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QUANTUM FIELD THEORY 2

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Der wackre Schwabe forcht' sich nit,
ging seines Weges Schritt vor Schritt.[1]

[1] Ludwig Uhland oft zitiert von Albert Einstein.

Sommersemester 2019, Version vom 2019-08-13

Contents

1 Foreword

These notes are not original but mostly a combination of results found in books and lecture notes. My contribution is here foremost in the selection and the perspective provided. I have made liberal use of the following references

The $\star$ indicates the technical level of the book. It is roughly proportional to the time needed per page for a full understanding. Note, that the Zee book in particular is written in a conversational tone, but covers more concepts than any of the other books.

- Peskin, Schroeder - *An Introduction to Quantum Field Theory* $\star\star$
- Schwartz - *Quantum Field Theory and the SM* $\star\star$
- Martin Beneke - *Lecture notes on QFT* $\star\star\star$,
- Mandl, Shaw - *Quantum Field Theory* $\star$
- Zee - *Quantum Field Theory in a Nutshell* $\star$
- Srednicki - *Quantum Field Theory* $\star\star\star$
- Weinberg - *Quantum Field Theory I-III* $\star\star\star\star$
- Ramond - *Field Theory: a Modern Primer* $\star\star\star$
- Itzykson, Zuber - *Quantum Field Theory* $\star\star\star$
- Shifman, *Advanced topics in Quantum Field Theory* $\star\star\star$

and many more. I strongly suggest that you find a book (or books) you like from the ones above and study it as a complement to these lecture notes.

This course provides a hopefully gentle excursion into the beautiful world of advanced quantum field theory, building on the foundations we set during the last semester. You will learn to tackle the theoretical framework that underlies all of nature.

And remember: If you don't make mistakes, you're not working on hard enough problems. And that's a big mistake.

Let's start.

2
Introduction

In QFT1, we ended by defining QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{\frac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\text{gauge fixing, (more later)}} + \overline{\psi}(i\gamma^\mu D_\mu - m)\psi \tag{2.1}$$

with $D_\mu = \partial_\mu - ieA_\mu(x)$. Eq. (2.1) has a **global** U(1) symmetry:

$$\begin{cases} \psi(x) \to e^{i\alpha}\psi(x), \quad \alpha = \text{const.} \\ A_\mu(x) \to A_\mu(x) \end{cases}$$

and a gauge redundancy:

$$\begin{cases} \psi(x) \to e^{i\cdot e\cdot\alpha(x)}\psi(x) \\ A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x). \end{cases}$$

The A_μ propagator is

$$\frac{i}{k^2+i\epsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \tag{2.2}$$

where $\xi = 1$ is the Feynman gauge.
The propagator for massless spin 1 does **not** have $\sim \frac{k_\mu k_\nu}{k^2}$ like massive spin 1, which was responsible for the bad UV behavior.
Therefore QED is renormalizable by power counting.
The propagator $F.T.\langle\Omega|TA_\mu(x)A_\nu(y)|\Omega\rangle \overset{k\to\infty}{\propto} \frac{1}{k^2}$ and the electron-photon interaction $\overline{\psi}\not{A}\psi$ has dimension four and therefore $\Delta_i = 0$.

2.0.1 Renormalization

Bare fields:

Does Z_ξ require a separate renormalization condition?

$$\begin{aligned} &\psi_0 = \sqrt{Z_2}\psi, \quad A_0^\mu = \sqrt{Z_3}A^\mu \\ &m_0 = Z_m m, \quad e_0 = Z_e e \\ &\xi_0 = Z_\xi \xi, \end{aligned}$$

which gives

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \overline{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \\ &- \frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu} - \left(\frac{Z_3}{Z_\xi}-1\right)\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &+ (Z_2-1)\overline{\psi}i\not{\partial}\psi - (Z_2Z_m-1)m\overline{\psi}\psi + \\ &+ (Z_eZ_2Z_3^{1/2}-1)e\overline{\psi}\not{A}\psi \qquad (2.3) \\ &= \mathcal{L}_{renormalized} + \mathcal{L}_{counterterm}. \qquad (2.4)\end{aligned}$$

The Ward-identities will preserve the gauge-redundancy at quantum level:

$$Z_e = \frac{1}{\sqrt{Z_3}} \tag{2.5}$$

which relates the renormalization constants. This way:

Question: how is gauge-redundancy still working in the presence of $\frac{1}{2\xi}(\partial_\mu A^\mu)$ gauge-fixing? $\Rightarrow$ see later.

Note: This only holds if the regularization preserves gauge redundancy, (see chapter Anomaly for counter-example).

$$\begin{aligned}\overline{\psi}_0\not{D}_0\psi_0 &= \overline{\psi}_0(\not{\partial} - ie_0\not{A}_0)\psi_0 \qquad (2.6) \\ &= Z_2\overline{\psi}(\not{\partial} - iZ_ee\sqrt{Z_3}\not{A})\psi. \qquad (2.7)\end{aligned}$$

We do **not** use all other possible terms up to dimension four to the renormalized Lagrangian:

$$\underset{\text{photon mass}}{\overset{\downarrow}{A_\mu A^\mu}}, \quad \underset{\substack{\uparrow \\ \text{photon self-interactions}}}{\partial_\mu A_\nu A^\mu A^\nu}, \quad A^\mu A^\nu A_\mu A_\nu \tag{2.8}$$

due to the gauge redundancy.

2.1 On-shell renormalization

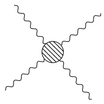
Most used scheme in high-energy processes $\sqrt{s} \gg m_e$ is $\overline{MS}$ (or MS), another possibility which uses physical quantities (like pole-mass/physical mass) is the on-shell scheme.

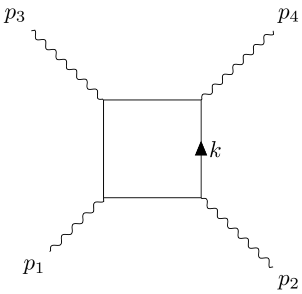
2.1.1 Z_2, Z_m from electron two-point-function

We define as usual

$$S_0^{(2)}(p) \equiv \int d^4(x-y)\, e^{ip(x-y)}\langle\Omega|T(\psi_{0\alpha}(x)\overline{\psi}_{0\beta}(y))|\Omega\rangle \tag{2.10}$$

$$\xrightarrow{p^2\to m^2} \underbrace{Z_2}_{\substack{\text{on-shell} \\ \text{field renormalization}}}\frac{i(\not{p}+\overbrace{m}^{\text{physical mass } 0.511\text{MeV}})}{p^2-m^2+i\epsilon} + \text{non-singular} \tag{2.11}$$

The fact, that we can ignore terms in Eq. (2.8), implies e.g. that  is finite, otherwise we would have to add it to the Lagrangian. **But** the Ward-identity tells us $\epsilon_\mu(p) \to p_\mu$ should make this four-point amplitude vanish. The correct scaling at high momenta is:



$$\xrightarrow{k\to\infty} \int d^4k \frac{1}{(\not{k}-p_i)^4} \sim$$

$$\sim \int_0^\Lambda d^4k \, \frac{1}{\not{k}^4} \left(\underset{\substack{\uparrow \\ 0}}{(\dots)} \, 1 + \underset{\substack{\uparrow \\ 0}}{\dots} + \underset{\substack{\uparrow \\ \text{non-zero}}}{\frac{p_i^4}{k^4}} + \dots \right) \tag{2.9}$$

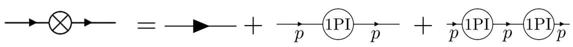
$$= \text{finite}$$

This is simpler to see: it must be a gauge invariant operator with 4 photon fields, which means we know that it is

$$\overset{\substack{\mathcal{O}(1)\text{ number} \\ \downarrow}}{c} \frac{e^4}{16\pi^2} \underset{\substack{\uparrow \\ \text{The Fourier transform of this term gives} \\ \text{the ext. momentum dependence in } Eq.\ (2.9)}}{\frac{1}{m^2}} (F_{\mu\nu}F^{\mu\nu})^2 \sim (\partial_\mu A_\nu)^4 \quad (\leftarrow \text{four powers of external momenta!})$$

In general, there is also $(F_{\mu\nu}\tilde{F^{\mu\nu}})^2$ but the same argument applies.

Renormalized perturbation theory:

$S^{(2)}(p) =$  $+ \dots$

$$\tag{2.12}$$

$$= \frac{i}{\not{p} - m + i\varepsilon} + \frac{i}{\not{p} - m + i\varepsilon} \underset{\substack{\uparrow \\ \text{renormalized self-energy} \\ \text{including counter-term}}}{(-i\Sigma(p,m))} \frac{i}{\not{p} - m + i\varepsilon} + \dots \tag{2.13}$$

$$= \frac{i}{\not{p} - m - \Sigma(p,m)} \xrightarrow{p^2\to m^2} \underset{\substack{\uparrow \\ \text{This term of renormalized} \\ \text{two point function defines the OS-scheme}}}{\frac{i}{\not{p} - m + i\varepsilon}} \tag{2.14}$$

Renormalization condition:

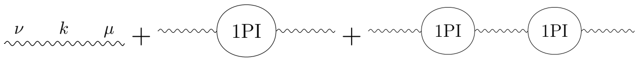
$$\Sigma(p,m) = \Sigma_1(p^2,m)\infty + \Sigma_2(p^2,m)\not{p} \tag{2.15}$$

$$\overset{p^2\to m^2}{=} 0 \quad (\text{pole at } m_{phys}) \tag{2.16}$$

and

$$\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{p^2=m^2} = 0 \quad (\text{residue} = 1) \tag{2.17}$$

2.1.2 Z_3, Z_ξ from photon two-point-function:

$G^{(2)}(k) =$ 

$$= \frac{i}{k^2+i\varepsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)$$

$$+ \frac{i}{k^2+i\varepsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\rho}{k^2}\right) \underset{\substack{\uparrow \\ \text{renormalized self-energy}}}{i\Pi_{\rho\sigma}(k)} \left(-\eta_{\sigma\nu} + (1-\xi)\frac{k_\sigma k_\nu}{k^2}\right)_k + \ldots$$

If $Z_e = \frac{1}{\sqrt{Z_3}}$: we get for Eq. (2.7):

$$\overline{\psi}_0 \not{D}_0 \psi_0 = Z_2 \overline{\psi}(\not{\partial} - ie\not{A})\psi \tag{2.18}$$

The gauge redundancy is preserved in the renormalized Lagrangian with

$$A_\mu \to A_\mu + \partial_\mu \alpha \tag{2.19}$$

$$\psi \to e^{\underset{\substack{\uparrow \\ \text{renormalized charge}}}{ie}\alpha(x)}\psi \tag{2.20}$$

also because of gauge-redundancy:

$$Z_\xi = Z_3 \tag{2.21}$$

with extra counter-term vertex for $\frac{1}{2\xi}(\partial_\mu A_\mu)^2$ (no physical measurement needed to determine this parameter $\to$ it would be bad since it is an unphysical parameter).

Most general form:

$$\Pi_{\mu\nu} = (k^2\eta_{\mu\nu} - k_\mu k_\nu)\Pi(k^2) + k_\mu k_\nu \Pi_2(k^2) \tag{2.22}$$

$\Pi_2(k^2)$ vanishes at 1-loop. We will show that for general gauge theories

$$\Pi_2(k^2) \equiv 0$$

true to any order (see BRST chapter).

We get for the two-point function after resumming the geometric series

$$G^{(2)}(k) = \frac{i}{k^2+i\varepsilon}\left[\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi(k^2)} - \xi\frac{k_\mu k_\nu}{k^2}\right] \tag{2.23}$$

$$\overset{k^2\to 0}{\longrightarrow} \frac{i}{k^2+i\varepsilon}(-\eta_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}) + \text{non-sing.} \tag{2.24}$$

We determine Z_3 from $\Pi(k^2 = m_A^2 = 0) = 0$.

We see that

$$Z_\xi = Z_3 \tag{2.25}$$

using the unrenormalized two-point-function

$\Pi_0(k^2)$ is the unrenormalized self-energy

$$G_0^{(2)} = \frac{i}{k^2+i\varepsilon}\left[\left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi_0(k^2)} - \xi_0\frac{k_\mu k_\nu}{k^2}\right] \tag{2.26}$$

$$\overset{k^2\to 0}{\longrightarrow} Z_3\frac{i}{k^2+i\varepsilon}\left(-\eta_{\mu\nu} + (1-\xi Z_\xi)\frac{k_\mu k_\nu}{k^2}\right) + \text{non-sing.} \tag{2.27}$$

Note: $\xi_0 = Z_\xi \xi$.

It follows:

$$\frac{1}{Z_3} = 1 - \Pi_0(k^2 = 0) \tag{2.28}$$

and

$$Z_\xi = Z_3 \tag{2.29}$$

2.1.3 Renormalization of the coupling strength Z_e

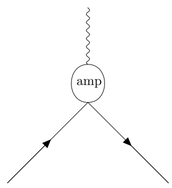
Renormalization of the coupling strength $\mathbf{Z_e}$ defined at small momentum transfer $|\vec{q}| \ll m$

$$\begin{aligned}&\langle e^-(p', s')\gamma(q, \lambda); \text{out}|e^-(p, s); \text{in}\rangle \\ &\equiv (2\pi)^4\delta^{(4)}(p' + q - p) \cdot ie\bar{u}(p', s')\gamma^\mu u(p, s)\epsilon_\mu(q, \lambda)\end{aligned} \tag{2.30}$$

with $q^\mu \to 0$.

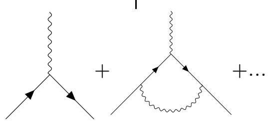
We require that this is not renormalized at any order for $q \to 0$

Similar to scalar $\lambda\phi^4$ before.

$\langle e^-(p', s')\gamma(q, \lambda); \text{out}|e^-(p, s); \text{in}\rangle = \sqrt{Z_3}Z_2 \cdot$  $\cdot$ (polarization terms)

(2.31)

$$= (2\pi)^4\delta^{(4)}(p' + q - p)\sqrt{Z_3}Z_2 ie_0\Gamma_0^\mu(p', p)u(p, s)\epsilon^\star_\mu(q, \lambda) \tag{2.32}$$



where Γ_0^μ is a 4×4 matrix in Dirac spinor space.

A basis for Γ_0^μ is:

$$\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, [\gamma^\mu, \gamma^\nu] \equiv \frac{2}{i}\sigma^{\mu\nu} \tag{2.33}$$

Γ_0^μ must have one open index and contain p and p'.

We can easily show[1], that only $\gamma_\mu f(q^2) + (p_\mu + p'_\mu)\gamma(q^2)$.

Use Gordon-Identity:

$$\bar{u}(p', s')\gamma^\mu u(p, s) = \bar{u}(p', s')\left[\frac{(p' + p)^\mu}{2m} - \frac{i\sigma^{\mu\nu}(p - p')_\nu}{2m}\right]u(p, s) \tag{2.34}$$

to replace $(p + p')_\mu g(q^2)$:

$$\Gamma_0^\mu(p, p') = \gamma^\mu \underset{\uparrow \atop ①}{F_1^0(q^2, m)} + \frac{i\sigma^{\mu\nu}(p' - p)_\nu}{2m}\underset{\uparrow \atop ②}{\overset{\text{“magnetic” formfactor} \atop \downarrow}{F_2(q^2, m)}} \tag{2.35}$$

[1] We use parity conservation (no γ_5) and the Ward-Id $\partial_\mu\Gamma_0^\mu = 0$. For a derivation see e.g. Peskin/Schroeder, p. 186.

② vanishes for $q \to 0$ and so the $q \to 0$ renormalization condition only involves ① with Eq. (2.30) and Eq. (2.31):

$$\frac{1}{Z_e} = \sqrt{Z_3} Z_2 F_1^0(q^2 = 0, m) \tag{2.36}$$

or

$$F_1(q^2 = 0, m) \quad \text{for renormalized vertex} \tag{2.37}$$

2.1.4 Ward-Identity

$F_1^0(q^2 = 0, m) = \frac{1}{Z_2}$ which implies

$$Z_e = \frac{1}{\sqrt{Z_3}} \tag{2.38}$$

relating charge and photon field renormalization.

Proof:

Use Ward-identity from generating functional for Noether current j_μ.

$$\partial_\mu \langle \Omega | T(J_0^\mu(x) \psi_{0\alpha}(z_1) \overline{\psi}_{0\beta}(z_2)) | \Omega \rangle \tag{2.39}$$

$$\begin{aligned} &= \delta^{(4)}(x - z_1) \langle \Omega | T(\psi_{0\alpha}(z_1) \overline{\psi}_{0\beta}(z_2)) | \Omega \rangle \\ &- \delta^{(4)}(x - z_2) \langle \Omega | T(\psi_{0\alpha}(z_1) \overline{\psi}_{0\beta}(z_2)) | \Omega \rangle \end{aligned} \tag{2.40}$$

with

$$J_0^\mu = -\overline{\psi}_0 \gamma^\mu \psi_0 \tag{2.41}$$

$$F_\psi = i\psi \tag{2.42}$$

$$F_{\overline{\psi}} = -i\overline{\psi} \tag{2.43}$$

$$\delta\psi = \epsilon F_\psi . \tag{2.44}$$

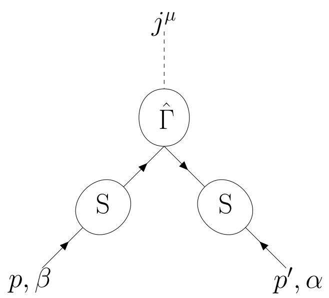
Take Fourier-transformation:

$$\underset{\substack{\uparrow \\ \text{to get } \overline{\psi}\gamma^\mu\psi_0 \text{ as in } \overline{\psi}\not{A}\psi}}{(-1)} \int d^4x d^4z_1 d^4z_2 \; e^{iqx + ip'z_1 - ipz_2} \tag{2.45}$$

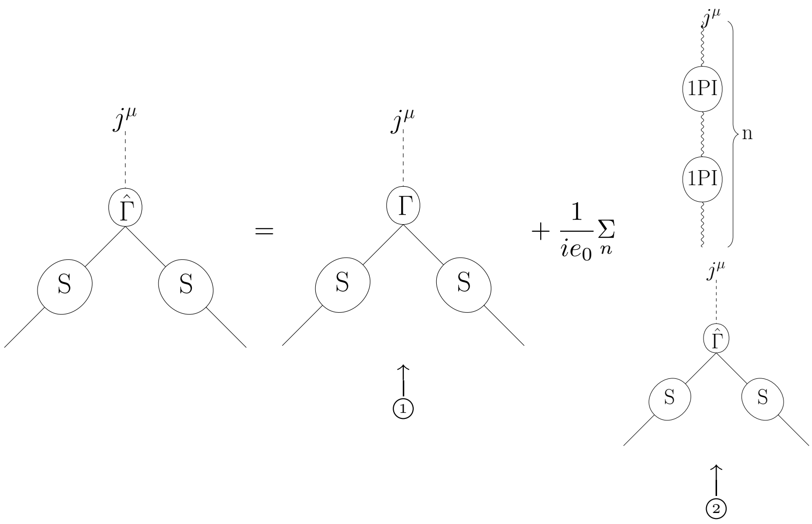
LHS of Eq. (2.30):

$$(2\pi)^4 \delta^4(q + p' - p)(-iq_\mu) S_{0\alpha\alpha'}(p') \hat{\Gamma}_{0\alpha'\beta'} S_{0\beta\beta'}(p) =$$

$=$

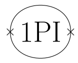


Note: $\hat{\Gamma}$ is not yet the vertex factor Γ. We can however relate them:



Extra contribution ② vanishes after contraction with q_μ

$$q_\mu \cdot \text{1PI} = 0. \tag{2.46}$$



Ignore the difference between Γ and $\hat{\Gamma}$!
RHS of Eq. (2.30):

$$(2\pi)^4\delta^{(4)}(q+p'-p)(-1)(S_{0\alpha\beta}(p)-S_{0\alpha\beta}(p')), \tag{2.47}$$

Multiply Eq. (2.47) by $S_0^{-1}(p)\cdot S_0^{-1}(p')$:

$$(p'-p)\Gamma_0^\mu(p',p) = iS_0^{-1}(p') - iS_0^{-1}(p) \tag{2.48}$$

Take $q\to 0$ limit, expand RHS of Eq. (2.30) around p, to derive

$$\Gamma_0^\mu(p',p)_{p'\to p} = i\frac{\partial}{\partial p_\mu}S_0^{-1}(p) \tag{2.49}$$

Take OS-limit $p^2\to m^2$ for which $S_2(p)\to\frac{iZ_2}{\not{p}-m+i\varepsilon}$

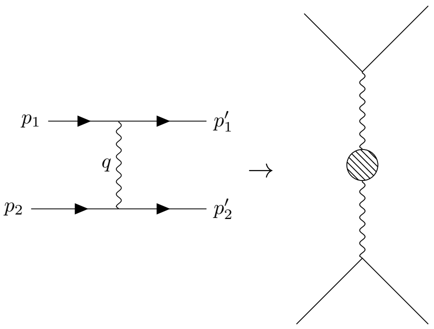
$$\gamma_\mu F_1^0(q^2=0,m) = i\frac{1}{iZ_2}\frac{\partial}{\partial p_\mu}(\not{p}-m) = \frac{1}{Z_2}\gamma_\mu \tag{2.50}$$

2.2 Scale dependence of the QED coupling

In the OS-scheme e is not scale-dependent at $q\to 0$. Its value has been measured as

$$\alpha_{em} = \frac{e^2}{4\pi} = \frac{1}{137.026\ldots}. \tag{2.51}$$

We find a scale-dependence of the interaction strength through the photon vacuum polarization.



← tree and 1-loop and higher

$$ \tag{2.52} $$

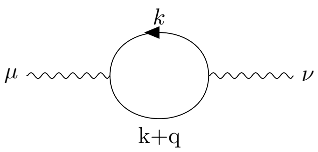
It contains

$$\frac{e^2}{q^2} \quad (q = p_1' - p_1) \tag{2.53}$$

respectively

$$\frac{e^2}{q^2} \cdot \frac{1}{1 - \Pi(q^2)} \equiv \frac{e^2_{eff}(q^2)}{q^2}. \tag{2.54}$$

At one-loop we have:



which means for the vacuum polarization

$$\begin{aligned} i\Pi(q^2)_{\mu\nu} = (-1)(ie)^2 \tilde{\mu}^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \, \frac{\mathrm{Tr}(\gamma_\mu \mathrm{i}(\not{k} - \mathrm{m})\gamma_\nu \mathrm{i}(\not{k} + \not{q} + \mathrm{m}))}{(k^2 - m^2 + i\varepsilon)((k+q)^2 - m^2 + i\varepsilon)} \\ + (-i)(Z_3 - 1)(q^2 \eta_{\mu\nu} - q_\mu q_\nu) \end{aligned} \tag{2.55}$$

and the counterterm

$$(\text{from } - \frac{(Z_3 - 1)}{4} F_{\mu\nu} F_{\mu\nu}) \tag{2.56}$$

So we get

$$\Pi(q^2) = -\frac{2\alpha_{em}}{\pi} \Gamma(\epsilon) \int_0^1 dx \; x\bar{x} \left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{4\pi\tilde{\mu}^2} \right)^{-\epsilon} - (Z_3 - 1) \tag{2.57}$$

⇒The effective interaction $e_{eff}(\mathbf{q}^2)$ increases with $\mathbf{q}$, i.e. e^- and e^+ attract each other more strongly at higher momentum.

$$\tilde{V}(-\mathbf{q})_{\text{tree}} = \frac{e^2}{-\mathbf{q}^2} \overset{\text{F.T.}}{\rightarrow} V(\mathbf{r}) = \int \frac{d^3 q}{(2\pi)^3} \, e^{i\mathbf{qr}} \frac{e^2}{-\mathbf{q}^2} = -\overset{\substack{\text{Coulomb!} \\ \downarrow}}{\frac{\alpha_{em}}{r}} \tag{2.58}$$

Correction at low-energy:

$$\tilde{V}(-\mathbf{q}) = \frac{e^2}{-\mathbf{q}^2}\left(1 + \frac{2\alpha_{em}}{\pi}\frac{1}{30}\frac{\mathbf{q}^2}{m^2} + \dots\right) \tag{2.59}$$

$$= \frac{e^2}{-\mathbf{q}^2} - \frac{e^4}{60\pi^2}\cdot\frac{1}{m^2} + \dots \tag{2.60}$$

For the Fourier transform use $\int \frac{d^3q}{(4\pi)^3} e^{i\mathbf{q}\cdot\mathbf{r}} \cdot 1 = \delta^{(3)}(\mathbf{r})$.

$$V(r) = -\frac{\alpha_{em}}{r} - \underbrace{\frac{4\alpha_{em}^2}{15}\delta^{(3)}(\mathbf{r})}_{\Delta V \text{ Ühling term}} + \dots \tag{2.61}$$

which only acts at $\mathbf{r} = 0$ and will affect $n = 0$ orbitals of hydrogen. It contributes to Lamb-shift:

$$\Delta E = \langle \psi_i | \Delta V | \psi_i \rangle \tag{2.62}$$

with ψ_i as the hydrogen wave function splitting of $2P_{1/2}$, $2P_{1/2}$ by 27 MHz lowering $2S_{1/2}$.

Z_3 must be determined from $\Pi(q^2 = 0) \overset{\substack{\text{OS-scheme}\\ \downarrow}}{=} 0$.
This gives

$$Z_3 = 1 - \frac{2\alpha_{em}}{\pi}\Gamma(\epsilon)\int_0^1 dx\, x\, \bar{x}\left(\frac{m^2}{4\pi\tilde{\mu}^2}\right)^{-\epsilon} \tag{2.63}$$

$$= 1 - \frac{\alpha_{em}}{3\pi}\left(\frac{1}{\epsilon} - \ln\left(\frac{m^2}{\tilde{\mu}^2}\right)\right) \tag{2.64}$$

and

$$\Pi(q^2) = \frac{2\alpha_{em}}{\pi}\int_0^1 dx\, x\bar{x}\ln\left(\frac{m^2 - x\bar{x}q^2 - i\varepsilon}{m^2}\right) \tag{2.65}$$

Consider a non-relativistic scattering with $|\vec{p}_{el}| \ll m$:

$$q_\mu = \begin{pmatrix} E_{p'} - E_p \\ \mathbf{p_1'} - \mathbf{p_1} \end{pmatrix} \simeq \begin{pmatrix} 0 \\ \mathbf{q} \end{pmatrix} \to q^2 \approx -\mathbf{q}^2 < 0 \tag{2.66}$$

$$\Pi(-\mathbf{q}^2) = \frac{2\alpha_{em}}{\pi}\int_0^1 dx\, x\bar{x}\ln\left(1 + x\bar{x}\frac{\mathbf{q}^2}{m^2}\right) \tag{2.67}$$

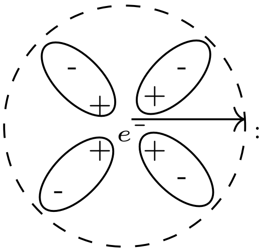
$$\simeq \frac{2\alpha_{em}}{\pi}\left[\frac{1}{30}\frac{\mathbf{q}^2}{m^2} - \frac{1}{280}\left(\frac{\mathbf{q}^2}{m^2}\right)^2 + \dots\right]. \tag{2.68}$$

$$\tilde{V}(q^2)\frac{e_{eff}(q^2)}{q^2} = \frac{e^2}{q^2}\frac{1}{1 - \Pi(q^2)}. \tag{2.69}$$

Thus we get

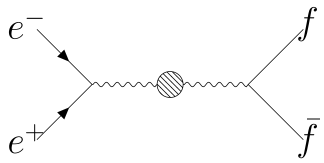
$$\tilde{V}(q^2) = \frac{e^2}{q^2}\left(1 + \frac{2\alpha_{em}}{\pi}\int_0^1 dx\, x(1-x)\ln\left(1 + x\bar{x}\frac{\mathbf{q}^2}{m^2}\right) + \dots\right). \tag{2.70}$$

We can interpret the effect on $e_{eff}(\mathbf{q}^2)$ if we think of a vacuum as a medium which screens charge through the creation of e^+e^--pairs.



$: \quad e_{eff} < e_{eff}(r \to 0).$

We have measured this effect very precisely e.g at the Large Electron Positron (LEP) collider at CERN:



(2.71)

from which we can extract

$$\alpha_{eff}(0)^{-1} = 137.026 \quad \longrightarrow \quad \alpha_{eff}(M_Z = 91\text{GeV})^{-1} = 128.927.$$

and we see the scale dependence explicitly.

2.3 Anomalous magnetic moment

In the non-relativistic limit, the Dirac Hamiltonian in the presence of a magnetic field is

$$H = \frac{\mathbf{p}^2}{2m} + V(r) + \frac{e}{2m}\mathbf{B} \cdot (\mathbf{L} + g\mathbf{S}) \tag{2.72}$$

For the Hamiltonian see e.g. QM2. You might have encountered the magnetic dipole moment also in this form, showing the interaction energy

$$H = -\boldsymbol{\mu} \cdot \boldsymbol{B}$$

in discussions of magnetism in statistical physics with

$$\boldsymbol{\mu} = -g\frac{e}{2m}\mathbf{S}.$$

See e.g. my Stat-Mech script, chapter 7.

acting on doublets $|\psi\rangle = \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}$ and $\mathbf{S} = \frac{\boldsymbol{\sigma}}{2}$.

In the Schrödinger equation, g is a free parameter, however the Dirac equation implies $g = 2$.

How do we calculate quantum corrections to g?

2.3.1 Extracting the magnetic moment

We want to get to calculate radiative corrections to g without taking the non-relativistic limit.

See e.g. Schwartz 10.4 or QM2.

The EOM is

$$(i\not{D} - m)\psi = 0 \tag{2.73}$$

Multiplied by $(i\not{D} + m)$

$$(\not{D}^2 + m^2)\psi = 0. \tag{2.74}$$

Use[2]:

[2] Show this!

$$\not{D}^2 = D_\mu D^\mu + \frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu} \tag{2.75}$$

with $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.
Interestingly, this describes concisely the difference between covariant derivatives on scalars and on spinors.
We find:

$$(D_\mu D^\mu + m^2 + \frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu})\psi = 0 \tag{2.76}$$

The scalar field obeys:

$$(D_\mu D^\mu + m^2)\phi = 0 \tag{2.77}$$

In the Weyl-representation:

$$\frac{e}{2} F_{\mu\nu}\sigma^{\mu\nu} = -e \begin{pmatrix} (\mathbf{B} + \mathbf{iE})\sigma & 0 \\ 0 & (\mathbf{B} - \mathbf{iE})\sigma \end{pmatrix} \tag{2.78}$$

Note:

$$F_{0i} = E_i$$
$$F_{ij} = -\epsilon_{ijk} B_k$$

In momentum space $(\not{D}^2 + m^2)\psi = 0$ implies

$$(E - eA_0)^2\psi = \left(m^2 + (\mathbf{p} - e\mathbf{A})^2 - 2e\mathbf{B}\cdot\mathbf{S}\right)\psi \tag{2.79}$$

Ignoring the electric field $\mathbf{E} \equiv 0$ in the following. Note, that it appears in the relativistic definition of the magnetic moment.

with $\mathbf{S} = \frac{\boldsymbol{\sigma}}{2}$ or

$$\frac{(E - eA_0)^2}{2m}\psi = \left(\frac{m}{2} + \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} - 2\frac{e}{2m}\mathbf{B}\cdot\mathbf{S}\right)\psi \tag{2.80}$$

which we can directly compare to Eq. (2.72) to read off:

$$g = 2. \tag{2.81}$$

Our strategy is now:

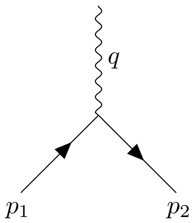
1. We can extract the magnetic moment from

$$g\frac{e}{4} F_{\mu\nu}\sigma^{\mu\nu} \quad \text{term} \tag{2.82}$$

2. We look for loop contributions to Eq. (2.82)

The leading order interaction of a fermion with a gauge boson is

$iM_0^\mu =$



$$= -ie\bar{u}(p_1)\gamma^\mu u(p_2). \tag{2.83}$$

which we can write with the help of the Gordon-identity as

$$M_0^\mu = -e\underbrace{\left(\frac{p_1^\mu + p_2^\mu}{2m}\right)}_{①}\bar{u}(p_2)u(p_1) - \frac{e}{2m}\underbrace{i\bar{u}(p_2)q_\nu\sigma^{\mu\nu}u(p_1)}_{②}. \tag{2.84}$$

① Klein-Gordon-like photon interaction

② Spin-dependent →magnetic moment

The Klein-Gordon photon interactions are contained in

$$(D_\mu\phi)^\dagger D^\mu\phi = ieA^\mu\,(\phi^\dagger\partial_\mu\phi - \phi\partial_\mu\phi^\dagger) + \dots$$
$$\to \propto (p_\mu + p'_\mu)\epsilon_\mu(q).$$

which after Fourier transform gives the interaction in ①.

g is $\frac{4m}{e}$ times the coefficient of ②.

Back to the vertex correction using Eq. (2.35)

$$\Gamma_\mu(q)^{(0),1-loop} = \gamma^\mu F_1(q^2, m) + i\frac{\sigma^{\mu\nu}}{2m} q_\nu F_2(q^2, m) \tag{2.85}$$

It follows

$$g = 2 + 2F_2(0) \tag{2.86}$$

where we measure the coupling at $q \to 0$.

Calculating the vertex (see Schwartz 17.4)

$$F_2(0) = \frac{\alpha}{2\pi} \tag{2.87}$$

Thus:

$$g = 2 + \frac{\alpha}{\pi} = 2.00232\ldots \tag{2.88}$$

$g - 2$ has been calculated to 5-loop order! The current best measurement is $g = 2.0023193043617 \pm (3 \cdot 10^{-13})$.
We cannot compare the experiment to the theory here (use $g-2$ of μ) since we cannot measure α any other way which is as precise. $g - 2$ is used to define the renormalized value of the fine-structure constant.

$$\frac{1}{\alpha} = 137.035999070 \pm (9.8 \cdot 10^{-10}) \tag{2.89}$$

This is one of the biggest triumphs of QFT.

3
Non-abelian gauge invariance

Here we still discuss gauge invariance generalized to non-abelian symmetry groups. They are of particular importance since they appear in the Standard Model (SM) of particle physics whose gauge group is

$$G = SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{3.1}$$

Interestingly, each gauge group appears in a different aggregate state at low-energy:
SU(3)$_C$: confined
SU(2)$_L \times U(1)_Y$: screened (spontaneous symmetry breaking)
$U(1)_{em}$: Coulomb (only long range force). [1]

[1] Gravity is another long range force, which is a gauge theory of the Lorentz group $SO(1,3)$.

3.1 Massless Spin - One - Particles

Please read chapter 15 of QFT1 script if you are unfamiliar with massless spin 1 representations and $U(1)$ gauge theories.

Reminder: Spin 1 fields are in a $(\frac{1}{2}, \frac{1}{2})$ Lorentz representation. Massless representations of the Poincaré group differ from massive ones in that they are one-dimensional , containing only states with helicity j or $-j$. A parity invariant theory requires two dimensional representations containing both possible helicities $(j, -j)$. It is difficult to represent in local fields eg. the four-vector field $A_\mu(x)$.

How to go from $4 \to 2$?

① A^0 is not dynamical: no kinetic term $\dot{A}_0$.
Fix A_i and $\dot{A}_i$ at some initial time t_0.
$\hookrightarrow t_0$ fixed!

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{3.2}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$\delta S = 0 \Leftrightarrow \partial_\mu F^{\mu\nu} = 0. \tag{3.3}$$

Set $\nu = 0$:

$$\nabla^2 A_0 + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = 0. \tag{3.4}$$

$$A_0(\mathbf{x}) = \int d^3x' \frac{\nabla'_{\mathbf{x}} \cdot \frac{\partial \mathbf{A}'(\mathbf{x}')}{\partial t}}{4\pi |\mathbf{x} - \mathbf{x}'|}. \tag{3.5}$$

We see, that A_0 is clearly a dependent variable, we do not get to fix it specifying initials conditions.

② Redundancy : gauge-invariance

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x). \tag{3.6}$$

Symmetries lead to selection rules (or conservation laws). Do we have infinitely many conserved currents here? **No**. Gauge symmetry is a **redundancy**. Two states connected by a gauge symmetry have to be identified: **Same** physical state. Compare to symmetry: Two different physical states with same properties. It is also obvious from Maxwell's equation.
EOM is **not** sufficient to specify time evolution of A_μ.

$$(\eta_{\mu\nu}(\partial^\lambda \partial_\lambda) - \partial_\mu \partial_\nu) A^\nu = 0. \tag{3.7}$$

This operator is **not** invertible, zero eigenvectors for:

$$A_\mu \sim \partial_\mu \lambda(x) \tag{3.8}$$

Fixing initial data we cannot uniquely determine A_μ, since there is no way to distinguish A_μ and $A_\mu + \partial_\mu \lambda$.
$\hookrightarrow$ Same physical state.

Need to pick unique representation from gauge orbits:

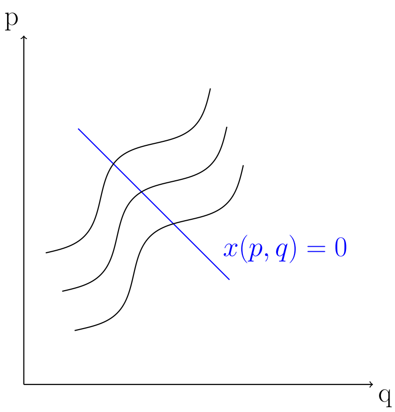


Figure 3.1: Three gauge orbits and a gauge fixing condition $x(p,q) = 0$.

Example: Coulomb gauge:

$$\nabla \cdot \mathbf{A} = 0 \tag{3.9}$$

This does not completely fix the gauge redundancy: we can still use harmonic functions which satisfy $A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda$ and $\nabla^2 \lambda(x) = 0$. However, A^0 is fixed, see above Eq. (3.5). With $\nabla \cdot \mathbf{A} = 0$ we get

$$A_0 \equiv 0. \tag{3.10}$$

We find 3 independent entries in $A_\mu = (0, A_1, A_2, A_3)$ now subject to a single constraint $\nabla \cdot \mathbf{A} = 0$ or $\mathbf{k} \cdot \boldsymbol{\epsilon}(\mathbf{k}) = 0$
$\hookrightarrow$ There are only 2 transverse polarizations describing the physical polarizations of a massless spin 1 field.

See QFT1, chapter 15.

3.1.1 One more comment on unphysical gauge-invariance:

Consider

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2\left(A_\mu + \partial_\mu \pi\right)^2, \tag{3.11}$$

which is called the Stueckelberg Lagrangian.
Gauge-invariant:

$$A_\mu \to A_\mu + \partial_\mu \alpha(x) \tag{3.12}$$

$$\pi(x) \to \pi(x) - \alpha(x). \tag{3.13}$$

We can use gauge symmetry to set $\pi(x) \equiv 0$ everywhere. We thus get a massive photon:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{m^2}{2}A_\mu A^\mu. \tag{3.14}$$

Let's integrate out π! This is crazy because π is massless, but the result is interesting

$$\mathcal{L} \supset \frac{m^2}{2}\left(\partial_\mu \pi \partial^\mu \pi + 2A_\mu \partial^\mu \pi + A_\mu A^\mu\right). \tag{3.15}$$

$$\delta S = 0 \quad \Leftrightarrow \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \pi} = 0 \quad \Leftrightarrow \quad \Box\, \pi = -\partial^\mu A_\mu \tag{3.16}$$

$$\pi = -\frac{1}{\Box}\partial^\mu A_\mu. \tag{3.17}$$

Plug back into $\mathcal{L}$:

$$\frac{m^2}{2}\left[\partial_\mu \frac{1}{\Box}(\partial^\lambda A_\lambda)\partial^\mu \frac{1}{\Box}(\partial^\nu A_\nu) - 2A_\mu \partial^\mu \frac{1}{\Box}(\partial^\nu A_\nu) + A_\mu A^\mu\right] \tag{3.18}$$

$$\overset{\text{P. I.}}{=} \frac{m^2}{2}\left[-\partial^\mu \cancelto{1}{\partial_\mu \frac{1}{\Box}}(\partial^\lambda A_\lambda)\frac{1}{\Box}(\partial^\nu A_\nu) + 2\partial^\mu A_\mu \frac{1}{\Box}(\partial^\nu A_\nu) + A_\mu A^\mu\right] \tag{3.19}$$

$$= \frac{m^2}{2}\left[+(\partial^\lambda A_\lambda)\frac{1}{\Box}(\partial^\nu A_\nu) + A_\mu A^\mu\right], \tag{3.20}$$

which is the same as

$$-\frac{1}{4}F_{\mu\nu}\frac{m^2}{\Box}F^{\mu\nu} = -\frac{m^2}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{1}{\Box}(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{3.21}$$

$$= -\frac{m^2}{2}\left[\partial_\mu A_\nu \frac{1}{\Box}\partial^\mu A_\nu - \partial_\mu A_\nu \frac{1}{\Box}\partial^\nu A_\mu\right] \tag{3.22}$$

$$= -\frac{m^2}{2}\left[-A_\nu \partial^\mu \cancelto{1}{\partial_\mu \frac{1}{\Box}} A^\nu - \partial_\mu A_\nu \frac{1}{\Box}\partial^\nu A^\mu\right] \tag{3.23}$$

$$= \frac{m^2}{2}\left[A_\nu A^\nu + \partial_\mu A_\nu \frac{1}{\Box}\partial^\nu A^\mu\right]. \tag{3.24}$$

This gives

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\left(1 + \frac{m^2}{\Box}\right)F_{\mu\nu}, \tag{3.25}$$

which is also gauge-invariant! The additional term $\frac{m^2}{\Box}$ gets increasingly important at larger distances. $V \sim m^2 r^2$.
$\Rightarrow$ This will lead to poles without particles.
$\hookrightarrow$ Not perturbatively unitary (later)

Very strange theory since it looks **non-local** due to $\frac{1}{\Box}-$ derivative.

Question: Why not quantize $F_{\mu\nu}$?

It would be gauge-invariant, but need to couple to currents $A_\mu \bar{\psi}\gamma_\mu\psi$ or generally $\sim \mathcal{L}_{int} = A_\mu J^\mu$ with J_μ as a conserved current. We can get A_μ back with $A_\mu \sim \frac{1}{\Box}\partial^\nu F_{\mu\nu}$, but this leads to theories which are not manifestly local, need to check at every stage.

Summary of digression

- Gauge invariance is unphysical
- Spectrum is physical: massive vs. massless (3 polarizations vs. 2 polarizations
- Gauge invariance is a mere redundancy of the description, but it makes our description a lot easier.

3.2 Geometry of gauge-invariance

Assume complex-valued Dirac field $\psi = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and require gauge redundancy:

$$\psi(x) \to e^{i\alpha(x)}\psi(x). \tag{3.26}$$

Question: How to write invariant Lagrangians?

Naive try: write same terms invariant under global phase rotation Works for the mass term

$$m\bar{\psi}(x)\psi(x) \checkmark \tag{3.27}$$

However: what about derivatives? Derivatives compare the field at different points in space.
Derivative of $\psi(x)$ in direction of n^μ

$$n^\mu \partial_\mu \psi \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\psi(x + \epsilon n) - \psi(x)\right]. \tag{3.28}$$

In a theory with local phase invariance, not sensible, because $\psi(x+\epsilon n)$ and $\psi(x)$ have different phase transformation properties! Introduce compensating factor: Scalar $U(y, x)$ which transforms as

$$U(y, x) \to e^{i\alpha(y)}U(y, x)e^{-i\alpha(x)}. \tag{3.29}$$

We set

$$U(y, y) = 1 \tag{3.30}$$

and in general, we can require $U(y, x)$ to be a pure phase:

$$U(y, x) = e^{i\phi(y,x)}. \tag{3.31}$$

Now: $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation laws. We can define a sensible derivative

$$n^\mu D_\mu \psi \equiv \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right], \tag{3.32}$$

(**covariant**) derivative.

3.2.1 The gauge field as a connection

Expand comparator $U(x+\epsilon n, x)$ for infinitesimal ϵ:

$$U(x+\epsilon n, x) \equiv 1 - ig\cdot\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2), \tag{3.33}$$

where g is a constant (for later convenience) and $A_\mu(x)$ is a continuous function. This vector field $A_\mu(x)$ is called a **connection**. Connection: infinitesimal limit of comparator of local symmetry. We can write covariant derivative as

$$D_\mu\psi(x) = \partial_\mu\psi(x) + ig\cdot A_\mu(x)\psi(x). \tag{3.34}$$

By using the transformation $U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)}$ and Eq. (3.33), we find

$$\begin{aligned}
U(x+\epsilon n, x) &\to e^{i\alpha(x+\epsilon\eta)}\, U(x+\epsilon n, x)\, e^{-i\alpha(x)} \\
&= e^{i\alpha(x)}(1 + i\epsilon n^\mu\partial_\mu\alpha(x))\left[1 - ig\cdot\epsilon n^\mu A_\mu(x)\right]e^{-i\alpha(x)} \\
&= 1 - ig\cdot\epsilon n^\mu\left(A_\mu - \frac{1}{g}\partial_\mu\alpha\right) + \mathcal{O}(\epsilon^2).
\end{aligned}$$

And therefore:

$$A_\mu(x) \to A_\mu(x) - \frac{1}{g}\partial_\mu\alpha(x) \tag{3.35}$$

which is exactly how we know a $U(1)$ gauge field is supposed to transform.

Let's see how $D_\mu\psi(x)$ transforms:

$$D_\mu\psi(x) \to \left[\partial_\mu + ig(A_\mu - \frac{1}{g}\partial_\mu\alpha)\right]\, e^{i\alpha(x)}\psi(x) \tag{3.36}$$

$$= e^{i\alpha(x)}(\partial_\mu + igA_\mu)\psi(x) \tag{3.37}$$

$$= e^{i\alpha(x)}D_\mu\psi(x) \tag{3.38}$$

same as $\psi(x)$:

$$D_\mu\psi(x) \to e^{i\alpha(x)}D_\mu\psi(x). \tag{3.39}$$

What is $U(y,x)$?

$$U(y,x) \equiv U_{P(x\to y)} \equiv e^{-ig\int_x^y A^\mu(x')dx'_\mu} = e^{-ig\int_0^1 A^\mu(x'_\mu(s))\dot{x'}_\mu(s)ds}.$$

where $P(x \to y)$ is a path from $x \to y$, parameterized by $x^\mu(0) = x^\mu$, $x^\mu(1) = y^\mu$ and $\dot{x}^\mu(s) = \frac{dx^\mu}{ds}$.
In fact: $D_\mu\psi(x) \to e^{i\alpha(x)} D_\mu\psi(x)$ is equivalent to the statement that the **parallel-transported field** along any path is covariantly constant.

$\psi_P(x)$ is the parallel transported field along the path P.

Proof:

$$\begin{aligned}
\frac{dx^\mu}{ds} D_\mu \psi_P(x) = 0 \quad &\Leftrightarrow \\
\frac{dx^\mu}{ds}\left(\frac{\partial}{\partial x^\mu} + igA_\mu\right)\psi_P(x) = 0 \quad &\Leftrightarrow \\
\frac{d\psi_P\,(x(s))}{ds} = -igA_\mu(x(s))\frac{dx^\mu}{ds}\cdot\psi_P(x(s)) \quad &\Leftrightarrow \\
\psi_P(x(s)) = e^{-ig\int_0^s A^\mu(\tilde{x}(t))\dot{\tilde{x}}_\mu(t)dt}\,\psi_P(x(0)).&
\end{aligned}$$

3.3 Kinetic energy term for A_μ

Render any matter Lagrangian $\mathcal{L}_{matter}(\psi, \partial_\mu\psi)$ invariant under global $U(1)$, invariant under **local** $U(1)$ by $\partial_\mu \to D_\mu$

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m - gA\!\!\!/)\psi. \tag{3.40}$$

$\mathcal{L}_{matter}(\psi, \partial_\mu, A_\mu)$ does not contain derivatives of the gauge field.
If we treat A_μ as a dynamical variable[2], then its equation of motion is a constraint on the matter field, which sets the current to zero:

[2] Without a kinetic term so far.

$$\delta S_{A_\mu} = 0 \Leftrightarrow \frac{\partial\mathcal{L}}{\partial A_\mu} = 0 \Leftrightarrow J_\mu = 0. \tag{3.41}$$

$$\mathcal{L}_{matter} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + J_\mu A_\mu. \tag{3.42}$$

with $J_\mu = g\bar{\psi}\gamma_\mu\psi$. .

The $U(1)$ Noether current!

3.3.1 The field strength

To allow the current to flow we must make A_μ dynamical!
If we set $x = y$ in $U(x, y)$, but choose a non-trivial path

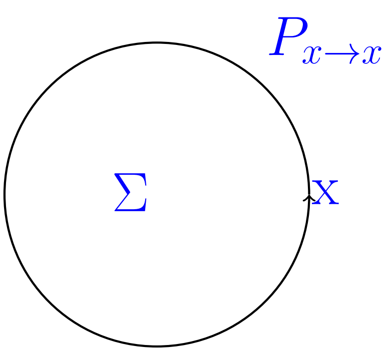


Figure 3.2: $P_{x\to x}$, loop encloses the surface Σ.

we get a *Wilson-loop*:

$$U(x, x)^{\text{loop}} = e^{-ig\oint_p A_\mu dx^\mu}. \tag{3.43}$$

The Wilson loop is gauge invariant, trivial by Eq. (3.29).

By Stoke's theorem, this can be written as:

$$U^{\text{loop}}(x,x) = e^{-ig/2\int_\Sigma F_{\mu\nu}d\sigma^{\mu\nu}} = 1 - i\frac{g}{2}\int_\Sigma F_{\mu\nu}d\sigma^{\mu\nu} + \mathcal{O}(g^2) \quad (3.44)$$

The non-relativistic version should be very familiar

$$\oint_p A_i\, dx^i = \int_\Sigma d\sigma^i \epsilon^{ijk}\partial_j A_k = \int_\Sigma d\sigma^i B_i$$

which says that the magnetic flux through a surface is the integral of the magnetic vector potential around the loop enclosing the surface.

with $d\sigma^{\mu\nu}$ being the surface element of Σ that P bounds with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since the Wilson-loop is gauge-invariant, so is $F_{\mu\nu}$. Next, since $D_\mu\psi$ transforms nicely, so does $D_\mu D_\nu \psi$ and so

$$[D_\mu, D_\nu]\,\psi(x) \to e^{i\alpha(x)}\,[D_\mu, D_\nu]\,\psi(x). \quad (3.45)$$

However, the commutator is not itself a derivative

$$\begin{aligned} [D_\mu, D_\nu]\,\psi &= ([\partial_\mu, \partial_\nu] + ig\,[\partial_\mu, A_\nu] - ig\,[\partial_\nu, A_\mu])\,\psi & (3.46)\\ &= ig\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\cdot\psi & (3.47)\end{aligned}$$

We find the field strength as the commutator of two covariant derivatives

$$\boxed{[D_\mu, D_\nu] = igF_{\mu\nu}.} \quad (3.48)$$

Since $\psi(x)$ accounts for the entire transformation law (Eq. (3.45)), the factor $F_{\mu\nu}$ must be gauge invariant (which it is of course) !

Nice geometric interpretation: D_μ, D_ν is the difference between

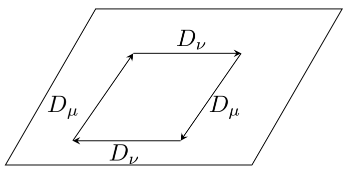


Figure 3.3: Geometric interpretation of a commutator of covariant derivatives.

$D_\mu D_\nu$ = fields separated in $\nu-$ direction followed by $\mu-$direction.
$D_\nu D_\mu$ = fields separated in $\mu-$ direction followed by $\nu-$direction.

This is equivalent to a Wilson loop around an infinitesimal rectangular path.

Wilson-loops have simple discretizations known as plaquettes, which are used to construct lattice actions.

3.4 Non-abelian symmetries

Consider kinetic Lagrangian with N-Dirac fermions

$$\mathcal{L} = \sum_{j=1}^{N} \bar{\psi}_j(i\partial\!\!\!/ - m)\psi_j. \quad (3.49)$$

Under global transformations this is invariant under

$$\psi_j(x) \to \underbrace{\left(e^{i\alpha^a T^a}\right)_{ij}}_{V_{ij}} \psi_j,\ \alpha^a \in \mathbb{R},$$

where the T^a are the SU(N) generators in the fundamental representation.

$$\mathrm{SU}(N) : V^\dagger V = \mathbb{1}_{n\times n}, \quad \det V = 1.$$

Generators satisfy

$$[T_a, T_b] = i f_{abc} T_c \tag{3.50}$$

where f_{abc} are structure constants.

We have factorized trivial $U(1)$ factor to focus on non-abelian part of the transformation.

$$U(N) = U(1)\cdot \mathrm{SU}(N)$$

The action is naturally $U(N)$ invariant, but we do not want to discuss trivial $U(1)$ here.

SU(2) example:

$$T_a = \frac{\sigma_a}{2}, \quad a = 1,2,3 \qquad \text{with} \quad f_{abc} = \epsilon_{abc}$$

we can generalize to arbitrary $SU(N)$ algebras.

$SU(N)$ **Lie-Algebra**

T_a are hermitian, traceless with $a = 1, \ldots, n^2 - 1$.

Normalization

$$\mathrm{Tr}\,[\mathrm{T_a}, \mathrm{T_b}] = \frac{1}{2}\delta_{\mathrm{ab}}$$

Hermitian:

$$\begin{aligned} V^\dagger V &= 1 \\ \Leftrightarrow \left(e^{-i\alpha^a T^a}\right)^\dagger \left(e^{-i\alpha^a T^a}\right) &\overset{!}{=} 1 \\ \Rightarrow 1 + i\alpha^a (T^a - T^{a\dagger}) + \ldots &= 1 \quad \forall \alpha^a \\ \Rightarrow T^a &= T^{a\dagger} \end{aligned}$$

Trace-less:

$$\begin{aligned} 1 &\overset{!}{=} \det V \\ \Rightarrow 0 &= \ln\det V = \ln(v_1 \cdot \underset{\text{V eigenvalues}}{\overset{\downarrow}{v_2}} \ldots v_n) \\ &= \sum_i \ln v_i = \mathrm{Tr}\ln \mathrm{V} = \mathrm{i}\alpha^{\mathrm{a}}\mathrm{Tr}(\mathrm{T^a}), \end{aligned}$$

which is true for $\forall\alpha^a$ and $\Rightarrow$ Tr $\mathrm{T^a} = 0$.

Eli Cartan classified all possible semi-simple Lie-algebras.[3]

[3] See central tutorial.

Name	Abbreviation	# of gen's		Symmetry
Unitary	$SU(N)$	N^2-1	$N>2$	$V^\dagger V = 1$
Orthogonal	$SO(N)$	$\frac{N(N-1)}{2}$	$N>2$	$V^\dagger V = 1$
Symplectic	$Sp(2N)$	$N(2N+1)$	$N>1$	$V^T J_+ V = J_+$
Exceptional	$G_2(14), F_4(52),$	$E_6(78), E_7(133),$	$E_8(248)$	

Table 3.1: Classification of all possible semi-simple Lie-algebras with

With

$$J_+ = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

3.4.1 Local symmetries

Generalize to local symmetries

$$\psi(x) \Rightarrow V(x)\psi(x).$$

To compare the field at different points, parallel transport

$$\psi_{P(x\to y)} = U(y,x)\psi(x), \tag{3.51}$$

where $U(x,y)$ is the parallel transport **matrix**. If $\psi_{P(x\to y)}$ is to transform as a field at y under local gauge transformation, then $U(y,x)$ must transform

$$U(y,x) \to V(y)U(y,x)V^{-1}(x). \tag{3.52}$$

Can we construct the parallel transport matrix $U(y,x)$ in terms of a local vector field?
Infinitesimal shift gives the connection:

$$U(x+dx,x) = \mathbb{1} - i\mathbf{A}_\mu dx^\mu. \tag{3.53}$$

The connection or gauge field A_μ is an element of the Lie-Algebra and can be expanded in generators

$$\mathbf{A}_\mu \equiv A^a_\mu T^a. \tag{3.54}$$

$\mathbf{A}_\mu$ is a matrix field and will transform under **global** transformation in an **adjoint** representation of SU(N), with Eq. (3.52) we get for $V = const.$,

$$\mathbf{A}_\mu \to V\mathbf{A}_\mu V^{-1}. \tag{3.55}$$

Note, no charge has appeared (or gauge coupling).
How does the gauge field transform? Using Eq. (3.52) and Eq. (3.53), we get

$$V(x+dx)(\mathbb{1} - i\mathbf{A}_\mu dx^\mu)V^{-1}(x) \tag{3.56}$$
$$= (V(x) + \partial_\mu V(x)dx^\mu)\,(\mathbb{1} - i\mathbf{A}_\mu dx^\mu)V^{-1}(x) \tag{3.57}$$
$$= \mathbb{1} - i\mathbf{A}^V_\mu(x)dx^\mu. \tag{3.58}$$

And we find the *gauge transformation of a non-abelian vector field:*

$$\boxed{\mathbf{A}_\mu \to \mathbf{A}^V_\mu = V(x)\mathbf{A}_\mu(x)V^{-1}(x) + i(\partial_\mu V(x))V^{-1}(x).} \tag{3.59}$$

The two terms are

$$\mathbf{A}_\mu = V(x)\underset{\substack{\uparrow\\ 1}}{\mathbf{A}_\mu}V^{-1}(x) + i(\partial_\mu V(x)\underset{\substack{\uparrow\\ 2}}{)})V^{-1}(x).$$

1: $\mathbf{A}_\mu$ transforms as adjoint, survives even in global case, i.e. for $V(x) = V = $ const, see Eq. (3.55).

②: is specific to gauge field or geometrically speaking, the connection.

Check $U(1)$:
With $V = e^{i\alpha(x)}, V^{-1} = e^{-i\alpha(x)}$
one gets:

$$A_\mu \to V(x)A_\mu(x)A_\mu V^{-1}(x) + i(\partial_\mu V(x))V^{-1}(x)$$
$$= A_\mu(x) - \partial_\mu\alpha(x). \quad \checkmark$$

Consider an infinitesimal gauge-transformation:

$$V(x) = \mathbb{1} + i\omega^a(x)T^a + \dots \quad \omega^a \ll 1, \tag{3.60}$$

then:

$$\mathbf{A}_\mu \to \mathbf{A}_\mu(x) + i\omega^a(x)\left[T^a\omega^a, \mathbf{A}_\mu(x)\right] - \partial_\mu\omega_a(x)T^a + \dots \tag{3.61}$$

and the components transforms as ?
We can project onto the them with $2\,\mathrm{Tr}\left[\mathrm{T^c}, \mathbf{A}_\mu\right] = \mathrm{A}^{\mathrm{c}}_\mu$ where we have used $\mathrm{Tr(T_aT_b)} = \frac{\delta_{\mathrm{ab}}}{2}$.
We find,

$$A^c_\mu(x) \to A^c_\mu(x) - \partial_\mu\omega^c(x) + 2i\omega^a\mathrm{Tr}\left[\left[\mathrm{T^a}, \mathbf{A}_\mu(\mathrm{x})\right]\mathrm{T^c}\right] + \dots$$

The generators obey the Lie-algebra:

$$\left[T^a, T^b\right] = if^{abc}T^c \tag{3.62}$$

and we obtain

$$\boxed{A^c_\mu \to A^c_\mu - \partial_\mu\omega^c - \omega^a A^b_\mu f^{abc} + \dots} \tag{3.63}$$

The infinitesimal gauge transformation can be written very elegantly: Under a V transformation:

$$\omega^a T^a \equiv \boldsymbol{\omega} \to V\boldsymbol{\omega}V^{-1}(\text{adjoint}). \tag{3.64}$$

The covariant derivative for a field in the adjoint is[4]

[4] See exercise.

$$D^{adj}_\mu\boldsymbol{\omega} \equiv \partial_\mu\boldsymbol{\omega} + i\left[\mathbf{A}_\mu, \boldsymbol{\omega}\right]. \tag{3.65}$$

Comparing with Eq. (3.61)

$$\delta\mathbf{A}_\mu = \mathbf{A}^\omega_\mu - \mathbf{A}_\mu = i\left[T^a\omega^a, \mathbf{A}_\mu(x)\right] - \partial_\mu(\omega_a T^a), \tag{3.66}$$

$$\boxed{\delta\mathbf{A}_\mu = -D^{adj}_\mu\boldsymbol{\omega}.} \tag{3.67}$$

The connection allows us to define a covariant derivative for non-abelian fields:

$$D_\mu\psi(x) = \left(\partial_\mu + i\mathbf{A}_\mu(x)\right)\psi(x). \tag{3.68}$$

It transforms as

Using $\psi(x) \to V(x)\psi(x)$

$$\begin{aligned}
&D^V_\mu V(x)\psi(x) = (\partial_\mu + iA^V_\mu(x))V(x)\psi(x)\\
&= (\partial_\mu + iVA_\mu V^{-1} - (\partial_\mu V)V^{-1})V\psi\\
&= V(\partial_\mu + iA_\mu)\psi + \cancel{(\partial_\mu V)\psi} - \cancel{(\partial_\mu V)V^{-1}V\psi}\\
&= V(x)D^V_\mu\psi.
\end{aligned}$$

Just as the abelian case, the covariant derivative has nice transformation properties, and behaves like the fermion field itself, i.e.

$$D_\mu \psi(x) \to V(x) D_\mu \psi(x)$$

We can use D_μ to define the field-strength[5]:

$$\mathbf{F}_{\mu\nu} = i \left[\mathbf{D}_\mu, \mathbf{D}_\nu \right]. \tag{3.69}$$

[5] We will get increasingly sloppy and won't use $\mathbf{D}_\mu$ for the covariant derivative, even tough it is matrix valued.

We know that this has good transformation properties, since D_μ does [6]

$$\mathbf{F}_{\mu\nu}(x) \to V(x) \mathbf{F}_{\mu\nu}(x) V^{-1}(x). \tag{3.70}$$

[6] The non-abelian field strength is **not** gauge-invariant!

With the covariant derivative in the fundamental representation

$$\mathbf{F}_{\mu\nu} = i \left[D_\mu, D_\nu \right] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + i \left[\mathbf{A}_\mu, \mathbf{A}_\nu \right]. \tag{3.71}$$

In components

$$\begin{aligned} \mathbf{F}_{\mu\nu} &= F^a_{\mu\nu} \cdot T^a \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu. \end{aligned}$$

We know that non-abelian field strength is **not** gauge-invariant and to form an invariant function, we must take the trace with respect to the group indices of $F_{\mu\nu}$ or the trace of products of field strengths. We can easily generalize the abelian procedure to arrive at locally invariant Lagrangians: $\partial_\mu \to D_\mu$, e.g.

$$\mathcal{L}_{matter} = \bar{\psi}(i \not{D} - m)\psi = \bar{\psi}(i \not{\partial} - \not{A}^a T^a - m)\psi \tag{3.72}$$

is invariant under local SU(N). An invariant kinetic term quadratic in derivatives is

$$\mathcal{L} = -\frac{1}{2g^2} \mathrm{Tr} \left[\mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu} \right] + \mathcal{L}_{\text{matter}}. \tag{3.73}$$

with the normalization $\mathrm{Tr}\left[\mathrm{T^a T^b}\right] = \frac{1}{2}\delta^{\mathrm{ab}}$,

$$= -\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \mathcal{L}_{matter}(\psi, D_\mu \psi). \tag{3.74}$$

We can recover a more conventional normalization

$$A_\mu \to g A_\mu \qquad \text{(rescaling)} \tag{3.75}$$

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \mathcal{L}_{matter}(\psi, (\partial_\mu + igA_\mu)\psi) \tag{3.76}$$

with

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + ig \left[A_\mu, A_\nu \right]. \tag{3.77}$$

The Yang-Mills equations of motion follow (later). The $F^a_{\mu\nu}$ are not all independent, because of the Bianchi identities:

$$D_\mu F_{\rho\sigma} + D_\rho F_{\sigma\mu} + D_\sigma F_{\mu\rho} = 0. \tag{3.78}$$

which are a direct consequence of the Jacobi identity, which commutators trivially satisfy:

$$0 = \left[D_\mu, \left[D_\rho, D_\sigma \right] \right] + \left[D_\rho, \left[D_\sigma, D_\mu \right] \right] + \left[D_\sigma, \left[D_\mu, D_\rho \right] \right]. \tag{3.79}$$

3.4.2 *Dual field strength tensor and Chern-Simmons current*

Why did we not consider the other invariant?

$$I = \mathrm{Tr}\left(\epsilon^{\mu\nu\rho\sigma} \mathrm{F}_{\mu\nu} \mathrm{F}_{\rho\sigma}\right) \tag{3.80}$$

as a kinetic term. It is Lorentz- and gauge-invariant has two derivatives!

Answer: We can write it as a total derivative with the dual field strength:

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}. \tag{3.81}$$

Proof:
Use $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + iA_{[\mu} A_{\nu]}$ with $f^{[\alpha} g^{\beta]} \equiv f^\alpha g^\beta - f^\beta g^\alpha$ (antisymmetry).

$$\begin{aligned}
\mathrm{Tr}(\mathrm{F}_{\mu\nu}\widetilde{\mathrm{F}}^{\mu\nu}) &= \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\left(\partial^{[\mu}\mathrm{A}^{\nu]} + \mathrm{i}\mathrm{A}^{[\mu}\mathrm{A}^{\nu]}\right)\left(\partial^{[\rho}\mathrm{A}^{\sigma]} + \mathrm{i}\mathrm{A}^{[\rho}\mathrm{A}^{\sigma]}\right)\right] \\
&= 2\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\left(\partial^\mu \mathrm{A}^\nu + \mathrm{i}\mathrm{A}^\mu\mathrm{A}^\nu\right)\left(\partial^\rho\mathrm{A}^\sigma + \mathrm{i}\mathrm{A}^\rho\mathrm{A}^\sigma\right)\right] \\
&= 2\,\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\partial^\mu\mathrm{A}^\nu\partial^\rho\mathrm{A}^\sigma + 2\mathrm{i}(\partial^\mu\mathrm{A}^\nu)\mathrm{A}^\rho\mathrm{A}^\sigma - \mathrm{A}^\mu\mathrm{A}^\nu\mathrm{A}^\rho\mathrm{A}^\sigma] \\
&= 2\,\partial^\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left[\mathrm{A}^\nu\partial^\rho\mathrm{A}^\sigma + \frac{2\mathrm{i}}{3}\mathrm{A}^\nu\mathrm{A}^\rho\mathrm{A}^\sigma\right] \\
&= 2\,\partial^\mu J^{CS}_\mu,
\end{aligned}$$

which is called the Chern-Simons-current. By taking

$$\frac{1}{2}\mathrm{Tr}\left[\mathrm{F}^{\mu\nu}\tilde{\mathrm{F}}_{\mu\nu}\right] = \partial_\mu \mathrm{J}^\mu_{\mathrm{CS}} \tag{3.82}$$

as the kinetic Lagrangian, we could not generate any EOM for A_μ. It would only affect the action at its endpoints.
However:

- This will be crucial in the discussion of anomalies, the strong CP problem of QCD and the topology of the vacuum.
- This does not contribute at any order in perturbation theory, but it can have physical effects due to non-perturbative contributions – this term e.g. leads to a CP-violating dipole moment of the neutron if we add it to the QCD Lagrangian.[7]

We use the cyclicity of the trace to show that the last term vanishes

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\rho\mathrm{A}_\sigma) \\
&= \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\mathrm{A}_\sigma\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\rho) \\
&= -\epsilon^{\sigma\mu\nu\rho}\mathrm{Tr}(\mathrm{A}_\sigma\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\rho) \\
&= -\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\mathrm{A}_\mu\mathrm{A}_\nu\mathrm{A}_\rho\mathrm{A}_\sigma) \\
&= 0.
\end{aligned}$$

[7] And it is a big mystery why its coefficient θ is so small $\frac{1}{2}\theta\mathrm{Tr}[\mathrm{F}^{\mu\nu}\tilde{\mathrm{F}}_{\mu\nu}]$. The limit on the EDM of the neutron tells us $\theta < 10^{-9}$.

3.5 *Pure Yang-Mills theory*

We will study some classical properties of Yang-Mills-theories.

$$S^{YM} = -\frac{1}{2g^2}\int d^4x\ \mathrm{Tr}\left[\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right] \tag{3.83}$$

with

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + i\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right] \tag{3.84}$$

and

$$\mathbf{A}_\mu = A^a_\mu T^a. \tag{3.85}$$

The T^a matrices generate the Lie-algebra:

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{3.86}$$

with $a, b, c = 1, \ldots, k$ (dimension of the Lie-algebra).

As a consequence of Eq. (3.86), the T^a are traceless.

$$\mathrm{Tr}(\left[\mathrm{T^a}, \mathrm{T^b}\right]) = \mathrm{Tr}(\mathrm{T^aT^b} - \mathrm{T^bT^a}) \stackrel{\text{cyclic tr.}}{=} \mathrm{Tr}(0) = \mathrm{i}f^{\mathrm{abc}}\mathrm{Tr}(\mathrm{T^c}).$$

Every element in algebra can be written as a sum of commutators:

$$\mathrm{Tr}(\mathrm{T^a}) = 0$$

The Cartan-killing metric is non-singular for semi-simple groups!

3.5.1 Equation of motion

The full YM-action in components, and rescaling to canonical kinetic terms with $A \to gA$:

Again in non-canonical normalization

$$S^{YM} = \int d^4x \left[\underbrace{-\frac{1}{2}\partial_\mu A_\nu^a \partial^\mu A^{\nu a} + \frac{1}{2}\partial_\mu A_\nu^a \partial^\nu A^{\mu a}}_{1} \right] + \tag{3.87}$$

$$+ \underbrace{g f^{abc} A_\mu^b A_\nu^c \partial^\mu A^{a\nu} - \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{d\mu} A^{e\nu}}_{2} \tag{3.88}$$

1 is same as in Maxwell (leads to linear EOM)
2 non-linear self-interactions of gauge fields (cubic, quartic)

We get the EOM by varying the action Eq. (3.83)

$$\delta S = -\frac{1}{g^2} \int d^4x \, \mathrm{Tr}\left[\mathbf{F}_{\mu\nu} \delta \mathbf{F}^{\mu\nu}\right] \tag{3.89}$$

with

$$\delta \mathbf{F}_{\mu\nu} = \partial_\mu \delta \mathbf{A}_\nu + i \delta \mathbf{A}_\mu \mathbf{A}_\nu + i \mathbf{A}_\mu \delta \mathbf{A}_\nu - (\mu \leftrightarrow \nu). \tag{3.90}$$

Using the antisymmetry of $\mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}$:

$$\delta S = -\frac{2}{g^2} \int d^4x \, \mathrm{Tr}\left[\mathbf{F}_{\mu\nu}(\partial^\mu \delta \mathbf{A}^\nu + \mathrm{i}\delta \mathbf{A}^\mu \mathbf{A}^\nu + \mathrm{i}\mathbf{A}^\mu \delta \mathbf{A}^\nu)\right]. \tag{3.91}$$

Integrating the first term by parts and using the cyclic property of the trace:

$$\delta S = \frac{2}{g^2} \int d^4x \, \mathrm{Tr}\left[(\partial^\mu \mathbf{F}_{\mu\nu} + \mathrm{i}\left[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}\right])\delta \mathbf{A}^\nu\right] \tag{3.92}$$

We can read the EOM in matrix form:

$$\boxed{\partial^\mu \mathbf{F}_{\mu\nu} + i\left[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}\right] = 0.} \tag{3.93}$$

Since $F_{\mu\nu}$ transforms as an adjoint, we can write this :

$$\boxed{D^\mu_{adj} \mathbf{F}_{\mu\nu} = 0,} \tag{3.94}$$

which shows that the EOM is covariant.
In addition, the fields satisfy the Bianchi identities Eq. (3.78), which are in compact form

$$D^\mu_{adj} \widetilde{\mathbf{F}}_{\mu\nu} = 0 \tag{3.95}$$

This is not an EOM since it is trivially solved by expressing $\mathbf{F}_{\mu\nu}$ in terms of $\mathbf{A}_\mu$, thanks to the Jacobi identity.

3.5.2 Currents and charges

We can define a conserved current: $\partial_\mu j^\mu = 0$

$$\mathbf{j}_\nu = -\partial^\mu \mathbf{F}_{\mu\nu} \overset{EOM}{=} i\left[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}\right], \tag{3.96}$$

which corresponds to the **Noether** current.

This conserved current implies conserved charges

All in matrix notation: $\mathbf{Q} = Q^a T^a, \mathbf{j}_\nu = J^a_\nu T^a, \dots$.

$$\mathbf{Q} \equiv \int d^3x\, \mathbf{j}_0 \tag{3.97}$$

$$= -\int d^3x\, \partial^i \mathbf{F}_{i0} \tag{3.98}$$

$$= -\oint_{\Sigma_\infty} d^2\sigma^i \mathbf{F}_{i0} \tag{3.99}$$

We see that contrary to abelian gauge theories like QED, non-abelian currents and charges are different

Σ_∞ is a surface at spatial infinity $|\vec{x}| \to \infty$.

The current is **not gauge-invariant** (and **not gauge covariant**) and neither are the charges.

There is no such thing as a classical current, like a wire with quarks instead of electrons.

There is, however, a special class of gauge transformations, that leave the charges to transform covariantly:

$$\mathbf{Q} \to \mathbf{Q}' = -\oint_{\Sigma_\infty} d^2\sigma^i\, V \mathbf{F}_{i0} V^\dagger \tag{3.100}$$

with $V(x) \overset{|\vec{x}|\to\infty}{\longrightarrow} V = \text{const.}$

3.5.3 External currents are shady business

We can couple the gauge-field by adding an external source:

$$\frac{2}{g}\int d^4x\, \mathrm{Tr}(\mathbf{A}^\mu \mathbf{J}^{\mathrm{ex}}_\mu) \tag{3.101}$$

with $\mathbf{J}^{ex}_\mu = J^{ex,a}_\mu\, T^a$, which we take as a non-dynamical field.[8]
The EOM are then:

[8] We will do that later, when we e.g. couple to fermions and scalars.

$$D^\mu \mathbf{F}_{\mu\nu} = \mathbf{J}^{ex}_\nu \tag{3.102}$$

We require covariant transformations of $\mathbf{J}^{ex}_\nu$

$$\mathbf{J}^\mu_{ex} \to V \mathbf{J}^\mu_{ex} V^\dagger. \tag{3.103}$$

The external current source must be **covariantly** conserved due to the EOM:

$$D^\mu_{adj} \mathbf{J}^{ex}_\mu = \partial^\mu \mathbf{J}^{ex}_\mu + i[\mathbf{A}^\mu, \mathbf{J}^{ex}_\mu] = 0. \tag{3.104}$$

Proof:
EOM gives

$$D^{\mu}_{adj}\mathbf{F}_{\mu\nu} = \mathbf{J}^{ex}_{\nu} \tag{3.105}$$

Now we show

$$D^{\nu}_{adj}\mathbf{J}^{ex}_{\nu} = D^{\nu}_{adj}D^{\mu}_{adj}\mathbf{F}_{\mu\nu} \overset{!}{=} 0. \tag{3.106}$$

Drop “adj” $D^{\mu}_{adj} \overset{here!}{=} D^{\mu}$.

$$\begin{aligned}
D^{\nu}D^{\mu}\mathbf{F}_{\mu\nu} &= D^{\nu}\left[\partial_{\mu}\mathbf{F}^{\mu\nu} + i\left[\mathbf{A}_{\mu}, \mathbf{F}^{\mu\nu}\right]\right] \\
&= \cancelto{0}{\partial_{\nu}\partial_{\mu}F^{\mu\nu}} + i\left[\partial_{\nu}\mathbf{A}_{\mu}, F^{\mu\nu}\right] + i\left[\mathbf{A}_{\mu}, \cancel{\partial_{\nu}F^{\mu\nu}}\right]^{0} - \left[\mathbf{A}_{\nu}, \left[\mathbf{A}_{\mu}, \mathbf{F}^{\mu\nu}\right]\right] + i\left[\cancel{\mathbf{A}_{\mu}, \partial_{\mu}\mathbf{F}^{\mu\nu}}\right]^{0} \\
&= i\left[\partial_{\nu}\mathbf{A}_{\mu}, \mathbf{F}^{\mu\nu}\right] - \left[\mathbf{A}_{\nu}, \left[\mathbf{A}_{\mu}, \mathbf{F}^{\mu\nu}\right]\right] \\
&= -\frac{i}{2}\left[\partial_{\mu}\mathbf{A}_{\nu} - \partial_{\mu}\mathbf{A}_{\nu}, \mathbf{F}^{\mu\nu}\right] - \left(\mathbf{A}_{\nu}\mathbf{A}_{\mu}\mathbf{F}^{\mu\nu} - \mathbf{A}_{\nu}\cancel{\mathbf{F}^{\mu\nu}\mathbf{A}_{\mu}} - \cancel{\mathbf{A}_{\mu}\mathbf{F}^{\mu\nu}\mathbf{A}_{\nu}} + \mathbf{F}^{\mu\nu}\mathbf{A}_{\mu}\mathbf{A}_{\nu}\right) \\
&= -\frac{i}{2}\left[\partial_{\mu}\mathbf{A}_{\nu} - \partial_{\mu}\mathbf{A}_{\nu} + i\left[\mathbf{A}_{\mu}, \mathbf{A}_{\nu}\right], \mathbf{F}^{\mu\nu}\right] \\
&= -\frac{i}{2}\left[\mathbf{F}^{\mu\nu}, \mathbf{F}_{\mu\nu}\right] = 0.
\end{aligned}$$

It follows

There is a simpler version of this using the Jacobi-Identity. Try it!

$$\boxed{D^{\nu}_{adj}\mathbf{J}^{ex}_{\nu} = 0.}$$

Important:
$\mathbf{J}^{ex}_{\mu}$ is not a Noether current!
Noether: $j_{\mu} = -\partial^{\rho}\mathbf{F}_{\rho\mu} + \mathbf{J}^{ex}_{\mu}$

We see now that $\frac{1}{g}\int d^4x \ \mathrm{Tr}\left[\mathbf{A}^{\mu}\mathbf{J}^{\mathrm{ex}}_{\mu}\right]$ is **not** gauge invariant!

$$\delta_{\omega}\int d^4x \ \mathrm{Tr}(\mathbf{A}_{\mu}\mathbf{J}^{\mathrm{ex}}_{\mu}) = -\int \mathrm{d^4x} \ \mathrm{Tr}\left[\mathbf{J}^{\mu}_{\mathrm{ex}}\partial_{\mu}\boldsymbol{\omega}\right] \tag{3.107}$$

$$= +\int d^4x \ \mathrm{Tr}\left[\partial_{\mu}\mathbf{J}^{\mu}_{\mathrm{ex}} \cdot \boldsymbol{\omega}\right] \tag{3.108}$$

where we used

$$\mathrm{Tr}(\mathbf{A}_{\mu}\mathbf{J}^{\mu}) \to \mathrm{Tr}(\mathrm{U}\mathbf{A}_{\mu}\mathrm{U}^{\dagger}\mathrm{U}\mathbf{J}^{\mu}\mathrm{U}^{\dagger} + \mathrm{i}\mathrm{U}\partial_{\mu}\mathrm{U}^{\dagger}\mathrm{U}\mathbf{J}^{\mu}\mathrm{U}^{\dagger}).$$

Gauge-invariance can be restored, if $\partial_{\mu}\mathbf{J}^{\mu}_{ex} = 0$, but this is not a **covariant** statement. This requirement breaks gauge-invariance.
Coupling Y-M theories to non-dynamical external sources is shady.

3.6 Parallel transport in non-abelian gauge-theories

We know

$$\psi(x + dx) = \psi(x) - idx^{\mu}A^{a}_{\mu}(x)T^{a}\psi(x) + \dots \tag{3.109}$$

but $[A^{a}_{\mu}(x)T^{a}, A^{b}_{\mu}T^{b}] \neq 0$ in general.
We need a path ordering!

$$\psi(y) = \left[Pe^{-i\int_x^y dx^{\mu}\mathbf{A}_{\mu}}\right]\psi(x) \tag{3.110}$$

Weinberg Witten theorem
The observation about currents follow from a general theorem. *this section needs to be re-written – see also the notes from the central tutorial*

A theory with a global non-abelian symmetry under which spin-1 particles are charged does not admit gauge-invariant conserved current. Another formulation without reference to (unphysical because of redundancy) gauge-invariance:
There cannot be a conserved Lorentz covariant current in a theory with massles spin-1 with non-vanishing charges with respect to the current.

Lorentz-covariance → gauge-invariance because polarizations of massless spin-1 transforms non-covariantly under some Lorentz

$$\epsilon_\mu(p) \to \epsilon_\mu(p) + p_\mu$$

Similarly for massless spin-2:
A theory with a conserved and Lorentz-covariant energy momentum tensor can never have a massless particle of spin 2.

Again: Lorentz-covariance cannot be a gauge-field associated with a local symmetry.

In SM:
It is not a problem since conserved currents are not gauge-covariant. **But** the Weinberg/Witten theorem forbids emergent gravity.
An example: A theory with scalars, spinors which has a conserved energy momentum cannot go through phase-transition to produce a massless graviton, since energy-momentum tensor could no longer exist.

However in AdS/CFT emergent gravity in 5D from conformal theory in 4D (see e.g. Maldacena or Randall/Sundrum).

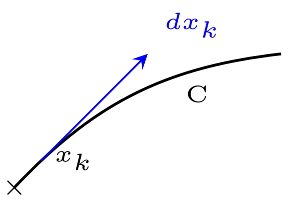


Figure 3.4: Parallel transport along a curve C.

where we define:

$$Pe^{-i\int_x^y dx^\mu \mathbf{A}_\mu} \equiv \Pi_k(1 - dx_k^C \cdot \mathbf{A}(x_k)) \tag{3.111}$$

along the curve C.
This clearly satisfies $Pe^{-i\int_x^y dz^\mu \mathbf{A}_\mu(z)} \to U(y)e^{-i\int_x^y dz^\mu \mathbf{A}_\mu(z)}U(x)^\dagger$

$$\mathrm{Tr}\left[\mathrm{e}^{-\mathrm{i}\oint_{\mathrm{x}} \mathrm{dx}'^\mu \mathbf{A}_\mu(\mathrm{x}')}\right] \text{ is gauge invariant over a closed loop.}$$

Before we quantize, we will study solutions of classical Y-M-theory.

3.7 Euclidean space Yang Mills Theories

It is a bit too technical, but important later on.

Motivation: Euclidean space = Minkowski with imaginary time.

In QM:
Processes with imaginary time evolution formally correspond to tunneling which happen instantly in real time,

't Hooft:
non-singular solutions of pure Y-M in euclidean space: Instantons, which describe tunneling phenomena in QFT, characterized by topological quantum numbers.

We know that

$$\mathrm{Tr}\left[(\mathbf{F}_{\mu\nu} - \widetilde{\mathbf{F}}_{\mu\nu})(\mathbf{F}_{\mu\nu} - \widetilde{\mathbf{F}}_{\mu\nu})\right] \geq 0 \tag{3.112}$$

because it is a sum of squares:

$$a_\mu a_\mu = \sum_{i=0}^{3}(a_i a_i)$$

with an Euclidean metric:

$$\eta_{\mu\nu}^{Eucl} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

and

$$\mathrm{Tr}[\mathrm{A^a T^a A^a T^a}] = \frac{1}{2}\sum_{\mathrm{a=1}}^{\mathrm{n^2-1}}(\mathrm{A^a})^2$$

From Eq. (8.102) it follows that

$$\mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu} + \widetilde{\mathbf{F}}_{\mu\nu}\widetilde{\mathbf{F}}_{\mu\nu}] \geq 2\,\mathrm{Tr}[\mathbf{F}_{\mu\nu}\widetilde{\mathbf{F}}_{\mu\nu}] \tag{3.113}$$

with $\mathrm{Tr}[\widetilde{\mathbf{F}}_{\mu\nu}\widetilde{\mathbf{F}}_{\mu\nu}] = \mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}]$, where the following relation was used

$$\epsilon_{\mu\nu\rho\sigma}\epsilon_{\rho\sigma\alpha\beta} \overset{Eucl.}{=} 2\,(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}).$$

Check:
$$\widetilde{\mathbf{F}}_{\mu\nu}\widetilde{\mathbf{F}}_{\mu\nu} = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\epsilon_{\mu\nu\lambda\kappa}\mathbf{F}_{\rho\sigma}\mathbf{F}_{\lambda\kappa}.$$
Example: $\mu = 0$, $\nu = 1$:
$$\begin{aligned}&\frac{1}{4}\epsilon_{01\rho\sigma}\epsilon_{01\lambda\kappa}\mathbf{F}_{\rho\sigma}\mathbf{F}_{\lambda\kappa}\\ &= \frac{1}{4}(\mathbf{F}_{23} - \mathbf{F}_{32})(\mathbf{F}_{23} - \mathbf{F}_{32})\\ &= \mathbf{F}_{23}\mathbf{F}_{23},\end{aligned}$$
when summed over all μ, ν.

We get

$$\mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}] \geq \mathrm{Tr}[\mathbf{F}_{\mu\nu}\widetilde{\mathbf{F}}_{\mu\nu}]. \tag{3.114}$$

So we find **lower bound** for the Y-M-Euclidean action. Equality is achieved for selfdual solutions:[9]

$$\mathbf{F}_{\mu\nu} = \widetilde{\mathbf{F}}_{\mu\nu} \tag{3.115}$$

and for antiselfdual solutions:

$$\mathbf{F}_{\mu\nu} = -\widetilde{\mathbf{F}}_{\mu\nu}. \tag{3.116}$$

We know from Eq. (3.82) that we can write

$$\int d^4x\,\mathrm{Tr}(\mathbf{F}_{\mu\nu}\widetilde{\mathbf{F}}_{\mu\nu}) = 2\int \mathrm{d^4x}\,\partial_\mu\,\mathbf{j}^\mu_{\mathrm{CS}} \tag{3.117}$$

with

$$\mathbf{j}_\mu^{CS} = \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}(\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma + \frac{2\mathrm{i}}{3}\mathbf{A}_\nu\mathbf{A}_\rho\mathbf{A}_\sigma), \tag{3.118}$$

[9] These are also called BPS states, after Bogomolny –Prasad–Sommerfield.

such that

$$S_E^{YM} = \frac{1}{2g^2}\int d^4x\ \mathrm{Tr}[\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}] \geq \frac{1}{\mathrm{g}^2}\oint_\Sigma \mathrm{d}^3\sigma_\mu\ \mathbf{j}_{\mathrm{CS}}^\mu \tag{3.119}$$

where we integrate over boundary at Eucledian infinity (Σ).

$\hookrightarrow$ Min(S_E) will therefore depend on A_μ at $|x| \to \infty$.

Further, we want a finite S_E^{YM}, therefore $F_{\mu\nu}^a$ has to vanish at $|x| \to \infty$.

$$F_{\mu\nu}^a(x) \to 0, \quad \text{for } |x_\mu^2| \to \infty \tag{3.120}$$

sufficiently fast. This can be achieved for $\mathbf{A}_\mu(x) \to 0$ or gauge-equivalent of 0:

$$\mathbf{A}_\mu(x) \to -iU\partial_\mu U^\dagger \qquad \text{for } x^2 \to \infty \tag{3.121}$$

Substituting Eq. (3.121) into j_μ^{CS} defined in Eq. (3.118), we get for the first of the two terms

Where we have used $UU^\dagger = \mathbb{1}$ and $\partial_\rho(UU^\dagger) = 0$ and so

$$\partial_\rho U = -U(\partial_\rho U^\dagger)U$$

$$\begin{aligned}\epsilon_{\mu\nu\rho\sigma}\cdot \mathrm{Tr}[\mathbf{A}_\nu\partial_\rho\mathbf{A}_\sigma] &= -\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\mathrm{U}(\partial_\nu \mathrm{U}^\dagger)\partial_\rho[\mathrm{U}\partial_\sigma \mathrm{U}^\dagger]] \\ &= -\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\mathrm{U}(\partial_\nu \mathrm{U}^\dagger)(\partial_\rho \mathrm{U})(\partial_\sigma \mathrm{U}^\dagger)] \\ &= \epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\mathrm{U}(\partial_\nu \mathrm{U}^\dagger)\mathrm{U}(\partial_\rho \mathrm{U}^\dagger)\mathrm{U}(\partial_\sigma \mathrm{U}^\dagger)]\end{aligned}$$

Therefore on a surface at infinity ($x^2 \to \infty$):

$$j_{\mu,\infty}^{CS} = \frac{1}{3}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\mathrm{U}(\partial_\nu \mathrm{U}^\dagger)\mathrm{U}(\partial_\rho \mathrm{U}^\dagger)\mathrm{U}(\partial_\sigma \mathrm{U}^\dagger)] \tag{3.122}$$

Hence:

$$S_E^{YM} \geq \frac{1}{3g^2}\int_{\Sigma_\infty} d^3\sigma_\mu\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}[\mathrm{U}(\partial_\nu \mathrm{U}^\dagger)\mathrm{U}(\partial_\rho \mathrm{U}^\dagger)\mathrm{U}(\partial_\sigma \mathrm{U}^\dagger)]. \tag{3.123}$$

The lower bound on the euclidean action depends entirely on group element $U(x)$ at the boundary. The minimum is independent on the details of the field configuration at finite x (or in the 'interior' of space)! This is the characteristic of a topological quantity.

3.7.1 Winding number for a $SU(2)$ gauge theory

Let's work out the consequences for $U \in SU(2)$.
Mapping of boundary of $\mathbb{R}^4 \sim S_\infty^3$ ($\sum_{i=0}^3 x_i^2 = R,\ R \to \infty$) into SU(2) group.

Geometry of $SU(2)$?

We can write elements $U \in SU(2)$ as $\begin{pmatrix}\alpha & -\beta^* \\ \beta & \alpha^*\end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$ and

$$|\alpha|^2 + |\beta|^2 = 1. \tag{3.124}$$

This satisfies $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

Check this!

Eq. (3.124) shows that with $\alpha = x_1 + ix_2$ and $\beta = x_3 + \mathrm{i}x_4$ ($x_i \in \mathbb{R}$) we get

$$1 = |\alpha|^2 + |\beta|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{3.125}$$

We conclude:[10]

[10] Diffeomorphism is an isomorphism for smooth maps.

$$\boxed{SU(2) \text{ is diffeomorphic to } S^3!} \tag{3.126}$$

The minimum of action Eq. (3.123) is a mapping of

It is an invertible smooth map.

$$\boxed{S^3_\infty \to S^3_{SU(2)}.} \tag{3.127}$$

These mappings are characterized by homotopy classes.
Write

$$\partial_\mu U^\dagger = \sum_{a=1}^{3} \frac{\partial \phi_a}{\partial x^\mu} \frac{\partial}{\partial \phi_a} U^\dagger = \partial_\mu \phi^a \partial_a U^\dagger \tag{3.128}$$

where ϕ_a are angles parameterizing a SU(2) element.

$$S_E^{YM} \geq \frac{1}{3g^2} \oint_{S^3_\infty} d^3\sigma_\mu \underbrace{\epsilon_{\mu\nu\rho\sigma} \partial_\nu \phi^a \partial_\rho \phi^b \partial_\sigma \phi^c}_{\text{Jacobian of } S^3_\infty \to S^3_{SU(2)}} \mathrm{Tr}[\mathrm{U}(\partial_\nu \mathrm{U}^\dagger)\mathrm{U}(\partial_\rho \mathrm{U}^\dagger)\mathrm{U}(\partial_\sigma \mathrm{U}^\dagger)]. \tag{3.129}$$

By using a parametrization of $SU(2)$ Euler-angles, one can show that this measures how many times S^3_∞ is mapped into $S^3_{\mathrm{SU}(2)}$[11]:

[11] For the experts: this classification is called homotopy theory and the equivalence classes of mappings $\Pi^k(\mathcal{M}) : S^k \to \mathcal{M}$ form a group, the so-called homotopy group. In our case $\Pi^n(\mathcal{S}^n) = \mathbb{Z}$ which is the winding number. It can be negative because the winding has an orientation. Here $\Pi^3(\mathcal{S}^3) = \mathbb{Z}$ which is n

$$\boxed{S_E^{YM} \geq \frac{8\pi^2}{g^2} n} \tag{3.130}$$

where n is called the *Pontryagin index* or winding number.

$$\boxed{n \equiv \frac{1}{16\pi^2} \int d^4x \; \mathrm{Tr}[\mathbf{F}_{\mu\nu} \widetilde{\mathbf{F}}_{\mu\nu}].} \tag{3.131}$$

One can show that n is invariant under smooth deformations and is a so-called topological quantum number. It measures a topological property of a mapping.

An intuitive analogy is given by $U(1)$ transformations on a circle S^1 with $\phi \in [0, 2\pi]$

$$U(\phi) = e^{i\alpha(\phi)}$$

where the group $U(1)$ also has the geometry of S^1, we can define a simple and intuitive *winding number* [12]

[12] For the experts: this is $\Pi^1(\mathcal{S}^1) = \mathbb{Z}$

$$n \equiv \frac{i}{2\pi} \int_0^{2\pi} d\phi \, U \partial_\phi U^\dagger$$

This looks very similar to Eq. (3.129), except for the complications of the non-abelian group.
A transformation $U_m(\phi) = \exp(im\phi)$ clearly is a mapping $S^1 \to S^1$ which winds m times around the circle. The winding number n is invariant under smooth deformations δU (see Ex!).

SU(2) **Example:**
't Hooft found the SU(2) Instanton

$$A_\mu(x) = \frac{-ix^2}{x^2 + \lambda^2} U \partial_\mu U^\dagger \tag{3.132}$$

with

$$U = \frac{1}{\sqrt{x^2}}(x_0 - i\vec{x} \cdot \vec{\sigma}) \tag{3.133}$$

where $\vec{\sigma}$ are the Pauli-matrices and $x^2 = x_0^2 + \vec{x}^2$.

Let us see if it satisfies the properties we need

- We see that $A_\mu(x) \overset{x^2 \to \infty}{\longrightarrow} -iU\partial_\mu U^\dagger$.
- It can be shown it is self-dual $\mathbf{F}_{\mu\nu} = \widetilde{\mathbf{F}}_{\mu\nu}$. *Show this!*
- It has Pontryagin-index $n = +1$.

This is the end of our discussion of classical Yang-Mills. There are many more topics, e.g. monopoles and generalizations of instanton solutions which would be fascinating to study, see my QFT3 lectures. (E.g. in Georgi-Glashow model: 't Hooft-Polyakov monopole: $SU(2)/U(1) = G/H$ with $\Pi_2(G/H) = \Pi_1(H) = \Pi_1(U(1)) = \mathbb{Z}$ It contains monopoles with charges $\mathbb{Z}$.)
If you want to learn more about this beautiful subject, I recommend Sidney Coleman -*Aspect of Symmetries*, which are some of the best written lectures and which every serious QFT student should have worked through once.

4
Quantizing Yang-Mills theories

4.1 Warm-Up

We treat a toy example for the following lecture.
We want to integrate

$$I = \int_{\mathbb{R}^2} d^2x \; e^{-\vec{x}^T \mathbf{A} \vec{x}} \tag{4.1}$$

with

$$\vec{x} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{4.2}$$

and

$$\mathbf{A} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

$\mathbf{A}$ is not invertible, $\det \mathbf{A} = 0$ and so

$$I \neq \sqrt{\frac{\pi^2}{\det \mathbf{A}}}. \tag{4.4}$$

We use the following trick:

$$f(\xi) = \int_{-\infty}^{\infty} d\pi \; \delta(b - \xi\pi) = \frac{1}{\xi}. \tag{4.5}$$

We can apply this trick to Eq. (4.1)

$$\begin{aligned} I &= \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db \; e^{-\alpha a^2} && (4.6) \\ &= \xi \int_{-\infty}^{\infty} d\pi \int_{-\infty}^{\infty} da \int_{-\infty}^{\infty} db \; \delta(b - \xi\pi) e^{-\alpha a^2} && (4.7) \\ &= \underbrace{\left[\xi \int_{-\infty}^{\infty} d\pi \right]}_{\text{b-independent}} \int_{-\infty}^{\infty} da \underbrace{\int_{-\infty}^{\infty} db \; \delta(b) e^{-\alpha a^2}}_{\text{calculable, gauge fixed}} && (4.8) \end{aligned}$$

where we used the gauge-transformation: $b \to b + \xi\pi$ and $db \to db'$.

Reminder: U(1) path integral for massless spin 1
Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + J_\mu A^\mu \tag{4.9}$$

EOM:

$$J_\nu = (k^2 g_{\mu\nu} - k_\mu k_\nu) A^\mu \tag{4.10}$$

Not invertible because the operator $k^2 g_{\mu\nu} - k_\mu k_\nu$ has the determinant 0.[1]
Physical reason:
We cannot uniquely solve for A_μ as a functional of J_μ because of gauge redundancy.

[1] Eigenvector k^μ with Eigenvalue 0.

$$A_\mu \to A_\mu + \partial_\mu \alpha(x). \tag{4.11}$$

Many vector-fields correspond to same J_μ current.

Fix: Add a gauge-fixing term: $\frac{1}{2\xi}(\partial_\mu A^\mu)^2$.
We will show now that any matrix element of **gauge-invariant operator** will be independent of ξ and therefore **justify gauge-fixing**.

Proof:
The correlation function with general fields ϕ_i:

$$\begin{aligned}&\langle\Omega| T\left\{\mathcal{O}(x_1,\ldots,x_n)\right\}|\Omega\rangle = \\ &= \frac{1}{Z[0]}\int \mathcal{D}A_\mu\ \mathcal{D}\phi_i\ \mathcal{D}\phi_i^* e^{i\int d^4x\ \mathcal{L}[A,\phi_i]} \cdot \mathcal{O}(x_1,\ldots,x_n)\end{aligned} \tag{4.12}$$

where $\mathcal{O}(x_1,\ldots,x_n)$ is any gauge-invariant collection of fields, e.g.:

$$\mathcal{O}(x_1,\ldots,x_n) = \phi^*(x_1)\phi(x_1)\phi^*(x_2)\phi(x_2) \tag{4.13}$$

with $\phi(x) \to e^{i\alpha(x)}\phi(x)$,
but **not** $A_\mu(x_1)A_\mu(x_2)$ or $\phi^*(x_1)\phi(x_2)\phi^*(x_3)\phi(x_4)$, which are not gauge-invariant.
We can always go to gauge where

$$\partial_\mu A^\mu \equiv 0. \tag{4.14}$$

Assume $\partial_\mu A'_\mu \neq 0$, then gauge-transform:

$$\partial_\mu A'_\mu = f(x) \ \to\ \partial_\mu A_\mu(x) + \partial_\mu\partial_\mu\alpha = f \tag{4.15}$$

Use

$$\alpha(x) = \frac{+1}{\Box} f(x) \quad \text{to satisfy Eq. (4.14)} \tag{4.16}$$

or

$$\alpha(x) = \frac{1}{\Box}\partial_\mu A'_\mu(x). \tag{4.17}$$

Consider:

$$f(\xi) = \int \mathcal{D}\pi\ e^{-i\int d^4x\ \frac{1}{2\xi}(\Box\pi)^2} \tag{4.18}$$

This represents path integral over gauge orbits which will factor out of P.I.

Why?
Shift:

$$\pi(x) \to \pi(x) - \alpha(x) = \pi(x) - \frac{1}{\Box}\partial_\mu A_\mu. \tag{4.19}$$

This leaves the integration measure invariant, since it is just a shift

$$f(\xi) = \int \mathcal{D}\pi \, \exp\left(-i \int d^4x \, \frac{1}{2\xi} \left(\Box\pi - \partial_\mu A_\mu\right)^2\right). \tag{4.20}$$

Same $f(\xi)$ as in Eq. (4.18)!

Multiply and divide the correlation function Eq. (4.12).

$$\langle\Omega|T\mathcal{O}(x_1, \dots, x_n)|\Omega\rangle = \frac{1}{Z[0]}\frac{1}{f(\xi)} \int \mathcal{D}\pi\mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* \, \mathcal{O}(x_1, \dots, x_n) \times \exp\left(i \int d^4x \, \mathcal{L}\left[A, \phi_i\right] - \frac{1}{2\xi}(\Box\pi - \partial_\mu A_\mu)^2\right). \tag{4.21}$$

Use Stückelberg-Trick and gauge-transform by

$$A_\mu = A'_\mu + \partial_\mu\pi \text{ and } \phi_i = e^{i\pi}\phi'_i \tag{4.22}$$

So we get

$$\Box\pi - \partial_\mu A_\mu \to \partial_\mu A'_\mu. \tag{4.23}$$

$$\begin{cases} \text{The measure: } \mathcal{D}\pi\mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* \quad \text{unchanged} \\ \text{The Lagrangian: } \mathcal{L}[A, \phi_i] \quad \text{invariant (b/c gauge-invariant!)} \\ \text{The Operator: } \mathcal{O}(x_1, \dots, x_n) \quad \text{invariant (also b/c gauge-invariant!)} \end{cases}$$

We can perform the same manipulation on normalisation $Z[0]$.

$$Z[0] = \left[\frac{1}{f(\xi)} \int \mathcal{D}\pi\right] \int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* \; e^{i\int d^4x \; \mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A_\mu)^2} \tag{4.24}$$

and

$$\boxed{\langle\Omega|T\mathcal{O}(x_1, \dots, x_n)|\Omega\rangle = \frac{1}{Z[0]}\left[\frac{1}{f(\xi)} \int \mathcal{D}\pi\right] \int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x \; \mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A_\mu)^2} \mathcal{O}(x_1, \dots, x_n). \quad (4.25)}$$

The term in the brackets $[\dots]$ cancels out between $Z[0]$ and the numerator!

We have shown :

$$\langle\Omega|T\mathcal{O}(x_1, \dots, x_n)|\Omega\rangle = $$

$$= \frac{\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x \; \mathcal{L}[A,\phi_i]}\mathcal{O}(x_1,\dots,x_n)}{\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x \; \mathcal{L}[A,\phi_i]}} \tag{4.26}$$

$$= \frac{\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x \; \mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A_\mu)^2} \mathcal{O}(x_1, \dots, x_n)}{\int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi_i^* e^{i\int d^4x \; \mathcal{L}[A,\phi_i] - \frac{1}{2\xi}(\partial_\mu A_\mu)^2}}. \tag{4.27}$$

Conclusion:
Gauge-fixed correlation functions give the same results as the correlation functions calculated with gauge-invariant Lagrangian (= ξ-independence).

However:
Gauge-covariant correlation functions like

$$\langle\Omega|T\left\{\overline{\psi}(x_1)\psi(x_2)\right\}|\Omega\rangle \qquad \text{or} \qquad \langle\Omega|T\left\{A_\mu A_\nu\right\}|\Omega\rangle \tag{4.28}$$

will depend on ξ!

Remaining $\xi-$invariance is cancelled by the **Ward-identity** :
Gauge-fixed propagator is:

$\nu; b$ p

$$= \frac{-i}{p^2+i\varepsilon}[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}]$$

thanks to now invertible kinetic term.
$\xi-$independence comes from $i\Pi_{\mu\nu}$ **photon propagator**.
General matrix element:

$$\mathcal{M} = (\ldots)\int d^4k_1 \cdots \Pi_{\mu_1\nu_1}(k_1)\cdots\mathcal{M}^{\mu_1\nu_1}(k_i) \tag{4.29}$$

Replace $\Pi_{\mu\nu}(k) \to \Pi_{\mu\nu}(k) + \xi k_\mu k_\nu$.
The correction will vanish because $k_\mu\mathcal{M}^\mu = 0$ even for the Ward-identity does not require $p^2 = 0$ of photons, and therefore holds also for the internal propagator. q.e.d.

4.2 Faddeev-Popov procedure

We will now compute the gluon propagator in covariant R_ξ-gauge.
We will find a new feature: ghost fields which are anticommuting Lorentz-scalars!
Repeat the procedure form $U(1)$-theory!
In SU(N):
Now need:

1. $N^2 - 1$ fields π^a

2. Gauge-transformation with $|\pi^a| \ll 1$:

$$A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu\pi^a + f^{abc}A_\mu^b\pi^c, \tag{4.30}$$

which we showed can be written as :

Note: Compare to Eq. (3.63)
$$\begin{cases}\pi^a = -\omega^a \\ A_\mu \to gA_\mu \text{ canonical normalisation.}\end{cases}$$

$$A_\mu^a \to A_\mu^a + \frac{1}{g}D_\mu^{adj}\pi^a \tag{4.31}$$

with

$$D_\mu^{adj}\pi^a = \partial_\mu\pi^a + gf^{abc}A_\mu^b\pi^c. \tag{4.32}$$

The function $f(\xi)$ is now more complicated (see Eq. (4.18)).

$$f(\xi, A) = \int \mathcal{D}\pi \exp\left(-i\int d^4x \, \frac{1}{2\xi}(\partial_\mu D_\mu \pi^a)^2\right) \tag{4.33}$$

We can still define gauge-transformation parameters which take any gauge configuration to Lorentz-gauge:

$$\partial_\mu A^a_\mu \overset{!}{=} 0. \tag{4.34}$$

Solve

$$\partial_\mu A^a_\mu = \frac{1}{g}\partial_\mu D^{ab}_\mu \alpha^b[A] \tag{4.35}$$

Shifting the π^a by this gauge-transformation then gives

$$f[A, \xi] = \int \mathcal{D}\pi \exp\left(-i\int d^4x \, \frac{1}{2\xi}(\partial_\mu A^a_\mu - \partial_\mu D_\mu \pi^a)^2\right) \tag{4.36}$$

We can write:

$$\begin{aligned}
&\int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \; e^{i\int d^4x\, \mathcal{L}[\phi_i, A]} \\
&= \int \mathcal{D}\pi \, \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \; \frac{1}{f[A,\xi]} \exp\left(i\int d^4x \; \mathcal{L}[A, \phi_i] - \frac{1}{2\xi}(\partial_\mu A^a_\mu - \partial_\mu D_\mu \pi^a)^2\right) \\
&= \left[\int \mathcal{D}\pi\right] \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \; \frac{1}{f[A,\xi]} \exp\left(i\int d^4x \; \mathcal{L}[A, \phi_i] - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right),
\end{aligned} \tag{4.37}$$

where we used $A^a_\mu \to A^a_\mu + D_\mu \pi^a$.
This removes π from exponent and $[\int \mathcal{D}\pi]$ is just an unphysical constant, similar to the abelian case (see Eq. (4.25)), **but** $f[A, \xi]$ depends on A!

4.2.1 Intermediate summary

Gauge-fixing the path integral: additional π^a modes appear. These π^a modes have $\sim \pi^a(\partial^4 + \partial^2 + \ldots)\pi^a$ kinetic terms and are **ghost-like**[2]:

[2] wrong sign kinetic term

$$\mathcal{L}_\pi \propto +(\partial_\mu A^a_\mu - \partial_\mu D^{ab}_\mu \pi^b)^2 \sim -a\pi\Box^2\pi - b\pi\Box\pi + \ldots \tag{4.38}$$

To be precise: we will find this form after gauge transformation:

$$A^a_\mu \to A^a_\mu - \frac{D_\mu \pi^a}{\Box}$$

and

$$(\partial_\mu A^a_\mu - \partial_\mu D^{ab}_\mu \pi^b)^2 \to$$

$$(\partial_\mu A^a_\mu - \frac{\cancel{\partial_\mu}\partial_\mu \pi^a}{\cancel{\Box}} - \partial_\mu \partial_\mu \pi^b + \ldots)^2$$

In momentum space:

$$\begin{aligned}
\pi_\pi &= \frac{-1}{2ak^4 - 2bk^2} && \text{(propagator)} \\
&= \frac{1}{2b}\left[\frac{1}{k^2} - \frac{a}{ak^2 - b}\right] && (4.39)
\end{aligned}$$

$\hookrightarrow \pi$ really represents two fields, one of which has **negative norm**: the ghost field.

We could now just divide out the diagrams involving π, like in case of vacuum bubbles when calculating connected Green's functions.

We will use an alternative approach which allows us to *add* diagrams instead of *subtracting* them. We can simplify $f[\xi, A]$ in Eq. (4.36), since it is quadratic in π^a.

A gaussian integral in coordinate space is

$$\int dx\, e^{-a^2x^2} = \sqrt{\frac{\pi}{a^2}} \tag{4.40}$$

Here as a path-integral, we have with $a = \partial_\mu D_\mu$

$$f[\xi, A] = \int \mathcal{D}\pi \exp\left(-i \int d^4x\, \frac{1}{2\xi}(\partial_\mu A^a_\mu - \partial_\mu D_\mu \pi^a)^2\right) \tag{4.41}$$

$$= \text{const.} \times \frac{1}{\sqrt{\det(\partial_\mu D_\mu)^2}}, \tag{4.42}$$

such that Eq. (4.37) becomes [3]

$$Z[0] = \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \; [\det(\partial_\mu D_\mu)] \cdot \exp\left(i \int d^4x\, \mathcal{L}[A, \phi] - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2\right).$$

[3] We drop the *const.* pre-factors whenever we want, since they will drop out of connected Green's functions.

We can also write a functional determinant as a path integral over Grassmann-valued fields:

$$\det(\mathcal{O}) = \int \mathcal{D}\overline{\psi}\mathcal{D}\psi \exp\left(-i \int d^4x\, \overline{\psi}\mathcal{O}\psi\right) \tag{4.43}$$

which is here

$$\det(\partial_\mu D_\mu) = \int \mathcal{D}c\mathcal{D}\bar{c} \exp\left(i \int d^4x\, \bar{c}(-\partial_\mu D_\mu)c\right), \tag{4.44}$$

Finally we get a gauged-fixed path integral:

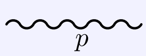
$$Z[0] = \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \mathcal{D}c\mathcal{D}\bar{c} \cdot \exp\left(i \int d^4x\, \mathcal{L}[A, \phi] - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 - \bar{c}^a(\partial_\mu D_\mu)c^a\right).$$

This is the *Faddeev–Popov*-Lagrangian:[4]

[4] Faddeev, L D and Popov, V N (1967). Feynman Diagrams for the Yang-Mills Field. Phys. Lett. B25: 29–30.

$$\mathcal{L}_{R_\xi} = -\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + (\partial_\mu \bar{c}^a)(\delta^{ac}\partial_\mu + g f^{abc} A^b_\mu)c^c \tag{4.45}$$

with the propagator:



$$\nu; b \quad \underset{p}{\sim\sim\sim} \quad \mu; a = i\frac{-\eta^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon}\delta^{ab} \tag{4.46}$$

This is the same as the photon propagator in R_ξ-gauge, except for the additional δ^{ab} in color space.

4.3 BRST invariance

We know that in QFT we need to add all terms consistent with symmetries, otherwise will be generated a quantum level and needed for counter terms.
We choose to have only terms which agree with gauge-invariance.

Question:
How can this be justified in presence of a gauge-fixing term?
This is not gauge-invariant: does it generate other gauge-inv. violating operators?

Answer:
Even including ghosts (or gauge-fixing): $\mathcal{L}$ has an exact global symmetry! This is the so-called BRST symmetry[5].

[5] Becchi, Rouet, Stora and Tyutin

4.3.1 Warm-up: BRST invariance in QED

Faddeev-Popov-Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + |D_\mu\phi_i|^2 - m^2|\phi_i|^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \bar{c}\Box c \tag{4.47}$$

where the first 3 terms are gauge-invariant.The gauge-fixing term breaks the gauge-redundancy.

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu\alpha(x) \tag{4.48}$$

up to $\alpha(x)$ satisfying $\Box\alpha(x) = 0$.
So instead of gauge-transforming with $\alpha(x)$, we transform with

Note, EOM for QED ghosts:

$$\Box c = 0$$
$$\Box \bar{c} = 0$$

$$\alpha(x) = \theta \cdot c(x) \tag{4.49}$$

The Lagrangian is invariant under:

θ are Grassman numbers obeying the following relations:

$$\theta c = -c\theta$$
$$\theta^2 = 0.$$

$$\phi_i \to e^{i\theta c(x)}\phi_i(x) \overset{\theta^2=0=c^2}{=} \phi_i(x) + i\theta c(x)\phi(x) \tag{4.50}$$

$$A_\mu(x) \to A_\mu(x) + \frac{1}{g}\theta\partial_\mu c(x) \tag{4.51}$$

if we can use $\Box c = \Box\bar{c} = 0$.

If we **do not** use EOM only $\mathcal{L}_{\text{gauge-inv.}}$ is invariant

$$(\partial_\mu A_\mu)^2 \to (\partial_\mu A_\mu)^2 + \frac{2}{g}(\partial_\mu A_\mu)(\theta\Box c) + \frac{1}{g^2}\underbrace{\cancel{(\theta\Box c)(\theta\Box c)}}_{0}. \tag{4.52}$$

Now we see that if

$$\bar{c}(x) \to \bar{c}(x) - \frac{1}{g}\theta\frac{1}{\xi}\partial_\mu A_\mu(x) \tag{4.53}$$

then the Lagrangian is invariant even after gauge-fixing and without the EOM!

Because:

$$
\begin{aligned}
&-\frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \bar{c}\Box c && (4.54)\\
&\to -\frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \cancel{\frac{\cancel{2}}{\cancel{2}\xi g}(\partial_\mu A_\mu)(\theta\Box c)} - \bar{c}\Box c + \cancel{\frac{1}{g\xi}\theta\partial_\mu A_\mu\Box c} && (4.55)\\
&= -\frac{1}{2\xi}(\partial_\mu A_\mu)^2 - \bar{c}\Box c. \quad \checkmark && (4.56)
\end{aligned}
$$

This is called BRST invariance!
We use $\theta c(x)$ instead of scalar $\alpha(x)$, which allows us to find a generalization of gauge-invariance, which holds even in presence of gauge-fixing.

4.4 *BRST invariance for non-abelian theories*

The Lagrangian is now:

$$\mathcal{L}_{FP} = \mathcal{L}[A^a_\mu, \phi_i] - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + (\partial_\mu \bar{c}^a)(D_\mu c^a). \tag{4.57}$$

where $D^{ab}_\mu c^b = (\partial_\mu \delta^{ab} + g f^{acb} A^c_\mu) c^b$.

We can define the non-abelian BRST transformations as:

$$
\begin{aligned}
\phi_i &\to \phi_i + i\theta c^a T^a_{ij}\phi_j && (4.58)\\
A^a_\mu &\to A^a_\mu + \frac{1}{g}\theta D_\mu c^a && (4.59)\\
\bar{c}^a &\to \bar{c}^a - \frac{1}{g}\xi^{-1}\theta\partial_\mu A^a_\mu && (4.60)
\end{aligned}
$$

The original $\mathcal{L}[A^a_\mu, \phi_i]$ is invariant because of gauge invariance and the transformation of $(\partial_\mu A^a_\mu)^2$ cancels like in Sec. 4.3.1 against the $\partial_\mu \bar{c}^a$ transformation. We need to be more careful with the transformation of the *covariant derivative* now

$$D_\mu c^a \xrightarrow{T^a\alpha^a(x)=\theta c^a(x)T^a} D_\mu c^a + \theta f^{abc}(D_\mu c^b)c^c. \tag{4.61}$$

because of the A^a_μ dependence of D^{ab}_μ, as we can see from

$$
\begin{aligned}
D^{ab}_\mu c^b &\to \left(\partial_\mu\delta^{ab} + g f^{acb}(A^c_\mu + \frac{1}{g}\theta D_\mu c^c)c^b\right) && (4.62)\\
&= D_\mu c^a + \theta f^{abc}(D_\mu c^b)c^c. && (4.63)
\end{aligned}
$$

From the following relation we know

$$
\begin{aligned}
e^{i\alpha^a T^a} &= e^{i\theta c^a T^a}\\
&= 1 + i\theta c^a T^a
\end{aligned}
$$

because $c^2 = \theta^2 = 0$.

Compared to QED, we need to work a bit harder and we will now need to transform c^a as well:

$$c^a \to c^a - \frac{1}{2}\theta f^{abc} c^b c^c. \tag{4.64}$$

Let us check the cancellation:

Note,
c^a and $\bar{c}^a$ are not related (unlike ψ, $\bar{\psi}$ e.g. by charge conjugation), we choose different transformations for both.

$$D_\mu c^a \xrightarrow{BRST} \widetilde{D}_\mu \tilde{c}^a + \theta f^{abc}(D_\mu c^b)\overset{\overset{= c^c \text{ because } \theta^2=0}{\downarrow}}{\tilde{c}^c} \tag{4.65}$$

$$= D_\mu c^a + \theta f^{abc}(D_\mu c^b)c^c$$

$$- \theta f^{abc}\left[\underbrace{\frac{1}{2}(\partial_\mu c^b)c^c + \frac{1}{2}c^b(\partial_\mu c^c)}_{0} + \underbrace{\frac{g}{2}A^b_\mu f^{cde}c^d c^e}_{1}\right]. \tag{4.66}$$

Let's massage this expression:

0

Grassmann:

$$f^{abc}(\partial_\mu c^b)c^c = -f^{abc}c^c(\partial_\mu c^b) \tag{4.67}$$

$$= f^{abc}c^b(\partial_\mu c^c) \qquad \text{because } f^{abc} = -f^{acb}. \tag{4.68}$$

1

The Jacobi-identity ($d \leftrightarrow c$): $f^{abd}f^{dce} + f^{bcd}f^{dae} + f^{cad}f^{dbe} = 0$.

$$f^{abc}f^{cde}A^b_\mu c^d c^e \overset{\overset{\text{Jacobi-Identity}}{\downarrow}}{=} -f^{bdc}f^{cae}A^b_\mu c^d c^e - f^{dac}f^{cbe}A^b_\mu c^d c^e \tag{4.69}$$

$$= 2f^{abc}f^{bed}A^e_\mu c^d c^c. \tag{4.70}$$

e.g. first term: $a \to a$, $b \to c$, $d \to d$, $c \to b$, $e \to c$ to get $-f^{edb}f^{bac}A^e_\mu c^a c^c = f^{abc}f^{bed}A^e_\mu c^d c^c$.

Finally, we get

$$\begin{aligned} D_\mu c^a \xrightarrow{BRST} & D_\mu c^a + \theta f^{abc}(D_\mu c^b)c^c - \theta f^{abc}[(\partial_\mu c^b)c^c + g f^{bed}A^{bed}A^e_\mu c^d c^c] \\ =& D_\mu c^a + \theta f^{abc}(D_\mu c^b)c^c - \theta f^{abc}(D_\mu c^b)c^c \\ =& D_\mu c^a \qquad \checkmark \end{aligned}$$

The Faddeev-Popov Lagrangian is invariant under a global BRST symmetry, parametrized by θ. BRST is an exact global symmetry and will be preserved at quantum level. We can use it to derive a generalized Ward-identity.

4.5 Axial Gauges

Ghosts appear in covariant gauges to remove unphysical d.o.f. from 4-vector A_μ. We can also choose a **non-covariant** gauge, where **ghosts decouple** from physical fields: we can ignore them. It is straight-forward to see once we have dealt with the gauge fixing in this case.

4.5.1 Generalized Faddeev-Popov

First we generalize the Faddeev-Popov procedure. Define an arbitrary gauge-fixing:

$$\overset{\overset{\text{functional}}{\downarrow}}{G}[A] = 0. \tag{4.71}$$

Lorentz-gauge:

$$G[A] = \partial_\mu A^a_\mu. \tag{4.72}$$

We call $\hat{A}^a_\mu$ all gauge fields which satisfy

$$G[\hat{A}] = 0. \tag{4.73}$$

An arbitrary gauge field A^a_μ can be written as:

$$A^a_\mu = \hat{A}_\mu + D_\mu \pi^a \tag{4.74}$$

We can split the path integral into an integral over $\hat{A}^a_\mu$ and over π^a.
Observe:

$$1 = \int \mathcal{D}\pi \, \delta(G[A^a_\mu - D_\mu \pi^a]) \det\left(\frac{\delta G[A^a_\mu - D_\mu \pi^a]}{\delta \pi^b}\right). \tag{4.75}$$

Repeating for Lorentz-gauge:

$$\begin{aligned}\det\left(\frac{\delta G[A^a_\mu - D_\mu \pi^a]}{\delta \pi^b}\right) &= \det(\partial^\mu D_\mu) = \frac{1}{f[A]} \\ &= \int \mathcal{D}c\mathcal{D}\bar{c} \, \exp\left(i \int d^4x \, \bar{c}^a(-\partial^\mu D_\mu)c^a\right)\end{aligned}$$

... as before! If we include the "1" of Eq. (4.75) into the path-integral:

$$\begin{aligned}Z[0] = \text{const.} \times &\int \mathcal{D}\pi \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \, \delta\left(G[A^a_\mu - D_\mu \pi^a]\right) \\ &\times \det\left(\frac{\delta G[A^a_\mu - D_\mu \pi^a]}{\delta \pi^b}\right) \exp\left(i \int d^4x \, \mathcal{L}[\phi, A]\right)\end{aligned}$$

Now we shift again:

$$A^a_\mu \to A^a_\mu + D_\mu \pi^a \tag{4.76}$$

which leaves the measure invariant.

$$\begin{aligned}Z[0] =&\text{const.} \times \left[\int \mathcal{D}\pi\right] \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \delta\left(G[A^a_\mu]\right) \\ &\times \det\left(\frac{\delta G[A_\mu - D_\mu \pi^a]}{\delta \pi^b}\right)\Bigg|_{\pi\to 0} \exp\left(i \int d^4x \, \mathcal{L}\right)\end{aligned} \tag{4.77}$$

where the first bracket is infinite const.
If we shift G by a constant, the determinant does not change and we see:

$$\int \mathcal{D}\chi \, \exp\left(-i \int d^4x \, \frac{\chi^2}{2\xi}\right) \delta[G[A^a_\mu] - \chi] = \exp\left(-i \int d^4x \, \frac{1}{2\xi} G[A^a_\mu]^2\right).$$

And so:

$$\begin{aligned}Z[0] = \text{const.} \times &\int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \det\left(\frac{\delta G[A^a_\mu - D_\mu \pi^a]}{\delta \pi^b}\right)\Bigg|_{\pi\to 0} \\ &\times \exp\left(i \int d^4x \, \mathcal{L}[A, \phi] - \frac{1}{2\xi} G[A^a_\mu]^2\right),\end{aligned}$$

which reproduces the Lagrangian in covariant gauge above.

4.5.2 Axial gauges and decoupling of ghosts

In axial-gauge

$$G[A] = n^\mu A^a_\mu. \tag{4.78}$$

where n^μ is an arbitrary, constant 4-vector, with e.g.

$$n^\mu = (0,0,0,1) \qquad \text{and} \qquad A^a_3 = 0 \tag{4.79}$$

which is the *Arnowitt-Fickler gauge*.

Use Eq. (4.77):

$$Z[0] = \text{const.} \times \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \; \delta[G[\mathbf{A}_\mu]] \det\left(\frac{\delta G[\mathbf{A}_\mu - D_\mu \pi^a]}{\delta\pi}\right)\bigg|_{\pi\to 0} \times \exp\left(i\int d^4x \; \mathcal{L}[A,\phi]\right).$$

and

$$\begin{aligned}\frac{\delta G[A^a_\mu - D_\mu \pi^a]}{\delta \pi^b}\bigg|_{\pi\to 0} &= \frac{\delta}{\delta\pi^b}\left[n^\mu A^a_\mu - n^\mu \partial_\mu \pi^a - n^\mu A^d_\mu f^{adc}\pi^c\right] \\ &= n^\mu\partial_\mu - f^{adb}\underset{\substack{\uparrow \\ \text{The bracket vanishes.}}}{(n^\mu A^d_\mu)}\end{aligned}$$

With the δ-function: $\det(\ldots)$ is $\mathbf{A}_\mu$-independent!

$$n^\mu \mathbf{A}_\mu = 0. \tag{4.80}$$

So we get:

$$Z[0] = \text{const.} \times \int \mathcal{D}\mathbf{A}_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \; \delta[n^\mu A_\mu] \det(n^\mu\partial_\mu) e^{i\int d^4x \; \mathcal{L}[\phi,A]}$$

The ghost-fields decouple from $\mathbf{A}_\mu$!

The propagator reads

Show! → exercise.

$$i\Pi_{axial}(k)^{\mu\nu}_{ab} = -\frac{i}{k^2}\left(g^{\mu\nu} + \frac{k^\mu k^\nu}{(\mathbf{k}\cdot\mathbf{n})^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{\mathbf{k}\cdot\mathbf{n}}\right)\delta^{ab}. \tag{4.81}$$

This is clearly **not** manifestly Lorentz-invariant.[6] S-matrix elements however will be Lorentz-invariant.

[6] Because it depends on a preferred direction n^μ and it contains three-vector scalar products $(\mathbf{k}\cdot\mathbf{n})$.

The propagator satisfies:

$$k_\mu \Pi_{axial}(k)^{\mu\nu}_{ab} = 0 \quad \text{and} \quad n_\mu \Pi_{axial}(k)^{\mu\nu}_{ab} = 0. \tag{4.82}$$

Conclusion:
Axial gauge shows, ghosts are not strictly necessary to describe non-abelian gauge theories. However, unless a natural n_μ exists in the problem at hand[7], calculations become difficult.

[7] E.g. spinor helicity formalism.

4.6 BRST and Slavnov operator

We can define an operator $\mathcal{Q}_{\text{BRS}}$ which generates BRST transformations:

$$\begin{aligned}
\phi_i &\to \phi_i + \theta\, \mathcal{Q}_{\text{BRS}}\, \phi_i \\
A^a_\mu &\to A^a_\mu + \theta\, \mathcal{Q}_{\text{BRS}}\, A^a_\mu \\
c^a &\to c^a + \theta\, \mathcal{Q}_{\text{BRS}}\, c^a \\
\bar{c}^a &\to \bar{c}^a + \theta\, \mathcal{Q}_{\text{BRS}}\, \bar{c}^a
\end{aligned}$$

where we mean as in Sec. 4.4,

We have redefined $\theta \to g\theta$

$$\theta \mathcal{Q}_{\text{BRS}}\, \phi_i = ig\theta c^a T^a_{ij} \phi_j \tag{4.83}$$

$$\theta \mathcal{Q}_{\text{BRS}}\, A^a_\mu = \theta D_\mu c^a. \tag{4.84}$$

$$\theta \mathcal{Q}_{\text{BRS}}\, \bar{c}^a = -\frac{1}{\xi}\theta \partial_\mu A^a_\mu \tag{4.85}$$

$$\theta \mathcal{Q}_{\text{BRS}}\, c^a = -g\frac{1}{2}\theta f^{abc} c^b c^c. \tag{4.86}$$

We can show that $\mathcal{Q}_{\text{BRS}}$ is **nilpotent**

Exercise!

$$\mathcal{Q}^2_{\text{BRS}} = 0. \tag{4.87}$$

Further, this transformation is a symmetry of $\mathcal{L} + \mathcal{L}_{\text{gauge-fix}}$, and so as for any global symmetry, this gives rise to a conserved Noether current j^μ_{BRS} with the charge Q_{BRS}

Note the difference in Notation between the charge Q_{BRS} and the operator $\mathcal{Q}_{\text{BRS}}$.

$$\delta_{\text{BRS}}\phi_i = \theta\, \mathcal{Q}_{\text{BRS}}\phi_i = [\theta\, Q_{\text{BRS}}, \phi_i] \tag{4.88}$$

and so

$$Q_{\text{BRS}} = \int d^3x\, j^0_{\text{BRS}}(x) \tag{4.89}$$

and so the BRS charge Q_{BRS} commutes with the Hamiltonian H:

$$[Q_{\text{BRS}}, H] = 0. \tag{4.90}$$

Our gauge-fixed Lagrangian, loses gauge invariance, but possesses a **residual global fermionic symmetry**, the BRST symmetry. We will now try to understand the implications of this fermionic symmetry[8] .

[8] The generators are Grassmann variables and anticommute, unlike the generators of ordinary symmetries which are scalars. This whole structure rhymes with supersymmetry

4.6.1 Set of physical states – Q-cohomology

The eigenstates of Hilbert space of H can be divided into three subspaces:

$$H_0 \quad : \quad Q_{\text{BRS}}\, |\psi_0\rangle = 0 \tag{4.91}$$

$$H_1 \quad : \quad Q_{\text{BRS}}\, |\psi_1\rangle \neq 0 \tag{4.92}$$

$$H_2 \quad : \quad |\psi_2\rangle = Q_{\text{BRS}}\, |\psi_1\rangle. \tag{4.93}$$

and so $Q_{\text{BRS}}\, |\psi_2\rangle = Q^2_{\text{BRS}}\, |\psi_1\rangle = 0$.

The inner products in H_2 vanish

$$\langle\psi_{2,a}|\psi_{2,b}\rangle = \langle\psi_{1,a}|Q_{\text{BRS}}^2|\psi_{1,b}\rangle = 0. \tag{4.94}$$

$\mathcal{Q}_{\text{BRS}}$ is self-adjoint. Argue why this is the case!

as do the inner products between states in H_0 and H_2,

$$\langle\psi_{2,a}|\psi_{0,b}\rangle = \langle\psi_{1,a}|Q_{\text{BRS}}|\psi_{0,b}\rangle = 0. \tag{4.95}$$

This looks abstract, but it is important to understand the physics of non-abelian gauge theories and the meaning of BRST.

If we consider single-particle states of a non-Abelian gauge theory in the limit of zero coupling, one can show that according to the transformations in Eq. (4.83), that Q_{BRS} tells us where the various states belong.

Table 4.1: BRST subspaces

Transverse polarization and ϕ	H_0
Forward gauge polarization and anti-ghost	H_1
Backward gauge polarization and ghost	H_2

where the *physical subspace* is H_0 – as it should be, since these are the transverse polarizations which contain the physical degrees of freedom.

$$|\text{phys}\rangle \in H_0 \qquad Q_{\text{BRS}}|\text{phys}\rangle = 0.$$

This relationship has an intuitive explanation: BRST is the quantum version of gauge transformation, $\mathcal{Q}_{\text{BRS}}$ is the generator of infinitesimal transformations. $\mathcal{Q}_{\text{BRS}}|\text{phys}\rangle = 0$ means that physical states are gauge-invariant.

Again, we consider single-particle states of a non-Abelian gauge theory in the limit of zero coupling $g = 0$. Let us see if we can make sense of Table 4.1 using examples.
Starting from Eq. (4.83), we see that the RHS of the transformation of the matter field ϕ_i contains only multi-particle states, so asymptotically $c^a T^a_{ij}\phi_j \to 0 + (\ldots)$, where the terms in the brackets contain the multi-particle state which gets projected out once we consider S-matrix elements. The asymptotic field $\phi_i^{in/out}$ is therefore a BRS invariant and in H_0.
We can similarly argue using the BRST transform of the ghost c^a

Recall

Forward polarization: $\epsilon^+_\mu(k) = \frac{1}{\sqrt{2}|\mathbf{k}|}\begin{pmatrix} k_0 \\ \mathbf{k}\end{pmatrix}$

Backward polarization: $\epsilon^-_\mu(k) = \frac{1}{\sqrt{2}|\mathbf{k}|}\begin{pmatrix} k_0 \\ -\mathbf{k}\end{pmatrix}$

where

$$\epsilon^{tr}_i \cdot \epsilon^{tr}_j = -\delta_{ij}$$
$$\epsilon^{\pm} \cdot \epsilon^{tr}_j = 0$$
$$\epsilon^{+} \cdot \epsilon^{-}_j = 1$$
$$(\epsilon^{\pm})^2 = 0.$$

1. From Eq. (4.86), we read: $\mathcal{Q}_{\text{BRS}}\, c^a = O$ and (multi-particle)

   $\mathcal{Q}_{\text{BRS}}$ annihilates the one-ghost state (limit of $g \to 0$, single-particle states)

   $\Rightarrow c^a$ is either in H_0 or in H_2. Which one is it?

2. Using Eq. (4.84), we get

$$\mathcal{Q}_{\text{BRS}} A^a_\mu \sim \theta D_\mu c^a \sim \partial_\mu c^a \sim k_\mu c^a.$$

We see that the RHS in momentum space only contains the forward longitudinal polarization $\epsilon^+_\mu(k)$. (For the same reason, the transverse asymptotic field is annihilated by $\mathcal{Q}_{\text{BRS}}$, since the RHS does not contain this polarization.)

$\mathcal{Q}_{\text{BRS}}$ converts forward component of A^a_μ to ghost state

$$|\text{ghost}\rangle = Q_{\text{BRS}}|\text{forward gauge}\rangle.$$

$\Rightarrow$ ghost c^a is in H_2.

4.7 S-matrix and BRST

The S-matrix time-evolves asymptotic states. If we take a collection of transversely polarized states (in past): does this only evolve polarized states in far future? If not: S-matrix between transversely polarized states would not be unitary!
Let $|A;tr\rangle$ be a (asymptotic) state of **transverse** only gauge polarizations[9], which is therefore annihilated by $\mathcal{Q}_{\text{BRS}}$.

[9] no ghosts, anti-ghosts or other polarizations.

We want to show the unitarity of S on the physical (transverse) subspace:

$$\sum_{c,tr.} \left\langle A;tr|S^\dagger|C;tr\right\rangle \left\langle C;tr|S|B;tr\right\rangle = \left\langle A;tr|\mathbb{1}|B;tr\right\rangle . \tag{4.96}$$

Physical states are in H_0 (they span H_0)

$$|\text{phys}\rangle \in H_0 : \quad \mathcal{Q}_{\text{BRS}}|\text{phys}\rangle = 0. \tag{4.97}$$

All states in the physical subspace are annnihilated by $\mathcal{Q}_{\text{BRS}}$. Since $\mathcal{Q}_{\text{BRS}}$ commutes with the Hamiltonian, time evolution of such a state is also annihilated by $\mathcal{Q}_{\text{BRS}}$.
Thus:

$$\mathcal{Q}_{\text{BRS}} S|A;tr\rangle = 0. \tag{4.98}$$

This means that with Eq. (4.91), that $S|A;tr\rangle$ can be in H_0 **or** in H_2.
States in H_2 have zero inner product with states in H_0 (and zero inner product with states in H_2), see Eq. (4.95). Therefore the overlap of any two states of the form $S|\ A;tr\rangle$ comes only from components in H_0.
Therefore:

$$\begin{aligned} \left\langle A;tr|S^\dagger S|B;tr\right\rangle &= \sum_{C;all} \left\langle A;tr|S^\dagger|C;all\right\rangle \left\langle C;all|S|B;tr\right\rangle & (4.99)\\ &= \sum_{C;tr} \left\langle A;tr|S^\dagger|C;tr\right\rangle \left\langle C;tr|S|B;tr\right\rangle & (4.100)\end{aligned}$$

Since full S-matrix is unitary this implies that restricted S-matrix, the S-matrix of $|\text{phys}\rangle$, is unitary!

More mathematically, $\mathcal{Q}_{\text{BRS}}$ defines a cohomology.[10] For our nil-

[10] Recall, the "kernel" of a linear map Q between two vector spaces, is the set of all elements for which

$$\ker(Q) = \{\mathbf{v} \in V \mid Q(\mathbf{v}) = \mathbf{0}\}\,.$$

whereas the "image" is the set of elements, you get once apply the transformation Q to all elements of the vector space potent operator we have

$$\ker(\mathcal{Q}_{\text{BRS}}) \supset \text{im}(\mathcal{Q}_{\text{BRS}}) \tag{4.101}$$

because $\mathcal{Q}_{\text{BRS}}(\mathcal{Q}_{\text{BRS}}|\psi\rangle) = 0$. The $\mathcal{Q}_{\text{BRS}}-$cohomology is

$$H_{\text{phys}} = \ker(\mathcal{Q}_{\text{BRS}})/\text{im}(\mathcal{Q}_{\text{BRS}}) \tag{4.102}$$

The quotient space of the kernel of $\mathcal{Q}_{\text{BRS}}$ modded by the image of $\mathcal{Q}_{\text{BRS}}$ is the physical Hilbert space.

Suggested reading: "A Brst Primer" D. Nemeschansky, C. R. Preitschopf and M. Weinstein. 10.1016/0003-4916(88)90233-3 Annals Phys. 183, 226 (1988) and Kugo "Eichtheorie" chapter 5.8 and 5.9

4.8 BRS charge and Slavnov-Taylor Identities

The generalized Ward Identities result from the invariance under global, fermionic BRS transformations. To obtain these relations between Green's functions, we study the generating functional. First, we introduce more convenient real Grassmann ghost fields ρ_a and σ_a defined by

$$c_a = \frac{1}{\sqrt{2}}(\rho_a + i\,\sigma_a) \tag{4.103}$$

$$\bar{c}_a = \frac{1}{\sqrt{2}}(\rho_a - i\,\sigma_a) \tag{4.104}$$

Their BRST transformations are

$$\delta_{\text{BRS}}\,\phi_i = ig\theta\sigma^a T^a_{ij}\phi_j \tag{4.105}$$

$$\delta_{\text{BRS}}\,A^a_\mu = \theta D_\mu\sigma^a. \tag{4.106}$$

$$\delta_{\text{BRS}}\,\rho^a = -\frac{1}{\xi}\theta\partial_\mu A^a_\mu \tag{4.107}$$

$$\delta_{\text{BRS}}\,\sigma^a = -g\frac{1}{2}\theta f^{abc}\sigma^b\sigma^c. \tag{4.108}$$

It turns out that to derive the identities, it is useful to additionally introduce generalized sources $\kappa^a_\mu, \nu^a, \lambda^i$ for the composite operators $D_\mu\sigma^a,\ f^{abc}\sigma^b\sigma^c, \sigma^a T^a_{ij}\phi_j$, respectively.
The generating functional in terms of the Faddeev-Popov Lagrangian is therefore

$$Z[J,\alpha,\beta,\chi,\bar{\chi},\kappa,\nu,\lambda,\bar{\lambda}] = \int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi^*_i\mathcal{D}\sigma\mathcal{D}\rho\ \exp\left\{i\int d^4x\ (\mathcal{L}+\Sigma)\right\}$$

with the source Σ given by

$$\begin{aligned}\Sigma(x) = J^a_\mu A^{a,\mu} &+ \alpha^a\rho^a + \beta^a\sigma^a + (\bar{\chi}\phi + (\phi)^\dagger\chi) \\ &+ \kappa^a_\mu D^\mu\sigma^a + \frac{1}{2}\nu^a f^{abc}\sigma^b\sigma^c + (\bar{\lambda}T^a\sigma^a\phi + \phi^\dagger T^a\sigma^a\lambda).\end{aligned}$$

where we have suppressed some of the indices. Since the Faddeev-Popov Lagrangian is invariant under global BRST and the measure is too[11], we can do as in QFT1 and derive

[11] Show this! Or see e.g.Ryder - QFT chapter 7.6

$$\int d^4x \int \mathcal{D}A_\mu\mathcal{D}\phi_i\mathcal{D}\phi^*_i\mathcal{D}\sigma\mathcal{D}\rho\ e^{i\int d^4x'\ (\mathcal{L}[x']+\Sigma[x'])}\,(\delta\Sigma(x)) = 0$$

where $\delta_{\text{BRS}}\Sigma(x)$ is the infinitesimal change of the source terms due to the BRST transformation

$\delta = \delta_{\text{BRS}}$ from now on.

$$\delta\Sigma(x) = J^a_\mu \delta A^{a,\mu} + \alpha^a \delta\rho^a + \beta^a \delta\sigma^a + \bar{\chi}\delta\phi + (\delta\phi)^\dagger \chi \\ + \kappa^a_\mu \delta(D^\mu \sigma^a) + \frac{1}{2}\nu^a f^{abc}\delta(\sigma^b\sigma^c) + \bar{\lambda}T^a\delta(\sigma^a\phi) + \delta(\phi^\dagger T^a \sigma^a)\lambda$$

One can show that the terms in the second line vanish, i.e. all the infinitesimal transformations of the composite operators, due to the nil-potency of the BRST operator and thus we can write

Since they are in the image of $\mathcal{Q}_{\mathrm{BRS}}$. See exercise sheet.

$$0 = \theta \int d^4x \int \mathcal{D}A_\mu \mathcal{D}\phi_i \mathcal{D}\phi_i^* \mathcal{D}\sigma\mathcal{D}\rho \; e^{i\int d^4x' \, (\mathcal{L}[x'] + \Sigma[x'])} \\ \times \left(J^a_\mu D_\mu \sigma + i\frac{1}{\xi}\alpha^a \partial_\mu A^a_\mu + \frac{g}{2}\beta^a f^{abc}\sigma^b\sigma^c + ig\bar{\chi}\sigma^a T^a \phi - ig\phi^\dagger \sigma^a T^a \chi \right)$$

This equation is rather cumbersome and very non-linear. Now, it becomes clear, why the generalized sources are useful: we can replace these non-linear terms by functional derivatives

$$0 = \theta \int d^4x \left(J^a_\mu \frac{\delta}{\delta\kappa^a_\mu} + i\frac{1}{\xi}\alpha^a \partial_\mu \frac{\delta}{\delta J^a_\mu} + \frac{g}{2}\beta^a \frac{\delta}{\delta\nu^a} + ig\bar{\chi}\frac{\delta}{\delta\lambda} - ig\frac{\delta}{\delta\bar{\lambda}}\chi \right) Z[J, \ldots, \bar{\lambda}] \tag{4.109}$$

This expression generates one form of the *Slavnov-Taylor identities* which relate different types of Green's functions. We can now simply differentiate Eq. (4.109) by $J_\mu, \alpha, \ldots$ and then set them to zero, which will give relations between n-point functions.
A simpler way to get to the content of Eq. (4.109), is to use the fact that $\delta_{\mathrm{BRS}} Z[J, \ldots, \bar{\lambda}] = 0$ implies that Green functions are invariant under BRS transformations.[12]

[12] See e.g. also Cheng and Li – "Gauge theory of elementary particle physics", chapter 9.3

$$\delta_{\mathrm{BRS}} \left(\langle \Omega | T\{A^a_\mu(x) A^b_\mu(0)\} | \Omega \rangle \right) = 0$$

where you just have to replace the $\delta_{\mathrm{BRS}} A^a_\mu$ in the correlator by the RHS of Eq. (4.106) to derive a relation between the gauge field the ghost fields. You can check that you will clearly get the same result if you take functional derivatives of Eq. (4.109) w.r.t. $J^a_\mu(x)$ and $J^b_\mu(0)$ and then set all the sources to zero.

4.9 *Transversity of gauge boson self-energy to all orders from BRST symmetry*

This has been discussed in the central tutorial.

4.9.1 *Introductory considerations*

The discussion largely follows Prof. Beneke's notes for quantum field theory II.

Our starting point is the gauge-fixed Lagrangian, which for the purpose of exploiting BRST is best written using the auxiliary bosonic field B^a: in a covariant gauge, we have

$$\tilde{\mathcal{L}} = \overset{\substack{\text{gauge+matter} \\ \downarrow}}{\mathcal{L}} + B^a \partial_\mu A^{\mu a} + \frac{\xi}{2}(B^a)^2 + (\partial_\mu \bar{c})^a (D^\mu c)^a, \tag{4.110}$$

where

The convention is $D_\mu = \partial_\mu - igA_\mu$.

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu. \tag{4.111}$$

Let us see explicitly how this $\tilde{\mathcal{L}}$ arises.
We start from a naive functional integral:

$$\langle\Omega|T\{\mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\dots\}|\Omega\rangle = |N|^2\int \mathcal{D}A\, e^{iS[A]}\mathcal{O}_i\mathcal{O}_j\dots. \tag{4.112}$$

Throughout the discussion, "T" actually means "$T^\star$" as usual when using the path integral. This allows us, in particular, to pull derivatives in front of the T.

Now we must eliminate the redundancy through a gauge fixing. (Pick one representative for each orbit).
Insert the identity as

$$1 = \int \mathcal{D}\epsilon\, \det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)\delta\left(f[A_\epsilon]-F\right) \tag{4.113}$$

where f will be our gauge-fixing function, and A_ϵ is the gauge field transformed by a small ϵ:

$$A^a_{\mu\epsilon} = A^a_\mu + \frac{1}{g}D^{ac}_\mu\epsilon^c. \tag{4.114}$$

Inserting the identity Eq. (4.113) we get for Eq. (4.112):

$$= |N|^2\int \mathcal{D}A_\epsilon\mathcal{D}\epsilon\, e^{iS[A]}\det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)\delta\left(f[A_\epsilon]-F\right)\mathcal{O}_i\mathcal{O}_j\dots \tag{4.115}$$

Now S and $\mathcal{O}_{i,j,\dots}$ are gauge invariant, and also the measure satisfies

$$\mathcal{D}A = \mathcal{D}A_\epsilon, \tag{4.116}$$

so Eq. (4.115) can be written as

$$= |N|^2\int \mathcal{D}A\mathcal{D}\epsilon\, e^{iS[A_\epsilon]}\det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)\delta\left(f[A_\epsilon]-F\right)\mathcal{O}_i\mathcal{O}_j\dots \tag{4.117}$$

Redefining $A_\epsilon \to A$ as an integration variable we obtain

$$= |N|^2\left(\int\mathcal{D}\epsilon\right)\int \mathcal{D}\,A e^{iS[A]}\det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)_{\epsilon=0}\delta\left(f[A]-F\right)\mathcal{O}_i\mathcal{O}_j\dots \tag{4.118}$$

$$= |N'|^2\int \mathcal{D}A\, e^{iS[A]}\det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)_{\epsilon=0}\delta\left(f[A]-F\right)\mathcal{O}_i\mathcal{O}_j\dots \tag{4.119}$$

where in the last step we used the fact that the integral over $\mathcal{D}\epsilon$ just gives an (infinite) overall constant.
Now the current expression does not depend on F (recall Eq. (4.113)), so let us multiply times the constant

$$\int\mathcal{D}F\,\tilde{G}[F] \tag{4.120}$$

with a functional $\tilde{G}[F]$ of our choice.
Pick then

$$\tilde{G}[F] = \int\mathcal{D}B^a\, e^{i\int d^4x B^a\left(F^a+\frac{\xi}{2}B^a\right)}, \tag{4.121}$$

which in fact is equivalent to the choice we made when deriving $\tilde{\mathcal{L}}$ in the formulation **without** the auxiliary field: complete the square

$$\tilde{G}[F] = \underbrace{\int\mathcal{D}B^a\, e^{i\int d^4x\,\frac{\xi}{2}\left(B^a+\frac{F^a}{\xi}\right)^2}}_{\text{constant}} e^{id^4x\left(-\frac{F^{a2}}{2\xi}\right)} \tag{4.122}$$

$$= \text{const.}\;\cdot e^{i\int d^4x\left(-\frac{F^{a2}}{2\xi}\right)} = \text{const.}\;\cdot G[F] \tag{4.123}$$

where $G[F]$ leads to the formulation without B^a. Then Eq. (4.119) becomes

$$|N'|^2 \int \mathcal{D}A\, e^{i\int d^4x\,\mathcal{L}} \det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)_{\epsilon=0} \delta\left(f[A]-F\right) \int \mathcal{D}F\mathcal{D}B^a\, e^{i\int d^4x\, B^a\left(F^a+\frac{\xi}{2}B^a\right)} \mathcal{O}_i\mathcal{O}_j\ldots \tag{4.124}$$

$$= |N'|^2 \int \mathcal{D}A\mathcal{D}B^a\, e^{i\int d^4x\left(\mathcal{L}+B^a f^a+\frac{\xi}{2}(B^a)^2\right)} \det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)_{\epsilon=0} \mathcal{O}_i\mathcal{O}_j\ldots. \tag{4.125}$$

Now we write the determinant as

$$\det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)_{\epsilon=0} = \int \mathcal{D}[c^a,\bar{c}^a] e^{i\int d^4x d^4y\, \bar{c}^a(x)(-i)\, \frac{\delta f^a[A_\epsilon(x)]}{\delta\epsilon^b(y)}\Big|_{\epsilon=0} c^b(y)} \tag{4.126}$$

and for the covariant gauges, $f^a = \partial_\mu A^{\mu a}$, we obtain

$$\left.\frac{\delta f^a[A_\epsilon(x)]}{\delta\epsilon^b(y)}\right|_{\epsilon=0} = \frac{1}{g}\partial_\mu D^{\mu ab}\delta^{(4)}(x-y) \tag{4.127}$$

so

$$i\int d^4x d^4y\,\bar{c}^a(x)(-i)\left.\frac{\delta f^a[A_\epsilon(x)]}{\delta\epsilon^b(y)}\right|_{\epsilon=0} c^b(y) = i\int d^4x\,\bar{c}^a(x)(-i)\frac{1}{g}\partial_\mu D^{\mu ab}c^b(x)$$

integrating by parts and redefining

$$\frac{ic^a}{g} \to c^a, \tag{4.128}$$

we obtain

$$i\int d^4x\,\left(\partial_\mu\bar{c}^a\right)\left(D^\mu c\right)^a. \tag{4.129}$$

Hence the path integral becomes

$$|N'|^2 \int \mathcal{D}A\mathcal{D}B^a\mathcal{D}[c,\bar{c}]\, e^{i\int d^4x\,\left(\mathcal{L}+B^a\partial_\mu A^{\mu a}+\frac{\xi}{2}(B^a)^2+\partial_\mu\bar{c}^a(D^\mu c)^a\right)} \mathcal{O}_i\mathcal{O}_j\ldots \tag{4.130}$$

where the bracket in the exponent is $\tilde{\mathcal{L}}$ (gauge-fixed Lagrangian).
So far we did not include matter fields. Their effect simply goes into $\mathcal{L}$, provided the integration measure for fermions is invariant under $\psi \to \psi_\epsilon$. We proceed under this assumption (non-anomalous gauge symmetry).

4.10 Definition of the BRST transformation

The gauge-fixed Lagrangian is invariant under BRST transformations

$$\delta_\theta\tilde{\mathcal{L}} = 0, \tag{4.131}$$

where the action of BRST is

θ is the Grassmann number
Δ is the Slavnov operator.

$$\phi \to \phi + \underbrace{\theta\Delta\phi}_{\delta_\theta\phi} \qquad \text{for any } \phi \tag{4.132}$$

The action on individual fields is:

$$\delta_\theta \psi = ig\theta c^a T^a \psi \tag{4.133}$$

$$\delta_\theta A^a_\mu = \theta (D_\mu c)^a \tag{4.134}$$

$$\delta_\theta c^a = -\frac{1}{2} g\theta f^{abc} c^b c^c \tag{4.135}$$

$$\delta_\theta \bar{c}^a = \theta B^a \tag{4.136}$$

$$\delta_\theta B^a = 0 \tag{4.137}$$

with the properties:

- Δ is nilpotent, i.e. $\delta_\theta(\Delta\phi) = 0$

See Exercise Sheet 2.

4.11 Ward-identities and BRST transformation

Now the invariance under BRST gives rise to **Ward identities** obeyed by the Green's functions of the non-Abelian gauge theory:

$$\sum_{k=1}^{m} \langle \Omega | T\{\phi_{n_1}(x_1) \cdots \phi_{n_{k-1}}(x_{k-1}) \theta \Delta(\phi_{n_k}(x_k)) \phi_{n_{k+1}}(x_{k+1}) \cdots \phi_{n_m}(x_m)\} | \Omega \rangle = 0 \tag{4.138}$$

where ϕ_n are **any** of the fields in the theory:

$$\phi_n \in \{\psi, A^a_\mu, c^a, \bar{c}^a, B^a \ldots\}. \tag{4.139}$$

This identity is obtained assuming BRST is non-anomalous, which happens if the gauge symmetry is non-anomalous. We will also exploit the following EOM identities

$$\langle \Omega | T \left\{ \frac{\delta S[\phi_{n'}]}{\delta \phi_n(x)} \phi_{n_1} \cdots \phi_{n_m}(x_m) \right\} | \Omega \rangle \tag{4.140}$$

$$= i \sum_{i=1}^{m} \delta^{(4)}(x - x_i) \delta_{n n_i} \langle \Omega | T\{\phi_{n_1}(x_1) \cdots \underbrace{\hat{\phi}_{n_i}(x_i)}_{\substack{\text{means} \\ \text{“omitted from list”}}} \cdots \phi_{n_m}(x_m)\} | \Omega \rangle. \tag{4.141}$$

It vanishes unless $n = n_i$ and $x = x_i$ for one the $\phi_{n_1} \cdots \phi_{n_m}$. These identities can be obtained by appropriate manipulations of the generating functional $Z[\phi_n]$.[13]

[13] It is reviewed in class, but it will not be repeated here, see QFT1.

4.12 Transversity of the gauge boson self-energy

Take the two-point function

$$\langle \Omega | T\{A^a_\mu(x) A^b_\nu(y)\} | \Omega \rangle = \int \frac{d^4k}{(2\pi)^4} \, e^{-ik(x-y)} G^{ab}_{\mu\nu}(k), \tag{4.142}$$

where the most general form of the Fourier transform is

$$G^{ab}_{\mu\nu}(k) = \delta^{ab} \frac{i}{k^2 + i\epsilon} \left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) \underset{\substack{\uparrow \\ \text{transverse} \\ \text{part}}}{A(k^2)} - \frac{k_\mu k_\nu}{k^2} \underset{\substack{\uparrow \\ \text{longitudinal} \\ \text{part}}}{B(k^2)} \right]. \tag{4.143}$$

At lowest order in the coupling constant g this is the gauge boson propagator, so we know

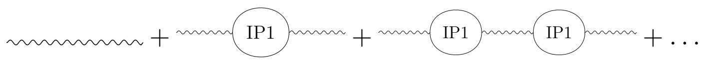
The propagator is

$$-\frac{i}{k^2+i\epsilon}\left(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right). \tag{4.144}$$

$$A = 1+\mathcal{O}(g^2) \tag{4.145}$$

$$B = \xi+\mathcal{O}(g^2). \tag{4.146}$$

We will show that $\mathbf{B}=\boldsymbol{\xi}$ at **all orders in** g.
If this is true, then we can write the two-point function as



with

$$\Pi_{\mu\nu}(k) = (k^2 g_{\mu\nu}-k_\mu k_\nu)\Pi(k^2)+k_\mu k_\nu \Pi_2(k^2). \tag{4.147}$$

It is easy to convince oneself that the fact that B does not get corrected at any order in g, is equivalent to

$$\begin{aligned}&\Pi_2 = 0\\ &\text{the self-energy is transverse: } k^\mu \Pi_{\mu\nu} = 0.\end{aligned} \tag{4.148}$$

because the first term already gives $B=\xi$!
Then it is also easy to resum the series

$$= \frac{i}{k^2}\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)(1+\Pi+\Pi^2+\ldots)+\frac{i}{k^2}\left(\underset{\substack{\uparrow\\ \text{from first}\\ \text{term only}}}{-\xi\frac{k_\mu k_\nu}{k^2}}\right) \tag{4.149}$$

$$= \frac{i}{k^2}\left[\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi}-\xi\frac{k_\mu k_\nu}{k^2}\right] \tag{4.150}$$

Hence

$$A(k^2) = \frac{1}{1-\Pi(k^2)}. \tag{4.151}$$

4.13 Proof of $B=\xi$ (at all orders)

Now let us prove that $B=\xi$ at all orders.
We use the EOM identity in the form

fields do not match, RHS of *Eq.* (4.141) is zero.

$$0 \overset{\downarrow}{=} \langle\Omega|T\{\frac{\delta S}{\delta B^a(x)}\partial^\mu A^b_\mu(y)\}|\Omega\rangle = \langle\Omega|T\{(\partial_\nu A^{\nu a}(x)+\xi B^a(x))\partial^\mu A^b_\mu(y)\}|\Omega\rangle. \tag{4.152}$$

where we used

$$\frac{\delta S}{\delta B^a(x)} = \partial_\nu A^{\nu a}(x)+\xi B^a(x). \tag{4.153}$$

We rewrite Eq. (4.152) as

$$\partial_\nu^{(x)}\partial_\mu^{(y)}\langle\Omega|T\{A^{\nu a}(x)A^{\mu b}(y)\}|\Omega\rangle \tag{4.154}$$

$$= -\xi\langle\Omega|T\{\underset{\Delta\bar{c}^a(x)}{\overset{\downarrow}{B^a}}(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle \tag{4.155}$$

$$= -\xi\langle\Omega|T\{\Delta\bar{c}^a\partial^\mu A^b_\mu(y)\}|\Omega\rangle. \tag{4.156}$$

Now look at Eq. (4.156): the LHS is already $\partial_\mu\partial_\nu$ (two-point function), but we need to work on the RHS.
Let us use the Ward identity for two fields. Eq. (4.138) reads

$$\begin{aligned}0 &= \langle\Omega|T\{\theta\Delta\bar{c}^a(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle + \langle\Omega|T\{\bar{c}^a(x)\theta\Delta(\partial^\mu A^b_\mu(y))\}|\Omega\rangle \\ &= \langle\Omega|T\{\theta\Delta\bar{c}^a(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle - \langle\Omega|T\{\theta\bar{c}^a(x)\Delta(\partial^\mu A^b_\mu(y))\}|\Omega\rangle.\end{aligned}$$

Now we have

$$\Delta(\partial^\mu A^b_\mu(y)) = \partial^\mu(\Delta A^b_\mu(y)) = \partial^\mu(D_\mu c(y))^b \tag{4.157}$$

but also

$$\frac{\delta S}{\delta\bar{c}^b(y)} \underset{\substack{\text{IBP before}\\ \text{taking derivative}}}{\overset{}{=}} -\partial^\mu D^{bc}_\mu c^c(y). \tag{4.158}$$

Hence

$$0 = \langle\Omega|T\{\Delta\bar{c}^a(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle + \langle\Omega|T\{\bar{c}^a(x)\frac{\delta S}{\delta\bar{c}^b(y)}\}|\Omega\rangle \tag{4.159}$$

and Eq. (4.156) becomes

$$\begin{aligned}\partial_\nu^{(x)}\partial_\mu^{(y)}\langle\Omega|T\{A^{\nu a}(x)A^{\mu b}(y)\}|\Omega\rangle &= \xi\langle\Omega|T\{\bar{c}^a(x)\frac{\delta S}{\delta\bar{c}^b(y)}\}|\Omega\rangle \\ &= -\xi\langle\Omega|T\{\frac{\delta S}{\delta\bar{c}^b(y)}\bar{c}^a(x)\}|\Omega\rangle.\end{aligned}$$

The last step is precisely the case where the RHS of the EOM identity Eq. (4.141) does **not** vanish!
We arrive at

$$\partial_\nu^{(x)}\partial_\mu^{(y)}\langle\Omega|T\{A^{\nu a}(x)A^{\mu b}(y)\}|\Omega\rangle = -\xi i\delta^{(4)}(x-y)\delta^{ab}\underbrace{\langle\Omega|\Omega\rangle}_{=1}. \tag{4.160}$$

The rest is easy:

$$\partial_\nu^{(x)}\partial_\mu^{(y)}\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}G^{ab}_{\mu\nu}(k) = -i\xi\delta^{ab}\underbrace{\int\frac{d^4k}{(2\pi)^4}\,e^{ik(x-y)}}_{\delta^{(4)}}. \tag{4.161}$$

It follows

$$(-ik^\nu)(ik^\mu)G^{ab}_{\mu\nu}(k) = -i\xi\delta^{ab} \tag{4.162}$$

$$k^\mu k^\nu G^{ab}_{\mu\nu}(k) = -i\xi\delta^{ab} \tag{4.163}$$

Then recall the form of $G^{ab}_{\mu\nu}$:

$$k^\mu k^\nu G^{ab}_{\mu\nu}(k) = \frac{i}{\cancel{k^2}}\delta^{ab}(-\cancel{k^2}B) = -i\xi\delta^{ab} \tag{4.164}$$

$$\Rightarrow B = \xi, \tag{4.165}$$

which is an **exact** result.

4.14 Addendum: some more justification for functional integral

We give a some more justification for the functional integral Eq. (4.112):

$$\frac{\delta f^a[A_\epsilon]}{\delta\epsilon^b} = \frac{\delta f^a[A_\epsilon]}{\delta A_\epsilon^c}\frac{\delta A_\epsilon^c}{\delta\epsilon^b} = f'^{ac}[A_\epsilon]\frac{1}{g}D_\mu^{cb} \tag{4.166}$$

Then we can write

$$\int \mathcal{D}A_\epsilon \mathcal{D}\epsilon\, e^{iS[A_\epsilon]} \det\left(f'^{ac}[A_\epsilon]\delta f[A_\epsilon]\right) \det\left(\frac{1}{g}D_\mu^{cb}\right) \delta\left(f[A_\epsilon]-F\right)\mathcal{O}_i\mathcal{O}_j\ldots \tag{4.167}$$

$$\det\left(\frac{1}{g}D_\mu\right) = \det\left(\frac{1}{g}U_\epsilon D_\mu U_\epsilon^\dagger\right) = \det\left(\frac{1}{g}D_\mu^{(\epsilon)}\right).$$

Remark: det is gauge invariant.

Hence renaming

$$A_\epsilon \to A \tag{4.168}$$

gives

$$\int \mathcal{D}A_\epsilon \mathcal{D}\epsilon\, e^{iS[A]} \det\left(f'^{ac}[A]\delta f[A_\epsilon]\right) \det\left(\frac{1}{g}D_\mu^{cb}\right) \delta\left(f[A]-F\right)\mathcal{O}_i\mathcal{O}_j\ldots \tag{4.169}$$

$$= \left(\int \mathcal{D}\epsilon\right)\int \mathcal{D}A\, e^{iS[A]} \det\left(\frac{\delta f[A_\epsilon]}{\delta\epsilon}\right)_{\epsilon=0} \delta\left(f[A]-F\right)\mathcal{O}_i\mathcal{O}_j\ldots \tag{4.170}$$

5
Renormalization of non-abelian gauge theories

We will now discuss the renormalization of non-abelian gauge theories, such as QCD. In QCD the gauge group is $G = SU(3)$ and the matter fields are n_f Dirac fermions in the fundamental representation of G

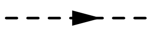
5.1 Feynman Rules

The kinetic terms

$$\mathcal{L}_{kin} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial_\mu A^{a\mu})^2 + \overline{\psi}_i(i\partial\!\!\!/ - m)\psi_i - \bar{c}^a \Box c^a. \tag{5.1}$$

give the propagators

$\nu; b$  $\mu; a$ $\quad = \quad i\dfrac{-\eta^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon}\delta^{ab}$ (gauge)

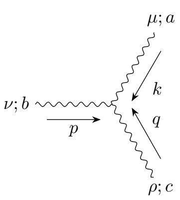
b  a $\quad = \quad \dfrac{i\delta^{ab}}{p^2 + i\varepsilon}$ (ghost)

j  i $\quad = \quad \dfrac{i\delta^{ij}}{p\!\!\!/ - m + i\varepsilon}$ (fermion)

where we will usually employ the Feynman-gauge $\xi = 1$.

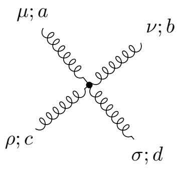
The interactions are

$$\begin{aligned} \mathcal{L}_{int} = &- g f^{abc}(\partial_\mu A_\nu)^a A^b_\mu A^c_\nu - \frac{1}{4}g^2(f^{gab}A^a_\mu A^b_\nu)(f^{gcd}A^c_\mu A^d_\nu) \\ &+ g f^{abc}(\partial_\mu \bar{c}^a)A^b_\mu c^c + g\overline{\psi}_i\gamma^\mu T^a_{ij}\psi_j A^a_\mu. \end{aligned} \tag{5.2}$$

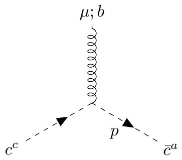


$$= g f^{abc}[\eta^{\mu\nu}(k-p)^\rho + \eta^{\nu\rho}(p-q)^\mu + \eta^{\rho\mu}(q-k)^\nu]$$

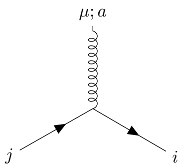
(triple gluon)



$$
\begin{aligned}
= & -ig^2[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\
& + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) \\
& + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})] \\
& \qquad \text{(four gluon)}
\end{aligned}
$$



$$= -\,gf^{abc}p^{\mu} \qquad \text{(ghost-vertex)}$$



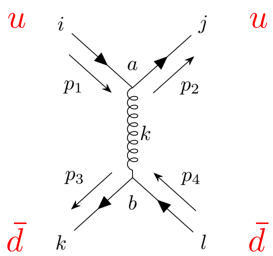
$$= ig\gamma^{\mu}T^a_{ij} \qquad \text{(fermion-vertex)}$$

5.2 A tree-level QCD amplitude

Let us first calculate a tree-level amplitude to examine how QCD differs from QED. Consider the process

$$u\bar{d} \to u\bar{d}. \tag{5.3}$$

Analogously to t-channel scattering in QED we can extract the QCD potential (at small g-coupling). The tree-level diagram for this elastic scattering is:



$$= (ig_s)^2 T^a_{ji}\delta_{ab}T^b_{kl} \times \bar{u}_j(p_2)\gamma^{\mu}u_i(p_1)\frac{-i}{k^2}\left[\eta_{\mu\nu} - (1-\xi)\frac{k_{\mu}k_{\nu}}{k^2}\right]\bar{v}_k(p_3)\gamma^{\nu}v_l(p_4)$$

With on-shell spinors, ξ-dependence drops out: $k = p_2 - p_1$, using the EOM

$$
\begin{aligned}
\bar{u}(p_2)\not{k}v(p_1) &= \bar{u}(p_2)(\not{p}_2 - \not{p}_1)u(p_1) \\
&= \bar{u}(p_2)(m-m)u(p_1) = 0 \qquad \text{(like in QED)}
\end{aligned}
$$

We find the amplitude identical to QED with $e \to -g_s$ up to the **color factor**:

$$T^a_{ji}T^a_{kl} \tag{5.4}$$

Each $u(\bar{d})$ can have red, green or blue (or anti-red, anti-green, anti-blue respectively). Let us examine their impact.

Example 1:

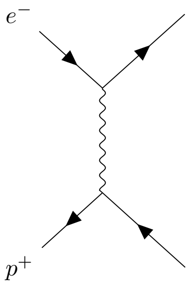
Try $\underset{i=1}{\underset{\uparrow}{u}}$ red and $\underset{k=2}{\underset{\uparrow}{\bar{d}}}$ anti-green:

$$T^a_{ji}T^a_{kl} = \frac{1}{2}(\delta_{jl}\delta_{ik} - \frac{1}{N}\delta_{ij}\delta_{kl}). \tag{5.5}$$

So we get

$$T^a_{j1}T^a_{2l} = -\frac{1}{2N}\delta_{1j}\delta_{2l}. \tag{5.6}$$

So the final state has red quark and anti-green also: color is conserved.
Compare to e^-p^+-scattering:



$$= (-ie)^2(\bar{u}\gamma^\mu u)\frac{-ig^{\mu\nu}}{k^2}(\bar{v}\gamma_\nu v). \tag{5.7}$$

Eq. (5.6) shows that this color amplitude has the opposite sign: This color combination leads to a **repulsive** interaction/potential.

Example 2:
u: red and $\bar{d}$: anti-red:[1]
$i = k = 1$

[1] Or one of the other two combinations: green and anti-green, or blue and anti-blue.

$$T^a_{j1}T^a_{1l} \overset{\overset{N=3}{\downarrow}}{=} \begin{pmatrix} 1/3 & & \\ & 1/2 & \\ & & 1/2 \end{pmatrix}_{jl} \tag{5.8}$$

Final state can be: (red, anti-red), (blue, anti-blue), (green, anti-green). The color factor is now > 0, which implies an **attractive potential**.
In general, we can decompose the color structure of $\bar{u}_i d_j$ according to

$$3 \otimes \bar{3} = \underset{\text{singlet}}{\underset{\uparrow}{1}} \oplus \overset{\overset{octet}{\downarrow}}{8} \tag{5.9}$$

Among the 9 color combinations of $u_i\bar{d}_j$, we find 8 with color and 1 color neutral[2] all of which will be left invariant under gluon exchange. We already saw that color-octet states (like red anti-green above in example 1) will be left invariant in the amplitude.

[2] See Eq. (5.10) for the definition.

The final state in example 1 is also red anti-green.

