Q3

(a) No. $\Rightarrow \Omega = \sum_{\{S_i\}} \delta(H(\{S_i\}) - E)$

(b) $Z = \sum_{S_i} \exp(\beta(J_1 \sum_i S_i S_{i+1} + J_2 \sum (1-S_i)(1-S_{i+1})))$

$= \sum_{S_1=0,1} \sum_{S_2=0,1} \cdots \sum_{S_N=0,1} \exp(\beta[J_1 S_1 S_2 + J_2(1-S_1)(1-S_2)])$

Transfer matrix.

$$M = \exp(\beta J_1 S_1 S_2 + \beta J_2 (1-S_1)(1-S_2))$$

$$= \begin{bmatrix} \exp(\beta J_2) & 1 \\ 1 & \exp(\beta J_1) \end{bmatrix}.$$

eigenvalues of M are.

$$\lambda^2 - (\exp(\beta J_1) + \exp(\beta J_2))\lambda + (\exp(\beta(J_1+J_2)) - 1) = 0.$$

$$\lambda_\pm = \frac{2u \pm \sqrt{v^2+1}}{2}$$

$$u = \frac{\exp(\beta J_1) + \exp(\beta J_2)}{2}$$

$$v^2 = [(e^{\beta J_1} + e^{\beta J_2})^2 - 4e^{\beta(J_1+J_2)}]$$

b) $Z = tr(M^L) = \lambda_+^L + \lambda_-^L$

For large L, $\frac{\lambda_-}{\lambda_+} < (1)$

$\Rightarrow Z \approx \lambda_+^L = (u + \sqrt{v^2+1})^L$

c)

Q.2

$a_{n+1} = a_n + c_n,$

$c_n = 3c_{n-1}, \quad a_0 = 4, \quad c_0 = c$

$V_n = \begin{bmatrix} a_n \\ c_n \end{bmatrix}, \quad \& \quad V_{n+1} = \begin{bmatrix} a_{n+1} \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} a_n + c_n \\ 3c_n \end{bmatrix}.$

transfer matrix.

$M = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$

$\Rightarrow \quad V_n = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ c_{n-1} \end{bmatrix}$

~~rope~~ recursion from a_0, c_0 gives.

$V_n = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}^{n-1} \begin{bmatrix} a_0 \\ c_0 \end{bmatrix}.$

Diagonalise this.

$\det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad \lambda = 1, 3.$

for $\lambda = 1$, $V_{eigen} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda = 3$, $V_{eigen} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad P^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow M^n = P D^n P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^n \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & \frac{3^{n}-1}{2} \\ 0 & 3^n \end{pmatrix}$

$\Rightarrow \begin{bmatrix} a_n \\ c_n \end{bmatrix} = \begin{bmatrix} 1 & -\left(\frac{3^n - 1}{2}\right) \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} a_0 \\ c_0 \end{bmatrix}.$

$= \begin{bmatrix} a_0 + \frac{c_0}{2}(3^n - 1) \\ c_0 + 3^n \end{bmatrix}.$

$\Rightarrow \underline{a_n = a_0 + \frac{c_0}{2}(3^n - 1)}$

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(Q9)

$$H = -J \sum_i \delta_{s_i,1}\, \delta_{s_{i+1},1}$$

$$Z = tr(e^{-\beta H}) = tr\left(e^{\beta J(\delta_{s_1,1}\delta_{s_2,1} + \dots + \delta_{s_N,1}\delta_{s_1,1})} \right)$$

... periodic

(I presume $\delta_{s_i,1}$ means that it is only non-zero if both s_i & s_{i+1} are equal to 1)

$$Z = tr\left(\prod_{i=1}^{N} T(i,i+1)\right)$$

$$T(i,i+1) = e^{\beta J(\delta_{s_i,1}\, \delta_{s_{i+1},1})}$$

three basis $\Rightarrow$ $|1\rangle, |0\rangle,$ ~~$|0\rangle$~~ $|-1\rangle$.

