Stellar Structure and Evolution

Part 1

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Stars come in different flavours

- Variation in stellar brightness is due to
  - variation in distance from us
  - variation in intrinsic brightness

- Variation in stellar “colour” is due to difference in surface temperature.

- Among several factors governing these variations, two are dominant:
  - total **mass** of the star
  - age of the star.

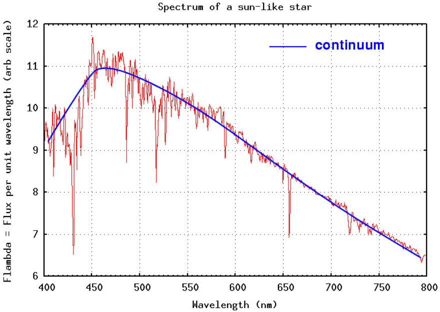
- Stars of different masses are built very differently on the inside.

Stars evolve

- As the star gets older, its "global properties" like radius, luminosity and surface temperature change.
- Internal structure also changes with age, accompanied by a corresponding change in the internal chemical composition (distribution of ions, atoms or molecules of different species).
- Stars evolve **very** slowly, so stellar evolution can be considered as a quasistatic process (except at certain short-lived phases of a star's life).
- We shall construct the equations of stellar structure for a "**static**" star, i.e. a star at one instant of its life.
- The four "global" properties used to characterize stars and compare them with models are its **Mass (M)**, **Radius (R)**, **Luminosity (L)** and **effective temperature (T_{eff})**.

Effective Temperature

- Stars emit continuum radiation at all wavelengths which resembles the spectrum of a **blackbody** very closely.



- The temperature of a blackbody whose radiation would mimic the spectrum of a star most closely is called the **effective temperature ($T_{\rm eff}$)** of the star.

$$L = 4\pi\sigma R^2 T_{\rm eff}^4$$

Basic Assumptions

- **Isolation**: A single star may be considered isolated in empty space, so that its structure (and evolution) depends only on its internal processes.
  - true of all single stars (e.g., the closest star to the Sun, Proxima Centauri is at a distance of 4.3 lightyears from us, which is larger than the solar diameter by a factor of 3×10^7)
  - Even for stars in binaries, this condition holds true for most part of a star's life.
  - However, in close binaries, it may happen in late stages of evolution that the gravitational and radiation effects of one star strongly influences the structure and evolution of the other.

- **Spherical Symmetry**: We shall assume our star to be spherically symmetric (gravitation is a central force). For example, the Sun bulges by only about 10 km at the equator, compared to its poles!
- Departure from spherical symmetry can arise for stars which undergo very fast rotation or have very strong magnetic fields.
- This assumption immediately reduces the problem to an one-dimensional problem (only r dependence, no θ, ϕ dependence of the structure.

Chemical Composition

- The main constituents of stars, like the rest of the universe, are hydrogen and helium.
- All other elements, heavier than helium, are termed as "metals" in astrophysics!
- Mass fraction of a species:

$$X_i = \frac{\text{mass of species } i \text{ in a given mass } m}{m}; \qquad \sum_i X_i = 1$$

- In particular, $X \equiv X_{\mathrm{H}},\ \ Y \equiv X_{\mathrm{He}},\ \ Z \equiv X_{\mathrm{metals}} \longrightarrow$ "metallicity".
- Clearly,

$$X + Y + Z = 1$$

For the Sun, $X \approx 0.70$, $Y \approx 0.28, Z \approx 0.02$.

Variables

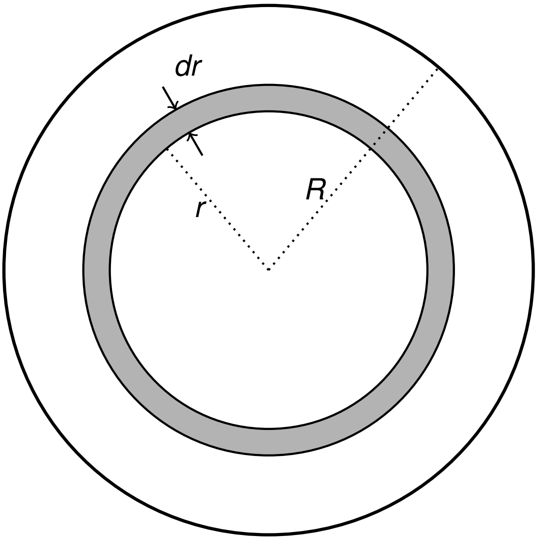
- Common variables:

