Problem set 1
DUE: Tue. September 8, 2015

Problem 1

We know that all the polynoms complex roots have pairs, thus P(x) has a pair for $x_1 = 1 - i\sqrt{7}$, that is $X_2 = 1 + i\sqrt{7}$
Also due to fundamental theorem of algebra P(x) = $(x-x_1)(x-x_2)(x-x_3)$. As far as we know x_1 and x_2 we can multiply them and get that P(x) = $(x^2-2x+8)(x-x_3)$.
Hence $x^3 - 2x^2 + 8x - x^2x_3 + 2xx_3 - 8x_3 = -24 + 14x - 5x^2 + x^3$.
$-8x_3 = -24$
$x_3 = 3$

Problem 2

$x^3 - 2x^2 + 2x - 1 = 0$
In a nightmare I have found that $x_1 = 1$ is a solution of equation above. We can easily prove it: 1 - 2 + 2 - 1 = 0
Then we can divide initial polynom by (x - 1) and according to fundamental theorem of algebra we will have $(x-x_3)(x-x_2)$
$(x-x_3)(x-x_2) = x^2 - x + 1$
Thus $x_{2,3} = \dfrac{1 \pm i\sqrt{3}}{2}$

Problem 3

$x^3 - x^2 + \dfrac{x}{3} - \dfrac{1}{27} = 0$
Lama in the hat had come to me and told that $x_1 = \dfrac{1}{3}$ is a root of previous equation.
Then we can divide initial polynom by (x - $\dfrac{1}{3}$) and according to fundamental theorem of algebra we will have $(x-x_3)(x-x_2)$
$(x-x_3)(x-x_2) = x^2 - \dfrac{2}{3}x + \dfrac{1}{9} = (3x-1)^2$
Thus $x_{2,3} = x_1 = \dfrac{1}{3}$

Problem 4

We know that geometry progression sum is $\dfrac{b_1(1-q^n)}{1-q}$.
So, we can rewrite initial equation in another form:
$\dfrac{1-z^{2015}}{1-z} = 0$
We can check that z = 0 is not a root if the equation and thus we can multiply this equation by (1 - z) and so $1 - z^{2015} = 0$.
$cos(2015\phi) + isin(2015\phi) = 1$
Defining $\phi = \dfrac{2\pi k}{2015}$, where $k \in N$
z = $cos\phi + isin\phi$ is a root of initial equation

Problem 5

P(x) $= x^3 - 2^{-\frac{2}{3}}zx + 1.$

This polynom has 3 roots. It means that we can rewrite it in the following form.

P(x) $= (x - x_1)(x - x_2)^2$

We can do it because $x_2 = x_3$.

Thus, opening brackets in a new form of our polynom we get:

P(x) $= x^3 - x^2(2x_2 + x_1) + x(x_2^2 + 2x_1x_2) - x_1x_2^2$

From initial form and a new one we can get a system:

$$\begin{cases} x_1x_2^2 = 1 \\ 2x_2 + x_1 = 0 \\ x_2^2 + 2x_1x_2 = -2^{-\frac{2}{3}}z \end{cases}$$

From this system we can get that:

$$\begin{cases} x_1 = -2x_2 \\ x_2^3 = -\dfrac{1}{2} \\ z = \dfrac{x_2^2 + 2x_1x_2}{-2^{-\frac{2}{3}}} \end{cases}$$

Hence:

$$\begin{cases} x_1 = -2x_2 \\ x_2^3 = -\dfrac{1}{2} \\ z = 2^{-\frac{2}{3}} \cdot 3x_2^2 \end{cases}$$

From this system we can get that: z should be equal to -3.

Problem 6

cosh(x) $= \dfrac{e^x + e^{-x}}{2}$

$cosh(log(x + \sqrt{x^2 - 1})) = \dfrac{e^{log(x+\sqrt{x^2-1})} + e^{-log(x+\sqrt{x^2-1})}}{2}$

Assuming that logarithm base is уб we can derive that initial function is equal to:

$$\frac{x + \sqrt{x^2 - 1} + \dfrac{1}{x + \sqrt{x^2 - 1}}}{2} = x$$

Problem 7

$sinh(x) = \dfrac{e^x - e^{-x}}{2}$

$cosh^2(t) - sinh^2(t) = 1$

By definition of hyperbolic functions:
$cosh^2(t) - sinh^2(t) = \dfrac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{4} = \dfrac{e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}}{4} = 1$
Nothing will change for any t because it doesn't depend on it.

