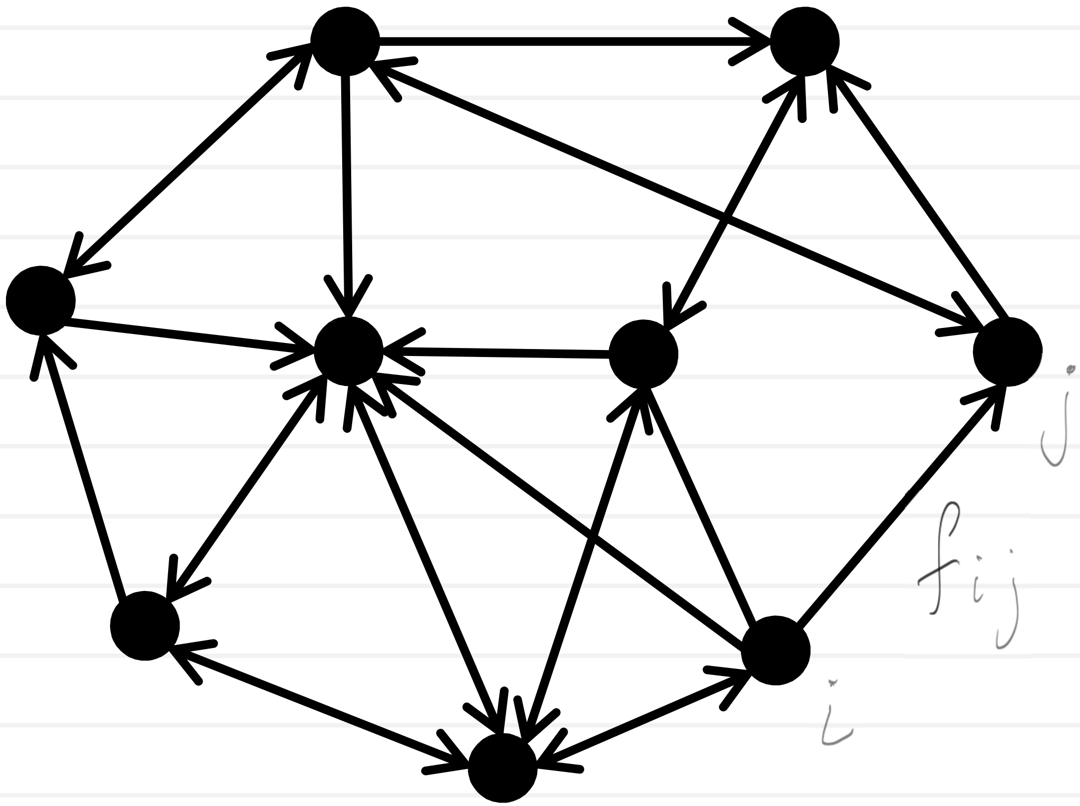
Lecture 7: Optimal network flows

Network flows



f_{ij} – the flow from i to j

$$f_{ij} \neq f_{ji}$$

u_{ij} – capacity of arc ij

b_i – inflow at vertex i (can be negative)

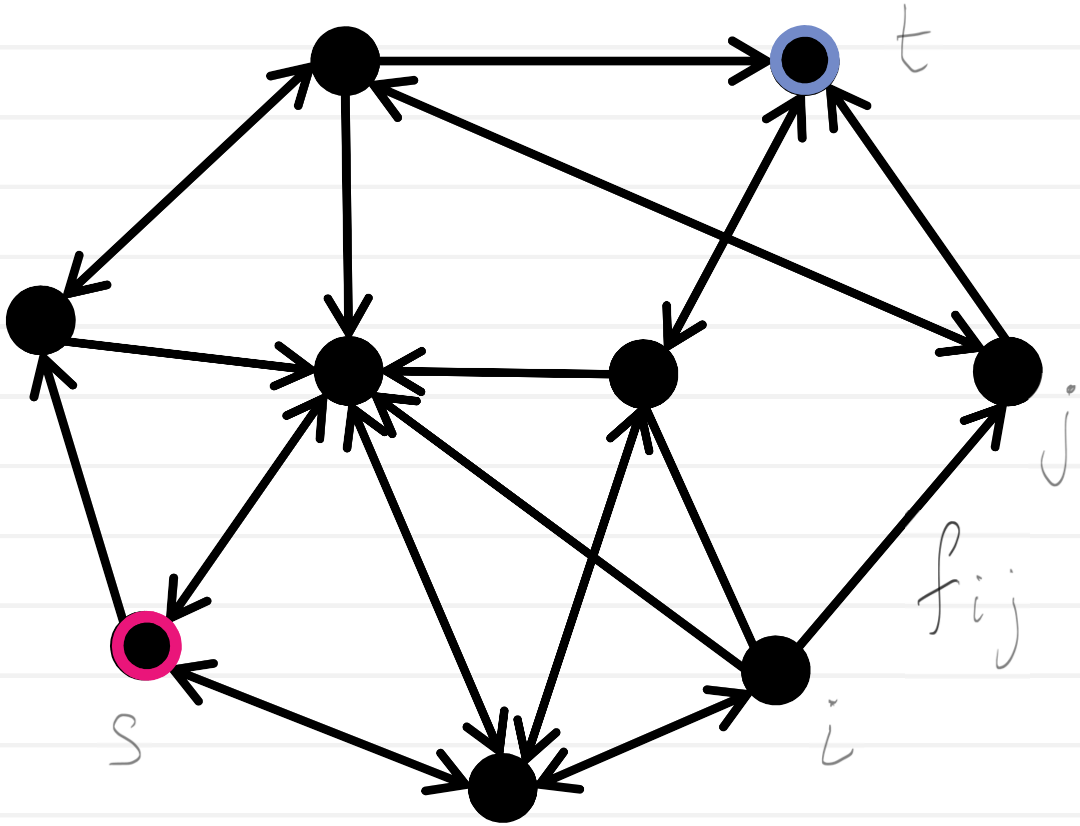
$I(i)$ – vertices with arcs directed to vertex i

$O(i)$ – vertices with arcs directed from vertex i

Definition: f is a feasible flow if

$0 \leq f_{ij} \leq u_{ij}$ and $b_i + \sum_{j \in I(i)} f_{ji} = \sum_{j \in O(i)} f_{ij}$

Total inflow



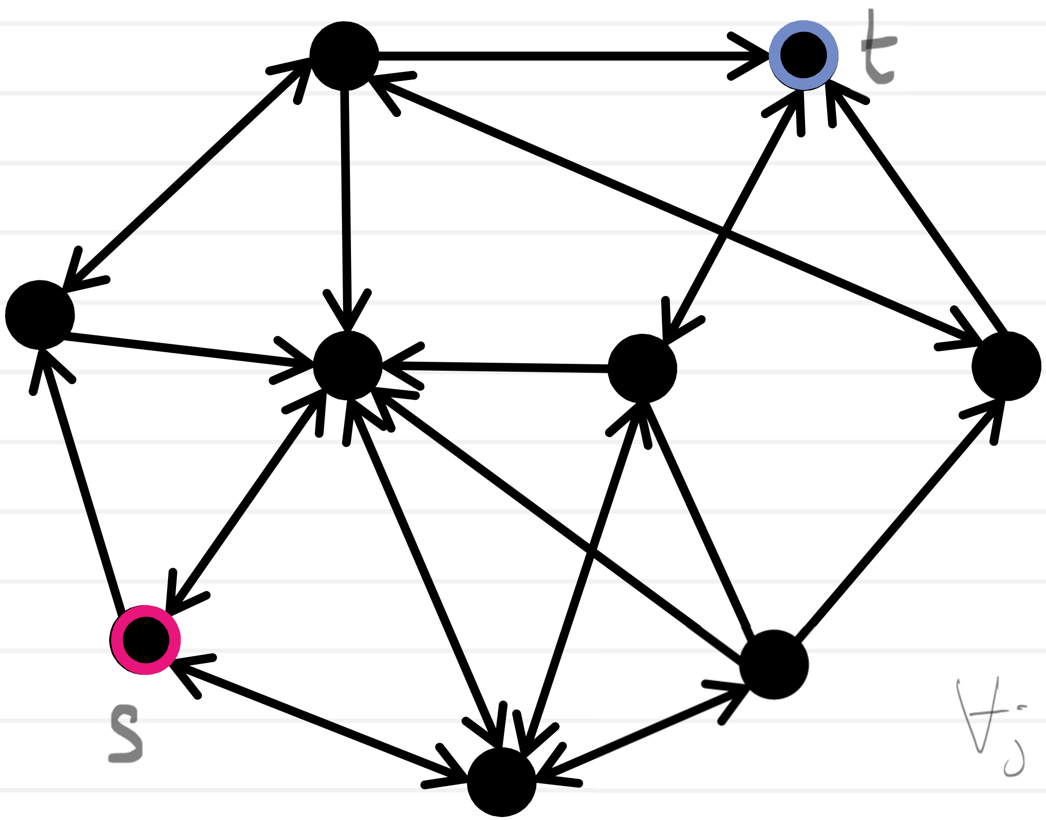
Sources are vertices with positive inflow, sinks are vertices with negative inflow.

Statement: if the network admits feasible flows then

$$\sum_{i=1}^{N} b_i = 0$$

Maximum flow (maxflow)

Maximum flow problem: given arc capacities, a single source vertex s, and a single sink vertex t, find a maximal flow (that maximizes the inflow at s and outflow at t).



$$\max_{b,f} \; b$$

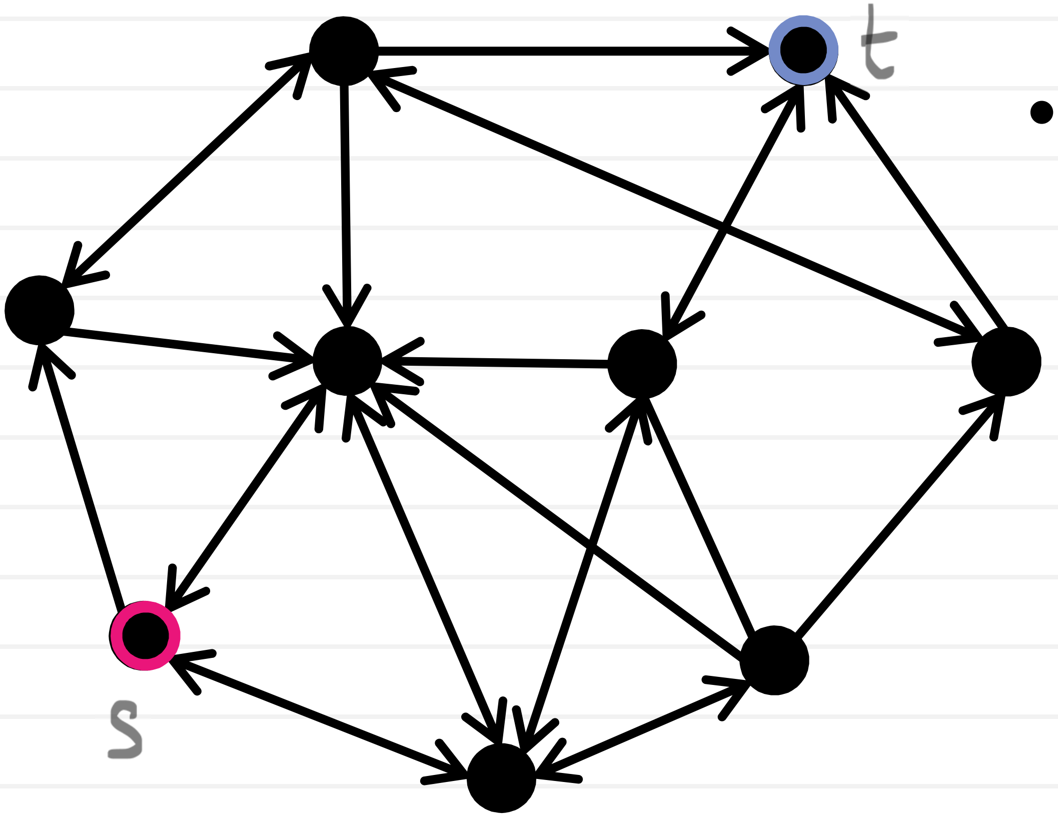
$$\text{s.t.:} \quad 0 \le f_{ij} \le u_{ij}$$

$$b + \sum_{i \in I(s)} f_{is} = \sum_{i \in O(s)} f_{si}$$

$$\sum_{i \in I(t)} f_{it} = \sum_{i \in O(t)} f_{ti} + b$$

$$\forall j \ne s,t \quad \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

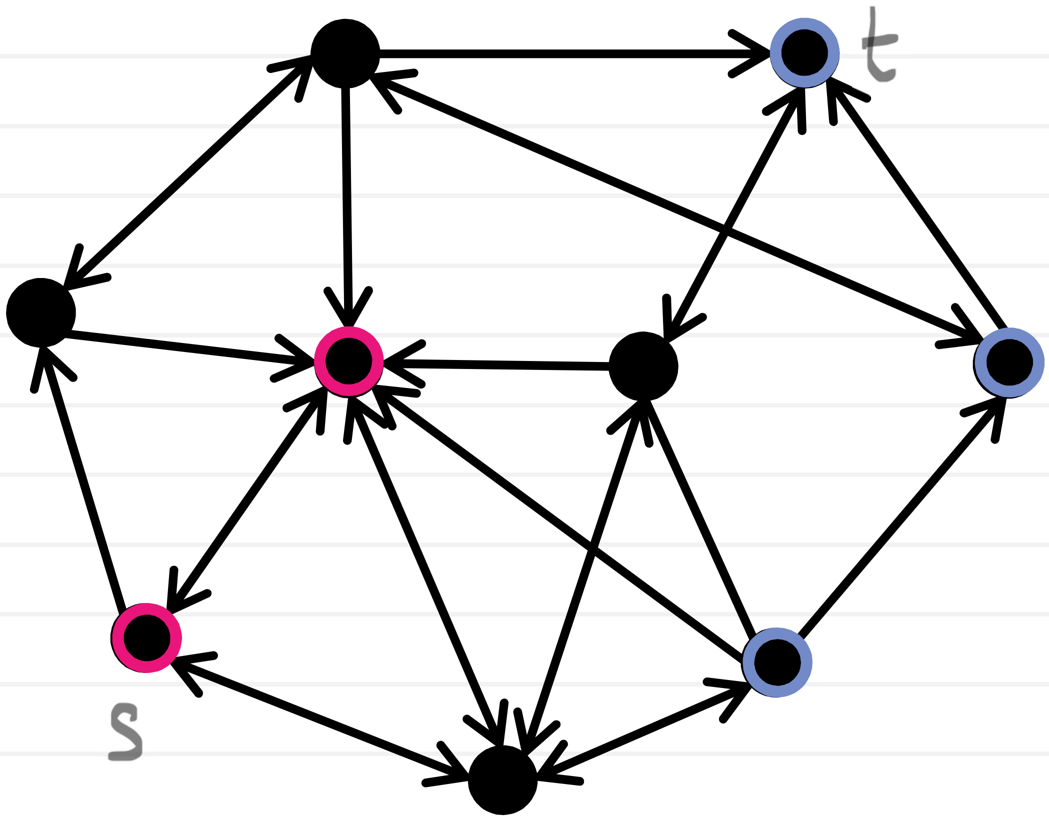
Maxflow interpretation



- Arc capacities are maximum amount of water that can be pumped.
- We try to pump as much water as possible from s to t.

Minimum cost flow (mincost flow)

Minimum cost flow: given arc capacities, a set of inflows, and *arc costs*, find a feasible flow that minimizes the total cost.

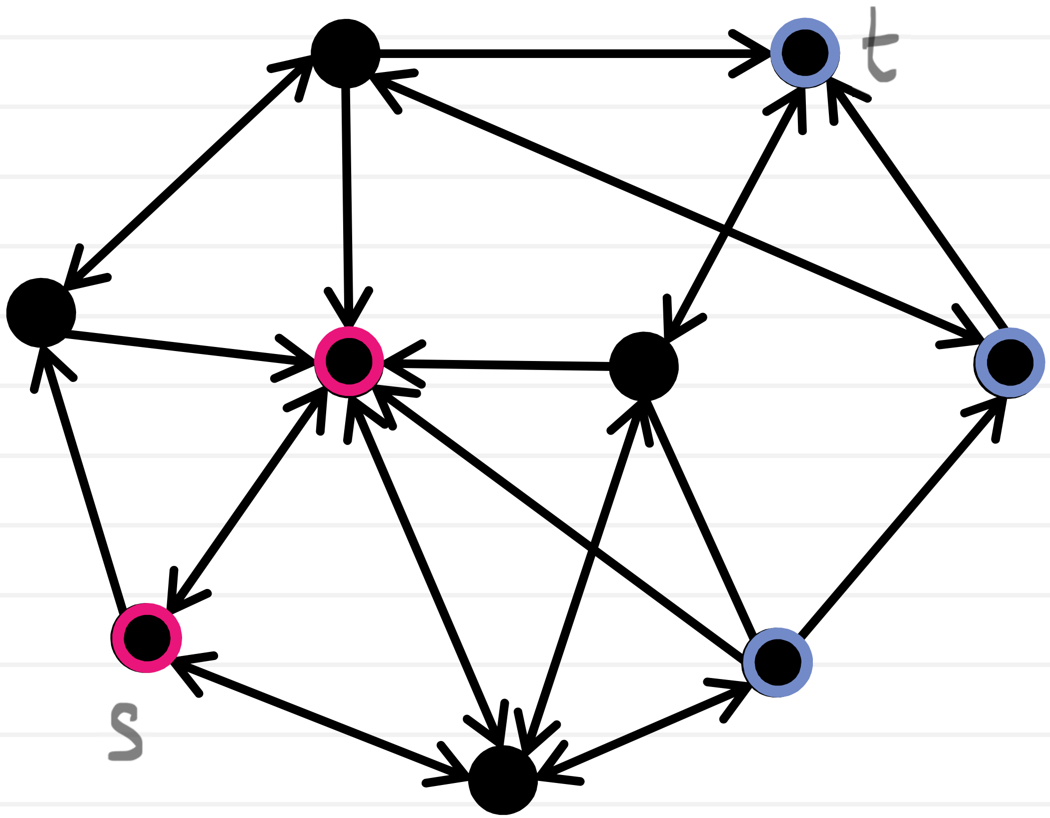


$$\min_f \; c^T f = \sum_{i,j} c_{ij} f_{ij}$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le u_{ij}$$

$$b_j + \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

Min-cost flow: transportation networks



$$\min_{f} c^T f = \sum_{i,j} c_{ij} f_{ij}$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le u_{ij}$$

$$b_j + \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

Deliver goods from producers to consumers subject to road capacities at the minimum transportation cost *(transshipment problem).*

Integrality

$$\max_{b,f} \; b$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le u_{ij}$$

$$b + \sum_{i \in I(s)} f_{is} = \sum_{i \in O(s)} f_{si}$$

$$\sum_{i \in I(t)} f_{it} = \sum_{i \in O(t)} f_{ti} + b$$

$$\forall j \ne s,t \quad \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

Corollary: if u_{ij} are integer, then all flow values are integer.

$$\min_{f} \; c^T f$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le u_{ij}$$

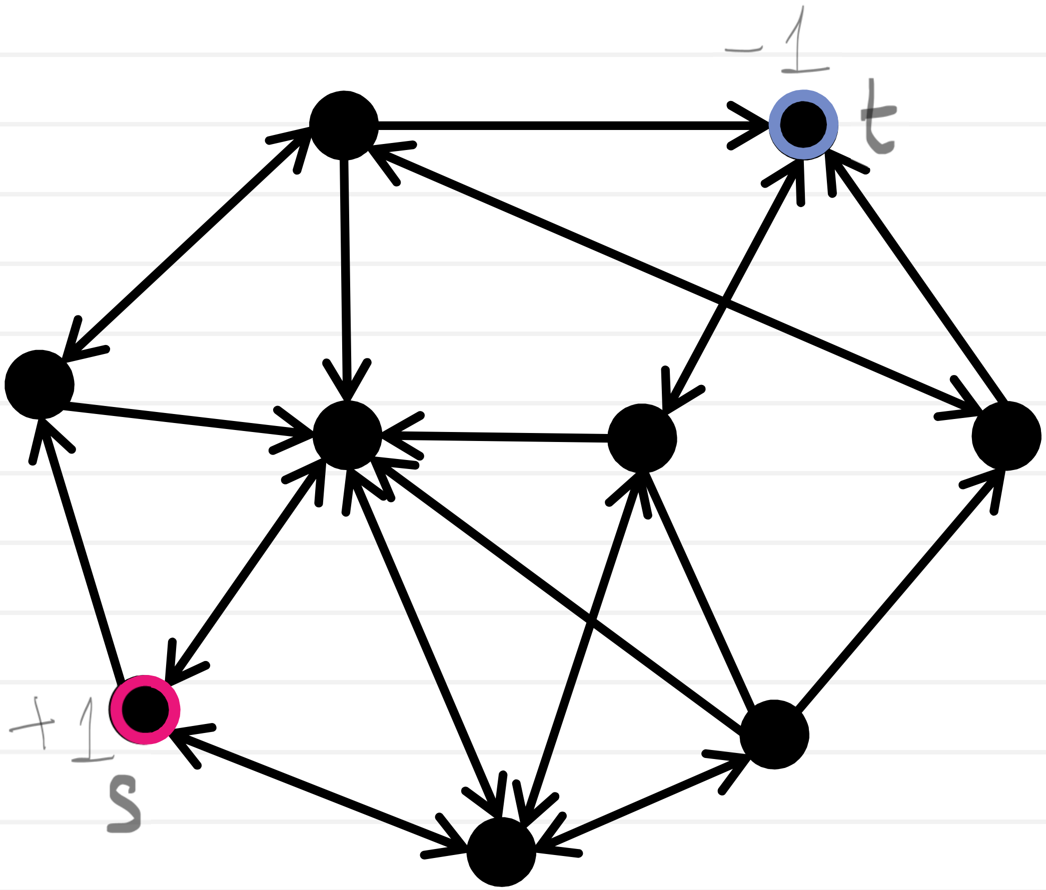
$$b_j + \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

Corollary: if b_j and u_{ij} are integer, then all flow values are integer.

