Assignment - 2 19MS151

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PAGE NO.:

DATE / /

Q1 $\psi(x,0) = Ax(a-x)$

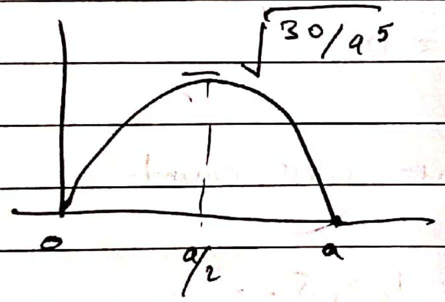
(a) $$\psi(x,0) = \begin{cases} Ax(a-x) & 0 \le x \le a \\ 0 & x<0 \text{ or } x>a \end{cases}$$

$$\int_{-\infty}^{\infty} |\psi(x,0)|^2 dx = 1$$

$$A^2 \int_0^a x^2(a-x^2)^2 dx = 1$$

$$\Rightarrow A^2 \int_0^a (x^2a^2 + x^4 - 2ax^3) dx = 1$$

$$A^2 = \frac{30}{a^5} \Rightarrow |A| = \sqrt{\frac{30}{a^5}}$$



(b) The closest reresemblance is to $n=1$ state of infinite square well of length a

$\phi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$ which also has no node

$P(n=1) = |c_1|^2$ $\quad \psi(x,0) = \sum c_n \phi_n(x)$

where ϕ_n is the square well basis.

$$c_1 = \int \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \cdot \sqrt{\frac{30}{a^5}}\, x(a-x)\, dx$$

$$= \frac{8\sqrt{15}}{\pi^3}$$

$$P(n=1) = |c_1|^2 = \frac{64 \times 15}{\pi^6} \approx \underline{\underline{0.9985}}$$

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(c) $\langle E \rangle = -\frac{\hbar^2}{2m} \int_0^a \psi^* \frac{\partial}{\partial x^2} \psi \, dx$

$= -\frac{\hbar^2}{2m} \times \frac{30}{a^5} (-2) \int_0^a x(a-x)\,dx = \frac{5\hbar^2}{ma^2}$

(d) $\langle x \rangle = A^2 \int_0^a x^3 (a-x)^2 dx = \frac{a}{2}$

$\langle x^2 \rangle = A^2 \int_0^a x^4 (a-x)^2 dx = \frac{2a^2}{7}$

$\Delta x = \sqrt{\left(\frac{2}{7} - \frac{1}{4}\right) a^2} = 0.2\,a$

$\langle p \rangle = A^2 \frac{\hbar}{i} \int_0^a x(a-x) \frac{d}{dx} (x(a-x))\,dx = 0$

$\langle p^2 \rangle = -A^2 \hbar^2 \int_0^a x(a-x) \frac{d^2}{dx^2} (a^2 - x^2)\,dx = \frac{20}{a^2} \hbar^2$

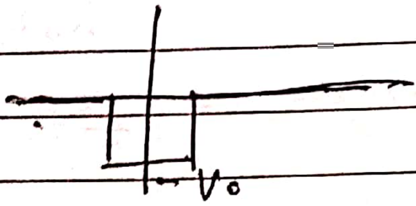
$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{2\hbar}{a}\sqrt{5}$

$\Delta x \Delta p = 0.2 \times 2 \times \frac{\hbar}{a} \times a\sqrt{5} \simeq 0.85\hbar > \frac{\hbar}{2}$

Heisenberg

Q2

$$V(x) = \begin{cases} 0 & -\infty \le x \le -a/2 \\ -V_0 & -a/2 \le x \le a/2 \\ 0 & a/2 \le x \le \infty \end{cases}$$



Equations = $\begin{cases} \dfrac{\partial^2 \psi}{\partial x^2} = -\dfrac{2m}{\hbar^2}(E+V_0)\psi = -k^2\psi & \cdots |x| < \frac{a}{2} \\ \dfrac{\partial^2 \psi}{\partial x^2} = -\dfrac{2m}{\hbar} E\psi = -q^2\psi & \cdots |x| > \frac{a}{2} \end{cases}$

$$\Rightarrow \begin{cases} \psi = A\sin kx + B\cos kx \\ Ce^{-qx} + De^{qx} = \begin{cases} Ce^{-qx} & x > a/2 \\ De^{qx} & \cdots x < a/2 \end{cases} \end{cases}$$

