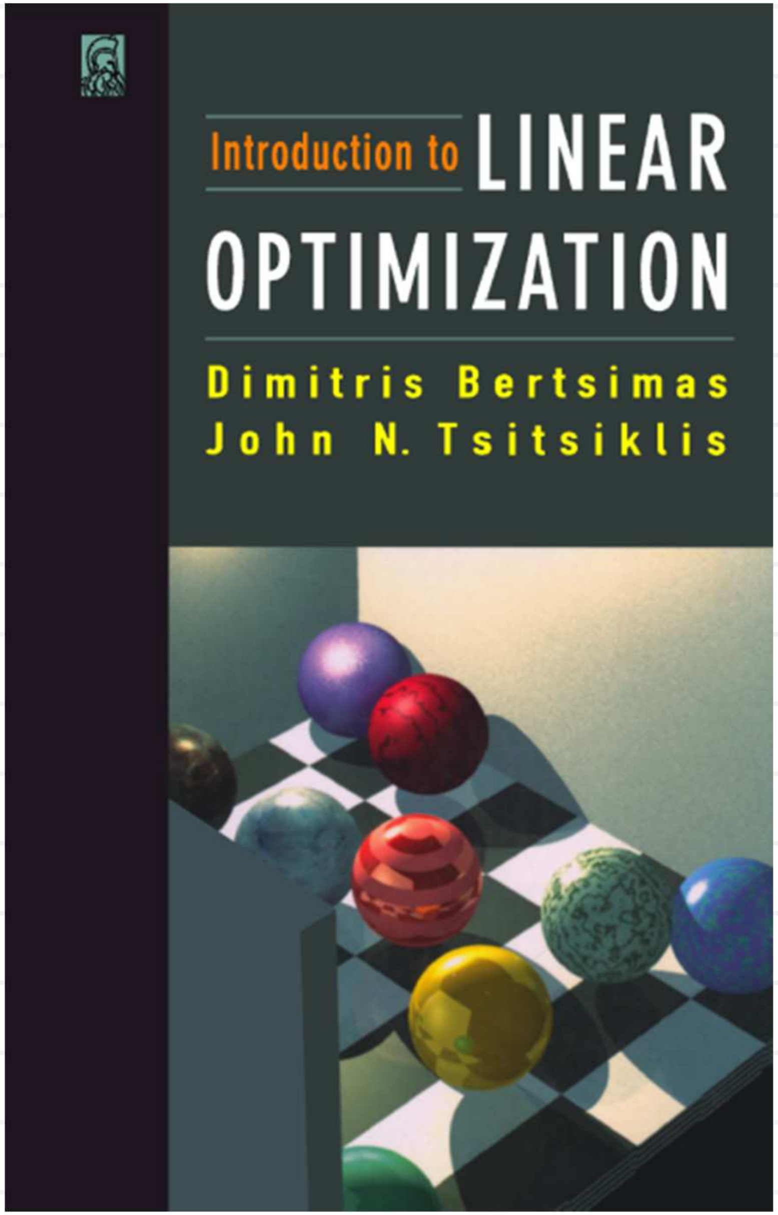
Lecture 5: Introduction to Linear Programming

LP Textbook


Introduction to LINEAR OPTIMIZATION
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John N. Tsitsiklis

Optimization problem/program

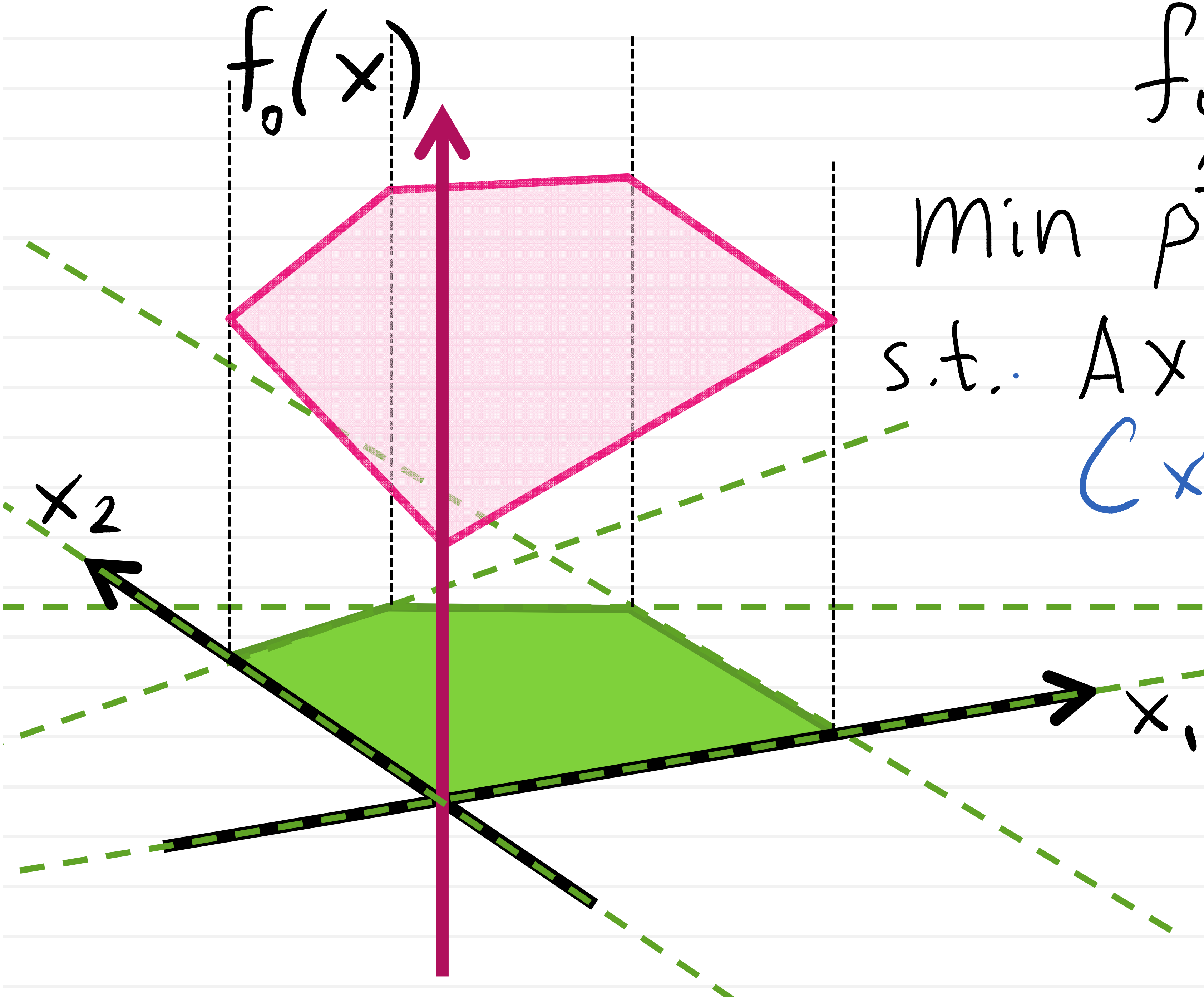
"minimize" objective

$$\min f_0(x)$$

$$\text{s.t.: } x \in D$$

"subject to" domain

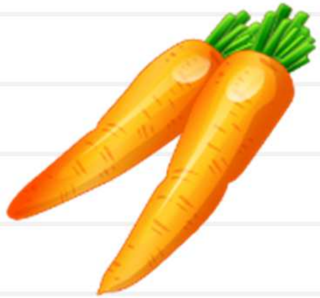
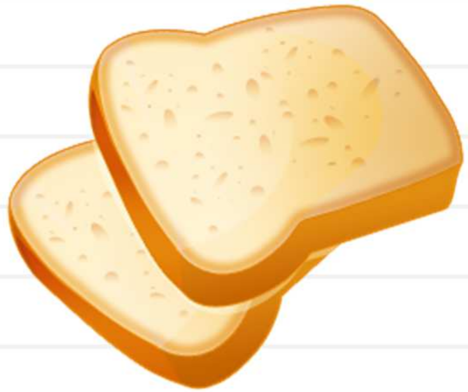
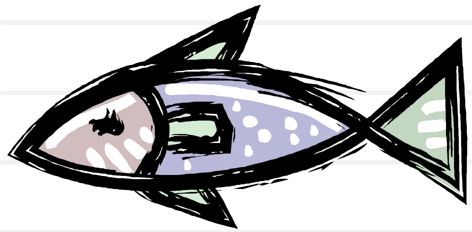
Linear program



$$f_0(x)$$
$$\parallel$$
$$\min p^T x$$
$$\text{s.t. } Ax \le b$$
$$Cx = d$$

Diet problem

Costs: c_1 c_2 . . . c_M

nutrient 1

nutrient 2

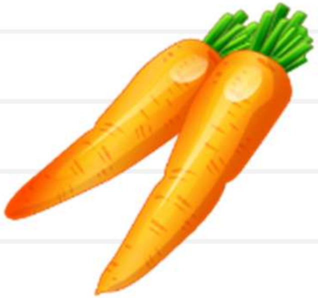
....

