Problem set 1
DUE: Tue. September 8, 2015

Problem 1 In general case no, because summation of two polynomials can give us another denominator. In case if fixed denominator it is obviously a linear space. Its dimension is n and the basis is $\left(\dfrac{1}{q(x)}, \dfrac{x}{q(x)}, ..., \dfrac{x^n}{q(x)}\right)$.

Problem 2

1. $v_4 + v_1 = v_1 + v_2 - (v_2 + v_3) + (v_3 + v_4)$, that means, that we can represent the last vector in terms of other. Thus this combination is not linearly independent.

2. $(v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) = -1 \cdot (v_4 - v_1)$ Thus this combination is not linearly independent.

3. This combination is linearly independent because I cannot represent the last vector in terms of others.

4. $-(v_1 + v_2) + (v_2 + v_3) = (v_3 - v_4) + (v_4 - v_1)$ Thus this combination is not linearly independent.

Problem 3

According to the fundamental theorem of algebra, we can represent such polynomials in the following form: $p(x) = (x - x_{-1})(x - x_0)(x - x_1) \cdot (x - x_2) \cdot ... \cdot (x - x_{n-1})$.

It means, that all vectors in this vector space will include first three guys of p(x). It means, that everything left is polynomial with degree not exceeding n - 1. Thus the dimension is n. The basis is $((x - x_{-1})(x - x_0)(x - x_1), x \cdot (x - x_{-1})(x - x_0)(x - x_1), ..., x^{n-1} \cdot (x - x_{-1})(x - x_0)(x - x_1))$

Problem 4

$A_I^1 = \begin{pmatrix} 1 & 1 \\ 1 & -\dfrac{1}{2} \end{pmatrix}$

$A_1^{II} = \begin{pmatrix} 1 & 2 \\ 2 & \text{-1} \end{pmatrix}$

$A_I^{II} = A_I^1 \cdot A_1^{II} = \begin{pmatrix} 3 & 1 \\ 0 & 2.5 \end{pmatrix}$

Problem 5

For some particular B there are several possible variants: if coordinates of B are not equal there are no solutions (example (1, -1)). Otherwise there are infinite number of solutions (example (1,1)).

$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot (x_1, x_2)^T = (x_1 + x_2, x_1 + x_2)^T$

Thus, ker(A) $= (0, 0)^T$

Problem 6

Yes, these matrices form a vector space. The basis is set of matrices of size (n × m) with one on A_{ii} place. This space has dimension min(n, m) because there are min(n, m) linearly independent rows.

Coordinates of this space have the following form.

$$\begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix}_{min(m,n)} .$$

Given matrices are obviously linear independent. Yes, they form a basis over matrices two by two.

$$A = \begin{pmatrix} \frac{1}{a} & 0 & 0 & \frac{1}{d} \\ 0 & \frac{1}{b} & \frac{1}{c} & 0 \\ 0 & \frac{1}{b} & -\frac{1}{c} & 0 \\ \frac{1}{a} & 0 & 0 & -\frac{1}{d} \end{pmatrix}$$

Problem 7

Blue whale brought me on his tale the following scalar product <C, D> = Tr($D^T C$). Occasionally it is well defined and according to this scalar product given basis is orthonormal. For the parametrized case we can introduce <C, D> = $A^{-1}Tr((AD)^T AC)$.