Shortest path

Shortest path can be cast as an instance of min-cost flow (with unit excess/demand).

arc costs without negative cycles



$$\min_f \; c^T f$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le +\infty$$

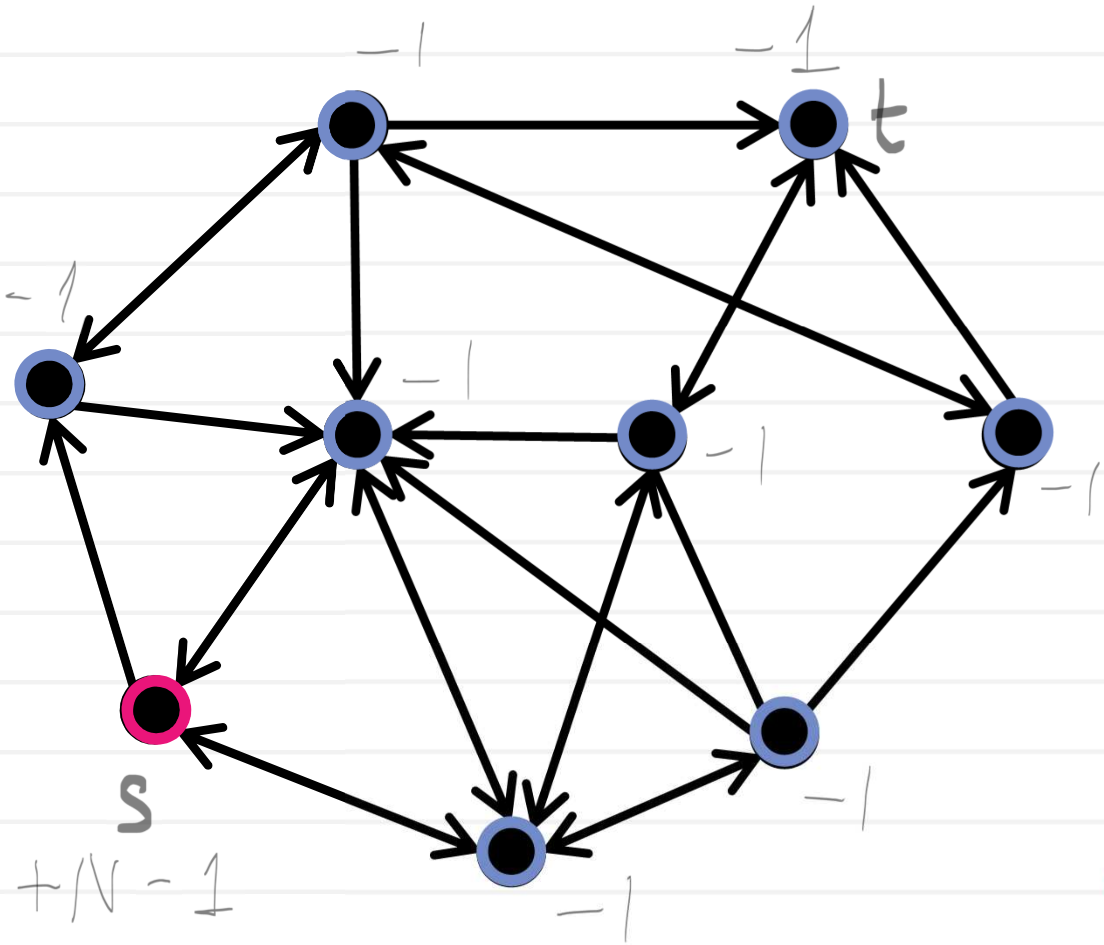
$$b_j + \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

$$b_s = +1 \qquad b_t = -1$$

Shortest paths to all vertices

Shortest path can be cast as an instance of min-cost flow (with unit excess/demand).

arc costs without negative cycles



$$\min_f \; c^T f$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le +\infty$$

$$b_j + \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

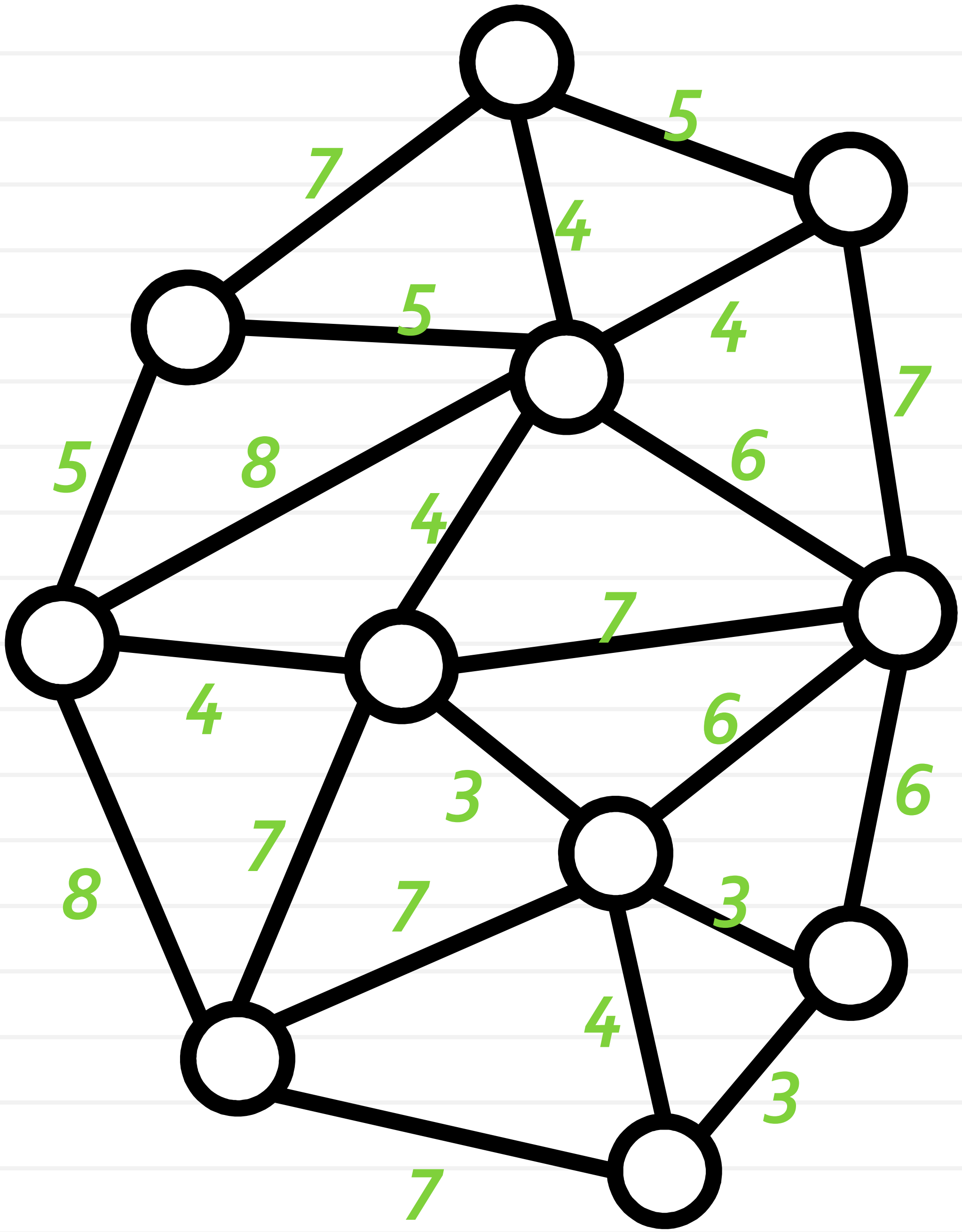
$$b_s = +N-1 \qquad \forall i \ne s \;\; b_i = -1$$

Finding optimal flows

How to solve network flow problems (min-cost flow, maxflow, shortest path)?

- Generic LP solver
- Optimized LP solver (e.g. *network simplex*) with integer arithmetics. Interestingly, basic solutions correspond to spanning trees containing all arcs with non-zero flow.
- Optimized algorithms specific to each problem (Dijkstra, Bellman-Ford, Ford-Fulkerson, push-relabel,…)

Dijkstra algorithm for shortest paths



non-negative

arc costs

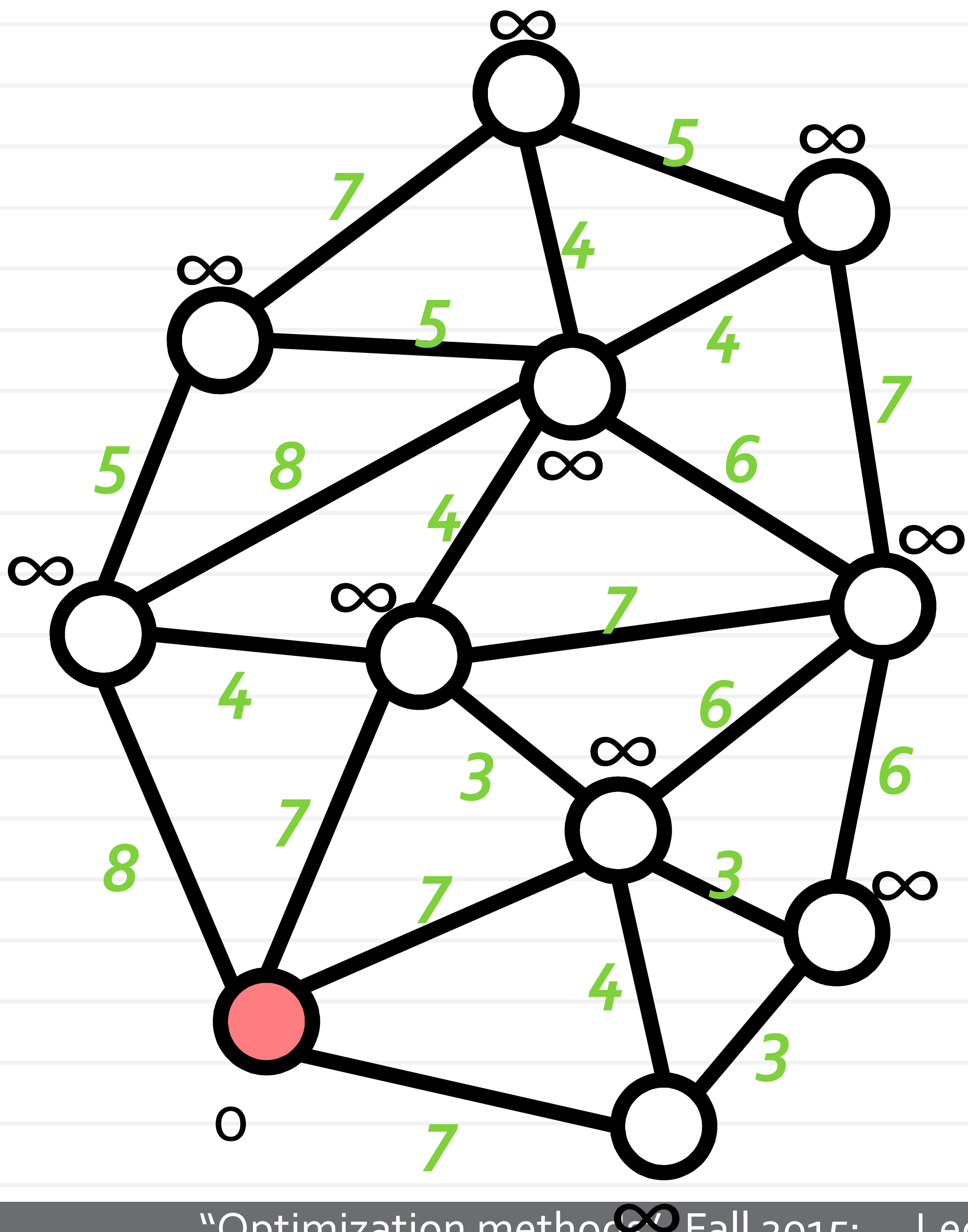
$$\min_f \; c^T f$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le +\infty$$

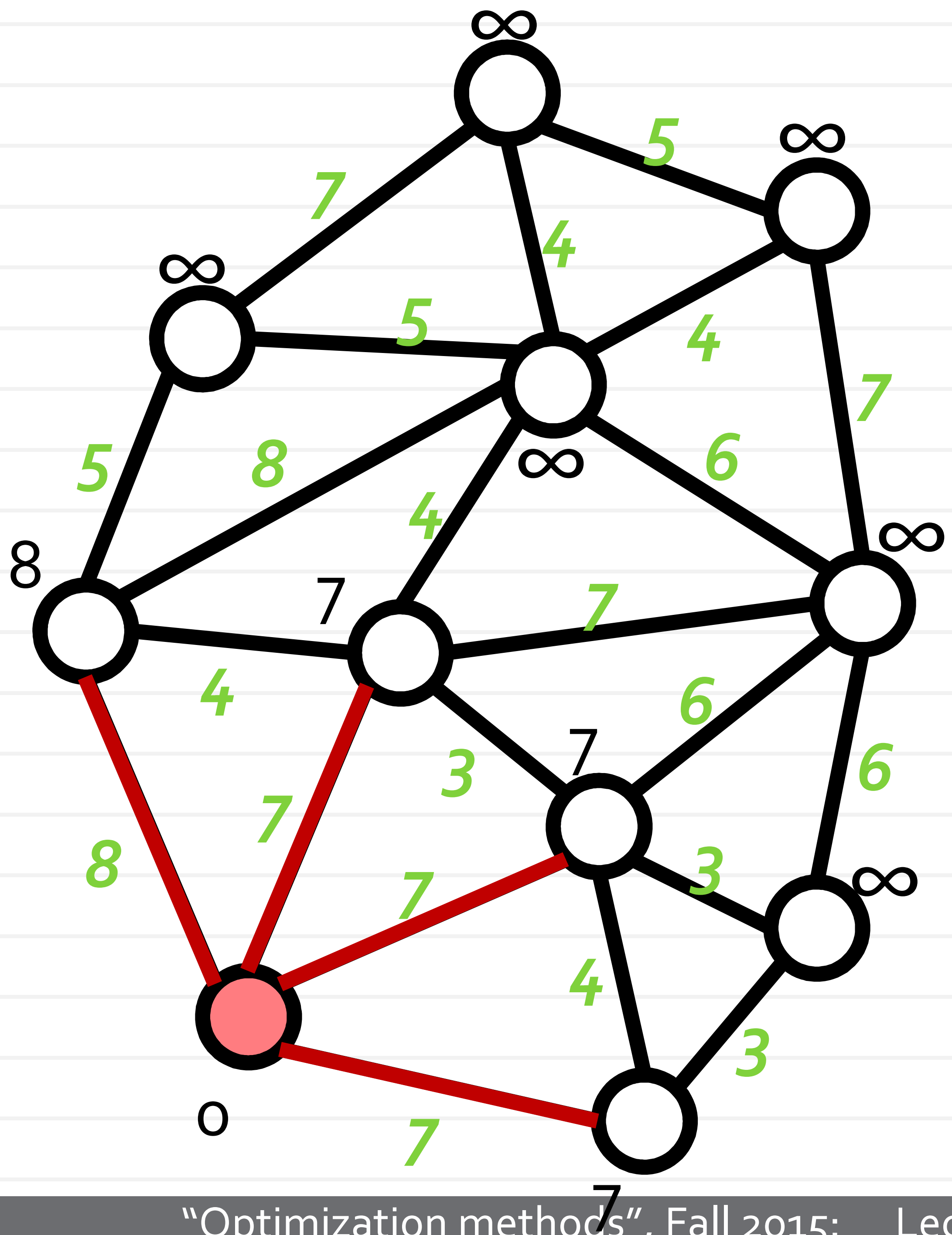
$$b_j + \sum_{i \in I(j)} f_{ij} = \sum_{i \in O(j)} f_{ji}$$