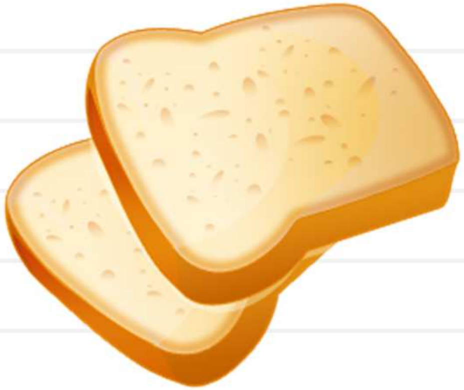
nutrient N

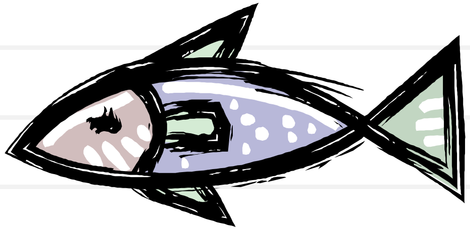
$w_{i,n}$

Content of nutrient n in a unit of a product i

Diet problem







Minimize the cost, so that all required nutrients are supplied:

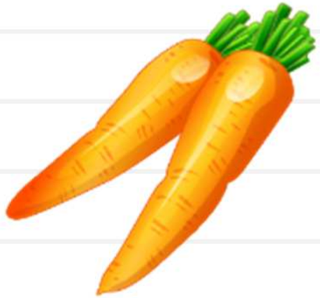
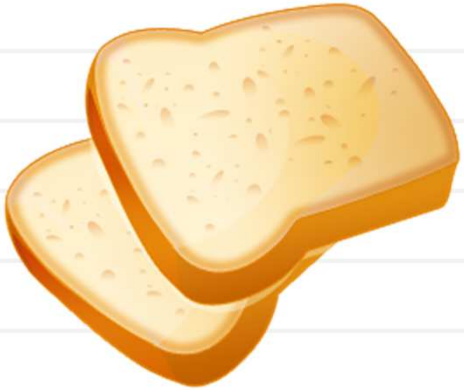
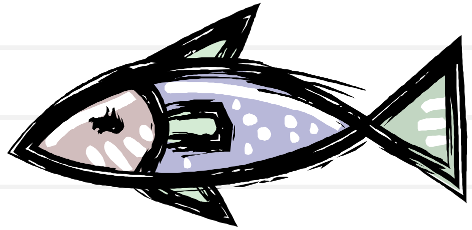
$$\min \sum_{i=1}^{M} x_i c_i$$

Required daily amount

$$\text{s.t.:} \quad \forall n \quad \sum_{i=1}^{M} x_i w_{i,n} \geq r_n$$

$$\forall i \quad x_i \geq 0$$

Diet problem

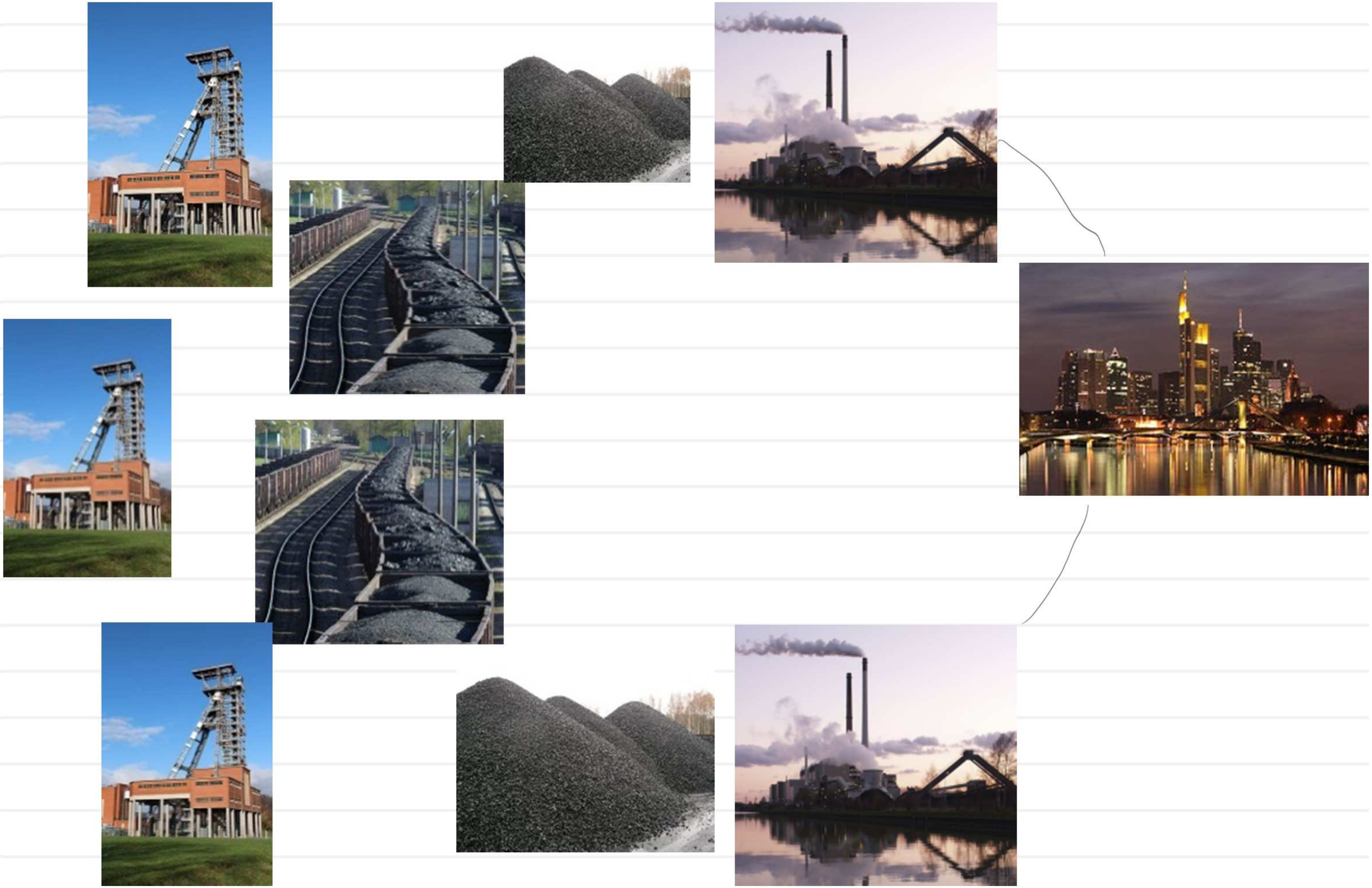
Minimize the cost, so that all required nutrients are supplied:

$$\min \quad c^T x$$

$$\text{s.t.:} \quad Wx \geq r$$

$$x \geq 0$$

Coal supply, delivery, and consumption



Mining

M mines

a_m max amount of coal per day

b_m cost per ton

x_m^t how much do we mine at day t at mine m

Energy production

S stations

l_s min burning amount (in tons of coal)

u_s max burning amount (in tons of coal)

y_s^t how much coal to burn at station s at day t

$p_s + q_s \cdot y_s^t$ produced power

$v_s + w_s \cdot y_s^t$ cost of production

Transportation and storage

$d_{m,s}$ transportation cost from mine m to station s

k_s maximum capacity of the storage at station s

z_s^0 initial amount of coal at station s

$f_{m,s}^t$ how much coal to send from m to s on day t

$z_{t,s}$ how much coal remains at station s after day t

Meeting the demand

e_t predicted demand at day t

Putting it all together: objective

$$\min_{x,f,z,y} \sum_{m,t} b_m \cdot x_m^t +$$

mining

$$+ \sum_{m,s,t} d_{m,s} f_{m,s}^t +$$

transportation

$$+ \sum_{s,t} \left(v_s + w_s y_s^t \right)$$

power generation

Putting it all together: constraints

$0 \leq x_m^t \leq a_m$ a mine has a limited output

$\sum_s f_{m,s}^t = x_m^t$ transporting everything mined

$f_{m,s}^t \geq 0$

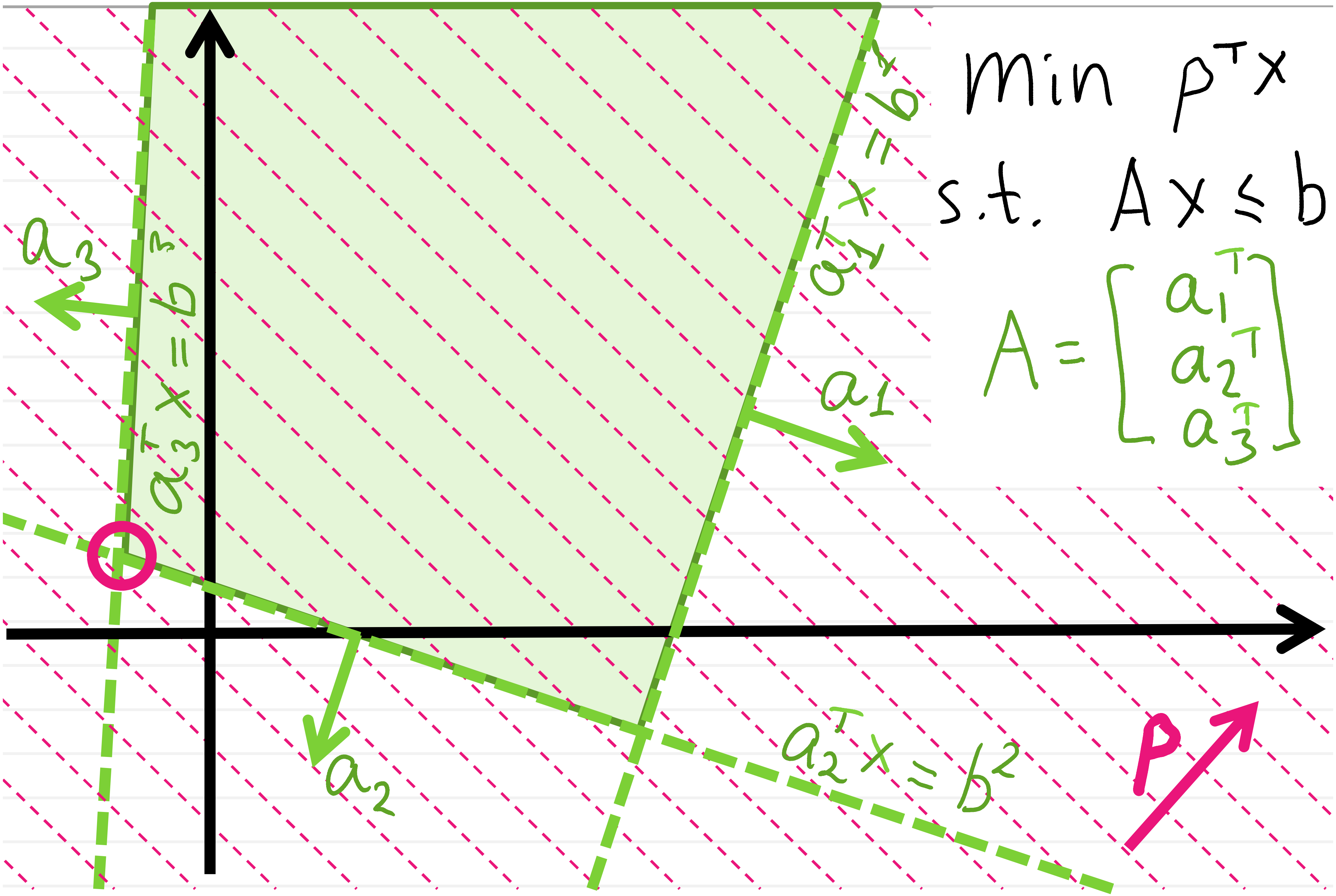
$z_s^t = z_s^{t-1} + \sum_m f_{m,s}^t - y_s^t$ coal balance

