Proof Portfolio

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Theorem 1. *Let $f : A \to B$ be a function and let A_1 and A_2 be subsets of A. Prove that if f is one-to-one then*

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2).$$

Proof. Suppose $f : A \to B$ is an injective function.

If $x \in f(A_1 \cap A_2)$, then there exists some $a \in A_1 \cap A_2$ such that $f(a) = x$. Since $a \in A_1 \cap A_2$, then a is also an element of A_1, which means $f(a)$ is an element of $f(A_1)$. In the same way, $f(a)$ is also an element of $f(A_2)$. As a result, $f(a)$ is an element of $f(A_1) \cap f(A_2)$ which implies $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

Next, let y be an element of $f(A_1) \cap f(A_2)$. Then, there is an $a_1 \in A_1$ and $a_2 \in A_2$ such that $f(a_1) = y$ and $f(a_2) = y$. From this we can conclude that $a_1 = a_2$ since $f(a_1) = f(a_2)$ and by the function being injective. This implies $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$.

The only way $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$ and $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ is if $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. ☺

Theorem 2. *Let a be a fixed real number. Then*

$$\sum_{i=0}^{n}(a+i) = \frac{1}{2}(n+1)(2a+n)$$

for every nonnegative integer n.

Proof. We proceed by induction.
Let $n = 0$ be the smallest nonnegative integer. Then,

$$\sum_{i=0}^{0}(a+i) = a = \frac{1}{2}(0+1)(2a+0).$$

Thus, the result holds when $n = 0$.
Assume that

$$\sum_{i=0}^{k}(a+i) = \frac{1}{2}(k+1)(2a+k)$$

for a nonnegative integer k. We show that $\sum_{i=0}^{k+1}(a+i) = \frac{1}{2}(k+2)(2a+k+1)$. Observe that

$$\begin{align}
\sum_{i=0}^{k+1}(a+i) &= (a+k+1) + \sum_{i=0}^{k}(a+i) \tag{1}\\
&= (a+k+1) + \frac{1}{2}(k+1)(2a+k) \tag{2}\\
&= \frac{1}{2}(2ak+4a+k^2+3k+2) \tag{3}\\
&= \frac{1}{2}(k+2)(2a+k+1), \tag{4}
\end{align}$$

where in (2), we use the Inductive Hypothesis.

Thus, by the Principle of Mathematical Induction, we conclude that $\sum_{i=0}^{n}(a+i) = \frac{1}{2}(n+1)(2a+n)$ for all nonnegative integers n. ☺

The **Fibonacci Sequence** $F_1, F_2, F_3, \ldots$ is defined by

$$F_n = \begin{cases} 1 & n = 1, 2 \\ F_{n-2} + F_{n-1} & n \geq 3 \end{cases}$$

Theorem 3. *Then nth Fibonacci Sequence is*

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

for every positive integer n.

Proof. We proceed by the Strong Principle of Mathematical Induction. Since $F_1 = 1 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^1 - \left(\frac{1-\sqrt{5}}{2} \right)^1 \right]$, the formula holds for n = 1. Assume, for a positive integer k, that $F_i = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^i - \left(\frac{1-\sqrt{5}}{2} \right)^i \right]$ for every i with $1 \leq i \leq k$. We show that $F_{k+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right]$. First, observe that when $k = 1$, $F_{k+1} = F_{1+1} = F_2 = 1 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1-\sqrt{5}}{2} \right)^2 \right]$ and so the formula holds. Hence we may assume that $k \geq 2$. Since $k + 1 \geq 3$, it follows by the recurrence relation that

$$\begin{aligned}
F_{k+1} &= F_{k-1} + F_k && (1) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \right] + \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k} - \left(\frac{1-\sqrt{5}}{2} \right)^{k} \right] && (2) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k} - \left(\frac{1-\sqrt{5}}{2} \right)^{k} \right] && (3) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(1 + \frac{1+\sqrt{5}}{2} \right) - \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(1 + \frac{1-\sqrt{5}}{2} \right) \right] && (4) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{3+\sqrt{5}}{2} \right) - \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{3-\sqrt{5}}{2} \right) \right] && (5) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{1-\sqrt{5}}{2} \right)^2 \right] && (6) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right], && (7)
\end{aligned}$$

where in (2), we use the Inductive hypothesis.
It therefore follows by the Strong Principle of Mathematical Induction that $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$ for all positive integers n. ☺

