

**Assignment 1;  
due Tuesday October 28**

**Part 1**

**Exercise 1**

**Solution**

- (a) R cannot be a rotation matrix because  $\det(R) \neq 1$
- (b)  $\det(R^{-1}) = \frac{1}{\det(R)} = -1$  So R cannot be the inverse of a rotation matrix
- (c)  $RR(RR)^T = RRR^T R^T = RIR^T = RR^T = I$   
 $\det(RR) = \det(R) * \det(R) = 1$   
So, RR can be a rotation matrix
- (d)  $2R2R^T = 4RR^T = 4I$   
So, 2R cannot be a rotation matrix

**Exercise 2**

**Solution**

- (a) R is yet undefined. General rotation matrix R is  $R_Y(\alpha)R_Z(\beta)R_Y(\gamma)$
- (b)  $R1 = R_Y(\alpha)R_Z(\beta)R_Y(\gamma) =$   
$$\begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\alpha & 0 \\ \sin\beta & 0 & 0 \\ 0 & \cos\alpha & 1 \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} =$$
  
$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\beta & \cos\gamma\sin\alpha + \sin\gamma\cos\alpha\cos\beta \\ \cos\gamma\sin\beta & \cos\beta & \sin\beta\sin\gamma \\ -\cos\beta\cos\gamma\sin\alpha - \cos\alpha\sin\gamma & \sin\alpha\sin\beta & \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma \end{pmatrix}$$
  
 $R2 = R_Y(\hat{\alpha})R_Z(\hat{\beta}) =$   
$$\begin{pmatrix} \cos\hat{\alpha} & 0 & \sin\hat{\alpha} \\ 0 & 1 & 0 \\ -\sin\hat{\alpha} & 0 & \cos\hat{\alpha} \end{pmatrix} \begin{pmatrix} \cos\hat{\beta} & \sin\hat{\alpha} & 0 \\ \sin\hat{\beta} & 0 & 0 \\ 0 & \cos\hat{\alpha} & 1 \end{pmatrix} = \begin{pmatrix} \cos\hat{\alpha}\cos\hat{\beta} & -\cos\hat{\alpha}\sin\hat{\beta} & \sin\hat{\alpha} \\ \sin\hat{\beta} & \cos\hat{\beta} & 0 \\ -\sin\hat{\alpha}\cos\hat{\beta} & \sin\hat{\alpha}\sin\hat{\beta} & \cos\hat{\alpha} \end{pmatrix}$$

We can see that  $R_{23}$  is zero, but  $R_{123}$  depends on  $\hat{\beta}$  and  $\hat{\gamma}$ . So we can choose  $\gamma$  and  $\beta$  such, that  $R_{123}$  wouldn't be zero. Thus we cannot represent any rotation matrix by only to rotations.

(c) We remember that  $R = R_Y(\alpha)R_Z(\beta)R_Y(\gamma) =$

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\sin\beta & \cos\gamma\sin\alpha + \sin\gamma\cos\alpha\cos\beta \\ \cos\gamma\sin\beta & \cos\beta & \sin\beta\sin\gamma \\ -\cos\beta\cos\gamma\sin\alpha - \cos\alpha\sin\gamma & \sin\alpha\sin\beta & \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma \end{pmatrix}$$

$$\text{And } R_Z(\hat{\beta}) = \begin{pmatrix} \cos\hat{\beta} & -\sin\hat{\beta} & 0 \\ \sin\hat{\beta} & \cos\hat{\beta} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can see that  $R_{23} = \sin\beta\sin\gamma$  it is not equal to 1 always. So, if  $\gamma = -\alpha$  it is still false that  $R = R_Z(\hat{\beta})$

$$(d) \quad R_Y(\hat{\alpha}) = \begin{pmatrix} \cos\hat{\alpha} & 0 & \sin\hat{\alpha} \\ 0 & 1 & 0 \\ -\sin\hat{\alpha} & 0 & \cos\hat{\alpha} \end{pmatrix}$$

We can see that  $R_{23}$  from the previous example is equal to  $\sin\beta\sin\gamma$  that is not zero always. So it is the fact that when  $\alpha = -\beta$   $R \neq R_Y(\hat{\alpha})$ . So, the assumption is false.