Problem 8

$a_0 = \int_0^1 x - x^2 dx = \frac{1}{6}$

$b_n = 2\int_0^1 (x - x^2) sin(2\pi n x) dx = 0$

$a_n = 2\int_0^1 (x - x^2) cos(2\pi n x) dx = \frac{-1}{(\pi n)^2}$

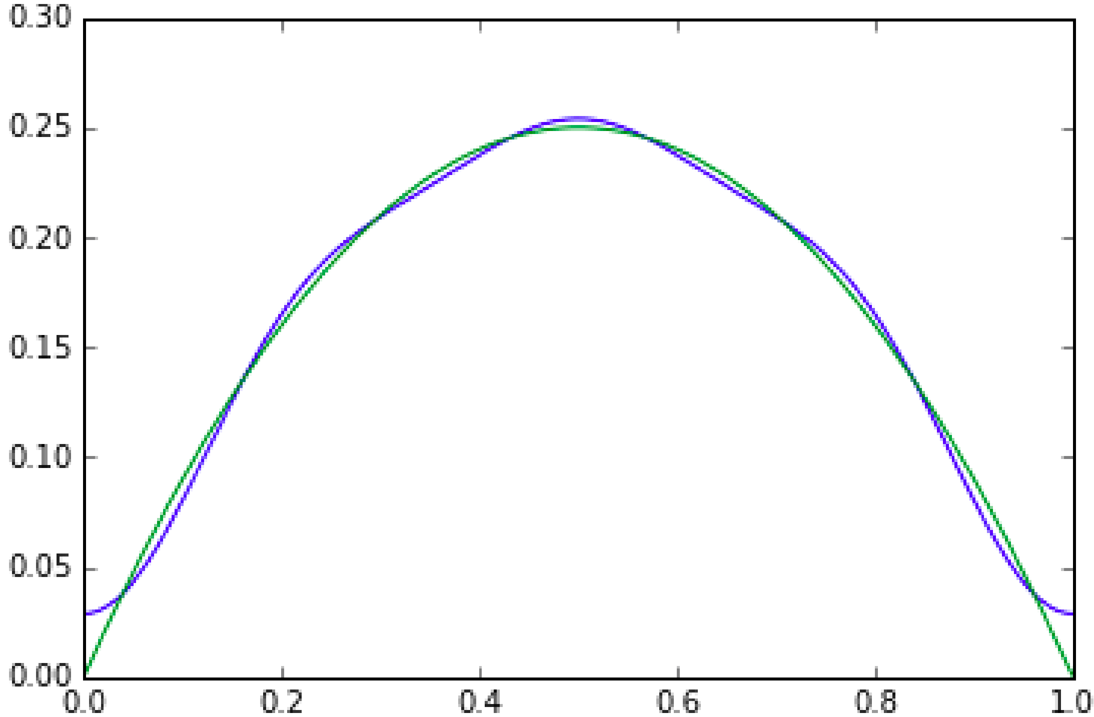

Рис. 1: "Fourier approximation"

