Signals are Vectors

Signals are Vectors

- Signals are mathematical objects
- Here we will develop tools to analyze the **geometry** of sets of signals
- The tools come from **linear algebra**
- By interpreting signals as vectors in a vector space, we will be able to speak about the length of a signal (its "strength," more below), angles between signals (their similarity), and more
- We will also be able to use matrices to better understand how signal processing systems work
- Caveat: This is not a course on linear algebra!

Vector Space

DEFINITION

A linear **vector space** V is a collection of vectors such that if $x, y \in V$ and α is a scalar then

$$\alpha\, x \in V \qquad \text{and} \qquad x + y \in V$$

- In words:
  - A rescaled vector stays in the vector space
  - The sum of two vectors stays in the vector space
- We will be interested in scalars (basically, numbers) α that either live in $\mathbb{R}$ or $\mathbb{C}$
- Classical vector spaces that you know and love
  - $\mathbb{R}^N$, the set of all vectors of length N with real-valued entries
  - $\mathbb{C}^N$, the set of all vectors of length N with complex-valued entries
  - Special case that we will use all the time to draw pictures and build intuition: $\mathbb{R}^2$

The Vector Space $\mathbb{R}^2$ (1)

- Vectors in $\mathbb{R}^2$: $x = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}, \; y = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix}, \; x[0], x[1], y[0], y[1] \in \mathbb{R}$
  - Note: We will enumerate the entries of a vector starting from 0 rather than 1 (this is the convention in signal processing and programming languages like "C", but not in Matlab)
  - Note: We will not use the traditional boldface or underline notation for vectors

- Scalars: $\alpha \in \mathbb{R}$

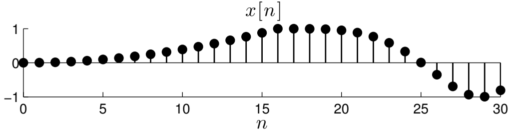
- Scaling: $\alpha\, x = \alpha \begin{bmatrix} x[0] \\ x[1] \end{bmatrix} = \begin{bmatrix} \alpha\, x[0] \\ \alpha\, x[1] \end{bmatrix}$

The Vector Space $\mathbb{R}^2$ (2)

- Vectors in $\mathbb{R}^2$: $x = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$, $y = \begin{bmatrix} y[0] \\ y[1] \end{bmatrix}$, $x[0], x[1], y[0], y[1] \in \mathbb{R}$
- Scalars: $\alpha \in \mathbb{R}$
- Summing: $x + y = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix} + \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} x[0] + y[0] \\ x[1] + y[1] \end{bmatrix}$

The Vector Space $\mathbb{R}^N$

- Vectors in $\mathbb{R}^N$: $x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad x[n] \in \mathbb{R}$

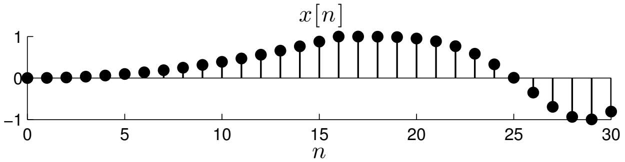


- This is exactly the same as a real-valued discrete time signal; that is, **signals are vectors**
  - Scaling αx amplifies/attenuates a signal by the factor α
  - Summing $x + y$ creates a new signal that mixes x and y

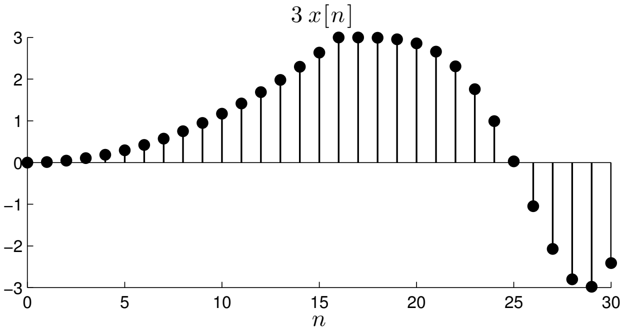
- $\mathbb{R}^N$ is harder to visualize than $\mathbb{R}^2$ and $\mathbb{R}^3$, but the intuition gained from $\mathbb{R}^2$ and $\mathbb{R}^3$ generally holds true with no surprises (at least in this course)

The Vector Space $\mathbb{R}^N$ – Scaling

- Signal $x[n]$

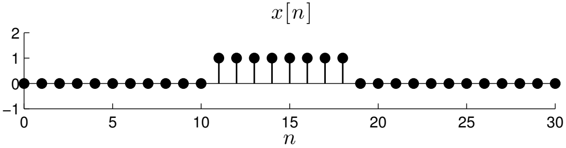


- Scaled signal $3\,x[n]$

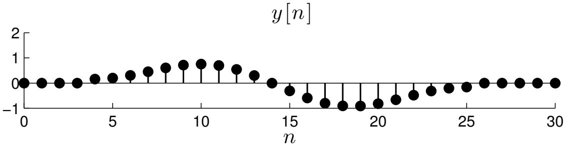


The Vector Space $\mathbb{R}^N$ – Summing

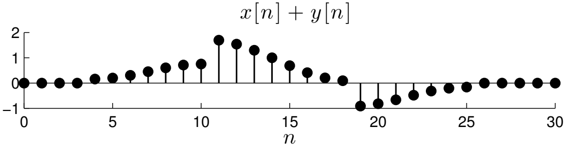
- Signal $x[n]$



- Signal $y[n]$



- Sum $x[n] + y[n]$



The Vector Space $\mathbb{C}^N$ (1)