For the color singlet states which can go to any color-anticolor combination, the invariant, normalized combination is

$$|\mathbb{1}_c\rangle = \frac{1}{\sqrt{3}}(|r\bar{r}\rangle + |b\bar{b}\rangle + |g\bar{g}\rangle) \tag{5.10}$$

e.g. for $|r\bar{r}\rangle \to |r\bar{r}\rangle$ the color factor is $\mathrm{Tr}(T^a_{j1}T^a_{1l}) = \frac{4}{3}$.
Combining all the pre-factors and the multiplicity N_c of the final state, we get relative to the QED amplitude:

$$|\mathbb{1}_c\rangle \to \text{ anything} \qquad \Rightarrow \qquad \text{pre-factor} = (\frac{1}{\sqrt{3}})^2 \frac{4}{3} \overset{\overset{3}{\downarrow}}{N_c}. \tag{5.11}$$

The tree-level potentials are

$$V(r) = \frac{1}{6}\frac{g_s^2}{4\pi r} \qquad \text{(color octet)}$$
$$V(r) = -\frac{4}{3}\frac{g_s^2}{4\pi r}. \qquad \text{(color singlet)}$$

Only the color singlet state is attractive.

This is consistent with the observation that we do not find colored mesons[3], but only color-neutral bound states!
$\hookrightarrow$ hadrons:

$$\begin{array}{ll} \text{mesons:} & \bar{q}^i q_i \\ \text{baryons:} & \epsilon^{ijk} q_i q_j q_k \end{array}$$

[3] e.g. a quark-antiquark color octet bound state

This tree-level potential is not very useful quantitatively because QCD at $E \sim m_{\text{hadron}} \sim$ GeV is strongly coupled $g_s \gg 1$, see Sec. 5.7.2. For a quantitative calculation of meson and hadron properties, one needs a numerical, non-perturbative lattice approach.

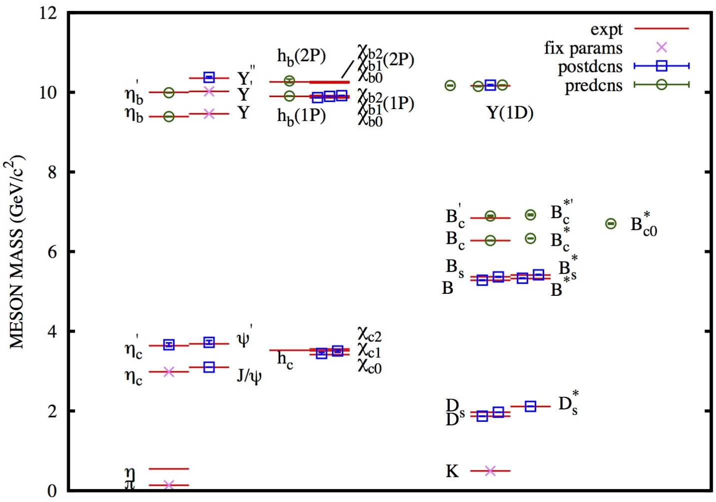


Figure 5.1: The masses of "gold-plated" mesons comparing the lattice QCD (HPQCD collaboration) results to experiment (an update of a figure that appeared in arxiv:hep-lat/1207.5149)

5.3 One-loop divergences of non-abelian gauge theories

Let's start with some 1-loop corrections and then discuss things more systematically. The goal is the running of $\alpha_s(\mu)$.

Reminder QED: renormalizable, only 4 counterterms.

$$-\frac{1}{4}\delta_3 F^2_{\mu\nu} + \overline{\psi}(i\delta_2 \partial\!\!\!/ - \delta_m)\psi - e\delta_1 \overline{\psi} \underset{\substack{\uparrow \\ \text{normalized fields}}}{A\!\!\!/} \psi$$

$$\begin{cases} \delta_3 = Z_3 - 1 & \delta_2 = Z_2 - 1 \\ \delta_m = Z_2 m_0 - m & \delta_1 = Z_1 - 1 \end{cases}$$

Ward-Identity: $\delta_2 = \delta_1$ or $Z_2 = Z_1$

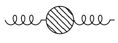
Charge e_0: $e = \frac{Z_2}{Z_1} Z_3^{1/2} e_0 \overset{Z_2=Z_1}{=} \sqrt{Z_3} e_0$ depends only on photon field strength renormalization.
Gauge-invariance requires

$$
\begin{aligned}
&\overline{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi && (5.12)\\
&\hookrightarrow \overline{\psi}(i\partial\!\!\!/\delta_2 - e\delta_1 A\!\!\!/)\psi && (5.13)\\
&\overset{!}{=} \delta\overline{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi \qquad \Leftarrow \; \delta_1 \overset{!}{=} \delta_2 && (5.14)
\end{aligned}
$$

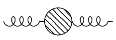
to preserve the symmetry:

$$
\psi \to e^{i\alpha(x)}\psi
$$
$$
A \to A_\mu + \frac{1}{e}\partial_\mu \alpha
$$

Recall, the QED gauge boson self-energy

$$q^u \Big($$  $$\Big) = 0,$$

which implies

 $$= i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2) \qquad (5.15)$$

and only possible divergence is logarithmic.

5.4 General considerations for renormalizing Yang-Mills theories

We start with the Lagrangian

$$
\begin{aligned}
\mathcal{L} = \mathcal{L}_{kin} - g f^{abc}(\partial_\mu A_\nu)^a A^b_\mu A^c_\nu - \frac{1}{4} g^2 (f^{gab} A^a_\mu A^b_\nu)(f^{gcd} A^c_\mu A^d_\nu)\\
+ g f^{abc}(\partial_\mu \bar{c}^a) A^b_\mu c^c + g\overline{\psi}_i \gamma^\mu T^a_{ij} \psi_j A^a_\mu. \qquad (5.16)
\end{aligned}
$$

For all the vertices we define again

$$
\Delta_i = \overset{\text{space-time dim.}}{\overset{\downarrow}{d}} - \underset{\underset{\#\text{ derivatives}}{\uparrow}}{a_i} - \sum_f \overset{\#\text{ fields of type f}}{\overset{\downarrow}{n_{if}}} \left(\frac{d}{2} - 1 + s_f\right)
$$

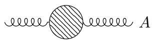
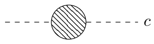
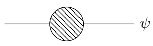
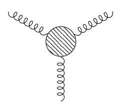
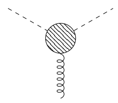
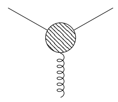
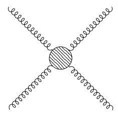
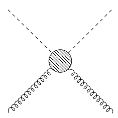
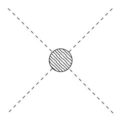
where as in QFT1 $s_f = 0$ for $A_\mu, c^a, \bar{c}^a$ and $s_f = \frac{1}{2}$ for ψ. You can check that all the vertices in Eq. (5.16) have

$$
\Delta_i = 0 \qquad \text{(QCD)}
$$

which means that the theory is power-counting renormalizable. The superficial degree of divergence for Green functions is:

$$
D = \overset{\text{space time dim.}}{\overset{\downarrow}{4}} - \frac{3}{2}\underset{\underset{\text{fermions}}{\uparrow}}{E_\psi} - \overset{\text{gauge bosons}}{\overset{\downarrow}{E_A}} - \underset{\underset{\text{ghosts, antighosts}}{\uparrow}}{E_{c,\bar{c}}}. \qquad (5.17)
$$

where the E_X count the number of external fields of type X.
We will need a regulator which satisfies BRS or gauge symmetry, such that the regularized theory still has the symmetry and contains the same number of independent terms as the tree-level Lagrangian. Dimensional regularization satisfies this requirement.
The *actual* degree of divergence is often less than the *superficial* one because of additional constraints coming from the Lorentz structure of the vertex and gauge invariance – it is at most logarithmic.

	superficial degree of divergence	actual degree of divergence	
	2	0	(gauge invariance)
	2	0	(ghost vertex $\propto p^\mu$ + Lorentz invariance)
	1	0	(Lorentz invariance)
	1	0	(Lorentz invariance)
	1	0	(p^μ in ghost vertex)
	0	0	
	0	0	
	0	finite	(p^μ in ghost vertex)
	0	finite	(p^μ in ghost vertex)

5.5 *Vacuum polarization*

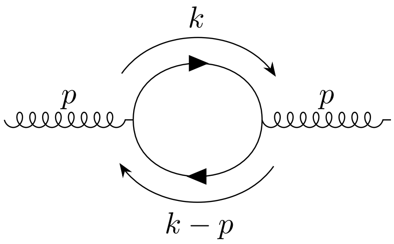
5.5.1 *Fermion bubble*

We will see we can discuss the fermion loop independently, whereas we will have to discuss the gauge and ghost loops together.

$$= (-1)\ \mathrm{Tr}(T^aT^b)(ig)^2 \int \frac{d^4k}{(2\pi)^4}\ \frac{i}{(p-k)^2-m^2}\frac{i}{k^2-m^2}\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p}+m)\gamma^\nu(\not{k}+m)\right]. \tag{5.18}$$

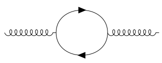
The result in Eq. (5.18) is the same as in QED except for $\mathrm{Tr}(T^aT^b) = T_F\delta^{ab}$ with $T_F = \frac{1}{2}$. If there are n_f species of fermions all in the rep-

The trace $\mathrm{Tr}(T^aT^b)$ can be understood intuitively: Think of QED with 2 fermions of charge Q_1 and Q_2, which would get $\sim Q_1^2 + Q_2^2 = \mathrm{Tr}(Q_i^2)$. Here we sum over all fermions of color, each one contributing with its "charge" T^aT^b.



resentation R the Feynman diagram is $\propto n_f \mathrm{Tr}[\mathrm{T^a T^b}] = \mathrm{n_f C(R)}\delta^{\mathrm{ab}}$.
We use dimensional regularization to find

$$\sum_{\text{fermions}}\left(\quad \right) \overset{\substack{QED\\ \downarrow}}{=} i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab} n_f C(R)\cdot$$



$$\cdot\left(\frac{-g^2}{(4\pi)^{d/2}}\int_0^1 dx\; 8x(1-x)\frac{\Gamma(2-d/2)}{(m^2-x(1-x)q^2)^{2-d/2}}\right)$$
$$= i(q^2 g^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(\frac{-g^2}{(4\pi)^2}\cdot\frac{4}{3}\, n_f\; C(r)\;\Gamma(2-\frac{d}{2}) + \ldots\right)$$

where we have used the QED result.

Remember: dim-reg see QFT1

$$\Delta I_n^0 = \int \frac{d^d l_{E(\text{ucldean})}}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^n}$$
$$= \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2}. \qquad (5.19)$$

$$\Delta I_n^2 = \int \frac{d^d l_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^n}$$
$$= \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-\frac{d}{2}-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-d/2-1}. \qquad (5.20)$$

e.g.

$$\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-d/2}$$
$$\underset{\substack{\uparrow\\ d=4-\epsilon}}{=} \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln(4\pi) + \mathcal{O}(\epsilon)\right). \qquad (5.21)$$

We will need the color factors

$$\mathrm{Tr}(\mathrm{T^a T^b}) = C(R)\delta^{ab} \qquad (5.22)$$
$$\sum_a (T^a T^a)_{ij} = C_2(R)\delta_{ij} \qquad (5.23)$$
$$f^{acd} f^{bcd} = C_2(G)\delta_{ab} \qquad (5.24)$$

with SU(N) relations:

$$C(R) = \frac{1}{2}$$
$$C_2(F) = \frac{N^2-1}{N}$$
$$C_2(G) = N.$$

which in particular implies $C(R) = C(F) = \frac{1}{2}$ for fermions in the fundamental (F), as in QCD.

$$\Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x); \quad \gamma = 0.5772\ldots$$
$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = 1 - \underbrace{(2-\frac{d}{2})}_{\frac{\epsilon}{2}}\ln\Delta + \ldots \qquad d\to 4$$

since

$$x^\alpha = e^{\ln x^\alpha} = e^{\alpha\ln x} \propto 1 + \alpha\ln x$$
$$\frac{1}{(4\pi)^{d/2}} = \frac{1}{(4\pi)^2}(1 + \frac{\epsilon}{2}\ln(4\pi) + \ldots)$$
$$\Gamma(2) = 1 \qquad x\Gamma(x) = \Gamma(x+1).$$

5.5.2 Trilinear gluon bubble

Now we will calculate the diagrams:

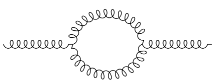


Figure 5.2: Trilinear gluon bubble

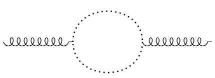


Figure 5.3: Ghost bubble

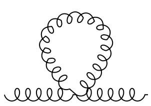
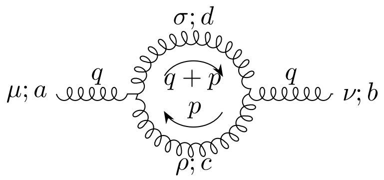


Figure 5.4: Four point gluon bubble

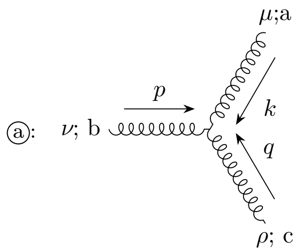
We start with the diagram generated by the trilinear gauge boson vertex, the trilinear gluon bubble



$$= \underbrace{\frac{1}{2}}_{\text{symmetry factor}} \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2} \frac{-i}{(p+q)^2} \; g^2 \; f^{acd} f^{bfe} \delta^{ce} \delta^{df} N^{\mu\nu}. \tag{5.25}$$

with the numerator

Feynman-rules referring to Eq. (5.26)



$$\begin{pmatrix} a \to d & p \to q & \nu \to \mu \\ c \to c & k \to -q-p & \rho \to \rho \\ b \to q & q \to q & \mu \to \sigma \end{pmatrix}$$

ⓑ:

$$\begin{pmatrix} a \to b & k \to -q & \mu \to \nu \\ c \to c & q \to -p & \rho \to \rho \\ b \to d & p \to q+p & \nu \to \sigma \end{pmatrix}$$

$$\begin{aligned} N^{\mu\nu} &= [\eta^{\sigma\mu}(-2q-p)^{\rho} + \overset{\text{ⓐ}}{\eta^{\mu\rho}}(q-p)^{\sigma} + \eta^{\rho\sigma}(2p+q)^{\mu}] \times \\ &\times [\eta^{\nu\sigma}(-2q-p)^{\rho} + \eta^{\sigma\rho}(q+2p)^{\nu} \underset{\substack{\text{minus sign}\\ \text{compared to}\\ \text{Peskin due to}\\ f^{bde}\text{ prefactor}}}{+} \eta^{\rho\nu}(q-p)^{\sigma}] \quad \leftarrow \text{ⓑ} \end{aligned} \tag{5.26}$$

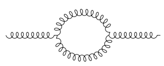
The structure functions are:

$$f^{acd} f^{bcd} = C_2(G)\delta^{ab}$$
$$C_2(N) = \frac{N^2-1}{2N} \overset{N=3}{=} \frac{4}{3}.$$

The denominators are combined using Feynman:

$$\frac{1}{p^2}\frac{1}{(p+q)^2} = \int_0^1 dx \, \frac{1}{((1-x)p^2 + x(p+q)^2)^2} = \int_0^1 dx \, \frac{1}{(P^2-\Delta)^2} \tag{5.27}$$

with $P = p + xq$ and $\Delta = -x(1-x)q^2$.
We find:



$$= -\frac{g^2}{2} C_2(G)\delta^{ab} \int_0^1 dx \int \frac{d^4P}{(2\pi)^4} \frac{1}{(P^2-\Delta^2)} \, N^{\mu\nu}. \tag{5.28}$$

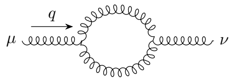
Now we eliminate p in favor of P and massage $N^{\mu\nu}$. First, we discard terms linear in P_μ

$$\int d^4P \; f(P^2) P_\mu = 0.$$

and we replace $P^\mu P^\nu \to \eta^{\mu\nu}\frac{P^2}{d}$ (by symmetry), to find

$$\begin{aligned} N^{\mu\nu} \overset{(\dots)}{\to} & -\eta^{\mu\nu} P^2 \, 6 \cdot (1 - \frac{1}{d}) - \eta^{\mu\nu} q^2 [(2-x)^2 + (1+x)^2] \\ & + q^\mu q^\nu \left[(2-d)(1-2x)^2 + 2(1+x)(2-x)\right]. \end{aligned} \tag{5.29}$$

We Wick-rotate: $l_E^0 = -il^0$; $\vec{l} = \vec{l}_E$ and use $I_n^0(\Delta)$ from Eq. (5.19) and $I_n^2(\Delta)$ from Eq. (5.20):



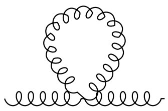
$$= \frac{ig_0^2}{(4\pi)^{d/2}} C_2(G)\delta^{ab} \int_0^1 dx \, \frac{1}{\Delta^{2-d/2}}$$

$$\times \left(\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{3}{2}(d-1)x(1-x)\right]\right.$$
$$+\Gamma\left(2-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2\right]-$$
$$\left.-\Gamma\left(2-\frac{d}{2}\right)q^\mu q^\nu\left[\left(1-\frac{d}{2}\right)(1-2x)^2+(1+x)(2-x)\right]\right) \quad (5.30)$$

Before we analyze this further, we will work out the other graphs. We expect that only the sum of all relevant graphs will satisfy the Ward identity.

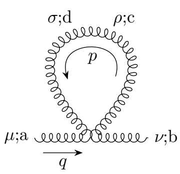
Note the pole at $d \to 2$, signifying a quadratic divergence in cut-off regularization and that the result is not transverse $q^\mu M_\mu \neq 0$ yet.

5.5.3 Four-point gluon bubble



$$\sim \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2} = 0 \quad \text{in dim-reg.} \qquad (5.31)$$

This is a scale-less integral which vanishes in dim-reg, with another regulator, like Pauli-Villars it would be quadratically divergent. As we know, quadratic divergences show up as poles in at $d \to 2$ in dim. regularization. This quadratic divergence is cancelled by $d \to 2$ pole in .✓
Although they add up to 0 trivially here $0+0=0$, it is maybe somewhat illuminating to consider that the cancellation (in other schemes) requires the coupling constants to be equal in both graphs. We find



$$\overset{\text{Symmetry}}{\downarrow}$$
$$= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\frac{-i\eta_{\rho\sigma}}{p^2}\delta^{cd}(-ig^2)\times$$

$$\times\left[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})+\right.$$
$$+f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho})$$
$$\left.+f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\right]$$
$$= -g^2 C_2(G)\delta^{ab}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2}\eta^{\mu\nu}(d-1).$$

where we used $f^{ace}f^{bce} = C_2(G)\delta^{ab}$ and $d = \eta_{\rho\sigma}\eta^{\rho\sigma}$. We want to show the cancellation of $d \to 2$ pole explicitly.
So multiply integrand by $1 = \frac{(q+p)^2}{(q+p)^2}$. Combine denominators using

Feynman trick as above by using $P = p + xq$. So we get

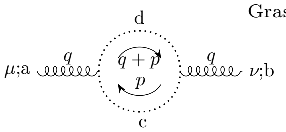
$$-g^2 C_2(G) d^{ab} \int_0^1 dx \int \frac{d^4P}{(2\pi)^4} \frac{1}{(P^2-\Delta)^2} \eta^{\mu\nu}(d-1)\left[P^2 + (1-x)^2 q^2\right] \tag{5.32}$$

$$\underset{\text{Wick-rot.}}{=} \frac{ig_0^2}{(4\pi)^{d/2}} \delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}}$$
$$\times \left(-\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{1}{2}d(d-1)x(1-x)\right] - \Gamma\left(2-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[(d-1)(1-x)^2\right]\right). \tag{5.33}$$

The poles at $d \to 2$ do not cancel. We need the ghost loop!

5.5.4 Ghost bubble

Grassmann-fermion loop



$$= (-1)\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2} \frac{i}{(p+q)^2} g_0^2 f^{dac}(p+q)^\mu f^{cbd} p^\nu .$$

Using the same trick we get

$$\frac{ig_0^2}{(4\pi)^{d/2}} C_2(G)\delta^{ab} \int_0^1 dx \frac{1}{\Delta^{2-d/2}}\, x \left(-\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2[(1-x)\frac{x}{2}] + \Gamma\left(2-\frac{d}{2}\right)q^\mu q^\nu[x(1-x)]\right). \tag{5.34}$$

Let's combine the three $\Gamma(1-\frac{d}{2})$-coefficients:

$$\Gamma(1-\frac{d}{2})\eta^{\mu\nu}q^2 x(1-x)\cdot \underbrace{[\underset{Eq.\ (5.30)}{3(d-1)} - \overset{Eq.\ (5.33)}{d(d-1)} - \underset{Eq.\ (5.34)}{1}]}_{(1-\frac{d}{2})(2-d)}\frac{1}{2} \tag{5.35}$$

So we can simplify the Γ-function:

$$(2-d)\Gamma\left(1-\frac{d}{2}\right) = (2-d)\left(\frac{2}{2-d} - \gamma + \dots\right) \tag{5.36}$$

This cancels the $d \to 2$ divergence! The sum of the three diagrams has no quadratic divergence and no gauge boson mass renormalization. The ghost diagram is essential and that the **same g coupling** enters everywhere.

With this factor $(2-d)$ the Γ-function becomes

$$(2-d)\Gamma(1-d) = \Gamma(2-d) \tag{5.37}$$

like the remaining terms. It contributes to $q^2\eta^{\mu\nu}\Gamma(2-d)$.
The three pure gauge diagrams simplify to

$$\frac{ig_0^2}{(4\pi)^{d/2}} C_2(G)\delta^{ab} \int_0^1 dx \frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} (\eta^{\mu\nu}q^2 - q^\mu q^\nu)\left[\left(1-\frac{d}{2}\right)(1-2x)^2 + 2\right]$$

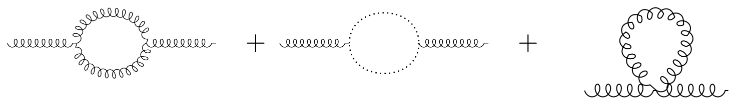
The integral gives $(\frac{5}{3} + \mathcal{O}(\epsilon))\Gamma\left(2-\frac{d}{2}\right)$

Note:
If we use the cut-off scheme

 $\propto \Lambda^2$

means a renormalization of the gauge boson mass $\delta m_A A_\mu A^\mu$ since this has no momentum dependence. It must be renormalized by pure field term $\delta m_A A_\mu A^\mu$ and not $-\delta_3 \frac{1}{4} F_{\mu\nu}$.

manifestly transverse (!) as required by the Ward-Identity.
The UV divergent part is the sum of



with $d \to 4$

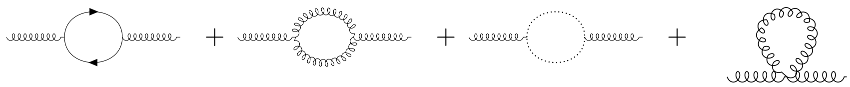
$\Delta^{2-d/2} \simeq 1 + \mathcal{O}(\epsilon)$

$$i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(\frac{-g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)C_2(G)\Gamma\left(2-\frac{d}{2}\right)+\dots\right) \tag{5.38}$$

Expanding in $4-\epsilon$: $\Gamma(2-\frac{d}{2}) = \frac{2}{\epsilon} - \gamma + \dots$ including $\int \frac{1}{\Delta}$

$$i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\frac{g^2}{16\pi^2}C_2(G)\left[\frac{2}{\epsilon}\cdot\frac{5}{3}+\frac{31}{9}+\frac{5}{3}\ln\frac{\tilde{\mu}^2}{-q^2}+\mathcal{O}(\epsilon)\right] \tag{5.39}$$

Adding the fermion bubble, we find for the divergent and logarithmic pieces:



$$\begin{aligned} i\Pi^{ab}_{\mu\nu}(q) = i\delta^{ab}\frac{g^2}{16\pi^2}\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\Bigg[C_2(G)\left[\frac{2}{\epsilon}\frac{5}{3}+\frac{5}{8}\ln\left(\frac{\tilde{\mu}^2}{-p^2}\right)\right] \\ -n_f C(r)\left[\frac{2}{\epsilon}\frac{4}{3}+\frac{4}{3}\ln\left(\frac{\tilde{\mu}^2}{-p^2}\right)\right]+\dots\Bigg]. \end{aligned} \tag{5.40}$$

The result has the correct momentum structure and is logarithmically divergent, as expected.

5.6 Renormalization at 1-loop

Before moving on to the full 1-loop renormalization, we need to understand how the theory is renormalized. As usual, in terms of the **renormalized fields and couplings**:

We will omit the subscript R for renormalized fields to simplify the notation below.

$$\mathcal{L} = -\frac{1}{4}Z_3(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + Z_2\overline{\psi}_i(i\not{\partial} - Z_m m)\psi_i \tag{5.41}$$

$$- Z_{3_c}\overline{c}^a\Box c^a - \mathbf{g_R}Z_{A_3}f^{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu \tag{5.42}$$

$$- \frac{1}{4}\mathbf{g_R}^2 Z_{A_4}(f^{eab}A^a_\mu A^b_\nu)(f^{ecd}A^c_\mu A^d_\nu) \tag{5.43}$$

$$+ \mathbf{g_R}Z_1 A^a_\mu\overline{\psi}_i\gamma^\mu T^a_{ij}\psi_j + \mathbf{g_R}Z_{1_C}f^{abc}(\partial_\mu\overline{c}^a)A^b_\mu c^c. \tag{5.44}$$

$\mathbf{g_R}$ appears in front of 4 different terms. We will see later that charge universality imposes relations among Z_α.

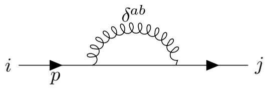
What about $-Z_{gf}\frac{1}{2\xi}(\partial_\mu A^a_\mu)^2$? In central tutorial: Ward-identity guarantees that ξ is not renormalized: $Z_\xi = Z_3$ Here: $\xi^0 = Z_\xi \xi$. It follows $Z_{gf} = 1$ to all orders!

Counterterms: $-\frac{1}{4}F^{a2}_{\mu\nu}\delta_3 + \overline{\psi}(i\delta_2\not{\partial} - \delta_m)\psi$ for the two-point functions. Our calculation above gives us:

$$Z_3 = 1 - \frac{2}{\epsilon}\frac{g^2}{16\pi^2}\left(-\frac{5}{3}C_2(G) + n_f C(r)\cdot\frac{4}{3}\right) + \text{ finite} \tag{5.45}$$

Our goal is the β-function, we need the 3-point function at 1-loop and the fermion 2-point function.

Two point function: self-energy fermions



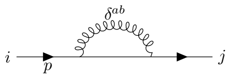
$$= i\Sigma_2^{ij}(\not p) = iC_2(r)\delta^{ij}\Sigma_2^{\text{QED}}(\not p). \quad (5.46)$$

same as QED up to the color factor, which is using Eq. (5.23),

$$\sum_{a,b}\sum_{k,l}(T^a)_{ki}(T^b)_{jl}\delta^{ab}\delta_{kl} = \sum_a \underbrace{(T^aT^a)_{ij}}_{\text{quadratic Casimir}} = C_2(r)\cdot\delta_{ij} = \frac{N^2-1}{2N}\cdot\delta_{ij}.$$

We will just quote the QED results.

Conventions as in Schwartz and Peskin!



$$= i\delta^{ij}\left(\frac{g^2}{16\pi^2}(2\not p - 8m)\frac{1}{\epsilon}C_2(r) + \delta_2\not p - (\delta_m+\delta_2)m\right).$$

So we get

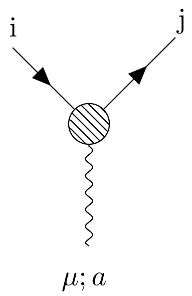
$$\boxed{\begin{aligned}\delta_2 &= \frac{1}{\epsilon}\frac{g^2}{16\pi^2}(-2C_2(r)) \quad (5.47)\\ \delta_m &= \frac{1}{\epsilon}\frac{g^2}{16\pi^2}(-6C_2(r)). \quad (5.48)\end{aligned}}$$

The ghost two-point function gives

See exercise !

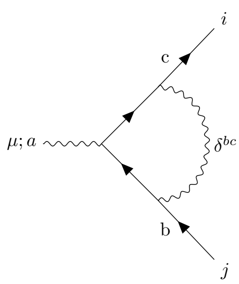
$$\boxed{\delta_{3_C} = \frac{1}{\epsilon}\frac{g^2}{16\pi^2}C(G).} \quad (5.49)$$

We need the coupling renormalization, we could get from all trilinear vertices. We choose gauge-fermion-fermion vertex:



$$= ig(\Gamma_\mu^a)_{ij}. \quad (5.50)$$

At tree-level: $\Gamma_{ij}^{a\mu}(q_1,q_2,p) = \gamma^\mu T_{ij}^a$.



$$= \sum_{b,c} ig(T^bT^aT^b)_{ij}\delta^{bc}\underbrace{\Gamma^\mu_{(2A)}}_{\text{Same as in QED}} \quad (5.51)$$

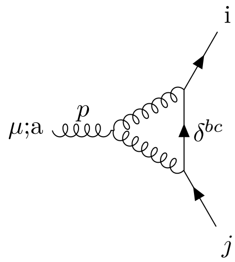
Simplyfy the color factors:

$$\begin{aligned}T^bT^aT^b &= T^bT^bT^a + T^b\underbrace{[T^a,T^b]}_{if^{abc}T^c}\\ &= C_2(r)T^a + if^{abc}T^bT^c\\ &= C_2(r)T^a + if^{abc}\frac{1}{2}[T^b,T^c]\\ &= C_2(r)T^a - \frac{1}{2}f^{abc}f^{bcd}T^d\\ &= (\underbrace{C_2(r)}_{C_F} - \frac{1}{2}\underbrace{C(G)}_{C_A})T^a.\end{aligned}$$

Use the result from QED: $-e \to g$

$$= ig\left(C_F - \frac{C_A}{2}\right) T^a_{ij}\gamma^\mu \left(\frac{g^2}{16\pi^2}\right)\left(\frac{2}{\epsilon} + \ln\frac{\tilde{\mu}^2}{-p^2} + \text{finite}\right). \qquad (5.52)$$

Focus on color factors:

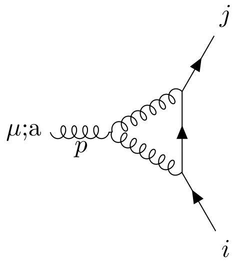


$$\propto T^c T^b f^{abc} = \frac{1}{2} f^{abc}[T^c, T^b] \qquad (5.53)$$

$$= \frac{i}{2}\underbrace{f^{abc} f^{cbd}}_{-\delta^{ad}} T^d \qquad (5.54)$$

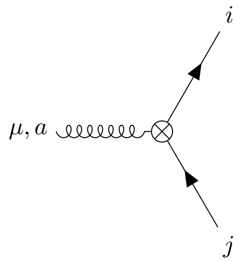
$$= -\frac{i}{2} C_2(G) T^a. \qquad (5.55)$$

The result is:



$$= igC_2(G)T^a_{ij}\gamma^\mu \left(\frac{g^2}{16\pi^2}\right)\left[\frac{3}{\epsilon} + \frac{3}{2}\ln\frac{\tilde{\mu}^2}{-p^2} + \dots\right] \qquad (5.56)$$

The counterterm is



$$= igT^a_{ij}\delta_1. \qquad (5.57)$$

Adding both vertex graphs, we find to cancel UV-divergence:

$$\delta_1 = \frac{1}{\epsilon}\left(\frac{g^2}{16\pi^2}\right)\left[-2C_2(r) - 2C_2(G)\right]. \qquad (5.58)$$

5.7 Running coupling in non-abelian gauge theories

We have all the results to calculate the β-function. The renormalization group equation is determined by independence of observables on UV-cutoff **or** subtraction point at which we renormalize, **or** scale μ of dimensional regularization. We use $\overline{MS}$ here.[4]

[4] Remember:
MS: remove $\frac{1}{\epsilon}$ pole
$\overline{MS}$: remove $\frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$.

Fermion-gauge-boson vertex:

$$\mathcal{L} = \overset{\text{such that } g_R \text{ is dim-less}}{\overset{\downarrow}{\mu^{\frac{4-d}{2}}}} g_R Z_1 A_\mu^a \overline{\psi}^i \gamma_\mu T^a_{ij} \underset{\text{Renormalized fields}}{\underset{\uparrow}{\psi_j}} \tag{5.59}$$

$$= \mu^{\frac{4-d}{2}} g_R \frac{Z_1}{Z_2\sqrt{Z_3}} A_\mu^{a(0)} \overline{\psi}_i^{(0)} T^a_{ij} \overset{\text{bare fields}}{\overset{\downarrow}{\psi_j^{(0)}}}. \tag{5.60}$$

The bare charge therefore:

$$g_0 = g_R \frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{4-d}{2}} \tag{5.61}$$

RGE from

$$0 \overset{!}{=} \mu \frac{d}{d\mu} g_0 = \mu \frac{d}{d\mu}\left[\frac{Z_1}{Z_2\sqrt{Z_3}} \mu^{\frac{4-d}{2}} g_R\right] \tag{5.62}$$

Recall in QED: $Z_1 = Z_2$, whereas here: $Z_1 \neq Z_2$.
We will see later that

$$\frac{Z_1}{Z_2} = \frac{Z_{1_C}}{Z_{3_C}} = \frac{Z_{A_3}}{Z_3} = \frac{\sqrt{Z_{A^4}}}{\sqrt{Z_3}}$$

Expanding perturbatively

$$Z_i = 1 + \overset{\mathcal{O}(g_R^2)}{\overset{\downarrow}{\delta_i}} \tag{5.63}$$

$$\frac{1}{\sqrt{Z_3}} = 1 - \frac{1}{2}\delta_3 + \dots \tag{5.64}$$

$$\frac{1}{Z_2} = 1 - \delta_2 + \dots \tag{5.65}$$

The renormalization group equation Eq. (5.62) becomes

$$\beta(g_R) = \mu \frac{d}{d\mu} g_R = \left[\underbrace{g_R (-\frac{\epsilon}{2})}_{\star} - \mu \frac{d}{d\mu}(\delta_1 - \delta_2 - \frac{\delta_3}{2}) + \dots\right] \tag{5.66}$$

Each δ_i depends only through $g_R(\mu)$ on μ

$$g_R \mu \frac{d}{d\mu} \delta_i = \underbrace{\mu \frac{d g_R}{d\mu}}_{\star} \frac{\partial \delta_i}{\partial g_R} g_R \tag{5.67}$$

Solve perturbatively using ($\star$)

$$\beta(g_R) = -\frac{\epsilon}{2} g_R + \overbrace{\frac{\epsilon}{2} g_R}^{\star} g_R \frac{\partial}{\partial g_R}(\delta_1 - \delta_2 - \frac{1}{2}\delta_3). \tag{5.68}$$

Using the 1-loop result and $\epsilon \to 0$

$$\beta(g_R) = -\frac{g_R^3}{16\pi^2}\left[\frac{11}{3} C_2(G) - \frac{4}{3} n_f C(r)\right] \tag{5.69}$$

In QCD (set N=3) $C_2(G) = \overset{\text{Adjoint Casimir}}{\overset{\downarrow}{C_A}} = N = 3$.

Fermions in fundamental: $C(r) = C(F) = T_F = \frac{1}{2}$

$$\mu \frac{d}{d\mu} \alpha_S(\mu) = -\frac{\alpha_S^2}{2\pi} \beta_0. \tag{5.70}$$

with $\alpha = \frac{g^2}{4\pi} \rightarrow d\alpha = d\frac{g^2}{4\pi} = \frac{g}{2\pi} dg$.

$$\beta_0^{QCD} = 11 - \frac{2n_f}{3} \tag{5.71}$$

As long as there are less than 17 flavors of quarks (we have found 6: u,d,c,s,b,t) $\beta_0 > 0$ and $\alpha_s(\mu)$ **decreases** with μ!

5.7.1 Asymptotic freedom

Compare to QED:

$$\beta_0^{QED} = -\frac{4}{3} n_f. \tag{5.72}$$

increases with $\mu \Rightarrow$ Landau-Pole. Gauge self-interaction decreases $\alpha_S(\mu)$ with increasing μ (antiscreening of charge)
$\Rightarrow$ UV fixed point in QCD
Solve

$\frac{4}{3}$ not $\frac{2}{3}$ because of $\mathrm{Tr}(\mathrm{T^a T^b}) = \frac{1}{2} \mathrm{f^{ab}}$.

$$\alpha_S(\mu) = \frac{2\pi}{\beta_0} \frac{1}{\ln \underbrace{\frac{\mu}{\Lambda_{QCD}}}_{\mu > \Lambda_{QCD}}} \tag{5.73}$$

where Λ_{QCD}: Landau-Pole in IR.
QCD is strong at long distances, weaker at higher energies.

5.7.2 Dimensional transmutation

Solve RGE

$$\frac{d}{d \ln \mu} \alpha_S(\mu) = -\frac{\alpha_S^2}{2\pi} \beta_0, \tag{5.74}$$

where we used the following relation:

$$d\left(\frac{1}{\alpha_S}\right) = -\frac{1}{\alpha_S^2} d\alpha_s = -\frac{1}{\alpha_S^2} \left(-\frac{\alpha_S^2}{2\pi} \beta_0\right) d \ln \mu. \tag{5.75}$$

It follows

$$\frac{d\alpha_S^{-1}}{d \ln \mu} = \frac{\beta}{2\pi} \qquad \text{(easy to solve)}$$

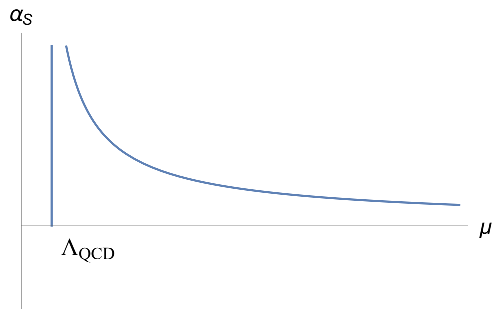
So we get

$$\alpha_S^{-1} = c + \frac{\beta_0}{2\pi} \ln \mu = \alpha_S^{-1}(\Lambda) + \frac{\beta_0}{2\pi} \ln\left(\frac{\mu}{\Lambda}\right) \tag{5.76}$$

with the boundary condition $\alpha_S^{-1}(\Lambda)$.

$$\alpha_S(\mu) = \frac{\alpha_S(\Lambda)}{1 + \alpha_S(\Lambda) \frac{\beta_0}{2\pi} \ln(\frac{\mu}{\Lambda})} \tag{5.77}$$

Get α_S by choosing Λ_{QCD}, so $\alpha_S(\Lambda_{QCD}) = \infty$.



With Eq. (5.73) we see that

$$\Lambda_{QCD} = \mu e^{-\frac{2\pi}{\alpha_S(\mu)\beta_0}} \qquad \text{(Dimensional transmutation)}$$

with $\Lambda_{QCD} = 100$ MeV.

We can trade the dimension-less $\alpha_S(\mu)$ for dimensionful Λ_{QCD}. It also explains the appearance of exponential scale hierarchies. The classical Lagrangian only contains dimensionless quantities.

5.7.3 Charge universality

QED: $Z_1 = Z_2$ (exact)
This implies universal electric charges, e.g. proton and electron charge are the same even after radiative corrections.
Two couplings:

$$\begin{aligned} & g_0 \overline{\psi}_p^{(0)} \gamma_\mu \psi_p^0 A_\mu^{(0)} + g_0 \overline{\psi}_{e^+}^{(0)} \gamma_\mu \psi_{e^+}^0 A_\mu^{(0)} && (5.78) \\ &= Z_1^P g_R^p \overline{\psi}_P \gamma_\mu A_\mu \psi_P + Z_1^{e^+} g_R^{e^+} \overline{\psi}_{e^+} \gamma_\mu \psi_{e^+} A_\mu && (5.79) \\ &= \frac{Z_1^P}{Z_2^P \sqrt{Z_3}} g_R \overline{\psi}_P^{(0)} \gamma_\mu \psi_P^{(0)} A_\mu + \frac{Z_1^{e^+} g_R^{e^+}}{Z_2^P \sqrt{Z_3}} \overline{\psi}_{e^+}^{(0)} \gamma_\mu \psi_{e^+}^{(0)} A_\mu. && (5.80) \end{aligned}$$

We see that charge universality is satisfied

$$\frac{g_R^P}{g_0} = \frac{Z_2^P}{Z_1^P} \sqrt{Z_3} \stackrel{?}{=} \frac{Z_2^{e^+}}{Z_1^{e^+}} \sqrt{Z_3} = \frac{g_R^{e^+}}{g_0}, \quad \text{ok}, \tag{5.81}$$

since $Z_2^i = Z_1^i$.

How about QCD?
We saw that $Z_1 \neq Z_2$ since we found that

$$\begin{aligned} Z_1 &= 1 + \frac{1}{\epsilon} \frac{1}{16\pi^2} (-2C_F - 2C_A) && (5.82) \\ Z_2 &= 1 + \frac{1}{\epsilon} \frac{1}{16\pi^2} (-2C_F). && (5.83) \end{aligned}$$

However, a weaker condition also would work. Using the example of an up and down quark, if

$$\frac{Z_2^u}{Z_1^u} = \frac{Z_2^d}{Z_2^d},$$

then we would also find charge universality in Eq. (5.81), which is exactly what is in fact the case.

We see in the exercise that the universal renormalization of g_R is satisfied and requires:

$$\frac{Z_1}{Z_2} = \frac{Z_{1_C}}{Z_{3_C}} = \frac{Z_{A^3}}{Z_3} = \frac{\sqrt{Z_{A^4}}}{\sqrt{Z_3}}. \tag{5.84}$$

All this corresponds to (1-loop):

$$\delta_1 - \delta_2 = \delta_{1_C} - \delta_{3_C} = +\frac{1}{\epsilon}\frac{g^2}{16\pi^2}(-2C_A). \tag{5.85}$$

which you can check explicitly.

5.7.4 Running of mass in QCD

Scale dependence of $m_R(\mu)$ since $Z_m \neq 0$ $(= 1 + \frac{1}{\epsilon}(\frac{g^2}{4\pi^2})(-6C_F))$![5]

[5] It can be quite significant $m_b(m_b) \simeq 4.2\text{GeV} \to m_b(M_Z) \simeq 2.8\text{GeV}$.

RGE:

$$0 = \mu\frac{d}{d\mu}m^0 = \mu\frac{d}{d\mu}(Z_m m_R) = Z_m m_R\left[\frac{\mu}{m_R}\frac{dm_\alpha}{d\mu} + \frac{\mu}{Z_m}\frac{dZ_m}{d\mu}\right]. \tag{5.86}$$

Define:

$$\begin{aligned}\gamma_m &= +\frac{\mu}{m_R}\frac{dm_R}{d\mu} && \text{(anomalous dimension)}\\ &\left(= -\frac{\mu}{Z_m}\frac{dZ_m}{d\mu}\right) && (5.87)\end{aligned}$$

It follows

$$\boxed{\mu\frac{dm_R}{d\mu} = \gamma_m \cdot m.} \tag{5.88}$$

Since Z_m only depends on μ through $g_R(\mu)$ we have

$$\gamma_m = -\frac{\mu}{Z_m}\frac{dZ_m}{d\mu} = -\frac{1}{Z_m}\frac{dZ_m}{dg_R}\mu\frac{dg_R}{d\mu} \tag{5.89}$$

where

$$\mu\frac{dg_R}{d\mu} = \beta(g_R) = -\frac{\epsilon}{2}g_R + \underbrace{\frac{\epsilon}{2}g_R^2\partial_{g_R}(\delta_1 + \delta_2 - \frac{\delta_3}{2})}_{\text{This summand is } \mathcal{O}(\epsilon^0)} \tag{5.90}$$

Eq. (5.89) does not depend on $m(\mu)$ itself since UV poles $\sim \frac{c}{\epsilon}$ arise from momenta $p, k \gg m$.
$\Rightarrow$ renormalization does not contain m $(\partial_m c = 0)$.

$$\begin{aligned}\gamma_m &= -\frac{1}{Z_m}\frac{dZ_m}{dg_R}\mu\frac{dg_R}{d\mu} && (5.91)\\ &= -\frac{1}{1+\delta_m}\left(\frac{2}{g_R}\delta_m\right)\left(-\frac{\epsilon}{2}g_R\right) + \dots && (5.92)\\ &= \delta_m \cdot \epsilon && (5.93)\\ &= -\frac{6g_R^2}{16\pi^2}C_F. && (5.94)\end{aligned}$$

where we used $Z_m = 1 + \frac{g_R^2}{16\pi^2}(-6C_F) + \dots$ and $\delta_m = Z_m - 1$.

This is similar to QED: $\gamma_m^{QED} = -\frac{3e_R^2}{8\pi^2}$ with the extra factor from $\mathrm{Tr}(\underbrace{\mathrm{T^a T^b}) = \frac{1}{2}}_{C_F}\delta^{ab}$.
See QFT I for explicit solution.

6 Gravity as a gauge theory

So far we have assumed **global** Lorentz-invariance and invariance under **global** translations (in space and time).Einstein understood that in order to incorporate gravity we can use the **Equivalance principle**:

> “a freely falling observer does not feel a gravitational force.”

Motivate with Newton’s EOM in constant gravitational field $\vec{g}$:

$$\underset{\substack{\uparrow \\ \text{Inertial mass}}}{m_I}\frac{d^2\vec{r}}{dt^2} = \overset{\substack{\text{Gravitational mass} \\ \downarrow}}{m_G}\vec{g} \tag{6.1}$$

To a very good accuracy: $m_I - m_G = 0 \ \pm (10^{-11}m)$.
We can write:

$$m\frac{d^2}{dt^2}[\vec{r}(t) - \frac{1}{2}\vec{g}t^2] = 0. \tag{6.2}$$

We can generate the gravitational force on the RHS, a change of frame:

$$\vec{r}(t) \to \vec{r}(t)' = \vec{r}(t) - \frac{1}{2}\vec{g}t^2 \tag{6.3}$$

A freely falling frame of reference does not experience gravity.

What about non-constant $\vec{g}(t, \vec{x})$?

Einstein postulated:

> “Gravitational fields are such that at each point in space-time,
> they allow themselves to be transformed away
> by choosing an appropriate set of coordinates.”

(= local free-fall).

6.1 Recipe:

1. Take **local** quality (like the Lagrange density $\mathcal{L}$) which is Lorentz-invariant (special relativity).
2. Coordinates in **local object** are identified with "freely falling" frame. In arbitrary frame, interaction with gravity will automagically appear.

Preferred coordinates	general coordinates
ξ^m	x^μ
use m, n, p, q	μ, ν, ρ, σ
No gravity	gravity

6.2 Example: scalar-field $\phi(x)$

In the absence of gravity:

$$S[\phi] = \int d^4x \, \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)\right) \tag{6.4}$$

with $x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix}$ and

$$\begin{aligned} \partial_\mu &\equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla}) \\ \partial^\mu &= \eta^{\mu\nu}\partial_\nu = (\partial_t, -\vec{\nabla}) \end{aligned}$$

with the inverse metric of SR:

$$\eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Local quantities:

1. $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$
2. Volume element: $d^4x = dx^0 dx^1 dx^2 dx^3$.

Equivalence principle: reinterpret x^μ as free-fall-coordinates.

$$\{x^\mu\} \to \underset{\text{“flat” coordinates}}{\underset{\uparrow}{\{\xi^m\}}},\ m = 0, 1, 2, 3. \tag{6.5}$$

with $ds^2 = \eta_{mn} d\xi^m d\xi^n$ and $\eta_{mn}\eta^{np} = \delta^p_m$.

The "new" Lagrangian is then:

$$\mathcal{L} \to \frac{1}{2}\eta^{mn}\partial_m \phi \partial_n \phi - V(\phi) \tag{6.6}$$

with $\partial_m = \frac{\partial}{\partial \xi^m}$.

Volume element: $d^4x \to d\xi^0 d\xi^1 d\xi^2 d\xi^3$.

Therefore, our action **including gravity** is given by

$$S[\phi] = \int d^4\xi \left[\frac{1}{2}\eta^{mn}\partial_m\phi\partial_n\phi - V(\phi)\right]. \tag{6.7}$$

Voilà! It looks identical, **but** the labels ξ^m vary from point to point.

Let's re-express this in non-inertial coordinates x^μ:

$$d\xi^m = \frac{\partial \xi^m}{\partial x^\mu} dx^\mu. \tag{6.8}$$

The transformation matrix between flat and arbitrary coordinates is

$$e^m_\mu(x) \equiv \frac{\partial \xi^m}{\partial x^\mu} \qquad \text{(Vierbein)}. \tag{6.9}$$

Inverse operation:

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^m} d\xi^m \equiv e^\mu_m(x) d\xi^m \tag{6.10}$$

Satisfies

$$d\xi^m = e^m_\mu dx^\mu = e^m_\mu e^\mu_n d\xi^n \tag{6.11}$$

and therefore

$$e^m_\mu e^\mu_n = \delta^m_n. \tag{6.12}$$

Similarly

$$e^\mu_n e^n_\nu = \delta^\mu_\nu. \tag{6.13}$$

Derivatives:

$$\frac{\partial}{\partial \xi^m} = \frac{\partial x^\mu}{\partial \xi^m}\frac{\partial}{\partial x^\mu} = e^\mu_m \partial_\mu. \tag{6.14}$$

This expresses special coordinates in an arbitrary frame.

Our scalar Lagrangian is now:

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu}(x)\partial_\mu\phi\partial_\nu\phi - V(\phi) \tag{6.15}$$

with $g^{\mu\nu}(x) = \eta^{mn} e^\mu_m(x) e^\nu_n(x)$ which is the metric.
A general line element is:

$$ds^2 = \eta_{mn} d\xi^m d\xi^n = \eta_{mn}\, e^m_\mu(x) e^n_\nu(x)\, dx^\mu dx^\nu \tag{6.16}$$
$$\equiv g_{\mu\nu}(x)\, dx^\mu dx^\nu. \tag{6.17}$$

This defines the metric tensor:

$$g_{\mu\nu}(x) = \eta_{mn} e^m_\mu(x) e^n_\nu(x). \tag{6.18}$$

It satisfies : $g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$.

What about the volume element?
We need the Jacobian!

$$d^4\xi = J(\xi, x) d^4x \tag{6.19}$$

with

$$J = \det\left(\frac{\partial \xi^m}{\partial x^\mu}\right) = \det(e_\mu^m) \equiv E. \tag{6.20}$$

This is $\sqrt{-\det(g_{\mu\nu})}$ because with $\det(AB) = \det(A)\det(B)$.

$$\begin{aligned}
\det(g_{\mu\nu}) &= \det(e_\mu^m e_\nu^n \eta_{\mu\nu}) && (6.21)\\
&= \det(e_\mu^m)\det(e_\nu^n)\det(\eta_{\mu\nu}) && (6.22)\\
&= J \cdot J \cdot (-1). && (6.23)
\end{aligned}$$

It follows

Remark:
In Eq. (6.22) no sum over indices!

$$J = \sqrt{-\det(g_{\mu\nu})}. \tag{6.24}$$

which we also write as $\sqrt{-\det g}$. Hence, we find the action of a scalar field with gravity as

$$S = \int d^4x \sqrt{-\det g}\left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\, \partial_\nu\phi - V(\phi)\right]. \tag{6.25}$$

Gravity is fully encoded in the metric.
Without gravity

$$e_\mu^m \to \delta_\mu^m \qquad \text{(no gravity)}. \tag{6.26}$$

The special, locally free-falling derivatives obey an algebra:

This makes sense: we now should see the effect of gravity, since we are comparing local inertial frames at (infinitesimally) different points.

$$[\partial_m, \partial_n] \neq 0\ ! \tag{6.27}$$

Mathematically speaking: the *Lie bracket* does not vanish.

Because:

$$\begin{aligned}
[\partial_m, \partial_n] &= e_m^\mu \partial_\mu (e_m^\nu \partial_\nu) - (m \leftrightarrow n) && (6.28)\\
&= e_m^\mu e_n^\nu \cancelto{0}{\partial_\mu \partial_\nu} + e_m^\mu (\partial_\mu e_n^\nu)\partial_\nu - (m \leftrightarrow n) && (6.29)\\
&= \left[(\partial_m e_n^\mu) - (\partial_n e_m^\mu)\right] e_\mu^\rho \partial_\rho && (6.30)
\end{aligned}$$

with $[\partial_\mu, \partial_\nu] = 0$.
We see, that the notation ∂_m or ∂_n is slightly misleading. They do not commute and are a **non-coordinate** basis[1], as are e.g. spherical derivatives.

[1] They don't satisfy $[\partial_\mu, \partial_\nu] = 0$

Example:
Orthonormal basis:

$$\begin{pmatrix} e_r \\ e_\theta \\ e_\phi \end{pmatrix} = \begin{pmatrix} \partial_r \\ \frac{1}{r}\partial_\theta \\ \frac{1}{r\sin\theta}\partial_\phi \end{pmatrix} \tag{6.31}$$

which are spherical coordinates in $\mathbb{R}^3, (r, \theta, \phi)$.

Commuting r and θ: using $f(r,\theta,\phi)$

$$(e_r e_\theta - e_\theta e_r)f = \partial_r(\frac{1}{r}\partial_\theta f) - \frac{1}{r}\partial_\theta(\partial_r f) \tag{6.32}$$

$$= -\frac{1}{r^2}\frac{\partial f}{\partial \theta} = -\frac{1}{r}e_\theta f. \tag{6.33}$$

Euclidean basis vectors (e_x, e_y, e_z) commute

$$[\partial_i, \partial_j] = 0 \quad \text{(coordinate basis).} \tag{6.34}$$

6.3 Dirac spinor and gravity

We will now use our formalism to generate gravitational interactions of a Dirac spinor:

$$\mathcal{L}_D = \frac{i}{2}\cdot\overline{\psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \psi \tag{6.35}$$

$$= \frac{1}{2}\overline{\psi}i\partial\!\!\!/\psi - \frac{1}{2}\overline{\psi}i\overleftarrow{\partial\!\!\!/}\psi \tag{6.36}$$

where $\overleftarrow{\partial\!\!\!/}$ acts on an object on the left.

Lorentz transformations act on Dirac spinors, which are $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ of $SO(1,3)$ via

$$\psi \to U\psi = \exp\left(\frac{i}{2}\epsilon^{\mu\nu}\sigma_{\mu\nu}\right)\psi \tag{6.37}$$

Note, we are using a slightly different convention from QFT1 for the Dirac spinor here. In QFT1, we had

$$D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right),$$

with

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

with $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$ and $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

In our special coordinate system at each space-time point P, the Dirac equation holds and is Lorentz invariant. Hence at P, with favored coordinates ξ^m, the invariance group is

Do not confuse the transformation parameters ϵ^m_n with the vierbeins e^m_μ!

$$\begin{cases} \xi^m \to \xi^m + \epsilon^m(x) & \text{Translations} \\ \xi^m \to \epsilon^m_n(x)\xi^n & \text{Lorentz} \\ \psi \to \exp(\frac{i}{2}\epsilon^{mn}(x)\sigma_{mn})\psi \equiv U(x)\psi & \text{Lorentz} \end{cases} \tag{6.38}$$

where the transformation parameters ϵ^m_n and ϵ^m must depend on the point P and its coordinate label x^μ, since our favored coordinate system varies from point to point.

To generalize the Dirac equation to include gravitational interactions, we must therefore require **local** invariance under Lorentz transformation!

Luckily we already know how to promote global symmetries to local redundancies: we use Yang-Mills techniques!

Special relativity:

$$\begin{cases} \psi \to U\psi \\ \partial_\mu\psi \to \Lambda_\mu{}^\nu U\partial_\nu\psi \end{cases} \tag{6.39}$$

These are **global** transformations.

How can we make them local?

We define a new covariant derivative:

The analogy to Yang-Mills should be obvious.

$$D_m \equiv e^\mu_m(\partial_\mu + i\omega_\mu) \tag{6.40}$$

and require that it transforms under local Lorentz transformations, see Eq. (6.38), the same way as under global transformations.
We require

$$D_m\psi \to \Lambda_m{}^n U(x) D_n \psi. \tag{6.41}$$

With Eq. (6.41) we can define

$$\mathcal{L} = i\bar{\psi}\gamma^m D_m \psi, \tag{6.42}$$

which now includes gravity.
The defining equation is Eq. (6.40).

$$D_m \to D'_m = \Lambda_m{}^n \underset{\substack{\uparrow \\ \text{matrix} \\ \text{in spinor space}}}{U(x)} \overset{\substack{\text{also has spinor index} \\ \downarrow}}{D_n} U^\dagger(x). \tag{6.43}$$

This is satisfied if

See Yang-Mills!

$$\omega_\mu \to \omega'_\mu = U\omega_\mu U^\dagger - iU(\partial_\mu U^\dagger) \tag{6.44}$$

and

$$e^\mu_m \to e^\mu_m{}' = \Lambda_m{}^n e^\mu_n. \tag{6.45}$$

Check:

$$\begin{aligned} D_m\psi &\to \Lambda_m{}^n D'_n U(x)\psi(x) \\ &= \Lambda_m{}^n e^\mu_n(\partial_\mu + i\omega'_\mu)U(x)\psi(x) \\ &= e^\mu_n{}'(\cancel{(\partial_\mu U)}\psi + U(\partial_\mu\psi) + iU\omega_\mu\psi + \cancel{U(\partial_\mu U^\dagger)}U\psi) \\ &= e^\mu_n{}' U(x)(\partial_\mu + i\omega_\mu)\psi(x), \end{aligned}$$

where we used $U^\dagger U = \mathbb{1}$ and $\partial_\mu(U^\dagger U) = 0$ and therefore

$$\to (\partial_\mu U^\dagger)U = -U^\dagger\partial_\mu U.$$

Expand $\omega_\mu(x)$ in terms of representation of Lorentz algebra for spinors:

$$\omega_\mu(x) \equiv \frac{1}{2}\omega_\mu{}^{mn}(x)\sigma_{mn} \tag{6.46}$$

This is in analogy to **connection** being spanned by elements of algebra.
Local $SO(1,3)$ transformations have therefore a **covariant** derivative for **spinors**:

$$D_p \equiv e^\mu_p(x)\left[\partial_\mu + \frac{i}{2}\omega^{mn}_\mu(x)\sigma_{mn}\right]. \tag{6.47}$$

D_p is a matrix in 4×4 space of Dirac spinors. The fields $\omega^{mn}_\mu(x)$ are the analogues of the Yang-Mills fields and the matrices σ_{mn} generate the action of $SO(1,3)$ on Dirac spinors.

We have suppressed the spinor indices.

For an arbitrary representation (V_m vector, T_{mn} tensor, ...) we will have:

$$D_p = e^\mu_p(x)\left[\partial_\mu + \frac{i}{2}\omega^{mn}_\mu\mathbf{X_{mn}}\right] \tag{6.48}$$

where $\mathbf{X_{mn}}$ are **SO(1,3)** generators in the representation of the field, whose indices we have suppressed.[2] The $\mathbf{X_{mn}}$ satisfy the Lorentz algebra:

$$[X_{mn}, X_{pq}] = -i\eta_{mp}X_{nq} + i\eta_{np}X_{mq} - i\eta_{nq}X_{mp} + i\eta_{mq}X_{np}, \quad (6.49)$$

which we have encountered in QFT1 already.

[2] As we did for the spinor indices of σ_{mn}.

6.4 The Dirac Lagrangian with gravity

We can now couple Dirac fields to gravity with

$$\mathcal{L}_D = \frac{i}{2}\bar{\psi}\,\gamma^p e^\mu_p(\partial_\mu + \frac{i}{2}\omega_\mu^{mn}\sigma_{mn})\,\psi + \text{h.c.} \quad (6.50)$$

What do we get if we evaluate the commutator $[D_p, D_q] =?$ We will now show that we will get two contributions

Recall, that it gave us the field strength in Yang-Mills theories.

$$[D_m, D_n] = \frac{i}{2}\underset{①}{\underset{\uparrow}{R_{mn}{}^{pq}}}X_{pq} + \underset{②}{\underset{\uparrow}{S^p_{mn}}}D_p \quad (6.51)$$

① Curvature tensor (Riemann in special frame)
② Torsion tensor (vanishes in pure Einstein or for only scalar field sources).
Both $R_{mn}{}^{pq}$ and S^p_{mn} have honest covariant transformations.

Now show Eq. (6.51). We first evaluate

The first term will contribute to the torsion

$$D_m D_n = e^\mu_m(D_\mu e^\rho_n)D_\rho + e^\mu_m e^\rho_n D_\mu D_\rho \quad (6.52)$$

and with

$$D_\mu e^\rho_n = \partial_\mu e^\rho_n + \frac{i}{2}\omega_\mu{}^{pq}X_{pq}\cdot e^\rho_n. \quad (6.53)$$

The $SO(1,3)$ generator X_{pq} is acting on "n"-index.
We know that for a Lorentz-vector the generator has to satisfy, see Eq. (6.54),

Reminder of **Lorentz-vector** transformations. We start with $u^\mu v_\mu = \eta_{\mu\nu}u^\mu v^\nu$ which is Lorentz-invariant. This requires

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu_\nu + \ldots \quad |\omega| \ll 1 \quad (6.54)$$

with $\omega_{\mu\nu} = -\omega_{\nu\mu}$.
Because:

$$\eta_{\mu\nu}u^\mu v^\nu \overset{!}{=} \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta u^\alpha u^\beta \quad (6.55)$$

$$\eta_{\alpha\beta} \overset{!}{=} \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta, \quad (6.56)$$

plugging in Eq. (6.54) gives

$$\eta_{\mu\nu} = \eta_{\mu\nu} + \underset{\overset{!}{=}0\ \Rightarrow\ \omega_{\mu\nu}=-\omega_{\nu\mu}}{\underset{\uparrow}{(\omega_{\mu\nu}+\omega_{\nu\mu})}} + \ldots \quad (6.57)$$

$$\frac{i}{2}(\omega_\mu)^{mn}(X_{mn})\cdot a_q \overset{!}{=} (\omega_\mu)_{qn}\; a^n \quad (6.58)$$

This is satisfied for[3]

$$(X_{pq})\cdot a_n = i\eta_{qn}a_p - i\eta_{pn}a_q \quad (6.59)$$

We also know that (greek indices!)

$$\begin{aligned} D_\mu D_\rho &= (\partial_\mu + \frac{i}{2}\omega_\mu{}^{mn}X_{mn})(\partial_\rho + \frac{i}{2}\omega_\rho{}^{pq}X_{pq}) \\ &= \frac{i}{2}\partial_\mu\omega_\rho^{mn}X_{mn} - \frac{1}{4}\omega_\mu{}^{mn}\omega_\rho{}^{pq}X_{mn}X_{pq} \\ &\quad + (\text{derivatives } \ \partial_\rho, \partial_\mu \text{ acting on } \to) \end{aligned}$$

[3] **Check:**

$$\begin{aligned} \frac{1}{2}(\omega_\mu)^{pq}(X_{pq})\cdot a_n &= \frac{1}{2}(\omega_\mu)^{pq}(i\eta_{qn}a_p - \eta_{pn}a_q) \\ &= \frac{i}{2}(\omega_\mu)^p{}_n a_p - \frac{i}{2}(\omega_\mu)_n{}^q a_q \\ &= -i(\omega_\mu)_{np}a^p. ✓ \end{aligned}$$

where the term in the brackets vanishes once we evaluate the commutator – as in the determination of $[D_\mu, D_\nu] = iF_{\mu\nu}$.
Using all this in Eq. (6.52) we see that with Eq. (6.49) for the commutator $[X_{mn}, X_{pq}]$, we finally get for Eq. (6.51),

$$[D_m, D_n] = \frac{i}{2} R_{mn}{}^{pq} X_{pq} + S^p_{mn} D_p$$

the form of the coefficients, as the *Riemann tensor*

$$\begin{aligned} R_{mn}{}^{pq} &= e^\mu_m e^\rho_n \left[\partial_\mu \omega_\rho{}^{pq} - \partial_\rho \omega_\mu{}^{pq} - \omega_\mu{}^{rp} \omega_{\rho r}{}^q + \omega_\rho{}^{rp} \omega_{\mu r}{}^q\right] \\ &= (e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)(\partial_\mu \omega_\rho{}^{pq} - \omega_\mu{}^{rp} \omega_{\rho r}{}^q) \end{aligned} \tag{6.60}$$

and the *Torsion tensor*

$$S_{mn}{}^p = e^p_\rho (e^\mu_m D_\mu e^\rho_n - e^\mu_n D_\mu e^\rho_m) \tag{6.61}$$
$$= \left[e^\mu_m (\partial_\mu e^\rho_n + \omega_{\mu n}{}^q e^\rho_q) - (m \leftrightarrow n)\right] e^p_\rho \tag{6.62}$$
$$= e^p_\rho (e^\mu_m \partial_\mu e^\rho_n - e^\mu_n \partial_\mu e^\rho_m) + e^\mu_m \omega_{\mu n}{}^p - e^\mu_n \omega_{\mu m}{}^p. \tag{6.63}$$

The torsion and the curvature transform covariantly since they have only latin indices.

What are the components in these quantities? We will discuss the physical degrees of freedom later.
Two invariants

$$R^\star = \epsilon^{mnpq} R_{mnpq}$$
$$R = R_{mn}{}^{mn} \qquad \text{scalar curvature}$$

where the last invariant is also called *Ricci scalar*.
We have introduced

$$\underset{\substack{\uparrow \\ \mu}}{4} \times \overset{\substack{\mathrm{m} \\ \downarrow}}{4} = 16 \quad \text{vierbeins} \quad e^\mu_m(x) \tag{6.64}$$

and

$$\underset{\substack{\uparrow \\ \mu}}{4} \times \overset{\substack{\text{antisymmetric } 4\times 4 \\ \downarrow}}{6} = 24 \quad \text{connections} \quad \omega_\mu{}^{mn}(x). \tag{6.65}$$

We must now indicate the dynamics obeyed by these degrees of freedom. As for the gauge field four-vector, we do not expect all of these fields to correspond to physical degrees of freedom.

As you might remember from QFT1, massless states are characterized by their helicity $\sigma = \pm|\sigma|$, which implies two helicity states. This means that the graviton, a massless spin 2 particle will have 2 physical degrees of freedom.

6.5 *Action for gravity*

We could build the kinetic term in analogy with Yang-Mills as

$$\int d^4\xi \; R_{mn}{}^{pq} R^{mn}{}_{pq}. \tag{6.66}$$

This is invariant and has dimension −4 for the Lagrangian, just like Y-M, but this does not give the right answer.[4]

Gravity, unlike Yang-Mills, has a fundamental **dimensionful** constant

$$G = 6.674 \times 10^{-11} \frac{\mathrm{m}^3}{\mathrm{kg\ s}^2}$$

or

$$G = \frac{1}{8\pi M_{Pl}^2}$$

with $M_{Pl} = 2.435 \times 10^{18}$GeV (Planck-Mass).

The simplest action should include G. We have two options

$$\int d^4\xi\, \epsilon^{mnpq} R_{mnpq} \quad \text{and} \quad \int d^4\xi\, R_{mn}{}^{mn}.$$

The Einstein action is the 2[nd] one:

$$S_E = \frac{1}{16\pi G} \int d^4x\, E R_{mn}{}^{mn} \tag{6.67}$$

with $E = \det(\mathrm{e}^{\mathrm{m}}_{\mu})$, which is the Jacobian of local inertial to general coordinates.

The EOMs can be derived by varying the vierbeins and the connections.

$$\delta S_E = \delta_e S_E + \delta_\omega S_E = 0 \tag{6.68}$$

Recall the form of the Riemann tensor

$$R_{mn}{}^{mn} = (e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)(\partial_\mu \omega_\rho{}^{mn} - \omega_\mu{}^{rm}\omega_{\rho r}{}^n) \tag{6.69}$$

and vary the connections

$$\begin{aligned}
\delta_\omega S_E &= \frac{1}{16\pi G} \int d^4x\ E\delta R \qquad (6.70)\\
&= \frac{1}{16\pi G} \int d^4x \cdot \delta\omega_\rho{}^{mn} \left[-\partial_\mu (E(e^\mu_m e^\rho_n - e^\rho_m e^\mu_n)) + \right. \qquad (6.71)\\
&\quad + E(e^\rho_m e^\mu_q - e^\mu_m e^\rho_q)\omega_{\mu n}{}^q\\
&\quad \left. + E(e^\mu_q e^\rho_n - e^\rho_q e^\mu_n)\omega_\mu{}^q{}_m \right].
\end{aligned}$$

We get that the variation has the form of a covariant derivative of a Lorentz two-tensor

$$D_\mu \left[E(e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)\right] = 0. \tag{6.72}$$

We can solve this for the connections:

$$\omega_\mu{}^{mn} = \frac{1}{2} e^q_\mu [T_q^{mn} - T^{mn}{}_q - T^n{}_q{}^m] \tag{6.73}$$

with

$$T^q{}_{mn} = (e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)\partial_\rho e^q_\mu. \tag{6.74}$$

[4] It leads to physical ghosts – positive energy states with negative norm which can be produced from physical states. At classical level, theories with higher derivative suffer the Ostrogradski instability: the Hamiltonian is not bounded from below. This doesn't prevent people from trying, see e.g. https://arxiv.org/pdf/1403.4226.pdf

We see that the connections are fully determined by e^μ_n! We can even show that in vacuum

$$S_{mn}{}^q \equiv 0 \quad \text{(Torsion)}. \tag{6.75}$$

$\omega_\mu{}^{mn} \to$ just auxiliary fields (like A^0 in QED).
See exercise!

Further

$$R_{mn}{}^{pq} = -R_{nm}{}^{pq} \tag{6.76}$$
$$R_{mn}{}^{pq} = -R_{mn}{}^{qp} \tag{6.77}$$
$$R_{mnpq} = R_{pqmn} \tag{6.78}$$

Let's vary with respect to e^μ_m:

$$\delta_e S_E = \frac{1}{16\pi G}\int d^4x\ [\delta E R_{mn}{}^{mn} + E(\partial_\mu \omega_\rho{}^{mn} - \omega_\mu{}^{rm}\omega_{\rho r}{}^n) \times \delta(e^\mu_m e^\rho_n - e^\mu_n e^\rho_m)]\,. \tag{6.79}$$

We need the variation of the determinant of vierbeins $E = \det(e^\mu_r)$. [5]

[5]
$$\begin{aligned}\delta\, \det M &= \det(M+\delta M) - \det(M)\\ &= e^{\mathrm{Tr}\ln(M+\delta M)} - e^{\mathrm{Tr}\ln(M)}\\ &= e^{\mathrm{Tr}\ln(M(1+M^{-1}\delta M))} - e^{\mathrm{Tr}\ln(M)}\\ &= e^{\mathrm{Tr}\ln M + \mathrm{Tr}\ln(1+M^{-1}\delta M)} - e^{\mathrm{Tr}\ln(M)}\\ &= e^{\mathrm{Tr}\ln M + \mathrm{Tr}(M^{-1}\delta M)} - e^{\mathrm{Tr}\ln(M)}\\ &= e^{\mathrm{Tr}\ln M}(1+\mathrm{Tr}(M^{-1}\delta M)+\ldots) - e^{\mathrm{Tr}\ln(M)}\\ &= e^{\mathrm{Tr}\ln M}\mathrm{Tr}(M^{-1}\delta M)\\ &= \det(M)\mathrm{Tr}(M^{-1}\delta M) \quad (6.80)\end{aligned}$$
where we need $\ln(1+x) = x + \ldots$.

We can apply Eq. (6.80) to our problem. So we get

$$\delta E = E e^\mu_m \delta e^m_\mu\,. \tag{6.81}$$

Further we will use

$$\delta e^\rho_m = -e^\mu_m e^\rho_n \delta e^n_\mu, \tag{6.82}$$

$\delta_e S = 0$:

$$0 = E e^\mu_m R_{ab}{}^{ab}\delta e^m_\mu + E\ B^{mn}_{\mu\rho}\underbrace{\delta(e^\mu_m e^\rho_n)}_{①} \tag{6.83}$$

which defines $B^{mn}_{\mu\rho}$ comparing to Eq. (6.79). We need the following relation

$$\delta e^\mu_m = -e^\lambda_m e^\mu_q \delta e^q_\lambda \tag{6.84}$$

where we used the relation $e^\mu_n e^n_\rho = \delta^\mu_\rho$. and $B^{mn}_{\mu\rho} = -B^{mn}_{\rho\mu}$, see Eq. (6.60).
So we get

$$\begin{aligned} ① &= (\delta e^\mu_m e^\rho_n + e^\mu_n \delta e^\rho_n) B^{mn}_{\mu\rho}\\ &= -(e^\lambda_m e^\mu_q e^\rho_n \delta e^q_\lambda + e^\lambda_n e^\rho_q e^\mu_m \delta e^q_\lambda) B^{mn}_{\mu\rho}\\ &= -\underbrace{e^\mu_q e^\rho_n B^{mn}_{\mu\rho}}_{R_{qn}{}^{mn} \text{ with } Eq.\ (6.60)} e^\lambda_m \delta e^q_\lambda - \underbrace{e^\rho_q e^\mu_n B^{nm}_{\mu\rho}}_{R_{qn}{}^{nm} \text{ with } Eq.\ (6.60)} e^\lambda_m \delta e^q_\lambda\\ &= -2R_{qn}{}^{mn} e^\lambda_m \delta e^q_\lambda \quad \text{since } R_{qn}{}^{mn} = -R_{qn}{}^{nm}. \end{aligned}$$

and therefore

$$\delta_e S = 0 \ \Leftrightarrow\ 0 = (R\delta^m_q - 2R_{qn}{}^{mn}) e^\lambda_m \delta e^q_\lambda$$

or

$$R_{qn}{}^{mn} - \frac{1}{2}\delta^m_q R = 0. \tag{6.85}$$

which is the Einstein EOM.

6.5.1 Coupling gravity to matter

We can couple to matter via

$$\delta S_M \equiv \int d^4x\, E\left[\frac{1}{2}\delta e_\mu^m T^\mu{}_m + \delta\omega_\mu{}^{mn} C_{mn}{}^\mu\right], \tag{6.86}$$

which defines $T^\mu{}_m$ and $C_{mn}{}^\mu$ as the sources which will appear on the RHS of the EOM. The full Einstein equations are:

$$R_{mn}{}^{pn} - \frac{1}{2}\delta_m^p\, R = 8\pi G\, T^p{}_m \tag{6.87}$$

where $T^p{}_m$ is the **energy-momentum tensor** of matter

Similarly, varying connections, we get

$$D_\mu\left[E(e_m^\mu e_n^\rho - e_n^\mu e_m^\rho)\right] = 16\pi G\, C_{mn}{}^\rho. \tag{6.88}$$

We can repeat the above procedure and solve for connections as a function of e_m^μ and $C_{mn}{}^\rho$.

6.5.2 Relation to usual formation:

Our goal is now to connect to the usual formulation of gravity using only "greek" indices, i.e. the general coordinates.
We proceed algebraically. Consider $e_\mu^m D_\rho v_m$, where $v_m = e_m^\mu v_\mu$ is a four-vector:

$$e_\mu^m D_\rho v_m = e_\mu^m(\partial_\rho v_m + \omega_{\rho m}{}^n v_n) \tag{6.89}$$

We get for Eq. (6.89):

$$e_\mu^m D_\rho v_m = \partial_\rho v_\mu + \underbrace{\left[e_\mu^m \partial_\rho e_m^\sigma + e_\mu^m \omega_{\rho m}{}^n e_n^\sigma\right]}_{\equiv \Gamma^\sigma{}_{\mu\rho} = e_\mu^m D_\rho e_m^\sigma} v_\sigma. \tag{6.90}$$

We calculate $\partial_\rho v_m$:
$\partial_\rho v_m = \partial_\rho(e_m^\sigma v_\sigma) = e_m^\sigma(\partial_\rho v_\sigma) + (\partial_\rho e_m^\sigma)v_\sigma$

This $\Gamma^\sigma{}_{\mu\rho} = e_\mu^m D_\rho e_m^\sigma$ is **not yet** the usual Christoffel symbol.
Define a covariant derivative:

$$\nabla_\rho v_\mu \equiv \partial_\rho v_\mu + \Gamma^\sigma{}_{\rho\mu} v_\sigma. \tag{6.91}$$

This is the usual covariant derivative of general relativity.[6] We can generalize this to tensors with latin **and** greek indices. Start with a general two-Tensor in free-fall coordinates:

[6] See e.g. Weinberg – Gravitation and Cosmology.

$$e_\mu^m D_\rho(T_{mn}) = e_\mu^m[\partial_\rho T_{mn} + \omega_{\rho m}{}^q T_{qn} + \omega_{\rho n}{}^q T_{mq}] \tag{6.92}$$

T_{mn} has two latin indices in SR-frame. Now use

$$T_{mn} = e_m^\sigma T_{\sigma n}, \tag{6.93}$$

which we plug into Eq. (6.92):

$$= e_\mu^m[\partial_\rho(e_m^\sigma T_{\sigma n}) + \omega_{\rho m}{}^q T_{\sigma n} e_q^\sigma + \omega_{\rho n}{}^q T_{\sigma q} e_m^\sigma)] \tag{6.94}$$

$$= \partial_\rho T_{\mu n} + T_{\sigma n} e_\mu^m \partial_\rho e_m^\sigma + \omega_{\rho m}{}^q T_{\sigma n} e_q^\sigma + \omega_{\rho n}{}^q T_{\sigma q} e_m^\sigma. \tag{6.95}$$

Use

$$e^m_\mu \partial_\rho e^\sigma_m \overset{\text{Def. } \Gamma^\sigma{}_{\mu\rho}}{=} -e^m_\mu \omega_{\rho m}{}^q e^\sigma_q + \Gamma^\sigma{}_{\mu\rho} \tag{6.96}$$

We get

$$\partial_\rho T_{\mu n} - \cancel{T_{\sigma n} e^m_\mu \omega_{\rho m}{}^q e^\sigma_q} + \cancel{\omega_{\rho m}{}^q e^\sigma_q T_{\sigma n} e^\sigma_q e^m_\mu} + e^m_\mu \omega_{\rho n}{}^q T_{\sigma q} e^\sigma_m + \Gamma_{\mu\rho}{}^\sigma T_{\sigma n} \tag{6.97}$$

which we identify with

$$\boxed{\nabla_\rho T_{\mu n} = \partial_\rho T_{\mu n} + \Gamma_{\mu\rho}{}^\sigma T_{\sigma n} + \omega_{\rho n}{}^q T_{\mu q}.} \tag{6.98}$$

which has the expected form: a Christoffel symbol for the greek index and the free falling connection for the latin index. This formula is useful once we apply it to the vierbein with $T_{\mu p} = \eta_{pq} e^q_\mu$:

$$\begin{aligned}
\nabla_\rho e_{\mu p} &= \partial_\rho e_{\mu p} + \omega_{\rho p}{}^q e_{\mu q} + \Gamma^\sigma_{\mu\rho} e_{\sigma p} && (6.99)\\
&= \partial_\rho e_{\mu p} + \omega_{\rho p \mu} + e^m_\mu (\partial_\rho e^\sigma_m) e_{\sigma p} + e^m_\mu \omega_{\rho m}{}^n e^\sigma_n e_{\sigma p} && (6.100)\\
&= \partial_\rho e_{\mu p} + \omega_{\rho p \mu} - e^m_\mu e^\sigma_m \partial_\rho e_{\sigma p} + \omega_{\rho \mu p} && (6.101)\\
&= \partial_\rho e_{\mu p} + \omega_{\rho p \mu} - \partial_\rho e_{\mu p} - \omega_{\rho p \mu}, && (6.102)\\
&= 0 && (6.103)
\end{aligned}$$

where we used the Christoffel-symbol from Eq. (6.90). We see that the covariant derivative of vierbeins vanishes identically

$$\nabla_\rho e_{\mu p} = 0. \tag{6.104}$$

The metric satisfies therefore:

$$\boxed{\nabla_\rho \, g_{\mu\nu} = 0.} \tag{6.105}$$

where $g_{\mu\nu} = \eta^{mn} e^\mu_m e^\nu_n$. This requires 40 equations to be solved. These equations define implicit restrictions on the metric made in our formalism – not surprising since we are only considering spaces which can be mapped at each point in space-time, via the vierbein, into the flat space-time of special relativity.

because $4 \times 10 = 40$ since ρ runs from $1 \ldots 4$ and a symmetric 4×4 tensor has 10 components.

Quoting some results in the standard notation (using “greek” tensors).TODO: CHECK CHRISTOFFEL CONVENTION.

$$[\nabla_\nu, \nabla_\mu] \, T_\rho = S_{\nu\mu}{}^\sigma \nabla_\sigma T_\rho + R_{\nu\mu}{}^\lambda{}_\rho T_\lambda \tag{6.106}$$

with $S_{\nu\mu}{}^\sigma = e^m_\nu e^n_\mu e^\sigma_q S_{mn}{}^q$ and $S_{\nu\mu}{}^\sigma = \Gamma_{\mu\nu}{}^\sigma - \Gamma_{\nu\mu}{}^\sigma$ (antisymmetry of greek connection). It can be shown that the Riemann tensor takes the form:

$$R_{\nu\mu}{}^\lambda{}_\rho = \partial_\nu \Gamma^\lambda{}_{\rho\mu} + \Gamma^\sigma{}_{\rho\nu} \Gamma^\lambda{}_{\sigma\mu} - (\mu \leftrightarrow \nu). \tag{6.107}$$

This is exactly the result of the usual formalism, except for $\Gamma_{\mu\nu}{}^\sigma \neq \Gamma_{\nu\mu}{}^\sigma$. In the absence of matter that can contribute to torsion

$$S_{mn}{}^p \equiv 0 \tag{6.108}$$

and the $\Gamma_{\mu\nu}{}^\sigma$ are the usual Christoffel symbols, meaning they are symmetric

$$\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu} \qquad \text{for} \qquad S_{mn}{}^p = 0$$

6.6 Gravity and gauge theories

Covariant derivative for local SO(1,3) and an internal gauge group G with gauge fields A_μ

$$D_m = e^\mu_m(\partial_\mu + \frac{i}{2}\omega_\mu{}^{mn}X_{mn} + iA_\mu) \tag{6.109}$$

The commutator is

$$[D_m, D_n] = S_{mn}{}^q D_q + \frac{i}{2}R_{mn}{}^{pq}X_{pq} + iF_{mn} \tag{6.110}$$

which gives for a $G = U(1)$-symmetry:

$$F_{mn} = e^\mu_m e^\rho_n(\partial_\mu A_\rho - \partial_\rho A_\mu) \tag{6.111}$$

compared to a naive guess $D_m A_n - D_n A_m$. We see that we need to apply the equivalence principle to gauge-covariant objects (like the field strength).

Fermions give a contribution to the torsion S:

$$S_{Dirac} = \frac{1}{2}\int d^4x E\,\overline{\psi}\gamma^p e^\mu_p(\partial_\mu + i\omega_\mu{}^{mn}\sigma_{mn})\psi. \tag{6.112}$$

This leads to a non-vanishing torsion:

$$C^\mu_{mn} = \frac{i}{4}Ee^\mu_p\overline{\psi}\gamma^p\sigma_{mn}\psi. \tag{6.113}$$

as you can directly see by varying w.r.t. to the connections $\omega_\mu{}^{mn}$.

6.7 Spin 2 fields and the QFT of gravity

We have encountered spin/helicity $0, 1/2, 1$. Massless fields with long-range forces can be at most of Spin 2 because of **soft-theorems** by Weinberg. The existence of massless spin 1 (like the photon) implies charge conservation (using only Lorentz and locality).
Further, a spin 2 particle with $m = 0$ couples **universally** to matter (like gravity!).

Spin 3/2 is important in supersymmetry (gravitino), but we will skip it here.

6.7.1 Massive Spin 2 fields

In the gravity chapter we found that dynamical field of pure gravity is $e^\mu_M(x)$ (vierbein) or $g_{\mu\nu} = e^M_\mu e^N_\nu \eta_{MN}$(metric).
The metric is traditionally used, an alternative would be e^μ_M (Palatini formalism). We will focus on $g_{\mu\nu}$ here.
Because of $g_{\mu\nu} = g_{\nu\mu}$ and the fact the metric is real we have 10 entries in 4D.

What is the corresponding Poincaré-representation?

