Natural Deduction Rules for Prop/Pred Logic

	introduction	*elimination*
$\wedge$	$\dfrac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge\mathrm{i}$	$\dfrac{\phi \wedge \psi}{\phi}\ \wedge\mathrm{e}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\ \wedge\mathrm{e}_2$
$\vee$	$\dfrac{\phi}{\phi \vee \psi}\ \vee\mathrm{i}_1 \qquad \boxed{\dfrac{}{\phi \vee \neg\phi}\ \mathrm{LEM}}$	$\dfrac{\phi \vee \psi \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\ \vee\mathrm{e}$
$\rightarrow$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\mathrm{i}$	$\dfrac{\phi \quad \phi \rightarrow \psi}{\psi}\ \rightarrow\mathrm{e} \qquad \boxed{\dfrac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \mathrm{MT}}$
$\neg$	$\dfrac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\mathrm{i} \qquad \boxed{\dfrac{\boxed{\begin{array}{c}\neg\phi\\ \vdots\\ \bot\end{array}}}{\phi}\ \mathrm{PBC}}$	$\dfrac{\phi \quad \neg\phi}{\bot}\ \neg\mathrm{e}$
$\bot$	(no introduction rule for $\bot$)	$\dfrac{\bot}{\phi}\ \bot\mathrm{e}$
$\neg\neg$	$\boxed{\dfrac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}}$	$\dfrac{\neg\neg\phi}{\phi}\ \neg\neg\mathrm{e}$

$$\dfrac{}{t = t}\ {=}\mathrm{i} \qquad \dfrac{t_1 = t_2 \quad \phi[t_1/x]}{\phi[t_2/x]}\ {=}\mathrm{e}.$$

$$\dfrac{\boxed{\begin{array}{l}x_0\\ \quad\vdots\\ \quad\phi[x_0/x]\end{array}}}{\forall x\, \phi}\ \forall x\,\mathrm{i}. \qquad \dfrac{\forall x\, \phi}{\phi[t/x]}\ \forall x\,\mathrm{e}.$$

$$\dfrac{\phi[t/x]}{\exists x \phi}\ \exists x\,\mathrm{i}. \qquad \dfrac{\exists x\, \phi \quad \boxed{\begin{array}{l}x_0\ \phi[x_0/x]\\ \quad\vdots\\ \quad\chi\end{array}}}{\chi}\ \exists\mathrm{e}.$$

$$\forall x\,(Q(x) \to R(x)),\ \exists x\,(P(x) \wedge Q(x)) \ \vdash\ \exists x\,(P(x) \wedge R(x))$$

1		$\forall x\,(Q(x) \to R(x))$	premise
2		$\exists x\,(P(x) \wedge Q(x))$	premise
3	x_0	$P(x_0) \wedge Q(x_0)$	assumption
4		$Q(x_0) \to R(x_0)$	$\forall x$ e 1
5		$Q(x_0)$	$\wedge$e$_2$ 3
6		$R(x_0)$	$\to$e 4, 5
7		$P(x_0)$	$\wedge$e$_1$ 3
8		$P(x_0) \wedge R(x_0)$	$\wedge$i 7, 6
9		$\exists x\,(P(x) \wedge R(x))$	$\exists x$ i 8
10		$\exists x\,(P(x) \wedge R(x))$	$\exists x$ e 2, 3–9

By natural deduction, show the validity of:

$$\forall x\, P(a,x,x),\ \forall x \forall y \forall z\, (P(x,y,z) \rightarrow P(f(x),y,f(z))) \ \vdash\ P(f(a),a,f(a))$$

t can be chosen to help with our proof

$$\frac{\forall x\, \phi}{\phi[t/x]}\ \forall x\, \mathrm{e}.$$

By natural deduction, show the validity of:

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1	$\forall x\, P(a,x,x)$	prem
2	$\forall x\, \forall y\, \forall z\, (P(x,y,z) \to P(f(x),y,f(z)))$	prem
3	$P(a,a,a)$	$\forall x\,$e 1
4	$\forall y\, \forall z\, (P(a,y,z) \to P(f(a),y,f(z)))$	$\forall x\,$e 2
5	$\forall z\, (P(a,a,z) \to P(f(a),a,f(z)))$	$\forall y\,$e 4
6	$P(a,a,a) \to P(f(a),a,f(a))$	$\forall z\,$e 5
7	$P(f(a),a,f(a))$	$\to$e 6, 3

t can be chosen to help with our proof

$$\frac{\forall x\, \phi}{\phi[t/x]}\ \forall x\,\mathrm{e}.$$

By natural deduction, show the validity of:

$$\forall y\, Q(b, y), \forall x \forall y\, (Q(x, y) \to Q(s(x), s(y))) \vdash \exists z\, (Q(b, z) \land Q(z, s(s(b))))$$

t can be chosen to help with our proof

$$\frac{\forall x\, \phi}{\phi[t/x]}\ \forall x\, \mathrm{e}. \qquad \frac{\phi[t/x]}{\exists x \phi}\ \exists x\, \mathrm{i}.$$

By natural deduction, show the validity of:

$$\forall y\, Q(b, y), \forall x \forall y\, (Q(x, y) \rightarrow Q(s(x), s(y))) \vdash \exists z\, (Q(b, z) \wedge Q(z, s(s(b))))$$

1	$\forall y\, Q(b, y)$	prem
2	$\forall x\, \forall y\, (Q(x, y) \rightarrow Q(s(x), s(y)))$	prem
3	$\forall y\, (Q(b, y) \rightarrow Q(s(b), s(y)))$	$\forall x$ e 2
4	$Q(b, s(b)) \rightarrow Q(s(b), s(s(b)))$	$\forall y$ e 3
5	$Q(b, s(b))$	$\forall x$ e 1
6	$Q(s(b), s(s(b)))$	$\rightarrow$e 4, 5
7	$Q(b, s(b)) \wedge Q(s(b), s(s(b))$	$\wedge$i 5, 6
8	$\exists z\, (Q(b, z) \wedge Q(z, s(s(b)))$	$\exists x$ i 7

t can be chosen to help with our proof $\quad \dfrac{\forall x\, \phi}{\phi[t/x]}\ \forall x\,\text{e.} \qquad \dfrac{\phi[t/x]}{\exists x \phi}\ \exists x\,\text{i.}$

The importance of both **proof theory** and **semantics**

proof theory

- useful for establishing assertions like ‘$\Gamma \vdash \psi$ is valid,’
  - we provide a proof of ψ from Γ
- not so useful for establishing assertions like ‘$\Gamma \vdash \phi$ is not valid.’
  - how can you show that there is no proof of something?

semantics

- useful for establishing assertions of the form ‘$\Gamma \vDash \psi$ is not valid.’
  - need only talk about one valuation/model
- not so useful for establishing assertions like ‘$\Gamma \vDash \psi$ is valid,’
  - need to talk about (infinitely) many models.

Statement	True	False
$\forall x \quad P(x)$	$P(x)$ is true for every x	There is at least one x for which $P(x)$ is false
$\exists x \quad P(x)$	There is at least one x for which $P(x)$ is true	$P(x)$ is false for every x

Models

How can we **evaluate** formulas in **predicate logic**?

The **truth value of a formula in predicate logic** depends on, and varies with, the **actual choice of values** and the **meaning of the predicate** and **function symbols** involved.

We require a **model** of all function and predicate symbols involved.

Definition 2.14 Let $\mathcal{F}$ be a set of function symbols and $\mathcal{P}$ a set of predicate symbols, each symbol with a fixed number of required arguments. A model $\mathcal{M}$ of the pair $(\mathcal{F}, \mathcal{P})$ consists of the following set of data:

1. A non-empty set A, the universe of concrete values;
2. for each nullary function symbol $f \in \mathcal{F}$, a concrete element $f^{\mathcal{M}}$ of A
3. for each $f \in \mathcal{F}$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$ from A^n, the set of n-tuples over A, to A; and
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a subset $P^{\mathcal{M}} \subseteq A^n$ of n-tuples over A.

Example

Given:

$\mathcal{F} \stackrel{\text{def}}{=} \{\mathtt{s(\cdot)},\ \mathtt{p(\cdot)},\ \oplus,\ \mathtt{zero}\}$

$\mathcal{P} \stackrel{\text{def}}{=} \{=,\ >,\ \mathtt{Even}\}$

a 'signature', two sets of symbols

The model $\mathcal{M}$ called Int:

domain:

$A \stackrel{\text{def}}{=} \mathbb{Z}$,

concrete functions:

$\mathtt{zero}$ is the number 0, s is the successor fn, p the predecessor fn

$\oplus$ is integer addition

concrete predicates:

$=,\ >,\ \mathtt{Even}$ are the usual predicates for integers

A sentence in the model $\mathcal{M}$:

$\exists x\,(x > \mathtt{zero})$ "there is an integer greater than 0"

