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19MS151

PH3102
Assignment - 3

Q1 (a). $f$ is analytic

$$f(a) = \sum_{n=0}^{\infty} k_n a^n$$

$$f'(a) = \sum_{n=0}^{\infty} n k_n a^{n-1}$$

$$[A, f(B)] = \sum k_n [A, B^n]$$

$$[A, B^n] = \sum_{k=1}^{n} B^{n-k} [A, B] B^{k-1}$$

$$[A, f(B)] = \sum_{0}^{\infty} k_n \sum_{1}^{n} B^{n-k} [A, B] B^{k-1}$$

$$[B, [A, B]] = 0$$

$$\Rightarrow [B^k, [A, B]] = 0$$

$$= B^{k-1} [B, [A, B]] + [B, [A, B]] B^{k-1} = 0$$

$$\Rightarrow B^{n-k} [A, B] B^{k-1} = B^{n-1} [A, B]$$

$$\Rightarrow [A, f(B)] = \sum k_n \sum_{1}^{n} [A, B] B^{n-1}$$

$$= \sum k_n n [A, B] B^{n-1}$$

$$= [A, B] \left( \sum k_n n B^{n-1} \right)$$

$$= [A, B] f'(B)$$

Q1 (b) $[x, e^{ip_x l/\hbar}] = [x, p_x] \frac{il}{\hbar} e^{ip_x l/\hbar}$

$\because [x, p_x] = i\hbar$

$\Rightarrow [x, e^{ip_x l/\hbar}] = -l e^{ip_x l/\hbar}$

(c) $[x, e^{ip_x a/\hbar}]|x'\rangle = -a e^{ip_x a/\hbar}|x'\rangle$

$x e^{ip_x a/\hbar}|x'\rangle - e^{ip_x a/\hbar} x|x'\rangle = -a e^{ip_x a/\hbar}|x'\rangle$

$\Rightarrow x e^{ip_x a/\hbar}|x'\rangle - e^{ip_x a/\hbar} x'|x'\rangle$

$\hat{x}|x\rangle = x|x\rangle$

$\Rightarrow x e^{ip_x a/\hbar}|x'\rangle = (x'-a) e^{ip_x a/\hbar}|x'\rangle$

$\Rightarrow e^{ip_x a/\hbar}|x'\rangle = |x'-a\rangle$

Q2

Unitary $\Rightarrow$ $U^{\dagger}U = \mathbb{1}$

(a) $\dfrac{1+iA}{1-iA}$ $\Rightarrow$ $\left(\dfrac{1+iA}{1-iA}\right)^{\dagger} = \left(\dfrac{1-iA^{\dagger}}{1+iA^{\dagger}}\right)$

$$\Rightarrow \left(\frac{1+iA}{1-iA}\right)^{\dagger}\left(\frac{1+iA}{1-iA}\right) = \frac{(1-iA)(1+iA)}{(1+iA)(1-iA)} = \mathbb{1}.$$

iff $A = A^{\dagger}$ $\Rightarrow$ A is Hermitian

(b). A, B be hermitian & $[A,B]=0$
i.e. A, B commute.

$$\left(\frac{A+iB}{\sqrt{A^2+B^2}}\right)^{\dagger}\left(\frac{A+iB}{\sqrt{A^2+B^2}}\right) = \frac{(A-iB)(A+iB)}{A^2+B^2}$$

$$= \frac{A^2+B^2+(AB-BA)i}{A^2+B^2} = \mathbb{1}$$

$\Rightarrow$ for $\dfrac{A+iB}{\sqrt{A^2+B^2}}$ to be unitary
A, B are hermitian
& $[A,B]=0$.

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Q10 $\frac{dA_H}{dt} = \frac{i}{\hbar}[H, A_H] + \left(\frac{\partial A_S}{\partial t}\right)_H$

$A_H = U^{+} A_S U$ $\quad U \Rightarrow$ time evolution

$\hat{H} = b\hat{J}_z$.

$\Rightarrow$ ~~$[H, J]$~~

let $J_+ = J_x + iJ_y$, $J_- = J_x - iJ_y$.

using it's commutator relation

$[H, J_-] = b[J_z, J_-] = -b\hbar J_-$

$[H, J_+] = b\hbar J_+$.

