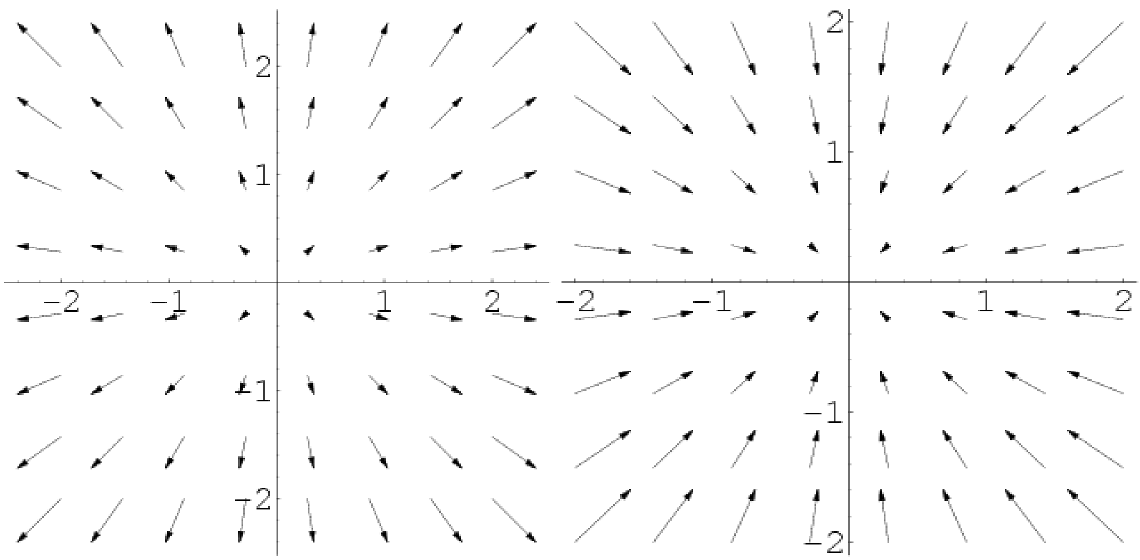
Elements of Vector Calculus : Divergence and Curl of a Vector Field

Lecture 3: Electromagnetic Theory

Professor D. K. Ghosh, Physics Department, I.I.T., Bombay

In the last lecture we had defined divergence of a vector field and had obtained an expression for the divergence in the Cartesian coordinates. We also derived the divergence theorem which connects the flux of a vector field with the volume integral of the divergence of the field.

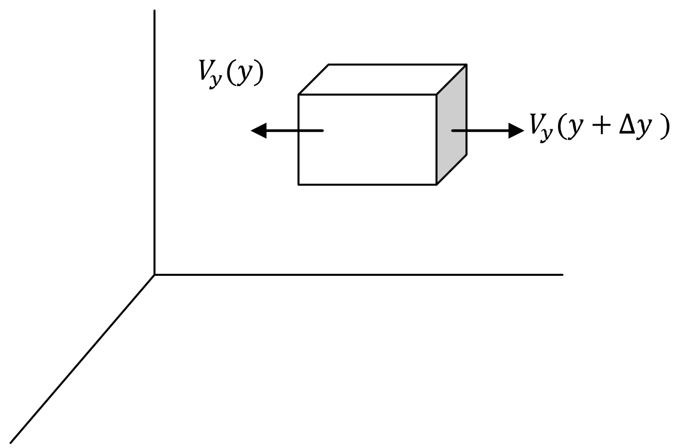
Physically, the divergence, as the name suggests is a measure of the amount of spread that the field has at a point.



For instance, in the figure above, the vector field shown to the left has a positive divergence while that to the right has a negative divergence. In electrostatics, we will see that the field produced by a positive charge has positive divergence while a negative charge produces an electrostatic field with negative divergence.