- $\mathbb{C}^N$ is the same as $\mathbb{R}^N$ with a few minor modifications

- Vectors in $\mathbb{C}^N$: $x = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad x[n] \in \mathbb{C}$

- Each entry $x[n]$ is a complex number that can be represented as

$$x[n] = \mathrm{Re}\{x[n]\} + j\,\mathrm{Im}\{x[n]\} = |x[n]|\, e^{j\angle x[n]}$$

- Scalars $\alpha \in \mathbb{C}$

The Vector Space $\mathbb{C}^N$ (2)

- Rectangular form

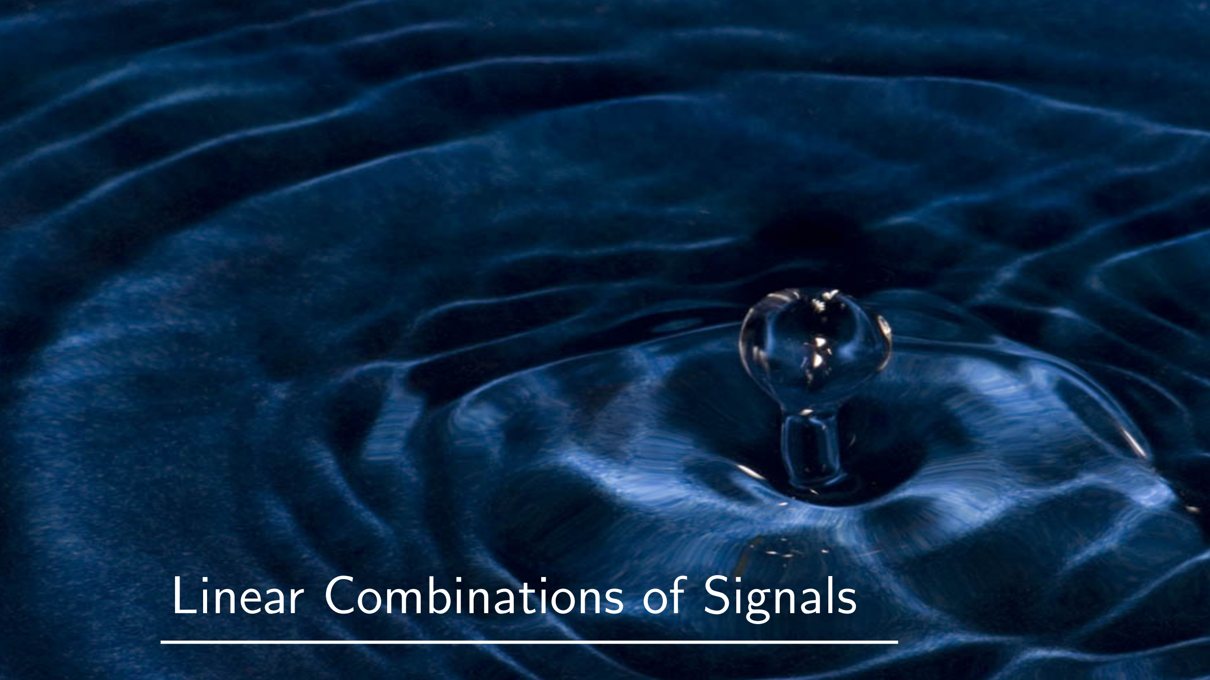
$$x = \begin{bmatrix} \mathrm{Re}\{x[0]\} + j\,\mathrm{Im}\{x[0]\} \\ \mathrm{Re}\{x[1]\} + j\,\mathrm{Im}\{x[1]\} \\ \vdots \\ \mathrm{Re}\{x[N-1]\} + j\,\mathrm{Im}\{x[N-1]\} \end{bmatrix} = \mathrm{Re}\left\{ \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \right\} + j\,\mathrm{Im}\left\{ \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \right\}$$

- Polar form

$$x = \begin{bmatrix} |x[0]|\, e^{j\angle x[0]} \\ |x[1]|\, e^{j\angle x[1]} \\ \vdots \\ |x[N-1]|\, e^{j\angle x[N-1]} \end{bmatrix}$$

Summary

- Linear algebra provides powerful tools to study signals and systems
- Signals are **vectors** that live in a vector space
- Of particular interest in signal processing are the vector spaces $\mathbb{R}^N$ and $\mathbb{C}^N$



Linear Combinations of Signals

Linear Combination of Signals

- Many signal processing applications feature **sums** of a number of signals

DEFINITION

Given a collection of M vectors $x_0, x_1, \dots x_{M-1} \in \mathbb{C}^N$ and M scalars $\alpha_0, \alpha_1, \dots, \alpha_{M-1} \in \mathbb{C}$, the **linear combination** of the vectors is given by

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{M-1} x_{M-1} = \sum_{m=0}^{M-1} \alpha_m x_m$$

- Clearly the result of the linear combination is a vector $y \in \mathbb{C}^N$

Linear Combination Example

- A recording studio uses a **mixing board** (or desk) to create a linear combination of the signals from the different instruments that make up a song

