Assignment 4

Q1) Prove the following commutation relations,

$$[\hat{L}_i, \hat{r}^2] = 0, \quad [\hat{L}_i, \hat{\vec{r}} \cdot \hat{\vec{p}}] = 0 \tag{1}$$

Q2) (a) Find the matrix representation of L_x, L_y, L_y for angular momentum $l = 1$ and $l = 2$.

(b) For the angular momentum 1 ($l = 1$) system, write down the matrix representation of the following Hamiltonian in simultaneous eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$,

$$\hat{H} = -\hat{L}_x + \frac{\hat{L}_z^2}{2I} \tag{2}$$

Q3) A particle in a spherically symmetric potential is in a state described by the wave packet

$$\psi(x, y, z) = C\,(xy + yz + zx)\,e^{-\alpha r^2} \tag{3}$$

(a) Write the wave-function in terms of the Spherical Harmonics $Y_l^m(\theta, \phi)$ with appropriate l and m.

(b) Find out the normalization constant C?

(c) Show that the above wave function is an eigenfunction of $\hat{L}^2$ with $l = 2$.

(d) What are the relative probabilities for $m = 2, 1, 0, -1, -2$?

Q4) Let $\hat{\mathbf{n}}$ be a unit vector in a direction specified by the polar angles (θ, ϕ). Show that the component of the angular momentum in the direction $\hat{\mathbf{n}}$ is

$$\hat{L}_n \quad = \quad \sin\theta\,\cos\phi \hat{L}_x + \sin\theta\,\sin\phi \hat{L}_y + \cos\theta\,\hat{L}_z \tag{4}$$

If the system is in simultaneous eigenstates of $\mathbf{L}^2$ and $\hat{L}_z$ belonging to the eigenvalues $l(l+1)\hbar^2$ and $m\hbar$,

(a) what are the expectation values of L_n and L_n^2?

Q5) a) Prove the following,

$$\mathbf{L}^2 = \mathbf{x}^2\mathbf{p}^2 - (\mathbf{x}\cdot\mathbf{p})^2 + i\hbar\,\mathbf{x}\cdot\mathbf{p} \tag{5}$$

where x and p are position and momentum vector operators.

(b) Using the position representation for the above $\langle \mathbf{x}'|\,\mathbf{L}^2\,|\alpha\rangle$, prove the following

$$\frac{1}{2m}\,\langle \mathbf{x}'|\,\mathbf{p}^2\,|\alpha\rangle = -\left(\frac{\hbar^2}{2m}\right)\left(\frac{\partial^2}{\partial r^2}\langle \mathbf{x}'|\alpha\rangle + \frac{2}{r}\frac{\partial}{\partial r}\langle \mathbf{x}'|\alpha\rangle - \frac{1}{\hbar^2\,r^2}\,\langle \mathbf{x}'|\,\mathbf{L}^2\,|\alpha\rangle\right) \tag{6}$$

Q6) a) Prove the following relation using the commutation and anti-commutation relation of Pauli matrices

$$\sigma_j\sigma_k = \delta_{jk}I + i\epsilon_{jkl}\sigma_l \tag{7}$$

(b) Using the above relation obtain the following identity

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})I + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} \tag{8}$$

where $\vec{a}$ and $\vec{b}$ are any arbitrary vectors.

Q7) (a) Show that any 2×2 matrix can be written in terms of Pauli matrices as,

$$A + \vec{B}\cdot\vec{\sigma} \tag{9}$$

(b) What are the conditions that A and $\vec{B}$ must satisfy if the matrix to be unitary. If it is to be Hermitian.

Q8) In the Holstein-Primakoff representation, one can express the Spin matrices in terms of the bosonic creation $\hat{a}^\dagger$ and annihilation operator $\hat{a}$ as,

$$\begin{aligned}
\hat{S}_z &= \hbar(S - \hat{a}^\dagger\hat{a}) && (10)\\
\hat{S}_+ &= \hbar\sqrt{2S - a^\dagger\hat{a}}\hat{a} && (11)\\
\hat{S}_- &= \hbar\hat{a}^\dagger\sqrt{2S - a^\dagger\hat{a}} && (12)
\end{aligned}$$

where the bosonic operators follows the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$.

(a) Using this representation show that, the Spin matrices follows commutation relation

$$[\hat{S}_i, \hat{S}_j] = i\hbar\epsilon_{ijk}\hat{S}_k \tag{13}$$

(b) Also show that the total spin angular momentum $\hat{S}^2 = S(S+1)\mathbb{I}$.

Q9) (a) Given an generalized Laguerre polynomial $L^\beta_\alpha(x)$, prove the following

$$\int_0^\infty e^{-x}x^\beta\, L^\beta_\alpha(x)L^\beta_{\alpha'}(x)dx = \frac{(\alpha+\beta)!}{\alpha!} \tag{14}$$

Hint! use the Generating function $e^{-xz/(1-z)}/(1-z)^{\beta+1} = \sum_{\alpha=0}^\infty L^\beta_\alpha(x)z^n$ and $(t+y)^r = \sum_{\alpha=0}^\infty {}^rC_\alpha t^{r-\alpha}y^\alpha$.

(b) The radial part of the wave function of the Hydrogen atom is given by the Laguerre polynomial $R_{nl}(r) = C(kr)^l e^{-kr/2}L^{2l+1}_{n-l-1}(kr)$, where $k = 2/na_0$. Calculate the normalization constant C using

$$\int_0^\infty r^2[R_{nl}(r)]^2\, dr = 1. \tag{15}$$

Hint! Use the following recursion relation $xL^\beta_\alpha = (2\alpha+\beta+1)L^\beta_\alpha - (\alpha+\beta)L^\beta_{\alpha-1} - (\alpha+1)L^\beta_{\alpha+1}$

(c) Using the properties of Laguerre polynomial evaluate $\langle r\rangle$ from

$$\langle r\rangle = \int_0^\infty r^2[R_{nl}(r)]^2\, dr. \tag{16}$$

Q10) Consider the Hamiltonian $\hat{H}(\lambda)$ which depends on some parameter λ, follows the eigenvalue equation

$$\hat{H}(\lambda)\,|\psi_n(\lambda)\rangle = E_n(\lambda)\,|\psi_n(\lambda)\rangle \tag{17}$$

(a) If we can consider λ (which for example can be chosen as $\hbar, m$ etc.) to be a continuous variable and perform derivative with respect to this parameter, show that

$$\frac{\partial E_n(\lambda)}{\partial\lambda} = \left\langle \frac{\partial\hat{H}(\lambda)}{\partial\lambda}\right\rangle \tag{18}$$

This relation is known as "Feynman-Hellmann" theorem.

(b) Consider the Hydrogen atom problem. Now using the Feynman-Hellmann theorem and by choosing the appropriate parameter λ show that,

$$\left\langle \frac{1}{r} \right\rangle_{nl} = \frac{Z}{a_0 n^2}, \quad \langle \hat{p}^2 \rangle = \frac{m^2 c^2 \alpha^2}{n^2} \tag{19}$$

(c) Using the above expectation values show that, $\langle \hat{T} \rangle = -\frac{1}{2} \langle \hat{V}(r) \rangle$, where $\hat{T}$ is kinetic energy operator.