Thus, A satisfies the polynomial $a_0 + a_1 x + \cdots + a_{n^2} x^{n^2} = 0$.

As any matrix satisfies some polynomial, for each $A \in M_n(F)$, there will be a unique monic polynomial of the smallest degree which A satisfies. This polynomial is called the minimal polynomial of A. If $A = [T]_\beta$ for $T : V \to V$ and some β, then

$$c_0 I_n + c_1 A + \cdots + c_k A^k = 0_n$$

implies that (and is implied by)

$$c_0 I_V + c_1 T + \cdots + c_k T^k = 0.$$

Thus, minimal polynomial of T and $[T]_\beta$ coincide.

Example (1) Let $P : V \to V$ be a projection map, i.e., $P^2 = P$. It satisfies $x^2 = x$ and we had seen (cf. lecture 28) that the eigenvalues are 0 and 1. Unless $P = 0$, the minimal polynomial of P is $x^2 - x = 0$.

(2) Let $A \in M_2(F)$ be given. If $p_A(x)$ is the characteristic polynomial of A, then

$$\begin{aligned} p_A(A) &= A^2 - \mathrm{trace}(A)A + \det(A)I_2 \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

is the zero matrix (exercise). Unless A is a scalar matrix, the characteristic polynomial is also the minimal polynomial for A.

Question *Is there an integer $N(n)$ such that any matrix $A \in M_n(F)$ satisfies a polynomial of degree $N(n)$?*

Theorem [Cayley-Hamilton]
A square matrix satisfies its characteristic polynomial.

How not to prove this: Consider the polynomial $p_A(x) = \det(A - xI_n)$ and simply plug in $x = A$!

Note that similar marices have the same characteristic and minimal polynomial.

Remark One proves that a generic matrix over an algebraically closed field is diagonalizable. Then one checks Cayley-Hamilton for diagonal matrices. This can then be extended to arbitrary rings because it is a polynomial identity.

Cayley-Hamilton Theorem implies that the minimal polynomial of A divides the characteristic polynomial. The converse is also true (over a field).