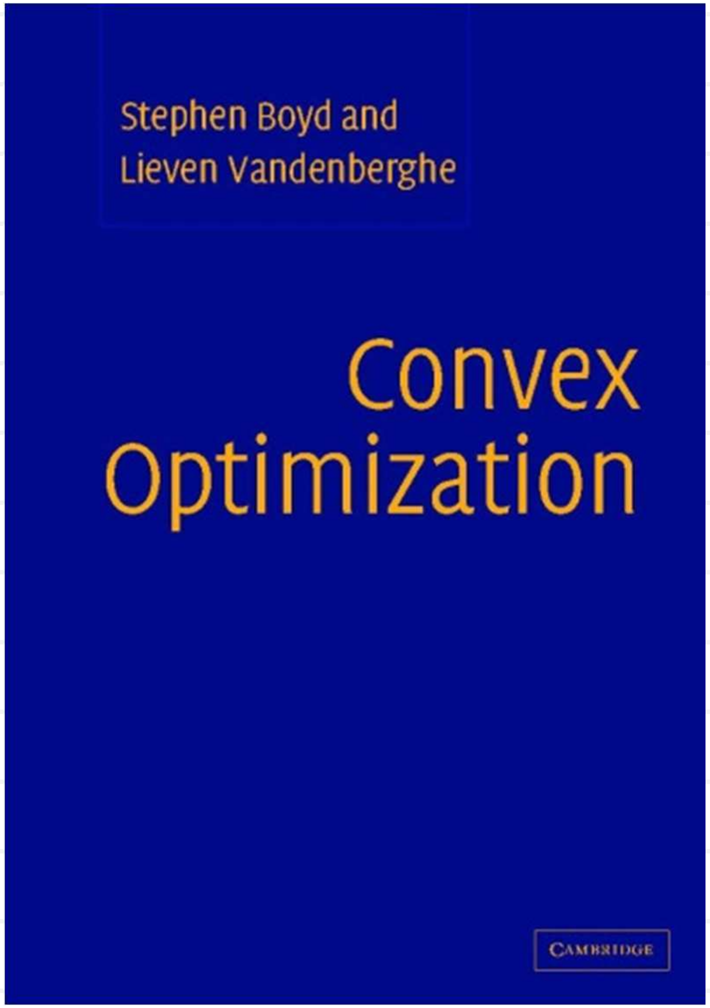
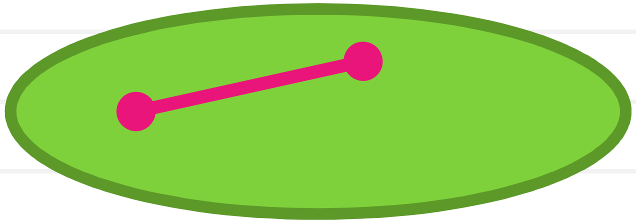
Lecture 12: Convexity

Textbook



http://www.stanford.edu/~boyd/cvxbook/

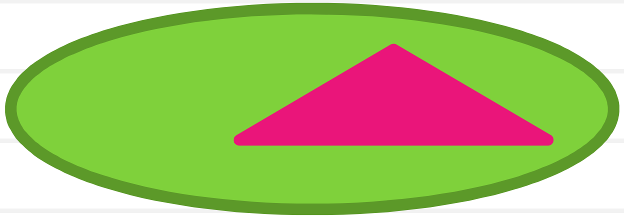
Convex set: definition



Definition: M is convex if

$$\forall x_1\ x_2 \in M,\ \forall \theta_1 \geq 0, \theta_2 \geq 0, \theta_1 + \theta_2 = 1$$

$$\theta_1 x_1 + \theta_2 x_2 \in M$$



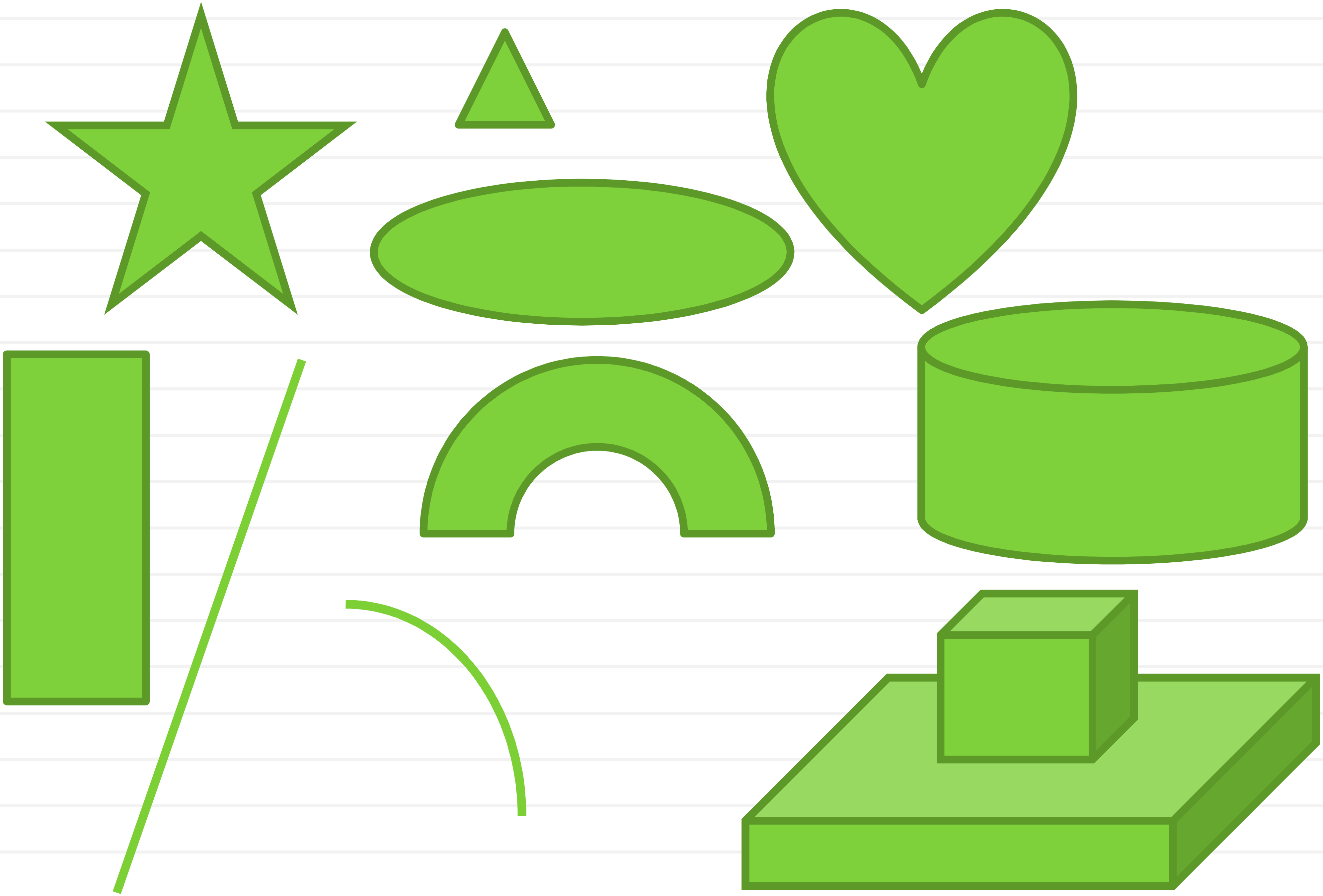
Definition: M is convex if

$$\forall x_1\ x_2 \ldots x_n,\ \forall \theta_1\ \theta_2 \ldots \theta_n : \sum \theta_i = 1$$

$$\theta_i \geq 0 \qquad \sum \theta_i x_i \in M$$

"convex combination"

Quiz



Examples of convex sets:

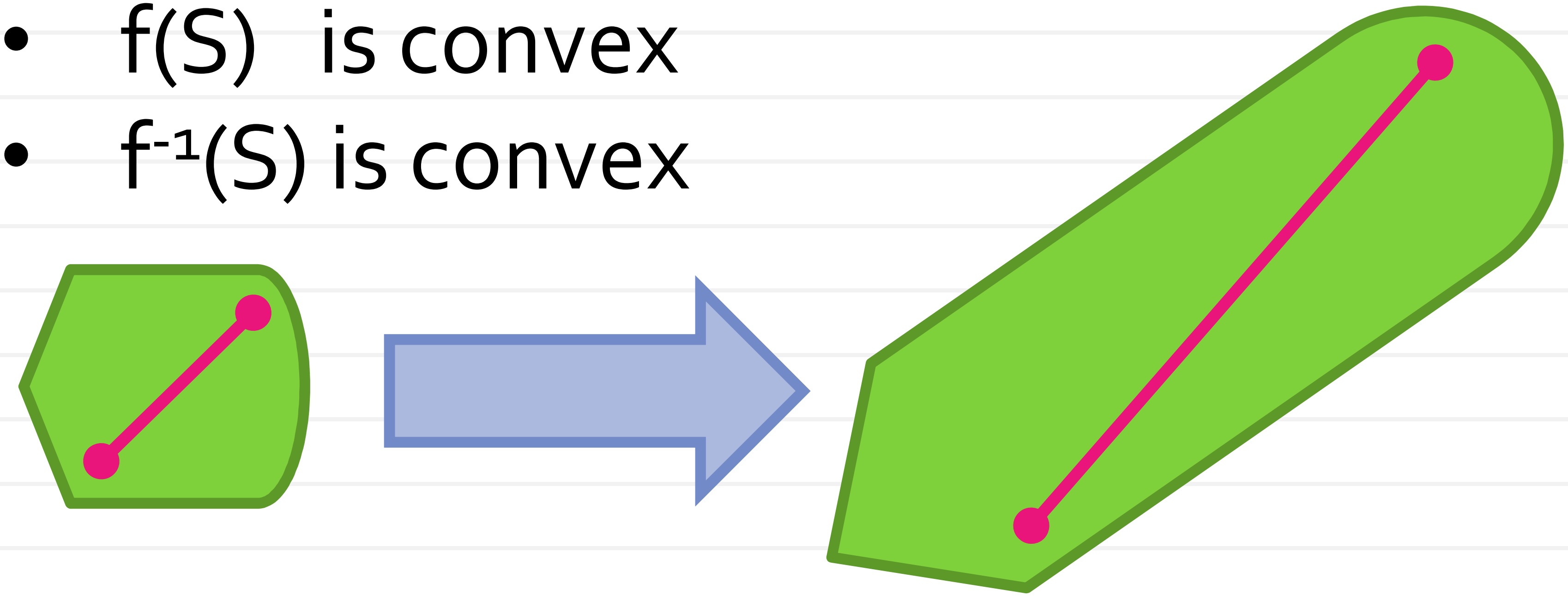
- Empty set
- Complete space
- Halfspace $\{x \mid \langle x, \theta \rangle \geq \nu\}$
- Affine set $\{c + \sum_{i=1}^{m} \theta_i x_i \mid \theta_i \in \mathbb{R}^m\}$
- Intersection of any number of convex sets
- Polygon $\{x \mid Ax \leq b, \ Cx = d\}$