$$b_s = +N-1 \qquad \forall i \neq s \;\; b_i = -1$$

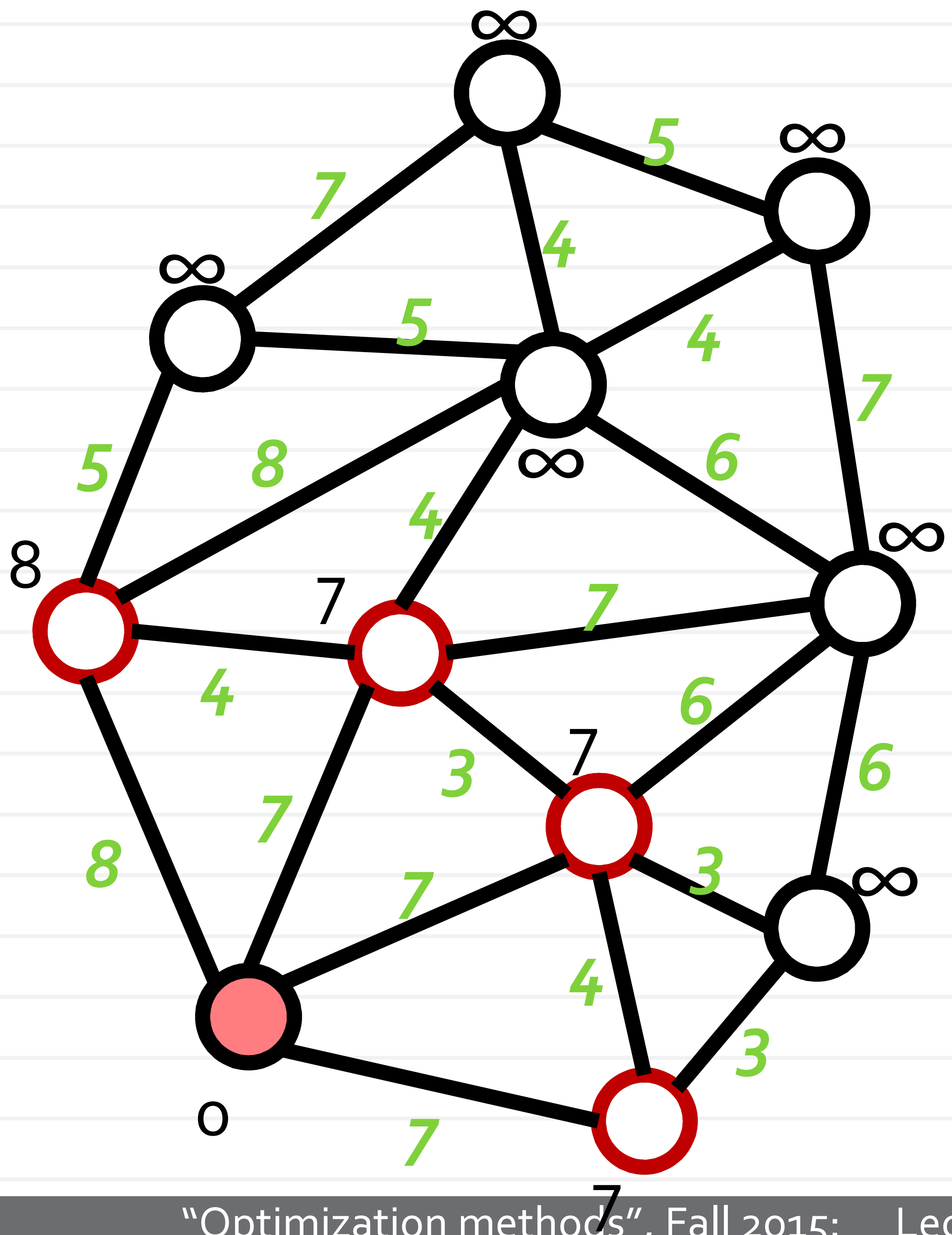
Dijkstra Algorithm



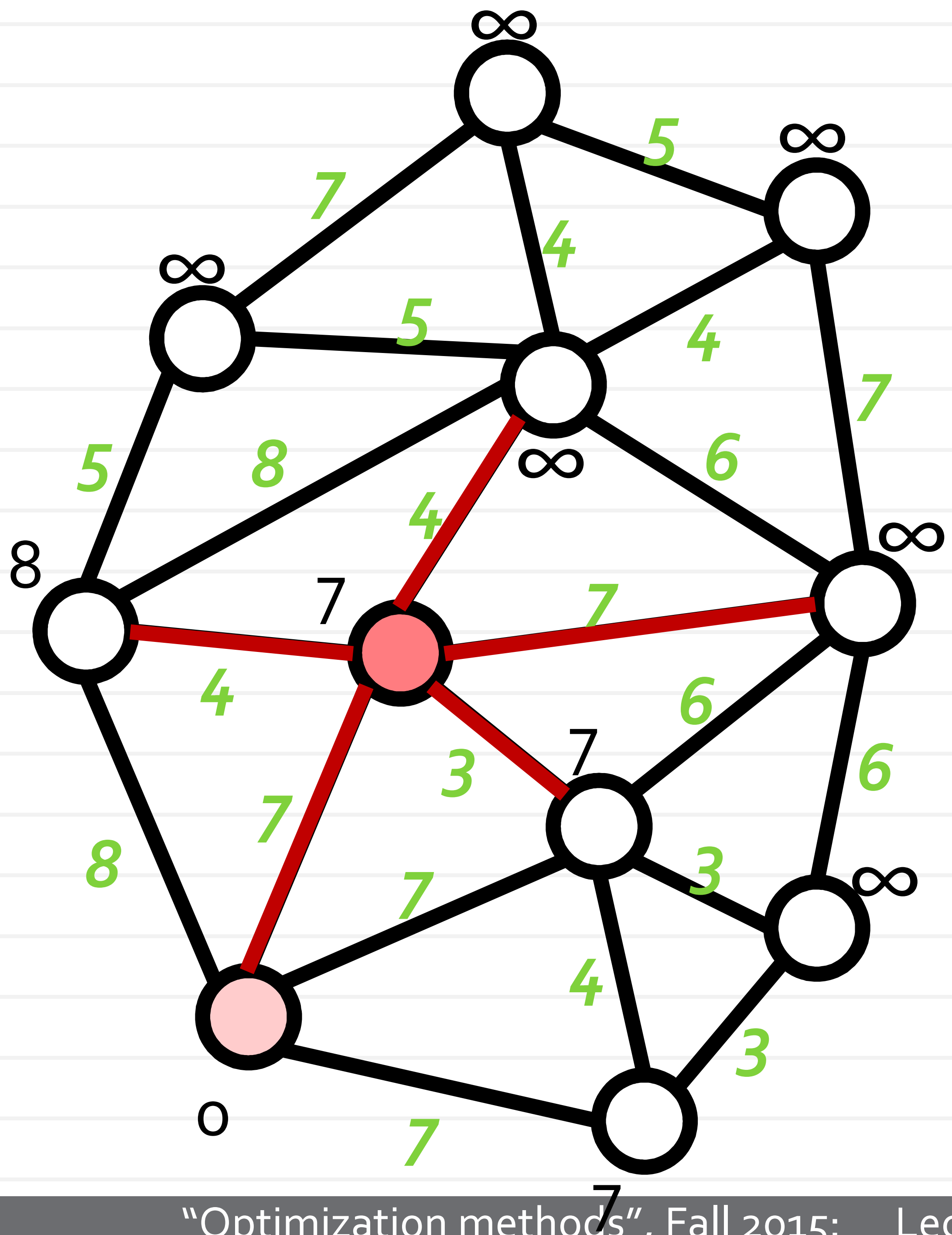
Dijkstra Algorithm



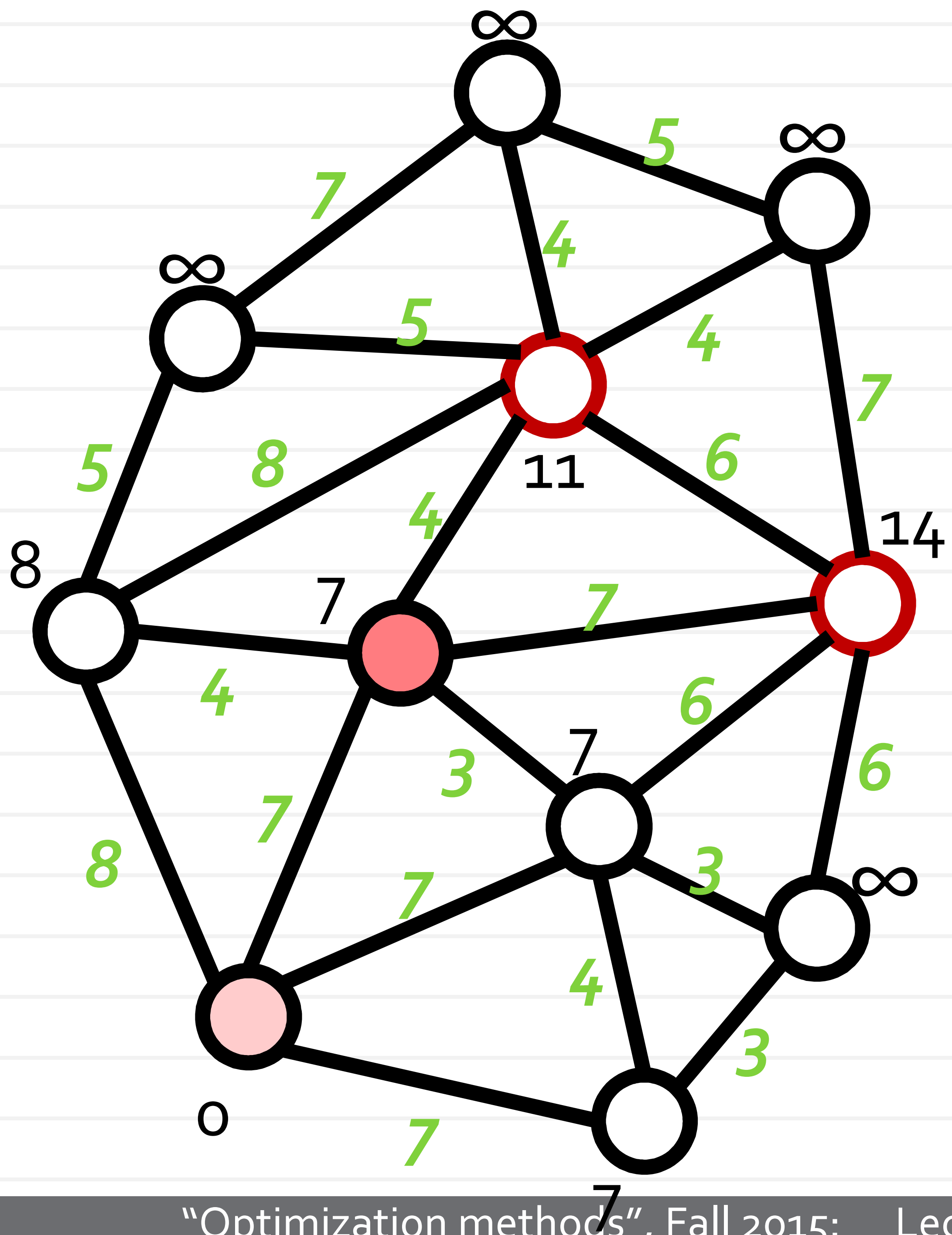
Dijkstra Algorithm



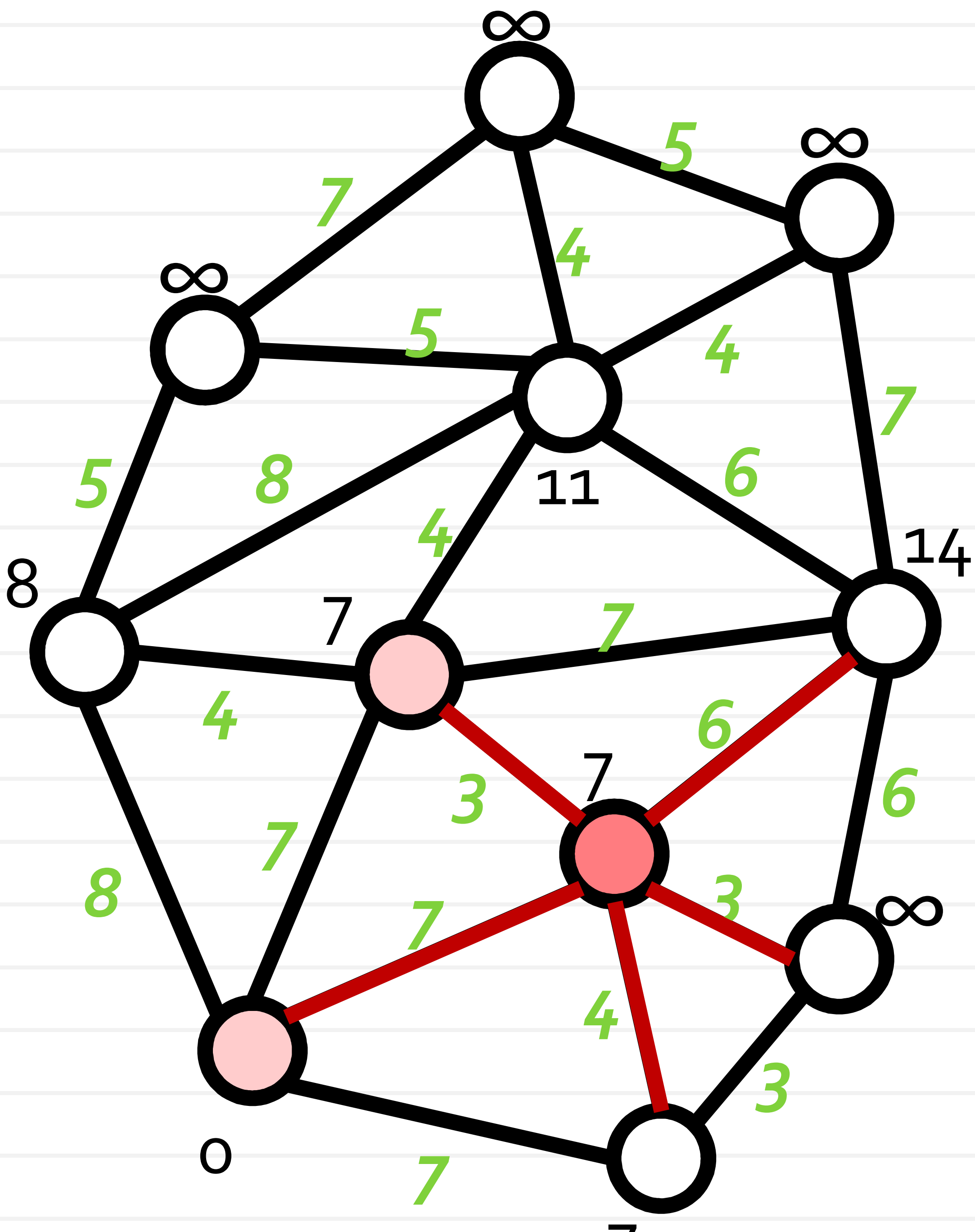
Dijkstra Algorithm



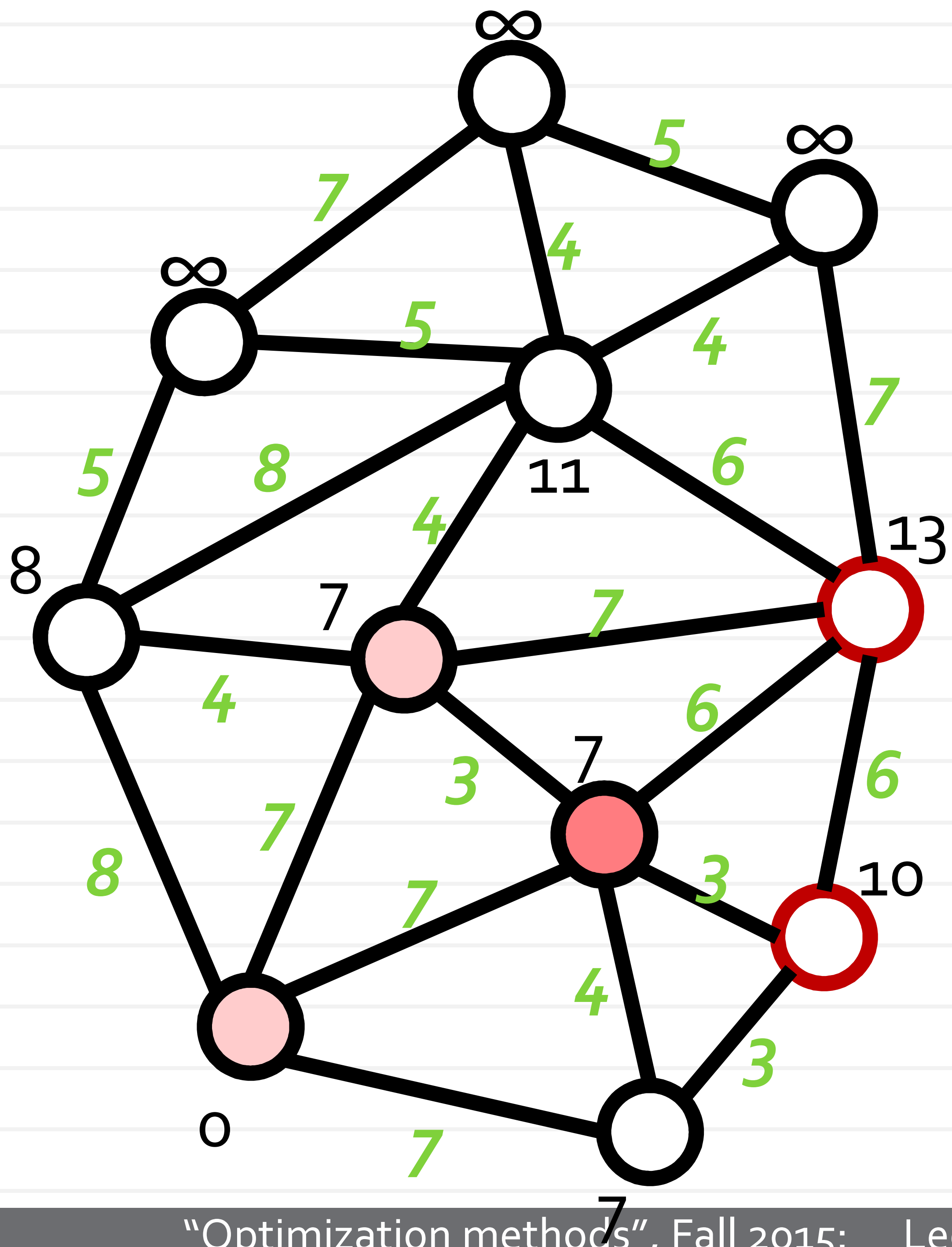
Dijkstra Algorithm



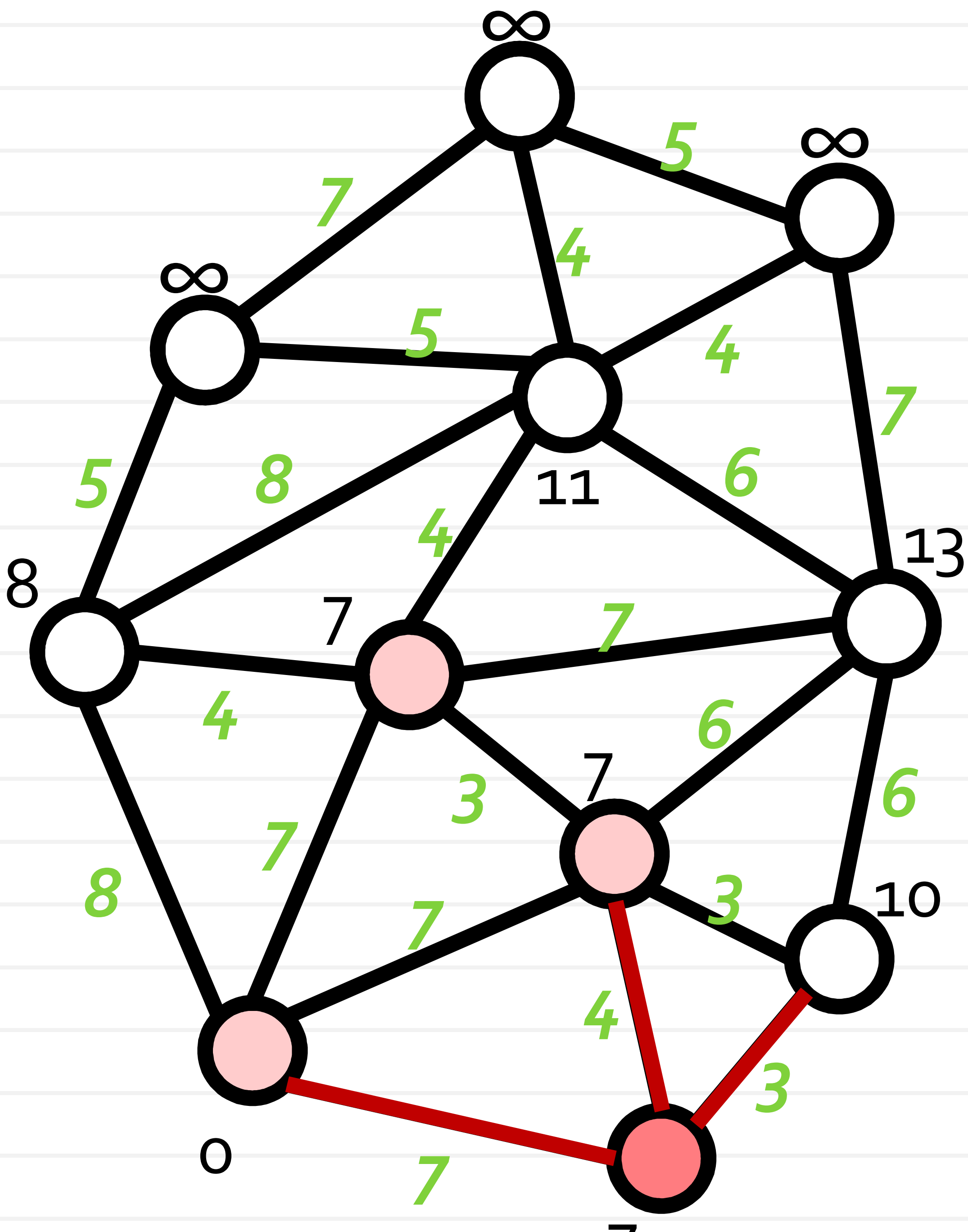
Dijkstra Algorithm



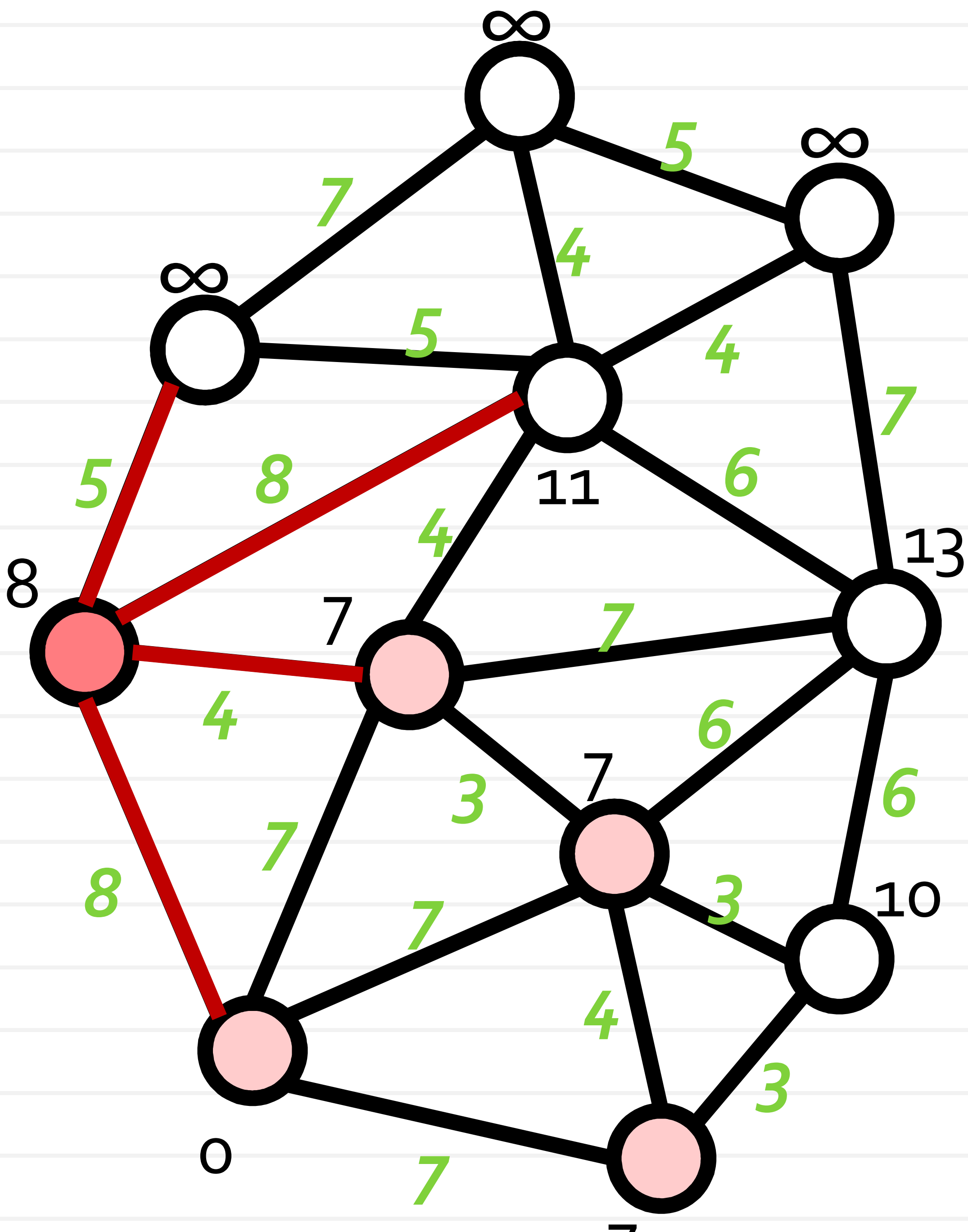
Dijkstra Algorithm



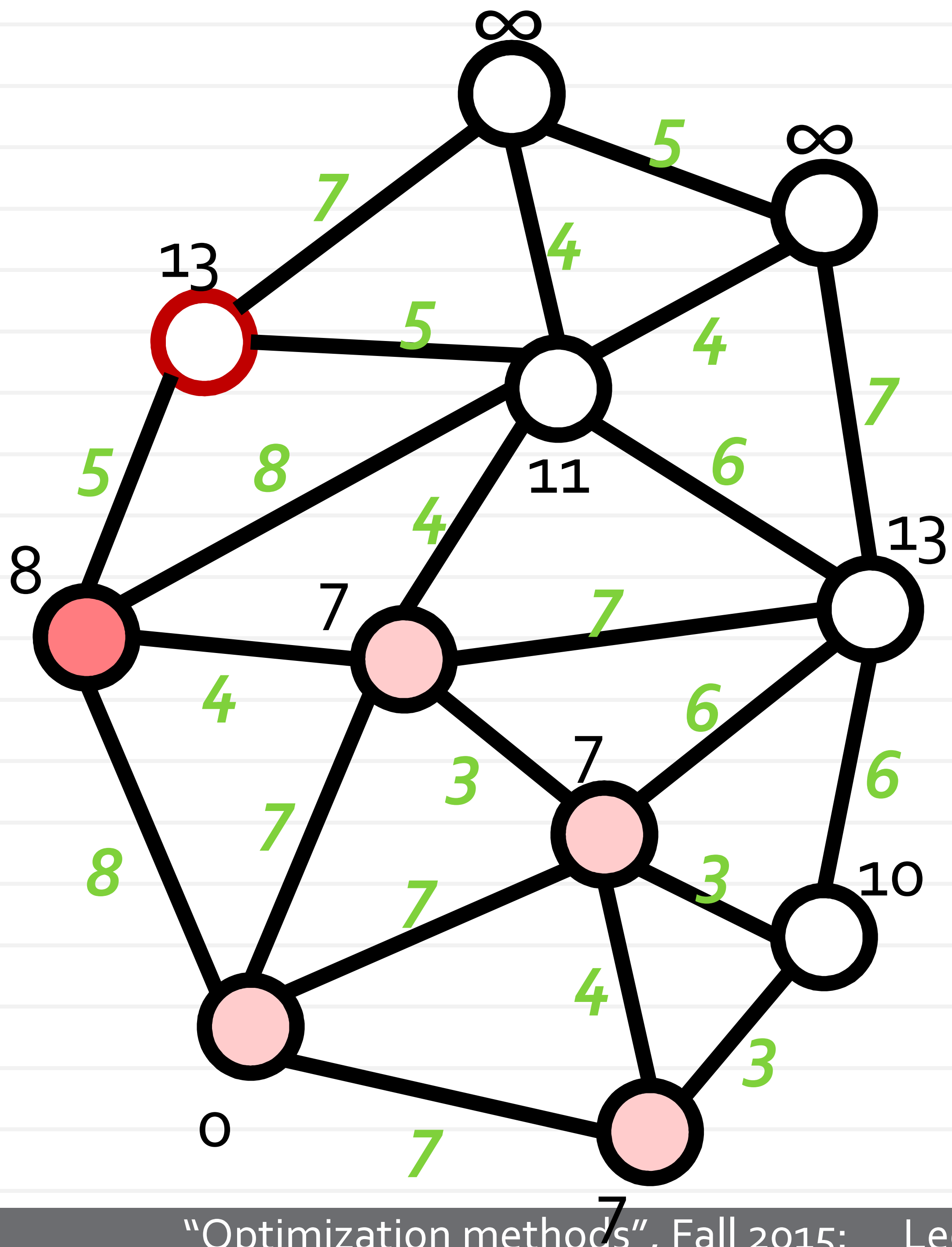
Dijkstra Algorithm



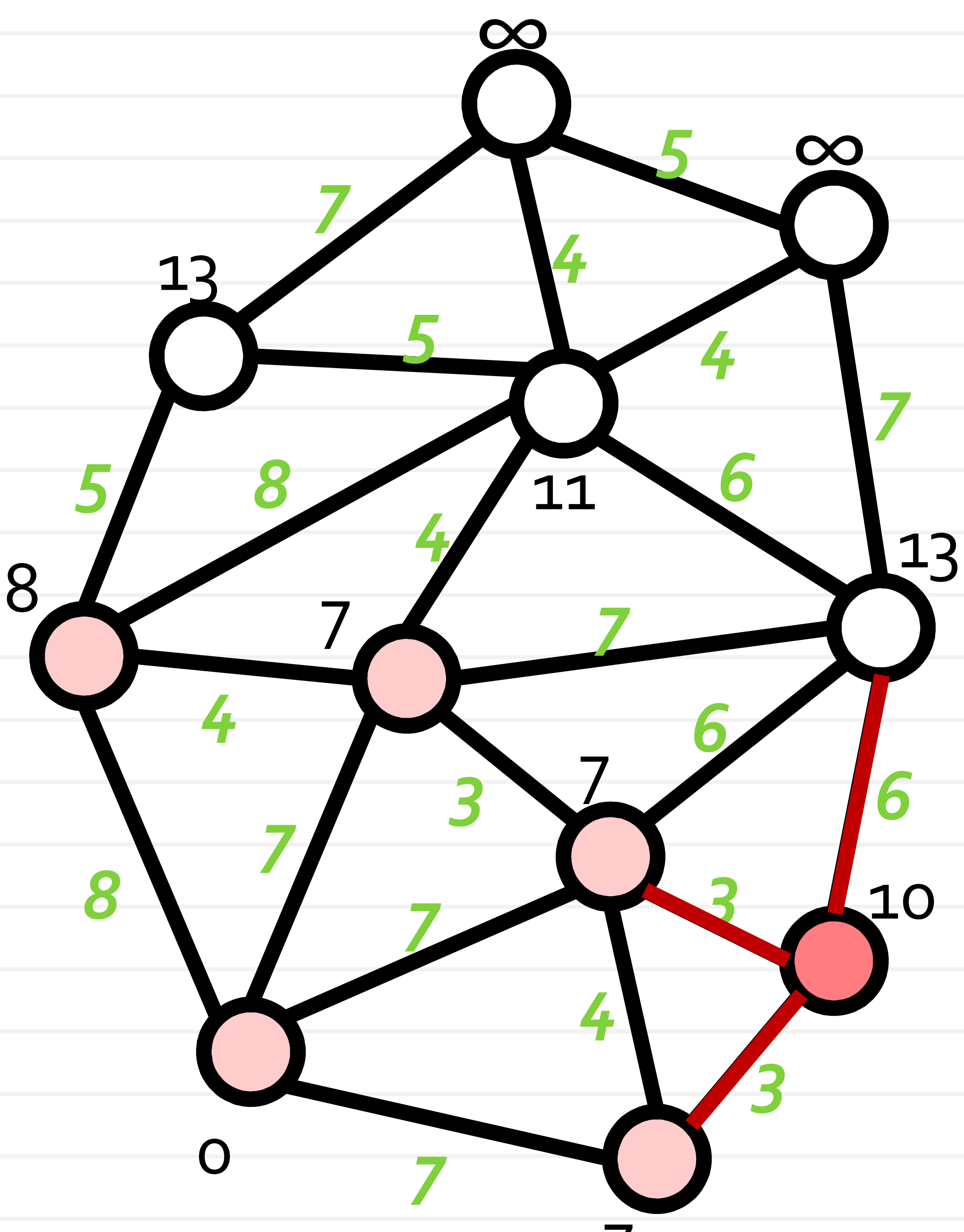
Dijkstra Algorithm



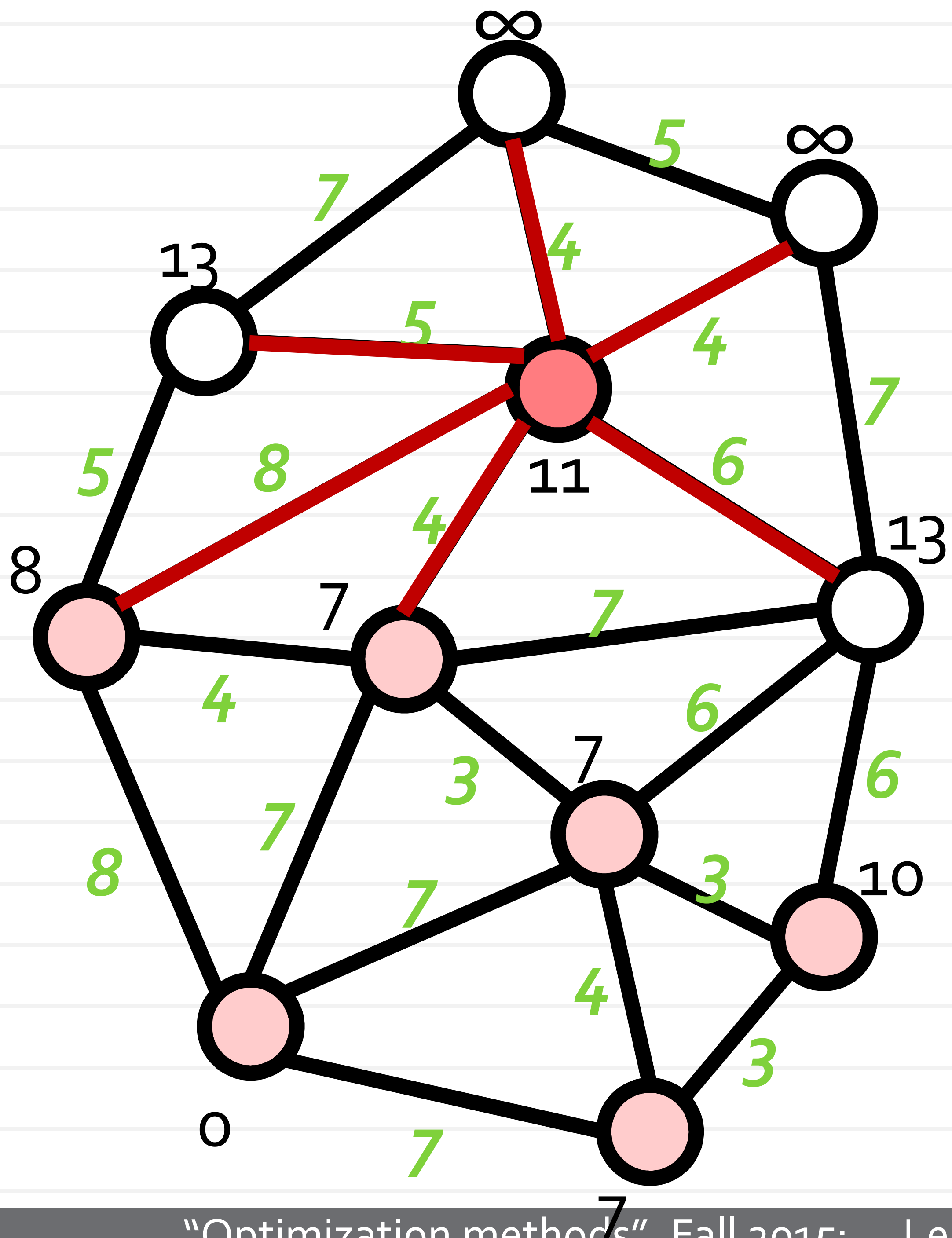
Dijkstra Algorithm



Dijkstra Algorithm



Dijkstra Algorithm