Vector field: A_μ massive vector field

$$(1/2, 0) \otimes (0, 1/2) = (1/2, 1/2) = A_\mu$$

The *little group* is the invariance group of the reference momentum $p_\mu = (m, 0, 0, 0)$ and we recall from QFT1

$$4 \overset{\substack{\text{Under little}\\ \text{group SO(3)}}}{=} \overset{\substack{\text{Scalar}\\ \downarrow}}{1} \oplus \underset{\substack{\uparrow\\ \text{Vector}}}{3} \tag{6.114}$$

$g_{\mu\nu}$ is the symmetric part of two vectors, so use

$$(1/2, 1/2) \otimes (1/2, 1/2) = B_{\mu\nu}. \tag{6.115}$$

It is a Lorentz-tensor with 16 real entries, but it is reducible. An easy way to find the irreps of tensor representations is to use the fact that they are eigenstates of the permutation operator. The well-known spin multiplication gives $1/2 \otimes 1/2 = 0_A \oplus 1_S$. We get for Eq. (6.115) the following decomposition:

$$(1/2, 1/2) \otimes (1/2, 1/2) = \underbrace{(0,0) \oplus (1,1)}_{\substack{\text{Symmetric tensor}\\ h_{\mu\nu}=h_{\nu\mu}\\ \text{with Tr}(h_{\mu\nu})=(0,0)}} \oplus \underbrace{(1,0) \oplus (0,1)}_{\substack{\text{Antisymmetric tensor}\\ \text{self-dual 6 dof}\\ \mathbf{F}_{\mu\nu}\text{or } \tilde{\mathbf{F}}_{mn}}} \tag{6.116}$$

Let us consider the little group SO(3) decomposition of $h_{\mu\nu}$:

$$\underset{\substack{\uparrow\\ 10}}{h_{\mu\nu}} \overset{SO(3)}{=} \overset{\substack{h_{00}\\ \downarrow}}{1} \oplus \underset{\substack{\uparrow\\ \text{Tr}(\text{h}_{\text{ij}})}}{1} \oplus \overset{\substack{h_{0i}\\ \downarrow}}{3} \oplus \underset{\substack{\uparrow\\ \text{traceless}\\ \text{part of}\\ h_{ij}}}{5} \tag{6.117}$$

Check comparing dimensions:

$$\text{Symmetric:} \begin{pmatrix} 1 & 5 & 6 & 7 \\ & 2 & 8 & 9 \\ & & 3 & 10 \\ & & & 4 \end{pmatrix}$$

$$= 10 = \underset{\substack{\uparrow\\ (0,0)}}{1^{\text{S}}_{\text{trace}}} + \overset{\substack{(1,1)\\ \downarrow}}{9^{\text{S}}_{\text{traceless}}}$$

$$\text{Antisymmetric:} \begin{pmatrix} 0 & 1 & 2 & 3 \\ & 0 & 4 & 5 \\ & & 0 & 6 \\ & & & 0 \end{pmatrix}$$

$= 6^{\text{A}}$. ✓

We know that a **massive** spin-s field has $(2s + 1)$ degrees of freedom, so h_{ij} would offer themselves as canonical coordinates for a massive $s = 2$ particle. We have been able to argue that in the analogous case of a massive spin-1 that only A_i are physical since $\pi_0 = \frac{\partial \mathcal{L}}{\partial \partial_t A_0}$ vanishes. Let us derive this in a different way before applying this to spin 2.

6.7.2 Longitudinal fields and spin 1

We can write any vector field $A_\mu(x)$ as

$$A_\mu(x) = \overset{\substack{\text{Transverse}\\ \downarrow}}{A^T_\mu(x)} + \underset{\substack{\uparrow\\ \text{Longitudinal}}}{\partial_\mu \pi(x)} \tag{6.118}$$

with

$$\partial_\mu A^T_\mu \equiv 0. \tag{6.119}$$

This split is **not** unique, since we can shift

$$A^T_\mu \to A^T_\mu + \partial_\mu \alpha(x) \tag{6.120}$$

$$\pi \to \pi - \alpha(x). \tag{6.121}$$

But we can always pick split such that $\partial_\mu A_\mu^T \equiv 0$ (Lorentz gauge). This decomposition is very helpful because it lets us find out whether the non-transverse polarizations are physical by analyzing the Lagrangian. We need therefore the most general Lorentz-invariant Lagrangian up to A_μ^2 and ∂_μ^2:

$$\mathcal{L} = aA_\mu \Box A_\mu + bA_\mu \partial_\mu \partial_\nu A_\nu + m^2 A_\mu A_\mu \tag{6.122}$$

with the substitution:

$$A_\mu = A_\mu^T + \partial_\mu \pi \text{ and } \partial_\mu A_\mu^T = 0. \tag{6.123}$$

So we get

$$\mathcal{L} = aA_\mu^T \Box A_\mu^T + m^2 (A_\mu^T)^2 - (a+b)\pi \Box^2 \pi - m^2 \pi \Box \pi, \tag{6.124}$$

where mixed terms $\sim \partial_\mu \pi \cdot A_\mu^T$ vanish because of Lorentz-gauge (after partial integration).
The theory is problematic for $a + b \neq 0$, because there are **ghosts** ($\leftarrow$ non-unitary!). We see this by looking at the π propagator, which has four-derivative terms $(a+b)\pi\Box^2\pi$! The π propagator is

$$\begin{aligned} \Pi_\pi &= \frac{-1}{2(a+b)k^4 - 2m^2k^2} &(6.125)\\ &= \frac{1}{2m^2}\left[\frac{1}{k^2} \underset{\substack{\uparrow \\ \text{Wrong sign}}}{-} \frac{1}{k^2 - \frac{m^2}{(a+b)}}\right]. &(6.126) \end{aligned}$$

π really represents two fields, one of which has **negative norm** for generic a, b. We can remove the dangerous 4-derivative terms by choosing $a = -b$, which gives the **unique** physical Lagrangian for a massive spin 1 field.

You can also see that you need 4 boundary conditions as initial data to fix classical dynamics $\dddot{\pi}(t_0)$, $\ddot{\pi}(t_0)$, $\dot{\pi}(t_0)$ and $\pi(t_0)$. Compared to 2 boundary conditions for Klein-Gordon: $\dot{\phi}(t_0), \phi(t_0)$.s

With $a = -b = \frac{1}{2}$ and $m^2 \to \frac{1}{2}m^2$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}A_\mu \Box A_\mu - \frac{1}{2}A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2}m^2 A_\mu^2 &(6.137)\\ &= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}m^2 A_\mu^2. &(6.138) \end{aligned}$$

The longitudinal terms $\partial_\mu \pi \sim A_\mu$ get a kinetic term from the mass term.
What happens in $m^2 \to 0$ limit?
The theory is also sick in general: the mode $\partial_\mu \pi$ has interactions, but **no** kinetic term.
Why is this problematic? Imagine

$$\mathcal{L} = Z_\pi \pi \Box \pi + \lambda \pi^3 \tag{6.139}$$

in canonical normalization $\pi_c = \sqrt{Z_\pi}\pi$, we find

$$\mathcal{L} = \pi_c \Box \pi_c + \frac{\lambda}{Z_\pi^{3/2}} \pi_c^3 \tag{6.140}$$

Why are higher derivative terms problematic?
Example:

$$\mathcal{L}_{\Box^2} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{a}{2\Lambda^2}\phi\Box^2\phi - V_{int}(\phi) \tag{6.127}$$

with $a = \pm 1$, Λ is some energy scale.
This system is plagued by ghosts and potential tachyons. We can reduce this to a purely two-derivative Lagrangian to read off the stability properties. Add an auxiliary scalar field χ with

$$\mathcal{L}' = \frac{1}{2}(\partial\phi)^2 - a\chi\Box\phi - \frac{a}{2}\Lambda^2\chi^2 - V_{int}(\phi) \tag{6.128}$$

after integrating out χ, we get back $\mathcal{L}_{\Box^2}$:
Use EOM for χ:

$$\frac{\delta S}{\delta\chi} = 0 \Leftrightarrow \frac{\partial\mathcal{L}}{\partial\chi} = 0 \tag{6.129}$$

$$-a\Box\phi - a\Lambda^2\chi = 0 \tag{6.130}$$

$$\chi = -\frac{\Box\phi}{\Lambda^2} \tag{6.131}$$

$$\tag{6.132}$$

back in $\mathcal{L}'$:

$$\mathcal{L}' = \frac{1}{2}(\partial\phi)^2 + a\frac{\Box\phi}{\Lambda^2}\Box\phi - \frac{a}{2}\Lambda^2\frac{(\Box\phi)^2}{\Lambda^4} - V_{int} \tag{6.133}$$

$$= \frac{1}{2}(\partial\phi)^2 + \frac{a}{2\Lambda^2}\phi\Box^2\phi - V_{int}(\phi) \quad \checkmark \tag{6.134}$$

We can diagonalize the quadratic part with the substitution

$$\phi = \phi' - a\chi \tag{6.135}$$

to get

$$\mathcal{L}' = \frac{1}{2}(\partial\phi')^2 - \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}a\chi^2\chi^2 - V_{int}(\phi',\chi). \tag{6.136}$$

χ has the wrong kinetic term and can also be a tachyon for $a = -1$ (exponentially growing modes). The problem of the wrong sign (kinetic term $\mathcal{L} \supseteq -\frac{1}{2}(\partial\chi)^2$) is that the **Hamiltonian** now is **negative**. When we couple this χ to "ordinary" matter like ϕ' through $V_{int}(\phi',\chi)$ the system is **unstable**: zero energy is required to excite both sectors $\Rightarrow$ instability of the vacuum.

What about $\phi\frac{\Box^2\phi}{\Lambda^2}$ terms in an effective field theory?

The ghost χ in Eq. (6.136) has a mass Λ and will only be relevant at $E \gtrsim \Lambda$ (the $\frac{\phi\Box^2\phi}{\Lambda^2}$ will then start dominating over $\phi\Box\phi$). We can use $\mathcal{L}_{\Box^2}$ for energies **below** Λ and assume that some UV physics with new degrees of freedom enters at Λ and takes care of the instability due to the ghost.

$\mathcal{L}_{\Box^2}$ makes perfect sense as an EFT with cutoff Λ! The harmful effects are deferred to the cutoff.

Taking $Z_\pi \to 0$ leads to infinitely strong interactions!
We need to remove π from interactions, which means that π never appears in the substitution $A_\mu \to A_\mu + \partial_\mu\pi$. This is true if A_μ interactions can be written as

$$\mathcal{L} = \ldots + gA_\mu J^\mu \tag{6.141}$$

with

$$\partial_\mu J_\mu = 0. \tag{6.142}$$

Eq. (6.141) is just another way of deriving the requirement of gauge-invariance for ($m^2 \to 0$) massless spin 1 fields! We can use this method to determine interactions iteratively by starting from $\mathcal{L}_{int}^{try} = g\,A_\mu(i\phi^\dagger\partial_\mu\phi + h.c.) + \ldots$ to show that $|D_\mu\phi|^2$ is the correct term.

See exercise for an application to spin1 and spin2.

6.7.3 *Longitudinal fields and spin 2*

The most general terms $h_{\mu\nu}^2$ and ∂_μ^2 can be written as:

$$\begin{aligned}\mathcal{L} =& a h_{\mu\nu}\Box h_{\mu\nu} + b h_{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\alpha} + c h\Box h\\ &+ d h\partial_\mu\partial_\nu h_{\mu\nu} + m^2(f h_{\mu\nu}h_{\mu\nu} + g h^2)\end{aligned} \tag{6.143}$$

with the trace $h = h^\mu{}_\mu$. We have omitted terms which are related by partial integration.

We decompose as before:

This is also called a **Stückelberg decomposition.**

$$h_{\mu\nu} = \overset{\substack{\text{transverse}\\ \downarrow}}{h^T_{\mu\nu}} + \partial_\mu\pi_\nu + \partial_\nu\pi_\mu. \tag{6.144}$$

and require

$$\partial_\mu h^T_{\mu\nu} \equiv 0 \qquad \text{(4 conditions)}$$

We can further decompose π_μ in transverse and longitudinal parts

$$\pi_\mu = \pi^T_\mu + \partial_\mu\pi^L \tag{6.145}$$

with $\partial_\mu\pi^T_\mu \equiv 0$.

Let us discuss the mass term first (check for 4 derivative terms). Therefore we insert Eq. (6.144) and Eq. (6.145) in Eq. (6.143). So we get

$$m^2(f h_{\mu\nu}^2 + g h^2) = 4m^2(f+g)\pi_L\Box^2\pi_L + \ldots \tag{6.146}$$

First we look at $h_{\mu\nu}^2$:

$$\begin{aligned}h_{\mu\nu}^2 &= (h^T_{\mu\nu} + \partial_\mu(\pi^T_\nu + \partial_\nu\pi^L) + \partial_\nu(\pi^T_\mu + \partial_\mu\pi^L))^2\\ &= h^{T\,2}_{\mu\nu} - 2\pi^T_\nu\Box\pi_\nu + \underbrace{(\partial_\mu\partial_\nu\pi^L)^2 + (\partial_\nu\partial_\mu\pi^L)^2 + 2\partial_\mu\partial_\nu\pi_L\partial_\nu\partial_\mu\pi^L}_{\substack{=4\pi_L\Box^2\pi_L\\ \text{(after IBP)}}}\end{aligned}$$

and now h^2:

$$\begin{aligned}h^2 &= (h^{T\mu}{}_\mu + 2\cancelto{0}{\partial^\mu\pi^T_\mu} + 2\Box\pi^L)^2 &(6.147)\\ &= (h^{T\mu}{}_\mu)^2 + 4\pi^L\Box^2\pi^L + 4h^{T\mu}{}_\mu\Box\pi^L. &(6.148)\end{aligned}$$

Eliminate the dangerous term with $f = -g$ in Eq. (6.146).
Similarly, we find (focus on $\Box^2$ and $\Box^3$ terms)

$$a h_{\mu\nu}\Box h_{\mu\nu} = -2a\pi^T_\nu\Box^2\pi^T_\nu + 4a\pi_L\Box^3\pi_L + \ldots \tag{6.149}$$

and

$$bh_{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\alpha} = -b\partial_\mu h_{\mu\nu}\partial_\alpha h_{\alpha\nu} + \dots \tag{6.150}$$
$$= -b(\Box\pi^T_\nu + 2\Box\partial_\nu\pi^L)^2 + \dots \tag{6.151}$$
$$= -b\pi^T_\nu\Box^2\pi^T_\nu + 4b\pi^2\Box^3\pi_L + \dots \tag{6.152}$$

and

$$ch\Box h = 4c\pi^L\Box^3\pi^L + 4ch^{T\mu}{}_\mu\Box^2\pi^L \tag{6.153}$$

and

$$dh\partial_\mu\partial_\nu h_{\mu\nu} = d(2\Box\pi^L)(2\Box^2\pi^L) + dh^{T\mu}{}_\mu(2\Box^2\pi^L) \tag{6.154}$$

and so in summary for $\Box^2-$ and $\Box^3-$terms:

$$\begin{aligned}\mathcal{L} &= \pi_L\Box^2\pi_L(4m^2(f+g)) + \pi_\nu\Box^2\pi_\nu(-b-2a)\\ &\quad + \pi_L\Box^3\pi_L(4a+4b+4c+4d) + h^{T\mu}{}_\mu\Box^2\pi^L(2d+4c)\\ &\quad + \dots\end{aligned} \tag{6.155}$$

To avoid ghosts/non-unitarity we need these terms to vanish, and so:

$$f+g=0 \quad -b=2a \quad -d=2c \quad b=-d \tag{6.156}$$

The kinetic term is canonically normalized for $a=-\frac{1}{2}$

$$\begin{aligned}-\mathcal{L} &= \frac{1}{2}h_{\mu\nu}\Box h_{\mu\nu} - h_{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\alpha} - \frac{1}{2}h\Box h + h\partial_\mu\partial_\nu h_{\mu\nu}\\ &\quad + \frac{1}{2}m^2(h^2_{\mu\nu} - h^2)\end{aligned}$$

which is the **Unique Lagrangian for the massive spin 2 field** (Fierz/Pauli 1939).
The leading terms in the **massless limit**

$$-\mathcal{L} = \frac{1}{2}h_{\mu\nu}\Box h_{\mu\nu} - h_{\mu\nu}\partial_\mu\partial_\alpha h_{\nu\alpha} - \frac{1}{2}h\Box h + h\partial_\mu\partial_\nu h_{\mu\nu} \tag{6.157}$$

are exactly what is contained in the **Einstein-Hilbert** action:

$$\int d^4x\,\sqrt{-g}RM^2_{Pl} \tag{6.158}$$

if expanded in $h_{\mu\nu}$ with $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{Pl}}h_{\mu\nu}$ and $\eta_{\mu\nu}$ as the Euclidean metric.
Similar as for massless spin 1, the $\dot{\pi}_\mu$ for massless spin 2 should never appear in interactions, e.g.

$$\mathcal{L}_{int} = h_{\mu\nu}T^{\mu\nu} \quad \text{with } T_{\mu\nu} = T_{\nu\mu}. \tag{6.159}$$

To remove π_ν when $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\pi_\nu + \partial_\nu\pi_\nu$ requires

$$\partial_\mu T^{\mu\nu} = 0. \tag{6.160}$$

So $h_{\mu\nu}$ must couple to a **conserved tensor current**. Only one symmetry has this property: the translation symmetry with charge $Q^\nu = P^\nu$

$$P^\nu = \int d^3x\, T^{0\nu}(x). \tag{6.161}$$

Thus the gauge-symmetry corresponds to **local** translations of space-time! [7] This also shows why it is not possible to construct massless fields of higher spin than 2. We would need to couple it to a current associated with a tensorial charge but there is no such charge (or if it existed the theory would be trivial (!) because the S-matrix would be overconstrained in this case.).

Massive fields of higher spin are not problematic and exist in nature, e.g. the QCD resonance $\omega_3(1670\text{MeV})$.

[7] In analogy to spin 1 couplings to conserved current $\partial_\mu J^\mu = 0$ and the gauge symmetry being local redundancy.

See exercise for spin 3.

6.7.4 *Scalar field example*

We can show that this requires the Lagrangian for a scalar field

$$\mathcal{L}_\phi = \sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)) \tag{6.162}$$

and

$$\mathcal{L}_g = M_{Pl}^2\sqrt{-g}R \quad \text{with} \quad g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{Pl}}h_{\mu\nu}.$$

and the transformation on ϕ from a local translation $x^\alpha \to x^\alpha + \pi^\alpha(x)$ is

$$\begin{aligned}\phi(x) &\to \phi(x^\alpha + \pi^\alpha) = \phi(x^\alpha) + \overset{\text{Taylor expansion in } \pi}{\overset{\downarrow}{\pi^\alpha\partial_\alpha\phi(x)}} + \dots \\ h_{\mu\nu} &\to (\eta_{\alpha\mu} + \partial_\alpha\pi_\mu)(\eta_{\beta\nu} + \partial_\beta\pi_\nu)(\eta_{\alpha\beta} + h_{\alpha\beta}\underset{\text{Taylor expansion}}{\underset{\uparrow}{[x^\gamma + \pi^\gamma]}}) - \eta_{\alpha\beta}.\end{aligned}$$

if we require invariance to all orders in π. This $h_{\mu\nu}$ transformation to leading order in π^μ is :

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\pi_\nu + \partial_\nu\pi_\mu + (\partial_\mu\pi^\alpha)h_{\alpha\nu} + (\partial_\nu\pi^\alpha)h_{\mu\alpha} + \pi^\alpha\partial_\alpha h_{\mu\nu}. \tag{6.163}$$

where we can identify a gauge part:

$$\underset{\substack{A_\mu \to A_\mu + \partial_\mu\alpha \\ \text{with } \alpha = \pi^\nu}}{\underset{\uparrow}{h_{\mu\nu} + \partial_\mu\pi_\nu + \partial_\nu\pi_\mu}} \tag{6.164}$$

the gauge connection analog to $V\partial_\mu V^\dagger$ in Eq. (3.59), and an infinitesimal general coordinate transformation of the tensor $h_{\mu\nu}$:

$$(\partial_\mu\pi^\alpha)h_{\alpha\nu} + (\partial_\nu\pi^\alpha)h_{\mu\alpha} + \pi^\alpha\partial_\alpha h_{\mu\nu}. \tag{6.165}$$

the gauge analog of being in a non-trivial representation of the gauge symmetry $VA_\mu V^\dagger$.

6.8 Equations of motion

Let us derive the classical equations of motions using Euler-Lagrange:

$$\delta S = 0 \;\Leftrightarrow\; \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \partial_\nu \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \partial_\nu \phi} + \ldots = 0. \tag{6.166}$$

Use the general form

$$\mathcal{L}_{\text{general form}} = h_{\rho\sigma} \partial_\alpha \partial_\beta h_{\gamma\delta}. \tag{6.167}$$

This is not a Lorentz scalar and is just used to derive a general functional derivative. The Lagrangian of course only contains Lorentz scalars.

and apply $\frac{\delta}{\delta h_{\mu\nu}}$

$$0 = \partial_\alpha \partial_\beta h_{\gamma\delta} + \partial_\alpha \partial_\beta h_{\rho\gamma}, \quad \text{(EOM)} \tag{6.168}$$

with $\rho = \mu, \sigma = \nu$ and $\gamma = \mu, \delta = \nu$.
So let's have a look at an example:

$$\mathcal{L} \supset -h_{\mu\nu} \partial_\mu \partial_\alpha h_{\nu\alpha} = -h_{\rho\sigma} \partial_\alpha \partial_\beta \eta^{\rho\alpha} \eta^{\beta\gamma} \eta^{\sigma\delta} h_{\gamma\delta} \tag{6.169}$$

So we get

$$0 = -\partial_\alpha \partial_\beta h_{\gamma\delta} \eta^{\rho\alpha} \eta^{\beta\gamma} \eta^{\sigma\delta} \eta_{\rho\mu} \eta_{\sigma\nu} - \partial_\alpha \partial_\beta h_{\rho\sigma} \eta^{\rho\alpha} \eta^{\beta\gamma} \eta^{\sigma\delta} \eta_{\gamma\mu} \eta_{\delta\nu}, \tag{6.170}$$

$$0 = -\partial_\mu \partial_\alpha h_{\alpha\nu} - \partial_\alpha \partial_\mu h_{\alpha\nu} \quad \text{etc.} \tag{6.171}$$

$$0 = -2\partial_\mu \partial^\alpha h_{\alpha\nu} \overset{\frac{\delta}{\delta h_{\nu\mu}}}{=} -\partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} \quad \text{and so on} \tag{6.172}$$

The EOM is:

$$\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} - \eta_{\mu\nu} \Box h \tag{6.173}$$
$$+ \eta_{\mu\nu} \partial^\lambda \partial^\kappa h_{\lambda\kappa} + \partial_\mu \partial_\nu h \tag{6.174}$$
$$= -m^2 (h_{\mu\nu} - \eta_{\mu\nu} h). \tag{6.175}$$

To get the Equation of motion multiply Eq. (6.175) by ∂^μ.
→LHS vanishes

$$0 = -m^2 \partial^\mu (h_{\mu\nu} - \eta_{\mu\nu} h). \tag{6.176}$$

It follows for $m^2 \neq 0$:

$$\partial^\mu h_{\mu\nu} = \partial_\nu h. \tag{6.177}$$

Plug back in EOM:

$$\partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h = -m^2 (h_{\mu\nu} - \eta_{\mu\nu} h). \tag{6.178}$$

Then take the trace of Eq. (6.175). → LHS vanishes.
It follows $m^2 \neq 0$:

$$h = 0. \tag{6.179}$$

Let's summarize what we have got:

$$h = 0 \tag{6.180}$$
$$\partial^\mu h_{\mu\nu} = 0 \tag{6.181}$$
$$(\Box + m^2)h_{\mu\nu} = 0. \tag{6.182}$$

It is the same as the massive vector field!

Counting degrees of freedom:

1. **Massive spin 2**: it should be $(2s+1) = 5$

$$\begin{aligned} h_{\mu\nu} = h_{\nu\mu} \quad &\to 10 \text{ dof} \quad \text{(symmetric)} \\ \partial^\mu h_{\mu\nu} = 0 \quad &\to -4 \text{ dof} \\ h = 0 \quad &\to -1 \text{ dof} \end{aligned}$$

   We have 5 dof.✓

2. **Massless spin 2**:
   This satisfies a gauge symmetry

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu + \dots \tag{6.183}$$

   The Pauli-Fierz Lagrangian ($m^2 = 0$) is gauge-invariant since we chose the $a, b, c, \dots$ such that for the decomposition

$$h_{\mu\nu} = h^T_{\mu\nu} + \partial_\mu \pi^T_\nu + \partial_\nu \pi^T_\mu + \partial_\mu \partial_\nu \pi. \tag{6.184}$$

   the terms π^T_ν, π drop out!
   We count again

$$\begin{aligned} h_{\mu\nu} = h_{\mu\nu} \quad &\to 10 \text{ dof} \quad \text{(symmetric)} \\ h_{\mu\nu} = h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\nu \pi_\mu \quad &\to -4 \text{ dof} \quad \text{(gauge)} \end{aligned}$$

   Here we get 6 dof.

Further inspection of the EOM shows that 4 components do **not** have time derivatives. You can check the $h_{0\mu}$ does not have ∂_t in the Lagrangian (like A_0), which means that they are just auxiliary fields - like A_0 in gauge theories. $\Rightarrow$ conjugate momentum vanishes.

Check for h_{00} and h_{01} is enough.

Counting:

$$\begin{aligned} h_{\mu\nu} = h_{\nu\mu} \quad &\to 10 \text{ dof} \\ +\cancel{\partial_t h_{0\mu}} \quad &\to -4 \text{ dof} \quad \text{(auxiliary fields)} \\ h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \pi_\nu + \partial_\mu \pi_\nu \quad &\to -4 \text{ dof} \quad \text{(gauge redundancy)} \end{aligned}$$

So we get 2 dof. We will discuss the interactions later including some discussion of quantum gravity. Heuristically, we can expand using $g = \eta + h/M_{Pl} + \dots$

As we know, a massless spin 2 particle has two polarizations.

$$\mathcal{L} = \sqrt{-g} M_{Pl}^2 R \tag{6.185}$$
$$\sim \left(\frac{1}{2} h \Box h + \frac{1}{M_{Pl}} \Box h^3 + \dots \right) \tag{6.186}$$

where we ignored the indices to show the structure.

This is a non-renormalizable theory, suppressed by $\frac{1}{M_{Pl}}$. We need either **very large energies** ($\sim M_{Pl} \sim 10^{19}$GeV) to overcome $\frac{\Box}{M_{Pl}}$ or **very large field values** with $h \geq M_{Pl}$. Very large field values are available, e.g. gravitation potential of the sun

$$u(r) \sim \phi_{Newton} \sim \frac{M_{sun}}{M_{Pl}} \frac{1}{r} \quad \text{with} \; \frac{M_{sun}}{M_{Pl}} \sim 10^{38}. \tag{6.187}$$

and we can see the non-linearities of GR e.g. in the anomalous precession of the perihelion of mercury.

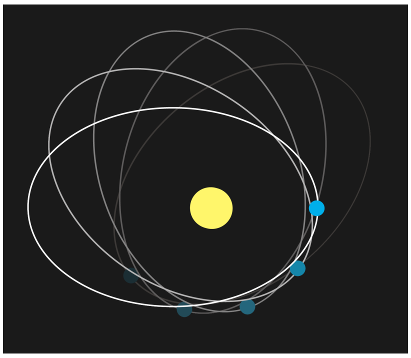


Figure 6.1: The perihelion precession of Mercury. GR accounts for about 10% of the total precession, the remainder being the gravitational tugs of other solar bodies. The measurement is very precise with 574.10 ± 0.65 arcsec/century

7 Spontaneous symmetry breaking

Consider the following Lagrangian:[1]

$$\mathcal{L} = \frac{1}{2}\left((\partial\vec{\phi})^2 - \mu^2\vec{\phi}^2\right) - \frac{\lambda}{4}(\vec{\phi}^2)^2 \tag{7.1}$$

with $\partial\vec{\phi}^2 = \partial_\mu\phi_i\partial^\mu\phi^i$ and $\vec{\phi} = (\phi_1, \ldots, \phi_N)$. The theory has an O(N) symmetry (rotations in N-dimensions[2]).
We could add explicit symmetry breaking terms like

$$\phi_1^2, \phi_1^4, \phi_1^2\vec{\phi}^2, \ldots, \tag{7.2}$$

which would break $O(N)$ to $O(N-1)$. This breaking would be **by hand**.
We can go further and break to $O(N-M)$, etc. The terms in Eq. (7.2) still respect a reflection symmetry

$$\phi_a \to -\phi_a \quad \forall a. \tag{7.3}$$

We could also break it by adding ϕ_a^3 or ϕ_a^5 terms. Explicit breaking is useful (in particular if small) but more interesting is

spontaneous symmetry breaking.

The system breaks itself.
Flip the sign of the $\vec{\phi}^2$ term:

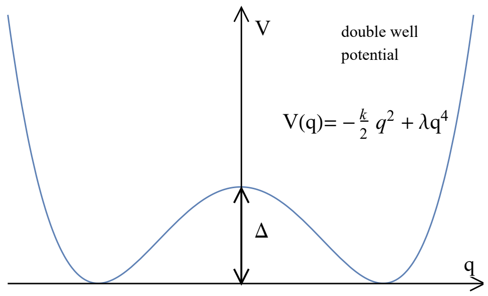
$$\mathcal{L} = \frac{1}{2}\left((\partial\vec{\phi})^2 {\color{red}+} \mu^2\vec{\phi}^2\right) - \frac{\lambda}{4}(\vec{\phi}^2)^2. \tag{7.4}$$

Naively, we predict that ϕ creates particles of mass $\sqrt{-\mu^2} = i\mu$ ($\to$probably not correct). It is similar to anharmonic oscillator with wrong-sign spring constant in **classical mechanics**:

$$L = \frac{1}{2}(\dot{q}^2 + kq^2) - \frac{\lambda}{4}q^4 = T - V. \tag{7.5}$$

[1] We will use it to introduce the concepts of spontaneous symmetry breaking.

[2] $\vec{\phi} \to V\vec{\phi}$ with $V \in O(N)$ and $V^TV = \mathbb{1}_{N\times N}$.



Potential energy: $V(q) = -\frac{1}{2}kq^2 + \frac{\lambda}{4}q^4$.
Two minima: $q_{min} = \pm v$ with $v = \left(\frac{k}{\lambda}\right)^{\frac{1}{2}}$. At low energies: choose one minimum and expand → study small oscillations. Committing to one (v or $-v$) however **breaks** reflection symmetry $q \to -q$!
In **quantum mechanics**, the particle can **tunnel** between the two minima. The tunnel barrier is

$$\Delta = V(0) - V(\pm v). \tag{7.6}$$

The probability of being in either one of the minima must be equal, thus respecting the reflection symmetry $q \to -q$ of the Hamiltonian! The ground state wave function will be **even**!

$$\psi(q) = \psi(-q). \tag{7.7}$$

How about **QFT**?

Let us try the same reasoning.
The energy functional is

$$E = \int d^3x \left[\frac{1}{2}(\partial_0\vec{\phi})^2 + \frac{1}{2}(\partial_i\vec{\phi})^2 \underbrace{- \frac{\mu^2}{2}\vec{\phi}^2 + \frac{\lambda}{4}\vec{\phi}^4}_{V} \right]. \tag{7.8}$$

Study $N = 1$ first ($\boldsymbol{\phi} = \phi_1$).
Ground state: field configuration $\phi(x)$ with minimal energy.

The case $N \geq 2$ will be dramatically different.

$$\underset{\text{time-independent}}{\underset{\uparrow}{\partial_t\phi(\vec{x},t) = 0}} \quad \text{and} \quad \underset{\text{homogeneous}}{\underset{\uparrow}{\partial_i\phi(\vec{x},t) = 0}}. \tag{7.9}$$

So we can conclude

$$\phi(\vec{x},t) = \text{const.} \tag{7.10}$$

$$\frac{\partial V}{\partial \phi} = 0 \Leftrightarrow -\mu^2\phi + \lambda\phi^3 = 0 \tag{7.11}$$

Solving for ϕ we get

$$\phi_0 = \pm\frac{\mu}{\sqrt{\lambda}} = \pm v. \tag{7.12}$$

If we want to transfer the field from one ground state, say $\phi_0 = v$ to $\phi_0 = -v$, we need to add energy which is proportional to the **volume of space**.

$$\Delta = \Omega v(\phi) \tag{7.13}$$

with $\Omega =$ Vol. of space.

So for a large system ($\Omega \to \infty$), we have to choose one of the ground states, and perturb around it. This breaks $\phi \to -\phi$ symmetry!
Compare to QM: homogeneous $\phi(\vec{x}, t) = \phi(t)$

$$S = \Omega \int dt \left(\frac{\dot{\phi}^2}{2} - V(\phi) \right). \tag{7.14}$$

It is the same as a particle with the mass:

$$M = \Omega \tag{7.15}$$

in a potential

$$U(\phi) = \Omega V(\phi). \tag{7.16}$$

The tunneling amplitude between ϕ_0 and $-\phi_0$:

$$\begin{aligned} A &\propto \exp\left(- \int_{-\phi_0}^{\phi_0} \sqrt{2M(U(\phi) - E)} d\phi \right) \qquad \text{(WKB)} &(7.17)\\ &\propto \exp\left(-\Omega \int_{-\phi_0}^{\phi_0} \sqrt{2V(\phi)} d\phi \right), &(7.18)\end{aligned}$$

where we used $E = 0$ since $V(\phi_0) = 0$. The amplitude tends exponentially to zero as $\Omega \to \infty$.
In quantum theory, it is legitimate to consider $\phi = \phi_0$ as the ground state, if the spatial volume is large. This is the situation in QFT.

$\Rightarrow$ Symmetry is spontaneously broken.

Let us choose as ground state: $\phi_0 = +v$.
Expand

$$\phi = v + \chi(x) \tag{7.19}$$

with $v = \frac{\mu}{\sqrt{\lambda}}$.

$$\begin{aligned} \partial_\mu (v + \chi) &= \partial_\mu \chi &(7.20)\\ V(v + \chi) &= \frac{\lambda}{4}((\chi + v)^2 - v^2)^2 &(7.21)\end{aligned}$$

So

$$V_\chi(\chi) = \lambda v^2 \chi^2 + (\lambda v)\chi^3 + \frac{\lambda}{4}\chi^4 + \underset{\substack{\uparrow \\ \text{Ignore}}}{\frac{\mu^4}{4\lambda}}. \tag{7.22}$$

$$\mathcal{L}_\chi = \partial\chi^2 - \mu^2\chi^2 - \sqrt{\lambda}\mu\chi^3 - \frac{\lambda}{4}\chi^4. \tag{7.23}$$

The shifted field χ creates a particle of mass

$$m_\chi = \sqrt{2}\mu > 0. \tag{7.24}$$

The last term is just $v(\phi)|_{\phi=v}$. We can drop it (zero-point energy).
$\mathcal{L}[\chi]$ is **not** invariant under $\chi \to -\chi$ (!), but under $\underset{\chi+v\to-\chi-v}{\chi \to -\chi - 2v}$,

which means symmetry is hidden in the relationship between

$$m_\chi, \alpha\chi^3, \beta\chi^4 \tag{7.25}$$

$$\begin{cases} m_\chi^2 = 2\mu^2 \\ \alpha = \sqrt{\lambda}\mu \\ \beta = \lambda \end{cases}$$

Exercise: Show that the Lagrangian Eq. (7.23) is equivalent to the one where we perturb around $\phi = -v$!

7.1 *Spontaneous breaking of U(1), Goldstone bosons*

Let's have a closer look at the case **N=2**:
Set

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \boldsymbol{\phi} = (\phi_1, \phi_2). \tag{7.26}$$

The Lagrangian is

$$\mathcal{L} = \partial_\mu\phi^\star\partial^\mu\phi - m^2\phi^\star\phi - \lambda(\phi^\star\phi)^2. \tag{7.27}$$

Global U(1)-symmetry:

$$\phi(x) \to \tilde{\phi}(x) = e^{i\alpha}\phi(x). \tag{7.28}$$

or

$$\begin{cases} \phi_1 \to \cos\alpha\ \phi_1 - \sin\alpha\ \phi_2 \\ \phi_2 \to \sin\alpha\ \phi_1 + \cos\alpha\ \phi_2 \end{cases}$$

In our case it is a $O(2)$-rotation.
The energy functional is

$$E = \int d^3x\ (\partial_0\phi^\star\partial^0\phi + \partial_i\phi^\star\partial_i\phi + V(\phi^\star, \phi)) \tag{7.29}$$

Groundstate: homogeneous and time-independent

with $V = m^2\phi^\star\phi + \lambda(\phi^\star\phi)^2$.
We can distinguish two cases:

1. $\mathbf{m^2 \geq 0}$:
   - Ground state $\phi = 0$
   - Excitations: ϕ_1, ϕ_2 equal mass, interactions respect U(1)
     $V = V(\phi_1^2 + \phi_2^2)$.
2. $\mathbf{m^2 < 0}$:
   - $m^2 = -\mu^2 \quad (\mu^2 > 0)$.
   - We now have a continuous set of minima:

   $$\phi = e^{i\alpha}\frac{\phi_0}{\sqrt{2}} \tag{7.30}$$

   with ϕ_0 from:

   $$\frac{\partial V}{\partial|\boldsymbol{\phi}|} = 0 \qquad \phi_0 = \frac{\mu}{\sqrt{\lambda}}. \tag{7.31}$$

- What are the particle states?

Let us focus on the case $m^2 = -\mu^2 < 0$. The so called *Mexican hat potential* is shown in Fig. 7.1. Different minima are now con-

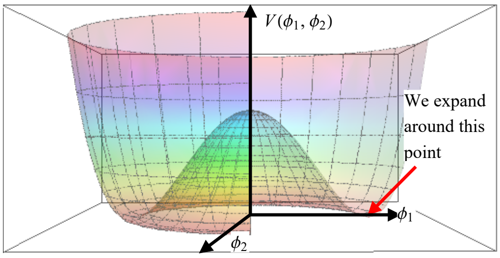


Figure 7.1: The Mexican hat potential: $V(\phi_1, \phi_2) = V(|\phi|)$ as function of $|\phi|^2 = \phi_1^2 + \phi_2^2$.

nected **without** energy barrier (move along $|\phi_0|-$circle). However, it still requires an infinite energy to the transition of $\Omega \to \infty$: For a homogeneous field $\phi(t)$ the kinetic energy is

$$E_{kin} = \Omega|\dot{\phi}|^2. \tag{7.32}$$

A change in the whole space requires infinite energy!
$\hookrightarrow$ It is sufficient to consider only one minimum.

$$\phi_0 = \frac{\phi_0}{\sqrt{2}}, \phi_1 = \phi_0,\ \phi_2 = 0. \tag{7.33}$$

7.1.1 Linear sigma model

Excitations:[3]

$$\phi_1(x) = \phi_0 + \chi(x) \tag{7.34}$$
$$\phi_2(x) = \theta(x). \tag{7.35}$$

[3] We will call this parametrization the *linear sigma model*.

$\partial_\mu\phi_1 = \partial_\mu\chi$ and $\partial_\mu\phi_2 = \partial_\mu\theta$.

$$\begin{aligned} V(\theta,\chi) = &-\frac{\mu^2}{2}((\phi_0+\chi)^2+\theta^2) \\ &+\frac{\lambda}{4}((\phi_0+\chi)^2+\theta^2)^2+\frac{\mu^2}{4\lambda}. \end{aligned} \tag{7.36}$$

We cancel the quadratic order

$$-\frac{\mu^2}{2}\theta^2 + \frac{\lambda}{4}2\overset{\overset{\phi_0=\frac{\mu^2}{\lambda}}{\downarrow}}{\phi_0^2}\theta^2 = 0. \tag{7.37}$$

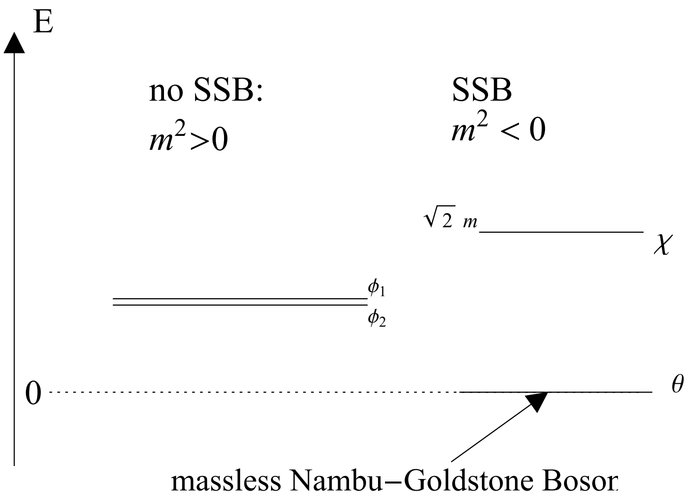
So we get

$$V_{\theta^2,\chi^2,\theta\chi} = \mu^2\chi^2. \tag{7.38}$$

$$\mathcal{L}^{(2)}_{\chi,\theta} = \frac{1}{2}(\partial_\mu\chi)^2 + \frac{1}{2}(\partial_\mu\theta)^2 - \mu^2\chi^2. \tag{7.39}$$

We find one massive degree of freedom with $m = \sqrt{2}\mu$ and one *massless* boson $m_\theta = 0$.
The following relations (sum rules) hold in both cases

$$m_{\phi_1}^2 + m_{\phi_2}^2 = 2m^2 \qquad m_\chi^2 + m_\theta^2 = 2m^2. \tag{7.40}$$



7.1.2 *Non-linear sigma model*

We can reparametrize the field ϕ to make the "radial vs. goldstone" splitting more transparent.
Use

$$\phi = \frac{1}{\sqrt{2}}(v + r(x))e^{i\pi(x)/v}. \tag{7.41}$$

This is the simplest version of a **non-linear sigma model**. This seems to be a bit strange to redefine the fields in this way, but remember: QFT is *reparametrization independent*! We can replace any field ϕ using a non-linear reparametrization $\phi = \chi F(\chi)$ with some function of the new fields χ which satisfies $F(0) = 1$. The same observables are predicted if we use $\mathcal{L}(\phi)$ or $\mathcal{L}(\chi F(\chi))$. The S-matrices will have the same single-particle poles since $F(0) = 1$.[4]
The condition $F(0) = 1$ is satisfied:

[4] You can convince yourself that this does not change the single particle poles and in the path-integral we are simply changing integration variables by $\phi \to \chi$.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}}(v + r(x))[1 + i\frac{i\pi(x)}{v} + \ldots] \tag{7.42}$$

$$= \frac{i\pi(x)}{\sqrt{2}} + \frac{r(x)}{\sqrt{2}} + \frac{v}{\sqrt{2}} + \ldots \tag{7.43}$$

Rewrite the Lagrangian. Potential:

$$V(|\phi|) = V(r(x)) \quad \leftarrow \text{independent of } \pi(x). \tag{7.44}$$

Kinetic term:[5]

[5] The fact that $e^{i\pi/v}$ drops out is unique to the $U(1)$ goldstone boson.

$$\partial_\mu\phi = (\partial_\mu r(x))e^{i\pi/v} + (v + r(x))e^{i\pi/v} i\partial_\mu\pi/v. \tag{7.45}$$

$$\partial_\mu\phi^\star\partial_\mu\phi = \frac{1}{2}\left[(\partial_\mu r - (v+r)i\frac{\partial_\mu\pi}{v})e^{-i\pi/v}\right]\cdot\left[(\partial_\mu r + (v+r)i\frac{\partial_\mu\pi}{v})e^{i\pi/v}\right]$$

$$= \frac{1}{2}\partial_\mu r\partial^\mu r + \frac{1}{2}\left(1 + \frac{r(x)}{v}\right)^2 \partial_\mu\pi\partial^\mu\pi \tag{7.46}$$

We can write the Lagrangian:

$$\mathcal{L}(\pi(x), r(x)) = \frac{1}{2}(\partial_\mu r)^2 + \frac{1}{2}\left(1 + \frac{2r(x)}{v} + \frac{r(x)^2}{v^2}\right)\partial_\mu\pi\partial^\mu\pi + V(r(x)). \tag{7.47}$$

1. We find interactions of the radial mode $r(x)$ and Goldstone boson $\pi(x)$!
2. We have only "derivative-interactions" of the Goldstone boson.

What happened to the $U(1)$-symmetry in the non-linear sigma model? Linear sigma model: $\phi \to e^{i\alpha}\phi$ becomes,

$$\phi \to e^{i\alpha}\phi(x) = e^{i\alpha}e^{i\pi(x)/v}(v + r(x)) \tag{7.48}$$

$$r(x) \to r(x) \tag{7.49}$$

$$\pi(x) \to \pi(x) + \alpha \cdot v \tag{7.50}$$

We find

$$\text{Goldstone bosons satisfy a shift symmetry } \pi(x) \to \pi(x) + \alpha \cdot v\ . \tag{7.51}$$

Our Lagrangian obeys it since $\pi(x)$ only appears in derivative terms

$$\mathcal{L}[r(x), \partial_\mu r(x), \underset{\substack{\uparrow \\ \text{Only derivative terms!}}}{\partial_\mu\pi(x)}]. \tag{7.52}$$

This is also called a *non-linear realization* of the symmetry.[6] Spontanous symmetry breaking leads to shift-symmetric realization of the symmetry.

[6] Once we move on to non-abelian groups, we will see that the goldstone transformations are only so simple for infinitesimal shifts and will involve complicated non-linear functions in the general case – much like non-abelian gauge transformations.

7.2 General remarks on spontaneous symmetry breaking

A symmetry G associated with variations δG is spontaneously broken, if there is an operator $A(\phi)$ for which[7]

$$\boxed{\langle\Omega|\delta^G A(\phi)|\Omega\rangle \neq 0.} \tag{7.53}$$

[7] If G is not spontaneously broken, it should be manifest linearly on the states and their expectation values

$$\langle\Omega|A(\phi)|\Omega\rangle \xrightarrow{G} \langle\Omega|A(\phi)|\Omega\rangle + \langle\Omega|\delta^G A(\phi)|\Omega\rangle$$

For the transformed matrix element to be the same as the original, we need $\langle\Omega|\delta^G A(\phi)|\Omega\rangle = 0$.

$\delta^G A(\phi)$ is called "order parameter". Let us define

$$b(x) \equiv \langle\Omega|\delta^G A(\phi)|\Omega\rangle \tag{7.54}$$

and

$$B(x) \equiv \delta^G A(\phi) \tag{7.55}$$

Example: $G = U(1)$, $\phi \to e^{i\alpha}\phi$, $\delta^G\phi = i\alpha\phi$.

$$\mathcal{L} = |\partial\phi|^2 + \frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4. \tag{7.56}$$

$\langle\phi\rangle$ is

$$\langle\phi\rangle = \langle\Omega|\phi|\Omega\rangle = v \tag{7.57}$$

with $\frac{\partial V(\phi)}{\partial|\phi|} = 0 \Leftrightarrow v = \frac{\mu}{\sqrt{\lambda}}$ (at tree-level).
So we get

$$\delta^G\phi = i\alpha\phi \quad \Rightarrow \quad \langle\Omega|\delta^G\phi|\Omega\rangle \neq 0. \quad \checkmark \tag{7.58}$$

7.2.1 General properties

We assume in the following that Poincaré is not spontaneously broken.

$$U(x,a)|\Omega\rangle = 0. \tag{7.59}$$

1. This means that $b(x) = \langle\Omega|\delta^G A(\phi)|\Omega\rangle$ has to be a **scalar** and **constant**.[8]

   Proof: Let's take the translation $x^\mu \to x^\mu + a^\mu$ as an example.

$$0 \overset{\text{by assumption}}{\downarrow\!=} \langle\Omega|\delta^a B(x)|\Omega\rangle = a_\mu\langle\Omega|\partial^\mu B(x)|\Omega\rangle \tag{7.60}$$
$$= a_\mu\partial^\mu\langle\Omega|B(x)|\Omega\rangle \quad \Rightarrow \text{independent of } x \tag{7.61}$$

   Clearly $\delta^G A(\phi)$ must not have a Lorentz-index, otherwise, e.g. $\langle\Omega|B_\mu|\Omega\rangle = v_\mu \neq 0$, which would break the Lorentz-symmetry.

2. SSB can only occur if the vacuum is degenerate.[9]

   Proof:

$$\delta\phi = i[\epsilon^a Q^a, \phi] \tag{7.62}$$

   and

$$\phi' = e^{i\epsilon^a Q^a}\phi(x)e^{-i\epsilon^a Q^a}. \tag{7.63}$$

   Suppose $|\Omega\rangle$ is not degenerate.
   Since Q^a is a conserved charge, $U(g) = e^{i\epsilon^a Q^a}$ with $U(g)$ (unitary) representation of $g \in G$, we have

$$[H, U(g)] = 0. \tag{7.64}$$

   So $U(g)|\Omega\rangle$ has the same E_0 as $|\Omega\rangle$ since by assumption:

$$e^{i\epsilon^a Q^a}|\Omega\rangle \propto |\Omega\rangle, \tag{7.65}$$

   $|\Omega\rangle$ is an eigenstate of Q^a with some real (Q^a is hermitian) eigenvalue q^a.

   For an arbitrary operator $A(x)$:

$$\langle\Omega|\delta^G A(x)|\Omega\rangle = i\langle\Omega|Q^A A(x) - A(x)Q^A|\Omega\rangle \tag{7.66}$$
$$= i(q^a - q^a)\langle\Omega|A(x)|\Omega\rangle = 0. \tag{7.67}$$

   It follows that all symmetries are unbroken.

[8] Note, that this can also mean $B = \bar{\psi}\psi$

[9] Recall the two discrete minima for $N = 1$ or the bottom ridge of the mexican hat for $N = 2$.

If the ground state is degenerate, $U(g)|\Omega\rangle \propto$ some other vacuum state.
Example:

$$\mathcal{L} = \frac{1}{2}\partial\phi^2 + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{7.68}$$

Here we have $G = \mathbb{Z}_2$.
The ground states are

$$\phi_0 = \pm v = \pm\frac{\mu}{\sqrt{\lambda}} \quad U(g)|+v\rangle = |-v\rangle. \tag{7.69}$$

So

$$U(g)\phi(x) = -\phi(x). \tag{7.70}$$

The vacuum is charged under the symmetry G.
Previously, we discussed $U(1)$-symmetry, which gave: 1 Goldstone Boson (GB). How about higher symmetries? We expect 1 GB per broken continuous symmetry.

7.3 Partial spontaneous symmetry breaking

Take $G = U(2) = SU(2) \times U(1)$ and

$$\mathcal{L} = |\partial\Phi|^2 + \frac{\mu^2}{2}|\Phi|^2 - \frac{\lambda}{4}|\Phi|^4. \tag{7.71}$$

with $\Phi = \begin{pmatrix}\Phi_1 \\ \Phi_2\end{pmatrix}$ and $\Phi_i \in \mathbb{C}$. The Lagrangian is invariant under

$$U(g)\phi \to e^{i\alpha^a\sigma^a/2}\phi \tag{7.72}$$

where $a = 0, 1, 2, 3$ and σ^a are the Pauli-matrices with $a = 1, 2, 3$ and $\sigma^0 = \mathbb{1}$. The minimum is at $|\Phi|^2 = v^2$ with $v = \frac{v}{\sqrt{\lambda}}$.
The ground states are on 4-sphere: $|\Phi_1|^2 + |\Phi_2|^2 = v^2$.
Choose

$$\langle\Phi\rangle = \begin{pmatrix}0 \\ v\end{pmatrix} \tag{7.73}$$

The vacuum vector does not break $SU(2) \to \emptyset$, but leaves $U(1)$ invariant $\phi_1 \to e^{i\alpha}\phi_1$. Is this the fulls story?
Let's look at the spectrum expanding Φ

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 + v\sqrt{2}\end{pmatrix}$$

with ϕ_i real. We get

$$\mathcal{L} = \frac{1}{2}(\partial\phi_i)^2 - \mu^2\phi_3^2 + \mathcal{O}(\phi_i^3) \tag{7.74}$$

So we have

$$m_{\phi_3}^2 = 2\mu^2 \qquad m_{\phi_1} = m_{\phi_2} = m_{\phi_4} = 0. \tag{7.75}$$

This means that we have 1 massive scalar and 3 Goldstones.

Summary
We started with 4 symmetry generators. 3 of them were spontaneously broken and 1 was left unbroken, so 1 GB for **each broken generator**.

This suggests a theorem? Yes! But wait a minute, we'll derive it below.

Let us reexamine the above example: We did not derive the full picture. Check $\mathcal{L}$ again including interactions,

$$\begin{aligned}\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}(\phi_3) - \mu\frac{\lambda}{2\sqrt{2}}\phi_3(\phi_1^2+\phi_2^2+\phi_4^2) \\ -\frac{\lambda}{8}\phi_3^2(\phi_1^2+\phi_2^2+\phi_4^2) - \frac{\lambda}{16}(\phi_1^2+\phi_2^2+\phi_4^2)^2. \end{aligned} \tag{7.76}$$

This is $SO(3)$ symmetric in the vector $\vec{\phi}_{(3)}$ with

$$\vec{\phi}_{(3)} = \begin{pmatrix}\phi_1\\ \phi_2\\ \phi_4\end{pmatrix}$$

Question 1: Could we have seen this from the start?
Question 2: Is the symmetry breakdown $SU(2)\to SO(3)$? No, it does not make sense because of the same number of generators.

We missed larger symmetry of $\mathcal{L}$

The $\mathcal{L}$ is in fact $SO(4)$ symmetric. With the parametrization

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\varphi_1 + i\varphi_2\\ \varphi_3 + i\varphi_4\end{pmatrix}$$

We can write

$$|\phi|^2 = \varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2. \tag{7.77}$$

This is a useful insight in the Standard Model since the Higgs sector (before gauging) has this symmetry. The Higgs field is a complex doublet $H = (h_1, h_2)$ with the same Lagrangian as Eq. (7.71). The additional $SU(2)$ is called a *custodial symmetry* which leads to additional selection rules and constraints on extensions of the SM.

and we see that $\mathcal{L} = \mathcal{L}[|\phi|^2, |\partial\phi|^2]$ is $SO(4)$ invariant. This is a larger symmetry than $SU(2)$ since

$$SO(4) \simeq SU(2)_1 \times SU(2)_2$$

and

$$\langle\phi\rangle = \begin{pmatrix}0\\0\\0\\v\end{pmatrix}$$

breaks $SO(4)$ to $SO(3)$.

This vev is $SO(3)$ rotation invariant

$$\langle\phi\rangle = \begin{pmatrix}0\\0\\0\\v\end{pmatrix}\begin{matrix}\to\phi_1\\ \to\phi_2\\ \to\phi_4\\ \to\phi_3\end{matrix}$$

In fact:

$$\begin{matrix} & \text{unbroken after SSB} \\ & \downarrow \\ G & \to H \\ \uparrow & \\ \text{global symmetry of } \mathcal{L} & \end{matrix}$$

which is here $SO(4)\to SO(3)$.This explains the additional structure that we found in the Lagrangian Eq. (7.76) after expanding around the vev.

Let us count generators. $SO(N)$ is generated by a $N\times N$ antisymmetric matrix $T_a^{ij} = -T^{ij}$.

$$\underset{\text{6 generators}}{\overset{\text{SO(4)}}{\begin{pmatrix}0 & 1 & 2 & 3\\ & 0 & 4 & 5\\ & & 0 & 6\\ & & & 0\end{pmatrix}}} \to \underset{\text{3 generators}}{\overset{\text{SO(3)}}{\begin{pmatrix}0 & 1 & 2\\ & 0 & 3\\ & & 0\end{pmatrix}}}$$

7.4 Goldstone Theorem

Goldstone Theorem:
There is a massless scalar for each spontaneously broken generator of internal symmetry of $\mathcal{L}[\phi^a]$.

A theory with ϕ^a scalar fields: the vev

$$\phi_0^a = \langle\Omega|\phi^a|\Omega\rangle$$

minimizes the potential

$$\frac{\partial}{\partial\phi^a}V(\phi)\bigg|_{\phi^a(x)=\phi_a^0} = 0. \tag{7.78}$$

The mass matrix is given by the 2nd derivatives.

We will deal with quantum corrections to the tree-level potential in the next section.

$$\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\bigg|_{\phi_a^0} = m_{ab}^2. \tag{7.79}$$

This is a symmetric matrix (eigenvalues = masses^2).We know that in the **minimum** $\frac{\partial^2}{\partial\phi^a\partial\phi^b}$ cannot have negative eigenvalues.
We need to show **Goldstone's theorem**

> Every continuous symmetry of $\mathcal{L}$ which is not a symmetry of the vacuum ϕ_0^a leads to a zero eigenvalue of m_{ab}^2.

which means that there is a massless scalar boson associated with each spontaneously broken generator.

A general continuous transformation is

$$\phi^a \to \phi^a + \underset{\substack{\uparrow\\ \text{infinitesimal}}}{\epsilon}\overset{\substack{\text{some function of } \phi^a(x)\\ \downarrow}}{\Delta^a(\phi)} + \ldots. \tag{7.80}$$

We know that the $\Delta^a(\phi)$ generate the symmetries of $\mathcal{L}$ (and v) and therefore for constant ϕ^a the potential must be invariant.

$$V(\phi^a) = V(\phi^a + \epsilon\Delta^a(\phi)) \tag{7.81}$$

or

$$0 = \Delta^a(\phi)\frac{\partial V}{\partial\phi_a} \tag{7.82}$$

where we expanded in ϵ.
We take the derivative of Eq. (7.82) with respect to ϕ^b and evaluate at $\phi^a = \phi_0^a$.

$$0 = \frac{\partial\Delta^a(\phi)}{\partial\phi^b}\bigg|_{\phi_0^a} \underbrace{\frac{\partial V}{\partial\phi_a}\bigg|_{\phi_0^a}}_{\substack{=0\\ \text{because } \phi_0^a \text{ is the minimum}}} + \Delta^a(\phi_0)\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}\bigg|_{\phi_0^a}. \tag{7.83}$$

$$\boxed{0 = \Delta^a(\phi_0^a)\frac{\partial^2 V}{\partial\phi^a\partial\phi^b}\bigg|_{\phi_0^a}.} \tag{7.84}$$

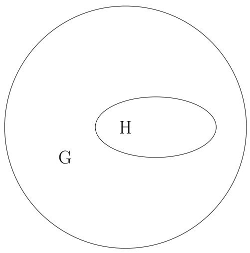
1. If the transformation leaves ϕ_0^a invariant:

$$\phi_0^a \to \phi_0^a + \epsilon\Delta^a(\phi_0^a) = \phi_a^0 \tag{7.85}$$
$$\Rightarrow \Delta^a(\phi_0^a) = 0 \quad \text{and} \quad \text{Eq. (7.84) trivially satisfied.} \tag{7.86}$$

2. If the symmetry is spontaneously broken

$$\Delta^a(\phi_0^a) \neq 0 \tag{7.87}$$

This is the desired vector with the eigenvalue 0 because of Eq. (7.84)
$\hookrightarrow$ massless field in m^2_{ab}.



G : global continuous symmetry of $\mathcal{L}$
H : remaining subgroup after SSB
$n(\text{Goldstones}) = n(G) - n(H)$

Example:

$$G/H = SU(N)/SU(N-1) = N^2 - 1 - ((N-1)^2 - 1) \tag{7.88}$$
$$= (2N-1) \quad \text{goldstones!} \tag{7.89}$$

7.5 The effective action

We will now generalize the previous tree-level considerations for the zeros of the scalar mass matrix to a quantum corrected potential. The relevant object to consider is the so-called *effective action* $\Gamma[\phi]$.
Reminder: generating functional

$$Z[J] \equiv \frac{\int \mathcal{D}\phi e^{iS[\phi]+i\int d^4x\, J(x)\phi(x)}}{\int \mathcal{D}\phi\, e^{iS[\phi]}} \tag{7.90}$$

and

$$\langle\Omega| T\left(\phi(x_1)\dots\phi(x_n)\right)|\Omega\rangle = \prod_{i=1}^n \frac{1}{i}\frac{\delta}{\delta J(x_i)}\, Z[J]|_{J=0} \tag{7.91}$$
$$= G(x_1,\dots,x_n). \tag{7.92}$$

Generating a functional $W[J]$

$$Z[J] = e^{iW[J]} \tag{7.93}$$

of connected green's functions

$$G^C(x_1,\dots,x_n) = \Pi_{i=1}^n \frac{1}{i}\frac{\delta}{\delta J(x_i)}\, W[J]|_{J=0}\,. \tag{7.94}$$

We can also consider Green's functions at $J \neq 0$. $\Rightarrow$ G_J^C and $G_J(x_1,\dots,x_n)$.
We define the effective action:

We can generate Green's functions of arbitrary operators $\mathcal{O}(x)$ by adding $i\int d^4x\, J_0(x)\mathcal{O}(x)$.

$$\boxed{\Gamma[\phi] \equiv W[J] - \int d^4x\, \phi(x)J(x)} \tag{7.95}$$

It is a functional of a classical field $\phi(x)$ (not an operator). This is a functional Legendre transformation. For a given $\phi(x)$ we can determine $J(x)$ by solving

$$\phi(x) = \frac{\delta W[J]}{\delta J(x)} \qquad \text{for} \qquad J(x) \tag{7.96}$$

and inserting in the RHS of Eq. (7.95).
We expand $\Gamma[\phi(x)]$ around $\phi_0(x)$ with $\hat{\phi} = \phi - \phi_0$

$$\Gamma[\phi] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \, \Gamma^{(n)}_{\phi_0(x)}(x_1, \ldots, x_n) \hat{\phi}(x_1) \cdots \hat{\phi}(x_n) \tag{7.97}$$

with $\Gamma^{(n)}_{\phi_0(x)}(x_1, \ldots, x_n) = \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_n)} \Gamma(\phi)|_{\phi=\phi_0}$.

Assuming that $\phi(x)$ is slowly varying, we can perform a *gradient expansion*:

$$\Gamma(\phi) = \int d^4x \left(-V_{\text{eff}}(\phi(x)) + \frac{1}{2} Z[\phi] \partial_\mu \phi \partial^\mu \phi + \ldots \right) \tag{7.98}$$

This defines the **effective potential** $V_{eff}(\phi)$

7.5.1 Properties of $\Gamma(\phi)$

1. A possible vacuum expectation value is given as

$$\phi = \langle \Omega | \phi(x) | \Omega \rangle_J = \frac{1}{i} \frac{\delta}{\delta J(x)} i \, W[J]|_J \,. \tag{7.99}$$

If $\phi \neq 0$ for $J = 0$ (no external sources), then this corresponds to a field configuration where $\Gamma[\phi]$ is stationary:

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} |_{\phi=\phi_{cl}} = 0. \tag{7.100}$$

Proof:
With

$$\langle \Omega | \phi(x) | \Omega \rangle = \phi_{cl}(x) = \frac{\delta W[J]}{\delta J(x)} \tag{7.101}$$

We will not be very careful about distinguishing between $\phi_{cl}(x)$ and $\phi(x)$. Whenever it is in between states $\langle \Omega | \phi(x) | \Omega \rangle$ it is an operator. Otherwise we mean the vacuum expectation value.

$$\frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}(y)} = \frac{\delta}{\delta \phi_{cl}} \left[W[J] - \int d^4x \; J \cdot \phi \right] \tag{7.102}$$

$$= \int d^4x \, \underset{\substack{\uparrow \\ \phi_{cl}(x)}}{\frac{\delta W}{\delta J(x)}} \frac{\delta J(x)}{\delta \phi_{cl}(y)} - \int d^4x \, \frac{\delta J(x)}{\delta \phi_{cl}(y)} \phi_{cl}(x) - J(y). \tag{7.103}$$

We get

$$\boxed{\frac{\delta \Gamma[\phi]}{\delta \phi_{cl}(y)} = -J(y)} \tag{7.104}$$

It is the "dual" relation to $\frac{\delta W}{\delta J} = \phi_{cl}$.

In the absence of external sources: $J(y) \equiv 0$ it follows

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = 0. \tag{7.105}$$

The vacuum expectation value (vev) should be Poincaré-invariant.

$$\phi(x) = \phi_0 = \text{const.} \tag{7.106}$$

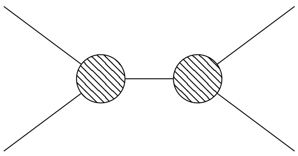
So

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)}|\phi_0 = -\frac{\partial V_{eff}}{\partial\phi}|\phi_0 = 0. \tag{7.107}$$

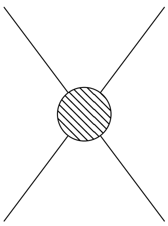
The vev is a stationary point of the effective potential.
In case of a linear symmetry $\delta^G\phi \propto \phi \hookrightarrow$ spontaneously broken, if stationary points of $\Gamma[\phi]$ occur only for $\phi(x) \neq 0$ since then also:

$$\langle\Omega|\delta^G\phi|\Omega\rangle \neq 0 \tag{7.108}$$

2. $\Gamma[\phi]$ is the generating functional of the 1PI (one-particle-irreducible) Green's functions, e.g.



1 particle **re**ducible connected 3 point



1PI 4 point function

3. In classical limit ($\hbar \to 0$):

$$\Gamma[\phi] = S[\phi]. \tag{7.109}$$

This means $\Gamma[\phi]$ contains the classical action and includes quantum corrections when $\hbar \neq 0$.
Also: $S[\phi]$ is a generating functional of 1PI tree (nothing else but vertices following from Γ_{int})

4. $V_{eff}(\phi)$ is the minimum of the **energy density** under constraint that $\phi = \langle\Omega|\phi(x)|\Omega\rangle$ takes a vev.
The equation

$$\frac{\delta\Gamma}{\delta\phi} = 0 \qquad (J = 0) \tag{7.110}$$

is the quantum version of Euler-Lagrange.
Indeed for $\hbar \to 0$: $\Gamma = S$ and Eq. (7.82) becomes

$$\frac{\delta S}{\delta\phi} = 0. \tag{7.111}$$

Comments on 2. to 4.

We explain **(2)**

1PI generating functional. We will not prove it here but we will show special cases.

Start with $n = 2$ first:

$$\delta(x-y) = \frac{\delta\phi(x)}{\delta\phi(y)} = \frac{\delta}{\delta\phi(y)}\frac{\delta W[J]}{\delta J(x)} \tag{7.112}$$

$$= i\int d^4z\, \underset{\substack{\uparrow \\ G^C(x,z)}}{\frac{\delta^2 iW[J]}{i\delta J(x) i\delta J(z)}}\;\underset{\substack{\uparrow \\ \text{use } -J=\frac{\delta\Gamma}{\delta\phi}}}{\frac{\delta J(z)}{\delta\phi(y)}} \tag{7.113}$$

$$= -\int d^4z\, \underset{\substack{\uparrow \\ \Delta_F(x-z)}}{G_J^{C(2)}(x,z)}\;\underset{\substack{\uparrow \\ i(\overleftarrow{\partial}_z^2+m^2)\delta(y-z) \\ \text{in tree theory } \Gamma=S}}{\frac{i\delta^2\Gamma}{\delta\phi(z)\delta\phi(y)}} \tag{7.114}$$

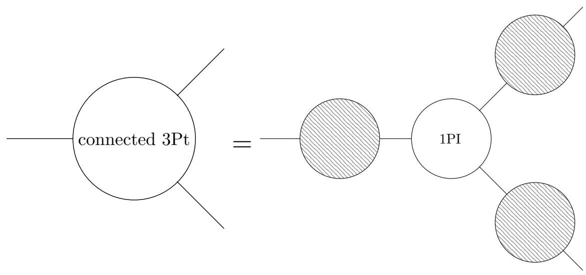
where we used the chain rule for functional derivatives.

We finally get

$$-i\Gamma_\phi^{(2)} = {G_{CJ}^{(2)}}^{-1} \tag{7.115}$$

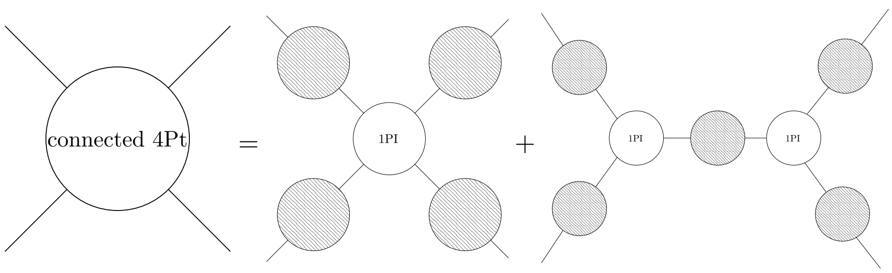
Similarly you can show (try!)

$$G_{C,J}^{(3)}(x_1,x_2,x_3) = \int d^4z_1 d^4z_2 d^4z_3 \frac{i\delta^3\Gamma}{\delta\phi(z_1)\delta\phi(z_2)\delta\phi(z_3)} G_{C,J}^{(2)}(x_1,z_1)G_{C,J}^{(2)}(x_2,z_2)G_{C,J}^{(2)}(x_3,z_3).$$



Any connected diagram contributing to $G_C^{(3)}$ can be written as on the RHS, which confirms that $\frac{\delta^3\Gamma}{\delta\phi_1\delta\phi_2\delta\phi_3}$ is a 1PI 3-point function.

The next order looks like



$+\, t, u$-channel topologies

such that $\frac{\delta^4\Gamma}{\delta\phi\delta\phi\delta\phi\delta\phi}$ is 4 Point 1PI function.

To show **(3)** reintroduce $\hbar$ in $Z[J]$

$$Z[J] = \int \mathcal{D}\phi\; e^{\frac{i}{\hbar}\left(S[\phi]+\int d^4x\; J\cdot\phi\right)} = e^{iW[J]} \tag{7.116}$$

We find

- The propagator is the inverse of the quadratic terms $\phi(\ldots)\phi$ in $S[\phi] \Rightarrow$ each P counts as $\hbar$.
- Every vertex V has $\frac{1}{\hbar}$.
- A connected diagram with P propagators and V vertices counts as

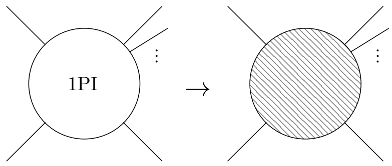
$$\boxed{\hbar^{P-V} = \hbar^{L-1}} \tag{7.117}$$

since the number of loops is $L = P - V + 1$ for any connected diagram.

See QFT1:

- each P $\int d^4 k_i$
- each V $\delta^{(4)}(x)$
- one $\delta-$function is trivial because of translational invariance =(P-V+1) number of independent $\int d^4 k_i$ integrations (which is the number of loops)

For $\hbar \to 0$, $L = 0$ (tree-level) dominates since it counts as $\frac{1}{\hbar}$!
Tree diagrams are built out of $S[\phi]$ and correspond to topologies of 1PI with replacement



[vertex from S if it exists] and so $\Gamma_{\hbar \to 0} = S$.

We show now **(4)** that V_{eff} is an energy-density given ϕ, in fact it is the minimum of the energy density under constraint

$$\phi = \langle \Omega | \phi(x) | \Omega \rangle. \tag{7.118}$$

Recall: path integral derivation is an amplitude:

$$\langle f | e^{-iHT} | i \rangle \qquad \text{transition} \quad |i\rangle \to |f\rangle \quad \text{in time T.}$$

Special case:

$$\underset{=\langle\Omega|\Omega\rangle=1}{\underset{\uparrow}{\langle \Omega | e^{-iHT} | \Omega \rangle}} \overset{T \to \infty}{=} N \int \mathcal{D}\phi \; e^{iS[\phi]} = 1 \tag{7.119}$$

since $H|\Omega\rangle = E_0 |\Omega_0\rangle$ with $E_0 = 0$.
Let's add a source term, then H becomes

$$H[J] \equiv H - \int J(\vec{x}) \phi(\vec{x}) d^3\vec{x} \tag{7.120}$$

and $S[\phi] \to S_J[\phi] \equiv S[\phi] + \int d^3x \; J(\vec{x})\phi(\vec{x})$. Adiabatically switching on and off J:

$$e^{-iE_J T} = \langle \Omega_J | e^{-iH[J]T} | \Omega_J \rangle = N \int \mathcal{D}\phi e^{iS_J[\phi]} = Z[J] = e^{iW[J]}.$$

We find $W[J] = -E_J T$ for this source and conclude

$$H|\Omega\rangle_J = E_J |\Omega\rangle_J + \int d^3\mathbf{x} \; \phi(\mathbf{x}) J(\mathbf{x}) |\Omega\rangle_J. \tag{7.121}$$

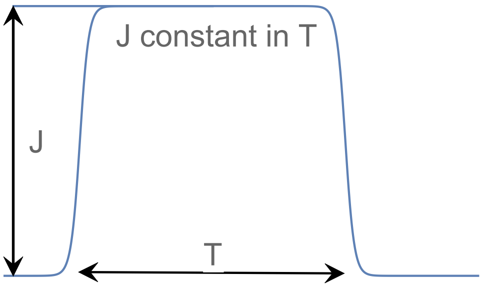


Figure 7.2: Adiabatically switching on and off J

Goal: Find $|\psi\rangle$ with $\langle\psi|\psi\rangle = 1$ such that $\langle\psi|H|\psi\rangle$ is minimized subject to $\langle\psi|\phi(x)|\psi\rangle = \phi(x)$. Use Lagrange multipliers λ, $\lambda_2(\vec{x})$ to minimize

$$\langle\psi|H|\psi\rangle - \lambda\langle\psi|\psi\rangle - \int d^3\vec{x}\ \lambda_2(\vec{x})\langle\psi|\phi(\vec{x})|\psi\rangle. \tag{7.122}$$

Compare to Eq. (7.121) and note that a solution is

$$|\psi\rangle = |\Omega\rangle_J \quad \text{with} \quad \lambda = E_J \quad \text{and} \quad \lambda_2(\vec{x}) = J(\vec{x}), \tag{7.123}$$

since Ω_J is the **lowest energy** state.
The minimal energy is

$$E_{min} = \langle\Omega_J|H|\Omega_J\rangle \overset{Eq.\ (7.121)}{=} E_J + \int d^3\vec{x}\ J(\vec{x})\phi(\vec{x}) \tag{7.124}$$

$$= \frac{1}{T}\left(-W[J] + \int d^4x\ J[\vec{x}]\phi(\vec{x})\right) \tag{7.125}$$

$$= -\frac{1}{T}\Gamma[\phi], \tag{7.126}$$

where we needed the following statement: Since $J = \lambda_2$ is determined such that $\langle\Omega_J|\phi(x)|\Omega_J\rangle = \phi(x)$. This corresponds to the definition of $\Gamma[\phi]$ being a Legendre transformation.
Choosing $\phi(\mathbf{x}) = \phi_0$ (and therefore $J(\mathbf{x})$) we get for the minimal energy density ϵ_{min} at $\langle\Omega|\phi(x)|\Omega\rangle = \phi$

$$\epsilon_{min} = \frac{E_{min}}{V} = -\frac{1}{VT}\Gamma[\phi] \underset{\uparrow \atop = -\int d^3\mathbf{x}\ V_{\text{eff}} = -VTV_{\text{eff}}}{=} V_{\text{eff}}(\phi). \tag{7.127}$$

where V is the volume of space. So we find the minimal energy density given

$$\phi = \langle\Omega_J|\phi(x)|\Omega_J\rangle \tag{7.128}$$

is indeed $V_{eff}(\phi)$ with ϕ =const.

7.5.2 Important property of the effective action

If $S[\phi]$ is invariant under a non-anomalous linear symmetry, then the effective action is also invariant.

Recall the derivation of the Ward-Identity,see QFT1.

$$\int \mathcal{D}\phi_n e^{S[\phi_n]+i\int d^4x J_n\phi_n} \int d^4x J_n(x) F_n^a[\phi_m(x)] = 0. \tag{7.129}$$

for a symmetry

$$\phi_n(x) \to \phi_n(x) + \underbrace{\epsilon^a F_n^a[\phi_m]}_{=\delta\phi_n} = \phi_n' \tag{7.130}$$

Start from $Z[J_n]$:

$$\begin{aligned}
Z[J_n] &= N \int \mathcal{D}\phi_n e^{iS[\phi_n]+i\int d^4x\, J_n\phi_n} \qquad (7.131)\\
&\underset{\phi\to\phi'}{\overset{\uparrow}{=}} N \int \mathcal{D}\phi_n' e^{iS[\phi_n']+i\int d^4x J_n\phi_n'}\\
&= N \int \mathcal{D}\phi_n \underbrace{\left|\det \frac{\delta\phi_m'}{\delta\phi_n}\right|}_{\substack{=1 \text{ since the}\\ \text{symmetry is}\\ \text{non-anomalous}}} \exp\Bigg(iS[\phi_n] + i\underbrace{\int d^4x \frac{\delta S}{\delta\phi_n}\delta\phi_n}_{\substack{=0\\ \text{since } \delta\phi_n \text{ is}\\ \text{symmetry of } S}} + i\int d^4x J_n(\phi_n+\delta\phi_n)\Bigg)\\
&\overset{\substack{\text{Expand exp}\\ \downarrow}}{=} N \int \mathcal{D}\phi_n e^{iS[\phi_n]+i\int d^4x\, J_n\phi_n} \Bigg(\underset{\substack{\uparrow\\ \text{same as}\\ \text{Eq. (7.131)}}}{1} + i\int d^4x\, J_n(x)\underset{\substack{\uparrow\\ \text{has to vanish for}\\ \text{each power in } \epsilon^a\\ \text{independently}}}{\epsilon^a} F^a[\phi_m] + \dots\Bigg)
\end{aligned}$$

For a linear symmetry

$$F_n^a(\phi_n) = i\underset{\substack{\uparrow\\ \text{Group generator}}}{T_{nm}^a}\phi_m(x) \tag{7.132}$$

The Ward-identity implies:

$$0 = \int d^4x\ J_n(x) iT_{nm}^a N \int \mathcal{D}\phi_n e^{iS[\phi_n]+i\int d^4x J_n\phi_n}\phi_m \tag{7.133}$$

$$= \int d^4x\ J_n(x) iT_{nm}^a \langle\Omega|\phi_m(x)|\Omega\rangle_J \tag{7.134}$$

$$= +i\int d^4x \left(-\frac{\delta\Gamma}{\delta\phi_n(x)}\right) T_{nm}^a\phi_m(x) \tag{7.135}$$

We can identify

$$-\frac{\delta\Gamma}{\delta\phi_n(x)} = J_n(x) \quad \text{and} \quad \phi_m(x) = \langle\Omega|\phi_m(x)|\Omega\rangle_J. \tag{7.136}$$

But this is precisely the change of $\Gamma[\phi]$ under symmetry transformation!

$$\Gamma[\phi'] = \Gamma[\phi+\delta\phi] = \Gamma[\phi] + \underbrace{\int d^4x \frac{\delta\Gamma}{\delta\phi_n(x)} \overset{\substack{iT_{nm}^a\phi_m(x)\\ \downarrow}}{\delta\phi_n(x)}}_{\substack{=0\\ \text{because of Eq. (7.135)}}} \tag{7.137}$$

It follows

$$\Gamma[\phi'] = \Gamma[\phi] \quad \to \text{invariant!} \tag{7.138}$$

We can now extend the tree-level discussion of Goldstone's proof using the mass-matrix

$$\left.\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}\right|_{\phi_0} = m^2_{ab}. \tag{7.139}$$

to the effective potential $\Gamma[\phi] \to V_{eff}(\phi)$.
We assume again that the theory is G-invariant which includes the assumption that the symmetry is non-anomalous. Therefore the effective action is invariant under G transformations:

$$0 = \int d^4x\, \frac{\delta\Gamma[\phi]}{\delta\phi_m} T^a_{nm}\phi_m(x) \quad \text{all } T^a \text{ in } g \tag{7.140}$$

g is a Lie-algebra of G.
For constant $\phi_m(x) = \phi_0 = \langle\Omega|\phi(x)|\Omega\rangle$

thanks to Poincaré-invariance

In Eq. (7.140) we can now replace $\frac{\partial^2 V}{\partial\phi_a\partial\phi_b}$ with $\frac{\partial^2 V_{eff}}{\partial\phi_a\partial\phi_b}$ and show Goldstone's theorem for quantum systems.
We will now show that

$$\left.\frac{\partial^2 V_{eff}}{\partial\phi_k\partial\phi_n}\right|_{\phi=\phi_0} \tag{7.141}$$

is the mass matrix with V_{eff}:

$$V_{eff}(\phi) = -\sum_n \frac{1}{n!}\phi_{k_1}\dots\phi_{k_n}\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(\phi_i = 0) \tag{7.142}$$

where $\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_1,\dots p_n)$ is the momentum space 1PI function.

$$\int d^4x_1\cdots d^4x_n\, e^{ip_1x_1+\ldots+ix_np_n}\Gamma^{(n)}_{k_1\dots k_n}(x_1,\dots,x_n) = (2\pi)^4\delta^{(4)}(p_1+\ldots+p_n)\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_1,\dots,p_n) \tag{7.143}$$

For constant ϕ:

$$-\int d^4x\, V_{eff}(\phi) = \Gamma[\phi] = \sum_n \frac{1}{n!}\int d^4x_1\cdots d^4x_n\, \phi_{k_1}\cdots\phi_{k_n}\Gamma^{(n)}_{k_1\dots k_n}(x_1,\dots,x_n) \tag{7.144}$$

$$= \sum_n \frac{1}{n!}\phi_{k_1}\dots\phi_{k_n}\underbrace{(2\pi)^4\delta^{(4)}(0)}_{=\int d^4x}\tilde{\Gamma}^{(n)}_{k_1\dots k_n}(p_1=0,\dots,p_n=0) \tag{7.145}$$

and so we have shown Eq. (7.143). It is clear that this corresponds to the non-derivative part of $\Gamma[\phi]$.

$$\left.\frac{\partial^2 V_{eff}}{\partial\phi_k\partial\phi_n}\right|_{\phi=\phi_0} = -\tilde{\Gamma}^{(2)}_{kn}(p=0) = -iG^{-1}_{C,kn}(p=0) \tag{7.146}$$

where $G^{-1}_{C,kn}(p=0)$ is the inverse of the connected two-point function $\langle\Omega|T(\phi_1,\phi_2)|\Omega\rangle$ in the momentum space at $p=0$.

$$G^{-1}_{C,kn}(p) = p^2 - m^2_{kn} - \Pi_{kn}(p) \overset{\overset{p\to 0}{\downarrow}}{=} -[m^2+\Pi(0)]_{kn} \tag{7.147}$$

with m^2_{kn} as the tree-level mass matrix and $m^2 + \Pi(0)$ including quantum corrections.
Hence the argument from before holds

$$M^2_{kn}\Delta^a_n(\phi_0) = 0 \tag{7.148}$$

Since the poles in Green's function correspond to the one-particle states (with $p^2 = m^2$) and since ϕ has to be a scalar we conclude that there must be at least N **massless** spin-0 particles, the Goldestone bosons.

