Systems

Systems are Transformations

DEFINITION

A discrete-time **system** $\mathcal{H}$ is a transformation (a rule or formula) that maps a discrete-time input signal x into a discrete-time output signal y

$$y = \mathcal{H}\{x\}$$

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y$$

- Systems manipulate the information in signals
- Examples:
  - A speech recognition system converts acoustic waves of speech into text
  - A radar system transforms the received radar pulse to estimate the position and velocity of targets
  - A functional magnetic resonance imaging (fMRI) system transforms measurements of electron spin into voxel-by-voxel estimates of brain activity
  - A 30 day moving average smooths out the day-to-day variability in a stock price

Signal Length and Systems

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y$$

- Recall that there are two kinds of signals: <u>infinite-length</u> and <u>finite-length</u>

- Accordingly, we will consider two kinds of systems:

  1. Systems that transform an infinite-length-signal x into an infinite-length signal y
  2. Systems that transform a length-N signal x into a length-N signal y

     (Such systems can also be used to process periodic signals with period N)

- For generality, we will assume that the input and output signals are complex valued

System Examples (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

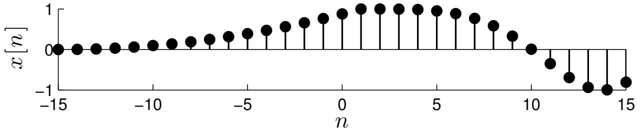
- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

System Examples (2)



- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n - m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n-1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n-1] \quad \forall n$$

Summary

- Systems transform one signal into another to manipulate information

- We will consider two kinds of systems:

  1. Systems that transform an infinite-length-signal x into an infinite-length signal y
  2. Systems that transform a length-N signal x into a length-N signal y

     (Such systems can also be used to process periodic signals with period N)

Linear Systems

Linear Systems

DEFINITION

A system $\mathcal{H}$ is (zero-state) **linear** if it satisfies the following two properties:

1. Scaling

$$\mathcal{H}\{\alpha\, x\} = \alpha\, \mathcal{H}\{x\} \quad \forall\ \alpha \in \mathbb{C}$$

$$x \longrightarrow \boxed{\mathcal{H}} \longrightarrow y \qquad\qquad \alpha\, x \longrightarrow \boxed{\mathcal{H}} \longrightarrow \alpha\, y$$

2. Additivity

$$\text{If} \quad y_1 = \mathcal{H}\{x_1\} \quad \text{and} \quad y_2 = \mathcal{H}\{x_2\} \quad \text{then}$$

$$\mathcal{H}\{x_1 + x_2\} = y_1 + y_2$$

$$x_1 \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1 \qquad\qquad x_2 \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_2$$

$$x_1 + x_2 \longrightarrow \boxed{\mathcal{H}} \longrightarrow y_1 + y_2$$

Linearity Notes

- A system that is not linear is called **nonlinear**
- To prove that a system is linear, you must prove rigorously that it has **both** the scaling and additivity properties for **arbitrary** input signals
- To prove that a system is nonlinear, it is sufficient to exhibit a **counterexample**

Example: Moving Average is Linear (Scaling)

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

- **Scaling:** (Strategy to prove – Scale input x by $\alpha \in \mathbb{C}$, compute output y via the formula at top, and verify that it is scaled as well)
  - Let
  $$x'[n] \;=\; \alpha x[n], \quad \alpha \in \mathbb{C}$$
  - Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)
  - Then
  $$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(\alpha x[n] + \alpha x[n-1]) \;=\; \alpha\left(\frac{1}{2}(x[n] + x[n-1])\right) \;=\; \alpha y[n] \;\checkmark$$

Example: Moving Average is Linear (Additivity)

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

- **Additivity:** (Strategy to prove – Input two signals into the system and verify that the output equals the sum of the respective outputs)
  - Let
$$x'[n] = x_1[n] + x_2[n]$$
  - Let $y'/y_1/y_2$ denote the output when $x'/x_1/x_2$ is input
  - Then
$$\begin{aligned} y'[n] &= \frac{1}{2}(x'[n] + x'[n-1]) = \frac{1}{2}(\{x_1[n] + x_2[n]\} + \{x_1[n-1] + x_2[n-1]\}) \\ &= \frac{1}{2}(x_1[n] + x_1[n-1]) + \frac{1}{2}(x_2[n] + x_2[n-1]) = y_1[n] + y_2[n] \;\; \checkmark \end{aligned}$$