Divergence, curl etc. were extensively used in fluid dynamics from which a lot of nomenclature have arisen. Let us consider a fluid flowing through an elemental volume of dimension $\Delta x \times \Delta y \times \Delta z$ with its sides oriented parallel to the Cartesian axes. In the figure below, we show only the y-component of the velocity of the fluid entering and leaving the elemental volume. Let the density of the fluid at (x, y, z) be $\rho(x, y, z)$ and the y-component of the velocity be v_y. We define a vector $\vec{V}(x, y, z) = \rho(x, y, z)\vec{v}(x, y, z)$. The mass of the fluid flowing into the volume per unit time through the left face which has an outward normal $\hat{n} = -\hat{j}$ is given by

$$\rho(x,y,z)v_y(x,y,z)\Delta x\Delta z = V_y(x,y,z)\Delta x\Delta z$$



Mass of the fluid flowing out is $V_y(x,y+\Delta y,z)\Delta x\Delta z$. Retaining only the first order term in a Taylor series expansion, we have

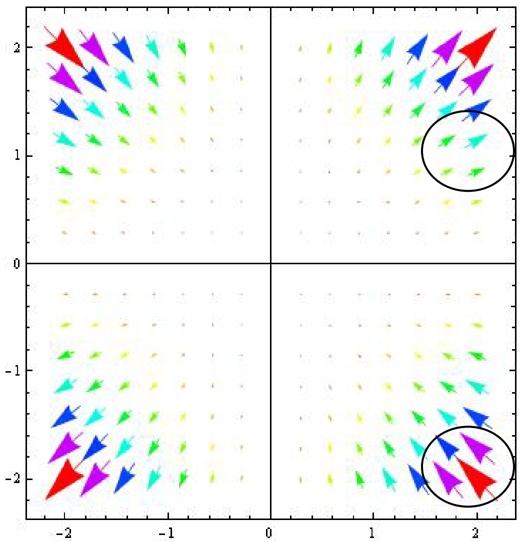
$$V_y(x,y+\Delta y,z) = V_y(x,y,z) + \frac{\partial V_y}{\partial y}\Delta y$$

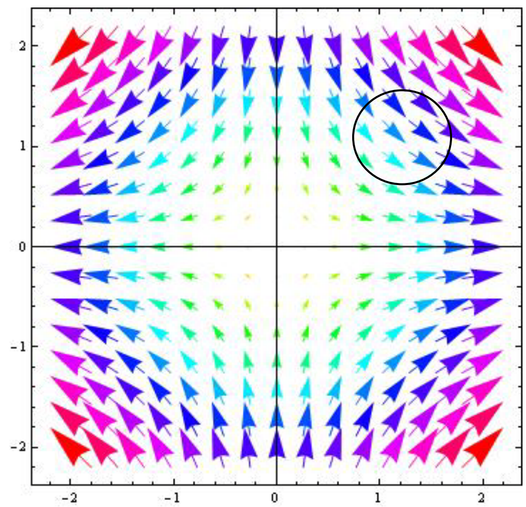
Thus the net increase in mass is $-\frac{\partial V_y}{\partial y}\Delta x\Delta y\Delta z$. This is the increase due to the y-component of the velocity. We can write similar expressions for the flow in the x and z directions. The net increase in mass per unit time is – $\left(\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}\right)\Delta x\Delta y\Delta z = -\vec{\nabla}\cdot\vec{V}\Delta\tau$, where we have put the volume element $\Delta\tau = \Delta x\Delta y\Delta z$ so as not to confuse with the vector $\vec{V}$ defined above.

Another of of talking about the net increase in mass is to realize that since the volume is fixed, the increase in mass is due to a change in the density alone. Thus the rate of increase of mass is $\frac{\partial\rho}{\partial t}\Delta\tau$. Equating these two expressions, we get what is known as the **equation of continuity** in fluid dynamics,

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{V} = 0}$$

To see what this equation implies, consider, for example a vector field given by $\vec{F} = x^2y\hat{\imath} + xy^2\hat{\jmath}$. The field has been plotted using Mathematica (see figure)





To the left is plotted $\vec{F} = x^2 y\hat{\imath} + xy^2\hat{\jmath}$. Note that in the first and the third quadrants the divergence is positive while in the other two quadrants it is negative. The figure to the right is for the force field $x\hat{\imath} - y\hat{\jmath}$ which has zero divergence.

The size of the arrow roughly represents the magnitude of the vector. The divergence is given by $4xy$. In the first and the third quadrants (x,y both positive or both negative) divergence is positive. One can see that in these quadrants, if you take any closed region the size of the arrows which are entering the region are smaller than those leaving it. Thus the density decreases, divergence is positive. Reverse is true in the even quadrants.

