Lecture 6: The simplex method

Variants of simplex algorithm

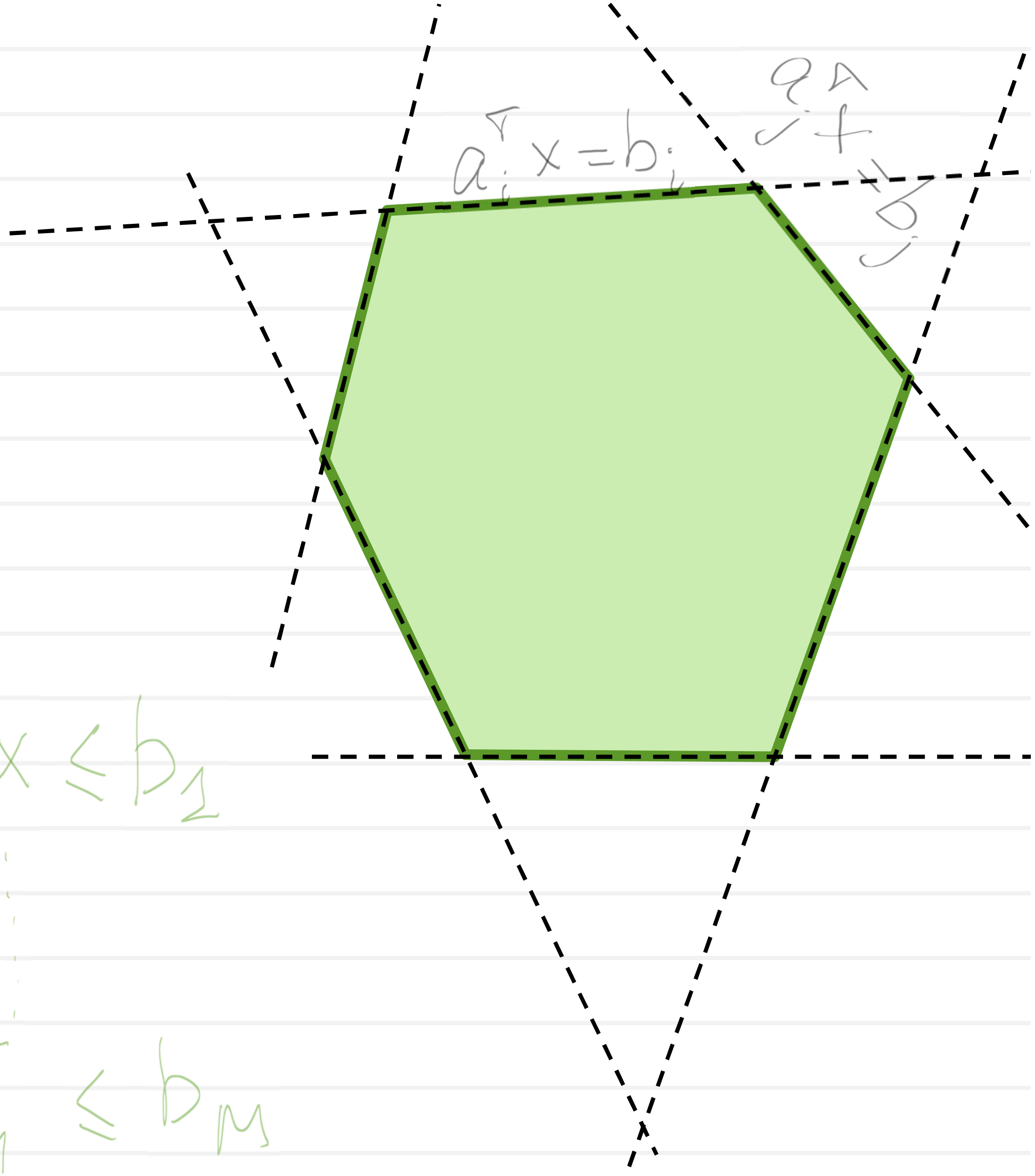
1. LP in inequality form (allows geometric intuition) – "Simplex-a"

2. LP in standard form (most implementations, most textbooks, Wikipedia) – "Simplex-b"

3. *(next lecture)* an idea of Simplex for *network LPs*

Simplex-a: setting

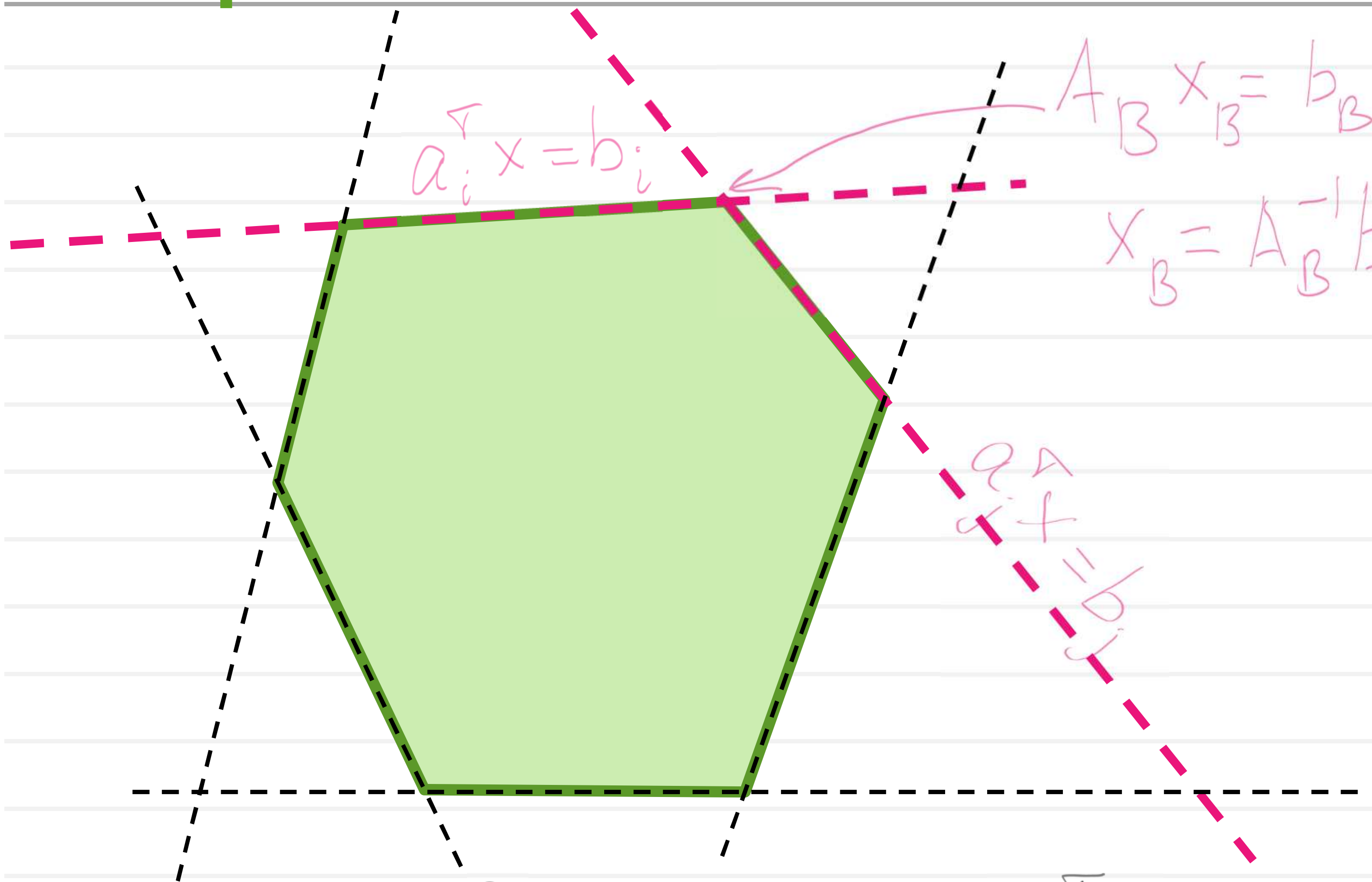
$$\min c^T x$$
$$\text{s.t. } Ax \le b$$



$x \in \mathbb{R}^n$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_M^T \end{bmatrix} \qquad \begin{cases} a_1^T x \le b_1 \\ \vdots \\ a_M^T \le b_M \end{cases}$$