$0 \leq z_s^t \leq k_s$ coal storage bounds

$l_s \leq y_s^t \leq u_s$ power plant operation constraint

$\sum_s (p_s + q_s y_s^t) \geq e^t$ meeting the demand

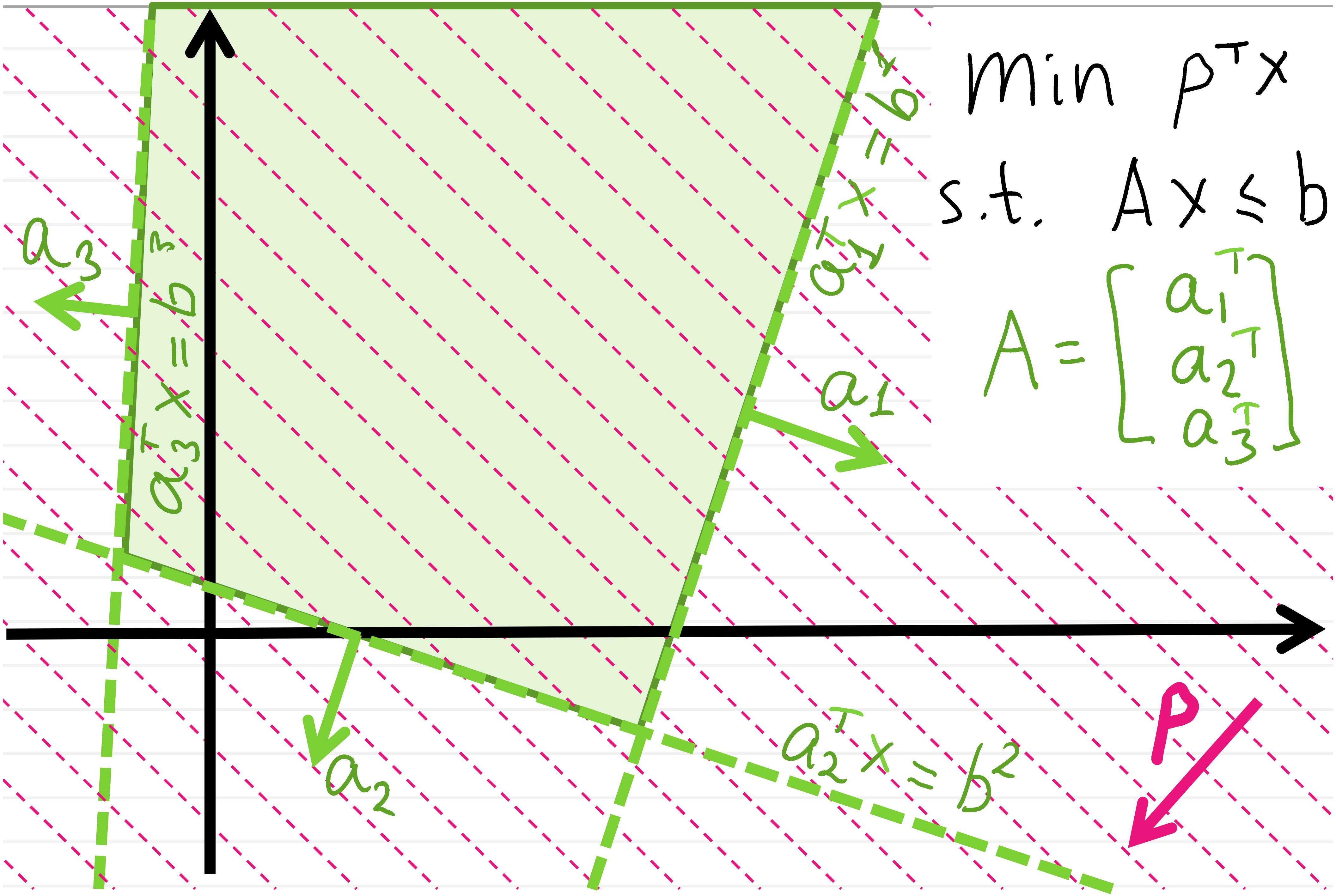
Overhead view



$$\min\ p^T x$$
$$\text{s.t. } Ax \le b$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$$

Overhead view



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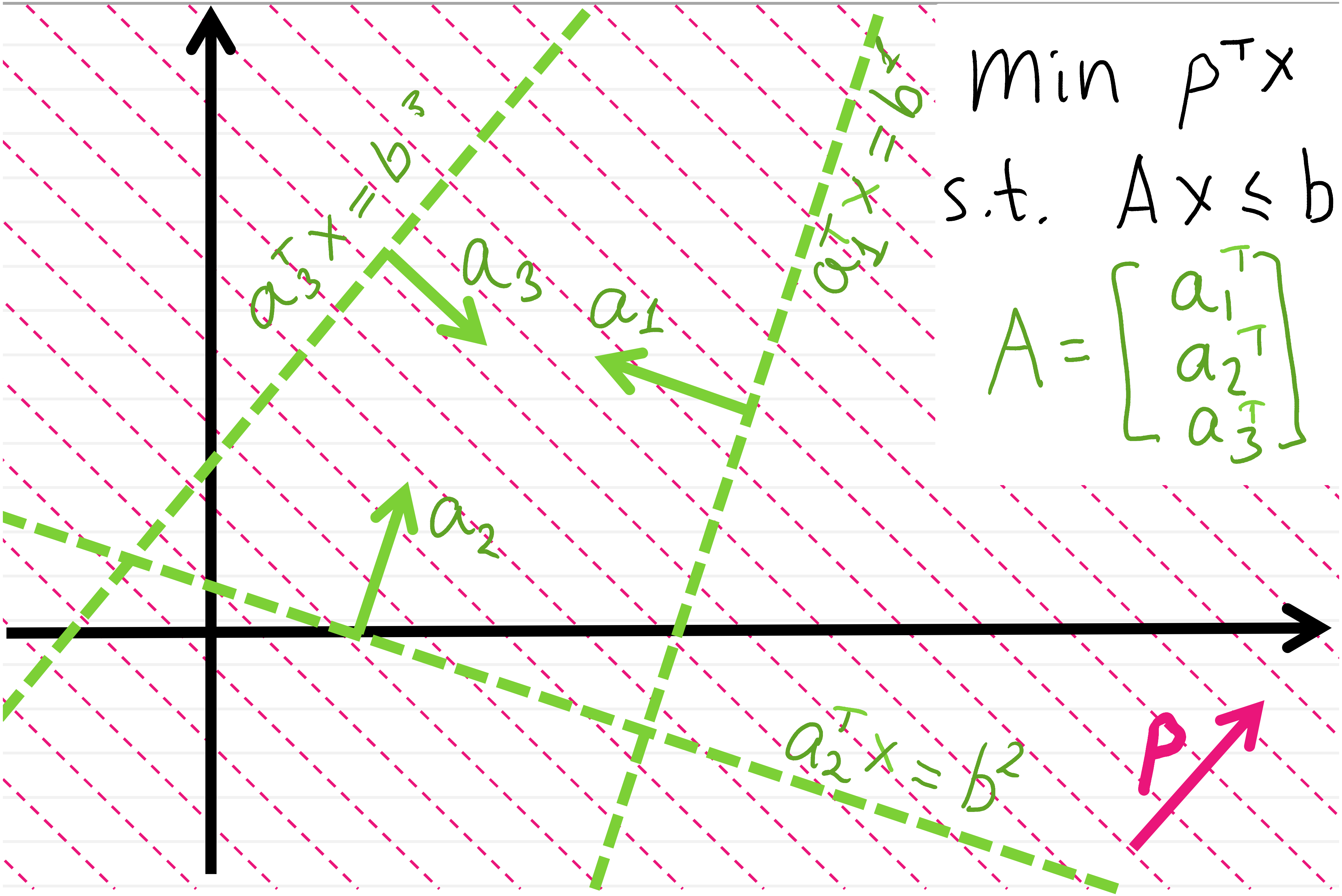

