Problem set 1
DUE: April 20, 2016

Problem 1

$p(x) = \begin{cases} Ae^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$
We know that

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

Hence

$$\int_{0}^{\infty} Ae^{-\lambda x}dx = 1$$

$$\int_{0}^{\infty} Ae^{-\lambda x}dx = -\frac{1}{\lambda}Ae^{-\lambda x}|_0^\infty = -\frac{1}{\lambda}A(e^{-\infty} - e^0) = \frac{1}{\lambda}A = 1$$

Thus $A = \lambda$

$$Ex = \int_{0}^{\infty} xAe^{-\lambda x}dx = A\int_{0}^{\infty} xe^{-\lambda x}dx = \%u = x, du = dx, dv = e^{-\lambda x}dx, v = -\frac{1}{\lambda}e^{-\lambda x}\% =$$

$$= A\left(-\frac{1}{\lambda}xe^{-\lambda x}|_0^\infty + \frac{1}{\lambda}\int_{0}^{\infty} e^{-\lambda x}dx\right) = A\frac{1}{\lambda}\int_{0}^{\infty} e^{-\lambda x}dx = -A\frac{1}{\lambda^2}(0-1) = A\frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$Varx = Ex^2 - (Ex)^2$$

$$Ex^2 = \int_{0}^{\infty} x^2Ae^{-\lambda x}dx = \%u = x^2, du = 2xdx, dv = e^{-\lambda x}dx, v = -\frac{1}{\lambda}e^{-\lambda x}\% =$$

$$= A\left(-\frac{1}{\lambda}x^2e^{-\lambda x}|_0^\infty + 2\int_{0}^{\infty} x\frac{1}{\lambda}e^{-\lambda x}dx\right) = 2A\frac{1}{\lambda}\int_{0}^{\infty} xe^{-\lambda x}dx = \frac{2}{\lambda^2}$$

Hence $Varx = \frac{1}{\lambda^2}$

$$G(k) = \lambda\int_{0}^{\infty} e^{(ik-\lambda)x}dx = \frac{\lambda}{ik-\lambda} = \left(1 - \frac{ik}{\lambda}\right)^{-1}$$

$$\mathrm{E}[X^m] = \frac{1}{i^m}\frac{\partial^m}{\partial k^m}G(k)|_{k=0}$$

$$E[X] = \left(1 - \frac{ik}{\lambda}\right)^{-2} \frac{1}{\lambda}|_{k=0} = \frac{1}{\lambda}$$

$$E[X^2] = 2\left(1 - \frac{ik}{\lambda}\right)^{-3} \frac{1}{\lambda^2}|_{k=0} = \frac{1}{\lambda^2}$$

$$E[X^m] = m!\left(1 - \frac{ik}{\lambda}\right)^{-(m+1)} \frac{1}{\lambda^m}|_{k=0} = \frac{m!}{\lambda^m}$$

Problem 2

According to CLT:

$$\frac{\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - Ex\right)}{Stdx} \xrightarrow{d} N(0,1)$$

$$Ex = \frac{1}{2}\cdot 0 + \frac{1}{3}\cdot 2 + \frac{1}{6}\cdot 26 = \frac{2}{3} + \frac{13}{3} = 5$$

$$Stdx = \sqrt{Varx}$$

$$Varx = Ex^2 - (Ex)^2$$

$$Ex^2 = \frac{1}{2}\cdot 0 + \frac{1}{3}\cdot 4 + \frac{1}{6}\cdot 26^2 = \frac{4}{3} + \frac{13^2\cdot 2}{3} = \frac{4 + 169\cdot 2}{3} = 114$$

$$Varx = 114 - 25 = 89$$

$$Stdx = \sqrt{89} \approx 9.43$$

$$E\sum_{i=1}^{n} x_i = nEx = 500$$

Due to the fact that x_i are independent:

$$Var\sum_{i=1}^{n} x_i = nVarx = 890$$

$$Std\sum_{i=1}^{n} x_i \approx 29.83$$

According to CLT:

$$\sum_{i=1}^{n} x_i \xrightarrow{d} N(500, 890)$$

Z-score is $\frac{200-500}{890} = -0.34$

$$P\left(\sum_{i=1}^{n} x_i \geq 200\right) = 1 - P\left(\sum_{i=1}^{n} x_i < 200\right) = 1 - 0.3669 = 63.31\%$$