Simplex-a: basis



$$A_B x_B = b_B$$

$$x_B = A_B^{-1} b_B$$

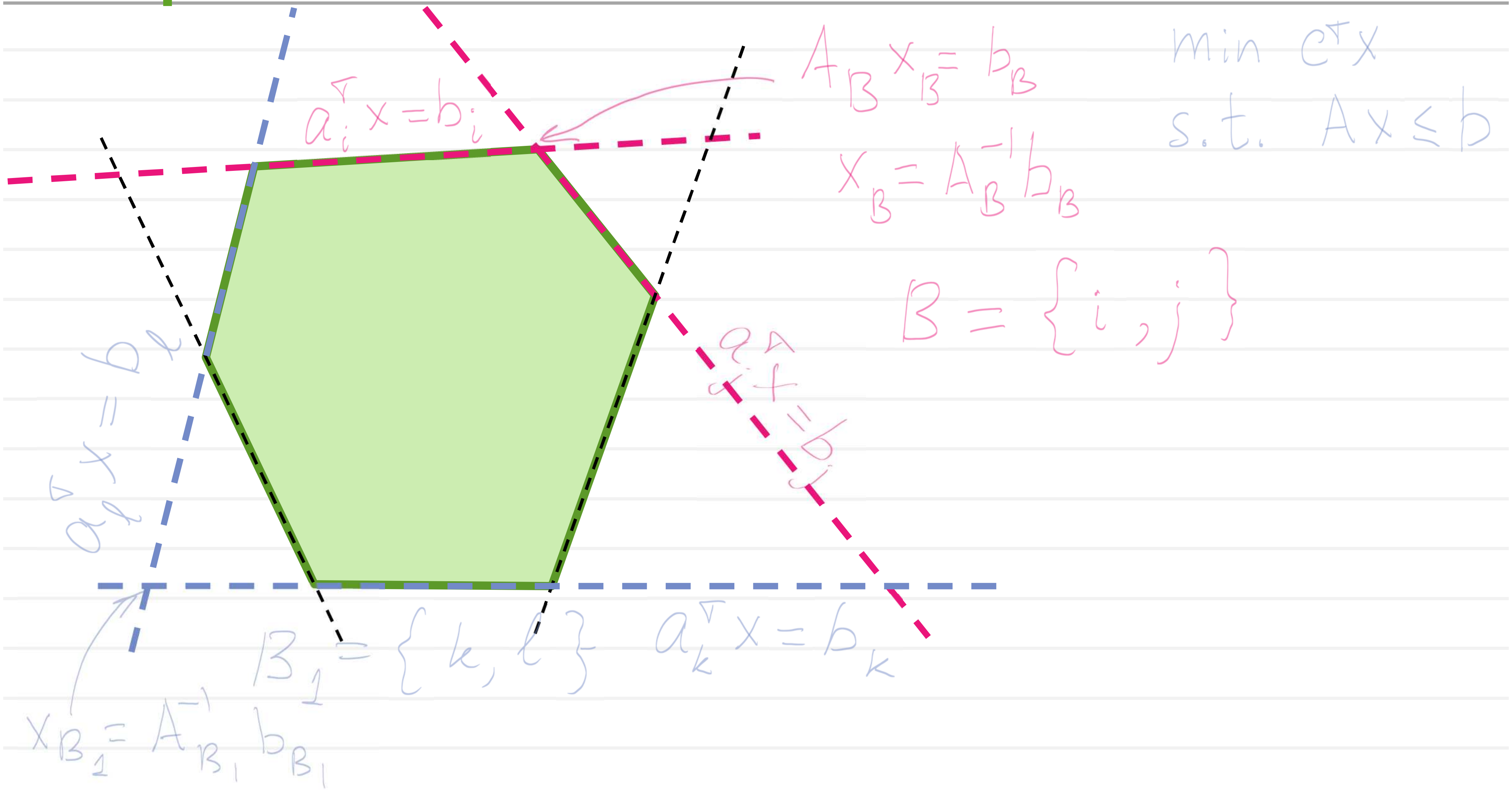
$$\min c^T x \quad \text{s.t. } Ax \le b$$

$$A = \begin{bmatrix} \vdots \\ a_i^T \\ \vdots \\ a_j^T \\ \vdots \end{bmatrix}$$

$$B = \{i, j\} \qquad A_B = \begin{bmatrix} a_i^T \\ a_j^T \end{bmatrix} \qquad b_B = \begin{bmatrix} b_i \\ b_j \end{bmatrix}$$

Definition: a *basis B* is a subset of n (integer) numbers between 1 and M, so that *rk* $A_B = n$.

Simplex-a: feasible basis



Definition: a basis B is *feasible* if x_B is feasible, i.e. $Ax_B \le b$.

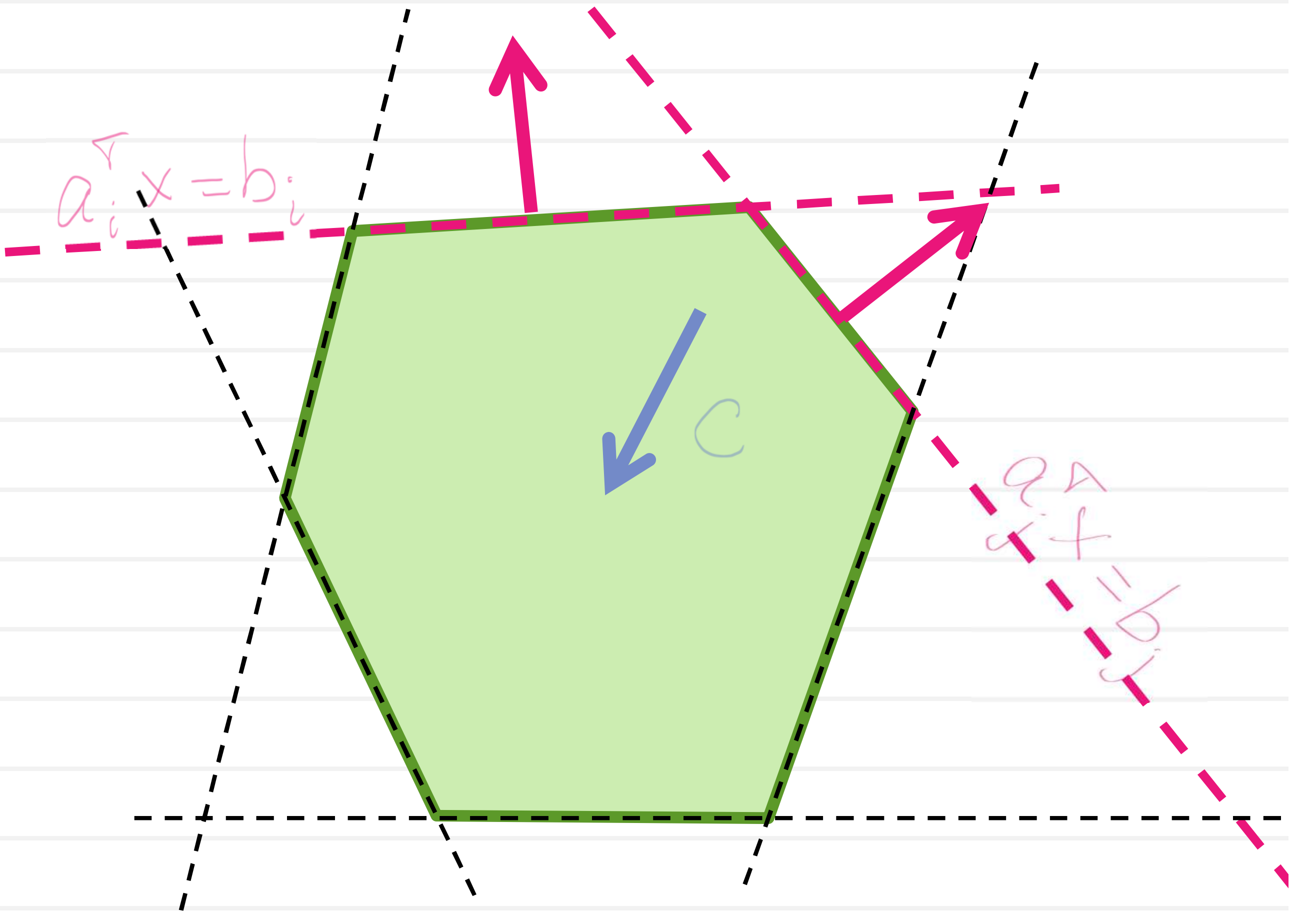
Simplex-a: optimal basis

Definition: a *basis B* is optimal if x_B is an optimum of the LP.

$$\min c^T x$$
$$\text{s.t. } Ax \le b$$

$$\lambda_B^T A_B = c^T$$

$$\lambda_B^T = c^T A_B^{-1}$$



Corollary: if all $\lambda_B \le 0$ and B is feasible then B is optimal.

Simplex-a: optimal basis

$$\lambda_B^T A_B = c^T \qquad \lambda_B^T = c^T A_B^{-1}$$

Corollary: if all $\lambda_B \leq 0$ and B is feasible then B is optimal.