$\langle 1|T|1\rangle = \exp(\beta J)$. , else zero.

$$T = \begin{bmatrix} \exp(\beta J) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$T^{L} = \begin{bmatrix} \exp(L\beta J) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(i) $Z = tr(T^{L}) = \cancel{\exp(N\beta J)} = \exp(L\beta J)$

(ii) Average energy.

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\partial(L\beta J)}{\partial \beta} = -LJ$$

(iii) ~~Does not change.~~ The average energy is constant with temperature

Q6) $H = -\sum(\delta_{S_i, S_{i+1}} + \delta_{S_{i+1} S_{i+2}} - 1)$

$S_i = \{Cu, Zn\} = \{1, 0\}.$

$\cancel{Z = \exp(\beta \sum}$

$Z = tr(\exp(\beta \sum(\delta_{S_i, S_{i+1}} + \delta_{S_{i+1}, S_{i+2}} - 1)))$

$= tr(\prod_{i=1}^{L} \exp(\beta(\delta_{S_i S_{i+1}} + \delta_{S_{i+1} S_{i+2}} - 1)))$

$= tr(\prod_{i=1}^{L} \exp(\beta \delta_{S_i S_{i+1}}) \prod_{i=1}^{L} \exp(\beta \delta_{S_{i+1} S_{i+2}}) \exp(-\beta L))$

$\cancel{T = \begin{bmatrix} \exp(\beta) & 0 \\ 0 & \exp(\beta) \end{bmatrix}}$ $\quad T = \begin{bmatrix} \exp(\beta) & 0 \\ 0 & \exp(\beta) \end{bmatrix}$

$\cancel{T^N}$ $\quad \cancel{T =}$ $\quad T^L = \begin{bmatrix} \exp(L\beta) & 0 \\ 0 & \exp(L\beta) \end{bmatrix}.$

$Z = tr(T^L . T^L \exp(-L\beta)) = tr(T^L T^L T^{-L})$

$= tr(T^L) = 2\exp(L\beta)$

i) $\underline{Z = 2\exp(L\beta)}$ -- partition function.

ii) The energy goes down when S_i, S_{i+1} & S_{i+2} are the same.
Therefore, at lower temperatures the lattice would get occupied by the same type of atoms, or form larger clusters of Cu or Zn atoms; at High $T \to \infty$, the Cu-Zn ratio will tend to 1 : 1. at it would be a disordered phase.

19MS /
Adwait Naravane

PH 3201

Q1 a) $h(s) = \int_0^{\infty} e^{-sx} f(x)\,dx.$

$$R(s) = \frac{1}{h(s)} \int_0^{\infty} s e^{-sx} f(x)\,dx.$$

$$h'(s) = -\int s e^{-sx} f(x)\,dx = -s\,h(s).$$

$h'(s) < 0$

$$R(s) = -\frac{h'(s)}{h(s)} \Rightarrow \cancel{R}$$

$$R'(s) = \frac{-h''(s)}{h(s)} + \frac{h'(s)^2}{h^2(s)} = s^2 - \frac{h''(s)}{h(s)}$$

$$\Rightarrow h''(s) = -h(s) + s^2 h(s)$$

$\Rightarrow R(s) < 0$ is monotonicaly decreasing

b) $\{a_n\} = \{-1^n\}$ $n = 0, \ldots \infty.$

$$g(x) = \sum a_n x^n = \sum (-1)^n x^n$$

~~$g'(x)$ $xg(x)$~~

$$xg(x) = -(g(x) - 1)$$

$$\Rightarrow g(x) = \frac{1}{x+1}$$

c) $E = -J S_1 S_2$.

$$Z = \sum e^{-\beta E(-1)} = \sum e^{+\beta J s_1 s_2}.$$

$$= 2 e^{\beta J} + 2 e^{-\beta J}$$

$$= 4 \cosh \beta J$$

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} = -J \tanh(\beta J).$$

$$\Rightarrow \langle E \rangle = -J \left(\frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} \right).$$

$\beta \to 0$, $T \to \infty$.

$$\cancel{\Rightarrow \langle E \rangle = -J \left(J e^{\beta J} + J e^{-\beta J} \right.}$$

$$\langle E \rangle = -J \left(\frac{1-1}{1+1} \right) = 0.$$

Average energy tends to 0.