MA2102: LINEAR ALGEBRA

Lecture 3: Linear Subpace

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Functions as vectors: Let S be a set and consider

$$\mathscr{F}(S,\mathbb{R}) := \{f : S \to \mathbb{R} \mid f \text{ is a function}\}.$$

- For $f, g \in \mathscr{F}(S,\mathbb{R})$, define $(f+g)(s) := f(s) + g(s)$.
- The zero function, i.e., $\mathtt{0}(s) = \mathtt{0}$ for any $s \in S$, is the additive identity.
- $(cf)(s) := cf(s)$ defines scaling.

Show that $\mathscr{F}(S,\mathbb{R})$ is a vector space over $\mathbb{R}$. Note that when $S = \{1, 2, \cdots, n\}$, then $\mathscr{F}(S,\mathbb{R})$ *looks like* $\mathbb{R}^n$.

Question *What is $\mathscr{F}(S,\mathbb{R})$ when $S = \{1, 2, \cdots\} = \mathbb{N}$?*

Answer The set of sequences of real numbers. This is a very *large* vector space.

Polynomials as vectors: Let n be a non-negative integer. Consider

$$P_n(\mathbb{R}) := \{p(x) \,|\, p \text{ is a polynomial with real coefficients, } \deg(p) \leq n\}.$$

Show that $P_n(\mathbb{R})$ is a vector space over $\mathbb{R}$.

Question *Is the set of polynomials of degree exactly n a vector space?*

Answer *No, for several reasons: not closed under addition, additive identity is not present, not closed under scaling.*

Look at $P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 \,|\, a_i \in \mathbb{R}\}$. This *looks like* $\mathbb{R}^3$ and a bijection is given as follows

$$f : P_2(\mathbb{R}) \to \mathbb{R}^3,\ a_0 + a_1 x + a_2 x^2 \mapsto (a_0, a_1, a_2).$$

This map is compatible with the the structures on both sides.

Polynomials as vectors (continued): We have the following inclusions

$$P_0(\mathbb{R}) \subset P_1(\mathbb{R}) \subset P_2(\mathbb{R}) \subset \cdots \subset P_n(\mathbb{R}) \subset P_{n+1}(\mathbb{R}) \subset \cdots$$

Consider the union

$$P(\mathbb{R}) := \cup_{n \geq 0} P_n(\mathbb{R}),$$

the set of all polynomials. Note that

$$\deg(p+q) \leq \max(\deg(p), \deg(q)).$$

Thus, $P(\mathbb{R})$ is closed under addition. Show that $P(\mathbb{R})$ is a vector space.

Question *Is $P(\mathbb{R})$ more than a vector space?*

Are these sets vector spaces? We shall look at three examples.

(1) $V = [0, \infty) = \{t \in \mathbb{R} | t \geq 0\}$

- no additive inverses
- not closed under scaling

(2) $U =$ unit vectors in $\mathbb{R}^3$

Remark This is a two dimensional sphere called the 2-sphere, denoted by S^2. It is 2 because intrinsically we have two dimensions.

- no additive identity
- not closed under scaling or addition

(3) Latitudes and longitudes

- no additive identity
- additive inverses do not make sense
- not closed under scaling or addition

Question *What are all the subsets of $\mathbb{R}^2$ which are vector spaces themselves?*

Answer We look for subsets apart from $W = \{(0,0)\}$ and $W = \mathbb{R}^2$. Consider

$$W_1 = \{(a,0) \in \mathbb{R}^2 \,|\, a \in \mathbb{R}\}, \;\; W_2 = \{(0,a) \in \mathbb{R}^2 \,|\, a \in \mathbb{R}\}.$$

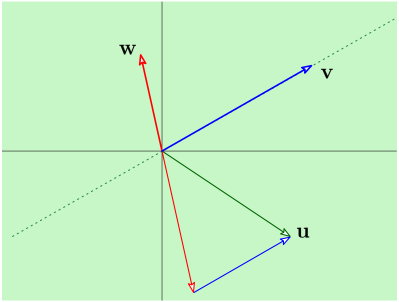
Note that we may also consider the lines through the origin, i.e.,

$$L = \{(x,y) \in \mathbb{R}^2 \,|\, y = mx\}.$$

Show that W_1, W_2 and L are vector spaces.

Are these all the subsets of $\mathbb{R}^2$ which are vector spaces? Yes!

Let $W \neq \{(0,0)\}$ be a vector space. Choose $\mathbf{v} \neq (0,0)$ such that $\mathbf{v} \in W$. By scaling, the line through $(0,0)$ and $\mathbf{v}$ is contained in W. If $W = \{\lambda \mathbf{v} \mid \lambda \in \mathbb{R}\}$, then W is a line through the origin. Otherwise, choose $\mathbf{w} \in W$ outside $\{\lambda \mathbf{v} \mid \lambda \in \mathbb{R}\}$.



Show that any vector $\mathbf{u}$ in $\mathbb{R}^2$ can be expressed as $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$. This will be proved later in a more general form. This implies that $W = \mathbb{R}^2$.

Definition [Vector subspace] A subset W of a vector space V is called a (vector) subspace if W is a vector space.

Note that if $\mathtt{0}' \in W$ is the additive identity, then choose any $\mathbf{w} \in W$. Since

$$\mathbf{w} + \mathtt{0}' = \mathbf{w} \text{ and } \mathbf{w} + \mathtt{0} = \mathbf{w}$$

by cancellation law $\mathtt{0}' = \mathtt{0}$. Show that the additive inverse of $\mathbf{w} \in W$ is the same in W as in V.

Remark A vector subspace is often called a linear subspace. We shall simply call it a subspace in subsequent lectures.

Proposition *A subset $W \subseteq V$ of a vector space V is a subspace if and only if the following holds:*

(i) $\mathbf{w}_1 + \mathbf{w}_2 \in W$ *whenever* $\mathbf{w}_1, \mathbf{w}_2 \in W$

(ii) $\lambda \mathbf{w} \in W$ *whenever* $\mathbf{w} \in W$ *and* $\lambda \in \mathbb{R}$.

The proposition is essentially a repackaging of the definition of a subspace. Among various consequences of it, we mention two.

- $\mathbf{0} = 0 \cdot \mathbf{w}$ for any $\mathbf{w} \in W$. By (ii), $\mathbf{0} \in W$.
- The vector $(-1) \cdot \mathbf{w}$ is in W by (ii). Now,

$$\mathbf{w} + (-1) \cdot \mathbf{w} = 1 \cdot \mathbf{w} + (-1) \cdot \mathbf{w} = (1 + (-1)) \cdot \mathbf{w} = 0 \cdot \mathbf{w} = \mathbf{0}$$

implies that $(-1) \cdot \mathbf{w}$ is the additive inverse of $\mathbf{w}$.

Non-examples (a) Consider the integers $\mathbb{Z} \subset \mathbb{R}$. It is closed under addition but not closed under scaling.

(b) Consider $S = x\text{-axis} \cup y\text{-axis} \subset \mathbb{R}^2$. It is closed under scaling but not closed under addition.

(c) Consider the unit circle S^1 with origin as its centre in $\mathbb{R}^2$. Both (i) and (ii) do not hold.