MA2102: LINEAR ALGEBRA

Lecture 31: Orthogonality

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Indian Institute of Science Education & Research Kolkata

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Recall that the length of $v \in V$ is given by $||v|| = \sqrt{\langle v, v \rangle}$. Moreover, we have $||v|| = 0$ if and only if $v = 0$ as well as

$$||cv|| = |c| \, ||v||.$$

The following summarizes some of the important properties.

Proposition Let V be an inner product space. Then

(i) [Cauchy-Schwarz inequality] $|\langle v, w \rangle| \leq ||v|| \, ||w||$

(ii) [Triangle inequality] $||v + w|| \leq ||v|| + ||w||$

(iii) [Parallelogram law] $||v - w||^2 + ||v + w||^2 = 2||v||^2 + 2||w||^2$

The proof is left as an exercise. Note that equality holds in (i) if and only if $v = cw$ for some scalar c, and in (ii) if and only if $v = cw$ for some non-negative real scalar c (exercise).

Remark Cauchy-Schwarz inequality allows us to define the angle between two vectors for real vector spaces. Since

$$-||v||\,||w|| \leq \langle v, w\rangle \leq ||v||\,||w||$$

we may write $\langle v, w\rangle = ||v||\,||w||\cos\theta$ for a unique angle $\theta \in [0, \pi]$. This θ is defined to be the angle between v and w.

Definition [Orthogonal] Let V be an inner product space. We say $v, w \in V$ are orthogonal to each other if $\langle v, w\rangle = 0$.

Sometimes we say v is perpendicular to w if $\langle v, w\rangle = 0$. We may also write $v \perp w$ to indicate the same. Being orthogonal is a symmetric relation but neither reflexive nor transitive. The standard basis of $\mathbb{R}^n$ are mutually orthogonal in the standard inner product.

Examples (1) Consider $P_3(\mathbb{R})$ with two inner products:

$$\langle p,q\rangle = \int_0^1 p(t)q(t)\,dt, \quad \langle p,q\rangle' = \int_{-1}^1 p(t)q(t)\,dt.$$

Note that

$$\left\langle 1, x-\tfrac{1}{2}\right\rangle = \int_0^1 1\cdot(t-\tfrac{1}{2})\,dt = \tfrac{t^2}{2}\Big|_0^1 - \tfrac{1}{2} = 0.$$

On the other hand

$$\left\langle 1, x-\tfrac{1}{2}\right\rangle' = \int_{-1}^1 1\cdot(t-\tfrac{1}{2})\,dt = \tfrac{t^2}{2}\Big|_{-1}^1 - 1 = -1.$$

Thus 1 and $x-\frac{1}{2}$ are not orthogonal to each other in $\langle\cdot,\cdot\rangle'$. Moreover, 1 has length $\sqrt{2}$ while $x-\frac{1}{2}$ has length $\sqrt{7/6}$ with respect to $\langle\cdot,\cdot\rangle'$.

(2) Consider $\mathbb{R}^2$ with

$$\langle (x_1, y_1), (x_2, y_2) \rangle := x_1 x_2 - x_1 y_2 - x_2 y_1 + 2 y_1 y_2.$$

Show that this defines an inner product. Observe that

$$\langle (1,1), (1,1) \rangle = 1, \quad \langle (1,0), (1,0) \rangle = 1$$

while $(1,1)$ and $(1,0)$ are orthogonal to each other.

If $\langle v, w \rangle = 0$, then $\langle w, v \rangle = 0$ and

$$||v + w||^2 = \langle v + w, v + w \rangle = ||v||^2 + ||w||^2.$$

We refer to this as the *Pythagoras Theorem*.

Proposition If $v_1, \ldots, v_k$ are mutually orthogonal, then

$$||v_1 + \cdots + v_k||^2 = ||v_1||^2 + \cdots + ||v_k||^2.$$

Proof.

We prove this by induction, the case $k = 1$ is vacuous and $k = 2$ was just proved. Assume that the generalized Pythagoras Theorem holds for $k-1$ vectors. As $\langle v_k, v_j \rangle = 0$ for $j = 1, 2, \ldots, k-1$, we conclude that

$$\left\langle v_k, v_1 + \cdots + v_{k-1} \right\rangle = \langle v_k, v_1 \rangle + \cdots + \left\langle v_k, v_{k-1} \right\rangle = 0.$$

By Pythagoras Theorem and induction hypothesis, we have

$$\begin{aligned} ||v_k + (v_1 + \cdots + v_{k-1})||^2 &= ||v_k||^2 + ||v_1 + \cdots + v_{k-1}||^2 \\ &= ||v_k||^2 + ||v_1||^2 + \cdots + ||v_{k-1}||^2. \end{aligned}$$

Rearranging the terms, we are done. $\square$

Definition [Orthonormal set] A collection $\{v_1, \ldots, v_k\} \subset V$ is called orthonormal if $\langle v_i, v_j \rangle = \delta_{ij}$.

Lemma If V is an inner product space of dimension n, then any orthonormal set has at most n elements. Moreover, any orthonormal set $\{v_1, \ldots, v_n\}$ is a basis.

Proof.

Let $S = \{v_1, \ldots, v_k\}$ be an orthonormal set. If $c_1 v_1 + \cdots + c_k v_k = 0$, then taking inner product with v_j we obtain

$$0 = c_1 \langle v_1, v_j \rangle + \cdots + c_k \langle v_k, v_j \rangle = c_j.$$

Thus, S is linearly independent, whence $k \leq n$. The last claim follows from Replacement Theorem. $\square$

Definition [Orthonormal basis] A basis β of V is called an orthonormal basis if it is an orthonormal set and a basis.

The standard basis of $\left(\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\text{std}}\right)$ is an orthonormal basis.

Revisit example (2) and conclude that $\{(1,1),(1,0)\}$ is an orthonormal basis. With respect to $\left(\mathbb{R}^2, \langle\cdot,\cdot\rangle_{\text{std}}\right)$,

$$\left\{\left(\tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}}\right), \left(\tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}}\right)\right\}$$

is an orthonormal basis. In example (1), the polynomials 1 and $\sqrt{12}(x-\frac{1}{2})$ form an orthonormal basis of $\left(P_3(\mathbb{R}), \langle\cdot,\cdot\rangle\right)$.

Observation If $\{v_1, \dots, v_n\}$ is an orthonormal basis of V, then for any $v \in V$

$$v = \langle v, v_1\rangle v_1 + \cdots + \langle v, v_n\rangle v_n.$$

Proof.

Write $v = c_1 v_1 + \cdots + c_n v_n$. Take inner product with v_j to obtain $\langle v, v_j\rangle = c_j$. $\square$