EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Electrical Engineering

5LMB0

Model Predictive Control

Quartile 3: 2023 - 2024

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1 Part. 1: Model Predictive Control (MPC) Design

1.1 Task 1: Designing an unconstrained MPC controller

Consider a discrete-time linear system,

$$x(k+1) = Ax(k) + Bu(k) \ , \qquad \forall k \in \mathbb{N} \tag{1}$$

where $x \in \mathbb{R}^2$ and $u \in \mathbb{R}$. The general solution for the discrete-time system 1 with a prediction horizon N, as introduced in **Lecture 1** slides is formulated as follows:

$$x_{N|k} = A^N x_{0|k} + \sum_{i=0}^{N-1} A^i B u_{N-1-i|k} \tag{2}$$

The individual predictions over multiple time steps can be systematically arranged into a matrix framework to obtain the compact form representation of the prediction model:

$$X_k = \Phi x(k) + \Gamma U_k \tag{3}$$

where

$$X_k = \begin{pmatrix} x_{1|k} \\ x_{2|k} \\ \vdots \\ x_{N|k} \end{pmatrix} \ , \quad U_k = \begin{pmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N-1|k} \end{pmatrix} \ , \quad x(k) = x_{0|k} \tag{4}$$

and the matrices Φ and Γ are defined as:

$$\Phi = \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} \ , \quad \Gamma = \begin{pmatrix} B & 0 & \cdots & 0 \\ AB & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{pmatrix} \tag{5}$$

Here, X_k and U_k represent the complete sequence of predicted states and control inputs respectively, while Φ and Γ are the prediction matrices. For the purposes of this assignment, these matrices have been utilized in their standard forms without any alterations.

The following quadratic cost function is used for the assignment:

$$J(x(k), U_k) = x_{N|k}^T P x_{N|k} + \sum_{i=0}^{N-1} \left(x_{i|k}^T Q x_{i|k} + u_{i|k}^T R u_{i|k} \right) \tag{6}$$

Where, P, Q and R are positive definite matrices, weighting terminal state, state trajectory, and control inputs respectively.

The cost function when written in a compact form becomes:

$$J(x(k), U_k) = x(k)^T Q x(k) + X_k^T \Omega X_k + U_k^T \Psi U_k \tag{7}$$

where

$$\Omega = \begin{pmatrix} Q & & & \\ & Q & & \\ & & \ddots & \\ & & & P \end{pmatrix}, \quad \Psi = \begin{pmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{pmatrix} \tag{8}$$

Substituting Equation (3) into Equation (7) gives a cost function $J(x(k), U_k)$, which is a quadratic function with respect to U_K. This quadratic cost function is expressed as:

$$J(x(k), U_k) = \frac{1}{2} U_k^T G U_k + U_k^T F x(k) + x(k)^T (Q + \Phi^T \Omega \Phi) x(k) \tag{9}$$

where

$$G = 2(\Psi + \Gamma^T \Omega \Gamma) \ , \quad F = 2\Gamma^T \Omega \Phi \tag{10}$$

Since the cost function $J(x(k), U_k)$ as specified in Equation (9) is a convex quadratic function with respect to U_k, the unique global minimum is achieved when the gradient of the cost function with respect to U_K is zero. This is mathematically represented as:

$$\nabla_{U_k} J(x(k), U_k) = G U_k + F x(k) = 0 \tag{11}$$

The optimal predicted input sequence is given by:

$$U_k^*(x(k)) = G^{-1} F x(k) \tag{12}$$

The unconstrained MPC control law is formulated as:

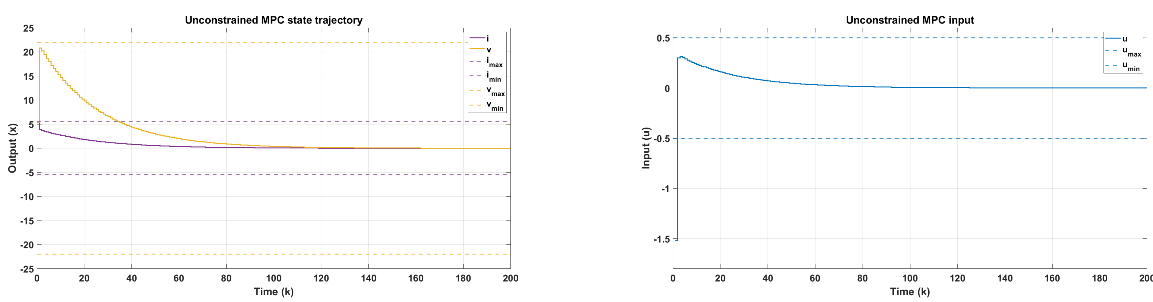
$$u_{0|k}^*(x(k)) = \begin{pmatrix} I_m & 0_{m \times m} & \cdots & 0_{m \times m} \end{pmatrix} U_k^*(x(k)) \tag{13}$$

Substituting Equation (12) into Equation (13),

$$u_{0|k}^*(x(k)) = -\begin{pmatrix} I_m & 0_{m \times m} & \cdots & 0_{m \times m} \end{pmatrix} G^{-1} F x(k) \tag{14}$$

$$K_{MPC} = \begin{pmatrix} I_m & 0_{m \times m} & \cdots & 0_{m \times m} \end{pmatrix} G^{-1} F \tag{15}$$

$$u_{0|k}^*(x(k)) = K_{MPC} x(k) \tag{16}$$



(a) (b)

Figure 1: 1(a) illustrates the state trajectories converging to the operating point and 1(b) displays the plot of the control input. On the plots, the states of the system are denoted as i and v for ease of understanding.

1.2 Task 2: Designing a constrained MPC controller

The compact formulation of the cost function is the same for the unconstrained and the constrained MPC and is given by Equation (9).

The general form of linear constraints in $x_{i|K}$ and $u_{i|k}$ can be written in the matrix form as given in **Lecture 2** and **Summary constrained MPC** slides:

$$M_i x_{i|k} + E_i u_{i|k} \leq b_i \ , \quad \forall i = 0, 1, \cdots, N-1 \tag{17}$$

$$M_N x_{N|k} \leq b_N \tag{18}$$

where

$$M_i = \begin{pmatrix} 0_{m\times n} \\ 0_{m\times n} \\ -I_n \\ I_n \end{pmatrix} \ , \quad E_i = \begin{pmatrix} -I_m \\ I_m \\ 0_{n\times m} \\ 0_{n\times m} \end{pmatrix} \ , \quad b_i = \begin{pmatrix} -u_{low} \\ u_{high} \\ -x_{low} \\ x_{high} \end{pmatrix} \ , \quad \forall i = 0, 1, \cdots, N-1 \tag{19}$$

$$M_N = \begin{pmatrix} -I_n \\ I_n \end{pmatrix} \ , \quad b_N = \begin{pmatrix} -x_{low} \\ x_{high} \end{pmatrix} \tag{20}$$

Grouping all these constraints together gives:

$$\mathcal{D}x(k) + \mathcal{M}X_k + \mathcal{E}U_k \leq c \tag{21}$$

where

$$\mathcal{D} = \begin{pmatrix} M_0 \\ 0_{2(m+n)\times n} \\ \vdots \\ 0_{2(m+n)\times n} \\ 0_{2n\times n} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0_{2(m+n)\times n} & \cdots & 0_{2(m+n)\times n} & 0_{2(m+n)\times n} \\ M_1 & \cdots & 0_{2(m+n)\times n} & 0_{2(m+n)\times n} \\ \vdots & \ddots & \vdots & \vdots \\ 0_{2(m+n)\times n} & \cdots & M_{N-1} & 0_{2(m+n)\times n} \\ 0_{2n\times n} & \cdots & 0_{2n\times n} & M_N \end{pmatrix}, \quad c = \begin{pmatrix} b_0 \\ b1 \\ \vdots \\ b_{N-1} \\ b_N \end{pmatrix} \tag{22}$$

$$\mathcal{E} = \begin{pmatrix} E_0 & 0_{2(m+n)\times m} & \cdots & 0_{2(m+n)\times m} \\ 0_{2(m+n)\times m} & E_1 & \cdots & 0_{2(m+n)\times m} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{2(m+n)\times m} & 0_{2(m+n)\times m} & \cdots & E_{N-1} \\ 0_{2n\times m} & 0_{2n\times m} & \cdots & 0_{2n\times m} \end{pmatrix} \tag{23}$$