We can also show Goldstone's theorem in a simpler way
With every continuous symmetry come a conserved charge Q and $[H, Q] = 0$.
The vacuum is $|\Omega\rangle$. Adding a constant to $H \to H + c$ we can always set

$$H|\Omega\rangle = 0. \tag{7.149}$$

If the symmetry is spontaneously broken the vacuum is not invariant

$$e^{i\theta Q}|\Omega\rangle \neq 0 \quad \text{and} \quad Q|\Omega\rangle \neq 0. \tag{7.150}$$

Consider the state $Q|\Omega\rangle$. What is its energy?

$$HQ|\Omega\rangle \overset{\overset{H|\Omega\rangle=0}{\downarrow}}{=} \underbrace{[H, Q]}_{\overset{\uparrow}{Q\text{ conserved}}}|\Omega\rangle = 0. \tag{7.151}$$

Thus we found another state $Q|\Omega\rangle$ with $E = 0$, same as E as $|\Omega\rangle$.
Consider another state:

$$|s\rangle = \int d^3x e^{-i\vec{k}\vec{x}} J^0(x)|\Omega\rangle. \tag{7.152}$$

Note:
There is no need for relativity or fields. In QFT we have local currents, and so

$$Q = \int d^3x\, J^0(x).$$

This has a spatial momentum $\vec{k}$ and $E_s(\vec{k}) \overset{\vec{k}\to 0}{\to} 0$.
With $\mathbf{p}|\Omega\rangle = 0$ we have

In a relativistic theory this means:
$|s\rangle$ describes a massless particle!
($E = \sqrt{m^2 + \vec{p}^2}$ and $m = 0$)

$$p^i|s\rangle = \int d^3x\, e^{-i\vec{k}\vec{x}} \underbrace{[p^i, J^0(\vec{x}, t)]}_{\overset{\uparrow}{\substack{\text{in general, the commutator:}\\ [p^\mu,\phi(x)]=-i\partial^\mu\phi(x)}}}|\Omega\rangle \overset{\overset{IBP}{\downarrow}}{=} k^i|s\rangle \tag{7.153}$$

Now multiply:

Same sign in the exponent $e^{-i\vec{p}\vec{x}}$ as in $\phi \sim a^\dagger_k e^{-i\vec{k}\vec{x}} + \dots$

$$|\pi(\vec{p})\rangle = \frac{-2i}{v}\int d^3x\, e^{-i\vec{p}\vec{x}} J_0(x)|\Omega\rangle \tag{7.154}$$

with

$$\langle\pi(\vec{q})| \tag{7.155}$$

and integrate over

$$\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{y}} \tag{7.156}$$

to obtain:

$$
\begin{aligned}
&-\frac{2i}{v}\int d^3x \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\vec{y}} \langle \pi(\vec{q})|J_0(x)|\Omega\rangle e^{-i\vec{p}\vec{x}} \\
&= \frac{-2i}{v}\langle \pi(\vec{q})|J_0(y)|\Omega\rangle
\end{aligned} \tag{7.157}
$$

where we used $\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = (2\pi)^3\delta^{(3)}(\vec{x}-\vec{y})$.
Now compare to

$$
\int \frac{d^3p}{(2\pi)^3} \underbrace{\langle \pi(\vec{q})|\pi(\vec{p})\rangle}_{\substack{\text{Normalisation of one-particle states} \\ =2\omega_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})}} e^{i\vec{p}\vec{y}} = 2\omega_{\vec{p}} e^{i\vec{q}\cdot\vec{y}} \tag{7.158}
$$

Since Eq. (7.157) = Eq. (7.158), we get

$$
\boxed{\langle \pi(\vec{q})|J_0(y)|\Omega\rangle = i\omega_{\vec{p}}\, v\, e^{i\vec{q}\vec{y}}.} \tag{7.159}
$$

The Lorentz-generalisation is:

$$
\langle \pi(\vec{q})|J_\mu(y)|\Omega\rangle = iq_\mu\, v\, e^{i\vec{q}\vec{y}}. \tag{7.160}
$$

It is useful to extract the GB
Check for U(1) example:

$$
\phi = (v + r(x))e^{i\pi(x)/v} \quad \text{with} \quad \pi, r \in \mathcal{R} \tag{7.161}
$$

and the Lagrangian $\mathcal{L}$:

$$
\mathcal{L} = |\partial_\mu\phi|^2 + \frac{\mu^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4. \tag{7.162}
$$

Decompose $r(x)$ ($m \to \infty, \lambda \to \infty$ and $v = \frac{\mu}{\lambda}$ fixed).

$$
\mathcal{L} = \partial_\mu\pi\partial_\mu\pi + (\text{heavy fields}) \tag{7.163}
$$

π has a shift symmetry $\pi \to \pi + \alpha \cdot v$.

$$
J_\mu(x) = \frac{\partial\mathcal{L}}{\partial\partial_\mu\pi(x)}\frac{\delta\pi}{\delta\alpha} = v\cdot\partial_\mu\pi(x) \tag{7.164}
$$

$|\pi\rangle$ is the one-particle state created and annihilated by $\pi(x)$ field:

$$
\langle\Omega|J^\mu(x)|\pi(p)\rangle = \langle\Omega|v\cdot\partial_\mu\pi(x)|\pi(p)\rangle \tag{7.165}
$$

with $|\pi(\mathbf{p})\rangle = \sqrt{2\omega_\mathbf{p}}a^\dagger_\mathbf{p}|\Omega\rangle$ and $[a_\mathbf{p}, a^\dagger_\mathbf{q}] = (2\pi)^3\delta(\mathbf{p}-\mathbf{q})$ and

$$
\pi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left(e^{-ipx}a_\mathbf{p} + e^{ipx}a^\dagger_\mathbf{p}\right) \tag{7.166}
$$

at leading order (free-field)

$$
\langle\Omega|v\partial_\mu\pi(x)|\pi(q)\rangle = \langle\Omega|v\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{q}}}(-ip_\mu)e^{-ipx}a_\mathbf{p}\sqrt{2\omega_\mathbf{q}}a^\dagger_\mathbf{q}|\Omega\rangle + (\text{vanishing terms}) \tag{7.167}
$$

$$
= \langle\Omega|v(-iq_\mu)e^{-iqx}|\Omega\rangle \overset{t=0}{=} v(-iq_\mu)e^{iq\mathbf{x}} \tag{7.168}
$$

Expanding the action to second order in the presence of a background field
We will calculate quantum fluctuations correcting the tree level potential. Let us first recall the functional Taylor series

$$S[\phi] + \int J\phi = S[\phi_s + \tilde{\phi}] + \int J\phi_s + \int J\tilde{\phi} \tag{7.169}$$

$$= S[\phi_s] + \left.\frac{\delta S[\phi]}{\delta\phi}\right|_{\phi_s} \tilde{\phi} + \frac{1}{2}\left.\frac{\delta^2 S}{\delta\phi^2}\right|_{\phi_s} \tilde{\phi}\tilde{\phi} + \ldots \tag{7.170}$$

Expand the Lagrangian including the source terms (ignore surface terms), for a background field configuration ϕ_s and quantum fluctuations $\tilde{\phi}$ with $\phi = \phi_s + \tilde{\phi}$ where ϕ_s satisfies the EOM with an external source J

$$\begin{aligned}
&\frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) + J\phi_s + J\tilde{\phi} \\
&= \frac{1}{2}(\partial_\mu\phi_s)^2 + \partial_\mu\tilde{\phi}\partial_\mu\phi_s + \frac{1}{2}\partial_\mu\tilde{\phi}^2 - V(\phi_s) - V'(\phi_s)\tilde{\phi} - V''(\phi_s)\frac{\tilde{\phi}^2}{2} + \mathcal{O}(\tilde{\phi}^3) + J\phi_s + J\tilde{\phi} \\
&= \underbrace{(\partial_\mu\phi_s)^2 - V(\phi_s)}_{S[\phi_s]} + \cancelto{=0,\,EOM}{\left(-\partial_\mu^2\phi_s - V'(\phi_s) + J\right)}\tilde{\phi} + \frac{1}{2}\partial_\mu\tilde{\phi}^2 - V''(\phi_s)\frac{\tilde{\phi}^2}{2} + \mathcal{O}(\tilde{\phi}^3) + J\phi_s
\end{aligned}$$

It follows:

$$S[\phi_s] + \int \frac{1}{2}\,\tilde{\phi}\left(-\partial^2 - V''(\phi_s)\right)\tilde{\phi} + \int J\phi_s + \mathcal{O}(\widetilde{\phi}^3) \tag{7.171}$$

If we now path integrate over the fluctuations $\tilde{\phi}$, the quadratic integral in Eq. (7.171) is a simple path integral of gaussian type, which is $\propto \mathrm{Tr}\ln(\partial^2 + \mathrm{V}''(\phi_\mathrm{s}))$, see below.

7.5.3 *Calculating the effective potential*

In the previous section we derived

$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4.$

$$\begin{aligned}
Z[J] = e^{\frac{i}{\hbar}W[J]} &= \int \mathcal{D}\phi\, e^{\frac{i}{\hbar}(S(\phi) + \int J\cdot\phi)} \\
&\simeq e^{\frac{i}{\hbar}\left[S[\phi_s] + \int_x J\phi_s\right]} \int \mathcal{D}\tilde{\phi}\, e^{\frac{i}{\hbar}\left[\int d^4x \frac{1}{2}(\partial\tilde{\phi})^2 - V''(\phi_s)\tilde{\phi_s}\right]}.
\end{aligned}$$

The path integral is a simple Gaussian integral in $\tilde{\phi}$ which we know how to evaluate. It is just:

$m^2_{\text{eff}}(\phi_s) = V''(\phi_s)$

$$I = \int \mathcal{D}\tilde{\phi}\, e^{i\int d^4x\left(\frac{1}{2}(\partial\tilde{\phi})^2 - m^2_{\text{eff}}\tilde{\phi}^2\right)} = C\left(\frac{1}{\det(\partial^2 + m^2_{\text{eff}})}\right)^{\frac{1}{2}} \tag{7.172}$$

$$= C\, e^{-\frac{1}{2}\mathrm{Tr}\ln(\partial^2 + \mathrm{m}^2_{\text{eff}})}. \tag{7.173}$$

We now have determined:

$$Z[J] = N\,\exp\left(\frac{i}{\hbar}\left(S[\phi_s] + \int J\phi_s\right) - \frac{1}{2}\mathrm{Tr}\ln\left(\partial^2 + \mathrm{m}^2_{\text{eff}}(\phi_\mathrm{s})\right) + \mathcal{O}(\hbar)\right)$$

or

$$W[J] = \left(S[\phi_s] + \int J\cdot\phi_s\right) + \frac{i\hbar}{2}\mathrm{Tr}\left(\ln\left(\partial^2 + \mathrm{V}''(\phi_\mathrm{s})\right)\right). \tag{7.174}$$

Let us now Legendre transform:

$$\phi = \frac{\delta W}{\delta J} = \cancelto{0}{\frac{\delta(S[\phi_s] + \int J\phi_s)}{\delta\phi_s}}\frac{\delta\phi_s}{\delta J} + \phi_s + \mathcal{O}(\hbar) \tag{7.175}$$

$$= \phi_s + \mathcal{O}(\hbar) \tag{7.176}$$

To leading order in $\hbar$, we therefore have $\phi = \phi_s$ and thus:

$$\Gamma[\phi] = S[\phi_s] + \frac{i\hbar}{2}\mathrm{Tr}\ln\left(\partial^2 + \mathrm{V}''(\phi)\right) + \mathcal{O}(\hbar^2). \tag{7.177}$$

The $\hbar$-correction is difficult to evaluate: we have to find the eigenvalues of $\left(\partial^2 + V''(\phi)\right)$ and take the logarithm and then sum. We can simplify it if we are studying ϕ =const.= ϕ_0.
Then $\left(\partial^2 + V''(\phi_0)\right)$ is translation-invariant and is easily calculable in momentum space.

$$\mathrm{Tr}\ln\left(\partial^2 + \mathrm{V}''(\phi)\right) = \mathrm{Tr}\ln\left(\mathrm{K}\right) \tag{7.178}$$

with

$$K_{x,y} = \left(\partial^2 + V''(\phi_0)\right)\delta^{(4)}(x-y). \tag{7.179}$$

First we diagonalize passing to momentum space:

$$\mathrm{Tr}(\mathcal{O}) = \int d^4x\, \langle x|\mathcal{O}|x\rangle = \int d^4x \int d^4p\, d^4q\, \langle x|p\rangle\langle p|\mathcal{O}|q\rangle\langle x|q\rangle.$$

$$K_{p,q} = \int \frac{d^4x}{(2\pi)^4}e^{-ip\cdot x}\frac{d^4y}{(2\pi)^4}e^{iq\cdot y}K_{x,y} \tag{7.180}$$

$$= \left(-p^2 + V''(\phi_0)\right)\delta^{(4)}(p-q). \tag{7.181}$$

The logarithm is then just the diagonal matrix with the logarithms along the diagonal:

$$\left[\ln(K)\right]_{p,q} = \ln\left(-p^2 + V''(\phi_0)\right)\delta^{(4)}(p-q) \tag{7.182}$$

and the trace is:

$$\mathrm{Tr}\left(\ln(\mathrm{K})_{\mathrm{p,q}}\right) = \int d^4p\, \left(\ln(K)\right)_{p,p} \tag{7.183}$$

$$= \frac{V\cdot T}{(2\pi)^4}\int d^4p\, \ln\left(-p^2 + V''(\phi_0)\right) \tag{7.184}$$

Now comparing to Eq. (7.177) we see:

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(\frac{k^2 - V''(\phi)}{k^2}\right) + \mathcal{O}(\hbar^2) \tag{7.185}$$

where we used the following relation:

$$\delta^4(0) = \int \frac{d^4x}{(2\pi)^4}\, e^{-ip\cdot x}\bigg|_{p=0} = \int\frac{d^4x}{(2\pi)^4} = \frac{V\cdot T}{(2\pi)^4}.$$

where VT is the space-time volume.

where we have added a ϕ-independent constant to make the argument of the logarithm dimensionless.
The integral is clearly (quadratically) divergent. We Wick-rotate:

$$V_{\text{eff}}(\phi) = V(\phi) + \underbrace{\frac{\hbar}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left(\frac{k_E^2 + V''(\phi)}{k_E^2}\right)}_{I(m_{eff}^2)} \tag{7.186}$$

We can just use a hard cut-off:

with $V''(\phi) = m_{\text{eff}}^2(\phi)$.

$$I(m_{\text{eff}}^2) = \frac{2\pi^2}{2(2\pi)^4}\hbar\int_0^\Lambda dk_E\, k_E^3\ln\left(1 + \frac{m_{\text{eff}}^2}{k_E^2}\right) + \text{const.} \tag{7.187}$$

$$= \frac{1}{128\pi^2}\left(2m_{\text{eff}}^2\Lambda^2 + 2m_{\text{eff}}^4\ln\left(\frac{m_{\text{eff}}^2}{\Lambda^2}\right) + \text{const.}\right) \tag{7.188}$$

The full effective potential is therefore:

$$V_{\text{eff}}(\phi) = V(\phi) + c_1 + c_2 m_{\text{eff}}^2(\phi) + \frac{1}{64\pi^2}m_{\text{eff}}^4\ln m_{\text{eff}}^2(\phi) + c_3 m_{\text{eff}}^4(\phi)$$

where c_1, c_2, c_3 are divergent, regulator-dependent, but **ϕ-independent** constants. We remove divergences by renormalization, the physics content resides as usual in the logarithmic term. We define renormalized values for λ_R, m_R^2 and the 'cosmological constant' Λ_R.

$c_2 = \frac{\Lambda^2}{64\pi^2}$ and so on.
Recall,

$$m_{\text{eff}}^2(\phi) = V''(\phi) = m^2 + \frac{\lambda}{2}\phi^2.$$

$$\Lambda_R \equiv c_1 + c_2 m^2 + c_3 m^4 \tag{7.189}$$

$$m_R^2 \equiv m^2 + \lambda c_2 + 2\lambda m^2 c_3 \tag{7.190}$$

$$\lambda_R = \lambda + 6\lambda^2 c_3 \tag{7.191}$$

where the coefficients are easily deduced by e.g. comparing

$$\left(m_{eff}^2\right)^2 c_3 = (m^4 + m^2\phi^2\lambda + \phi^4\frac{\lambda^2}{4})c_3 \qquad \text{vs.} \qquad \frac{\lambda\phi^4}{24}.$$

Finally, we get for the *Coleman-Weinberg* potential at one-loop:

$$V_{\text{eff}}^{\phi}(\phi) = \frac{1}{2}m_R^2\phi^2 + \frac{\lambda_R}{4!}\phi^4 + \hbar\frac{\left(m_R^2 + \frac{\lambda_R}{2}\phi^2\right)^2}{64\pi^2}\ln\left(m_R^2 + \frac{\lambda_R\phi^2}{2}\right). \tag{7.192}$$

where we have dropped the cosmological constant term.

The derivation for spinors is very similar, if we integrate over fermionic fluctuations coupling to the scalar ϕ via a Yukawa coupling:

$$\mathcal{L}_{\text{yukawa}} = y\,\phi\overline{\psi}\psi \tag{7.193}$$

which gives a ϕ-dependent mass

Compare to the quark and lepton masses in the Standard Model.

$$m_\psi(\phi) = m_\psi(0) + y\phi. \tag{7.194}$$

We can easily derive:

$$\begin{aligned} V_{\text{eff}}^{tot}(\phi) &= V_{\text{eff}}^{\phi}(\phi) + V_{\text{eff}}^{\psi}(\phi) \\ &= \Lambda_R + \frac{1}{2}m_R^2\phi^2 + \frac{\lambda_R}{4!}\phi^4 \\ &+ \hbar\frac{\left(m_R^2 + \frac{\lambda_R}{2}\phi^2\right)^2}{64\pi^2}\ln\left(m_R^2 + \frac{\lambda_R}{2}\phi^2\right) \\ &\underset{\substack{\uparrow \\ \text{negative sign} \\ \text{due to fermion loop}}}{-}\hbar\frac{(m_\psi(0) + y\phi)^4}{32\pi^2}\ln\left((m_\psi(0) + y\phi)^2\right) \end{aligned}$$

For large values of ϕ, the potential can turn negative:
e.g. in SM Higgs sector: the top-Higgs coupling drives the potential negative![10]

[10] See e.g. https://arxiv.org/pdf/1205.6497.pdf
Further reading: how to resum the contributions to reliably calculate the corrections for large field values: Schwartz 34.2.2.

7.6 The Anderson-Higgs mechanism

How does spontaneous symmetry breaking manifest itself in gauge theories?

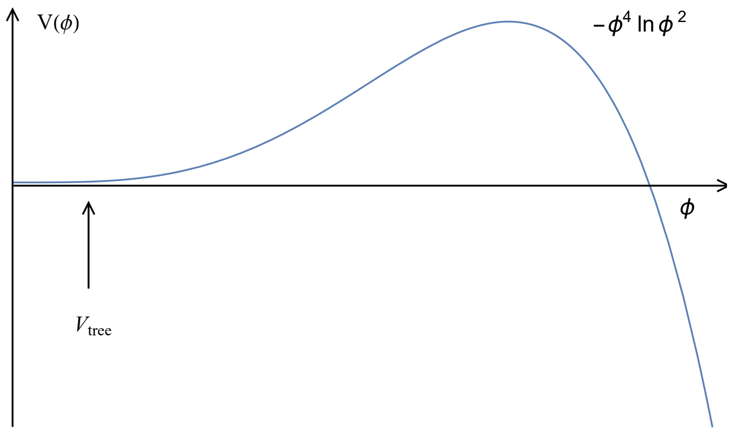


Figure 7.3: Potential for large values of ϕ.

Gauge transformation:

$$A_\mu \to A_\mu - \partial_\mu \Lambda \tag{7.195}$$

Promote global symmetry to local invariance, e.g.

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \tag{7.196}$$
$$A_\mu(x) \to A_\mu + \frac{1}{g}\partial_\mu\alpha(x). \tag{7.197}$$

We should use gauge-covariant derivatives:

$$\partial_\mu\phi \quad \text{becomes} \quad D_\mu\phi = (\partial_\mu - igA_\mu)\phi \tag{7.198}$$

$$D_\mu\phi \to (\partial_\mu - igA_\mu - i\partial_\mu\alpha(x))e^{i\alpha(x)}\phi(x) \tag{7.199}$$
$$= e^{i\alpha(x)}(\partial_\mu - igA_\mu)\phi(x) \tag{7.200}$$
$$= e^{i\alpha(x)}D_\mu\phi \tag{7.201}$$

The invariant Lagrangian is then:

$$\mathcal{L} = D_\mu\phi^\dagger D_\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{7.202}$$

7.6.1 Abelian Higgs mechanism

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D\phi)^\dagger D\phi - (-\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2) \tag{7.203}$$

For the ground state use the energy functional

$$E(A_\mu, \phi) = \int d^3x \left[\frac{1}{2}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2 + (D_0\phi)^\star D_0\phi + (D\phi_i)^\star D\phi_i + V(\phi^\star, \phi)\right] \tag{7.204}$$

where $\frac{1}{2}(F_{(0i)})^2 + \frac{1}{4}(F_{ij})^2$ are the gauge kinetic terms and $(D_0\phi)^\star D_0\phi + (D\phi_i)^\star D\phi_i$ are the scalar kinetic terms.
Now find the configuration of ϕ, A_μ which minimizes $E \to$ ground state. If $(A_\mu^{(0)}, \phi^{(0)})$ is a ground state, then $(A_\mu + \frac{1}{g}\partial_\mu\alpha(x), e^{i\alpha(x)}\phi(x))$ is also one!

Gauge kinetic terms are minimal if A_μ is a "pure gauge",

$$A_\mu = \frac{1}{g}\partial_\mu \alpha(x) \tag{7.205}$$

The **scalar kinetic** terms are minimized by

$$D_\mu \phi = (\partial_\mu - i\partial_\mu \alpha(x))\phi(x) = 0 \tag{7.206}$$

or

$$\phi(x) = e^{i\alpha(x)}\frac{\phi_0}{\sqrt{2}} \tag{7.207}$$

where ϕ_0 =const. We can get ϕ_0 from $V(\phi^\star, \phi)$

$$\phi_0 = \frac{\mu}{\sqrt{\lambda}}. \tag{7.208}$$

We need to pick one: $\alpha(x) \equiv 0$. So we get

$$A_\mu^{(0)} = 0 \quad \phi^{(0)} = \frac{\phi_0}{\sqrt{2}} \tag{7.209}$$

Excitations about the ground state:

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_0 + \chi(x) + \underset{\substack{\uparrow \\ \text{This was a massless GB in global U(1)}}}{i\theta(x)}) \quad \chi, \theta \text{ real fields} \tag{7.210}$$

To quadratic order we have

$$V(\chi, \theta) = \mu^2\chi^2 + (\text{field}^3) \tag{7.211}$$

$$D_\mu \phi = \frac{1}{\sqrt{2}}(\partial_\mu \chi + i\partial_\mu \theta - ie\phi_0 A_\mu) + (\text{field}^2) \tag{7.212}$$

Thus to quadratic order:

$$\mathcal{L}^{(2)} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2\chi^2 + \frac{e^2\phi_0^2}{2}\left(A_\mu - \frac{1}{e\phi_0}\partial_\mu \theta\right)^2 \tag{7.213}$$

It is unusual to have A_μ^2, $(\partial_\mu \theta)^2$, but also the cross term $A_\mu \partial^\mu \theta$.
Change the field variables:

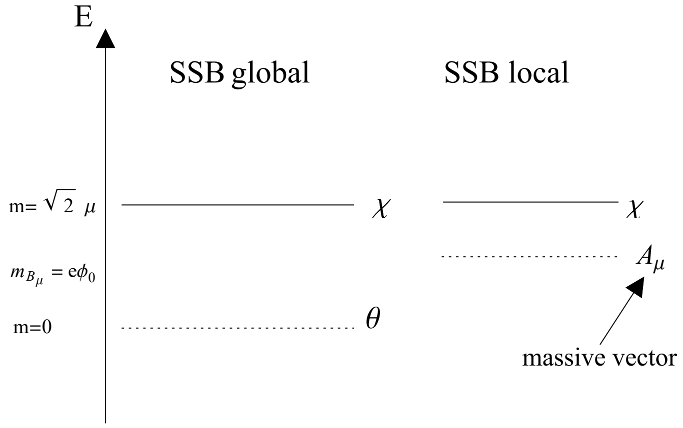
This clearly leaves the form of $F_{\mu\nu}^2$ invariant since it has the form of a gauge transformation of A_μ.

$$B_\mu = A_\mu - \frac{1}{e\phi_0}\partial_\mu \theta. \tag{7.214}$$

The Lagrangian is then:

$$\mathcal{L}^{(2)} = -\frac{1}{4}B_{\mu\nu}^2 + \frac{e^2\phi_0^2}{2}B_\mu B^\mu + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2\chi^2. \tag{7.215}$$

A mass for a vector field appears, and $\theta(x)$ is gone! **The vector field has "eaten" the GB.**



Go now to the non-linear sigma parametrization

$$\phi(x) = \frac{1}{\sqrt{2}}\rho(x)e^{i\beta(x)} \tag{7.216}$$

The gauge transformation:

$$\rho \to \rho \tag{7.217}$$
$$\beta(x) \to \beta(x) + \alpha(x) \tag{7.218}$$
$$\phi(x) \to e^{i\alpha(x)}\phi(x) \tag{7.219}$$

So $D_\mu\phi$ is

$$D_\mu\phi = \frac{1}{\sqrt{2}}(\partial_\mu\rho + i\rho(\partial_\mu\beta - eA_\mu))e^{i\beta(x)} \tag{7.220}$$

Thus the Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(\partial_\mu\rho)^2 - V(\rho) + \frac{e^2\rho^2}{2}(A_\mu - \frac{1}{e}\partial_\mu\beta)^2 \tag{7.221}$$

Re-parametrizing again $B_\mu = A_\mu - \frac{1}{e}\partial_\mu\beta$, then the full Lagrangian is independent of $\beta(x)$:

$$\mathcal{L} = -\frac{1}{4}B_{\mu\nu}^2 + \rho^2(x)\frac{e^2}{2}B_\mu^2 + \frac{1}{2}(\partial\rho)^2 - V(\rho) \tag{7.222}$$

B_μ is **gauge-invariant**!

We could have arrived here by fixing the gauge $\beta(x) = 0$

7.6.2 Non-abelian Higgs mechanism: SU(2)

SU(2) non-abelian gauge theory

$$\phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \quad \text{fundamental representation} \quad \phi_1, \phi_2 \quad \text{complex}$$

The Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - (-m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2) \tag{7.223}$$

The energy function in this case is $E(A^a_\mu, \phi)$.[11]

[11] It is the same as above, but now we have $F_{\mu\nu} \to F^a_{\mu\nu}$.

The ground state is

$$F^a_{\mu\nu} = 0 \tag{7.224}$$

and the pure gauge is

$$A_\mu(x) = \omega(x)\partial_\mu \omega^{-1}(x) \tag{7.225}$$

with $\omega(x) \in \mathrm{SU}(2)$.
Scalar: $D_\mu \phi(x) = 0$ and $\phi(x) = \omega(x)\phi^{(0)}$, with $\phi^{(0)}$ a complex two vector (constant).
We get $\phi^{(0)}$ from

$$\frac{\partial V}{\partial \phi^\dagger} = 0 = \frac{\partial V}{\partial \phi} \quad \text{(minimization)} \tag{7.226}$$

$$\phi^\dagger \phi = \frac{\mu^2}{2\lambda} \tag{7.227}$$

Choose:

$$A^{a(0)}_\mu = 0 \tag{7.228}$$

$$\phi^{(0)} = \begin{pmatrix} 0 \\ \frac{\phi_0}{\sqrt{2}} \end{pmatrix} \tag{7.229}$$

with $\phi_0 = \frac{\mu}{\sqrt{\lambda}} = \sqrt{2}v$.

We can write the excitations as

$$\phi(x) = \omega(x) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(\phi_0 + \chi(x)) \end{pmatrix} \tag{7.230}$$

Can we really do this? LHS of Eq. (7.230):

$$\begin{pmatrix} \xi_1(x) + i\xi_2(x) \\ \frac{\phi_0}{\sqrt{2}} + \xi_3(x) + i\xi_4(x) \end{pmatrix} \quad (\xi_i \text{ real}) \tag{7.231}$$

and

$$\omega(x) = e^{i\tau^a u^a(x)} \approx 1 + i\tau^a u^a(x) \tag{7.232}$$

RHS of Eq. (7.230):

$$\begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\chi}{\sqrt{2}} \end{pmatrix} + \overset{\begin{pmatrix} u^3 & u^1 - iu^2 \\ u^1 + iu^2 & u^3 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix}}{\overset{\downarrow}{i\tau^a u^a}} \begin{pmatrix} 0 \\ v \end{pmatrix} + \dots \tag{7.233}$$

$$= \begin{pmatrix} ivu^1(x) + vu^2(x) \\ v + \frac{\chi(x)}{\sqrt{2}} - ivu^3(x) \end{pmatrix} \tag{7.234}$$

Eq. (7.230) means any configuration of the field $\phi(x)$ close to the classical vacuum is gauge equivalent to:

We remove the $\omega(x)$ term by a gauge transformation.

$$\phi(x) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(\phi_0 + \chi(x)) \end{pmatrix} \quad \text{“unitary gauge”} \tag{7.235}$$

We expand linearly to arrive at the quadratic Lagrangian:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu \tag{7.236}$$

$$D_\mu \phi = \partial_\mu \phi - ig\frac{\tau^a}{2} A_\mu \phi \tag{7.237}$$

with $\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial_\nu A^a_\mu$.

So in quadratic order we have

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^a_{\mu\nu} + \frac{g^2\phi_0^2}{2} A^a_\mu A^a_\mu + \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2\chi^2. \tag{7.238}$$

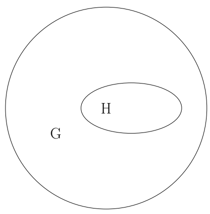
The Higgs mechanism:

1. Three massive vector fields with mass $m_{A_\mu} = g\phi_0$
2. One massive scalar χ with $m_\chi = \sqrt{2}\mu$

The three GB have been eaten by the gauge fields:

Before SSB	# d.o.f.	After SSB	# d.o.f.
A_μ, $m_A = 0 \rightarrow$	$2 \cdot 3$	$m_A \neq 0 \rightarrow$	$3 \cdot 3$
complex $\phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \rightarrow$	4	real $\chi \rightarrow$	1
	10		10

General theorem:



Global symmetry G broken to H

$$\underset{\uparrow}{n(G)} - \underset{\uparrow}{n(H)} = \# \text{ of GB} \tag{7.239}$$

number of generators

Symmetry gauged: G local.
The $n(G) - n(H)$ GB's are eaten by $(n(G) - n(H))-$gauge bosons, leaving $n(H)$ massless gauge bosons.
Examples:

1)	$G = U(1)$	$H = \emptyset$	$n(G) = 1$	$n(H) = 0$
2)	$G = SU(2)$	$H = \emptyset$	$n(G) = 3$	$n(H) = 0$
3)	$G = SO(3)$	$H = SO(2)$	$n(G) = 3$	$n(H) = 1$
4)	$G = SU(5)$	$H = SM = SU(3) \times SU(2) \times U(1)$	$n(G) = 24$	$n(H) = 12$

The case 3 is called **Georgi-Glashow model** which has heavy $W^\pm$ and a massless γ.

Note, that one can show that a theory with massive vector bosons, whose mass is generate by a Higgs mechanism or a linear sigma model, is indeed *renormalizable*. [12] This is intuitive in that in the UV, which means at high energies $E \gg m_A$, spontaneous symmetry breaking becomes a small effect. Thus at high energies, the linear sigma model is just a gauge theory coupled to a linearly transforming matter field, and is renormalizable for the same reason that non-Abelian gauge theories are renormalizable.

[12] 't Hooft, Veltman - https://www.nobelprize.org/prizes/physics/1999/summary/

8 Anomalies

Most of the time, a symmetry of the **classical** theory is also a symmetry of the **quantum** theory. If this is not the case, the symmetry is called **anomalous**.

8.1 Introduction

We found that when the measure of the path integral transforms trivially, that the symmetries $\phi_n \to \phi_n + \epsilon^a F_n^a[\phi_n; x]$ of the classical action

$$\delta_F S[\phi_n] = 0$$

translate into symmetries of the effective quantum action

$$\delta_F \Gamma[\phi] = 0$$

Additionally, they imply relations between n-point functions generated by $Z[J]$, which are the Ward-identities, like the QFT version of current conservation

Note, that we always mean the T^* product, which commutes with time derivatives.

$$\partial_\mu \langle \Omega | T\{j^\mu[\phi_n(x)]\} | \Omega \rangle = 0 \tag{8.1}$$

or in general

$$\begin{aligned}
&\partial_\mu^{(x)} \langle \Omega | T\{j_a^\mu[\phi_n(x)]\phi_{n_1}(x_1) \cdot \ldots \cdot \phi_{n_m}(x_m)\} | \Omega \rangle = \\
&= (-i) \sum_{k=1}^{m} \delta^{(4)}(x - x_k) \langle \Omega | T\{\phi_{n_1}(x_1) \cdot \ldots \cdot \phi_{n_{k-1}}(x_{k-1}) \cdot \\
&\qquad \cdot F_{n_k}^a[\phi_l(x_k)]\phi_{n_{k+1}}(x_{k+1}) \cdot \ldots \cdot \phi_{n_m}(x_m)\} | \Omega \rangle \, .
\end{aligned}$$

This can be spoiled by including $\hbar$ corrections.

8.2 Two types of anomalies: chiral and conformal

QFT knows of two anomalies

- The **conformal anomaly** where a classically scale invariant theory is not scale invariant after quantization. Classically scale invariant theories, are theories with Lagrangians without a dimensionful parameter, e.g. QCD with massless fermions or ϕ^4 theory with a potential like $V(\phi) = m^2\phi^2 + \lambda_3\phi^3 + \lambda_4\phi^4$ where $[m^2] = 2$

and $[\lambda_3] = 1$ are both vanishing. Most QFTs show this anomaly since renormalization always reintroduces an explicit scale and the parameters in the renormalized theory are scale-dependent, described e.g. by means of the β-function. The divergence of the dilatation current would then be proportional to the β function.

See exercise.

- **Chiral symmetries** can also be anomalous and will be the focus of this chapter. The divergence of the current will receive quantum corrections

$$\partial_\mu J_5^\mu = -\hbar \frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{8.2}$$

where we have already spoiled the punch line of this chapter. This is catastrophic for gauge theories: we require $J_5^\mu(x)$ to be conserved in order for a coupling $gJ_5^\mu(x)A_\mu(x)$ to be gauge-invariant. If $J_5^\mu(x)$ is not conserved, longitudinal polarizations will be produced and the gauge field receives a mass m_A, spoiling the renormalizability of the theory. The latter also implies that the theory becomes strongly coupled[1] at a scale $\Lambda = m_A/(g^3/(64\pi^3))$.

Analogously, the effective action $\Gamma[A_\mu]$ is no longer gauge invariant, since under

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha(x)$$

it transforms as

$$\delta_{\alpha(x)}\Gamma[A_\mu] \propto \int d^4x\, \alpha(x) \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

[1] See e.g. John Preskill – Gauge Anomalies in an Effective Field Theory

These **gauge-anomalies** are catastrophic, whereas anomalies of global symmetries do not lead to inconsistencies. In fact, many of the global symmetries of the SM, like baryon number B or lepton number L, are anomalous.
Anomalies have been discovered in the late 60's and have been confusing field theorists for a while. We now understand anomalies at the simple fact that when we change the integration variables in an integral, we better not forget the Jacobian.

Example: $U(1)_B$ (baryon number)

$$J_B^\mu = \sum_i \bar{q}_i \gamma_\mu q_i \quad \text{(q: quarks)} \tag{8.3}$$

is an anomalous, global symmetry of the SM.

8.3 Reminder: Dirac vs. Weyl representation

The Dirac spinor is not a fundamental representation of the Lorentz group.

$$\psi_{Dirac} \sim \underset{\substack{\uparrow \\ \psi_L}}{(2,1)} \oplus \underset{\substack{\uparrow \\ \psi_R}}{(1,2)} \quad \psi_{Dirac} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{8.4}$$

which means

$$\begin{aligned} \mathcal{L} &= \overline{\psi}_{Dirac}(i\partial\!\!\!/ - m)\psi_{Dirac} & (8.5)\\ &= \psi_L^\dagger i\sigma_\mu \partial_\mu \psi_L + \psi_R^\dagger i\bar{\sigma}_\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) & (8.6) \end{aligned}$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$$
$$\sigma_\mu = (\mathbf{1}, \sigma), \bar{\sigma}_\mu = (\mathbf{1}, -\sigma)$$
$$\gamma_0\gamma_\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} = \begin{pmatrix} \sigma_\mu & \\ & \bar{\sigma}_\mu \end{pmatrix},$$
$$\gamma_5 = \begin{pmatrix} -\mathbf{1} & \\ & \mathbf{1} \end{pmatrix}$$

In the massless limit ($m \to 0$), ψ_L and ψ_R decouple.

In fact, ψ_L and ψ_R are different particles in SM, e.g.

$$Q_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix}_{1/6} \quad SU(2) \text{ doublet}$$

$$(t_R)_{2/3}, \underset{\substack{\uparrow \\ U(1)_Y}}{(b_R)_{-1/2}} \quad SU(2) \text{ singlet}$$

This is a **chiral** representation because ψ_L and ψ_R carry different charges.
Compare to a **vector**-like representation: quarks in QCD or electrons in QED. ψ_L and ψ_R carry the same quantum number.

Axial and vector currents vs. chiral rotations
We can independently rotate two Weyl spinors:

$$\begin{cases} \psi_L \to e^{i\alpha_L \psi_L} \\ \psi_R \to e^{i\beta_R}\psi_R \end{cases}$$

both lead to two (classically) conserved currents:

$$J^L_\mu = \psi^\dagger_L \sigma_\mu \psi_L \quad \text{and} \quad J^R_\mu = \psi^\dagger_R \sigma_\mu \psi_R \tag{8.7}$$

alternatively we can use the Dirac formalism.

$$\text{vector} : \psi \to e^{i\delta_V}\psi = e^{i\delta_V} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{8.8}$$

$$\text{axial} : \psi \to e^{i\gamma_5 \rho_A}\psi = \begin{pmatrix} e^{-i\rho_A}\psi_L \\ e^{i\rho_A}\psi_R \end{pmatrix} \tag{8.9}$$

Compare:

$$\alpha_L = \delta_V - \rho_A \tag{8.10}$$

$$\beta_R = \delta_V + \rho_A. \tag{8.11}$$

Axial current:

$$J^5_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi. \tag{8.12}$$

Be careful when comparing to other literature how the axial current is defined, since $\overline{\psi}\gamma_\mu\gamma_5\psi = -\overline{\psi}\gamma_5\gamma_\mu\psi$. Classically:

$$\partial_\mu J^5_\mu = \overline{\psi}\overleftarrow{\not\partial}\gamma_5\psi - \overline{\psi}\gamma_5\overrightarrow{\not\partial}\psi \tag{8.13}$$

$$\overset{\text{EOM}}{=} 2mi\,\overline{\psi}\gamma_5\psi \tag{8.14}$$

$$= 2m\,J_5. \tag{8.15}$$

$$\overset{m\to 0}{=} 0 \tag{8.16}$$

Chiral density:

$$J_5 = i(\psi^\dagger_R\psi_R - \psi^\dagger_L\psi_L). \tag{8.17}$$

We will find that the axial current is indeed not conserved.

$$\partial_\mu j^\mu_5 = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \tag{8.18}$$

vs. classically:

$$\partial_\mu j^\mu_5 = 0 \quad (m \to 0) \tag{8.19}$$

8.4 Deriving the anomaly from the path integral measure

The chiral anomaly can be derived starting from the **functional integral for fermions**.[2] First of all, recall the standard form of functional integral:

[2] Fujikawa, Kazuo: *Phys.Rev.Lett.* **42** (1979), 1195.

$\{\varphi_n\}$: consider first scalar fields for simplicity.

$$Z_0 = Z[J=0] = \int \mathcal{D}[\varphi_n] e^{i\int d^4x\, \mathcal{L}[\varphi_n]}. \tag{8.20}$$

Recall, the master formula for the generating functional after a transformation of the fields:

$$0 = \int D\varphi_n e^{iS[\varphi_n]+i\int J_n\varphi_n} \int d^4x \Big\{ \sum_n \frac{\delta F_n[\varphi_n(x),x]}{\delta\varphi_n(x)} + i\Big(\frac{\delta S[\varphi_n]}{\delta\varphi_n(x)} + J_n(x)\Big) F_n[\varphi_n(x),x] \Big\}.$$

If the first term in the curly brackets does not vanish, we call the symmetry **anomalous**, that is, *if the path integral measure is not invariant.*

If the change of variables is a symmetry of the classical action, then we can replace $\frac{\delta S[\varphi_n]}{\delta\varphi_n(x)} F_n^a[\varphi_n(x),x] \to -\partial^\mu j_\mu^a(x)$ where $j_\mu^a(x)$ is the Noether current of this symmetry.

Remember that the first term in the curly brackets was derived from the Jacobian of the path integral measure:

$$\begin{aligned}
\left|\det\left[\frac{\delta\varphi_n'(y)}{\delta\varphi_n(x)}\right]\right| &= \exp\left[\mathrm{tr}\Big(\ln\Big(1+\epsilon\frac{\delta F_n[\varphi_n(y),y]}{\delta\varphi_n(x)}\Big)\Big)\right] \\
&= 1 + \epsilon\cdot\mathrm{tr}\Big(\frac{\delta F_{n'}[\varphi_n(y),y]}{\delta\varphi_n(x)}\Big) \\
&= 1 + \epsilon\int \mathrm{d}^4x \sum_n \left(\frac{\delta F_n[\varphi_n(x),x]}{\delta\varphi_n(x)}\right) + \epsilon^2\cdot\ldots,
\end{aligned}$$

where the trace in Eq. (8.21) is taken over n, n' and x, y, which translates to the sum and integral in the next line, setting $x=y$ and $n=n'$.

Now consider the **fermionic** case:

$$Z_0 = Z[J=0]\int \mathcal{D}[\psi,\overline{\psi}]\, e^{i\int d^4x\,\overline{\psi}(i\not{D})\psi}. \tag{8.21}$$

The appearance of anomaly is only related to fermions, it is enough to consider this.

8.4.1 Chiral rotation of the fermion path integral measure

We do a chiral transformation:

$$\psi \to (\mathbb{1}+i\alpha(x)\gamma_5)\psi = \psi' \tag{8.26}$$

$$\overline{\psi} = \psi^\dagger\gamma^0 \to \psi^\dagger(\mathbb{1}-i\alpha(x)\gamma_5)\gamma^0 = \overline{\psi}(\mathbb{1}+i\alpha(x)\gamma_5) = \overline{\psi}' \tag{8.27}$$

with $\alpha = \alpha(x)$.

We get naively

$$\begin{aligned}
\mathrm{tr}\Big(\frac{\delta F_{n'}[\varphi_n(y),y]}{\delta\varphi_n(x)}\Big) \to &(-1)\left[\mathrm{tr}\Big(\frac{\delta F[\psi;y]}{\delta\psi(x)}\Big) + \mathrm{tr}\Big(\frac{\delta F[\overline{\psi};y]}{\delta\overline{\psi}(x)}\Big)\right] \\
=&(-1)\cdot 2\,\underset{=0}{\underset{\uparrow}{\mathrm{tr}(\gamma_5)}}\cdot \underset{='\infty'}{\underset{\uparrow}{\delta^{(4)}(x-x)}}\cdot\int d^4x
\end{aligned}$$

Recall the **Noether-trick** using a space-time dependent transformation parameter. Consider a symmetry transformation:

$$\phi_e \to \phi_e + \alpha \Delta \phi_e. \tag{8.22}$$

For α =const. the action is invariant, and the Lagrangian changes by a total derivative $\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu J^\mu$.. Now make $\alpha \to \alpha(x)$ spacetime-dependent:

$$\begin{aligned} \mathcal{L} &\to \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \phi_e} \delta \phi_e + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_e)} \delta(\partial_\mu \phi_e) \\ &= \mathcal{L} + \underbrace{\alpha \left(\frac{\delta \mathcal{L}}{\delta \phi_e} \Delta \phi_e + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_e)} (\partial_\mu \Delta \phi_e) \right)}_{\substack{\text{already present for} \\ \alpha = \text{const.}}} + \underbrace{(\partial_\mu \alpha) \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_e)} \Delta \phi_e}_{\text{new term}} \end{aligned} \tag{8.23}$$

where we used for $\delta(\partial_\mu \phi_e)$ in Eq. (8.23):

$$\partial_\mu \phi_e \to \partial_\mu \phi_e + (\partial_\mu \alpha) \Delta \phi_e + \alpha \partial_\mu (\Delta \phi_e). \tag{8.24}$$

Hence for $\alpha = \alpha(x)$:

$$\mathcal{L} \to \mathcal{L} + \alpha(x) \partial_\mu K^\mu + (\partial_\mu \alpha(x)) \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_e)} \Delta \phi_e \tag{8.25}$$

and so the classical conservation equation is found

$$\begin{aligned} 0 &= \frac{\delta}{\delta \alpha(x)} \int d^4 y \, \mathcal{L}[\phi_e + \alpha(x) \Delta \phi] \\ &= \frac{\delta}{\delta \alpha(x)} \int d^4 y \left\{ \mathcal{L} + \alpha(x) \partial_\mu K^\mu - \alpha(x) \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_e)} \Delta \phi_e \right\} \\ &= \partial_\mu \underbrace{\left(K^\mu - \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi_e)} \Delta \phi_e \right)}_{= -J^\mu}. \end{aligned}$$

J^μ is the Noether current as we know. Applying this to a variable transformation of the path integral measure in the generating functional then results in the Ward Identities above, see QFT1.

where the (-1) comes from the fact that we get the *inverse* Jacobi determinant, see Eq. (8.30) below. The tr[...] is over x, y and the Dirac index. We have used $F[\psi; y] = \gamma_5 \psi(y)$ and $F[\overline{\psi}; y] = \gamma_5 \overline{\psi}$.

This is clearly ill-defined

$$0 \times \infty = ?$$

and not too surprising: we have not been very precise in the definition of the path integral measure so far. We will now define a basis for $\psi, \overline{\psi}$ using the eigenfunctions of the Dirac operator $(i\not{D})$, for a fixed gauge configuration $A_\mu(x)$. This basis exists since this is a hermitian operator. Strictly speaking this only holds in Euclidean space into which we can however Wick rotate.

Let us recall: for a unitary transformation U on the fermions

$$\psi \to U(x) \psi = \psi' \tag{8.28}$$

$$\overline{\psi} = \psi^\dagger \gamma^0 \to \psi^\dagger U^\dagger \gamma^0 = \overline{\psi} \underbrace{\gamma^0 U^\dagger \gamma^0}_{\overline{U}} = \overline{\psi}\, \overline{U} \tag{8.29}$$

Then we need the following relations:[3]

Consider an example with one variable. For **c-numbers** we have the following relations:

$$\int dx \, f(x).$$

With the transformation $x' = ax$ we get

$$\int dx' \left| \frac{dx}{dx'} \right| f\left(\frac{x'}{a} \right),$$

e.g. $f(x) = x$

$$\begin{aligned} &\int_0^1 dx x = \left. \frac{x^2}{2} \right|_0^1 = \frac{1}{2} \\ &\int_0^a dx' \, \frac{1}{a} \frac{x'}{a} = \frac{1}{a^2} \int_0^a dx' \, x' = \frac{1}{2}. \end{aligned}$$

For the fermionic example,see Eq. (8.32).

[3] Note, that these are determinants in functional space and not just in Dirac 4×4 – like the traces which are mostly functional except if we explicitly state that they are Dirac traces.

$$\mathcal{D}\psi' = (\det U)^{-1}\mathcal{D}\psi \tag{8.30}$$
$$\mathcal{D}\overline{\psi}' = (\det \overline{U})^{-1}\mathcal{D}\overline{\psi} \tag{8.31}$$

The transformation with the **inverse** Jacobian is due to the anti-commuting nature of the fermion fields, see Eq. (8.32).

For **Grassmann-numbers**:

$$\int d\theta\, f(\theta)$$

With $\theta' = a\theta$ we have

$$\int d\theta' \left|\frac{d\theta}{d\theta'}\right|^{-1} f\left(\frac{\theta'}{a}\right), \tag{8.32}$$

e.g. $f(\theta) = \theta$, we find

$$1 = \int d\theta\theta \to \int d\theta' \left(\frac{1}{a}\right)^{-1} \frac{\theta'}{a}$$
$$= \int d\theta'\theta' = 1.$$

Now we have

- Vector: $U = e^{i\alpha}$ and

$$\overline{U} = \gamma^0 U^\dagger \gamma^0 = \gamma^0 e^{-i\alpha}\gamma^0 = e^{-i\alpha} = U^{-1} \tag{8.33}$$

  It follows

$$D[\psi', \overline{\psi}'] = D[\psi, \overline{\psi}]\, \underbrace{(\det U)^{-1}(\det U^{-1})^{-1}}_{=1}. \tag{8.34}$$

  So we can conclude that the measure is invariant: we can safely derive the Ward-identities in the usual way when considering the vector transformations.

- Chiral: $U = e^{i\alpha\gamma_5}$ and

$$\overline{U} = \gamma^0 U^\dagger \gamma^0 = \gamma^0 e^{-i\alpha\gamma_5}\gamma^0 = e^{i\alpha\gamma_5} = U \tag{8.35}$$

  It follows

$$\mathcal{D}[\psi', \overline{\psi}'] = \mathcal{D}[\psi, \overline{\psi}] \quad \underbrace{(\det U)^{-2}}_{\text{not necessarily } =1} \quad . \tag{8.36}$$

  We must now compute this extra factor $(\det U)^{-2}$ carefully.

We begin by defining a basis of eigenstates of Dirac operator $i\not{D}$ (for a given configuration of A_μ):

$$(i\not{D})\phi_m = l_m \phi_m \tag{8.37}$$
$$\hat{\phi}_m(i\overleftarrow{\not{D}}) = l_m \hat{\phi}_m \tag{8.38}$$

We have to expand ψ and $\overline{\psi}$:

For $A_\mu = 0$: these are plane waves with definite momentum, and $l_m^2 = k^2 = k_0^2 - \vec{k}^2$.

$$\psi(x) = \sum_m \overset{\substack{\text{anticommuting coefficient}\\ \downarrow}}{a_m} \underset{\substack{\uparrow\\ \text{c-number}}}{\phi_m(x)} \tag{8.39}$$
$$\overline{\psi}(x) = \sum_m \hat{a}_m \hat{\phi}_m(x). \tag{8.40}$$

then the measure becomes

$$\mathcal{D}[\psi, \overline{\psi}] = \prod_m da_m d\hat{a}_m. \tag{8.41}$$

Now the action of the chiral transformation gives

$$\sum_m a'_m \phi_m = \psi' = (\mathbb{1} + i\alpha(x)\gamma_5)\psi = (\mathbb{1} + i\alpha(x)\gamma_5)\sum_n a_n \phi_n. \tag{8.42}$$

Take $\int d^4x\, \hat{\phi}_q(x)$ on both sides:

$$\sum_m a'_m \int d^4x\, \underbrace{\phi_m(x)\hat{\phi}_q(x)}_{=\delta_{mq}} = \int d^4x\, \hat{\phi}_q(\mathbb{1} + i\alpha(x)\gamma_5) \sum_n a_n \phi_n \quad (8.43)$$

So

$$a'_q = a_q + i \sum_n \int d^4x\, \alpha(x)\hat{\phi}_q \gamma_5 \phi_n(x) a_n \quad (8.44)$$

Where we have used that the eigenfunctions of a hermitian operator are orthogonal.

and the procedure is similar for $\hat{a}'_q$.
Now we define

$$a'_q = (\delta_{qn} + C_{qn})a_n \quad (8.45)$$

or in matrix notation:

$$a' = (\mathbb{1} + \mathbf{C})a \quad (8.46)$$

with $C_{qn} = i \int d^4x\, \alpha(x)\hat{\phi}_q \gamma_5 \phi_n(x)$.
Then we calculate:

$$\left| \det \frac{\delta[\psi', \overline{\psi}']}{\delta[\psi, \overline{\psi}]} \right|^{-1} = (\det(\mathbb{1} + \mathbf{C}))^{-1} \overset{\det M = e^{\mathrm{Tr}\ln \mathrm{M}}}{\overset{\downarrow}{=}} e^{-\mathrm{Tr}\ln(\mathbb{1}+\mathbf{C})} \quad (8.47)$$

$$\underset{\substack{\uparrow \\ \alpha \text{ is infinitesimal.}}}{\simeq} e^{-\mathrm{Tr}\,\mathbf{C}} \overset{\substack{\text{equal contribution from} \\ \psi \text{ and } \overline{\psi} \\ \downarrow}}{=} e^{-2i \sum_n \int d^4x\, \alpha(x)\hat{\phi}_n \gamma_5 \phi_n(x)} \quad (8.48)$$

Note that the exponent naively looks like $\mathrm{Tr}[\gamma_5] = 0$, but again preserving gauge invariance will be crucial: regulate

$$\sum_n \hat{\phi}_n(x)\gamma_5\phi_n(x) = \lim_{\Lambda\to\infty} \sum_n \hat{\phi}_n \gamma_5 \phi_n e^{l_n^2/\Lambda^2} \quad (8.49)$$

$$= \lim_{\Lambda\to\infty} \sum_n \hat{\phi}_n \gamma_5 e^{(i\not{D})^2/\Lambda^2} \phi_n. \quad (8.50)$$

Note: This is a gauge-invariant regulator, which will dampen the UV modes. This is where the dependence on A_μ will come from. One can show that the exact form of the regulator $f(l_m) = e^{l_m^2/\Lambda^2}$ is irrelevant. We could have chosen any function which satisfies $f(0) = 1$ and $f(\infty) = f'(\infty) = f'(\infty) = \ldots = 0$.

As we will also see when computing this diagrammatically using triangle diagrams, it is requiring gauge invariance[4] that forces the anomaly to appear.
Choosing to evaluate the trace in momentum space rather than on $\hat{\phi}, \phi$-basis, we write

$$\lim_{\Lambda\to\infty} \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \underbrace{\mathrm{Tr}\left[\gamma_5 \mathrm{e}^{(\mathrm{i}\not{D})^2/\Lambda^2}\right]}_{\text{Dirac trace only}} e^{-ik\cdot x} \quad (8.51)$$

Now we look at the term $(i\not{D})^2$:

$$\begin{aligned}(i\not{D})^2 &= \gamma^\mu\gamma^\nu iD_\mu iD_\nu = \left[\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right] iD_\mu iD_\nu \\ &= -D^2 + \frac{1}{i}\frac{i}{2}[\gamma^\mu, \gamma^\nu]\, iD_\mu iD_\nu = -D^2 + \frac{1}{2i}\sigma^{\mu\nu}[iD_\mu, iD_\nu] \\ &= -D^2 + \frac{1}{2i}\sigma^{\mu\nu} ieF_{\mu\nu} = -D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\end{aligned}$$

[4] Here: in the form of a gauge-invariant regulator.

hence Eq. (8.51) becomes

$$= \lim_{\Lambda\to\infty} \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\mathrm{Tr}\left[\gamma_5 \mathrm{e}^{\left(-\mathrm{D}^2+\frac{\mathrm{e}}{2}\sigma^{\mu\nu}\mathrm{F}_{\mu\nu}\right)/\Lambda^2}\right] \mathrm{e}^{-\mathrm{ik}\cdot\mathrm{x}}. \tag{8.52}$$

Now, to get a nonzero answer for the Dirac trace we need to lower at least 4 γ-matrices from the exponent, since $\mathrm{tr}[\gamma_5] = \mathrm{Tr}[\gamma_5\gamma^\alpha\gamma^\beta] = 0$ and

$$\mathrm{tr}\left[\gamma_\alpha\gamma_\beta\gamma_\delta\gamma_\gamma\gamma_5\right] = -4i\epsilon^{\alpha\beta\delta\gamma}. \tag{8.53}$$

Hence the leading term is $\propto \mathrm{Tr}[\gamma_5\frac{(\sigma^{\mu\nu}\mathrm{F}_{\mu\nu})^2}{\Lambda^4}]$ and we find

$$\int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\, \mathrm{Tr}\left[\gamma_5 \mathrm{e}^{\left(-\mathrm{D}^2+\frac{\mathrm{e}}{2}\sigma^{\mu\nu}\mathrm{F}_{\mu\nu}\right)/\Lambda^2}\right] \mathrm{e}^{-\mathrm{ik}\cdot\mathrm{x}}$$
$$= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x} \left(\mathrm{Tr}\left[\gamma_5\frac{1}{2!}\left(\frac{\mathrm{e}}{2}\right)^2\frac{1}{\Lambda^4}\sigma^{\mu\nu}\sigma^{\alpha\beta}\right]\mathrm{F}_{\mu\nu}\mathrm{F}_{\alpha\beta}\mathrm{e}^{-\frac{\partial^2}{\Lambda^2}} + \mathcal{O}\left(1/\Lambda^6\right)\right) e^{-ik\cdot x},$$

where we replaced $D \to \partial$ in the exponent, neglecting terms of higher order in e.

Now the leading term is including the calculation of the integral[5]

$$\int \frac{d^4k}{(2\pi)^4}\, e^{k^2/\Lambda^2}\,\frac{e^2}{8\Lambda^4}\mathrm{Tr}\left[\gamma_5\sigma^{\mu\nu}\sigma^{\alpha\beta}\right]\mathrm{F}_{\mu\nu}\mathrm{F}_{\alpha\beta} \tag{8.54}$$

$$= \frac{i}{16\pi^2}\quad \underbrace{\Lambda^4\frac{e^2}{8\Lambda^4}}_{\substack{\text{finite for }\Lambda\to\infty\\ \text{Subleading terms}\\ \mathcal{O}\left(\frac{1}{\Lambda^6}\right)\text{then vanish}}}\quad \mathrm{Tr}\left[\gamma_5\sigma^{\mu\nu}\sigma^{\alpha\beta}\right]\mathrm{F}_{\mu\nu}\mathrm{F}_{\alpha\beta} \tag{8.55}$$

$$= \frac{ie^2}{16\pi^2}\frac{1}{8}\mathrm{Tr}\left[\gamma_5\frac{\mathrm{i}}{2}\left[\gamma^\mu,\gamma^\nu\right]\frac{\mathrm{i}}{2}\left[\gamma^\alpha,\gamma^\beta\right]\right]\mathrm{F}_{\mu\nu}\mathrm{F}_{\alpha\beta} \tag{8.56}$$

$$= -\frac{ie^2}{16\pi^2}\frac{1}{32}\underbrace{\mathrm{Tr}\left[\gamma_5\left[\gamma^\mu,\gamma^\nu\right]\left[\gamma^\alpha,\gamma^\beta\right]\right]}_{4(-4i\epsilon^{\mu\nu\alpha\beta})}F_{\mu\nu}F_{\alpha\beta} \tag{8.57}$$

$$= -\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{8.58}$$

where we replaced $\partial \to -ik$.

[5] We use the Wick-rotation $k^0 = ik_E^0$ in Eq. (8.55) to get:

$$\int \frac{d^4k}{(2\pi)^4}\, e^{k^2/\Lambda^2} =$$
$$= i\int \frac{d^4k_E}{(2\pi)^4}\, e^{-k_E^2/\Lambda^2}$$
$$= i\frac{1}{(2\pi)^4}2\pi^2\int_0^\infty dk_E\, k_E^3 e^{-\frac{k_E^2}{\Lambda^2}}$$
$$= \frac{i}{16\pi^4}2\pi^2\frac{\Lambda^4}{2} = \frac{i\Lambda^4}{16\pi^2}.$$

where we see that the exponential has the correct sign and can be UV regulated as promised.

Summary:

$$\left|\det\frac{\delta[\psi',\overline{\psi}']}{\delta[\psi,\overline{\psi}]}\right|^{-1} = e^{-2i\int d^4x\,\alpha(x)\left[\sum_n\hat{\phi}_n(x)\gamma_5\phi_n(x)\right]}$$
$$= e^{-2i\int d^4x\,\alpha(x)\left[-\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]}. \tag{8.59}$$

Hence the transformed Z_0 is:

$$\int\mathcal{D}[\psi',\overline{\psi}']\, e^{i\int d^4x\,\overline{\psi}'(i\not{D})\psi'}$$
$$= \int\mathcal{D}[\psi',\overline{\psi}']\left|\det\frac{\delta[\psi',\overline{\psi}']}{\delta[\psi,\overline{\psi}]}\right|^{-1}\exp\left(i\int d^4x\,\overline{\psi}'(i\not{\partial})\psi'\right)$$
$$= \int\mathcal{D}[\psi',\overline{\psi}']\underbrace{\exp\left(-2i\int d^4x\,\alpha(x)\left[-\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]\right)}_{\text{from measure}}\underbrace{\exp\left(i\int d^4x\left[\overline{\psi}(i\not{D})\psi+\alpha(x)\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi)\right]\right)}_{\text{from }\mathcal{L}}$$
$$= \int\mathcal{D}[\psi',\overline{\psi}']\exp\left(i\int d^4x\left(\overline{\psi}(i\not{D})\psi+\alpha(x)\partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi)+\alpha(x)\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right)\right)$$

Now the change of variables should not change the result of the path integral. We apply $\frac{\delta}{\delta\alpha(x)}$ and can write

This is an abbreviation of the actual Ward-Identity, which is between vacuum states or in an n-point function including contact terms.

$$0 = \partial_\mu(\overline{\psi}\gamma^\mu\gamma_5\psi) + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{8.60}$$

So we get the quantum violation of the conservation of the chiral current:

We have temporarily restored the factors of $\hbar$.

$$\partial_\mu j_5^\mu = -\hbar\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}. \tag{8.61}$$

Note, that a similar, somewhat more tedious procedure exists where one shows the origin of the dialatation anomaly from the Jacobian of the measure, see K. Fujikawa, "Energy Momentum Tensor in Quantum Field Theory," Phys. Rev. D **23** (1981) 2262. doi:10.1103/PhysRevD.23.2262. For an alternative derivation, see exercise.

Comments:

- One can show that the anomaly is **exact at 1-loop**. For this reason the calculation we just did, which is inherently 1-loop, as can be seen e.g. from the fact that we got $\sim \frac{ie^2}{16\pi^2}$, gives the full answer.

- Since the quantum theory is not invariant under chiral transformations, the associated Noether current and charge are **not** conserved.
  An important example: Isospin singlet axial current in QCD
  For $n_{f(lavors)} = 2, Q = \begin{pmatrix} u \\ d \end{pmatrix}$
  The fermion condensate

$$\langle\Omega|\overline{Q}_L Q_R + \overline{Q}_R Q_L|\Omega\rangle \neq 0. \tag{8.62}$$

  is non vanishing in the vacuum. So only the vector symmetries remain unbroken[6].
  The orthogonal transformations (axial):

[6] These correspond to the isospin and baryon number.

$$\text{isospin singlet} \to J_S^\mu = \overline{Q}\gamma^\mu\gamma_5 Q \qquad J_5^{QM} = \overline{Q}\gamma^\mu\gamma_5\sigma^a Q \quad \leftarrow \text{isospin triplet} \tag{8.63}$$

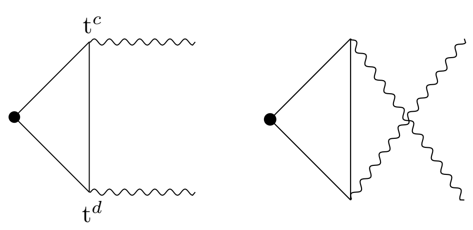
  are instead spontaneously broken. Then we would expect 4 GBs in the spectrum (approx. GBs in real world where $m_u, m_d \neq 0$, but $\ll \Lambda_{QCD}$). However, J_5^μ is the current associated to $U(1)_A$

$$Q_L \to e^{i\alpha}Q_L \tag{8.64}$$
$$Q_R \to e^{-i\alpha}Q_R \tag{8.65}$$

  or

$$Q \to e^{i\alpha\gamma_5}Q. \tag{8.66}$$

  and the same anomaly we just computed for QED shows up, but this time for coupling to gluons of color SU(3):



$$\partial_\mu J_5^\mu = -\frac{g_S^2 n_f}{16\pi^2}\overset{\text{trace over } \mathbb{1}}{\overset{\downarrow}{\mathrm{Tr}}}\left[\mathrm{t^c t^d}\right]\epsilon^{\mu\nu\alpha\beta}\mathrm{G}^{\mathrm{c}}_{\mu\nu}\mathrm{G}^{\mathrm{d}}_{\alpha\beta} \tag{8.67}$$

$$= -\frac{g_S^2 n_f}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}G^c_{\mu\nu}G^c_{\alpha\beta} \tag{8.68}$$

where we used $\mathrm{Tr}\left[\mathrm{t^c t^d}\right] = \frac{1}{2}\delta^{\mathrm{cd}}$.
Hence at quantum level $U(1)_A$ is not actually a symmetry. Then there should be no light GB associated to it. And in fact there is not: the lightest isospin singlet meson (singlet of $SU(n_f = 3)$ in fact is η', $m_{\eta'} \simeq 958\text{MeV}$) while $m_\pi \simeq 135\text{MeV}$. This is one example where the anomaly leaves important physical imprint.

- The functional determinant calculation can easily be generalized to **even dimension** $d = 2n$:
  We can construct

$$\gamma_5 = i\gamma_0\cdots\gamma_{2n-1}, \tag{8.69}$$

so that

$$\gamma_i\gamma_5 = \gamma_i i\gamma_0\cdots\gamma_{2n-1} = (-1)^{2n-1}i\gamma_0\cdots\gamma_{2n-1}\gamma_i \tag{8.70}$$

$$= -\gamma_5\gamma_i \quad \text{(anticommuting)} \tag{8.71}$$

and then we obtain

$$\partial_\mu j_5^\mu = (-1)^{n+1}\frac{2e^n}{n!(4\pi)^n}\epsilon^{\mu_1\cdots\mu_{2n}}F_{\mu_1,\mu_2}\ldots F_{\mu_{2n-1}\mu_{2n}} \tag{8.72}$$

Note: scaling dimensions:
For LHS:

$$[\psi] = \frac{d-1}{2} \tag{8.73}$$

$$\to [\partial_\mu j_5^\mu] = 1 + 2\frac{d-1}{2} = d \tag{8.74}$$

For the RHS:

$$[F_{\mu\nu}] = \frac{d-2}{2} + 1 = \frac{d}{2} \quad \text{and} \quad [e] = 2 - \frac{d}{2} \tag{8.75}$$

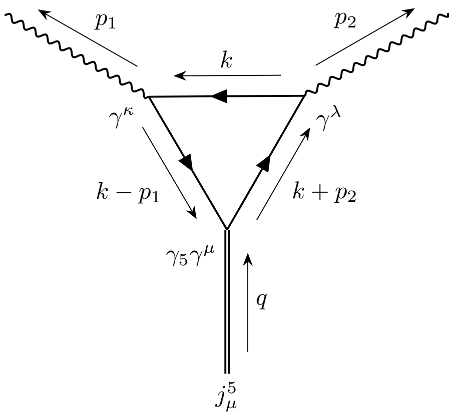
$$[e^n\epsilon F^n] = \frac{d}{2}\left(2 - \frac{d}{2}\right) + \frac{d}{2}\frac{d}{2} = d, \tag{8.76}$$

as it must be.

8.5 *Calculating the anomaly using Feynman diagrams*

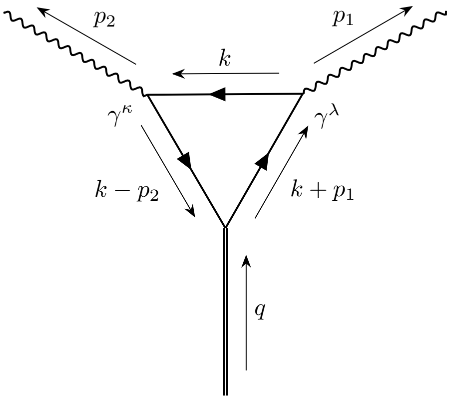
We must analyze the matrix element:

$$\int d^4x\, e^{-iqx}\langle p_1, p_2|j_5^\mu|0\rangle = (2\pi)^4\delta^4(p_1+p_2-q)\epsilon_k^\star(p_1)\epsilon_\lambda^\star(p_2)T_{k\lambda\mu}(p_1,p_2) \tag{8.77}$$



$$= \quad S_{\kappa\lambda\mu}(p_1,p_2).$$

and the diagram with $k \leftrightarrow \lambda$, $p_1 \leftrightarrow p_2$ exchanged



$= S_{\lambda\kappa\mu}(p_2, p_1)$

We find

$$T_{k\lambda\mu}(p_1,p_2) = S_{\kappa\lambda\mu}(p_1,p_2) + S_{\lambda\kappa\mu}(p_2,p_1) \tag{8.78}$$

$$S_{\kappa\lambda\mu}(p_1,p_2) = \underbrace{(-1)}_{\text{Fermion trace}}\overbrace{(-ie)^2}^{\text{Coupling}^2} \int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma_\kappa \frac{\mathrm{i}}{\not{k} - \not{p}_1 - \mathrm{m}}\gamma_\mu\gamma_5 \frac{\mathrm{i}}{\not{k} + \not{p}_2 - \mathrm{m}}\gamma_\lambda \frac{\mathrm{i}}{\not{k} - \mathrm{m}}\right] \tag{8.79}$$

For $m \to 0$ (and $e = 1$ for simplicity reasons) we get:

$$S_{\kappa\lambda\mu}(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}\left[\gamma_\kappa(\not{k} - \not{p}_1)\gamma_\mu\gamma_5(\not{k} + \not{p}_2)\gamma_\lambda\not{k}\right]}{(k-p_1)^2(k+p_2)^2k^2}. \tag{8.80}$$

We want to test the Ward-Identities:

A_1	$q_\mu T^{\kappa\lambda\mu} = (p_1+p_2)_\mu T^{\kappa\lambda\mu} = 0$	(axial)
V_1	$p^1_\kappa T^{\kappa\lambda\mu} = 0$	(vector)
V_2	$p^2_\lambda T^{\kappa\lambda\mu} = 0$	(vector)

Table 8.1: Ward-Identities if they are non-anomalous.

Observations:

1. $S_{\kappa\lambda\mu}$ is linearly divergent!

$$S_{\kappa\lambda\mu} \overset{k\to\infty}{\sim} \int^\Lambda d^4k\,\frac{1}{k^3} \sim \Lambda. \tag{8.81}$$

2. $S_{\kappa\lambda\mu}(p_1,p_2)$ is symmetric under exchange $(p_1,\kappa) \leftrightarrow (p_2,\lambda)$
   $\hookrightarrow$ enough to study $S_{\kappa\lambda\mu}(p_1,p_2)$ vs. $T_{\kappa\lambda\mu}$.

You can check now or find this after simplification later.

What about the axial Ward-identity $(p_1+p_2)^\mu S_{\kappa\lambda\mu}(p_1,p_2) = 0$?

$$\begin{aligned}(p_1+p_2)^\mu S_{\kappa\lambda\mu}(p_1,p_2) = \frac{1}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\Big[\gamma_\kappa\,\overbrace{(\not{k} - \not{p}_1)(\not{p}_1 + \not{p}_2)\gamma_5(\not{k} + \not{p}_2)}^{1}\,\gamma_\lambda\not{k}\Big] \\ \times \frac{1}{(k-p_1)^2(k+p_2)^2k^2}\end{aligned} \tag{8.82}$$

First simplify the $\gamma-$trace.[7]

Let's focus on ①:

$$① = -(\not{k}-\not{p}_1)(\not{k}-\not{p}_1)\gamma_5(\not{k}+\not{p}_2) - (\not{k}-\not{p}_1)\gamma_5(\not{k}+\not{p}_2)(\not{k}+\not{p}_2) \tag{8.83}$$

$$= -(k-p_1)^2\gamma_5(\not{k}+\not{p}_2) - (\not{k}-\not{p}_1)\gamma_5(k+p_2)^2. \tag{8.84}$$

Insert this into Eq. (8.82) and cancel with the denominator:

$$-(2\pi)^4(p_1+p_2)^\mu S_{\kappa\lambda\mu} = \int d^4k\, \frac{\mathrm{Tr}\left[\gamma_\kappa\gamma_5(\not{k}+\not{p}_2)\gamma_\lambda\not{k}\right]}{k^2(k+p_2)^2} + \int d^4k\, \frac{\mathrm{Tr}\left[\gamma_\kappa(\not{k}-\not{p}_1)\gamma_5\gamma_\lambda\not{k}\right]}{k^2(k-p_1)^2}. \tag{8.85}$$

By using a trace identity of the Dirac algebra[8] we find:

$$-4i\epsilon_{\kappa\lambda\alpha\beta}\Big[\underbrace{\int d^4k\, \frac{k^\alpha p_2^\beta}{k^2(k+p_2)^2}}_{\tilde{c}\,p_2^\alpha p_2^\beta} - \underbrace{\int d^4k\, \frac{k^\alpha p_1^\beta}{k^2(k+p_1)^2}}_{c\,p_1^\alpha p_1^\beta}\Big] \tag{8.87}$$

By Lorentz-invariance we know that resulting expression for the integrals has to have two vector indices which have to be made out of the only vector in appearing the two integrands (p_1^μ and p_2^μ, respectively) which when contracted with $\epsilon_{k\lambda\alpha\beta}$ vanishes.[9] Did we just show that $(p_1+p_2)^\mu T_{\kappa\lambda\mu}=0$ and that therefore the axial current is conserved? Not so fast![10]

The reason is that the linear divergence is quite subtle. Let us first study the Ward-Identity of the vector current and then get back to this.

The vector Ward-identity is:

$$p_1^k S_{k\lambda\mu}(p_1,p_2) \circledcirc \tag{8.88}$$

can be simplified similarly with $\not{p}_1 = \not{k} - (\not{k}-\not{p}_1)$:

$$\circledcirc - 4i\epsilon_{\lambda\alpha\mu\sigma}\int \frac{d^4k}{(2\pi)^4}\left(\underbrace{\frac{(k-p_1)_\alpha(k+p_2)_\sigma}{(k-p_1)^2(k+p_2)^2}}_{②} + \underbrace{\frac{k_\alpha(k+p_2)_\sigma}{k^2(k+p_2)^2}}_{\substack{=0\\ \text{b/c of Lorentz}\\ \epsilon_{\lambda\alpha\mu\sigma}p_2^\alpha p_2^\sigma c'=0}}\right). \tag{8.89}$$

Is ② also vanishing?

Let's shift the integral variable $k \to k + p_1$:

$$② \propto \epsilon_{\lambda\alpha\mu\sigma}\int d^4k\, \frac{k_\alpha(k+\tilde{p})_\sigma}{k^2(k+\tilde{p})^2} \propto \epsilon_{\lambda\alpha\mu\sigma}\tilde{p}_\alpha\tilde{p}_\sigma. \tag{8.90}$$

with $\tilde{p} = p_1 + p_2$. Too naive! We cannot just shift linearly divergent integrals.

Alternatively, we could regularize

1. **Dim-reg**: This is difficult since γ_5 is a 4D object $\{\gamma_5,\gamma_M\} =?$ It is possible, but technical (see e.g. Collins-Renormalization)
2. **Pauli-Villars**: massive fermion breaks chiral symmetry

We will make sense of this using no regularization, but we need to discuss linearly divergent integrals first.

[7] Use:
$$(\not{p}_1+\not{p}_2)\gamma_5 = -(\not{k}-\not{p}_1)\gamma_5 - \gamma_5(\not{k}+\not{p}_2)$$

[8]
$$\mathrm{Tr}\left[\gamma_\alpha\gamma_\beta\gamma_\delta\gamma_\gamma\gamma_5\right] = -4i\epsilon^{\alpha\beta\delta\gamma}. \tag{8.86}$$

[9] e.g. $\epsilon_{\kappa\lambda\alpha\beta}\,\tilde{c}\,p_2^\alpha p_2^\beta = 0$

[10] This isn't life in the fast lane, it's life in the oncoming traffic – Terry Pratchett

This shifting was subtle and confusing for many people for a long time.

8.5.1 Linearly divergent integrals

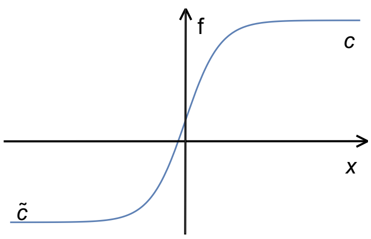
8.5.2 1D - Warm-up

Let's have a closer look at the 1D-case. Let us assume that the following integral is linearly divergent

$$\int_{-\infty}^{\infty} dx\, f(x). \tag{8.91}$$

Define a difference between this integral and an integral with a shifted integrand

$$\Delta(a) = \int_{-\infty}^{\infty} dx\, (\underset{\substack{\uparrow\\ ①}}{f(x+a)} - \underset{\substack{\uparrow\\ ②}}{f(x)}). \tag{8.92}$$



$$f(x) = \begin{cases} c & x \to \infty \\ \tilde{c} & x \to -\infty \end{cases}$$

① and ② are separately linearly divergent, but ① − ② is finite.
Expand around x:

$$\Delta(a) = \int_{-\infty}^{\infty} dx \left[f(x) + af'(x) + \frac{a^2}{2} f''(x) + \ldots - f(x) \right] \tag{8.93}$$

$$= \int_{-\infty}^{\infty} dx \left[af'(x) + \frac{a^2}{2} f''(x) + \ldots \right] = a\,(f(\infty) - f(-\infty)) \tag{8.94}$$

Higher derivatives do not contribute since the $f(\pm\infty) = \text{const.}$ and so e.g.

$$\frac{a^2}{2}\left[f'(\infty) - f'(-\infty)\right] = 0. \tag{8.95}$$

8.5.3 4D-generalization

$$\Delta^\alpha(a^\mu) = \int \frac{d^4k}{(2\pi)^4} \left[F^\alpha(k+a) - F^\alpha(k)\right]. \tag{8.96}$$

We now use the Wick-rotation:

$$\Delta^\alpha(a^\mu) = i \int \frac{d^4k_E}{(2\pi)^4} \left[F^\alpha(k_E + a) - F^\alpha(k_E)\right] \tag{8.97}$$

and Taylor-expand again:

$$(2\pi)^4 \Delta^\alpha(a^\mu) = i \int d^4k_E \left[a^\mu \frac{\partial}{\partial k_E^\mu} F^\alpha(k_E) + \frac{1}{2} a^\mu a^\nu \frac{\partial^2}{\partial k_E^\mu \partial k_E^\nu} F^\alpha(k_E) + \ldots \right] \tag{8.98}$$

We use the Gauss to integrate:

$$\Delta^\alpha(a^\mu) = ia^\mu \int \underbrace{\frac{d^3 S_\mu}{(2\pi)^4}}_{\substack{\uparrow \\ \text{normal to 4-sphere} \\ \text{at } |k_E|\to\infty}} F^\alpha(k_E) \tag{8.99}$$

and $d^3S_\mu = k^2 k_\mu d\Omega_4$.[11] Since the integral is linearly divergent, F^α has to scale as

[11] Dropping the subscript indicating that we are in Euclidean space.

$$\lim_{k_E\to\infty} F^\alpha(k_E) = A \cdot \frac{k_E^\alpha}{k_E^4}.$$

Thus,

$$\Delta^\alpha(a^\mu) = ia^\mu \lim_{|k|\to\infty} \int \frac{d^4\Omega_4}{(2\pi)^4} A \frac{k^\mu k^\alpha}{k^2}. \tag{8.100}$$

Using $k^\alpha k^\mu = \frac{1}{4}k^2\delta^{\mu\alpha}$ and $\Omega_4 = 2\pi^2$, we finally get

$$\Delta^\alpha(a^\mu) = \frac{i}{32\pi^2} A a^\alpha. \tag{8.101}$$

Now we can formulate the general result:

Linear divergent integrals that would vanish if we were allowed to shift are finite, with the result proportional to the necessary shift.

Here:

$$\int d^4k\, F^\alpha(k+a) - F(k) \tag{8.102}$$

could be made to "vanish" if we shift the second term of Eq. (8.102) by $k^\mu \to k^\mu + a^\mu$ and the actual result is finite and proportional to a^μ.

8.5.4 Back to the Ward-identity

Let us apply these new insights to the integral

$$p_1^\kappa S_{\kappa\lambda\mu}(p_1,p_2) = -4i\epsilon_{\lambda\alpha\mu\sigma} \int \frac{d^4k}{(2\pi)^4} \Bigg(\overbrace{k_\alpha k_\sigma}^{\substack{\text{anti-symmetric} \\ =0}} - p_\alpha^1 k_\sigma - \overbrace{p_\alpha^1 p_\sigma^2}^{\substack{\text{not lin. div.} \\ \sim\frac{1}{k^4}}} + k_\alpha p_\sigma^2 \Bigg) \times \frac{1}{(k-p_1)^2(k+p_2)^2} \tag{8.103}$$

Therefore:

$$F^\sigma(k) = -4i\epsilon_{\lambda\alpha\mu\sigma} \frac{(p_1+p_2)_\sigma k_\alpha}{(k-p_1)^2(k+p_2)^2} \tag{8.104}$$

$$\xrightarrow{k^2\to\infty} \underbrace{-4i\epsilon_{\lambda\alpha\mu\sigma}(p_1+p_2)_\sigma}_{A \text{ in Eq. (8.101)}} \frac{k_\alpha}{k^4} \tag{8.105}$$

Back to the axial current divergence:

$$\begin{aligned}&(2\pi)^4(p_1+p_2)^\mu S_{\kappa\lambda\mu}(p_1,p_2)\\&= -4i\epsilon_{\kappa\lambda\alpha\beta}\left(\int d^4k\,\underbrace{\frac{k^\alpha p_2^\beta}{k^2(k+p_2)^2}+\frac{k^\alpha p_1^\beta}{k^2(k+p_1)^2}}_{\star}\right)\end{aligned} \tag{8.106}$$

We use the general shift:

$$k^\mu \to k^\mu + \underbrace{b_1 p_1^\mu + b_2 p_2^\mu}_{a^\mu} \tag{8.107}$$

The linear divergent part is ($\star$):

$$\lim_{k\to\infty}\frac{k^\alpha}{k^4}\underbrace{(p_2^\beta+p_1^\beta)(-\epsilon_{k\lambda\alpha\beta}4i)}_{A} \tag{8.108}$$

Axial:

$$(p_1+p_2)^\mu S_{\kappa\lambda\mu}(p_1,p_2) = -\frac{1}{8\pi^2}\epsilon_{\kappa\lambda\alpha\beta}(b_1p_1+b_2p_2)^\alpha(p_2+p_1)^\beta \tag{8.109}$$

$$= -\frac{1}{8\pi^2}\epsilon_{\kappa\lambda\alpha\beta}(b_1-b_2)p_1^\alpha p_2^\beta. \tag{8.110}$$

8.5.5 Summary

$$\text{axial: } q^\mu S_{\kappa\lambda\mu} = -\frac{1}{8\pi^2}\,\epsilon_{\kappa\lambda\alpha\beta}\,p_1^\alpha p_1^\beta\,(b_1-b_2) \tag{8.111}$$

$$\text{vector: } p_1^\mu S_{\mu\kappa\lambda} = -\frac{1}{8\pi^2}\,\epsilon_{\kappa\lambda\alpha\beta}\,p_1^\alpha p_2^\beta\,(1-b_1+b_2) \tag{8.112}$$

Vector current conserved: We can find a momentum routing $b_1 - b_2 = 1$, which conserves the vector current, but violates the axial current (but not both!). The axial current is anomalous, with as we will see the same coefficient as in the calculation using the path integral.

The necessary shift is $a_\mu = p_\mu^1$. Choosing $b_2 = 0$ in Eq. (8.107).

Axial current conserved: Alternatively, we could choose $b_1 = b_2$ and we would conserve the axial current but the vector current would not be conserved:

$$p_k^1 S_{\kappa\lambda\mu}(p_1,p_2) = -4i\epsilon_{\lambda\alpha\mu\sigma}(p_1+p_2)_\sigma p_\alpha^1\frac{i}{32\pi^2} \tag{8.113}$$

using the antisymmetry of $\epsilon_{\lambda\alpha\mu\sigma}$, we would find a violation of the vector current conservation

$$p_1^k S_{k\lambda\mu} = -\frac{1}{8\pi^2}\epsilon_{k\lambda\mu\nu}p_1^k p_2^\nu. \tag{8.114}$$

This would imply that the vector Ward-Identity is violated? Is QED inconsistent?[12]

[12] Remember: vector U(1) is gauged!

Resolution:

Even though the integral is finite it depends on the shift a_μ of the loop momentum. The choice is arbitrary, but once we choose it, we have to stick with it and evaluate $S_{k\lambda\mu}$ once and for all - no matter what we contract it with $(p_1+p_2)^\mu S_{\kappa\lambda\mu}(p_1,p_2)$ or $p_1^\kappa S_{\kappa\lambda\mu}(p_1,p_2)$.
$\hookrightarrow$ We will choose a shift or k momentum routing that conserves the vector current!

If the fermions had a mass, that choice would be simple, since the mass term violates the conservation of the axial current.

1. We can write the general result as

$$\partial_\mu J_5^\mu = -\frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{8.115}$$

Exercise: check!