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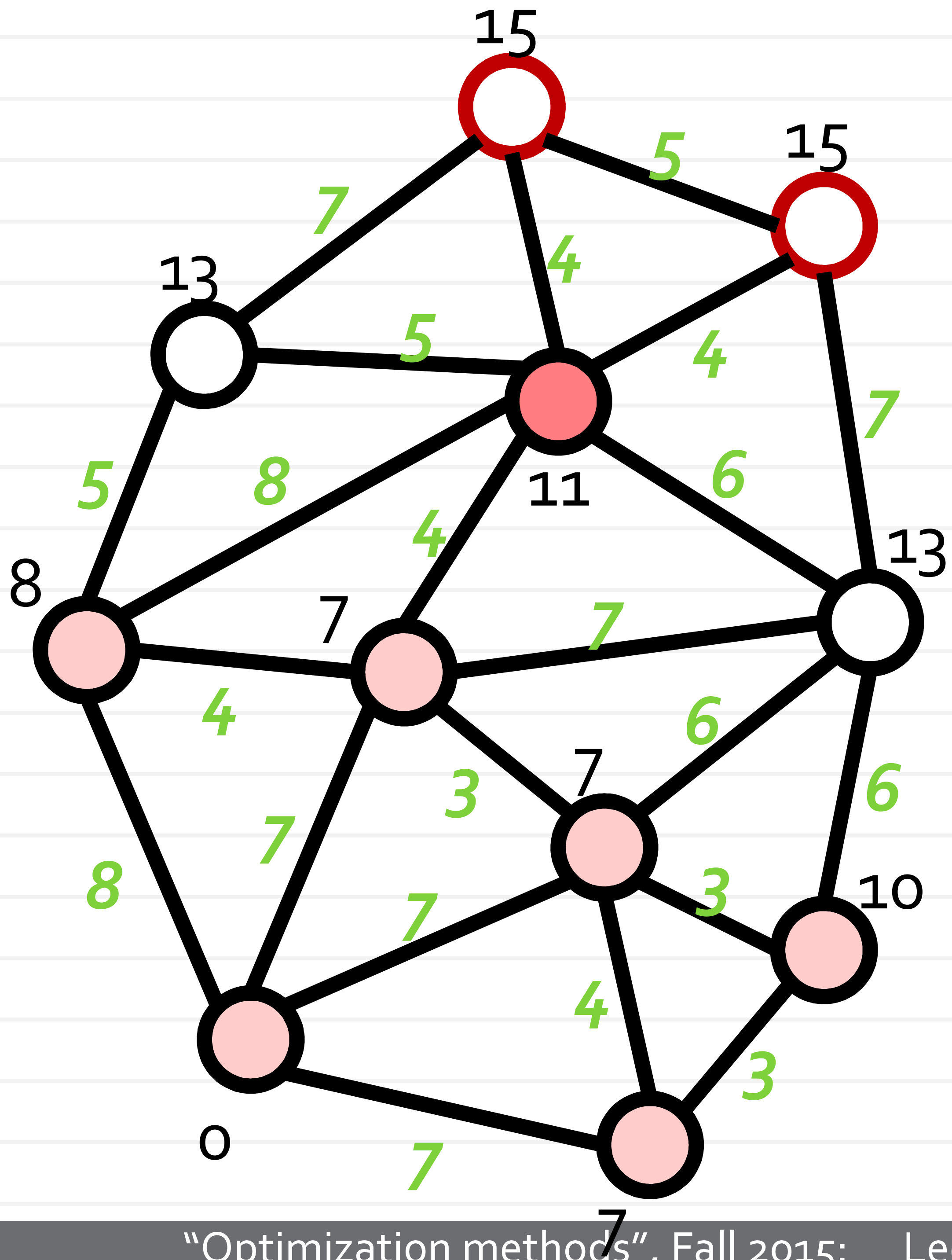
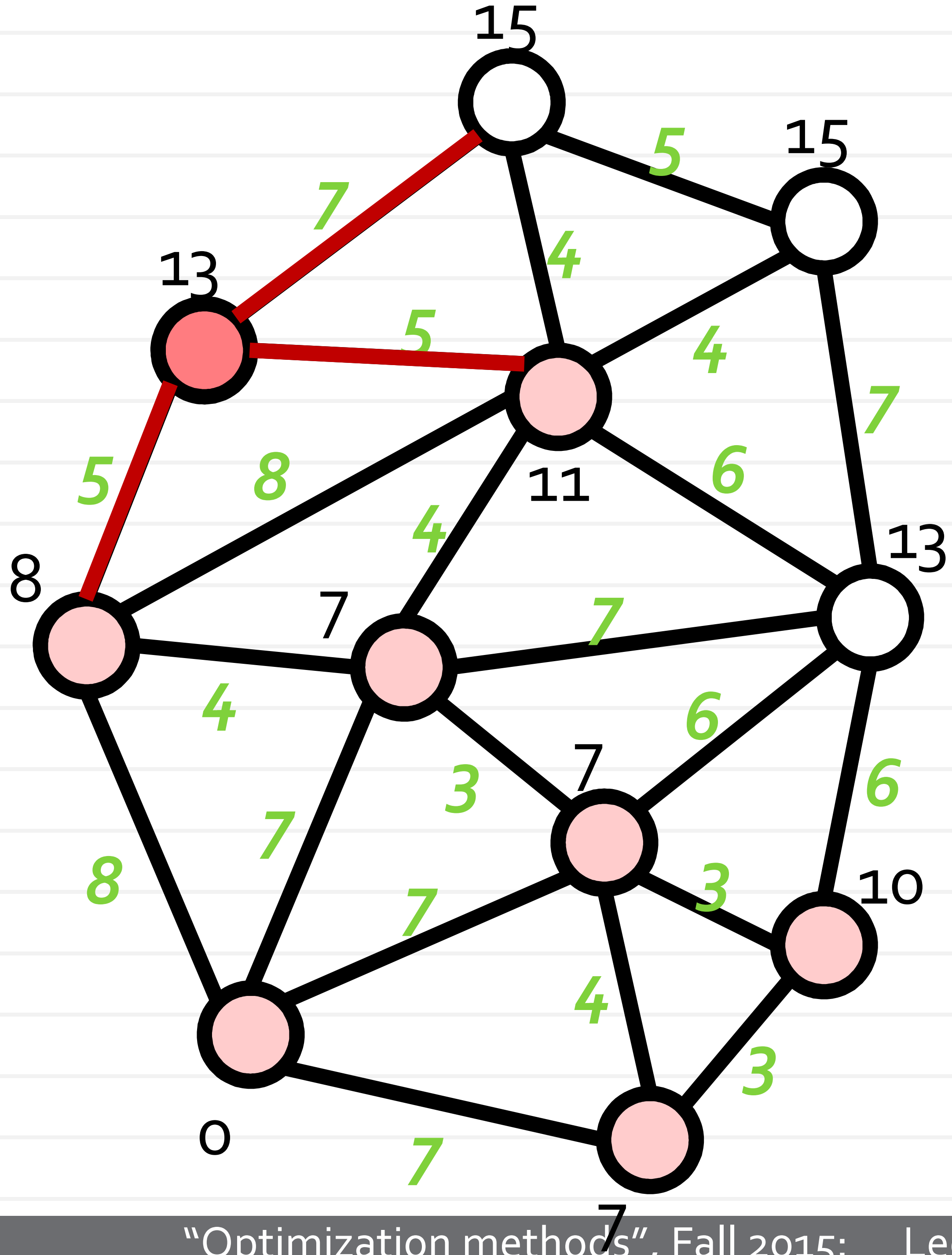
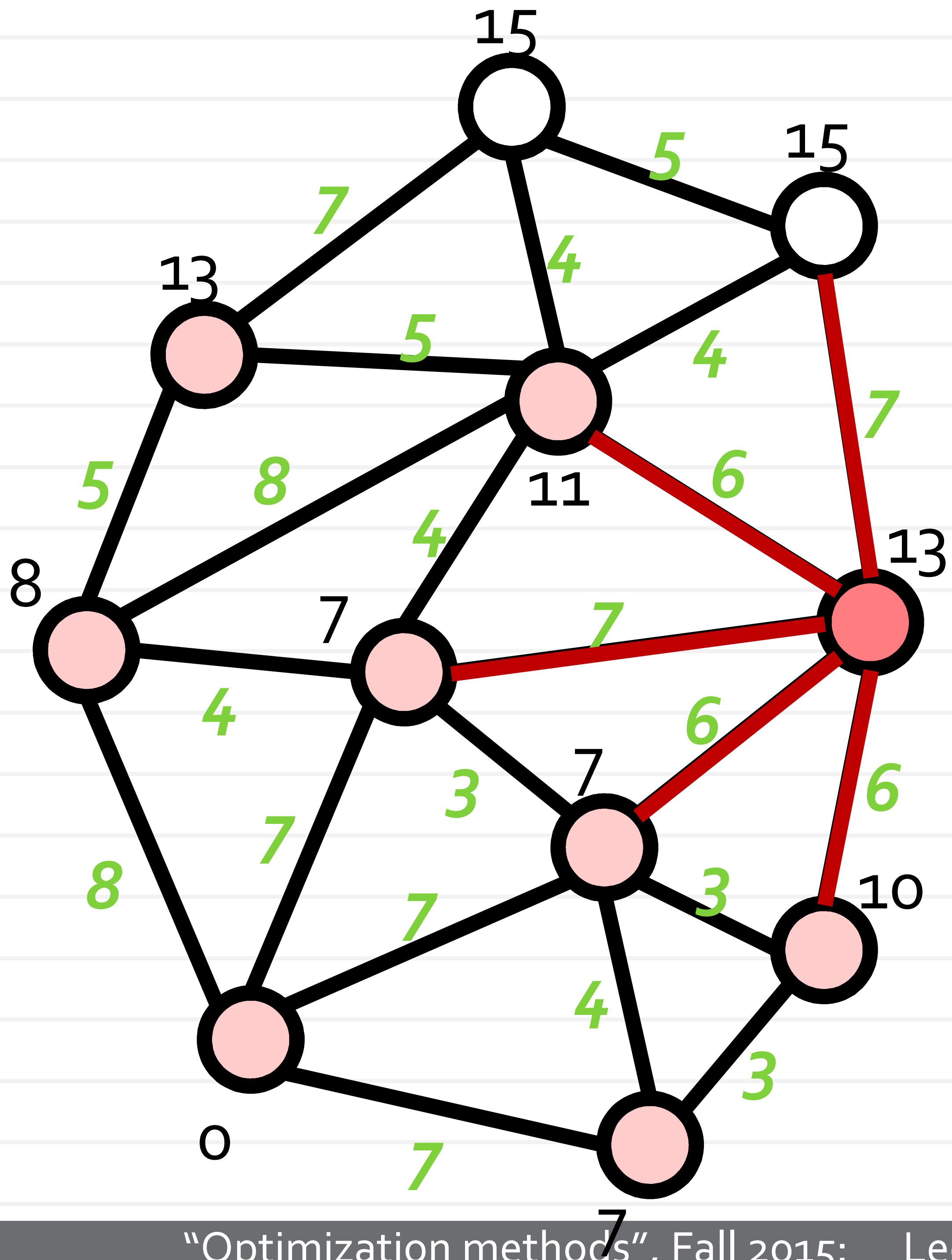


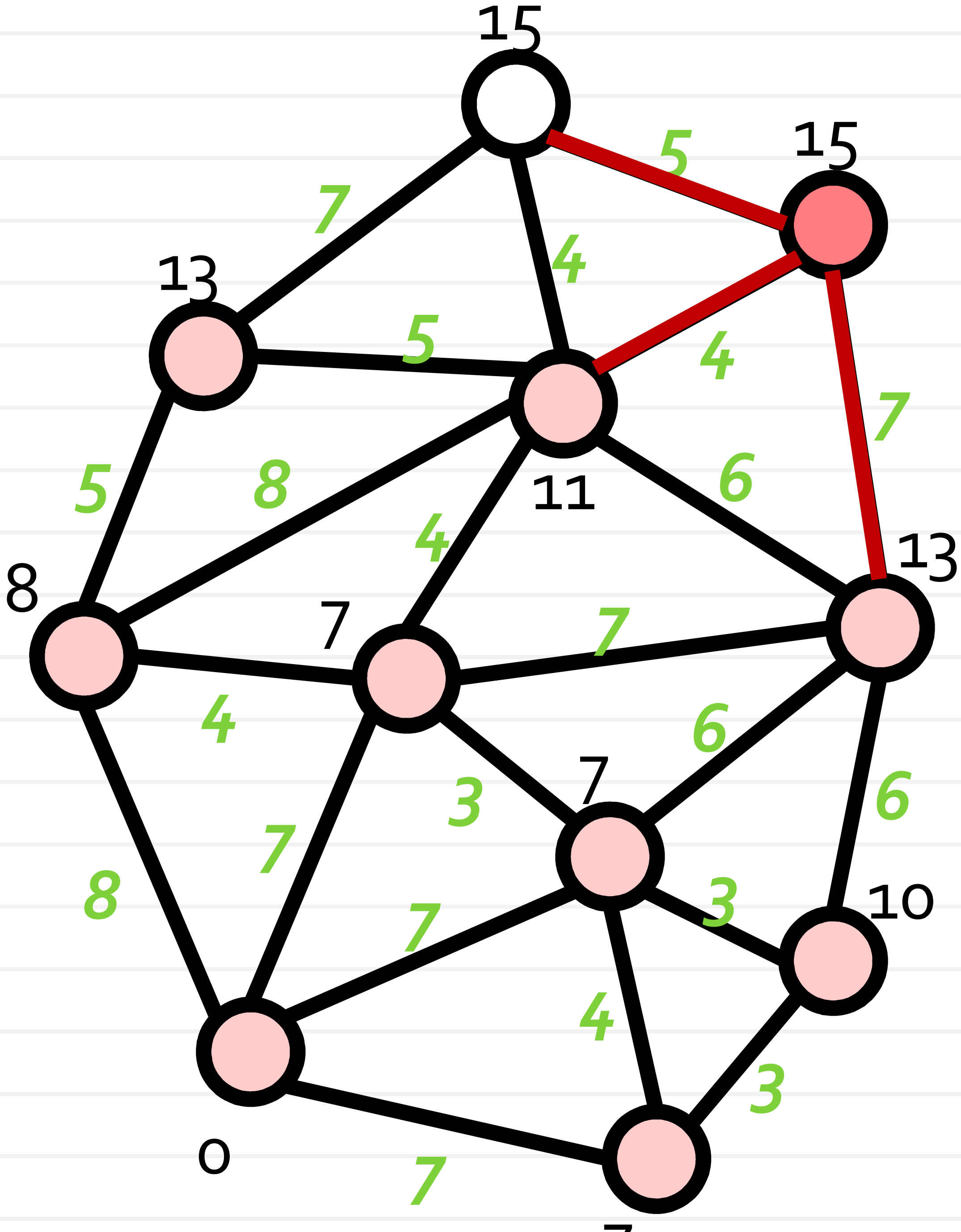
Dijkstra Algorithm



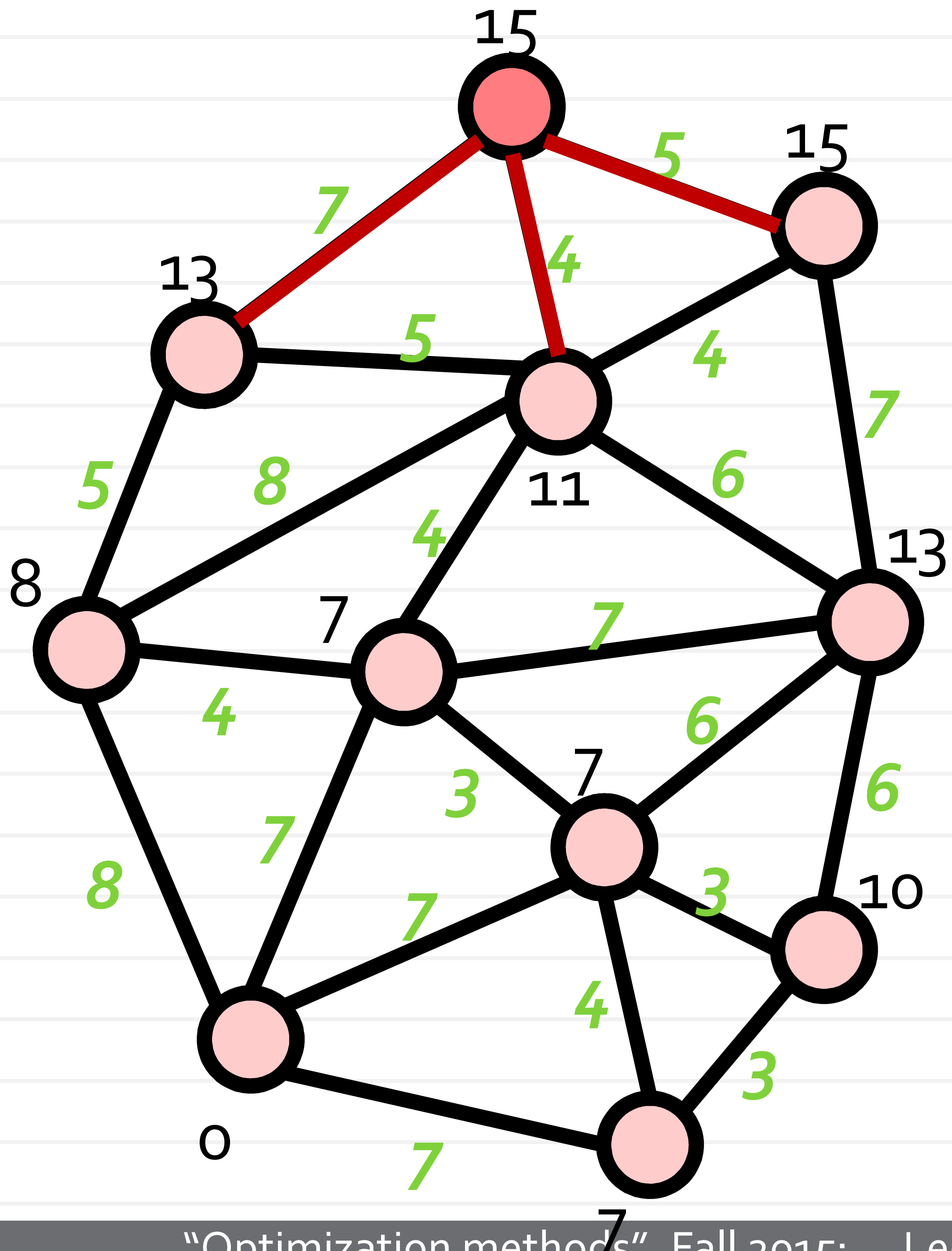
Dijkstra Algorithm



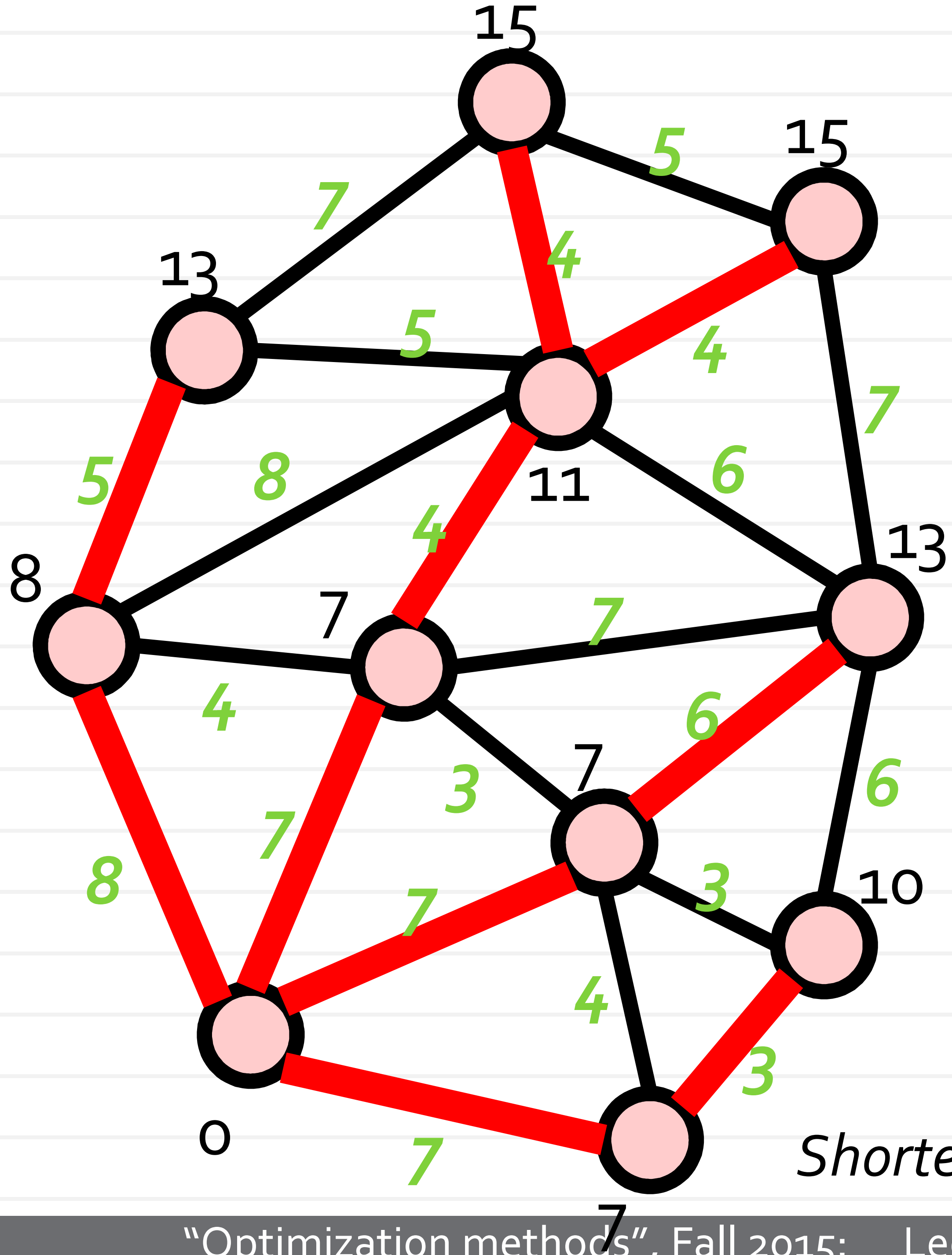
Dijkstra Algorithm



Dijkstra Algorithm

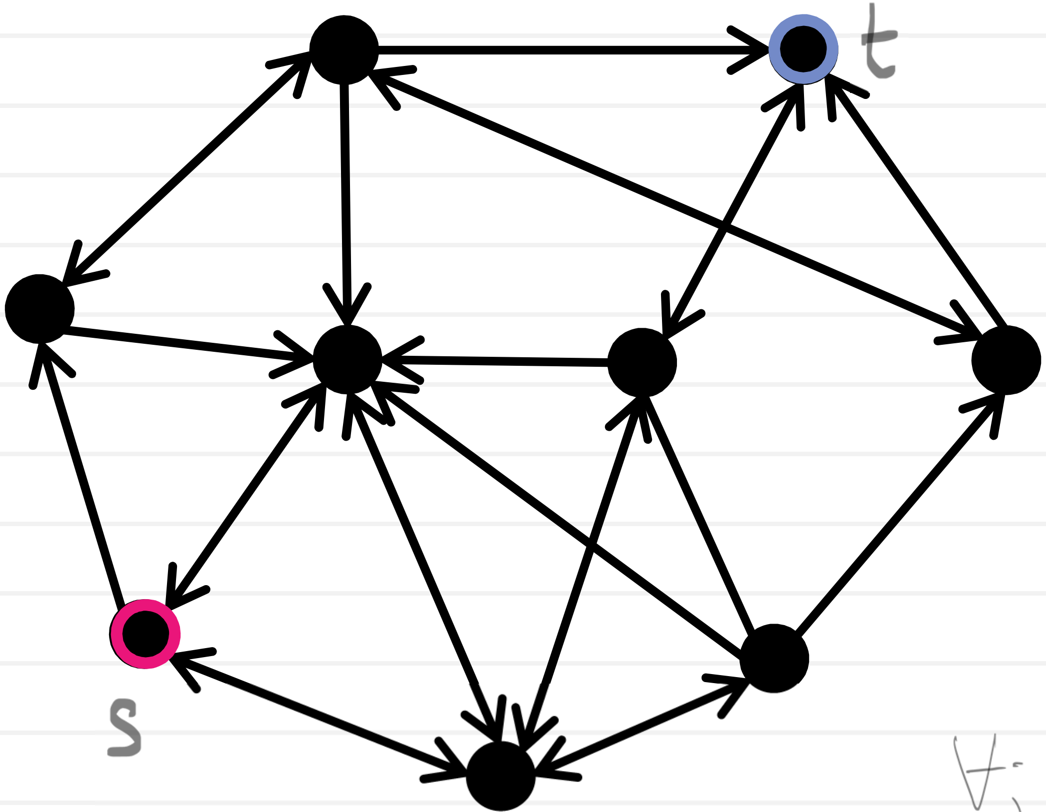


Dijkstra Algorithm



Shortest path spanning tree

Solving max-flow: Ford-Fulkerson



$$\max_{b,f} \; b$$

$$\text{s.t.:} \quad 0 \le f_{ij} \le u_{ij}$$

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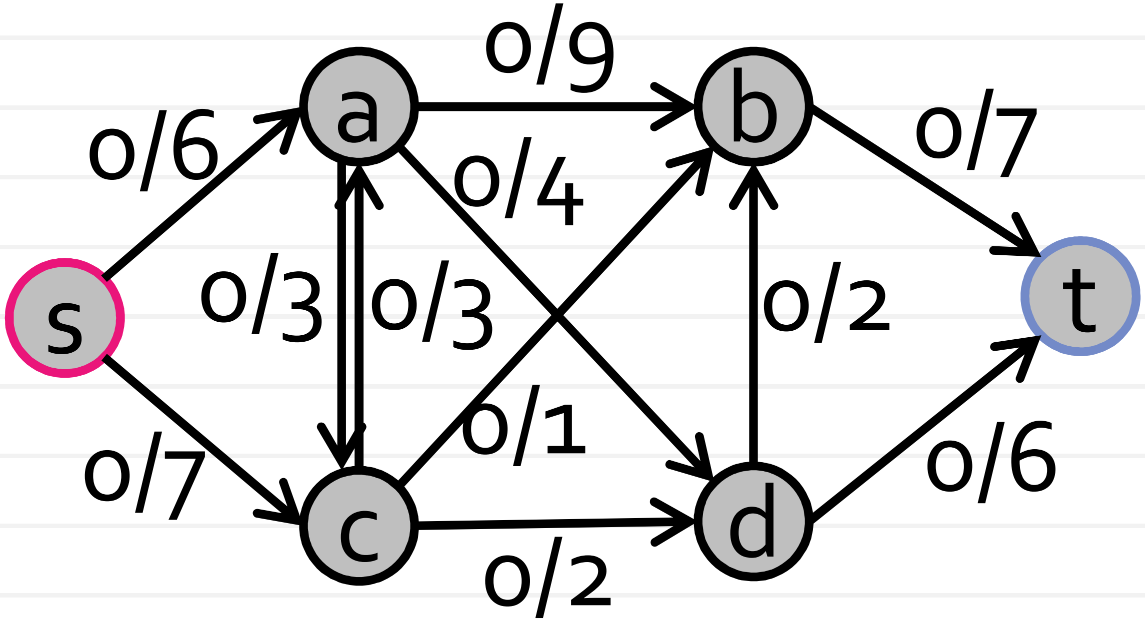
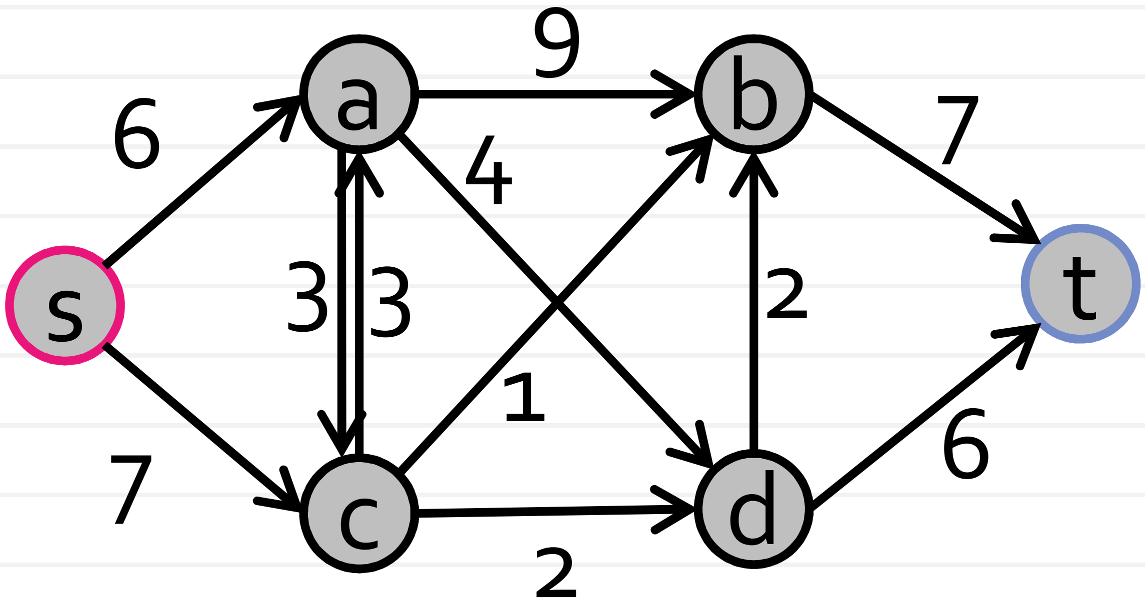
Solving max-flow: Ford-Fulkerson