Affine transform and convexity

Affine transform

$$f: \mathbb{R}^m \longrightarrow \mathbb{R}^n$$
$$f(x) = Ax + b$$

- $f(S)$ is convex
- $f^{-1}(S)$ is convex



Norm balls

Norm ball: $\{x \mid \|x - c\| \leq R\}$

Recall: the definition of the norm:

$\|\lambda x\| = |\lambda| \|x\|$

$\|x + y\| \leq \|x\| + \|y\|$

$\|x\| = 0 \Leftrightarrow x = 0$

The set of positive semi-definite matrices

$A \succeq 0 \qquad \{A \mid \forall x \;\; x^T A x \geq 0\}$

(A symmetric)

Is it convex? Why?

Geometry of PSD cone

$$S = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$$

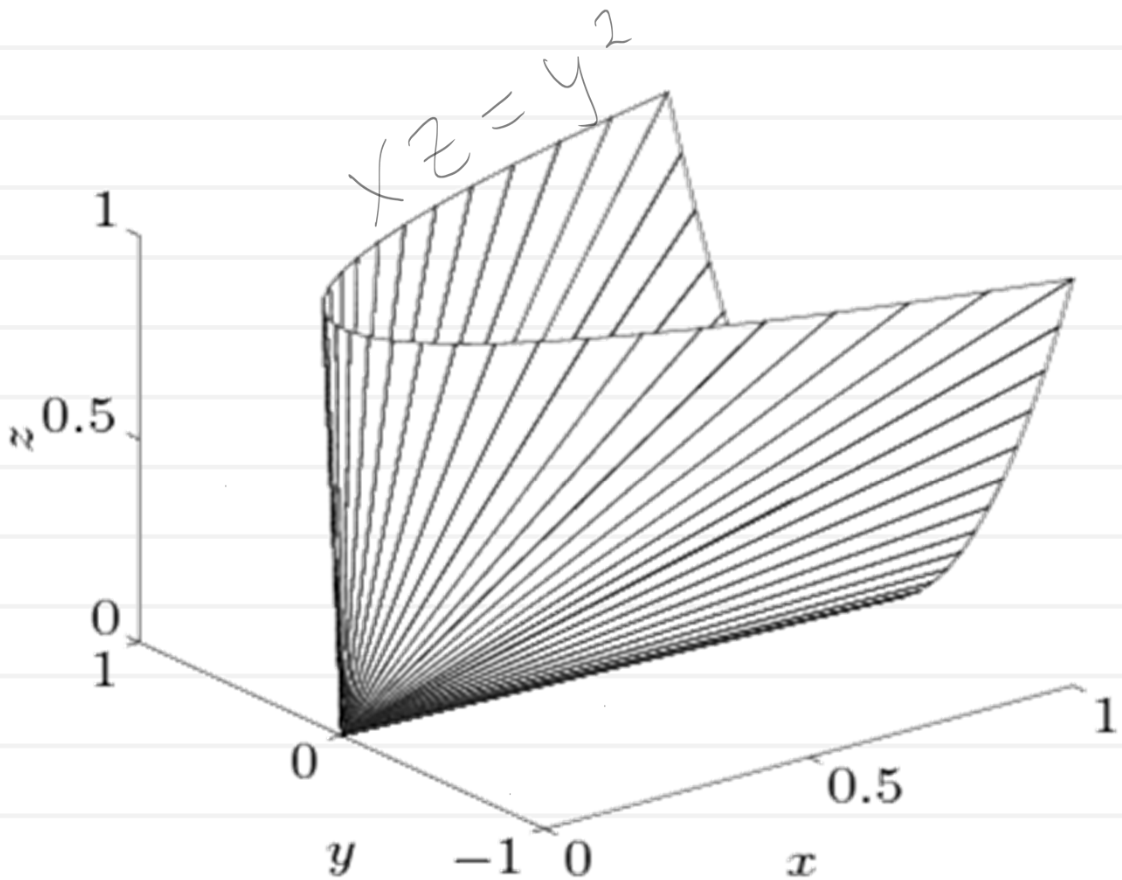


Image from BV

A related example

The set of all ellipses containing given *n* points

Quadric equation: $\frac{1}{2} x^T A x + b^T x + c$

What about the set of ellipses containing these three points?

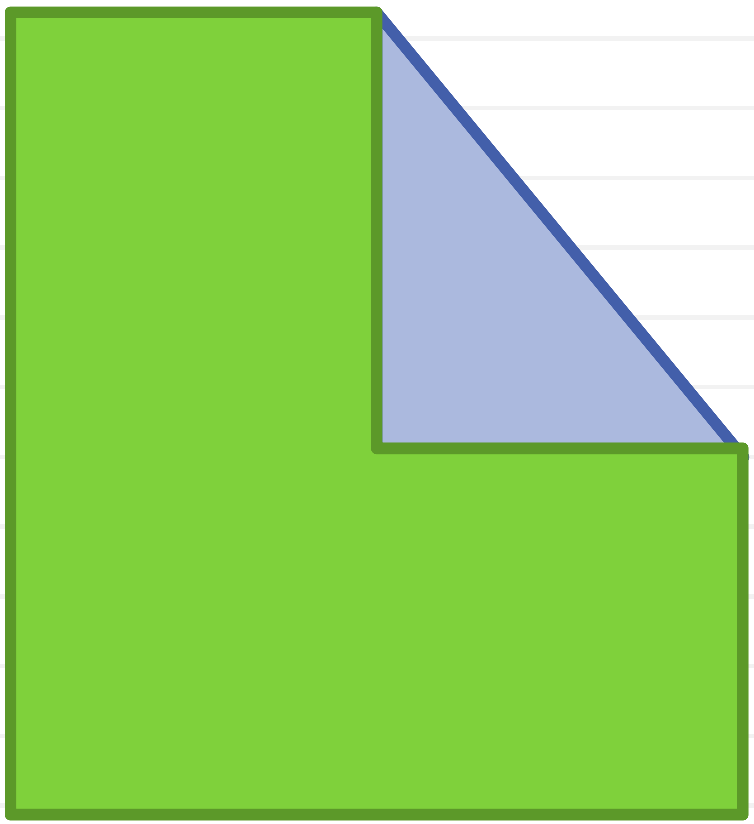
$$A \succ 0$$

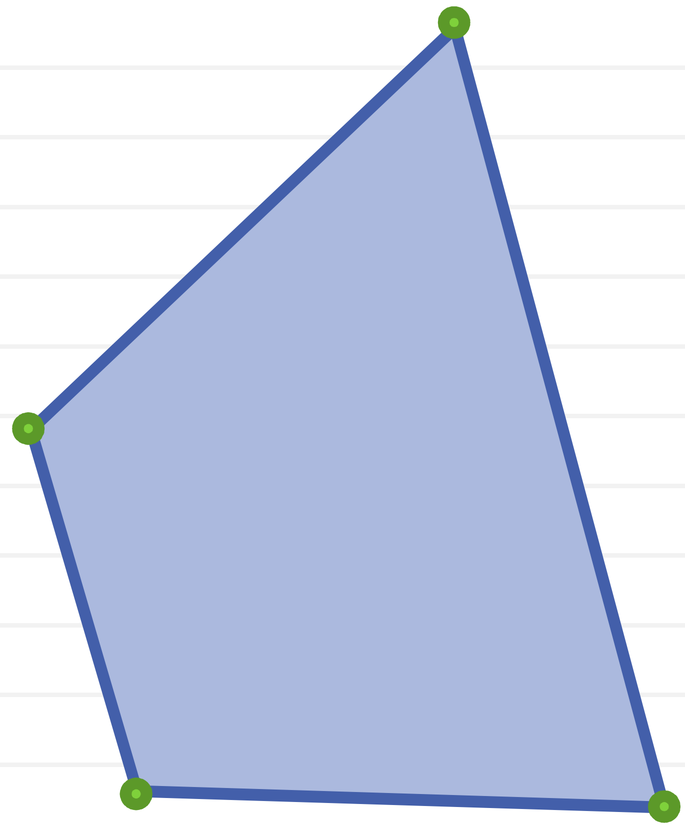
$$\frac{1}{2} p_i^T A p_i + b^T p_i + c \leq 0$$

The convex hull of M is….