### Exercise 3

#### Solution

First of all lets try to find all the eigenvalues.

We have to remember that  $R =$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

On the other hand, each column of this matrix is a rotation vector around each of axes.

$$\text{Thus } X = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} Y = \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} Z = \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

Moreover we know that:  $X = Y * Z \quad Y = Z * X \quad Z = X * Y$

So we can find that:

$$a_{11} = a_{22}a_{33} - a_{32}a_{23}$$

$$a_{22} = a_{11}a_{33} - a_{13}a_{31}$$

$$a_{33} = a_{11}a_{22} - a_{12}a_{21}$$

That are exactly some minors of matrix R.

We know that to find eigenvalues we have to solve the equation:

$$|R - \lambda E| = 0$$

$$\text{So, } |R - \lambda E| = -(\lambda - 1)[\lambda^2 - \lambda(a_{11} + a_{22} + a_{33} - 1) + 1] = 0$$

According to [1], we have to let  $\cos\phi = \frac{a_{11} + a_{22} + a_{33} - 1}{2}$ . That is the only thing that is not clear from [1]. I still cannot show why it is so. However when we let that we can easily show that  $\lambda_1 = 1$ ,  $\lambda_{2,3} = \cos\phi \pm i\sin\phi$

Since we made previous conversions it is very easy to answer the questions.

- (a) Wrong, because there are complex values in an answer
- (b) True
- (c) False because with some values of  $\theta$  we can reach more 1.
- (d) Due to the fact that we have complex values in an answer it is very easy to show that squared complex value is not always equals to 1.

#### Exercise 4

##### Solution

- (a) No. X could be a vector, laying on the plane of rotation. And the rotation is 360 degree by orthogonal to the plane.
- (b) No. It could be a rotation axis.
- (c) No. As mentioned in [a], it could lay on the plane of rotation if rotation is by 360 degrees of the orthogonal to the plane.
- (d)  $(Rx)^T = x^T$   
 $x^T R^T = x^T$   
 $x^T R^T R = x^T R$   
 $x^T I = x^T R$   
 $x^T = x^T R$  QED

## Part 2

#### Exercise 1

##### Solution

$$\begin{pmatrix} {}^A P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^A_B R & {}^A P_{BORG} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^B P \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} {}^A P \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

$${}^A P = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}$$

**Exercise 2**

**Solution**

$$\begin{pmatrix} {}^A P \\ 1 \end{pmatrix} = \begin{pmatrix} {}^A_B R & {}^A P_{BORG} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^B P \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} {}^A P \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$${}^A P = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

**Exercise 3**

**Solution**

a)  ${}^0_1 T$

$${}^0 P_{BORG} = (4, 2, 0).$$

Rotations:

- 1) Rotation around Y axis by  $\Pi$  anti-clockwise.
- 2) Rotation around Z axis by  $\frac{\Pi}{2}$  anti-clockwise.

The rotation matrix is:  $R = \begin{pmatrix} \cos \Pi & 0 & \sin \Pi \\ 0 & 1 & 0 \\ -\sin \Pi & 0 & \cos \Pi \end{pmatrix} \cdot \begin{pmatrix} \cos \frac{\Pi}{2} & -\sin \frac{\Pi}{2} & 0 \\ \sin \frac{\Pi}{2} & \cos \frac{\Pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence, transformation matrix is:  ${}^0_1 T = \begin{pmatrix} 0 & 1 & 0 & 4 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

b)  ${}^1_2 T$

$${}^2 P_{BORG} = (-4, 4, 0)$$

Rotations:

- 1) Rotation around X by  $\frac{3\Pi}{4}$  clockwise.
- 2) Rotation around Z by  $\frac{\Pi}{2}$  clockwise.

The rotation matrix is:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos - \frac{3\Pi}{4} & -\sin - \frac{3\Pi}{4} \\ 0 & \sin - \frac{3\Pi}{4} & \cos - \frac{3\Pi}{4} \end{pmatrix} \cdot \begin{pmatrix} \cos - \frac{\Pi}{2} & -\sin - \frac{\Pi}{2} & 0 \\ \sin - \frac{\Pi}{2} & \cos - \frac{\Pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

Hence, transformation matrix is:

$${}^1_2T = \begin{pmatrix} 0 & 0 & -1 & -4 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 4 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c)  ${}^2_3T$

$${}^2P_{BORG} = (4, 5\sqrt{2}, \sqrt{2})$$

Rotations:

- 1) Rotation around Z by  $\frac{\Pi}{2}$  clockwise.
- 2) Rotation around X by  $\frac{3\Pi}{2}$  clockwise.