2. We can measure the anomaly in $\pi^0 \to \gamma\gamma$

See chapter on CCWZ

$$\Gamma(\pi^0 \to 2\gamma) = \frac{N_c^2 \alpha m_T^2}{144\pi^3 F_\pi^2} = \left(\frac{N_c}{3}\right)^2 \cdot 1.11\cdot 10^{16}\frac{1}{\mathrm{s}} \tag{8.116}$$

$$\text{vs. } \Gamma_{\text{exp}} = (1.19 \pm 0.08)\cdot 10^{16}\frac{1}{\mathrm{s}} \quad \checkmark \tag{8.117}$$

3. Left-handed and right-handed fermion field give opposite contributions

$$\partial_\mu J_R^\mu = -\frac{1}{2}\frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{8.118}$$

$$\partial_\mu J_L^\mu = \frac{1}{2}\frac{e^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{8.119}$$

with $J^\mu_{L/R} = \overline{\psi}\gamma_\mu P_{L/R}\psi$ and $P_{L/R} = \frac{1}{2}(1\mp\gamma^5)$

4. Non-abelian case:

$$\overline{\psi} i\gamma^\mu(\partial_\mu - igA_\mu^a T^a)\psi \tag{8.120}$$

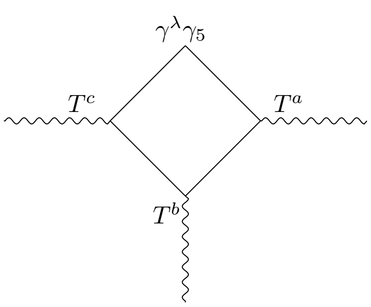
It follows

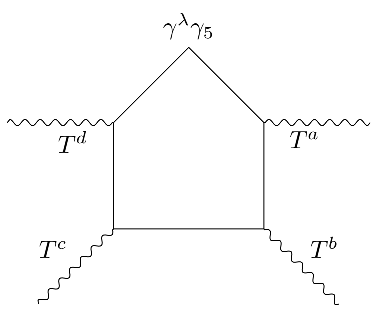
$$\partial_\mu J_5^\mu = -\frac{g^2}{(4\pi)^2}\epsilon^{\mu\nu\lambda\sigma}\mathrm{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}_{\lambda\sigma}) \tag{8.121}$$

with $\mathbf{F}_{\mu\nu} = F^a_{\mu\nu}T^a$.
The non-abelian field strength $\mathbf{F}_{\mu\nu}$ contains a non-linear piece:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \tag{8.122}$$

where $\left[T^a, T^b\right] = if^{abc}T^c$.
It follows that square and pentagon diagrams are anomalous:





5. Deep connections to topology. We will soon see why.

6. Non-renormalization of the anomaly!

8.6 Inconsistent gauge theories

A very important case: chiral symmetries of theories with **massless** fermions.

$$\mathcal{L} = i\overline{\psi}(\not{\partial} - ie\not{A})\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.123}$$

wich is invariant under the transformation

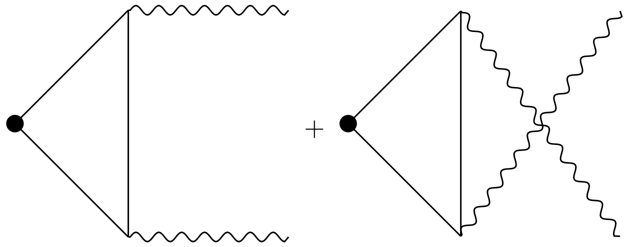
$$\psi \to e^{i\alpha}\psi \quad \text{(vector)} \to j_V^\mu = \overline{\psi}\gamma^\mu\psi \tag{8.124}$$

$$\psi \to e^{i\alpha\gamma_5} \quad \text{(chiral)} \to \text{Noether current is } \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta\psi, \text{so } j_5^\mu \equiv \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{8.125}$$

For the classical case, $m_\psi = 0$ we have $\partial_\mu j_5^\mu = 0$.
However, this equation is **not** compatible with gauge invariance, i.e. with the conservation of the vector current. As we saw, if we enforce the conservation of j_V^μ then a non-vanishing term appears at the RHS at quantum level:

$$\partial_\mu j_5^\mu = -\frac{e^2}{16\pi^2}\epsilon^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}, \tag{8.126}$$

which we got after evaluating the triangle diagrams:



This result has many important consequences.

A first example: QED with **one** left-handed Weyl fermion:

$$\mathcal{L} = i\overline{\psi}(\not{\partial} - ie\not{A})P_L\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{8.127}$$

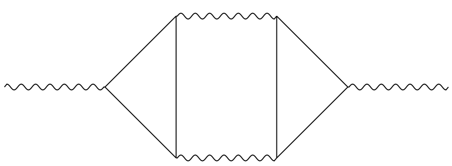
The photon couples to

$$\overline{\psi}\gamma^\mu P_L\psi = \frac{1}{2}(\underset{\substack{\uparrow\\ \text{vector}}}{\overline{\psi}\gamma^\mu\psi} - \underset{\substack{\uparrow\\ \text{axial}}}{\overline{\psi}\gamma^\mu\gamma_5\psi}) \tag{8.128}$$

We can state:

> We can conserve only one current:
> j_V^μ or j_5^μ, but not both!

In this theory gauge invariance is broken. The photon gets a mass via:



→ This theory is not consistent.

8.7 *Anomalies vs. zero-modes*

In the Fujikawa path integral discussion we saw that the anomaly comes from

$$\mathcal{D}\overline{\psi}'\mathcal{D}\psi' = (\det U)^{-2}\mathcal{D}\psi\mathcal{D}\overline{\psi} \tag{8.129}$$

defining the anomaly function $\mathcal{A}(x)$:

$$\ln\det(U) = i\int d^4x\,\alpha(x)\underbrace{\sum_n \phi_n^\dagger\gamma_5\phi_n(x)}_{\mathcal{A}(x)\text{: the anomaly}} \tag{8.130}$$

with the eigenfunction of $i\not{D}\phi_n = \lambda_n\phi_n$. For each eigenfunction ϕ_n of $i\not{D}$, we can find an eigenfunction with an opposite eigenvalue:

$$i\not{D}\gamma_5\phi_n = -\gamma_5 i\not{D}\phi_n = -\lambda_n\gamma_5\phi_n. \tag{8.131}$$

So:

$$\begin{aligned} \phi_n :&\quad \lambda_n \\ \gamma_5\phi_n :&\quad -\lambda_n \end{aligned}$$

For $\lambda \neq 0$, the hermicity of $i\not{D}$ tells us that

$$\phi_n \perp \gamma_5\phi_n \quad \text{(orthogonal eigenfunctions).} \tag{8.132}$$

The contribution to the anomaly function $\mathcal{A}(x)$ after $\int d^4x$ integration:

$$\int d^4x\,\underbrace{\phi_n^\dagger\gamma_5\phi_n(x)}_{=0\text{ for }\lambda_n\neq 0} \tag{8.133}$$

We conclude that only zero-modes ($\lambda_n = 0$) of $i\not{D}$ contribute to the anomaly.

We define zero-modes $|0,n\rangle_\pm$ and split them in γ_5-eigenfunctions

$$\gamma_5|0,n\rangle_\pm = \pm|0,n\rangle_\pm \qquad (\gamma_5\psi^{(0)}_{L/R} = \mp\psi^{(0)}_{L/R}) \tag{8.134}$$

We saw that the path integral measure is **not** invariant:

$$\psi \to e^{i\alpha(x)\gamma_5}\psi. \tag{8.135}$$

which gives with Eq. (8.130)

After regulating $\mathrm{Tr}(\gamma_5)\delta^{(4)}(0) \to \lim_{M^2\to\infty}\lim_{y\to x}\mathrm{Tr}\left(\gamma_5 \mathrm{e}^{-\left(\frac{\mathrm{i}\not{D}}{\mathrm{M}}\right)^2}\delta^{(4)}(\mathrm{x}-\mathrm{y})\right)$

$$\begin{aligned} \mathcal{D}\overline{\psi}'\mathcal{D}\psi' &= \mathcal{D}\psi\mathcal{D}\overline{\psi}\det\left(e^{-2i\alpha(x)\gamma_5}\right) \\ &= \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(-2i\int d^4x\,\alpha(x)\mathcal{A}(x)\right) \\ &= \mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(i\int d^4x\,\alpha(x)\left[\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]\right) \end{aligned}$$

A local transformation also leaves a term:

$$S[A,\psi',\overline{\psi}'] = S[A,\psi,\overline{\psi}] + \int d^4x\,J^5_\mu(x)\partial_\mu\alpha(x). \tag{8.136}$$

Using this definition:

$$\int d^4x\, \mathcal{A}(x) = \int d^4x\, \phi_n^\dagger(x)\gamma_5\phi_n(x) \tag{8.137}$$

$$= \sum_n \left(\langle 0,n|0,n\rangle_+ - \langle 0,n|0,n\rangle_-\right) \tag{8.138}$$

$$= n_+ - n_- \tag{8.139}$$

where n_+ and n_- denote the number of positive and negative chirality zero-modes of $i\not{D}$ for the given gauge configuration. This integer difference is also called the *index* of $i\not{D}$. This represents the famous Atiyah-Singer theorem

With $e = 1$.

$$\text{index}(i\not{D}) = -\frac{1}{32\pi^2}\int d^4x\, \left[\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right]$$

We already know since Sec. 3.4.2, that the RHS of the anomaly is a total derivative

$$\text{Tr}\left[\epsilon^{\mu\nu\alpha\beta}\text{F}_{\mu\nu}\text{F}_{\alpha\beta}\right] = 4\partial_\mu \text{K}^\mu \tag{8.140}$$

with

$$K^\mu = \epsilon^{\mu\nu\alpha\beta}A_\nu\partial_\alpha A_\beta \quad \text{(abelian)} \tag{8.141}$$

or

$$K_\mu = \epsilon_{\mu\nu\alpha\beta}\text{Tr}\left[\text{A}_\nu\partial_\alpha\text{A}_\beta + \frac{2\text{i}}{3}\text{A}_\nu\text{A}_\alpha\text{A}_\beta\right] \quad \text{(non-abelian).} \tag{8.142}$$

which can be identified with a *winding number* of the gauge configuration.[13] These winding numbers define equivalence classes of mappings of the boundary of space-time into the gauge group, characterized by so-called homotopy groups. We see that it makes sense, comparing to Eq. (8.139), that this quantity takes only discrete values.

[13] See my QFT3 lectures.

Can we define a new current, which is conserved even in the presence of the anomaly? Using $\partial_\mu J^5_\mu = \partial_\mu K_\mu$ we could define a new current

$$L_\mu = J^5_\mu - K_\mu \tag{8.143}$$

which would be conserved. **But:** this new current L_μ would not be gauge-invariant (even in the abelian case).

8.8 Anomalies and gauge groups

We will try to answer now: which representations are anomalous? Since only chiral symmetries are anomalous, it is best to express everything in one chirality only. *In this section we therefore work with left-handed Weyl fields only.* We can always express any right-handed field using a left-handed anti-field.
Recall:

$$\psi_L \to \left(1 - i\boldsymbol{\theta}\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\frac{\boldsymbol{\sigma}}{2}\right)\psi_L \tag{8.144}$$

$$\psi_R \to \left(1 - i\boldsymbol{\theta}\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\frac{\boldsymbol{\sigma}}{2}\right)\psi_R \tag{8.145}$$

Transformations are connected: ψ_R transforms as $\sigma_2\psi_L^\star$.

Proof: Using $\sigma_2\boldsymbol{\sigma}^\star = -\boldsymbol{\sigma}\sigma_2$
we can show

$$\sigma_2\psi_L^\star \to \sigma_2\left(1 + i\boldsymbol{\theta}\frac{\boldsymbol{\sigma}^\star}{2} - \boldsymbol{\beta}\frac{\boldsymbol{\sigma}^\star}{2}\right)\psi_L^\star \tag{8.146}$$

$$= \left(1 - i\boldsymbol{\theta}\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\frac{\boldsymbol{\sigma}}{2}\right)\sigma_2\psi_L^\star \tag{8.147}$$

which is the transformation of ψ_R, see Eq. (8.145).

We conclude: we can rewrite the Lagrangian containing ψ_L^i, ψ_R^j using ψ_L^i and the fields

$$\begin{cases} \psi'_{L_j} = \sigma_2 \psi^\star_{R_j} \\ \psi'^\dagger_{L_j} = \psi^T_{R_j}\sigma_2 \end{cases}$$

Example: kinetic terms

$$\psi_L^\dagger i\bar\sigma^\mu \partial_\mu \psi_L + \psi_R^\dagger i\sigma^\mu \partial_\mu \psi_R \tag{8.148}$$

The second term of Eq. (8.148) can be rewritten as LH-kinetic term,

$$\int d^4x\, \psi'^\dagger_L i\bar\sigma^\mu \partial_\mu \psi'_L = \int d^4x \psi^T_R\, \sigma_2 \bar\sigma^\mu \sigma_2\, i\partial_\mu \psi^\star_R \tag{8.151}$$

$$= \int d^4x\, \psi^T_{R,\alpha}(\sigma^\mu)^\star_{\alpha\beta} i\partial_\mu \psi^\star_{R,\beta} \tag{8.152}$$

$$= -\int d^4x\, i\partial_\mu \psi^\star_{R,\beta}(\sigma^\mu)^\star_{\alpha\beta}\psi_{R,\alpha} \tag{8.153}$$

$$\overset{\text{IBP}}{=} \int d^4x\, \psi_R^\dagger i\sigma_\mu \partial_\mu \psi_R \tag{8.154}$$

We use

$$\sigma_2\bar\sigma^\mu\sigma_2 = \sigma_2(1, -\boldsymbol{\sigma})\sigma_2 \tag{8.149}$$

$$= (1, +\boldsymbol{\sigma}^\star) = (\sigma^\mu)^\star. \tag{8.150}$$

with $(\sigma^\mu)^\star_{\alpha\beta} = (\sigma^\mu)_{\beta\alpha}$ since $\sigma^\dagger_\mu = \sigma_\mu$ and the first minus sign is due to the Grassmann nature of the spinor fields.

What if we repeat this for the covariant derivative?

$$\psi^\dagger_R i\sigma_\mu \left(\partial_\mu - igA^a_\mu t^a_{r(\text{epresentation})}\right)\psi_R \tag{8.155}$$

$$= \psi'^\dagger_L i\bar\sigma_\mu\left(\partial_\mu + igA^a_\mu (t^a_r)^T\right)\psi'_L \tag{8.156}$$

$$= \psi'^\dagger_L i\bar\sigma_\mu\left(\partial_\mu - igA^a_\mu t^a_{\bar r}\right)\psi'_L \tag{8.157}$$

There is a sign change in Eq. (8.156), since there is no derivative and therefore no IBP as in the pure kinetic term abov.

We denote $t^a_{\bar r}$ as the generators of the conjugate representation $\bar r$ to r.

$$t^a_{\bar r} = -(t^a_r)^\star, \tag{8.158}$$

using the fact that generators of unitary representations are hermitian, or $(t^a_r)^T = (t^a_r)^\star$..

Analogously for a scalar

$$\phi \to (1 + i\alpha^a t^a_r)\phi$$

then we have

$$\phi^\star \to (1 - i\alpha^a (t^a_r)^\star)\phi^\star$$
$$= (1 + i\alpha^a t^a_{\bar r})\phi^\star.$$

This is the same as $(t^a_r)^T = -t^a_{\bar r}$ since the generators of the unitary representations are hermitian.

Let us look at some examples.

QCD is a $SU(3)_C$ gauge theory with n_f Dirac fermions in in the **fundamental**

$$\underset{\substack{\uparrow \\ \text{fundamental} \\ \text{representation}}}{\psi \sim \mathbf{3}} \sim \begin{pmatrix} \psi_L \\ \psi_R, \end{pmatrix} \tag{8.159}$$

equivalent to SU(3) chiral gauge theory with

n_f fermions in $\mathbf{3}$ $\quad(\psi_L)$
n_f fermions in $\bar{\mathbf{3}}$ $\quad(\psi_R \sim \psi_L^\star)$

This a vector-like representation since it has the same number of r and $\bar r$ LH Weyl-fermions. A Dirac spinor in a $\mathbf{3}$ of SU(3) corresponds to the representation

$$R = r \oplus \bar r \tag{8.160}$$

This is a **real** representation since

$$\bar{R} = R \tag{8.161}$$

We will see that real (and pseudoreal) representations are anomaly free.

8.8.1 *Generally classification of gauge group representations*

- If the generators of a representation satisfy

$$t^a_{\bar{r}} = U\, t^a_r\, U^{-1} \tag{8.162}$$

  with U unitary, then the is a **real** representation.

- If there is a invertible transformation S

and $SS^{-1} = 1$

$$t^a_{\bar{r}} = S\, t^a_r\, S^{-1} \tag{8.163}$$

  the representation R is called **pseudoreal**.

- If the representation is neither real nor pseudoreal then it is **complex**.

If the representation is not complex, we can always find a matrix G_{ab} with two LH Weyl spinors η, ξ in R such that the mass term

$$G_{ab}\eta_a\sigma_2\xi_b + \text{ h.c.} \tag{8.164}$$

is Lorentz invariant and invariant under all internal symmetries. We can even turn this insight into a nice theorem: *no fermion that is allowed by a given symmetry to have a mass can contribute to the anomaly for that symmetry.*[14]

[14] This is quite intuitive, but if you want to read a proof, see e.g. Weinberg - QFT2.

Example : Dirac-spinor and QCD: we can write a mass term of the form in Eq. (8.164)

$$m\overline{\psi}_i\psi_i = m(\psi^\dagger_{R,i}\psi_{L,i} + \psi^\dagger_{L,i}\psi_{R,i}) \tag{8.165}$$

is a SU(3) invariant object and of the form Eq. (8.164).

$$m(\psi^\dagger_{R,i}\psi_{L,i} + \psi^\dagger_{L,i}\psi_{R,i}) \tag{8.166}$$

$$= m(\psi'^T_{L,i}\sigma_2\psi_{L,i} + \psi^\dagger_{L,i}\sigma_2\psi^\star_{L,i}) \tag{8.167}$$

This has the form of a mass term and we already now that a Dirac spinor in the fundamental is a real representation.

As you might know, such mass terms are not possible in the SM with weak interactions. The $SU(2) \times U(1)$ part of the gauge interactions is chiral.

The next largest representation is the **adjoint**. What is the contribution of chiral fermions in the adjoint representation? We find the adjoint is always *real*:

$$(T^a_{adj})^{bc} = -i\underset{\in\mathbb{R}}{\underset{\uparrow}{f^{abc}}} \tag{8.168}$$

Conjugate:

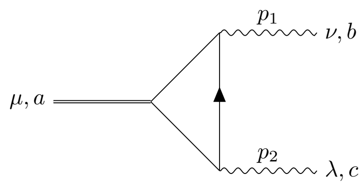
$$\left[(T^a_{adj})^{bc}\right]^\star = +if^{abc} = -\left(T^a_{adj}\right)^{bc} \tag{8.169}$$

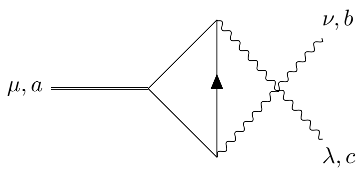
$$\bar{R} = R \qquad \text{(up to sign)} \tag{8.170}$$

8.8.2 Anomaly factor from group theory

The anomaly contribution for non-abelian gauge theories contains the two diagrams:

Recall that the two diagrams gave the same contribution - the only difference is now the order of the gauge generators.





$$\Rightarrow \qquad \mathrm{Tr}\left(\mathrm{t^a t^b t^c}\right) \qquad + \qquad \mathrm{Tr}\left(\mathrm{t^a t^c t^b}\right)$$

We get for the current

$$\partial_\mu j^a_{L\mu} \propto \mathcal{A}^{abc} F^b_{\mu\nu} \tilde{F}^c_{\mu\nu} \tag{8.171}$$

where $\mathcal{A}^{abc}$ now contains the group-theory factors:

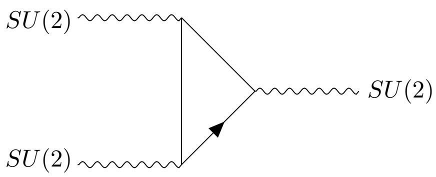
$$\mathcal{A}^{abc} = \mathrm{Tr}\left(\mathrm{t^a\{t^b, t^c\}}\right). \tag{8.172}$$

We will see that real (and pseudo-real) representations are anomaly free.

Unless $\mathcal{A}^{abc} = 0$, the current is not conserved and the chiral gauge anomaly exists.$\mathcal{A}^{abc}$ is a totally symmetric object.

Example: SU(2) chiral gauge theory with

1. SM-like two fermions: $Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$



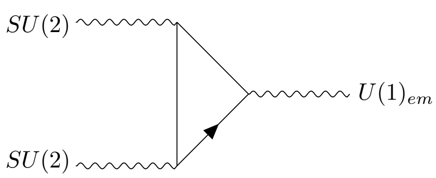
$$t^a = \frac{\sigma^a}{2} \qquad \{\sigma^a, \sigma^b\} = 2\delta^{ab}.$$

$$\mathcal{A}^{abc} = \frac{1}{8}\mathrm{Tr}\left(\sigma^{\mathrm{a}}\{\sigma^{\mathrm{b}}, \sigma^{\mathrm{c}}\}\right) = \frac{2}{8}\mathrm{Tr}\left(\sigma^{\mathrm{a}}\delta^{\mathrm{bc}}\right) = 0. \quad \checkmark \tag{8.173}$$

It is anomaly free!

2. What if these fermions also coupled to QED?
Mixed anomalies:

Alternatively, we could check if hypercharge $U(1)_Y$ is anomaly free. Since we know that pure QED is a vector-like theory (we can write masses for the electrons), it is anomaly free.



$$\propto \mathrm{Tr}\left(\mathrm{Q}\underbrace{\{\sigma^a, \sigma^b\}}_{2\delta^{\mathrm{ab}}}\right)$$

where Q is the electric charge.
So the anomaly is

$$\mathcal{A} = \frac{1}{2}\delta^{ab}\mathrm{Tr(Q)} \propto \sum_{\mathrm{i}} \mathrm{Q_i} \tag{8.174}$$

$$= \underset{\substack{\uparrow \\ N_C}}{3}(\underset{\substack{\uparrow \\ Q_u}}{2/3} - \underset{\substack{\uparrow \\ Q_d}}{1/3}) + (\underset{\substack{\uparrow \\ Q_\nu}}{0} - \underset{\substack{\uparrow \\ Q_e}}{1}) = 0. \tag{8.175}$$

where N_C denotes the number of colors. This is a subset of the cancellations that somewhat miraculously render the SM anomaly free.

8.8.3 General conditions for anomaly cancellation

If fermions are in the real representation R, then the anomaly vanishes.
If the representation is real (or pseudo-real)

$$t^a_{\bar{r}} = S t^a_r S^{-1} \quad \text{and} \quad -t^a_{\bar{r}} = (t^a_r)^T. \tag{8.176}$$

We can use this in the anomaly factor:

$$\begin{aligned}
\mathcal{A}^{abc} &= \mathrm{Tr}\left(\mathrm{t_a}\{\mathrm{t_b}, \mathrm{t_c}\}\right) \overset{\mathrm{SS^{-1}=1}}{=} \mathrm{Tr}\left(\mathrm{t^a_{\bar{r}}}\{\mathrm{t^b_{\bar{r}}}, \mathrm{t^c_{\bar{r}}}\}\right) && (8.177)\\
&= \mathrm{Tr}\left((-\mathrm{t^a_r})^\mathrm{T}\{(-\mathrm{t^b_r})^\mathrm{T}, (-\mathrm{t^c_r})^\mathrm{T}\}\right) && (8.178)\\
&= -\mathrm{Tr}\left(\{\mathrm{t^c_r}, \mathrm{t^b_r}\}\mathrm{t^a}\right) && (8.179)\\
&= -\mathrm{Tr}\left(\mathrm{t^a}\{\mathrm{t^b}, \mathrm{t^c}\}\right) = -\mathcal{A}^{\mathrm{abc}} && (8.180)\\
&\hookrightarrow \mathcal{A}^{abc} = 0 && (8.181)
\end{aligned}$$

In order to have a non-vanishing anomaly we need **complex** representations.

$SO(N)$	the fundamental N-representation is real
$SU(N)$	for $N > 2$ the fundamental representation is complex
$Sp(N)$	the fundamental representation is pseudo-real

Only the Lie-algebras $SU(N)$ with $N > 2$, $SO(4n+2)$, and E_6 with complex representations.

We additionally need the fully symmetric object $\mathcal{A}^{abc}$ to be non-vanishing. Of the Lie-algebras with complex representations only $SU(N)(N > 2)$ and $SO(6)(\sim SU(4))$ have a symmetric invariant like $\mathcal{A}^{abc}$. This implies that $SO(4n+2)$ with $n \geq 2$ and E_6 are anomaly free.

Let us discuss $\mathcal{A}^{abc}$ for $SU(N)$: consider the anti-commutator

$$\{t^a_N, t^b_N\} = \frac{1}{N}\delta^{ab} + d^{abc} t^c_N \tag{8.182}$$

only the last term on the RHS will contribute to the anomaly, due to the tracelessness of the generators. Thus

$$\mathrm{Tr}\left(\mathrm{t^a_r}\{\mathrm{t^b_r}, \mathrm{t^c_r}\}\right) = \frac{1}{2}\mathrm{A(r)d^{abc}} \tag{8.183}$$

with $A(\text{fund.}) = 1$ and $A(\bar{r}) = -A(r)$.

The SM can be embedded in a grand-unified theory (GUT) like $SO(10)$ or E_6. It would immediately explain anomaly freedom.
How about $SU(5)$ GUT? The SM fermions are in $\bar{5}$ and 10_A of $SU(5)$ with $A(\bar{5}) + A(10) = -1 + 1 = 0$✓

8.8.4 Summary of anomalous gauge groups

General gauge group

$$G = \underset{\substack{\uparrow \\ \text{compact, simple}}}{G_1} \otimes \ldots \otimes G_n \otimes U(1)^n \tag{8.184}$$

Table 8.2: Anomaly contributions of various $SU(N)$ representations

		dim r	$A(r)$
fundamental	$\square$	N	1
adjoint		N^2-1	0
anti-symmetric	$\boxminus$	$\frac{N(N-1)}{2}$	$N-4$
symmetric	$\square\square$	$\frac{N(N+1)}{2}$	$N+4$

where

$$G_i \in \{SU(N), SO(N), Sp(2N), \underbrace{G_2, F_4, E_6, E_7, E_8}_{\text{exceptional Lie groups}}\} \tag{8.185}$$

Of G_i these have only **real** (or pseudo-real) representations:

$$\begin{array}{cc} SU(2) & (\simeq SO(3)) \\ SO(N) & (\text{except } SO(4n+2)) \\ \text{Sp(N)} & \\ G_2, F_4, E_7, E_8 & \end{array}$$

Of the remaining ones with complex reps, the only ones whose $\mathcal{A}^{abc}$ does not vanish are $SU(N)$ with $N \geq 3$ and $U(1)$. There is also $SO(6)$ but since $SO(6) \simeq SU(4)$, it is included above.

8.9 Gravitational anomalies

We have seen that we can treat gravity as just another gauge theory with local translations (Lorentz). Coupling a chiral fermion to gravity we can generate anomalies on the gravity action, too. We have already derived the Dirac action, see Eq. (6.50). Here we use the Weyl action:

$$S_{\text{Weyl}} = \int d^4x\, E\overline{\psi}_L \gamma^\rho e^\mu_\rho i(\partial_\mu + i\omega_\mu{}^{mn}\sigma_{mn})\psi_L. \tag{8.186}$$

with $\psi_L = \left(\frac{1-\gamma_5}{2}\right)\psi$ and $E = \det\left(e^\mu_\rho\right)$ and the spin connection $\omega_\mu{}^{mn}$. We can now repeat the calculation above for gauge anomalies expanding the metric in

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{8.187}$$

which means

Since

$$g_{\mu\nu} = \eta_{\mu\nu} e^m_\mu e^n_\nu.$$

$$e^a_\mu = \delta^a_\mu + \frac{1}{2}h^a_\mu. \tag{8.188}$$

We then include the interaction:

$$\mathcal{L}_{int} = -\frac{1}{4} i h^{\mu\nu} \overline{\psi}\gamma_\mu \overleftrightarrow{\partial}_\nu P_L \psi + \ldots \tag{8.189}$$

where we neglect higher orders in $h_{\mu\nu}$. After some calculation or using the index-theorem, one finds

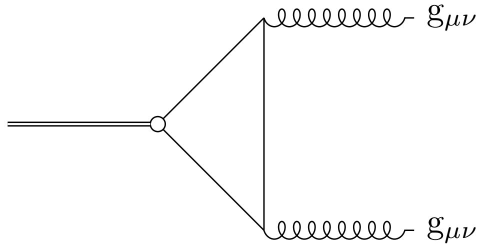
See e.g. Bertlmann - Anomalies in QFT

$$\partial_\mu j^5_\mu = -\frac{1}{384\pi^2} R\tilde{R} \tag{8.190}$$

with $R\tilde{R} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\sigma\tau}R^{\sigma\tau}_{\alpha\beta}$. The tensor $R_{\mu\nu\sigma\tau}$ is the Riemann tensor.

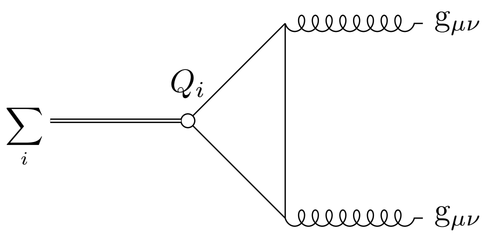
Non-abelian **mixed anomalies** automatically vanish, since the generators of semi-simple groups are traceless[15]

[15] No $U(1)$-factors.



$$\propto \ \mathrm{Tr}(\mathrm{t}^{\mathrm{a}}_{\mathrm{r}}) = 0$$

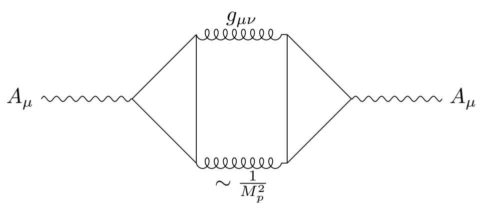
However, $U(1)$ groups are potentially problematic, since only if the sum of charges vanishes, do these gravitational anomalies go away



$$\propto \ \mathrm{Tr}(\mathrm{Q}) = \sum_{\mathrm{i}} \mathrm{Q}_{\mathrm{i}} \overset{!}{=} 0$$

If this condition is violated, then either gauge-invariance or general covariance is lost at quantum level.

What should a gravitational violation imply?



$$\Rightarrow \ m_\gamma^2 \sim \underbrace{G_N^2}_{\text{Newton's constant}} \overbrace{\Lambda^6}^{\substack{\text{cut-off} \\ \text{since the diagram is divergent}}}$$

Even for $\Lambda = 5\mathrm{TeV}$ this would give the photon a mass of the order $m_\gamma \sim 10^{-19}\mathrm{eV}$, which is close to the observed bound ($< 10^{-18}\mathrm{eV}$).

There are also **pure gravitational anomalies**, which violate the energy-momentum tensor conservation, scale invariance and symmetry of $T_{\mu\nu}$.

As we have seen in Sec. 8.8.4, the space-time group[16] can only have complex representations for $SO(4k+2)$. This means that only in $D = 6, 10, \ldots$ can there be pure gravitational anomalies. In particular, for matter fields coupled to gravity in four dimensions, the one-loop action always respects general covariance.

[16] we are now working in Euclidean space, e.g. in 4D we have $SO(1,3) \rightarrow SO(4)$

A spin $\frac{3}{2}$ and (self-dual) tensor field can contribute to the gravitational anomaly, and one finds e.g. that $10D$ superstring theory has the correct low-energy content to lead to anomaly free gravity. See e.g. Alvarez-Gaumé/Witten 1983.

8.10 Anomalous breaking of scale invariance

Quantum fields with no classical parameters still depend on a mass scale through the regularization of UV divergencies.[17] We will need the energy-momentum tensor for the following discussion. The

[17] This corresponds to the running of the coupling constants.

canonical energy-momentum tensor $T_{\mu\nu}$ derived from the Noether procedure might not be $T_{\mu\nu} = T_{\nu\mu}$ symmetric or gauge invariant. One can show it is always possible to convert to a symmetric and gauge-invariant $\Theta^{\mu\nu}$ by

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\lambda \Sigma^{\mu\nu\lambda} \tag{8.191}$$

where $\Sigma^{\mu\nu\lambda} = -\Sigma^{\lambda\nu\mu}$. This implies that $\Theta^{\mu\nu}$ is conserved if $T_{\mu\nu}$ is because

$$\partial_\mu \Theta^{\mu\nu} = \underbrace{\partial_\mu T^{\mu\nu}}_{=0} + \underbrace{\underbrace{\partial_\mu \partial_\lambda}_{\text{symm.}} \underbrace{\Sigma^{\mu\nu\lambda}}_{\text{antisymm. in } \mu\leftrightarrow\lambda}}_{=0} = 0. \tag{8.192}$$

The global P_μ (conserved charge) is also unchanged:

$$P^\nu = \int d^3x\, T^{0\nu} = \int d^3x\, \Theta^{0\nu} + \Delta \tag{8.193}$$

since $\Delta = 0$ because

$$\Delta = \int d^3x\, \partial_\lambda \Sigma^{0\nu\lambda} \tag{8.194}$$

$$= \int d^3x\, \partial_0 \underset{=0}{\underset{\uparrow}{\Sigma^{0\nu 0}}} + \underset{=0\,(\text{surface term!})}{\underset{\uparrow}{\int d^3x\, \partial_i \Sigma^{0\nu i}}}. \tag{8.195}$$

A scale transformation (or dilatation) are of the form:

$$\sigma : x \to e^{+\sigma} x \tag{8.196}$$

For a field theory, we can define:

$$\phi(x) \to e^{+D\sigma} \phi(e^{+\sigma} x) \tag{8.197}$$

with $D = 1$, the canonical mass dimension of ϕ (in $4D$). The infinitesimal form is:

So

$$\psi(x) \to e^{+3/2\sigma} \psi(e^{+\sigma} x)$$
$$A_\mu(x) \to e^{+1\sigma} A_\mu(e^{+\sigma} x)$$

$$\delta\phi = (D + x^\lambda \partial_\lambda)\phi \cdot \sigma. \tag{8.198}$$

If this is an invariance of the action, then there is a conserved current $\mathcal{D}^\mu$: the **dilatation** current.

8.10.1 Exampe 1: scalar ϕ^4 theory

Let us split the Lagrangian in marginal, scale-invariant terms $\mathcal{L}_d$ and a relevant 'deformation' $\mathcal{L}_m$

$$\mathcal{L}_d = \frac{1}{2}\partial_\mu \phi \partial_\mu \phi - \frac{\lambda}{4!}\phi^4 \tag{8.199}$$

$$\mathcal{L}_m = -\frac{m^2}{2}\phi^2 \tag{8.200}$$

$$\mathcal{L} = \mathcal{L}_d + \mathcal{L}_m. \tag{8.201}$$

We calculate $\delta\mathcal{L}_d$:

$$\delta\mathcal{L}_d = (4 + x_\lambda \partial_\lambda)\mathcal{L}. \tag{8.202}$$

This leads to a conserved current since

Check e.g. ϕ^4:

$$\frac{\lambda}{4!}\phi^4 \to \frac{\lambda}{4!}4\delta\phi\cdot\phi^3 = \frac{\lambda}{4!}(4\phi + 4x^\mu(\partial_\mu\phi))\phi^3 = (4 + x^\mu\partial_\mu)\lambda\frac{\phi^4}{4!} \quad \checkmark$$

$$\int d^4x\,\delta\mathcal{L}_d = \int d^4x\,(4 + x_\lambda\partial_\lambda)\mathcal{L} \tag{8.203}$$

$$\stackrel{\text{IBP}}{=} \int d^4x\,4\mathcal{L} - \underbrace{\partial_\lambda x^\lambda}_{=4}\mathcal{L} \tag{8.204}$$

$$= \int d^4x\,4\mathcal{L} - 4\mathcal{L} = 0. \tag{8.205}$$

On the other hand, the fields in $\mathcal{L}_m$ do have canonical dimension 2 and so

$$\delta\mathcal{L}_m = -\frac{1}{2}(2 + x_\lambda\partial^\lambda)m^2\phi^2. \tag{8.206}$$

Integrating by parts this becomes

$$\int d^4x\,\delta\mathcal{L}_m \stackrel{\text{IBP}}{=} \int m^2\phi^2 d^4x \neq 0. \tag{8.207}$$

Clearly the term $\mathcal{L}$ introduces an explicit dimensionful parameter (m^2) breaking the scale invariance.

8.10.2 Derivation of the dilatation current

We can derive the current using the GR energy-momentum-tensor:

$$\delta S_{matter} = \frac{1}{2}\int d^4x\sqrt{g}\,\Theta_{\mu\nu}\,\delta g^{\mu\nu} \tag{8.208}$$

$\Theta^{\mu\nu}$ defines the E-M-tensor, it also guarantees the covariant conservation. A scale transformation can be written as a change in the metric:

$$g_{\mu\nu}(x) \to e^{2\sigma}g_{\mu\nu}(x), \tag{8.209}$$

since for $x \to e^{+\sigma}x$, we get

$$g_{\mu\nu}dx^\mu dx^\nu \to g_{\mu\nu}e^{2\sigma}dx^\mu dx^\nu. \tag{8.210}$$

We now rescale $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$, which has the same effect as $x_\mu \to e^\sigma x_\mu$. Thus we have

$$\delta g_{\mu\nu} = 2\sigma g_{\mu\nu} \tag{8.211}$$

To perform the Noether-trick we promote the rescaling factor to a local $\sigma(x)$ field and plug it in Eq. (8.208)

$$\delta S = \frac{1}{2}\int d^4x\,\sqrt{g}\,\underbrace{\delta g^{\mu\nu}}_{2\sigma(x)g^{\mu\nu}}\,\Theta_{\mu\nu} \tag{8.212}$$

$$= \int d^4x\,\sqrt{g}\,\underbrace{\Theta^\mu_\mu}_{\partial_\mu\mathcal{D}^\mu}\,\sigma(x) = -\int d^4x\,\mathcal{D}_\mu\,\underbrace{\partial_\mu\sigma(x)}_{\substack{=0\\ \text{if } \sigma=\text{const.}}} \tag{8.213}$$

where in the last step, we have assumed that the metric is trivial, which is fine since do not care about gravity here. We see that there is a conserved current:

Classical scale-invariance: energy-momentum tensor is traceless!

$$\boxed{\partial_\mu \mathcal{D}^\mu = \Theta^\mu{}_\mu = 0.} \tag{8.214}$$

with $\mathcal{D}^\mu = \Theta^{\mu\nu} x_\nu$ since

$$\begin{aligned} \partial_\mu \mathcal{D}^\mu = \partial_\mu(\Theta^{\mu\nu} x_\nu) &= \partial_\mu \Theta^{\mu\nu} x_\nu + \Theta^{\mu\nu} \overset{\overset{\delta^\nu_\mu}{\downarrow}}{\partial_\mu x^\nu} && (8.215)\\ &= 0 + \Theta^\mu_\mu. && (8.216) \end{aligned}$$

Including **quantum corrections**, scale transformations are usually not a symmetry of the theory any, more even if there might be classical scale invariance.

Attention: We use g for the coupling and for the metric!

$$\frac{dg_R}{d\ln\mu} = \beta(g_R). \tag{8.217}$$

So we get for the coupling:

$\ln\mu' = \ln(\mu e^{-\sigma})$ from which $d\ln\mu = -d\sigma$ follows.

$$g_R \to g_R + \delta g_R = g_R - \beta(g_R)\delta\sigma. \tag{8.218}$$

The corresponding change in the Lagrangian is then:

$$\delta\mathcal{L} = \sigma \frac{\partial \mathcal{L}}{\partial \sigma} = -\sigma\beta(g)\frac{\partial}{\partial g}\mathcal{L}. \tag{8.219}$$

This means that quantum corrections change the dilatation current conservation! Using Eq. (8.213)

$$\delta S = \int d^4x \left(\partial_\mu \mathcal{D}_\mu - \beta(g)\frac{\partial}{\partial g}\mathcal{L}\right)\sigma(x)$$

and taking a functional derivative w.r.t. $\sigma(x)$ gives

$$\boxed{\partial_\mu \mathcal{D}^\mu = \Theta^\mu_\mu = \beta(g)\frac{\partial}{\partial g}\mathcal{L}} \tag{8.220}$$

8.10.3 Example 2: effective potential of scalar QED

$$\mathcal{L} = D_\mu\phi(D^\mu\phi)^* - \frac{1}{4}F^2_{\mu\nu}\,, \tag{8.221}$$

$$\mathcal{L} = -\phi\left(\Box + 2ieA^\mu\partial_\mu + ie(\partial_\mu A^\mu) - e^2A^2_\mu\right)\phi^* - \frac{1}{4}F^2_{\mu\nu}, \tag{8.222}$$

which we can write as

$$\mathcal{L} = -\phi(\Box + v(x))\phi^* - \frac{1}{4}F^2_{\mu\nu}. \tag{8.223}$$

with

$$v(x) = +2ieA^\mu\partial_\mu + ie(\partial_\mu A^\mu) - e^2A^2_\mu. \tag{8.224}$$

The generating functional is

$$Z = \int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu \exp\left\{-i\int d^4x\phi(\Box + v(x))\phi^* - \frac{i}{4}\int d^4x\, F^2_{\mu\nu}\right\}\,,$$

which we can perturbatively evaluate[18]

$$S = -\frac{1}{4}\int d^d x F_{\mu\nu}F^{\mu\nu}Z_3 - \frac{1}{4}be^2\int d^d x d^d y F_{\mu\nu}(x)L(x-y)F^{\mu\nu}(y)\,,$$

where $b = 1/(48\pi^2)$ is the leading coefficient in the beta function, and Z_3 is the wavefunction renormalization constant and

$$L(x-y) = \int \frac{d^d k}{(2\pi)^d}\ln\left(\frac{-k^2}{\mu^2}\right)e^{-ik(x-y)} \tag{8.225}$$

[18] See https://arxiv.org/abs/1702.00319

Or

$$L(x-y) \equiv \langle x|\ln\left(\frac{-k^2}{\mu^2}\right)|y\rangle.$$

Now rescale

$$A_\mu \to \frac{1}{e}A_\mu \tag{8.226}$$

to write

$$S = \int d^4 x\left(-\frac{1}{4}F_{\mu\nu}\left(\frac{1}{e^2(\mu)} - b\ln\left(\frac{\Box}{\mu^2}\right)\right)F^{\mu\nu}\right) \tag{8.227}$$

Scale transformation:

$$A_\mu(x) \to \lambda A_\mu(\lambda x). \tag{8.228}$$

We can see the trace anomaly in the $\hbar$ correction in the effective potential. First, we evaluate

Using $q\lambda = q'$.

$$\begin{aligned} L(\lambda x - \lambda y) &= \int \frac{d^4 q}{(2\pi)^4}e^{-iq(\lambda x-\lambda y)}\ln\left(\frac{-q^2}{\mu^2}\right) && (8.229)\\ &= \frac{1}{\lambda^4}\int\frac{d^4 q'}{(2\pi)^4}e^{-iq'(x-y)}\left[\ln\left(\frac{-q'^2}{\mu^2}\right) - \ln\left(\lambda^2\right)\right] && (8.230)\\ &= \frac{1}{\lambda^4}L(x-y) - \delta^{(4)}(x-y)\ln\left(\lambda^2\right). && (8.231)\end{aligned}$$

So we find

$$L(x-y) = \lambda^4\left(L(\lambda x - \lambda y)\right) + \ln\left(\lambda^2\right)\delta^{(4)}(x-y). \tag{8.232}$$

Now we use this in the rescaled effective action, where we transform $F_{\mu\nu}(x) \to \lambda^2 F_{\mu\nu}(x\lambda)$, focussing on the second term

$$\begin{aligned} &\int d^4 x\, d^4 y\, \lambda^2 F_{\mu\nu}(x\lambda)L(x-y)\lambda^2 F_{\mu\nu}(y\lambda)\\ &\int d^4 x\, d^4 y\, \lambda^4 F_{\mu\nu}(x\lambda)\left[\lambda^4 L(\lambda x - \lambda y) + \ln\left(\lambda^2\right)\delta^{(4)}(x-y)\right]F_{\mu\nu}(y\lambda)\\ &\overset{\substack{x\to x/\lambda\\ y\to y/\lambda}}{=} \int d^4 x\, d^4 y\, F_{\mu\nu}(x)L(x-y)F_{\mu\nu}(y) + \ln\left(\lambda^2\right)\int d^4 x\, F_{\mu\nu}^2(x)\end{aligned}$$

Together with the scale invariance of the classical action, we find that a scale transformation of the action and $\lambda = e^\sigma$

$$\delta_\lambda S = \int d^4 x\,\sigma\partial_\mu D_\mu - \frac{1}{4}\ln\lambda^2 be^2\int d^4 x\, F_{\mu\nu}^2(x) \tag{8.233}$$

We get

$$\partial_\mu D_\mu = \Theta^\mu_\mu = \frac{\beta e^2}{2}F_{\mu\nu}F^{\mu\nu}. \tag{8.234}$$

8.10.4 Example 3: QCD

$$S = -\frac{1}{4}\int d^4x\,\sqrt{g}\left[g^{\mu\lambda}g^{\nu\sigma}F^a_{\mu\nu}F^a_{\lambda\sigma}\right]. \tag{8.235}$$

We define the energy-momentum tensor by considering an infinitesimal variation:

$$g^{\mu\nu}(x) \to g^{\mu\nu}(x) + \delta g^{\mu\nu}(x) \tag{8.236}$$

under which

$$\delta S = \frac{1}{2}\int d^4x\sqrt{g}\Theta_{\mu\nu}(x)\delta g^{\mu\nu}(x). \tag{8.237}$$

We note that

$$g = -\det(g_{\mu\nu}) \tag{8.238}$$

So

$$\sqrt{g} \to \sqrt{g}\left(1 + \frac{1}{2}g_{\mu\nu}\delta g_{\mu\nu}\right). \tag{8.239}$$

$$\begin{aligned}\ln\det(M+\delta M) &= \mathrm{Tr}\ln(\mathrm{M}+\delta\mathrm{M})\\ &= \mathrm{Tr}\ln(\mathrm{M}) + \mathrm{Tr}\ln(1+\mathrm{M}^{-1}\delta\mathrm{M})\\ &= \mathrm{Tr}\ln(\mathrm{M}) + \mathrm{Tr}\ln(\mathrm{M}^{-1}\delta\mathrm{M}).\end{aligned}$$

Thus we get:

$$S \to \int d^4x\,\sqrt{g}\left[\left(1+\frac{1}{2}g_{\mu\nu}\delta g_{\mu\nu}\right)\left(-\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\right)\left(g^{\mu\lambda}+\delta g^{\mu\lambda}\right)\left(g^{\nu\sigma}+\delta g^{\nu\sigma}\right)\right] \tag{8.240}$$

or

$$S \to S + \frac{1}{2}\int d^4x\sqrt{g}\left(-F^a_{\mu\phi}F^{a\phi}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{a\lambda\sigma}F^a_{\lambda\sigma}\right)\delta g^{\mu\nu} \tag{8.241}$$

and therefore:

$$\Theta_{\mu\nu} = F^a_{\mu\phi}F^{a\phi}_{\nu} - \frac{1}{4}g_{\mu\nu}F^{\lambda\sigma}_aF^a_{\lambda\sigma}. \tag{8.242}$$

We skip the proof of symmetry and gauge-invariance since they are obvious.

Let's check the tracelessness:

$$\Theta^\mu_\mu = F^a_{\mu\phi}F^{a\phi\mu} - \frac{1}{4}\overset{\overset{=4}{\downarrow}}{g^\mu_\mu}F^{\lambda\sigma}_aF^a_{\lambda\sigma} = 0 \quad \checkmark$$

Now we add fermions:

$$\Theta^{\mu\nu} = -F^{\mu\lambda}F^\nu_\lambda + \frac{1}{4}(F_{\lambda\sigma})^2 + \frac{1}{2}\overline{\psi}i(\gamma^\mu D^\nu + \gamma^\nu D^\mu)\psi - g^{\mu\nu}\overline{\psi}(i\not{D} - m)\psi. \tag{8.243}$$

By using the Dirac equation we find the trace of $\Theta^{\mu\nu}$ given by

$$\Theta^\mu_\mu \overset{\overset{\text{EOM}}{\downarrow}}{=} m\overline{\psi}\psi, \tag{8.244}$$

which vanishes if $m = 0$.
In the case of massless QED e.g. we can rescale to remove the coupling constant from the covariant derivative

$$eA_\mu \to A_\mu. \tag{8.245}$$

As a consequence the coupling only appears in the term:

$$\mathcal{L} = -\frac{1}{4e^2}(F_{\lambda\sigma})^2 + \ldots \tag{8.246}$$

which means that

$$\frac{\partial}{\partial e}\mathcal{L} = \frac{1}{2e^3}(F_{\lambda\sigma})^2 \tag{8.247}$$

So Eq. (8.220) reads then using canonically normalized fields

$$\Theta^{\mu}_{\mu} = \frac{\beta(e)}{2e}(F_{\lambda\sigma})^2. \tag{8.248}$$

This is the **trace-anomaly**.In pure Yang-Mills the trace anomaly is

$$\Theta^{\mu}_{\mu} = \frac{\beta(g)}{2g}(F^{a}_{\lambda\sigma})^2. \tag{8.249}$$

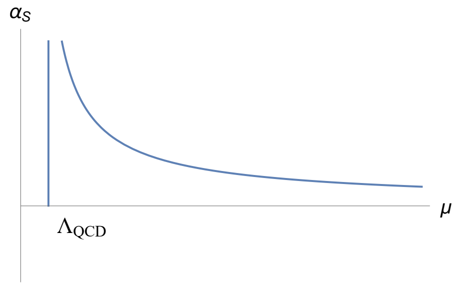
Note that $\Theta^{\mu}_{\mu} = 0$ at the fixed-point of the renormalization group $\beta(g_{\star}) = 0$. The quantum theory is scale-invariant.

In conclusion, a scale anomaly is **not** an inconsistency, but a fact of life in most QFTs.

9 General theory of Goldstones at low energy

S. R. Coleman, J. Wess and B. Zumino, "Structure of phenomenological Lagrangians. 1.," Phys. Rev. **177** (1969) 2239. doi:10.1103/PhysRev.177.2239, C. G. Callan, Jr., S. R. Coleman, J. Wess and B. Zumino, "Structure of phenomenological Lagrangians. 2.," Phys. Rev. **177** (1969) 2247. doi:10.1103/PhysRev.177.2247; and for a modern review: https://arxiv.org/pdf/1506.01961.pdf

We return to spontaneous symmetry breaking. We will now develop a theory of spontaneous symmetry breaking at energies $E < v$. This allows to write the most general low energy effective lagrangian, even for strongly interacting theories.
Important application: low Energy QCD pions, Kaons etc. predictive even strong $\alpha_{QCD}(\mu) \gg 1$.



As we will see, the relevant degrees of freedom won't be the quarks and gluons anymore but the Goldstone bosons of a spontaneous breaking of an approximate flavor symmetry $(m_u, m_d \ll \Lambda_{\text{QCD}})$.

9.1 Reminder: linear $\sigma-$model

N real scalar fields ϕ_n with $n = 1, \ldots, N$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_n \partial^\mu \phi_n - \frac{\lambda}{4}(\phi_n\phi_n - v^2)^2 \tag{9.1}$$

$O(N)$ symmetry with $O^T O = \mathbb{1}_{n\times n}$

$$\phi_n \to \underbrace{e^{i\epsilon^a T^a_{mn}}}_{SO(N)} \phi_m \tag{9.2}$$

with T^a hermitian, purely imaginary and dim $O(N) = \frac{N(N-1)}{2}$. For small coupling $(\lambda \ll 1)$ we can say

$$V_{eff}(\phi) \simeq V(\phi) \tag{9.3}$$

- $v^2 < 0$: minimum at $\phi_n = 0 \quad \forall n$
- $v^2 > 0$: degenerate minima $\phi_n\phi_n = v^2$. We can choose $\langle\phi_i\rangle = 0$ with $i = 1, \ldots, N-1$

$$\langle\phi_N\rangle = v \tag{9.4}$$

Hence: spontaneous breaking of $O(N) \to O(N-1)$. The vacuum manifold is: S^n since $\sum_n \phi_n^2 = v^2$. We get

$$\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = (N-1) \text{ GBs} \tag{9.5}$$

We parametrize

$$\phi_i = \pi_i(x) \quad \phi_N = v + \sigma(x) \tag{9.6}$$

with $i = 1, \dots, N-1$. The Lagrangian in these fields is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - 2\lambda v^2\sigma^2 + \partial_\mu\pi_i\partial^\mu\pi_i\right) - \lambda v\sigma(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi_i\pi_i)^2 \tag{9.7}$$

We find

- No mass term for π_i as expected
- Massive mode has $m_\sigma = \sqrt{2}\sqrt{\lambda}v$

9.1.1 Special case: $N = 4$

Note first $O(4) \simeq SU(2) \otimes SU(2)$[1]. We represent the 4−vector as 2×2 matrices:

[1] It is analogous to Lorentz group: $SO(1,3) \simeq SU(2)_L \otimes SU(2)_R$.

$$\Sigma(x) = \phi_4(x)\mathbb{1} + i\tau^k\phi_k(x) \tag{9.8}$$

with $k = 1,2,3$ and τ^k are the Pauli-matrices.

We now calculate the trace of $\mathrm{Tr}\,\Sigma^\dagger\Sigma$:

$$\mathrm{Tr}\,\Sigma^\dagger\Sigma = \mathrm{Tr}(\phi_4^2\mathbb{1} + \tau^{\mathrm{k}}\phi_{\mathrm{k}}\tau^{\mathrm{l}}\phi_{\mathrm{l}}) = 2\phi_{\mathrm{n}}\phi_{\mathrm{n}}. \tag{9.9}$$

So the Lagrangian is now:

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\,\partial_\mu\Sigma^\dagger\partial^\mu\Sigma - \frac{\lambda}{16}(\mathrm{Tr}\,\Sigma^\dagger\Sigma - 2\mathrm{v}^2)^2 \tag{9.10}$$

$O(4)$ is now represented as $SU(2)_L \otimes SU(2)_R$ with

$$\Sigma \to D_R\Sigma D_L^+ \tag{9.11}$$

with $D_R, D_L \in SU(2)$.
Again: for $v^2 > 0$ the symmetry is broken.

$$\mathrm{Tr}\,\Sigma^\dagger\Sigma = 2\mathrm{v}^2. \tag{9.12}$$

Choose a vacuum:

$$\langle\Omega|\Sigma|\Omega\rangle = v \cdot \mathbb{1}_{2\times2}, \tag{9.13}$$

which corresponds to $\langle\phi_4\rangle = v$ and $\langle\phi_i\rangle = 0$. This is invariant under $D_R = D_L$.[2]

[2] $\mathbb{1} \to D_R D_L^\dagger$.

The symmetry is thus broken to a **diagonal** subgroup:

$$\begin{array}{ccc} SU(R)_L \otimes SU(2)_R & \simeq & O(4) \\ \downarrow & & \downarrow \\ {\scriptstyle SU(2)_V} & & {\scriptstyle O(3)} \end{array} \tag{9.14}$$

9.1.2 Non-linear parametrization

We now use a non-linear parametrization to make the GB-field explicit:

$$\Sigma(x) = (v + \sigma(x))e^{i\boldsymbol{\tau}\,\boldsymbol{\pi}/v} \tag{9.15}$$

with $e^{i\boldsymbol{\tau}\,\boldsymbol{\pi}/v} = U(\boldsymbol{\pi})$. We know that the it transforms as

Although we use the same symbols σ, π^i are not the same as in the linear $\sigma-$model!

$$\begin{cases} \sigma \to \sigma \\ U(\boldsymbol{\pi}) \to D_R U(\boldsymbol{\pi}) D_L^\dagger \end{cases}$$

In terms of new fields we write:

$$\mathrm{Tr}\,\Sigma^\dagger\Sigma \overset{\overset{U^\dagger U=1}{\downarrow}}{=} 2(\mathrm{v} + \sigma(\mathrm{x}))^2. \tag{9.16}$$

$$\mathrm{Tr}\,\partial_\mu\Sigma^\dagger\partial^\mu\Sigma = 2\partial_\mu\sigma\partial^\mu\sigma + (\mathrm{v}+\sigma)^2\mathrm{Tr}(\partial_\mu \mathrm{U}^\dagger\partial^\mu \mathrm{U}), \tag{9.17}$$

where we used $\partial^\mu\Sigma = \partial^\mu\sigma U(\boldsymbol{\pi}) + (v+\sigma)\partial^\mu U(\boldsymbol{\pi})$ and $U^\dagger\partial^\mu(\partial^\mu U^\dagger)U = \partial^\mu(U^\dagger U) = 0$. Also the linear term vanishes.
The Lagrangian is then:

$$\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}(\partial_\mu \mathrm{U}(\boldsymbol{\pi})^\dagger\partial^\mu \mathrm{U}(\boldsymbol{\pi}))\left(1 + \frac{\sigma(\mathrm{x})}{\mathrm{v}}\right)^2 \tag{9.18}$$

$$+\frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma - 2\lambda v^2\sigma^2(x)\right) - \lambda v\sigma^3 - \frac{\lambda}{4}\sigma^4. \tag{9.19}$$

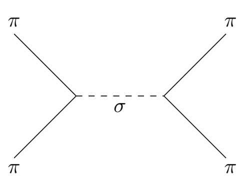
This Lagrangian is different from the linear $\sigma-$model; here in the non-linear $\sigma-$model: the only derivative terms of GB fields. $\hookrightarrow$ **all interactions of GB's are proportional to their momentum and become weak at low energies**.

9.1.3 Goldstone EFT

When we set $E \ll m_\sigma$, we can integrate out the $\sigma(x)-$field into the EFT. Since $\sigma(x)$ does **not** transform the EFT retains the $SU(2)_L \otimes SU(2)_R$ symmetry. $\hookrightarrow$ This is an advantage of the non-linear representation relative to the linear $\sigma-$representation.

Now we consider scattering with E, momenta $\ll m_\sigma = \sqrt{2\lambda}v$, then the $\sigma(x)$ field only appears as an internal line.

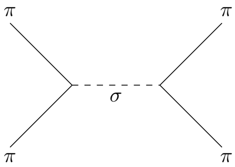
1. The $\sigma_{\pi\pi}$-vertex



$$\frac{v^4}{4}2\partial_\mu\boldsymbol{\pi}\partial^\mu\boldsymbol{\pi}\frac{1}{v^2}\frac{2\sigma(x)}{v} \tag{9.20}$$

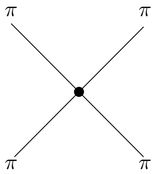
$$= \frac{1}{v}\sigma(x)\partial^\mu\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi} \tag{9.21}$$

We get



$$\underset{\frac{1}{q^2-m_\sigma^2}\simeq -\frac{1}{m_\sigma^2}}{\underset{\uparrow}{\sim}} \left(\frac{1}{v}E^2\right)^2 \frac{1}{m_\sigma^2} \sim \frac{E^2}{m_\sigma^2}\frac{E^2}{v^2} \tag{9.22}$$

2. Compare to



$$\sim \frac{E^2}{v^2} \tag{9.23}$$

from $\frac{1}{v^2}\partial_\mu \boldsymbol{\pi}\partial^\mu \boldsymbol{\pi}\cdot\boldsymbol{\pi}^2$ after expanding $\mathrm{Tr}\,(\partial_\mu \mathrm{U}^\dagger \partial^\mu \mathrm{U})$.
When comparing these two calculations we conclude

$$\frac{E^2}{m_\sigma^2}\frac{E^2}{v^2} \ll \frac{E^2}{v^2} \tag{9.24}$$

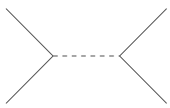
$\hookrightarrow$ The leading GB interactions are given by

$$\mathcal{L}^{(0)} = \frac{v^4}{4}\mathrm{Tr}(\partial_\mu \mathrm{U}(\boldsymbol{\pi})^\dagger \partial_\mu \mathrm{U}(\boldsymbol{\pi})), \tag{9.25}$$

which remains $SU(2)_L \otimes SU(2)_R$ invariant.
We can systematically improve the Lagrangian by matching to UV theory with $\sigma-$ diagrams

$$\mathcal{L}_{eff}^{(\boldsymbol{\pi})} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \dots \tag{9.26}$$

e.g.



$$\sim E^4 \tag{9.27}$$

is reproduced by the local vertex with 4 derivatives

$$\mathcal{L}^{(2)} \supset \frac{v^2}{8m_\sigma}\mathrm{Tr}\left(\partial_\mu \mathrm{U}(\boldsymbol{\pi})^\dagger \partial_\mu \mathrm{U}(\boldsymbol{\pi})\right)^2 \tag{9.28}$$

9.2 Goldstone boson interactions

The idea is: GB are the lightest particles in the spectrum, so we can use an EFT for GBs alone,since $E \ll$ mass of other particles.
Further, we can constrain this Lagrangian enormously by the underlying (spontaneously broken) global symmetry. This will be a generalization of the $O(4)$ $\sigma-$model.

Starting point: identify the degrees of freedom $\rightarrow$ one GB for each broken generator.

Tricky aspect: the vacuum manifold is curved (in general).

9.2.1 *Helping your intuition with a simple example*

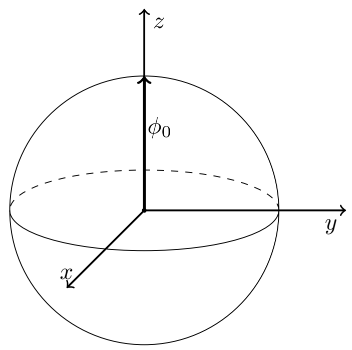
Example: $\overset{\overset{G}{\downarrow}}{SO(3)}/\overset{\overset{H}{\downarrow}}{SO(2)}$ and $\boldsymbol{\phi}_0 \;=\; \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$ The non-linear sigma parametrization in this case is

$$\boldsymbol{\phi} = \left(1 + \frac{\sigma(x)}{v}\right) e^{i(J_1\pi_1(x)+J_2\pi_2(x))}\boldsymbol{\phi}_0 \tag{9.29}$$

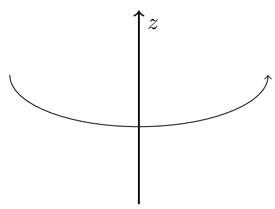
which in the low energy limit gives:

$$\boldsymbol{\phi}(x) \overset{GB}{\simeq} \exp(iJ_1\pi_1(x) + iJ_2\pi_2(x))\boldsymbol{\phi}_0. \tag{9.30}$$

The generators of the rotations: $J_1 : x$-, $J_2 : y$- and $J_3 : z$-rotation.



The unbroken group: rotations around the $z-$axis, generator J_3, $H = O(2)$



We want to parametrize the GB excitations in the following form

$$\phi(x) = \Sigma(x)\boldsymbol{\phi}_0 = e^{i\hat{T}^{\hat{A}}\pi^{\hat{A}}(x)/v}\boldsymbol{\phi}_0 \tag{9.31}$$

Note, that we changed notation compared to the introductory parts. Now: $U(\boldsymbol{\pi}) = \Sigma_{\text{now}}(\boldsymbol{\pi})$, since we will ignore the radial fluctuations.

with $\boldsymbol{\phi}_0$ as the vacuum expectation value (vev). This should hold even after G transformations.
Under a global symmetry transformation $g \in G$:

$$\boldsymbol{\phi} \to g\boldsymbol{\phi}(x) \tag{9.32}$$

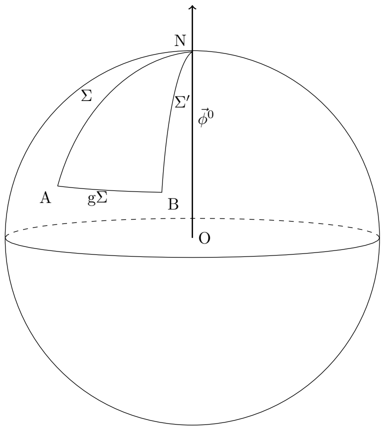
Thus

$$\Sigma(x) \to g\Sigma(x), \tag{9.33}$$

which is in general however not of the form Eq. (9.31) any more, because $g\Sigma(x)$ is a generic element of G and as such it can not be expressed as the exponential of broken generators only. We can write however

$$g\Sigma(\boldsymbol{\pi}) = \Sigma(\boldsymbol{\pi}')h \tag{9.34}$$

with $h \in H$.
The two matrices $g\Sigma(\boldsymbol{\pi})$ and $\Sigma(\boldsymbol{\pi}')$ describe the same field configuration which differs by H transformation. The vacuum manifold is $\boldsymbol{\phi}^2 = v^2$ which defines the surface of a sphere. The Goldstone dynamics happens solely on this manifold



$$\begin{aligned} NB &: \Sigma'\boldsymbol{\phi}_0 \\ NA \to AB &: g\Sigma\boldsymbol{\phi}_0 \\ NA &: \Sigma \\ AB &: g\Sigma. \end{aligned}$$

h is non-trivial $\to$ property of rotations in 3D. Rotate from $N \to A \to B$, not the same rotation as $N \overset{\Sigma'}{\to} B$ directly, but we can write Σ' as a rotation around $\boldsymbol{\phi}_0$ (or ON) followed by a rotation from $N \to B$.

The transformation h is non-trivial because the Goldstone manifold G/H is curved.

9.2.2 *Transformation of the standard parametrization*

We can write Eq. (9.34) as

$$\Sigma(\boldsymbol{\pi}) \to g\Sigma(\boldsymbol{\pi})h^{-1}(g, \Sigma(\boldsymbol{\pi})), \tag{9.35}$$

where we have made clear the implicit x-dependence of h through its dependence on g and $\Sigma(x)$.

The main idea is:

$$g[\alpha^A] = e^{i\alpha^A T^A} = \underset{\substack{\uparrow \\ \text{broken}}}{e^{f_{\hat{a}}[\alpha]\hat{T}^{\hat{a}}}} \underset{\substack{\uparrow \\ \text{unbroken}}}{e^{if_i[\alpha]T^i}}. \tag{9.36}$$

One can easily show that any group element $g \in G$ into broken and unbroken generators in the form above.

This is trivial for infinitesimal $|\alpha^A| \ll 1$:

$$1 + i\left(\alpha^{\hat{a}}\hat{T}^{\hat{a}} + \alpha^i T^i\right) = \left(1 + if_{\hat{a}}[\alpha]\hat{T}^{\hat{e}_f}\right)\left(1 + if_i[\alpha]T^i + \dots\right) \tag{9.37}$$

It follows

$$\begin{cases} \alpha^{\hat{a}} = f_{\hat{a}}[\alpha] + \mathcal{O}(\alpha^2) \\ \alpha^i = f_i[\alpha] + \mathcal{O}(\alpha^2) \end{cases}$$

It can be extended to finite group elements:

$$\boxed{\boldsymbol{\phi}(x) = e^{i\pi_{\hat{a}}(x)\hat{T}^{\hat{a}}/v}\boldsymbol{\phi}_0} \tag{9.38}$$

with $e^{i\pi_{\hat{a}}(x)\hat{T}^{\hat{a}}/v} = \Sigma(\boldsymbol{\pi})$.
The standard form under g transformation:

$$g\boldsymbol{\phi} = g\Sigma(\boldsymbol{\pi})\boldsymbol{\phi}_0 \tag{9.39}$$

$$= \Sigma(\boldsymbol{\pi}')h\boldsymbol{\phi}_0 = \Sigma(\boldsymbol{\pi}')\boldsymbol{\phi}_0 \tag{9.40}$$

It is still in the standard form with transformed

$$\boldsymbol{\pi} \to \boldsymbol{\pi}'. \tag{9.41}$$

9.3 General G/H coset

We will decompose the generators $\{T^A\}$ into

$$\underset{\substack{\uparrow \\ \text{Generates } G \\ A=1,\dots,\dim G}}{\{T^A\}} = \overset{\substack{\text{unbroken} \\ \text{generats } H \\ i=1,\dots,\dim H \\ \downarrow}}{\underset{\substack{\uparrow \\ \text{broken} \\ \text{generates } G/H \\ \hat{a}=1,\dots,\dim G - \dim H}}{\{T^i, \hat{T}^{\hat{a}}\}}}. \tag{9.42}$$

with the classification

$$T^i\boldsymbol{\phi}_0 = 0 \qquad \text{(unbroken)} \tag{9.43}$$

$$T^{\hat{a}}\boldsymbol{\phi}_0 \neq 0 \qquad \text{(broken)} \tag{9.44}$$

The algebra is:

$$[T^A, T^B] = if^{AB}{}_C T^C. \tag{9.45}$$

$SO(3)/SO(2)$ example.

$$T^i = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = J_3$$

$$J_3\phi_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

So we write:

$$\left[T^i, T^j\right] = i f^{ij}{}_k T^k + \cancel{i f^{ij}{}_{\hat{a}} \hat{T}^{\hat{a}}} \quad ① \tag{9.46}$$

$$\left[T^i, \hat{T}^{\hat{b}}\right] = i f^{i\hat{b}}{}_{\hat{c}} \hat{T}^{\hat{c}} + \cancel{i f^{i\hat{b}}{}_j T^j} \quad ② \tag{9.47}$$

$$\left[\hat{T}^{\hat{a}}, \hat{T}^{\hat{b}}\right] = i f^{\hat{a}\hat{b}}{}_c T^c + \underbrace{i f^{\hat{a}\hat{b}}{}_{\hat{c}} \hat{T}^{\hat{c}}}_{③}. \tag{9.48}$$

①: H is a subgroup and the algebra closes among the generators of H.

$$i f^{ij}{}_k = \left(T^i_{adj}\right)^j{}_k. \tag{9.49}$$

②: no $T^i \in H$ can appear on RHS since ① implies $f^{ij}{}_{\hat{a}} = 0$ and thus $f^{i\hat{a}}{}_j = 0$ by the antisymmetry of $f^{AB}{}_C$. We define the structure constants

$$i f^{i\hat{b}}{}_{\hat{c}} = \left(T^i_\pi\right)^{\hat{b}}{}_{\hat{c}}. \tag{9.50}$$

since we can also identify them as the unbroken generators in the adjoint representation acting on the G/H generators.
Special cosets have ③ $= 0$ or $f^{\hat{a}\hat{b}}{}_{\hat{c}} = 0$. If there is an algebra automorphism $\hat{T} \to -\hat{T}$, then we call it a *symmetric coset*. We can see that $\hat{T} \to -\hat{T}$ implies ③ $= 0$. $\Rightarrow \mathbb{Z}_2$ parity of GB

Using the standard normalization:

$$\mathrm{Tr}\left(\mathrm{T^a T^b}\right) = \frac{1}{2}\delta^{\mathrm{ab}} \tag{9.51}$$

we get

$$\mathrm{Tr}\left(\mathrm{T^i T^j}\right) = \frac{1}{2}\delta^{\mathrm{ij}} \tag{9.52}$$

$$\mathrm{Tr}\left(\mathrm{T^i \hat{T}^{\hat{a}}}\right) = 0 \tag{9.53}$$

$$\mathrm{Tr}\left(\mathrm{\hat{T}^{\hat{a}} \hat{T}^{\hat{b}}}\right) = \frac{1}{2}\delta^{\mathrm{\hat{a}\hat{b}}}. \tag{9.54}$$

9.4 *Transformation properties under G/H and H*

The behaviour of the goldstone fields is very different if we transform using the unbroken subgroup or using the G/H elements.

9.4.1 *Transforming under the unbroken group H*

Unbroken subgroup: H and $h = e^{iT^i \alpha^i}$

$$\boldsymbol{\phi} \to h\boldsymbol{\phi} \tag{9.55}$$

So we have

$$\Sigma(\boldsymbol{\pi})\boldsymbol{\phi}_0 \to h\Sigma(\boldsymbol{\pi})\boldsymbol{\phi}_0 \tag{9.56}$$

$$= h\Sigma(\boldsymbol{\pi})h^{-1}\underbrace{h\boldsymbol{\phi}_0}_{\boldsymbol{\phi}_0} \tag{9.57}$$

$$= \Sigma(h\boldsymbol{\pi}h^{-1})\boldsymbol{\phi}_0 \tag{9.58}$$

Since

We will drop the hat on $\hat{T}^{\hat{b}} = T^{\hat{b}}$ for the broken generators now, we will distinguish the broken and unbroken generators by means of the hat on the index.

$$\begin{aligned}
h\Sigma(\boldsymbol{\pi})h^{-1} &= h\left(\exp(i\pi^{\hat{a}}T^{\hat{a}})\right)h^{-1} \\
&= h\left(1 + i\pi^{\hat{a}}T^{\hat{a}} + \frac{\left(i\pi^{\hat{a}}T^{\hat{a}}\right)^2}{2} + \dots\right)h^{-1} \\
&= \left(1 + ih\pi^{\hat{a}}T^{\hat{a}}h^{-1} + \frac{(ih\pi^{\hat{a}}T^{\hat{a}}h^{-1}ih\pi^{\hat{a}}T^{\hat{a}}h^{-1})}{2} + \dots\right) \\
&= \exp(ih\pi^{\hat{a}}T^{\hat{a}}h^{-1}) \\
&= \Sigma(h\boldsymbol{\pi}h^{-1})
\end{aligned}$$

We identify $h = e^{i\alpha^i T^i}$ and $Y = \pi^{\hat{a}}T^{\hat{a}}$ in the BCH-formula and can further simplify:

We use the BCH-formula:

$$e^X Y e^{-X} = \sum_{m=0}^{\infty} \frac{1}{m!}[X,Y]_m$$

with $[X,Y]_m = [X,[X,Y]_{m-1}$ and $[X,Y]_0 = Y$.

$$[T^i, T^{\hat{a}}]_1 = (T^i_\pi)^{\hat{a}}{}_{\hat{c}}\, T^{\hat{c}} \tag{9.59}$$

where we have used Eq. (9.50). Then,

$$\begin{aligned}
[T^i, T^{\hat{a}}]_2 &= [T^i, [T^i, T^{\hat{a}}]] && (9.60)\\
&= (T^i_\pi)^{\hat{a}}{}_{\hat{c}}[T^i, T^{\hat{c}}] && (9.61)\\
&= (T^i_\pi)^{\hat{a}}{}_{\hat{c}}(T^i_\pi)^{\hat{c}}{}_{\hat{d}}T^{\hat{d}} && (9.62)
\end{aligned}$$

So we get

$$\begin{aligned}
h\,\pi^{\hat{a}}T^{\hat{a}}\,h^{-1} &= e^{i\alpha^i T^i}\,\pi^{\hat{a}}T^{\hat{a}}\,e^{-i\alpha^j T^j} && (9.63)\\
&= \sum_{n=0}^{\infty} \frac{(i\alpha^i\,T^i_\pi)^n}{n!}\pi^{\hat{a}}T^{\hat{a}} && (9.64)\\
&= e^{i\alpha^i T^i_\pi}\pi^{\hat{a}}T^{\hat{a}}. && (9.65)
\end{aligned}$$

In conclusion, we find

$$\begin{aligned}
\boldsymbol{\phi} &\to h\boldsymbol{\phi} && (9.66)\\
\rightsquigarrow \Sigma(\boldsymbol{\pi})\boldsymbol{\phi}_0 &\to \Sigma(e^{i\alpha^i T^i_\pi}\boldsymbol{\pi})\boldsymbol{\phi}_0 && (9.67)
\end{aligned}$$

Which shows that the GB's have simple transformation properties under the unbroken subgroup.

The GBs transform in the R_π representation under H transformations. The generators are T^i_π. as clear from the use of Eq. (9.50).

The transformation under the unbroken subgroup H remains linear.

Let's look at an example: $SO(N)/SO(N-1)$

$$\boldsymbol{\pi} = \left(\begin{array}{c|c} \emptyset_{(N-1)\times(N-1)} & \boldsymbol{\pi} \\ \hline -\boldsymbol{\pi}^T & O \end{array}\right) \qquad \Sigma = e^{i\boldsymbol{\pi}/v}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$$

with $O^T O = 1 \rightsquigarrow (T^a)^T = -(T^a)$.

$$H \to O(N-1) = \begin{pmatrix} \hat{O}(N-1) & 0 \\ 0 & 1 \end{pmatrix} \tag{9.68}$$

and

$$\boldsymbol{\pi} \xrightarrow{H} O(N-1)\boldsymbol{\pi}O(N-1)^T = \begin{pmatrix} \emptyset & \hat{O}\boldsymbol{\pi} \\ -\boldsymbol{\pi}^T\hat{O}^T & 0 \end{pmatrix}. \tag{9.69}$$

Therefore we can write

$$\boldsymbol{\pi} \to \hat{O}(N-1)\boldsymbol{\pi}, \tag{9.70}$$

which is clearly a linear transformation. The Goldstone bosons transform in the fundamental of $\hat{O}(N-1)$. This shows a subtlety: the transformation law $\boldsymbol{\pi} \to h\boldsymbol{\pi}h^{-1}$ looks like that of an adjoint representation but the Goldstones clearly do not fill out an adjoint[3]. From the explicit discussion in this example we see that the Goldstone are here in the fundamental of H.

[3] which would have dim(adj)= $N(N-1)/2$ for $SO(N)$

In general, we know that the representation R_π of the linear transformation under H is uniquely determined by the structure of G. Later, we will encounter $G = SU(N) \times SU(N)$ with $H = SU(N)$ being the diagonal subgroup, the representation of H would be in the adjoint of $SU(N)$.

We can identify R_π by analyzing the decomposition of the adjoint of G under H, using

$$\text{adj}(G) = \text{adj}(H) \oplus R_\pi$$

9.4.2 Transformation under G/H, broken generators

The rotations along the broken directions generate very different transformation and we will see that there is no simple way to write them explicitly. The action of a G/H transformation is however relatively simple if we restrict ourselves to infinitesimal rotations:

$$\boldsymbol{\phi} \to e^{i\alpha^{\hat{a}}T^{\hat{a}}}\boldsymbol{\phi} \tag{9.71}$$

$$\rightsquigarrow \quad \Sigma(\boldsymbol{\pi})\boldsymbol{\phi}_0 \to e^{i\alpha^{\hat{a}}T^{\hat{a}}}e^{i\pi(x)^{\hat{b}}T^{\hat{b}}/v}\boldsymbol{\phi}_0 = \exp\left(i\frac{\pi'^{\hat{b}}(x)}{v}T^{\hat{b}}\right)\boldsymbol{\phi}_0 \tag{9.72}$$

We expand Σ and $|\alpha| \ll 1$:

$$(1 + i\pi^{\hat{a}}T^{\hat{a}}/v + \dots) \to \left(1 + i\left(\frac{\pi^{\hat{b}}}{v} + \alpha^{\hat{b}}\right)T^{\vec{\hat{b}}} + \dots\right) \tag{9.73}$$

This is the famous **shift symmetry**, which forbids non-derivative terms in the GB-Lagrangian.

$$\frac{\pi^{\hat{b}}}{v} \to \frac{\pi^{\hat{b}}}{v} + \alpha^{\hat{b}} \tag{9.74}$$

The higher order terms are a very complicated function of the $\alpha^{\hat{a}}$ (non-linear realization)

$$\pi^{\hat{a}}(x) \to \underset{\substack{\uparrow \\ \text{The transformed GB field}}}{\pi^{\hat{b}'}}(x). \tag{9.75}$$

9.5 The Maurer-Cartan-Form and the d and e symbol

We will now identify composite objects which will have simpler transformation properties:

$$d_\pi^{\hat{a}}(\boldsymbol{\pi}) \quad \text{in} \quad R_\pi$$

and

$$e_\mu^i(\boldsymbol{\pi}) \quad \text{in} \quad H$$

The d and e symbols are defined by decomposing on G algebra, using the *Maurer-Cartan-form.*

$$i\Sigma(\boldsymbol{\pi})^{-1}\partial_\mu\Sigma(\boldsymbol{\pi}) \equiv d_\mu^{\hat{a}}(\boldsymbol{\pi})\hat{T}^{\hat{a}} + e_{\mu,i}(\boldsymbol{\pi})T^i. \tag{9.76}$$

Under a $g \in G$ transformation, we have

$$\Sigma \to g\Sigma(x)h^{-1}(g, \Sigma(x)), \tag{9.77}$$

which means for Eq. (9.76):

$$i\Sigma^{-1}\partial_\mu\Sigma \to ih\Sigma^{-1}g^{-1}\partial_\mu(g\Sigma(x)h^{-1}(x)) \tag{9.78}$$
$$= h(i\Sigma^{-1}\partial_\mu\Sigma)h^{-1} + ih\partial_\mu h^{-1}. \tag{9.79}$$

We see that the second term of Eq. (9.79), the inhomogeneous one, is of the Maurer-Cartan form associated with the subgroup H. This means that this term is in the Lie-algebra of H.
We find:

$$d_\mu^{\hat{a}}(\boldsymbol{\pi})T^{\hat{a}} \to h[\boldsymbol{\pi}, g]d_\mu^{\hat{a}}T^{\hat{a}}h^{-1}[\boldsymbol{\pi}, g] \tag{9.80}$$
$$e_\mu^i(\boldsymbol{\pi})T^i \to h[\boldsymbol{\pi}, g]\left(e_\mu^i T^i + i\partial_\mu\right)h^{-1}[\boldsymbol{\pi}, g]. \tag{9.81}$$

The d_μ symbol clearly transforms linearly and is a simple rotation of the $\hat{a}$ index in the R_π representation.

$$d_\mu^{\hat{a}} \xrightarrow{g} d_\mu^{\hat{a}\prime} = \left(e^{i\beta^i T_\pi^i}\right)^{\hat{a}}{}_{\hat{b}}\, d_\mu^{\hat{b}} \tag{9.82}$$

We see that $d_\mu^{\hat{a}}$ transforms like the goldstones (under the subset of $H \subset G$ transformations).
This is further supported by the fact that expanding Σ

Compare the transformation properties under H in Eq. (9.4).

$$i\Sigma^{-1}\partial_\mu\Sigma = i(1 + \ldots)\partial_\mu(i\pi^{\hat{a}}T^{\hat{a}}/v + \ldots) \tag{9.83}$$
$$\simeq -\frac{1}{v}\partial_\mu\pi^{\hat{a}}T^{\hat{a}} \qquad (\widehat{=}\, d_\mu^{\hat{a}}\hat{T}^{\hat{a}} + e_\mu^i T^i) \tag{9.84}$$

and so

$$d_\mu^{\hat{a}} \simeq -\frac{1}{v}\partial_\mu\pi^{\hat{a}} + \mathcal{O}\left(\frac{\partial\pi}{v}\cdot\frac{\pi^2}{v^2}\right). \tag{9.85}$$

However, d_μ is useful for invariants since it keeps transforming linearly even under full G transformations (unlike $\partial_\mu\pi^{\hat{a}}$) which only has this nice behavior for H transformations).

The e^i_μ-symbol resembles a **gauge-transformation** under H:
Compare for this purpose:

$$A_\mu \to U A_\mu U^{-1} + iU\partial_\mu U^{-1} \tag{9.86}$$

vs.

$$e_\mu \to h e_\mu h^{-1} + ih\partial_\mu h^{-1}. \tag{9.87}$$

This means we can use e_μ to construct covariant derivative (or field strengths). We can obtain all G-invariant operators of the GB-EFT using d_μ, e_μ and derivatives ∂_μ.[4] We now have to worry only about H-invariance when using d_μ and e_μ. The leading term in a derivative expansion is:

[4] The anomaly term is an exception, which we will see later.

$$\mathcal{L}^{(2)} = \frac{v^2}{2} d_{\mu,\hat{a}} d^{\mu,\hat{a}} = \frac{1}{2}\partial_\mu \pi_{\hat{a}} \partial^\mu \pi_{\hat{a}} + \mathcal{O}\left(\frac{\partial\pi}{v}\frac{\pi^2}{v^2}\right). \tag{9.88}$$

1. The parameter v^2 defines a unique infinite set of two derivative interactions fixed by the symmetry.

2. The $\pi^{\hat{a}}$ might form a **reducible** representation under H:
   Example:
   $SO(6)/SO(4)$, which leads to 9 GB's.
   In $R_\pi = 4 \oplus 4 \oplus 1$ of $H = SO(4)$.

$$\mathcal{L}^{(2)} = \sum_i \frac{f_i^2}{2} d_{\mu,\hat{a}} d^{\mu,\hat{a}_i} \tag{9.89}$$

It is a sum over the irrep.'s, multiplied by the f^i-vevs.