$$\min p^T x$$
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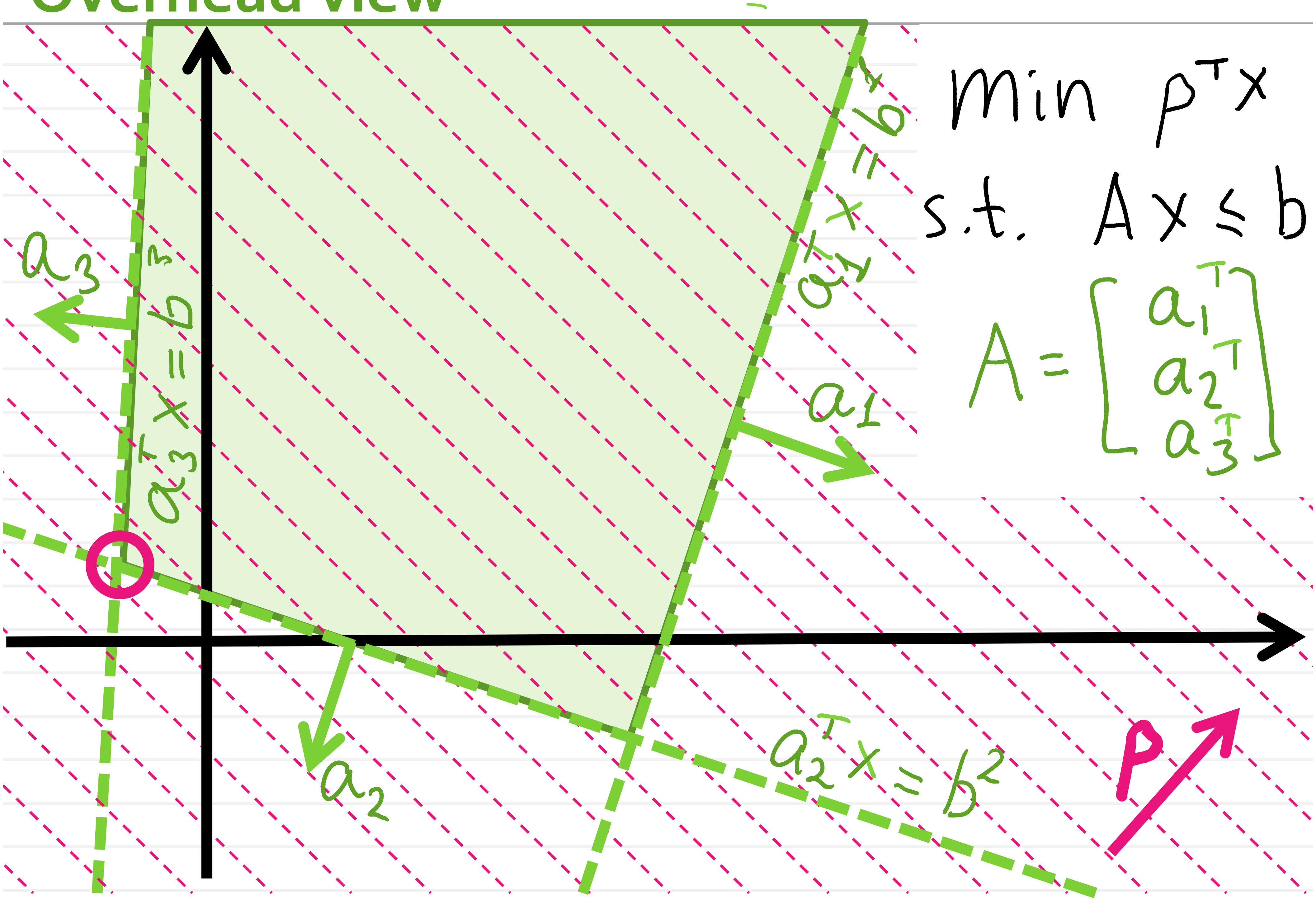
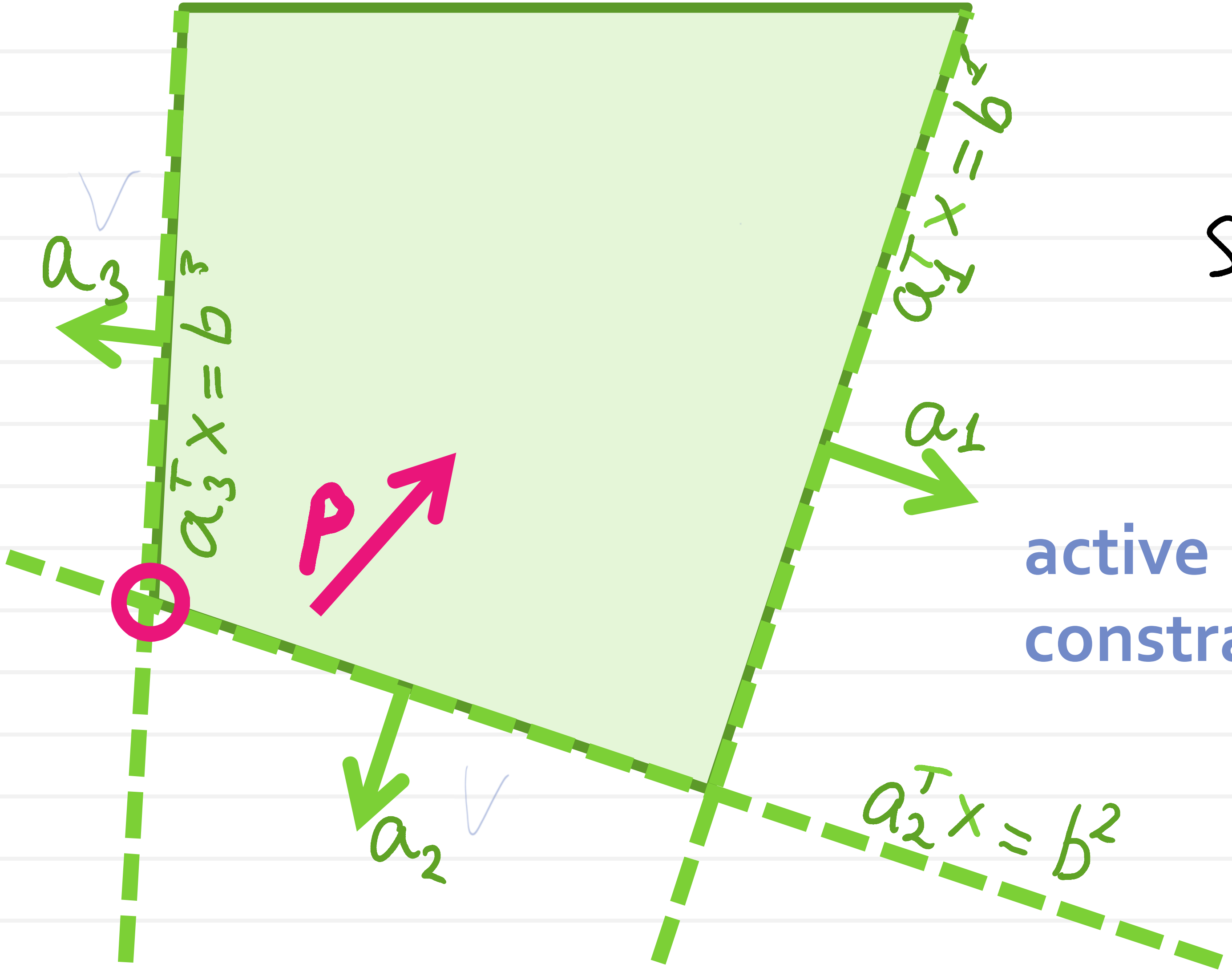
Overhead view



$$\min \ p^T x$$
$$\text{s.t. } Ax \le b$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$$

Optimality and active constraints



$$\min \, p^T x$$
$$\text{s.t. } Ax \le b$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$$

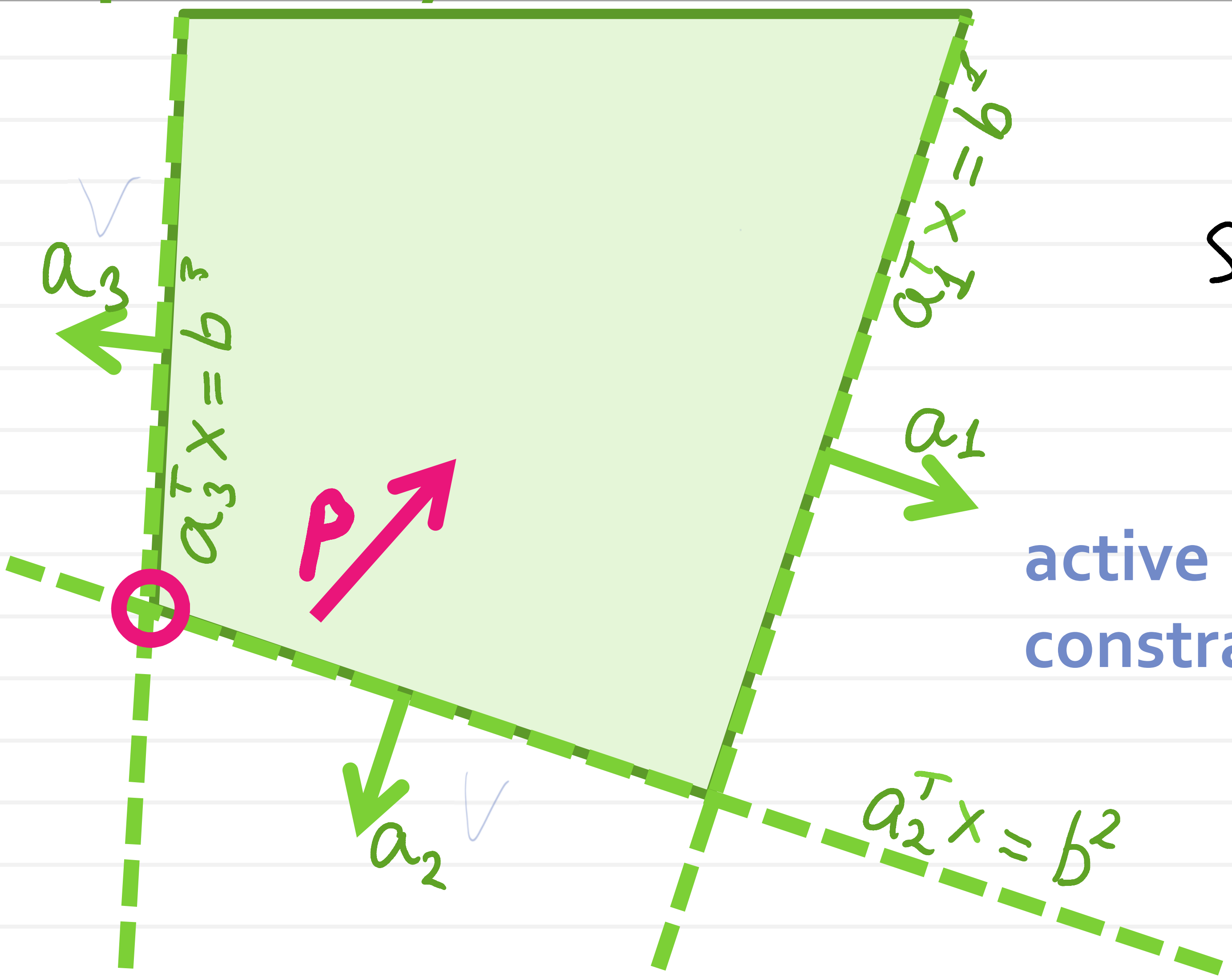
active constraints

$$\bar{A} = \begin{bmatrix} a_2^T \\ a_3^T \end{bmatrix}$$

$$\mathrm{rk}\,\bar{A} = 2$$

Corollary 1: if LP is feasible and bounded, then there exist an optimal point with full rank $\bar{A}$.

Optimality and active constraints



$$\min p^T x \quad \text{s.t.} \quad Ax \le b$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}$$