Replacing X_K in Equation (21) with the prediction matrices from Equation (3) gives the compact formulation of the constraint matrices:

$$LU_k \leq c + Wx(k) \tag{24}$$

where

$$L = \mathcal{M}\Gamma + \mathcal{E} \ , \quad W = -\mathcal{D} - \mathcal{M}\Phi \tag{25}$$

Since the MPC cost function takes a quadratic form and the constraints on it are linear, quadratic programming (QP) is a suitable method for finding the optimal sequence of control inputs. The constrained linear MPC optimization problem can be written as,

$$\begin{aligned} \min_{U_k} \quad & \frac{1}{2} U_k^T G U_k + U_k^T F x(k) \\ \text{subject to:} \quad & LU_k \leq c + Wx(k) \end{aligned} \tag{26}$$

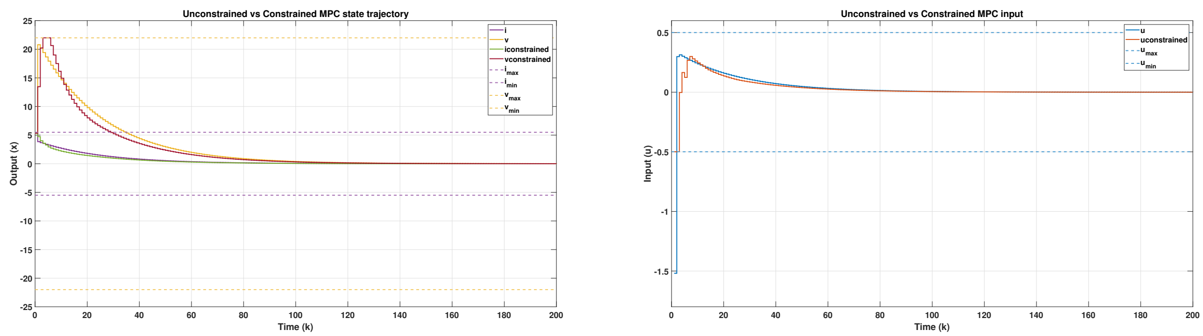
Solving this optimization problem provides a sequence of optimal control inputs $U_k^*(x(k))$. The first optimal control input from this set is then applied to the system, resulting in

$$u_{MPC}(x(k)) = u_0^*(k) = \begin{pmatrix} I_m & 0_{m\times m} & \cdots & 0_{m\times m} \end{pmatrix} U_k^*(x(k))) \tag{27}$$

In MATLAB, the **quadprog** function is used to solve the optimization problem,

$$\text{quadprog}(G, Fx(k), L, c + Wx(k)) \tag{28}$$

At every simulation timestep, this optimization is performed to calculate an optimal input sequence, provided the optimization problem is feasible, as the feasibility of the MPC QP problem may become compromised over time as the state $x(k)$ evolves. The plots comparing the control inputs and the states of both the unconstrained and constrained MPC are shown in Figure 2. They show the constrained MPC's effectiveness in adhering to operational constraints, thus ensuring the system's performance remains within predefined limits.



(a) (b)

Figure 2: State trajectories and control inputs for both unconstrained and constrained MPC strategies.

