Problem set 1
DUE: April 20, 2016

Problem 1

$p(x) = \begin{cases} Ae^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$
We know that

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Hence

$$\int_{0}^{\infty} Ae^{-\lambda x}dx = 1$$

$$\int_{0}^{\infty} Ae^{-\lambda x}dx = -\frac{1}{\lambda}Ae^{-\lambda x}|_0^{\infty} = -\frac{1}{\lambda}A(e^{-\infty} - e^0) = \frac{1}{\lambda}A = 1$$

Thus $A = \lambda$

$$Ex = \int_{0}^{\infty} xAe^{-\lambda x}dx = A\int_{0}^{\infty} xe^{-\lambda x}dx = \%u = x, du = dx, dv = e^{-\lambda x}dx, v = -\frac{1}{\lambda}e^{-\lambda x}\% =$$

$$= A\left(-\frac{1}{\lambda}xe^{-\lambda x}|_0^{\infty} + \frac{1}{\lambda}\int_{0}^{\infty} e^{-\lambda x}dx\right) = A\frac{1}{\lambda}\int_{0}^{\infty} e^{-\lambda x}dx = -A\frac{1}{\lambda^2}(0-1) = A\frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$Varx = Ex^2 - (Ex)^2$$

$$Ex^2 = \int_{0}^{\infty} x^2Ae^{-\lambda x}dx = \%u = x^2, du = 2xdx, dv = e^{-\lambda x}dx, v = -\frac{1}{\lambda}e^{-\lambda x}\% =$$

$$= A\left(-\frac{1}{\lambda}x^2e^{-\lambda x}|_0^{\infty} + 2\int_{0}^{\infty} x\frac{1}{\lambda}e^{-\lambda x}dx\right) = 2A\frac{1}{\lambda}\int_{0}^{\infty} xe^{-\lambda x}dx = \frac{2}{\lambda^2}$$

Hence $Varx = \frac{1}{\lambda^2}$

$$G(k) = \lambda\int_{0}^{\infty} e^{(ik-\lambda)x}dx = \frac{\lambda}{ik-\lambda} = \left(1 - \frac{ik}{\lambda}\right)^{-1}$$

$\mathrm{E}[X^m] = \frac{1}{i^m}\frac{\partial^m}{\partial k^m}G(k)|_{k=0}$

Problem 2

According to CLT:

$$\frac{\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - Ex\right)}{Stdx} \xrightarrow{d} N(0,1)$$

$$Ex = \frac{1}{2}\cdot 0 + \frac{1}{3}\cdot 2 + \frac{1}{6}\cdot 26 = \frac{2}{3} + \frac{13}{3} = 5$$

$$Stdx = \sqrt{Varx}$$

$$Varx = Ex^2 - (Ex)^2$$

$$Ex^2 = \frac{1}{2}\cdot 0 + \frac{1}{3}\cdot 4 + \frac{1}{6}\cdot 26^2 = \frac{4}{3} + \frac{13^2\cdot 2}{3} = \frac{4 + 169\cdot 2}{3} = 114$$

$$Varx = 114 - 25 = 89$$

$$Stdx = \sqrt{89} \approx 9.43$$

$$E\sum_{i=1}^{n} x_i = nEx = 500$$

Due to the fact that x_i are independent:

$$Var\sum_{i=1}^{n} x_i = nVarx = 890$$

$$Std\sum_{i=1}^{n} x_i \approx 29.83$$

According to CLT:

$$\sum_{i=1}^{n} x_i \xrightarrow{d} N(500, 890)$$

Z-score is $\frac{200-500}{890} = -0.34$

$$P\left(\sum_{i=1}^{n} x_i \geq 200\right) = 1 - P\left(\sum_{i=1}^{n} x_i < 200\right) = 1 - 0.3669 = 63.31\%$$

Problem 2