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Class test - I PH3103

I Naravane Adwait Bipinchandra (19MS151) agree not to use unfair means in this test

1) fifth root of $z = 8 + i4$.

$|z| = \sqrt{64+16} = \sqrt{80} = 4\sqrt{5}$

$z = 4\sqrt{5}\left(\frac{2}{\sqrt{5}} + i\frac{1}{\sqrt{5}}\right)$

$\cos\theta = \frac{2}{\sqrt{5}}, \quad \sin\theta = \frac{1}{\sqrt{5}}$

$\tan(\theta) = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right) = \tan^{-1}\left(\frac{y}{x}\right)$

$\Rightarrow z = 4\sqrt{5}\left(\cos\left(\tan^{-1}\left(\frac{1}{2}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\right)$

$$z^{1/5} = (4\sqrt{5})^{1/5}\left(\cos\left(\frac{1}{5}\tan^{-1}\left(\frac{1}{2}\right) + \frac{2\pi n}{5}\right) + i\sin\left(\frac{1}{5}\tan^{-1}\left(\frac{1}{2}\right) + \frac{2\pi n}{5}\right)\right)$$

where $n = 0, 1, 2, 3, 4$.

$\Rightarrow$ There are 5 roots of $z = 8 + i4$.

Which are

$(4\sqrt{5})^{1/5}\left(\cos\left(\frac{\theta}{5}\right) + i\sin\left(\frac{\theta}{5}\right)\right)$

$(4\sqrt{5})^{1/5}\left(\cos\left(\frac{\theta+2\pi}{5}\right) + i\sin\left(\frac{\theta+2\pi}{5}\right)\right)$

$(4\sqrt{5})^{1/5}\left(\cos\left(\frac{\theta+4\pi}{5}\right) + i\sin\left(\frac{\theta+4\pi}{5}\right)\right)$

$(4\sqrt{5})^{1/5}\left(\cos\left(\frac{\theta+6\pi}{5}\right) + i\sin\left(\frac{\theta+6\pi}{5}\right)\right)$

$(4\sqrt{5})^{1/5}\left(\cos\left(\frac{\theta+8\pi}{5}\right) + i\sin\left(\frac{\theta+8\pi}{5}\right)\right)$

where $\theta = \tan^{-1}\left(\frac{1}{2}\right)$

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2) $f(z) = \dfrac{-2z^2 - z}{-z}$

find $f(z)$ is continuous at $z = z_0 = -1 + i5$

$$f(z) = \frac{-2z^2 - z}{-z} = \frac{2z+1}{} \quad \text{for } z \neq 0.$$

$$f(z_0) = \frac{2z^2+z}{z} = 2z_0+1 = 2(-1+i5)+1$$
$$= -2 + i10 + 1$$
$$= -1 + 10i$$

$$\lim_{z \to z_0} f(z) = 2(x+iy)+1$$
$$= (2x+1) + i(2y)$$

$$\lim_{\substack{x=-1 \\ y \to 5}} f(z) = -1 + \lim_{y \to 5}(i2y) = -1 + 10i$$

$$\lim_{\substack{y=5 \\ x \to -1}} f(z) = \lim_{x \to -1}(2x+1) + i10$$
$$= (-2+1) + 10i = -1 + 10i$$

$$\Rightarrow \lim_{\substack{x = x_0 \\ y \to y_0}} f(z) = \lim_{\substack{y = y_0 \\ x \to x_0}} f(z) = f(z_0)$$

$\therefore$ $f(z)$ is continuous at $z_0 = -1 + i5$.

3) $f(z) = \dfrac{-z^2 - 2z}{-\bar{z}}$

$$\cancel{= \frac{z^2 + 2z}{\bar{z}} = \frac{(x+iy)^2 + 2(x+iy)}{x-iy}}$$

$$\cancel{= \frac{x^2 - y^2 + i2xy + 2x + i2y}{x-iy}}$$

$$\cancel{= \frac{\left(x^2 + 2x - y^2 + 2iy(x+i)\right)(x+iy)}{(x^2+y^2)}}$$

$$= \frac{z^2 + 2z}{\bar{z}} = \frac{(z^2 + 2z)\, z}{|z|^2}$$

$z = re^{i\theta}$

$$\Rightarrow \frac{z^3 + 2z^2}{r^2} = \frac{r^3 e^{3i\theta} + 2e^{2i\theta} r^2}{r^2}$$

$$f(z) = re^{3i\theta} + 2e^{2i\theta}$$

for $z_0 = -3 - i4$, $x = -3$, $y = -4$,

$\sqrt{3^2 + 4^2} = r = 5$, $\cos(\theta) = -\frac{3}{5}$, $\sin\theta = -\frac{4}{5}$.