The rotation matrix is:

$$R = \begin{pmatrix} \cos - \frac{\Pi}{2} & -\sin - \frac{\Pi}{2} & 0 \\ \sin - \frac{\Pi}{2} & \cos - \frac{\Pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos - \frac{3\Pi}{2} & -\sin - \frac{3\Pi}{2} \\ 0 & \sin - \frac{3\Pi}{2} & \cos - \frac{3\Pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

Hence, transformation matrix is:

$${}^3_2T = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 4 \\ -1 & 0 & 0 & 5\sqrt{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \sqrt{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

#### Exercise 4

#### Solution

a)  ${}^0_1T$

$${}^0P_{BORG} = (0, 0, -9).$$

Rotations:

1) Rotation around Z axis by  $\alpha^\circ$  clockwise.

$$\text{The rotation matrix is: } R = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, transformation matrix is:

$${}^0_1T = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 & 0 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b)  ${}^1_2T$

$${}^1P_{BORG} = (0, 0, -3).$$

Rotations:

1) Rotation around X axis by  $\alpha^\circ$  clockwise.

2) Rotation around Y axis by  $\frac{\Pi}{2}$  clockwise.

$$\text{The rotation matrix is: } R = \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos - \frac{\Pi}{2} & 0 & \sin - \frac{\Pi}{2} \\ 0 & 1 & 0 \\ -\sin - \frac{\Pi}{2} & 0 & \cos - \frac{\Pi}{2} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & \sin(\alpha) & -\cos(\alpha) \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 1 & 0 & 0 \end{pmatrix}$$

Hence, transformation matrix is:

$${}^2_1T = \begin{pmatrix} 0 & \sin(\alpha) & -\cos(\alpha) & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

### Exercise 5

#### Solution

a)  ${}^0_1T$

$${}^0P_{BORG} = (3 - 3\cos(\alpha), 2, -3\sin(\alpha)).$$

Rotations:

- 1) Rotation around Z axis by  $\alpha^\circ + \frac{\Pi}{2}$  clockwise.
- 2) Rotation around X axis by  $\frac{\Pi}{2}$  clockwise.

The rotation matrix is:

$$R = \begin{pmatrix} \cos\left(-\left(\alpha + \frac{\Pi}{2}\right)\right) & -\sin\left(-\left(\alpha + \frac{\Pi}{2}\right)\right) & 0 \\ \sin\left(-\left(\alpha + \frac{\Pi}{2}\right)\right) & \cos\left(-\left(\alpha + \frac{\Pi}{2}\right)\right) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos - \frac{\Pi}{2} & -\sin - \frac{\Pi}{2} \\ 0 & \sin - \frac{\Pi}{2} & \cos - \frac{\Pi}{2} \end{pmatrix} =$$

$$\begin{pmatrix} -\sin(\alpha) & 0 & \cos(\alpha) \\ \cos(\alpha) & 0 & \sin(\alpha) \\ 0 & -1 & 0 \end{pmatrix}$$

Hence, transformation matrix is:

$${}^0_1T = \begin{pmatrix} -\sin(\alpha) & 0 & \cos(\alpha) & 3 - 3\cos(\alpha) \\ \cos(\alpha) & 0 & \sin(\alpha) & 2 \\ 0 & -1 & 0 & -3\sin(\alpha) \end{pmatrix}$$

b)  ${}^1_2T$

$${}^1P_{BORG} = (0.2, 0, 1).$$

Rotations:

1) Rotation around X axis by  $\frac{\Pi}{2}$  anti-clockwise.

The rotation matrix is:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{\Pi}{2} & -\sin\frac{\Pi}{2} \\ 0 & \sin\frac{\Pi}{2} & \cos\frac{\Pi}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hence, transformation matrix is:

$${}^2_1T = \begin{pmatrix} 1 & 0 & 0 & 0.2 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

#### List of references

[1] <http://robotics.caltech.edu/~jwb/courses/ME115/handouts/rotation.pdf>