Problem 3

According to recitations, for Z-chanel we have:

$$p(y = 0|x = 0) = 1$$
$$p(y = 1|x = 0) = 0$$
$$p(y = 1|x = 1) = 1 - f = 0.85$$
$$p(y = 0|x = 1) = f = 0.15$$

We know, that

$$\sum_{j=1}^{n} P(y|x_j)P(x_j)$$

Hence

$$P(y) = P(y|x = 0)P(x = 0) + P(y|x = 1)P(x = 1) = P(y|x = 0)0.9 + P(y|x = 1)0.1$$

P(x = 0) = 0.9
P(x = 1) = 0.1
So we can now compute:
$P(y = 1) = 0.1(1 - f) = 0.1 - 0.1f = 0.1 - 0.1 \cdot 0.15 = 0.1 - 0.015 = 0.085$
$P(y = 0) = 0.9 + 0.1f = 0.9 + 0.1 \cdot 0.15 = 0.915$

$$P(x|y) = \frac{P(y|x)P(x)}{P(y)}$$

Hence

$$P(x = 1|y = 0) = \frac{0.1f}{0.9 + 0.1f} = \frac{0.15}{0.915} = \frac{1}{61} \approx 0.016$$

$$I(X; Y) = S(Y) - S(Y|X)$$

$S(Y) = -P(Y=0)log(P(Y=0)) - P(Y=1)log(P(Y=1)) = -0.915log(0.915) - 0.085log(0.085)$

$S(Y|X) = -\sum_{i=1}^{n_x} P(x_i) \sum_{j=1}^{n_y} P(y_j|x_i) log(P(y_j|x_i)) = -P(x=0)(P(y=0|x=0)log(P(y=0|x=0)) - P(y=1|x=0)log(P(y=1|x=0))) + P(x=1)(P(y=0|x=1)log(P(y=0|x=1)) + P(y=1|x=1)log(P(y=1|x=1))) = -0.1(0.15log(0.15) + 0.85log(0.85)) = -0.015log(0.15) - 0.085log(0.85)$

Therefore

$$I(X;Y) = -0.915log(0.915) - 0.085log(0.085) + 0.015log(0.15) + 0.085log(0.85) =$$

$$= log\left(\frac{0.15^{0.015} \cdot 0.85^{0.085}}{0.915^{0.915} \cdot 0.085^{0.085}}\right) \approx log(1.28) \approx 0.36$$

The capacity of the channel $C(Q) = max_{P(x)} I(X;Y)$

In general $I(X;Y) = S_{binary}(P(x=1)(1-f)) - P(x=1)S_{binary}(y|x=1) = S_{binary}(P(x=1)(1-f)) - P(x=1)S_{binary}(f)$

Now we can differentiate I(X;Y) with respect to p.

$$\frac{\partial I(X;Y)}{\partial p} = \frac{-p(1-f)log_2(p(1-f)) - (1-p(1-f))log_2(1-p(1-f))}{\partial p} - S_{binary}(f) =$$

$= \frac{f-1}{ln(2)}\left(ln(p(1-f)) + (1-f)\frac{p}{p(1-f)}\right) + \frac{1}{ln(2)}\left((1-f)ln(1-p(1-f)) + \frac{p(1-f)-1}{1-p(1-f)}(f-1)\right) - S_{binary}(f) =$

$$= \frac{f-1}{ln(2)}(ln(p(1-f)) + 1) + \frac{1-f}{ln(2)}(ln(1-p(1-f)) + 1) - S_{binary}(f)$$