$\frac{d}{dt} J_-(H) = \frac{i}{\hbar}[H, J_-(H)] = \frac{i}{\hbar} U^{+}[H, J_-(S)] U$

$= -ibJ_-(H)$

$\frac{d}{dt} J_+(H) = ibJ_+(H)$

$\Rightarrow J_-(H) = J_-(0) e^{-ibt}$, $J_+(H) = J_+(0) e^{ibt}$.

$A_S(0) = A_H(0)$ .....

$J_x = \frac{J_+ + J_-}{2}$, $J_y = \frac{J_+ - J_-}{2i}$

$\Rightarrow J_x = \frac{J_+(0)}{2} e^{ibt} + \frac{J_-(0)}{2} e^{-ibt}$.

$J_y = \frac{J_+(0)}{2i} e^{ibt} - \frac{J_-(0)}{2i} e^{-ibt}$

Q3 $A^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{T^*} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = A$ ∴

$B^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = B$ ⇒ Hermitian

$\det(A - \lambda \mathbb{I}) = 0$

$\Rightarrow \lambda^3 - 2\lambda = 0 \Rightarrow \lambda = 0, \pm\sqrt{2}$ are eigenvalues.

$A\alpha = 0 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = 0$

$\Rightarrow \alpha_2 = 0 \; , \; \alpha_1 + \alpha_3 = 0$

$\alpha^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ ... eigenvector with $\lambda = 0$.

$A\alpha = \sqrt{2}\alpha \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \sqrt{2}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$

$\Rightarrow \alpha_2 = \sqrt{2}\alpha_1 \; , \; \alpha_1 + \alpha_3 = \sqrt{2}\alpha_2$

take $\alpha^2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$ for $\lambda = \sqrt{2}$

for $\lambda = -\sqrt{2}$, $A\alpha = -\sqrt{2}\alpha \Rightarrow \alpha_2 = -\sqrt{2}\alpha_1 = -\sqrt{2}\alpha_3$

$\alpha_1 + \alpha_3 = -\sqrt{2}\alpha_2$

$\Rightarrow \alpha^3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$ for $\lambda = -\sqrt{2}$

Similarly for $B$, $\det(B - \lambda\mathbb{I}) = 0$

$\Rightarrow \lambda = 0, 1, -1$.

(I am not doing any ~~…~~)

with eigenvector

$B\beta = 0 \Rightarrow \beta^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ for $\lambda = 0$

$\beta^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ for $\lambda = 1$

& $\beta^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda = -1$

Q3 (b) $\begin{pmatrix}1\\0\\0\end{pmatrix} = c_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix} + c_3\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$

Solving this gives.

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}1/2\\1/4\\1/4\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = b_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + b_2\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix} + b_3\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{bmatrix}0\\\frac{1}{2\sqrt{2}}\\-\frac{1}{2\sqrt{2}}\end{bmatrix}$$

& $\begin{pmatrix}0\\0\\1\end{pmatrix} = a_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + a_2\begin{pmatrix}1\\\sqrt{2}\\1\end{pmatrix} + a_3\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}-1/2\\1/4\\1/4\end{pmatrix}$$

Unitary transform here is

$$U = \begin{pmatrix}1/2 & 0 & -1/2\\1/4 & 1/2\sqrt{2} & 1/4\\1/4 & -1/2\sqrt{2} & 1/4\end{pmatrix}$$

$\cancel{Q = U}^{+}$ $\quad U^{-1} = U^{\dagger}$

$U^{\dagger}$

$Q9$ (a) $\langle L_x\rangle = \langle L_y\rangle = 0.$

$$\langle L_x\rangle = \frac{1}{2}\langle l,m|L_+|l,m\rangle + \frac{1}{2}\langle l,m|L_-|l,m\rangle$$

$$= 0$$

$\langle L_y\rangle = 0$ similarly. $L_\pm = L_x \pm iL_y.$

$$L_\pm|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}$$

$$L_x^2 = \frac{1}{4}(L_+^2 + L_-^2 + L_+L_- + L_-L_+)$$

$$L_+L_+ = L_x^2 + L_y^2 - \hbar L_z.$$

$$\langle l,m|L_x^2|l,m\rangle = \frac{1}{4}\langle l,m|L_+L_- + L_-L_+|l,m\rangle$$