Proof: assume $\exists x^* : Ax^* \leq b \quad c^T x^* < c^T x_B$

$$A_B x^* \leq b_B$$

$$\lambda_B^T A_B x^* \geq \lambda_B^T b_B$$

$$c^T x^* \geq \lambda_B^T A_B x_B$$

$$c^T x^* \geq c^T x_B$$

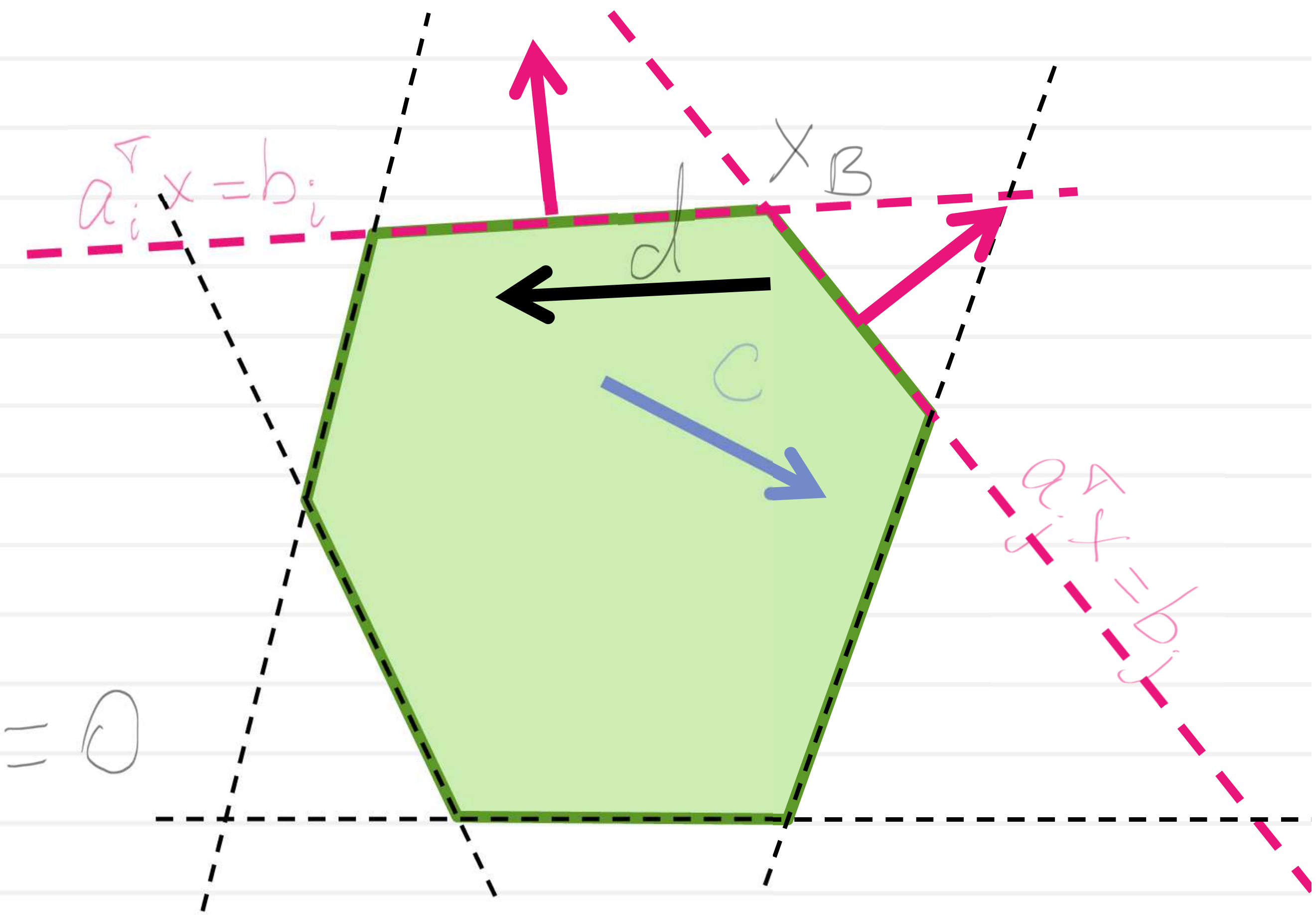
Simplex-a: changing basis

$$\lambda_B^T A_B = c^T$$

$$\lambda_B^T = c^T A_B^{-1}$$

$$k : \lambda_B^k > 0$$

$$\begin{cases} A_{B \setminus \{B_k\}} \cdot d = 0 \\ a_{B_k}^T \cdot d = -1 \end{cases}$$

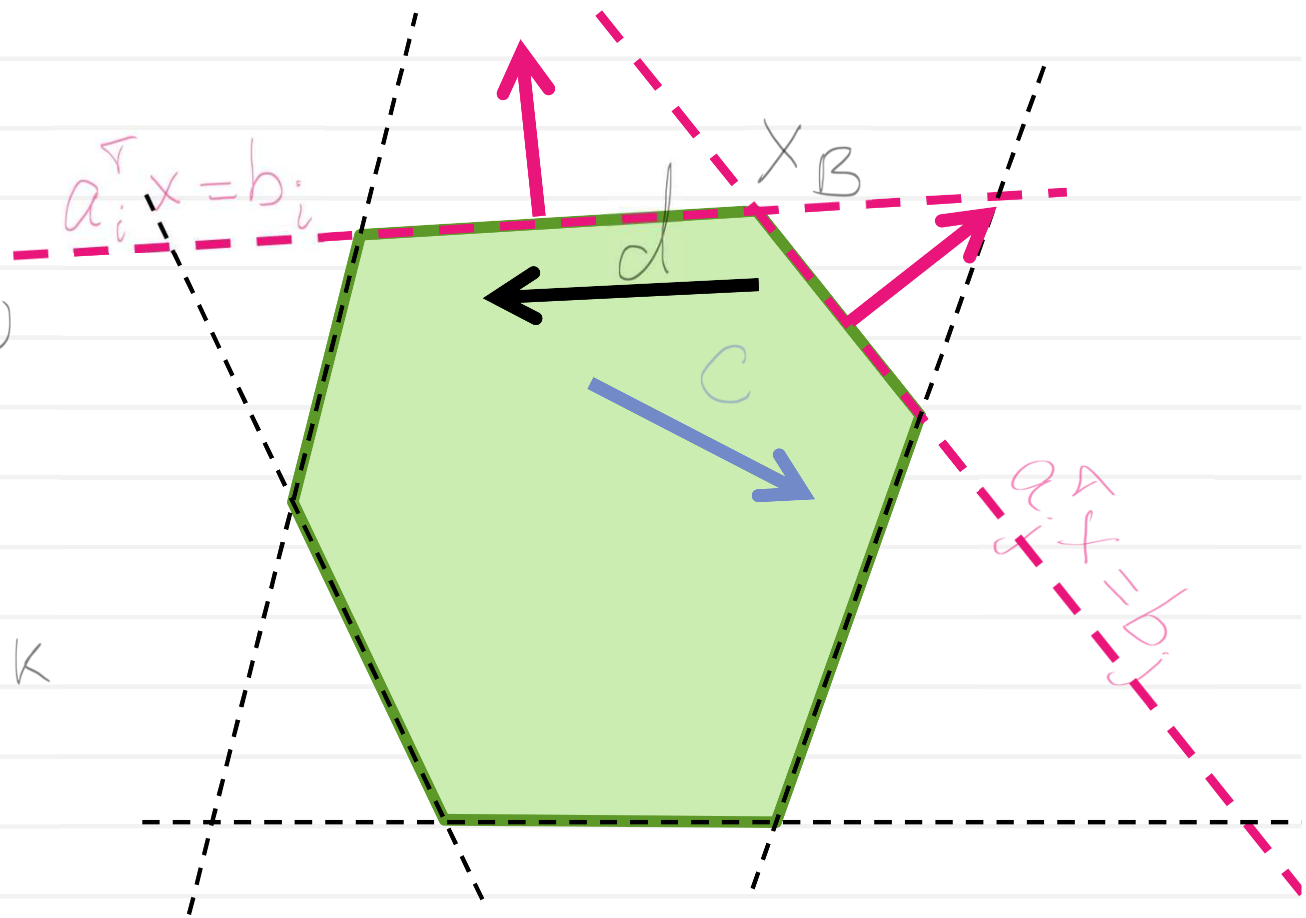


$$x_B \Longrightarrow x_B + \mu \cdot d$$

Simplex-a: changing basis

$$\lambda_B^T A_B = c^T$$

$$c^T d = \lambda_B^T \underbrace{A_B d}_{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} k}$$



$$c^T d < 0$$

$$c^T x_B \overset{>}{\Longrightarrow} c^T(x_B + \overset{\overset{0}{\wedge}}{\mu} \cdot d)$$