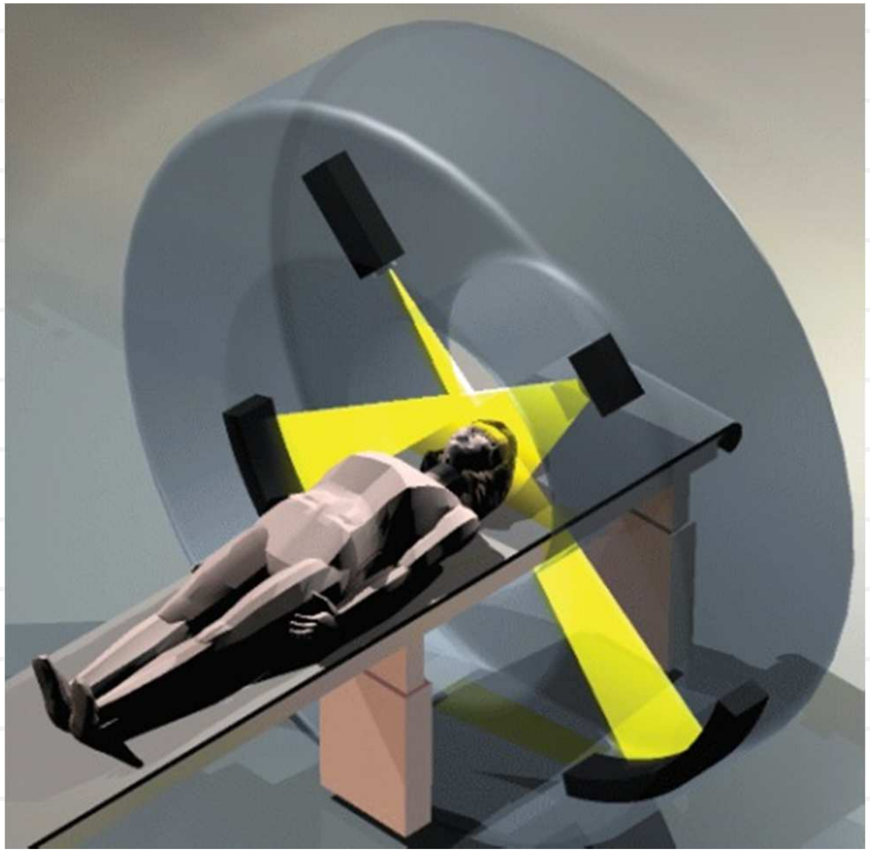
active constraints

$$\bar{A} = \begin{bmatrix} a_2^T \\ a_3^T \end{bmatrix}$$

$$\operatorname{rk} \bar{A} = 2$$

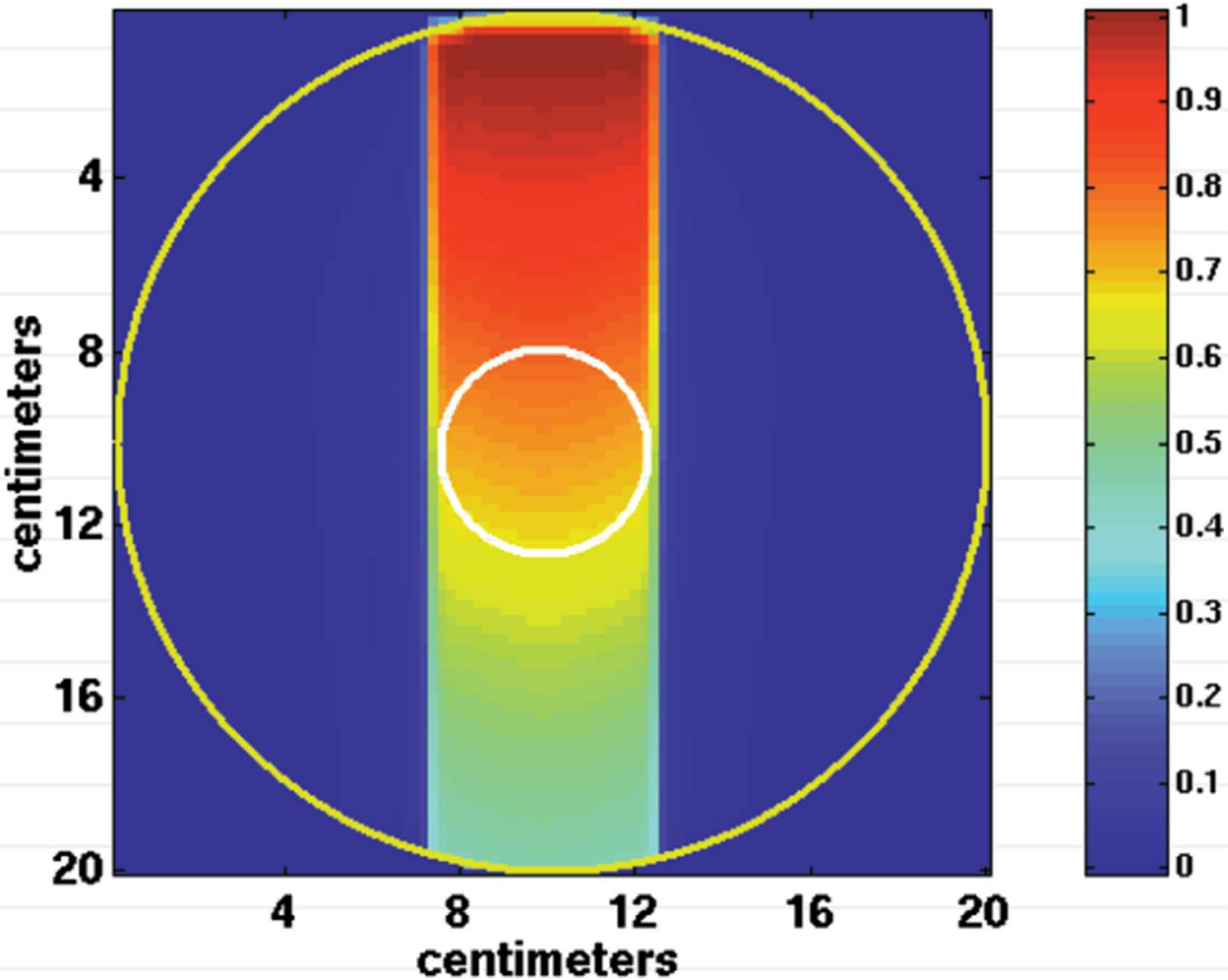
Corollary 2: if x is feasible w.r.t. the polytope $Ax \le b$ and its rank of $\bar{A}$ is full, then there exist p, so that x is optimal.

Radiation treatment



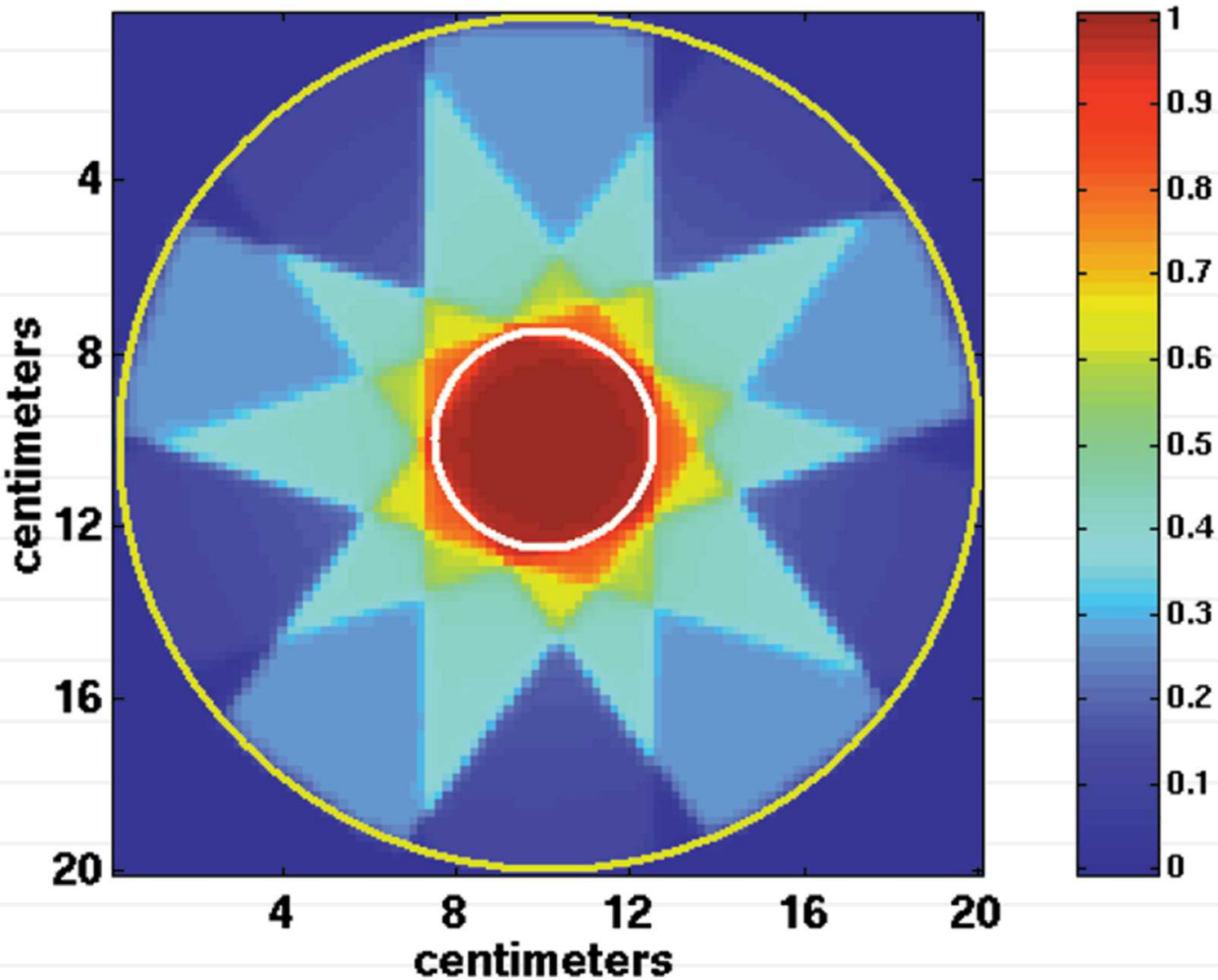
- One of the main cancer treatment therapies (600,000+ a year in the US)
- Multiple beams of radiation
- Radiation kills both healthy and cancer cells (healthy cell have better self-repair)

Radiation intensity



[Shepard et al. Optimizing the delivery of radiation therapy to cancer patients. SIAM Rev'99]

