PH3103 - End Sem

19MS151
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Q2

$$U(x,y) = e^x(x\cos y - y\sin y).$$

$$\frac{\partial U}{\partial x} = e^x\cos y + xe^x\cos y - ye^x\sin y.$$

$$\frac{\partial^2 U}{\partial x^2} = 2e^x\cos y + xe^x\cos y - ye^x\sin y.$$

$$\frac{\partial U}{\partial y} = e^x(-x\sin y - \sin y - y\cos y).$$

$$\frac{\partial^2 U}{\partial y^2} = ye^x\sin y - e^x x\cos y - 2e^x\cos y.$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 2e^x\cos y + xe^x\cos y - ye^x\sin y + (-2e^x\cos y - xe^x\cos y + ye^x\sin y)$$

$$= 0$$

$U(x,y)$ is Harmonic.

$V(x,y)$ is Harmonic conjugate.

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \& \quad \frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

$$\frac{\partial V}{\partial y} = e^x\cos y + xe^x\cos y - ye^x\sin y.$$

$$V = \int \frac{\partial V}{\partial y}\,dy = e^x\sin y + xe^x\sin y - e^x\int y\sin y\,dy$$

$$= e^x\sin y + xe^x\sin y + ye^x\cos y - \int\cos y\,dy\,e^x$$

$$= e^x\sin y + xe^x\sin y + ye^x\cos y + C(x).$$

$$= xe^x\sin y + ye^x\cos y$$

$$\frac{\partial V}{\partial x} = e^x(x\sin y + \sin y + y\cos y).$$

$$\Rightarrow V = e^x\sin y + ye^x\cos y + \sin y\int x\,dx\,e^x$$

$$= ye^x\cos y + xe^x\sin y + D(y).$$

Both are same, $C(x) = D(y) = 0$.

$$\Rightarrow V(x,y) = xe^x\sin y + ye^x\cos y.$$

which is $U(x,y)$'s Harmonic conjugate.

Q6

$$f(x) = \begin{cases} 1 & -1 \leq x \leq -1/2 \\ 0 & -1/2 \leq x < 1/2 \\ 1 & 1/2 \leq x \leq 1. \end{cases}$$

fourier series on $[-1, 1]$.

$$a_0 = \int_{-1}^{1} f(x).dx$$

$$= \int_{-1}^{-1/2} 1\,dx + \int_{-1/2}^{1/2} 0\,dx + \int_{1/2}^{1} 1\,dx$$

$$= x\Big|_{-1}^{-1/2} + \cancel{\ } x\Big|_{1/2}^{1}$$

$$= +1 \cancel{8} -1/2 + 1 - 1/2$$

$$= p$$

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx = \int_{-1}^{-1/2} \cos(n\pi x)dx + \int_{\frac{1}{2}\beta}^{1} \cos(n\pi x)\,dx$$

$$= \frac{\sin(n\pi x)}{n\pi}\Big|_{-1}^{1/2} + \frac{\sin(n\pi x)}{n\pi}\Big|_{1/2}^{1}$$

$$= -\frac{\sin(n\pi/2)}{n\pi} + \cancel{\frac{\sin(n\pi)}{n\pi}} + \cancel{\frac{\sin(n\pi)}{n\pi}} - \frac{\sin(n\pi/2)}{n\pi}$$

$$= -\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = \text{even} \\ -\frac{2}{n\pi} & n = \text{odd}. \end{cases}$$

$$b_n = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{-1/2} \sin(n\pi x)dx + \int_{1/2}^{1} \sin(n\pi x)dx$$

$$= -\frac{\cos(n\pi/2)}{n\pi} + \frac{\cos(n\pi)}{n\pi} - \frac{\cos(n\pi)}{n\pi} + \frac{\cos(n\pi/2)}{n\pi}$$

$$= 0.$$

$$\Rightarrow f(x) = 1 + \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \left(-\frac{2}{n\pi}\right) \cos(n\pi x).$$

Q3

$$\bar{g}(f) = \frac{2}{1+f^2}$$

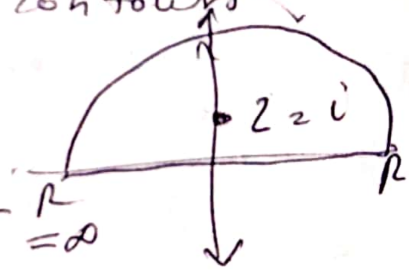
$$F^{-1}(g(f)) = g(x).$$

$$g(x) = \int_{-\infty}^{\infty} \frac{2e^{i2\pi f x}}{1+f^2}\, df$$

$$\Rightarrow g(x) \propto \int_{-\infty}^{\infty} \frac{2}{1+f^2} e^{ifx}\, df.$$