others are zero

$$L_y^2 = -\frac{1}{4}(L_+^2 + L_-^2 - L_-L_+ - L_+L_-)$$

$$\langle l,m|L_y^2|l,m\rangle = \frac{1}{4}\langle l,m|L_-L_+ + L_+L_-|l,m\rangle$$

$$\langle L_x^2\rangle + \langle L_y^2\rangle + \langle L_z^2\rangle = \langle L^2\rangle$$

$$2\langle L_x^2\rangle = \hbar^2 l(l+1) - \hbar^2 m^2$$

$$\Rightarrow \langle L_x^2\rangle = \langle L_y^2\rangle = \frac{\hbar^2}{2}(l(l+1) - m^2)$$

$$\Delta L_x = \langle L_x^2\rangle - \langle L_x\rangle^2$$

$$\Delta L_x \Delta L_y = \frac{\hbar^2}{2}(l(l+1) - m^2)$$

Q9 (b) $l >> m$, $\Rightarrow l(l+1) - m^2 >> m(m+1) - m^2$

$\Rightarrow l(l+1) - m^2 \geq m$.

$\Delta L_x \Delta L_y \geq \frac{\hbar^2 m}{2}$

$\Delta L_x \Delta L_y \geq \frac{|\langle [L_x, L_y] \rangle|}{2} = \frac{i\hbar \langle L_z \rangle}{2} = \frac{\hbar}{2} |\langle L_z \rangle|$.

now $\langle l, m | L_z | l, m \rangle = m\hbar$.

$\Rightarrow \langle L_z \rangle = m\hbar$.

$\Rightarrow \frac{\hbar}{2} \langle L_z \rangle = \frac{\hbar^2 m}{2}$

$\Rightarrow \Delta L_x \Delta L_y \geq \frac{\hbar}{2} |\langle L_z \rangle| = \frac{\hbar^2 m}{2}$

Q8 (a) ~~$|\alpha\rangle = \sum \langle n|\alpha\rangle |n$~~

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\alpha\rangle$$

$$f(n) = \langle n|\alpha\rangle$$

$$\langle n| a = \sqrt{n+1}\langle n+1|$$

$$\langle n|a|\alpha\rangle = \langle n|\alpha|\alpha\rangle = \sqrt{n+1}\langle n+1|\alpha\rangle$$

$$\langle n|\alpha\rangle = \frac{\sqrt{n+1}}{\alpha}\langle n+1|\alpha\rangle \ldots\ldots$$

$$\langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}}\langle n-1|\alpha\rangle.$$ and so on,

$$\langle n|\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle$$

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum \frac{\alpha^n}{n!}|n\rangle.$$

for coherent state $|\alpha\rangle = D(\alpha)|0\rangle$

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$

$$\langle 0|\alpha\rangle = \langle 0|D(\alpha)|0\rangle$$

$$[\alpha a^\dagger, \alpha^* a] = \alpha\alpha^*[a^\dagger, a] = -|\alpha|^2$$

Now $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{\alpha^* a}$

BCH formula

$$\langle 0|D(\alpha)|0\rangle = e^{-|\alpha|^2/2}\langle 0|(1+\alpha a^\dagger + \frac{\alpha^2 a^{\dagger 2}}{2} + \ldots)(1+\alpha^* a + \frac{\alpha^{*2} a^2}{2} + \ldots)|0\rangle$$

$$= e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}}\langle n| \sum \frac{(\alpha^*)^n}{\sqrt{n!}}|n\rangle$$

$$= e^{-|\alpha|^2/2}$$

$$p(n) = |f|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}$$

Q8 (b) $D(\alpha+\beta) = \exp(\alpha a^{\dagger} - \alpha^{*} a + \beta a^{\dagger} - \beta^{*} a)$

$$= \exp(\alpha a^{\dagger} - \alpha^{*} a)\exp(\beta a^{\dagger} - \beta^{*} a)$$

$$\exp\left(-\tfrac{1}{2}[\alpha a^{\dagger} - \alpha^{*} a, \beta a^{\dagger} - \beta^{*} a]\right).$$

