MA2102: LINEAR ALGEBRA

Lecture 34: Adjoint

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Given a finite dimensional inner product space V ,for each $v \in V$ we have a linear map

$$\langle \cdot, v \rangle : V \to F, \;\; w \mapsto \langle w, v \rangle.$$

Thus, we have a map

$$\Phi_V : V \to V^*, \;\; v \mapsto \langle \cdot, v \rangle.$$

Note that $\Phi_V(v_1 + v_2) = \Phi_V(v_1) + \Phi_V(v_2)$ and if $\Phi_V(v_1) = \Phi_V(v_2)$, then

$$\langle w, v_1 \rangle = \langle w, v_2 \rangle \text{ for any } w \in V$$

whence $v_1 = v_2$. Thus, Φ is injective. It is also surjective as the following result implies.

Theorem [Finite dimensional Riesz representation] *Let V be a finite dimensional inner product space. If $T : V \to F$ is a linear map, then there exists $v_0 \in V$ such that $T(v) = \langle v, v_0 \rangle$ for $v \in V$.*

Proof.

Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V. We may write

$$v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n$$

which implies that

$$\begin{aligned} T(v) &= \langle v, v_1 \rangle T(v_1) + \cdots + \langle v, v_n \rangle T(v_n) \\ &= \langle v, \overline{T(v_1)} v_1 \rangle + \cdots + \langle v, \overline{T(v_n)} v_n \rangle \\ &= \langle v, \overline{T(v_1)} v_1 + \cdots + \overline{T(v_n)} v_n \rangle. \end{aligned}$$

We may set $v_0 = \overline{T(v_1)} v_1 + \cdots + \overline{T(v_n)} v_n$. $\square$

Recall that any map $T : V \to W$ induces a dual map $T^* : W^* \to V^*$. If V and W are finite dimensional inner product spaces, then have canonical bijections $\Phi_V : V \to V^*$ and $\Phi_W : W \to W^*$.

Definition [Adjoint] The map from $W \to V$ defined as the composite

$$W \xrightarrow{\Phi_W} W^* \xrightarrow{T^*} V^* \xrightarrow{(\Phi_V)^{-1}} V$$

is called the adjoint of T.

Let us unravel what it means. By definition of Φ_V, it is additive but skew-linear, i.e.,

$$\Phi_V(cv) = \langle \cdot, cv \rangle_V = \bar{c} \langle \cdot, v \rangle_V = \bar{c}\,\Phi_V(v).$$

Since the adjoint is a composition of three additive maps, it is additive. Let us compute the effect of T^* on $\langle \cdot, w \rangle_W$, i.e.,

$$\begin{aligned} T^*\big(\langle \cdot, w \rangle_W\big)(v) &= (\langle \cdot, w \rangle_W)(T(v)) \\ &= \langle T(v), w \rangle_W \\ &= \langle T(\cdot), w \rangle_W(v). \end{aligned}$$

It follows that

$$cw \xrightarrow{\Phi_W} \langle \cdot, cw\rangle_W = \bar{c}\langle \cdot, w\rangle_W \xrightarrow{T^*} \bar{c}\langle T(\cdot), w\rangle_W \xrightarrow{(\Phi_V)^{-1}} c(\Phi_V)^{-1}\big(\langle T(\cdot), w\rangle_W\big).$$

Thus, the adjoint is a linear map.

Notation The adjoint of T will be denoted by T^*. There should be no confusion between the dual and the adjoint, although they are denoted by the same symbols, since the domains are different.

Question *Is there a cleaner description of the adjoint?*

Let $w \in W$ and consider $T^*(w)$. By construction,

$$(\Phi_V)^{-1}\big(\langle T(\cdot), w\rangle_W\big) = T^*(w).$$

Applying Φ_V to both sides we obtain

$$\langle T(\cdot), w\rangle_W = \langle \cdot, T^*(w)\rangle_V.$$

Definition [Adjoint] Given a map $T : V \to W$ between inner product spaces, a linear map $T^* : W \to V$ satisfying

$$\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V \tag{1}$$

for any $v \in V, w \in W$ is called the adjoint of T.

We have seen that such a T^* exists previously. Show that there is only one such linear map satisfying (1).

Examples (1) The adjoint of the identity map $I_V : V \to V$ is I_V. The adjoint of cI_V is $\bar{c}I_V$ (exercise).

(2) The adjoint of the zero map $0 : V \to W$ is the zero map $0 : W \to V$.

(3) Consider the linear map

$$T : \mathbb{R}^3 \to \mathbb{R}^2, \quad T(x, y, z) = (x + 2y + 3z, 4x + 5y + 6z).$$

To evaluate $T^*(a,b)$ we compute

$$\begin{aligned}
\langle T(x,y,z),(a,b)\rangle_{\mathbb{R}^2} &= (x+2y+3z,4x+5y+6z)\cdot(a,b) \\
&= (a+4b)x+(2a+5b)y+(3a+6b)z. \\
&= \langle (x,y,z),(a+4b,2a+5b,3a+6b)\rangle_{\mathbb{R}^3}.
\end{aligned}$$

The above quantity must equal $\langle (x,y,z),T^*(a,b)\rangle_{\mathbb{R}^3}$, whence

$$T^*(a,b)=(a+4b,2a+5b,3a+6b).$$

(4) Consider the trace map $\mathrm{trace}: M_2(\mathbb{R})\to\mathbb{R}$. To evaluate $\mathrm{trace}^*(\mu)$ we compute

$$\begin{aligned}
\langle \mathrm{trace}(A),\mu\rangle_{\mathbb{R}} &= (a+d)\mu \\
&= \mathrm{trace}(\mu I_2 A) = \mathrm{trace}((\mu I_2)^t A) \\
&= \langle A,\mu I_2\rangle_{M_2(\mathbb{R})}.
\end{aligned}$$

Therefore, $\mathrm{trace}^*(\mu)=\mu I_2$.

Proposition Let V be a finite dimensional inner product space and β be an orthonormal basis of V. If $T : V \to V$, then $[T^*]_\beta = [T]_\beta^*$.

Proof.

If $\beta = \{v_1, \ldots, v_n\}$, then any vector can be written as

$$v = \langle v, v_1\rangle v_1 + \cdots + \langle v, v_n\rangle v_n.$$

In particular, for any linear map $L : V \to V$,

$$L(v_j) = \langle L(v_j), v_1\rangle v_1 + \cdots + \langle L(v_j), v_n\rangle v_n.$$

Thus, the $(i,j)^{\text{th}}$ entry of $[L]_\beta$ is given by $\langle L(v_j), v_i\rangle$. Note that

$$\big([T^*]_\beta\big)_{ij} = \langle T^*(v_j), v_i\rangle = \overline{\langle v_i, T^*(v_j)\rangle} = \overline{\langle T(v_i), v_j\rangle} = \overline{([T]_\beta)_{ji}}.$$

This completes the proof. $\square$

Compare this result with example (1).