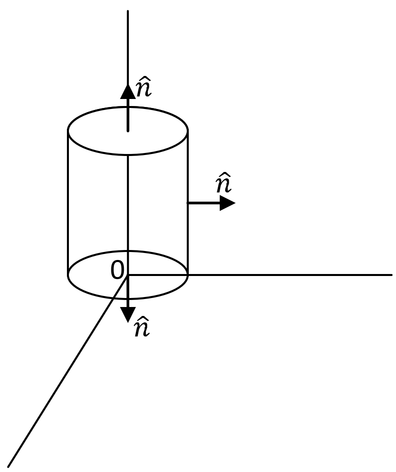
In the figure to the right, the force field $x\hat{\imath} - y\hat{\jmath}$ has zero divergence. If you take a closed region in this figure, you find as many vectors are getting in as are going out. The field is a solenoidal (zero divergence) field.

Divergence Theorem – Examples :

Recall divergence theorem :$\oint_S \vec{F}\cdot d\vec{S} = \int_V \mathrm{div}\vec{F}\ dV$, where the surface integral is taken over a closed surface defining the enclosed volume. As an example consider the surface integral of the position vector $\vec{r}$ over the surface of a cylinder of radius *a* and height *h*.

Evaluating the surface integral by use of the divergence theorem is fairly simple. Divergence of position vector has a value 3 because $\nabla \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$. Thus the volum integral of the divergence is simply three times the volume of the cylinder which gives $3\pi a^2 h$.

Direct calculation of the surface integral can be done as follows. For convenience, let the base of the cylinder be in the x-y plane with its centre at the origin.



There are three surface of the cylinder, a top cap, a bottom cap and the curved surface. For the top cap, the normal vector $\hat{n}$ is along the $\hat{k}$ direction, so that $\vec{r} \cdot \hat{n} = z$. On the top surface z is constant and is given by z=h. Thus $\int_S \vec{r} \cdot \hat{n} dS = \pi a^2 h$. For the bottom cap, $\hat{n}$ is along the $-\hat{k}$ direction so that $\vec{r} \cdot \hat{n} = -z$. However, the value of z on this surface being zero, the flux vanishes. We are now left with the curved surface for which the outward normal is parallel to the x-y plane. The unit vector on this surface is $\frac{\hat{\imath}x + \hat{\jmath}y}{\sqrt{x^2+y^2}}$. However, on the surface $\sqrt{x^2 + y^2} = a$. Thus $\vec{r} \cdot \hat{n} = \left(\hat{\imath}x + \hat{\jmath}y + \hat{k}z\right) \cdot \frac{\hat{\imath}x + \hat{\jmath}y}{a} = \frac{x^2+y^2}{a} = a$. Thus the surface integral is $\int_S \vec{r} \cdot \hat{n} dS = a \int_S dS = a(2\pi a h) = 2\pi a^2 h$. Adding to this the contribution from the top and the bottom face, the surface integral works out to $3\pi a^2 h$, as was obtained from the divergence theorem

As a second example, consider a rather nasty looking vector field $\vec{F} = (2x + z^5)\hat{\imath} + (y^2 + \sin^2 xz)\hat{\jmath} + (xz + y^3 e^{-x^2})\hat{k}$ over the surface of a cubical box $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. We will not attempt to calculate the surface integral directly. However, the divergence theorem gives a helping hand.

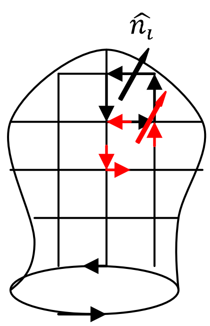
$$\text{div}\vec{F} = \frac{\partial}{\partial x}((2x + z^5) + \frac{\partial}{\partial y}(y^2 + \sin^2 xz) + \frac{\partial}{\partial z}(xz + y^3 e^{-x^2})$$
$$= 2 + 2y + x$$

We need to calculate the triple integral $\iiint(2 + 2y + x)dxdydz$. As the integrand has no z dependence, the z-integral evaluates to 1. The volume integral is $\int_0^1\int_0^1(2 + 2y + x)dxdy = 2 + 2.\frac{1}{2} + \frac{1}{2} = \frac{7}{2}$

Curl of a Vector Field :

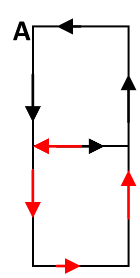
We have seen that the divergence of a vector field is a scalar field. For vector fields it is possible to define an operator which acting on a vector field yields another vector field. The name curl comes from "circulation" which measures how much does a vector field "curls" about a point.