* Even solution $\psi = B\cos(kx)$

$$B\cos\left(k\frac{a}{2}\right) = Ce^{-qa/2}$$

$$-Bk\sin\left(k\frac{a}{2}\right) = -Cq\,e^{-qa/2}.$$

$$\Rightarrow \tan\left(\frac{ka}{2}\right) = \frac{q}{a}$$

* Odd sol^n $\psi = A\sin(kx)$

$$\cancel{B}\ A\sin\left(\frac{ka}{2}\right) = \cancel{D}Ce^{-qa/2}$$

$$Ak\sin\left(\frac{ka}{2}\right) = -Cq\,e^{-qa/2}$$

$$\Rightarrow \tan\left(\frac{ka}{2}\right) = -\frac{a}{q}$$

* if $\theta = \frac{ka}{2}$, and $y(\theta) = \tan\theta$.
let $\theta_0 = \frac{k_0 a}{2}$ and $k_0^2 = \frac{2mV_0}{\hbar}$

$$y(\theta) = \sqrt{\frac{-E}{E+V_0}} = \sqrt{\frac{V_0}{E+V_0} - 1}$$

$$\Rightarrow \tan\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}$$

$$E = \frac{\hbar^2 k^2}{2m} - V_0 \quad \text{or} \quad E = \frac{\hbar^2 q^2}{2m}$$

* for odd soln $\Rightarrow$ $\cot\theta = \sqrt{\frac{\theta_0^2}{\theta^2} - 1}$.

For bound state, $V_0 = -\alpha\delta(x)$

$$\psi = \begin{cases} A e^{kx} & x < 0 \\ B e^{-kx} & x > 0. \end{cases}$$

$A = B \Rightarrow$ $E < 0 = E = -\frac{\hbar^2 k^2}{2m}$

(a) $V_0 \to \infty$ as $a \to 0$
$V_0 \to -\alpha\delta(x)$.

$\therefore E = -\frac{\hbar^2 k^2}{2m}$ only bound ground state

(b) $\cot\theta_0 > 0$ $\rightarrow$ $\theta_0 < \frac{\pi}{2}$ has no solutions.

$$\frac{2mV_0}{\hbar^2}\frac{a^2}{4} < \frac{\pi^2}{4} \Rightarrow V_0 < \frac{\pi^2\hbar^2}{2ma} = V_c$$
critical

(6) ψ_i be energy eigenstates

$X_{mn} = \langle \psi_m | x | \psi_n \rangle$.

~~$a = \sqrt{\frac{m\omega}{2\hbar}}(x -$~~ $\quad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$

$$X_{mn} = \sqrt{\frac{\hbar}{2m\omega}}\ \langle m | a + a^\dagger | n \rangle = \langle m | \left(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\right)$$

$$P_{mn} = -i\sqrt{\frac{m\omega\hbar}{2}}\ \langle m | (a - a^\dagger) | n \rangle$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\ \langle m | \left(\sqrt{n}\,|n-1\rangle - \sqrt{n+1}\,|n+1\rangle\right)$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}\left(\sqrt{n}\,\delta_{m,n-1} - \sqrt{n+1}\,\delta_{m,n+1}\right)$$

$$X = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ \vdots & 0 & \sqrt{3} & 0 & \ddots \end{pmatrix}$$

$$p = i\sqrt{\frac{m\omega\hbar}{2}}\begin{pmatrix} 0 & -\sqrt{1} & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ \vdots & 0 & \sqrt{3} & 0 & \ddots \end{pmatrix}$$

DATE / /

~~Q4~~ $V(x) = \begin{cases} -\infty & x \leq 0 \\ \frac{1}{2}\omega^2 m x^2 & x > 0 \end{cases}$

~~$\psi(x)=0$~~ $\psi(0) = 0$

$\psi(x \leq 0) = 0$

for $x > 0$

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\psi + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

which gives.

$$\psi = c\, e^{-y^2/2} H_n(y).$$

where $y \propto x$ with some scale.

for $n = $ odd, $H_n(x) = 0$.

only $n =$ odd solutions will count.