- Say $x_0 =$ drums, $x_1 =$ bass, $x_2 =$ guitar, $\ldots$, $x_{22} =$ saxophone, $x_{23} =$ singer $(M = 24)$

- Linear combination (output of mixing board)

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{23} x_{23} = \sum_{m=0}^{23} \alpha_m x_m$$

- Changing the α_m's results in a different "mix" y that emphasizes/deemphasizes certain instruments

Linear Combination = Matrix Multiplication

- Step 1: Stack the vectors $x_m \in \mathbb{C}^N$ as column vectors into an $N \times M$ matrix

$$X = \left[x_0 | x_1 | \cdots | x_{M-1} \right]$$

- Step 2: Stack the scalars α_m into an $M \times 1$ column vector

$$a = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$$

- Step 3: We can now write a linear combination as the matrix/vector product

$$y = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_{M-1} x_{M-1} = \sum_{m=0}^{M-1} \alpha_m x_m = \left[x_0 | x_1 | \cdots | x_{M-1} \right] \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = X a$$

Linear Combination = Matrix Multiplication (The Gory Details)

- M vectors in $\mathbb{C}^N$: $x_m = \begin{bmatrix} x_m[0] \\ x_m[1] \\ \vdots \\ x_m[N-1] \end{bmatrix}, \quad m = 0, 1, \ldots, M-1$
- $N \times M$ matrix: $X = \begin{bmatrix} x_0[0] & x_1[0] & \cdots & x_{M-1}[0] \\ x_0[1] & x_1[1] & \cdots & x_{M-1}[1] \\ \vdots & \vdots & & \vdots \\ x_0[N-1] & x_1[N-1] & \cdots & x_{M-1}[N-1] \end{bmatrix}$
- Note: The row-n, column-m element of the matrix $[X]_{n,m} = x_m[n]$
- M scalars α_m, $m = 0, 1, \ldots, M-1$: $a = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix}$
- Linear combination $y = Xa$

Linear Combination = Matrix Multiplication (Summary)

- Linear combination $\quad y = Xa$

- The row-n, column-m element of the $N \times M$ matrix $[X]_{n,m} = x_m[n]$

$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots & x_m[n] & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \alpha_m \\ \vdots \end{bmatrix} = Xa$$

- Sum-based formula for $y[n]$

$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n]$$

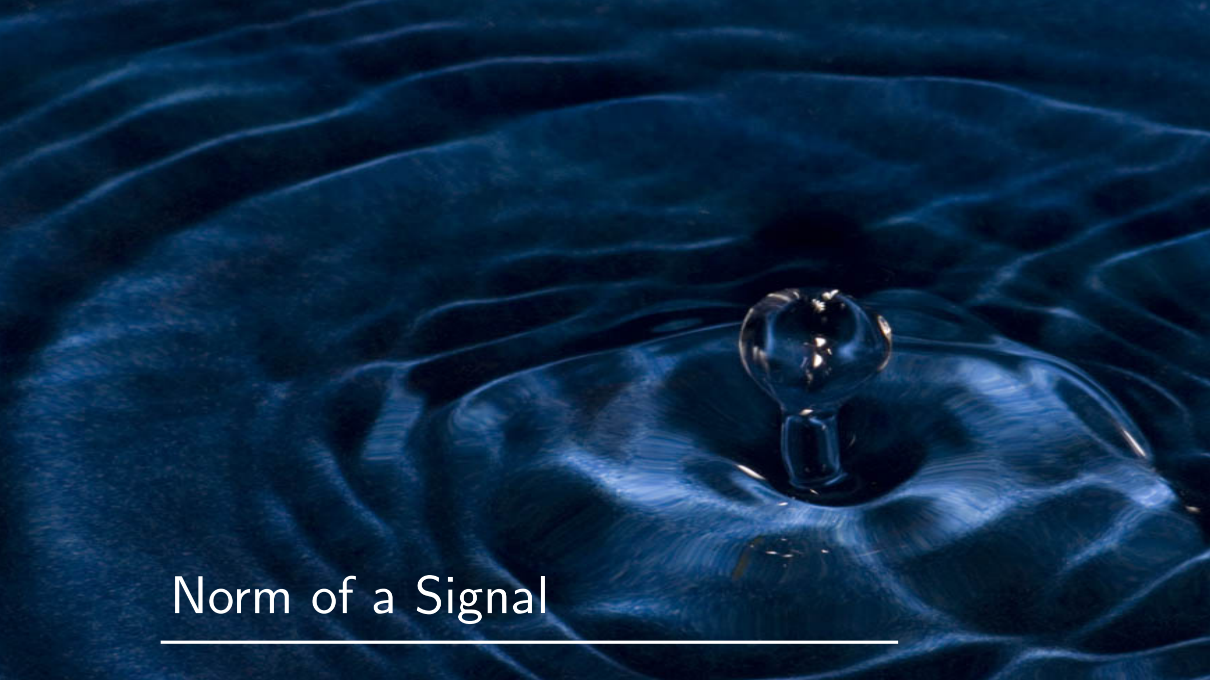
Linear Combination as Matrix Multiplication (Matlab)

- Linear combination Matlab demo

Summary

- Linear algebra provides power tools to study signals and systems
- Signals are **vectors** that live in a vector space
- We can combine several signals to form one new signal via a **linear combination**
- Linear combination is basically a matrix/vector multiplication