Consider an open surface of the type shown – something like an inverted pot with a rim. We wish to calculate the surface integral of a vector field defined over this surface.



Let us divide the surface into a large number of small segments and calculate the line integral over the boundary of such an elemental surface. One such elemental surface with boundary is shown. If we define the bounding curve of this surface in the counterclockwise direction, the normal to such a surface will be outward. If we consider each such segment i and take the line integral over the bounding curve ΔC_i, the contribution to the line integral from adjacent regions will cancel because the line integral from the boundary are traversed in opposite sense (look at the black and the red arrows on the boundary of two segments). Considering all such segments, we will be only left with the uncompensated line integral at the rim. This is illustrated in the following figure :

The line integral on the boundary for ABCD + line integral for The boundary of CDEF is equal to the line integral for ABEF because the integral is traversed along CD in the former and along DC in the latter.



Thus we have,

$$\oint_C \vec{F}\cdot d\vec{l} = \sum_i \left(\oint_{C_i}\vec{F}\cdot d\vec{l}\right) = \sum_i \left(\frac{\oint_{C_i}\vec{F}\cdot d\vec{l}}{\Delta S_i}\right)\Delta S_i \qquad (1)$$

The quantity in the parenthesis in the last expression is defined as the curl of the vector field $\vec{F}$ when the limit of the elemental surface goes to zero. Curl being a vector, its direction is specified as the outgoing normal to the surface element.

$$\textbf{Curl}\vec{\boldsymbol{F}} = \lim_{\Delta S_i\to 0}\frac{\oint_{C_i}\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{l}}}{\Delta S_i}\widehat{\boldsymbol{n}_i} \qquad \textbf{(2)}$$

.It may be noted that because the definition is valid in the limit of the surface area going to zero, it is a point relationship. Using this definition, we can write the previous equation (1) as

$$\oint_C \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{l}} = \int_S \textbf{Curl}\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{S}}$$

This equation relates the surface integral of the curl of a vector field with the line integral of the vector field and is known as "**Stoke's Theorem**".

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Tutorial

1. Calculate the flux of the vector field $\vec{F} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$ over the surface of a unit cube whose edges are parallel to the axes and one of the corners is at the origin. Use this result to illustrate the divergence theorem.
2. Calculate the flux of the vector field $\vec{F} = z\hat{\imath} + y\hat{\jmath} + x\hat{k}$ over the surface of a unit sphere. Use this result to illustrate the divergence theorem. (Use spherical coordinates).
3. Calculate the flux of the vector $\vec{F} = x^4y\hat{\imath} - 2x^3y^2\hat{\jmath} + z^2\hat{k}$ over the surface of a right circular cylinder of radius R bounded by the surfaces z=0 and z=h. Calculate it directly as well as by use of the divergence theorem.
4. Calculate the flux of the position vector $\int_S \vec{r}\cdot\hat{n}dS$ through a torus of inner radius *a* and outer radius *b*. Use the result to illustrate divergence theorem. (* This is a hard problem).
5. Calculate the flux of the vector field $\vec{F} = xy^2\hat{\imath} + yz\hat{\jmath} + yx^2\hat{k}$ over the surface defined by $z = 4 - x^2 - y^2; z \geq 3$.

Solution to Tutorial Problems :

1. The geometry of the cube along with the direction of surface normalsare shown in the figure. Consider the base of the cube which is the plane z=0. On this face $\vec{F} = xy\hat{\imath}$. Since the normal is along the -z direction flux from this face is zero. Similarly, the flux from the other two faces which meet at the origin are also zero. Consider the top face where z=1. On this face $\vec{F} = xy\hat{\imath} + y\hat{\jmath} + x\hat{k}$. The normal is in the +z direction, so that the flux is $\int_0^1 xdx \int_0^1 dy = 1/2$. Likewise, the flux from the other two faces are also ½ each. The total flux, therefore, is 3/2. The divergence of the field $\vec{F} = xy\hat{\imath} + yz\hat{\jmath} + zx\hat{k}$ is