Combining beams



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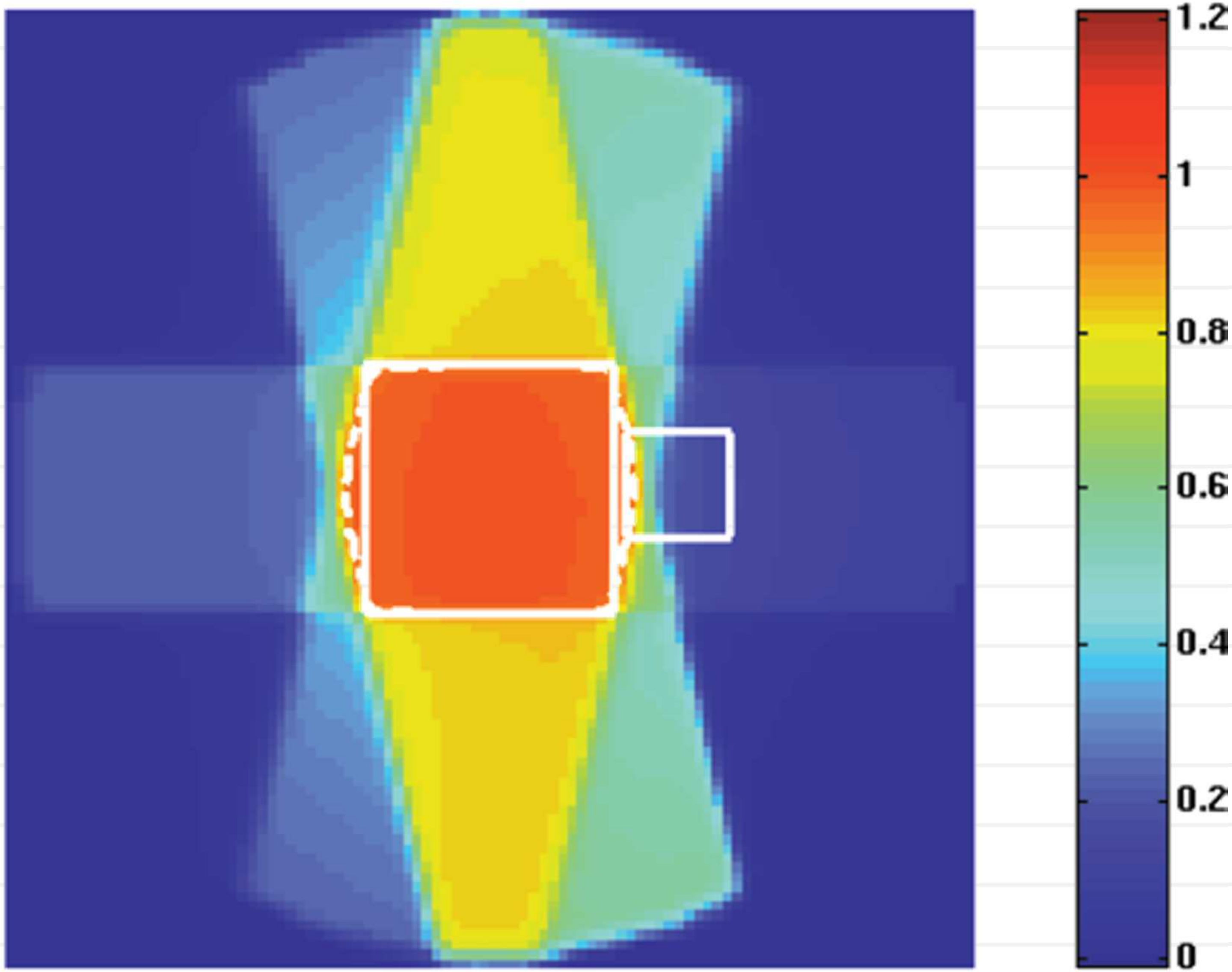
Radiation intensity



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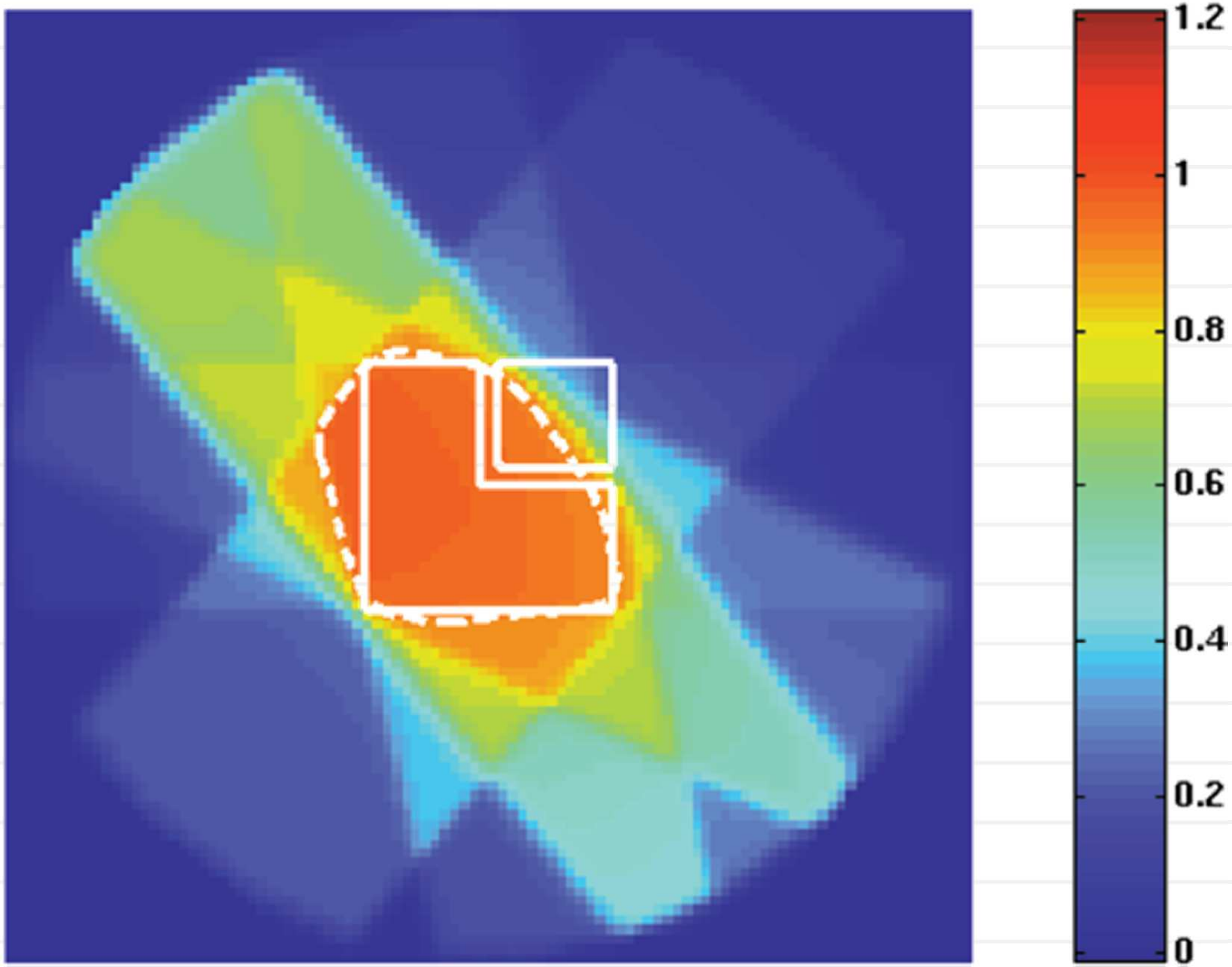
Radiation intensity



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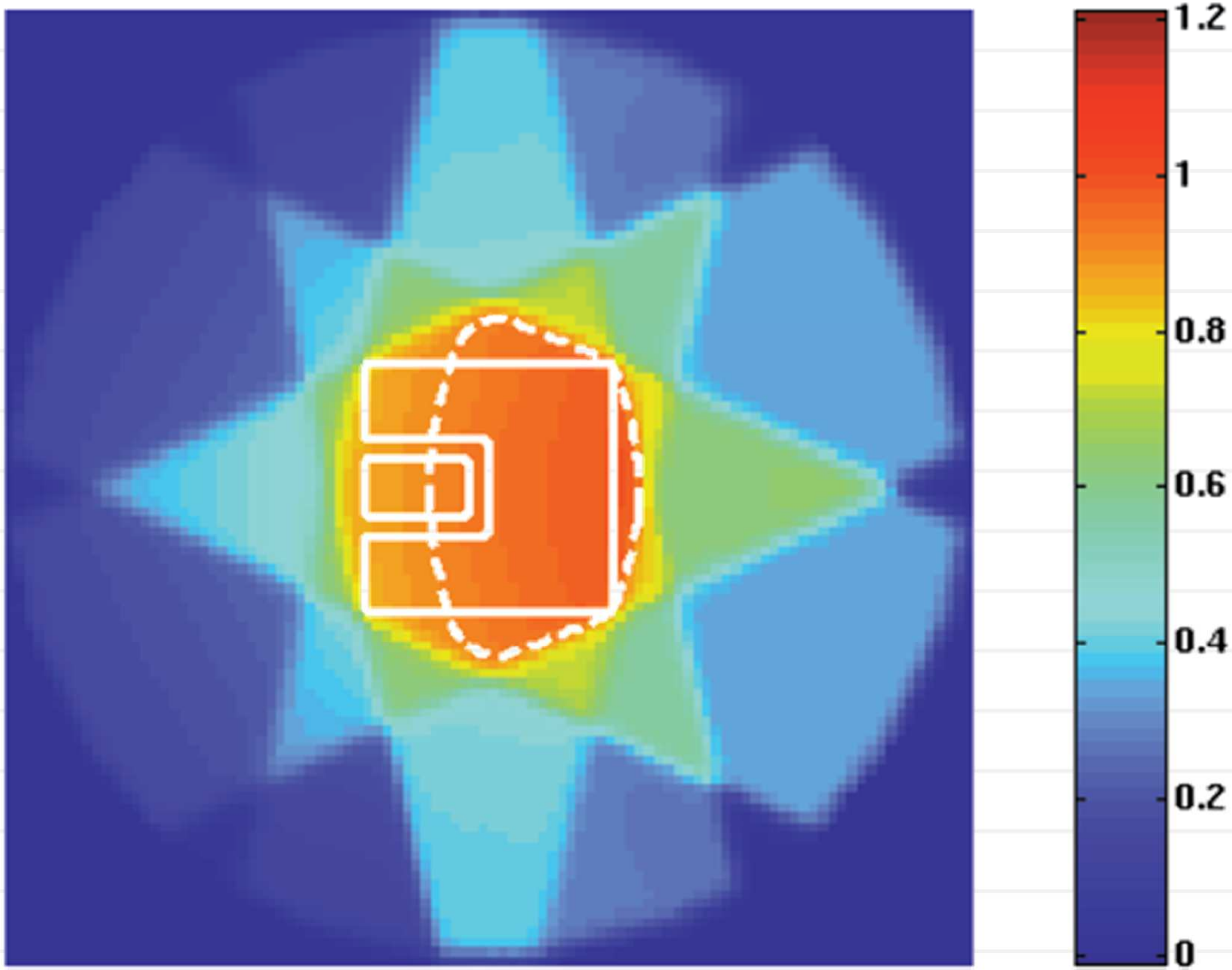
Radiation intensity