(I like using $e^{i\omega x}$ insted of $e^{i2\pi f x}$ to skip the 2π)

We can solve this using contours



$$\frac{2}{1+f^2} e^{ifx} \to \frac{2}{1+z^2} e^{izx}$$

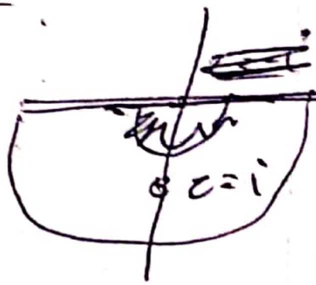
poles at $z = \pm i$

for $x > 0$, we pick contour in upper half of plane.

$$\Rightarrow \int_{-\infty}^{\infty} df\, \frac{e^{ifx}}{1+f^2} = (2\pi i)\, \underset{z=i}{\mathrm{Res}} \left(\frac{e^{izx}}{1+z^2} \right)$$

$$= 2\pi i \cdot \frac{e^{-x}}{i2} = \pi \cdot e^{-x}.$$

for $x < 0$.
we pick contour in lower half.



$$\int_{-\infty}^{\infty} df \, \frac{e^{ifx}}{1+f^2} = (2\pi i) \operatorname{Res}_{z=-i} \left(\frac{e^{izx}}{1+z^2} \right).$$

$$= 2\pi i \cdot \frac{e^{x}}{i2}$$

$$= \pi e^{x}. \quad \text{-- } \underline{\underline{x < 0}}.$$

$$\Rightarrow \int_{-\infty}^{\infty} df \, \frac{e^{ifx}}{1+f^2} = \underline{\underline{\pi e^{-|x|}}}$$

$$\Rightarrow \quad F^{-1}(g(f)) = \pi e^{-|x|}$$

... (The answer might be different on how one weighs their frequency like $\omega = 2\pi\nu$)

Q1

1. $$\int_0^\infty \frac{1-\cos x}{x^2}\,dx$$

$$f(z) = \int_0^\infty \frac{1-e^{iz}}{z^2}\,dz$$

second order pole at $z=0$.

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \cancel{4}\,2\int_0^\infty \frac{1-\cos x}{x^2}\,dx$$

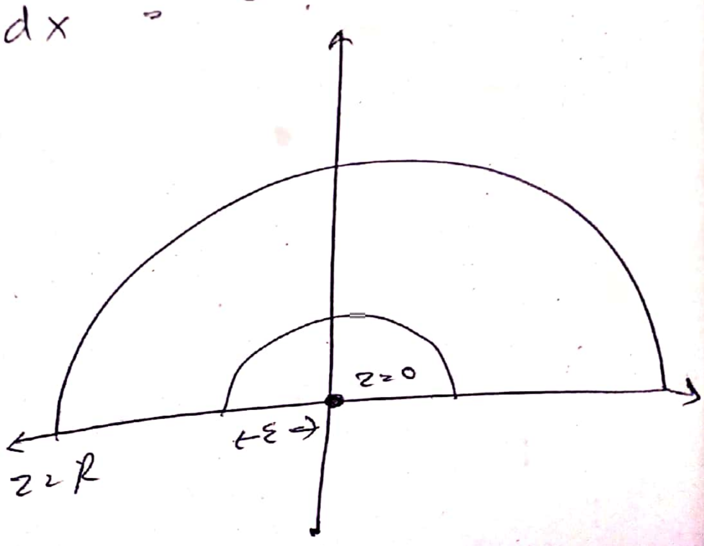
$$= 4\int_{-\infty}^{\infty} \frac{\cos^2(x/2)}{x}\,dx$$

$$\Rightarrow f(z) = \int_{-\infty}^{\infty} \frac{e^{iz}}{z^2}\,dz = \int_{-\infty}^{\infty} e^{iz}\,dz$$

$$\operatorname*{Res}_{z=0}(e^{iz}) = \lim_{z\to 0}\frac{d}{dz}e^{iz} = i.$$

$$\Rightarrow \operatorname{Re}(\operatorname{Res}(e^{iz})) = \operatorname{Res}(\cos/z)) = 0.$$

$$\Rightarrow PV\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = 0.$$

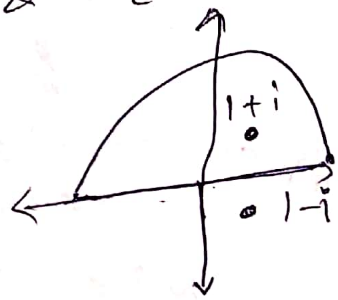


Q1

2. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx$ PV.