Problem 8

Using L'Hopital's rule several times we can find that:

$$\lim_{x\to 0} \frac{isin(-ix) + sin(x) - 2x}{log(1 + x^5)} = \lim_{x\to 0} \frac{(cos(ix) + cos(x) - 2)(1 + x^5)}{5x^4} =$$

$$\lim_{x\to 0} \frac{(cos(ix) + cos(x) - 2) \cdot 5x^4 + (-isin(ix) - sin(x))(1 + x^5)}{20x^3} =$$

$$\lim_{x\to 0} \frac{(cos(ix) + cos(x) - 2) \cdot 20x^3 + (-isin(ix) - sin(x)) \cdot 5x^4 + +(-isin(ix) - sin(x)) \cdot 5x^4 + (cos(ix) - cos(x))(1 + x^5)}{60x^2} =$$

$$\lim_{x\to 0} \frac{(cos(ix) + cos(x) - 2) \cdot 60x^2 + (-isin(ix) - sin(x)) \cdot 20x^3 + (-isin(ix) - sin(x)) \cdot 40x^3 + +(cos(ix) - cos(x)) \cdot 10x^4 + (cos(ix) - cos(x)) \cdot 5x^4 + (-isin(ix) + sin(x))(1 + x^5)}{120x} =$$

$$\lim_{x\to 0} \frac{(cos(ix) + cos(x) - 2) \cdot 120x + (-isin(ix) - sin(x)) \cdot 60x^2 + (-isin(ix) - sin(x)) \cdot 180x^3 + (cos(ix) - cos(x)) \cdot 60x^3 + (cos(ix) - cos(x)) \cdot 60x^3 + (-isin(ix) + sin(x)) \cdot 15x^4 + +(-isin(ix) + sin(x)) \cdot 5x^4 + (cos(ix) + cos(x))(1 + x^5)}{120} = \frac{2}{120} = \frac{1}{60}$$

Problem 9

By definition of cosh we can derive that:
cosh(in arccos(x)) $= \dfrac{e^{in \cdot arccos(x)} + e^{-in \cdot arccos(x)}}{2}$
According to Euler's formula:

$$\frac{e^{in \cdot arccos(x)} + e^{-in \cdot arccos(x)}}{2} = \frac{cos(n \cdot arccos(x)) + isin(n \cdot arccos(x)) + +cos(n \cdot arccos(x)) - isin(n \cdot arccos(x))}{2} = cos(n \cdot arccos(x))$$

When n = 0:
$cos(n \cdot arccos(x) = cos(0) = 1$
When n = 1:
$cos(n \cdot arccos(x) = cos(arccos(x) = x$
When n = 2:
$cos(n \cdot arccos(x) = cos(2 \cdot arccos(x) = 2cos^2(arccos(x)) - 1 = 2x^2 - 1$
When n = 3:
$cos(n \cdot arccos(x) = cos(3 \cdot arccos(x) = 4cos^3(arccos(x)) - 3cos(arccos(x)) = 4x^3 - 3x$

Problem 10

z = a + ib
w = c + id
|z-w| = $\sqrt{(a-c)^2+(b-d)^2}$
|z| = $\sqrt{a^2+b^2}$
|w| = $\sqrt{c^2+d^2}$
||z| - |w|| = $|\sqrt{a^2+b^2}-\sqrt{c^2+d^2}|$
Now we can rewrite : $|z-w| \geq ||z|-|w||$
as:
$\sqrt{(a-c)^2+(b-d)^2} \geq |\sqrt{a^2+b^2}-\sqrt{c^2+d^2}|$
$a^2+b^2+c^2+d^2-2ac-2bd \geq a^2+b^2+c^2+d^2-2\sqrt{(a^2+b^2)(c^2+d^2)}$
$\sqrt{(a^2+b^2)(c^2+d^2)} \geq ac+bd$
$(a^2+b^2)(c^2+d^2) \geq (ac)^2+(bd)^2+2(ac)(bd)$
$(ac)^2+(ad)^2+(bc)^2+(bd)^2 \geq (ac)^2+(bd)^2+2(ac)(bd)$
$(ad)^2+(bc)^2 \geq 2(ac)(bd)$
$\frac{(ad)^2+(bc)^2}{2} \geq \sqrt{(ac)^2(bd)^2}$
The last equality always holds, QED.

Problem 11

$\frac{\partial f}{\partial x_1} = \frac{\partial f_1}{\partial x_1} + i\frac{\partial f_2}{\partial x_1}$
$\frac{\partial f}{\partial x_2} = \frac{\partial f_1}{\partial x_2} + i\frac{\partial f_2}{\partial x_2}$
$\frac{\partial^2 f_1}{\partial x_1^2} + \frac{\partial^2 f_1}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial f_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial f_2}{\partial x_1}\right) = 0$

Problem 12

$x \cdot e^{2\pi i} = x(cos(2\pi)+isin(2\pi)) = x$, QED.
Due to the fact, that $x \cdot e^{2\pi i} = x$, $log(xe^{2\pi i}) - log(x) = log(x) - log(x) = 0$