The intersection of all convex sets containing the set M,

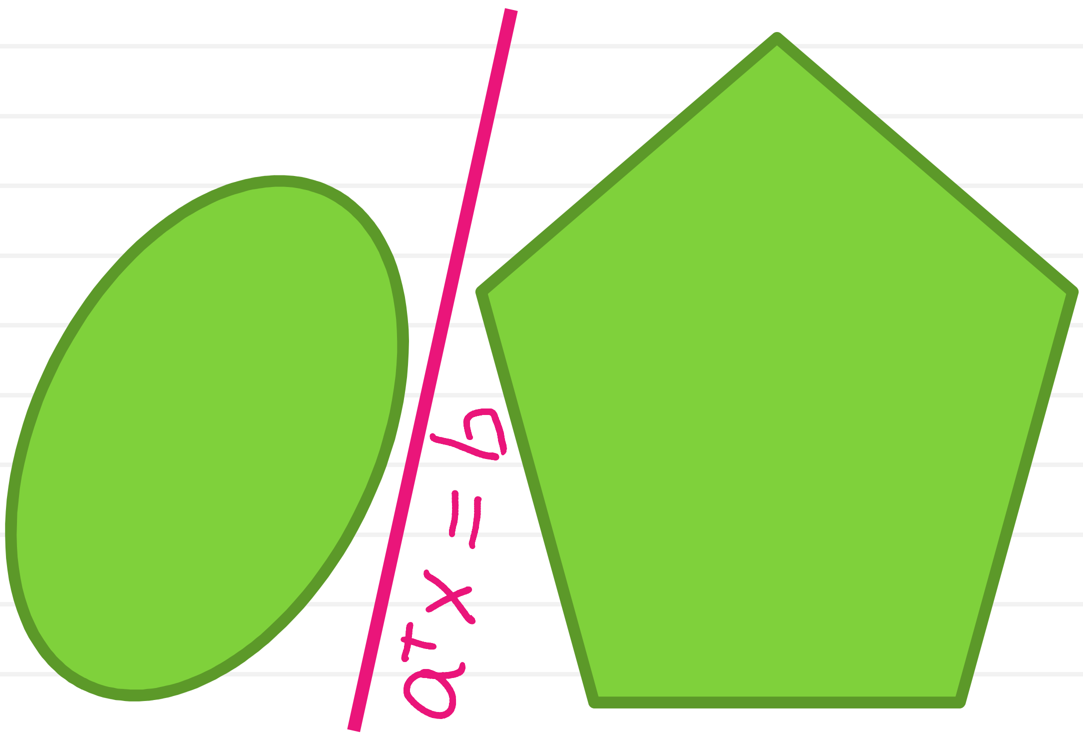
or $\left\{ \sum_{i=1}^{n} \theta_i x_i \;\middle|\; \forall n \; \forall \theta : \sum \theta_i = 1 \;\; \forall x : x_i \in M \right\}$





Separating hyperplane

Theorem. if C and D are non-intersecting convex sets in R^n, then there exists a in R^n and b in R: $\forall x \in C \;\; a^T x \leq b, \; \forall x \in D \;\; a^T x \geq b$



Could we require strict separability?

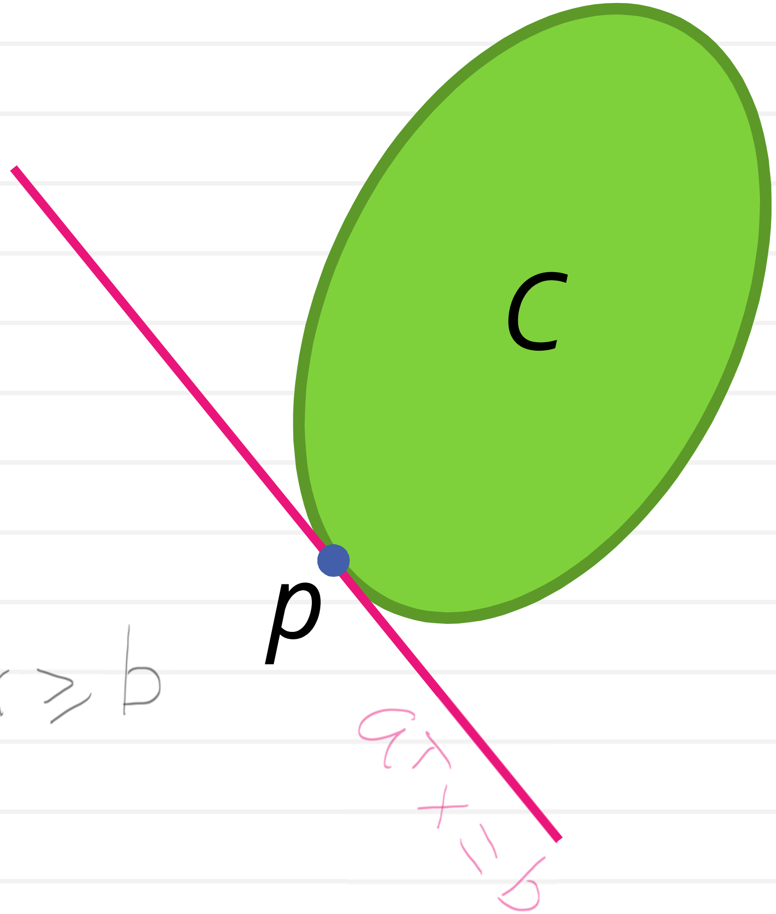
Supporting hyperplane

Theorem:

- Let C be a convex set.
- Let p ∈ *bd* C

Then: $\exists\ a \in \mathbb{R}^m,\ b:$

$a^T p = b,\ \forall x \in C\ \ a^T x \geq b$



(this hyperplane is called ***supporting hyperplane*** *to C at p)*

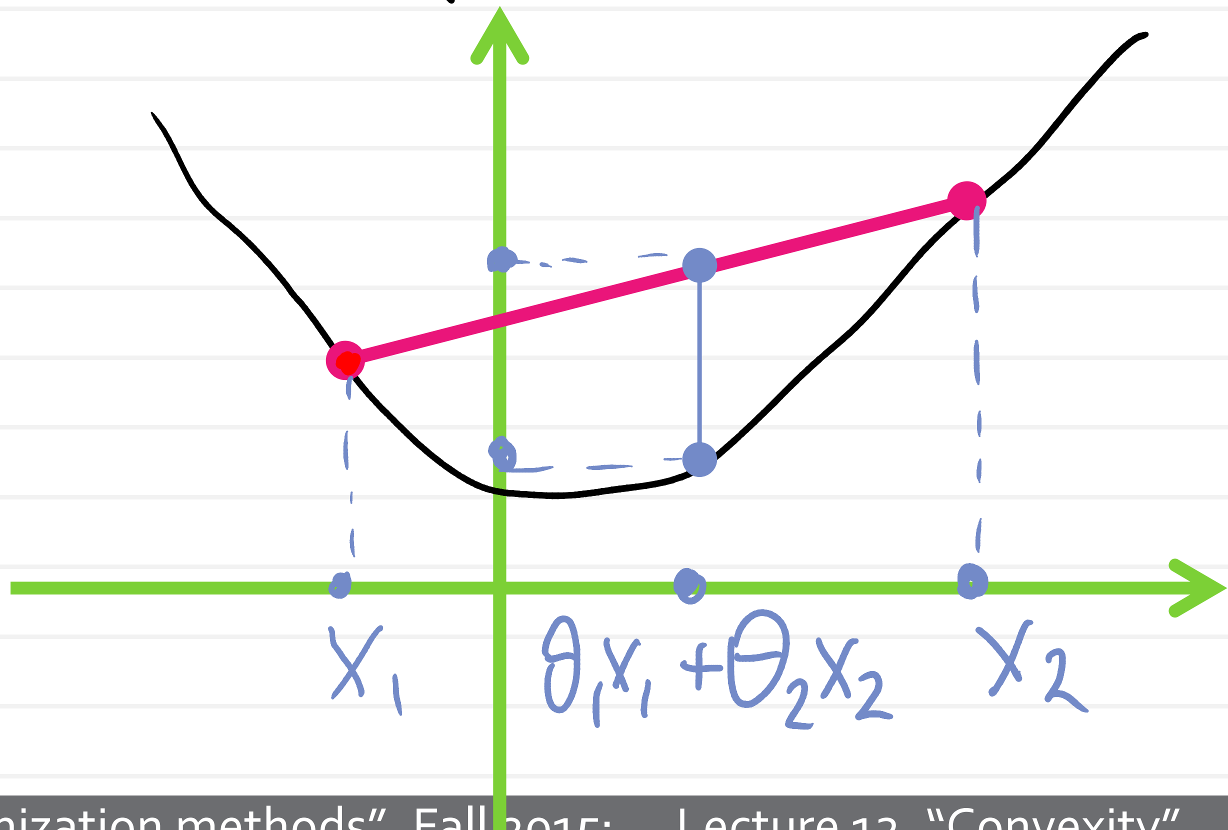
Proof (idea): separating plane to *p* and *int C.*

Convex function

Definition. f is convex, if:

- it is defined over a convex set
- $\forall x_1, x_2, \theta_1 \geq 0, \theta_2 \geq 0 \;\; \theta_1 + \theta_2 = 1$

$$f(\theta_1 x_1 + \theta_2 x_2) \leq \theta_1 f(x_1) + \theta_2 f(x_2)$$

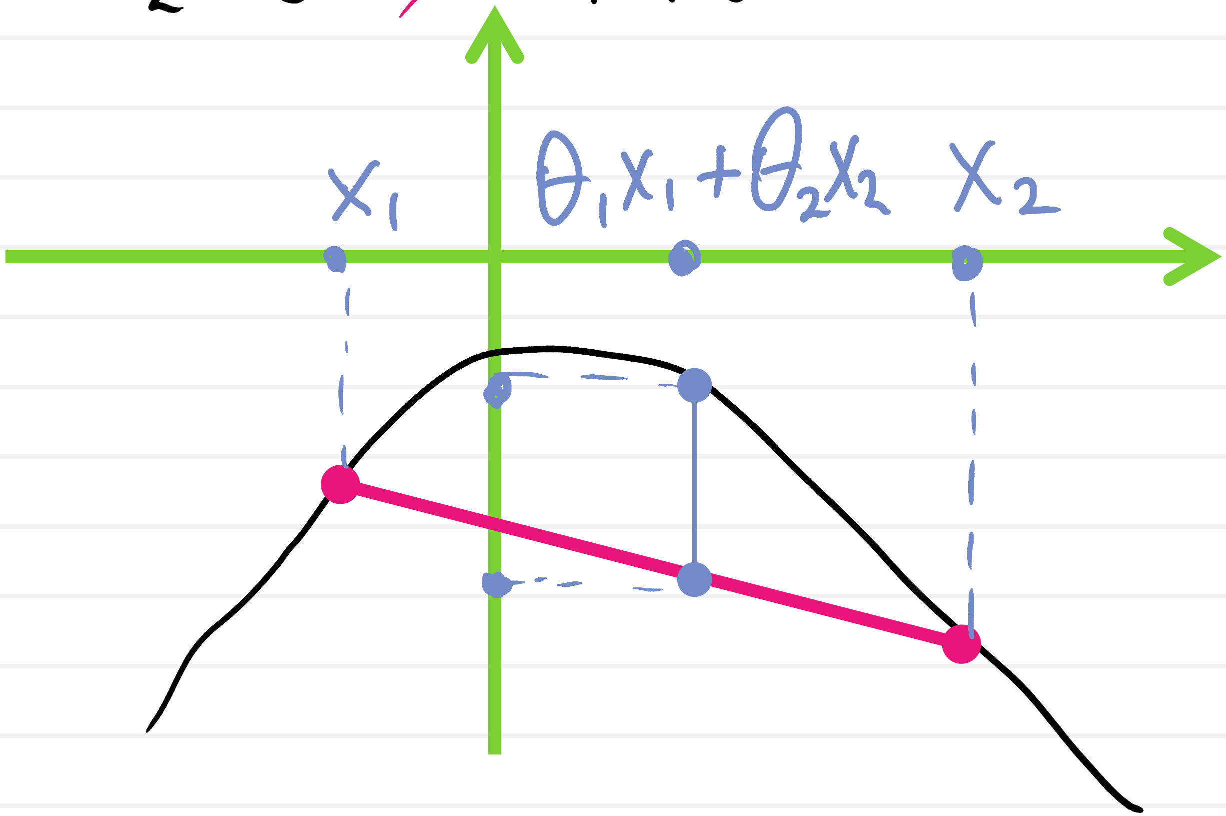


Concave function

Definition. f is concave, if:

- it is defined over a convex set
- $\forall x_1, x_2, \theta_1 \geqslant 0, \theta_2 \geqslant 0 \quad \theta_1 + \theta_2 = 1$

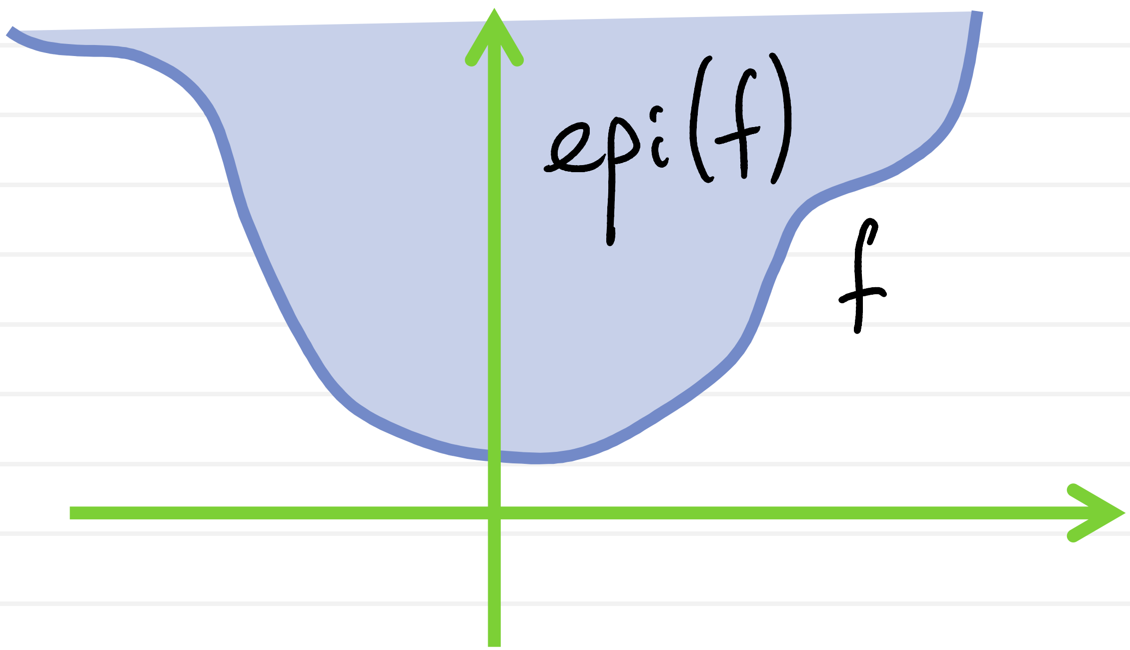
$$f(\theta_1 x_1 + \theta_2 x_2) \geqslant \theta_1 f(x_1) + \theta_2 f(x_2)$$



Quiz

Epigraphs and convexity

$$epi(f) = \{(x,t) \mid x \in \operatorname{dom} f,\ f(x) \leq t\}$$



Corollary: f is convex **iff** $epi\, f$ is convex.

Operations preserving convexity

- Sum: $g(x) = f_1(x) + f_2(x)$
- Multiplication by a positive scalar:

$$g(x) = \alpha f(x)$$

$$g(x) = \sum_i \alpha_i f_i(x) \qquad \alpha_i \geqslant 0$$

- Composition with an affine mapping:

$$g(x) = f(Ax+b)$$

- Pointwise maximum/supremum:

$$g(x) = \max\{f_1(x), f_2(x)\}$$

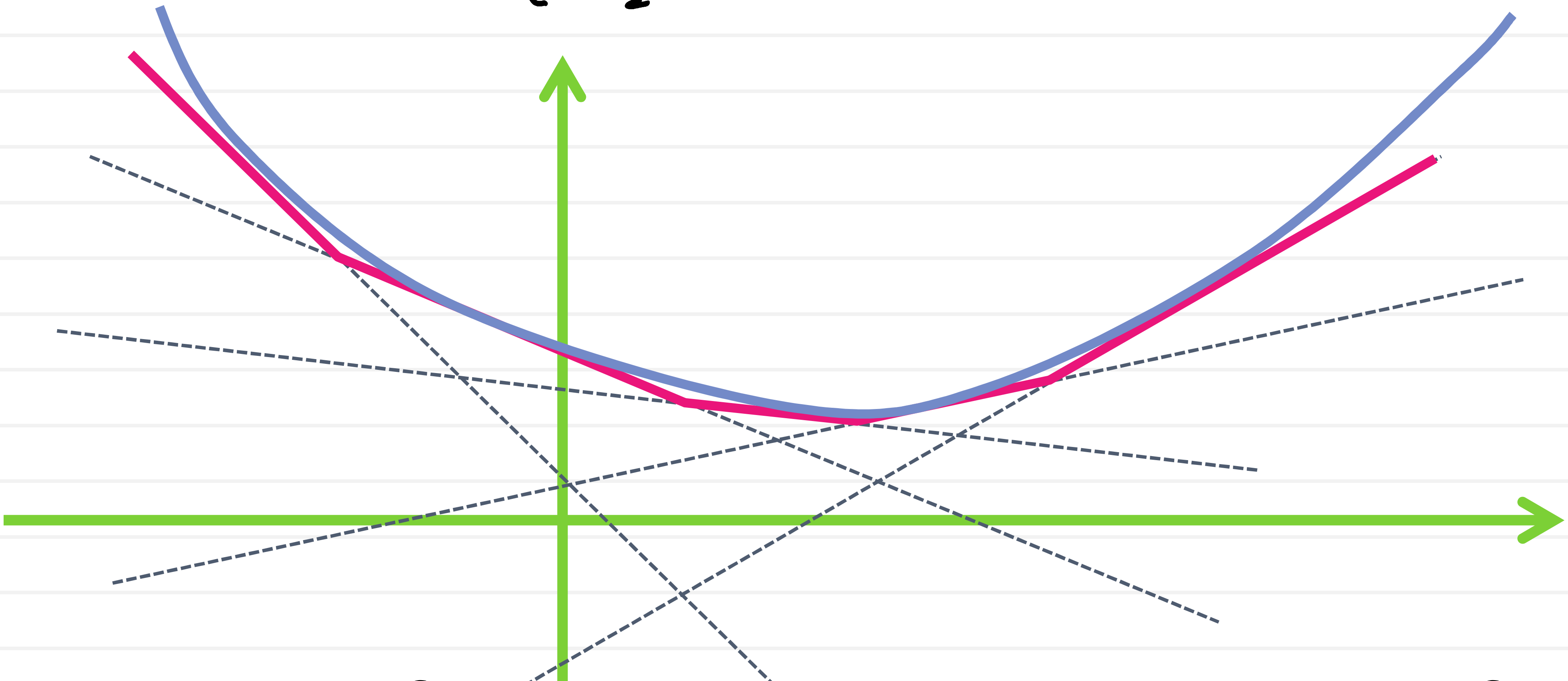
$$g(x) = \sup_{\alpha \in A} f_\alpha(x)$$