Example: Squaring is Nonlinear

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = (x[n])^2$$

- **Additivity:** Input two signals into the system and see what happens
  - Let
$$y_1[n] = (x_1[n])^2, \qquad y_2[n] = (x_2[n])^2$$
  - Set
$$x'[n] \;=\; x_1[n] + x_2[n]$$
  - Then
$$y'[n] \;=\; (x'[n])^2 \;=\; (x_1[n] + x_2[n])^2 \;=\; (x_1[n])^2 + 2x_1[n]x_2[n] + (x_2[n])^2 \;\neq\; y_1[n] + y_2[n]$$
  - Nonlinear!

Linear or Nonlinear? You Be the Judge! (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

Linear or Nonlinear? You Be the Judge! (2)

- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n-m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n-1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n-1] \quad \forall n$$

Matrix Multiplication and Linear Systems

- Matrix multiplication (aka Linear Combination) is a fundamental signal processing system

- **Fact 1:** Matrix multiplications are linear systems (easy to show at home, but do it!)

$$y = \mathbf{H}\,x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\,x[m]$$

(Note: This formula applies for both infinite-length and finite-length signals)

- **Fact 2:** *All* linear systems can be expressed as matrix multiplications

- As a result, we will use the matrix viewpoint of linear systems extensively in the sequel

- Try at home: Express all of the linear systems in the examples above in matrix form

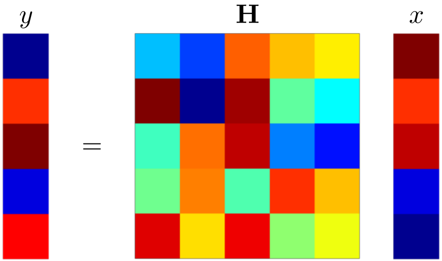
Matrix Multiplication and Linear Systems in Pictures

- Linear system

$$y = \mathbf{H}\,x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\, x[m] = \sum_m h_{n,m}\, x[m]$$

where $h_{n,m} = [\mathbf{H}]_{n,m}$ represents the row-n, column-m entry of the matrix $\mathbf{H}$



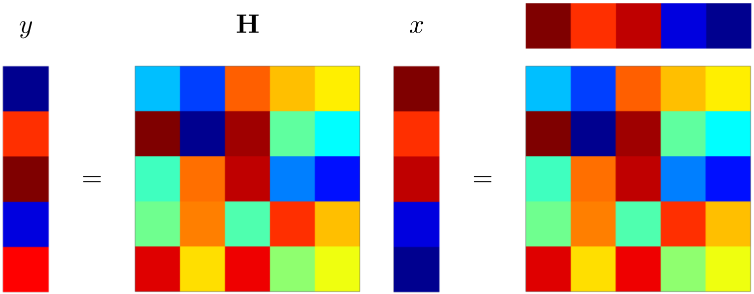
System Output as a Linear Combination of Columns

- Linear system

$$y = \mathbf{H}\,x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\,x[m] = \sum_m h_{n,m}\,x[m]$$

where $h_{n,m} = [\mathbf{H}]_{n,m}$ represents the row-n, column-m entry of the matrix $\mathbf{H}$



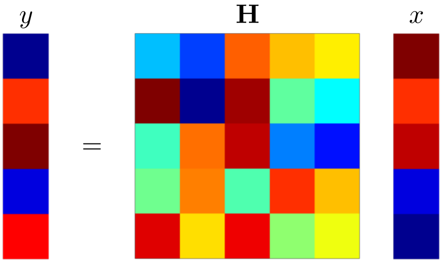
System Output as a Sequence of Inner Products

- Linear system

$$y = \mathbf{H}\,x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\, x[m] = \sum_m h_{n,m}\, x[m]$$

where $h_{n,m} = [\mathbf{H}]_{n,m}$ represents the row-n, column-m entry of the matrix $\mathbf{H}$



Summary

- Linear systems satisfy (1) scaling and (2) additivity
- To show a system is linear, you have to prove it rigorously assuming arbitrary inputs (work!)
- To show a system is nonlinear, you can just exhibit a counterexample (often easy!)
- Linear systems $\equiv$ matrix multiplication
  - Justifies our emphasis on linear vector spaces and matrices
  - The output signal y equals the linear combination of the columns of $\mathbf{H}$ weighted by the entries in x
  - Alternatively, the output value $y[n]$ equals the inner product between row n of $\mathbf{H}$ with x