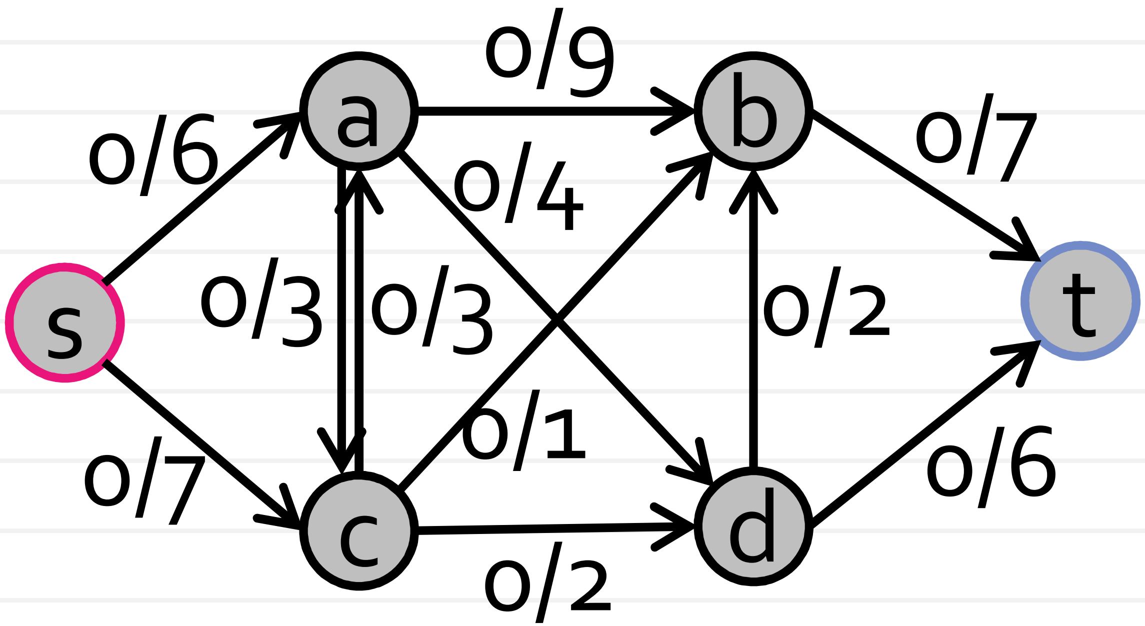
Original graph

Flow = 0



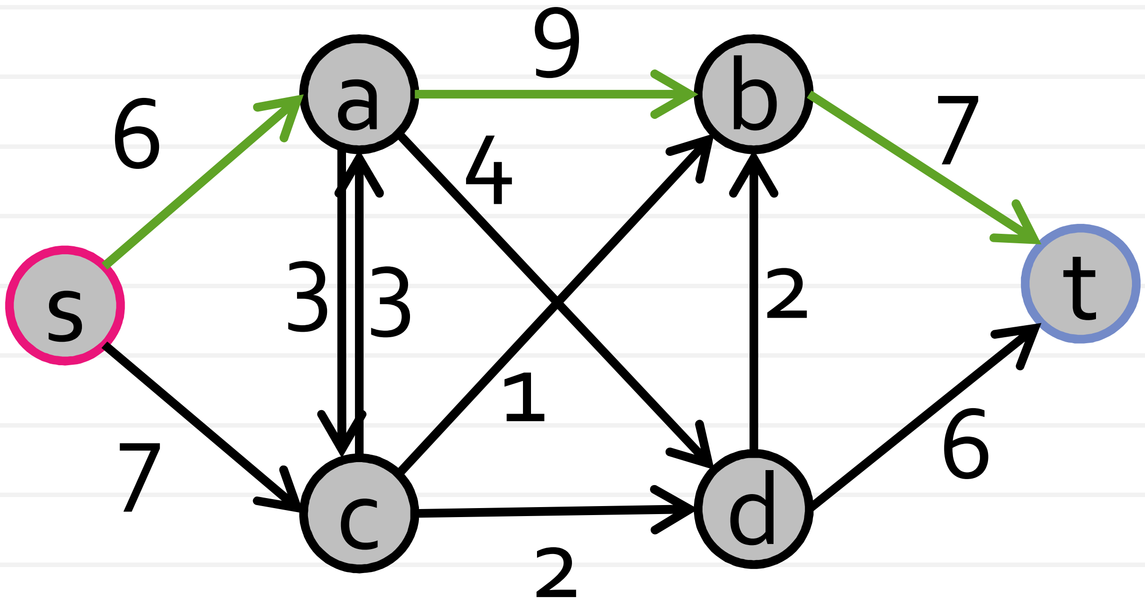
Residual graph

Solving max-flow: Ford-Fulkerson



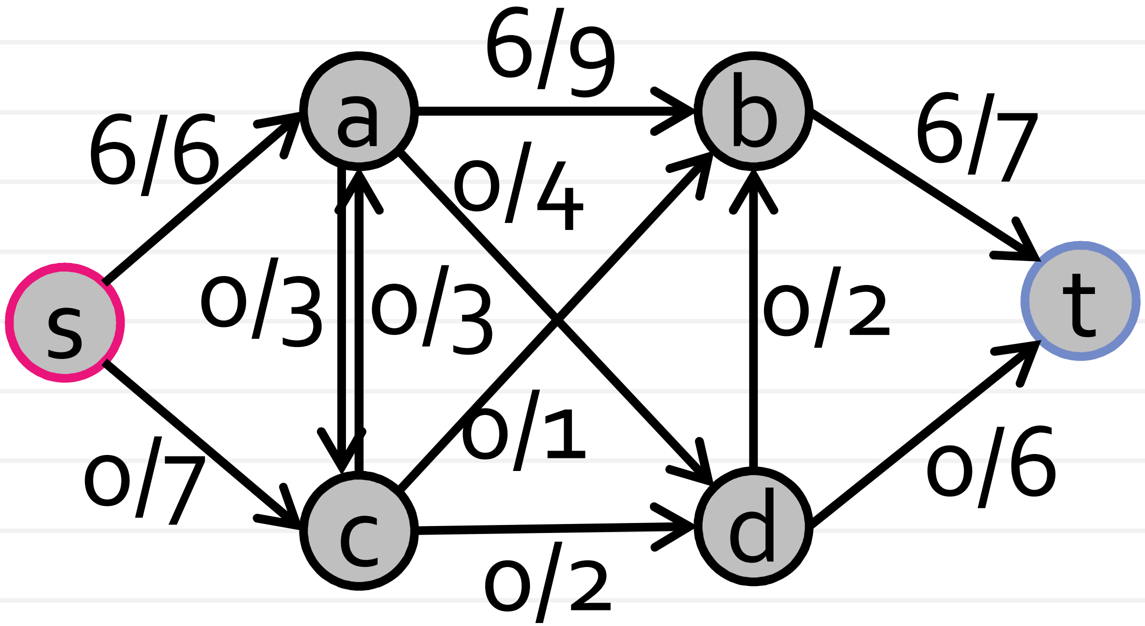
Original graph

Flow = 0



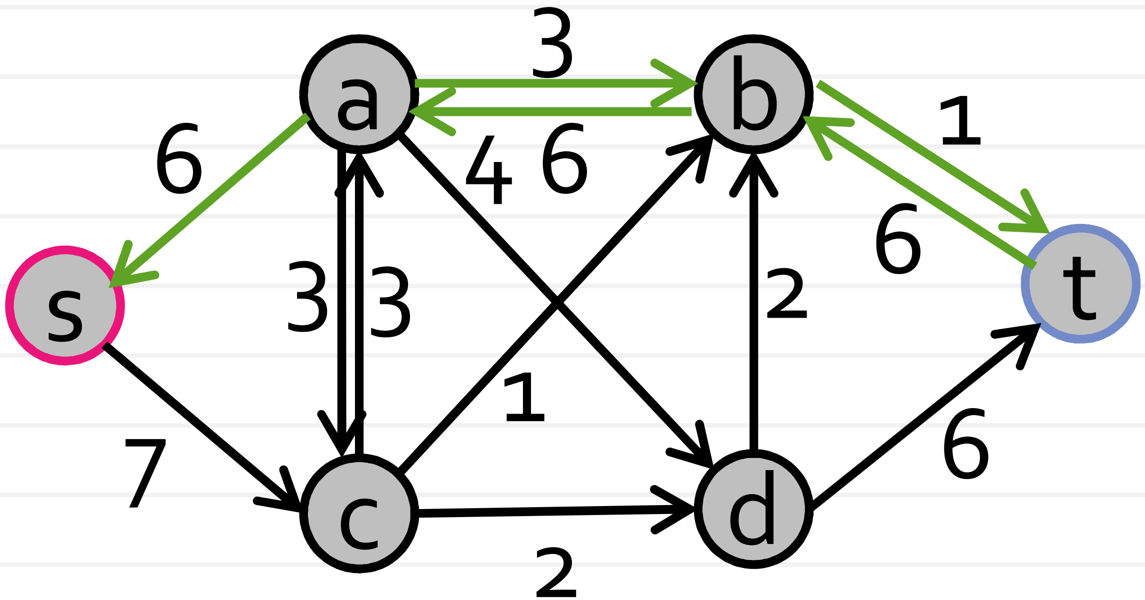
Residual graph

Solving max-flow: Ford-Fulkerson



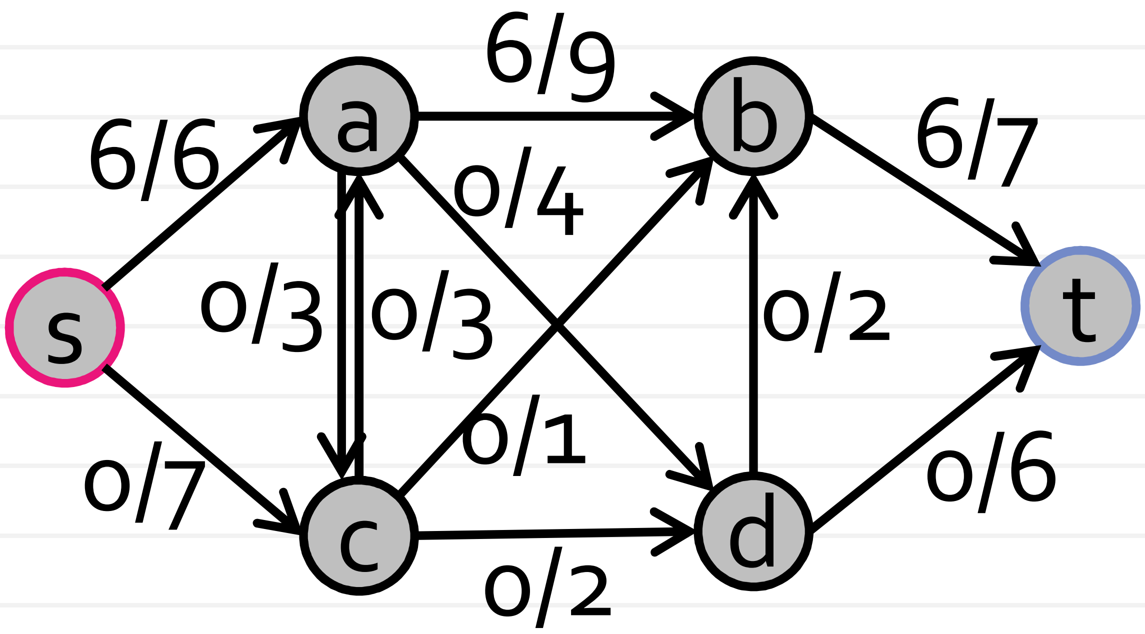
Original graph

Flow = 6



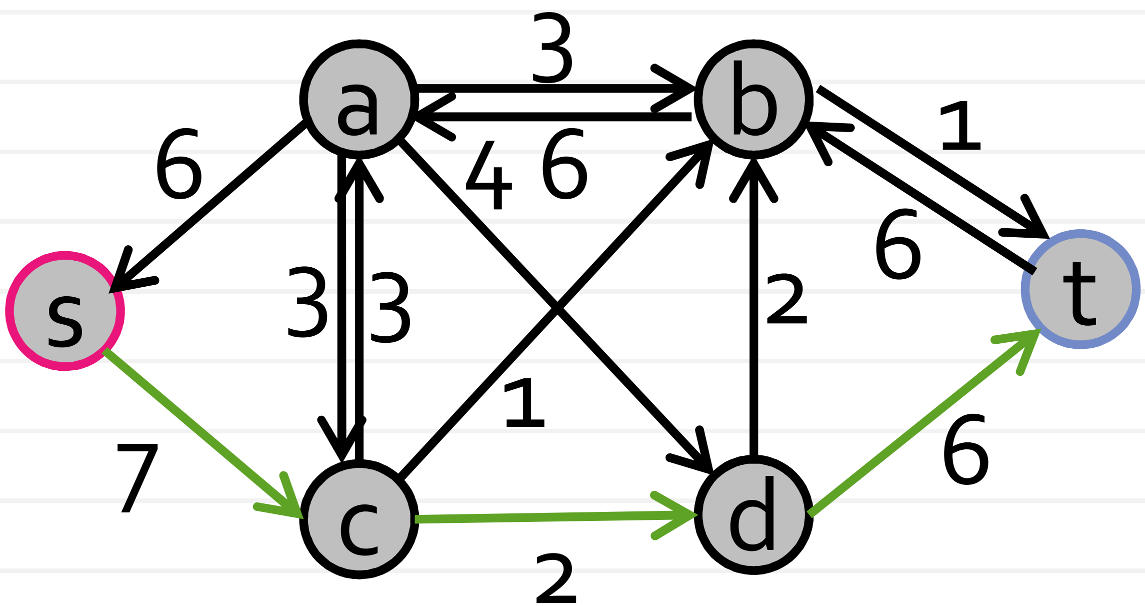
Residual graph

Solving max-flow: Ford-Fulkerson



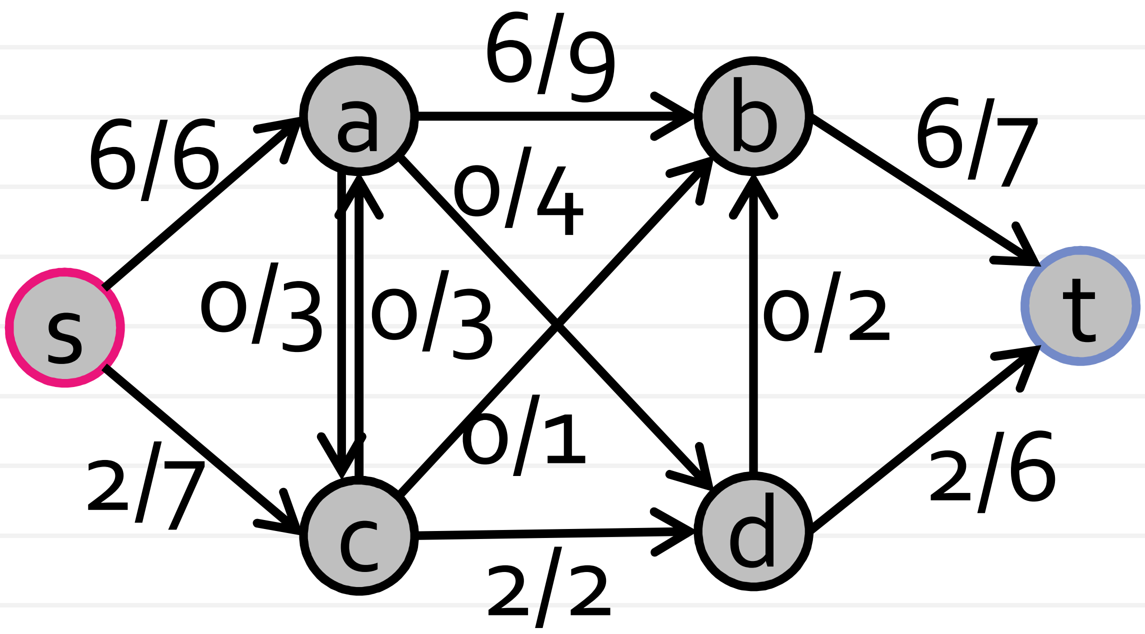
Original graph

Flow = 6



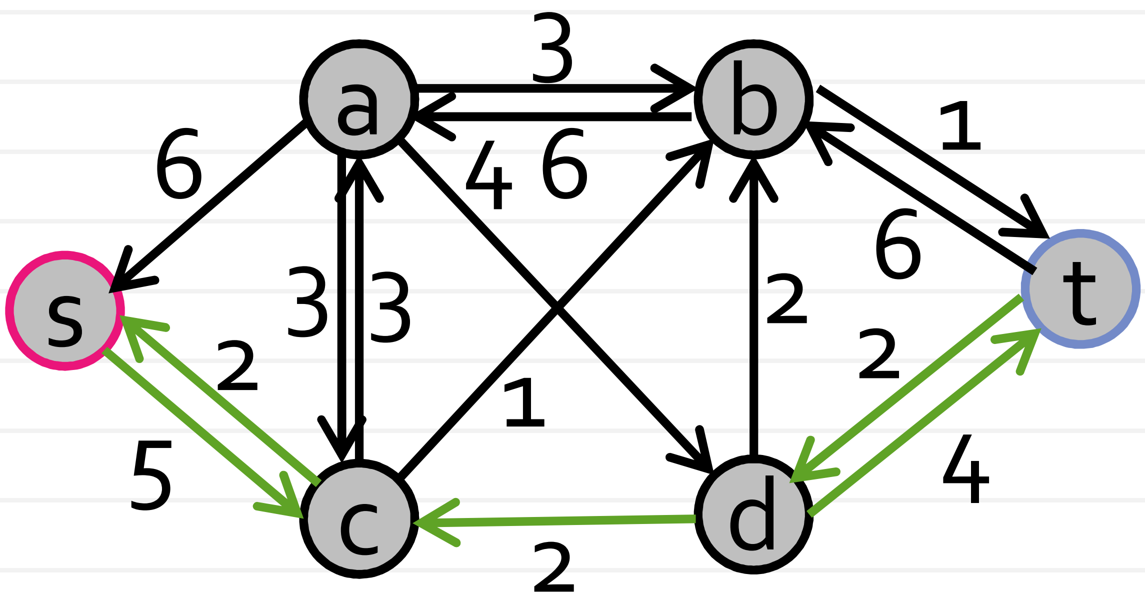
Residual graph

Solving max-flow: Ford-Fulkerson



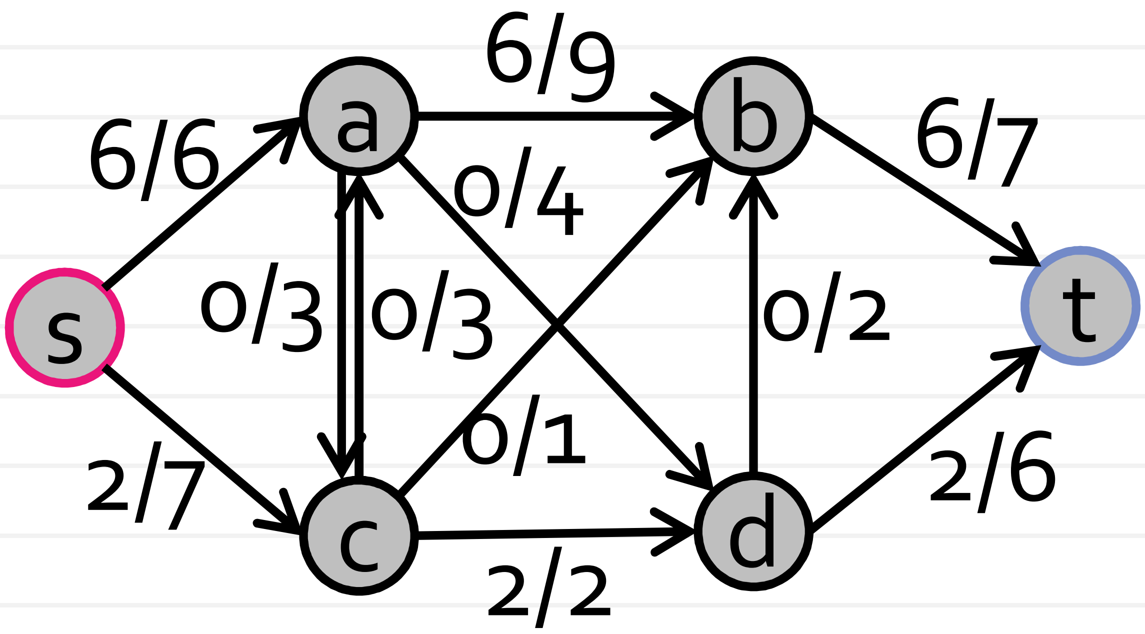
Original graph

Flow = 8



Residual graph

Solving max-flow: Ford-Fulkerson



Original graph

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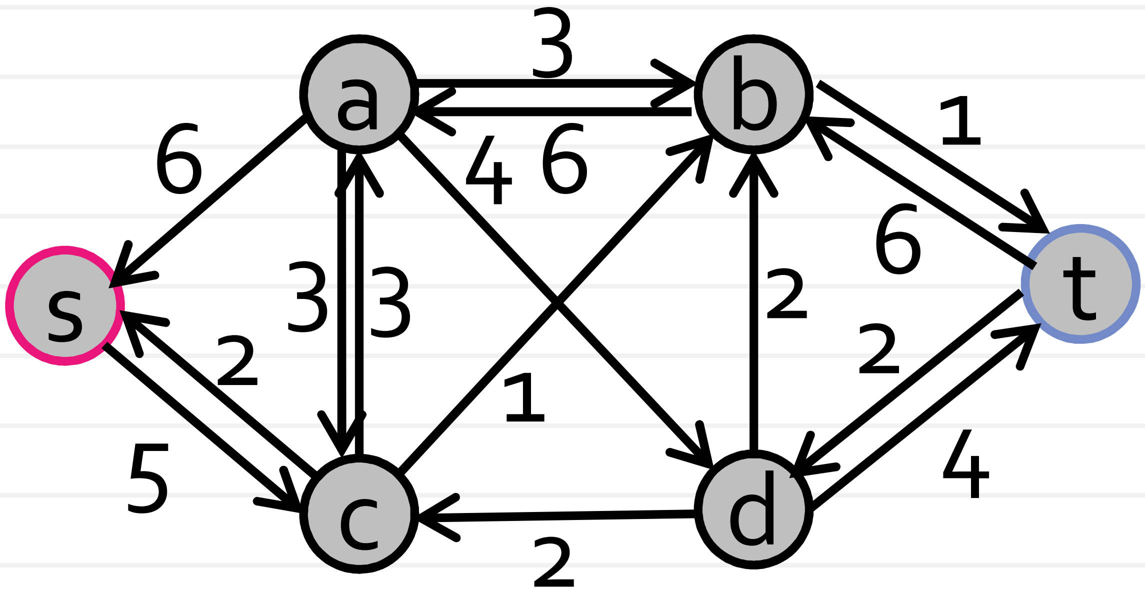
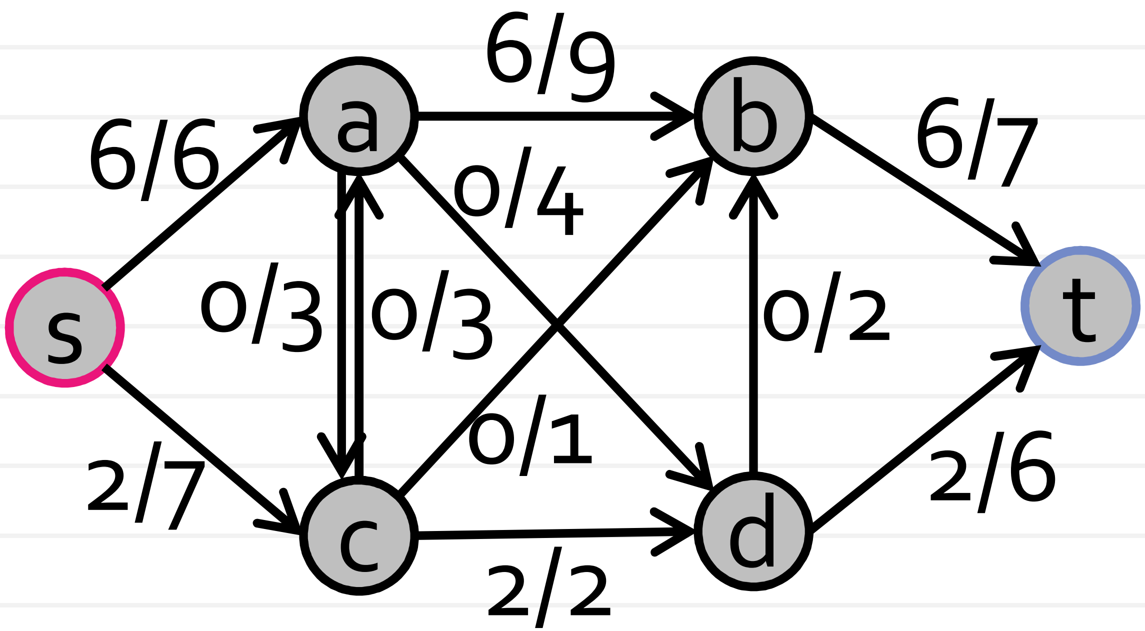