$f(z) = \dfrac{1}{z^2-2z+2}$

poles at $z = 1 \pm i = z_1$ & z_2



$f(z) = \dfrac{1}{(z-z_1)(z-z_2)}$.

simple poles.

$$\oint \frac{dz}{(z-z_1)(z-z_2)} = \int_{-R \to \infty}^{R} \frac{dx}{x^2-2x+2} + \underbrace{\int_{C_R} \frac{dz}{(z-z_1)(z-z_2)}}_{\to 0 \text{ Jordans lemma.}} = 0$$

$\Rightarrow \text{Res}(f(z), z=z_1) = \lim\limits_{z \to z_1} \dfrac{1}{(z-z_2)} = \dfrac{1}{2i}$

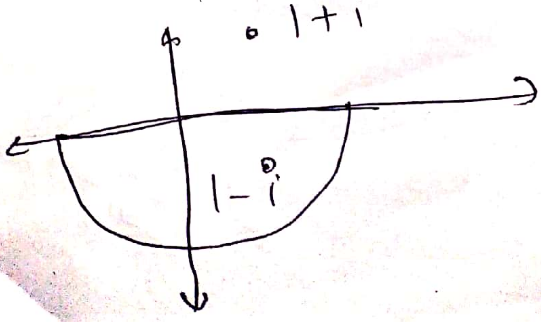
$\text{Res}(f(z), z=z_2) = \lim\limits_{z \to z_2} \dfrac{1}{z-z_1} = -\dfrac{1}{2i}$

$$\Rightarrow PV \int \frac{dx}{x^2-2x+2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2-2x+2}$$

$= 2\pi i \sum \text{Res}(f(z), z = z_i)$

$= \dfrac{2\pi i}{2i} = \pi$ for upper contour

& $PV \displaystyle\int \frac{dx}{x^2-2x+2} = -\frac{2\pi i}{2i} = -\pi$ for lower contour.



Q4 $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0.$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1-\frac{n^2}{x^2}\right)y = 0.$$

$$P(x) = \frac{1}{x}, \quad Q(x) = 1-\frac{n^2}{x^2}.$$

a) at $x = 0$.

$$\left.\begin{aligned} x\,P(x) &= 1 \\ x^2 Q(x) &= x^2-n^2 \end{aligned}\right\} \text{both are analytic}$$

$x=0$ point is proper singularity.

b) take $x = \frac{1}{a}$, tend $a \to 0$ later,

$$\frac{dy}{dx} = \frac{da}{dx}\frac{dy}{da} = -a^2\frac{dy}{da}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = a^4\frac{d^2y}{da^2} + 2a^3\frac{dy}{da}.$$

for. $\cancel{x^2}\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1-\frac{n^2}{x^2}\right)y = 0.$

$$\frac{d^2y}{da^2} + \frac{1}{a}\frac{dy}{da} + \left(\frac{1}{a} - \frac{n^2}{a^2}\right)y = 0.$$

at $a = 0$.

$$\left(\frac{1}{a^4} - \frac{n^2}{a}\right)\cdot x a^2 = \left(\frac{1}{a^2} - n^2\right) \text{ is not analytic in } a.$$

So $x = \infty$ is improper singular point

c) take $y(x) = x^s \sum_{k=0}^{\infty} a_k x^k$.

$$y' = \sum_{k=0}^{\infty} (k+s)\, a_k\, x^{k+s-1}$$

$$y'' = \sum_{k=2}^{\infty} (k+s)(k+s-1)\, a_k\, x^{k+s-2}$$

Bessel eq. (substitute).

$$\Rightarrow \sum (k+s)(k+s-1) a_k x^{k+s-2} + \cancel{\tfrac{1}{x}} \sum (k+s) a_k x^{k+s-2} + \left(1 - \frac{n^2}{x^2}\right) \sum a_k x^{k+s}$$

$$\Rightarrow \sum (k+s)(k+s-1) a_k x^{k+s-2} + \sum (k+s) a_k x^{k+s-2} + \sum_{k=2}^{\infty} a_{k-2} x^{k+s-2} \cancel{A} - n^2 \sum a_k x^{k+s-2} = 0.$$

$$\Rightarrow \sum_{k=2}^{\infty} \left((k+s)(k+s-1) a_k + (k+s) a_k + a_{k-2} \cancel{A} - n^2 a_k\right) x^{k+s-2}$$

$$+ s(s-1) a_0 x^{s-2} + s(s+1) a_1 x^{s-1} + s a_0 x^{s-2} + (s+1) a_1 x^{s-1} - n^2 a_0 x^{s-2} - n^2 a_1 x^{s-1} = 0$$

$\Rightarrow$ power series for low power $s-2$.