9.5.1 *The global H invariance as a gauge redundancy*

Gauging and local invariance

$$e_\mu \to h e_\mu h^{-1} + ih\partial_\mu h^{-1}. \tag{9.90}$$

We can define a covariant derivative:

$$\nabla_\mu \equiv \partial_\mu + ie_\mu \tag{9.91}$$

and a field strength:

$$e_{\mu\nu} = \partial_\mu e_\nu - \partial_\nu e_\mu + i[e_\mu, e_\nu] \tag{9.92}$$

and

$$e_{\mu\nu} \to h(\boldsymbol{\pi}, g) e_{\mu\nu} h^{-1}(\boldsymbol{\pi}, g) \tag{9.93}$$

The covariant variables d_μ, $e_{\mu\nu}$ and those formed by acting with ∇_μ are the building blocks of an invariant Lagrangian.
Let's look at some examples:

1. Higher dimensional operators in a derivative expansion can be written as

$$\mathcal{O}_1 = \mathrm{Tr}\left[\mathrm{d}_\mu \mathrm{d}_\mu\right]^2 \tag{9.94}$$

and

$$\mathcal{O}_2 = \mathrm{Tr}\left[\mathrm{d}_\mu \mathrm{d}_\nu\right] \mathrm{Tr}\left[\mathrm{d}_\mu \mathrm{d}_\nu\right] \tag{9.95}$$

and more.
This affects the $\pi\pi \to \pi\pi$ scattering.

$$\mathcal{L} = \frac{v^2}{4}\mathrm{Tr}\ \left[\mathrm{d}_\mu \mathrm{d}_\mu\right] + \sum_{\mathrm{i}} \mathrm{c_i}\mathcal{O}_{\mathrm{i}} + \dots \tag{9.96}$$

$$\mathcal{A}(\pi^a\pi^b \to \pi^c\pi^d) = A(s,t,u)\delta^{ab}\delta^{cd} + A(t,s,u)\delta^{ac}\delta^{bd} + A(u,t,s)\delta^{ad}\delta^{bc}.$$

A straight-forward calculation gives

$$A(s,t,u) = \frac{s}{v^2} + \frac{4}{v^4}(2c_1 s^2 + c_2(t^2+u^2)) + \dots \tag{9.97}$$

2. We can couple to matter fields, e.g.

$$\left(\nabla_\mu \psi\right)_r = \partial_\mu \psi_r - ie_{\mu,i}(t^i)_r{}^s \psi_s \tag{9.98}$$

where the matter fields transforms

$$\psi_r \to \psi_r^{(g)} = \underset{\substack{\uparrow \\ \text{where this is in the representation of } \psi}}{h[\pi,g]_r{}^s} \psi_S \tag{9.99}$$

The Lagrangian is:

$$\mathcal{L} = i\overline{\psi}\gamma_\mu \nabla_\mu \psi - m_\psi \overline{\psi}\psi \tag{9.100}$$

coupling Goldstones to fermionic matter.

9.6 Spontaneous breaking of chiral symmetry

QCD: $SU(3)$: non-abelian gauge theory with Dirac fermions, **quarks** in fundamental representation.
It has the fundamental fields: A_μ^A, gluon $A = 1, \dots 8$. The quarks $q_{\alpha a f}$ are characterized by:

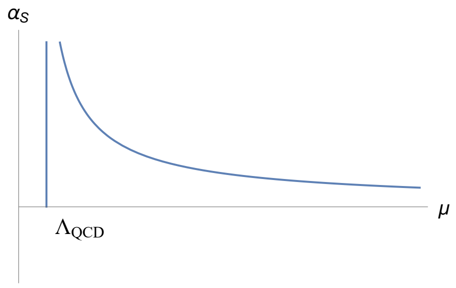
$$\begin{aligned} &\alpha = 1, \dots 4 \\ &\text{Dirac spinor} : q = \begin{pmatrix} q_L \\ q_R \end{pmatrix} \\ &\text{Colour index} : a = 1,2,3 \\ &\text{Quark species `flavor'} : f = u,d,c,s,b,t. \end{aligned}$$

Even though q_L, q_R have different weak-interactions the quantum number does not matter for pure QCD.

All flavors have the same gauge interactions, just differ by the mass.

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^A_{\mu\nu} + \sum_{f=u,d,c,s,b,t} \bar{q}_f (i\not{D} - m_f) q_f + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghost} \tag{9.101}$$

Confinement:



where $\lambda_{QCD} \simeq 1\text{GeV}$.
Quarks form bound states ("hadrons") that use color-singlets. Color charge is **confined** in neutral, composite objects. This is an empirical fact and has not been shown analytically.

It is a millenium problem!

Mesons: $\bar{q}_a q_a$: $\bar{3} \otimes 3 = \overset{\text{singlet}}{\overset{\downarrow}{1}} \oplus 8$.
Baryons: $\epsilon_{abc} q_a q_b q_c$.
$q_b q_c : 3 \otimes 3 = \bar{3} \oplus 6$ and $q_a q_b q_c : 3 \otimes \bar{3} = 1 \oplus 8$.
The proton is made of (uud) connected by gluons. It has $m_{photon} = 938\text{MeV}$, $2m_u + m_d \simeq 20\text{MeV}$.
$\hookrightarrow$ Most of the mass is from binding energy, only possible for very strong interactions.

9.7 *Approximate global symmetries of QCD*

It is interested in the low energy regime $Q \lesssim 1\text{GeV}$

f	u	d	s	c	b	t
m_f[MeV]	5	10	90	1300	4200	175000

where u, d, s are the light quarks ($m_f \leq \Lambda_{QCD}$) and c, b, t are the heavy quarks ($m_f \geq \Lambda_{QCD}$).
We will ignore the heavy quarks in the following.
In the approximation $m_f = 0$, we can rotate q_{L_f} and q_{R_f} separately (axial and vector currents) by unitary matrices in flavor space:

$$q_{L_f} \to U_{Lff'} q_{L'_f} \tag{9.102}$$

$$q_{R_f} \to U_{Rff'} q_{R'_f} \tag{9.103}$$

with $U_L, U_R \in U(3)$.

Global flavor symmetry (for $m_f = 0$):

$$U(3)_L \otimes U(3)_R = \underset{\substack{\uparrow \\ q_f \to e^{i\beta} q_f \\ \text{same } \beta\ \forall f}}{U(1)_B} \otimes \overset{\substack{q_f \to e^{i\alpha\gamma_5} q_f \\ \text{same } \alpha\ \forall f \\ \downarrow}}{U(1)_A} \otimes \underset{\substack{\uparrow \\ q_f \to e^{i\beta^a T^a} q_f}}{SU(3)_V} \otimes \overset{\substack{q_f \to e^{i\alpha^a T^a \gamma_5} q_f \\ \downarrow}}{SU(3)_A}. \tag{9.104}$$

with $f = u, d, s$.
Comments:

- $\mathbf{U(1)}_B$: unbroken even in presence of the quark mass: $\sum m_f \bar{q}_f q_f$, $q_f \to e^{i\beta} q_f\, \forall f$.
  $\hookrightarrow$ Exact global symmetry (baryon number): it guarantees that the lightest baryon (proton) is stable.

Exercise-sheet: $SU(2)_W$ anomaly? It is a small non-perturbative effect at $T = 0$. The decay rate: $\Gamma_{decay} \propto e^{-\frac{2\pi}{\alpha_w(M_W)}} \sim 10^{-77} \gg$ (life time of the universe)$^{-1}$ with $\alpha_w \simeq 0.04 \ll 1$.

- $\mathbf{U(1)_A}$:
  This symmetry is $SU(3)_C$ anomalous, i.e. not a symmetry of the $SU(3)_c$ quantum theory (contrary to $SU(2)_W$ anomaly, α_{QCD} can be large.)

$$\delta_{U(1)_A} S = 0, \quad \text{but } \delta_{U(1)_A} \underset{\substack{\uparrow \\ \text{effective action:} \\ \int \underbrace{\mathcal{D}\psi\mathcal{D}\psi\mathcal{D}A_\mu}_{\text{QCD}} e^{iS}}}{\Gamma} \neq 0 \tag{9.105}$$

- $\mathbf{SU(3)_V}$ and $\mathbf{SU(3)_A}$: observe the QCD resonance spectrum. We focus on u, d: $SU(3)_V$ and $SU(3)_A$ first.

M		$SU(2)_V$	J^P
n, p	940MeV, 938MeV	doublet	$\frac{1}{2}^+$
η	547MeV	singlet	0^-
$K^0, \bar{K}, K^\pm$	493MeV, 497MeV	doublet: $\begin{smallmatrix} K^+ \\ K^0 \end{smallmatrix}$, $\begin{smallmatrix} K^- \\ \bar{K}^0 \end{smallmatrix}$	0^-
$\pi^0, \pi^\pm$	134MeV, 139MeV	triplet	0^-

The behavior is like $SU(2)_V$ representations.
We could explain the lightness of $\pi^0, \pi^\pm$ if they are goldstones.

They are $\underset{\substack{\uparrow \\ \text{Spin}}}{J}^{\overset{\text{Parity}}{\downarrow}\!P} = 0^-$. This would correspond to a **broken axial** symmetry, e.g.

$$j_\mu^A = \overline{\psi} \tau^a \gamma_\mu \gamma_5 \psi \tag{9.106}$$

with $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$.

These symmetries are indeed spontaneously broken and explicitly seen in lattice field theory calculations.
Further: light mesons $\pi^0, \pi^\pm, K^0, \bar{K}^0, K^\pm, \eta$ are 8 d.o.f. with 0^-.
$\hookrightarrow$ Could these be GBs of $\overline{\psi}\lambda^a\gamma_\mu\gamma_5\psi$ with $\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$
Spontaneous breaking 8 λ^a generators! $SU(3)_A$ SSB.

- $\mathbf{SU(3)_V}$: for $m_f = 0$ or $m_f = \mathbb{1} \cdot \alpha$. This would be an exact symmetry since

$$\mathcal{L} \supset m \sum_f \bar{q}_L^f q_R^f + \text{h.c.}, \tag{9.107}$$

which would be invariant under

$$q_L \to U q_L \tag{9.108}$$
$$q_R \to U q_R \tag{9.109}$$

Quarks:

$$\bar{q}\gamma_\mu\lambda_a q \qquad \text{for } q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \tag{9.110}$$

The quark mass differences explicitly break $SU(3)_V$. $SU(2)_V$ subgroup (u, d) is less strongly broken.

$$m_u, m_d \ll m_s$$

with $m_u = 5\text{MeV}$, $m_d = 10\text{MeV}$ and $m_s = 90\text{MeV}$.

- $\mathbf{SU(3)_A}$: We do not observe $SU(3)_A$ multiplets in the hadron spectrum $\to$ it would imply partners with $SU(3)$ generators λ^a with opposite parity for each hadron.

$$\bar{q}_f\gamma^\mu\gamma_5 T^a_{fg} q_g \tag{9.111}$$

has negative parity, and so has Q_A^a.
Hence if $|p\rangle$ e.g. is a proton state, then $Q_A^a|p\rangle$ is also an eigenstate of the Hamiltonian.

$$\underset{\substack{\uparrow \\ \text{parity transformation}}}{U_p}(Q_A^a|p\rangle) = U_p Q_A^a U_p^{-1} \underbrace{U_p|p\rangle}_{(+1)|p\rangle} \tag{9.112}$$

$$= -Q_A^a|p\rangle. \tag{9.113}$$

$\hookrightarrow$ These states are not observed.
$SU(3)_A$ is spontaneously broken through strong dynamics of QCD (non-perturbative effects).
$\delta_{SU(3)_A}\Gamma = 0$, but the vacuum is not invariant.
$SU(3)_A$ is more precisely a spontaneously broken **approximate** symmetry. The explicit breaking is small, especially for $SU(2)_A$ subgroup q_u, q_d.

9.8 The effective chiral Lagrangian

We can make a dynamical **assumption** that the effect of the QCD force is to form quark-antiquark condensates.

$$\langle\Omega|\bar{q}_L^i q_R^j + \text{h.c.}|\Omega\rangle \neq 0 \tag{9.114}$$

with $i = u, d, c, s, b, t$.
The symmetry $SU(3)_L \otimes SU(3)_R$ or $SU(3)_A \otimes SU(3)_V$ is therefore broken spontaneously to

$$\begin{aligned} SU(3)_V \otimes SU(3)_A &\to SU(3)_V \\ \text{or } SU(3)_L \otimes SU(3)_R &\to SU(3)_{L+R} \end{aligned}$$

Eq. (9.114) is left invariant by

$$q_L \to \overset{\overset{SU(3)_L}{\downarrow}}{L} q_L \tag{9.115}$$

$$q_R \to \underset{\underset{SU(3)_R}{\uparrow}}{R} q_R \tag{9.116}$$

with $L = R$.

$$\bar{q}_L q_R \to \bar{q}_L L^\dagger R q_R \tag{9.117}$$

$$\overset{L=R^\dagger}{=} \bar{q}_L R^\dagger R q_R = \bar{q}_L q_R \quad \checkmark \tag{9.118}$$

The QCD vacuum does not know about flavor

$$\langle\Omega|\bar{q}_L^i q_R^j + \bar{q}_R^j q_L^i|\Omega\rangle = v_{QCD}^3 \delta^{ij}. \tag{9.119}$$

Using CCWZ from the chapter Eq. (9.31) we can parametrize the GB-dof

$$\langle\phi\rangle = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} v_{QCD} \qquad \text{(vev)} \tag{9.120}$$

with $\phi \to L\phi R^+$ and $L \in SU(3)_L$, $R \in SU(3)_R$.
The unbroken transformations is: $L = R$ ("L+R")

$$\langle\phi\rangle \to L\langle\phi\rangle L^+ = \langle\phi\rangle LL^+ = \langle\phi\rangle. \tag{9.121}$$

Broken generators: $L = R^+$ ("L-R")

$$\Sigma(x) = L(x)\langle\phi\rangle L(x) \tag{9.122}$$

$$\Sigma(x) = \exp\left(\frac{2i}{f_\pi}\boldsymbol{\pi}(x)\right) \tag{9.123}$$

with

$$\boldsymbol{\pi}(x) = \overset{\overset{SU(3)\text{: Gellmann matrices}}{\downarrow}}{\lambda^a} \pi^a(x) \qquad a = 1, \dots, 8 \tag{9.124}$$

$$\text{or } \boldsymbol{\pi}(x) = \underset{\underset{SU(2)\text{: Pauli matrices}}{\uparrow}}{\tau^a} \pi^a(x) \qquad a = 1, \dots, 3 \tag{9.125}$$

with the following normalization:[5]

$$\mathrm{Tr}\,(\lambda^{\mathrm{a}}, \lambda^{\mathrm{b}}) = 2\delta^{\mathrm{ab}}. \tag{9.127}$$

[5] This is a slightly different normalization as before where we had:

$$\mathrm{Tr}(\mathrm{T}^{\mathrm{a}}, \mathrm{T}^{\mathrm{b}}) = \frac{1}{2}\delta^{\mathrm{ab}} \quad \lambda^{\mathrm{a}} = \frac{\mathrm{T}^{\mathrm{a}}}{2}, \tau^{\mathrm{a}} = \frac{\sigma^{\mathrm{a}}}{2}. \tag{9.126}$$

$$\Sigma \to \Sigma' = L\Sigma R^{-1} \tag{9.128}$$

with $L = e^{\frac{i}{2}\boldsymbol{\omega}_L}$ and $R = e^{\frac{i}{2}\boldsymbol{\omega}_R}$ where we have $\boldsymbol{\omega}_L = \omega_L^a\lambda^a$ and $\boldsymbol{\omega}_R = \omega_R^a\lambda^a$.
The H-transformation (unbroken group) is:

$$\delta\boldsymbol{\pi} = \frac{i}{2}\left[\boldsymbol{\omega}_V, \boldsymbol{\pi}(x)\right] + \ldots \qquad \text{(infinitesimal)} \tag{9.129}$$

with $\boldsymbol{\omega}_L = \boldsymbol{\omega}_R \equiv \boldsymbol{\omega}_V$ and $L = R$.
G/H transformations: $\boldsymbol{\omega}_L = -\boldsymbol{\omega}_R \equiv \boldsymbol{\omega}_A$

$$\delta\boldsymbol{\pi} = \underbrace{\frac{f_\pi}{2}\boldsymbol{\omega}_A}_{\substack{\text{shift symmetry forbids}\\ \text{mass terms for } \boldsymbol{\pi}}} - \frac{1}{v}[\boldsymbol{\pi}, [\boldsymbol{\pi}, \boldsymbol{\omega}_A]] + \ldots \tag{9.130}$$

The unique term with least derivatives is:

$$\mathcal{L} = \frac{f_\pi^2}{16}\mathrm{Tr}\,\left[\partial_\mu\Sigma\partial_\mu\Sigma^{-1}\right] \tag{9.131}$$

$$= \frac{1}{2}\partial_\mu\pi^a\partial_\mu\pi^a + \ldots \tag{9.132}$$

In terms of physical fields we get:

$$\boldsymbol{\pi} = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\pi_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\pi_8 & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\pi_8 \end{pmatrix} \tag{9.133}$$

with $\bar{K}^0 = K^{0+}$.
Explicit chiral symmetry breaking:
First we reintroduce quark masses:

$$\mathcal{L}_m = \bar{q}_L^i M_{ij} q_R^j + \text{h.c.} \tag{9.134}$$

with $M_{ij} = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$.
M_{ij} can be viewed as a “spurion” of flavor symmetry breaking.
We can make $\mathcal{L}_m$ formally invariant again by postulating a transformation of

$$M \to LMR^+ \tag{9.135}$$

with $q_L \to Lq_L$ and $q_R \to Rq_R$.
Think of M_{ij} as a very heavy field with the vev $\langle M\rangle = M_{ij}$. All effects of M_{ij} must be such that under the simultaneous transformation of q_L, q_R and M_{ij} the theory is invariant.
It is very useful for bookkeeping chiral symmetry breaking effects due to M_{ij}.

The L, R symmetry of the rest of the theory allows us to always go to the form:

$$M^{ij} \to LMR^+ = \overset{\substack{\text{diagonal}\\ \downarrow}}{D^{ij}} = \begin{pmatrix} \star & & \\ & \star & \\ & & \star \end{pmatrix}$$

$$\begin{pmatrix} m_u & \\ \end{pmatrix}$$

Inclusion in chiral perturbation theory:

$$\mathcal{L}_m = \frac{v_{QCD}^3}{3}\mathrm{Tr}\left(\mathrm{M}^\dagger\Sigma + \Sigma^\dagger\mathrm{M}\right) \tag{9.136}$$

Question: Why do we not have e.g. $i\mathrm{Tr}\left(\mathrm{M}^\dagger\Sigma - \Sigma^\dagger\mathrm{M}\right)$? [6] We do the $SU(2)$ calculation explicitly:

$$\Sigma = \exp\left(\frac{2i}{f_\pi}\pi^a\tau^a\right) \tag{9.137}$$

$$= 1 + \frac{2i}{f_\pi}\pi^a\tau^a - \frac{4}{2f_\pi^2}\pi^a\tau^a\pi^b\tau^b + \dots \tag{9.138}$$

[6] $\rightarrow$ think of parity!

which gives

$$\begin{aligned} v_{QCD}^3\mathrm{Tr}\left(\mathrm{M}^\dagger\Sigma + \Sigma^\dagger\mathrm{M}\right) &= v^3\mathrm{Tr}\left(\mathrm{M}(\Sigma+\Sigma^\dagger)\right) \\ &= -4\frac{v^2}{f_\pi^2}\mathrm{Tr}\left(\mathrm{M}\tau^\mathrm{a}\tau^\mathrm{b}\right)\pi^\mathrm{a}\pi^\mathrm{b} + \dots \\ &= -2\frac{v^3}{f_\pi^2}\mathrm{Tr}\left(\mathrm{M}\{\tau^\mathrm{a},\tau^\mathrm{b}\}\right)\pi^\mathrm{a}\pi^\mathrm{b} + \dots \\ &= -2\cdot 2\frac{v^3}{f_\pi^2}\mathrm{Tr}(\mathrm{M})\pi^\mathrm{a}\pi^\mathrm{a} + \dots \end{aligned}$$

where we used in the last step $\{\tau^a,\tau^b\} = \{\sigma_a,\sigma_b\} = 2\delta_{ab}$.
All three pions have the same mass!

$$\mathcal{L}_m \supset \underbrace{-\frac{v^3}{2f_\pi^2}\mathrm{Tr}(\mathrm{M})}_{\frac{m_\pi^2}{2}}\pi^a\pi^a + \dots \tag{9.139}$$

from which we get the famous result

$$\boxed{m_\pi^2 = (m_u + m_d)\frac{v^3}{f_\pi}.} \tag{9.140}$$

We can repeat the same calculation with $SU(3)$:

$$m_{\pi^\pm}^2 = m_0(m_u + m_d) \tag{9.141}$$

$$m_{K^\pm}^2 = m_0(m_u + m_s) \tag{9.142}$$

$$m_{K^0}^2 = m_0(m_d + m_s) \tag{9.143}$$

We can extract here the quark masses and e.g. see that $m_u < m_d$ since m_{K^0} (497.6 MeV) $> m_{K^\pm}$(493.6 MeV). The diagonal fields π^3 and π^8 mix and appear as

$$\frac{\mathcal{L}_m}{m_0} \supset \frac{1}{2}(m_u + m_d)\pi^3\pi^3 + \frac{1}{6}(m_u + m_d + 4m_s)\pi^8\pi^8 + \frac{1}{\sqrt{3}}(m_u - m_d)\pi^3\pi^8.$$

We need to rotate $\pi^3, \pi^8 \rightarrow \eta, \pi^0$ to diagonalize.

Let us assume for simplicity $\hat{m} = m_u = m_d$ (no mixing):

$$m_{\pi^0}^2 = m_0 \cdot 2\hat{m} \tag{9.144}$$

$$m_\eta^2 = m_0(2\hat{m} + 4m_s)\frac{1}{3} \tag{9.145}$$

To relate all the masses we have the Gell-Mann-Oakes-Renner relation:

$$m_\eta^2 + m_{\pi^0}^2 = \frac{2}{3}(m_{K^0}^2 + m_{K^+}^2 + m_{\pi^+}^2) \tag{9.146}$$

What is still missing is the electromagnetic correction:

$$m_{\pi^0}^2 = m_0\left(m_u + m_d - \frac{1}{4}\frac{(m_u - m_d)^2}{m_s} + \mathcal{O}\left(\frac{1}{m_s^2}\right)\right) \tag{9.147}$$

$$m_\eta^2 = m_0\left(\frac{4m_s + m_u + m_d}{3} + \frac{1}{4}\frac{(m_u - m_d)^2}{m_s} + \mathcal{O}\left(\frac{1}{m_s^2}\right)\right) \tag{9.148}$$

$$m_{\pi^+}^2 - m_{\pi^0}^2 \simeq \frac{1}{4}\frac{(m_u - m_d)^2}{m_s} m_0. \tag{9.149}$$

It is too small to explain the measured values

$$\pi^+(134.96\,\mathrm{MeV})$$
$$\pi^0(139.57\,\mathrm{MeV}).$$

We have not included the weak and electromagnetic gauge interactions.

9.9 Electro-weak Interactions

Quarks couple bi-linearly to W, Z, γ gauge bosons

$$\overline{\psi}\gamma_\mu\psi A_\mu = \partial_\mu \cdot A^\mu. \tag{9.150}$$

The strategy is to write currents J_μ in terms of Σ and couple it to gauge bosons (external fields). Currents of $SU(3)_L \otimes SU(3)_R$:

$$J_{\mu,L}^a = \frac{\partial\mathcal{L}}{\partial\partial_\mu\pi^b}\frac{\delta\pi^b}{\delta\omega_L^a} \quad \text{and } (L \leftrightarrow R) \tag{9.151}$$

With

$$\mathcal{L} = \mathcal{L}_0 + \ldots = \frac{f_\pi^2}{16}\mathrm{Tr}\left(\partial_\mu\Sigma\partial_\mu\Sigma^{-1}\right) \tag{9.152}$$

$$= \frac{1}{2}\partial_\mu\pi^a\partial_\mu\pi^a + \ldots \tag{9.153}$$

The transformation: $\Sigma \to L\Sigma R^{-1}$

$$1 + 2i\frac{\boldsymbol{\pi}}{f_\pi} + \ldots \to 1 + 2i\frac{\boldsymbol{\pi}}{f_\pi} + \frac{i}{2}\boldsymbol{\omega}_L - \frac{i}{2}\boldsymbol{\omega}_R \tag{9.154}$$

$$\boldsymbol{\pi} \to \boldsymbol{\pi} + \frac{f_\pi}{4}(\boldsymbol{\omega}_L - \boldsymbol{\omega}_R) \tag{9.155}$$

$$\frac{\delta\pi^a}{\delta\omega_L^b} = \frac{f_\pi}{4}\delta_b^a \tag{9.156}$$

$$\frac{\delta\pi^a}{\delta\omega_R^b} = -\frac{f_\pi}{4}\delta_b^a \tag{9.157}$$

It follows

$$J^{\mu,a}_{L,R} = \frac{f_\pi}{4}\partial_\mu \pi^a(x) + \dots \tag{9.158}$$

We can write the chiral lagrangian as:

$$\mathcal{L}_0 = -\frac{1}{f_\pi}\pi^a \partial^\mu (J^a_{\mu,L} - J^a_{\mu,R}). \tag{9.159}$$

We see that we can get the full current by

$$\mathrm{Tr}\left(\Sigma^{-1}\lambda^{\mathrm{a}}\partial_\mu \Sigma\right) \tag{9.160}$$

since it transforms as the $SU(3)_L$ current. To fix the normalization (compare to Eq. (9.158)):

$$J^a_{\mu,L} = -i\frac{f_\pi^2}{16}\mathrm{Tr}\left(\Sigma^{-1}\lambda^{\mathrm{a}}\partial_\mu \Sigma\right) \tag{9.161}$$

$$J^a_{\mu,R} = -i\frac{f_\pi^2}{16}\mathrm{Tr}\left(\Sigma\lambda^{\mathrm{a}}\partial_\mu \Sigma^{-1}\right) \tag{9.162}$$

Note: We have the same form as the Maurer-Cartan form above: $\mathrm{Tr}\left(\lambda^{\mathrm{a}}(\partial_\mu\Sigma)\Sigma^{-1}\right)$ and $\mathrm{Tr}\left(\lambda^{\mathrm{a}}(\partial_\mu\Sigma^{-1})\Sigma\right)$ just by projecting to π^a. There are two forms since here we have $\Sigma \to L\Sigma R^+$, two different contributions for L and R.

or

$$J^a_{\mu,L/R} = \pm\mathrm{Tr}\left(\lambda^{\mathrm{a}}\partial_\mu \boldsymbol{\pi}\right) - \frac{\mathrm{i}}{8}\mathrm{Tr}\left([\boldsymbol{\pi},\lambda^{\mathrm{a}}]\partial_\mu\boldsymbol{\pi}\right) \mp \frac{1}{12\mathrm{f}_\pi}\mathrm{Tr}\left([\boldsymbol{\pi},[\boldsymbol{\pi},\lambda^{\mathrm{a}}]]\partial_\mu\boldsymbol{\pi}\right) + \dots$$

9.10 Alternative derivation

The photon couples vectorially to the three light quarks with the strength $\propto Q_{electric}$. The photon coupling is determined requiring local invariance under the local transformation:

$$\Sigma \to e^{i\alpha(x)\tilde{Q}}\Sigma e^{-i\alpha(x)\tilde{Q}} \tag{9.163}$$

$$\delta\Sigma = i\alpha(x)[\tilde{Q},\Sigma] \tag{9.164}$$

with

$$\tilde{Q} = \begin{pmatrix} (Q_u =)\frac{2}{3} & & \\ & (Q_d =) -\frac{1}{3} & \\ & & (Q_s =) -\frac{1}{3} \end{pmatrix} \tag{9.165}$$

$$= \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8 \tag{9.166}$$

where λ_3 and λ_8 are

$$\lambda_3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} \quad \text{and} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{9.167}$$

The Lagrangian density is already invariant under this transformation, we simply replace:

You can check!

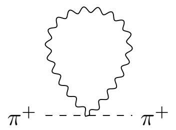
$$\partial_\mu\Sigma \to \partial_\mu + iA_\mu[Q,\Sigma] \tag{9.168}$$

We get

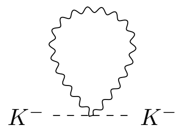
$$J^{em}_\mu = J^3_{\mu,L} + \frac{1}{\sqrt{3}}J^8_{\mu,L} + \frac{1}{\sqrt{3}}J^8_{\mu,R}, \tag{9.169}$$

which has the form of the current above.

We now understand how to include QED effects



or



(9.170)

into the meson mass determination.

$$m_{\pi^\pm} = m_0(m_u + m_d) + \Delta^2_{em} \tag{9.171}$$
$$m_{K^\pm} = m_0(m_u + m_s) + \Delta^2_{em} \tag{9.172}$$

(unknown, but equal contribution)
We can fit the meson masses from the experiment:

$$m_0 m_u = 6528\text{MeV}^2 \qquad m_0 m_d = 11720\text{MeV}^2$$
$$m_0 m_s = 23500\text{MeV}^2 \qquad \Delta^2_{em} = 1232\text{MeV}^2$$

So we can calculate the ratios of masses:

$$\frac{m_u}{m_d} \simeq \frac{1}{2} \qquad \frac{m_d}{m_s} \simeq \frac{1}{20} \tag{9.173}$$

It is an accurate determination of the light quark mass ratios. We find

$$m_\eta^{\chi PT} = 564 \text{ MeV vs. } m_\eta^{exp.} = 547.86\,\text{MeV}.$$

This is not bad! Given that we were only relying on symmetries and have used zero knowledge of non-perturbative QCD.

Note: We have ignored $\eta - \eta'$-mixing , which improves the result: $\sin\theta_{\eta\eta'} \simeq 0.2$.

9.11 Gauging $SU(2)_L \otimes U(1)_Y$

We can also include $SU(2)_L \otimes U(1)_Y$. Simpler for $SU(2)_L \otimes SU(2)_R$ χPT:

$$SU(2)_L^{weak} = SU(2)_L^{flavor}$$
$$U(1)_Y = \tau_{3R} + \frac{B-L}{2}.$$

$$\partial_\mu \to \partial_\mu - ig\omega^a_\mu \frac{\tau^a}{2} - ig'\frac{\tau^3}{2}B_\mu \quad \text{(fundamental)} \tag{9.174}$$
$$\hookrightarrow D_\mu \Sigma^i_j = (\partial_\mu - ig\omega^a_\mu(\tau^a)^i_{i'} + \frac{1}{2}ig'(\tau^3)^{j'}_j B_\mu)\Sigma^{i'}_{j'} \tag{9.175}$$

fixes the interactions of $\pi', W's, Z's, \gamma's$. This will lead to a mass term for W and Z!

9.11.1 Gauge boson masses

The kinetic terms of the pions lead to masses of the W, Z:

$$\frac{f_\pi^2}{16}\text{Tr}\left(\text{D}_\mu\Sigma(\text{D}_\mu\Sigma)^\dagger\right) = \frac{\text{f}_\pi^2}{16}\text{Tr}\left[\begin{pmatrix} \frac{gW_3}{2} - \frac{g'B}{2} & g\frac{W^1 - iW^2}{2} \\ g\frac{W^1+iW^2}{2} & -\frac{gW_3}{2} + \frac{g'B}{2}\end{pmatrix}^2\right] \tag{9.176}$$

where we've set $\boldsymbol{\pi} = 0 \to \Sigma = \mathbb{1}$. It follows

$$g^2 \frac{f_\pi^2}{16} W_\mu^\pm W_\mu^\pm + \frac{g^2 + g'^2}{2} \frac{f_\pi^2}{16} Z_\mu Z^\mu + 0 \cdot A_\mu A^\mu \tag{9.177}$$

We find gauge boson mass terms as in SM after Higgs-vev! **But** no Higgs here!

Important result: QCD itself breaks the EW symmetry. It gives a (small) **dynamical** mass to W, Z-bosons except this mass is too small:

$$f_\pi \simeq 200\text{MeV} \tag{9.178}$$

So the electro-weak boson masses are roughly:

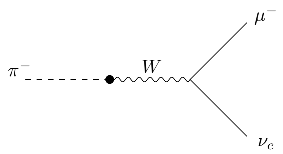
$$W, Z \sim \frac{g}{4} \cdot 200\text{MeV} \sim 50\text{MeV}$$

$\hookrightarrow$ about a factor 1000 too small.[7]

[7] This mechanism has been revived as an alternative to an elementary Higgs breaking EW symmetry and is known as *Technicolor*. The corresponding $f = v_{\text{EW}}$.

9.11.2 Pion decay constant f_π from $\pi^+ \to \mu^+ \nu_\mu$

We can compute f_π from the leptonic decay of the pion

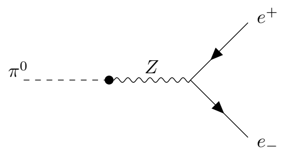


(9.179)

$$\Gamma(\pi^+ \to \mu^+ \nu_\mu) = \frac{m_\pi}{16\pi} \left(G_F f_\pi m_\mu V_{ud}\right)^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \tag{9.180}$$

The experimental value is $f_\pi^{exp} = 186$ MeV.

We can also calculate, e.g. $\pi^0 \to e^+ e^-$:

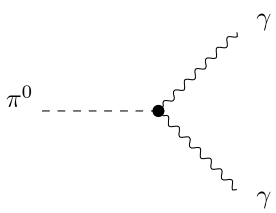


(9.181)

which is

$$\Gamma(\pi^0 \to e^+ e^-) = \frac{m_\pi}{16\pi} (G_F m_e f_\pi)^2 \left(\frac{1}{4} - \sin^2\theta_W + 2\sin^4\theta_W\right) \tag{9.182}$$

This is comparable to the charged decay (except for $\left(\frac{m_e}{m_\mu V_{ud}}\right)^2$ suppression) $\sim 1\%$. But: It has not been observed. **Reason**: neutral pions decays much faster into two photons.



(9.183)

Our Lagrangian does not contain such an interaction!

9.12 *Higher orders and cut-off scale.*

Chiral Lagrangian is an EFT, we are expanding in $\sim \frac{\text{momentum}}{f_\pi}$.
The leading order is:

$$\mathcal{L} = \frac{f_\pi^2}{16}\mathrm{Tr}(\partial_\mu\Sigma\partial_\mu\Sigma^\dagger) + \mathrm{m}_0^2\frac{\mathrm{f}_\pi^2}{8}\mathrm{Tr}\,(\Sigma^\dagger \mathrm{M}) + \mathrm{h.c.} \tag{9.184}$$

What about higher orders:

$$\mathrm{Tr}\,(\mathrm{D}^\mu\mathrm{D}_\nu\Sigma^\dagger\mathrm{D}_\mu\mathrm{D}_\nu\Sigma) \to \frac{\mathrm{f}^2}{\Lambda^2} \tag{9.185}$$

$$\mathrm{Tr}\,(\mathrm{D}^\nu\Sigma^\dagger\mathrm{D}_\nu\Sigma\Sigma^\dagger\mathrm{m}_0\mathrm{M}) \to \frac{\mathrm{f}^2}{\Lambda^2} \tag{9.186}$$

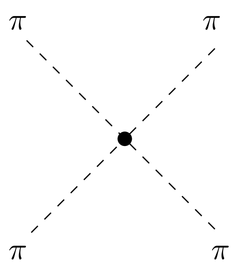
We expect more fields with higher suppression by $\frac{1}{\Lambda}$

$$\text{extra } \partial_\mu\partial^\mu \to \frac{1}{\Lambda^2}$$
$$\text{extra } m_0 M \to \frac{1}{\Lambda^2}$$

Expansion in reality in $\left(\frac{p^2}{\Lambda^2}\right)$, $\left(\frac{m_0 M}{\Lambda^2}\right)$. It is valid if $p^2, m_0 M \ll \Lambda^2$.

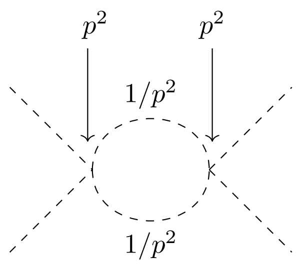
How large is the cut-off scale?
It is a $4-\pi$ interaction See Exercise.

$$\frac{f^2}{16}\mathrm{Tr}\left(\mathrm{D}^\mu\Sigma\mathrm{D}^\mu\Sigma^{-1}\right) \to \frac{\mathrm{p}^2}{\mathrm{f}^2}\pi^4 \tag{9.187}$$



(9.188)

The cut-off estimate by making loops are not bigger than tree-level terms.



$$\sim \frac{1}{(4\pi)^2}\int \frac{d^4p}{p^4}\frac{p^4}{f^4} \sim \frac{\Lambda^4}{f^4} \tag{9.189}$$

if all momenta are just loop momentum p.
This cannot be the result because this would violate the shift symmetry. The diagram (see Eq. (9.189)) will at least have two derivative operators.

$$\sim p_{ext}^2\frac{\Lambda}{(4\pi)^2 f^4}\pi^4(x) \tag{9.190}$$

We need to make sure:

$$\overset{\substack{\text{tree-level}\\ \downarrow}}{\frac{p_{ext}^2}{f^2}}\pi^4(x) > p_{ext}^2\overset{\substack{1-\text{loop}\\ \downarrow}}{\frac{\Lambda}{(4\pi)^2 f^4}}\pi^4(x) \tag{9.191}$$

It follows

$$\boxed{\Lambda \lesssim 4\pi f} \tag{9.192}$$

with $f_\pi \sim 200\text{MeV} \rightsquigarrow \Lambda \simeq (1-2)\text{GeV}$.
This give us the range of validity of our EFT:

$$0 \leq E \lesssim (1-2)\text{GeV} \tag{9.193}$$

9.13 Anomalies and Goldstone bosons

We have used global symmetries (like $SU(3)_V \otimes SU(3)_A$) to constrain the low-energy EFT of QFTs with spontaneous symmetry breaking. These "selection rules" greatly simplify the possible interactions - even in the presence of strong coupling.
It is important to discuss:
What if the underlying theory is anomalous with respect to global symmetries?
Question 1: Effect of global anomalies?
Question 2: What happens in presence of gauge fields?
Divergence of the current: $\partial_\lambda j_\lambda \propto \epsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ via chiral fermions.
But in QCD the low-energy EFT does not contain fermions, only $\Sigma(\boldsymbol{\pi})$!

9.13.1 $\pi_0 \to \gamma\gamma$

The interpretation and explanation of the decy rate $\Gamma(\pi_0 \to \gamma\gamma)$ have been crucial for the discovery and understanding of anomalies.
Simplest term:

$$\mathcal{L} \supset g\pi^0(x)\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{9.194}$$

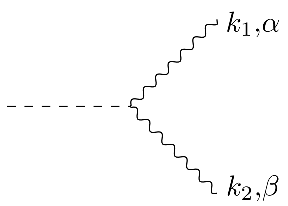
which is gauge-invariant ($\pi^0(x)$ being neutral does **not** transform) and is **parity conserving** :

$$\boxed{\text{QED and QCD conserve } P.} \tag{9.195}$$

$$P\pi^0P = -\pi^0 \quad PF\tilde{F}P = -F\tilde{F} \rightsquigarrow \pi^0F\tilde{F} \quad \text{even!}$$

We are suppressing derivative terms/quark masses.

We compute the decay rate:



(9.196)

$$4g\pi^0\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma \tag{9.197}$$

It leads to $\mathcal{A}$:

$$\mathbf{A} = 8g\epsilon^{\mu\nu\rho\sigma} ik_{1,\mu} ik_{2,\rho} \tag{9.198}$$

So it follows:

$$\underset{\substack{\text{sum over}\\ \text{photon polarizations}\\ \downarrow}}{|\mathcal{A}|^2} = 64g^2 k_{1\mu}k_{2\rho}k_{1\mu'}k_{2\rho'} \underbrace{\epsilon^{\mu\alpha\rho\beta}\epsilon^{\mu'\alpha'\rho'\beta'}(-g_{\alpha\alpha'})(-g_{\beta\beta'})}_{(-2)\left(g^{\mu\mu'}g^{\rho\rho'}-g^{\mu\rho'}g^{\rho\mu'}\right)} \tag{9.199}$$

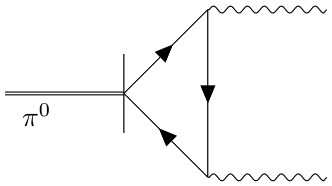
$$= 128g^2 \underbrace{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}_{=\frac{1}{2}(k_1+k_2)^2} \tag{9.200}$$

$$= 32g^2 m_\pi^4. \tag{9.201}$$

We get for the decay rate:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{1}{2m_\pi} \cdot \underset{\substack{\uparrow\\ \text{identical final states}}}{\frac{1}{2}} \cdot \frac{1}{8\pi} \cdot 32g^2 m_\pi^4 = \frac{g^2 m_\pi^3}{\pi} \tag{9.202}$$

In Eq. (9.194) g has mass-dimension 1 ($[g] = 1$) and contains a e^2. Now we estimate g using quark-level:



$$g = \frac{e^2}{16\pi^2} \cdot \frac{1}{f_\pi} = \frac{\alpha_{em}}{4\pi f_\pi} \tag{9.203}$$

with $f_\pi \simeq 100\text{MeV}$ we find

$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = \frac{1}{\tau} \simeq 4 \cdot 10^{-16} 1/s$$

So τ is: $\tau = 2.6 \cdot 10^{-17}s$ vs. $\tau_{exp} = 8.3 \cdot 10^{-17}s$.
Given the crude approximation this is reasonably close.

Problem: (Sutherland-Veltman-paradox)

The interaction $\pi_0 F\tilde{F}$ **cannot** appear in a chiral Lagrangian since it would break the **shift symmetry**![8] All interactions of π^0 must involve a derivative. This additional $\partial_\mu^2 \sim m_\pi$ suppresses the amplitude further

[8] This is clearly too naive, since $F\tilde{F} \sim \partial_\mu K^\mu$, but it is still not obvious how to write an invariant term using the Σ.

$$\mathcal{A}\left(\pi^0 \to \gamma\gamma\right)_{naive} \left(\frac{m_\pi}{4\pi f_\pi}\right)^2 \simeq A\left(\pi^0 \to \gamma\gamma\right)_{allowed} \tag{9.204}$$

$\hookrightarrow 10^{-4}\Gamma\left(\pi^0 \to \gamma\gamma\right)_{naive}$ in conflict with the observation.

9.13.2 Partial resolution of the Sutherland-Veltman-paradox

$\pi^0 \sim e^{i\gamma_5\tau_3}$ of $SU(2)_A$ which is spontaneously broken.
In the quark Lagrangian:

$$\overline{\psi}(i\not{D})\psi \qquad \text{with } \psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{9.205}$$

$$i\not{D} = i\partial_\mu + g_S \not{A}_{gluon} + Q_e \not{A}_{photon} \tag{9.206}$$

with $Q = \begin{pmatrix} \frac{2}{3} & \\ & -\frac{1}{3} \end{pmatrix}$.
In the Fujikawa method we can write:

$$\mathrm{Tr}\left(\gamma_5 \mathrm{e}^{(\mathrm{i}\not{D})^2/\Lambda^2}\right) \to \mathrm{Tr}\left(\gamma_5 \tau_3 \mathrm{e}^{(\mathrm{i}\not{D})^2/\Lambda^2}\right) \tag{9.207}$$

where the color and flavor are included.
Now expand $e^{(i\not{D})}$ to the second order.
We get an extra factor

$$\underset{\substack{\uparrow \\ \text{flavor}}}{\mathrm{Tr}\,(\tau^3 \mathrm{Q}^2)}\mathrm{Tr}(\mathbb{1}) = \frac{\mathrm{N_c}}{6} \tag{9.208}$$

and so

$$\delta_\Theta \mathcal{L}_{QCD} = \Theta \cdot \frac{N_C}{6} \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \tag{9.209}$$

The effective change of QCD Lagrangian in presence of an E-M-field must be reproduced by the Goldstone effective Lagrangian.

$$\int \mathcal{D}\,\Sigma(\pi) \quad \text{is invariant.} \tag{9.210}$$

$\Sigma(\pi)$ is bosonic, no Fujikawa term is generated.
Hence $\mathcal{L}[\pi]$ must transform like Eq. (9.209).
Recall:

$$\Sigma(\boldsymbol{\pi}) \xrightarrow{\substack{\omega_L=\Theta_A \\ \omega_R=-\Theta_A}} e^{i\Theta T^3} \Sigma(\boldsymbol{\pi}) e^{-i\Theta T^3} \tag{9.211}$$

with $\Sigma = e^{i\frac{\sqrt{2}}{f_\pi}\begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix}}$.
This implies infinitesimally:

$$\pi^0 \to \pi^0 - \frac{\Theta}{2} \overset{\substack{\text{Be careful with definition!} \\ \downarrow}}{f} \tag{9.212}$$

under τ^3 $SU(2)_A$ transformation.
Hence we obtain Eq. (9.209) for $\delta_\Theta \mathcal{L}^{(2)}[\pi]$ if we add

$$\Delta\mathcal{L}^{(2)} = -2\frac{N_C}{6} \frac{e^2}{16\pi^2 f} \pi^0(x) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \tag{9.213}$$

This is exactly the term needed in the beginning with

$$g = -\frac{N_C}{6} \frac{\alpha_{em}}{4\pi f} 2 \tag{9.214}$$

So we get for the decay rate:

$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = \left(\frac{N_C}{3}\right)^2 \frac{\alpha_{em}^2}{64\pi^3} \frac{m_\pi^3}{f_\pi^2} \cdot 4 \tag{9.215}$$

This is in a very good agreement with the experiment ($\Gamma^{Exp} \simeq 7.74 \pm 0.37\mathrm{eV}$) for $N_C = 3$.
Only place where N_C appears in the chiral Lagrangian: “IR probe of UV physics”!
But:

- What is the general form of $\Delta\mathcal{L}$?
- Do anomalies affect ungauged processes?

9.14 Anomalies and the chiral Lagrangian*

We now attempt a general description of anomalous in $\mathcal{L}[\Sigma(\boldsymbol{\pi})]$.
So far we have used $SU(3)_L \otimes SU(3)_R$ as a guide to infer the chiral effective Lagrangian, but we did not discuss **discrete** symmetries.
$\mathcal{L}_{QCD}$ is invariant under:

1. C (charge conjugation)
2. P (parity)

How does this appear in the chiral Lagrangian?

1. Charge conjugation:

$$\begin{aligned}&\pi^0 \to \pi^0\\&\eta \to \eta\\&\pi^\pm \to \pi^\mp\\&K^\pm \to K^\mp\\&K^0 \to K^0\end{aligned}$$

In terms of Σ equivalent to $\Sigma \to \Sigma^T$

$$\boldsymbol{\pi} = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\pi_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi_3 + \frac{1}{\sqrt{6}}\pi_8 & K^0 \\ K^- & \bar{K}^0 & -\sqrt{\frac{2}{3}}\pi_8 \end{pmatrix}$$

2. Parity: GB are bound states of quarks and antiquarks.

$$\overline{\psi}\gamma_5 T^a \psi \xrightarrow{P} -\overline{\psi}\gamma_5 T^a \psi \tag{9.216}$$

Thus under parity we have $\pi^a \to -\pi^a(-\mathbf{x}, t)$

$$\begin{cases}\mathbf{x} \xrightarrow{P} -\mathbf{x}\\ t \to t\\ \Sigma \xrightarrow{P} \Sigma^{-1}\end{cases}$$

We see that the chiral Lagrangian is invariant under C and P.

$$\mathcal{L}^{(2)} = \frac{f_\pi^2}{16}\mathrm{Tr}(\partial_\mu \Sigma \partial_\mu \Sigma^{-1}) \tag{9.221}$$

which is invariant under $\Sigma \to \Sigma^T$ or $\Sigma \to \Sigma^{-1}$ due to the cyclic property of the trace and $\mathrm{Tr}(\mathrm{A}^\mathrm{T}) = \mathrm{Tr}(\mathrm{A})$.

Problem: $\mathcal{L}^{(2)}$ is also invariant under simpler symmetry that is **not** present in QCD.
The simpler transformation is:

$$(-1)^{N_B} : \Sigma \to \Sigma^{-1} \quad (\text{without } \mathbf{x} \to -\mathbf{x}) \tag{9.222}$$

or via

$$j_A^{\mu,a} = \overline{\psi}\gamma^\mu\gamma^5 T^a \psi = f_\pi \partial_\mu \pi^a + \dots \tag{9.217}$$

Applying the parity operator we get

$$j_A^{\mu,a} \xrightarrow{P} -(-1)^\mu j_A^{\mu,a} \tag{9.218}$$

with $\partial_\mu \xrightarrow{P} (-1)^\mu \partial_\mu$ we get

$$\pi(\mathbf{x}) \xrightarrow{P} -\pi(-\mathbf{x}, t) \tag{9.219}$$

$$(-1)^\mu = \begin{cases}\mu = 0 : 1\\ \mu = i : -1\end{cases} \tag{9.220}$$

leaves $\mathcal{L}^{(2)}$ invariant. N_B is the number of bosons.
$\hookrightarrow$It forbids the transition of even number of boson to an odd number.
Example: $K^+K^- \to p^+\pi^-\pi^0$ is forbidden by $(-1)^{N_B}$, but **allowed** by QCD[9]
We conclude that the chiral EFT is not complete.
We **need** a term that violates $(-1)^{N_B}$, but preserves P.
The parity transformation is

[9] It can be measured, e.g. ϕ meson decays to both: K^+K^- and $\pi^+\pi^-\pi^0$.

$$P = \underset{\substack{\uparrow \\ \mathbf{x}\to-\mathbf{x}}}{P_0}\overset{\substack{\Sigma\to\Sigma^{-1} \\ \downarrow}}{(-1)^{N_B}} \tag{9.223}$$

So the term needs to break P_0 or $(-1)^{N_B}$ individually, whereas both conserve it as we will see.

The Euler-Lagrange equation is:

$$\frac{f_\pi^2}{2}\left(\partial^\mu\partial_\mu\Sigma + \Sigma(\partial_\mu\Sigma^{-1})\partial_\mu\Sigma\right) = 0. \tag{9.224}$$

$$\frac{f_\pi^2}{8}\Sigma\cdot\partial^\mu(\Sigma^{-1}\partial_\mu\Sigma) = 0 \tag{9.225}$$

$$\text{or } \frac{f_\pi^2}{8}\partial_\mu X^\mu = 0 \tag{9.226}$$

with $X^\mu \equiv \Sigma^{-1}\partial_\mu\Sigma$ which transforms as $X^\mu \to U_R X_\mu U^{-1}$.

Proof of Eq. (9.224):

$$\frac{16}{f_\pi^2}\mathcal{L}^{(2)} = \mathrm{Tr}(\partial_\mu\Sigma^{-1}\partial_\mu\Sigma) \tag{9.227}$$

Euler-Lagrange: $\delta S = 0$ (We will ignore surface terms in the following):

$$\delta\mathcal{L}^{(2)} = \mathrm{Tr}\left(\partial_\mu\delta\Sigma^{-1}\partial_\mu\Sigma + \partial_\mu\Sigma^{-1}\partial_\mu\delta\Sigma\right) \tag{9.228}$$
$$= -\mathrm{Tr}\left(\delta\Sigma^{-1}\partial_\mu\partial_\mu\Sigma + \delta\Sigma\partial_\mu\partial_\mu\Sigma^{-1}\right) \tag{9.229}$$
$$= -\mathrm{Tr}\left(\delta\Sigma^{-1}\left(\partial_\mu\partial_\mu\Sigma - \Sigma(\partial_\mu\partial_\mu\Sigma^{-1})\Sigma\right)\right) = 0 \tag{9.230}$$

where we used $\delta(\Sigma^{-1}\Sigma) = 0 \rightsquigarrow \delta\Sigma = -\Sigma\delta\Sigma^{-1}\Sigma$.
EOM:

$$\underbrace{\partial_\mu\partial_\mu\Sigma}_{①} - \underbrace{\Sigma(\partial_\mu\partial_\mu\Sigma^{-1})\Sigma}_{②} = 0 \tag{9.231}$$

Now we massage the second term in Eq. (9.231):

$$② = \Sigma(\partial_\mu(\partial_\mu\Sigma^{-1}))\Sigma \tag{9.232}$$
$$= -\Sigma(\partial_\mu(\Sigma^{-1}\cdot\partial_\mu\Sigma\cdot\Sigma^{-1}))\Sigma \tag{9.233}$$
$$= -\Sigma(\partial_\mu\Sigma^{-1})(\partial_\mu\Sigma) - \partial_\mu\partial_\mu\Sigma - \underbrace{(\partial_\mu\Sigma)(\partial_\mu\Sigma^{-1})\cdot\Sigma}_{③} \tag{9.234}$$

Now we have to look at the term ③:

$$(\partial_\mu\Sigma)(\partial_\mu\Sigma^{-1})\cdot\Sigma = \Sigma(\partial_\mu\Sigma^{-1})\Sigma\Sigma^{-1}(\partial_\mu\Sigma)\Sigma^{-1}\cdot\Sigma \tag{9.235}$$
$$= \Sigma(\partial_\mu\Sigma^{-1})(\partial_\mu\Sigma) \tag{9.236}$$

Plugging back in Eq. (9.231)(③ in ② and ② in Eq. (9.231)):

$$2\partial_\mu\partial_\mu\Sigma + 2\Sigma(\partial_\mu\Sigma^{-1})(\partial_\mu\Sigma) \quad \checkmark \tag{9.237}$$

We get:

$$\boxed{\frac{f_\pi^2}{8}\partial_\mu X^\mu = 0} \tag{9.238}$$

with $X_\mu = \Sigma^{-1}\partial_\mu\Sigma$ and $X^\mu \xrightarrow{G} U_R X_\mu U_R^{-1}$ ($G = SU(3)_L \otimes SU(3)_R$).

It transforms linearly since here also $\omega_L = \omega_R - \omega_V$, since ω_L does not appear.

A candidate for $(-1)^{N_B}$ or P_0 breaking:

$$e^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma = F \tag{9.239}$$
$$= X_0 \underbrace{X_1 X_2 X_3}_{\text{3 space derivatives}} + (\text{permutations}) \tag{9.240}$$

For the space derivatives it follows under $P_0(\vec{x} \to -\vec{x})$:

$$F \to -F. \tag{9.241}$$

Under $(-1)^{N_B}$:

$$X_\mu = \Sigma^{-1}\partial_\mu\Sigma \xrightarrow{(-1)^{N_B}} \Sigma\partial_\mu\Sigma^{-1} = -\partial_\mu\Sigma\cdot\Sigma^{-1} = -\Sigma X_\mu\Sigma^{-1} \tag{9.242}$$

RHS:

$$\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma \overset{(-1)^{N_B}}{\to} \Sigma\left(\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma\right)\Sigma^{-1} \tag{9.243}$$

LHS:

$$\partial_\mu X_\mu \overset{(-1)^{N_B}}{\to} -\partial_\mu\left(\Sigma X_\mu\Sigma^{-1}\right) \tag{9.244}$$
$$= -\partial_\mu\Sigma X_\mu\Sigma^{-1} - \Sigma\partial_\mu X_\mu\Sigma^{-1} - \Sigma X_\mu\partial_\mu\Sigma^{-1}. \tag{9.245}$$

We now show that the sum of the first and third term of Eq. (9.245) vanishes.

$$-\partial_\mu\Sigma\cdot X_\mu\cdot\Sigma^{-1} - \Sigma X_\mu\partial_\mu\Sigma^{-1} \tag{9.246}$$
$$= -\partial_\mu\Sigma\cdot\underbrace{\Sigma^{-1}\partial_\mu\Sigma\cdot\Sigma^{-1}}_{\partial_\mu\Sigma^{-1}} - \underbrace{\Sigma\cdot\Sigma^{-1}}_{=1}\partial_\mu\Sigma\cdot\partial_\mu\Sigma^{-1} \tag{9.247}$$
$$= \partial_\mu\Sigma\partial_\mu\Sigma^{-1} - \partial_\mu\Sigma\partial_\mu\Sigma^{-1} = 0 \quad \checkmark \tag{9.248}$$

EOM:

$$\frac{f_\pi^2}{8}\partial_\mu X^\mu - \lambda\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma = 0 \tag{9.249}$$

applying $(-1)^{N_B}$ we get

$$\Sigma\left(-\frac{f_\pi^2}{8}\partial_\mu X^\mu - \lambda\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma\right)\Sigma^{-1} = 0. \tag{9.250}$$

− is the wrong sign. It is not invariant except if we also transform by P_0. Now using P_0 we get

$$-\frac{f_\pi^2}{8}\partial_\mu X_\mu + \lambda\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma \tag{9.251}$$

$\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma$ changes the sign because this contains 3 spatial coordinates:

$$\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma \tag{9.252}$$
$$= X_0X_1X_2X_3 + (\text{permut.}) \tag{9.253}$$
$$\xrightarrow{P_0} X_0(-X_1)(-X_2)(-X_3) + (\text{permut.}) \tag{9.254}$$
$$= -X_0X_1X_2X_3 - (\text{permut.}) \tag{9.255}$$

It follows the invariance under combination:

$$P = P_0(-1)^{N_B}. \tag{9.256}$$

How do we write this in a Lagrangian?

$$\frac{f_\pi^2}{8}\partial_\mu X_\mu = \lambda\epsilon^{\mu\nu\rho\sigma}X_\mu X_\nu X_\rho X_\sigma. \tag{9.257}$$

It turns out that this is not easy. It needs to be $SU(3)_L \otimes SU(3)_R$ invariant.
An obvious candidate is:

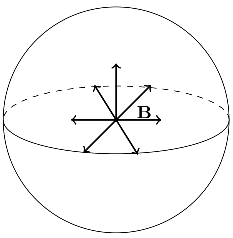
$$\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,(\mathrm{X}_\mu\mathrm{X}_\nu\mathrm{X}_\rho\mathrm{X}_\sigma) \tag{9.258}$$

which vanishes identically because of the cyclicity of the trace.

$$A = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,(\mathrm{X}_\mu\mathrm{X}_\nu\mathrm{X}_\rho\mathrm{X}_\sigma) = \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\mathrm{X}_\sigma\mathrm{X}_\mu\mathrm{X}_\nu\mathrm{X}_\rho) \tag{9.259}$$
$$= -\epsilon^{\sigma\mu\nu\rho}\mathrm{Tr}\,(\mathrm{X}_\sigma\mathrm{X}_\mu\mathrm{X}_\nu\mathrm{X}_\rho) \tag{9.260}$$
$$= -\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\,(\mathrm{X}_\mu\mathrm{X}_\nu\mathrm{X}_\rho\mathrm{X}_\sigma) \tag{9.261}$$
$$= 0. \tag{9.262}$$

9.14.1 Analogy in simpler system: magnetic monopole

Condier: charged point particle (classical) confined to sphere with a magnetic monopole at its center.



(9.263)

with $\mathbf{x}^2 = R^2$.
Point particle on a sphere

$$\mathcal{L} = \frac{m}{2}\dot{x}_i^2(t) + \underset{\substack{\uparrow\\ \text{Lagrange multiplier}}}{\lambda}(x_i(t)x_i(t) - R^2) \tag{9.264}$$

EOM:

$$m\ddot{x}_i^2 - 2\lambda x_i = 0. \tag{9.265}$$

We eliminate λ by $\sum_i x_i$ in Eq. (9.265):

$$0 = 2\lambda x_i x_i - m x_i \ddot{x}_i \tag{9.266}$$
$$0 = 2\lambda R^2 - m\frac{d}{dt}\left[x_i\dot{x}_i\right] + m\dot{x}_i\dot{x}_i \tag{9.267}$$
$$0 = 2\lambda R^2 + m\dot{x}_i\dot{x}_i. \tag{9.268}$$

where the bracket is zero since the tangent $\dot{x}_i$ on a sphere is always $\perp$ to the radius x_i.

$R^2 = x_i x_i \overset{\frac{d}{dt}}{\to} 0 = 2\dot{x}_i x_i.$

We can solve for λ in Eq. (9.268) and plug it in Eq. (9.265):

$$m\ddot{x}_i + m x_i \frac{\dot{x}_j\dot{x}_j}{R^2} = 0. \tag{9.269}$$

It is invariant under two separate symmetries.

$$\frac{d}{dt} \to -\frac{d}{dt}$$
$$x_i \to -x_i$$

$$t \to -t \tag{9.270}$$
$$\text{and } \mathbf{x} \to -\mathbf{x} \tag{9.271}$$

1:

$$x_i \to x_i, \dot{x}_i \to -\dot{x}_i$$
$$\ddot{x}_i \to \ddot{x}_i$$

2:

$$x_i \to -x_i, \dot{x}_i \to -\dot{x}_i$$
$$\ddot{x}_i \to -\ddot{x}_i.$$

Suppose we want independence only under **combined**:

$$\begin{cases} t \to -t \\ \mathbf{x} \to -\mathbf{x} \end{cases}$$

We can add a new force:

	$m\dot{x}_i$	$+$	$\frac{m}{R^2}\dot{x}_l\dot{x}_l\dot{x}_i$	$=$	$\alpha\epsilon_{ijk}x_j\dot{x}_k$	
$\vec{x} \to -\vec{x}$	$-$		$---$	$\Rightarrow -$	$--$	$\Rightarrow +$
$t \to -t$	$+$		$--+$	$\Rightarrow +$	$+-$	$\Rightarrow -$
combined	$-$		$++-$	$\Rightarrow -$	$-+$	$\Rightarrow -$

This cannot be derived form the Lagrangian (in analogy with chiral EFT), since the obvious candidate

$$\mathcal{L} \supset \epsilon_{ijk}x_i x_j \dot{x}_k \tag{9.272}$$

vanishes identically.

Solution:

We can get this kind of "force" as a Lorentz-Force due to the magnetic monopole:

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B} \tag{9.273}$$

in components:

$$F_i = e\epsilon_{ijk}\dot{x}_j B_k \tag{9.274}$$

with

$$B_j = g\frac{1}{(x_i x_i)} \cdot \frac{x_i}{\sqrt{x_i x_i}} \tag{9.275}$$
$$= g\frac{x_i}{R^3} \tag{9.276}$$

It is the same as the electric point charge $\mathbf{E} = \frac{1}{\mathbf{r}^2}\tilde{\mathbf{r}}e$.
So we put the new force to the RHS of the EOM:

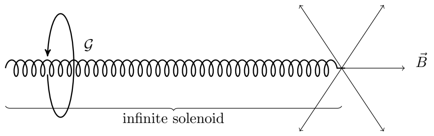
$$m\ddot{x}_i + \frac{m}{R^2}\dot{x}_l\dot{x}_l \cdot x_i = \alpha\epsilon_{ijk}x_j\dot{x}_k \tag{9.277}$$

which can be interpreted as a magnetic monopole with the charge

$$g = \alpha\frac{R^3}{e} \tag{9.278}$$

located at the center of the sphere of radius R.

The interaction of the particle however is determined by the vector potential $\mathbf{A}$. It is well-known that for magnetic monopole $\mathbf{A}$ is not everywhere well-defined. We can have magnetic monopole even without magnetic charges.



(9.279)

From electrodynamics we know the solution:

$$\mathbf{A} = g\frac{1-\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}} \tag{9.280}$$

We can choose the Dirac string anywhere and e.g. use two different coordinate systems and patch them together.
Consider:

Singular at negative $\theta = \pi$ ($x = y = 0, z < 0$) and not singular at $\theta = 0$. This is called the **Dirac string**.
This is not a "real" monopole because quantum mechanics allows us to see the solenoid via Ahoronov-Bohm, only if the flux inside the string is zero, then the Dirac quantization condition.

$$L = \frac{1}{2}m\dot{x}_i^2 + e\dot{x}_iA_i - \lambda(x_ix_i - R^2) \tag{9.281}$$

where A_i is defined using two different coordinate systems. It is no longer gauge-invariant due to the Dirac singularity of the string.

Quantum amplitudes from the sum over paths weighted by

$$e^{iS[x,\dot{x}]}. \tag{9.282}$$

Consider a closed loop C and a part of the path-integral from Eq. (9.281) and $C : x_i[t_0] = x_i[t_1]$:

$$\int \mathcal{D}e^{iS} \supset \exp\left(ie\int_{t_0}^{t_1} dt\, A_i\dot{x}_i(t)\right) \tag{9.283}$$

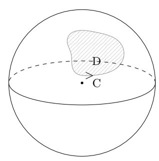
$$= \exp\left(ie\oint_C dx_i\, A_i(x)\right) \tag{9.284}$$

over a close path on the sphere. Using Stokes

$$\exp\left(ie\oint_C dx_i\, A_i(x)\right) = \exp\left(ie\int_D F_{ij}d\sigma^{ij}\right) \tag{9.285}$$

where D is a disk bounded by the orbit C. There is an ambiguity, meaning that we have two types of disks.

D: disk bounded by the orbit C



$$\exp\left(ie\oint_C dx_i\, A_i(x)\right) = \exp\left(ie\int_D F_{ij}d\sigma^{ij}\right) \tag{9.286}$$

$$= \exp\left(-ie\int_{\bar{D}} F_{ij}d\sigma^{ij}\right) \tag{9.287}$$

Figure 9.1: D: inside C and $\bar{D}$: outside C.

The minus-sign can be explained because of the charge which is in the orientation of G (bounds it right-hand sense: $G \to D$ vs. bounds it left-hand sense: $G \to \bar{D}$).
Consistency between the choices requires them to be equal! In other words:

$$\exp\left(ie \oint_{D+\bar{D}} dx_i\, A_i(x)\right) = 1. \tag{9.288}$$

But $D + \bar{D}$ is nothing but the whole sphere surrounding the magnetic monopole. Using Gauss's law:

F_{ij} is well-defined everywhere and gauge-invariant. We should have started with this term in action. **But** the ambiguity D vs. $\bar{D}$ forces quantization. Then everything is ok.

$$\oint A_i dx_i = \int_{D+\bar{D}=S^2} d\boldsymbol{\sigma}(\mathbf{D} \times \mathbf{A}) = \int_{S^2} d\boldsymbol{\sigma} \cdot \mathbf{B} = 4\pi \cdot g. \tag{9.289}$$

were we use $\mathbf{B} = g \frac{\mathbf{r}}{R} \frac{1}{R^2}$.
We see that

$$ie \int_{D+\bar{D}} F_{ij} d\sigma^{ij} = 4\pi i e g = ix. \tag{9.290}$$

For this to give $e^{ix} = 1 \rightsquigarrow x = 2\pi n$.

$$\boxed{e \cdot g = \frac{n}{2}} \tag{9.291}$$

with $n = \pm 1, \pm 3, \ldots$ and e as the electric charge and g as the magnetic charge.
This is the famous **Dirac quantization condition**. It implies that there is a single magnetic monopole. The electric charge is quantized!

What do we learn from this excursion?

1. Derive an extra term in the chiral action as 5D integral of a total 5 dimensional divergence.[10]

2. Expect coefficient in front of the extra term to be **quantized** (analogous to Dirac quantization condition)

[10] In the monopole example:instead

$$\int_{\partial_\mu D} dt A_i \dot{x}_i(x)$$

$$\text{use} \int_D d\sigma^{ij} F_{ij}.$$

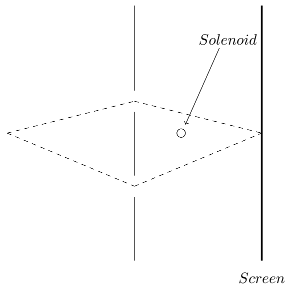
Alternative view of Dirac quantization condition.

E.g.

$$\frac{1}{2m} \left(\boldsymbol{\nabla} + ie\mathbf{A}(\mathbf{r})\right)^2 \psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{9.292}$$

Solve with

$$\psi(\mathbf{r}) = \exp\left(-ie \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{A}(\mathbf{r}')d\mathbf{r}'\right) \underset{\substack{\uparrow \\ \text{solution of the Schrödinger equation} \\ \text{with } \mathbf{A}\equiv 0.}}{\psi_{A=0}(\mathbf{r})} \tag{9.293}$$



$$ \tag{9.294} $$

$$\Delta\phi = e \oint_{\mathcal{G}} \mathbf{A} d\mathbf{r} = e \int \boldsymbol{\nabla} \times \mathbf{B} d\mathbf{s} = e\Phi. \tag{9.295}$$

It is a real monopole only if $\Delta\phi$ around the circle outside of the monopole vanishes (or is $2\pi n$).
This phase has important consequences:

$$e\Phi = 2\pi n. \tag{9.296}$$

Spacetime $x^\mu = 4D$ sphere M.
The coset field $\Sigma(\boldsymbol{\pi}(x))$ is a mapping:

$$\Sigma : M \to SU(3) \quad \text{(group manifold)} \tag{9.297}$$

No subleties of the topological nature $\pi_4(SU(3)) = 0$ or $SU(3)$ is 8-dimensional. It can shrink any mapping to a point.

We are looking for a term:

$$\Gamma = \int_Q d\sigma^{ijklm} \Omega_{ijklm} \tag{9.298}$$

where Q is a 5D-disk bounding M and $d\sigma^{ijklm}$ as a surface element. Ω_{ijklm} is a fifth rank tensor, invariant under $SU(3)_L \otimes SU(3)_R$. One can show it since Ω_{ijklm} is in the Lie-algebra:

$$\Omega_{ijklm} \sim (8 \otimes 8 \otimes 8 \otimes 8 \otimes 8)_{A(ntisymmetric)} = 1 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 \tag{9.299}$$

We do not expect Q (bounding disk) to be unique (as the curve C was not for a monopole). The sum of the two disks Q (inside) and $\bar{Q}$ (outside) is a closed 5D-sphere S_{5D}. There is no ambiguity in the

Figure 9.2: Q: inside of S_{5D} Q: outside of S_{5D}

path integral if

$$\int_{S_{5D}} d\sigma^{ijklm} \Omega_{ijklm} = 2\pi N \tag{9.300}$$

with $N = \pm 1, \pm 2, \pm 3, \ldots$
The mapping of $S_{5D} \to SU(3)$ is not topologically trivial, it has a winding number corresponding to N ($\Pi_5(SU(3)) = \mathbb{Z}$)

This means not all spheres are equal but we can find a "basic sphere" where the flux is normalized:

$$\Gamma_0 = \int_{S_0} d\sigma^{ijklm} \Omega_{ijklm} = 2\pi. \tag{9.301}$$

The other 5D-spheres are then just an integer multiplier of the result of using S_0 because of Eq. (9.300):

$$\int_S d\sigma^{ijklm} \Omega_{ijklm} = N \int_{S_0} d\sigma^{ijklm} \Omega_{ijklm} \tag{9.302}$$

The chiral action is then including an extra term:

$$\mathcal{L} = \mathcal{L}^{(2)} + N\Gamma_0. \tag{9.303}$$

We will skip the mathematical problem of finding an invariant tensor which satisfies the normalization condition Eq. (9.301).[11]

[11] See Witten, *Nucl. Phys.* B 223 (1983) 422 and Bott/Seeley, *Comm. Math. Phys.* 62 (1978) 235.

$$\Omega_{ijklm} = -\frac{i}{240\pi^2} \mathrm{Tr}\left(\mathrm{X_{[i} X_j X_k X_l X_{m]}}\right) \tag{9.304}$$

with the brackets indicating the total antisymmetrization,
and

$$X_i = \Sigma^{-1} \frac{\partial}{\partial y^i} \Sigma \tag{9.305}$$

are the I-coordinates of the disk.
Expanding in $\frac{\boldsymbol{\pi}}{f}$:

$$X_i = 1 + \frac{2i}{f_\pi} \lambda^a \partial_i \pi^a + \dots \tag{9.306}$$

we get:

$$\Omega_{ijklm} = \frac{2}{14\pi^2 f_\pi^4} \partial_{[i} \left(\mathrm{Tr}\left(\boldsymbol{\pi} \partial_\mathrm{j} \boldsymbol{\pi} \partial_\mathrm{k} \boldsymbol{\pi} \partial_\mathrm{l} \boldsymbol{\pi} \partial_{\mathrm{m}]} \boldsymbol{\pi}\right)\right) + \dots \tag{9.307}$$

Since this is a total divergence its contribution leads to a $4D$ term in the action:

$$\Delta\Sigma = \Sigma_{WZ} = \frac{2N}{15\pi^2 f_\pi^5} \int d^4x\, e^{\mu\nu\rho\sigma} \mathrm{Tr}\left(\boldsymbol{\pi} \partial_\mu \boldsymbol{\pi} \partial_\nu \boldsymbol{\pi} \partial_\rho \boldsymbol{\pi} \partial_\sigma \boldsymbol{\pi}\right) + \dots \tag{9.308}$$

This clearly includes a vertex that can produce

$$K^+ K^- \to \pi^+ \pi^- \pi^0. \tag{9.309}$$

Everything is fixed up to mysterious integer N. This is the **Wess-Zumino** term (1971).
Unfortunately the gauging of the WZ-term is tricky. The integral is not invariant under the local chiral group.
Use trickle down-technique[12]

[12] See e.g. supergravity.

1. Gauge naively, check what is left-over.
2. Invent a new term to fix leak
3. Go to 2. and add more terms (fixing leaks of leak-fixing terms)

WZ is invariant under global charge transformation, but local

$$\delta\Sigma = i\alpha(x)\,[Q,\Sigma] \tag{9.310}$$

does not leave it invariant. We get

$$\delta S_{WZW(itten)} = -N\int d^4x\, J_\mu \partial_\mu \alpha(x) \tag{9.311}$$

with

$$J_\mu = \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\Sigma^{-1}\mathrm{Q}\partial_\nu\Sigma\mathrm{X}_\rho\mathrm{X}_\sigma + \mathrm{Q}\mathrm{X}_\nu\mathrm{X}_\rho\mathrm{X}_\sigma\right) \tag{9.312}$$

$$= \frac{1}{32\pi^2 f_\pi^3}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}\left(\mathrm{Q}\partial_\nu\boldsymbol{\pi}\partial_\rho\boldsymbol{\pi}\partial_\sigma\boldsymbol{\pi}\right) + \ldots \tag{9.313}$$

This is $\frac{1}{f_\pi^3}$ which we could not have found varying 4D integrand. We only see it varying the 4D integral directly.
Under gauge transformation: $\delta A_\mu = -\frac{1}{e}\partial_\mu\alpha(x)$.
Consider the improved action:

$$S'_{WZW} = S_{WZW} + eN\int d^4x\, A_\mu J_\mu \tag{9.314}$$

Now the current itself is not gauge-invariant and we need to fix its leakage.
Its variation is:

$$\delta J^\mu = i\epsilon^{\mu\nu\rho\sigma}\partial_\rho k_\sigma \partial_\nu\alpha(x) \tag{9.315}$$

with

$$k_\sigma = \frac{1}{24\pi^2}\mathrm{Tr}\left(\Sigma^{-1}\mathrm{Q}^2\partial_\sigma\Sigma + \mathrm{Q}^2\mathrm{X}_\sigma\right) \tag{9.316}$$

$$= \frac{i}{12 f_\pi\pi^2}(\partial_\sigma\pi^3 + \frac{1}{\sqrt{3}}\partial_\sigma\pi_\sigma) \tag{9.317}$$

which we can write after IBP:

$$\delta S'_{WZW} = -\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\mu k_\sigma \partial_\nu\alpha(x). \tag{9.318}$$

So in order to plug this leak we need to add another A_μ term to finally get:

$$\boxed{S_{\mathrm{WZW}} + Ne\int d^4x\, A_\mu J_\mu - iNe^2\epsilon^{\mu\nu\rho\sigma}\int d^4x\, \partial_\rho A_\mu A_\nu k_\sigma.} \tag{9.319}$$

which is gauge-invariant.

Remarks:

1. The term $Ne\int d^4x\, A_\mu J_\mu$ contains $J_\mu \sim \frac{1}{f_\pi^3}\mathrm{Tr}\left(\mathrm{Q}\partial_\nu\boldsymbol{\pi}\partial_\rho\boldsymbol{\pi}\partial_\sigma\boldsymbol{\pi}\right)$

$$eA_\mu J_\mu \Rightarrow \pi^+\pi^- \to \gamma\pi^0 \tag{9.320}$$

   is strongly suppressed and only of the order $1/f_\pi^3$.

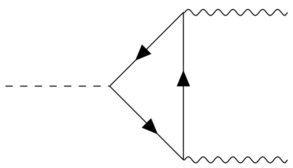
2. The term $iNe^2\epsilon^{\mu\nu\rho\sigma}\int d^4x\,\partial_\rho A_\mu A_\nu k_\sigma$ is more interesting and contains

$$\partial_\rho A_\mu A_\nu k_\sigma \tag{9.321}$$

with $k_\sigma \sim \frac{\partial_\rho \pi_3}{f_\pi} + \frac{1}{\sqrt{3}}\frac{\partial_\sigma \pi_8}{f_\pi}$ leading to vertex:

$$\frac{e^2}{48\pi^2}\frac{N}{f_\pi}\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{9.322}$$

π^0 is associated with $\bar{u}\gamma_5 u - \bar{d}\gamma_5 d$



$\sim N_C$ as seen in chapter $\pi \to \gamma\gamma$.

(9.323)

This fixes $N = N_C$. It explains again why N is quantized. However, at chiral level N appears only as (integer) parameter to be fixed by a measurement.

10
Scattering amplitudes

10.1 Motivation

You have never been asked to calculate an amplitude that contains more than four or five particles, even at tree level. The number of diagrams grows exponentially ($\sim$ factorial): e.g.[1]

$gg \to gg$	4 diagrams
$gg \to ggg$	25 diagrams
$gg \to gggg$	220 diagrams

[1] And there are $\sim$ 10000 terms in the matrix element for $gg \to ggg$...

Even though this would be a character building exercise, there are massive simplifications possible in certain cases, e.g. the so called maximally helicity violating amplitudes, built for simple color structures multiplying expressions like:

$$\mathcal{A}(1^+ \dots i^- \dots j^- \dots k^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle k1 \rangle}. \tag{10.1}$$

with $\sigma \propto |\mathcal{A}|^2$. At the end of this chapter you will understand the formalism to derive and to express the amplitude this way. We seem to go from a **simple** action

$$S = \int d^4x \left(-\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} \right) \tag{10.2}$$

to incredibly complicated, mostly **redundant** Feynman diagrams to **simple** amplitudes.

Can we skip the intermediate step? Why this complication?

One reason is certainly that we have to rely on gauge redundancy:

$$A_\mu \to A_\mu + \partial_\mu \alpha \qquad \text{(gauge)} \tag{10.3}$$
$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \theta_\nu + \partial_\nu \theta_\mu \qquad \text{(diffeomorphism)}. \tag{10.4}$$

Because we want a local, Lorentz-invariant action which manifests symmetries.

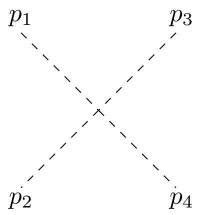
This redundancy is actually **endemic to the action principle**. Consider this example with a scalar field

$$\mathcal{L}_k = k(\phi) \partial_\mu \phi \partial_\mu \phi \tag{10.5}$$

with a complicated interaction

$$k(\phi) = 1 + \lambda_1\phi + \frac{\lambda_2}{2!}\phi^2 + \frac{\lambda_3}{3!}\phi^3 + \dots \tag{10.6}$$

Naively, we expect a complicated S-matrix. Really? Let us evaluate the $\phi\phi \to \phi\phi$ amplitude. We know that the derivative terms lead to contractions of external momenta



$$= \mathcal{A} \propto (\dots) = \sum_{i\neq j} p_i p_j \propto s + t + u = 0 \tag{10.7}$$

with

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{10.8}$$

and

$$t = (p_1 + p_4)^2 = (p_2 + p_3) \tag{10.9}$$

and

$$u = (p_1 + p_3)^2 = (p_2 + p_4) \tag{10.10}$$

It is crucial for this result, that the amplitude is evaluated on-shell respecting momentum conservation

$$\sum p_i = 0 \tag{10.11}$$

$$p_i^2 = 0 \tag{10.12}$$

$\Rightarrow$ **kinematics**!

We can show that the e.g. 17 particle amplitudes also vanish. $\rightarrow$ Why?
$\mathcal{L}_k$ is secretly a theory of a free scalar. The S-matrix is invariant under field redefinitions:

$$\phi \to f(\phi) \tag{10.13}$$

with

$$\underset{\substack{\uparrow \\ \text{in weak field limit interpolate} \\ \text{into canonically normalized field}}}{f'(0) = 1.} \tag{10.14}$$

The transformation is explicitly[2]

$$\phi = \hat{\phi}\left(1 - \frac{\lambda_2}{12}\hat{\phi}^2\right) + \mathcal{O}(\lambda_2^2) \quad \text{yields freely.} \tag{10.15}$$

The S-matrix is invariant under the **non-symmetry** of the action. Hindsight: it is not surprising ϕ is an **integration variable**.[3] We

[2] Where we have set $\lambda_1 = 0, \lambda_3 = 0$ for simplicity

[3] e.g.

$$Z[J] \equiv \frac{\int \mathcal{D}\phi e^{iS[\phi]+i\int d^4x\, J(x)\phi(x)}}{\int \mathcal{D}\phi\, e^{iS[\phi]}}$$

conclude

$$\text{Different actions} \quad \leftrightarrow \quad \text{Same S-matrix, same physics}$$

What defines a theory? (Not the action). We will find that on-shell amplitudes will allow us to specify the physical content of a theory and furthermore, they are restricted by certain principles which will only allow a small set of possible terms.

10.1.1 Zoology of QFTs

Scalars are very complicated:

$$\mathcal{L} = \partial_\mu \phi^a \partial_\mu \phi^a + \frac{1}{3!}\omega_{abc}\phi_a\phi_b\phi_c + \frac{1}{4!}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d \tag{10.16}$$

We need to specify ω_{abc} and λ_{abcd}. The theory might be simple with respect to Feynman diagrams we need to calculate, but it is not predictive in the slightest.

If, on the other hand, the fields are vectors, then there is almost no freedom and the only theory permitted is Yang-Mills theory.

One can show that an interacting renormalizable theory of massless vectors has to have the form of Yang-Mills. The fact that the structure constants f_{abc} of the gauge group are antisymmetric and satisfy the Jacobi identities is then an output.

$$\mathcal{L}_{gauge} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} \tag{10.17}$$

One can also show that we cannot write down a renormalizable interacting theory of a massless tensor. Instead, the leading allowed interactions are nonrenormalizable and coincide precisely with the Einstein-Hilbert action,

$$\mathcal{L}_{gravity} = \frac{\sqrt{g}}{16\pi G} \cdot R \tag{10.18}$$

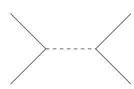
One parameter:

$$\underset{\text{gauge}}{\underset{\uparrow}{g}} \quad \text{or} \quad \underset{\text{gravity}}{\underset{\uparrow}{\frac{1}{M_{Pl.}}}} \tag{10.19}$$

Instead of the "Feynman method" we follow the amplitude philosophy. We construct a **general ansatz** for the S-matrix and use

1. **Dimensional analysis**: The amplitude mass dimension is consistent with mass dimension of the coupling.
2. **Lorentz-invariance**: s, t, u for $2 \to 2$ and we will see: covariant with respect to the little group.
3. **Locality**: kinematic singularities are consistent with the factorization and the unitary which encodes locality.[4]

[4] No $\frac{1}{s^2}$ pole in



(10.20)

only $\frac{1}{s}$ allowed (see QFT1).

Remarkably we can **bootstrap** an enormous amount of physics. On-shell recursion relations (BCFW[5]), generalized unitarity, dual conformal invariance, "gravity = (gauge)^2", twistors.

[5] Britto-Cachazo-Feng-Witten

We need to learn about the kinematics first. We will use the **spinor helicity formalism**.