BCH

$$[\ldots] = [\alpha a^{\dagger}, \beta a^{\dagger}] - [\alpha^{*} a, \beta a^{\dagger}] - [\alpha a^{\dagger}, \beta^{*} a] + [\alpha^{*} a, \beta^{*} a]$$

$$= 2i \,\mathrm{im}(\alpha\beta^{*})$$

$$D(\alpha+\beta) = D(\alpha) D(\beta) \exp(-i \,\mathrm{im}(\alpha\beta^{*}))$$

Q.11 Take the creation-annihilation operators

(a)

$$\frac{da}{dt} = i[H, a] = i[\omega(a^\dagger a + \tfrac{1}{2}), a]$$

$$= -i\omega a$$

$$\Rightarrow a_H(t) = a(0) e^{-i\omega t} = a_s e^{-i\omega t}.$$

Similarly. $\frac{da^\dagger}{dt} = i[H, a^\dagger] = i(\omega(a^\dagger a + \tfrac{1}{2}), a^\dagger] = i\omega a^\dagger$

$$a^\dagger_H(t) = a^\dagger(0) e^{i\omega t} = a_s^\dagger e^{i\omega t}$$

$a_s, a_s^\dagger$ are schrodinger operators

$$\hat{X}_H = \sqrt{\frac{\hbar}{2m\omega}} (a_H^\dagger + a_H), \quad \hat{p}_H = i\sqrt{\frac{m\hbar\omega}{2}} (a_H^\dagger - a_H).$$

$$\hat{X}_H = \sqrt{\frac{\hbar}{2m\omega}} (a_s^\dagger e^{i\omega t} + a_s e^{-i\omega t}), \quad \hat{p}_H = i\sqrt{\frac{m\hbar\omega}{2}} (a_s^\dagger e^{i\omega t} - a_s e^{-i\omega t})$$

now write $a_s, a_s^\dagger$ in terms of $x_s, x_p$

& we get.

$$X_H(t) = X_s \cos(\omega t) + \frac{p_s}{m\omega} \sin(\omega t)$$

$$p_H(t) = p_s \cos(\omega t) - m\omega X_s \sin(\omega t).$$

(b)

$$\cancel{[X_H(t_1), p_H(t_2)] = X_s p_s \cos^2(\omega t) + \frac{p_s^2}{m\omega} \sin(\omega t)\cos(\omega t)}$$

$$\cancel{[X_H(t_1), p_H(t_2)] = \frac{i\hbar}{2} ((a_{H t_1}^\dagger + a_{H t_1})(a_{H t_2}^\dagger - a_{H t_2}) - (a_{H t_2}^\dagger - a_{H t_2})(a_{H t_1}^\dagger + a_{H t_1}))}$$

$$\cancel{= \frac{i\hbar}{2} ( a_{H t_1}^\dagger a_{H t_2}^\dagger - a_{H t_1}^\dagger a_{H t_2} + a_{H t_1} a_{H t_2}^\dagger - a_{H t_1} a_{H t_2} - a_{H t_2}^\dagger a_{H t_1}^\dagger}$$

$$[X_H(t_1), p_H(t_2)] = \cos(\omega t_1)\cos(\omega t_2)[X, P] - \sin(\omega t_2)\sin(\omega t_1)[P, X]$$

$$= i\hbar(\cos(\omega t_1)\cos(\omega t_2) + \sin(\omega t_1)\sin(\omega t_2))$$

$$[X_H(t_1), p_H(t_2)] = i\hbar \cos(\omega(t_1 - t_2))$$

$$[x_H(t_1), x_H(t_2)] = \frac{\sin(\omega t_1)\cos(\omega t_2)}{m\omega}[p,x] + \frac{\cos(\omega t_1)\sin(\omega t_2)}{m\omega}[x,p]$$

$$= \frac{i\hbar}{m\omega}\sin(\omega(t_2-t_1))$$

$$[p_H(t_1), p_H(t_2)] = [p\cos(\omega t_1) - m\omega x\sin(\omega t_1),\ p\cos(\omega t_2) - m\omega x\sin(\omega t_2)]$$

$$= im\omega\hbar\sin(\omega(t_1-t_2))$$

(e) $\langle n| x_H(t), x_H(0)|n\rangle = \langle n|\left(x\cos(\omega t) + \frac{p}{m\omega}\sin(\omega t)\right)x|n\rangle$

$$= \cos(\omega t)\langle n|x^2|n\rangle + \sin(\omega t)\langle n|px|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|a^2 + a^{+2} + aa^+ + a^+a|n\rangle + \frac{i\hbar}{2}\langle n|(a+a^+)(a-a^+)|n\rangle$$