  r : radius (independent variable)
  $m(r)$: mass
  $\rho(r)$: density
  $P(r)$: pressure ($P_{\text{gas}} \gg P_{\text{rad}}$ in low-mass stars)
  $T(r)$: temperature
  $l(r)$: luminosity
  $g(r) = Gm/r^2$: acceleration due to gravity
  $m(R) = M$: total mass
  $l(R) = L$: surface luminosity
  $T(R) \approx T_{\text{eff}}$: effective temperature

- Alternatively, m is taken as the independent variable.

Conservation of mass

- Consider a thin spherical shell of radius r and thickness dr inside the star.



Conservation of mass

- The mass contained inside this shell is

$$dm = m(r + dr) - m(r) = 4\pi r^2 dr\, \rho(r)$$

$$\implies \boxed{\frac{dm}{dr} = 4\pi r^2 \rho(r)}$$

This is the first equation of stellar structure called the **Mass Conservation Equation**.

- With m as independent variable,

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho(r)}$$

Hydrostatic equilibrium

- The star does not collapse under gravity or does not blow up due to pressure is because there is perfect balance of these forces.
- For the thin shell of mass dm,
  - inward force of gravity is $\dfrac{Gmdm}{r^2}$.
  - outward force due to pressure is the difference in pressure forces between the inner surface at radius r, and the outer surface at radius $r + dr$.
- In general, $P = P_{\text{gas}} + P_{\text{rad}}$.

  For low mass stars, $P_{\text{gas}} \gg P_{\text{rad}}$ (for Sun, $P_{\text{rad}}/P_{\text{gas}} \approx 10^{-4}$).

Hydrostatic equilibrium

For no motion of the layer (i.e., hydrostatic equilibrium):

$$P(r)(4\pi r^2) - P(r+dr)(4\pi r^2) - \frac{Gmdm}{r^2} = 0$$

$$\implies -\frac{dP}{dr}dr(4\pi r^2) - \frac{Gm(4\pi\rho r^2 dr)}{r^2} = 0$$

$$\implies \boxed{\frac{dP}{dr} = -\frac{Gm}{r^2}\rho = -g\rho}$$

This is the second equation of stellar structure, called the **Hydrostatic Balance equation**

The first two equations alone shall lead to some very interesting results for the stellar interior.

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho = -g\rho$$

Since both g and ρ are positive quantities, the pressure gradient must be negative, i.e. the pressure inside a star must decrease outwards.

Hydrostatic balance is maintained to a high degree everywhere inside a star, on average.
Small perturbations create oscillations $\longrightarrow$ **Asteroseismology**!

Central Pressure

- What is the pressure $P_c \equiv P(0)$ at the centre of a star?
- Pressure vanishes at the surface of the star, i.e. $P(R) = 0$.
- Try integrating Eqn of Hydrostatic Balance:

$$P(R) - P(0) = -P_c = \int_0^R dP = -\int_0^R \frac{Gm\rho}{r^2} dr$$

- Problem: we do not know $\rho(r)$, the density distribution inside the star!

Central Pressure

- Alternative approach: recast in terms of m as the independent variable, by using mass equation:

$$\frac{dP}{dm} = \frac{dP}{dr} \bigg/ \frac{dm}{dr} = -\frac{Gm\rho}{r^2} \bigg/ 4\pi r^2 \rho = -\frac{Gm}{4\pi r^4}$$

  and integrate

$$P(M) - P(0) = -P_c = \int_0^M dP = -\int_0^M \frac{Gm}{4\pi r^4} dm$$

$$\implies P_c = \int_0^M \frac{Gm}{4\pi r^4} dm$$

- But still cannot compute the integral because we do not know how the radius varies with mass inside the star: $r(m)$.