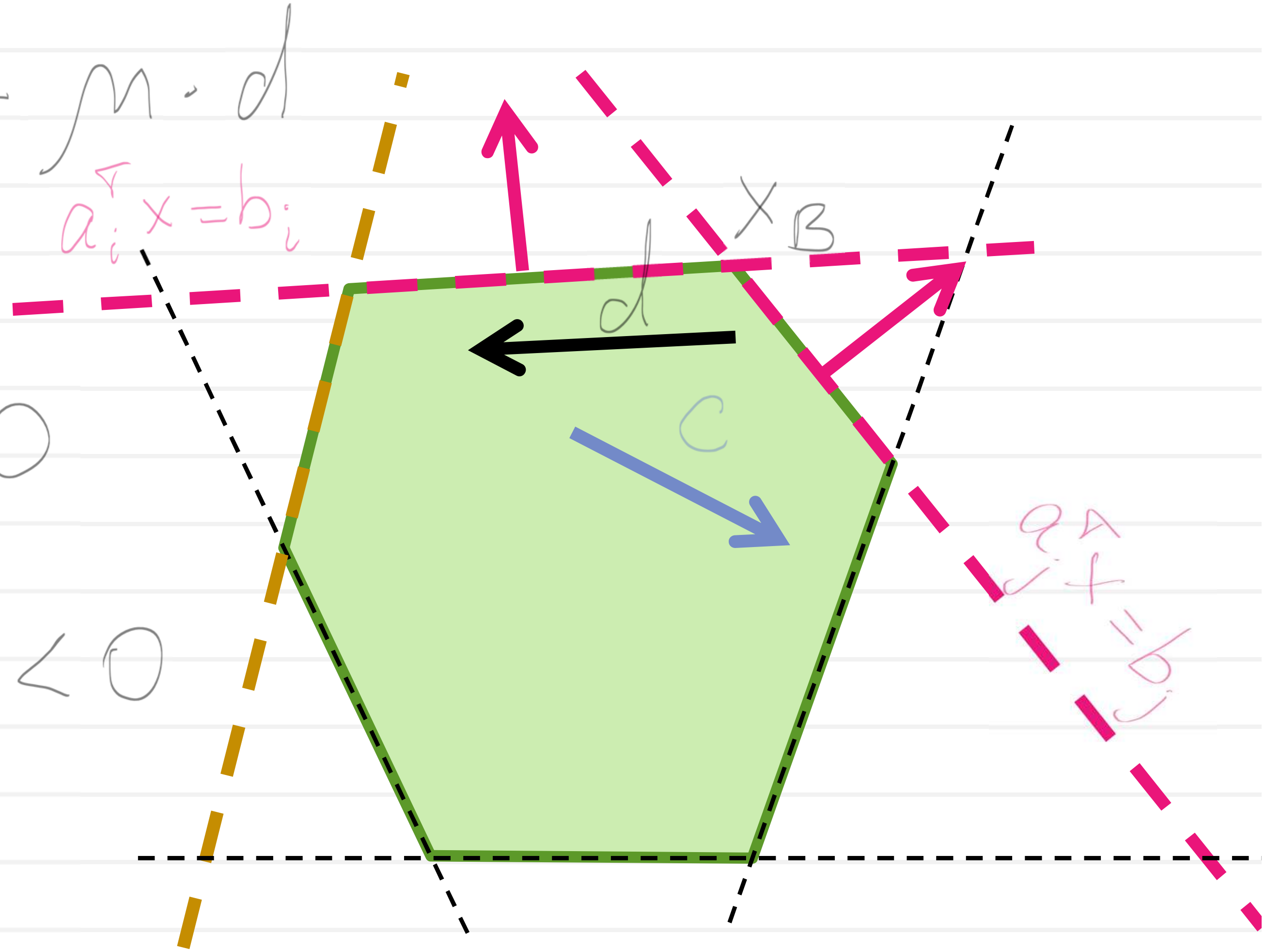
Simplex-a: changing basis

$$x_B \Longrightarrow x_B + \mu \cdot d$$

$$c^T d < 0$$

$$A_{B \setminus \{B_k\}} \cdot d = 0$$

$$a_{B_k}^T \cdot d = -1 < 0$$



$j \notin B$

$$\mu_j = \frac{b_j - a_j^T x_B}{a_j^T \cdot d}$$

How far can we go, till we hit the constraint

Simplex-a: changing basis

$$x_B \Longrightarrow x_B + \mu \cdot d$$

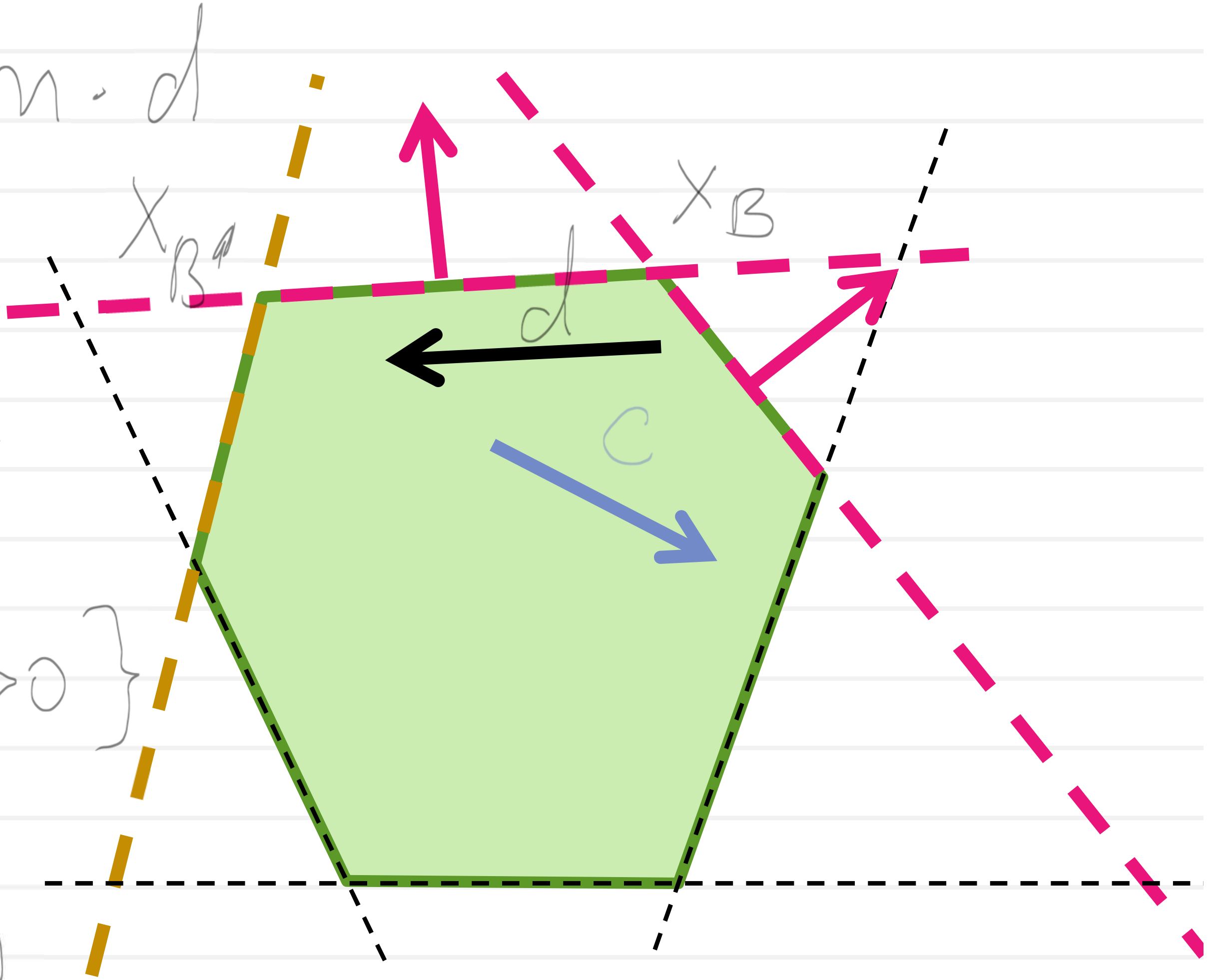
$$j \notin B$$

$$\mu_j = \frac{b_j - a_j^T x_B}{a_j^T \cdot d}$$

$$t = \operatorname{argmin}\{\mu_j \mid \mu_j > 0\}$$

$$B' := B \setminus \{B_k\} \cup \{t\}$$

$$x_{B'} = x_B + \mu_t d = A_{B'}^{-1} \cdot b_{B'}$$



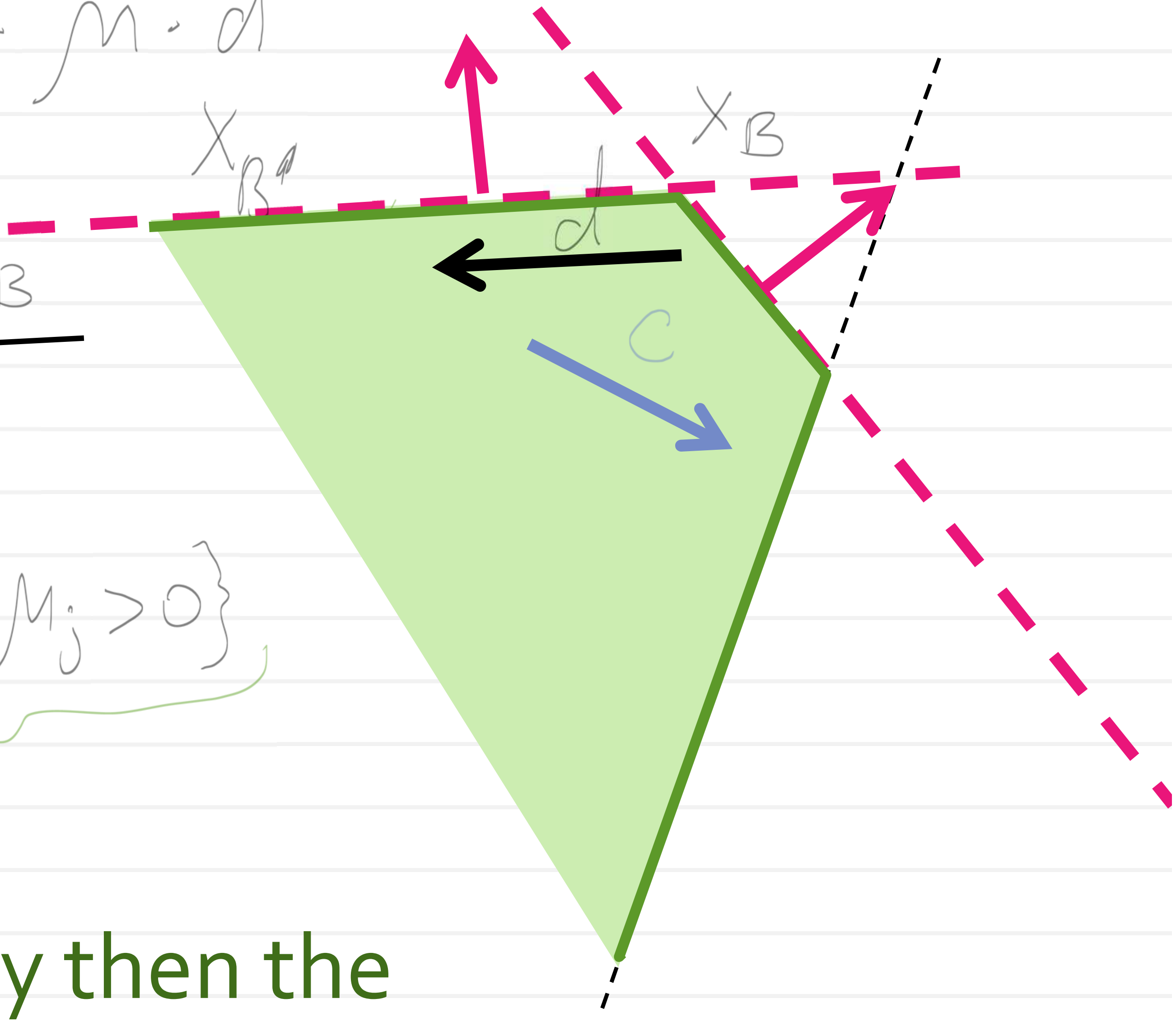
Simplex-a: unbounded case

$$x_B \Longrightarrow x_B + \mu \cdot d$$

$j \notin B$

$$\mu_j = \frac{b_j - a_j^T x_B}{a_j^T \cdot d}$$

$$t = \operatorname{argmin} \underbrace{\{\mu_j \mid \mu_j > 0\}}$$



If this set is empty then the LP is unbounded

Simplex-a algorithm: outline

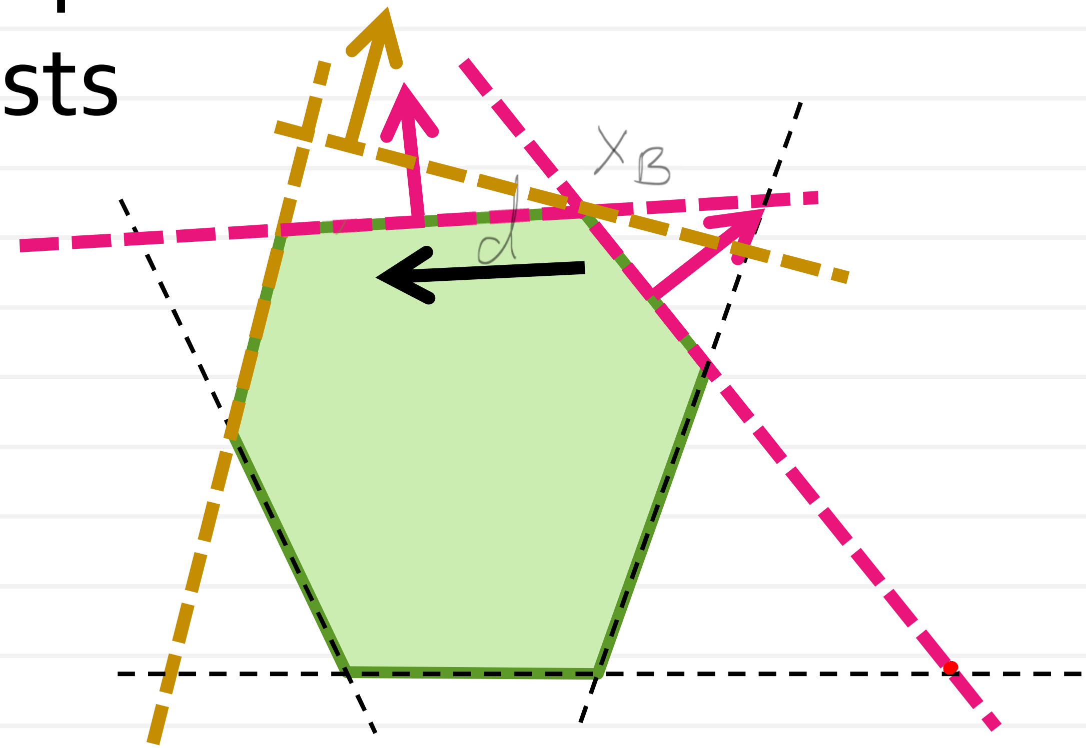
Initialize the feasible basis (TODO)

Iterate

- Solve $A_B{}^T \lambda_B = c$
- If all λ_B are non-positive, **done (solution found)**
- Pick the most positive λ_k
- Find d, find μ
- If all μ are negative, **done (unbounded)**
- Find the smallest μ_t
- Update the basis (k exists, t enters)

Simplex-a: convergence

- There is a finite number of vertices.
- We improve by $\mu_t c^T d$ on each iteration. This is positive if no degeneracy exists



- The convergence may take an exponential (in N) number of steps

Initializing the simplex-a method

- We have now discussed how the simplex method can proceed given that it is already at the corner of the domain (i.e. given a feasible basis)
- But how do we get the initial point?
- Idea: solve an auxiliary (*Phase-1*) program that will give us a feasible point.

LP reformulation

Initial problem:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \le b \end{aligned}$$

New equivalent problem:

$$\begin{aligned} \min\ & c^T (y - z) \\ \text{s.t.: } & Ay - Az \le b \\ & y \ge 0 \quad z \ge 0 \end{aligned}$$

Given the solution of the top problem, the solution of the bottom problem can be recovered:

$$y_i = \max(x_i, 0) \qquad z_i = \max(-x_i, 0)$$

LP reformulation

Initial problem:

$$\min c^T x \quad \text{s.t. } Ax \le b$$

New equivalent problem:

$$\min c^T(y - z) \quad \text{s.t.: } Ay - Az \le b, \quad y \geq 0 \quad z \geq 0$$

Given the solution of the bottom problem, the solution of the first problem can be recovered:

$$x = y - z$$

Phase-1 problem for Simplex-a

Phase-2
(Main LP)

$$\min \; c^T(y - z)$$
$$\text{s.t.: } Ay - Az \le b$$
$$y \ge 0 \quad z \ge 0$$

Phase-1

$$\min \sum_{i=1}^{M} \xi_i$$
$$\text{s.t.: } Ay - Az \le b + \xi$$
$$y \ge 0, \; z \ge 0, \; \xi \ge 0$$

Statement 1: if Phase-2 (main LP) problem has a feasible solution, then Phase-1 optimum is zero (i.e. all slacks are zero). **Proof:** trivial check.

Phase-1 problem for Simplex-a

Phase-2 (Main LP)

$$\min\ c^T(y-z)$$
$$\text{s.t: } Ay - Az \le b$$
$$y \ge 0 \quad z \ge 0$$

Phase-1

$$\min \sum_{i=1}^{M} \xi_i$$
$$\text{s.t: } Ay - Az \le b + \xi$$
$$y \ge 0,\ z \ge 0,\ \xi \ge 0$$

Statement 2: if Phase-1 optimum is zero (i.e. all slacks are zero) then we get a feasible basis point for Phase-2. **Proof:** trivial check.

Solving Phase-1 problem

- Now we know that if we can solve a Phase-1 problem then we will either find a starting point for the simplex method in the original method (if slacks are zero) or verify that the original problem was infeasible (if slacks are non-zero).

- But how to solve the Phase-1 problem?

Solving Phase-1 problem for simplex-a

Phase-1

$$\min \sum_{i=1}^{M} \xi_i$$

$$\text{s.t.: } Ay - Az \le b + \xi$$

$$y \ge 0, \; z \ge 0, \; \xi \ge 0$$

Phase 1 problem has 2*N+M variables.

Here is a feasible basic solution!

$z = 0$ $y = 0$ $\xi_i = \max(0, -b_i)$

N active constraints

N active constraints

M active constraints

We can now start the simplex algorithm and solve it!