Norm of a Signal

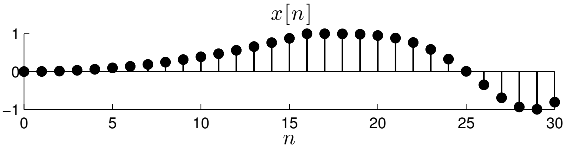


Strength of a Vector

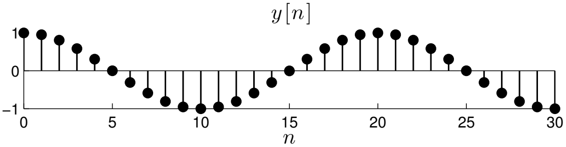
- How to quantify the "strength" of a vector?

- How to say that one signal is "stronger" than another?

- Signal x



- Signal y



Strength of a Vector: 2-Norm

DEFINITION

The **Euclidean length**, or 2-**norm**, of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2}$$

The **energy** of x is given by $(\|x\|_2)^2 = \|x\|_2^2$

- The norm takes as input a vector in $\mathbb{C}^N$ and produces a **real number** that is ≥ 0
- When it is clear from context, we will suppress the subscript "2" in $\|x\|_2$ and just write $\|x\|$

2-Norm Example

- Ex: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- ℓ_2 norm

$$\|x\|_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Strength of a Vector: p-Norm

- The Euclidean length is not the only measure of "strength" of a vector in $\mathbb{C}^N$

DEFINITION

The p-**norm** of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_p = \left(\sum_{n=0}^{N-1} |x[n]|^p \right)^{1/p}$$

DEFINITION

The 1-**norm** of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_1 = \sum_{n=0}^{N-1} |x[n]|$$

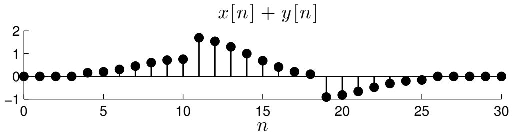
Strength of a Vector: ∞-Norm

DEFINITION

The ∞-**norm** of a vector $x \in \mathbb{C}^N$ is given by

$$\|x\|_\infty = \max_n |x[n]|$$

- $\|x\|_\infty$ is simply the largest entry in the vector x (in absolute value)

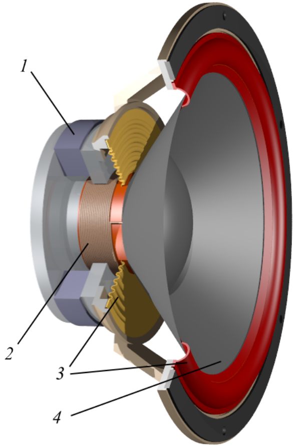


- While $\|x\|_2^2$ measures the **energy** in a signal, $\|x\|_\infty$ measures the **peak** value (of the magnitude); both are very useful in applications

- Interesting mathematical fact: $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$

Physical Significance of Norms (1)

- Two norms have special physical significance
  - $\|x\|_2^2$: energy in x
  - $\|x\|_\infty$: peak value in x
- A **loudspeaker** is a transducer that converts electrical signals into acoustic signals
- Conventional loudspeakers consist of a paper cone (4) that is joined to a coil of wire (2) that is wound around a permanent magnet (1)
- If the energy $\|x\|_2^2$ is too large, then the coil of wire will melt from excessive heating
- If the peak value $\|x\|_\infty$ is too large, then the large back and forth excursion of the coil of wire will tear it off of the paper cone



Physical Significance of Norms (2)

- Consider a **robotic car** that we wish to guide down a roadway
- How to measure the amount of deviation from the center of the driving lane?
- Let x be a vector of measurements of the car's GPS position and let y be a vector containing the GPS positions of the center of the driving lane
- Clearly we would like to make the <u>error signal</u> $y - x$ "small"; but how to measure smallness?
- Minimizing $\|y - x\|_2^2$, energy in the error signal, will tolerate a few large deviations from the lane center (not very safe)
- Minimizing $\|y - x\|_\infty$, the maximum of the error signal, will not tolerate any large deviations from the lane center (much safer)

Normalizing a Vector

DEFINITION

A vector x is **normalized** (in the 2-norm) if $\|x\|_2 = 1$

- Normalizing a vector is easy; just scale it by $\frac{1}{\|x\|_2}$
- Ex: $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad ||x||_2 = \sqrt{\sum_{n=0}^{N-1} |x[n]|^2} = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$$x' = \frac{1}{\sqrt{14}} x = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \qquad ||x'||_2 = 1$$

Summary

- Linear algebra provides power tools to study signals and systems
- Signals are **vectors** that live in a vector space
- Norms measure the “strength” of a signal; we introduced the 2- 1-, and ∞-norms

Inner Product

The Geometry of Signals

- Up to this point, we have developed the viewpoint of "signals as vectors" in a vector space
- We have focused on quantities related to individual vectors, ex: norm (strength)
- Now we turn to a quantity related to pairs of vectors, the **inner product**
- A powerful and ubiquitous signal processing tool

Aside: Transpose of a Vector

- Recall that the **transpose** operation ^{T} converts a column vector to a row vector (and vice versa)