$\forall y\, \neg(y = \mathtt{zero}) \rightarrow \exists x\,(x > y)$ "for any integer not equal to 0, there exists an integer greater than it"

Example

Given:

$\mathcal{F} \stackrel{\text{def}}{=} \{\texttt{s(·), p(·), ⊕, zero}\}$

$\mathcal{P} \stackrel{\text{def}}{=} \{\texttt{=, >, Even}\}$

a 'signature', two sets of symbols

The model $\mathcal{M}$ called Nat3:

domain:

$A \stackrel{\text{def}}{=} \mathbb{N}$ modulo 3,

concrete functions:

`zero` is the number 0, s is the successor modulo 3, p the predecessor modulo 3, $\oplus$ is addition modulo 3

concrete predicates:

`=`, `>`, `Even` are the usual predicates for natural numbers

A sentence in the model $\mathcal{M}$:

$\exists x\,(x > \texttt{zero})$ "there is a natural number modulo 3 greater than 0"

$\forall y\,\neg(y = \texttt{zero}) \rightarrow \exists x\,(x > y)$ "for any nn mod 3 not equal to 0, there exists a nn mod 3 greater than it"

Example

Given:

$\mathcal{F} \stackrel{\text{def}}{=} \{\mathtt{s(\cdot)},\ \mathtt{p(\cdot)},\ \oplus,\ \mathtt{zero}\}$

$\mathcal{P} \stackrel{\text{def}}{=} \{=,\ >,\ \mathtt{Even}\}$

a 'signature', two sets of symbols

The model $\mathcal{M}$ called Pres5:

domain:

A $\stackrel{\text{def}}{=}$ the last 5 US presidents = {Biden, Trump, Obama, Bush, Clinton}

concrete functions:

$\mathtt{zero} \stackrel{\text{def}}{=}$ Biden, s is the successor, p the predecessor, $\oplus$ is the president who took office latest

concrete predicates:

$=$ is identity, $>$ is 'took office later than', $\mathtt{Even}$ is true for presidents who held office for an even number of years

A sentence in the model $\mathcal{M}$:

$\exists x\,(x > \mathtt{zero})$ "there is a president who took office later than Biden"

$\forall y\,\neg(y = \mathtt{zero}) \rightarrow \exists x\,(x > y)$ "for any president not Biden, there exists a president who took office later"

Let $\mathcal{F} \stackrel{\text{def}}{=} \{+, *, -\}$ and $\mathcal{P} \stackrel{\text{def}}{=} \{=, \leq, <, \texttt{zero}\}$, where $+$, $*$, $-$ take 2 arguments, and where $=$, $\leq$, $<$ are predicates with 2 arguments, and $\texttt{zero}$ is a predicate with 1 argument.

The model $\mathcal{M}$:

1. The non-empty set A is the set of real numbers.
2. The function $+^{\mathcal{M}}$, $*^{\mathcal{M}}$, and $-^{\mathcal{M}}$ take two real numbers as arguments and return their sum, product, and difference, respectively.
3. The predicates $=^{\mathcal{M}}$, $\leq^{\mathcal{M}}$, and $<^{\mathcal{M}}$ model the relations equal to, less than, and strictly less than, respectively. The predicate $\texttt{zero}^{\mathcal{M}}$ holds for r iff r equals to 0.

Example formula:

$$\forall x \forall y (\texttt{zero}(y) \rightarrow x * y = y)$$

Let $\mathcal{F} \stackrel{def}{=} \{e, \cdot\}$, and $\mathcal{P} \stackrel{def}{=} \{\leq\}$, where e is a constant, $\cdot$ is a function of 2 arguments and $\leq$ is a predicate with 2 arguments.

The model $\mathcal{M}$:

1. A is the set of binary strings over the alphabet $\{0, 1\}$, including the empty string ε.
2. The interpretation of $\cdot^{\mathcal{M}}$ is the concatenation of strings.
3. $\leq^{\mathcal{M}}$ is the prefix ordering of strings, that is the set $\{(s_1, s_2) \,|\, s_1 \text{ is a prefix of } s_2\}$.

$\forall x\,((x \leq x \cdot e) \wedge (x \cdot e \leq x))$

Every word is a prefix of itself concatenated with the empty word

$\exists y \forall x\,(y \leq x)$

There exists a word s that is the prefix of every word (in fact it is ε).

$\forall x \exists y\,(y \leq x)$

Every word has a prefix.