$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad n = 1, 3, 5 \ldots$

$\psi_n = c\, H_n(y)\, e^{-y^2/2} \quad n = 1, 3, 5 \ldots$ eigenstates.

PAGE NO.:
DATE / /

Q3 $V(x) = -\alpha\left(\delta(x+a) + \delta(x-a)\right)$

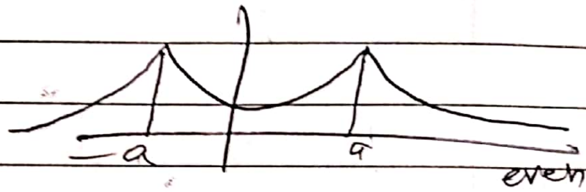
$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi$

$$\psi = \begin{cases} Ae^{kx} & x < -a \\ A(e^{kx} + e^{-kx}) & |x| \leq a \\ Ae^{-kx} & x > a \end{cases}$$

Boundary

$Ae^{-ka} = B(e^{ka} + e^{-ka})$

$A = B(1 + e^{2ka})$



$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d}{dx}\left(\Delta\frac{\partial\psi}{\partial x}\right)dx = \int_{-\varepsilon}^{\varepsilon}(E-V)\psi\, dx$$

$\varepsilon \to 0 \Rightarrow -\frac{\hbar^2}{2m}\Delta\frac{\partial\psi}{\partial x} = \alpha\psi.$

$\Rightarrow -kAe^{-ka} - kB(e^{ka} - e^{-ka}) = -\frac{2m\alpha}{\hbar^2}e^{-ka}$

$e^{-2ka} = \frac{\hbar^2 k}{m\alpha} - 1$ $\qquad k^2 = -\frac{2mE}{\hbar^2}.$

if $z = 2ka,$ $\quad E = -\frac{z^2}{8}\left(\frac{\hbar^2}{ma^2}\right)$

for another odd sol^n

$$\psi = \begin{cases} -Ae^{kx} & x < -a \\ B(e^{kx} - e^{-kx}) & |x| < a \\ Ae^{kx} & x > a. \end{cases}$$

Continuity & discontinuity on derivative.

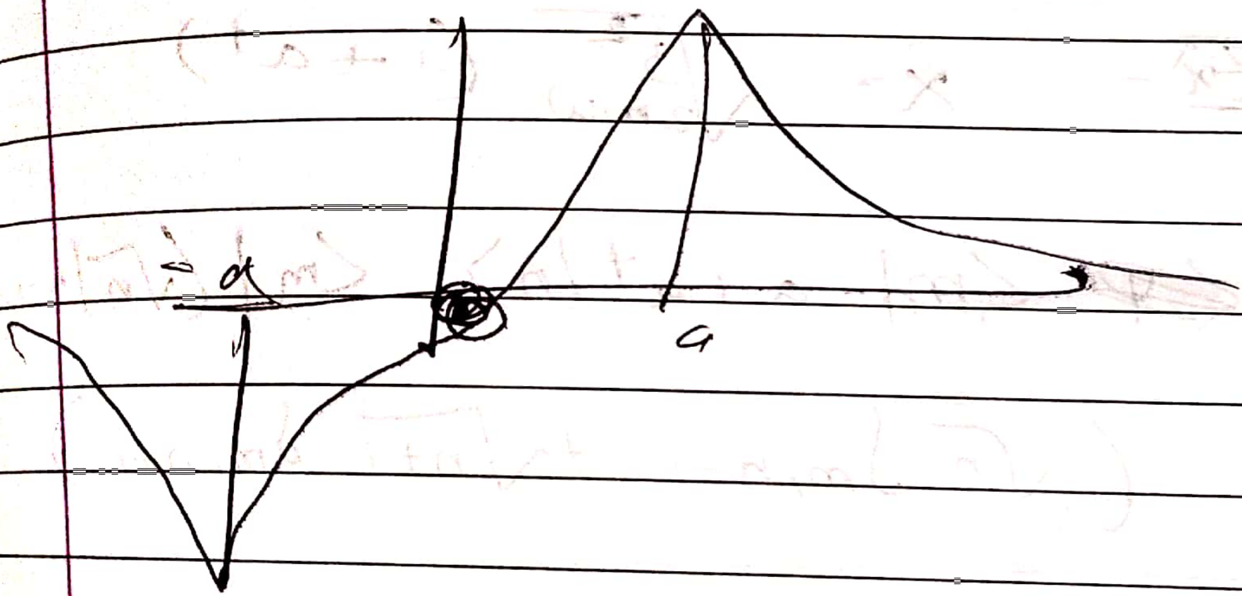
$e^{-2ka} = 1 - \frac{\hbar^2 k}{m\alpha}$, $z = 2ka$

