MA2102: Linear Algebra

Lecture 1: Introduction

18th August 2020

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Geometry: Consider a vector $\mathbf{v} = (x_1, \ldots, x_n)$. The *length* or *norm* of $\mathbf{v}$ is given by

$$||\mathbf{v}|| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Given a vector, we may rotate it, scale it. Given two vectors, we can add them. Some properties of this addition:

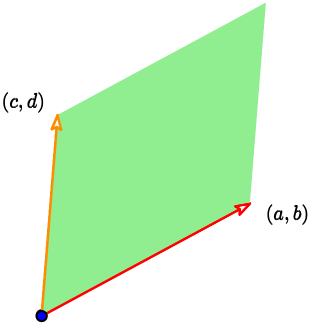
[commutative] $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$

[associative] $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

[identity] $\mathbf{v} + \mathtt{0} = \mathbf{v}$

[distributive] $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$

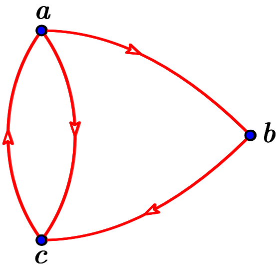
Specialize to $n = 2$, i.e., vectors on the plane $\mathbb{R}^2$. Consider vectors given by (a, b) and (c, d).



$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \longrightarrow ad - bc$$

The determinant is the signed area of the parallelogram spanned by the vectors.
It is also the dot product of (a, b) with $(d, -c)$, where $(d, -c)$ is (c, d) rotated clockwise by $\pi/2$.

Graph Theory Consider a network with 3 nodes and inflow/outflow directions as given:



$$T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The matrix T is called the adjacency matrix, its $(i,j)^{\text{th}}$ entry is the number of ways to go from node i to node j in *one* directed step. Here a is node 1, b is node 2 and c is node 3.

$$T^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The $(i,j)^{\text{th}}$ entry is the number of ways to go from node i to node j in *two* directed steps.

Question *What does $T + T^2$ given by*

$$T + T^2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

measure?

Answer The number of distinct paths, of at most 2 steps, from node i to node j.

Communication Let us look at the following table:

	H	K	Te	N	T	M	B
Satbhav	**1**	0	**1**	0	0	0	0
Temjeninla	0	0	0	**1**	0	0	0
Naravane	**1**	0	0	0	0	**1**	0
Karthikeyan	**1**	0	0	0	**1**	0	0
Hema Padmaja	**1**	**1**	**1**	0	0	0	0
Ananya	**1**	0	0	0	0	0	**1**

H : Hindi, K : Kannada, Te : Telugu, N : Nagamese,
T : Tamil, M : Marathi, B : Bengali

Let us denote the underlying 6×7 matrix by C.

Question *What does CC^T measure?*

Answer Note that

$$CC^T = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

is a symmetric 6×6 matrix.

- The diagonal entries are the number of languages a person can self-converse in.
- The off-diagonal entries denote the number of languages person i has in common with person j.

Let us write the matrix C and C^T as

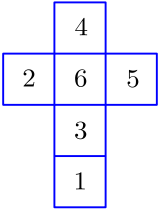
$$C = \begin{pmatrix} - & \mathbf{R}_1 & - \\ - & \mathbf{R}_2 & - \\ & \vdots & \\ - & \mathbf{R}_k & - \end{pmatrix}, \quad C^T = \begin{pmatrix} | & | & & | \\ \mathbf{R}_1^T & \mathbf{R}_2^T & \cdots & \mathbf{R}_k^T \\ | & | & & | \end{pmatrix}$$

The $(j,j)^{\text{th}}$ entry of CC^T is $\mathbf{R}_j^T \cdot \mathbf{R}_j^T$. More generally, the $(i,j)^{\text{th}}$ entry of CC^T is $\mathbf{R}_i^T \cdot \mathbf{R}_j^T = \left\langle \mathbf{R}_i^T, \mathbf{R}_j^T \right\rangle$.

We will later study $n \times n$ square matrices C satisfying $CC^T = I_n$. These are examples of isometries and are called orthogonal matrices.

Question *What do the entries of $C^T C$ measure?*

Probability Consider a regular die with faces numbered 1 through 6.



Replace the number on each face with the average of the number on it and its adjacent faces. For example, 6 gets replaced by $\frac{2+5+4+3+6}{5} = 4$. The face with 1 now gets assigned the number $\frac{2+5+3+4+1}{5} = 3$.

The transformation can be succinctly captured by the matrix

$$P = \begin{pmatrix} 0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 0 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 0 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 0 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 0 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \end{pmatrix}$$

as P acting on $\mathbf{v}_0 = (6\ 5\ 4\ 3\ 2\ 1)^T$ produces the new face numbers (for faces labelled 6 through 1).

Question *What happens in the limit when we iterate this renumbering process?*

We have to compute $\lim_{n\to\infty} P^n \mathbf{v}_0$. Any guesses otherwise?

Linear algebra tells us $P = UDU^T$, where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/5 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

Observe that $U^T U = I_6$ implies that

$$\lim_{n\to\infty} P^n \mathbf{v}_0 = \lim_{n\to\infty} (UDU^T)^n \mathbf{v}_0 = \lim_{n\to\infty} UD^n U^T \mathbf{v}_0.$$

Note that

$$\lim_{n\to\infty} UD^n U^T = \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \end{pmatrix}$$

The limiting values of the faces are 3.5 each.

The space we live in The 3-dimensional space is denoted by $\mathbb{R}^3$, where each point is specified by (x, y, z). We can do the *usual* operations and the scalars come from $\mathbb{R}$.

A few important rules/axioms:

- $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_2, y_2, z_2) + (x_1, y_1, z_1)$

Both equal $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$*; follows from commutativity of* $+$ *in* $\mathbb{R}$.

- $(x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3)) = ((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3)$

Follows from associativity of $+$ *in* $\mathbb{R}$.

- $(0, 0, 0) + (x, y, z) = (x, y, z)$

Follows from $0 + a = a$ *in* $\mathbb{R}$.

- $(-x, -y, -z) + (x, y, z) = (0, 0, 0)$

Follows from $-a + a = 0$ *in* $\mathbb{R}$.

- $1 \cdot (x,y,z) = (x,y,z)$

Follows from $1 \cdot a = a$ *in* $\mathbb{R}$*.*

- $(\lambda\mu) \cdot (x,y,z) = \lambda \cdot (\mu \cdot (x,y,z))$

Follows from $(\lambda\mu) \cdot a = \lambda \cdot (\mu \cdot a)$ *in* $\mathbb{R}$*; associativity of multiplication.*

- $\lambda \cdot ((x_1,y_1,z_1) + (x_2,y_2,z_2)) = \lambda \cdot (x_1,y_1,z_1) + \lambda \cdot (x_2,y_2,z_2)$

Follows from $a \cdot (b+c) = ab + ac$ *in* $\mathbb{R}$*; distributive law.*

- $(\lambda + \mu) \cdot (x,y,z) = \lambda \cdot (x,y,z) + \mu \cdot (x,y,z)$

Follows from distributive law and commutativity of multiplication.

The first four rules concern the vector addition in $\mathbb{R}^3$, while the last four rules concern properties of $\mathbb{R}$ in conjunction with vector addition.

We will study sets V (think of $\mathbb{R}^3$ or $\mathbb{R}^n$) with a binary operation called *addition*, defined over a set F (think of $\mathbb{R}$), having *addition* and *multiplication*, such that the eight axioms are satisfied.