Bicycle Model

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1 Derivation

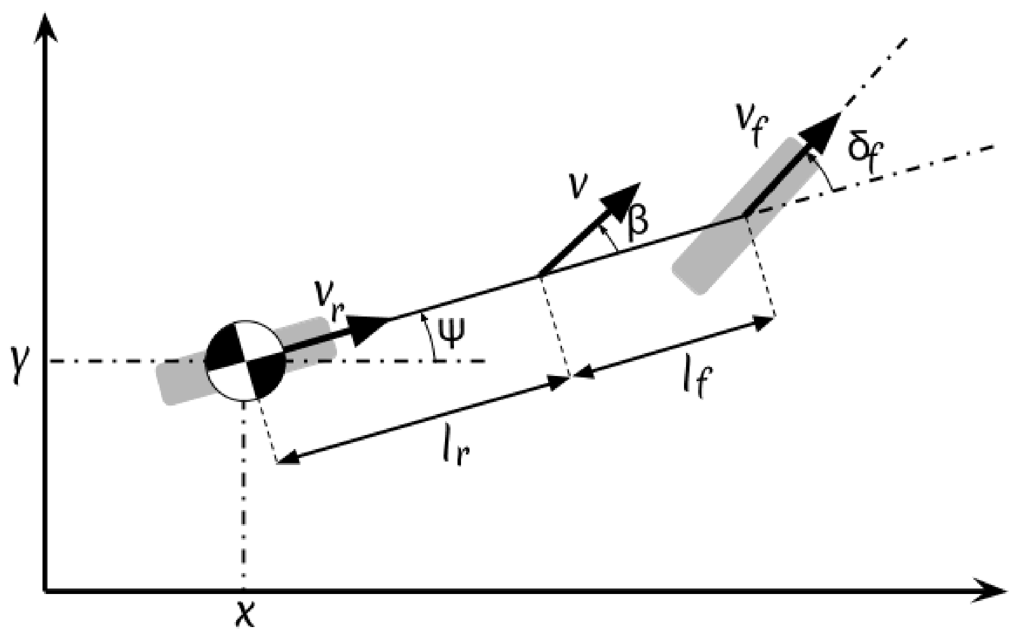

Figure 1: Bicycle model state variable diagram.

Some preliminary relationships:
State variables:

$$z = \begin{bmatrix} x \\ y \\ \psi \\ \delta_f \\ v_f \end{bmatrix}$$

We will use v_f as the state variable for velocity to avoid discontinuities.

$$v_r = v_f \cos(\delta_f)$$

$$\dot{\psi} = v_f \sin(\delta_f)$$

We will assume a front wheel drive vehicle with wheel torque τ and wheel radius r, generating a force of $\frac{\tau}{r}$ as an input. Assuming a mass m and an inertia I for the vehicle defined as the inertia at the wheel base, not the center of mass, we can derive the following equation for the acceleration of the front wheel:

$$a_f = \frac{\tau}{r}\left(\frac{1}{m\cos(\delta_f)} + \frac{((l_r + l_f)\sin(\delta_f))^2}{I}\right)$$

We will also assume that the steering angle is controlled by some steering rate input w

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{\delta_f} \\ \dot{v_f} \end{bmatrix} = \begin{bmatrix} v_f \cos(\delta_f)\cos(\psi) \\ v_f \cos(\delta_f)\sin(\psi) \\ \frac{v_f \sin(\delta_f)}{l_r + l_f} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ a_f \end{bmatrix}$$

The linearized dynamics for this system are defined as:

$$\nabla f(X) = \begin{bmatrix} 0 & 0 & -v_f \cos(\delta_f)\sin(\psi) & -v_f \sin(\delta_f)\cos(\psi) & \cos(\delta_f)\cos(\psi) \\ 0 & 0 & v_f \cos(\delta_f)\cos(\psi) & -v_f \sin(\delta_f)\sin(\psi) & \cos(\delta_f)\sin(\psi) \\ 0 & 0 & 0 & \frac{v_f \cos(\delta_f)}{l_r + l_f} & \frac{\sin(\delta_f)}{l_r + l_f} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ a_f \end{bmatrix}$$

Alternatively, we can clamp the vehicle velocity and steering angle with a sigmoid function: The sigmoid is defined as

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

The derivative of the sigmoid is

$$\frac{d}{dz}\sigma(z) = \sigma(z)(1 - \sigma(z))$$

We will use the following constraints to clamp velocity and steering angle:

$$\delta_{fapplied} = 2\delta_{fmax}\left(\sigma(\delta_f) - \frac{1}{2}\right)$$

$$v_{fapplied} = 1.5 v_{fmax}\left(\sigma(v_f) - \frac{1}{3}\right)$$

Notice that we have clamped the vehicles reverse velocity. This results in the following dynamics:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{\delta_f} \\ \dot{v_f} \end{bmatrix} = \begin{bmatrix} 1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\cos(\psi) \\ 1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\sin(\psi) \\ \frac{1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)\sin\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)}{l_r+l_f} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ a_f \end{bmatrix}$$

The linearized dynamics for this system are defined as:

$$\nabla f_{02}(X) = -1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\sin(\psi)$$

$$\nabla f_{03}(X) = -1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)2\delta_{fmax}\sigma(\delta_f)(1-\sigma(\delta_f))\sin\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\cos(\psi)$$

$$\nabla f_{04}(X) = 1.5v_{fmax}\sigma(v_f)(1-\sigma(v_f))\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\cos(\psi)$$

$$\nabla f_{12}(X) = 1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\cos(\psi)$$

$$\nabla f_{13}(X) = -1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)2\delta_{fmax}\sigma(\delta_f)(1-\sigma(\delta_f))\sin\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\sin(\psi)$$

$$\nabla f_{14}(X) = 1.5v_{fmax}\sigma(v_f)(1-\sigma(v_f))\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)\sin(\psi)$$

$$\nabla f_{23}(X) = \frac{1.5v_{fmax}\left(\sigma(v_f)-\frac{1}{3}\right)2\delta_{fmax}\sigma(\delta_f)(1-\sigma(\delta_f))\cos\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)}{l_r+l_f}$$

$$\nabla f_{24}(X) = \frac{1.5v_{fmax}\sigma(v_f)(1-\sigma(v_f))\sin\left(2\delta_{fmax}\left(\sigma(\delta_f)-\frac{1}{2}\right)\right)}{l_r+l_f}$$