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Assignment 1

Exercises 1

In a class of 23 students, what is the probability that at least two people have the same birthday?

Solution Estimate the probability that no one matched birthdays at the group - $\overline{p}(n)$ Fix the random person from the group. Take second random person from group. The birthdays are not the same. Probability of this event is $1-\frac{1}{365}$. Take third random person. Probability is $1-\frac{2}{365}$. Take fourth and so on.

$\overline{p}\left(n\right)=\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\ldots\left(1-\frac{n-1}{365}\right)=\frac{365!}{365^{n}\left(365-n\right)!}$

If there is 23 students at the group:

$\overline{p}\left(n\right)=\frac{365!}{365^{23}\left(365-23\right)!}=0.4927$

The probability that at least two people have the same birthday is:

$p\left(n\right)=1-\overline{p}\left(n\right)=0.5073$

Answer: $p\left(n\right)=0.5073$

Exercises 2

From a group of families with two children, one family is selected. Describe the space of elementary events. Assuming all elementary events equally probable, consider the random event A: there are a boy and a girl in that family, and the random event B: there is no more than one girl in the family. Calculate $P\left(A\right)$, $P\left(B\right)$, and $P\left(A\cap B\right)$. Are the events A and B independent?

Solution

Calculate $P\left(B|A\right)$.

The probability that the child is boy is equal $\frac{1}{2}$.

The probability that the child is girl is equal $\frac{1}{2}$.

The probability of A is:

$P\left(A\right)=P\left(boy,girl\right)+P\left(girl,boy\right)=\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

If event A is occurred it means that there is one boy and one boy in the family. It B occurred always if A had occurred. Thus

$P\left(B|A\right)=1$

Hence,

$P(A\cap B)=P\left(B|A\right)\cdot\left(A\right)=1\cdot\dfrac{1}{2}=\dfrac{1}{2}$.

Answer: $P(A\cap B)=\dfrac{1}{2}$.

Exercises 3

Three fair dice are rolled. What is the probability of obtaining at least one 6 if it is known that all the three dice showed different faces?

Solution

The number of three dices with different faces:

$C_6^3=\dfrac{6!}{3!\cdot 3!}=\dfrac{4\cdot 5\cdot 6}{6}=20$

The number of three dices with different faces and without face 6:

$C_5^3=\dfrac{5!}{2!\cdot 3!}=\dfrac{4\cdot 5}{2}=10$

The probability of three dices with different faces and without face 6:

$\overline{p}=\dfrac{C_5^3}{C_6^3}=\dfrac{1}{2}$

The probability of probability of obtaining at least one 6 if it is known that all the three dice showed different faces is:

$p=1-\overline{p}=1-\dfrac{1}{2}=\dfrac{1}{2}$

Answer: $p=\dfrac{1}{2}$

Exercises 4

Suppose a breathalyzer has 5% false positives and 8% false negatives. That is, only 5% of the time will it indicate that a person is drunk when he is actually sober and 8% of the time will it indicate that a person is sober when the person is in fact drunk. Using this test, the police spot test a population of drivers, 99% of whom are sober. What is the chance that a person, who tests as drunk, is actually sober?

Solution

$p\left(D\right)$ - the test "drunk"probability is:

$p\left(D\right)=p\left(D|drunk\right)\cdot\left(drunk\right)+p\left(D|sober\right)\cdot p\left(sober\right)=\left(1-0.08\right)\cdot 0.01+0.05\cdot 0.99=0.0587$

According to Bayes' rule:

$p\left(D\right)=\dfrac{p(D|sober)\cdot p\left(sober\right)}{p\left(D\right)}=\dfrac{0.05\cdot 0.99}{0.0587}=0.843$

Answer: $p\,(D) = 0.843$

Exercises 5

Two fair dice are rolled. Let X_1 denote the number of points shown by the first die and X_2 denote the number of points shown by the second die. Consider the following events:

$A_1 = \{X_1 \textit{ is divisible by } 2,\ X_2 \textit{ is divisible by } 3\}$
$A_2 = \{X_1 \textit{ is divisible by } 3,\ X_2 \textit{ is divisible by } 2\}$
$A_3 = \{X_1 \textit{ is divisible by } X_2\}$
$A_4 = \{X_2 \textit{ is divisible by } X_1\}$
$A_5 = \{X_1 \ +\ X_2 \textit{ is divisible by } 2\}$
$A_6 = \{X_1 \ +\ X_2 \textit{ is divisible by } 3\}$