Now let's find optimum:

$$\frac{f-1}{ln(2)}(ln(p(1-f)) + 1) + \frac{1-f}{ln(2)}(ln(1-p(1-f)) + 1) - S_{binary}(f) = 0$$

$$ln(p(1-f)) + 1 - ln(1-p(1-f)) - 1 = S_{binary}(f)\frac{ln(2)}{f-1}$$

$$ln(p(1-f)) - ln(1-p(1-f)) = S_{binary}(f)\frac{ln(2)}{f-1}$$

$$ln\left(\frac{p(1-f)}{1-p(1-f)}\right) = ln(2^{S_{binary}(f)(f-1)})$$

$$\frac{1}{p(1-f)} - 1 = 2^{\frac{S_{binary}(f)}{1-f}}$$

$$p = \frac{1}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}} + 1)}$$

Now we can find

$$C(Q) = S_{binary}(P(x=1)(1-f)) - P(x=1)S_{binary}(f)|p = \frac{1}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}} + 1)}$$

$$C(Q) = S_{binary}\left(\frac{1}{2^{\frac{S_{binary}(f)}{1-f}} + 1}\right) - \frac{S_{binary}(f)}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}} + 1)}$$

$$C(Q) = \left(\frac{1}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(2^{\frac{S_{binary}(f)}{1-f}} + 1\right) - \left(1 - \frac{1}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(1 - \frac{1}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) - \frac{S_{binary}(f)}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}}+1)}$$

$$C(Q) = \left(\frac{1}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(2^{\frac{S_{binary}(f)}{1-f}} + 1\right) - \left(\frac{2^{\frac{S_{binary}(f)}{1-f}}}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(\frac{2^{\frac{S_{binary}(f)}{1-f}}}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) - \frac{S_{binary}(f)}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}}+1)}$$

$$C(Q) = \left(\frac{1}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(2^{\frac{S_{binary}(f)}{1-f}} + 1\right) - \left(\frac{2^{\frac{S_{binary}(f)}{1-f}}}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(2^{\frac{S_{binary}(f)}{1-f}}\right) + \left(\frac{2^{\frac{S_{binary}(f)}{1-f}}}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) log_2\left(2^{\frac{S_{binary}(f)}{1-f}} + 1\right) - \frac{S_{binary}(f)}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}}+1)}$$

$$C(Q) = log_2\left(2^{\frac{S_{binary}(f)}{1-f}} + 1\right) - \left(\frac{2^{\frac{S_{binary}(f)}{1-f}}}{2^{\frac{S_{binary}(f)}{1-f}}+1}\right) \frac{S_{binary}(f)}{1-f} - \frac{S_{binary}(f)}{(1-f)(2^{\frac{S_{binary}(f)}{1-f}}+1)}$$

$$C(Q) = log_2\left(2^{\frac{S_{binary}(f)}{1-f}} + 1\right) - \frac{S_{binary}(f)}{1-f}$$

$$C(Q) = log_2\left(1 + 2^{-\frac{f}{1-f}log_2(f) - log_2(1-f)}\right) + \frac{f}{1-f}log_2(f) + log_2(1-f)$$

$$C(Q) = log_2\left(1 + \frac{f^{\frac{f}{f-1}}}{1-f}\right) + \frac{f}{1-f}log_2(f) + log_2(1-f)$$

$$C(Q) = log_2\left(1 - f + f^{\frac{f}{f-1}}\right) + \frac{f}{1-f}log_2(f)$$

$$C(Q) = log_2\left(f^{\frac{f}{1-f}} - f^{\frac{1}{1-f}} + 1\right)$$

Problem 4

$$\begin{pmatrix} & GG & Gg & gg \\ GG & 0.5 & 0.25 & 0 \\ Gg & 0.5 & 0.5 & 0.5 \\ gg & 0 & 0.25 & 0.5 \end{pmatrix}$$

MC is irreducible because there are no states that could become unreached from any state on any time step. Due to the fact, that MC contains self-loops it is aperiodic.

Let us define the matrix P in a general form:

$$\begin{pmatrix} & GG & Gg & gg \\ GG & p_{00} & p_{01} & p_{02} \\ Gg & p_{10} & p_{11} & p_{12} \\ gg & p_{20} & p_{21} & p_{22} \end{pmatrix}$$

$\mu_1(GG) = p_{01} = 0.25$
$\mu_1(Gg) = p_{11} = 0.5$
$\mu_1(gg) = p_{21} = 0.25$