- Norms (e.g. spectral norm)

Example 1

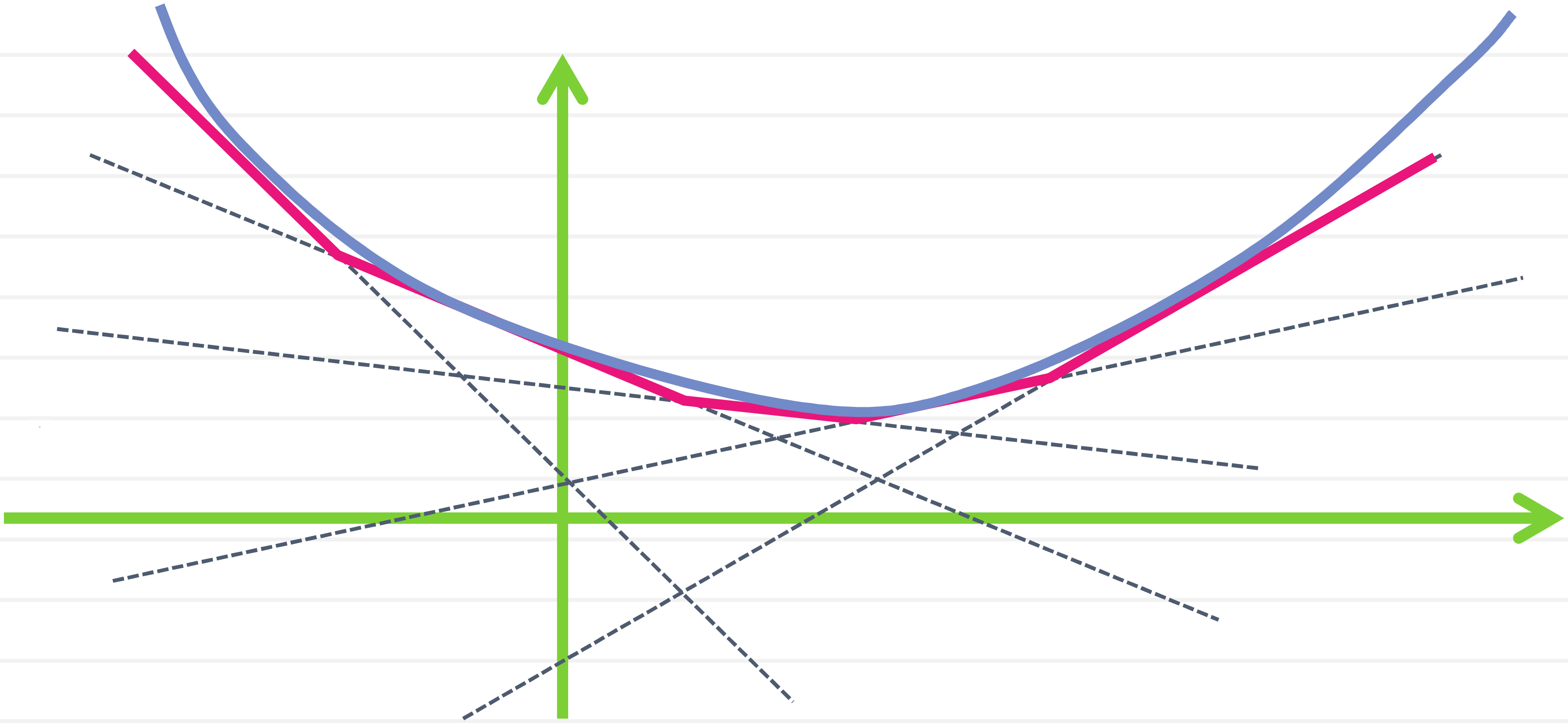
"upper envelope"

$$f(x) = \max\{a_1^T x + b, a_2^T x + b, \ldots, a_n^T x + b\}$$



Any convex function is an upper envelope of linear functions!

Supporting hyperplane for smooth functions



Corollary: if f is convex and differentiable at x, then the only supporting hyperplane at x is:

$$h(y) = \nabla f(x)^T \cdot (y - x) + f(x)$$

More examples

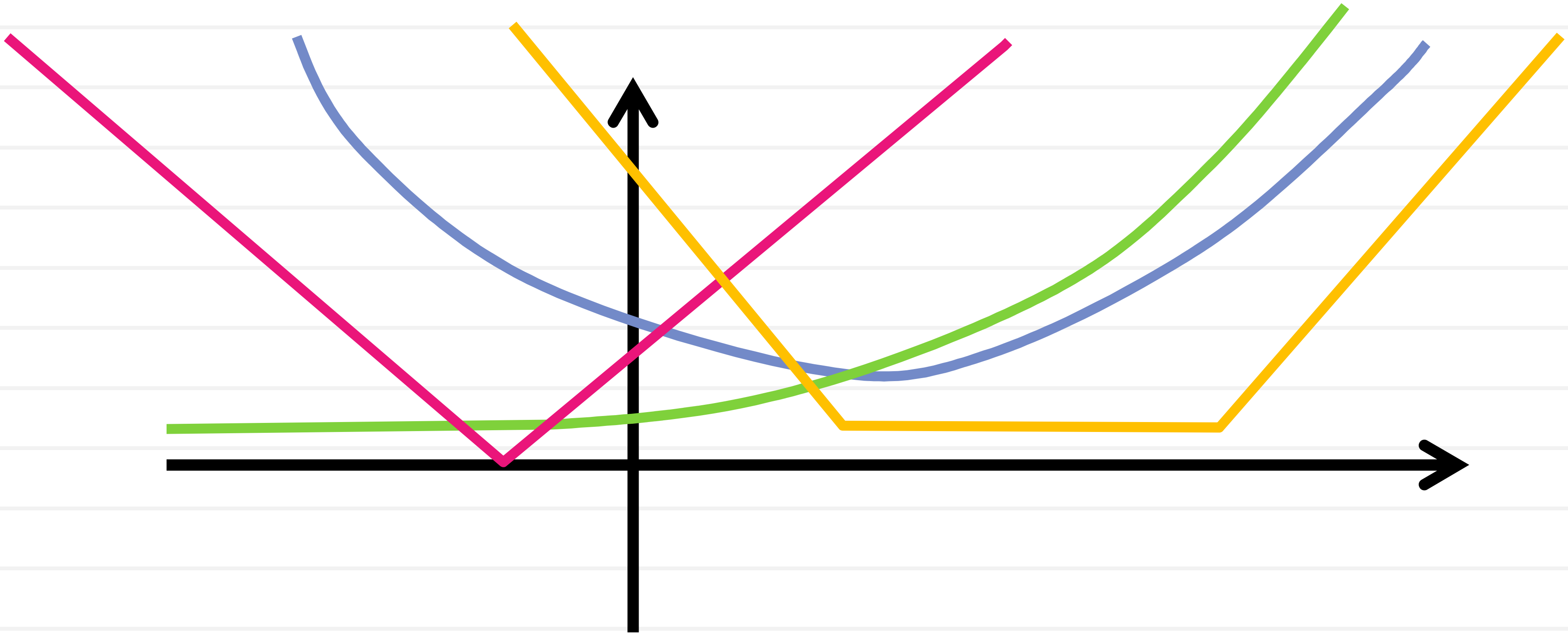
1. Distance to the furthest point in a set
2. The sum of three largest components
3. The sum of three largest squared components
4. The maximum eigenvalue of a symmetric matrix

Strict convexity

Definition. f is <u>strictly</u> convex, if:

- it is defined over a convex set
- $\forall x_1, x_2, \theta_1 > 0, \theta_2 > 0 \;\; \theta_1 + \theta_2 = 1$