$$+ \frac{i\hbar}{2}\langle n|[a^+,a]|n\rangle$$

$$= \frac{\hbar}{2m\omega}\langle n|aa^+ + a^+a|n\rangle$$

$$= \begin{cases} \frac{\hbar}{2m\omega}\left(\frac{1}{n+1}\right), & n=0 \\ \frac{\hbar}{2m\omega}\left(\frac{1}{n+1}+\frac{1}{n}\right), & n\neq 0 \end{cases} \quad - \frac{i\hbar}{2}$$

$$\Rightarrow \langle n|x_H(t)\, x_H(0)|n\rangle = \begin{cases} \frac{\hbar}{2m\omega}\cos(\omega t) - \frac{i\hbar}{2}\sin\omega t & n=0 \\ \frac{\hbar}{2m\omega}\left(\frac{2n+1}{n^2+n}\right)\cos\omega t - \frac{i\hbar}{2}\sin(\omega t) & n\neq 0 \end{cases}$$

Q5 $i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$

$$\psi(x,t) = \int\frac{d^3p}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\phi(p,t).$$

$$i\hbar\frac{\partial}{\partial t}\int\frac{d^3p}{\sqrt{2\pi\hbar}^3} e^{ipx/\hbar}\phi(p,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\int\frac{d^3p}{\sqrt{2\pi\hbar}^3} e^{ipx/\hbar}\phi(p,t) + V(x)\int\frac{d^3p}{\sqrt{2\pi\hbar}^3} e^{ipx/\hbar}\phi(p,t).$$

do Fourier trans.

$$\hookrightarrow \phi(p,t) = \int\frac{d^3x}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\psi(x,t).$$

$$V(x)\int\frac{d^3p}{\sqrt{2\pi\hbar}^3} e^{ipx/\hbar}\phi(p,t) = \int\frac{d^3p}{\sqrt{2\pi\hbar}^3} e^{ipx/\hbar}\left(\int\frac{d^3x}{\sqrt{2\pi\hbar}^3} e^{-ipx/\hbar} V(x')\psi(x',t)\right)$$

this is a ~~contraction~~ convolution. *

$$\Rightarrow \quad = \int\frac{dp}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\left(V(p)*\phi(p,t)\right)$$

$$\int\frac{dp}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\left(i\hbar\frac{\partial}{\partial t}\phi(p,t)\right) = -\int dp\,\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\phi(p,t)\right) + \int\frac{dp}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\left(V(p)*\phi(p,t)\right)$$

$$\Rightarrow \quad i\hbar\frac{\partial}{\partial t}\phi(p,t) = \frac{p^2}{2m}\phi(p,t) + V(p)*\phi(p,t)$$

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$$[x,p] = i\hbar \Rightarrow \langle k|[x,p]|\psi\rangle$$

$$i\hbar\langle k|\psi\rangle = \langle k|xp - px|\psi\rangle$$

$$\langle k|xp|\psi\rangle = \int_{-\infty}^{\infty}\langle k|x|x\rangle\langle x|p|\psi\rangle dx = i\hbar\int_{-\infty}^{\infty}dx\frac{\partial}{\partial x}(x e^{ikx/\hbar})\psi(x)$$

$$= i\hbar\psi(k) - k\int_{-\infty}^{\infty}dx\, x e^{ikx}\psi(x).$$

$$x e^{ikx} = \frac{\partial}{\partial k}e^{ikx}\, i\hbar, \Rightarrow \langle k|xp|\psi\rangle = i\hbar\psi(k) + i\hbar k\frac{\partial}{\partial k}\psi(k)$$

~~$\langle k|x|\psi\rangle = i\hbar\partial$~~ $\quad i\hbar\cancel{\langle k|\psi\rangle} = \cancel{i\hbar\psi(k)} + i\hbar k\frac{\partial}{\partial k}\psi(k) - k\langle k|x|\psi\rangle$

$$\Rightarrow \langle k|x|\psi\rangle = i\hbar\frac{\partial}{\partial k}\psi(k)$$

$$E = ($$

4) $H = E b^{+} b$

$$= (b^{+} \; b)\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b^{+} \\ b \end{pmatrix}$$

$$H = \varepsilon_0 a^{+} a + \Delta (a^2 + a^{+2})$$

$$= (a^{+} \; a)\begin{pmatrix} \Delta & \varepsilon_0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} a^{+} \\ a \end{pmatrix}$$

