Project 1: Classical Mechanics

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Similarities between bead on a hoop and cyclotron and extending it to Electric fields.

Abstract

In this paper, we shall analyse the physical and mathematical similarities between a bead constrained on a horizontal circular hoop and a charged particle moving with a constant velocity in a plane perpendicular to a static uniform Magnetic field. We will approach this problem using Lagrangian dynamics and basic Energy conservation laws. We shall also extend this to Electric fields and discuss about Electric fields that might lead to similar equations of motion.

1 Introduction

A bead on a hoop, is a particle on a frictionless circular hoop on a horizontal plane. The particle therefore experiences no forces on the plane and is constrained to move around the hoop. We can restrict this problem to 2D as gravity will play no role in our problem.

The only constraint here is that of moving along the hoop of radius R. Let $g_\lambda(r)$ be the constraint.

$$g_\lambda(r) = \lambda(r - R) = 0 \tag{1}$$

The particle experiences no friction and therefore the only term of interest in our problem is the above constraint. We can analyse the constraint and solve the problem using the lagrange multiplier's method for both Newtonian and Lagrangian approaches.

As for cylclotron, i.e. a charged particle in a magnetic field, we launch a charged particle in a chamber where the particle experiences no drag other than a static, uniform magnetic field switched on along the z-axis, the magnetic field does no work as it acts perpendicular to the particle's velocity in the horizontal $(x-y)$ plane.

We will explore the Physical and Mathematical similarities between these two problems.

Finally we shall extend this to Electric fields, which on the other hand, accelerate the charged particle in the direction of the field and give non-trivial results. We shall observe similarities between equations of motions for some "physically interesting" Electric fields.

2 Analysis & Explanations

2.1 Bead on a Hoop

Formulating a problem in Lagrangian Mechanics requires no analysis of forces. Instead, the energy of the system and principle of extremal action are used to determine the equations of motion. The principle of extremal actions states that the action, defined as

$$S = \int_{t_1}^{t_2} L(q^i, \dot{q}^i, t)dt \tag{2}$$

of a particle moving between two points is extremized. The Lagrangian $L(q^i, \dot{q}^i, t)$ is a function of generalised coordinates q^i and generalised velocites $\dot{q}^i$ is defined as

$$L = T - U \tag{3}$$

And the Action S is minimised for a Lagrangian that satisfies the Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{4}$$

As we restrict our particle to a horizontal plane, we only need 2 dimensions.

$$T = \underbrace{\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)}_{\text{Cartesian coordinates}} = \underbrace{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)}_{\text{Polar coordinates}}$$

For the sake of simplicity, we shall only use polar coordinates.

$$L = T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

as we do not have any potential.
We can incorporate the constraints into the Lagrangian by realising that the Lagrangian is not unique and can be replaced with

$$L' = L + \sum_i \lambda_i g_i(q_i, t)$$

This Lagrangian still minimises the action given that g_i is holonomic (or semi-holonomic).
In our case, the Euler-Lagrange equations become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{\partial g_\lambda}{\partial q_i}$$

Where $g_\lambda(r) = \lambda(r - R)$ The equations of motion are,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \lambda$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$m\ddot{r} - mr\dot{\theta}^2 = \lambda \tag{5}$$

$$mr^2\ddot{\theta} = 0 \tag{6}$$

Putting in the constraint of $r = R$,

$$mR^2\ddot{\theta} = 0$$

$$mR\dot{\theta} = -\lambda$$

As $r = R$ implies $\ddot{r} = \dot{r} = 0$.

$$\ddot{\theta} = 0 \implies \dot{\theta} = \omega \implies \theta = \omega t + \phi$$

And the Lagrange multiplier,

$$\lambda = -mR\omega^2$$

ω here is some constant which happens to be the angular velocity of the bead around the center.
The Constraint force here is

$$Q_i = \frac{\partial g_\lambda}{\partial q_i} \implies Q_r = \lambda \frac{\partial (r - R)}{\partial r} = -mR\omega^2 \tag{7}$$

Which happens to be the centripetal force. This makes sense, considering that the hoop keeps the bead moving in a uniform circle by exerting the centripetal force on it and preventing it from leaving the hoop and moving elsewhere on the plane.

2.2 Cyclotron

For a non-relativistic charged particle in an Electromagnetic field. The Lagrangian is

$$L = \frac{1}{2}mv^2 - q(\phi - \vec{v}.\vec{A})$$

where ϕ and $\vec{A}$ are the scalar and vector potential respectively with,

$$E = -\nabla\phi - \frac{\partial A}{\partial t}$$

$$B = \nabla \times A$$

For a uniform, static magnetic field in the z-direction, $B = B_0\hat{k}$, we can pick the Coulomb gauge of $\nabla.\vec{A} = 0$.
From this we can choose a simple vector potential like

$$\vec{A} = (0, 0, xB_0) \text{ or } \vec{A} = (-yB_0, 0, 0)$$

but we shall pick a vector potential that we can easily use in cylindrical coordinates,

$$\vec{A} = \frac{B_0}{2}(-y, x)$$

Which gives the required Magnetic field by

$$\nabla \times A = \frac{B_0}{2}(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}) = B_0$$

The Lagrangian here is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB_0}{2}(x\dot{y} - y\dot{x})$$

which in cylindrical coordinates becomes

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + \frac{qB_0}{2}r^2\dot{\theta} \tag{8}$$

The canonical momenta for the cyclotron system in cylindrical coordinates

$$\underbrace{p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2(\dot{\theta} + \frac{qB_0}{2m})}_{\text{These momenta are constant.}}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

The momenta along θ and z are constant as the coordinates are cyclic i.e. they do not appear in the Lagrangian explicitly. We can use this later to pick up initial conditions.
We can perform a legendre transformation on the Lagrangian to get the Hamiltonian of the system,

$$H = \sum_i p_i\dot{q}_i - L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) = T \tag{9}$$

An interesting thing to note here is that the velocity depending vector potential term does not appear here as Magnetic fields do no work.
Also,

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = 0 \implies H = E$$

We now use energy conservation (E is constant) to get,

$$\dot{r} = \pm\sqrt{\frac{2E}{m} - \frac{p_z^2}{m^2} - r^2\dot{\theta}^2}$$

We can now choose a system by setting $r(0) = 0$ and $p_\theta(0) = 0$,
Here we are left with two choices, either $\dot{\theta} = -\frac{qB_0}{2m}$ or $r(t) = 0$.
We shall pick the first, the second one isn't very interesting as in that case

the particle just moves straight along the z-axis.

$$\dot{r} = \pm\sqrt{\frac{2E}{m} - \frac{p_z^2}{m^2} - \frac{q^2B_0^2r^2}{4m^2}}$$

integrating this gives,

$$r(t) = \pm\frac{2m}{qB_0}\sqrt{\frac{2E}{m} - v_z^2}\sin\left(\frac{qB_0t}{2m}\right) \tag{10}$$

The motion along the other directions being,

$$z(t) = z(0) + v_zt, \quad \theta(t) = -\frac{qB_0}{2m}t + \theta(0)$$

where angular velocity $\omega = -\frac{qB_0}{2m}$ and velocity along the z-axis $v_z = \frac{p_z}{m}$
In this setup, the charged particle starts at the origin with an velocity $R\omega$, around a circle with it's axis displaced to the point

$$O = (0, -\frac{m}{qB_0}\sqrt{\frac{2E}{m} - v_z^2})$$

and $R = \frac{m}{qB_0}\sqrt{\frac{2E}{m} - v_z^2}$.

The physical and Mathematical similarities between the bead on a hoop and cyclotron are that they both move around a circle of fixed radius on the horizontal plane uniformly. The radius of the bead's motion is fixed and is a constraint while the radius of the charged particle in the Magnetic field, depends on the initial velocity with which the particle enters the chamber (magnetic field).
In both cases, the constraint force is the centripetal force, with it being fixed for the bead on a hoop ($Q_r = -mR\omega^2$) and for cyclotron, the centripetal force is the Lorentz force, from one of the equations of motion

$$mr\omega^2 + qB_0r\omega \implies Q_r = -qvB_0 = -qB_0R\omega$$

towards the center around the displaced axis.
The particle follows a helical motion.

2.3 Electric fields

Electric fields on the other hand are a bit tricky. In any Electric field, the charged particle will accelerate along the field and therefore the uniform circular motion will not be maintained.
However we can extend uniform circular motion to electric fields, given a certain initial conditions.
One form of electric fields, for which we can get circular motion are central force fields which are "well-like" i.e. they attain their minimum value at the centre.
An example of this would be a 2d harmonic oscillator-like potential,

$$\phi = \frac{1}{2}k(x^2 + y^2)$$

which give the equations of motion,

$$m\ddot{x} = -qkx, \quad m\ddot{y} = -qky$$

with initial condition, $x = 0, y = R, \dot{x} = v, \ddot{y} = 0$, we get uniform circular motion with $(x, y) = (R\cos(\omega t), R\sin(\omega t))$ with $\omega = v/R = \sqrt{\frac{qk}{m}}$.
Such a field can arise, if we have an ambient surface charge distribution of $\sigma = 2\epsilon_0$ on the horizontal plane.(From the Maxwell's equation, $\nabla.E = \rho/\epsilon_0$)

We can incorporate this along side the magnetic field, the initial conditions could be kept the same as above with the initial velocity v following the condition,

$$-m\frac{v^2}{r} = qvB_0 + qE(r) \tag{11}$$

given that $E(r)$ gives a well-like central force as mentioned above and v is the tangential velocity.
The key point here is that any central force $F(r)$ will give rise to a uniform circular motion if it follows the condition,

$$-m\frac{v^2}{r} = F(r)$$

As for the above case, for a combination of electric and magnetic fields,

$$m\frac{v^2}{r} - qB_0 v - qkr = 0$$

With tangential velocity,

$$v = \frac{qB_0 + \sqrt{q^2 B_0^2 + 4kqm}}{2m/r}$$

This expression recovers the expression for both the cases of just a static magnetic field and just a harmonic electric field as discussed before while maintaining uniform circular motion.

3 Conclusion

We have analysed the problem of a bead on a hoop and also the cyclotron problem i.e. the problem of a a charged particle in a uniform and static magnetic field. We discussed the physical and mathematical similarities between both which happens to be a uniform circular motion. We extended this to Electric fields by discussing an example of a harmonic oscillator-like electric potential. We also extended the same to a mixture of both an Electric and a static Magnetic field. We also confirmed that the equations of motion for the hybrid case gives us the equations of motion for the individual fields in the limit that the other field tends to 0,