Time-Invariant Systems

Time-Invariant Systems (Infinite-Length Signals)

DEFINITION

A system $\mathcal{H}$ processing infinite-length signals is **time-invariant** (shift-invariant) if a time shift of the input signal creates a corresponding time shift in the output signal

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$$

$$x[n-q] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n-q]$$

- Intuition: A time-invariant system behaves the same no matter when the input is applied

- A system that is not time-invariant is called **time-varying**

Example: Moving Average is Time-Invariant

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = \tfrac{1}{2}(x[n] + x[n-1])$$

- Let

$$x'[n] = x[n-q], \quad q \in \mathbb{Z}$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then

$$y'[n] \;=\; \frac{1}{2}(x'[n] + x'[n-1]) \;=\; \frac{1}{2}(x[n-q] + x[n-q-1]) \;=\; y[n-q] \quad \checkmark$$

Example: Decimation is Time-Varying

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n] = x[2n]$$

- This system is time-varying; demonstrate with a counter-example

- Let
$$x'[n] = x[n-1]$$

- Let y' denote the output when x' is input (that is, $y' = \mathcal{H}\{x'\}$)

- Then
$$y'[n] \;=\; x'[2n] \;=\; x[2n-1] \;\neq\; x[2(n-1)] \;=\; y[n-1]$$

Time-Invariant or Time-Varying? You Be the Judge! (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

Time-Invariant or Time-Varying? You Be the Judge! (2)

- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n-m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n-1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n-1] \quad \forall n$$

Time-Invariant Systems (Finite-Length Signals)

DEFINITION

A system $\mathcal{H}$ processing length-N signals is **time-invariant** (shift-invariant) if a circular time shift of the input signal creates a corresponding circular time shift in the output signal

$$x[n] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[n]$$

$$x[(n-q)_N] \longrightarrow \boxed{\mathcal{H}} \longrightarrow y[(n-q)_N]$$

- Intuition: A time-invariant system behaves the same no matter when the input is applied
- A system that is not time-invariant is called **time-varying**

Summary

- Time-invariant systems behave the same no matter when the input is applied
- Infinite-length signals: Invariance with respect to any integer time shift
- Finite-length signals: Invariance with respect to a circular time shift
- To show a system is <u>time-invariant</u>, you have to prove it rigorously assuming arbitrary inputs (work!)
- To show a system is <u>time-varying</u>, you can just exhibit a counterexample (often easy!)

Linear Time-Invariant Systems

Linear Time Invariant (LTI) Systems

DEFINITION

A system $\mathcal{H}$ is **linear time-invariant** (LTI) if it is both linear and time-invariant

- LTI systems are the foundation of signal processing and the main subject of this course

LTI or Not? You Be the Judge! (1)

- Identity

$$y[n] = x[n] \quad \forall n$$

- Scaling

$$y[n] = 2\,x[n] \quad \forall n$$

- Offset

$$y[n] = x[n] + 2 \quad \forall n$$

- Square signal

$$y[n] = (x[n])^2 \quad \forall n$$

- Shift

$$y[n] = x[n+2] \quad \forall n$$

- Decimate

$$y[n] = x[2n] \quad \forall n$$

- Square time

$$y[n] = x[n^2] \quad \forall n$$

LTI or Not? You Be the Judge! (2)

- Shift system ($m \in \mathbb{Z}$ fixed)

$$y[n] = x[n-m] \quad \forall n$$

- Moving average (combines shift, sum, scale)

$$y[n] = \frac{1}{2}(x[n] + x[n-1]) \quad \forall n$$

- Recursive average

$$y[n] = x[n] + \alpha\, y[n-1] \quad \forall n$$

Matrix Multiplication and LTI Systems (Infinite-Length Signals)

- Recall that all linear systems can be expressed as matrix multiplications

$$y = \mathbf{H}\,x$$

$$y[n] = \sum_m [\mathbf{H}]_{n,m}\,x[m]$$

  Here $\mathbf{H}$ is a matrix with infinitely many rows and columns

- Let $h_{n,m} = [\mathbf{H}]_{n,m}$ represent the row-n, column-m entry of the matrix $\mathbf{H}$

$$y[n] = \sum_m h_{n,m}\,x[m]$$

- When the linear system is also shift invariant, $\mathbf{H}$ has a special structure

