2

Infinite Series

2·1. Definitions

A series containing an infinite number of terms $u_1, u_2, u_3, \ldots, u_n, \ldots$ that occur according to some definite law, is called *infinite series* and is denoted by

$$u_1+u_2+u_3+\ldots u_n+\ldots \text{ or } \sum_{n=1}^{\infty} u_n \text{ or simply } \Sigma u_n.$$

The sum of the first n terms of the series is denoted by S_n.

$$\therefore \quad S_n=u_1+u_2+u_3+\ldots+u_n.$$

Convergence and Divergence of Series

Consider an infinite series

$$\Sigma u_n=u_1+u_2+u_3+\ldots+u_n+\ldots$$

and let

$$S_n=u_1+u_2+u_3+\ldots+u_n.$$

The given series Σu_n is said to be *convergent* if $\underset{n\to\infty}{\text{Lt}} S_n$ is finite quantity and the given series is said to be divergent if $\underset{n\to\infty}{\text{Lt}} S_n$ tends to infinity.

However if $\underset{n\to\infty}{\text{Lt}} S_n$ does not tend to a definite limit the given series is known as oscillatory, sometimes called non-convergent series.

Example 1. *Discuss the convergence of the series,*

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots$$

Sol. Here

$$S_n=1+\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots+\frac{1}{3^{n-1}}$$

$$=\frac{1\cdot(1-1/3^n)}{1-1/3} \qquad \text{(Sum of G.P.)}$$

$$=\frac{3}{2}\left(1-\frac{1}{3^n}\right)$$

$$=\frac{3}{2}-\frac{1}{2\cdot 3^{n-1}}$$

Now $\underset{n\to\infty}{\text{Lt}} S_n=\underset{n\to\infty}{\text{Lt}}\left(\frac{3}{2}-\frac{1}{2\cdot 3^{n-1}}\right)=\frac{3}{2}$

Since $\underset{n\to\infty}{\text{Lt}}\ S_n$ is a finite quantity, the given series is convergent.

Example 2. *Investigate the series for convergence or divergence,*

$$1+2+3+\ldots+n+\cdots$$

Sol. Here

$$S_n=1+2+3+\ldots+n$$

$$=\frac{n(n+1)}{2}$$

(Sum of first n natural numbers)

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ S_n=\underset{n\to\infty}{\text{Lt}}\ \frac{n(n+1)}{2}\to\infty$$

Since $\underset{n\to\infty}{\text{Lt}}\ S_n$ is infinite, the given series is divergent.

Example 3. *Discuss the nature of the series*

$$1-1+1-1+\cdots$$

Sol. Obviously S_n is 1 or 0 according as n is odd or even. Therefore, S_n cannot tend to a definite limit, hence the given series is oscillatory.

EXERCISE 2 (*a*)

Discuss the nature of the following series.

1. $1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$
2. $1^2+2^2+3^2+\cdots$
3. $1^3+2^3+3^3+\cdots$
4. $2+6+18+\cdots$
5. $3+5+7+\cdots$
6. $k-k+k-k+\cdots$

2.2. Fundamental Theorem on Infinite Series

The convergence or divergence of an infinite series remains unaffected by removal or addition of a finite number of terms.

Since the sum of a finite number of terms is finite, so their removal from the series or their addition to the series will not change the convergence or divergence of the series.

2.3. Convergence of a Geometric Series

Show that the geometric series

$$a+ar+ar^2+\ldots\ldots+ar^{n-1}+\ldots$$

(*i*) Converges if $|r|<1$

(*ii*) Diverges if $r\geqslant 1$

(*iii*) Oscillates if $r\leqslant -1$

Let $S_n = a + ar^2 + \dots + ar^{n-1}$ ...(1)

Case I. When $|r| < 1$.

Now
$$S_n = \frac{a(1-r^n)}{1-r}$$
$$= \frac{a}{1-r} - \frac{ar^n}{1-r}$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} \left(\frac{a}{1-r} - \frac{ar^n}{1-r}\right)$$

Since $|r| < 1$, therefore, $\operatorname*{Lt}_{n\to\infty} r^n = 0$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \frac{a}{1-r} \quad (a \text{ finite quantity})$$

Since $\operatorname*{Lt}_{n\to\infty} S_n$ is a finite quantity, the given series converges if $|r| < 1$.

Case II When $r \geqslant 1$

We first consider the case when $r > 1$

Here
$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} \left(\frac{ar^n}{r-1} - \frac{a}{r-1}\right)$$

Since $r > 1$, $\operatorname*{Lt}_{n\to\infty} r^n \to \infty$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n \to \infty$$

Since $\operatorname*{Lt}_{n\to\infty} S_n$ tends to infinity the given series diverges if $r > 1$.

When $r = 1$, we have from (1),

$$S_n = a + a + a + \dots n \text{ times}$$
$$= na$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} S_n = \operatorname*{Lt}_{n\to\infty} na \to \infty.$$

Hence the given series is divergent when $r = 1$.

Thus the given series diverges if $r \geqslant 1$,

Case III. When $r \leqslant -1$.

We first consider the case when $r = -1$, we have the given series from (1),

$$S_n = a - a + a - \dots n \text{ times}$$

Obviously S_n oscillates between a and 0, according as n is odd or even.

When $r < -1$, let $r = -K$ (where $K > 1$)

$$\therefore \quad S_n = \frac{a(1-r^n)}{1-r} = \frac{a[1-(-K)^n]}{1+K}$$

$$= \frac{a[1+(-1)^{n+1} K^n]}{1+K}$$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a}{1+K} [1+(-1)^{n+1} K^n]$ which oscillates between $-\infty$ and $+\infty$, according as n is even or odd.

Therefore, the given series oscillates when $r \leq -1$

2·4. Convergence or Divergence of p-Series

Show that the series

$$\Sigma n^{-p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

(*i*) Converges if $p > 1$

(*ii*) Diverges if $p \leq 1$

(*i*) **Case I.** When $p > 1$

Let the terms of the given series be grouped in such a manner that first, second, third.. groups contain 1 term, 4 terms, 8 terms,...respectively. Thus the given series may be written as

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots$$

Now $\qquad \frac{1}{1^p} = 1$

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \left(\frac{1}{2}\right)^{p-1} \quad \left[\because \frac{1}{3^p} < \frac{1}{2^p}\right]$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p}$$

$$= \left(\frac{1}{2}\right)^{2(p-1)}$$

Similarly $\frac{1}{8^p} + \frac{1}{9^p} + \ldots + \frac{1}{15^p} < \frac{8}{8^p} = \left(\frac{1}{2}\right)^{3(p-1)}$ and so on.

Adding corresponding sides, we get

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right)$$
$$+ \left(\frac{1}{8^p} + \frac{1}{9^p} + \ldots \frac{1}{15^p}\right) + \ldots < 1 + \left(\frac{1}{2}\right)^{p-1} + \left(\frac{1}{2}\right)^{2(p-1)}$$
$$+ \left(\frac{1}{2}\right)^{3(p-1)} + \ldots$$

or $\qquad \Sigma n^{-p} < 1 + (\tfrac{1}{2})^{p-1} + (\tfrac{1}{2})^{2(p-1)} + (\tfrac{1}{2})^{3(p-1)} + \ldots$

The series on the right hand side is a geometric series with common ratio $(\frac{1}{2})^{p-1}<1$ (when $p>1$), hence convergent.