Simplex-a: our first LP solver

1. Eliminate equalities, split variables into negative and positive parts
2. Formulate Phase-1 LP
3. Find the solution of Phase-1 LP (Simplex)
4. If the objective is non-zero, then the initial LP is infeasible
5. Use the solution of Phase-1 LP to initialize the Simplex algorithm for the main LP.
6. Solve main LP (or prove unboundedness)

The standard form of LP

$$\boxed{\begin{aligned} \max\ & c^T x \\ \text{s.t.:}\ & Ax = b \\ & x \geq 0 \end{aligned}}$$

$b \geq 0$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \leq b \\ & x \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & c^T x \\ \text{s.t.:}\ & Ax + \xi = b \\ & x \geq 0,\ \xi \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.:}\ & Ax = b \end{aligned} \quad \Rightarrow \quad \begin{aligned} \max\ & c^T x^+ - c^T x^- \\ & Ax^+ - Ax^- = b \\ & x^+, x^- \geq 0 \end{aligned}$$

LP: standard form

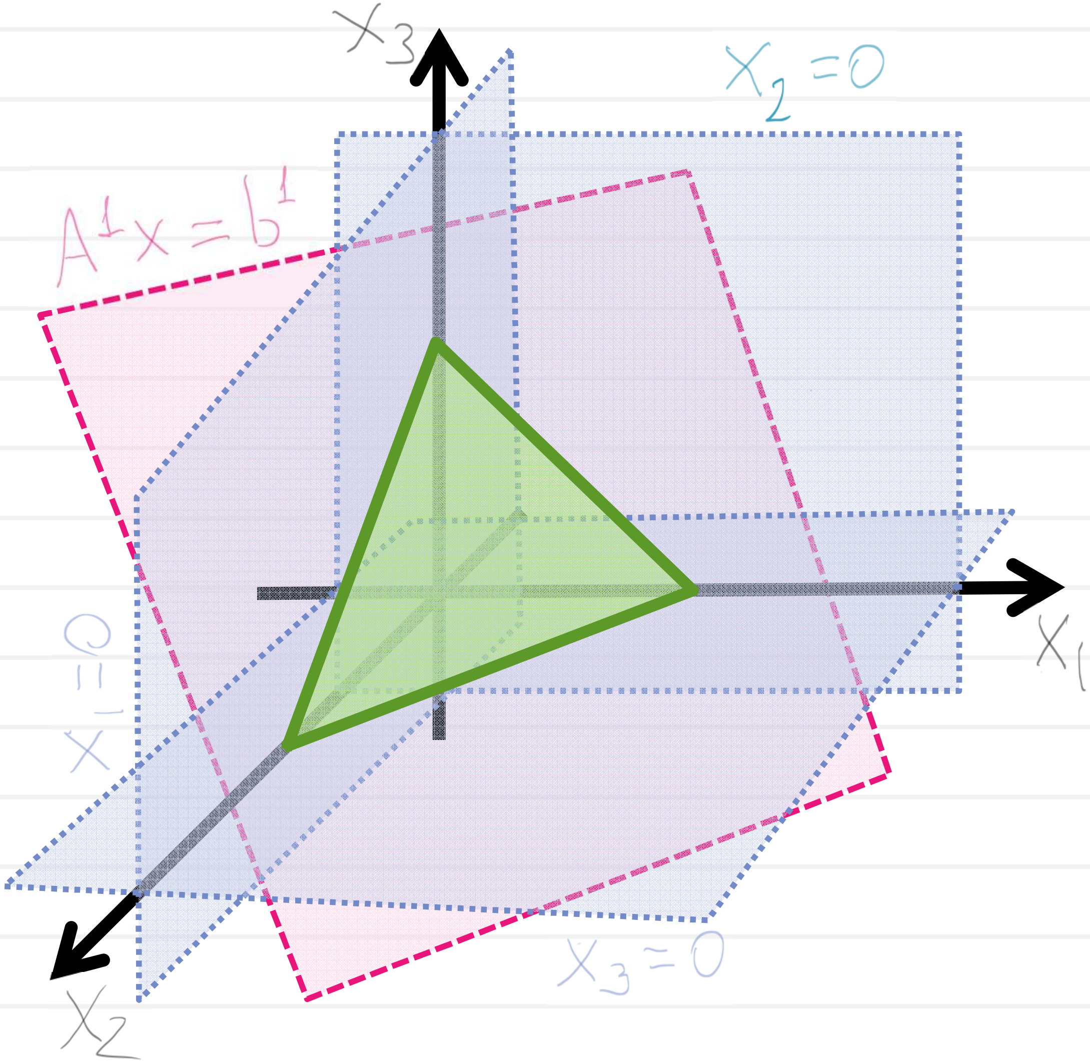
$$\min \ c^T x$$
$$\text{s.t.:} \ Ax = b$$
$$x \geq 0$$

