# MA2102: Linear Algebra

## Lecture 36: Isometry

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Recall the following characterization of real self-adjoint maps (cf. lecture 35).

**Theorem**
*Let $V$ be a finite dimensional real vector space. If $T : V \to V$ is a linear map, then $T$ is self-adjoint if and only if there exists an orthonormal eigenbasis.*

We hade seen that a self-adjoint map $T : V \to V$ has real eigenvalues even when $V$ is a complex vector space. There is a partial characterization of self-adjoint maps.

**Theorem** [Spectral Theorem for self-adjoint operators]
*Let $V$ be a finite dimensional complex inner product space. If $T : V \to V$ is a self-adjoint map, then there exists an orthonormal eigenbasis.*

The converse is not true.

**Examples** (1) Consider the scaling map $T = \lambda I_v : V \to V$. Any basis of $V$ is an eigenbasis as $E_\lambda = V$. For any choice of inner product on $V$, consider an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$. It follows that $\beta$ is an orthonormal eigenbasis. As $T^* = \overline{\lambda} I_V$, $T$ is self-adjoint if and only if $\lambda$ is real (cf. lecture 35). Note that $TT^* = T^*T$.

(2) Fix $\theta \in [0, 2\pi)$ and consider the linear map

$$T : \mathbb{C}^2 \to \mathbb{C}^2,\ \ (z_1, z_2) \mapsto (z_1 \cos\theta + z_2 \sin\theta, -z_1 \sin\theta + z_2 \cos\theta).$$

Here $\mathbb{C}^2$ is equipped with the standard inner product. With respect to the standard orthonormal basis $\beta = \{(1,0),(0,1)\}$, the matrix of $T$ looks like

$$[T]_\beta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which is not Hermitian if $\theta \neq 0, \pi$. Thus, $T$ is not self-adjoint.

Note that $e^{i\theta}$ and $e^{-i\theta}$ are eigenvalues of $T$. Show that $(i, 1)$ and $(1, -i)$ are eigenvectors of $T$. In fact, these eigenvectors form an orthogonal basis. Finally, note that $TT^* = T^*T = I_2$.

Seeking for the right class of operators which can be characterized as those admitting an orthonormal eigenbasis, we are led to the notion of normal operators.

**Definition** [Normal] A linear map $T : V \to V$ is called normal if $TT^* = T^*T$.

There is the following characterization.

**Theorem** [Spectral Theorem for normal operators]
*Let $V$ be a finite dimensional complex inner product space. If $T : V \to V$ is a linear map, then $T$ is normal if and only if there exists an orthonormal eigenbasis.*

**Remark** Normal operators are well studied. Normal operators contain many important classes of operators:

Self-adjoint operators : $T^* = T$

Skew-Hermitian operators : $T^* = -T$

Positive operators : $T = SS^*$ for some operator $S$

Unitary operators : $TT^* = I_V = T^*T$

In finite dimensions, for unitary operators $T$, the condition $T^*T = I_V$ guarantees $TT^* = I_V$. This is not always true in infinite dimensions.

Motivated by geometry, we study *distance preserving* maps.

**Definition** [Isometry] The distance between vectors $v$ and $w$ is defined to be $||v - w||$.

A linear map $T : V \to V$ is called an isometry if it is distance preserving, i.e.,

$$||T(v) - T(w)|| = ||v - w||. \tag{1}$$

**Remark** There are non-linear distance preserving maps. For instance, translations in $\mathbb{R}^n$ by a fixed vector are non-linear but preserves distance.

Observe that $T$ is an isometry if and only if $||T(v)|| = ||v||$ for any $v \in V$. Expanding both sides of

$$\langle T(v+w), T(v+w)\rangle = \langle v+w, v+w\rangle$$

and using conjugate symmetry, we obtain

$$\mathrm{Re}\langle T(v), T(w)\rangle = \mathrm{Re}\langle v, w\rangle.$$

Using $v + iw$ instead of $v + w$, show that $\mathrm{Im}\langle T(v), T(w)\rangle = \mathrm{Im}\langle v, w\rangle$. Thus, an isometry satisfies

$$\langle T(v), T(w)\rangle = \langle v, w\rangle. \tag{2}$$

It is clear that any $T$ satisfying (2) satisfies (1).

By the definition of adjoint, we have

$$\langle v, w\rangle = \langle T(v), T(w)\rangle = \langle v, T^* T(w)\rangle.$$

As $T^* T(w) - w$ is a vector orthogonal to all of $V$, it must be zero, i.e., $T^* T = I_V$. Thus, an isometry can also be defined as a map satisfying

$$T T^* = I_V. \tag{3}$$

**Remark** For finite dimensional vector spaces, this condition is equivalent to $T T^* = I_V$. Thus, the notion of isometries and unitary operators coincide for finite dimensional inner product spaces.

**Examples** (1) If $T : \mathbb{C}^n \to \mathbb{C}^n$ is an isometry with respect to the standard inner product, then for any orthonormal basis $\beta$ we have

$$[T]_\beta [T^*]_\beta = I_n = [T^*]_\beta [T]_\beta$$

As $[T^*]_\beta = [T]_\beta^*$, we see that $[T]_\beta$ is an unitary matrix.

Conversely, if $[T]_\beta$ is an unitary matrix and $\beta$ is an orthonormal basis, then

$$[TT^*]_\beta = [T]_\beta [T^*]_\beta = [T]_\beta [T]_\beta^* = I_n$$

implies that $TT^* = I_V$. Similarly, $T^*T = I_V$ and $T$ is an isometry/unitary map.

(2) The same setup as in (1), except $T : \mathbb{R}^n \to \mathbb{R}^n$. In this case $[T]_\beta$ is an orthogonal matrix, i.e.,

$$[T]_\beta [T]_\beta^t = I_n = [T]_\beta^t [T]_\beta$$

if and only if $T$ is unitary/isometry.

(3) Let $P \in M_n(\mathbb{C})$ be a unitary matrix, i.e., $P^*P = I_n$. Consider the conjugation

$$\mathrm{Ad}_P : M_n(\mathbb{C}) \to M_n(\mathbb{C}), \ \ A \mapsto PAP^*.$$

Note that

$$\begin{aligned}
\langle \mathrm{Ad}_P(A), \mathrm{Ad}_P(B) \rangle &= \operatorname{trace}\big((\mathrm{Ad}_P(B))^* \mathrm{Ad}_P(A)\big) \\
&= \operatorname{trace}(PB^*P^*PAP^*) \\
&= \operatorname{trace}(PB^*AP^*) \\
&= \operatorname{trace}(B^*AP^*P) \\
&= \langle A, B \rangle.
\end{aligned}$$

Thus, $\mathrm{Ad}_P$ is an isometry/unitary.