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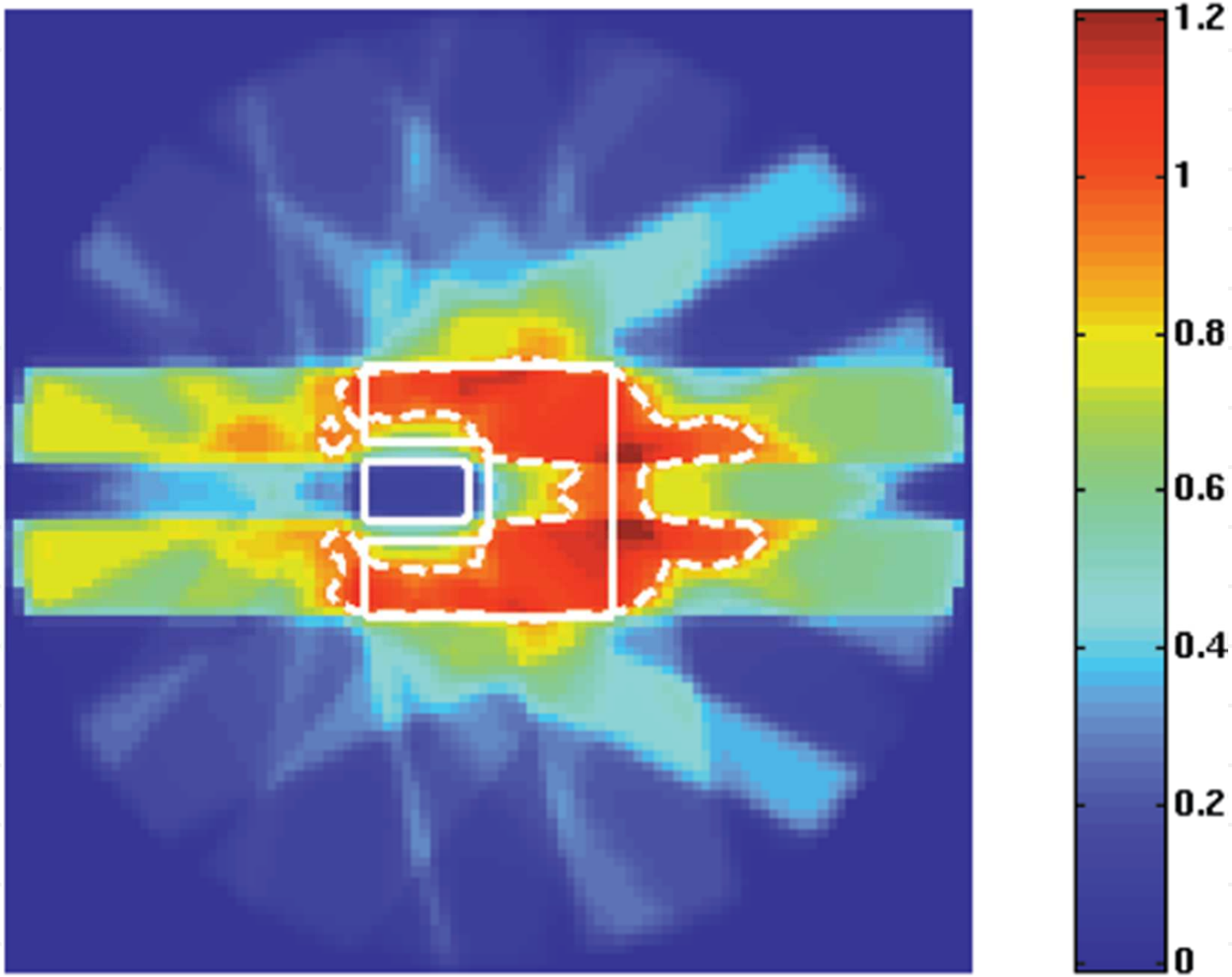
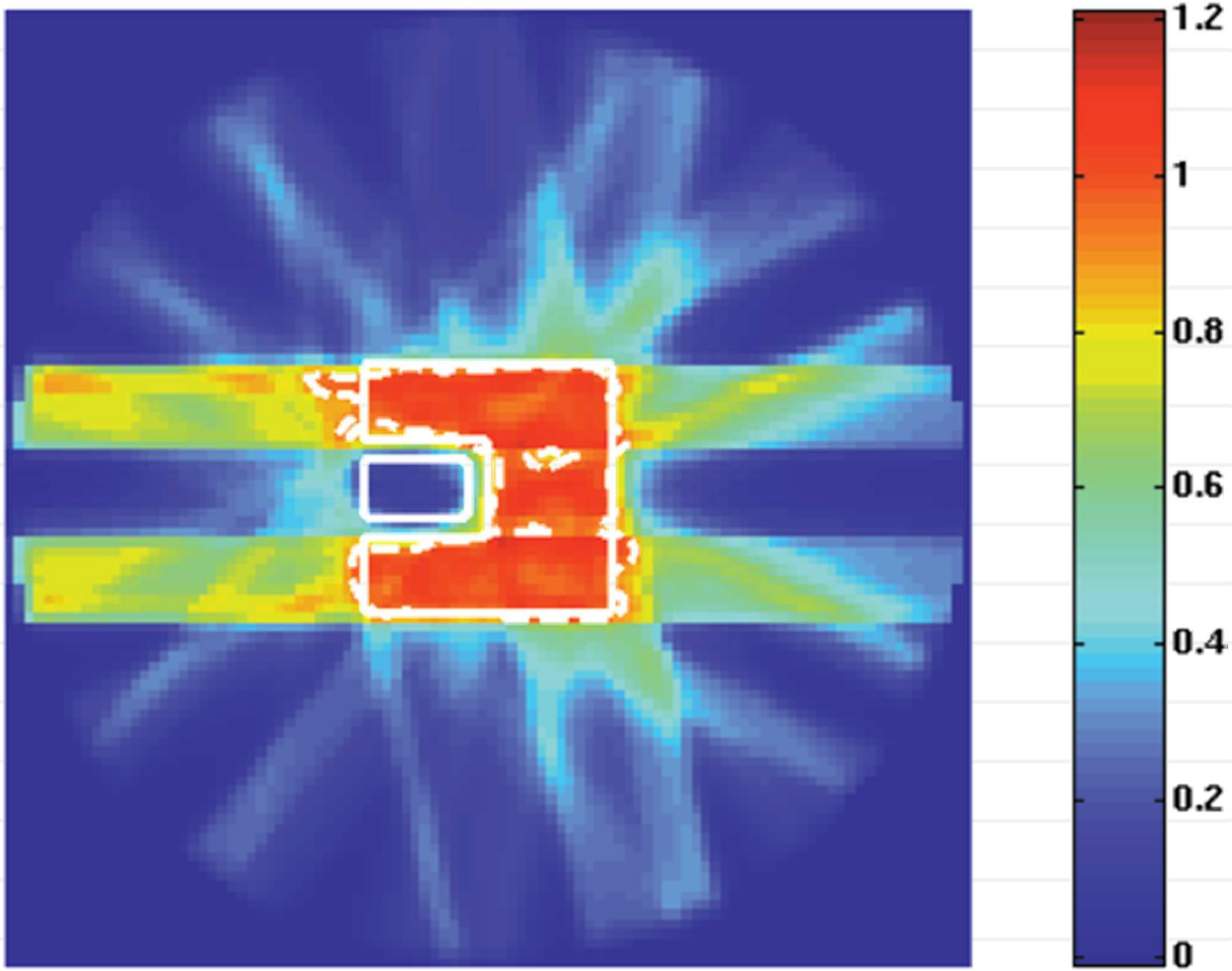
Segmented beam delivery



[Shepard et al. Optimizing the delivery of radiation therapy to cancer patients. SIAM Rev'99]

IMRTdelivery



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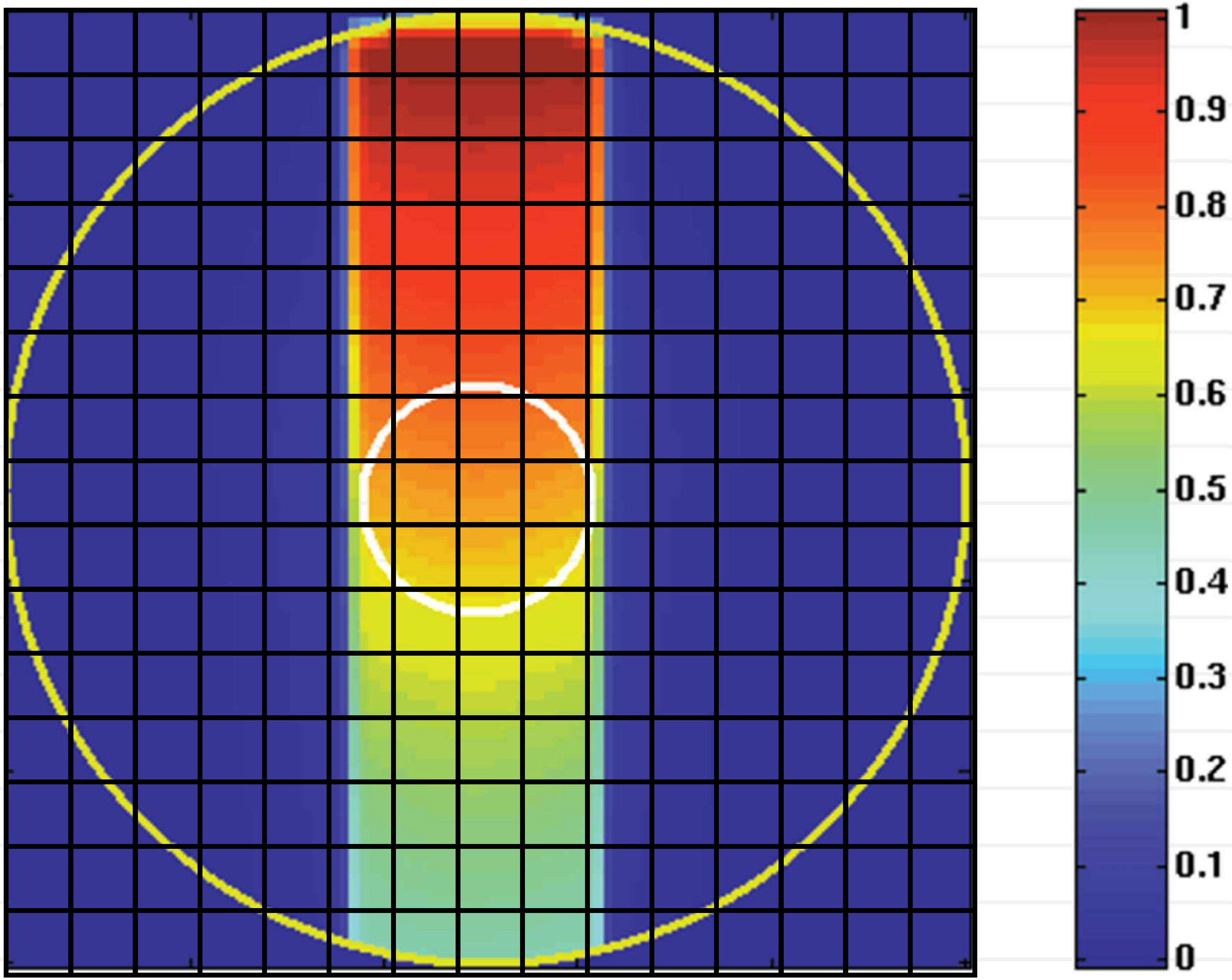
LP formulation



d_{ip} - amount of radiation (in Gy) delivered by beam *i* to cell *p* if the beam is emitted at a unit power

