Project 2: Classical Mechanics

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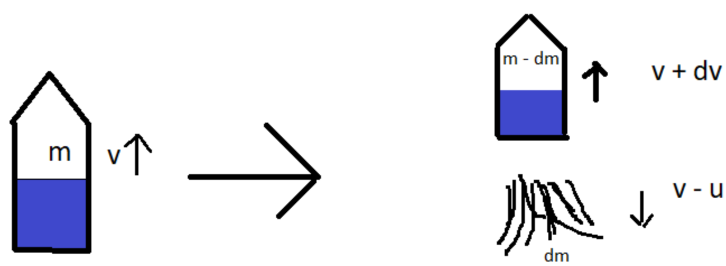
October 26, 2021

The Rocket Equation

Abstract

In this paper, we shall analyse a specific case of a variable mass system, namely a rocket. We shall derive the Tsiolkovsky rocket equation and approach this problem both from a Newtonian Mechanics and Lagrangian Dynamics perspective.



1 Introduction

A simple rocket is modelled as a variable mass system which ejects mass at a constant speed in it's own frame. This ejection generates a force which pushes the rocket further above the ground by countering the gravitational force.

This simple model can be generalised further to include aerial drag, variations in the planet's gravity with height and also the rotation of the rocket.

We will explore the derivations of the simple rocket equations using both the Newtonian and the Lagrangian approach.

2 Analysis & Explanations

2.1 Newtonian approach

For the Newtonian approach, we can use Newton's second law to obtain the equation of motion. We shall assume that the motion is in one dimensions and the velocity of the ejecta is constant with respect to the rocket(v_{ej}).

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \sum F \implies p_f(t+dt) - p_i(t) = \sum F dt$$

If the mass at time 't' is taken as m and velocity as v, then the initial momentum becomes $p_i = mv$.

After time dt, the rocket loses fuel of mass dm at a velocity of $v - v_{ej}$ upwards in the ground frame, this boosts the rocket's velocity by a small dv.

The final momentum is $p_f = (m - dm)(v + dv) + dm(v - v_{ej})$.

$$p_f - p_i = mdv - v_{ej}dm = \sum F dt$$

As the rocket is losing mass,

$$dm = \dot{m}dt = -\frac{\mathrm{d}m}{\mathrm{d}t}dt$$

As for the Force F, we can have anything from drag to the gravitational force, here our force will be uniform gravity $F = -mg$. The equations of motion become,

$$mdv - v_{ej}dm = -mgdt$$

$$mdv = -(v_{ej}\frac{\mathrm{d}m}{\mathrm{d}t} + mg)dt \tag{1}$$

Integrating the equation of motion from time $t = 0$ to $t = t$.

$$\implies \int_{v_0}^{v_t} dv = -v_{ej}\int_{m_0}^{m_t} \frac{dm}{m} - \int_0^t g dt$$

$$\Delta v = -v_{ej}ln(\frac{m_0}{m}) - gt \tag{2}$$

This is the famous Tsiolkovsky rocket equation, the trajectory of the rocket now depends on how the mass is being ejected with time, but overall the velocity only depends on the initial and final masses.
If we take the rate of ejection to be constant, then

$$m = m_0 - \alpha t$$

and integrate the equations of motion

$$y = \int_0^{t_u} v dt$$

$$y = v_{ej}(t - \frac{m_0}{\alpha})ln(m_0 - \alpha t) - v_{ej}(l + ln(m))t - \frac{1}{2}gt^2$$

2.2 Lagrangian approach

The Euler Lagrange Equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q_q$$

We take y as our coordinate for height,
The Lagrangian can be taken as

$$L = T - V = \frac{1}{2}m(t)\dot{y}^2 + mgy$$

and the generalised force can be taken as $Q_q = \dot{m}(v - v_{ej})$ Q_q here is the Exhaust Force which will be the same as that of the exhaust that we took into account in the Newtonian approach. The velocity of the fuel w.r.t. ground is $v_{ejg} = v - v_{ej}$.

Note that, we could have derived this force by taking the total Kinetic energy of the fuel and the rocket system. The Kinetic energy of the fuel (T_{fuel}) would be an integral over the histories of all the ejecta, as the rate of mass loss and the velocity of the rocket is not constant.

$$T_{fuel}(t \longrightarrow t + dt) = \frac{1}{2}dm(v - v_{ej})^2$$

as mass of dm gets ejected every time dt.

$$T_{fuel} = \frac{1}{2}\int_0^t dt'\dot{m}(v - v_{ej})^2$$

Here $\frac{\partial T_{fuel}}{\partial v} \propto \int \dot{m}(v - v_{ej})dt' = f$ and for the derivative wrt. time we use the Leibniz's rule,
$\frac{\mathrm{d}f}{\mathrm{d}t} = \dot{m}(t)(v(t) - v_{ej}) = Q_q$ which is the expression that we took above.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} = \dot{m}\dot{y} + m\ddot{y}$$

$$\frac{\partial L}{\partial y} = mg$$

adding all these up we get our equation motion,

$$m\ddot{y} = \dot{m}v_{ej} - mg$$

$$\implies mdv = -(v_{ej}\frac{\mathrm{d}m}{\mathrm{d}t} + mg)dt$$

Which the same as the equation derived from the Newtonian approach.

As states before, the trajectories depend on how mass changes with time ($m(t)$), but the final velocities depend on the initial and final masses.

3 Conclusion

We did a short analysis on the problem of a simple rocket and derived the Tsiolkovsky rocket equation. We have also confirmed that the Newtonian approach and the Lagrangian approach yields the same answer. We also derived the trajectory for a case of constant fuel ejection.