Thus the given series is convergent when $p>1$.

(*ii*) **Case II.** When $p=1$

The given series becomes,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots$$

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$$

$$1+\frac{1}{2}=1+\frac{1}{2}$$

Now $$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$$

Similarly $$\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{16}>\frac{1}{16}+\frac{1}{16}+\ldots 8 \text{ times}=\frac{1}{2} \text{ and so on.}$$

Adding corresponding sides, we get

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

The series on the right hand side (ignoring the first term) is a geometric series with common ratio unity hence divergent.

Thus the given series is divergent when $p=1$.

Case III. When $p<1$.

$$\frac{1}{1^p}=1$$

$$\frac{1}{2^p}>\frac{1}{2}$$

$$\frac{1}{3^p}>\frac{1}{3}$$

$$\frac{1}{4^p}>\frac{1}{4} \text{ and so on.}$$

Adding corresponding sides, we get

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots>1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

The series on right hand side is divergent (by case II)

Thus the given series is divergent when $p<1$.

2·5. Positive Term Series

An infinite series in which all the terms from and after some term are positive, is called a positive term infinite series or **positive term series.** For example $-1-2-3+1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, is a positive term series, because from 4th term onward all the terms of the series are positive.

Convergence or Divergence of a positive term Series

We have so far discussed the convergence or divergence of a series by evaluating the sum of first n terms of the series. However, it is not easy to calculate the sum of first n terms of every series. Thus it is not possible to determine the nature of every series by direct application of definition. Hence various other methods have been derived to determine the convergence of a series.

Necessary Condition for Convergence of a Series

Theorem. If Σu_n is a convergent series of positive terms, then $\underset{n\to\infty}{\text{Lt}}\ u_n=0$.

Let $S_n=u_1+u_2+u_3+u_4+\ldots+u_n$

Since Σu_n is convergent,

$$\underset{n\to\infty}{\text{Lt}}\ S_n=K \text{ (say)}$$

Also $\underset{n\to\infty}{\text{Lt}}\ S_{n-1}=K$

Now $u_n=S_n-S_{n-1}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n=\underset{n\to\infty}{\text{Lt}}\ (S_n-S_{n-1})$$

$$=\underset{n\to\infty}{\text{Lt}}\ S_n-\underset{n\to\infty}{\text{Lt}}\ S_{n-1}$$

$$=K-K=0$$

or $\underset{n\to\infty}{\text{Lt}}\ u_n=0$

Hence the condition.

Note 1. It must be noted carefully that converse of the above theorem is not true, *i.e.* even if $\underset{n\to\infty}{\text{Lt}}\ u_n=0$, the series Σu_n may not converge.

For example, let $\Sigma u_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$

Here $u_n=\frac{1}{n}$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}=0$$

But we have seen that the given series is divergent.

Note 2. The result can effectively be applied to show that the given series is divergent if $\underset{n\to\infty}{\text{Lt}}\ u_n \neq 0$.

For example, let $\Sigma u_n = 1+2+3+\dots+n+\dots$

Here $u_n = n$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\ u_n = \underset{n\to\infty}{\text{Lt}}\ n \neq 0$$

Hence the given series is divergent.

2·6. Tests for Convergence and Divergence of a Series

Comparison Test

If Σu_n and Σv_n be two positive term series, such that from and after some particular term $\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = k$ (a non-zero finite quantity), then Σu_n and Σv_n either both converge or both diverge.

Proof. Let the series from and after the particular term be. $u_1+u_2+u_3+\dots+u_n+\dots$

Now for all values of n,

let $$\underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = k$$

Therefore by definition of limit, there exists a positive number ε, such that

$$\left|\frac{u_n}{v_n} - k\right| < \varepsilon \text{ for all } n$$

i.e. $$-\varepsilon < \frac{u_n}{v_n} - k < \varepsilon$$

or $$k - \varepsilon < \frac{u_n}{v_n} < k + \varepsilon \quad \dots (1)$$

Case I. When Σv_n is convergent.

From (1), we have

$$\frac{u_n}{v_n} < k + \varepsilon \quad \text{for all } n.$$

or $$u_n < (k+\varepsilon)\, v_n$$
or $$\Sigma u_n < \Sigma (k+\varepsilon)\, v_n$$
or $$\Sigma u_n < (k+\varepsilon)\, \Sigma v_n$$

Each term of the series Σu_n is term by term less than the corresponding terms of $(k+\varepsilon)\,\Sigma v_n$. Since Σv_n is convergent, therefore, Σu_n is also convergent.

Case II. When Σv_n is divergent

From (1), we have

$$\frac{u_n}{v_n} > (k - \varepsilon) \text{ for all } n.$$

$\therefore \qquad u_n > (k-\epsilon)\, v_n$

or $\qquad \Sigma\, u_n > (k-\varepsilon)\, \Sigma\, v_n$

Each term of Σu_n is term by term greater than the corresponding terms of $(k-\varepsilon)\,\Sigma v_n$. Since Σv_n is divergent, therefore, $\Sigma\, u_n$ is also divergent.

Note 1. This test is very useful when degree of n, in u_n can readily be determined.

Note 2. To select auxiliary series $\Sigma\, v_n = \Sigma\, \frac{1}{n^p}$, it should be noted that p=Difference in degree of n in denominator and numerator of u_n.

Example 1. *Test for convergence of the series,*

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\ldots+\frac{1}{n^2+1}+\ldots$$

Sol. Here $\qquad u_n = \frac{1}{n^2+1}$

Let $\qquad \Sigma v_n = \Sigma\, \frac{1}{n^2}$ be an auxiliary series,

(See note 2, Art. 2·6)

$$\therefore \qquad v_n = \frac{1}{n^2}$$

Now, $$\frac{u_n}{v_n} = \frac{n^2}{n^2+1} = \frac{1}{\left(1+\frac{1}{n^2}\right)}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\, \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\, \frac{1}{\left(1+\frac{1}{n^2}\right)}$$

$$=1 \qquad \text{(a non-zero, finite quantity)}$$

Hence Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma\, \frac{1}{n^2}$ is convergent (being p series with $p>1$), therefore, Σu_n is also convergent.

Example 2. *Test the convergence or divergence of the series,*

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\infty$$

Sol. Here $\qquad u_n = \frac{n+1}{n^p}$

$\therefore \quad$ Now let $\qquad \Sigma v_n = \Sigma\, \frac{1}{n^{p-1}}$ be the auxiliary series,

$$\therefore \qquad v_n = \frac{1}{n^{p-1}}$$

$$\frac{u_n}{v_n} = \frac{n+1}{n^p} \cdot n^{p-1}$$

$$= \frac{n+1}{n} = \left(1 + \frac{1}{n}\right)$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \left(1 + \frac{1}{n}\right)$$

$$= 1 \qquad \text{[a non-zero finite quantity]}$$

Hence Σu_n and Σv_n converge or diverge together.