10.2 Massless fermions

We have the Dirac matrices:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} , \qquad \gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{10.21}$$

with

$$\sigma^\mu = (1, \vec{\sigma}) \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}) \tag{10.22}$$

We have

$$U_R(p) = \begin{pmatrix} 0 \\ u_R(p) \end{pmatrix} , \qquad U_L(p) = \begin{pmatrix} u_L(p) \\ 0 \end{pmatrix} \tag{10.23}$$

The **two-component** spinors satisfy the following relation:

$$p \cdot \sigma \; u_R = 0 \; , \qquad p \cdot \bar{\sigma} \; u_L = 0 \; . \tag{10.24}$$

We have the following relation:

$$\epsilon_{ij}(0, 1/2)^*_j = (1/2, 0)_i \quad \text{or} \tag{10.25}$$

$$u_R(p) = i\sigma^2 u_L^*(p) \tag{10.26}$$

Via Eq. (10.26) the spinors $u_R(p)$ and $u_L(p)$ are related to each other. This leads to a phase convention for $u_R(p)$!
For massless particles we have same Eq. (10.24) as for anti-particles:

$$V_R(p) \quad \text{creates LH anti-fermion} \tag{10.27}$$

$$V_L(p) \quad \text{creates RH anti-fermion} \tag{10.28}$$

We set

$$\bar{U}_L(p) = \langle p \; , \quad \bar{U}_R(p) = [p \; , \quad U_L(p) = p] \; , \quad U_R(p) = p\rangle \; , \tag{10.29}$$

Lorentz-invariant spinor products will then be:

$$\bar{U}_L(p) U_R(q) = \langle pq \rangle \quad \text{(angle brackets)} \tag{10.30}$$

$$\bar{U}_R(p) U_L(q) = [pq] \quad \text{(square brackets)} \; . \tag{10.31}$$

We can relate the spinors to light-like ($p^2 = 0$) 4-vectors:

$$p\rangle[p = U_R(p)\bar{U}_R(p) = \not{p} \left(\frac{1 - \gamma_5}{2} \right) \tag{10.32}$$

$$p]\langle p = U_L(p)\bar{U}_L(p) = \not{p} \left(\frac{1 + \gamma_5}{2} \right) \tag{10.33}$$

We now derive some important properties:

1. The brackets are conjugate to each other: $\langle pq \rangle = [qp]^*$
   Since

$$[qp]^* = \bar{U}_R(q)^* U_L^*(p) \tag{10.34}$$

$$= (u_R^\dagger(q) u_L(p))^* \tag{10.35}$$

$$= u_R(q)_\alpha u_L(p)^*_\alpha = \bar{U}_L(p) U_R(q) \tag{10.36}$$

$$= \langle pq \rangle \quad \checkmark \tag{10.37}$$

2. The brackets are "square roots" of the Lorentz-vector product:

$$\langle pq\rangle[qp] = 2p\cdot q \tag{10.38}$$

Because

$$\begin{aligned}\langle pq\rangle[qp] &= \mathrm{Tr}\left[\not{q}\left(\frac{1-\gamma_5}{2}\right)\not{p}\left(\frac{1+\gamma_5}{2}\right)\right] &(10.39)\\ &= \mathrm{Tr}\left[\not{q}\,\not{p}\left(\frac{1+\gamma_5}{2}\right)\right] &(10.40)\\ &= 2p\cdot q, &(10.41)\end{aligned}$$

where we've used the relation $\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_5)=0$.

3. The bracket are **anti-symmetric**.
Using

$$\langle pq\rangle = u_L^\dagger(p)u_R(q) = u_{La}^*(p)(i\sigma^2)_{ab}u_{Lb}^*(q) \tag{10.42}$$

we find

$$\begin{aligned}\langle pq\rangle &= -\langle qp\rangle &(10.43)\\ [pq] &= -[qp] &(10.44)\end{aligned}$$

since $i\sigma_2$ is antisymmetric.[6]

[6] $i\sigma_2 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

The vector currents are:

$$\begin{aligned}\bar U_L(p)\gamma^\mu U_L(q) &= u_L^\dagger(p)\bar\sigma^\mu u_L(q) &(10.45)\\ \bar U_R(p)\gamma^\mu U_R(q) &= u_R^\dagger(p)\sigma^\mu u_R(q)\ . &(10.46)\end{aligned}$$

We can rearrange:

Recall:

$$(i\sigma^2)^2 = -\mathbb{1}. \tag{10.47}$$

$$\begin{aligned}u_L^\dagger(p)\bar\sigma^\mu u_L(q) &= u_L^\dagger(p)\bar\sigma^\mu(-(i\sigma^2)^2)u_L(q) &(10.48)\\ &\overset{\bar\sigma_\mu^\dagger=\bar\sigma_\mu\quad \bar\sigma_\mu^*=\bar\sigma_\mu^T=-\epsilon\sigma_\mu\epsilon\quad \epsilon=-i\sigma_2\quad \epsilon^2=-1}{\downarrow}\\ &= u_L^\dagger(p)(-i\sigma^2)\sigma^{\mu T}(i\sigma^2)u_L(q) &(10.49)\\ &= u_R^T(p)\sigma^{\mu T}u_R^*(q) &(10.50)\\ &= u_R^\dagger(q)\sigma^\mu u_R(p). &(10.51)\end{aligned}$$

So the following relation can be deduced

$$\langle p\gamma^\mu q] = [q\gamma^\mu p\rangle\ . \tag{10.52}$$

With the Fierz-identity we can write the identity of the σ-matrices:

$$\begin{aligned}(\bar\sigma^\mu)_{ab}(\bar\sigma_\mu)_{cd} &= 2(i\sigma^2)_{ac}(i\sigma^2)_{bd} &(10.53)\\ (\sigma^\mu)_{ab}(\sigma_\mu)_{cd} &= 2(i\sigma^2)_{ac}(i\sigma^2)_{bd}. &(10.54)\end{aligned}$$

From the relation Eq. (10.54) we can find

$$\begin{aligned}\langle p\gamma^\mu q]\langle k\gamma_\mu \ell] &= u_L^\dagger(p)\bar\sigma^\mu u_L(q)u_L^\dagger(k)\bar\sigma^\mu u_L(\ell) &(10.55)\\ &= 2u_L^\dagger(p)(i\sigma_2)u_L^*(k)u_L(p)(i\sigma_2)u_L(\ell) &(10.56)\\ &= -2u_L^\dagger(p)u_R(k)u_R^\dagger(q)u_L(\ell) &(10.57)\\ &= -2\langle pk\rangle[q\ell] &(10.58)\\ &= 2\langle pk\rangle[\ell q] &(10.59)\end{aligned}$$

Further we have:

$$\langle p \not{k} q] = \langle pk\rangle [kq] \tag{10.60}$$

$$\tag{10.61}$$

The Lorentz products are:

$$s_{ij} \equiv (p_i + p_j)^2 = 2p_i \cdot p_j \tag{10.62}$$

for massless particles.

10.2.1 $e^-_L e^+_R \to \mu^-_L \mu^+_R$ *in QED*

Let us have a look at an **example**: $e^-_L e^+_R \to \mu^-_L \mu^+_R$ in QED.

$$\begin{aligned} i\mathcal{M} &= (-ie)^2 \, \frac{-i}{q^2} \, \bar{U}_L(3)\gamma^\mu U_L(4) \, \bar{U}_L(2)\gamma_\mu U_L(1) && (10.63)\\ &= \frac{ie^2}{q^2} \langle 3\gamma^\mu 4] \langle 2\gamma_\mu 1] && (10.64)\\ &= \frac{2ie^2}{q^2} \langle 32\rangle [14] && (10.65) \end{aligned}$$

We know

$$\begin{aligned} |\langle 32\rangle|^2 = |[14]|^2 &= (k_2 + k_3)^2 = (k_1 + k_4)^2 && (10.66)\\ &= s_{23} = \underset{\text{Mandelstam}}{\underset{\uparrow}{u}} && (10.67) \end{aligned}$$

Now we recall in the center of mass frame:

$$\begin{aligned} u &= -2E^2(1+\cos\theta) \quad \text{and} && (10.68)\\ q^2 &= s = 4E^2. && (10.69) \end{aligned}$$

So we get:

$$|i\mathcal{M}|^2 = e^4(1+\cos\theta)^2 \ . \tag{10.70}$$

This is surprisingly easy to derive with our new technique. We can simplify Eq. (10.65) further. The denominator of Eq. (10.65) is

Remember:
Before we had to $\sum \bar{M}M$ we took the trace.

$$q^2 = 2k_1 \cdot k_2 = \langle 12\rangle [21] \ . \tag{10.71}$$

By multiplying the numerator and the denominator of Eq. (10.65) by $\langle 32\rangle$ we get

$$i\mathcal{M} = 2ie^2 \frac{\langle 32\rangle [14]}{\langle 12\rangle [21]} = 2ie^2 \frac{\langle 32\rangle [14] \langle 32\rangle}{\langle 12\rangle [21] \langle 32\rangle} \ . \tag{10.72}$$

The denominator can be written:

$$\begin{aligned} [21]\langle 32\rangle &= [12]\langle 23\rangle && (10.73)\\ &= \langle 32\rangle [21] && (10.74)\\ &= \langle 3 \not{2} \, 1] && (10.75)\\ &= \langle 3(-\not{1} - \not{3} - \not{4})1] && (10.76)\\ &\overset{\not{1}1]=0 \quad \text{and} \quad \langle 3\not{3}=0}{\overset{\downarrow}{=}} -\langle 3 \not{4} 1] && (10.77)\\ &= -\langle 34\rangle [41] && (10.78)\\ &= -[14]\langle 43\rangle && (10.79) \end{aligned}$$

and finally using Eq. (10.79)

$$i\mathcal{M} = 2ie^2 \frac{\langle 32\rangle[14]\langle 32\rangle}{\langle 12\rangle[21]\langle 32\rangle} \tag{10.80}$$

$$= 2ie^2 \frac{\langle 32\rangle\cancel{[14]}\langle 32\rangle}{\langle 12\rangle\cancel{[14]}\langle 43\rangle}(-1) \tag{10.81}$$

$$= 2ie^2 \frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle} \ . \tag{10.82}$$

or in square brackets:

$$i\mathcal{M} = 2ie^2 \frac{[14]^2}{[12][34]}. \quad \checkmark \tag{10.83}$$

10.3 Polarizations (massless photons)

First, we define

$$\epsilon_R^{*\mu}(k) = \frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu k]}{\langle rk\rangle} \tag{10.84}$$

$$\epsilon_L^{*\mu}(k) = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu k\rangle}{[rk]} \tag{10.85}$$

with the reference momentum r_μ,[7] γ-momentum k_μ. Now we show that ϵ_R^μ and ϵ_L^μ satisfy the normal conditions.

[7] Only requirement on r_μ:

$$r^2 = 0 \quad \text{(light-like)} \tag{10.86}$$

$$r \cdot k \neq 0 \quad \text{(not allowed)} \tag{10.87}$$

10.3.1 Massless photons and polarizations

We will now show that we can write the polarizations from Eq. (10.84) and Eq. (10.85) as such.

Proof: we first realize that Eq. (10.84) and Eq. (10.85) satisfy the following relation:

1.

$$[\epsilon_R^*(k)]^* = \epsilon_L^*(k) \tag{10.88}$$

because

$$[rk]^* = -[rk] \tag{10.89}$$

and

$$(\langle r\gamma^\mu k])^* = [r\gamma_\mu k\rangle \tag{10.90}$$

In Eq. (10.90) we used the following considerations to prove this relation

$$\langle r\gamma_\mu k]^* = \left(u_L^\dagger(r)\bar{\sigma}^\mu u_L(k)\right)^* \tag{10.91}$$

$$= u_L^T(r)\bar{\sigma}^{\mu T} u_L(k)^* \tag{10.92}$$

$$= u_L^\dagger(k)\bar{\sigma}^\mu u_L(r) \tag{10.93}$$

$$= \langle k\gamma_\mu r] = [r\gamma_\mu k\rangle \tag{10.94}$$

where we have used Eq. (10.52).

2.

$$k_\mu \epsilon^{*\mu}_{R,L}(k) = 0 \; . \tag{10.95}$$

Eq. (10.95) follows from the fact that

$$k_\mu \epsilon^{*\mu}_{R,L}(k) = \frac{1}{\sqrt{2}} \frac{\langle r \not{k} k]}{\langle rk \rangle} \overset{\not{k}k]=0}{=} 0 \tag{10.96}$$

3.

$$\epsilon^*_R(k) \cdot (\epsilon^*_L(k))^* = \epsilon^*_R(k) \cdot \epsilon^*_R(k) = 0 \tag{10.97}$$

follows from

$$\epsilon^*_R(k) \cdot \epsilon^*_L(k) \propto \langle r\gamma_\mu k]\langle r\gamma_\mu k] = 2\underbrace{\langle rr\rangle}_{=0}[kk] = 0. \tag{10.98}$$

4.

$$|\epsilon^*_R(k)|^2 = \epsilon^*_R(k)_\mu \cdot \epsilon_R(k)^\mu = -1 \tag{10.99}$$

since

$$|\epsilon^*_R(k)|^2 = \frac{1}{2} \frac{\langle r\gamma^\mu k]\langle k\gamma_\mu r]}{\langle rk\rangle[kr]} = \frac{\langle rk\rangle[rk]}{\langle rk\rangle[kr]} = -1 \quad \checkmark \tag{10.100}$$

Furthermore we get from 1)

$$|\epsilon^*_L(k)|^2 = -1. \tag{10.101}$$

5.

$$\epsilon^*_R(k) \cdot (\epsilon^*_L(k))^* = 0 \tag{10.102}$$

This property is satisfied for polarization vectors, e.g.

Check this property for $k_\mu = (k, 0, 0, k)$.

$$\epsilon^\mu_R(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \qquad \epsilon^\mu_L(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \tag{10.103}$$

For a particular choice of r_μ, we evaluate $\epsilon^*_{L,R}(k)$:

$$k_\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix} \quad r = \begin{pmatrix} r \\ 0 \\ 0 \\ -r \end{pmatrix} \tag{10.104}$$

We can check that

$$r \cdot k \neq 0 \quad \checkmark \quad r^2 = 0 \quad \checkmark \tag{10.105}$$

The associated spinors are (in this basis):

$$u_L(k) = \sqrt{2k} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad u_R(k) = \sqrt{2k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \quad u_L(r) = \sqrt{2r} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \; . \tag{10.106}$$

Then we can write

$$\langle r\gamma^\mu k] = u_L^\dagger(r)\bar{\sigma}^\mu u_L(k) = \sqrt{4kr}\ (0,1,-i,0)^\mu \langle rk\rangle \tag{10.107}$$
$$= u_L^\dagger(r) u_L(k) = -\sqrt{4kr} \tag{10.108}$$

This choice of r_μ results in a manifestly right-handed polarization vector

$$\epsilon_R^{*\mu}(k) = -\frac{1}{\sqrt{2}}(0,1,i,0)^* \tag{10.109}$$

How does the change of the reference vector affect the definition? It is a change to a light-like vector s

$$\sqrt{2}\left(\epsilon_R^{*\mu}(k;r) - \epsilon_R^{*\mu}(k;s)\right) = \left(\frac{\langle r\gamma^\mu k]}{\langle rk\rangle} - \frac{\langle s\gamma^\mu k]}{\langle sk\rangle}\right) \tag{10.110}$$
$$= \frac{1}{\langle rk\rangle\langle sk\rangle}(-\langle r\gamma^\mu k]\langle ks\rangle + \langle s\gamma_\mu k]\langle kr\rangle) \tag{10.111}$$
$$= \frac{1}{\langle rk\rangle\langle sk\rangle}(-\langle r\gamma^\mu\ \not{k}s\rangle + \langle s\gamma^\mu\ \not{k}r\rangle) \tag{10.112}$$
$$= \frac{1}{\langle rk\rangle\langle sk\rangle}\langle s(\not{k}\gamma^\mu + \underset{\substack{\uparrow \\ \{\gamma_\mu,\gamma_\nu\}=2\eta_{\mu\nu}}}{\gamma^\mu}\ \not{k})r\rangle \tag{10.113}$$
$$= \frac{1}{\langle rk\rangle\langle sk\rangle}\cdot 2k^\mu\langle sr\rangle \tag{10.114}$$

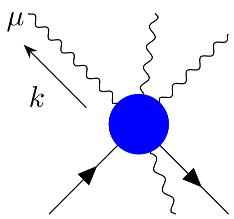
with $k]\langle k = \underset{\substack{\uparrow \\ P_R}}{\not{p}\frac{1+\gamma_5}{2}}$, $s\rangle = u_R(s)$.

We find

$$\epsilon_R^{*\mu}(k;r) - \epsilon_R^{*\mu}(k;s) = f(r,s)k^\mu, \tag{10.115}$$

where $f(r,s)$ is a function of the two reference vectors r, s.

We know that when this expression is dotted into an on-shell photon or gluon amplitude, the Ward-Identity tells us that this is zero.

$k^\mu \cdot$ 

$= 0. \quad (10.116)$

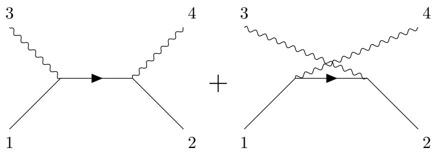
We can therefore ignore the reference momentum r, s and use whatever is convenient.

10.4 Color ordering

Next time ...

10.5 Application: $e^+e^- \to \gamma\gamma$

Now we want to illustrate the application of these polarization vectors. For this purpose we compute the amplitude for $e^+e^- \to \gamma\gamma$.



(10.117)

So we get for the value of the amplitude:

$$i\mathcal{M} = (-ie)^2 \langle 2 \left\{ \gamma \cdot \epsilon(4) \frac{i(\not{2} + \not{4})}{s_{24}} \gamma \cdot \epsilon(3) + \gamma \cdot \epsilon(3) \frac{i(\not{3} + \not{2})}{s_{23}} \gamma \cdot \epsilon(4) \right\} 1] \ . \tag{10.118}$$

where we have suppressed the stars and indices in the polarizations $\epsilon(i) = \epsilon^*_\mu(p_i)$ which will be L or R, see below.

What are the possible photon polarizations? There are 4 options: LL, LR, RL, RR.

1. $\gamma_R\gamma_L$ which is related by $3 \leftrightarrow 4$ exchange to $\gamma_L\gamma_R$

2. $\gamma_R\gamma_R \overset{\text{P}}{\leftrightarrow} \gamma_L\gamma_L$ (parity)

3. The amplitude $\gamma_R\gamma_R$ and $\gamma_L\gamma_L$ are actually zero. Take therefore $\gamma_R\gamma_R$ and use $r = 2$

$$\epsilon^*_{R\mu}(3) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma_\mu 3]}{\langle 23 \rangle} \ , \qquad \epsilon^*_{R\mu}(4) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma_\mu 4]}{\langle 24 \rangle} \ . \tag{10.119}$$

with the identity

$$\langle p\gamma_\mu q]\langle k\gamma_\mu \ell] = 2\langle pk\rangle[\ell q] \tag{10.120}$$
$$\langle p\gamma_\mu q][k\gamma_\mu \ell\rangle = 2\langle p\ell\rangle[kq] \ . \tag{10.121}$$

We get

$$\langle 2\gamma \cdot \epsilon(4) \ \propto \langle 2\gamma_\mu \ldots \langle 2\gamma 4] \overset{Eq.\ (10.121)}{\propto} \cancelto{0}{\langle 22\rangle}[4 = 0. \tag{10.122}$$

We have the same cancellation for

$$\langle 2\gamma \cdot \epsilon(3) \tag{10.123}$$

$e^+_L e^-_R \to \gamma_L\gamma_L$ and $e^+_L e^-_R \to \gamma_R\gamma_R$ vanish!

4. Now we compute $\gamma_R\gamma_L$. We choose therefore

$$\epsilon_{R\mu}(3) = \frac{1}{\sqrt{2}} \frac{\langle 2\gamma_\mu 3]}{\langle 23 \rangle} \ , \qquad \epsilon_{L\mu}(4) = -\frac{1}{\sqrt{2}} \frac{[1\gamma_\mu 4\rangle}{[14]} \ . \tag{10.124}$$

The first diagram gives (we forget about slashes now!):

$$i\mathcal{M}_1 = \frac{-ie^2}{s_{24}}\frac{1}{(-2)}\underbrace{\langle 2\gamma_\mu \frac{[1\gamma_\mu 4\rangle}{[14]}}_{\frac{1}{[14]}2[1\langle 24\rangle}(2+4)\underbrace{\gamma_\mu \frac{\langle 2\gamma_\mu 3]}{\langle 23\rangle}1]}_{2\frac{1}{\langle 23\rangle}2\rangle[31]} \tag{10.125}$$

$$= \frac{2ie^2}{s_{24}\langle 23\rangle[14]}\langle 24\rangle[31][1(2+4)2\rangle \tag{10.126}$$

$$\overset{22\rangle=0}{\downarrow} \frac{2ie^2}{s_{24}}\frac{\langle 24\rangle[31]}{[14]\langle 23\rangle}[14]\langle 42\rangle \tag{10.127}$$

$$\overset{s_{24}=(p_2+p_4)^2=(p_1+p_3)^2=\langle 13\rangle[31]}{\downarrow} 2ie^2\frac{\langle 24\rangle\cancel{[31]}\cancel{[14]}\langle 42\rangle}{\langle 13\rangle\cancel{[31]}\cancel{[14]}\langle 23\rangle} \tag{10.128}$$

$$= 2ie^2\frac{(\langle 24\rangle)^2}{\langle 23\rangle\langle 31\rangle} \tag{10.129}$$

We now use

$$s_{23} = u \qquad s_{13} = s_{24} = t \tag{10.130}$$

Finally we get

$$|i\mathcal{M}|^2 = 4e^4\frac{t}{u} = 4e^4\frac{1-\cos\theta}{1+\cos\theta} \tag{10.131}$$

We can generalize this method to include massive external particles like W/Z bosons and massive fermions. Now we want to discuss a powerful tool "**Little group invariance**".

10.6 Little group (LG) scaling

$p\rangle$ and $q]$ are solutions to the massless Weyl equation:

$$\not{p}p\rangle = 0 \qquad p^2 = 0 \tag{10.132}$$

$$\not{q}q] = 0 \qquad q^2 = 0 \tag{10.133}$$

$$p\rangle[q = \not{p}p_L \tag{10.134}$$

These relations are invariant under the little group rotation

$$p\rangle = u_R(p) \qquad p] = u_L(p) \tag{10.135}$$

The reference vector $k = (k, \underbrace{0, 0}_{SO(2)}, k)$. The eigenvalue J_z

See Poincaré-representation using induced representations and little group.

$$e^{i\alpha J_z}u_{L/R}(p) = e^{\pm i\frac{\alpha}{2}}u_{L/R}(p) \tag{10.136}$$

For real momenta t i a complex phase

$$p\rangle \to tp\rangle \tag{10.137}$$

$$p] \to t^{-1}p] \tag{10.138}$$

with $|t| = 1$ (pure phase) or $t = e^{i\frac{\alpha}{2}}$ as in Eq. (10.136).[8]

[8] For complex momenta it can be an arbitrary number.

$$u_R(p) = i\sigma_2 u_L^*(p) \tag{10.139}$$

We know that a little-group transformation has to be a phase otherwise Eq. (10.139) could not be satisfied.

Let us explore the consequences for amplitudes

- a Feynman-diagram consists of propagators, vertices and external line rules
- For massless particles, we can rewrite in terms of $[ij]$ and $\langle ij\rangle$ brackets
- The propagator and the vertices cannot scale under LG only **external lines**
  - a **scalar** is constant (singlet under LG)
  - $p]$ and $q\rangle$ transform as **spinors** with t^{-2h} with $h=\pm\frac{1}{2}$
  - massless **vectors** transform as

$$\epsilon_R^{*\mu}(k)=\frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu k]}{\langle rk\rangle}\overset{\text{LG}}{\overset{\downarrow}{\longrightarrow}}t^{-2h}\epsilon_R^{*\mu}(k)\quad\text{for } h=1 \tag{10.140}$$

$$\epsilon_L^{*\mu}(k)=-\frac{1}{\sqrt{2}}\frac{[r\gamma^\mu k\rangle}{[rk]}\overset{\text{LG}}{\overset{\downarrow}{\longrightarrow}}t^{-2h}\epsilon_L^{*\mu}(k)\quad\text{for } h=-1. \tag{10.141}$$

The scattering amplitude is **little group covariant**[9]

[9] It is not invariant!

$$\boxed{A(1^{h_1}\dots n^{h_n})\to\prod_i t_i^{-2h_i}A(1^{h_1}\dots n^{h_n})} \tag{10.142}$$

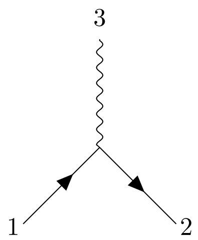
with

$$A(1^{h_1}\dots n^{h_n})=e_{\mu_1}^{h_1}\cdots e_{\mu_n}^{h_n}A^{\mu_1\cdots\mu_n}. \tag{10.143}$$

Why not invariant? Physical observables should be invariant. This is no contradiction: on real kinematics the little group parameters t_i are **pure phases** which trivially cancel in $S\sim|\mathcal{M}|^2$!

10.6.1 3-particle amplitude example

Let us calculate the simple 3 point example $e_L^-e_R^+\to\gamma_R$. Later we will show how to derive this result directly from the symmetry properties of the amplitude.



$$e_L^-e_R^+\to\gamma_R \tag{10.144}$$

$$iA_3=(-ie)\bar{U}_L(1)\gamma\epsilon(3)U_L(2) \tag{10.145}$$

$$=(-ie)\langle 1\gamma_\mu 2]\frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu 3]}{\langle r3\rangle} \tag{10.146}$$

$$=i\underset{\substack{\uparrow\\ -\sqrt{2}e}}{\tilde{e}}\langle 1r\rangle\frac{[32]}{\langle r3\rangle}, \tag{10.147}$$

where in the last step we used

$$\langle p\gamma_\mu k]\langle \ell\gamma_\mu q] = 2\langle pq\rangle[\ell k]\ . \tag{10.148}$$

Let us now remove the r-dependence. First we insert $1 = \frac{[12]}{[12]}$. So Eq. (10.147) becomes

$$= i\tilde{e}\langle 1r\rangle\frac{[32][12]}{\langle r3\rangle[12]} \tag{10.149}$$

$$= i\tilde{e}\frac{[32][23]\cancel{\langle 3r\rangle}}{\cancel{\langle r3\rangle}[12]} = i\tilde{e}\frac{[23]^2}{[12]}. \tag{10.150}$$

We get

$$\boxed{A_3 = \tilde{e}\frac{[23]^2}{[12]}.} \tag{10.151}$$

Auxiliary calculation for Eq. (10.150):

$$[12]\langle 1r\rangle = -[21]\langle 1r\rangle = -[2\not{1}r\rangle$$
$$= [2(\not{2}+\not{3})r\rangle \underset{\substack{\uparrow\\ [2\not{2}=0}}{=} [2\not{3}r\rangle = [23]\langle 3r\rangle\ .$$

We can see that we find *only square brackets.*

10.6.2 Three-particle kinematics

It is a consequence of the special 3-particle kinematics which we will now explore. Four-momentum conservation requires

$$p_1^\mu + p_2^\mu + p_3^\mu = 0. \tag{10.152}$$

Hence it follows

$$(p_1+p_2)^2 = \langle 12\rangle[12] = p_3^2 = 0 \tag{10.153}$$
$$(p_2+p_3)^2 = \langle 23\rangle[23] = p_1^2 = 0 \tag{10.154}$$
$$(p_3+p_1)^2 = \langle 31\rangle[31] = p_2^2 = 0. \tag{10.155}$$

We can distinguish two cases

- If $\langle 12\rangle \neq 0$ we find

  $$[12] = 0. \tag{10.156}$$

  Additionally, we can see

  $$\langle 12\rangle[23] = \langle 1\not{2}3] \overset{\substack{1+2+3=0\\ \downarrow}}{=} -\langle 1\not{1}3] - \langle 1\not{3}3] \overset{\substack{\langle 1\not{1}=0 \quad \not{3}3]=0\\ \downarrow}}{=} 0. \tag{10.157}$$

  So we get

  $$[23] = 0 \quad \text{similarly } [31] = 0. \tag{10.158}$$

- If however $[12] \neq 0$ then

  $$\langle 12\rangle = \langle 23\rangle = \langle 31\rangle = 0. \tag{10.159}$$

  by following a similar line of arguments.

We summarize:
The three particle amplitude has support on two possible kinematic configurations.

$$\text{holomorphic: } [ij] = 0 \qquad 1] \propto 2] \propto 3] \tag{10.160}$$
$$\text{anti-holomorphic: } \langle ij\rangle = 0 \qquad 1\rangle \propto 2\rangle \propto 3\rangle \tag{10.161}$$

Since for real momenta, angle and square brackets are complex conjugate of each other: the on-shell 3-particle amplitude of only massless particles can only be non-vanishing in **complex momenta**. We will use complex momentum 3 point amplitudes as building blocks for higher point amplitudes.

Without loss of generality, the 3-particle amplitude in holomorphic configuration is some general polynomial in the angle brackets

$$A_3(1^{h_1}2^{h_2}3^{h_3}) = c\langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2}. \tag{10.162}$$

This form is the only one possible, since little group covariance imposes homogeneous scaling behaviour

$$2\rangle \to t\cdot 2\rangle \tag{10.163}$$
$$1^{h_1} \to t^{-2h_1}1^{h_1} \tag{10.164}$$

and so on. We see

$$-2h_1 = n_2 + n_3 \tag{10.165}$$
$$-2h_2 = n_3 + n_1 \tag{10.166}$$
$$-2h_3 = n_1 + n_2, \tag{10.167}$$

which can be solved for the helicities h_i of the external particles

$$n_1 = h_1 - h_2 - h_3 \tag{10.168}$$
$$n_2 = h_2 - h_3 - h_1 \tag{10.169}$$
$$n_3 = h_3 - h_1 - h_2. \tag{10.170}$$

We find the analogous relations to Eq. (10.168) (and following) for $\langle\rangle \to []$ if we replace

$$n_i \to -n_i$$

We can conclude that the helicity structure fixes the amplitude up to an overall constant c![10]

[10] In a conformal field theory, the 3-point correlation functions are determined by the scaling dimensions of the operators.

Now we check the result from Sec. 10.6.1, which we have explicitly derived

$$A_3(\overset{1}{\downarrow}{e_L^-}\,\overset{2}{\downarrow}{e_L^+} \to \overset{3}{\downarrow}{\gamma_R}) = \tilde{e}\frac{[23]^2}{[12]}. \tag{10.171}$$

First we calculate the n_i using the equations for angle brackets:

$$e_L^- \quad h_1 = -\frac{1}{2} \qquad \to n_1 = -\frac{1}{2} - \frac{1}{2} - 1 = -2 \tag{10.172}$$
$$e_R^+ \quad h_2 = +\frac{1}{2} \qquad \to n_2 = \frac{1}{2} - 1 + \frac{1}{2} = 0 \tag{10.173}$$
$$\gamma_R \quad h_3 = +1 \qquad \to n_3 = 1 + \frac{1}{2} - \frac{1}{2} = 1. \tag{10.174}$$

which translates to square brackets relations

$$n_1^{[]} = 2 \qquad n_2^{[]} = 0 \qquad n_3^{[]} = -1. \tag{10.175}$$

Hence

$$A_3^{①}(e_L^- e_R^+ \to \gamma_R) = c\frac{[23]^2}{[12]} \tag{10.176}$$

Let us consider an amplitude with three different (aka gluons) gauge bosons:

$$A_3^{②}(g_1^- g_2^- g_3^+) = c\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 23\rangle} \tag{10.177}$$

where we **assumed** it only depends on angle brackets.

What if it only depended on square brackets, $n_i \to -n_i$? $\to$ the scaling is the opposite.

$$A_3(g_1^- g_2^- g_3^+) = \tilde{c}\frac{[13][23]}{[12]^3} \tag{10.178}$$

How can we distinguish Eq. (10.176) and Eq. (10.177)? $\to$ We use dimensional analysis

$$p] \quad \text{and} \quad q\rangle \to \text{dim } \frac{1}{2} \tag{10.179}$$

$$[ij] \quad \text{and} \quad \langle ij\rangle \to \text{dim } 1. \tag{10.180}$$

For the amplitude in Eq. (10.176), we have a momentum dependence p^1:

$$[A_3^{①}] = 1 \quad \to \quad \underset{\substack{\uparrow \\ A_\mu A_\mu \partial_\mu A_\nu \\ \text{interaction in } \mathrm{Tr}\left(\mathrm{F}^a_{\mu\nu}\mathrm{F}^a_{\mu\nu}\right)}}{p^1 \text{ momentum dependence}}\ , \tag{10.181}$$

whereas in Eq. (10.177) we have a momentum dependence p^{-1}:

$$[A_3^{②}] = -1 \quad \to \quad \underset{\substack{\uparrow \\ g' AA\frac{\partial}{\Box}A \\ \text{no such term in a local Lagrangian}}}{p^{-1} \text{ momentum dependence}}\ . \tag{10.182}$$

The little group and locality uniquely fix massless 3-particle amplitudes! The mass dimension cannot be negative, as this would signal non-locality due to the inverse powers of momenta. The assumption of **locality** assumes

$$[A(1^{h_1}2^{h_2}3^{h_3})] = n_1 + n_2 + n_3 = -(h_1 + h_2 + h_3) = -h \geq 0. \tag{10.183}$$

The brackets Eq. (10.183) denote the mass dimension without considering the intrinsic dimensionality of coupling constants. We see

$$\text{holomorphic kinematics applies } \to h \leq 0$$
$$\text{anti-holomorphic kinematics applies} \to h \geq 0.$$

Under the little group covariance and locality we find a general form for the 3-particle amplitude of massless particles in 4D

$$A(1^{h_1}2^{h_2}3^{h_3}) = \begin{cases} \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}, & h \leq 0 \\ [12]^{h_1+h_2-h_3}[23]^{h_2+h_3-h_1}[31]^{h_3+h_1-h_2}, & h \geq 0 \end{cases} \tag{10.184}$$

This formula is as practical and elegant as the "tuxedo t-shirt". We derived it from symmetry and it holds **non-perturbatively**.[11] This can be enough to fix **all** amplitudes (at tree level and together with recursion relations) in some theories!

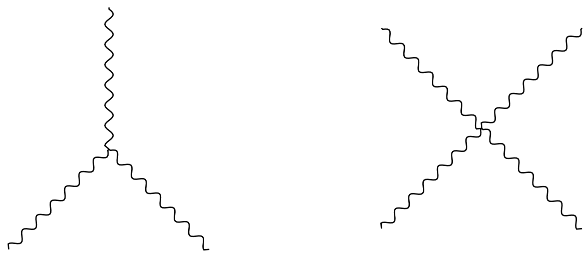
[11] Loops would generate more complicate functions like $\ln(p_1 \cdot p_2)$, but those are either vanishing or singular.

10.7 Fun with polarizations - the MHV classification

The aim of this chapter is to study the gluon scattering amplitudes. The Yang-Mills Lagrangian contains two types of interaction terms

$$\mathrm{tr} F_{\mu\nu}F^{\mu\nu} \longrightarrow AA\partial A \;+\; A^4. \tag{10.185}$$

In a typical gauge (Feynman-gauge) we have



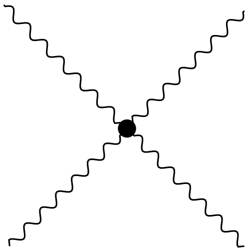
cubic $\mathcal{O}(\text{mass}^1)$ quartic $\mathcal{O}(\text{mass}^0)$

We consider **tree** diagrams with only **cubic** vertices (=trivalent) with n **external** legs. The number of vertices and propagators grow linearly with
The mass dimension of the diagram

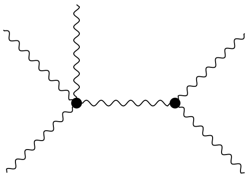
$$[A_n] \;\sim\; \frac{(\text{mass})^{n-2}}{(\text{mass}^2)^{n-3}} \;\sim\; (\text{mass})^{4-n}. \tag{10.186}$$

AAny diagram with cubic and quartic vertices has mass dimension

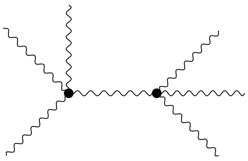
$(\text{mass})^{4-n}$.



$$n = 4 \quad [A_4] = \text{mass}^0 = \text{mass}^{4-n} \quad \checkmark \tag{10.187}$$



$$n = 5 \quad [A_5] = \frac{(\text{mass}^0)^1(\text{mass}^1)^1}{(\text{mass}^2)^1} = \text{mass}^{4-n} \quad \checkmark \tag{10.188}$$



$$n = 6 \quad [A_6] = \frac{(\text{mass}^0)^2}{\text{mass}^2} = \text{mass}^{4-n} \quad \checkmark \tag{10.189}$$

Important: powers in the numerator cannot exceed

$$n - 2 = V_3. \tag{10.190}$$

Use interchangeably $R = +$ and $L = -$-

The gluon tree amplitude is:

$$A_n \sim \sum_{\text{diagrams}} \frac{\sum \left(\prod(\epsilon_i.\epsilon_j)\right)\left(\prod(\epsilon_i.k_j)\right)\left(\prod(k_i.k_j)\right)}{\prod P_I^2}. \tag{10.191}$$

All-plus tree gluon amplitudes vanish:

$$A_n(1^+2^+ \ldots n^+) = 0. \tag{10.192}$$

Proof of Eq. (10.192): We know

$$\langle r_1 \gamma^\mu q_1] \langle r_2 \gamma_\mu q_2] = \langle r_1 r_2 \rangle [q_1 q_2]. \tag{10.193}$$

$$\epsilon_i \cdot \epsilon_j \propto \langle r_i r_j \rangle \tag{10.194}$$

$$\epsilon_i \cdot \epsilon_j \propto [r_i r_j] \tag{10.195}$$

$$\epsilon_i \cdot \epsilon_j \propto \langle h_i r_j \rangle [h_j r_i]. \tag{10.196}$$

For all-R amplitude: we choose all $r_i = r$ then we have:

$$\epsilon_i^R \cdot \epsilon_j^R = 0. \tag{10.197}$$

That means the only way to absorb Lorentz-indices of all ϵ_i^R is through:

$$\epsilon_i^R \cdot k_j \quad \rightarrow \quad \text{n powers of momentum in the numerator.} \tag{10.198}$$

But: we have just argued that no more than $(n-2)$-powers of momenta is possible in any gluon diagram. Hence we have

$$\boxed{A_n(1^+2^+ \ldots n^+) = 0.} \tag{10.199}$$

This argument would be **very** hard to do in general for Feynman diagram calculation.

Try!

Let us flip one gluon polarization:

$$A_n(1^- 2^+ \dots n^+). \tag{10.200}$$

We choose

$$r_2 = r_3 = \dots = r_n = k_1. \tag{10.201}$$

Hence

$$\epsilon_i^+ \cdot \epsilon_j^+ \overset{\overset{j,i\geq 2}{\downarrow}}{=} 0 \tag{10.202}$$

and

$$\epsilon_i^- \cdot \epsilon_j^- = \langle k_1 r_2 \rangle [k_2 r_1] \propto \langle k_1 k_2 \rangle = 0. \tag{10.203}$$

Again we would need n-factors:

$$\epsilon_i^+ \cdot k_j, \epsilon_i^- \cdot k_j. \tag{10.204}$$

We conclude again:

$$\boxed{A_n(1^- 2^+ \dots n^+) = 0.} \tag{10.205}$$

The first non-vanishing amplitudes $A_n(1^- 2^+ \dots n^+) \neq 0$ is called <u>m</u>aximally <u>h</u>elicity <u>v</u>iolating (MHV).

10.8 Bootstrapping amplitudes

Let us consider some familiar examples.

10.8.1 Scalars

In case of identical scalars we have:

$$A(123) = \omega \tag{10.206}$$

where all helicities vanish:

$$h_i \equiv 0 \quad i = 1, 2, 3. \tag{10.207}$$

With multiple species of scalars we find

$$A(1_a 2_b 3_c) = \omega_{abc}. \tag{10.208}$$

The subscript Eq. (10.208) run over flavors and with the fact that the external states are bosons we conclude that the coupling constant ω_{abc} is symmetric in its indices.

This is the **only** on-shell amplitude of 3 particles.
What about $\phi(\partial\phi)^2$?
$\rightarrow$ amplitudes vanish since they can only depend on products of momenta:

$$p_1^2 = p_2^2 = p_3^2 = p_1 p_2 = p_2 p_3 = p_3 p_1 = 0. \tag{10.209}$$

We can make this manifest in the action by a field redefinition (see introduction).

10.8.2 Vectors

If we have identical vectors the helicities are:

$$h_i = \pm 1. \tag{10.210}$$

We see that

$$\langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2}. \tag{10.211}$$

n_i-exponents are **odd** integers:

$$n_1 = h_1 - h_2 - h_3. \tag{10.212}$$

The 3-particle amplitude is **odd** under exchange of two particles even though they are **bosons**(!), e.g.

$$1 \leftrightarrow 2 \tag{10.213}$$
$$n_1 \quad n_2 \tag{10.214}$$
$$n_2 \overset{1\leftrightarrow 2}{\longrightarrow} n_1 \tag{10.215}$$
$$n_3 \quad n_3. \tag{10.216}$$

and

$$\langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2} \overset{1\leftrightarrow 2}{\longrightarrow} \overset{\text{odd}}{\langle 21\rangle^{n_3}}\langle 13\rangle^{n_2}\langle 32\rangle^{n_1} \tag{10.217}$$
$$= -\langle 12\rangle^{n_3}\langle 23\rangle^{n_1}\langle 31\rangle^{n_2}. \tag{10.218}$$

$$A(1^{h_1}2^{h_2}3^{h_3}) = -A(2^{h_2}1^{h_1}3^{h_3}). \tag{10.219}$$

It is in agreement with e.g. QED: the 3-particle photon amplitude vanishes because of charge symmetry!

A **cubic** self-interaction of a **single** vector is forbidden. **But** interactions among multiple species (also called colors) of vectors are allowed.

$$A(1_a^- 2_b^- 3_c^+) \overset{h\leq 0}{=} f_{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} \tag{10.229}$$

and

$$A(1_a^+ 2_b^+ 3_c^-) \overset{h\geq 0}{=} f_{abc}\frac{[12]^3}{[13][32]}, \tag{10.230}$$

where we just had to apply $n_i \to -n_i$ in last step.
Now f_{abc} has to be antisymmetric such that the amplitude is **even** under exchange of bosons.

We know from Feynman diagrams:

$$A(1_a 2_b 3_c) = f_{abc}(e_1 e_2)(p_1 e_3 - p_2 e_3) + \text{cyclic perm.} \tag{10.231}$$

We calculate n_i in Eq. (10.229):

$$n_1 = h_1 - h_2 - h_3 \tag{10.220}$$
$$= -1 + 1 - 1 \tag{10.221}$$
$$= -1 \tag{10.222}$$

$$n_2 = h_2 - h_1 - h_3 \tag{10.223}$$
$$= h_2 - h_1 - h_3 \tag{10.224}$$
$$= -1 \tag{10.225}$$

$$n_3 = h_3 - h_1 - h_2 \tag{10.226}$$
$$= 1 + 1 + 1 \tag{10.227}$$
$$= 3. \tag{10.228}$$

In case of all helicities are +1 or all −1 we obtain

$$A(1_a^- 2_b^- 3_c^-) = f'_{abc}\langle 12\rangle\langle 23\rangle\langle 31\rangle \tag{10.232}$$
$$A(1_a^+ 2_b^+ 3_c^+) = f''_{abc}[12][23][31] \tag{10.233}$$

Note: The MHV argument which predicts vanishing $+++$ does **not** apply because this is not tree-level.

f'_{abc} andf''_{abc} are again antisymmetric. The mass-dimension is 3. It must involve 3 momenta/derivatives.

$$f_{abc}F^a_{\mu\nu}F^b_{\nu\rho}F^c_{\rho\mu}, \tag{10.234}$$

a higher dimensional operator, which is induced by **loops** of heavy colored particles.

10.8.3 Tensors

In this case we have identical tensors with the helicities

$$h_i = \pm 2 \qquad \text{(graviton)} \tag{10.235}$$

The exponent n_i is **even**. So the amplitude is invariant under exchange of two bosons. We get for the 3-particle amplitude:

$$A(1^{--}2^{--}3^{++}) = \frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 32\rangle^2} \tag{10.236}$$

and

$$A(1^{++}2^{++}3^{--}) = \frac{[12]^6}{[13]^2[32]^2}, \tag{10.237}$$

where we just used $h_i \to 2h_i$ in Eq. (10.220). It is quite simple when we compare this expression with the Feynman calculation:

$$A(123) = -\frac{1}{2}\,(e_1e_2)^{\,2}\,(p_1e_3)\,(p_2e_3) + (e_1e_2)\,(e_2e_3)\,(p_1e_3)\,(p_2e_1) + \text{perm.}$$

where we wrote the tensor polarization as a product of the vector. For the amplitude for tensors which scatter in the all minus or all plus configuration we get:

$$A(1^{--}2^{--}3^{--}) = \langle 12\rangle^2\langle 23\rangle^2\langle 31\rangle^2 \tag{10.238}$$

and

$$A(1^{++}2^{++}3^{++}) = [12]^2[23]^2[31]^2. \tag{10.239}$$

Counting the dimension $\left[[ij]^6\right] = 6$, we see that both amplitudes originate from a six-derivative interaction by the curvature-cubed operator:

$R_{\mu\nu}{}^{\rho\sigma} \sim \partial_\mu\partial_\nu h^{\rho\sigma}$

$$R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}. \tag{10.240}$$

Why not a **quadratic** curvature invariant? Historic surprise of first quantum gravity calculations

$$R^2 : R^2, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}. \tag{10.241}$$

By naive power counting we can state that 1-loop graviton digrams should produce divergent contributions corresponding to local curvature-squared counterterms mentionned in Eq. (10.241). None of these operators contribute to on-shell scattering amplitudes in 4D! For a better understanding we rewrite $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ in terms of the Gauss-Bonet combination:

$$R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \tag{10.242}$$

which is a total derivative in 4D. With the pure gravity on-shell relations

$$R = 0 = R_{\mu\nu} \tag{10.243}$$

we can eliminate all operators of the form:

By using a field re-definition in the EFT

$$R, R_{\mu\nu}^2, R_{\mu\nu\alpha\beta}^2 \tag{10.244}$$

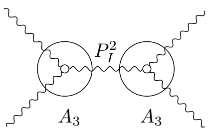
and we get

Pure gravity is finite at 1-loop!

There is a two-loop divergence in $R_{\mu\nu\alpha\beta}R_{\alpha\beta\kappa\lambda}R_{\kappa\lambda\mu\nu}$.

10.9 4-particle amplitudes and factorization

That the 3-point vertices are fixed by symmetry is maybe not surprising. → They are generated by pure contact vertices. For a 4-particle amplitudes we need an additional ingredient: **locality**. We can tune the external kinematics such that $p_I^2 = 0$ (on-shell).

A_4  (10.245)

The tree-level amplitude has a $\frac{1}{p^2}$-singularity which signals the particle propagation over **macroscopic** in space-time distances – compared to virtual particles with $p_I^2 \neq 0$.

In the factorization limit we have

$$p_I^2 = 0 = s. \tag{10.246}$$

We have two on-shell sub-amplitudes, which means that we have two different processes occurring at disparate points. So we have stringent conditions on the structure of the amplitude:

We focus here on the example of an s-channel singularity, but there are of course also u, t singularities, which can and will exploit.

$$\lim_{s\to 0} sA_4 = A_3 \cdot A_3. \tag{10.247}$$

The mass dimension is

$$[A_4] = 2[A_3] - \overset{\overset{s}{\downarrow}}{2}. \tag{10.248}$$

The relation Eq. (10.248) looks trivial but together with Eq. (10.247), it will be helpful in determining the structure of 4 point amplitudes from 3 point building blocks.

10.9.1 Scalars

We already know from locality

$$[A_3] \geq 0. \tag{10.249}$$

With Eq. (10.248) we find:

$$[A_3] \geq 0 \qquad \rightarrow \qquad [A_4] \geq -2. \tag{10.250}$$

bounded from below (locality on point vertex without intermediate particles requires $[A_4] \geq 0$)

$$\begin{aligned}
\phi^3 &: \quad A_4 = s^{-1} + t^{-1} + u^{-1} && (10.251)\\
\phi^4 &: \quad A_4 = 1 && (10.252)\\
(\partial\phi)^2\phi^2 &: \quad A_4 = s + t + u = 0 && (10.253)\\
(\partial\phi)^4 &: \quad A_4 = s^2 + t^2 + u^2 && (10.254)\\
(\partial\partial\phi)^2(\partial\phi)^2 &: \quad A_4 = s^3 + t^3 + u^3 && (10.255)
\end{aligned}$$

and so on.

10.9.2 Vectors

We already know

$$[A_3] = 1 \tag{10.256}$$

So we find with Eq. (10.248):

$$[A_3] = 1 \qquad \rightarrow \qquad [A_4] = 2[A_3] - 2 = 0, \tag{10.257}$$

which means that the 4-amplitude is dimensionless.

Let us find an ansatz with respect to little group covariance

$$A(1_a^- 2_b^- 3_c^+ 4_d^+) \rightarrow t_1^2 t_2^2 t_3^{-2} t_4^{-2} A(1_a^- 2_b^- 3_c^+ 4_d^+). \tag{10.258}$$

We factor out the full little group weight of the amplitude and get for a general ansatz:

$$A(1_a^- 2_b^- 3_c^+ 4_d^+) = \langle 12\rangle^2 [34]^2 F(s,t,u) \tag{10.259}$$

with

$$F(s,t,u) = \frac{c_{st}}{st} + \frac{c_{tu}}{tu} + \frac{c_{us}}{us}, \tag{10.260}$$

which has only simple poles in (s,t,u) and has a proper mass dimension[12]

Demanding factorization of the s-channel we set

$$t \overset{\substack{s\to 0\\ \downarrow}}{=} -u \tag{10.261}$$

since we have

$$s + t + u = 0. \tag{10.262}$$

[12] Mass dimension is zero:

$$\begin{aligned}
\langle 12\rangle^2[34]^2 &= 4\\
[F] &= -4\\
\rightarrow \left[\langle 12\rangle^2[34]^2 F\right] &= 0 \quad \checkmark
\end{aligned}$$

$P = -(p_1 + p_2) = p_3 + p_4.$

$$\begin{aligned}
\lim_{s\to 0} s\, A(1_a^- 2_b^- 3_c^+ 4_d^+) &= \langle 12\rangle^2 [34]^2 \frac{1}{t}\,(c_{st} - c_{us}) \\
&= \sum_{h=\pm}\sum_e A(1_a^- 2_b^- P_e^h) A(3_c^+ 4_d^+ P_e^{-h}) && (10.263) \\
&= \sum_e A(1_a^- 2_b^- P_e^+) A(3_c^+ 4_d^+ P_e^-) \\
&= \sum_e f_{abe} f_{cde} \frac{\langle 12\rangle^3}{\langle 1P\rangle\langle P2\rangle}\frac{[34]^3}{[3P][P4]} && (10.264) \\
&= \sum_e f_{abe} f_{cde} \langle 12\rangle^2 [34]^2 \frac{1}{t}. && (10.265)
\end{aligned}$$

$[24]\langle 24\rangle = 2p_2 \cdot p_4 = t$
$\langle 1P\rangle[P4] = -\langle 12\rangle[24]$
$\langle P2\rangle[3P] = \langle 42\rangle[34].$

Comparing we get:

$$c_{st} - c_{us} = \sum_e f_{abe} f_{cde} \tag{10.266}$$

We require u- and t-channel factorization:

$$c_{tu} - c_{st} = \sum_e f_{bce} f_{ade} \tag{10.267}$$

$$c_{us} - c_{tu} = \sum_e f_{cae} f_{bde} \tag{10.268}$$

Summing all three equations we find the Jacobi-identity

$$\sum_e f_{abe} f_{cde} + f_{bce} f_{ade} + f_{cae} f_{bde} = 0. \tag{10.269}$$

In Yang-Mills theory this is the result of the Lie-algebra, just

$$\mathrm{Tr}([\mathrm{T_a}, \mathrm{T_b}][\mathrm{T_c}, \mathrm{T_d}]) + \mathrm{Tr}([\mathrm{T_b}, \mathrm{T_c}][\mathrm{T_a}, \mathrm{T_d}]) + \mathrm{Tr}([\mathrm{T_c}, \mathrm{T_a}][\mathrm{T_b}, \mathrm{T_d}]) = 0.$$

Here the Jacobi-identity is a **consistency condition** which arose from the factorization of the 4-particle amplitude.

10.10 Recursion relations

If the entire S-matrix (at tree level) is defined from a finite set of amplitudes this theory is called “on-shell reconstructible”. The main work comes from Britto, Cachazo, Feng and Witten (BCFW).
The on-shell recursion can be considered as a systematic procedure for relating an amplitude to its values at singular kinematics. It is different from algorithmic off-shell Feynman-diagrams since everything is in terms of **on-shell** amplitudes.

10.10.1 Momentum shifts

Here we want to the kinematics mentioned in the previous section by deforming the external momenta in the following way:

$$p_i \to p_i(z) = p_i + z q_i \qquad z \in \mathbb{C}. \tag{10.270}$$

From the shifted momenta we get a deformation of the amplitude

$$A \to A(z). \tag{10.271}$$

The momentum shifts are restricted by two conditions:

1. Total momentum conservation

$$\sum_i p_i(z) = 0 \qquad \longrightarrow \qquad \sum_i q_i = 0 \tag{10.272}$$

2. On-shell conditions

$$p_i(z)^2 = 0 \qquad \overset{\forall z}{\longrightarrow} \qquad q_i^2 = q_i p_i = 0. \tag{10.273}$$

Eq. (10.272) and Eq. (10.273) are generally easy to satisfy.

10.10.2 Factorization from analyticity

We start by the original amplitude $A(0)$ which can be obtained from the residue of $A(z)$ at $z = 0$ using Cauchy's formula

https://en.wikipedia.org/wiki/Holomorphic_function

$$A(0) = \frac{1}{2\pi i} \oint_{z=0} dz\, \frac{A(z)}{z} = -\sum_I \operatorname*{Res}_{z=z_I} \left[\frac{A(z)}{z}\right] + B_\infty \tag{10.274}$$

where we need the Cauchy's theorem to express the residue at $z = 0$ in terms of the residues at all other poles. Furthermore we have used the boundary condition:

$$B_\infty = 0 \quad \text{if} \quad A(z) \to 0\,(z \to \infty). \tag{10.275}$$

Each z_I labels a kinematic singularity and we know that this necessarily coincides with a **physical factorization** channel.

$$P_I = \underbrace{\sum_{i \in I}}_{\substack{\text{subset} \\ \text{of external particles}}} p_i \qquad \to \qquad P_I(z) = \sum_{i \in I} p_i(z). \tag{10.276}$$

The total momentum is:

$$P_I(z) = P_I + z Q_I \tag{10.277}$$

with $Q_I = \sum_{i \in I} q_i$.
Normally we have:

$$P_I^2 \neq 0 \qquad \to \qquad P_I \quad \text{off-shell}. \tag{10.278}$$

The deformed momentum $P_I(z)$ is on-shell if

$$0 = P_I^2(z_I) = P_I^2 + 2 z_I Q_I P_I \qquad \to \qquad z_I = -\frac{P_I^2}{2 Q_I P_I} \tag{10.279}$$

where we have assumed that

$$q_i q_j = 0 \tag{10.280}$$

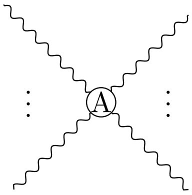
such that $Q_I^2 = 0$ for any subset of legs I.

At the point z_I the amplitude should factorize

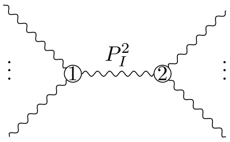
$$\lim_{z\to z_I} P_I^2(z)A(z) = \underset{\substack{\uparrow \\ \text{lower point amplitudes}}}{A_L(z_I)\,A_R(z_I)} \tag{10.281}$$

which implies for the residue:

Here we are considering generic n-particle amplitudes, denoted by the ... in the diagram, and not just the 4 particle to 3 particle factorization. The number of external legs is clearly the same before and after factorization.



$$\operatorname*{Res}_{z=z_I}\left[\frac{A(z)}{z}\right] = \frac{1}{z_I}\left(\frac{dP_I^2(z_I)}{dz_I}\right)^{-1} \cdot \lim_{z\to z_I}\left[P_I^2(z)A(z)\right]$$



$$= -\frac{2Q_I P_I}{P_I^2}\cdot\frac{1}{2z_I Q_I\cdot P_I}A_1(z_I)A_2(z_I).$$

plugging back in Eq. (10.274):

$$A(z=0) = \sum_I \frac{1}{P_I^2}A_L(z_I)A_R(z_I) + B_\infty, \tag{10.282}$$

which is the general formula for on-shell recursion. B_∞ depends on the momentum shift, good choices of momentum shifts satisfy

$$B_\infty = 0 \quad \checkmark. \tag{10.283}$$

10.11 Outlook

A particularly useful complex momentum shift is given by directly shifting spinors i, j:

See exercise!

$$i\rangle \to i\rangle \tag{10.284}$$
$$j\rangle \to j\rangle - z i\rangle \tag{10.285}$$
$$j] \to j] \tag{10.286}$$
$$i] \to i] + z\, j] \tag{10.287}$$

automatically preserves the on-shell conditions and one can show that B_∞ vanishes in gauge theory and gravity! Yang-Mills is on-shell-reconstructible with BCFW gravity.
Just as in Yang-Mills, the tree-level S-matrix can be reduced via on-shell-recursion to primordial 4-point vertex! An infinite tower of interactions in graviton vertices serve no other purpose than to manifest diffeomorphism invariance. With on-shell recursion we can e.g. proof the formula MHV

$$A(1_-\dots i_-\dots j_+\dots k_-) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle k1\rangle}. \tag{10.288}$$

More: color kinematics duality: graviton = gluon^2.

11 Appendix

11.1 Group theory: Basic definitions

Useful reference for group theory: H. Georgi: *Lie Algebras in Particle Physics*.

A group is a set G equipped with one operation $\cdot$ that combines any two elements a, b to form another element $a \cdot b$.
$(G, \cdot)$ must satisfy:

1. closure: $\forall a, b \in G, a \cdot g \in G$
2. associativity: $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (\cdot b \cdot c)$
3. identity: there exists one element $e \in G : \forall a \in G, e \cdot a = a \cdot e = a$
4. inverse: $\forall a \in G$ there exists $b \in G$ such that $a \cdot b = b \cdot a = e$ (b is denoted a^{-1})

A **Lie group** is a group with ∞ number of elements that are also a differentiable manifolds. Any group element continuously connected with $\mathbb{1}$ can be written as

$$U = e^{i\alpha^a T^a} \cdot \mathbb{1} \tag{11.1}$$

where α^a are numbers that parametrize the group elements and T^a are the group generators.
If we know the explicit form of the group element U, we can deduce the form of the T^a by expanding around $\mathbb{1}$.

Let's have a look at an example: orthochronous Lorentz, $SO(1,3)$.
The boost along the x^1-axis is

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \tag{11.2}$$

with $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{11.3}$$

$$t' = \gamma(t + \beta x) \tag{11.4}$$
$$x' = \gamma(\beta t + \gamma x) \tag{11.5}$$
$$y' = y \tag{11.6}$$
$$z' = z \tag{11.7}$$

Expand for small β at $O(\beta)$

$$\Lambda^{\mu}{}_{\nu} = \left(\begin{array}{cc|cc} 1 & \beta & & \\ \beta & 1 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) = \delta^{\mu}{}_{\nu} + \beta \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & & \\ & & & \end{array}\right). \tag{11.8}$$

Now we have

$$U \simeq 1 + i\alpha^a T^a \tag{11.9}$$

So we can identify by comparing Eq. (11.9) and Eq. (11.8):

$$\alpha^a \to \beta \tag{11.10}$$

$$iT^a \to \omega^{\mu}{}_{\nu} = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & & \\ & & & \end{array}\right) \tag{11.11}$$

The generators of T^a form a **Lie algebra**, defined through the commutation relations

$$\left[T^a, T^b\right] = i\underset{\substack{\uparrow \\ \text{structure constants}}}{f^{abc}} T^c \tag{11.12}$$

The group is called

- **Abelian** if $f^{abc} = 0$
- **non-abelian**, otherwise

e.g. SU(2):

$$f^{abc} = \underset{\substack{\uparrow \\ \text{totally antisymmetric with } \epsilon^{123}=1}}{\epsilon^{abc}}. \tag{11.16}$$

$$[A, B] = AB - BA \tag{11.13}$$

$$[A, [B, C]] = [A, BC - CB] \tag{11.14}$$

$$= ABC - ACB - BCA + CBA. \tag{11.15}$$

The Jacobi identity is:

$$\boxed{[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.} \tag{11.17}$$

Now we derive the Jacobi identity for structure constants. We set therefore

$$A \to a \tag{11.18}$$

$$B \to b \tag{11.19}$$

$$C \to c. \tag{11.20}$$

We then write

$$\left[T^a, \left[T^b, T^c\right]\right] = \left[T^a, \left[if^{bcd}, T^d\right]\right] \tag{11.21}$$

$$= if^{bcd}\left[T^a, T^d\right] = if^{bcd} if^{ade} T^e \tag{11.22}$$

$$= -f^{bcd} f^{ade} T^e. \tag{11.23}$$

Then we can write Eq. (11.17) as

$$\left(-f^{bcd}f^{ade} - f^{cad}f^{bde} - f^{abd}f^{cde}\right)T^e = 0. \tag{11.24}$$

So the Jacobi-identity for structure constants follows:

$$f^{bcd}f^{ade} + f^{cad}f^{bde} + f^{abd}f^{cde} = 0. \tag{11.25}$$

The commutator relations completely determine the structure of the group sufficiently close to $\mathbb{1}$; if we go far away the the global aspects matter (e.g. $SU(2)$ and $O(3)$, which have the same commutator relations but differ globally).[1]

An **ideal** is a sub-algebra $I \subset \mathcal{G}$ such that

$$[g, i] \in \mathcal{I} \quad \text{for any} \begin{cases} g \in \mathcal{G} \\ i \in \mathcal{I} \end{cases} \qquad \text{(invariant subalgebra)} \tag{11.29}$$

$\{0\}$ and the whole $\mathcal{G}$ are trivial ideals; an algebra that does not admit a non-trivial ideal is a SIMPLE algrebra.

e.g. $SU(N)$, $SO(N)$

A semi-simple algebra is the direct sum of simple algebras.[2]

Theorem:

All finite-dimensional representations of semisimple algebras are Hermitian.

We can construct theories where the symmetry is a **unitary** transformation on fields.[3] We are also interested in the case where the number of generators are **finite**. This leads to **compact** algebras. [4] The requirement of being **simple** and **compact** is very stringent.

[1] This is **not** relevant for the introductory description of non-Abelian gauge theories.
Note that f^{abc} are antisymmetric in the first 2 indices:

$$\left[T^a, T^b\right] = if^{abc}T^c \tag{11.26}$$

$$-\left[T^b, T^a\right] = -if^{bac}T^c. \tag{11.27}$$

It follows:

$$f^{abc} = -f^{bac}. \tag{11.28}$$

[2] e.g. the Standard Model (SM): $SU(3) \oplus SU(2) \oplus U(1)$.

[3] Unitary theories conserve the probabilities.

[4] because the corresponding Lie group is a finite-dimensional compact manifold.

11.2 Classification of simple and compact groups

11.2.1 Unitary groups $U(N)$

The unitary group $U(N)$ defining representation acts on a space of N-dimensional complex vectors.

$$U = e^{i\alpha T} \tag{11.30}$$

with T Hermitian ($T^\dagger = T$).

$$U^\dagger U = \left(e^{i\alpha T}\right)^\dagger \left(e^{i\alpha T}\right) = e^{-i\alpha T^\dagger}e^{i\alpha T} \tag{11.31}$$

$$= e^{-i\alpha T + i\alpha T} = \mathbb{1}. \tag{11.32}$$

A complex inner product is preserved: take ψ, χ states, then

$$[U\psi]^\dagger U\chi = \psi^\dagger \underbrace{U^\dagger U}_{\mathbb{1}}\chi = \psi^\dagger\chi. \tag{11.33}$$

Now $U(N)$ contains the **pure phase** transformations

$$U = e^{i\alpha}, \tag{11.34}$$

which form a $U(1)$ subgroup. They are removed by requiring

$$\det U = 1 \tag{11.35}$$

rather than a generic complex phase with $|\cdot| = 1$, which gives the **simple group** $SU(N)$, whose dimension is

$$d(SU(N)) = N^2 - 1. \tag{11.36}$$

$$\begin{array}{ccc} N \times N \text{ complex matrix} & \to & 2N^2 \\ U^\dagger U = \mathbb{1} & \to & N^2 \text{ conditions} \\ \det U = 1 & \to & 1 \text{ condition} \\ & & \Rightarrow 2N^2 - N^2 - 1 = N^2 - 1 \end{array}$$

11.2.2 Orthogonal groups

Orthogonal groups preserve a real inner product.
ψ, χ are vectors

$$O\psi \cdot O\chi = \psi^T \underbrace{O^T O}_{\mathbb{1}} \chi = \psi^T \chi. \tag{11.37}$$

with

$$O^T O = \mathbb{1}. \tag{11.38}$$

The number of generators: $N \times N$ matrix satisfying

$$O^T O = \mathbb{1}. \tag{11.39}$$

So we have

$$N + \frac{N^2 - N}{2} \tag{11.40}$$

independent conditions.
The dimension of the orthogonal group is

$$d(O(N)) = N^2 - \left[N + \frac{N^2 - N}{2} \right] \tag{11.41}$$

$$= \frac{N(N-1)}{2}. \tag{11.42}$$

Now we set

$$\det O = \pm 1 \quad \text{take } +1 \quad \text{to get } SO(N), \tag{11.43}$$

which is the rotation group in N dimensions. It is clear that

$$d(SO(N)) = \frac{N(N-1)}{2} : \tag{11.44}$$

Number of planes in N dimensions.

11.2.3 Symplectic groups $Sp(N)$

The symplectic group is a subgroup of the unitary $N \times N$ transformations (N is even) preserving an antisymmetric inner product

$$\psi^T \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}}_{J} \chi \tag{11.45}$$

where J is in $\left(\frac{N}{2} \times \frac{N}{2}\right)$ block form.
The number of conditions from

$$S^T J S = J \tag{11.46}$$

is

$$\frac{N^2 - N}{2} \tag{11.47}$$

since

$$(S^T J S)^T = S^T J^T S \tag{11.48}$$
$$= -S^T J S \qquad \text{(antisymmetry)} \tag{11.49}$$

So we can calculate the dimension of the symplectic group

$$d(Sp(N)) = N^2 - \left[\frac{N^2 - N}{2}\right] \tag{11.50}$$
$$= \frac{N(N+1)}{2} \tag{11.51}$$

11.2.4 Exceptional groups: G_2, F_4, E_6, E_7, E_8

The subscript denotes the rank of the group. It is the dimension of the Cartan subgroup, which is the maximal commuting subgroup; This completes the classification of compact simple Lie algebras.

Note that if there is any Hermitian representation (as for semisimple algebras) then the structure constants are real.

$$\left[T^a, T^b\right] = i f^{abc} T^c. \tag{11.52}$$
$$\underbrace{\left[T^a, T^b\right]^\dagger}_{(T^aT^b - T^bT^a)^\dagger} = -i f^{abc\star} T^c \tag{11.53}$$
$$\left[T^b, T^a\right] = i f^{bac} T^c \tag{11.54}$$
$$= -i f^{abc} T^c. \tag{11.55}$$

where we used the fact that f^{abc} is antisymmetric in the first 2 indices. Hence we have obtained

$$f^{abc} = f^{abc\star}. \tag{11.56}$$

11.3 General properties of representations

Representations of an algebra are constructed by embedding the generators into matrices. One can show that for compact semi-simple Lie algebras, for any representation R we can choose a basis for the generators such that

$$\mathrm{Tr}\left(\mathrm{T}_{\mathrm{R}}^{\mathrm{a}}\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\right)=\underset{\substack{\uparrow \\ \text{constant for each } R}}{\mathrm{T}(\mathrm{R})}\delta^{\mathrm{ab}} \tag{11.57}$$

now combined with

$$\left[T_R^a,T_R^b\right]=if^{abc}T_R^c. \tag{11.58}$$

This implies

$$\left[T_R^a,T_R^b\right]T_R^c=if^{abd}T_R^dT_R^c. \tag{11.59}$$

We take the trace:

$$\mathrm{Tr}\left[\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{a}},\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\right],\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\right]=if^{abd}\mathrm{Tr}\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{d}}\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\right] \tag{11.60}$$

$$=if^{abd}T(R)\delta^{dc}=iT(R)f^{abc}. \tag{11.61}$$

It follows

$$f^{abc}=-\frac{i}{T(R)}\mathrm{Tr}\left[\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{a}},\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\right]\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\right]. \tag{11.62}$$

Now this implies that

$$f^{abc}=f^{bca} \tag{11.63}$$

because

$$\mathrm{Tr}\left[\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{a}},\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\right]\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\right]=\mathrm{Tr}\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{a}}\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}-\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\mathrm{T}_{\mathrm{R}}^{\mathrm{a}}\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\right] \tag{11.64}$$

$$=\mathrm{Tr}\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\mathrm{T}_{\mathrm{R}}^{\mathrm{a}}-\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\mathrm{T}_{\mathrm{R}}^{\mathrm{b}}\mathrm{T}_{\mathrm{R}}^{\mathrm{a}}\right] \tag{11.65}$$

$$=\mathrm{Tr}\left[\left[\mathrm{T}_{\mathrm{R}}^{\mathrm{b}},\mathrm{T}_{\mathrm{R}}^{\mathrm{c}}\right]\mathrm{T}_{\mathrm{R}}^{\mathrm{a}}\right], \tag{11.66}$$

where we used the cyclicity of the trace.
Recall that

$$f^{abc}=-f^{bac} \tag{11.67}$$

It follows that

$$f^{abc} \text{ is completely antisymmetric.} \tag{11.68}$$

Now for a given R, an infinitesimal transform under the group is

$$\phi\to(1+i\alpha^aT_R^a)\phi \tag{11.69}$$

$$\phi^\star\to(1-i\alpha^aT_R^{a\star})\phi^\star, \tag{11.70}$$

but also we can define the **conjugate representation**:

$$\phi^\star\to(1+i\alpha^aT_{\bar{R}}^a)\phi^\star \tag{11.71}$$

Hence

$$T_{\bar{R}}^a=-(T_R^a)^\star=-(T_R^a)^T. \tag{11.72}$$

Now $\bar{R}$ is **equivalent** to R, if there exists a unitary transform U such that

$$T^a_{\bar{R}} = U T^a_R U^\dagger. \tag{11.73}$$

In this case, R is a **real** representation.
Then we can always find a matrix E_{ab} such that if $\psi, \xi \in \mathcal{R}$ then $\psi_a E_{ab} \chi_b$ is **invariant**.

$$E_{ab} \begin{cases} \text{symmetric} \to \text{strictly real} \\ \text{antisymmetric} \to \text{pseudoreal} \end{cases}$$

11.4 Representation of SU(N) groups

A free theory of N complex massless fields is automatically invariant under

$$U(N) \simeq SU(N) \otimes U(1). \tag{11.74}$$

The two most important representations are: **fundamental** and **adjoint**.

11.4.1 Fundamental representation

The fundamental representation acts on a space of $N-$dim complex vectors. The smallest non-trivial representation of the algebra. Dimension is N. For $SU(N)$, it is a set of $N \times N$ Hermitian, **traceless** matrices.

Why traceless?

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{T}}. \tag{11.75}$$

Diagonalize

$$V\vec{\alpha}\cdot\vec{T}V^{-1} = D, \tag{11.76}$$

then we get

$$\det U \overset{\substack{\text{det is basis-independent.} \\ \downarrow}}{=} \det e^{iD} = \Pi_j e^{i[D]_{jj}} = e^{i\sum_j [D]_{jj}} \tag{11.77}$$

$$= e^{i\mathrm{Tr\,D}} \tag{11.78}$$

$$\underset{\substack{\uparrow \\ \text{Tr is basis-independent}}}{=} e^{i\mathrm{Tr}(\vec{\alpha}\cdot\vec{\mathrm{T}})} \tag{11.79}$$

hence if

$$\mathrm{Tr\,T^a} = 0 \tag{11.80}$$

then

$$\det U = 1. \tag{11.81}$$

Note that the tracelessness argument applies for **any** semisimple algebra.

For $SU(2)$:

$$T^a = \frac{\sigma^a}{2} \tag{11.82}$$

with σ^a as the Pauli-matrices. They satisfy

$$\left[T^a, T^b\right] = \frac{1}{4}[\sigma^a, \sigma^b] \tag{11.83}$$

$$= \frac{1}{4} 2i\epsilon^{abc}\sigma^c = i\underset{\substack{\uparrow \\ \text{structure constants}}}{\epsilon^{abc}} T^c. \tag{11.84}$$

For $SU(3)$:

$$T^a = \frac{\lambda}{2} \tag{11.85}$$

with λ^a are the Gell-Mann matrices.
Now for the 2 of $SU(2)$ we have

$$T^a_R = \frac{\sigma^a}{2} \quad \text{and} \quad T^a_{\bar{R}} = -\frac{\sigma^{a\star}}{2} \tag{11.86}$$

but we know that

$$\sigma^2 \sigma^a \sigma^2 = -\sigma^{a\star} \tag{11.87}$$

hence

$$T^a_{\bar{R}} = -\frac{\sigma^{a\star}}{2} = \sigma^2 \left(\frac{\sigma^a}{2}\right) \sigma^2 = \sigma^2 T^a_R \sigma^2 \tag{11.88}$$

i.e. real representation with $U = \sigma^2$, also

$$E_{ab} = (i\sigma^2)_{ab} = \epsilon_{ab} \tag{11.89}$$

so it is **pseudoreal**.
For $N > 2$, the N of $SU(N)$ is instead complex (so in particular: $3 \neq \bar{3}$ for $SU(3)$).

11.4.2 Adjoint representation

The adjoint representation acts on the space spanned by the generators themselves

$$\left(T^a_{adj}\right)_{bc} = q f^{abc} \tag{11.90}$$

where q is purely imaginary.
The dimension is $N^2 - 1$ (number of generators), e.g. for $SU(2)$:

Find q such that T^a_{adj} satisfy the commutator relations of the algebra.

$$(T^a_{adj})_{bc} = q\epsilon^{abc} \to T^1_{adj} = q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{etc.} \tag{11.91}$$

The adjoint representation is important because it is the representation where non-Abelian gauge fields transform.

The conjugate representation of the adjoint is

$$T^a_{\overline{adj}} = -\left(T^a_{adj}\right) = -\left(q f^{abc}\right)^\star = +q f^{abc} \underset{\substack{\uparrow \\ q \in Im \\ f^{abc} \text{ real}}}{=} T^a_{adj} \tag{11.92}$$

hence the adjoint is always **real**.

11.5 Casimir operator

For any representation T^a_R, we know that

$$T^2 \equiv T^a_R T^a_R \quad \text{(sum ovsser a)} \tag{11.93}$$

commutes with all the generators:

$$\begin{aligned} \left[T^2, T^b_R\right] &= \left[T^a_R T^a_R, T^b_R\right] && (11.94) \\ &= T^a_R \left[T^a_R, T^b_R\right] + \left[T^a_R, T^b_R\right] T^a_R && (11.95) \\ &= T^a_R i f^{abc} T^c_R + i f^{abc} T^c_R T^a_R && (11.96) \\ &= \underset{\substack{\uparrow \\ \text{antisym.}}}{i f^{abc}} \underset{\substack{\uparrow \\ \text{sym.}}}{\{T^a_R, T^c_R\}} = 0. && (11.97) \end{aligned}$$

Hence T^2 must be proportional to the identity by Schur's lemma. So

$$T^a_R T^a_R = C_2(R) \cdot \mathbb{1} \tag{11.98}$$

$C_2(R)$ is the quadratic Casimir of R. As we discussed we also have

Familiar example: $SU(2)$:

$$J^2 = T^a_R T^a_R \tag{11.99}$$

is the Casimir with eigenvalues $j(j+1)$ (total spin).

for an appropriate choice of the basis

$$\mathrm{Tr}\left(\mathrm{T_R^a T_R^a}\right) = \mathrm{T(R)}\delta^{\mathrm{ab}} \tag{11.100}$$

with $T(R)$ as the index of R (sometimes called $C(R)$). Now we contract Eq. (11.100) with δ^{ab}:

$$\underset{\substack{\uparrow \\ C_2(R)d(R)}}{\mathrm{Tr}\left(\mathrm{T_R^a T_R^b}\right)} = T(R)d(G) \tag{11.101}$$

which leads to

$$d(R)C_2(R) = T(R)d(G), \tag{11.102}$$

which is useful to compute $C_2(R)$.

11.5.1 Fundamental of $SU(N)$

It is typical to choose

$$T(\mathrm{fund}) = \frac{1}{2} \equiv T_F \tag{11.103}$$

as in the case of $SU(2)$.

$$C_2(\mathrm{fund}) = \frac{T(\mathrm{fund})d(G)}{d(\mathrm{fund})} = \frac{1}{2}\frac{N^2-1}{N} \equiv C_F \tag{11.104}$$

11.5.2 Adjoint of $SU(N)$

First note that

$$d(R) = d(G), \tag{11.105}$$

so

$$C_2(\mathrm{adj}) = T(\mathrm{adj}). \tag{11.106}$$

How do we compute $C_2(\mathrm{adj})$?

It is instructive to do it by building the adjoint as a product of fundamental and antifundamental:
given 2 representations r_1 and r_2, their **direct product** is a representation of the dimension

$$d(r_1) \cdot d(r_2). \tag{11.107}$$

Objects transforming under it are **tensors** ξ_{pq} with $p \in r_1$ and $q \in r_2$. The product representation can in general be decomposed as a direct sum of irrepresentations:

$$r_1 \times r_2 = \sum_i r_i. \tag{11.108}$$

The matrices are

$$t^a_{r_1 \times r_2} = t^a_{r_1} \otimes \mathbb{1} + \mathbb{1} \otimes t^a_{r_2}. \tag{11.109}$$

Now the quadratic Casimir of the product is:

$$t^a_{r_1\times r_2} t^a_{r_1\times r_2} = (t^a_{r_1})^2 \otimes \mathbb{1} + \mathbb{1} \otimes (t^a_{r_2})^2 + 2t^a_{r_1} \otimes t^a_{r_2}. \tag{11.110}$$

Take the trace of Eq. (11.110):

$$\begin{aligned}
\mathrm{Tr}\left({t^a_{r_1\times r_2}}^2\right) &= \mathrm{Tr}({t^a_{r_1}}^2)\mathrm{d}(r_2) + \mathrm{d}(r_1)({t^a_{r_2}}^2) + \underset{\substack{\uparrow \\ t^a_{r_i} \\ \text{are traceless.}}}{0} && (11.111)\\
&= C_2(r_1)d(r_1)d(r_2) + d(r_1)C_2(r_2)d(r_2) && (11.112)\\
&= \left[C_2(r_1) + C_2(r_2)\right] d(r_1)d(r_2). && (11.113)
\end{aligned}$$

From Eq. (11.108) we get

$$\mathrm{Tr}\left({t^a_{r_1\times r_2}}^2\right) \underset{\substack{\uparrow \\ \text{block-diagonal form}}}{=} \sum_i \mathrm{Tr}\left(t^2_{r_i}\right) = \sum_i C_2(r_i)d(r_1) \tag{11.114}$$

Now we apply to the case:

$$r_1 = N, r_2 = \bar{N} \quad \text{in } SU(N) \tag{11.115}$$

$$N \times \bar{N} = \overset{\substack{\xi_{pq}=\delta_{pq} \\ \downarrow}}{1} + \underset{\substack{\uparrow \\ \text{remaining pieces: } N\times N \text{ traceless tensor} \\ \to \text{Adjoint representation}}}{(N^2-1)} \tag{11.116}$$

From Eq. (11.113) = Eq. (11.114) we get

$$\left[C_2(N) + C_2(\bar{N})\right] N^2 = C_2(\mathrm{adj})d(\mathrm{adj}). \tag{11.118}$$

For the singlet representation we have

$$T^a_1 = 0 \quad \to C_2(1) = 0. \tag{11.117}$$

We find

$$C_2(\mathrm{adj}) = 2N^2 \frac{N^2-1}{2N} \frac{1}{N^2-1} = N \equiv C_A, \tag{11.119}$$

where we used $2C_2(N)N^2 = C_2(\mathrm{adj})(N^2-1)$.
Hence we can state

$$\begin{aligned}
C_2(\mathrm{adj}) &= C_A = N && (11.120)\\
T(\mathrm{adj}) &= T_A = N. && (11.121)
\end{aligned}$$

So we have seen for the **fundamental** of $SU(N)$ that

$$\mathrm{Tr}\left(t^a t^b\right) = T_F \delta^{ab} \qquad T_F = \frac{1}{2} \tag{11.122}$$

$$\sum_a \left(t^a t^a\right)_{ij} = C_F \delta_{ij} \qquad C_F = \frac{N^2-1}{2N}. \tag{11.123}$$

Another useful relation is

$$\sum_a \left(t^a\right)_{ij} \left(t^a\right)_{kl} = T_F \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right). \tag{11.124}$$

To prove Eq. (11.124) we have to begin with the fact that for any $N \times N$ complex matrix M we can write

$$M = \underbrace{M_0}_{\text{complex}}\mathbb{1} + \underbrace{M^a}_{\text{coefficients}}\overbrace{t^a}^{\text{traceless}}. \tag{11.125}$$

Then we calculate

$$\mathrm{Tr(M) = M_0 N} \tag{11.126}$$

$$\mathrm{Tr(Mt^b) = Tr\left((M_0\mathbb{1} + M_a^a t^a)\, t^b\right) \overset{t^b \text{ is traceless}}{=} M_a \underbrace{Tr\left(t^a t^b\right)}_{=T_F\delta^{ab}} = T_F M^b.} \tag{11.127}$$

It follows

$$M_0 = \frac{1}{N}\mathrm{Tr(M)} \tag{11.128}$$

$$M^b = \frac{1}{T_F}\mathrm{Tr(Mt^b)}, \tag{11.129}$$

which we plug back into Eq. (11.125).

$$M = \frac{1}{N}\mathrm{Tr(M)}\mathbb{1} + \frac{1}{\mathrm{T_F}}\mathrm{Tr(Mt^a)t^a} \tag{11.130}$$

$$M_{ij} = \frac{1}{N}M_{kk}\delta_{ij} + \frac{1}{T_F}M_{lk}(t^a)_{kl}(t^a)_{ij}. \tag{11.131}$$

We can now write

$$\delta_{il}\delta_{jk}M_{lk} = \frac{1}{N}M_{lk}\delta_{lk}\delta_{ij} + \frac{1}{T_F}M_{lk}(t^a)_{kl}(t^a)_{ij}. \tag{11.132}$$

Since M is an arbitrary matrix, the coefficient of M_{lk} must vanish.

$$\delta_{il}\delta_{jk} = \frac{1}{N}\delta_{ij}\delta_{kl} + \frac{1}{T_F}(t^a)_{ij}(t^a)_{kl}, \tag{11.133}$$

which is what we wanted.

Another useful result is

$$\boxed{\sum_{a,b} f^{abc}f^{abd} = C_A\delta^{cd},} \tag{11.134}$$

which is just the definition of the quadratic Casimir for the adjoint:

$$(T^aT^a)_{cd} = C_2(\mathrm{adj})\delta_{cd}, \tag{11.135}$$

but now we saw that

$$(T^a)_{bc} = qf^{abc} \qquad (q \in Im) \tag{11.136}$$

$$= (T^a)_{cb}(T^a)_{bd} \tag{11.137}$$

$$= qf^{acb}qf^{abd} \tag{11.138}$$

$$= -f^{acb}f^{abd} = +f^{abc}f^{abd} \tag{11.139}$$

These results will be useful for calculations in non-abelian gauge theories.