Tutorial 3
Adwait Naravane 19MS151

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Q1

A Particle is initially ($t < 0$) prepared in the ground state of an infinite, one-dimensional potential well with walls at $x = 0$ and $x = a.$, (i.e.)

$$V(x) = 0, \qquad 0 \leq x \leq a$$

$$V(x) = +\infty, \qquad o.w.$$

Now assume the wall at $x = a$ is *suddenly* (at $t = 0$) moved to $x = 8a$. Using *sudden approximation*, calculate the probability of finding the particle in

The energy eigenstates for this system are,

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

with eigenvalues $E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$.

The system is prepared in the ground state at $t = 0$.
$\Psi(x, t = 0) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) = |1_{old}\rangle$
The walls instantly go from $x = a$ to $x = 8a$ at time $t = 0$.

For the new system, the eigenstates are.

$$\Psi_n(x) = \frac{1}{2\sqrt{a}} \sin\left(\frac{n\pi x}{8a}\right) = |N_{new}\rangle$$

The transition amplitude,

$$\mathcal{A}_{1\to N} = \langle N_{new}|1_{old}\rangle = \int_0^a dx \frac{1}{a\sqrt{2}} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{n\pi x}{8a}\right) = -\frac{2^{11/2} \sin\left(\frac{n\pi}{8}\right)}{\pi(n^2 - 64)}$$

The probability amplitude would be,

$$P(1_{old} \to N_{new}) = \mathcal{A}^2 = \frac{2^{11} \sin^2(\frac{n\pi}{8})}{\pi^2(n^2 - 64)^2}$$

i) ground to ground

$$P(1_{old} \rightarrow 1_{new}) = \frac{2^{11}\sin^2(\frac{\pi}{8})}{\pi^2 63^2}$$

ii) ground to first

$$P(1_{old} \rightarrow 2_{new}) = \frac{2^{11}\sin^2(\frac{\pi}{4})}{\pi^2 60^2}$$

iii) ground to second

$$P(1_{old} \rightarrow 3_{new}) = \frac{2^{11}\sin^2(\frac{3\pi}{8})}{\pi^2 55^2}$$

Q2

Question No.2 (Quantum Quench of an Harmonic Oscillator)

Consider a particle of unit mass is trapped in a one dimensional harmonic oscillator potential $V_0 = \frac{1}{2}\omega_0 x^2$. Initial Hamiltonian is given by,

$$H_0 = \frac{p^2}{2} + \frac{1}{2}\omega_0 x^2$$

Now within a very tiny time scale $\tau << \frac{\hbar}{\omega_0}$, the spring constant is doubled such that $V = \frac{1}{2}\omega x^2$. This phenomena is known as quantum quench.

i) Assume that the particle is in the ground state initially. If we measure the post quench energy of the particle, what is the probability for finding the particle in the ground state of the new potential V ?

ii) Show that after some time $T = \frac{2\pi}{\omega}$, the system returns back to it's initial state with a negative sign.

iii) Find the correlator $< \psi_0|x(t_1)x(t_2)|\psi_0 >$ (*work in the Heisenberg picture,*), where ψ_0 is the ground state of the pre-quench Hamiltonian H_0. Is the correlation function time translationally invariant ? Justify your answer. What does it signify ?

The system was initially with the Hamiltonian, $H_0 = \frac{p^2}{2} + \frac{1}{2}\omega_0 x^2 = \omega_0(a_0^\dagger a_0 + \frac{1}{2})$ and in a small time, goes to $H = \omega(a^\dagger a + \frac{1}{2})$.
(Assume $\hbar = m = 1$)

$x = \frac{1}{\sqrt{2\omega_0}}(a_0 + a_0^\dagger) = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $p = i\sqrt{\frac{\omega_0}{2}}(a_0^\dagger - a_0) = i\sqrt{\frac{\omega}{2}}(a^\dagger - a)$

From this we can write $a, a^\dagger$ and $a_0, a_0^\dagger$ in terms of each other.

$$a = \frac{a_0}{2}\left(\sqrt{\frac{\omega}{\omega_0}} + \sqrt{\frac{\omega_0}{\omega}}\right) + \frac{a_0^\dagger}{2}\left(\sqrt{\frac{\omega}{\omega_0}} - \sqrt{\frac{\omega_0}{\omega}}\right)$$

Now, let the ground state of the old system be $|\Omega\rangle$.
Therefore, $a_0 |\Omega\rangle = 0$ which implies $(\omega_0 + \omega)a |\Omega\rangle = (\omega - \omega_0)a^\dagger |\Omega\rangle$.
Let $|n\rangle$ be the eigenbasis of the new system, if we write the old ground in terms of the new basis we get $|\Omega\rangle = \sum_{n=0} \alpha_n |n\rangle$

$(\omega - \omega_0) \langle 0| a^\dagger |\Omega\rangle = 0 = (\omega + \omega_0) \langle 1|\Omega\rangle$
$\sqrt{3}(\omega + \omega_0) \langle 3|\Omega\rangle = (\omega - \omega_0)\sqrt{2} \langle 1|\Omega\rangle = 0$
We can see that all odd terms are 0.