But $\qquad \Sigma v_n = \Sigma \frac{1}{n^{p-1}}$ is,

(*i*) Convergent if $p-1 > 1$ i.e. $p > 2$
(*ii*) Divergent if $p-1 \leqslant 1$ i.e. $p \leqslant 2$
$\therefore$ Σu_n converges if $p > 2$ and diverges if $p \leqslant 2$

Example 3. *Discuss the nature of the series,*

$$\Sigma\, u_n = \Sigma\left(\frac{2}{5n+1}\right)$$

Sol. Here $u_n = \dfrac{2}{5n+1}$

Let $\qquad \Sigma v_n = \Sigma \dfrac{1}{n}$

$$v_n = \frac{1}{n}$$

Now $\qquad \dfrac{u_n}{v_n} = \dfrac{2n}{5n+1}$

$$= \frac{2}{\left(5 + \frac{1}{n}\right)}$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}} \frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}} \frac{2}{\left(5 + \frac{1}{n}\right)}$$

$$= \frac{2}{5} \qquad \text{[a non-zero finite quantity]}$$

Hence Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \dfrac{1}{n}$ is known to be divergent, therefore, the given series is also divergent.

2.7. Comparison of Ratios

If Σu_n and Σv_n be two positive term series such that from and after some particular term,

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

and if Σv_n is convergent, then Σu_n is also convergent.

Proof. Let the two series Σu_n and Σv_n, from and after the particular term be,

$u_1+u_2+u_3+\dots$ and $v_1+v_2+v_3+\dots$, respectively.

Since $$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

$$\frac{u_1}{u_2} > \frac{v_1}{v_2}, \quad \frac{u_2}{u_3} > \frac{v_2}{v_3}, \quad \frac{u_3}{u_4} > \frac{v_3}{v_4} \text{ and so on}$$

or $$\frac{u_2}{u_1} < \frac{v_2}{v_1}, \quad \frac{u_3}{u_2} < \frac{v_3}{v_2}, \quad \frac{u_4}{u_3} < \frac{v_4}{v_3} \text{ and so on} \quad \dots(1)$$

Now $$u_1+u_2+u_3+\dots = u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\dots\right)$$

$$= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\dots\right)$$

$$< u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\dots\right)$$

$$= \frac{u_1}{v_1}(v_1+v_2+v_3+\dots)$$

$$\therefore \quad u_1+u_2+u_3+\dots < \frac{u_1}{v_1}(v_1+v_2+v_3+\dots)$$

Now series Σv_n is convergent, therefore, Σu_n is also convergent.

Note. Similarly we can show Σu_n is divergent if

(*i*) Σv_n is divergent.

and (*ii*) from and after some particular term,

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

2.8. D' Alembert's Ratio Test

If Σu_n be a positive term series such that from and after some particular term,

$$\operatorname*{Lt}_{n\to\infty} \frac{u_{n+1}}{u_n} = k, \text{ then } \Sigma u_n$$

(*i*) Converges, if $k < 1$

(*ii*) Diverges, if $k > 1$.

Proof. Let the series from and after the particular term be

$$u_1+u_2+u_3+\ldots+u_n+\ldots$$

Case I. When $k < 1$.

By definition of a limit, a positive number $\lambda (k < \lambda < 1)$ can be found such that

$$\frac{u_{n+1}}{u_n}<\lambda, \qquad \text{for all } n \qquad \ldots(1)$$

Thus $$\frac{u_2}{u_1}<\lambda, \frac{u_3}{u_2}<\lambda, \frac{u_4}{u_3}<\lambda, \ldots\text{and so on.} \qquad \ldots(2)$$

Now $u_1+u_2+u_3+\cdots+u_n+\cdots$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\cdots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\cdots\right)$$

$$<u_1(1+\lambda+\lambda^2+\lambda^3+\cdots) \qquad \text{[By (2)]}$$

$$=\frac{u_1}{1-\lambda}, \text{ a finite quantity.}$$

Hence Σu_n is convergent.

Case II. When $k>1$.

By definition of limit, we have

$$\frac{u_{n+1}}{u_n}>1.$$

Now $$S_n=u_1+u_2+u_3+\ldots+u_n$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots+\frac{u_n}{u_1}\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots+\frac{u_n}{u_{n-1}}\cdot\frac{u_{n-1}}{u_{n-2}}\cdots\frac{u_2}{u_1}\right)$$

$\therefore$ $$S_n>u_1(1+1+1+\cdots n \text{ terms})$$

or $$S_n>nu_1$$

Now $\lim\limits_{n\to\infty} S_n > \lim\limits_{n\to\infty} nu_1$ which tends to infinity.

Hence Σu_n is divergent.

Note. If $\underset{n\to\infty}{\text{Lt}} \dfrac{u_{n+1}}{u_n}=1$, the test fails to give any information about convergence or divergence of the series and therefore further tests are needed. For example let

$$\Sigma u_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}+\ldots$$

Here $\qquad u_n=\dfrac{1}{n}, \qquad u_{n+1}=\dfrac{1}{(n+1)}$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{n}{n+1}=\frac{1}{(1+1/n)}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\text{Lt}} \frac{1}{(1+1/n)}=1$$

The series Σu_n is known to be divergent, being p-series with $p=1$.

Further, let $\Sigma u_n=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots$

Here $\qquad u_n=\dfrac{1}{n^2}$ and $u_{n+1}=\dfrac{1}{(n+1)^2}$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{n^2}{(n+1)^2}=\frac{1}{(1+1/n)^2}$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=1$$

But the given series is convergent, being p-series with $p>1$.

Thus we see that when

$$\underset{n\to\infty}{\text{Lt}} \frac{u_{n+1}}{u_n}=1,$$

D' Alembert's ratio test fails.

Example 1. *Test for convergence the series,*

$$\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\ldots$$

Sol. Here $\qquad u_n=\dfrac{n}{1+2^n}$

$$\therefore \quad u_{n+1}=\frac{n+1}{1+2^{n+1}}$$

$$\therefore \quad \frac{u_{n+1}}{u_n}=\frac{n+1}{1+2^{n+1}}\times\frac{1+2^n}{n}$$

$$=\frac{n+1}{n}\cdot\frac{1+2^n}{1+2^{n+1}}$$

$$=\left(1+\frac{1}{n}\right)\cdot\frac{\left(1+\frac{1}{2^n}\right)}{\left(2+\frac{1}{2^n}\right)}$$

$$\therefore \underset{n\to\infty}{\text{Lt}}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\text{Lt}}\frac{\left(1+\frac{1}{n}\right)\cdot\left(1+\frac{1}{2^n}\right)}{\left(2+\frac{1}{2^n}\right)}=\frac{1}{2}<1$$

Hence the given series is convergent.

Example 2. *Test for convergence the series*

$$1+3x+5x^2+7x^3+\ldots \qquad (x>0)$$

Sol. Here $\qquad u_n=(2n-1)x^{n-1}$

$$u_{n+1}=(2n+1)x^n$$

$$\therefore \qquad \frac{u_{n+1}}{u_n}=\frac{(2n+1)}{(2n-1)}\frac{x^n}{x^{n-1}}$$

$$=\frac{(2+1/n)}{(2-1/n)}\cdot x$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\frac{u_{n+1}}{u_n}=\underset{n\to\infty}{\text{Lt}}\frac{(2+1/n)}{(2-1/n)}x=x$$

The given series converges if $x<1$ and diverges for $x>1$.

If $x=1$, the test fails and we apply further tests.

When $\qquad x=1$, we have

$$u_n=(2n-1)$$

$$\underset{n\to\infty}{\text{Lt}}\ u_n=\underset{n\to\infty}{\text{Lt}}\ (2n-1)\neq 0$$

Since $\underset{n\to\infty}{\text{Lt}}\ u_n\neq 0$, the given series diverges for $x=1$.

Hence the given series converges for $x<1$ and diverges for $x>1$.

Example 3. *Discuss the convergence and divergence of the series*

(i) $\Sigma[\sqrt[3]{(n+1)}-\sqrt[3]{n}]$

(ii) $\Sigma \sin \dfrac{1}{n}$.

Sol. (i) Here $u_n=(n+1)^{1/3}-n^{1/3}$

$$=n^{1/3}\left(1+\frac{1}{n}\right)^{1/3}-n^{1/3}$$

$$=n^{1/3}\left[\left(1+\frac{1}{n}\right)^{1/3}-1\right]$$

$$=n^{1/3}\left[\left\{1+\frac{1}{3}\cdot\frac{1}{n}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2\,!}\cdot\frac{1}{n^2}+\ldots\right\}-1\right]$$

$$=n^{1/3}\left(\frac{1}{3n}-\frac{1}{9n^2}+\ldots\right)$$

$$=\frac{1}{n^{2/3}}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\right)$$

Let $v_n=\dfrac{1}{n^{2/3}}$ · Applying the comparison test, we have

$$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\frac{1}{n^{2/3}}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\right)\Big/\frac{1}{n^{2/3}}$$

$$=\underset{n\to\infty}{\text{Lt}}\left(\frac{1}{3}-\frac{1}{9n}+\ldots\right)=\frac{1}{3},$$

which is a non-zero finite quantity. Therefore, both the series Σu_n and Σv_n either converge or diverge together. But $\Sigma v_n (p=\frac{2}{3}<1)$ is known to be divergent. As such Σu_n is also divergent.

(ii) Here $u_n=\sin\dfrac{1}{n}$.

Let $v_n=\dfrac{1}{n}$

Now $$\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=\underset{n\to\infty}{\text{Lt}}\left(\sin\frac{1}{n}\Big/\frac{1}{n}\right)$$

$$=\underset{x\to 0}{\text{Lt}}\frac{\sin x}{x}=1$$

where $\dfrac{1}{n}=x$.

But Σv_n is a divergent series $(p=1)$.

Therefore $\Sigma u_n = \Sigma \sin \dfrac{1}{n}$ is also a divergent series.

EXERCISE 2 (*b*)

Test for convergence of the series.

1. $\frac{1}{2}+\frac{2}{5}+\frac{3}{10}+\dots+\frac{n}{n^2+1}+\dots$

2. $\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\dots+\sqrt{\frac{n}{n+1}}+\dots$ $\frac{u_n}{1/n}$

3. $\frac{1}{2^2}+\frac{1}{5^2}+\frac{1}{8^2}+\dots$

4. $\frac{1}{1\cdot3}+\frac{1}{3\cdot5}+\frac{1}{5\cdot7}+\dots$ $\frac{u_n}{v_n}$

5. $\frac{3}{2^2.3^2}+\frac{5}{3^2.4^2}+\frac{7}{4^2.5^2}+\dots$ ″

6. $\frac{1}{2}+\frac{\sqrt[3]{2}}{3\sqrt{2}}+\frac{\sqrt[3]{3}}{4\sqrt{3}}+\dots$ ″

7. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\dots$ ″

8. $\frac{1}{2}+\frac{\sqrt{2}}{5}+\frac{\sqrt{3}}{10}+\frac{\sqrt{4}}{17}+\dots$ ″

9. $\frac{1}{2}+\frac{1}{2\cdot2^2}+\frac{1}{3\cdot2^3}+\dots\infty$

10. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\dots+\frac{n^2-1}{n^2+1}x^n+\dots \quad (x>0)$

11. $\frac{x}{1\cdot2}+\frac{x^2}{2\cdot3}+\frac{x^3}{3\cdot4}+\dots$

12. $x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\infty$

13. $1+3x+5x^2+7x^3+\dots$

14. $1^2+2^2.x+3^2x^2+\dots \quad (0<x<1)$

15. $1+\frac{2^p}{2\,!}+\frac{3^p}{3\,!}+\frac{4^p}{4\,!}+\dots\infty$

16. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\dots\infty$

17. $\frac{1}{3}+\frac{2^2}{3^2}+\frac{3^2}{3^3}+\dots\infty$

18. $\frac{2}{1}+\frac{2.5.8}{1.5.9}+\frac{2.5.8.11}{1.5.9.13}+\dots$

19. $1+\frac{1^2.2^2}{1.3.5}+\frac{1^2.2^2.3^2}{1.3.5.7.9}+\ldots$

20. $\frac{1}{e}+\frac{8}{e^3}+\frac{27}{e^5}+\ldots$

Discuss the nature of the following seaies

21. $\sum\frac{1}{\sqrt{n+1}+\sqrt{n}}$

22. $\sum\frac{(n+1)}{n^p}$

23. $\Sigma\left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$

24. $\Sigma\left(\sqrt{n^2+1}-n\right)$

25. $\sum\frac{n}{n^2+1}$

26. $\sum\frac{1}{\sqrt{n}}\tan\frac{1}{n}$

(**Hint.** Let $v_n=\frac{1}{n^{3/2}}$)

27. $\sum_{n=2}^{\infty}\frac{1}{\log n}$

28. $\sum\sin\left(\frac{1}{n^2}\right)$

29. $\sum\left(1-\cos\frac{\pi}{n}\right)$

30. Show that the following series

(*a*) $\frac{x}{x+1}+\frac{x^2}{x+2}+\frac{x^3}{x+3}+\ldots$

(*b*) $\frac{x}{a\,.\,1^2+b}+\frac{2x^2}{a\,.\,2^2+b}+\frac{3x^3}{a\,.\,3^2+b}+\ldots$

converge when $x<1$ and diverge when $x\geq 1$.

2·9. Raabe's Test. [*Higher Ratio Test*]

If Σu_n be a positive term series, such that from and after some particular term

$$\operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=k, \text{ then } \Sigma u_n,$$

(*i*) Converges if $k>1$

(*ii*) Diverges if $k<1$

Proof. Let the series from and after the particular term be

$$u_1+u_2+u_3+\ldots+u_n+\ldots$$

Case I. When $k>1$

Let p be a positive number such that $k>p>1$ and compare the given series Σu_n with an auxiliary series.

$$\Sigma v_n = \Sigma \frac{1}{n^p},$$

which is convergent when $p>1$. Now Σu_n converges if,

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \text{(See Art. 2·7)}$$

i.e. if $$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p \qquad \left[\because\ v_n = \frac{1}{n^p}\right]$$

or if $$\frac{u_n}{u_{n+1}} > 1 + p\cdot\frac{1}{n} + \frac{p(p-1)}{1\ .\ 2}\cdot\left(\frac{1}{n}\right)^2 + \cdots$$

or if $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{1\ .\ 2}\cdot\frac{1}{n} + \cdots\cdots$$

or if $$\operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > \operatorname*{Lt}_{n\to\infty}\left[p + \frac{p(p-1)}{2n} + \cdots\right]$$

or if $$\operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > p$$

i.e. if $k>p$, which is true.

$$\left[\because\ \operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = k\right]$$

Hence the given series Σu_n is convergent.

Case II. When $k<1$.