Solution

It is obvious that:

$$\mathrm{P}(A_1) = \frac{1}{6}\ \mathrm{P}(A_2) = \frac{1}{6}\ \mathrm{P}(A_3) = \frac{7}{18}\ \mathrm{P}(A_4) = \frac{7}{18}\ \mathrm{P}(A_5) = \frac{1}{2}\ \mathrm{P}(A_6) = \frac{1}{3}$$

So let's find all jointly probabilities

$$\begin{array}{lllll}
\mathrm{P}(A_1 \cap A_2) = \frac{1}{36} & \mathrm{P}(A_2 \cap A_3) = \frac{1}{18} & \mathrm{P}(A_3 \cap A_4) = \frac{1}{6} & \mathrm{P}(A_4 \cap A_5) = \frac{1}{4} & \mathrm{P}(A_5 \cap A_6) = \frac{1}{6} \\
\mathrm{P}(A_1 \cap A_3) = \frac{1}{36} & \mathrm{P}(A_2 \cap A_4) = \frac{1}{18} & \mathrm{P}(A_3 \cap A_5) = \frac{1}{4} & \mathrm{P}(A_4 \cap A_6) = \frac{1}{6} & \\
\mathrm{P}(A_1 \cap A_4) = \frac{1}{36} & \mathrm{P}(A_2 \cap A_5) = \frac{1}{36} & \mathrm{P}(A_3 \cap A_6) = \frac{1}{6} & & \\
\mathrm{P}(A_1 \cap A_5) = \frac{1}{36} & \mathrm{P}(A_2 \cap A_6) = \frac{1}{36} & & & \\
\mathrm{P}(A_1 \cap A_6) = \frac{1}{36} & & & &
\end{array}$$

Answer:

By definition of independence only $\{A_1 \textit{ and } A_2\}$, $\{A_5 \textit{ and } A_6\}$ are independent.

There is no reason to check triples and another combinations because it will include not independent events and so will not be independent.

Exercises 6

A) Let X be a random variable with a uniform distribution, i.e. with pdf equal to 1 for x $\in$ [0, 1] and 0 otherwise. Find the pdf of Y = exp(tX) for a fixed t. (This transformation is known as the moment generation function)

B) Find the pdf for this transformation (exp(t*x)) if the pdf of X is equal to exp(-x) for x positive and 0 otherwise.

Solution

According to the theorem from George Casella book

$f_y(y) = f_x(g^{-1}(y)) * \dfrac{g^{-1}(y)}{dy}$ if g(x) increases

$f_y(y) =$ -$f_x(g^{-1}(y)) * \dfrac{g^{-1}(y)}{dy}$ if g(x) decreases

A)

$$f_x(x) = \begin{cases} 1, \ x \in [0,1] \\ 0 \ otherwise \end{cases}$$

x $\in$ [0, 1] so y $\in$ [1, e^t]

g(x) $= e^{tx}$

$g^{-1}(y) = \dfrac{ln(y)}{t}$

$\dfrac{g^{-1}(y)}{dy} = \dfrac{1}{ty}$

$f_y(y) = \dfrac{1}{ty}$

B)

$$f_x(x) = \begin{cases} e^{-x}, \ x > 0 \\ 0 \ otherwise \end{cases}$$

g(x) $= e^{tx}$

$g^{-1}(y) = \dfrac{ln(y)}{t}$

$\dfrac{g^{-1}(y)}{dy} = \dfrac{1}{ty}$

$f_y(y) = e^{-\dfrac{ln(y)}{t}} * \dfrac{1}{ty}$

Answer:
A)

$$f_y(y) = \begin{cases} \dfrac{1}{ty}, y \in [1, e^t] \\ 0 \; otherwise \end{cases}$$

B)

$$f_y(y) = \begin{cases} \dfrac{y^{-\frac{1}{t}-1}}{t}, x > 0 \\ 0 \; otherwise \end{cases}$$

Exercises 7

Solution

Answer: