Proof Portfolio Bonus

Anupam Bhakta

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Definition 1 (Divisible)**.** *An integer n is* **divisible** *be an integer d is there exists an integer k such that $n = d \times k$.*

Theorem 1. *For all nonnegative integers n,*

$$2^{2n} - 1 \textit{ is divisible by } 3.$$

Proof. We proceed by induction. Since $2^{2 \cdot 0} - 1 = 0 = 3 \cdot 0$, the result holds for $n = 0$. Assume that $2^{2k} - 1$ is divisible by 3 for all nonnegative integers k. By the definition of divisibility, this means that $2^{2k} - 1 = 3x \implies 2^{2k} = 3x + 1$ for some integer x. We show that $2^{2(k+1)} - 1$ is divisible by 3 for all nonnegative integers. Observe that

$$\begin{aligned}
2^{2(k+1)} - 1 &= 2^2 \cdot 2^{2k} - 1 &(1)\\
&= 4(3x + 1) - 1 &(2)\\
&= 12x + 4 - 1 &(3)\\
&= 3(4x - 1), &(4)
\end{aligned}$$

where in (2), we use the Inductive Hypothesis. Since x is an integer, $3(4x - 1)$ is divisible by 3. Thus, $2^{2n} - 1$ is divisible by 3 for all nonnegative integers n. $\square$