$\forall x \forall y \forall z\,((x \leq y) \rightarrow (x \cdot z \leq y \cdot z))$

If s_1 is a prefix of s_2, then $s_1 s_2$ is a prefix of $s_1 s_3$ (doesn't hold).

$\neg \exists x \forall y\,((x \leq y) \rightarrow (y \leq x))$

There is no word s such that whenever s is a prefix of some other word s_1, it is the case that s_1 is a prefix of s as well.

Given a formula $\forall x \Phi$, or $\exists x \Phi$, we intend to check whether Φ holds for all, respectively some, value a in our model. We have no way of expressing this in our syntax.

We are forced to interpret formulas relative to an ***environment (look-up table)***, that is, a mapping from variable symbols to concrete values.

$$l : \mathtt{var} \mapsto A$$

Definition (Updated Look-Up Tables): Let l be a look-up table $l : \mathtt{var} \mapsto A$, and let $a \in A$. We denote by $l[x \mapsto a]$ the look-up table which maps x to a and any other variable y to $l(y)$.

Definition 2.18 Given a model $\mathcal{M}$ for a pair $(\mathcal{F}, \mathcal{P})$ and given an environment l, we define the satisfaction relation $\mathcal{M} \vDash_l \phi$ for each logical formula ϕ over the pair $(\mathcal{F}, \mathcal{P})$ and look-up table l by structural induction on ϕ. If $\mathcal{M} \vDash_l \phi$ holds, we say that ϕ computes to T in the model $\mathcal{M}$ with respect to the environment l.

- P: If ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, then we interpret the terms $t_1, t_2, \ldots, t_n$ in our set A by replacing all variables with their values according to l. In this way we compute concrete values $a_1, a_2, \ldots, a_n$ of A for each of these terms, where we interpret any function symbol $f \in \mathcal{F}$ by $f^{\mathcal{M}}$. Now $\mathcal{M} \vDash_l P(t_1, t_2, \ldots, t_n)$ holds iff $(a_1, a_2, \ldots, a_n)$ is in the set $P^{\mathcal{M}}$.
- $\forall x$: The relation $\mathcal{M} \vDash_l \forall x\, \psi$ holds iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for all $a \in A$.
- $\exists x$: Dually, $\mathcal{M} \vDash_l \exists x\, \psi$ holds iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for some $a \in A$.
- $\neg$: The relation $\mathcal{M} \vDash_l \neg\psi$ holds iff it is not the case that $\mathcal{M} \vDash_l \psi$ holds.
- $\vee$: The relation $\mathcal{M} \vDash_l \psi_1 \vee \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_1$ or $\mathcal{M} \vDash_l \psi_2$ holds.
- $\wedge$: The relation $\mathcal{M} \vDash_l \psi_1 \wedge \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_1$ and $\mathcal{M} \vDash_l \psi_2$ hold.
- $\rightarrow$: The relation $\mathcal{M} \vDash_l \psi_1 \rightarrow \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_2$ holds whenever $\mathcal{M} \vDash_l \psi_1$ holds.

We sometimes write $\mathcal{M} \nvDash_l \phi$ to denote that $\mathcal{M} \vDash_l \phi$ does not hold.

Example

$\mathcal{F} \stackrel{\text{def}}{=} \{\texttt{alma}\}$ alma is a constant

$\mathcal{P} \stackrel{\text{def}}{=} \{\texttt{loves}\}$ loves is a predicate with two arguments

The model $\mathcal{M}$:

$A \stackrel{\text{def}}{=} \{\texttt{a, b, c}\}$,

$\texttt{alma}^{\mathcal{M}} \stackrel{\text{def}}{=} \texttt{a}$ a constant function

$\texttt{loves}^{\mathcal{M}} \stackrel{\text{def}}{=} \{\texttt{(a,a), (b,a), (c,a)}\}$ a predicate

We want to check whether the model $\mathcal{M}$ satisfies:

None of Alma's lovers' lovers love her.

1. $\forall x \forall y\, (\, \texttt{loves}(x, \texttt{alma}) \wedge \texttt{loves}(y, x) \rightarrow \neg \texttt{loves}(y, \texttt{alma})\,)$
2. We choose a for x and b for y.
Since (a,a) is in the set loves$^{\mathcal{M}}$ and (b,a) is in the set $\texttt{loves}^{\mathcal{M}}$,
we would need that the latter does not hold since it is the interpretation of $\texttt{loves}(y$, alma); this cannot be. <u>The sentence does not hold in the model.</u>

What if $\texttt{loves}^{M} \stackrel{\text{def}}{=} \{(b, a), (c, b)\}$?