~~PH302~~

PH3202

E & M

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19MS151.

Q1 $a^\mu = \frac{dU^\mu}{d\tau}$, U^μ is 4-velocity ($\vec{V}$)

$$\bar{a} = \frac{d\bar{U}}{d\tau} = \gamma(v)\cdot\frac{d}{dt}\,\gamma(v).(c,\vec{v})$$

($\vec{v}$ is 3-velocity)

$$\frac{d}{dt}\gamma = \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$

$$= -\frac{1}{2}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}\left(-\frac{v}{c^2}\cdot\frac{dv}{dt}\right).$$

... (dot product)

$$= \gamma^3(v)\ \frac{v\cdot a}{c^2}$$

$$\Rightarrow \bar{a} = \left(\gamma^4\frac{a\cdot v}{c},\ \gamma^2(v)\,a + \gamma^4(v)\frac{a\cdot v}{c^2}\,v\right)$$

time component ↑ (space comp) ↑ (a is 3-acceler)

Now to show $\bar{a}$ & $\bar{V}$ are perpendicular

~~$\bar{a}\cdot\bar{V} = \gamma^5(v)(a\cdot v) - \gamma^3(v)(a\cdot v) - \gamma^5(v)$~~

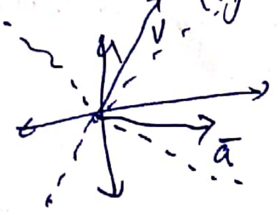
$$\bar{a}\cdot\bar{V} = \gamma^5(v)(a\cdot v) - \gamma^3(v)(a\cdot v) - \gamma^5(v)\frac{(a\cdot v)}{c^2}|v|^2$$

$$= \gamma^3(v)(a\cdot v)\left(\frac{1-v^2/c^2}{1-v^2/c^2} - 1\right)$$

$$= 0$$

$\Rightarrow$ $\bar{a} \perp \bar{V}$ are perpendicular.

$\bar{a}^\nu$ is spacelike as it lies outside the lightcone. as $\bar{V}^\nu$ is timelike;



Q2

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = 4\pi J^\beta$$

$$\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0.$$

$$g_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = 4\pi j^\beta \Rightarrow g_{\alpha\beta}\frac{\partial^2 F_{\alpha\beta}}{\partial x^\alpha \partial x^\beta} = 4\pi \frac{\partial j^\beta}{\partial x^\beta}$$

$$\Rightarrow g_{\alpha\beta}\frac{\partial^2 F_{\alpha\beta}}{\partial x^\alpha \partial x^\beta} + g_{\beta\alpha}\frac{\partial^2 F_{\beta\alpha}}{\partial x^\alpha \partial x^\beta} = 8\pi \frac{\partial j^\alpha}{\partial x^\alpha}.$$

as $g_{\alpha\beta} = g_{\beta\alpha}$ and $F_{\alpha\beta} = -F_{\beta\alpha}$.

$$0 = 8\pi \frac{\partial j^\alpha}{\partial x^\alpha} \Rightarrow \underline{\underline{\frac{\partial J^\alpha}{\partial x^\alpha} = 0}}$$

Q4

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{e^{-\alpha r}}{r} = A\frac{e^{-\alpha r}}{r}.$$

$$E = -\nabla\phi = -A\frac{\partial}{\partial r}\left(\frac{e^{-\alpha r}}{r}\right)\hat{r}$$

$$= Ae^{-\alpha r}(1+\alpha r)\frac{\hat{r}}{r^2}$$

$$\rho = \varepsilon_0 \nabla\cdot E = \varepsilon_0 A e^{-\alpha r}(1+\alpha r)\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) + \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\alpha r}(1+\alpha r)\right)$$

Now. $\varepsilon_0 A e^{-\alpha r}(1+\alpha r)\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \varepsilon_0 A e^{-\alpha r}(1+\alpha r)4\pi\delta^3(r)$

$$= \underline{\varepsilon_0 A\, 4\pi\delta^3(r)}$$

$$\varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\nabla\left(e^{-\alpha r}(1+\alpha r)\right) = \varepsilon_0 A\frac{\hat{r}}{r^2}\cdot\frac{\partial}{\partial r}\left(e^{-\alpha r}(1+\alpha r)\right)\hat{r}$$

$$= -\varepsilon_0 A\frac{\alpha^2}{r}e^{-\alpha r}$$

$$\Rightarrow \rho(r) = \varepsilon_0 A\left(4\pi\delta^3(r) - \frac{\alpha^2}{r}e^{-\alpha r}\right)$$

$$\rho(r) = \delta^3(r) - \frac{\alpha^2}{4\pi r}e^{-\alpha r}$$

Q3 m_0 is rest mass of both.

let $c=1$, $\Rightarrow v = \frac{3}{5}$

$\gamma = (1-v^2)^{-1/2} = 5/4$.

Four momentum of this particle

$\Rightarrow p_{1\mu} = (\gamma m_0, \gamma m_0 v, 0, 0)$

Four momentum of particle at rest.

$p_{2\mu} = (\cancel{\gamma} m_0, 0, 0, 0)$

Four momentum of resulting

$\Rightarrow p_\mu = p_{1\mu} + p_{2\mu} = (m_0(\gamma+1), m_0\gamma v, 0, 0)$

Mass of resulting would be.

$m_{result}^2 = p_\mu^2 = m^2(\gamma+1)^2 - m^2\gamma^2 v^2 = 2(\gamma+1)m_0^2$.

$= \cancel{3}\ \frac{9}{2} m_0^2$

$\Rightarrow m_{res} = \frac{3}{\sqrt{2}} m_0$

Scanned with CamScanner

Q5 Boundary conditions are.

$V = 0$ at $x = 0$, $V = 0$ at $x = a$

$V = 0$ at $y = 0$, $V = 0$ at $y = b$.

$V = 0$ at $z = 0$, $V = V_0$ at $z = c$.

$\nabla^2 V = 0 \Rightarrow V = X(x)\, Y(y)\, Z(z)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -k, \quad \frac{1}{Y^2}\frac{dY^2}{dy^2} = -l^2$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 + l^2$$

$X = A\sin(kx) + B\cos(kx)$.

$Y = C\sin(ly) + D\cos(ly)$

$Z = E e^{\sqrt{k^2+l^2}\,z} + G\, e^{-\sqrt{k^2+l^2}\,z}$

Apply boundary conditions

$B = 0, \Rightarrow k = \frac{n\pi}{a}$,

$D = 0$ and $l = \frac{m\pi}{b}$

$V = 0$ at $z = 0 \Rightarrow Z(z) = E\sinh(\sqrt{k^2+l^2}\,z)$.

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\,z\right)$$

and $\therefore$ $$V_0 = \sum_n\sum_m C_{nm}\sinh\left(\pi c\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right),$$

Fourier transform.

$$C_{nm}\sinh\left(\pi c\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\right) = \frac{4V_0}{ab}\int_0^a \sin\left(\frac{n\pi x}{a}\right)dx \int_0^b \sin\left(\frac{m\pi y}{b}\right)dy$$

$$= \frac{16 V_0}{\pi^2 nm}(1-\cos n\pi)(1-\cos m\pi)$$

$$\Rightarrow C_{nm} \sinh\left(\pi c \sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\right) = \begin{cases} 0 & m \text{ or } n \text{ is even.} \\ \frac{16 V_0}{\pi^2 nm} & \text{odd } m \text{ \& } n. \end{cases}$$

~~Solution.~~