MA2102: Linear Algebra

Lecture 6: Linear Independence

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As observed in previous examples, we may eliminate an element from a linearly dependent set without changing the span.

Observation *Let S be a linearly dependent subset of a vector space V. Then there exists $v \in S$ such that* $\operatorname{span}(S) = \operatorname{span}(S - \{v\})$.

Proof.

If $\mathbf{0} \in S$, then $\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{0}\})$. Now suppose that $\mathbf{0} \notin S$. As S is linearly dependent, there exists $v_1, \cdots, v_k \in S$ and scalars c_i's, not all zero, such that

$$c_1 v_1 + \cdots + c_k v_k = \mathbf{0}.$$

By rearranging the indices, we may assume that $c_1 \neq 0$. Then

$$v_1 = -\tfrac{c_2}{c_1} v_2 - \cdots - \tfrac{c_k}{c_1} v_k.$$

Now it follows that $\operatorname{span}(S) = \operatorname{span}(S - \{v_1\})$. $\square$

The observation implies that if S is a set such that no proper subset of it spans $\text{span}(S)$, then S is linearly independent. Note that a subset S is linearly independent if and only if there is no non-trivial linear combination of elements in S that equal $\mathtt{0}$.

Show that if S is a linearly independent set, then any proper subset T of S will span a strictly smaller subspace of $\text{span}(S)$, i.e., $\text{span}(T) \subset \text{span}(S)$.

Convention Recall that we had defined $\text{span}(\varnothing) = \{\mathtt{0}\}$.

(i) The empty set $\varnothing$ is linearly independent, as linearly dependent sets are necessarily non-empty.

(ii) If we put $S_1 = \varnothing$ in

$$\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$$

which was proved for non-empty sets, then we get

$$\operatorname{span}(S_2) = \operatorname{span}(\varnothing) + \operatorname{span}(S_2)$$

implying $\operatorname{span}(\varnothing) = \{\mathbf{0}\}$.

(iii) If we want $\operatorname{span}(S)$ to be a subspace, then $\mathbf{0} \in \operatorname{span}(S)$.

Theorem *Let S be a linearly independent subset of a vector space V. Given $v \in V - S$, the set $S \cup \{v\}$ is linearly independent if and only if $v \notin \operatorname{span}(S)$.*

Proof.

Assume that $S \cup \{v\}$ is linearly independent. If $v \in \operatorname{span}(S)$, then write

$$v = a_1 v_1 + \cdots + a_k v_k$$

with scalars a_i and $v_i \in S$. Thus,

$$a_1 v_1 + \cdots + a_k v_k + (-1)v = \mathbf{0}$$

expresses zero as a non-trivial linear combination of elements in $S \cup \{v\}$, contradicting linear independence of $S \cup \{v\}$. This implies that $v \notin \operatorname{span}(S)$.

Assume that $v \notin \operatorname{span}(S)$. If $S \cup \{v\}$ is linearly dependent, then

$$a_1 v_1 + \cdots + a_k v_k + \lambda v = 0$$

for $v_i \in S$ and the scalars a_i, λ cannot all be zero. If $\lambda = 0$, then we can represent zero as a non-trivial linear combination of elements in S, contradicting linear independence of S. If $\lambda \neq 0$, then rewrite

$$v = -\frac{a_1}{\lambda} v_1 - \frac{a_2}{\lambda} v_2 - \cdots - \frac{a_k}{\lambda} v_k$$

whence $v \in \operatorname{span}(S)$, a contradiction. $\square$

We may either start with a set that spans V and keep removing elements from it until it fails to span V, or we may start with a linearly independent set and keep adding elements to it, increasing the span until it spans V. Both these process lead to the following notion.

Definition [Basis] A basis β of a vector space V is a subset such that β is linearly independent and spans V.

Examples (1) The vector space $\{\mathbf{0}\}$ has $\emptyset$ as a basis.

(2) The vector space $\mathbb{R}$ (over $\mathbb{R}$) has any non-zero real number as a basis.

(3) The vector space $\mathbb{R}^n$ (over $\mathbb{R}$) has $\{e_1, \ldots, e_n\}$ as a basis. It is called the *standard basis*.

Show that $\{e_1, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + \cdots + e_n\}$ is a basis for $\mathbb{R}^n$.

(4) The vector space $M_2(\mathbb{R})$ (over $\mathbb{R}$) has

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

as a basis (exercise).

(5) The vector space $P(\mathbb{R})$ (over $\mathbb{R}$) has $\beta = \{1, x, x^2, \ldots\}$ as a basis. Any polynomial is in the span of β. If β is linearly dependent, then for some $x^{k_1}, \cdots, x^{k_n}$ we must have

$$a_1 x^{k_1} + \cdots + a_n x^{k_n} = 0$$

where 0 stands for the zero polynomial. Any real number is the root of the zero polynomial while if at least some of the a_i's are non-zero, then the left hand side is a non-zero polynomial, which has finitely many roots, a contradiction. Thus, β is linearly independent.

Lemma Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V. Then every vector $v \in V$ can be expressed uniquely as $v = a_1 v_1 + \cdots + a_n v_n$ for scalars a_i.

Proof.

As a basis spans V, every v can be written as claimed. If $v = c_1 v_1 + \cdots + c_n v_n$ is another expression, then

$$a_1 v_1 + \cdots + a_n v_n = v = c_1 v_1 + \cdots + c_n v_n.$$

Therefore, $(a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n = \mathbf{0}$ coupled with linear independence of β, implies that $a_j = c_j$. $\square$

Proposition Let V be a vector space that admits a finite subset S that spans V. Then there exists a subset $\beta \subseteq S$ which is a basis for V.

Remark The space $P(\mathbb{R})$ does not have a finite spanning set. If it did, let $S = \{p_1, \ldots, p_k\}$ be such a set. Let $n_k = \deg(p_k)$ and $N = \max\{n_1, \ldots, n_k\}$. Then any linear combination of p_j's will result in a polynomial of degree at most N. Thus, x^{N+1} will not be in the span.