Matrix Structure of LTI Systems (Infinite-Length Signals)

- Linear system for infinite-length signals can be expressed as

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \sum_{m=-\infty}^{\infty} h_{n,m}\, x[m], \quad -\infty < n < \infty$$

- Enforcing time invariance implies that for all $q \in \mathbb{Z}$

$$\mathcal{H}\{x[n-q]\} \;=\; \sum_{m=-\infty}^{\infty} h_{n,m}\, x[m-q] \;=\; y[n-q]$$

- Change of variables: $n' = n - q$ and $m' = m - q$

$$\mathcal{H}\{x[n']\} \;=\; \sum_{m'=-\infty}^{\infty} h_{n'+q,m'+q}\, x[m'] \;=\; y[n']$$

- Comparing first and third equations, we see that for an LTI system

$$h_{n,m} \;=\; h_{n+q,m+q} \quad \forall\, q \in \mathbb{Z}$$

LTI Systems are Toeplitz Matrices (Infinite-Length Signals) (1)

- For an LTI system with infinite-length signals

$$h_{n,m} = h_{n+q,m+q} \quad \forall\, q \in \mathbb{Z}$$

$$\mathbf{H} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h_{-1,-1} & h_{-1,0} & h_{-1,1} & \cdots \\ \cdots & h_{0,-1} & h_{0,0} & h_{0,1} & \cdots \\ \cdots & h_{1,-1} & h_{1,0} & h_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h_{0,0} & h_{-1,0} & h_{-2,0} & \cdots \\ \cdots & h_{1,0} & h_{0,0} & h_{-1,0} & \cdots \\ \cdots & h_{2,0} & h_{1,0} & h_{0,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}$$

- Entries on the matrix diagonals are the same – **Toeplitz matrix**

LTI Systems are Toeplitz Matrices (Infinite-Length Signals) (2)

- All of the entries in a Toeplitz matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$
  - **Time-reversed** 0-**th row**: $h[m] = h_{0,-m}$

$$
\mathbf{H} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h_{0,0} & h_{-1,0} & h_{-2,0} & \cdots \\ \cdots & h_{1,0} & h_{0,0} & h_{-1,0} & \cdots \\ \cdots & h_{2,0} & h_{1,0} & h_{0,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & h[0] & h[-1] & h[-2] & \cdots \\ \cdots & h[1] & h[0] & h[-1] & \cdots \\ \cdots & h[2] & h[1] & h[0] & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix}
$$

- Row-n, column-m entry of the matrix $[\mathbf{H}]_{n,m} = h_{n,m} = h[n-m]$

LTI Systems are Toeplitz Matrices (Infinite-Length Signals) (3)

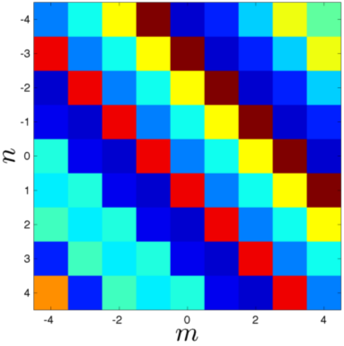
- All of the entries in a Toeplitz matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$ (this is an infinite-length signal/column vector; call it h)
  - **Time-reversed** 0-**th row**: $h[m] = h_{0,-m}$

- Example: Snippet of a Toeplitz matrix

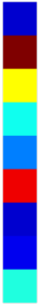
$$\begin{aligned}[\mathbf{H}]_{n,m} &= h_{n,m} \\ &= h[n-m]\end{aligned}$$

- Note the diagonals!

