Classtest -2 PH3103.

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Q1 (a) Sturm Liouville form.

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y = 0.$$

$$p(x)\frac{d^2}{dx^2}y + \frac{dy}{dx}p'(x) + q(x)y = 0.$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

$$\Rightarrow x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \left(x - \frac{n^2}{x}\right)y = 0 \quad \text{divide by } x.$$

$$\Rightarrow \frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(x - \frac{n^2}{x}\right)y = 0$$

$$p(x) = x, \quad q(x) = x - \frac{n^2}{x}$$

(b) Normal form — we transform $y \rightarrow u$.

$$u'' + v(x)u = 0 \quad \text{is normal.}$$

define $$u(x) = \frac{y(x)}{\exp\left(\int^x -\frac{1}{2}P(t)dt\right)}$$

and $$v(x) = Q(x) - \frac{1}{4}P^2(x) - \frac{1}{2}P'(x)$$ to convert Bessel eq into normal.

for $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0.$$

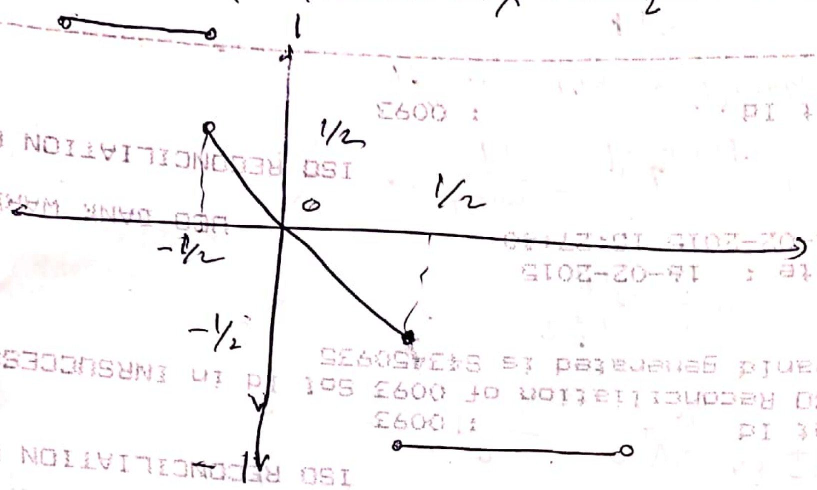
where, $P(x) = \frac{1}{x}$, $Q(x) = 1 - \frac{n^2}{x^2}$.

$$\Rightarrow v(x) = 1 - \frac{n^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2} = 1 + \frac{1-4n^2}{4x^2}$$

$$\Rightarrow u'' + \left(1 + \frac{1-4n^2}{4x^2}\right)u = 0$$

Q2 Fourier series

$$f(x) = \begin{cases} -1 & x > \frac{1}{2} \\ -x & x \in [-\frac{1}{2}, \frac{1}{2}]. \\ +1 & x < -\frac{1}{2} \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum a_n \cos n\pi x + \sum b_n \cos n\pi x.$$

$f(x)$ is odd $\Rightarrow$ $a_n = 0$.

$$b_n = \int_{-1}^{1} f(x) \sin(n\pi x)\,dx = 2\int_{0}^{1} f(x) \sin n\pi x\,dx$$

$$= -2\left(+\int_0^{1/2} x \sin(n\pi x)\,dx + \int_{1/2}^{1} \sin(n\pi x)\,dx \right)$$

$$= -2\left(\left. -\frac{x\cos n\pi x}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2} \right|_0^{1/2} - \left. \frac{\cos n\pi x}{n\pi} \right|_{1/2}^{1} \right)$$

$$b_n = -2\left(\frac{\cos(n\pi/2)}{2n\pi} + \frac{\sin(n\pi/2)}{n^2\pi^2} - \frac{\cos n\pi}{n\pi} \right)$$

$n =$ odd $\Rightarrow$ $b_n = -2\left(\frac{(-1)^n}{n^2\pi^2} + \frac{1}{n\pi} \right)$.

$n =$ even $=$ $b_n = -2\left(\frac{(-1)^{n/2}}{n^2\pi^2} - \frac{1}{n\pi} \right)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$= \sum_{n=1}^{\infty} \left(\left(\frac{2}{n\pi}\left((-1)^n - \frac{1}{2}\cos\frac{n\pi}{2}\right) - \frac{2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \right) \sin(n\pi x) \right)$$

Q1 $m\frac{d^2y}{dx^2} + \gamma\frac{dy}{dx} + \kappa y = 0$

Ansatz : $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$, $y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$.