A complex function is differentiable iff it obeys C-R conditions

$$f(z) = r\cos(3\theta) + ir\sin(3\theta) + 2\cos(2\theta) + 2i\sin(2\theta)$$

$$= \underbrace{\left(r\cos(3\theta) + 2\cos(2\theta)\right)}_{u(r,\theta)} + i\underbrace{\left(r\sin(3\theta) + 2\sin(2\theta)\right)}_{v(r,\theta)}$$

$$\frac{\partial u}{\partial r} = \cos(3\theta), \quad \frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{1}{r}\left(3r\cos(3\theta) + 4\cos(2\theta)\right)$$

$$\frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) = \frac{1}{r}\left(-3r\sin(3\theta) - 4\sin(2\theta)\right), \quad -\frac{\partial v}{\partial r} = -\sin(3\theta)$$

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~~$\cos$~~ $\frac{\partial u}{\partial r} = 4\cos^3(\theta) - 3\cos(\theta)$

$\cos\theta = -\frac{3}{5}$

$\Rightarrow 4\left(\frac{-27}{125}\right) + 3\left(\frac{3}{5}\right) = -\frac{108}{125} + \frac{9}{5} = \frac{117}{125}$

$\frac{1}{r}\left(\frac{\partial v}{\partial \theta}\right) = \frac{1}{5}\left(15\left(\frac{117}{125}\right) + 4\left(1 - 2.\frac{16}{25}\right)\right)$

$= \frac{1}{5}\left(15\left(\frac{117}{125}\right) - \frac{140}{125}\right) = \frac{323}{125}$

$\Rightarrow$ C-R conditions not obeyed at z_0.

$f(z) = -\frac{z^2 - 2z}{\bar{z}}$ is not differentiable

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4) $U(x,y) = e^{x^2-y^2}\sin(2xy) + x - 4y$

$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$

$\frac{\partial}{\partial x}U = \frac{d}{dx}e^{x^2-y^2}\sin(2xy) + e^{x^2-y^2}\frac{d}{dx}(\sin(2xy)) + 1$

$= 2x.\sin(2xy).e^{x^2-y^2} + 2e^{x^2-y^2}y\cos(2xy) + 1$

$\frac{\partial^2}{\partial x^2}U = 2\Big(\frac{\partial x}{\partial x}.e^{x^2-y^2}\sin(2xy) + x\frac{d}{dx}(e^{x^2-y^2})\sin(2xy)$
$+ xe^{x^2-y^2}\frac{d}{dx}(\sin(2xy))\Big)$

$+2y\Big(\frac{d}{dx}e^{x^2-y^2}.\cos(2xy) + e^{x^2-y^2}\frac{d}{dx}(\cos(2xy))\Big)$

$= 2e^{x^2-y^2}\Big((2x^2-2y^2+1)\sin(2xy) + 4xy.\cos(2yx)\Big)$

$\frac{\partial}{\partial y}U = \frac{d}{dy}(e^{x^2-y^2}).\sin(2xy) + e^{x^2-y^2}\frac{d}{dy}(\sin(2xy)) - 4$

~~Skipping steps~~

$\cancel{= e^{x^2-y^2}\left(\frac{d}{dx}(x^2)\right)\sin(2xy) + e^{x^2-y^2}\cos(2xy).2x\frac{d}{dy}y - 4}$

$= -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy) - 4$

$\frac{\partial^2}{\partial y^2}U = -2\Big(1.e^{x^2-y^2}\sin(2xy) + y.\frac{d}{dy}(e^{x^2-y^2}).\sin(2xy)$
$+ ye^{x^2-y^2}\frac{d}{dy}(\sin(2xy))\Big) + 2x\Big(\frac{d}{dy}(e^{x^2-y^2}).\cos(2xy)$
$+ e^{x^2-y^2}\frac{d}{dy}(\cos(2xy))\Big) + 0$

$= 2e^{x^2-y^2}\Big((2y^2-2x^2-1)\sin(2xy) - 4xy\cos(2xy)\Big)$

$\rightarrow \frac{\partial^2}{\partial x^2}U + \frac{\partial^2}{\partial y^2}U = 2e^{x^2-y^2}\Big((2x^2-2y^2+1+2y^2-2x^2-1)\sin(2xy)$
$\cancel{+ 4xy\cos(2xy) - 4xy\cos(2xy)}\Big)$

$= 0$

$\Rightarrow \nabla^2 U = 0 \Rightarrow U$ is harmonic

Harmonic conjugate "V" will obey C-R conditions

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \Rightarrow$$

$$V = \int \left((2x\sin(2xy))e^{x^2-y^2} + 2e^{x^2-y^2} y\cos(2xy) + 1 \right) dy$$

$$= e^{x^2-y^2}\cos(2xy) + y + f(x)$$

Doing this integral is hard
but by guessing we find that
$e^{x^2-y^2}\cos(2xy)$ is the harmonic conjugate of
$e^{x^2-y^2}\sin(2xy)$.

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$V = -\int \left(-2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy) - 4\right) dx$$

$$= e^{x^2-y^2}\cos(2xy) + 4x + g(y)$$

$$\Rightarrow V = e^{x^2-y^2}\cos(2xy) + 4x + y$$

is the harmonic conjugate of $U(x,y)$.

if $u = e^{x^2-y^2}\cos(2xy)$, & $v = e^{x^2-y^2}\sin(2xy)$

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy)) = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y\cos(2xy) - x\sin(2xy)) = -\frac{\partial v}{\partial x}$$

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