Let p be a positive number such that $k<p<1$ and compare the given series with an auxiliary series

$$\Sigma v_n = \Sigma \frac{1}{n^p},$$

which is divergent when $p<1$.

Now Σu_n diverges if

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}$$

or if $$\frac{u_n}{u_{n+1}} < \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

or if $$\frac{u_n}{u_{n+1}} < 1 + \frac{p}{n} + \frac{p(p-1)}{1\ .\ 2}\cdot\left(\frac{1}{n}\right)^2 +$$

or if $$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{1\ .\ 2}\cdot\frac{1}{n} + \cdots$$

or if $\underset{n\to\infty}{\text{Lt}}\ n\left(\dfrac{u_n}{u_{n+1}}-1\right) < \underset{n\to\infty}{\text{Lt}}\left[p+\dfrac{p(p-1)}{1\ .\ 2}\left(\dfrac{1}{n}\right)+\ldots\right]$

or if $k<p$ which is true.

Hence the given series Σu_n diverges.

Note 1. The higher ratio test fails if $k=1$

Note 2. This test is applied when the ratio test fails.

Example 1. *Test for convergence the series,*

$$\frac{2^2}{3\ .\ 4}+\frac{2^2\ .\ 4^2}{3\ .\ 4\ .\ 5\ .\ 6}+\frac{2^2\ .\ 4^2\ .\ 6^2}{3\ .\ 4\ .\ 5\ .\ 6\ .\ 7\ .\ 8}+\ldots\infty$$

Sol. Here $$u_n=\frac{2^2\ .\ 4^2\ .\ 6^2\ldots\ldots(2n)^2}{3\ .\ 4\ .\ 5\ldots\ldots(2n+2)}$$

and $$u_{n+1}=\frac{2^2\ .\ 4^2\ .\ 6^2\ldots\ldots(2n)^2\ .\ (2n+2)^2}{3\ .\ 4\ .\ 5\ldots\ldots(2n+2)(2n+3)(2n+4)}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{2^2\ .\ 4^2\ .\ 6^2\ldots(2n)^2}{3\ .\ 4\ .\ 5\ldots(2n+2)}$$

$$\times\frac{3\ .\ 4\ .\ 5\ .\ (2n+2)(2n+3)(2n+4)}{2^2\ .\ 4^2\ .\ 6^2\ldots(2n)^2(2n+2)^2}$$

$$=\frac{(2n+3)(2n+4)}{(2n+2)^2}$$

$$=\frac{(2+3/n)(2+4/n)}{(2+2/n)^2}$$

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\ \frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\ \frac{(2+3/n)(2+4/n)}{(2+2/n)^2}=1$$

Hence the ratio test fails and we apply higher ratio test.

Now $$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+3)(2n+4)}{(2n+2)^2}-1\right]$$

$$=\frac{6n^2+8n}{(2n+2)^2}$$

$$=\frac{(3+4/n)}{2(1+1/n)^2}$$

$$\therefore\ \underset{n\to\infty}{\text{Lt}}\ n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\ \frac{(3+4/n)}{2\ (1+1/n)^2}=\frac{3}{2}\ (>1)$$

Hence the given series converges.

Example 2. *Discuss the convergence of the series.*

$$\frac{1}{3}\ x+\frac{1.2}{3.5}\ x^2+\frac{1.2.3}{3.5.7}\ x^3+\ldots\infty\quad (x>0)$$

Sol. Here $$u_n=\frac{1.2.3\ldots n}{3.5.7\,.\,(2n+1)}\,x^n$$

and $$u_{n+1}=\frac{1.2.3\ldots n.(n+1)}{3.5.7\ldots(2n+1)(2n+3)}\,x^{n+1}$$

$$\therefore\quad \frac{u_n}{u_{n+1}}=\frac{1.2.3\ldots n}{3.5.7\,.\,(2n+1)}\,x^n\times\frac{3.5.7\ldots(2n+1)(2n+3)}{1.2.3\ldots n(n+1)}\times\frac{1}{x^{n+1}}$$

$$=\frac{(2n+3)}{(n+1)}\cdot\frac{1}{x}$$

$$=\frac{\left(2+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x}$$

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\frac{u_n}{u_{n+1}}=\underset{n\to\infty}{\text{Lt}}\frac{\left(2+\frac{3}{n}\right)}{\left(1+\frac{1}{n}\right)}\cdot\frac{1}{x}$$

$$=\frac{2}{x}$$

The given series converges if $\frac{2}{x}>1$, *i.e.* $2>x$ or $x<2$ and diverges if $\frac{2}{x}<1$ or $2<x$ or $x>2$.

The test fails at $x=2$

When $x=2$,

$$\frac{u_n}{u_{n+1}}=\frac{(2n+3)}{(n+1)}\cdot\frac{1}{2}$$

$$\therefore\quad n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{(2n+3)}{2(n+1)}-1\right]$$

$$=\frac{n}{2n+2}$$

$$=\frac{1}{\left(2+\frac{2}{n}\right)}$$

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\,n\left(\frac{u_n}{u_{n+1}}-1\right)=\underset{n\to\infty}{\text{Lt}}\frac{1}{\left(2+\frac{2}{n}\right)}=\frac{1}{2}\,(<1)$$

Thus the series diverges when $x=2$.

Hence the given series converges when $x < 2$ and diverges for $x \geq 2$.

2·10. Logarithmic Test

If Σu_n be a positive term series, such that from and after some particular term,

$$\underset{n\to\infty}{\text{Lt}}\; n \log \frac{u_n}{u_{n+1}} = k, \text{ then } \Sigma u_n$$

(*i*) converges if $k > 1$

(*ii*) diverges if $k < 1$.

Proof. Let from and after the particular term the series be

$$u_1+u_2+u_3+\ldots\ldots+u_n+\ldots\ldots$$

Case I. When $k > 1$.

Let p be a positive number such that $k > p > 1$ and compare the given series with an auxiliary series

$$\Sigma v_n = \Sigma \frac{1}{n^p},$$

which is convergent when $p > 1$.

Now Σu_n is convergent if

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

or if
$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p$$

or if
$$\log \frac{u_n}{u_{n+1}} > p \log \left(1+\frac{1}{n}\right)$$

$$\left[\because \log\left(1+\frac{1}{n}\right)^p = p \log\left(1+\frac{1}{n}\right)\right]$$

r if
$$\log \frac{u_n}{u_{n+1}} > p\left(\frac{1}{n} - \frac{1}{2n^2} + \ldots\ldots\right)$$

or if
$$n \log \frac{u_n}{u_{n+1}} > p - \frac{p}{2n} + \ldots$$

or if
$$\underset{n\to\infty}{\text{Lt}}\; n \log \frac{u_n}{u_{n+1}} > p$$

or if $k > p$ which is true.

Hence the given series is convergent.

Case II. The case when $k < 1$ can be proved similary.

Note 1. The above test fails when $k=1$.

