MA2102: LINEAR ALGEBRA

Lecture 11: Null Space

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We continue with our list of example of linear maps.

(10) Differentiation: Consider the derivative as a map

$$D : P_n(\mathbb{R}) \to P_n(\mathbb{R}), \;\; D(p) = p'.$$

Note that

$$D(p(x)+q(x)) = (p(x)+q(x))' = p'(x)+q'(x) = D(p(x))+D(q(x)).$$

Similarly, $D(cp(x)) = (cp(x))' = cp'(x) = cD(p(x))$, implies that D is a linear map.

Remark The codomain can be $P_{n-1}(\mathbb{R})$ as differentiation lowers the degree by one, provided we set $P_{-1}(\mathbb{R}) = \{0\}$.

Show that if $T_1, T_2 : V \to W$ are linear maps, then $T_1 + cT_2 : V \to W$ is a linear map for any scalar c.

(11) Integration: Let $C(\mathbb{R})$ denote the set of all continuous real valued functions on $\mathbb{R}$. We know from analysis that it is a vector space over $\mathbb{R}$. Since $P(\mathbb{R}) \subset C(\mathbb{R})$, it follows that $C(\mathbb{R})$ is infinite dimensional. Consider integration over $[0,1]$ as a map

$$\mathscr{I} : C(\mathbb{R}) \to \mathbb{R}, \quad \mathscr{I}(f) = \int_0^1 f(t)\,dt.$$

Note that

$$\mathscr{I}(f+g) = \int_0^1 (f(t)+g(t))\,dt = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = \mathscr{I}(f) + \mathscr{I}(g).$$

Similarly,

$$\mathscr{I}(cf) = \int_0^1 cf(t)\,dt = c\int_0^1 f(t)\,dt = c\mathscr{I}(f)$$

implies that $\mathscr{I}$ is a linear map.

Consider the integration map

$$\mathscr{I} : P(\mathbb{R}) \to P(\mathbb{R}),\ \ p(x) \mapsto \int_0^x p(t)\,dt.$$

Show that this is a linear map. We see that $T(x^k) = \frac{x^{k+1}}{k+1}$. As an example, consider the map

$$T : P_2(\mathbb{R}) \to P_3(\mathbb{R}),\ \ p \mapsto 2p' + 3\int_0^x p(t)\,dt.$$

This is a well-defined linear map.

(12) Linear Combination: Given vectors $v_1, \ldots, v_n \in V$, where V is a vector space over $\mathbb{R}$, consider the map

$$T : \mathbb{R}^n \to V,\ \ T(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

This is a linear map and is an injective map if $S = \{v_1, \dots, v_n\}$ is linearly independent (exercise). Moreover, if S spans V, then any $v = c_1 v_1 + \cdots + c_n v_n$, whence $T(c_1, \dots, c_n) = v$ and T is surjective.

(13) Quotient: Given a subspace W of a vector space V, we consider the quotient map

$$Q : V \to V/W, \quad Q(v) := [v] = v + W.$$

Note that

$$Q(v_1 + v_2) = [v_1 + v_2] = [v_1] + [v_2] = Q(v_1) + Q(v_2).$$

Similarly, $Q(cv) = [cv] = c[v] = c\,Q(v)$ and we conclude that Q is a linear map. By construction, Q is a surjective map.

There are natural subspaces associated to any linear map.

Definition [Null Space and Range] Let $T : V \to W$ be a linear map between vector spaces (over a field F). The set

$$N(T) := \{v \in V \mid T(v) = 0_W\}$$

is called the *null space* of T. The set

$$R(T) := \{T(v) \in W \mid v \in V\}$$

is called the *range* of T.

We may think of $N(T)$ as the amount of information lost and $R(T)$ as the amount of information retained, if T is transfer of information. Use linearity of T to conclude that $R(T)$ is a subspace of W.

If $v_1, v_2 \in N(T)$, then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0_W + 0_W = 0_W$$

implies that $v_1 + v_2 \in N(T)$. Moreover, $T(cv_1) = cT(v_1) = 0_W$. Thus, $N(T)$ is a subspace of V.

Example Consider the differentiation map

$$D : P(\mathbb{R}) \to P(\mathbb{R}), \;\; p \mapsto p'.$$

Observe that $N(T)$ consists of constant polynomials while

$$D(a_0 x + \tfrac{a_1}{2} x^2 + \cdots + \tfrac{a_k}{k+1} x^{k+1} = a_0 + a_1 x + \cdots + a_k x^k.$$

Thus, D is surjective and $R(D) = P(\mathbb{R})$.

Definition [Nullity & Rank] Let $T : V \to W$ be a linear map. If V is finite dimensional, then nullity of T is defined to the dimension of $N(T)$ and the rank of T is defined to be the dimension of $R(T)$.

Remark Let $n = \dim_F(V)$ and let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. The set $S = \{T(v_1), \ldots, T(v_n)\}$ spans $R(T)$. Therefore, a subset of S is a basis of $R(T)$.

Observe that if T is injective, then $N(T) = \{\mathtt{0}_V\}$. Conversely, if $N(T) = \{\mathtt{0}_V\}$ and $T(v_1) = T(v_2)$, then

$$\mathtt{0}_W = T(v_1) - T(v_2) = T(v_1 - v_2)$$

implies that $v_1 - v_2 \in N(T)$. This forces T to be injective.