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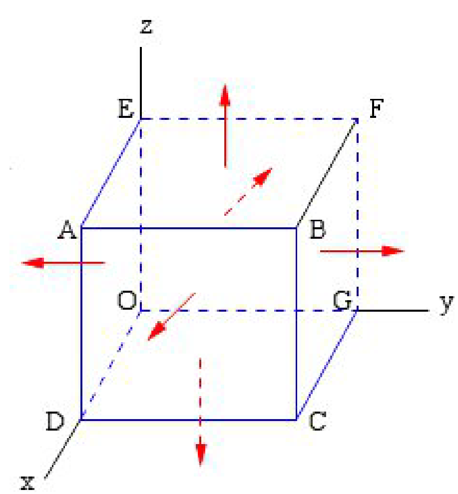


$$\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz)$$
$$= y + z + x$$

The volume integral of the divergence is

$$\iiint (x + y + z)dxdydz$$

By symmetry, this is 3 times $\iiint xdxdydz = \int_0^1 xdx \int_0^1 dy \int_0^1 dz = 1/2$. Thus the volume integral of the divergence is 3/2.

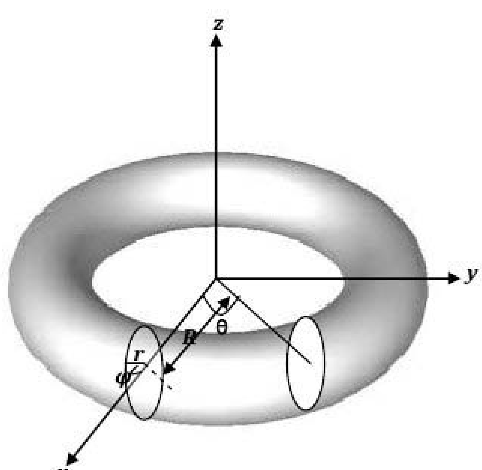
2. Divergence of the field is $0 + 1 + 0 = 1$. Thus the volume integral of the divergence over the surface of a unit sphere is just $\frac{4\pi}{3}$. To calculate the surface integral we note that the normal on the surface of the sphere is along the radial direction and is given by $\hat{n} = \frac{\hat{\imath}x + \hat{\jmath}y + \hat{k}z}{R}$, where R=1 is the radius of the sphere. Thus $\vec{F} \cdot \hat{n} = (2xz + y^2)$. Since the surface element on the sphere is $R^2 \sin\theta d\theta d\varphi = \sin\theta d\theta d\varphi$, we have , substituting (in spherical polar) $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$,

$$\int_S \vec{F}\cdot\hat{n} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta (2\sin\theta\cos\theta\cos\varphi + \sin^2\theta\sin^2\varphi)$$

The first term gives zero because of vanishing of the integral over φ. We are left with

$$\int_S \vec{F}\cdot\hat{n} = \int_0^{2\pi} \sin^2\varphi\, d\varphi \int_0^{\pi} \sin^3\theta\, d\theta = \pi\cdot\left[-\cos x + \frac{\cos^3 x}{3}\right]_0^{\pi} = \frac{4\pi}{3}$$

3. Let the base of the cylinder be at z=0 and the top at z=h. The origin is at the centre of the base. The cylinder has three surfaces. For the bottom surface, the direction of the normal is along $-\hat{k}$ and on this surface z=0. The surface integral for this surface is $\int_S -z^2 dS = 0$. For the top surface, the normal is along $+\hat{k}$, z=h, the surface integral is $\int_S z^2 dS = h^2 \int_S dS = h^2\pi R^2$. For the curved surface the direction of the normal is outward radial direction in the x-y plane which is $\frac{\hat{\imath}x+\hat{\jmath}y}{R}$, so that the surface integral is $\int_S \frac{x^5y-2x^3y^3}{R} dS$. The integral is done in the cylindrical coordinates by polar substitution $x = R\cos\theta, y = R\sin\theta$. The surface element is $dS = Rd\theta dz$. Thus the integral become $R^6\int (\cos^5\theta\sin\theta - 2\cos^3\theta\sin^3\theta)d\theta dz$. The angle integral in both cases gives zero. Thus the total flux is only contributed by the top surface and is πR^2h^2.
   This can also be seen by the divergence theorem. $\nabla\cdot\vec{F} = 2z$. The volume integral is $\iiint 2zdxdydz = 2\pi R^2\int_0^h zdz = \pi R^2h^2$
4. Geometrically a torus is obtained by taking a circle, say in the x-z plane and rotating it about the z-axis to obtain a solid of revolution. Let us define the mean radius of the torus to be $R = \frac{a+b}{2}$ and the radius of the circle which is being revolved about the z-axis to be $r = \frac{b-a}{2}$. The position vector of an arbitrary point on the torus is defined as follows.



