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PH3102
Quantum II

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Assignment - II

1) (a) Say we are in a cubic cavity of volume $V = L^3$ (for simplicity).

Due to the existence of the walls of this finite cavity, the only wavelengths that can exist inside are of order $n = \frac{2L}{\lambda}$.

(standing waves).
and for the 3 directions modes are
$$n_x = \frac{2L}{\lambda}, \quad n_y = \frac{2L}{\lambda}, \quad n_z = \frac{2L}{\lambda}$$

λ: wavelength of ~~mode~~ standing wave.
No. of waves in $\lambda \to \lambda + d\lambda$) is Volume of the sphere in k-space divided by 8 for one quadrant.

$$\Rightarrow \text{number} = 2 \times \frac{1}{8} 4\pi n^2 dn = \left(\frac{2L}{c}\right)^2 \frac{2L}{c} d\nu$$

8 comes from the one quadrant of the sphere
2 comes from the 2 polarisations our wave can have.
Alternatively, one can get this from the wave equation

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} + k^2 E = 0$$
, E is electric field.

$E = E(x,y,z)$ follows wave equation.
The solution inside a cavity will be a standing wave.

$$E = A \sin(k_x x)\sin(k_y y)\sin(k_z z).$$
$$k^2 = k_x^2 + k_y^2 + k_z^2$$
and $k_x = \frac{\pi n_x}{L}$, $k_y = \frac{\pi n_y}{L}$, $k_z = \frac{\pi n_z}{L}$, n_x, n_y, n_z are the modes

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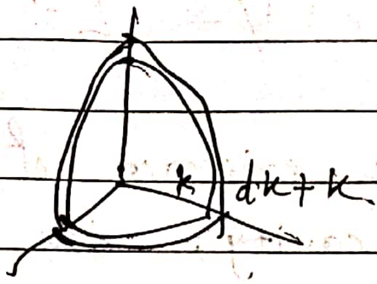
in k-space

Volume of octant in
$k \to k+dk$



$= \dfrac{\frac{1}{8}(4\pi k^2)\,dk}{(\pi/L)^3}$

$= \dfrac{L^3 k^2\,dk}{2\pi^2}$: No. of standing waves from $k \to k+dk$

$\Rightarrow$ No. of standing waves in volume $V = L^3$

$= \dfrac{k^2\,dk}{2\pi^2}$

Now $k = \dfrac{2\pi}{\lambda}$. No $= \dfrac{8\pi}{\lambda^2}\,d\lambda$

$\Rightarrow$ $n(\lambda)\,d\lambda = \dfrac{8\pi}{\lambda^4}\,d\lambda.$

or $k = \dfrac{2\pi\nu}{c}$ $\Rightarrow$ $n(\nu)\,d\nu = \dfrac{8\pi\nu^2}{c^3}\,d\nu$ per volume

$\Rightarrow$ No. of modes of EM radiation in volume
$V = \dfrac{8\pi\nu^2}{c^3}(V)$ in $\nu \to \nu + d\nu$.

(b) intensity of radiation at frequency ν at temperature T.

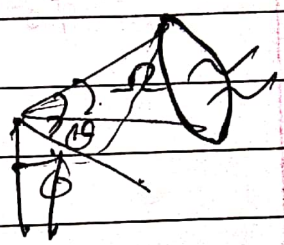
$I(\nu,T) = \dfrac{2h\nu^3}{c^2}\,\dfrac{1}{e^{h\nu/k_BT}-1}$

Power radiated over solid angle $d\Omega$ for frequency between $\nu \to \nu + d\nu$ over area A is

$dP = I(\nu,T)\,A\,d\nu\,d\Omega$

$\frac{P}{A} \Rightarrow$ power radiated per area

$$\frac{P}{A} = \int_0^\infty I(\nu,T)\, d\nu \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$



$$= 2\pi \cdot \frac{1}{2} \int_0^\infty I(\nu,T)\, d\nu$$

$$= \pi \frac{2h}{c^2} \int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{k_B T}}-1}\, d\nu$$

$$= \frac{2\pi k^3 T^3}{h^2 c^2} \int_0^\infty \frac{\left(\frac{h\nu}{k_B T}\right)^3}{e^{\left(\frac{h\nu}{k_B T}\right)^3}-1}\, d\nu =$$

Looking over at a table of integrals.

$$\int_0^\infty \frac{u^3}{e^u-1}\, du = \Gamma(4)\,\zeta(4) = \frac{\pi^4}{15}$$

$$\Rightarrow \frac{P}{A} = \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 = \underline{\sigma T^4}$$

σ: stefan boltzmann constant

$P = A\sigma T^4$ is Stefan Boltzmann law.