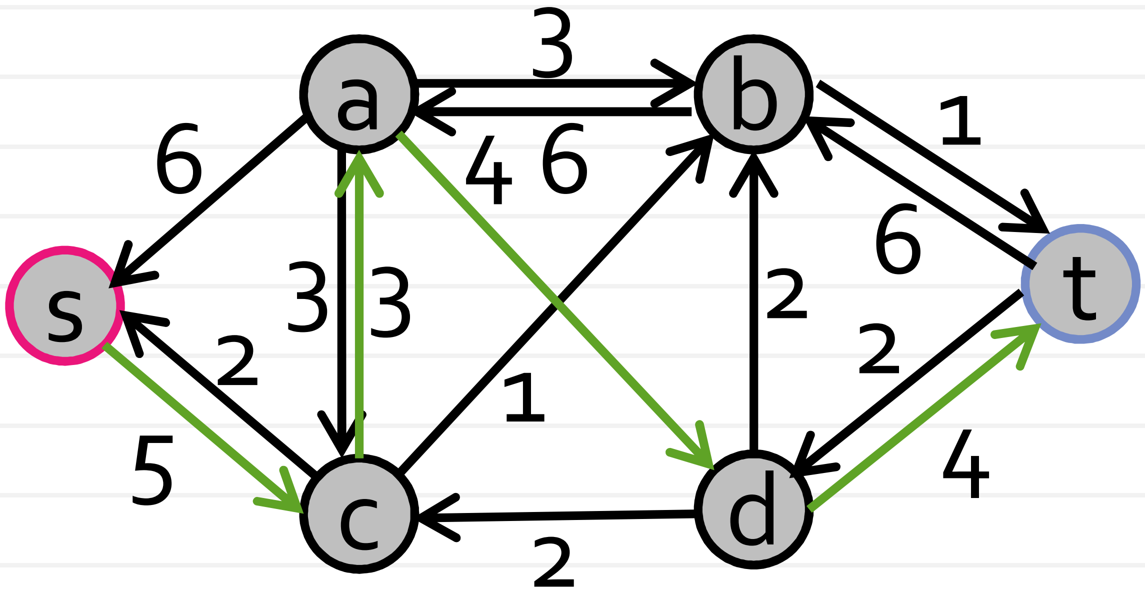
Residual graph

Solving max-flow: Ford-Fulkerson



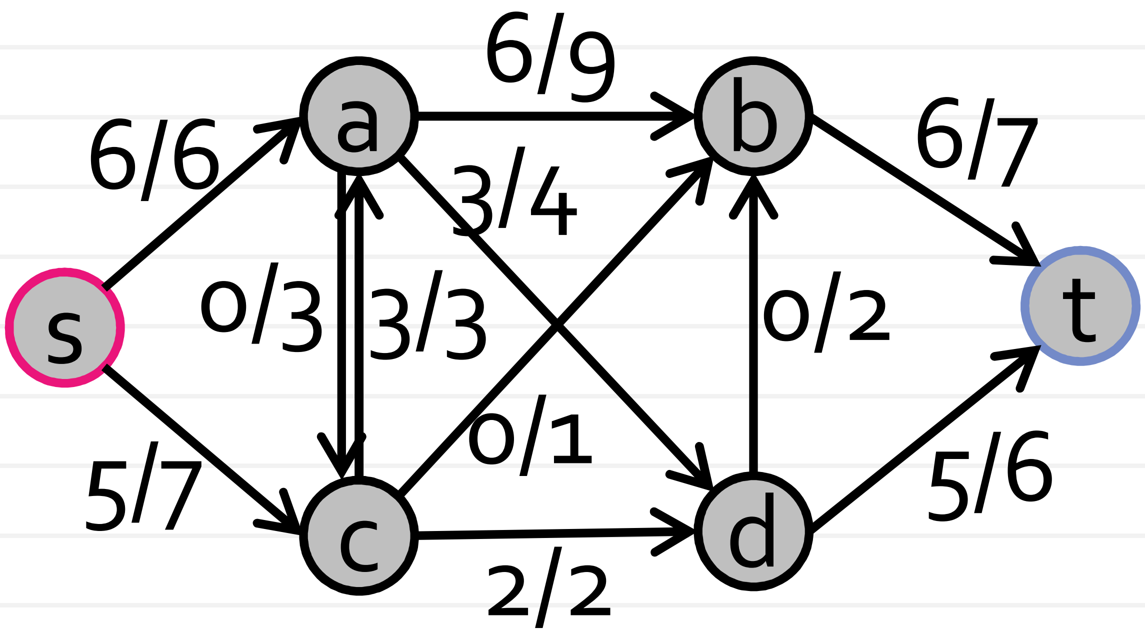
Original graph

Flow = 8



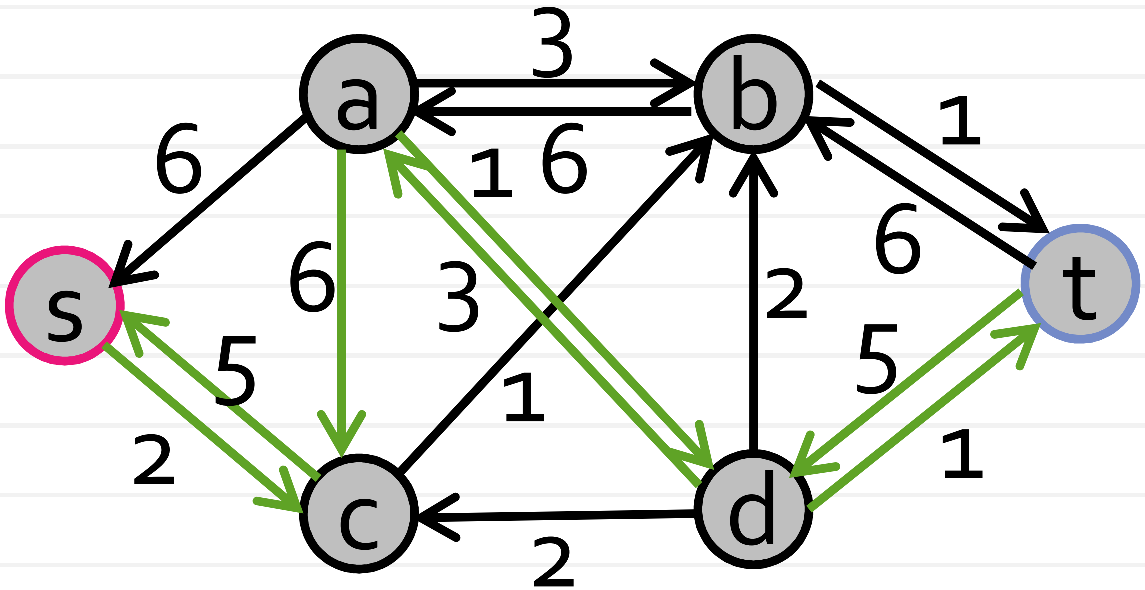
Residual graph

Solving max-flow: Ford-Fulkerson



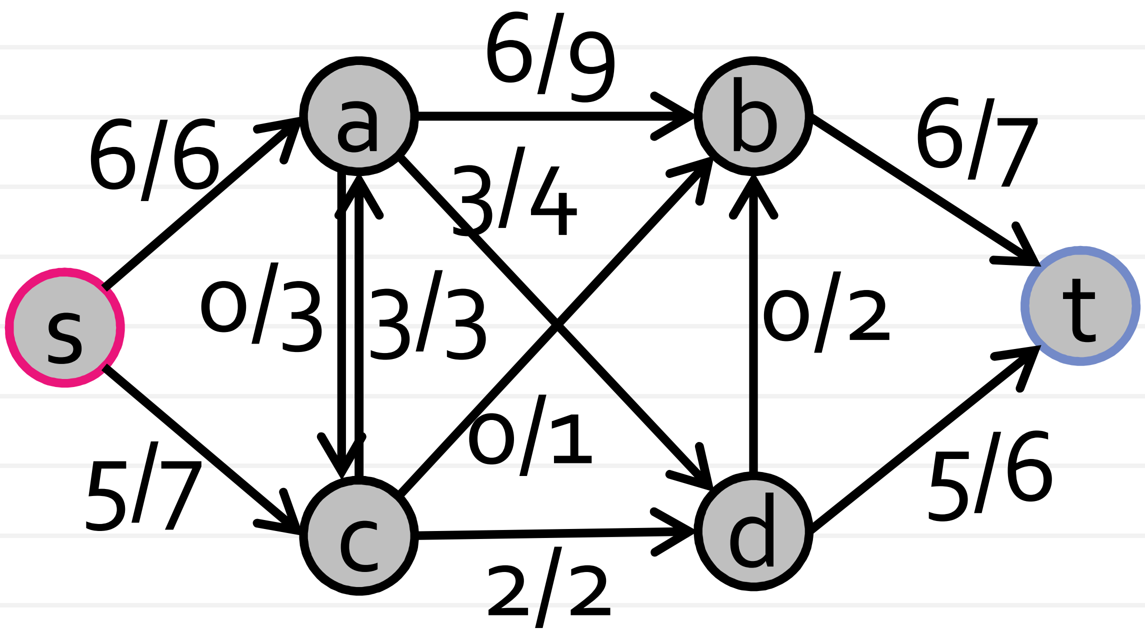
Original graph

Flow = 11



Residual graph

Solving max-flow: Ford-Fulkerson



Original graph

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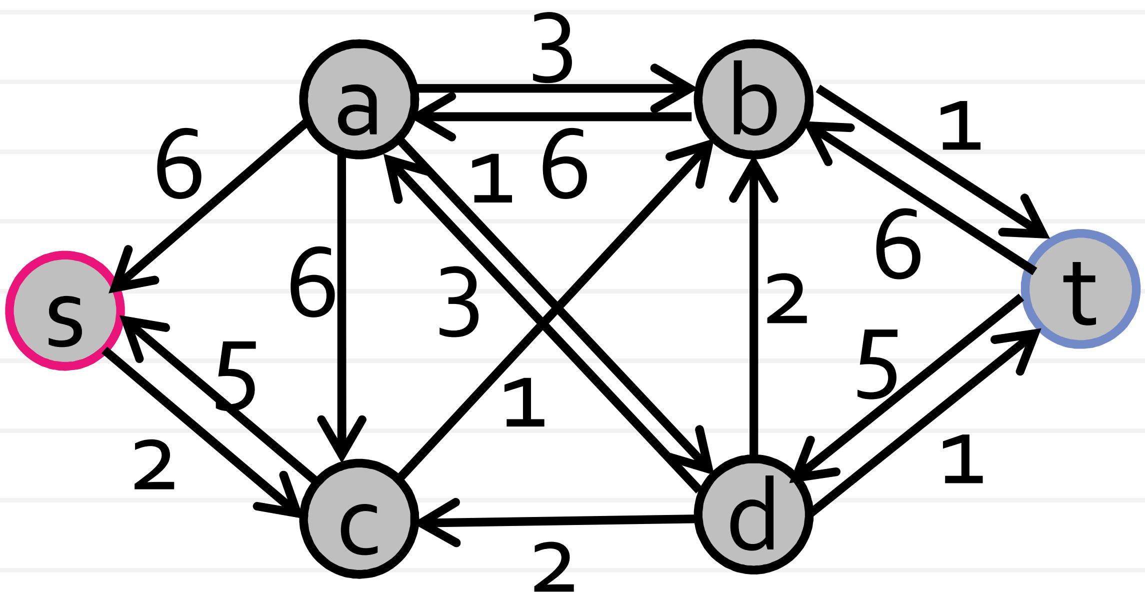
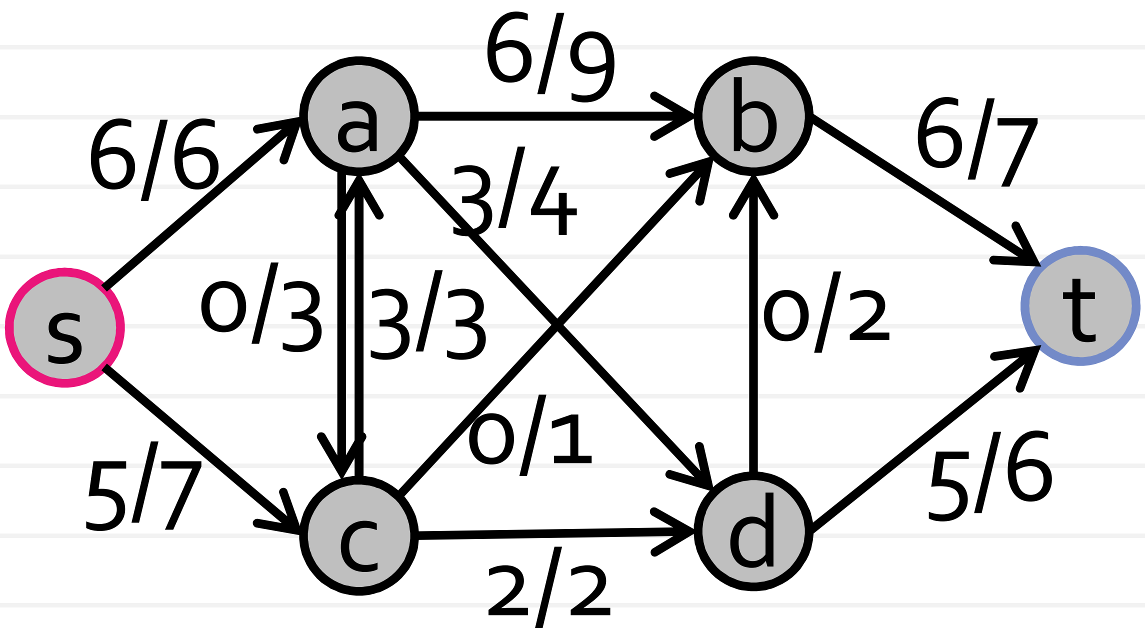


Residual graph

Solving max-flow: Ford-Fulkerson



Original graph

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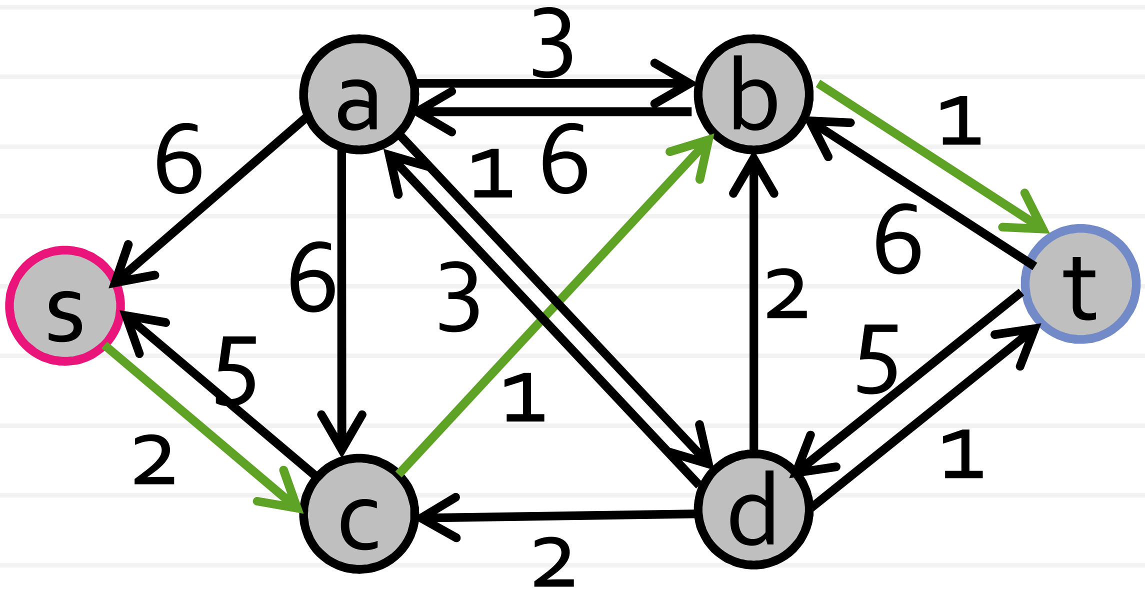
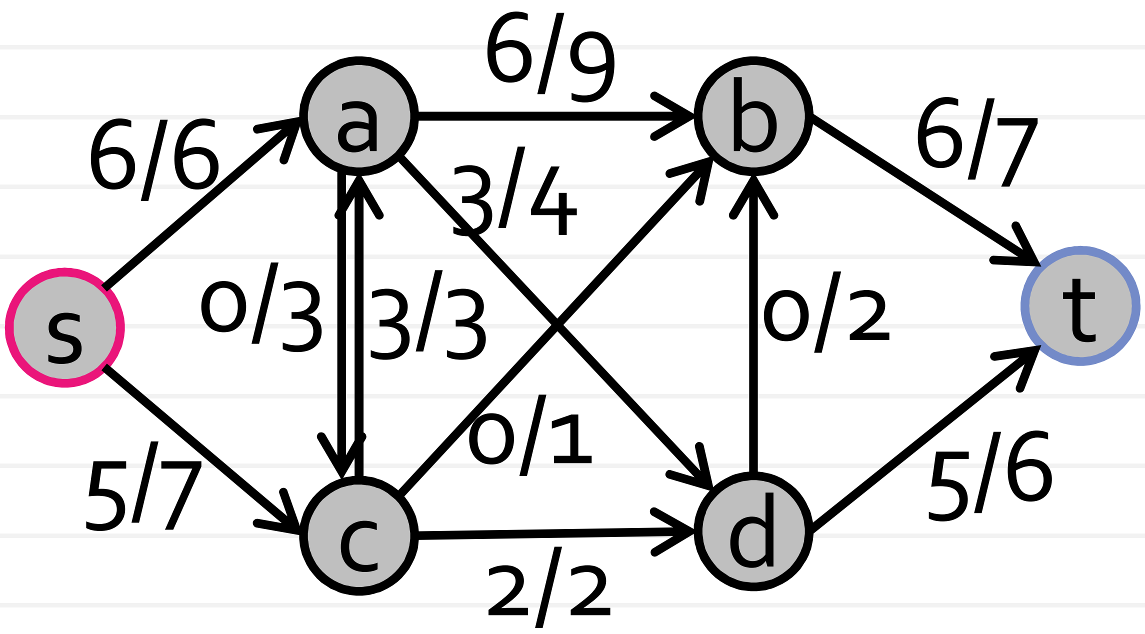


Residual graph

Solving max-flow: Ford-Fulkerson



Original graph

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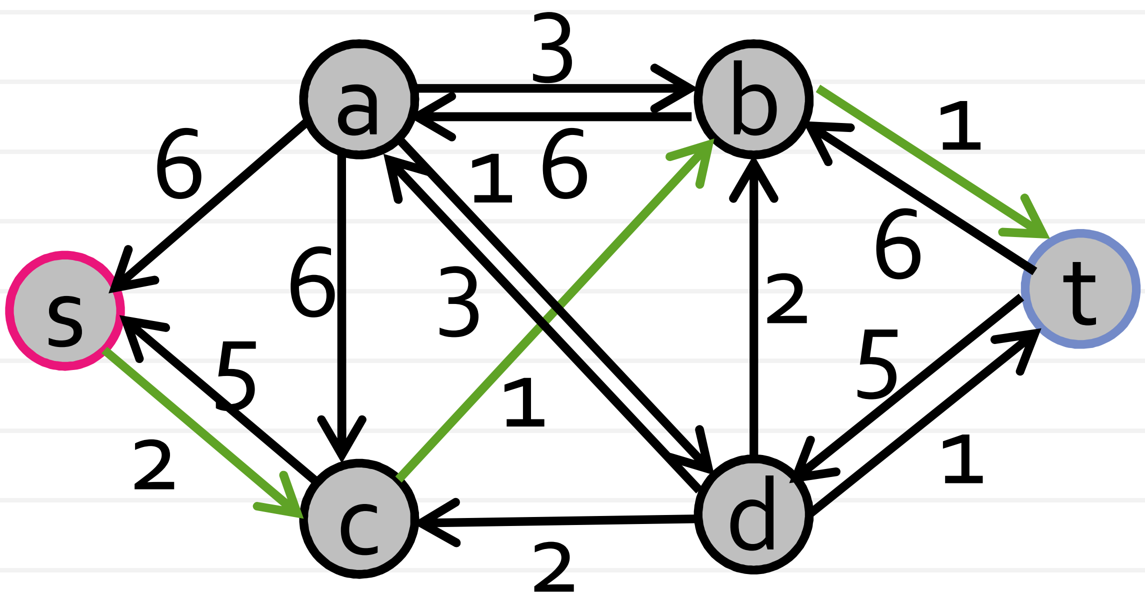
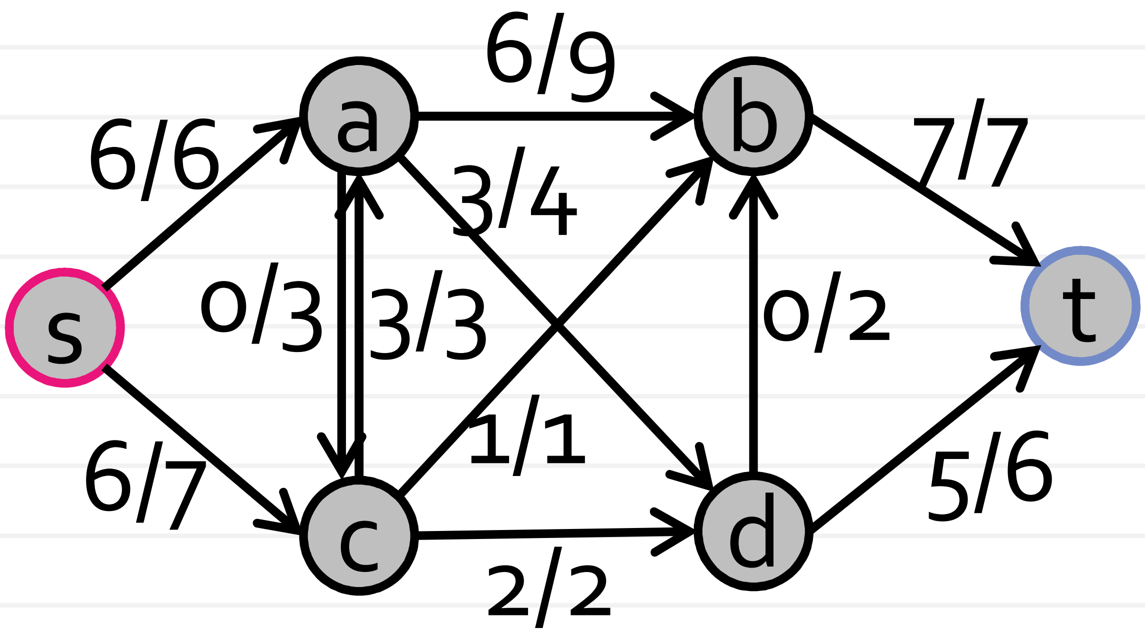


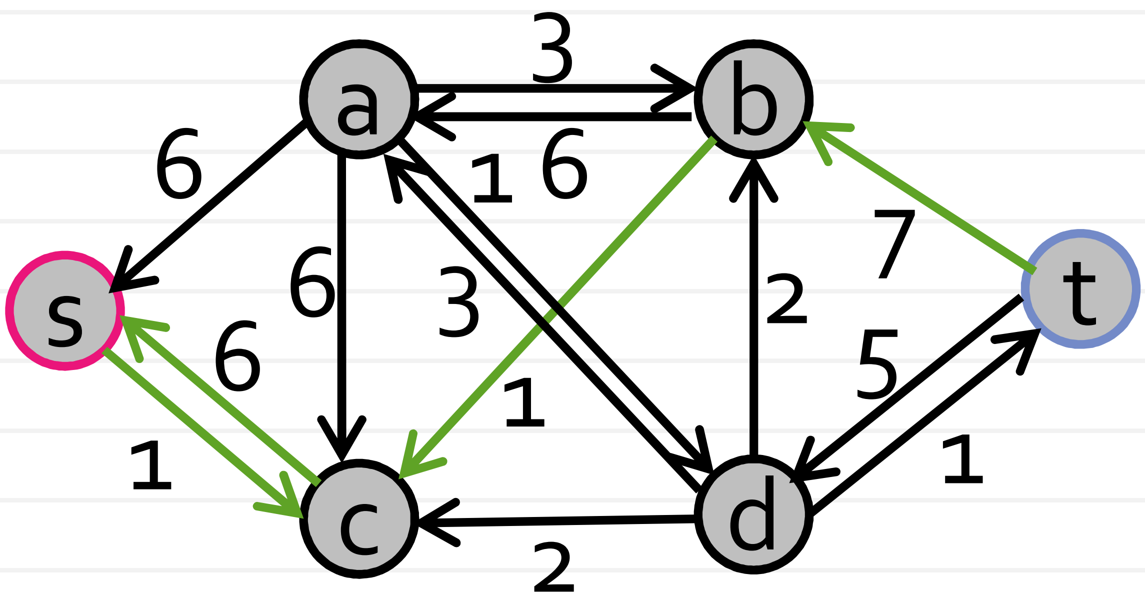
Residual graph

Solving max-flow: Ford-Fulkerson



Original graph

Flow = 12



Residual graph

Solving max-flow: Ford-Fulkerson

How to choose the augmented path?