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}^T = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}$$

- In addition to transposition, the **conjugate transpose** (aka Hermitian transpose) operation ^{H} takes the complex conjugate

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}^H = \begin{bmatrix} x[0]^* & x[1]^* & \cdots & x[N-1]^* \end{bmatrix}$$

Inner Product

DEFINITION

The **inner product** (or dot product) between two vectors $x, y \in \mathbb{C}^N$ is given by

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

- The inner product takes two signals (vectors in $\mathbb{C}^N$) and produces a single (complex) number
- **Angle** between two vectors $x, y \in \mathbb{R}^N$

$$\cos\theta_{x,y} = \frac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2}$$

- **Angle** between two vectors $x, y \in \mathbb{C}^N$

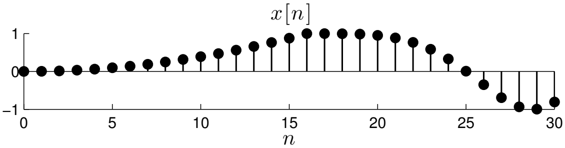
$$\cos\theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

Inner Product Example 1

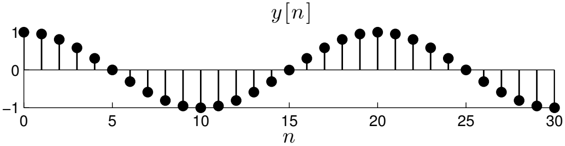
- Consider two vectors in $\mathbb{R}^2$: $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- $\|x\|_2^2 = 1^2 + 2^2 = 5, \qquad \|y\|_2^2 = 3^2 + 2^2 = 13$
- $\theta_{x,y} = \arccos\left(\frac{1\times 3 \ + \ 2\times 2}{\sqrt{5}\sqrt{13}}\right) = \arccos\left(\frac{7}{\sqrt{65}}\right) \approx 0.519 \text{ rad} \approx 29.7°$

Inner Product Example 2

- Signal x



- Signal y



- Inner product computed using Matlab: $\langle x, y \rangle \; = \; y^T x \; = \; 5.995$

- Angle computed using Matlab: $\theta_{x,y} \; = \; 64.9°$

2-Norm from Inner Product

- Question: What's the inner product of a signal with itself?

$$\langle x, x \rangle = \sum_{n=0}^{N-1} x[n]\, x[n]^* = \sum_{n=0}^{N-1} |x[n]|^2 = \|x\|_2^2$$

- Answer: The 2-norm!

- Mathematical aside: This property makes the 2-norm very special; no other p-norm can be computed via the inner product like this

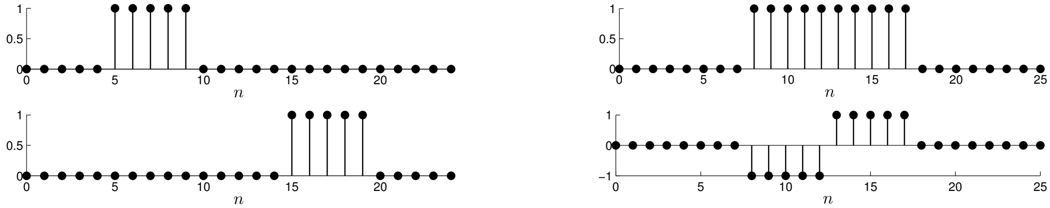
Orthogonal Vectors

DEFINITION

Two vectors $x, y \in \mathbb{C}^N$ are **orthogonal** if

$$\langle x, y \rangle = 0$$

- $\langle x, y \rangle = 0 \;\Rightarrow\; \theta_{x,y} = \frac{\pi}{2}\ \mathrm{rad} = 90^\circ$
- Ex: Two sets of orthogonal signals



Harmonic Sinusoids are Orthogonal

$$s_k[n] = e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, \;\; 0 \leq n \leq N-1, \;\; 0 \leq k \leq N-1$$

- Claim: $\langle s_k, s_l \rangle = 0, \quad k \neq l$ (a key result for the DFT)
- Verify by direct calculation

$$\begin{aligned}
\langle s_k, s_l \rangle &= \sum_{n=0}^{N-1} s_k[n]\, s_l^*[n] = \sum_{n=0}^{N-1} e^{j\frac{2\pi k}{N}n} \left(e^{j\frac{2\pi l}{N}n}\right)^* = \sum_{n=0}^{N-1} e^{j\frac{2\pi k}{N}n}\, e^{-j\frac{2\pi l}{N}n} \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(k-l)n} \quad \text{let } r = k - l \in \mathbb{Z}, r \neq 0 \\
&= \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}rn} = \sum_{n=0}^{N-1} a^n \quad \text{with } a = e^{j\frac{2\pi}{N}r}, \text{then use } \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \\
&= \frac{1 - e^{j\frac{2\pi r N}{N}}}{1 - e^{j\frac{2\pi r}{N}}} = 0 \;\; \checkmark
\end{aligned}$$

Harmonic Sinusoids are Orthogonal (Matlab)

- **Click here** to view a MATLAB demo exploring the orthogonality of harmonic sinusoids.

Normalizing Harmonic Sinusoids

$$s_k[n] = e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, \;\; 0 \le n \le N-1, \;\; 0 \le k \le N-1$$

- Claim: $\|s_k\|_2 = \sqrt{N}$
- Verify by direct calculation

$$\|s_k\|_2^2 \;=\; \sum_{n=0}^{N-1} |s_k[n]|^2 \;=\; \sum_{n=0}^{N-1} |e^{j\frac{2\pi k}{N}n}|^2 \;=\; \sum_{n=0}^{N-1} 1 \;=\; N \quad \checkmark$$

- **Normalized harmonic sinusoids**

$$\widetilde{s}_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi k}{N}n}, \quad n, k, N \in \mathbb{Z}, \;\; 0 \le n \le N-1, \;\; 0 \le k \le N-1$$

Summary

- **Inner product** measures the similarity between two signals

$$\langle x, y\rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

- Angle between two signals

$$\cos\theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y\rangle\}}{\|x\|_2\,\|y\|_2}$$

- Harmonic sinusoids are **orthogonal** (as well as periodic)

Matrix Multiplication and Inner Product

Recall: Matrix Multiplication as a Linear Combination of Columns

- Consider the (real- or complex-valued) matrix multiplication $y = Xa$
- The row-n, column-m element of the $N \times M$ matrix $[X]_{n,m} = x_m[n]$
- We can compute y as a **linear combination** of the **columns of** X weighted by the elements in a

$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_0[n] & x_1[n] & \cdots & x_{M-1}[n] \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = Xa$$

- Sum-based formula for $y[n]$

$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n], = \sum_{m=0}^{M-1} \alpha_m \, (\text{column } m \text{ of } X), \quad 0 \le n \le N-1$$

Matrix Multiplication as a Sequence of Inner Products of Rows

- Consider the **real-valued** matrix multiplication $y = Xa$

- The row-n, column-m element of the $N \times M$ matrix $[X]_{n,m} = x_m[n]$

- We can compute each element $y[n]$ in y as the **inner product** of the n**-th row of** X with the vector a

$$y = \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ x_0[n] & x_1[n] & \cdots & x_{M-1}[n] \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{M-1} \end{bmatrix} = Xa$$

- Can write $y[n]$

$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n] = \langle (\text{row } n \text{ of } X)^T, \, a \rangle, \quad 0 \leq n \leq N-1$$

Matrix Multiplication as a Sequence of Inner Products of Rows

- What about **complex-valued** matrix multiplication $y = Xa$?

- The same interpretation works, but we need to use the following "**inner product**"

$$y[n] = \sum_{m=0}^{M-1} \alpha_m \, x_m[n] \neq \langle (\text{row } n \text{ of } X)^T, a \rangle, \quad 0 \leq n \leq N-1$$

- Note: This is *nearly* the inner product for complex signals except that is lacking the complex conjugation

- We will often abuse notation by calling this an inner product

Summary

- Given the matrix/vector product $y = Xa$, we can compute each element $y[n]$ in y as the **inner product** of the n**-th row of** X with the vector a

- Not strictly true for complex matrices/vectors, but the interpretation is useful nevertheless

Cauchy Schwarz Inequality

Comparing Signals

- Inner product and angle between vectors enable us to **compare signals**

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

$$\cos \theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

- The Cauchy Schwarz Inequality quantifies the comparison

- A powerful and ubiquitous signal processing tool

- Note: Our development will emphasize intuition over rigor

Cauchy-Schwarz Inequality (1)

- Focus on real-valued signals in $\mathbb{R}^N$ (the extension to $\mathbb{C}^N$ is easy)
- Recall that $\cos\theta_{x,y} = \frac{\langle x,y\rangle}{\|x\|_2\,\|y\|_2}$
- Now, use the fact that $0 \leq |\cos\theta| \leq 1$ to write

$$0 \;\leq\; \left|\frac{\langle x,y\rangle}{\|x\|_2\,\|y\|_2}\right| \;\leq\; 1$$

- Rewrite as the **Cauchy-Schwarz Inequality** (CSI)

$$0 \;\leq\; |\langle x,y\rangle| \;\leq\; \|x\|_2\,\|y\|_2$$

- Interpretation: The inner product $\langle x,y\rangle$ measures the **similarity** of x to y

Cauchy-Schwarz Inequality (2)

$$0 \;\leq\; |\langle x, y\rangle| \;\leq\; \|x\|_2 \, \|y\|_2$$

- Interpretation: The inner product $\langle x, y\rangle$ measures the **similarity** of x to y
- Two extreme cases:
  - Lower bound: $\langle x, y\rangle = 0$ or $\theta_{x,y} = 90^\circ$: x and y are most <u>different</u> when they are <u>orthogonal</u>
  - Upper bound: $\langle x, y\rangle = \|x\|_2\|y\|_2$ or $\theta_{x,y} = 0^\circ$: x and y are most <u>similar</u> when they are <u>collinear</u> (aka <u>linearly dependent</u>, $y = \alpha x$)
- It is hard to understate the importance and ubiquity of the CSI!

Cauchy-Schwarz Inequality Applications

- How does a digital communication system decide whether the signal corresponding to a "0" was transmitted or the signal corresponding to a "1"? (Hint: CSI)

- How does a radar or sonar system find targets in the signal it receives after transmitting a pulse? (Hint: CSI)

- How does many computer vision systems find faces in images? (Hint: CSI)

Cauchy-Schwarz Inequality (Matlab)

- **Click here** to view a MATLAB demo illustrating the Cauchy-Schwarz inequality.

Summary

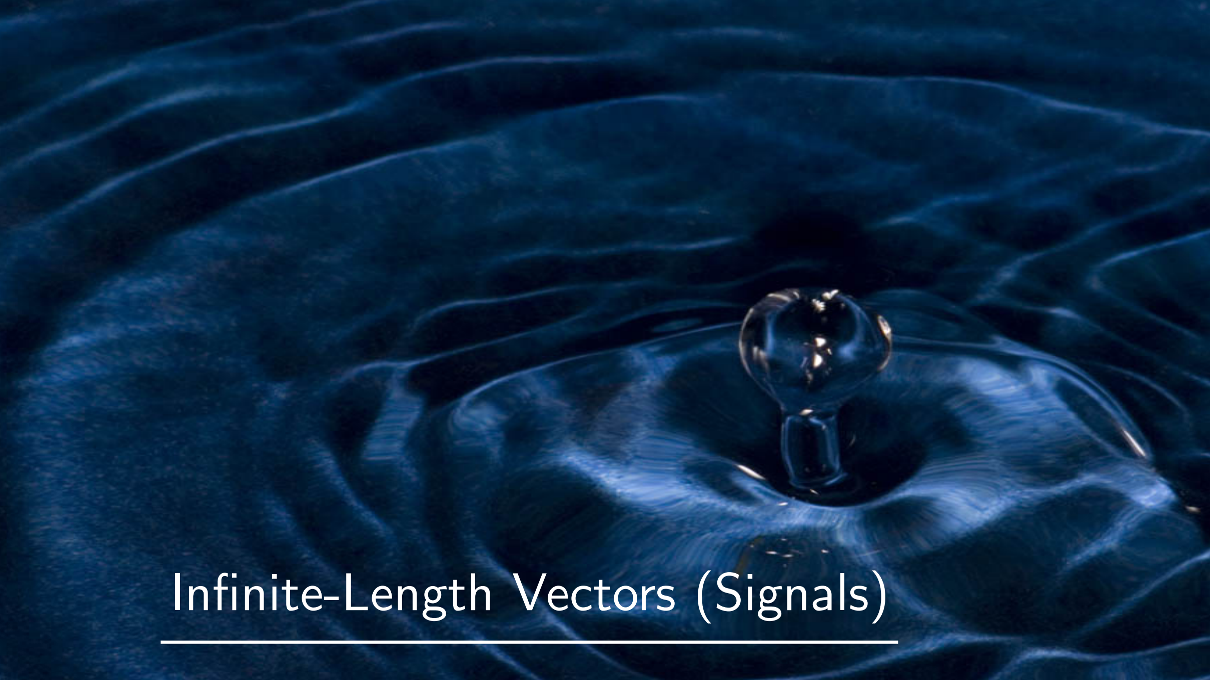
- **Inner product** measures the similarity between two signals

$$\langle x, y \rangle = y^H x = \sum_{n=0}^{N-1} x[n]\, y[n]^*$$

- **Cauchy-Schwarz Inequality** (CSI) calibrates the inner product

$$0 \leq \left| \frac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2} \right| \leq 1$$

  - Similar signals – close to upper bound (1)
  - Different signals – close to lower bound (0)

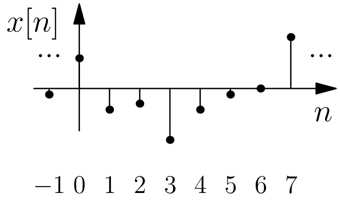


Infinite-Length Vectors (Signals)

From Finite to Infinite-Length Vectors

- Up to this point, we have developed some useful tools for dealing with finite-length vectors (signals) that live in $\mathbb{R}^N$ or $\mathbb{C}^N$: Norms, Inner product, Linear combination
- It turns out that these tools can be generalized to infinite-length vectors (sequences) by letting $N \to \infty$ (infinite-dimensional vector space, aka Hilbert Space)

$$x[n], \quad -\infty < n < \infty, \qquad x = \begin{bmatrix} \vdots \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}$$



- Obviously such a signal cannot be loaded into Matlab; however this viewpoint is still useful in many situations
- We will spell out the generalizations with emphasis on what changes from the finite-length case

2-Norm of an Infinite-Length Vector

DEFINITION

The 2-**norm** of an infinite-length vector x is given by

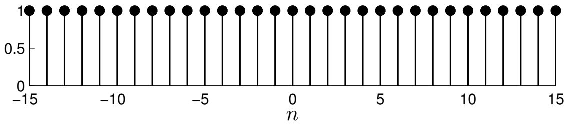
$$\|x\|_2 = \sqrt{\sum_{n=-\infty}^{\infty} |x[n]|^2}$$

The **energy** of x is given by $(\|x\|_2)^2 = \|x\|_2^2$

- When it is clear from context, we will suppress the subscript "2" in $\|x\|_2$ and just write $\|x\|$
- What changes from the finite-length case: Not every infinite-length vector has a finite 2-norm

ℓ_2 Norm of an Infinite-Length Vector – Example

- Signal: $x[n] = 1, \quad -\infty < n < \infty$



- 2-norm:

$$\|x\|_2^2 \; = \; \sum_{n=-\infty}^{\infty} |x[n]|^2 \; = \; \sum_{n=-\infty}^{\infty} 1 \; = \; \infty$$