Example 1. *Test the convergence of the series*

$$\sum \frac{(n-1)^{n-1}}{n^n}$$

Sol. Here $u_n = \dfrac{(n-1)^{n-1}}{n^n}$

$$u_{n+1} = \frac{n^n}{(n+1)^{n+1}}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{(n+1)^{n+1}}{n^n}$$

$$= \frac{n^{n-1}\left(1-\frac{1}{n}\right)^{n-1} \cdot n^{n+1} \cdot \left(1+\frac{1}{n}\right)^{n+1}}{n^{2n}}$$

$$= \left(1-\frac{1}{n}\right)^{n-1} \cdot \left(1+\frac{1}{n}\right)^{n+1}$$

or

$$\frac{u_n}{u_{n+1}} = \left(1-\frac{1}{n}\right)^{n-1} \cdot \left(1+\frac{1}{n}\right)^{n} \cdot \left(1+\frac{1}{n}\right)$$

$$\therefore \; n \log \frac{u_n}{u_{n+1}} = n\left[(n-1)\log\left(1-\frac{1}{n}\right) + \log\left(1+\frac{1}{n}\right)^n + \log\left(1+\frac{1}{n}\right)\right]$$

$$= n\left[(n-1)\left(-\frac{1}{n}-\frac{1}{2n^2}-\frac{1}{3n^3}+\ldots\right) + \log\left(1+\frac{1}{n}\right)^n + \left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\right)\right]$$

$$= n\left[\left(-1+\frac{1}{n}-\frac{1}{2n}+\ldots\right)\right] + \log\left(1+\frac{1}{n}\right)^n + \left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\right)\Big]$$

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\, n \left[\left(-1+\frac{1}{n}-\frac{1}{2n}+\ldots\right)\right.$$
$$\left. + \log\left(1+\frac{1}{n}\right)^n + \left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\right)\right]$$
$$= \underset{n\to\infty}{\text{Lt}}\, n \left[\left(-1+\frac{1}{2n}-\ldots\right)\right.$$
$$\left. + \log e + \left(\frac{1}{n}-\frac{1}{2n^2}+\ldots\right)\right]$$
$$= \underset{n\to\infty}{\text{Lt}}\, n \left[-1+\frac{3}{2n}+1+\ldots\right]$$
$$= \frac{3}{2} > 1. \qquad [\because \log e = 1]$$

Hence the given series converges.

Example 2. *Test the convergence of the series*

$$1+\frac{2x}{2\,!}+\frac{3^2x^2}{3\,!}+\frac{4^3x^3}{4\,!}+\frac{5^4x^4}{5\,!}+\ldots$$

Sol. Here $u_n = \dfrac{n^{n-1}\,x^{n-1}}{n\,!}$

and $u_{n+1} = \dfrac{(n+1)^n\,x^n}{(n+1)\,!}$

$$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{(n+1)^n}{n^{n-1}} \cdot \frac{x}{n+1}$$
$$= \left(1+\frac{1}{n}\right)^n \cdot \frac{n}{n+1} \cdot x.$$

Thus $\underset{n\to\infty}{\text{Lt}}\, \dfrac{u_{n+1}}{u_n} = \underset{n\to\infty}{\text{Lt}} \left[\left(1+\dfrac{1}{n}\right)^n \cdot \dfrac{1}{1+\frac{1}{n}} \cdot x\right] = ex$

Therefore, the series is convergent when $ex<1$ and divergent when $ex>1$. When $ex=1$ or $x=1/e$ we apply further test.

$$\underset{n\to\infty}{\text{Lt}}\, n \log \frac{u_n}{u_{n+1}} = \underset{n\to\infty}{\text{Lt}}\, n \log \left\{\frac{n^{n-1}}{(n+1)^n} \cdot e(n+1)\right\}$$
$$= \underset{n\to\infty}{\text{Lt}}\, n \log \left[\frac{e}{\left(1+\frac{1}{n}\right)^{n-1}}\right]$$
$$= \underset{n\to\infty}{\text{Lt}}\, n \left\{\log e - (n-1)\log\left(1+\frac{1}{n}\right)\right\}$$

$$= \underset{n\to\infty}{\text{Lt}}\, n\left\{1-(n-1)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right)\right\}$$

$$= \underset{n\to\infty}{\text{Lt}}\left\{n-(n-1)+\frac{(n-1)}{2n}-\frac{(n-1)}{3n^2}+\cdots\right\}$$

$$= \underset{n\to\infty}{\text{Lt}}\left\{\frac{3}{2}-\frac{1}{2n}-\frac{(n-1)}{3n^2}+\cdots\right\}=\frac{3}{2}>1$$

As such the given series is convergent when $x=1/e$.

EXERCISE 2 (c)

Test the following series for convergence or divergence.

1. $\frac{1}{3}x+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^3+\ldots\infty \qquad (x>0)$

2. $\frac{2}{5}x+\frac{2.4}{5.8}x^2+\frac{2.4.6}{5.8.11}x^3+\ldots\infty \qquad (x>0)$

3. $\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\ldots\infty \qquad (x>0)$

4. $\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots\infty$

5. $\frac{1}{2}+\frac{1.3}{2.4}\cdot\frac{1}{2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3}+\ldots\infty$

6. $1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma.(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$

2·11. Cauchy Root Test

If Σu_n be a positive term series, such that from and after some particular term,

$$\underset{n\to\infty}{\text{Lt}}(u_n)^{1/n}=k, \text{ then } \Sigma u_n$$

(*i*) converges if $k<1$

(*ii*) diverges if $k>1$.

Proof. Let the series from and after the particular term be

$$u_1+u_2+u_3+\ldots+u_n+\ldots$$

Case. I. When $k<1$.

By definition of a limit a positive l $(k<l<1)$ can be found such that

$$(u_n)^{1/n}<l$$

$\therefore \qquad (u_n) < l^n$

or $\qquad \Sigma u_n < \Sigma l^n$

The series on the right hand side is a geometric series with common ratio l (< 1), hence convergent.

$\therefore \quad \Sigma u_n$ is also convergent.

Case II. When $k > 1$.

By definition of limit

$$(u_n)^{1/n} > 1$$

i.e. $\qquad u_1 > 1,\ u_2 > 1,\ u_3 > 1,\ \dots,\ u_n > 1$

$\therefore \quad u_1+u_2+u_3+\dots u_n+\dots > 1+1+1+\dots+1+\dots$

The series on the right hand sides is obviously divergent.

Therefore the given series Σu_n is also divergent.

Note. The test fails if $k=1$.

Example 1. *Discuss the convergence of the series,*

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\dots+\frac{1}{n^n}+\dots$$

Sol. Here $\qquad u_n=\frac{1}{n^n} \quad$ or $\quad (u_n)^{1/n}=\frac{1}{n}$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n}=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}=0<1$$

Hence the given series is convergent.

Example 2. *Discuss the convergence of the series.*

$$\sum\left(1+\frac{1}{n}\right)^{n^2}$$

Sol. Here $\qquad u_n=\left(1+\frac{1}{n}\right)^{n^2}$

$$\therefore \qquad (u_n)^{1/n}=\left(1+\frac{1}{n}\right)^n$$

$$\therefore \qquad \underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n}=\underset{n\to\infty}{\text{Lt}}\ \left(1+\frac{1}{n}\right)^n=e>1$$

Therefore the given series is divergent.