$x \in \mathbb{R}^N \qquad c \in \mathbb{R}^N$

$A \in \mathbb{R}^{M \times N} \qquad b \in \mathbb{R}^M$

M

N



$M = 1$

$N = 3$

$A = [A_1^1 \ A_2^1 \ A_3^1]$

LP: standard form

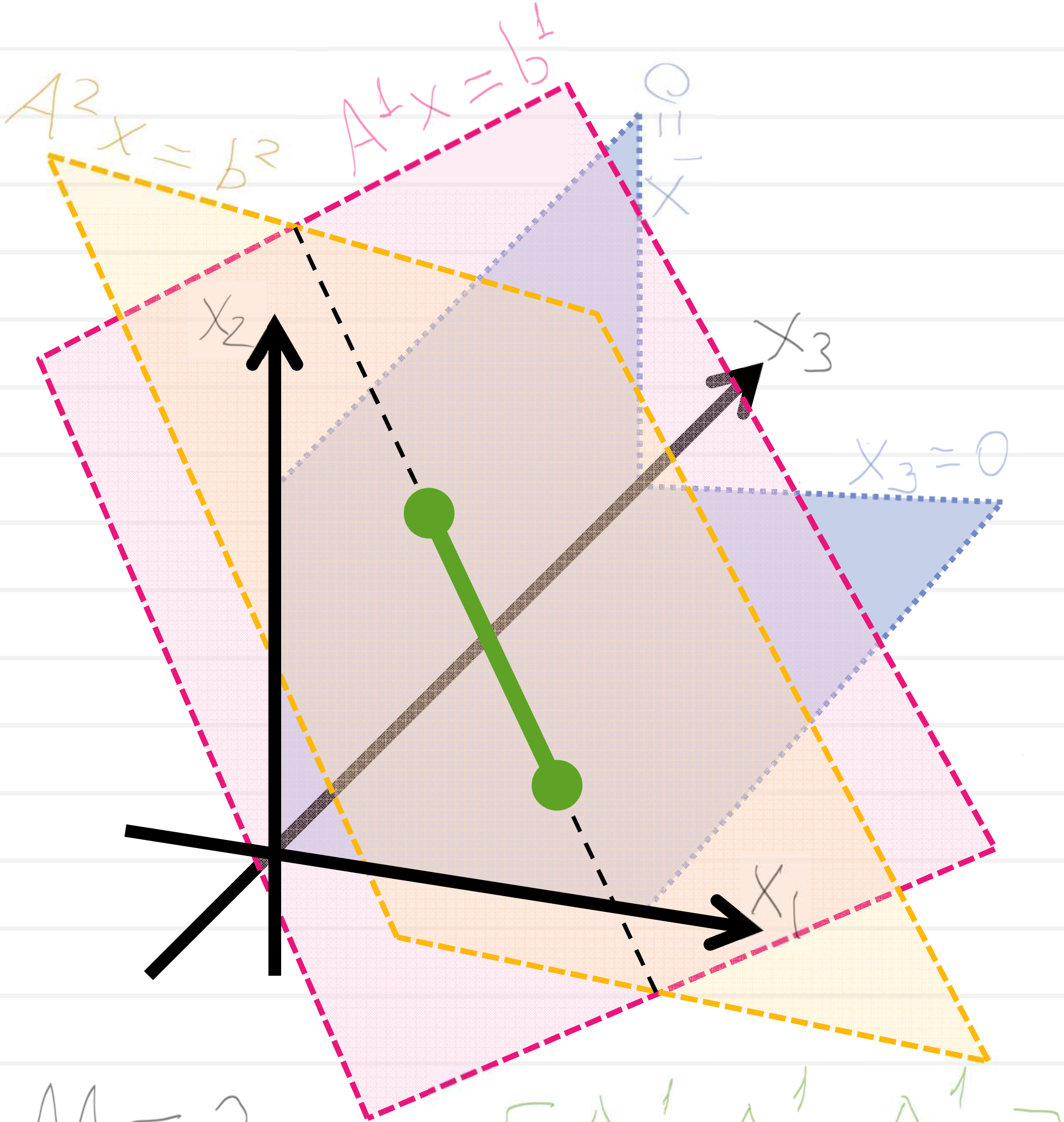
$$\min\ c^T x$$

$$\text{s.t.:}\ Ax = b$$

$$x \geq 0$$

$x \in \mathbb{R}^N \qquad c \in \mathbb{R}^N$

$A \in \mathbb{R}^{M \times N} \qquad b \in \mathbb{R}^M$

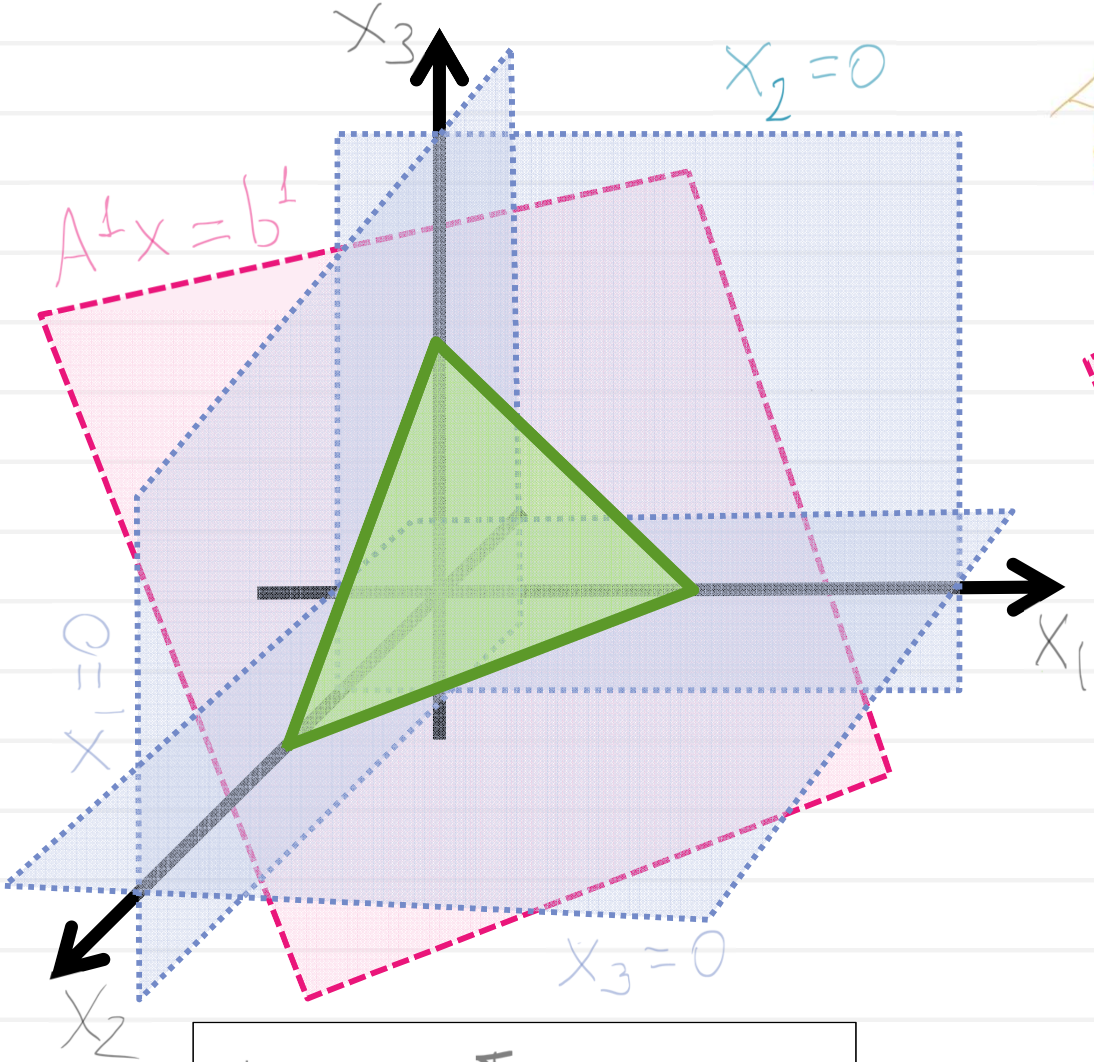


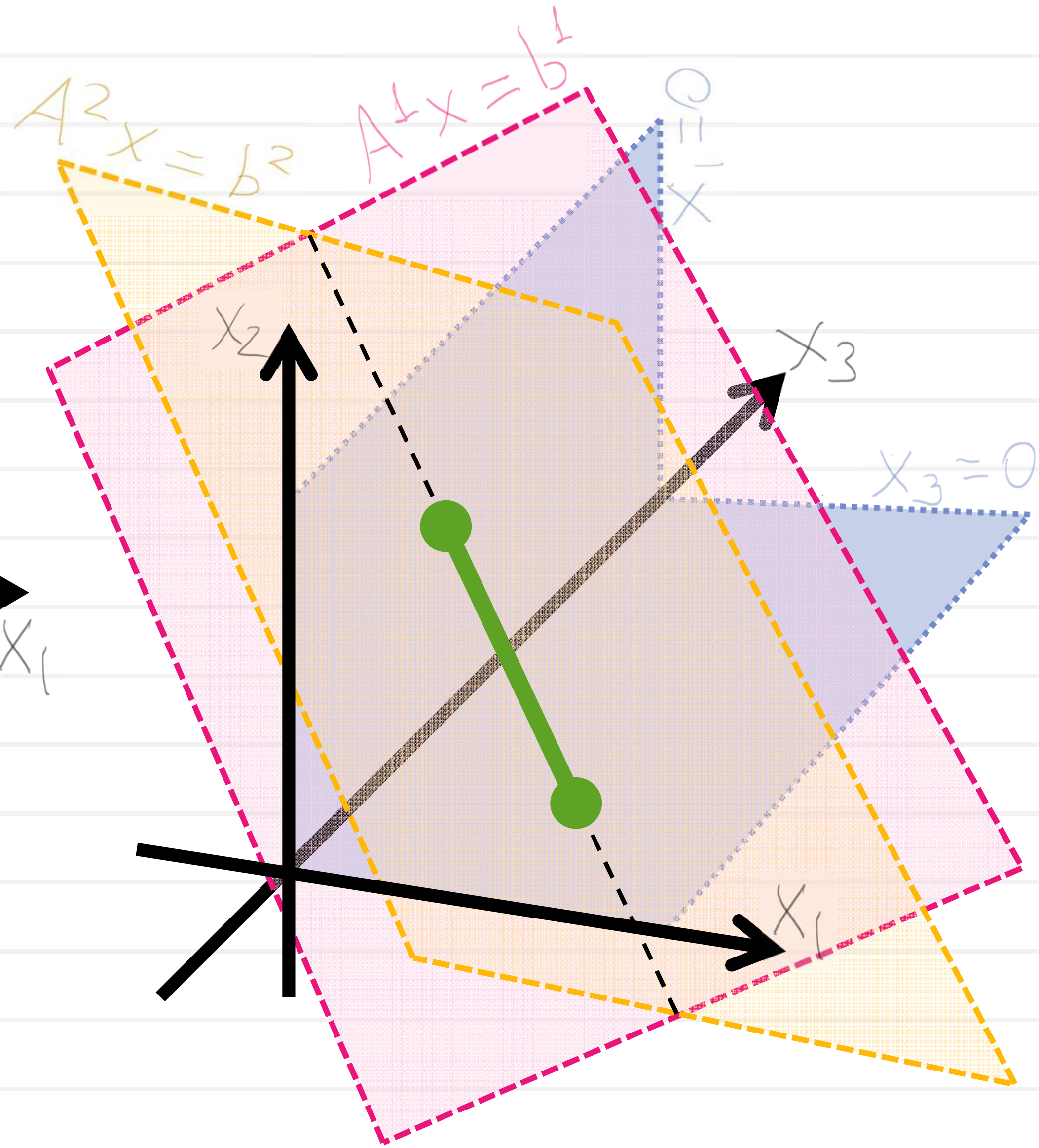
$M = 2$

$N = 3$

$$A = \begin{bmatrix} A_1^1 & A_2^1 & A_3^1 \\ A_1^2 & A_2^2 & A_3^2 \end{bmatrix}$$

LP: standard form





$$\min c^T x$$
$$\text{s.t.: } Ax = b$$
$$x \geq 0$$

Equivalent transformations

$$\max\ c^T x \quad \text{s.t.:}\quad a_1^T x = b_1,\ a_2^T x = b_2,\ \ldots,\ a_p^T x = b_p,\ x \geq 0$$

$$\max\ c^T x \quad \text{s.t.:}\quad a_1^T x = b_1,\ (a_2^T + \alpha a_1^T) x = b_2 + \alpha b_1,\ \ldots,\ a_p^T x = b_p,\ x \geq 0$$