$\Rightarrow m\sum a_n n(n-1)x^{n-2} + \gamma\sum n a_n x^{n-1} + \kappa\sum a_n x^n = 0.$

coefficients for each power.

x^0 : $2ma_2 + \gamma a_1 + \kappa a_0 = 0$.

x^1 : $6ma_3 + 2\gamma a_2 + \kappa a_1 = 0$.

x^2 : $12ma_4 + 3\gamma a_3 + \kappa a_2 = 0$

x^3 : $20ma_5 + 5\gamma a_4 + \kappa a_3 = 0$

$a_2 = \frac{-\gamma a_1 - \kappa a_0}{2m}$, $\Rightarrow a_3 = \frac{\gamma\kappa a_0 + (\gamma^2 - m\kappa)a_1}{6m^2}$

and so on.

$\Rightarrow y(x) = a_0 + a_1 x - \frac{\gamma a_1 + \kappa a_0}{2m}x^2 + \frac{\gamma\kappa a_0 + (\gamma^2 - m\kappa)a_1}{6m^2}x^3 + \cdots$

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Q3 $f(x) = \dfrac{\sin \pi x}{\pi x}$

$$\hat{f}(\nu) = \int_{-\infty}^{\infty} dx \cdot \frac{\sin \pi x}{\pi x} e^{-i(2\pi\nu x)}.$$

Fourier trans

$$\sin \pi x \cdot \frac{e^{i\pi x} - e^{-i\pi x}}{2i}$$

$$\Rightarrow \hat{f}(\nu) = \int_{-\infty}^{\infty} dx \cdot \frac{e^{i\pi x} - e^{-i\pi x}}{2i\pi x} \cdot e^{-i 2\pi \nu x}$$

$$= \frac{1}{2i\pi}\left[\int_{-\infty}^{\infty} dx \frac{e^{i\pi(1-2\nu)x}}{x} + \int_{-\infty}^{\infty} dx \frac{e^{-i\pi(1+2\nu)x}}{x}\right]$$

Contour integral.

$$\oint_C dz \frac{e^{i\pi(1-2\nu)z}}{z}.$$

take contour.

$$\Rightarrow \int_{C_R} dz \frac{e^{i\pi(1-2\nu)z}}{z} + \int_{C_r} dz \frac{e^{i\pi(1-2\nu)z}}{z} + \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) dx \cdot \frac{e^{i\pi(1-2\nu)x}}{x}.$$

$$\int_{C_R} dz \frac{e^{i\pi(1-2\nu)x}}{x} = i\int_0^{\pi} d\theta \cdot e^{i\pi(1-2\nu)R(\cos\theta + i\sin\theta)} = 0$$

$R \to \infty$ by Jordan's lemma

for $1 - 2\nu > 0$

$$\Rightarrow \int_{-\infty}^{\infty} dx\, e \frac{e^{i\pi(1-2\nu)x}}{x} + \int_{C_r} dz \frac{e^{i\pi(1-2\nu)x}}{x} = 0.$$

$$+ i\int d\theta\, e^{i\pi(1-2\nu)\varepsilon e^{i\theta}} = -i\pi$$

$\varepsilon \to 0$

$$\Rightarrow \int_{-\infty}^{\infty} dx \frac{e^{i\pi(1-2\nu)x}}{x} = i\pi$$

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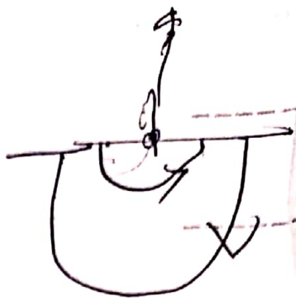
for $1 - 2\nu < 0$.

$\oint$ pick a flipped contour.



and $\int_{C_r} \to \int_0^{\pi} = i\pi$

$$\Rightarrow \int_{-\infty}^{\infty} dx \, \frac{e^{i\pi(1-2\nu)x}}{x} = -i\pi \qquad 1 - 2\nu < 0.$$

$$\Rightarrow \int_{-\infty}^{\infty} dx \, \frac{e^{i\pi(1-2\nu)x}}{x} = \begin{cases} -i\pi & 1 - 2\nu < 0 \\ i\pi & 1 - 2\nu > 0 \end{cases}$$

& similarly

$$\int_{-\infty}^{\infty} dx \, \frac{e^{-i\pi(1+2\nu)x}}{x} = \begin{cases} -i\pi & 1 + 2\nu < 0 \\ i\pi & 1 + 2\nu > 0. \end{cases}$$

$$\Rightarrow \hat{f}(\nu) = \int_{-\infty}^{\infty} dx \, \frac{\sin \pi x}{\pi x} \, e^{-i2\pi\nu x} = \begin{cases} 1 & |\nu| < \frac{1}{2} \\ 0 & |\nu| > \frac{1}{2} \end{cases}$$

Fourier transform

--- (different sources take different fourier trans. some take $f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty} e^{-i\omega x} f(x) \, dx$ etc).

in that case

$$f(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$$