Therefore, $|\Omega\rangle = \sum_{n=0} \alpha_{2n} |2n\rangle$.
with $\alpha_{2n+2} = \sqrt{\frac{2n+1}{2n+2}(\frac{\omega-\omega_0}{\omega+\omega_0})}\alpha_{2n}$.

i) The probability of finding the state in the new ground would be,
$| \langle 0|\Omega\rangle |^2$.
$\langle 0|\Omega\rangle = \int_\infty^\infty dx \frac{1}{\sqrt{\pi}}(\omega\omega_0)^{1/4} e^{-\frac{(\omega+\omega_0)}{2}x^2} = \sqrt{\frac{2\sqrt{\omega\omega_0}}{\omega+\omega_0}}$

Therefore, Probability $P = \frac{2\sqrt{\omega\omega_0}}{\omega+\omega_0}$

iii) $x(t) = \frac{1}{\sqrt{2\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t})$
$\langle\Omega| x(t_2)x(t_1) |\Omega\rangle = \frac{1}{2\omega} \langle\Omega| (a^2 e^{-i\omega(t_1+t_2)} + a^{\dagger 2} e^{i\omega(t_1+t_2)} + aa^\dagger e^{i\omega(t_1-t_2)} + a^\dagger a e^{i\omega(t_2-t_1)}) |\Omega\rangle$
Now we can write these ladder operators in terms of the old ones, to get

$$\langle\Omega| x(t_2)x(t_1) |\Omega\rangle = \frac{1}{8\omega}((\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2})(e^{-i\omega(t_1+t_2)} + e^{i\omega(t_1+t_2)}) + (\frac{\omega}{\omega_0} + \frac{\omega_0}{\omega} + 2)(e^{i\omega(t_1-t_2)} + e^{i\omega(t_2-t_1)}))$$

Finally we get,

$$\langle\Omega| x(t_2)x(t_1) |\Omega\rangle = \frac{1}{2\omega}(e^{-i\omega(t_2-t_1)} + \frac{(\omega-\omega_0)^2 \cos(\omega(t_2-t_1)) + (\omega^2-\omega_0^2)\cos(\omega(t_2+t_1))}{2\omega\omega_0})$$

Q3

Question No.3 (Landau-Zener Tunneling)

Consider a Hamiltonian written in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$\mathcal{H} = \frac{\Delta}{2}\sigma_x + \frac{vt}{2}\sigma_z$$

where $v > 0$ & $\Delta > 0$ are some constants such that $v << \pi\Delta^2/2$ (assuming $\hbar = 1$) and σ_i are the Pauli's sigma matrices. Note that at times $t \to \pm\infty$, $\mathcal{H}$ is dominated by the term proportional to σ_z. The solution for the Schrodinger equation (i.e., wavefunction at t^{th} instant of time) corresponding to the time dependent Hamiltonian $\mathcal{H}$ can be written in general as

$$|\psi(t)\rangle = c_1(t) |\uparrow\rangle + c_2(t) |\downarrow\rangle$$

such that $|c_1(t)|^2 + |c_2(t)|^2 = 1 \,\forall t$.

$H = \begin{pmatrix} vt/2 & d/2 \\ d/2 & -vt/2 \end{pmatrix}$ with $|\Psi(t)\rangle = c_1(t)\,|\uparrow\rangle + c_2(t)\,|\downarrow\rangle$

The Schrodinger's equation,
$i\hbar\partial_t\,|\Psi(t)\rangle = \hat{H}\,|\Psi(t)\rangle$

i) This ends up being,

$$\begin{pmatrix} \dot{c_1} \\ \dot{c_2} \end{pmatrix} = \begin{pmatrix} -i\frac{vt}{2}c_1 - i\frac{\Delta}{2}c_2 \\ -i\frac{\Delta}{2}c_1 + i\frac{vt}{2}c_2 \end{pmatrix}$$

ii) If we manipulate the above, we can get the following ode for c_1.

$$\ddot{c}_1 + (i\frac{v}{2} + \frac{v^2t^2 + \Delta^2}{4})c_1 = 0$$

iii) The coefficient of c_1 above, is analogous to the square of the frequency. The imaginary part of the frequency, which cause exponential growth would get smaller as $t \to \infty$. The modulus of c_1 would become constant as $t \to \infty$. i.e. $c_1 \approx |c|e^{-if(t)}$

$-\dot{f}^2 - i\ddot{f} + i\frac{v}{2} + \frac{v^2t^2+\Delta^2}{4} = 0$
Seperate the real and imaginary sides,
$\ddot{f} = \frac{v}{2}$
$\dot{f} = \pm\frac{1}{2}\sqrt{v^2t^2 + \Delta^2}$
$\dot{f} = \frac{vt}{2}\sqrt{1 + \frac{\Delta^2}{v^2t^2}} \approx \frac{vt}{2}(1 + \frac{\Delta^2}{2v^2t^2}) = \frac{vt}{2} + \frac{\Delta^2}{4vt}$

$$\frac{\dot{c_1}}{c_1}_{t\to\infty} = -i\dot{f}_{t\to\infty} = -i(\frac{vt}{2} + \frac{\Delta^2}{4vt})$$

iv)

$$\int_{-\infty}^{\infty} \frac{\dot{c_1}}{c_1}dt = -\int_C \frac{\dot{c_1}}{c_1}(z)dz = \int_{-\infty}^{\infty} d(c_1) = \ln\left(\frac{c_1(\infty)}{c_1(-\infty)}\right)$$
$$= i\lim_{R\to\infty}\int_0^{\pm\pi}(i\frac{vR^2e^{2i\theta}}{2} + \frac{i\Delta^2}{4v})d\theta = \mp\frac{\pi\Delta^2}{4v}$$

The probability of Transition,
$P = |c_1(\infty)|^2 = |c_1(-\infty)|^2\exp\left(\mp\frac{\pi\Delta^2}{2v}\right)$
as $c_1(-\infty) = 1$, the transition probability is

$$P = \exp\left(-\frac{\pi\Delta^2}{2v}\right)$$

v) If $v \ll \frac{\pi\Delta^2}{2}$, the transition probability through avoided crossing is nearly zero. i.e. if one starts in c_1, they cross into c_2.