$\mu_2(GG) = \mu_1(GG)p_{00} + \mu_1(Gg)p_{01} + \mu_1(gg)p_{02} = p_{01}p_{00} + p_{11}p_{01} = 0.25 \cdot 0.5 + 0.5 \cdot 0.25 = 0.25$
$\mu_2(Gg) = \mu_1(GG)p_{10} + \mu_1(Gg)p_{11} + \mu_1(gg)p_{12} = p_{01}p_{10} + p_{11}p_{11} + p_{21}p_{12} = 0.25 \cdot 0.5 + 0.25 + 0.5 \cdot 0.25 = 0.5$
$\mu_2(gg) = \mu_1(GG)p_{20} + \mu_1(Gg)p_{21} + \mu_1(gg)p_{22} = p_{11}p_{21} + p_{21}p_{22} = 0.5 \cdot 0.25 + 0.25 \cdot 0.5 = 0.25$

$\mu_3(GG) = \mu_2(GG)p_{00} + \mu_2(Gg)p_{01} + \mu_2(gg)p_{02} = 0.25 \cdot 0.5 + 0.25 \cdot 0.5 = 0.25$
$\mu_3(Gg) = \mu_2(GG)p_{10} + \mu_2(Gg)p_{11} + \mu_2(gg)p_{12} = 0.25 \cdot 0.5 + 0.5 \cdot 0.5 + 0.25 \cdot 0.5 = 0.5$
$\mu_3(gg) = \mu_2(GG)p_{20} + \mu_2(Gg)p_{21} + \mu_2(gg)p_{22} = 0.5 \cdot 0.25 + 0.25 \cdot 0.5 = 0.25$

One can see that μ_n doesn't depend on n.

$$P_1 = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.325 & 0.25 & 0.125 \\ 0.5 & 0.5 & 0.5 \\ 0.125 & 0.25 & 0.375 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.325 & 0.25 & 0.125 \\ 0.5 & 0.5 & 0.5 \\ 0.125 & 0.25 & 0.375 \end{pmatrix} \begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.3125 & 0.25 & 0.1875 \\ 0.5 & 0.5 & 0.5 \\ 0.1875 & 0.25 & 0.3125 \end{pmatrix}$$

$$P^n = \begin{pmatrix} 0.125\sum_{i=0}^{n-1} 0.5^i + 0.5^{n+1} & 0.25 & 0.5 - 0.125\sum_{i=0}^{n-1} 0.5^i - 0.5^{n+1} \\ 0.5 & 0.5 & 0.5 \\ 0.5 - 0.125\sum_{i=0}^{n-1} 0.5^i - 0.5^{n+1} & 0.25 & 0.125\sum_{i=0}^{n-1} 0.5^i + 0.5^{n+1} \end{pmatrix}$$

$P\pi^* = \pi^*$

$$\begin{pmatrix} 0.5 & 0.25 & 0 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.25 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Hence:

$$\begin{cases} 0.5x + 0.25y = x \\ 0.5x + 0.5y + 0.5z = y \\ 0.25y + 0.5z = z \end{cases}$$

$$x = 0.5y$$
$$z = 0.5y$$
$$0.25y + 0.5y + 0.25y = y$$

So $\pi^* = (0.5y, y, 0.5y)^T$
We know, that $\sum_i \pi^* = 1$, so 2y = 1, hence y = 0.5.
$\pi^* = (0.25, 0.5, 0.25)^T$
It is easy to see, that the detailed balance holds.

Problem 5

$\lambda = 10$
$P(n \geq 20) = 1 - \sum_{i=0}^{19} P(i, 1)$
$P(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$
Hence, $P(n \geq 20) \approx 0.34\%$

Due to the fact, that arrival follows the Poison law. We can find $\lambda_1 = (1-p)\lambda$ that is woman arrival rate.

Thus, $P(n = 10, 1) = \frac{(\lambda_1 t)^n}{n!} e^{-\lambda_1 t} = \frac{((1-p)10)^{10}}{10!} e^{(p-1)10}$, where n is amount of women came.

For men we have $\lambda_2 = p\lambda$

For the probability distribution of the inter-arrival time one obtains $P(t) \approx pe^{-pt}$. So, expected inter-arrival time of men is $\frac{1}{p}$.

$P(n = 0, 2) = e^{-20p}$, where n is amount of male customers