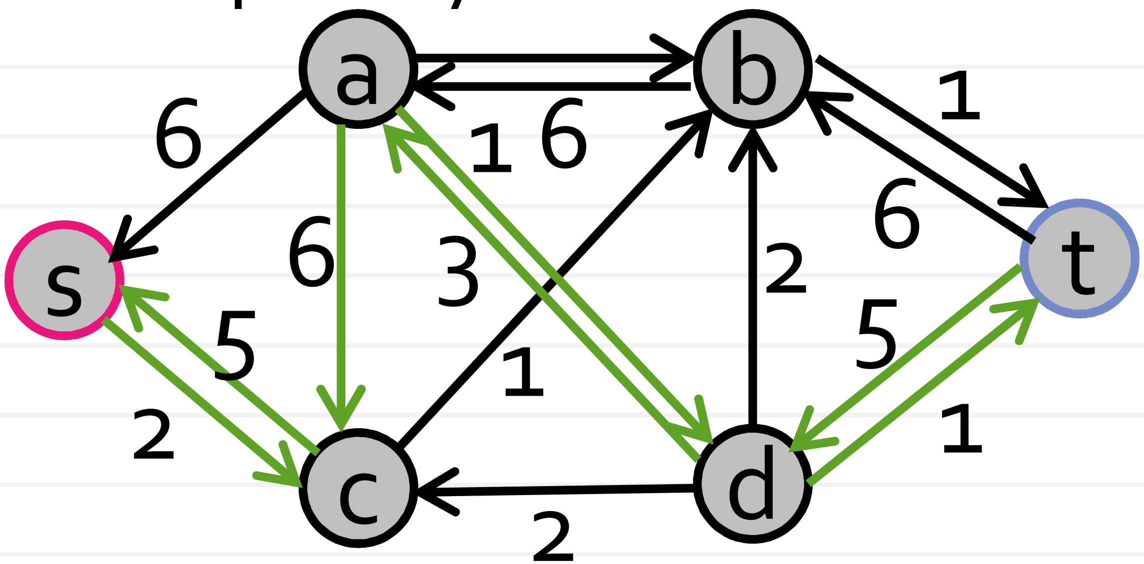
"Thickest" path: Ford-Fulkerson algorithm

- O(FE) – pseudopolynomial complexity

"Shortest" path: Edmonds-Karp algorithm

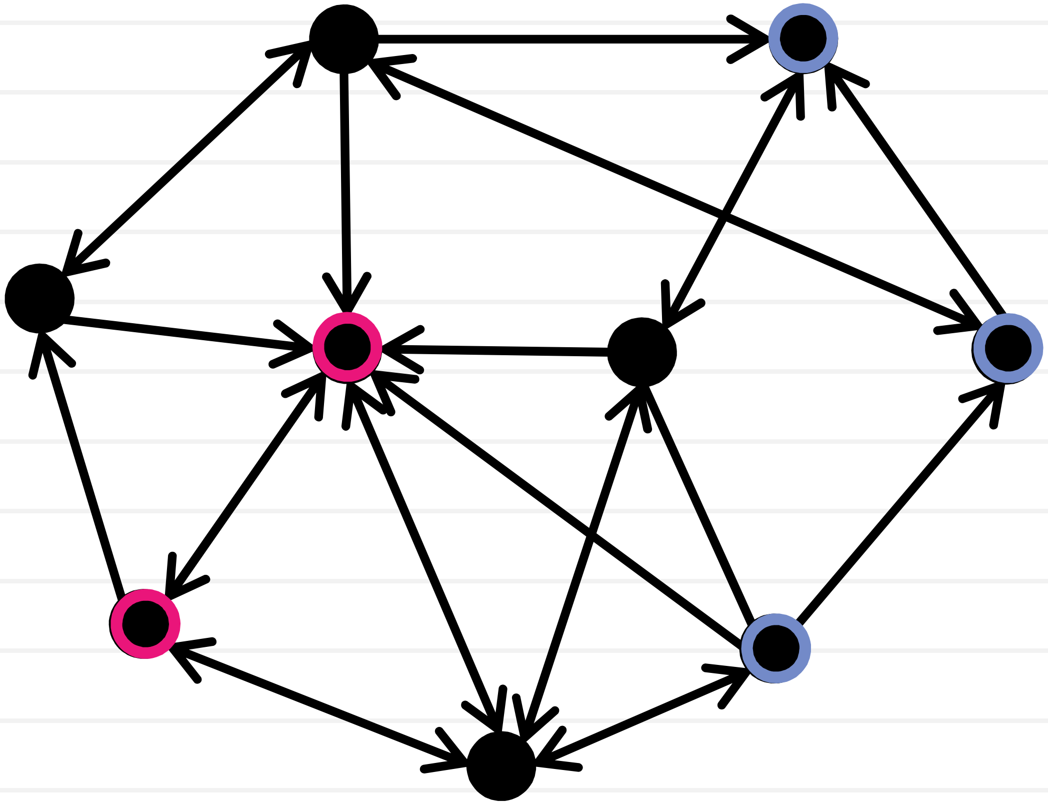
- $O(VE^2)$ – polynomial complexity

Current best: O(VE)



Network simplex for min-cost flow

Min-cost flow (*uncapacitated* version):



$$\min_f c^T f = \sum_{i,j} c_{ij} f_{ij}$$

$$\text{s.t.: } Af = b$$

$$\sum_{i \in O(j)} f_{ji} - \sum_{i \in I(j)} f_{ij} = b_j$$

$$f \geq 0$$

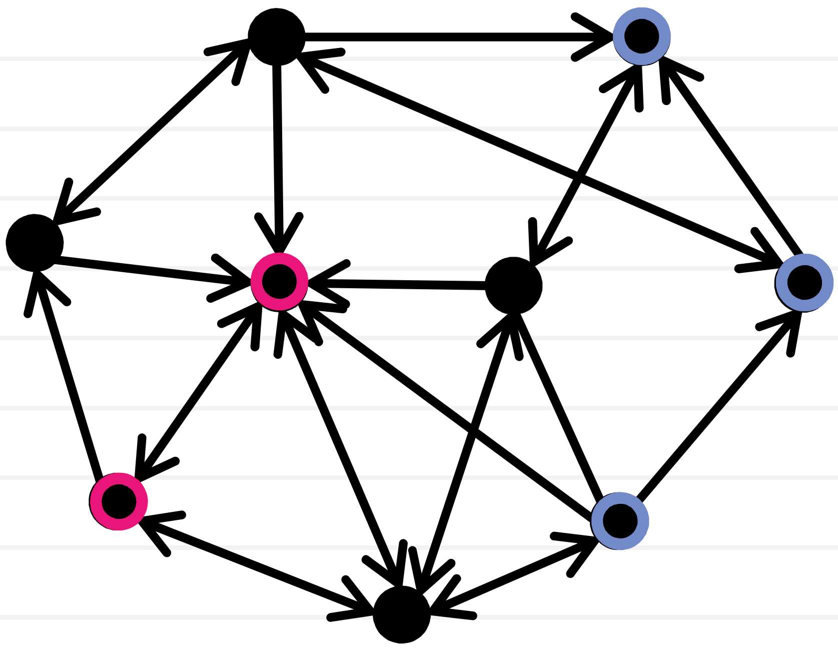
This is LP in a standard form!

|A| variables.

|V|-1 linearly-independent equations.

How do basic feasible solutions look like?

Network simplex ideas



$$\min_f c^T f = \sum_{i,j} c_{ij} f_{ij}$$

$$\text{s.t: } Af = b \quad f \geq 0$$

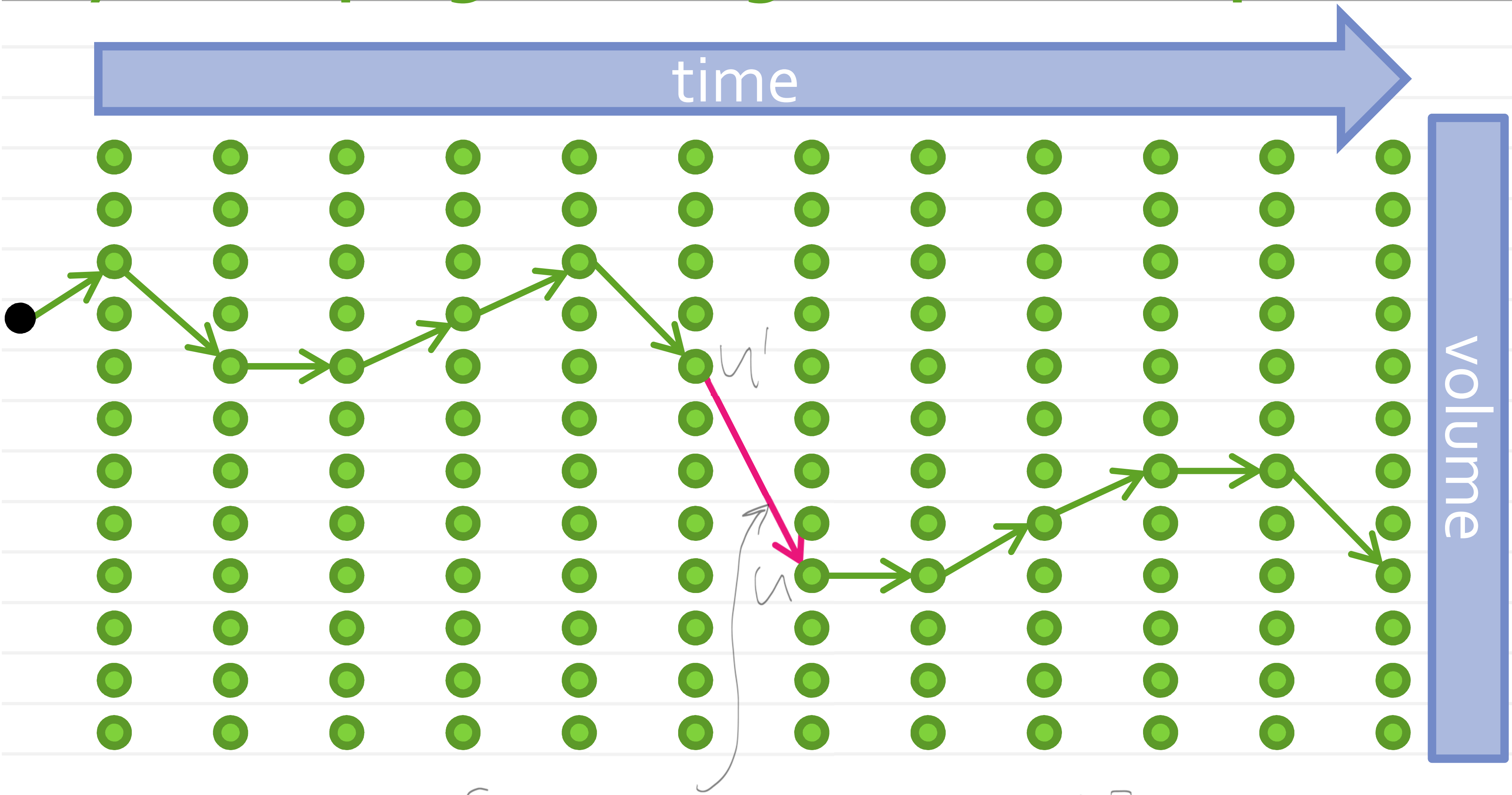
$$\sum_{i \in O(j)} f_{ji} - \sum_{i \in I(j)} f_{ij} = b_j$$

- Network simplex operates with basic solutions/canonical forms corresponding to spanning trees (of arcs with non-zero flow)
- Each time, an entering variable corresponds to an arc not in a tree
- During pivoting an arc is added to the tree creating a cycle, then one of the arcs in the cycle leaves the basic feasible solution

Networks and optimization

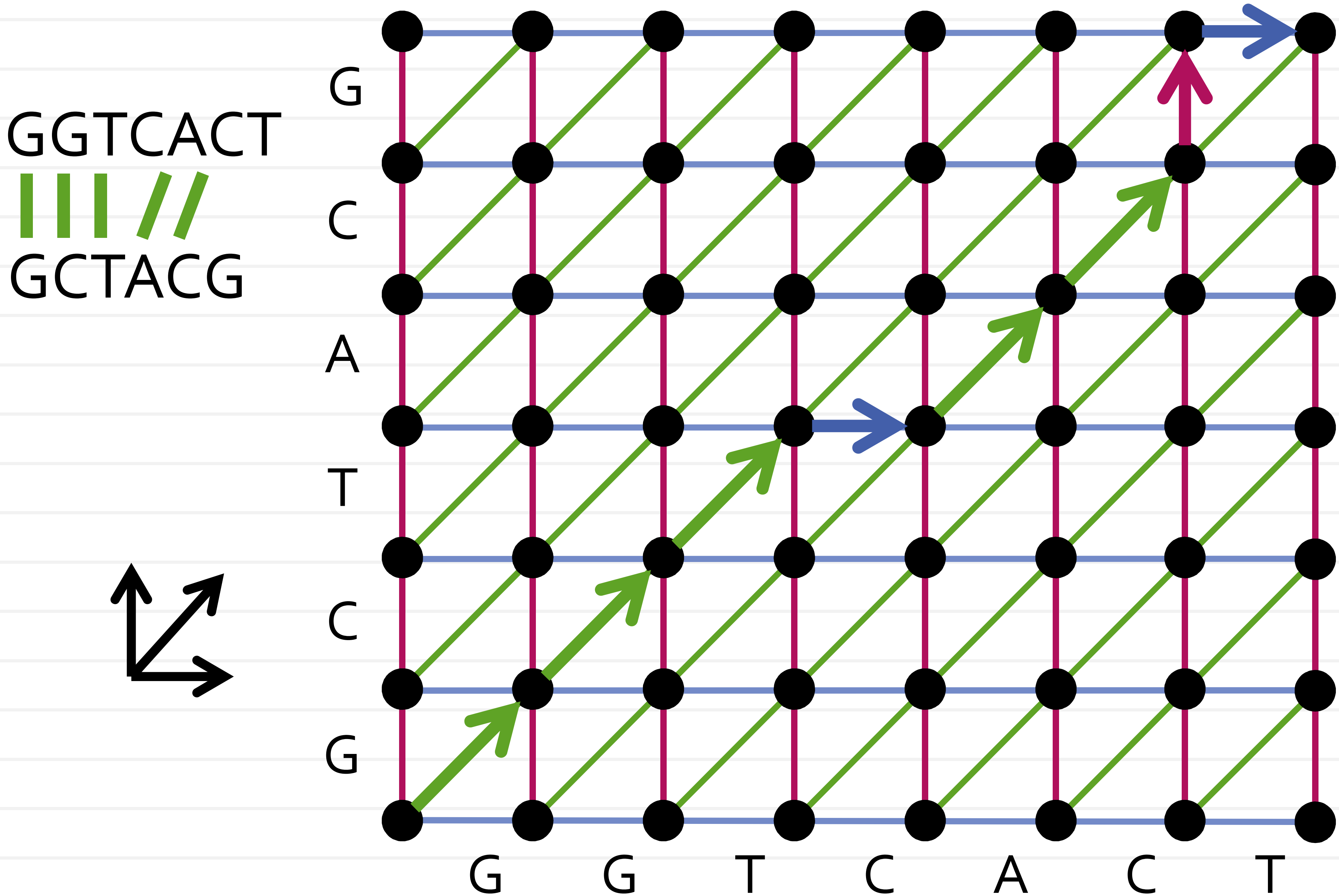
- A lot of optimization problems can be reduced to network optimization
- Due to integrality property, such reductions are very useful for integer programming

Dynamic programming and shortest paths

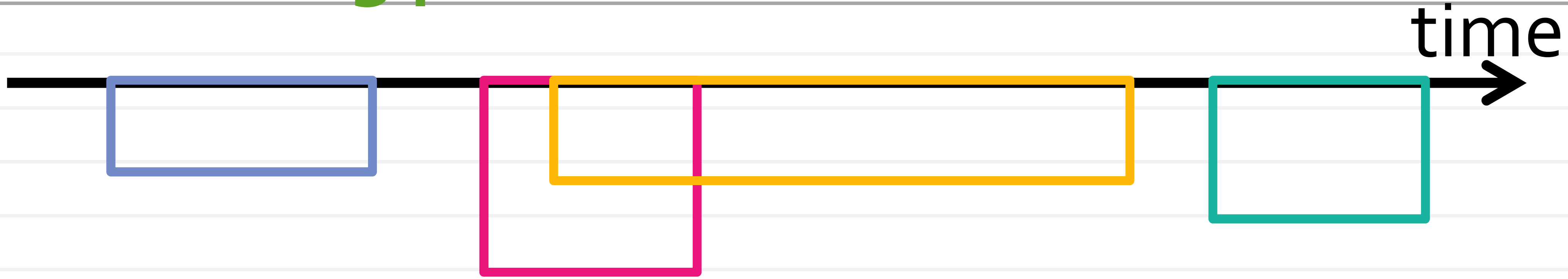


$$\varphi\left[d_n - \psi\left(u - u' + f_n - s(u')\right)\right]$$

Dynamic programming and shortest paths

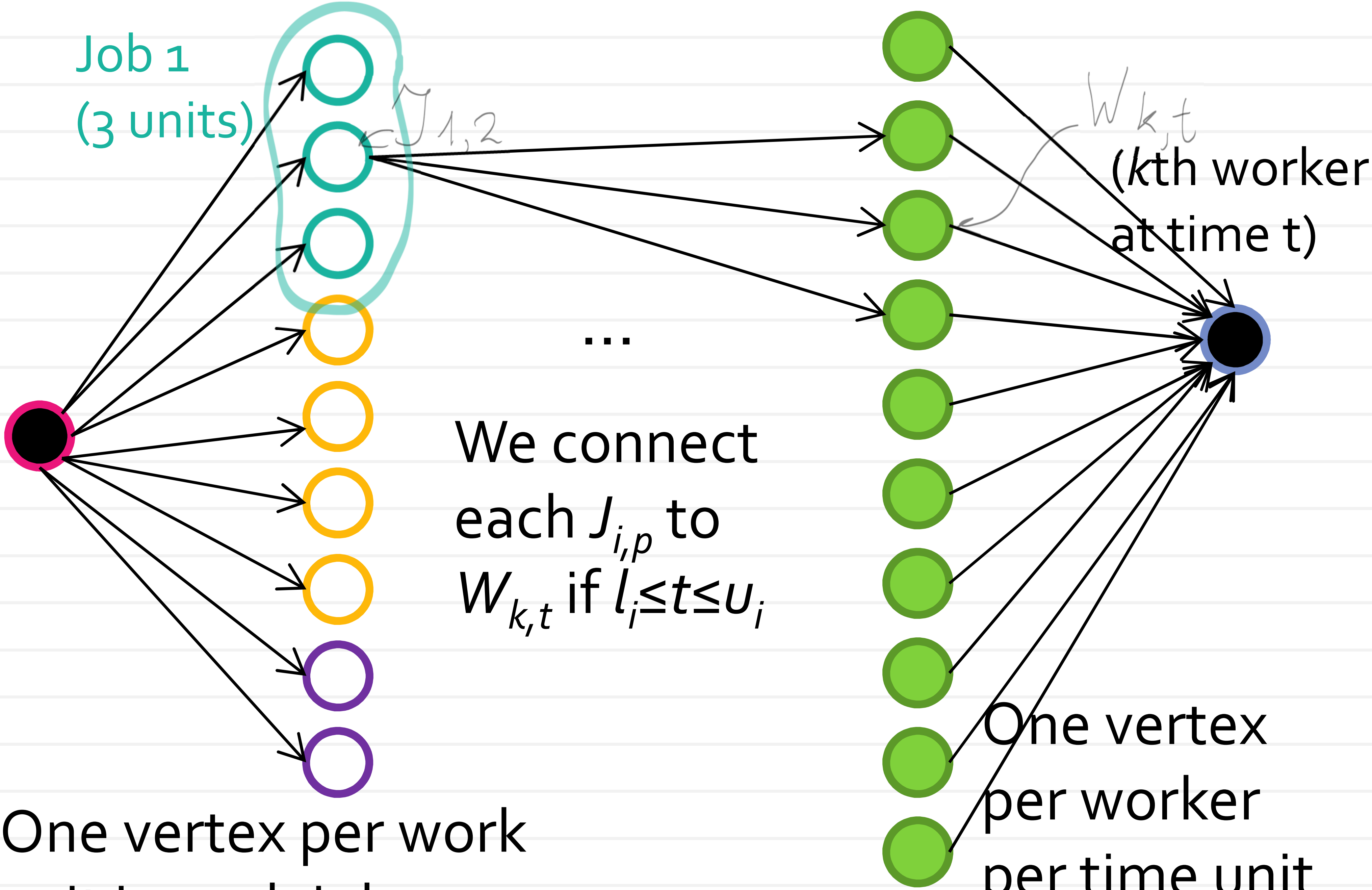


Scheduling problem



- We have a sequence of jobs
- Each job *i* is characterized by the amount of work w_i (in work units), the release date l_i, and the due date u_i
- We have workers, each of which work on a certain schedule (can be arbitrary) and can do one unit of work per unit time
- How to schedule to get as much work done as possible?