Consider the coordinate of an arbitrary point on the circle which is in the x-z plane. Let the position of the point make an angle φ with the x-axis. The coordinate of this point is

$(R + r\cos\varphi, 0, r\sin\varphi)$. When the circle is rotated about the z-axis by an angle θ, the z coordinate does not change. However, the x and y coordinates change and become

$$x = (R + r\cos\varphi)\cos\theta$$
$$y = (R + r\cos\varphi)\sin\theta$$

Thus an arbitrary point on the torus can be parameterized by φ, θ with x, y and z given by the above expressions. A surface element on the torus is then obtained by the area formed by an arc obtained by incrementing φ by dφ and the arc formed by incrementing θ by $d\theta$. The area element is therefore given by the cross product $d\vec{S} = \frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\, d\theta\, d\varphi$. (The unit vector on the surface is directed along the direction of the cross product.). The partial derivatives are given by

$$\frac{\partial \vec{r}}{\partial \theta} = -\hat{\imath}\,(R + r\cos\varphi)\sin\theta + \hat{\jmath}\,(R + r\cos\varphi)\cos\theta$$
$$\frac{\partial \vec{r}}{\partial \varphi} = -\hat{\imath}\, r\sin\varphi\cos\theta - \hat{\jmath}\, r\sin\varphi\sin\theta + \hat{k}\, r\cos\varphi$$

Thus

$$d\vec{S} = \hat{\imath}\,(R + r\cos\varphi) r\cos\theta\cos\varphi + \hat{\jmath}\,(R + r\cos\varphi) r\sin\theta\cos\varphi + \hat{k}\,(R + r\cos\varphi) r\sin\varphi$$

(the order of the cross product determines the outward normal). Thus , substituting $\vec{r} = \hat{\imath}x + \hat{\jmath}y + \hat{k}z$

$$\int_S \vec{r}\cdot d\vec{S} = \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi\, [(R + r\cos\theta)^2\, r\cos^2\theta\cos\varphi + (R + r\cos\theta)^2\, r\sin^2\theta\cos\varphi + r^2\,(R + r\cos\varphi)\sin^2\varphi] d\theta d\varphi$$
$$= \int_0^{2\pi} d\theta \int_0^{2\pi} d\varphi\, [r(R + r\cos\varphi)^2\cos\varphi + r^2(R + r\cos\varphi)\sin^2\varphi]$$

Since the integrand is independent of θ, the integral over it gives 2π . The integral over the remaining angle is straightforward. We can simplify the integrand as follows :

$$\int_S \vec{r}\cdot d\vec{S} = 2\pi r \int_0^{2\pi} [(R + r\cos\varphi)(R\cos\varphi + r\cos^2\varphi + r\sin^2\varphi]\, d\varphi$$
$$= 2\pi r \int_0^{2\pi} (R + r\cos\varphi)(r + R\cos\varphi\,) d\varphi$$
$$= 2\pi r \int_0^{2\pi} (Rr + r^2\cos\varphi + R^2\cos\varphi + Rr\cos^2\varphi) d\varphi$$

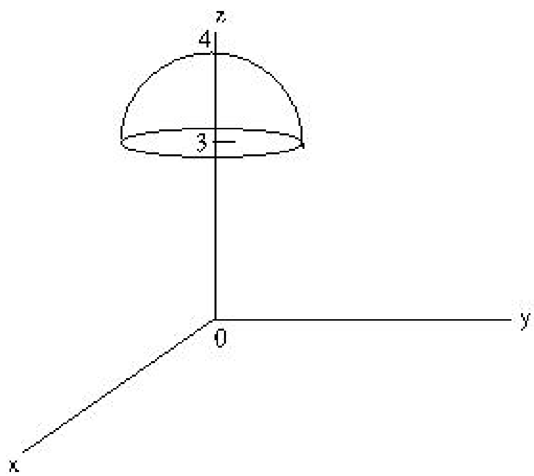
The second and the third term in the integral vanish, the remaining two terms give $3\pi Rr$, which makes the total contribution to the surface integral as $6\pi^2 Rr^2$.