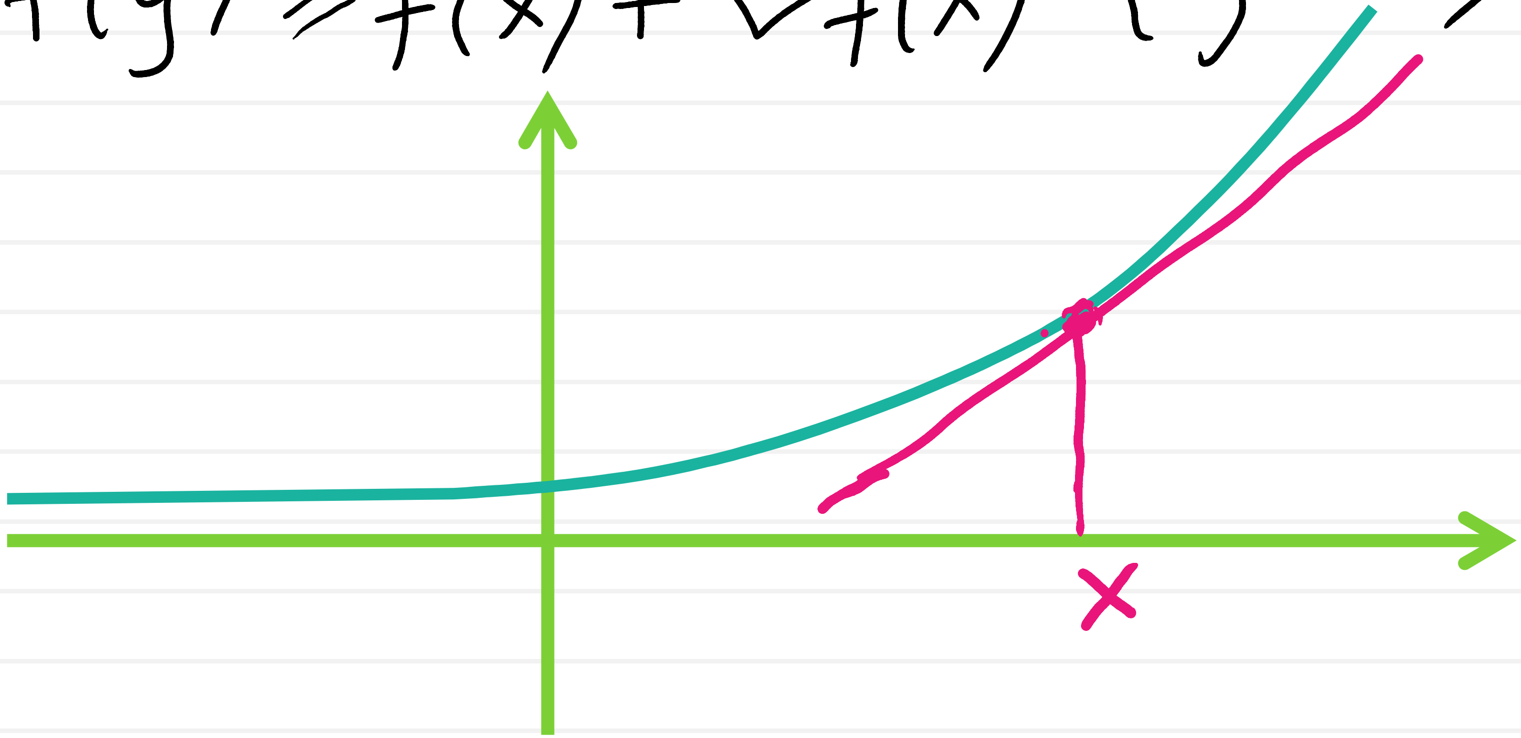
$$f(\theta_1 x_1 + \theta_2 x_2) < \theta_1 f(x_1) + \theta_2 f(x_2)$$



Convexity for differentiable functions

Corollary. Assume that f is defined over a convex set and is differentiable, then f is convex **iff** $\forall x, y \in \mathrm{dom} f$

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

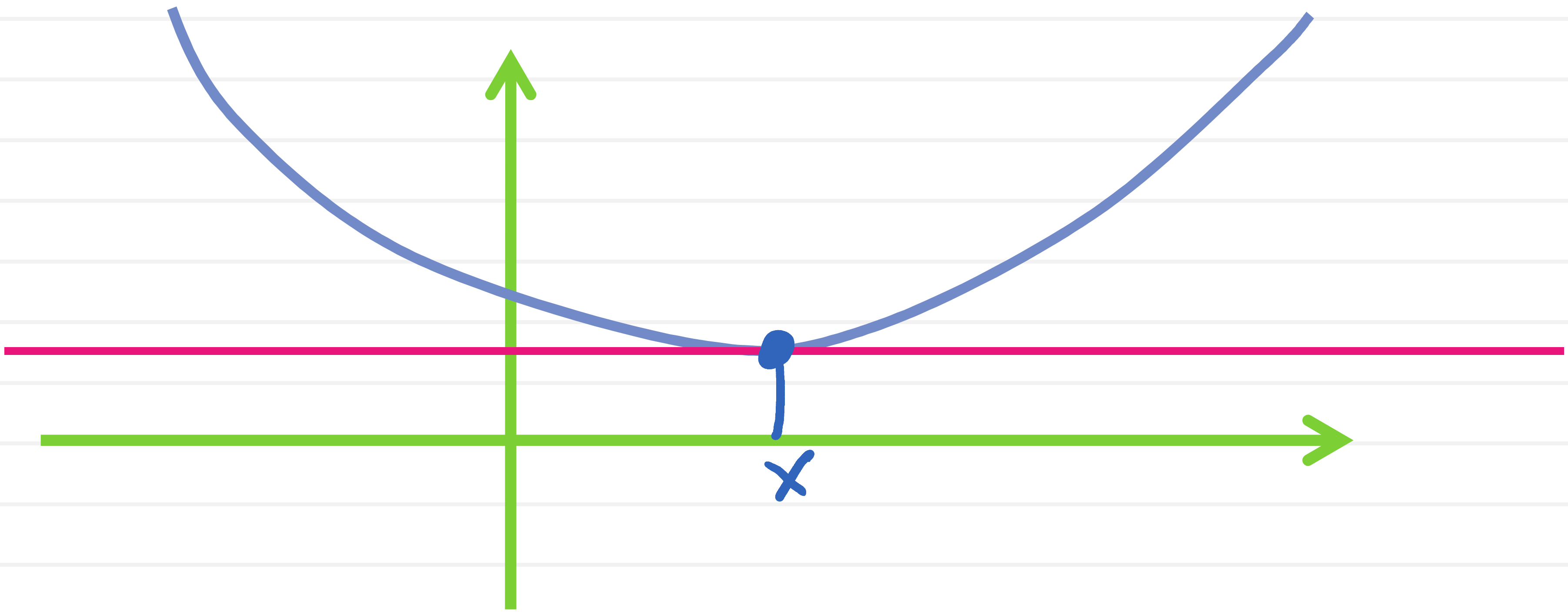


Optimizing a convex differentiable function

Corollary. If f is convex and $\nabla f(x) = 0$
then x is a global minimum of f.

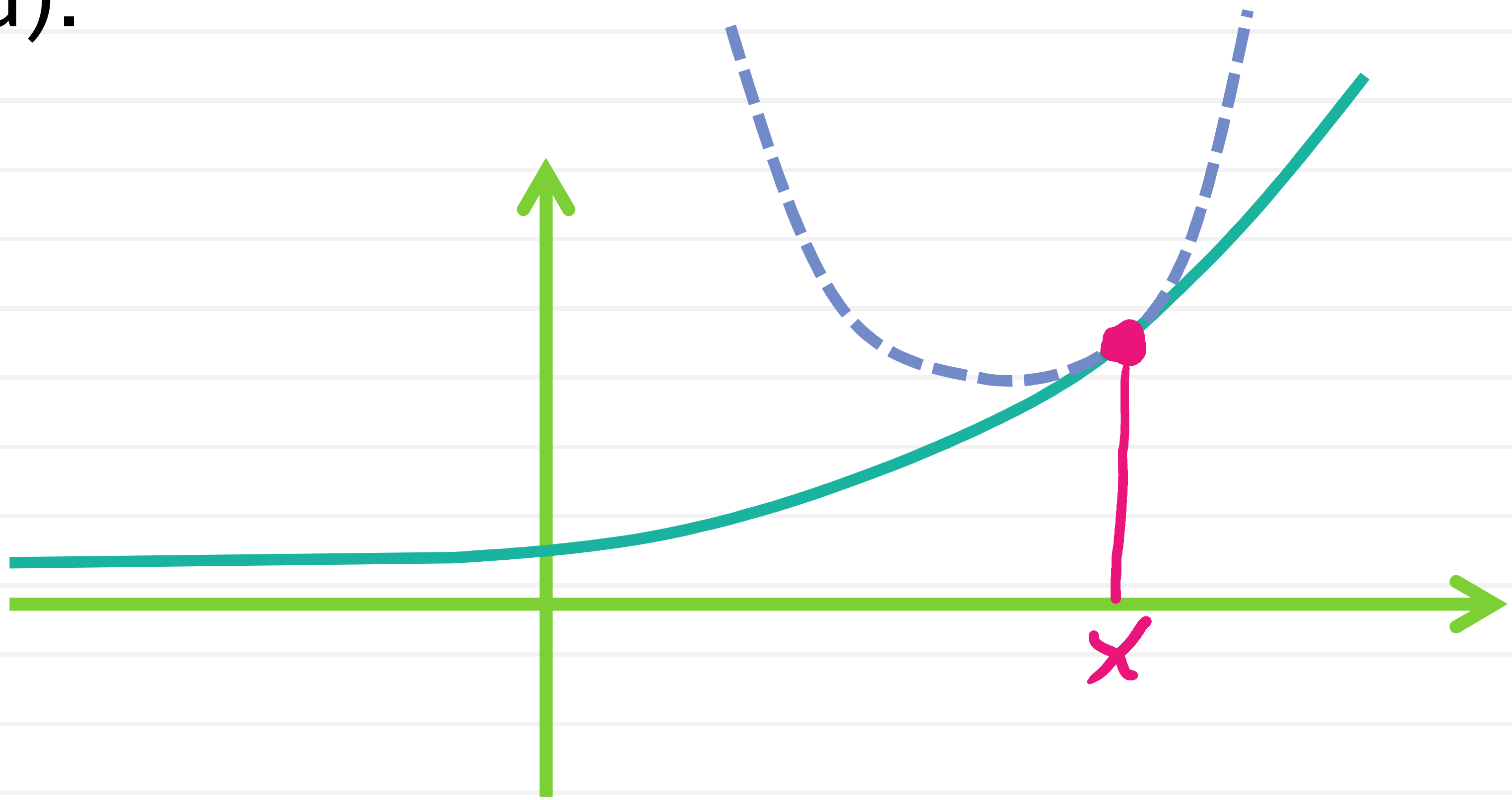
Proof: $\forall x, y \in \mathrm{dom}\, f$