Example 3. *Test the series for convergence and divergence,*

$$\frac{1}{2}\cdot x+\left(\frac{2}{3}\right)^4\cdot x^2+\left(\frac{3}{4}\right)^9\cdot x^3+\dots+\left(\frac{n}{n+1}\right)^{n^2}\cdot x^n+\dots$$

Sol. Here $$u_n=\left(\frac{n}{n+1}\right)^{n^2}\cdot x^n$$

$\therefore$ $$(u_n)^{1/n}=\left(\frac{n}{n+1}\right)^n\cdot x$$
$$=\left(\frac{1}{1+1/n}\right)^n\cdot x$$
$$=\frac{x}{(1+1/n)^n}$$

$\therefore$ $$\underset{n\to\infty}{\text{Lt}}\ (u_n)^{1/n}=\underset{n\to\infty}{\text{Lt}}\ \frac{x}{(1+1/n)^n}=\frac{x}{e}$$

The given series converges if $\frac{x}{e}<1$, *i.e.* $x<e$ and diverges if $x/e>1$, *i.e.* $x>e$.

The above test fails at $x=e$.

When $x=e$

$$u_n=\left(\frac{n}{n+1}\right)^{n^2}\cdot e^n$$

Now $\underset{n\to\infty}{\text{Lt}}\ u_n\neq 0$, therefore, the given series diverges at $x=e$.

Hence the given series converges for $x<e$ and diverges for $x\geqslant e$.

EXERCISE 2 (*d*)

Discuss the convergence and divergence of the following series :

1. $\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\ldots+\left(\frac{n}{2n+1}\right)^n+\ldots$
2. $\frac{1}{2}+\left(\frac{2}{5}\right)^3+\left(\frac{3}{8}\right)^5+\ldots$
3. $\frac{3}{4}+\left(\frac{5}{7}\right)^2+\left(\frac{7}{10}\right)^3+\ldots$
4. $x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots$
5. $(x+3)+2^2(x+3)^2+3^3(x+3)^3+\ldots$

2·12. Alternating Series

An infinite series in which from and after some particular term, the terms are alternately positive and negative is called an alternating series.

The general form of the series is

$$u_1 - u_2 + u_3 - u_4 + \cdots \qquad [u_n > 0]$$

For example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is an alternating series.

Leibnit'z Test for Convergence of Alternating Series

An alternating series

$$u_1 - u_2 + u_3 - u_4 + \cdots \qquad [u_n > 0]$$

is convergent, if for all values of n,

(*i*) $u_{n+1} < u_n$

(*ii*) $\underset{n \to \infty}{\text{Lt}} \; u_n = 0$

Proof. Let the sum of first $2n$ terms of the series be denoted by S_{2n}.

$$\therefore \; S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots + (u_{2n-1} - u_{2n}) \qquad \ldots(1)$$

Since $\qquad u_n > u_{n+1},$

$$u_1 > u_2 > u_3 > \ldots \qquad \ldots(2)$$

From (1) and (2) it follows that the expression in each of the brackets in (1) is positive.

Hence the sum S_{2n} is positive, *i e.* $S_{2n} > 0$

Now $S_{2n+1} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2n} - u_{2n+1})]$

The expression in each of the parenthesis is positive and subtracted from u_1.

$\therefore \qquad S_{2n+1} \leqslant u_1$

Also $\qquad S_{2n+1} = S_{2n} + u_{2n+1}$

$$\therefore \; \underset{n \to \infty}{\text{Lt}} S_{2n+1} = \underset{n \to \infty}{\text{Lt}} S_{2n} + \underset{n \to \infty}{\text{Lt}} u_{2n+1} \qquad \ldots(3)$$

The given series is convergent if

$$\underset{n \to \infty}{\text{Lt}} S_{2n+1} = \underset{n \to \infty}{\text{Lt}} S_{2n} \qquad \ldots(4)$$

Therefore, from (3) and (4), we have

$$\underset{n \to \infty}{\text{Lt}} \; u_{2n+1} = 0 \text{ for all } n.$$

Replacing $(2n+1)$ by n, we have

or $$\underset{n \to \infty}{\text{Lt}} \; u_n = 0$$

Thus the given series is convergent if

(*i*) $u_1 > u_2 > u_3 > \ldots$

(*ii*) $\underset{n \to \infty}{\text{Lt}} \; u_n = 0$

Note 1. It must be noted carefully that both the conditions are satisfied simultaneously for convergence of an alternating series.

Note 2. An alternating series if not convergent is oscillatory.

Example 1. *Discuss the convergence of the series,*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Sol. The terms of the series are alternately positive and negative, and also each term is numerically less than the preceding term,

i.e. $$u_n > u_{n+1} \qquad \text{for all } n \qquad \ldots(1)$$

Also $$\underset{n \to \infty}{\text{Lt}} \; u_n = \underset{n \to \infty}{\text{Lt}} \; \frac{1}{n} = 0 \qquad \ldots(2)$$

From the conditions (1) and (2), the given series converges.

Example 2. *Test the convergence of the series,*

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$$

Sol. The terms of the series are alternately positive and negative and also each term is numerically less than the preceding term, *i.e.*,

$$u_n > u_{n+1} \qquad \text{for all } n \qquad \ldots(1)$$

Also $$u_n = \frac{1}{(2n-1)^2}$$

$\therefore$ $$\underset{n \to \infty}{\text{Lt}} \; u_n = \underset{n \to \infty}{\text{Lt}} \; \frac{1}{(2n-1)^2} = 0 \qquad \ldots(2)$$

From the conditions (1) and (2), the given series converges.

Example 3. *Test the convergence of the series,*

$$x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \ldots$$

Sol. Here we shall apply D' Alembert test,

$$u_n = (-1)^{n-1} \frac{x^n}{n^2}$$

and $$u_{n+1} = (-1)^n \frac{x^{n+1}}{(n+1)^2}$$

$$\frac{u_{n+1}}{u_n}=-\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}$$

$$=-\frac{1}{(1+1/n)^2}\cdot x$$

$$\left|\frac{u_{n+1}}{u_n}\right|=\left|\frac{-1}{(1+1/n)^2}\cdot x\right|=\frac{1}{(1+1/n)^2}\,|\,x\,|$$

$$\therefore\quad \underset{n\to\infty}{\mathrm{Lt}}\left|\frac{u_{n+1}}{u_n}\right|=\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{(1+1/n)^2}\right]|\,x\,|=|\,x\,|$$

Therefore the given series converges, when $|\,x\,|<1$ and diverges for $|\,x\,|>1$.

When $|\,x\,|=1$, D' Alembert's ratio test fails.

Now when $|\,x\,|=1$, the series can be shown to be convergent by Leibnitz's test.

Hence the given series is convergent for $|\,x\,|\leqslant 1$ and divergent for $|\,x\,|>1$.

EXERCISE 2 (*e*)

Examine each of the following series for convergence or divergence :

1. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\dots$
2. $\frac{1}{2}-\frac{2}{3}\cdot\frac{1}{2^3}+\frac{3}{4}\cdot\frac{1}{3^3}-\frac{4}{5}\cdot\frac{1}{4^3}+\dots$
3. $\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\dots$
4. $\frac{1}{2}-\frac{1}{2\,.\,3}+\frac{1}{2\,.\,3\,.\,4}-\frac{1}{2\,.\,3\,.\,4\,.\,5}+\dots$
5. $\sum\frac{(-1)^{n-1}}{n\,!}$
6. $\sum\frac{(-1)^{n-1}}{\log n}$
7. $1-\frac{x}{1^2}+\frac{x^2}{2^2}-\frac{x^3}{3^2}+\dots$
8. $\frac{1}{\sqrt{2}-1}-\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}+1}+\dots$

2·13. Absolute Convergence

A series Σu_n is said to be absolutely convergent if Σu_n and $\Sigma\,|\,u_n\,|$ both converge.