However, the problem is straightforward if we apply the divergence theorem. The divergence of the position vector is 3. Thus by divergence theorem, the surface integral is 3 times the volume of the toroid. The volume of the toroid is rather easy to calculate if we note that if we cut it along a section, the toroid becomes a cylinder of radius r and length $2\pi R$. Thus the volume of the toroid is $\pi r^2 . 2\pi R = 2\pi^2 R r^2$. Thus 3 times the volume is $6\pi^2 R r^2$, consistent with our direct evaluation of the surface integral.

5. The surface is sketched below. Since $x^2 + y^2 \geq 0$, the region of interest is $3 \geq z \geq 4$. Notice that the divergence theorem is not directly applicable because the surface is not closed. However, one can close the surface by adding a cap to the surface at z=3. We will calculate the flux by applying the divergence theorem to this closed surface and then explicitly subtract the surface integral over the cap.

The divergence of the field is given by
$2(x^2 + y^2 + z) = 2(4 - z + z) = 8.$

Thus the surface integral over the closed surface is thus given by $\iiint 8dV$. To evaluate this, consider a disk lying between z and z+dz. The circular disk of width dz has a volume $dV = \pi r^2 dz = \pi(4 - z)dz.$ This gives the surface integral to be 4π. We have to now subtract from this the surface integral over the cap that was added by us, which is directed along $-\hat{k}$ direction. On this surface z=3 so that the radius of the disk is 1. This integral $\int x^2 y \, dS$ can be easily calculated and shown to have value zero. Thus the required surface integral has a value 4π.



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Self Assessment Quiz

In the following questions calculate the flux both by direct integration and also by application of the divergence theorem.

1. Calculate the flux of the field $\hat{\imath} z + \hat{\jmath} x + \hat{k}\, y$ over the surface of a right circular cylinder of radius R and height h in the first octant, i.e. in the region (x>0, y>0, z>0).

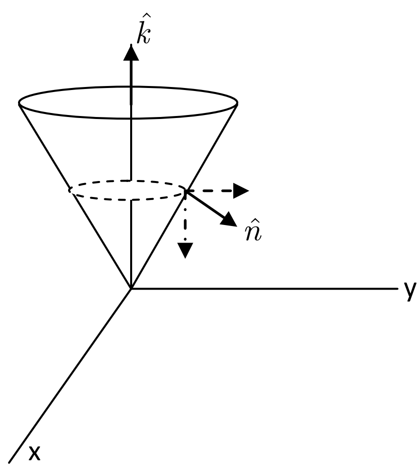
2. Evaluate the surface integral of the vector field $y\hat{\imath} + 2xy\hat{\jmath} + 3yz\hat{k}$ over the surface of a unit cube with the origin being at one of the corners.
3. Calculate the flux of $x^3\hat{\imath} + y^3\hat{\jmath} + z^3\,\hat{k}$ over the surface of a sphere of radius R with its centre at the origin.
4. Calculate the flux of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ through the surface defined by a cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 1$.
5. Evaluate the flux through an open cone $z = \sqrt{x^2 + y^2}$ with $z \le 2$ for the field $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solutions toSelf Assessment Quiz