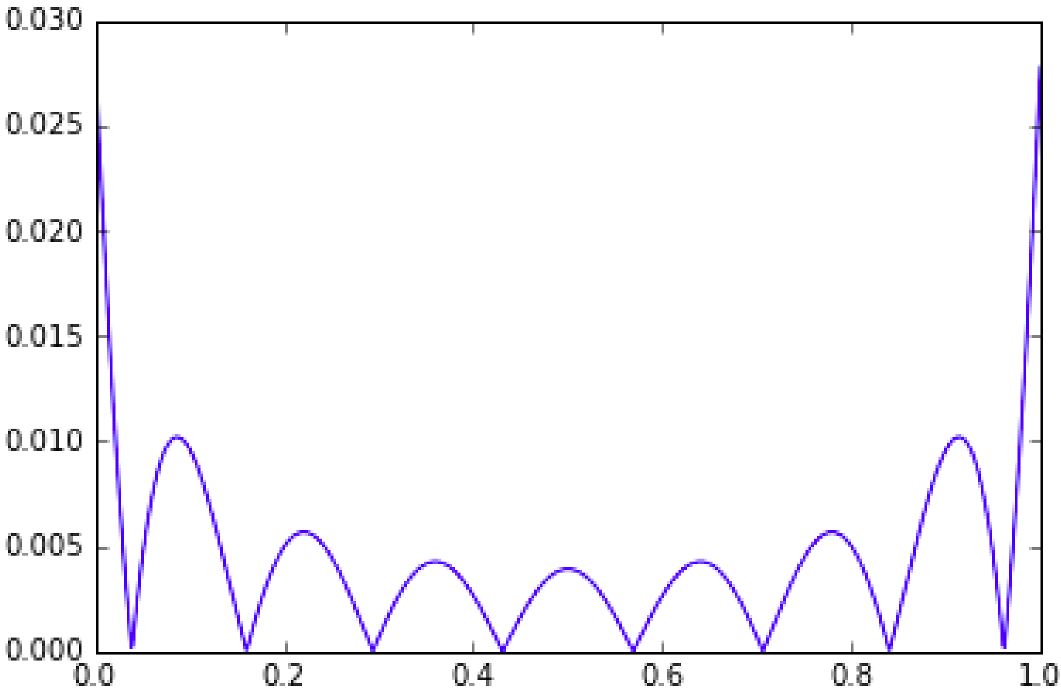

Рис. 2: "Deviation"

Problem 9

$a_0 = \int_0^1 x dx = \frac{1}{2}$

$a_n = 2\int_0^1 x cos(2\pi n x)dx = 0$

$b_n = 2\int_0^1 x sin(2\pi n x)dx = \frac{-cos(2\pi n)}{n\pi}$

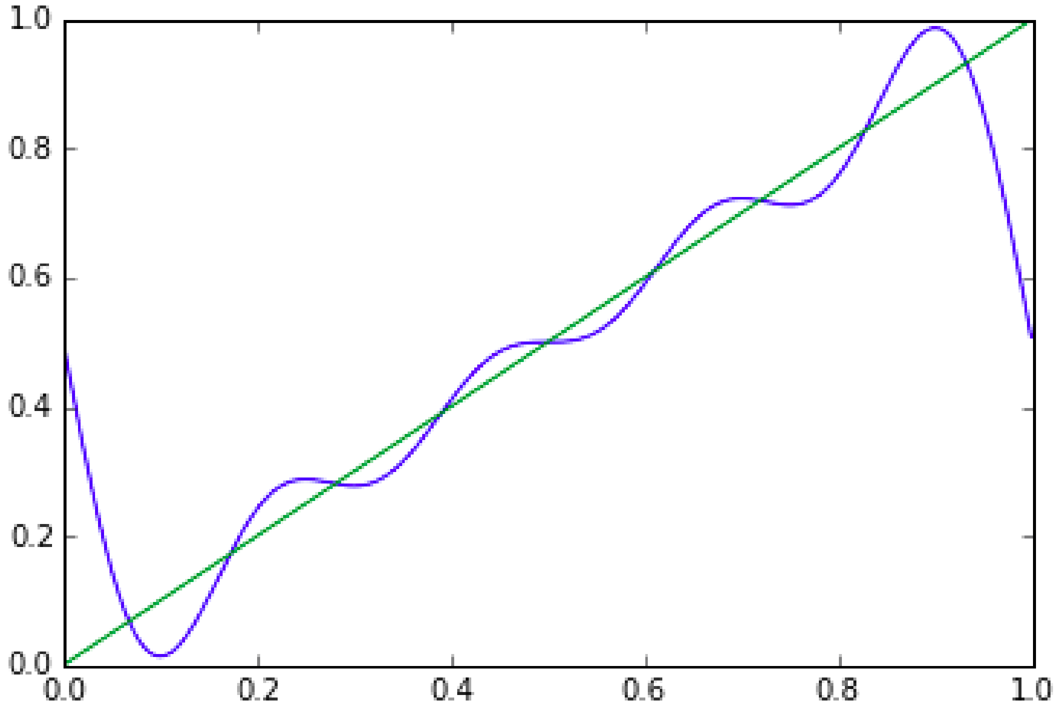

Рис. 3: "Fourier approximation"

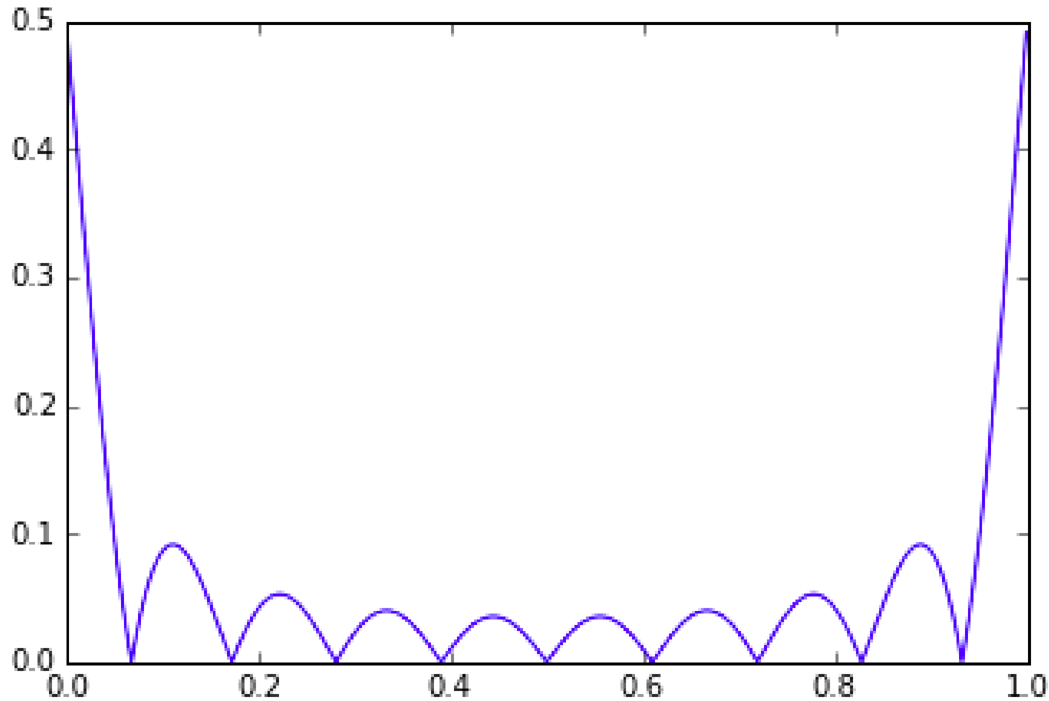

Рис. 4: "Deviation"

Problem 10

(a b) A (a $b)^T = a^2 + b^2 + 2\lambda ab$ Thus only if $\lambda = 0$ there are no such a and b that can make A not positive definite. Eigenvector of A with $\lambda = 0$ is (a, $b)^T$ for any real a and b.

Problem 11

Finding the characteristic polynomial gives us: $\lambda^3 - 9\lambda^2 - 6\lambda = 0$

Thus $\lambda_1 = 0$, $\lambda_{2,3} = \dfrac{9 \pm \sqrt{105}}{2}$

For $\lambda = 0$ eigenvector is $(1\ \text{-2}\ 1)^T$

For $\lambda = \dfrac{9 + \sqrt{105}}{2}$ eigenvector is $(\dfrac{\sqrt{105} - 5}{10}\ \dfrac{\sqrt{105} + 5}{20}\ 1)^T$

For $\lambda = \dfrac{9 - \sqrt{105}}{2}$ eigenvector is $(\dfrac{-\sqrt{105} - 5}{10}\ \dfrac{-\sqrt{105} + 5}{20}\ 1)^T$

$$\text{Thus } A = \begin{pmatrix} 1 & \dfrac{\sqrt{105} - 5}{10} & \dfrac{-\sqrt{105} - 5}{10} \\ \text{-2} & \dfrac{\sqrt{105} + 5}{20} & \dfrac{-\sqrt{105} + 5}{20} \\ 1 & 1 & 1 \end{pmatrix}.$$

Now we can represent initial matrix as A D A^{-1}. Where D is diag(0 $\dfrac{9+\sqrt{5}}{2}$ $\left.\dfrac{9-\sqrt{5}}{2}\right)$

Given matrix cannot be a scalar product because it is not commutative for multiplication.

Problem 12 For M_1:

$M = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix}.$

$A = \begin{pmatrix} 1 & 1 \\ 1 & \text{-1} \end{pmatrix}.$

For M_2:

Its eigenvectors are zero and so it is impossible to diagonalize it.