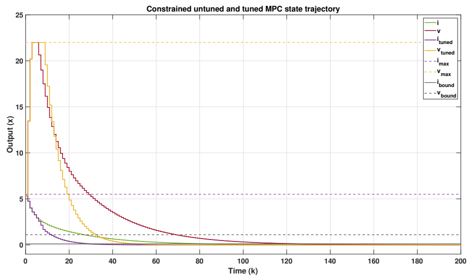
1.3 Task 3: Tuning the constrained MPC controller

In tuning the MPC controller for desired performance, the following two parameters are adjusted: the weighting matrix (Q), which shapes the cost of deviations between predicted outputs and their references; and the weighting matrix (R), which regulates the cost associated with changes in control inputs. For the revised weighting matrices, Q and R is set to:

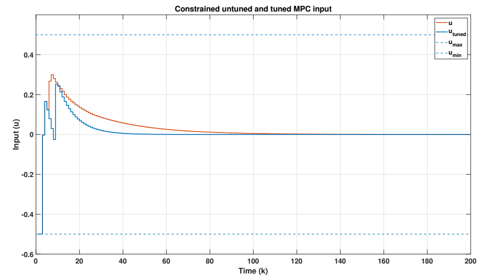
$$Q = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix} \ , \quad R = \begin{pmatrix} 0.2 \end{pmatrix} \tag{29}$$

In Task 3, the objective was to tune the weighting matrices to speed up the convergence of the constrained MPC from Task 2. To achieve this, the penalty in the R matrix was decreased to allow more assertive control actions, while adjustments in the Q matrix increased the penalty on the first state more than the second. This strategy led to a faster convergence to the operating point by emphasizing the importance of the first state, demonstrating that a focused penalization on deviations in the first state significantly enhances the system's convergence speed to the operating point. The plots of state and control input trajectories for the tuned constrained MPC are shown in Figure 3, where Figure 3(a) illustrates the state trajectories converging to the operating point for both the unmodified constrained and modified(tuned) MPC and Figure 3(b) displays the plot of the control inputs for both the strategies. Additionally, the following bounds on the states are also plotted in the figure,

$$\tilde{x}_1(k) \leq 0.11 \ , \quad \tilde{x}_2(k) \leq 1.1 \tag{30}$$



(a)



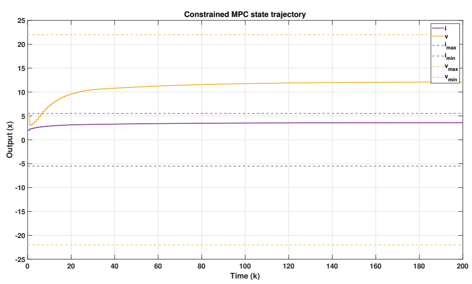
(b)

Figure 3: State and Control inputs trajectories

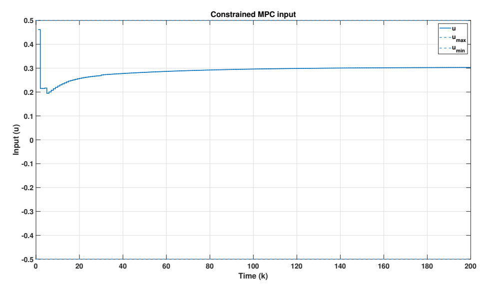
2 Part. 2: Stability of the MPC

2.1 Task 4: Designing MPC controller with a shorter prediction horizon

In this task, a new initial condition and prediction horizon are stated $x(0) = [2 \quad 5]^\top$ and $N = 10$ respectively, while keeping the weighting matrices and constraints the same, where $R = 1$ and $P = Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.



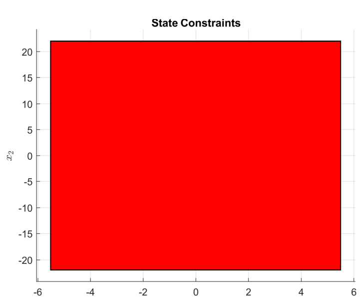
(a)



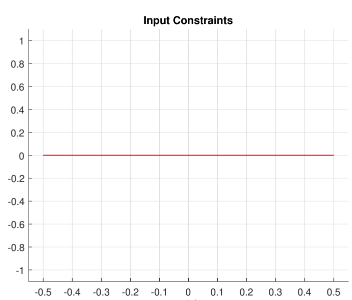
(b)

Figure 4: (a) State trajectories and (b) input values at every time step(k) for finite horizon $N = 10$

In Figure 4(a), it is observed that the MPC controller does not result in the state trajectories successfully converging to the origin. This outcome suggests that the closed-loop system is not asymptotically stable, but instead only Lyapunov stable. One reason could be that the terminal cost matrix P does not guarantee asymptotic stability. Another reason could be that the reduced prediction horizon value might result in the controller not being able to predict system dynamics properly. The state and input constraints are depicted in Figure 5(a) and 5(b) respectively.



(a)



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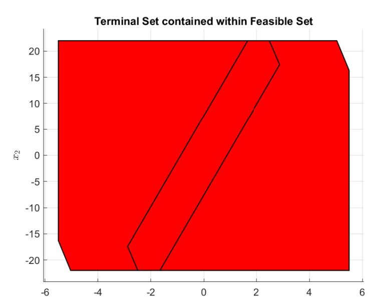


(c)

Figure 5

2.2 Task 5: Designing a constrained stabilising MPC controller

To ensure the system's closed-loop stability by the MPC controller and to address the problem encountered in Task 4 where the controller failed to stabilize the system around the designated operating point, the terminal cost P has to be computed as given in the **Lecture 5** slides which satisfy,

$$(A+BK)^{\top}P(A+BK)-P \preceq -Q-K^{\top}RK \tag{31}$$

where $K \in \mathbb{R}^{2X1}$ with $\rho(A+BK)<1$ and $Q,R \succ 0$.

The above equation can be pre- and post-multiplied by P^{-1}, which results in,

$$P^{-1}-(AP^{-1}+BKP^{-1})^{\top}P(AP^{-1}+BKP^{-1})-P^{-1}QP^{-1}-P^{-1}K^{\top}RKP^{-1} \succcurlyeq 0$$

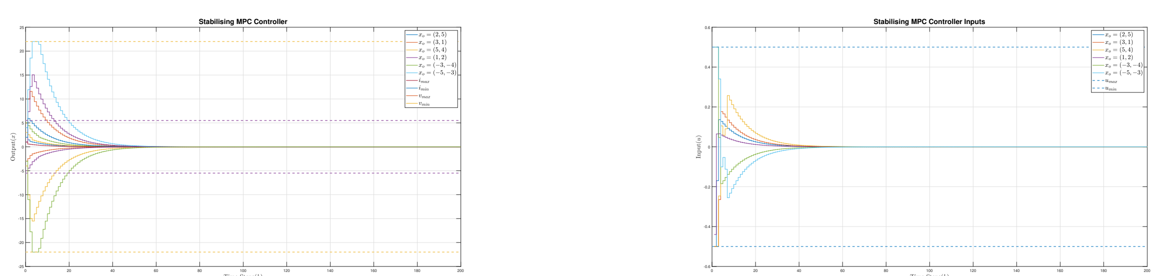
Taking the Schur complement of (31) results in,

$$\begin{pmatrix} O & (AO+BY)^{T} & O & Y^{T} \\ (AO+BY) & O & 0 & 0 \\ O & 0 & Q^{-1} & 0 \\ Y & 0 & 0 & R^{-1} \end{pmatrix} \succcurlyeq 0 \tag{32}$$

where $P=O^{-1}$ & $K=YO^{-1}$.
By solving the LMI (32), P and K are obtained as follows (using the YALMIP toolbox),

$$P=\begin{pmatrix} 179.147 & 11.452 \\ 11.452 & 2.416 \end{pmatrix} \qquad K=\begin{pmatrix} -0.518 & 0.0671 \end{pmatrix}$$

The terminal set $\mathbb{X}_T=\{\xi \in \mathbb{R}^n : M_N\xi \leq b_N\}$ holds to be invariant, $M_N(A+BK)\xi \leq b_N$ and constraint admissible $(M+EK)\xi \leq b$ for the closed-loop system.
The feasible set containing the terminal set is visualized in Figure 5(c). It should be noted that the feasible set of states has shrunk since the terminal constraints are applied when compared with the set of state constraints from Figure 5(a). Using the computed values of P and K, any state starting in the feasible set shall converge to the origin asymptotically. As shown in Figure-6, the closed-loop system converges to the origin for many different initial conditions while obeying the specified constraints.



(a) (b)

Figure 6: (a) State Trajectories and (b) Control input values of different initial conditions for finite horizon $N=10$