$$s(s-1) a_0 + s a_0 \cancel{*} - n^2 a_0 = 0 \quad \text{-- first indicial eq.}$$

$$\Rightarrow (s^2 - n^2) a_0 = 0$$

$$s = \pm n$$

for $s-1$

$$s(s+1)a_1 + (s+1)a_1 - n^2 a_1 = 0.$$

$$a_1(s^2+2s+1-n^2) = 0.$$

$$a_1(2s+1) = 0$$

$a_1 = 0$ as s is integer.

d) Recurrence relation.

$$(k+s)(k+s-1)a_k + (k+s)a_k + a_{k-2} - n^2 a_k = 0,$$

e) $a_k = \dfrac{-a_{k-2}}{k(k+2n)}$. for $s=n$

$a_k = \dfrac{-a_{k-2}}{k(k-2n)}$ for $s=-n$.

Q5

$\nabla^2 \phi = 0$

$\phi = \phi(x, y)$.

Seperate variable

$\phi(x,y) = \phi_x(x)\,\phi_y(y)$.

$\Rightarrow \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0$

$\Rightarrow \left(\frac{\partial^2}{\partial x^2}\phi_x\right)\phi_y + \phi_x \frac{\partial^2}{\partial y^2}\phi_y = 0$.

Divide by $\phi_x \phi_y$.

$\Rightarrow \frac{1}{\phi_x}\frac{\partial^2}{\partial x^2}\phi_x + \frac{1}{\phi_y}\frac{\partial^2}{\partial y^2}\phi_y = 0$.

$\Rightarrow \frac{1}{\phi_x}\frac{\partial^2}{\partial x^2}\phi_x = -\lambda^2, \quad \frac{1}{\phi_y}\frac{\partial^2}{\partial y^2}\phi_y = \lambda^2$.

$\frac{\partial^2}{\partial x^2}\phi_x = -\lambda^2\phi_x, \quad \frac{\partial^2}{\partial y^2}\phi_y = \lambda^2\phi_y$.

$\Rightarrow \phi_x = A\sin(\lambda x) + B\cos(\lambda x)$.

As $\phi(x=0, y) = \phi_x(0)\,\phi_y(y) = 0$

$B = 0$.

$\phi(x=a, y) = \phi_x(a)\,\phi_y(y) = 0$.

$\Rightarrow A\sin(\lambda a) = 0$

$\lambda a = n\pi$.

$\Rightarrow \lambda = \frac{n\pi}{a}$

$\phi_x(x) = A\sin\left(\frac{n\pi}{a}x\right)$

$\phi_y = C\cosh(\lambda y) + D\sin(\lambda y)$.

As $\phi(x, y=0) = \phi_x(x)\,\phi_y(0) = 0$.

$C = 0$.

~~$\phi_y(y=b) = D\sinh\left(\frac{n\pi}{a}b\right) = D\cdot\frac{e^{n\pi} - e^{-n\pi}}{2} = V$.~~

~~$\phi(x,b) = A\sin\left(\frac{n\pi}{a}x\right) D\, e^{n\pi} - e^{-n\pi} = V$~~

$\phi_y(y=b) = D\sin\left(\frac{n\pi b}{a}\right)^2$.

let $AD = A$. .. absorb constant.

$$\phi_n(x,y) = A \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right).$$

$$\Rightarrow \phi(x,y) = \sum A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right).$$

$$\phi(x,a) = \sum A_n \cancel{\left(\frac{e^{n\pi} - e^{-n\pi}}{2}\right)} \sinh(n\pi b/a) \sin\left(\frac{n\pi x}{a}\right).$$

$$\rightarrow V.$$

$\cancel{a/2}$

$$\cancel{A_n\left(\frac{e^{n\pi} - e^{-n\pi}}{2}\right)} = \frac{2}{a}\int_0^a V \sin\left(\frac{n\pi}{a}x\right) dx$$

$$A_n \cdot \sinh\left(\frac{n\pi b}{a}\right).$$

$$= -\frac{2V}{a}\cdot\frac{a}{n\pi}\cos\left(\frac{n\pi x}{a}\right)\Big|_0^a$$

$$= -\frac{2V\cancel{a}}{n\pi}\left((-1)^n - 1\right)$$

$$A_n\left(\frac{e^{n\pi} - e^{-n\pi}}{2}\right) = \frac{2V}{n\pi}\left(1 - (-1)^n\right)$$

$$\Rightarrow \cancel{A_n = \frac{4V\left(1-(-1)^n\right)}{n\pi\left(e^{n\pi} - e^{-n\pi}\right)}} \quad A_n = \frac{4V}{n\pi}\frac{\left(1-(-1)^n\right)}{\sinh\left(\frac{n\pi b}{a}\right)},$$

and

$$\phi(x,y) = \sum A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi x}{a}\right).$$