MA2102: LINEAR ALGEBRA

Lecture 19: Dual Maps

2nd October 2020

Indian Institute of Science Education & Research Kolkata

Proposition Let $T : V \to W$ be a linear map. If β, γ are ordered bases of V, W respectively, then

$$\left([T]_\beta^\gamma\right)^t = [T^*]_{\gamma^*}^{\beta^*}.$$

Recall that β^*, γ^* are the dual bases, i.e., $\beta^* = \{v_1^*, \ldots, v_n^*\}$ such that $v_i^*(v_j) = \delta_{ij}$. Similarly,

$$T^* : W^* \to V^*, \quad T^*(L) := L \circ T$$

is the dual of T.

Proof.
Let $\beta = \{v_1, \ldots, v_n\}$ and $\gamma = \{w_1, \ldots, w_m\}$. Suppose that

$$T(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m,$$

where $(i,j)^{\text{th}}$ entry of $[T]_\beta^\gamma$ is given by a_{ij}.

Check that

$$T^*(w_i^*)(v_j) = w_i^*(T(v_j)) = w_i^*(a_{1j}w_1 + \cdots + a_{mj}w_m) = a_{ij}.$$

Thus,

$$\begin{aligned}
T^*(w_i^*)(v) &= T^*(w_i^*)(c_1v_1 + \cdots + c_nv_n) \\
&= c_1T^*(w_i^*)(v_1) + \cdots + c_nT^*(w_i^*)(v_n) \\
&= c_1a_{i1} + \cdots + c_na_{in} \\
&= a_{i1}c_1v_1^*(v_1) + \cdots + a_{in}c_nv_n^*(v_n) \\
&= a_{i1}v_1^*(c_1v_1) + \cdots + a_{in}v_n^*(c_nv_n) \\
&= a_{i1}v_1^*(v) + \cdots + a_{in}v_n^*(v) \\
&= \left(a_{i1}v_1^* + \cdots + a_{in}v_n^*\right)(v).
\end{aligned}$$

Thus, the $(j,i)^{\text{th}}$ entry of $[T^*]_{\gamma^*}^{\beta^*}$ is given by a_{ij}. $\square$

Remark We shall see later that the rank of T can be computed by the row (or column) rank of $[T]_\beta^\gamma$. However, it follows that

$$\begin{aligned}\text{row rank of } [T]_\beta^\gamma &= \text{column rank of } [T]_\beta^\gamma \\ &= \text{row rank of } \left([T]_\beta^\gamma\right)^t \\ &= \text{row rank of } [T^*]_{\gamma^*}^{\beta^*}.\end{aligned}$$

Therefore, rank of T equals rank of T^*.

Examples (1) Consider the trace map trace : $M_2(\mathbb{R}) \to \mathbb{R}$. Consider

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and $\gamma = \{1\}$ as ordered bases. Show that $1^* : \mathbb{R} \to \mathbb{R}$ is the identity map.

Note that

$$[\mathrm{trace}]_\beta^\gamma = (1 \quad 0 \quad 0 \quad 1).$$

We also have

$$\mathrm{trace}^*(1^*)(A) = \mathrm{id}(\mathrm{trace}(A)) = \mathrm{trace}(A).$$

If $\beta^* = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ is the dual basis, then

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \pi_1(A) = a,\ \pi_2(A) = b,\ \pi_3(A) = c,\ \pi_4(A) = d.$$

Since $\mathrm{trace}^*(1^*) = \pi_1 + \pi_4$, we have

$$[\mathrm{trace}^*]_{\gamma^*}^{\beta^*} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(2) Consider the conjugation map

$$T : \mathbb{C} \to \mathbb{C}, \ \ z \mapsto \bar{z}.$$

We may also consider this as a map

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \ \ (x,y) \mapsto (x,-y).$$

Let $\beta = \{e_1 = (1,0), e_2 = (0,1)\}$ be the standard basis. The dual $e_1^* : \mathbb{R}^2 \to \mathbb{R}$ is characterized by $e_1^*(e_j) = \delta_{1j}$, i.e.,

$$e_1^*(x,y) = e_1^*(xe_1 + ye_2) = xe_1^*(e_1) + ye_1^*(e_2) = x.$$

Show that $e_2^* : \mathbb{R}^2 \to \mathbb{R}$ is given by $e_2^*(x,y) = y$. Observe that

$$\begin{aligned}
T^*(e_1^*)(e_1) &= e_1^*(T(e_1)) = e_1^*(e_1) = 1 \\
T^*(e_1^*)(e_2) &= e_1^*(T(e_2)) = e_1^*(-e_2) = 0 \\
T^*(e_2^*)(e_1) &= e_2^*(T(e_1)) = e_2^*(e_1) = 0 \\
T^*(e_2^*)(e_2) &= e_2^*(T(e_2)) = e_2^*(-e_2) = -1.
\end{aligned}$$

As $T^*(e_j^*)$ is determined by its values on e_1, e_2, in our case

$$[T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad [T^*]_{\beta^*} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(3) Let V be a vector space of dimension n. Consider a projection map

$$T : V \to V, \; T \circ T = T.$$

If $v \in R(T) \cap N(T)$, then $v = T(v')$ for some $v' \in V$ and $T(v) = 0_V$. Thus,

$$0_V = T(v) = T(T(v')) = T(v') = v.$$

This implies that $R(T) \oplus N(T)$ is a direct sum, of dimension

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = k + (n - k) = n.$$

In other words, $V = R(T) \oplus N(T)$.

Let $\beta = \{v_1, \ldots, v_k\}$ and $\gamma = \{u_1, \ldots, u_{n-k}\}$ be bases of $R(T)$ and $N(T)$ respectively. Show that $\beta \cup \gamma$ is a basis of V. Note that T is identity on $R(T)$ as $T \circ T = T$. The associated matrix is given by

$$[T]_{\beta \cup \gamma} = \begin{pmatrix} I_k & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & 0_{n-k} \end{pmatrix}.$$

Let v_i^* and consider the effect of T^* on β^*, i.e.,

$$\begin{aligned} T^*(v_i^*)(v_j) &= v_i^* \circ T(v_j) = v_i^*(v_j) = \delta_{ij} \\ T^*(v_i^*)(u_j) &= v_i^* \circ T(u_j) = v_i^*(0_V) = 0_V. \end{aligned}$$

With respect to the basis $\beta^* \cup \gamma^*$ of V^*, it can be seen that

$$[T^*]_{\beta^* \cup \gamma^*} = \begin{pmatrix} I_k & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & 0_{n-k} \end{pmatrix}.$$