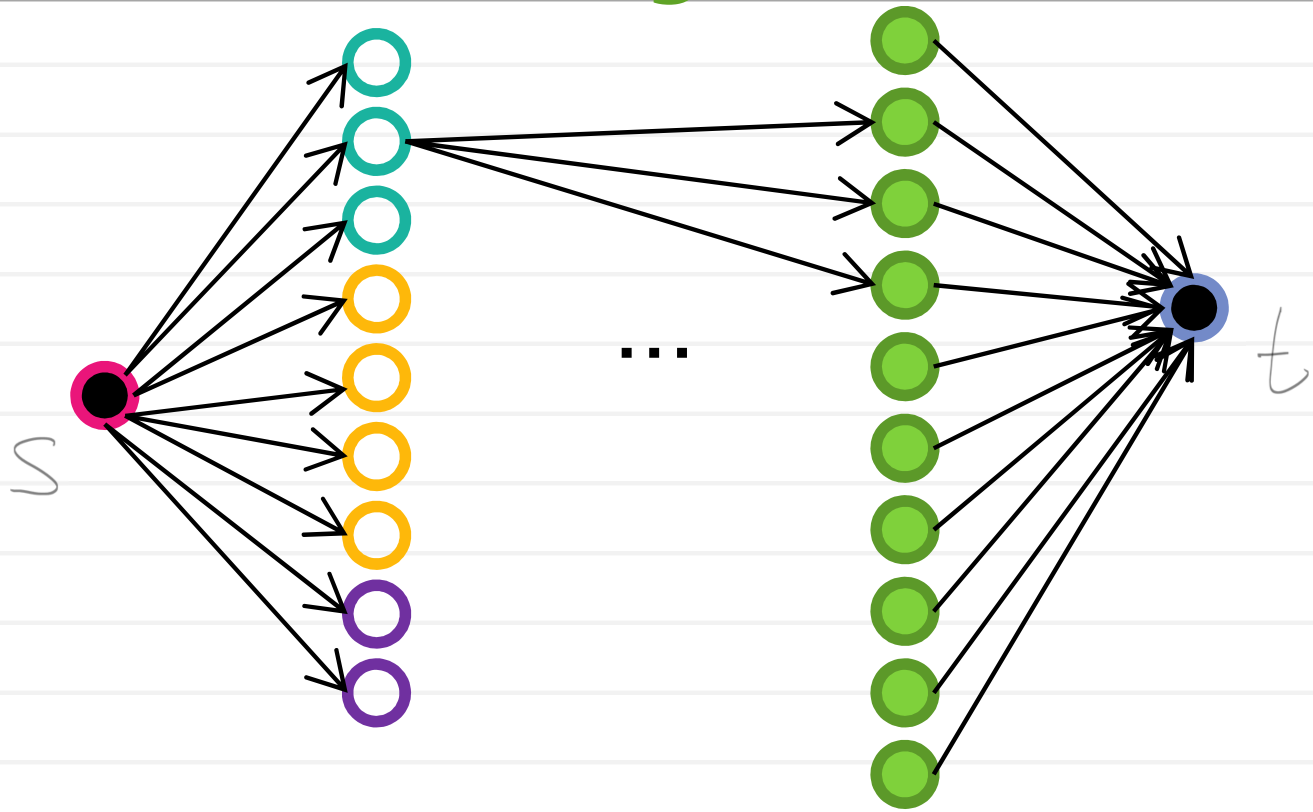
Solution to scheduling



We connect each $J_{i,p}$ to $W_{k,t}$ if $l_i \le t \le u_i$

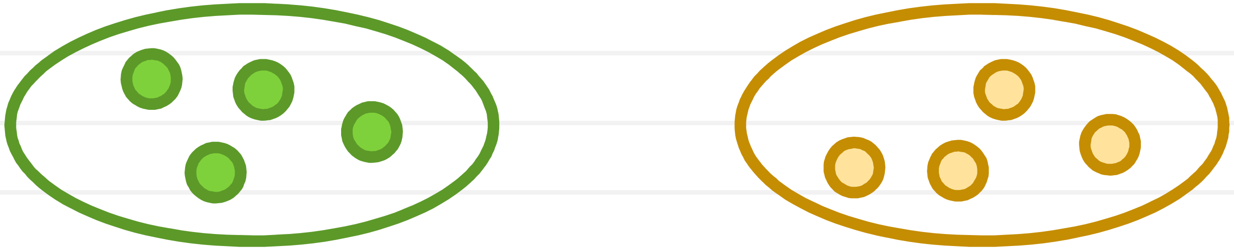
One vertex per work unit in each job

Solution to scheduling



- All arc capacities are set to one
- The maximum flow gives the schedule (each worker knows what to do at each time)
- Can maximize work for very large problems
- Cannot maximize the number of completed jobs

Linear assignment problem



- One-to-one matching two sets M and N of equal size (workers and projects, marriages, points at time t and t+1, etc.)
- a_{ij} is a cost of matching m_i and n_j

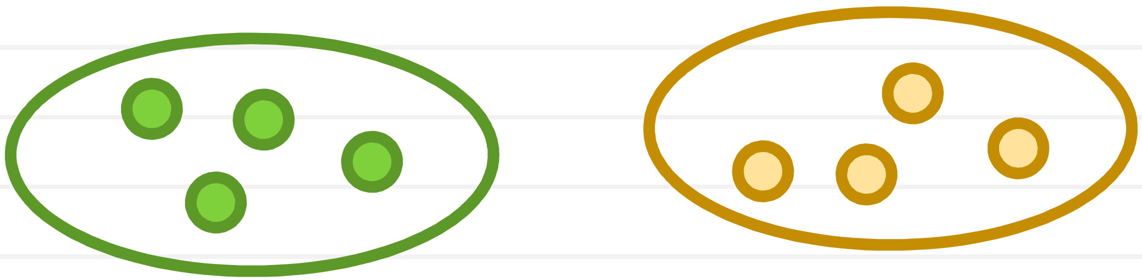
$$\min \sum_{i=1}^{|M|} a_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1$$

$$\sum_j x_{ij} = 1$$

$$x_{ij} \in \{0;1\}$$

Reduction to mincost flow



$$\min \sum_{i=1}^{|M|} a_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_i x_{ij} = 1$$

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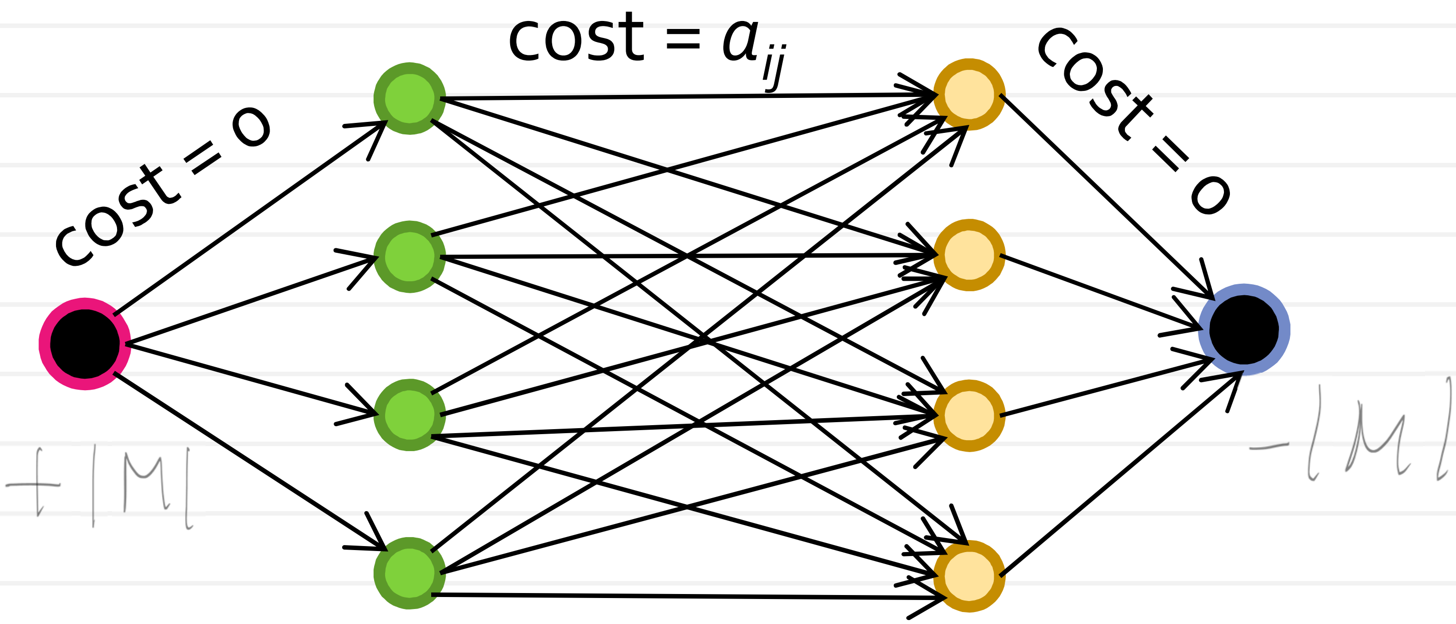


(all capacities=1)

- More efficient specialized algorithms exist (Hungarian)
- What about allowing not to match (at a cost)?

Optimal Cargo Transport

(aka “hopping airplane”)

- a_{ij} containers can be delivered from i to j
- Delivering each gives a profit c_{ij}
- The ship capacity is S
- Maximize profit by choosing which containers to transport

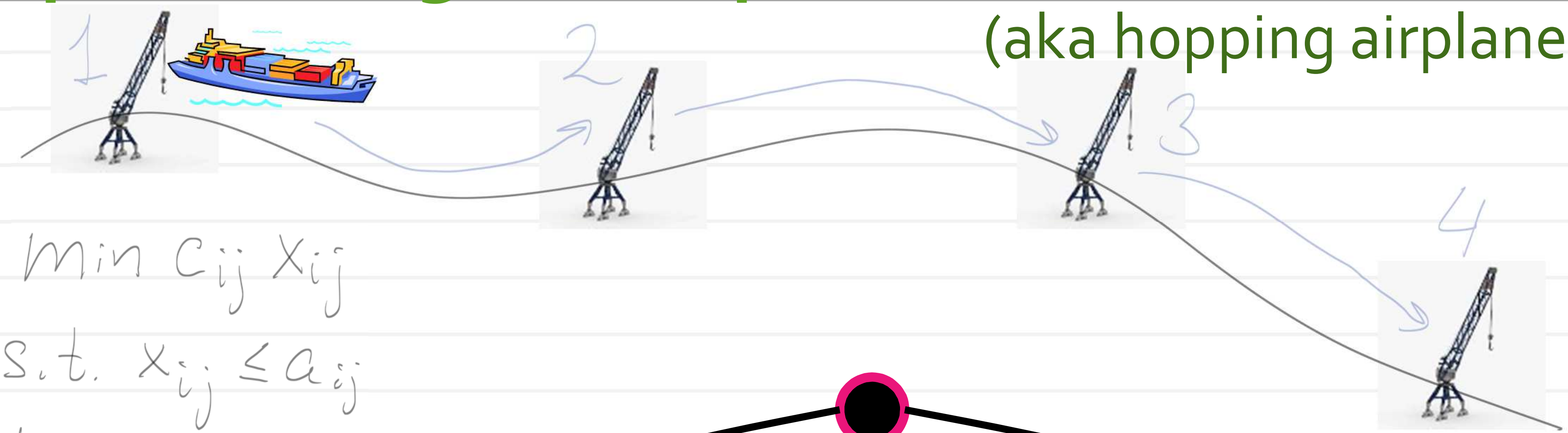
$$\min c_{ij} x_{ij}$$

$$\text{s.t. } x_{ij} \le a_{ij} \qquad x_{ij} \in \mathbb{Z}_+$$

$$\forall t \quad \sum_{\substack{i \le t \\ j \ge t+1}} x_{ij} \le S$$

Optimal Cargo Transport

(aka hopping airplane)



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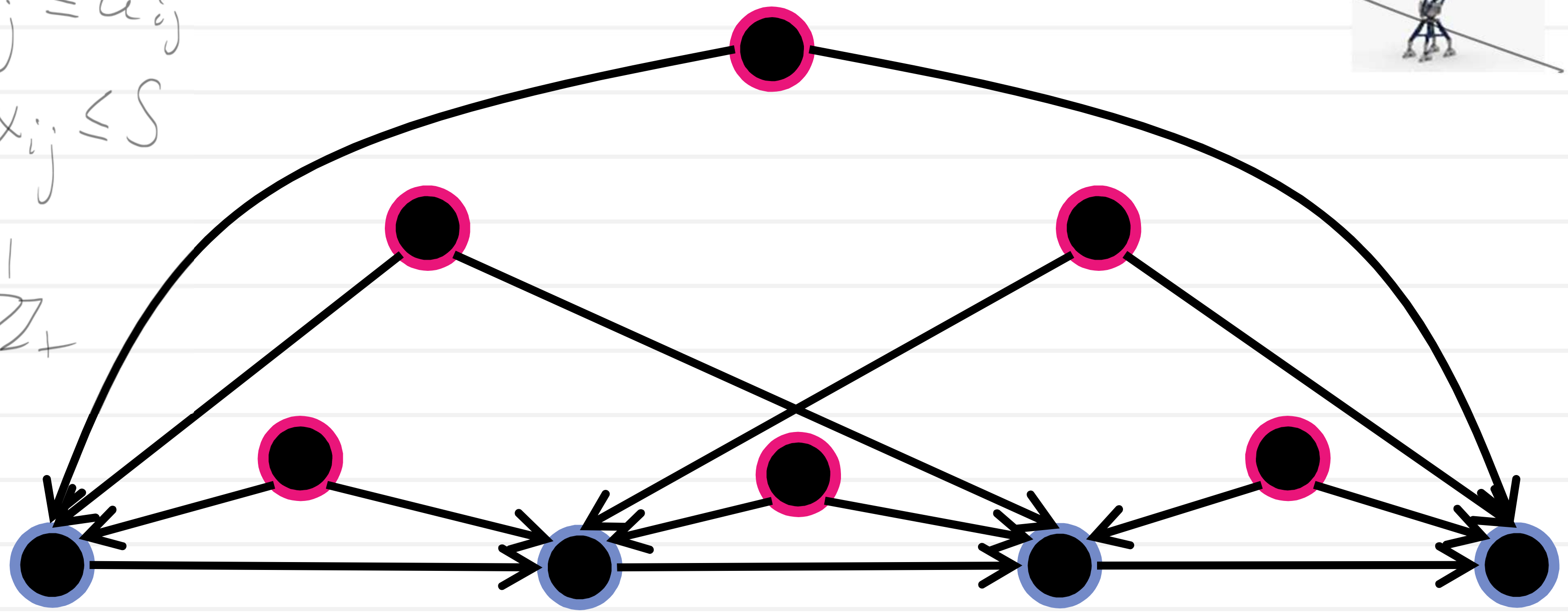
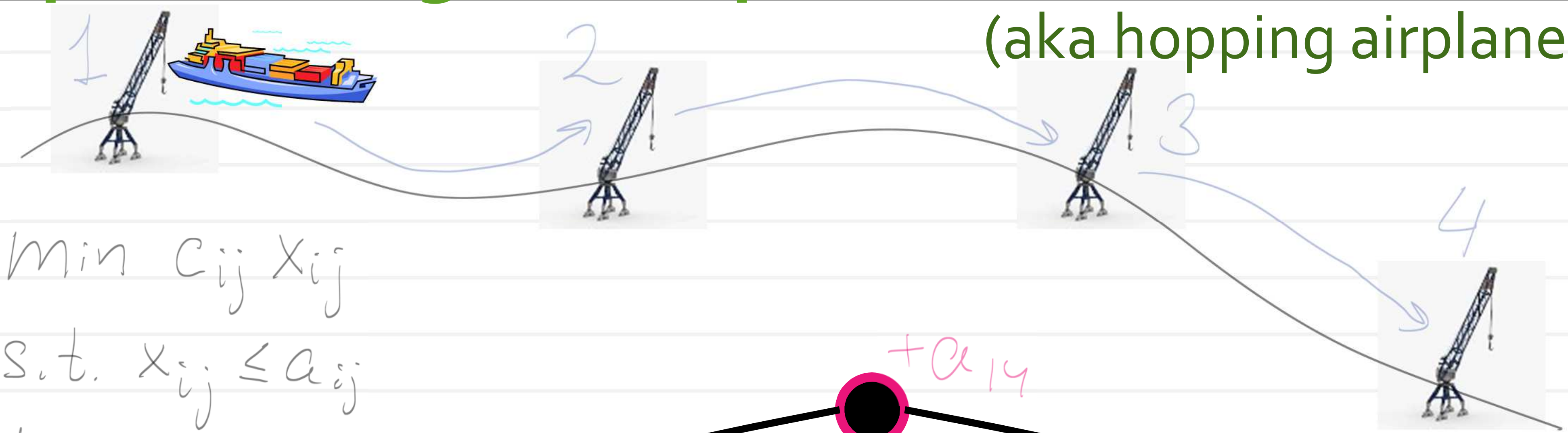


Optimal Cargo Transport

(aka hopping airplane)



$$\min \; c_{ij} x_{ij}$$

$$\text{s.t. } x_{ij} \le a_{ij}$$

$$\forall t \quad \sum_{\substack{i \le t \\ j \ge t+1}} x_{ij} \le S$$

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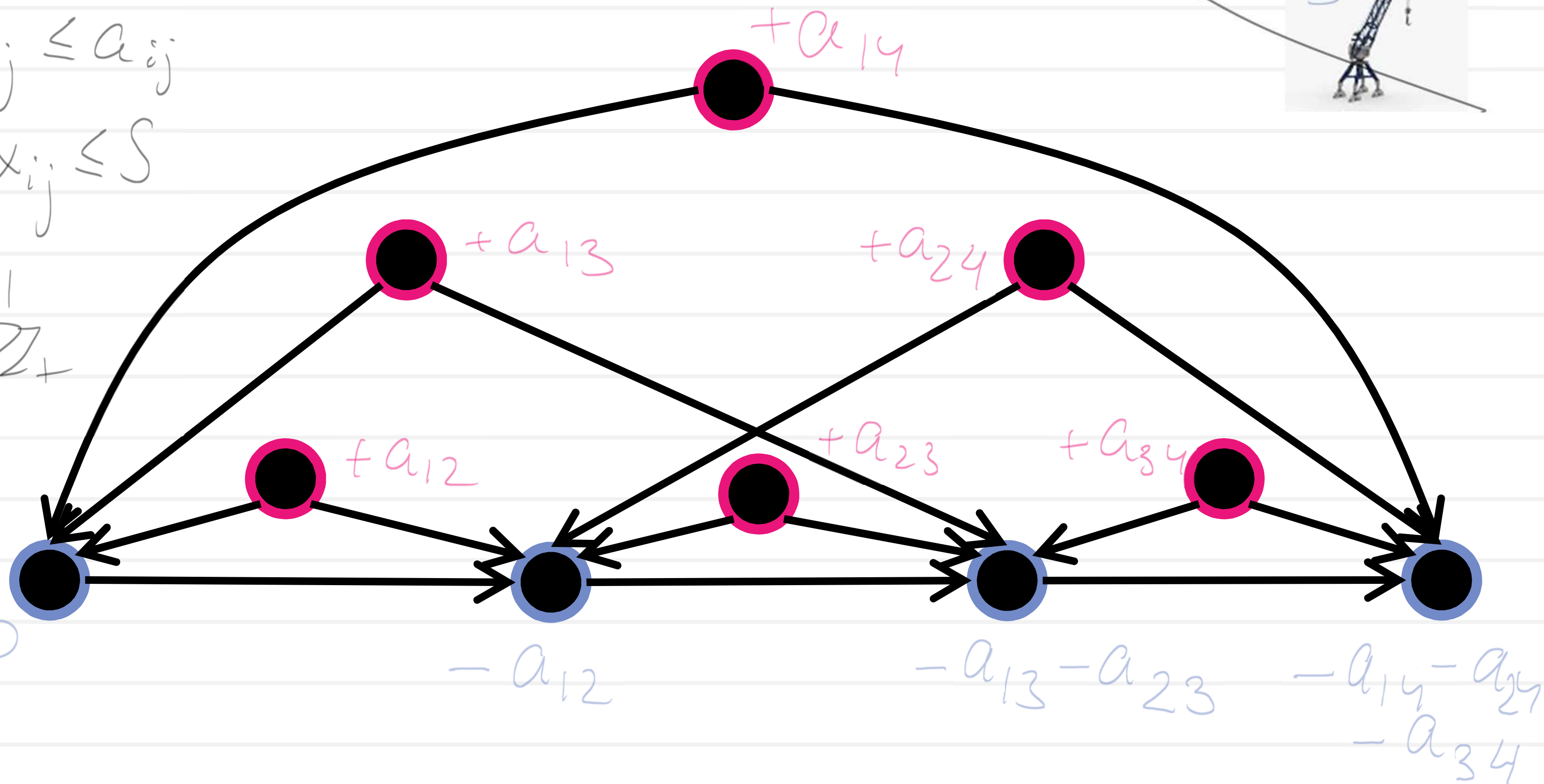

Optimal Cargo Transport

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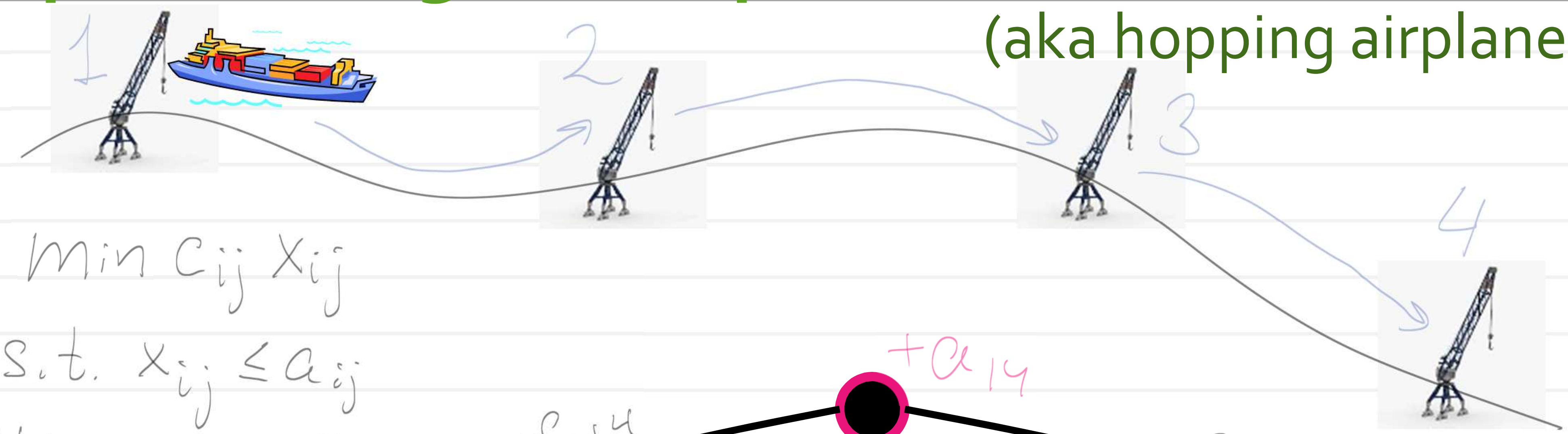

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$$\text{s.t. } x_{ij} \le a_{ij}$$

$$\forall t \sum_{\substack{i \le t \\ j \ge t+1}} x_{ij} \le S$$

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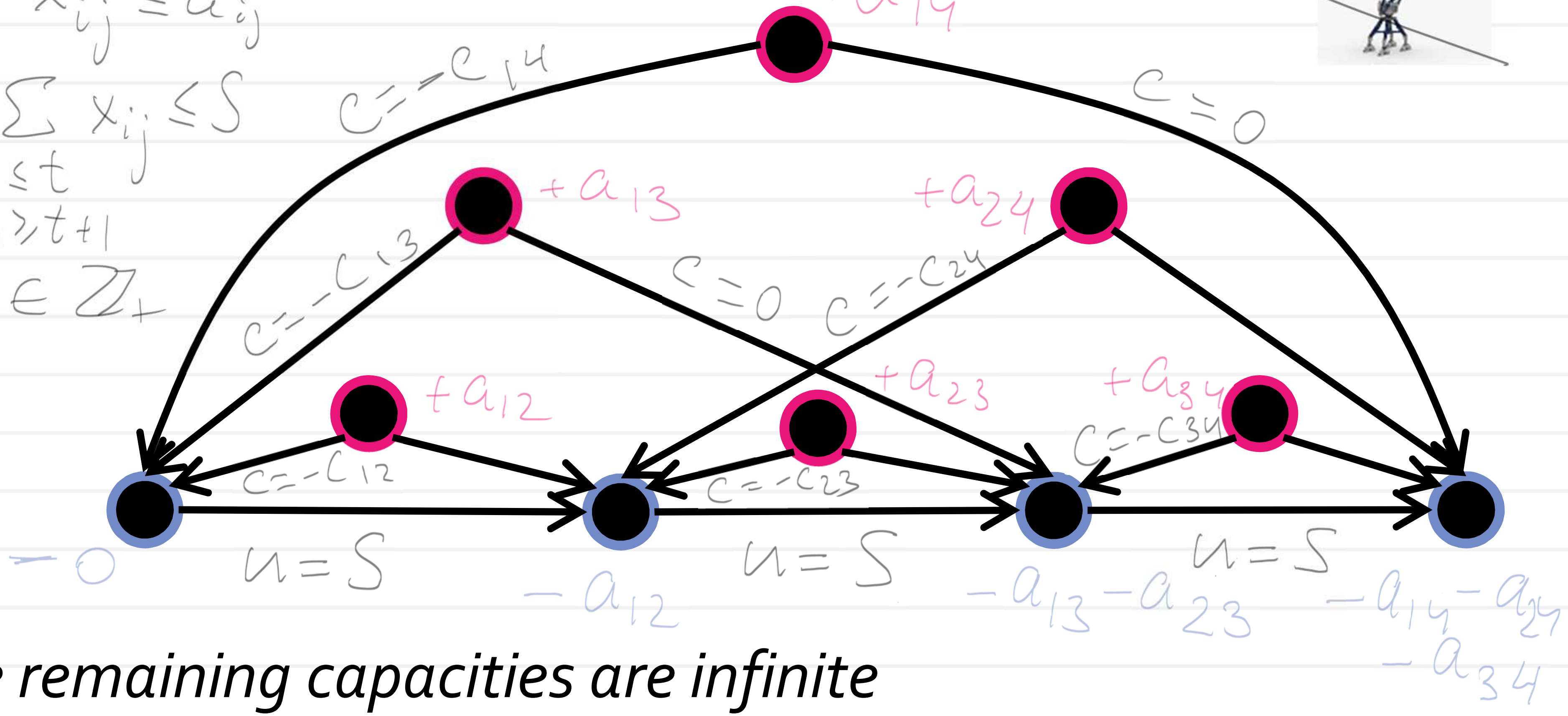

The remaining capacities are infinite

The remaining costs are zeros