Assignment 3

Q1) (a) For any two observables $\hat{A}$ and $\hat{B}$ satisfying the commutation $[\hat{B}, [\hat{A}, \hat{B}]] = 0$, prove that $[\hat{A}, f(\hat{B})] = [\hat{A}, \hat{B}]f'(\hat{B})$, where f is an analytic function.

(b) Using the above relation, evaluate the commutator $[\hat{x}, e^{i\hat{p}_x l/\hbar}]$

(c) Given that $\hat{x}\,|x'\rangle = x'\,|x'\rangle$, prove that $e^{i\hat{p}_x a/\hbar}\,|x'\rangle = |x' - a\rangle$ using the relation obtained from (b).

Q2) Find the condition(s) for which the following operators to be unitary. (a)$(I + i\hat{A})/(I - i\hat{A})$ (b) $(\hat{A} + \hat{B})/\sqrt{\hat{A}^2 + \hat{B}^2}$

Q3) Consider two matrices A and B given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{1}$$

(a) Check whether A and B are Hermitian or not. Calculate the eigenvalues and eigenvectors of both the matrices.

(b) Find a suitable unitary transformation U, which will change the eigenvectors of matrix A to those of B.

Q4) Prove that $e^{\lambda\hat{a}} f(\hat{a}^\dagger)\,|0\rangle = f(\hat{a}^\dagger + \lambda)\,|0\rangle$, where f is an analytical function and $[\hat{a}, \hat{a}^\dagger] = 1$. (Hint: you may use $[\hat{A}, f(\hat{B})] = [\hat{A}, \hat{B}]f'(\hat{B})$ for $[\hat{B}, [\hat{A}, \hat{B}]] = 0$)

Q5) Write down the Schrödinger equation in momentum representation. Also find the momentum representation of the position operator.

Q6) Consider the ground state of 1D harmonic oscillator with frequency ω. Suddenly the harmonic confinement is switched off ($\omega = 0$), so that the wavefunction evolves freely with time t.

(a) Find out $\langle x|\psi(t)\rangle$.

(b) Evaluate $\sigma_x(t)$, $\sigma_p(t)$ and $\sigma_x(t)\sigma_p(t)$.

Q7) (a) Find an expression for the position and momentum Heisenberg operator $\hat{x}_{\mathrm{H}}(t)$ and $\hat{p}_{\mathrm{H}}(t)$ for a 1D harmonic oscillator, in terms of the Schrödinger operators $\hat{x}$ and $\hat{p}$.

(b) Using the above obtained expressions, evaluate the following commutators: i) $[\hat{x}_{\mathrm{H}}(t_1), \hat{p}_{\mathrm{H}}(t_2)]$**, ii)** $[\hat{x}_{\mathrm{H}}(t_1), \hat{x}_{\mathrm{H}}(t_2)]$**, iii)** $[\hat{p}_{\mathrm{H}}(t_1), \hat{p}_{\mathrm{H}}(t_2)]$

(c) Evaluate the quantity $\langle n| \hat{x}_{\mathrm{H}}(t)\hat{x}_{\mathrm{H}}(0) |n\rangle$ **for the** n^{th} **excited state of a 1D harmonic oscillator.**

Q8) Given that $|\alpha\rangle$ **is a coherent state** ($a|\alpha\rangle = \alpha|\alpha\rangle$) **of 1D harmonic oscillator, then prove the following,**

(a) By writing $|\alpha\rangle = \sum_{n=0}^{\infty} f(n)|n\rangle$**, prove that** $|f(n)|^2 = |\alpha|^{2n} e^{-|\alpha|^2/2}/n!$**. Find the most probable value of** n **and hence of the total energy** E**.**

(b) Given that $D(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^\star \hat{a}}$**, prove that** $D(\alpha+\beta) = D(\alpha)D(\beta)e^{-i\,\mathrm{Im}\{\alpha\beta^\star\}}$

Q9) Given that $\hat{L}$ **is the angular momentum operator,**

(a) Show that $\Delta\hat{L}_x\Delta\hat{L}_y = \hbar^2[l(l+1) - m^2]/2$**, where** $\Delta L_i = \sqrt{\langle \hat{L}_i^2 - \left\langle \hat{L}_i \right\rangle^2}$**.**

(b) Show that this relation is consistent with $\Delta\hat{L}_x\Delta\hat{L}_y \geq (\hbar/2)\left|\left\langle \hat{L}_z \right\rangle\right| = \hbar^2 m/2$

Q10) Consider the angular momentum operator $\hat{L}$ **and the Hamiltonian** $\hat{H} = b\hat{L}_z$**, then evaluate the operators** $\hat{L}_y(t)$ **and** $\hat{L}_x(t)$ **at time** $t \neq 0$**.**

Q11) Find the eigenvalue of Hamiltonian $\hat{H} = \mathcal{E}_0\hat{a}^\dagger\hat{a} + \Delta(\hat{a}^2 + \hat{a}^{\dagger 2})$**, where** $[\hat{a}, \hat{a}^\dagger] = 1$ **and** $\mathcal{E}_0$**,** Δ **are real.**
(Hint: consider $\hat{a} = A\hat{b} + B\hat{b}^\dagger$**, where** A **and** B **are generally complex, such that** $\hat{H}$ **can be written in a diagonal form** $\hat{H}_{\mathrm{D}} = E\hat{b}^\dagger\hat{b}$ **with** E **being the eigenvalue. Note that the new operators** b **and** $b^\dagger$ **must satisfy** $[\hat{b}, \hat{b}^\dagger] = 1$**)**