$$f(y) \geqslant f(x) + \nabla f(x)^T (y - x)$$



Convexity for 2-differentiable functions

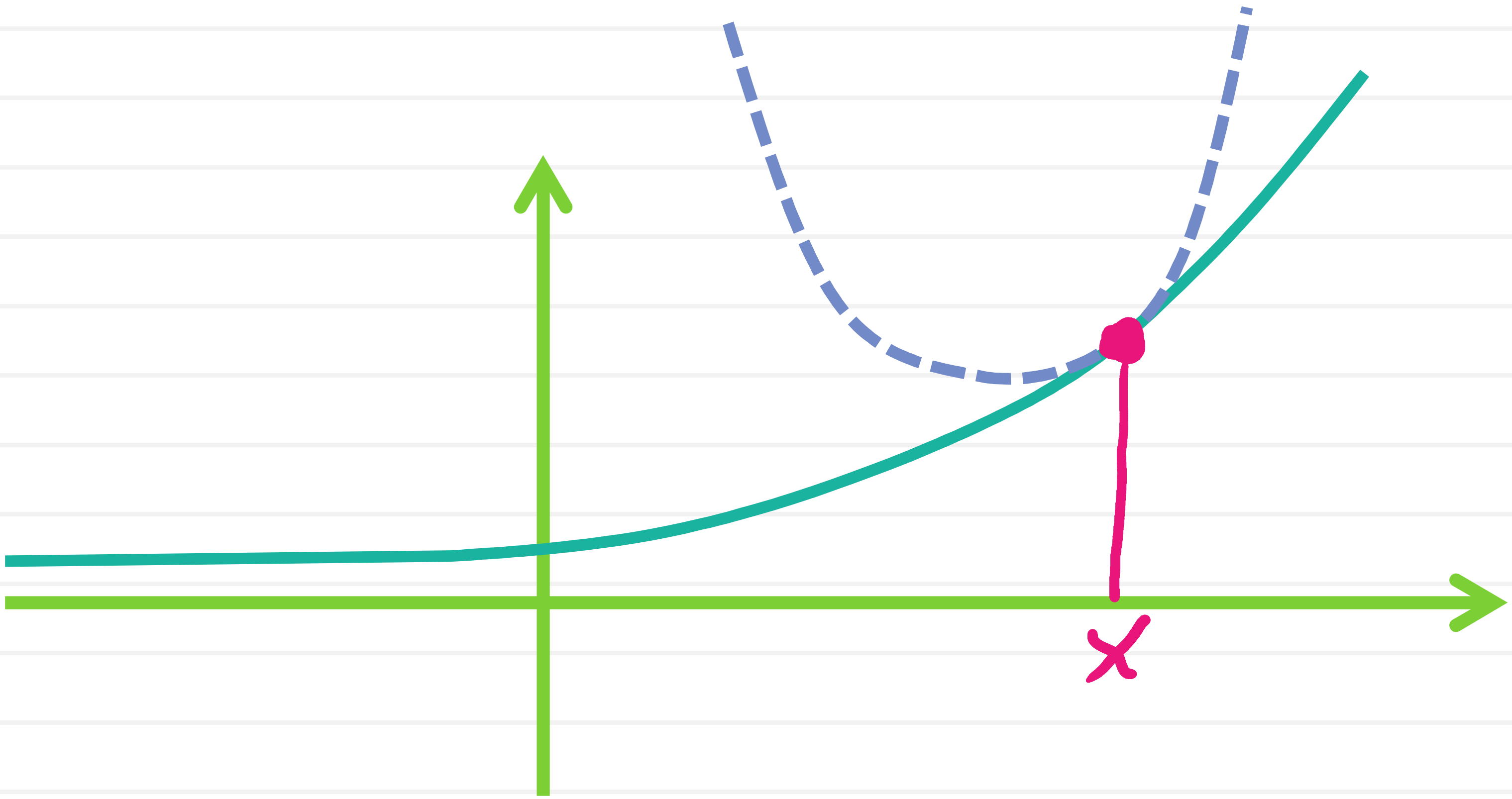
Corollary. Assume that f is defined over a convex set and is twice differentiable, then f is convex **iff** $\nabla^2 f(x) \succcurlyeq 0$ (the Hessian is p.s.d).



Convexity for 2-differentiable functions

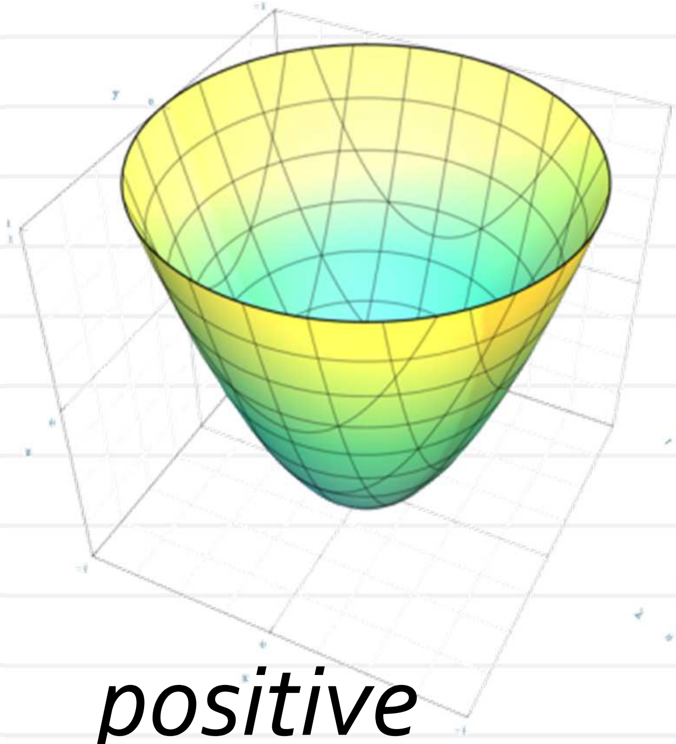
Corollary. Assume that f is defined over a convex set and the Hessian is p.d.

$\nabla^2 f(x) > 0$ Then f is strictly convex.

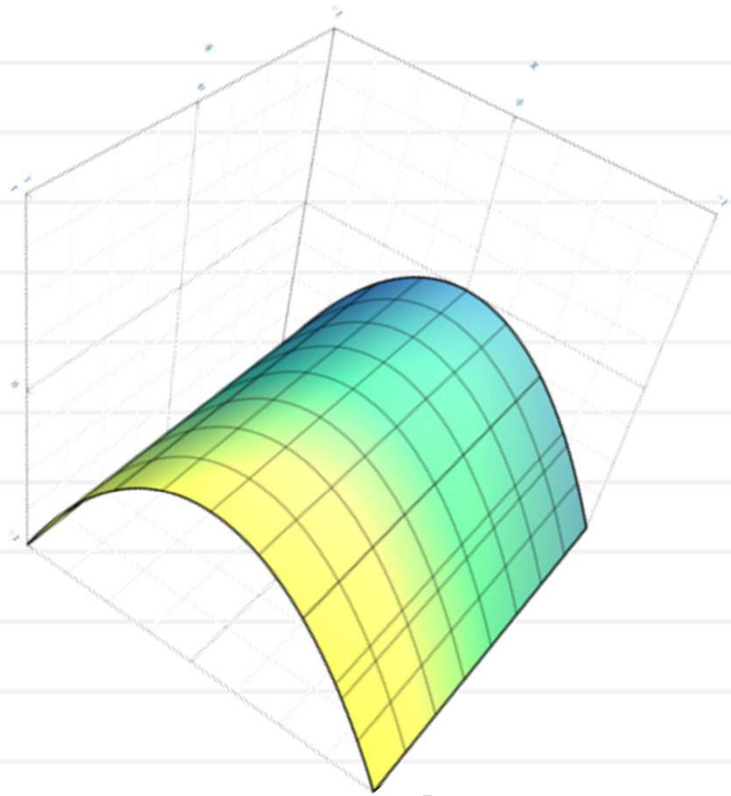


Examples for 2D functions

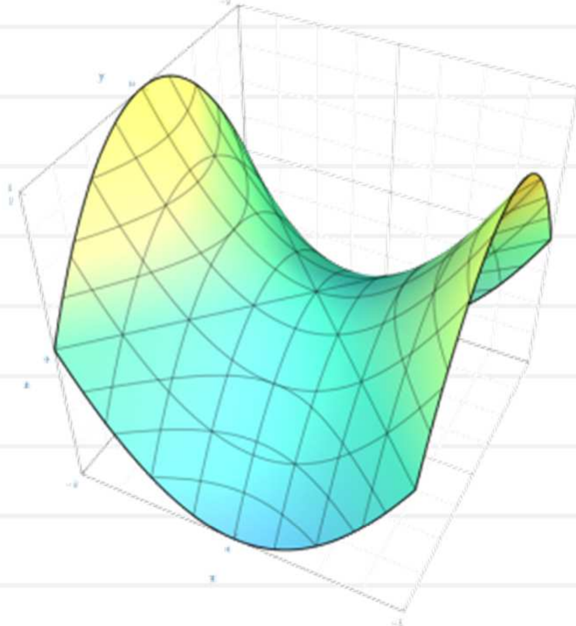
$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$



positive definite



negative semi-definite



indefinite

Images from Wikipedia

Checking convexity using the Hessian

What is positive (semi-)definiteness?

$$\forall x \quad x^T A x \geqslant 0$$

$$\forall x \quad x^T A x > 0$$

How to check it?

- Look at the eigenvalues?
- Look at principal minors (positive definiteness)?
- Run Cholesky (until it fails…)

Convexity quiz 2

$$F(\omega) = \frac{1}{2} \| X\omega - y \|^2$$

$$f(x, y) = \frac{x^2}{y} \qquad \begin{array}{l} x \geq 0 \\ y > 0 \end{array}$$

Composition rules

When is f(g(x)) convex?

$$g: \mathbb{R}^m \to \mathbb{R} \qquad f: \mathbb{R} \to \mathbb{R}$$

$$\nabla f(g(x)) = f'(g(x)) \nabla g(x)$$

$$\nabla^2 f(g(x)) = f''(g(x)) \cdot \nabla g(x) \nabla g(x)^T +$$

$$+ f'(g(x)) \nabla^2 g(x)$$

Sufficient condition 1: f is convex and non-decreasing, g is convex.