The program

$$\min_{x,y} \sum_p y_p$$

$$\text{s.t. } \forall p \quad y_p = \sum_{i=1}^{N} d_{i,p} \cdot x_i$$

$$\forall p \in T \quad y_p \geq \Delta_T$$

minimal dose to kill cancer cells

$$\forall p \in C \quad y_p \leq \Delta_C$$

maximal dose that is safe for critical regions

$$\forall i \quad x_i \geq 0$$

The robustified program

$$\min_{x,y,\xi,\eta} \sum_p y_p + \theta_T \sum_{p\in T} \xi_p + \theta_C \sum_{p\in C} \eta_p$$

$$\text{s.t. } \forall p \quad y_p = \sum_{i=1}^{N} d_{i,p} \cdot x_i$$

$$\forall p\in T \quad y_p \geq \Delta_T - \xi_p$$

$$\forall p\in C \quad y_p \leq \Delta_C + \eta_p$$

$$\forall i \quad x_i \geqslant 0 \qquad \forall p\in T \quad \xi_p \geqslant 0$$

$$\forall p\in C \quad \eta_p \geq 0$$

Piece-wise linear costs

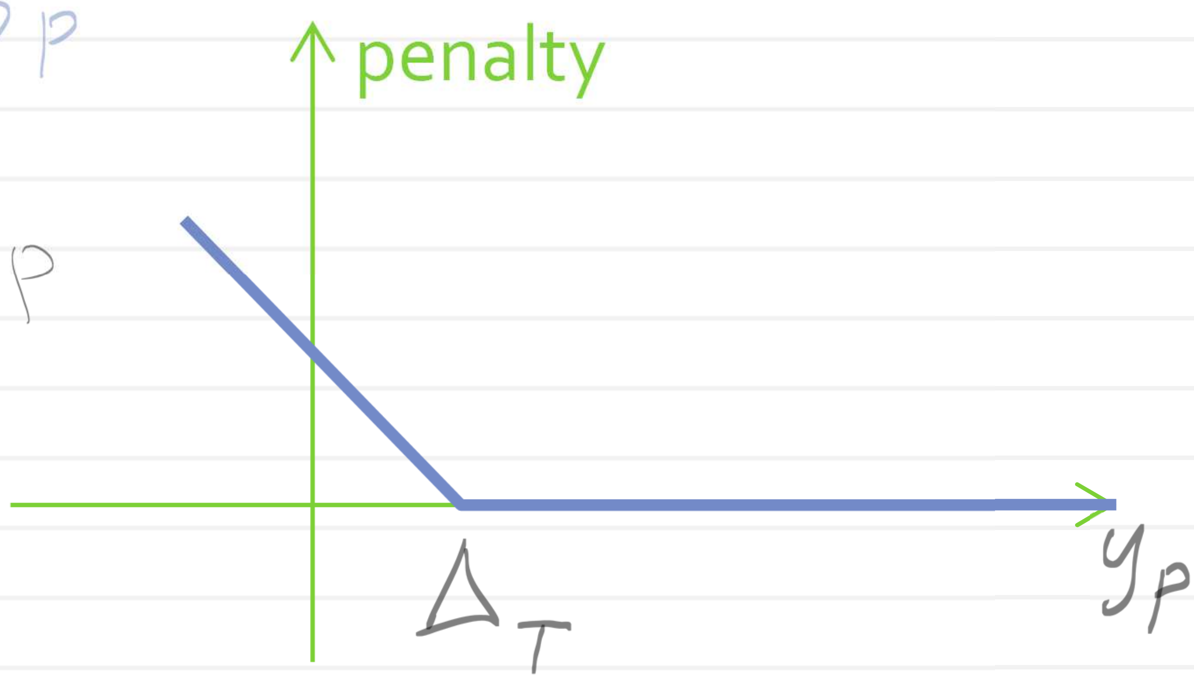
$$\min_{x,y,\xi,\eta} \sum_p y_p + \theta_T \sum_{p\in T} \xi_p + \theta_C \sum_{p \in C} \eta_p$$

$$\text{s.t.: } \ldots \quad \xi_p \geq 0$$

$$y_p \geq \Delta_T - \xi_p$$

$$\xi_p \geq \Delta_T - y_p$$

$$\xi_p \geq 0$$



Piece-wise linear costs

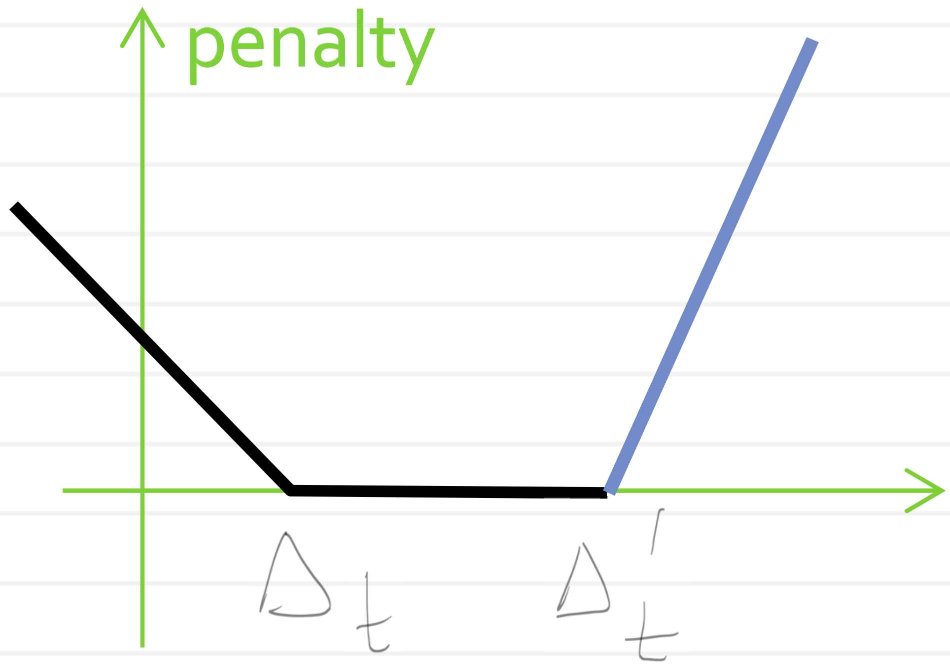
$$\min_{x,y,\xi,\eta} \sum_p y_p + \theta_T \sum_{p \in T} \xi_p + \theta_C \sum_{p \in C} \eta_p$$

s.t.:

$$\xi_p \geq \Delta_T - y_p$$

$$\xi_p \geq 0$$

$$\xi_p \geq 2y_p - 2\Delta'_t$$



LP for piecewise-linear functions

$$\min_x \sum_i g_i(x)$$

$$\text{s.t. } Ax \le b$$

$$Cx = d$$

$$h(x) \le 0$$

$$g_i(x) = \max_{j=1}^{n_i} (a_{i,j}^T x + a_{i,j}^0)$$

$$h(x) = \max_j a_{h,j}^T x + a_{h,j}^0$$

Equivalent LP:

$$\min_{x,\zeta} \sum_i \zeta_i$$

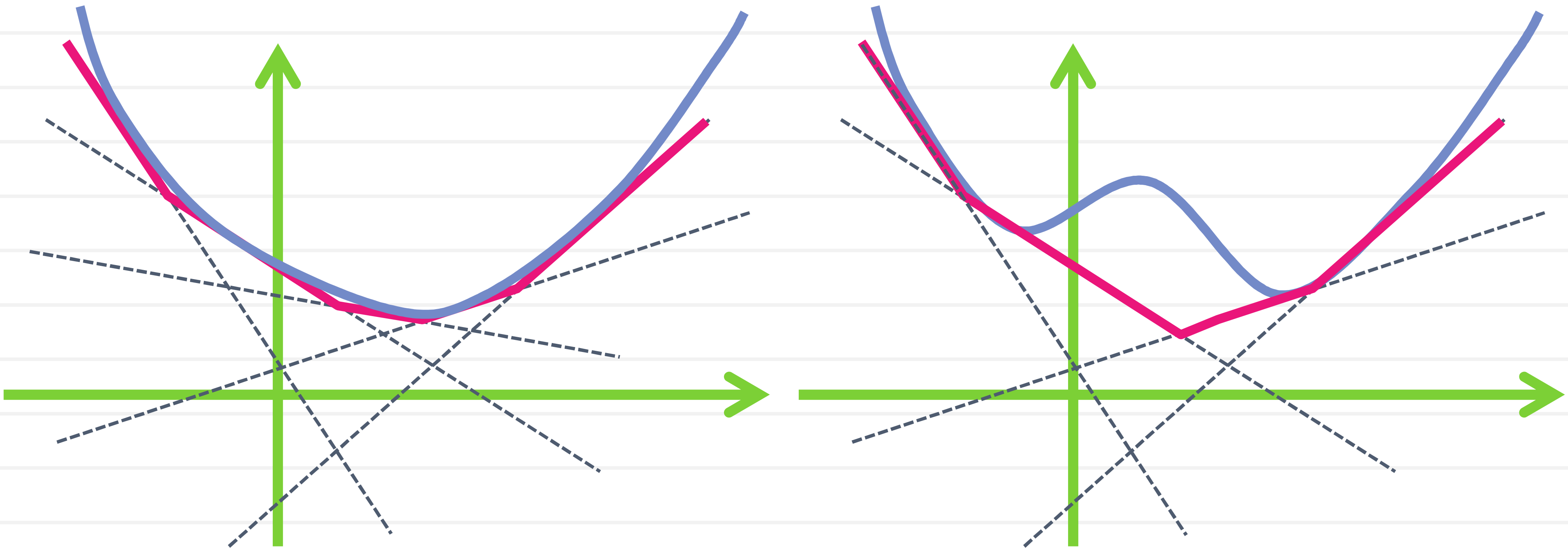
$$\text{s.t. } Ax \le b$$

$$Cx = d$$

$$\forall i,j \quad \zeta_i \ge a_{i,j}^T x + a_{i,j}^0$$

$$\forall j \quad a_{h,j}^T x + a_{h,j}^0 \le 0$$

Functions and upper envelopes



- f is convex iff it can be represented as an upper envelope of linear functions
- In high dimensions, the approximation may require too many functions
- More efficient convex optimizers (not reducing to LP) exist