Take home exam;
Due: January 7

Exercise 1

(a) We know that: $\int_{-\infty}^{\infty} PDF(x)dx = 1$, as far that we know that the 0 < x < 4, we can change integration limits, so $\int_0^4 \frac{c}{\sqrt{x}}dx = 1$. Finally we can conclude that: c = $\frac{1}{4}$

(b) $CDF(x) = \int_{-\infty}^{x} PDF(x)dx$

CDF(x) = $\begin{cases} 0, x < 0 \\ \frac{\sqrt{x}}{2}, 0 \leqslant x \leqslant 4 \\ 1, x > 4 \end{cases}$

(c) P(x < 0.25) = $CDF(0.25) = 0.25$
P(x > 1) = $1 - CDF(1) = 0.5$

(d) $Y = \sqrt{X}$
$F_Y(y) = P(Y < y) = P(\sqrt{X} < y) = P(X < y^2) = CDF(y^2)$

$F_Y(y) = \begin{cases} 0, y < 0 \\ \frac{y}{2}, 0 \leqslant y \leqslant 2 \\ 1, y > 2 \end{cases}$

(e) $E(y) = \int_{-\infty}^{\infty} y \cdot PDF(y)dy$
$E(y) = \int_0^2 \frac{y}{2}dy = 1$
$Var(y) = E(y^2) - E^2(y) = \int_0^2 \frac{y^2}{2}dy - 1 = \frac{1}{3}$

Exercise 2

(a) We know from the CLT that, $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - Ex\right) \sim N(0, \sigma^2)$, so we can derive, that: $\overline{X} \sim N\left(Ex, \frac{\sigma^2}{n}\right)$

From the initial distribution we can find Ex and σ^2:

$\text{PDF(x)} = \dfrac{dF(x)}{dx} = 4x^{-5}$

$E(x) = \int_{-\infty}^{\infty} x \cdot PDF(x)dx = \int_{1}^{\infty} \dfrac{4}{x^{-4}}dx = \dfrac{4}{3}$

$\sigma^2 = Ex^2 - E^2x = \int_{1}^{\infty} \dfrac{4}{x^{-3}}dx = \dfrac{2}{9}$, and so:

$\overline{X} \sim N\left(\dfrac{4}{3}, \dfrac{2}{9 \cdot n}\right)$

(b) Let g(x) = ln(x)

We can use delta method, because $g'(Ex) \neq 0$ and g(x) has a derivative equals to $\dfrac{1}{x}$

As far as we know that:

$\sqrt{n}\left(\overline{X} - \dfrac{4}{3}\right) \sim N(0, \sigma^2)$

We can conclude from delta method:

$\sqrt{n}\left(g(\overline{X}) - g\left(\dfrac{4}{3}\right)\right) \sim N\left(0, \sigma^2\left[g'\left(\dfrac{4}{3}\right)\right]^2\right)$, and so:

$ln\,(x) \sim N\left(ln\left(\dfrac{4}{3}\right), \dfrac{1}{8 \cdot n}\right)$

(c) We know that:

$\dfrac{3 \cdot \sqrt{n}}{\sqrt{2}}\left(\overline{X} - \dfrac{4}{3}\right) \sim N(0,1)$

Thus, $\dfrac{9 \cdot n}{2}\left(\overline{X} - \dfrac{4}{3}\right)^2 \sim \chi_1^2$

$n \cdot \left(\overline{X} - \dfrac{4}{3}\right)^2 \sim \dfrac{2}{9} \cdot \chi_1^2$

$n \cdot \left(\overline{X} - \dfrac{4}{3}\right)^2 = n \cdot \left(\overline{X} - 0.8 + 0.8 - \dfrac{4}{3}\right)^2 = n \cdot \left(\overline{X} - 0.8\right)^2 - 2 \cdot \dfrac{8 \cdot n}{15} \cdot \left(\overline{X} - 0.8\right) + \dfrac{64}{15^2}$

$2 \cdot \dfrac{8 \cdot n}{15} \cdot \left(\overline{X} - 0.8\right) \sim N\left(\dfrac{32}{15}, \dfrac{15^2}{16^2 \cdot n^2}\right)$

So, $n \cdot \left(\overline{X} - 0.8\right)^2 \sim \dfrac{2}{9} \cdot \chi_1^2 + N\left(\dfrac{416}{225}, \dfrac{15^2}{16^2 \cdot n^2}\right)$

Exercise 3

(a) 1. From Chebyshev-Markov inequality we know that:

$P\left(|\overline{X} - \mu| > \varepsilon\right) \leq \dfrac{\sigma^2}{n \cdot \varepsilon^2}$
We can find that $\varepsilon = 1.4$ and so, the probability that $\overline{X}$ is not in interval less than 25 %

2. From CLT we know that $\overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$
To find the probability that $\overline{X}$ is not in the interval we have to find the value of Laplass function.
$P\left(|N(128, \dfrac{6.3^2}{81}) - \mu| > \varepsilon\right) = 2 \cdot \Phi\left(\dfrac{\mu + \varepsilon}{\sigma^2}\right) = 0.02394$
And so, the probability that $\overline{X}$ in not in the interval is 1 - 0.02394 = 97.6%

(b) $129 \pm 1.96 \cdot 6.3 = [116.652, 141.348]$

Exercise 4

(a)

(b)

(c)

Exercise 5

(a)

(b)

(c)

Exercise 6

(a)

(b)

(c)