- Infinite energy!

p- and 1-Norms of an Infinite-Length Vector

DEFINITION

The p-**norm** of an infinite-length vector x is given by

$$\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{1/p}$$

DEFINITION

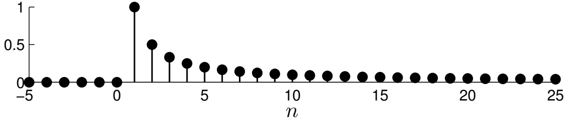
The 1-**norm** of an infinite-length vector x is given by

$$\|x\|_1 = \sum_{n=-\infty}^{\infty} |x[n]|$$

- What changes from the finite-length case: Not every infinite-length vector has a finite p-norm

1- and 2-Norms of an Infinite-Length Vector – Example

- Signal: $x[n] = \begin{cases} 0 & n \leq 0 \\ \frac{1}{n} & n \geq 1 \end{cases}$



- 1-norm

$$\|x\|_1 \;=\; \sum_{n=-\infty}^{\infty} |x[n]| \;=\; \sum_{n=1}^{\infty} \frac{1}{n} \;=\; \infty$$

- 2-norm

$$\|x\|_2^2 \;=\; \sum_{n=-\infty}^{\infty} |x[n]|^2 \;=\; \sum_{n=1}^{\infty} \left|\frac{1}{n}\right|^2 \;=\; \sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{\pi^2}{6} \approx 1.64 \;<\; \infty$$

∞-Norm of an Infinite-Length Vector

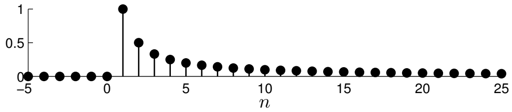
DEFINITION

The ∞-**norm** of an infinite-length vector x is given by

$$\|x\|_\infty = \sup_n |x[n]|$$

- What changes from the finite-length case: "sup" is a generalization of $\max$ to infinite-length signals that lies beyond the scope of this course

- In both of the above examples, $\|x\|_\infty = 1$



Inner Product of Infinite-Length Signals

DEFINITION

The **inner product** between two infinite-length vectors x, y is given by

$$\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x[n]\, y[n]^*$$

- The inner product takes two signals and produces a **single (complex) number**
- **Angle** between two real-valued signals

$$\cos\theta_{x,y} = \frac{\langle x, y \rangle}{\|x\|_2 \, \|y\|_2}$$

- **Angle** between two complex-valued signals

$$\cos\theta_{x,y} = \frac{\mathrm{Re}\{\langle x, y \rangle\}}{\|x\|_2 \, \|y\|_2}$$

Linear Combination of Infinite-Length Vectors

- The concept of a linear combination extends to infinite-length vectors

- What changes from the finite-length case: We will be especially interested in linear combinations of infinitely many infinite-length vectors

$$y = \sum_{m=-\infty}^{\infty} \alpha_m x_m$$

Linear Combination = Infinite Matrix Multiplication

- Step 1: Stack the vectors x_m as column vectors into a "matrix" with infinitely many rows and columns

$$X = \left[\cdots|x_{-1}|x_0|x_1|\cdots\right]$$

- Step 2: Stack the scalars α_m into an infinitely tall column vector $a = \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix}$

- Step 3: We can now write a linear combination as the matrix/vector product

$$y = \sum_{m=-\infty}^{\infty} \alpha_m x_m = \left[\cdots|x_{-1}|x_0|x_1|\cdots\right] \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix} = Xa$$

Linear Combination = Infinite Matrix Multiplication (The Gory Details)

- Vectors: $x_m = \begin{bmatrix} \vdots \\ x_m[-1] \\ x_m[0] \\ x_m[1] \\ \vdots \end{bmatrix}$, $-\infty < m < \infty$, and Scalars: $a = \begin{bmatrix} \vdots \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix}$
- Infinite matrix: $X = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & x_{-1}[-1] & x_0[-1] & x_1[-1] & \cdots \\ \cdots & x_{-1}[0] & x_0[0] & x_1[0] & \cdots \\ \cdots & x_{-1}[1] & x_0[1] & x_1[1] & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$
- Note: The row-n, column-m element of the matrix $[X]_{n,m} = x_m[n]$
- Linear combination $= Xa$

Linear Combination $=$ Infinite Matrix Multiplication (Summary)

- Linear combination $\quad y = Xa$

- The row-n, column-m element of the infinitely large matrix $[X]_{n,m} = x_m[n]$

$$y \;=\; \begin{bmatrix} \vdots \\ y[n] \\ \vdots \end{bmatrix} \;=\; \begin{bmatrix} & \vdots & \\ \cdots & x_m[n] & \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ \alpha_m \\ \vdots \end{bmatrix} \;=\; Xa$$

- Sum-based formula for $y[n]$

$$y[n] \;=\; \sum_{m=-\infty}^{\infty} \alpha_m \, x_m[n]$$

Summary

- Linear algebra concepts like norm, inner product, and linear combination work apply as well to infinite-length signals as with finite-length signals

- Only a few changes from the finite-length case

  - Not every infinite-length vector has a finite 1-, 2-, or ∞-norm

  - Linear combinations can involve infinitely many vectors