2)

(a)

$$V(x) = \begin{cases} 0 & -\infty \leq x \leq -a/2 \\ V_0 & -a/2 \leq x \leq a/2 \\ 0 & a/2 \leq x \leq \infty. \end{cases}$$

$$\bar{V}(k) = \int_{-\infty}^{\infty} V(x)\, e^{-ikx}\, dx = \int_{-a/2}^{a/2} V_0\, e^{-ikx}\, dx$$

$$= -\frac{V_0}{ik}\left(e^{-ikx}\right)_{-a/2}^{a/2} = -\frac{V_0}{ik}\cdot\left(e^{-i\frac{ka}{2}} - e^{i\frac{ka}{2}}\right)$$

$$= \frac{2V_0}{k}\sin\left(\frac{ka}{2}\right)$$

(b) $V_0 a = 1$

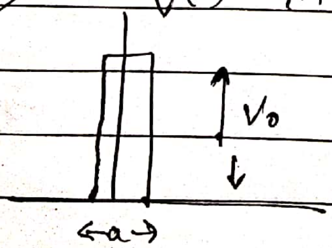
$\Rightarrow V_0 = \frac{1}{a}$

$$\Rightarrow \bar{V}(k) = \frac{2V_0}{k}\sin\left(\frac{ka}{2}\right) = \frac{2}{ka}\sin\left(\frac{ka}{2}\right) = \frac{\sin\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)}$$

$V_0 \to \infty \Rightarrow a \to 0$,

and $\lim_{a\to 0} \bar{V}(k) = \frac{\sin\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)} = 1$

$\Rightarrow \bar{V}(k)$ for $a \to 0$, $V_0 \to \infty$ $= 1$.



$V_0 \to \infty$, $V_0 a = 1$, $a \to 0$ is one way to arrive at the Dirac-delta function.

$\lim_{\substack{a\to 0 \\ V_0 \to \infty}} \bar{V}(k) = 1$ is the fourier transform of $\delta(x) = V(x)$.

4) (a) $\frac{L_z^2}{2I}\psi = E\psi$

$$L_z^2 = -\hbar^2\frac{\partial^2}{\partial\theta^2}$$

$$-\frac{\hbar^2}{2I}\cdot\frac{\partial^2}{\partial\theta^2}\psi = E\psi$$

$$\frac{\partial^2\psi}{\partial\theta^2} = -\frac{2IE}{\hbar^2}\psi$$

$$\psi = a e^{im\theta} + b e^{-im\theta}$$

$$\psi(\theta) = \psi(\theta + 2\pi)$$

$$\Rightarrow e^{i2\pi m} = 1$$

$$m^2 = \frac{2IE}{\hbar^2} \Rightarrow m = \frac{\sqrt{2IE}}{\hbar}$$

$$m \in I$$

$$\Rightarrow E = \frac{m^2\hbar^2}{2I}$$ energy eigenvalues

(b) $p = \frac{\sqrt{2IE}}{\hbar}$ $\quad q = \theta$

$$\Rightarrow \int_0 \sqrt{2IE}\, d\theta = 2\pi n\hbar$$

$$\Rightarrow \sqrt{2IE}\, \cancel{2\pi} = \cancel{2\pi} n\hbar$$

$$\Rightarrow E = \frac{n^2\hbar^2}{2I}$$ are the levels we get from Bohr-Sommerfield

($H = \frac{L_z^2}{2I}$ L_z is the angular momentum and therefore our canonical momentum p. θ is the canonical coordinate.)

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5) (a) $V = V_0 \left(\frac{r}{a}\right)^k$.

$$\Rightarrow \frac{p^2}{2m} + V_0\left(\frac{r}{a}\right)^k = E$$

$$\Rightarrow \sqrt{2mE - 2mV_0\left(\frac{r}{a}\right)^k} = p.$$

$$\Rightarrow \int p \, d\cancel{x} = \sqrt{2m} \int_0^{a(E/V_0)^{1/k}} \sqrt{E - V_0\left(\frac{r}{a}\right)^k} \, dr$$

$$E = V_0\left(\frac{r}{a}\right)^k$$

$$\Rightarrow \cancel{aE} \; \left(\frac{E}{V_0}\right)^{1/k} a$$

$$\sqrt{2mE} \int^{\left(\frac{E}{V_0}\right)^{1/k}a} \sqrt{1 - \frac{V_0}{E}\left(\frac{r}{a}\right)^k} \, dr$$

$$u = \frac{V_0^{\frac{1}{k}}}{E^{1/k}} \frac{r}{a}$$

$$du = \frac{V_0^{1/k}}{E^{1/k}} \frac{1}{a} dr$$

$$\Rightarrow \sqrt{2mE} \cdot \frac{a E^{1/k}}{V_0^{1/k}} \int_0^1 \sqrt{1-u^k} \, du$$

~~This integral turns out to be~~ from a calculator.

$$\Rightarrow \sqrt{2mE}\, a \left(\frac{E}{V_0}\right)^{1/k} \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(1+\frac{1}{k}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{k}\right)} = 4\sqrt{\pi}\, n \hbar.$$

$$E = \left(\frac{4\sqrt{\pi}\, n\hbar}{\sqrt{2m}\, a} \frac{\Gamma\left(\frac{3}{2}+\frac{1}{k}\right)}{\Gamma\left(1+\frac{1}{k}\right)}\right)^{k/k+1} (V_0)^{1/k+1}$$

energy levels for power law potential

(b) For $k=2$,

integral is

$$\sqrt{2mE}\cdot a\frac{\sqrt{E}}{\sqrt{V_0}}\int_0^1 \sqrt{1-u^2}\,du$$

$\Rightarrow$ ~~t = sin~~ $u = \sin t$.