$\mathbf{H} =$



$h =$



Matrix Structure of LTI Systems (Finite-Length Signals)

- Linear system for signals of length N can be expressed as

$$y[n] \;=\; \mathcal{H}\{x[n]\} \;=\; \sum_{m=0}^{N-1} h_{n,m}\, x[m], \quad 0 \le n \le N-1$$

- Enforcing time invariance implies that for all $q \in \mathbb{Z}$

$$\mathcal{H}\{x[(n-q)_N]\} \;=\; \sum_{m=0}^{N-1} h_{n,m}\, x[(m-q)_N] \;=\; y[(n-q)_N]$$

- Change of variables: $n' = n - q$ and $m' = m - q$

$$\mathcal{H}\{x[(n')_N]\} \;=\; \sum_{m'=-q}^{M-1-q} h_{(n'+q)_N,(m'+q)_N}\, x[(m')_N] \;=\; y[(n')_N]$$

- Comparing first and third equations, we see that for an LTI system

$$h_{n,m} \;=\; h_{(n+q)_N,(m+q)_N} \quad \forall\, q \in \mathbb{Z}$$

LTI Systems are circulant Matrices (Finite-Length Signals) (1)

- For an LTI system with length-N signals

$$h_{n,m} = h_{(n+q)_N,(m+q)_N} \quad \forall\, q \in \mathbb{Z}$$

$$\begin{bmatrix} h_{0,0} & h_{0,1} & h_{0,2} & \cdots & h_{0,N-1} \\ h_{1,0} & h_{1,1} & h_{1,2} & \cdots & h_{1,N-1} \\ h_{2,0} & h_{2,1} & h_{2,2} & \cdots & h_{2,N-1} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{N-1,0} & h_{N-1,1} & h_{N-1,2} & \cdots & h_{N-1,N-1} \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{N-1,0} & h_{N-2,0} & \cdots & h_{1,0} \\ h_{1,0} & h_{0,0} & h_{N-1,0} & \cdots & h_{2,0} \\ h_{2,0} & h_{1,0} & h_{0,0} & \cdots & h_{3,0} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{N-1,0} & h_{N-2,0} & h_{N-3,0} & \cdots & h_{0,0} \end{bmatrix}$$

- Entries on the matrix diagonals are the same + circular wraparound – **circulant matrix**

LTI Systems are circulant Matrices (Finite-Length Signals) (2)

- All of the entries in a circulant matrix can be expressed in terms of the entries of the
  - **0-th column**: $h[n] = h_{n,0}$
  - **Circularly time-reversed 0-th row**: $h[m] = h_{0,(-m)_N}$

$$\begin{bmatrix} h_{0,0} & h_{N-1,0} & h_{N-2,0} & \cdots & h_{1,0} \\ h_{1,0} & h_{0,0} & h_{N-1,0} & \cdots & h_{2,0} \\ h_{2,0} & h_{1,0} & h_{0,0} & \cdots & h_{3,0} \\ \vdots & \vdots & \vdots & & \vdots \\ h_{N-1,0} & h_{N-2,0} & h_{N-3,0} & \cdots & h_{0,0} \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix}$$

- Row-n, column-m entry of the matrix $[\mathbf{H}]_{n,m} = h_{n,m} = h[(n-m)_N]$

LTI Systems are circulant Matrices (Finite-Length Signals) (3)

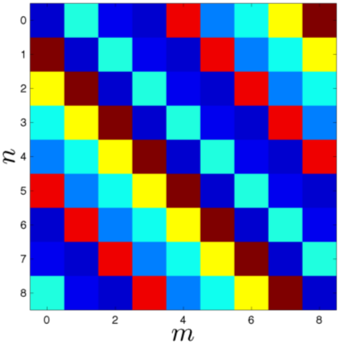
- All of the entries in a circulant matrix can be expressed in terms of the entries of the
  - 0-**th column**: $h[n] = h_{n,0}$ (this is a signal/column vector; call it h)
  - **Circularly time-reversed** 0-**th row**: $h[m] = h_{0,-m}$

- Example: circulant matrix

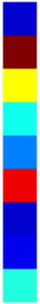
$$
\begin{aligned}
[\mathbf{H}]_{n,m} &= h_{n,m} \\
&= h[(n-m)_N]
\end{aligned}
$$

- Note the diagonals and circulant shifts!

$\mathbf{H} =$



$h =$



Summary

- LTI = Linear + Time-Invariant

- Fundamental signal processing system (and our focus for the rest of the course)

- Infinite-length signals: System = Toeplitz matrix $\mathbf{H}$
  - $[\mathbf{H}]_{n,m} \;=\; h_{n,m} \;=\; h[n-m]$

- Finite-length signals: System = circulant matrix $\mathbf{H}$
  - $[\mathbf{H}]_{n,m} \;=\; h_{n,m} \;=\; h[(n-m)_N]$