Now $a = A\hat{b} + B b^{+}$, $a^{+} = A^{*} b^{+} + B^{*} b$

$$\Rightarrow \begin{pmatrix} a^{+} \\ a \end{pmatrix} = \begin{pmatrix} A^{*} & B^{*} \\ B & A \end{pmatrix}\begin{pmatrix} b^{+} \\ b \end{pmatrix}$$

$$(b^{+} \; b)\begin{pmatrix} A^{*} & B \\ B^{*} & A \end{pmatrix} = (a^{+} \; a)$$

Bogulibov trans

$$a a^{+} - a^{+} a = 1 = [a, a^{+}]$$

$$\Rightarrow AA^{*} b b^{+} + AB^{*} b^{2} + BA^{*} b^{+2} + BB^{*} b^{+} b$$

$$- AA^{*} b^{+} b - A^{*} B b^{+2} - AB^{*} b^{2} - BA^{*} b b^{+}$$

$$= (AA^{*} - BB^{*})([b, b^{+}]) = \Rightarrow$$

$$\underline{|A|^2 - |B|^2 = 1}$$

$$H = (b^{+} b)\begin{pmatrix} A^{*} & B \\ B^{*} & A \end{pmatrix}\begin{pmatrix} \Delta & \varepsilon_0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} A^{*} & B^{*} \\ B & A \end{pmatrix}\begin{pmatrix} b^{+} \\ b \end{pmatrix}$$

$$= (b^{+} b)\begin{pmatrix} (A^{*})^2\Delta + A^{*} B \varepsilon_0 + B^2 \Delta & A^{*} B^{*} \Delta + |A|^2 \varepsilon_0 + AB\Delta \\ B^{*} A^{*} \Delta + |B|^2 \varepsilon_0 + AB\Delta & (B^{*})^2 \Delta + B^{*} A \varepsilon_0 + A^2 \Delta \end{pmatrix}\begin{pmatrix} b^{+} \\ b \end{pmatrix}$$

$$= (b^{+} b)\begin{pmatrix} 0 & E \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b^{+} \\ b \end{pmatrix}$$

$$E = \left(|A|^2 - |B|^2\right)\varepsilon_0 \Rightarrow \underline{\underline{E = \varepsilon_0}}$$

Energy eigenvalues

Q6 $\psi(x,0) = \psi^{SHO}_{n=0} \equiv \frac{1}{\sqrt{2^0 0!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$ ...

(a) $\langle x|\psi(t)\rangle = \psi(x,t) = ?$

$$\phi(k,0) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, \psi(x,0)\, e^{-ikx}$$ ... go to momentum space.

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \cdot e^{-ikx} \cdot e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\frac{\pi\, 2\hbar}{m\omega}\right)^{1/2}$$

... $\left(\int_{-\infty}^{\infty} dx\, e^{-ikx} e^{-ax^2} = \sqrt{\frac{\pi}{a}} \text{ identity}\right)$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{k^2\hbar}{2m\omega}} \left(\frac{\cancel{\pi}\, 2\hbar}{2\cancel{\pi} m\omega}\right)^{1/2}$$

$$= \left(\frac{m\omega}{\pi\hbar}\cdot\frac{\hbar^2}{m^2\omega^2}\right)^{1/4} e^{-\frac{k^2\hbar}{2m\omega}}$$

$$\phi(k,0) = \left(\frac{\hbar}{\pi m\omega}\right)^{1/4} \cdot e^{-\frac{\hbar k^2}{2m\omega}}$$

Now for $H = \frac{p^2}{2m}$, the momentum space is the eigenspace

$\Rightarrow$ Time evolution is just $\phi(k,0)\, e^{-\frac{iE_k t}{\hbar}}$

$E_k = \frac{\hbar^2 k^2}{2m}$

$$\Rightarrow \psi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, \phi(k)\, e^{ikx} \cdot e^{-iE_k t/\hbar}$$

for $\psi(x,0) = \left(\frac{2a}{\pi}\right)^{1/4} e^{-ax^2}$

$\phi(k) = \left(\frac{1}{2\pi a}\right)^{1/4} e^{-\frac{k^2}{4a}}$

exponent $\Rightarrow -\frac{k^2}{4a}\left(1 + \frac{2i\hbar a t}{m}\right) \Rightarrow b = \frac{1}{2a}\left(1 + \frac{2i\hbar a t}{m}\right)$