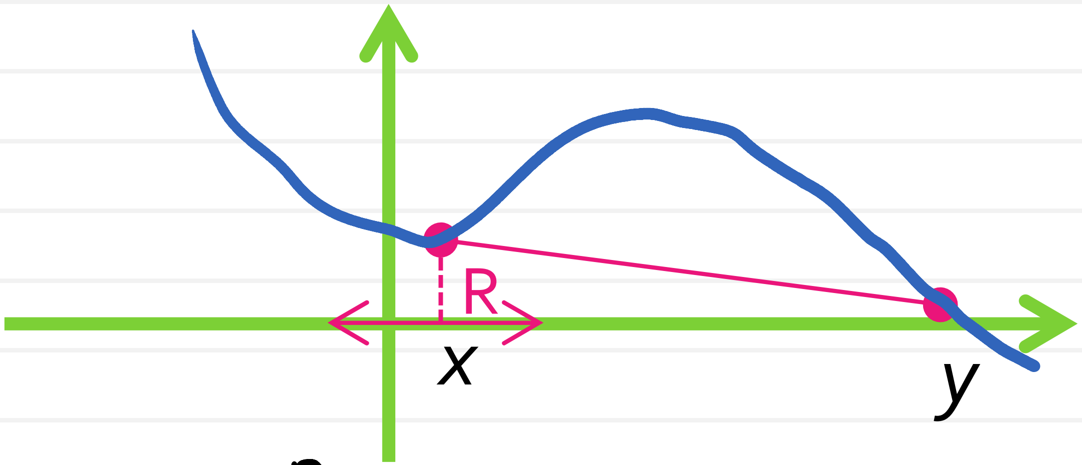
Sufficient condition 2: f is convex and non-increasing, g is concave.

Note: no sorts of converse statements are true, e.g.

$$f(x_1, x_2 \ldots x_n) = \log(e^{x_1} + e^{x_2} + \ldots + e^{x_n})$$

All minima of convex function are global

Proof: Assume x is a local minimum and is not a global minimum:



$f(y) < f(x)$

$$\theta_1 = \frac{R}{2\|y-x\|} \qquad \theta_2 = 1 - \theta_1$$

$$z = \theta_1 y + \theta_2 x$$

$$\|z - x\|_2 = \frac{R}{2} < R$$

$$f(z) = f(\theta_1 y + \theta_2 x) \le \theta_1 f(y) + \theta_2 f(x) \le f(x)$$

Convex programming

Convex objective

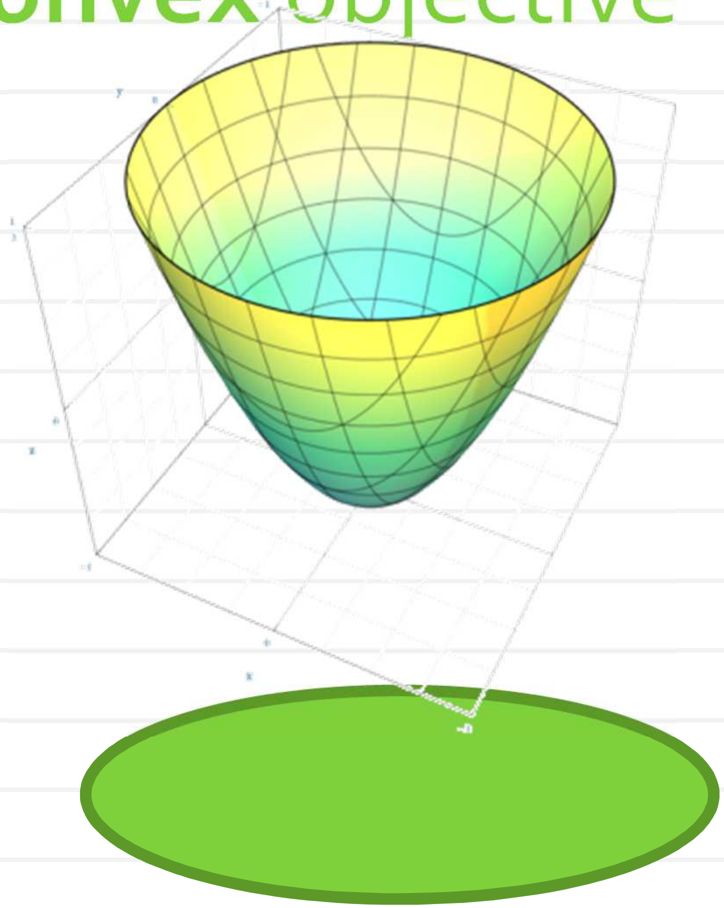
"minimize"

$$\min \; f_0(x)$$

$$\text{s.t.}: \; x \in D$$

"subject to"

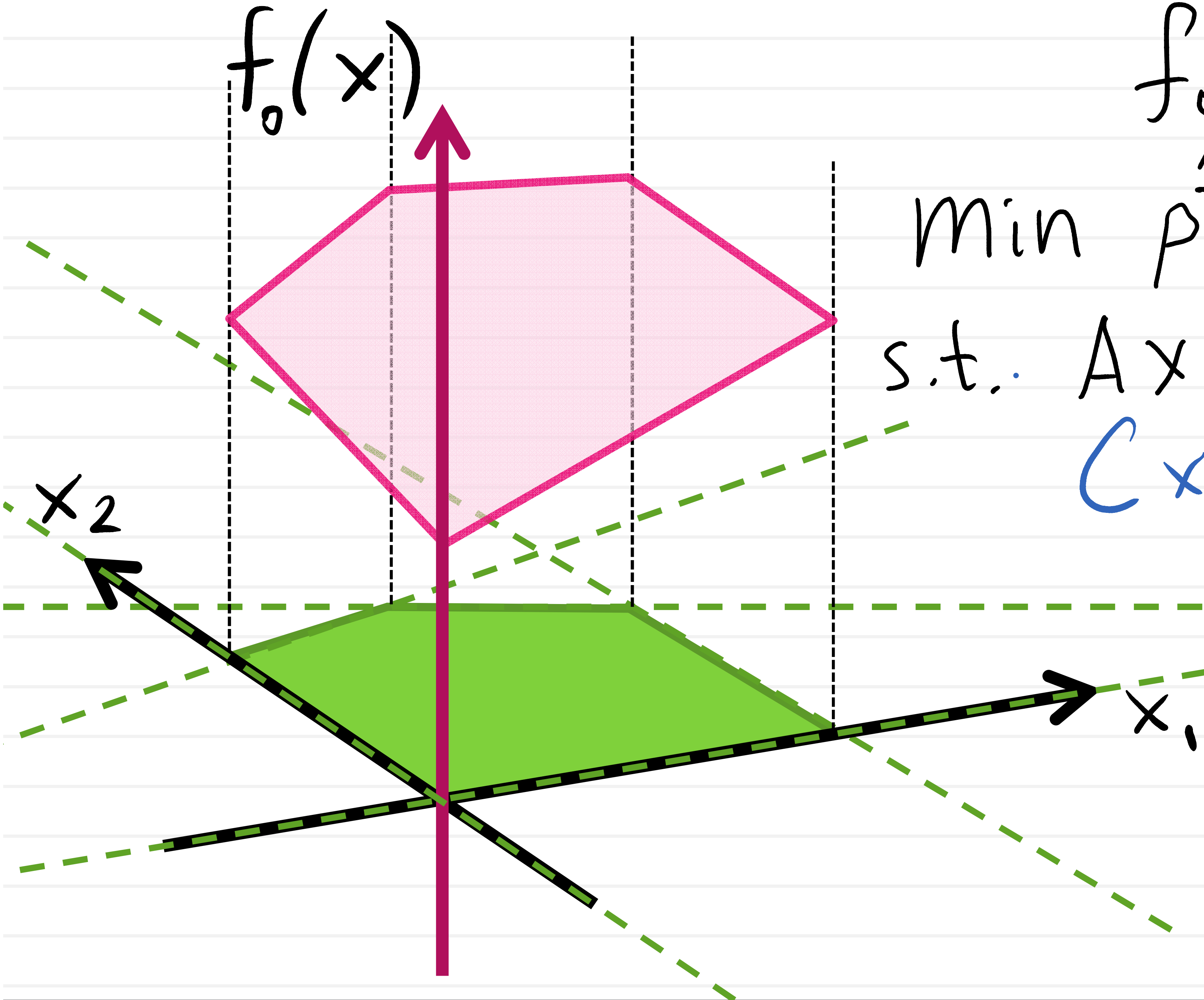
Convex domain



Corollary: any local optimum is global

Proof: see previous slide

Linear program



$$f_0(x) = p^T x$$

$$\begin{aligned} \min \quad & p^T x \\ \text{s.t.} \quad & Ax \le b \\ & Cx = d \end{aligned}$$

Sublevel sets

Sublevel set: $S_\alpha = \{x \in \mathrm{dom}\, f \mid f(x) \le \alpha\}$

Corollary: if f is convex then all sublevel sets are convex.

Is the opposite true?

Quasiconvex functions

Definition: the function is quasiconvex if all sublevel sets are convex.

Non-trivial example: $length(x) = max_i\,[x_i \neq 0]$

Solving quasiconvex problems

$$\min \; f_0(x)$$
$$\text{s.t.:} \quad x \in D$$

Solution: "bisection" over t

$$\boxed{\begin{array}{ll} \min & 1 \\ \text{s.t.:} & x \in D \\ & f_0(x) \le t \end{array}}$$