Intermediate Quantum Mechanics(PH3102) End. Sem. Exam

time 2hr 30min

[*Read the questions carefully. Extra credit for clean and precise answers. Marks will be deducted (even no marks) for final answers without any logical steps or physical arguments. Part marking (even full marks) for correct logical steps even if final answer maybe wrong*]

1. a) Observables are represented by what type of operators and why?

b) Write down the eigenstates of free particle Hamiltonian in 1D which are also eigenstates of parity.

c) A mixed ensemble of spin 1/2 system contains 60% up-spin state $|\uparrow\rangle$ and 40% down spin $|\downarrow\rangle$. Here up-spin and down-spin states are eigenstates of $\hat{S}_z$. Write down the density matrix representing this ensemble and find out the average value of x component of spin $\langle \hat{S}_x \rangle$.

d) Consider a particle in infinite potential well from $x = 0$ to $x = L$. When the particle is in the ground state with wavefunction $\psi(x) = \sqrt{\frac{2}{L}} \sin(\frac{\pi}{L}x)$, write down corresponding density matrix in the position representation. Find out average of position $\langle x \rangle$ and average of momentum $\langle p \rangle$.

e) Write down the operator which can transform the wavefunction $\psi(x)$ to $\psi(x + a)$.

(2+2+3+3+2)

2. a)Write down the time dependent Schrödinger equation in momentum representation for $\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$.

b) Show that the state $\mathcal{N}(a^\dagger)^n|0\rangle$ is eigenstate of the Hamiltonian $H = \hbar\omega(a^\dagger a + 1/2)$, where $a = \frac{1}{\sqrt{2}}(\frac{\hat{x}}{l} + i\frac{l}{\hbar}\hat{p})$ and $l = \sqrt{\frac{\hbar}{m\omega}}$. The ground state is denoted by $|0\rangle$.
Find out the eigenvalue and normalisation constant $\mathcal{N}$.

c) From the ladder operator approach obtain the wavefunction $\langle x|0\rangle$ of the ground state.

d) Show that the state $\mathcal{N}e^{\alpha a^\dagger}|0\rangle$ is eigenstate of a, and write the eigenvalue. $(4 + 6 + 3 + 2)$

3. The state $|l, m\rangle$ is simultaneous eigenstate of total angular momentum and its z-component, $\hat{L}^2|l, m\rangle = \hbar^2 l(l + 1)|l, m\rangle$, and $\hat{L}_z|l, m\rangle = \hbar m|l, m\rangle$

a) Show that $\hat{L}_{\pm}|l,m\rangle = \hbar\sqrt{l(l+1)-m(m\pm 1)}|l,m\rangle$

b) Calculate $\langle l,m|\hat{L}_x|l,m\rangle$ and $\langle l,m|\hat{L}_x^2|l,m\rangle$.

c) Write down the matrix representation of $\hat{L}_x$ in the basis of $|l,m\rangle$ for a fixed value $l=1$.

d) Find out eigenvalue of the Hamiltonian $H = \frac{1}{I_1}(\hat{L}_x^2 + \hat{L}_y^2) + \frac{1}{I_2}\hat{L}_z^2$. (4 + 3 + 2 + 2)

4. (a) Show that for spherical symmetric potential $V(r)$ the stationary state Schrödinger equation (in position representation) can be reduced to effective one dimensional form corresponding to the radial coordinate. Write down and draw the effective 1D potential for Hydrogen atom with $V(r) = -\frac{e^2}{r}$

(b) Obtain the average value $\langle \frac{1}{r} \rangle_{nl}$ for eigenstates of Hydrogen atom.

(Hint: Use Feynman-Hellmann theorem)

(c) Using the formalism of question 4(a), find out the ground state energy and wavefunction of spherical hard wall potential well where,

$$\begin{aligned} V(r) &= 0 \quad for\ r \le a \\ &= \infty \quad for\ r > \ a \end{aligned}$$

(5+3+4)

Mathematical formula:

$\nabla^2 = [\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\}]$

In position representation: $\langle\theta\phi|l,m\rangle = Y_{lm}(\theta,\phi)$.

Pauli matrices: $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Bohr radius: $a_0 = \frac{\hbar^2}{e^2 m}$ Ionization energy : $\frac{me^4}{2\hbar^2}$

$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$