$\Rightarrow -\frac{bk^2}{2}$

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$$\psi(x,t) = \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}}\, e^{-bk^2/2}\, e^{ikx}$$

$$= \left(\frac{1}{2\pi a}\right)^{1/4} \frac{1}{\sqrt{b}} \cdot e^{-\frac{x^2}{2b}}$$

$$\text{--}\left( \int_{-\infty}^{\infty} dk\, e^{-bk^2/2} e^{ikx} = \sqrt{\frac{2\pi}{b}}\, e^{-\frac{x^2}{2b}} \quad \text{identity} \right)$$

$$\Rightarrow \psi(x,t) = \sqrt{\frac{2a}{\pi}}\, \frac{1}{\sqrt{1+\left(\frac{2\hbar a t}{m}\right)^2}}\, e^{-\frac{2ax^2}{1+\left(\frac{2\hbar a t}{m}\right)^2}}$$

$$a = \frac{m\omega}{2\hbar}$$

$$\psi(x,t) = \sqrt{\frac{m\omega}{\pi\hbar}}\, \frac{1}{\sqrt{1+\left(\frac{2\hbar}{m}\frac{m\omega}{2\hbar}t\right)^2}}\, e^{-\frac{2\cdot\frac{m\omega}{2\hbar}x^2}{1+(\omega t)^2}}$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\, \frac{1}{\sqrt{1+\omega^2t^2}}\, e^{-\frac{m\omega x^2}{\hbar(1+(\omega t)^2)}}$$

(b) for a gaussian $e^{-\frac{x^2}{\sigma^2}}$ $\quad \frac{1}{2\sigma^2} = \langle x^2\rangle$

$$\Rightarrow \sigma_x^2(t) = \langle x^2\rangle - \langle x\rangle^2 = \langle x^2\rangle = \frac{1}{4a}\left(1+\left(\frac{2\hbar a t}{m}\right)^2\right)$$

$$\langle k^2\rangle = a \quad \Rightarrow \langle p^2\rangle = \hbar^2 a$$

$$\Rightarrow \sigma_p^2(t) = \hbar^2 a$$

$$\Rightarrow \sigma_x^2(t) = \frac{\hbar}{2m\omega}(1+\omega^2t^2), \quad \sigma_p^2 = \frac{m\omega\hbar}{2}$$

$$(\sigma_x\sigma_p)(t) = \frac{\hbar}{2}\sqrt{1+\omega^2t^2}$$

at $t=0$, it's a gaussian with minimum uncertainty

Q4 $[A, f(B)] = [A, B] f'(B)$ $f$ is analytic

now $[e^{-\lambda \hat{a}}, a^\dagger] = -[a^\dagger, e^{-\lambda a}] = \frac{d}{da} e^{-\lambda a} = -\lambda e^{-\lambda a}$.

$[e^{-\lambda a^\dagger}, a] = -\frac{d}{da^\dagger} e^{-\lambda a^\dagger} = \lambda e^{-\lambda a^\dagger}$

$a f(a^\dagger)|0\rangle = ([a, f(a^\dagger)] + f(a^\dagger) a)|0\rangle = \left(\frac{d}{da^\dagger} f(a^\dagger)\right)|0\rangle$

$= \left(\frac{df}{dx}\right)_{x=a^\dagger} |0\rangle$

$e^{\lambda \hat{a}} f(a^\dagger)|0\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} a^n f(a^\dagger)|0\rangle$

repeat process

$= |0\rangle + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left(\frac{df}{dx}\right)_{x=a^\dagger} \quad a^{n-1} \left(\frac{df}{dx}\right)_{x=a^\dagger} |0\rangle = \left(\cdots a^{n-2} \frac{\partial^2 f}{\partial x^2}\right)$

$= \cdots$ repeat $\forall n$ $= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\frac{d^n f}{dx^n}\right)_{x=a^\dagger} |0\rangle = f(a^\dagger + \lambda)|0\rangle$

Taylor expansion of $f(a^\dagger + \lambda)$ around $a^\dagger$.