For example let

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots \infty$$

Here Σu_n is convergent by Leibnitz's rule.

Also $$\Sigma \mid u_n \mid = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \infty$$

is convergent. being a geometric series with common ratio $\frac{1}{2}$ (<1)

Hence the given series is absolutely convergent.

2·14. Conditional Convergence

A series Σu_n is said to be conditionally convergent if Σu_n converges but $\Sigma \mid u_n \mid$ does not converge.

For example let

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \infty$$

Here Σu_n is convergent by Leibnitz's rule

Now $\Sigma \mid u_n \mid = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \infty$, is a divergent series being a p-series with $p=1$.

Thus Σu_n converges but $\Sigma \mid u_n \mid$ diverges.

Hence the given series Σu_n is conditionally convergent.

EXERCISE 2 (*f*)

Find out which of the following series converge absolutely or conditionally :

1. $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \dots$

2. $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \dots$

3. $\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2^2} + \frac{1}{3} \cdot \frac{1}{2^3} - \dots$

4. $\frac{1}{\sqrt[5]{2}} - \frac{1}{\sqrt[5]{3}} + \frac{1}{\sqrt[5]{4}} - \dots + (-1)^n \frac{1}{\sqrt[5]{n}} + \dots$

5. $\sum \frac{(-1)^{n-1}}{\sqrt{n}}$

6. $\sum \frac{(-1)^{n-1}\, n}{n^2+1}$

7. $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

Exercise 1 (c) (Page 34—35)

4. (i) $\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

(ii) $\dfrac{\sin 2\alpha}{\cosh 2\beta-\cos 2\alpha}+i\dfrac{\sinh 2\beta}{\cosh 2\beta-\cos 2\alpha}$

(iii) $2\left[\dfrac{\sin\alpha\cosh\beta}{\cosh 2\beta-\cos 2\alpha}-i\dfrac{\cos\alpha\sinh\beta}{\cosh 2\beta-\cos 2\alpha}\right]$

(iv) $\cosh\alpha\cos\beta+i\sinh\alpha\sinh\beta$

(v) $\dfrac{\sinh 2\alpha}{\cosh 2\alpha+\cos 2\beta}+i\dfrac{\sin 2\beta}{\cosh 2\alpha+\cos 2\beta}$

(vi) $e^{\cosh x\cos y}[\cos(\sinh x\sin y)+i\sin(\sinh x\sin y)]$

Exercise 1 (f) (Page 42—43)

1. (i) $\cos^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{\sin\theta}+\sqrt{1+\sin\theta}\}$

(ii) $\sin^{-1}(\sqrt{\sin\theta})+i\log\{\sqrt{1+\sin\theta}-\sqrt{\sin\theta}\}$

(iii) $\dfrac{1}{2}\tanh^{-1}\left\{\dfrac{2x}{(1+x^2+y^2)}\right\}+\dfrac{i}{2}\tan^{-1}\left\{\dfrac{2y}{(1-x^2-y^2)}\right\}$

12. $e^{2n\pi+\alpha}[\cos(\log\sec\alpha)-i\sin(\log\sec\alpha)]$

13. $\operatorname{Exp}\left(-\dfrac{\pi^2}{8}\right)\cos\left(\dfrac{\pi}{4}\log 2\right)$

Exercise 1 (g) (Page 48)

1. $\dfrac{4\sin\alpha}{5-4\cos\alpha}$.

2. $\dfrac{1-x\cos y}{1-2x\cos y+y^2}$.

3. $\cot\alpha$

4. $e^{x\cos\beta}\cdot\cos(\alpha+x\sin\beta)$.

5. $e^{-\cos\theta}\sin(\sin\theta)$.

6. $\frac{1}{2}\tan^{-1}\left[\dfrac{2c\sin\alpha}{1-c^2}\right]$, $(c\neq 1, \alpha\neq n\pi)$.

7. $\dfrac{\pi}{4}$ or $\dfrac{-\pi}{4}$ according as $\cos\alpha$ is positive or negative.

8. $\left(2\sin\dfrac{\alpha}{2}\right)^{-n}\cos\left(\dfrac{n\pi}{2}-\dfrac{n\alpha}{2}\right)$

9. $\dfrac{x\sin\theta+(-1)^{n+1}x^{n+1}\{x\sin n\theta+\sin(n+1)\theta\}}{x^2+2x\cos\theta+1}$

10. $1+\left(2\cos\dfrac{\theta}{2}\right)^n\cdot\sin\dfrac{n\theta}{2}$.

Exercise 2 (a) (Page 50)

1. Convergent
2. Divergent
3. Divergent
4. Divergent
5. Divergent
6. Oscillatory

Exercise 2 (b) Page (63—64)

1. Divgt.
2. Divgt.

3. Convgt. 4. Convgt.
5. Convgt. 6. Convgt.
7. Convgt. 8. Convgt.
9. Convgt. 10. Convgt. for $x<1$; Divgt. for $x \geqslant 1$
11. Convgt. for $x \leqslant 1$; Divgt. for $x>1$.
12. Convgt. for $x<1$; Divgt. for $x \geqslant 1$.
13. Convgt. for $x<1$; Divgt. for $x \geqslant 1$.
14. Convgt. 15. Convgt.
16. Convgt. 17. Convgt.
18. Convgt. 19. Divgt.
20. Congvt. 21. Divgt.
22. Convgt if $p>2$ and Divgt. for $p \leqslant 2$.
23. Convgt. 24. Divergent.
25. Divgt. 26. Convgt.
27. Divgt. 28. Convgt.
29. Convgt.

Exercise 2. (*c*) (Page 71)

1. Convgt. for $x<2$ and Divgt. for $x \geqslant 2$
2. Convgt. for $x<\frac{3}{2}$ and Divgt. for $x \geqslant \frac{3}{2}$
3. Convgt. for $x \leqslant 1$ and Divgt. for $x>1$
4. Convgt. for $x \leqslant 1$ and Divgt. for $x>1$
5. Convgt.
6. Convgt for $x<1$, Divgt. for $x>1$; when $x=1$, Convgt. if $\gamma-\alpha-\beta>0$ and Divgt. for $\gamma-\alpha-\beta \leqslant 0$,

Exercise 2 (*d*) (Page 73)

1. Convgt. 2. Convgt.
3. Convgt.
4. Convgt. for $x \leqslant 1$ and Divgt. for $x>1$. 5. Divgt.

Exereise 2 (*e*) (Page 76)

1. Convgt. 2. Convgt.
3. Convgt. 4. Convgt.
5. Convgt. 6. Convgt.
7. Convgt. for $|x| \leqslant 1$ and Divgt. for $|x|>1$.
8. Divgt.

Exercise 2 (*f*) (Page 77)

1. Abs. convgt. 2. Conditionally convgt.
3. Abs. convgt. 4. Conditionally convgt.
5. Conditionally convgt. 6. Conditionally convgt.
7. Conditionally convgt.

Exercise 3 (Page 84—85)

1. $\frac{4}{2x-3}-\frac{1}{x-1}$ 2. $1-\frac{2}{x-3}+\frac{6}{x-4}$

3. $\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x+4}$ 4. $\frac{3}{5(s-1)}-\frac{4}{15(s+4)}-\frac{1}{3(s-2)}$