Central Pressure

- Notice that if we replace r by R in the denominator in the integral,

$$P_c = \int_0^M \frac{Gm}{4\pi r^4} dm > \int_0^M \frac{Gm}{4\pi R^4} dm$$

$$\Rightarrow P_c > \frac{GM^2}{8\pi R^4}$$

- We have obtained at least a lower bound for the **central pressure** in a star.
- For the Sun, we plug in the values of solar mass and radius to obtain $P_{c,\odot} > 4.4 \times 10^{13}\,\mathrm{N/m^2}$. The actual pressure at the centre of the Sun is about $10^{15}\,\mathrm{N/m^2}$.
- Possible to obtain by dimensional analysis also.

Central Temperature

- Need an equation of state, which shall connect the pressure with temperature.
- Reasonable approximation for the equation of state is the ideal gas law ($PV = NkT$).
- The temperatures inside a star are so high (something that can be checked post facto) that beneath the outermost layers, all the atoms are in ionised state, and we have a plasma of positive ions and free electrons.
- The thermal energy of the ions and electrons dominate by a large factor any (electromagnetic) interaction potential, which means that we have essentially a population of non-interacting particles — an ideal gas.

Mean Molecular Weight

- Define a quantity called the "**mean molecular weight**" which is the average mass per particle in the gas, expressed in terms of proton mass.
- The mean molecular weight μ of a gas of N particles and total mass m is defined to be

$$\mu m_{\rm p} = \frac{m}{N} \qquad \text{i.e. } \mu = \frac{m}{N m_{\rm p}}$$

μ is a number.

- For example, for a fully ionized hydrogen gas of N_{H^+} ions of H^+ and equal number N_e of electrons,

$$
\begin{aligned}
\mu &= \frac{1}{m_\mathrm{p}} \frac{N_{\mathrm{H}^+} m_\mathrm{p} + N_\mathrm{e} m_\mathrm{e}}{N_{\mathrm{H}^+} + N_\mathrm{e}} \\
&= \frac{1}{m_\mathrm{p}} \frac{N_{\mathrm{H}^+} m_\mathrm{p}}{2 N_{\mathrm{H}^+}} \qquad \text{since } m_e \ll m_\mathrm{p} \\
&= \frac{1}{2}
\end{aligned}
$$

Ideal Gas equation

- Then, the density of a gas can be expressed as

$$\rho = \frac{m}{V} = \frac{N(\mu m_{\mathrm{p}})}{V}$$

- The ideal gas law can be written as

$$\boxed{P = \frac{NkT}{V} = \frac{\rho kT}{\mu m_{\mathrm{p}}}}$$

- Then the central temperature, T_{c} is

$$T_{\mathrm{c}} = \frac{P_{\mathrm{c}} \mu m_{\mathrm{p}}}{k \rho_{\mathrm{c}}}$$

- We need estimates of P_c (made earlier), ρ_c and μ.
- For ρ_c, we make the simplifying, although unrealistic, assumption that the density inside the star is uniform, i.e. the density is the average density, $\overline{\rho} = \dfrac{3M}{4\pi R^3}$ everywhere, including at the centre.
- In reality, the central density will be $\rho_c \sim \alpha\overline{\rho}$, where α is a multiplicative factor typically lying between 1 and 10.
- Further, let us assume the star to be composed purely of ionized hydrogen, i.e. $\mu = 1/2$. Again, the real value of μ will differ from this by a factor of the order of unity.

Central Temperature

- Under these assumptions, for the Sun, we obtain the central temperature to be

$$T_{c,\odot} = \frac{P_c \mu m_p}{k \rho_c} = \frac{G M_\odot^2}{8\pi R_\odot^4} \times \frac{\mu m_p}{k} \times \frac{4\pi R_\odot^3}{3 M_\odot} = \frac{G \mu m_p M_\odot}{6 k R_\odot}$$
$$\approx 19 \times 10^6\,\mathrm{K}$$

- For the Sun, the real value of T_c is determined to be about 15 million kelvin. Even with the gross simplifying assumptions, we could arrive at a close enough value.
- The important thing is that we find that the central temperature of stars is typically millions of kelvin, which is indeed high enough for nuclear fusion to take place.