1. For the curved surface of the cylinder, the unit vector is $\frac{\hat{\imath}x+\hat{\jmath}y}{R}$ which gives $\vec{F}\cdot\hat{n} = \frac{zx+xy}{R}$. Parameterize $x = R\cos\theta, y = R\sin\theta$, Since we are confined to the first octant $0 \le \theta \le \frac{\pi}{2}$. The flux through the slant surface is $\int_0^{\frac{\pi}{2}} d\theta \int_0^h R\,(z\cos\theta + R\sin\theta\cos\theta)\,dz = \frac{Rh^2+R^2h}{2}$. The top and the bottom caps are in the $\pm\hat{k}$ directions, the contribution from these two give zero by symmetry. There are two more surfaces if we consider the first octant, they are the positive x-z plane and the positive y-z plane., the normal to the former being in the direction of $-\hat{\jmath}$ while that for the latter is along $-\hat{\imath}$ directions. The flux from the former is $-\int_0^R xdx \int_0^h dz = -\frac{R^2h}{2}$, while that from the latter is $-\int_0^R dy \int_0^h zdz = -\frac{Rh^2}{2}$. Adding up all the contributions, the total flux from the closed surface is zero. This is consistent with the fact that the divergence of the field is zero.
2. There are six faces. For the face at x=0, since the surface is directed along $-\hat{\imath}$ direction, the surface integral is $-\int ydydz = -1/2$. The face at x=1 gives +1/2. The faces at y=0 and that at z=0 gives zero because the field is proportional to y and z respectively. The contribution to flux from y=1 is 1 and that from z=1 is 3/2. Adding, the flux is 5/2 units. This can also be done by the divergence theorem. Divergence of the field is 2x+3y, so that the volume integral is $\int (2x+3y)dxdydz = 5/2$.
3. The divergence of the given vector field is $3(x^2 + y^2 + z^2) = 3r^2$. Thus, by divergence theorem, the flux is $3\int r^2 dV = 3.4\pi \int_0^R r^4 dr = \frac{12\pi R^5}{5}$. We can show this result by direct integration. The unit normal on the surface of the sphere is given by $\hat{r} = \frac{\hat{i}x + \hat{j}y + \hat{k}z}{R}$, so that the flux is $\int \vec{F}\cdot\hat{r}dS = \int \frac{x^4 + y^4 + z^4}{R} dS$. This integral can be conveniently evaluated

in a spherical polar coordinates with $x = R\sin\theta\cos\varphi, y = R\sin\theta\sin\varphi, z = R\cos\theta$. The surface element on the sphere is $dS = R^2 \sin\theta d\theta d\varphi$. By symmetry, the flux can be seen to be $3\int \frac{z^4}{R} dS = \frac{3.2\pi}{R}\int_0^\pi (R\cos\theta)^4 R^2 \sin\theta d\theta$. (Note that we decided to do the integral involving z^4 rather than x^4 or y^4 because the azimuthal integral gives 2π in this case. The integral is easy to perform with the substitution $\mu = \cos\theta, d\mu = -\sin\theta d\theta$, which gives the flux to be $6\pi R^5 \int_{-1}^{1} \mu^4 d\mu = \frac{12\pi R^5}{5}$.

4. The divergence of the field is 3. The flux, therefore, is 3 times the volume of the cone which is $(1/3)\pi R^2 h = \pi/3$. The flux is thus π. The direct calculation of the flux involves two surfaces, the slant surface and the cap, as shown in the figure. The cap is in the xy plane and has an outward normal $\hat{n} = \hat{k}$. The flux $\int \vec{F}\cdot\hat{n} dS = \int z dxdy = \int dxdy = \pi$ (because on the cap z=1 and the cap is a disk of unit radius). Thus it remains to be shown that the flux from the slanted surface vanishes. At any height z, the section parallel to the cap is a circle of radius z. Since, the height and the radius of the cap are 1 each, the semi angle of the cone is 45^0.



Thus the normal to the slanted surface has a component $-1/\sqrt{2}$ along the z direction and $1/\sqrt{2}$ in the x-y plane. The component in the xy plane can be parameterized by the azimuthal angle φ and we can write $\hat{n}(z,\varphi) = \frac{\cos\varphi\hat{i} + \sin\varphi\hat{j} - \hat{k}}{\sqrt{2}}$. The area element can be written as $(zd\theta)(\sqrt{2}dz)$, the factor $\sqrt{2}$ appears because the length element is along the slant. Thus the contribution from the slanted surface is

$\int \vec{F} \cdot \hat{n} dS = \int_0^{2\pi} d\varphi \int_0^1 z dz \left(x \cos\varphi + y \sin\varphi - z \right)$. Using $x = z\cos\varphi, y = z\sin\varphi$, this integral can be evaluated and shown to be zero.

5. This problem is to be attempted similar to the problem 5 of the tutorial, i.e., by closing the cap and subtracting the contribution due to the cap. The divergence being 3, the flux from the closed cone is 3 times the volume of the cone which gives $8\pi.$ The contribution from the top face (which is a disk of radius 2) is $\int \vec{F} \cdot \hat{n} dS = \int z dx dy = \int 2 dx dy = 8\pi$. Thus the net flux is zero. (You can also try to get this result directly as done in problem 4, where we showed that the flux from the curved surface is zero).