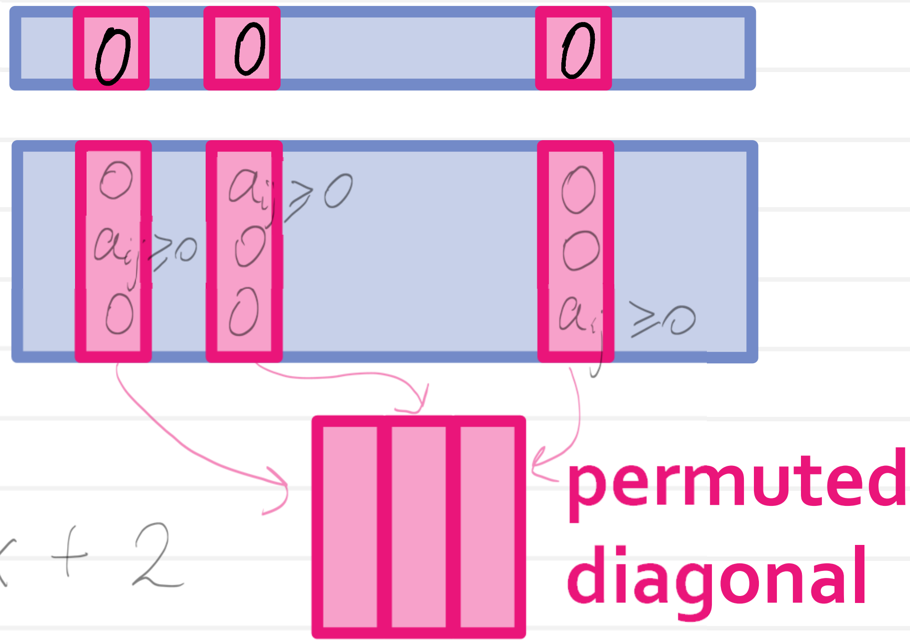
$$\max\ (c^T + \alpha a_1^T) x - \alpha b_1 \quad \text{s.t.:}\quad a_1^T x = b_1,\ a_2^T x = b_2,\ \ldots,\ a_p^T x = b_p,\ x \geq 0$$

Simplex-b: Canonical form of LP

$$\max \; c^T x + c_0$$
$$\text{s.t.:} \quad Ax = b \qquad b \geq 0$$
$$x \geq 0$$

basic variables

non-basic variables



Example:

$$\max \; [0 \; -3 \; 0 \; 1 \; 6 \; 0]\, x + 2$$

$$\text{s.t.:} \quad \begin{array}{cccccc} \begin{bmatrix} \boxed{0} & -3 & \boxed{1} & 1 & 3 & \boxed{0} \\ \boxed{4} & 1 & \boxed{0} & 0 & 9 & \boxed{0} \\ \boxed{0} & 1 & \boxed{0} & -10 & 0 & \boxed{2} \end{bmatrix} \end{array} x = \begin{bmatrix} 3 \\ 10 \\ 10 \end{bmatrix}$$

$x_1 \qquad x_3 \qquad x_6 \qquad x \geq 0$

basis: {1,3,6}

Basic feasible solution:

$x_2 = x_4 = x_5 = 0$

$x_1 = 10/4$

$x_3 = 3/1$

$x_6 = 10/2$

Simplex-b: changing basis (pivoting)

$$\max \begin{bmatrix} 0 & -3 & 0 & 1 & \boxed{6} & 0 \end{bmatrix} x + 2$$

basis: {1,3,6}

$$\text{s.t.:} \begin{bmatrix} 0 & -3 & 1 & 1 & 3 & 0 \\ 4 & 1 & 0 & 0 & 9 & 0 \\ 0 & 1 & 0 & -10 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ 10 \\ 10 \end{bmatrix}$$

$3/3$

$10/9$

$$x \geq 0$$

$$\max \begin{bmatrix} 0 & 3 & -2 & -1 & 0 & 0 \end{bmatrix} x + 8$$

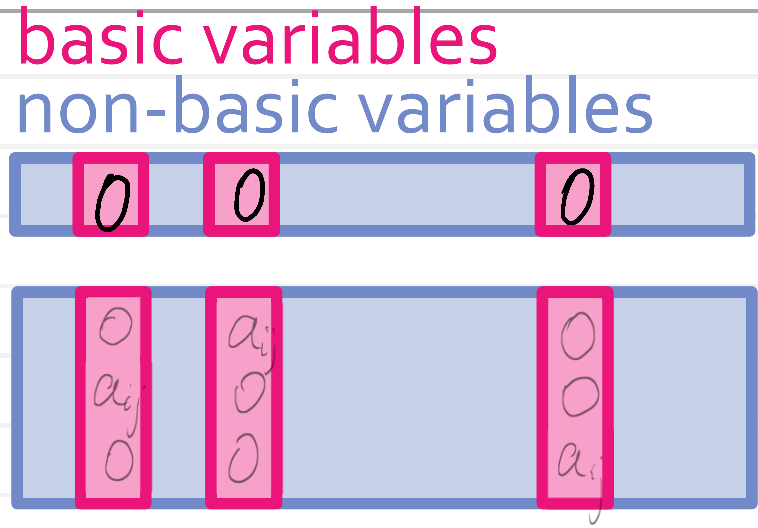
basis: {1,5,6}

$$\text{s.t.:} \begin{bmatrix} 0 & -3 & 1 & 1 & 3 & 0 \\ 4 & 8 & -3 & -3 & 0 & 0 \\ 0 & 1 & 0 & -10 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix}$$

$$x \geq 0$$

Simplex-b: pivoting

$$\max \; c^T x + c_0$$
$$\text{s.t.:} \quad Ax = b \qquad b \geq 0$$
$$x \geq 0$$



Pivoting (change of basis) step:

1. Pick a non-basic variable t with positive objective coefficient (*entering variable*)
2. Pick the row in A with the minimal positive ratio b_r / a_{tr} *(if none, then problem unbounded)*
3. Nullify all other entries in the column t by subtracting the scaled row
4. Update the objective by subtracting the scaled row a_r (update the constant accordingly)

Simplex-b: pivoting

$$\max \begin{bmatrix} 0 & -3 & 0 & 1 & \boxed{6} & 0 \end{bmatrix} x + 2$$

$$\text{s.t.:} \begin{bmatrix} \boxed{0} & -3 & \boxed{1} & 1 & \boxed{3} & \boxed{0} \\ \boxed{4} & 1 & \boxed{0} & 0 & \boxed{9} & \boxed{0} \\ \boxed{0} & 1 & \boxed{0} & -10 & \boxed{0} & \boxed{2} \end{bmatrix} x = \begin{bmatrix} 3 \\ 10 \\ 10 \end{bmatrix}$$

$$x \geq 0$$

$$\max \begin{bmatrix} 0 & 3 & -2 & -1 & 0 & 0 \end{bmatrix} x + 8$$

$$\text{s.t.:} \begin{bmatrix} \boxed{0} & -3 & 1 & 1 & \boxed{3} & \boxed{0} \\ \boxed{4} & 8 & -3 & -3 & \boxed{0} & \boxed{0} \\ \boxed{0} & 1 & 0 & -10 & \boxed{0} & \boxed{2} \end{bmatrix} x = \begin{bmatrix} 3 \\ 1 \\ 10 \end{bmatrix}$$

$$x \geq 0$$

Simplex-b: pivoting using *tableau*

0	-3	0	1	6	0	-2
0	-3	1	1	3	0	3
4	1	0	0	9	0	10
0	1	0	-10	0	2	10

0	3	-2	-1	0	0	-8
0	-3	1	1	3	0	3
4	8	-3	-3	0	0	1
0	1	0	-10	0	2	10

Phase 1 for Simplex-b

$$\boxed{\begin{aligned} \max \quad & c^T x + c_0 \\ \text{s.t.:} \quad & Ax = b \\ & x \geq 0 \end{aligned}}$$

$b \geq 0$

Must have optimum = 0 (otherwise initial problem infeasible)

Phase-1:

$$\boxed{\begin{aligned} \max \quad & 0^T x - \mathbb{1}^T y \\ & Ax + I_{p \times p}\, y = b \\ & x, y \geq 0 \end{aligned}}$$

$-y = Ax - b$

Phase-1 (canonical):

$$\boxed{\begin{aligned} \max \quad & \mathbb{1}^T A x + 0^T y - \mathbb{1}^T b \\ & Ax + I_{p \times p}\, y = b \\ & x, y \geq 0 \end{aligned}}$$

Phase 1 for Simplex-b

Phase-1 (canonical):

$$\boxed{\begin{aligned} \max\ & \mathbb{1}^T A x + 0^T y - \mathbb{1}^T b \\ & A x + I_{p\times p}\, y = b \\ & x, y \geq 0 \end{aligned}}$$

- Must have optimum = 0, y = 0 (otherwise initial problem infeasible)
- Hence, after the final iteration, y-variables are non-basic
  $\tilde{A} x + \tilde{C} y = \tilde{b}$
- The resulting $\tilde{A}$ and $\tilde{b}$ are equivalent to the original ones (define the same set) and are in the canonical form; *x* is a basic feasible solution

Simplex: final considerations

- Complexity: exponential
- Practical performance: excellent (still competitive!)
- Degenerate cases are tricky (special ordering rules enables convergence)
- “Tableau” notation for the standard form LP is the easiest to handle

Simplex-b solver

1. Turn arbitrary LP into a standard form LP
2. Formulate Phase-1 LP in a canonical form
3. Find the solution of Phase-1 LP (Simplex)
4. If the objective is non-zero, then the initial LP is infeasible
5. Use the solution of Phase-1 LP to initialize the Simplex algorithm for the main LP.
6. Iterate pivoting, until all objective coefficients are negative

Active set methods

0. Make a guess on *active set (AS)* of constraints.

Iterate:

1. Solve for optimum given current AS
2. Update AS by add/remove/swap

Until optimal

- **Simplex method** – an active set method for LP
- Discovered by George Dantzig in 1947
- Kick-started studies in optimization (in the West)