$t = \sin^{-1}(u) \qquad du = \cos t\,dt$.

$$\int \cos t\sqrt{1-\sin^2 t}\,dt = \int \cos^2 t\,dt$$

Now $\int \cos^n(t)\,dt = \frac{n-1}{n}\int \cos^{n-2}(t)\,dt + \frac{\cos^{n-1}(t)\sin t}{n}$

$$\Rightarrow \frac{\cos t\sin t}{2} + \frac{1}{2}\int 1\,dt$$

$$= \frac{\cos t\cdot\sin t}{2} + \frac{t}{2}$$

$$= \left.\frac{\sin^{-1}(u)}{2} + \frac{u\sqrt{1-u^2}}{2}\right|_0^1$$

$$= \frac{\sin^{-1}(1)}{2} = \frac{\pi}{4}$$

$$\Rightarrow \sqrt{\frac{2m}{2V_0}}\,aE\frac{\pi}{4} = \cancel{2\pi} n\hbar$$

$$\Rightarrow E = \frac{n\hbar}{a}4\sqrt{2}\sqrt{\frac{V_0}{m}}$$

Which is the same as the Harmonic oscillator energy levels.

$E = n\hbar\omega$ with shifted vaccum.

here $\omega = \frac{4\sqrt{2}}{a}\sqrt{\frac{V_0}{m}}$

(6) Uncertainty principle $\Rightarrow$ $\Delta x \, \Delta p \geq \hbar$.

Assume for $x=0$, Δx is the uncertainty in position

then $\Delta p = \frac{\hbar}{\Delta x}$ at $p=0$.

Kinetic energy ; $T = \frac{(\Delta p)^2}{2m}$

$V = \frac{1}{2} m\omega^2 (\Delta x)^2$

$E = T + V = \frac{\hbar^2}{2m(\Delta x)^2} + \frac{1}{2} m\omega^2 (\Delta x)^2$

For ground state, E is minimised.

$\frac{\partial E}{\partial (\Delta x)} = 0 \Rightarrow -\frac{\hbar^2}{m(\Delta x)^3} + m\omega^2 \Delta x = 0$

$(\Delta x)^4 = \frac{\hbar^2}{m^2\omega^2}$

put this into E above

and we get

$E = \hbar\omega$ which is of the same order

as the real value $E = \frac{\hbar\omega}{2}$.

3) (a) $p_\sigma(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$

Fourier transform

$$\overline{P_\sigma}(k) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 - 2i\sigma^2 x k)}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x^2 - 2(i\sigma^2 k x) + (i\sigma^2 k)^2 - (i\sigma^2 k)^2)}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x - i\sigma^2 k)^2}\, dt \right) e^{\frac{1}{2\sigma^2}(i\sigma^2 k)^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2}\; \cancel{e^{2\pi}}\; e^{-\frac{\sigma^2 k^2}{2}}$$

$$\Rightarrow \overline{P_\sigma}(k) = e^{-\frac{\sigma^2 k^2}{2}}$$

3) (b) Standard deviation.

$$\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} dx}{\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx} = \frac{\int x^2 e^{-\frac{x^2}{2\sigma^2}} dx}{\sqrt{2\pi}\,\sigma}$$

$$= \cancel{-\sigma^2 x\, e^{-x^2/2\sigma^2}\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} \sigma^2 e^{-x^2/2\sigma^2} dx$$

$$= \int \frac{\sqrt{\pi}\,\sigma^3}{\sqrt{2}} \frac{2 e^{-u^2}}{\sqrt{\pi}} du$$

$$u = \frac{x}{\sqrt{2}\sigma}.$$

$$= \sigma^3 \int_{-\infty}^{\infty} e^{-u^2} du = \frac{\sigma^3}{\sqrt{2\pi\sigma^2}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \sigma^2$$

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x\, e^{-x^2/2\sigma^2} dx}{\sqrt{2\pi}\,\sigma} = 0.$$

; x changes sign in $(-\infty, 0)$ and $(0, \infty)$ cancelling both

$$\Rightarrow \Delta x = \sqrt{\langle x^2 \rangle} = \sigma.$$

σ : standard deviation.

(c) $$\langle k^2 \rangle = \frac{\int_{-\infty}^{\infty} k^2 e^{-\frac{\sigma^2 k^2}{2}} dk}{\int_{-\infty}^{\infty} e^{-\frac{\sigma^2 k^2}{2}} dk} = \frac{\frac{\sqrt{2\pi}}{4\sigma^3}}{\frac{\sqrt{2\pi}}{\sigma}} = \frac{1}{4\sigma^2}$$

$\langle k \rangle = 0$ for k is an odd function cancelling the $\int_0^{\infty}$ side with $\int_{-\infty}^{0}$ side.

$$\Delta k = \frac{1}{2\sigma}$$

(d) $$\Rightarrow \Delta x \Delta k \geq \frac{1}{2}$$

or $\Delta p = \hbar \Delta k$

$$\Rightarrow \Delta x \Delta p \geq \hbar/2$$

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