Solution for $-ze^{-z} > 1 - cz \Rightarrow c < 1$ or $\alpha > \frac{\hbar^2}{2ma}$

$\alpha > \frac{\hbar^2}{2ma}$ $\Rightarrow$ one even, one odd bound.

$\alpha \leq \frac{\hbar^2}{2ma}$ $\Rightarrow$ one even.

$E_{even} = -0.615 \frac{\hbar^2}{ma^2}$, $E_{odd} = -0.317 \frac{\hbar^2}{ma^2}$.



Q5

(a) $\psi_n = C\, H_n(y) e^{-y^2/2}$

~~$\langle x \rangle_n = A \int_{-\infty}^{\infty} \psi_n x \psi_n dx = A \int H_n(y)$~~

$A \Rightarrow$ some normalisation constant which absorbes all scales

$\langle x \rangle = A \int \psi_n x \psi_n dx = A \int_{-\infty}^{\infty} H_n(y) e^{-y^2/2} \cdot y \cdot H_n(y) e^{-y^2/2} dy$

x & y are both some units of distance, so they are the same with some scale factor

$= A \int_{-\infty}^{\infty} y H_n(y) H_n(y) dy$

$H_n = 2y H_{n-1} - 2(n-1) H_{n-2}$

$\Rightarrow y H_n = \frac{1}{2} H_{n+1} + n H_{n-1}$

$\Rightarrow = A \int H_n(y) \cdot \left(\frac{1}{2} H_{n+1} + n H_{n-1} \right) dy$

Now Hermite polynomials are orthogonal & follow

$\int \psi_m \psi_n = \int H_m H_n e^{-y^2/2} dy = \sqrt{\pi}\, 2^n n!\, \delta_{mn}$

$\Rightarrow \langle x \rangle = A \left(\frac{1}{2} \delta_{n+1,n} + n \delta_{n-1,n} \right) = 0$

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Similarly:

$$\langle x^2 \rangle = A \int y^2 H_n(y) H_n(y) e^{-y^2/2} \, dy.$$

$$y H_n = \frac{1}{2} H_{n+1} + n H_{n-1}$$

$$\Rightarrow y H_n \, y H_n = \left(\frac{1}{2} H_{n+1} + n H_{n-1} \right)^2$$

$$= \frac{1}{4} H_{n+1}^2 + n^2 H_{n-1}^2 + n H_{n+1} H_{n-1}$$

$$\Rightarrow \langle x^2 \rangle \propto \int (y H_n)(y H_n) e^{-y^2/2} \, dy$$

$$\propto \int_{-\infty}^{\infty} \left(\frac{1}{4} H_{n+1} \cdot H_{n+1} e^{-y^2/2} + n^2 H_{n-1} H_{n-1} e^{-y^2/2} + n H_{n+1} H_{n-1} e^{-y^2/2} \right) dy$$

$$\propto \left(\frac{1}{4} 2^{n+1} (n+1)! \, \delta_{n+1,n+1} + n^2 \cdot 2^{n-1} (n-1)! \, \delta_{n-1,n-1} \right)$$

$$\propto 2^{n+1} \frac{(n+1)!}{4} + n \, n! \, 2^{n-1}$$

$$\propto 2^{n-1} n! \left(\frac{n+1}{4} \cdot 2^2 + n \right)$$

$$\Rightarrow \langle x^2 \rangle \propto (2n+1).$$

$$\Rightarrow \langle x^2 \rangle = A^2 (2n+1)$$

$$\Rightarrow \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \propto \sqrt{2n+1}$$

Now the constant

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{2n+1}$$