Stochastic Modeling and Computations
Skolkovo Institute of Science and Technology — March 28, 2016
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Problem Set I

This problem set is due by **Wed Apr 20**, 11:59 pm Moscow time.
Solutions should be turned in through the course web-site in an electronic format. Remember, your goal is to communicate. Full credit will be given only to the correct solution which is described clearly.

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1. **(10 points)** *Exponential Distribution*

   The probability density function of an exponential distribution is

   $$p(x) = \begin{cases} Ae^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1}$$

   where the parameter $\lambda > 0$.

   (i) Calculate the normalization constant A of the distribution.
   (ii) Calculate the *mean value* and the *variance* by direct integration.

   The *characteristic function* of a distribution is

   $$G(k) = \int_{-\infty}^{+\infty} e^{ikx} p(x) dx. \tag{2}$$

   It can be used to calculate high-order moments of the distribution.

   (iii) Calculate the characteristic function $G(k)$ of the exponential distribution.
   (iv) Using the function $G(k)$, calculate the $m-$th moment of the distribution.

2. **(10 points)** *Dice game*

   Assume that you play a dice game 100 times. Awards for the game are as follows

   1, 3 or 5: 0$

   2 or 4: 2$

   6: 26$

   (i) Estimate expected value of winnings

   (ii) Estimate standard deviation of winnings

   (iii) Estimate probability of winning at least 200$

   [hint: use central limit theorem]

3. **(10 points)** *Z channel*

   Consider the Z channel with $f = 0.15$ and the following probability distribution of the input symbols: $P(x = 0) = 0.9$, $P(x = 1) = 0.1$.

   (i) Compute the probability distribution of output $P(y)$.

   (ii) Compute the probability of $x = 1$ given $y = 0$.

   (iii) Compute the mutual information $I(X; Y)$.

   (iv) What is the capacity of the channel?

4. **(15 points)** *Hardy-Weinberg Law*

   Consider an experiment with rabbits matting. WLet us follow evolution of a particular gene that appears in two types, G or g. A rabbit has a pair of genes, either GG (dominant), Gg (hybrid — the order is irrelevant, so gG is the same as Gg) or gg (recessive). In the result of a single mating the offspring inherits a gene from each of its parents with equal probability. Thus, if a dominant parent (GG) mates with a hybrid parent (Gg), the offspring is dominant with probability 1/2 or hybrid with probability 1/2. Start with a rabbit of given character (GG, Gg, or gg) and assume that she mates with a hybrid. The offspring produced again mates with a hybrid, and the process is repeated for a number of generations.

   Note: The first experiment of such kind was conducted in 1858 by Gregor Mendel. He started to breed garden peas in his monastery garden and analysed the offspring of these matings.

   (i) Write down the transition matrix P of the Markov chain thus defined. Is the Markov chain irreducible and aperiodic?

   (ii) Assume that we start with a hybrid rabbit. Let μ_n be the probability distribution of the character of the rabbit of the n-th generation. In other words, $\mu_n(GG)$, $\mu_n(Gg)$, $\mu_n(gg)$ are the probabilities that the n-th generation rabbit is GG, Gg, or gg, respectively. Compute μ_1, μ_2, μ_3. Is there a some kind of law/rule emerging?

   (iii) Calculate P^n for general n. How does the moment, μ_n, depend on n?

   (iv) Calculate the stationary distribution of the Markov chain. Is detailed balance hold?

5. **(10 points)** *Splitting of Poisson process*

   Customers arrive at a store with the Poisson rate of 10 per hour. Each is either male or female with probability p and $1 - p$, respectively.

   (i) Compute probability that at least 20 customers have entered between 10 and 11 am.

   (ii) Compute probability that exactly 10 woman entered between 10 am and 11 am.

   (iii) Compute the expected inter-arrival time probability of men.

(iv) Compute the probability that there will be no male customers between 2 pm and 4 pm.