Theorem 4. *Let A and B be two sets and let $f : A \to B$ be a function. If $|A| > |B|$ then f is not injective (one-to-one).*

Proof. We prove the contrapositive. Suppose that a function $f : A \to B$ is injective for two sets A and B. Since f is injective, then different elements of A must have different images in B. Therefore, if $|A| = n$, then the elements of A have n images in B. As a result, $|B| \geq n$. Thus, if $f : A \to B$ is injective, then $|A| \leq |B|$. ☺

Theorem 5. *A graph G is regular if and only if $\overline{G}$ is regular.*

Proof. Asumme a graph G is r-regular with n vertices. By definition, the complement of G is $\overline{G}$ with $V(\overline{G}) = V(G)$ such that two distinct vertices u and v of G are adjacent in $\overline{G}$ if and only if u and v are not adjacent in G. This means that every vertex $v \in V(\overline{G})$ is adjacent to $(n-1) - r$ vertices. As such, $\overline{G}$ is a $[(n-1)-r]$-regular graph. Thus, if G is regular then $\overline{G}$ is also regular.

Assume a graph $\overline{G}$ is r-regular. We know that the complement of $\overline{G}$ is also regular due to the previous reasoning. By definition, the complement of a complement is the original object so G is also regular. Thus, we can conclude a graph G is regular if and only if $\overline{G}$ is regular. ☺

Theorem 6. *A 3-regular graph G has a cut-vertex if and only if G has a bridge.*

Proof. Assume G is a 3-regular graph with a cut-vertex v. Then $G - v$ can be seperated into two components G_1 and G_2 or three components G_1, G_2, and G_3. Consider the case where there are two components. Since G was originally a 3-regular graph, without loss of generality v has a vertex u that is adjacent in G_1 and two vertices that are adjacent in G_2. As such, there is a bridge uv between the two components. Next, consider the case in which there are three components. Then, v has vertex u, x, and y that is adjacent in G_1, G_2, and G_3 respectively. As such, there are bridges uv, xv, and yv.

Assume G is a 3-regular graph with a bridge uv. Then the vertices of the bridge, u and v, are also cut-vertices since removing them would remove their edges too. The bridge is a part of the edges that will be removed. Next, assume G has bridges uv, xv, and yv. By the same logic used previously, the vertices of the bridges are also cut-vertices.

Thus we can conclude that a 3-regular graph G has a cut-vertex if and only if G has a bridge. ☺

Definition 1 (Divisible)**.** *An integer n is* **divisible** *be an integer d if there exists an integer k such that $n = d \times k$.*

Theorem 7. *For all nonnegative integers n,*

$$2^{2n} - 1 \text{ is divisible by } 3.$$

Proof. We proceed by induction. Since $2^{2 \cdot 0} - 1 = 0 = 3 \cdot 0$, the result holds for $n = 0$. Assume that $2^{2k} - 1$ is divisible by 3 for all nonnegative integers k. By the definition of divisibility, this means that $2^{2k} - 1 = 3x \implies 2^{2k} = 3x + 1$ for some integer x. We show that $2^{2(k+1)} - 1$ is divisible by 3 for all nonnegative integers. Observe that

$$\begin{aligned} 2^{2(k+1)} - 1 &= 2^2 \cdot 2^{2k} - 1 && (1) \\ &= 4(3x + 1) - 1 && (2) \\ &= 12x + 4 - 1 && (3) \\ &= 3(4x - 1), && (4) \end{aligned}$$

where in (2), we use the Inductive Hypothesis. Since x is an integer, $3(4x - 1)$ is divisible by 3. Thus, $2^{2n} - 1$ is divisible by 3 for all nonnegative integers n. ☺