Proof Portfolio

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The **Fibonacci Sequence** $F_1, F_2, F_3, \ldots$ is defined by

$$F_n = \begin{cases} 1 & n = 1,2 \\ F_{n-2} + F_{n-1} & n \geq 3 \end{cases}$$

Theorem 1. *Then nth Fibonacci Sequence is*

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

for every positive integer n.

Proof. We proceed by the Strong Principle of Mathematical Induction. Since $F_1 = 1 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^1 - \left(\frac{1-\sqrt{5}}{2} \right)^1 \right]$, the formula holds for n = 1. Assume, for a positive integer k, that $F_i = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^i - \left(\frac{1-\sqrt{5}}{2} \right)^i \right]$ for every i with $1 \leq i \leq k$. We show that $F_{k+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right]$. First, observe that when $k = 1$, $F_{k+1} = F_{1+1} = F_2 = 1 = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1-\sqrt{5}}{2} \right)^2 \right]$ and so the formula holds. Hence we may assume that $k \geq 2$. Since $k + 1 \geq 3$, it follows by the recurrence relation that

$$\begin{aligned}
F_{k+1} &= F_{k-1} + F_k && (1) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \right] + \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k} - \left(\frac{1-\sqrt{5}}{2} \right)^{k} \right] && (2) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} + \left(\frac{1+\sqrt{5}}{2} \right)^{k} - \left(\frac{1-\sqrt{5}}{2} \right)^{k} \right] && (3) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(1 + \frac{1+\sqrt{5}}{2} \right) - \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(1 + \frac{1-\sqrt{5}}{2} \right) \right] && (4) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{3+\sqrt{5}}{2} \right) - \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{3-\sqrt{5}}{2} \right) \right] && (5) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{1-\sqrt{5}}{2} \right)^2 \right] && (6) \\
&= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right], && (7)
\end{aligned}$$

where in (2), we use the Inductive hypothesis.

It therefore follows by the Strong Principle of Mathematical Induction that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ for all positive integers n. ■