and now, the answers.

Most answers come with solutions, which match the problem in level of difficulty. You don't need much prior knowledge to understand solutions to easy problems; more scientific background is assumed for solutions to harder problems. If a solution doesn't make sense, don't hesitate to ask for help—from your friends, your teachers, the test writers, or the vast fields of human knowledge called the Internet.

Multiple choice

1. See **Appendix A** for the full solution of this problem.

2. D – because the interference of waves has resulted in an increase in intensity, this interference is constructive. As a side note, yes, it is possible for the combined intensity to be greater than the sum of its parts. This is because intensity is proportional to the *average* power, and usually, two waves will interfere both constructively and destructively, so the new average power is the sum of the two old averages. However, if two waves are identical or almost identical, they can be aligned so that they only interfere constructively and never destructively, resulting in an average that draws from data that has increased everywhere. In a case where waves interfere completely or almost completely constructively, the new average power is greater than the sum of the individual average powers.

3. B – compressions are regions of high particle density in longitudinal waves.

4. E – because $f = v/\lambda$ and v is the same for all waves, the wave with the shortest wavelength will have the highest frequency. From the figure, we see that wave B and wave C have the same wavelength. Even though the number of repeated cycles is greater for C, the length of any repeated portion is the same in both. Meanwhile, the length of a single cycle in wave A is shorter than length of a single cycle in either B or C, so wave A has the shortest wavelength, and therefore, the highest frequency.

5. D – because $v = \lambda f$ and the frequency is the same for all waves, the wave with the longest wavelength will have the highest wave speed.

6. B – a compressional wave and longitudinal wave are the same thing, although the term "compressional wave" has fallen into disuse, while most modern scientists favor "longitudinal wave." (A) is not correct because a harmonic is an integer multiple of the fundamental frequency, while an overtone is any tone greater than (but not equal to) the fundamental frequency of a tone. (C) is incorrect because particle speed is proportional to the displacement amplitude of the wave, while the speed of sound doesn't depend on amplitude. (D) is incorrect because rarefactions and compressions are antonyms, not synonyms. (E) is wrong because in general, brass instruments are made from brass, but very few wind instruments are made from brass.

7. B – because the maximum displacement of a node is always zero, the frequency according to this definition would always be infinite because the point would always be at its maximum at any given second in time. This is clearly false. (A) always works with this definition, so you can eliminate (A). For a sinusoidal wave, (B) is compatible with this definition too, so eliminate (B). (A root-mean-square value is a typical value of the displacement, and for sinusoidal wave, it equals $\sqrt{2}$ times the displacement of the antinodes.) There is nothing special about the midpoint in every case—if the string is vibrating at an odd harmonic, then (D) becomes a special case of (A). Therefore, we can eliminate (D). I completely made up (E), so we can cross that off too.

Alternative way to solve the problem: even if you couldn't visualize or imagine in your head what the definition meant, you may realize that the definition assumes that displacement happens. Only at nodes, there is no displacement, so this assumption must be false. Therefore, (B) must be always false.

8. B – in music, tuning forks are used as a baseline of comparison when tuning a musical instrument. Other purposes of tuning forks exist (doctors may use them to check how good your hearing is), but this is the primary purpose of tuning forks in music. (A) is incorrect as tuning forks are rarely used to make an instrument start playing. (C) is incorrect as tuning forks are neither wrenches nor silverware, and they don't make physical contact with the objects they are used with. (D) is incorrect because tuning forks make sound, rather than absorbing it. (E) is incorrect because most audiences would be annoyed if a fork were whacked to boost their perception, especially in the middle of a performance.

9. B – the wave speed on a string is $v = \sqrt{\frac{T}{\mu}}$, where T is the string's tension and $\mu = m/L$ is the mass density (mass per length) of the string. In this problem, the tension is three times greater in the new string, but the mass density μ doesn't change, as the new string has the same amount of mass per length. So, the wave speed on the new string is $\sqrt{3}\,v$.

10. A – let $f = 2F$. We can then rewrite this sequence of notes as $2F$, $3F$, $4F$, $5F$, and $6F$. By the missing fundamental effect, a human observer must hear the greatest common factor of all of these notes, or a frequency of F. We know that $F = 0.5f$, so the correct answer is choice (A). You can see why this effect is called the missing fundamental effect: while harmonics two through six are present and the fundamental is missing, the person still hears the fundamental.

11. D – we know nothing about the mass density of each string, nor do we know which string is under greater tension. Therefore, we can't tell which string will allow for faster waves.

12. D – Short explanation: in string P, tension is provided by the walls. In string Q, tension is provided by gravity. When the temperature rises, the ends of string P get closer to the wall, so the force of the wall on the string weakens, and so the tension in string P weakens too. Temperature has no effect on gravity, so the tension in Q remains the same.

Long explanation: solid objects grow in size when they warm up. This is known as *thermal expansion*. Next, *tension* is the force that describes how much a string pulls on something. Like a rubber band, the more you pull on a string, the more the string pulls back, and therefore, the higher the tension. As you can imagine, you require a greater force to stretch a 9 cm rubber band to 10 cm than you do to stretch a 1 cm rubber band to 10 cm, so the tension in the 1 cm band is greater. Similarly, when the temperature of the room rises, the natural length of the strings will increase (but only by a small, unnoticeable bit). For string P, because the natural relaxed length of the string is greater due to the thermal expansion, less force is required to hold the string rigid between the floor and ceiling of the room. While obviously nowhere as dramatic, string P becomes less like the 1 cm rubber band, and more like the 9 cm rubber band. Because it takes less force to stretch string P when the temperature rises, the tension of the string itself must also fall. Meanwhile, the force of tension from string Q results entirely from the string pulling back on the weight of the box, so the tension doesn't change in string Q. Even though string Q will expand like string P, string Q behaves differently because its tension doesn't come from being suspended the fixed length from the ceiling to the floor; instead, string Q will always stretch as far as the box wants to go, so its tension will always be in equilibrium with the weight mg of the box.

13. C – the Doppler Effect describes the shift in frequency due to moving sources and observers.

14. A – refraction is the phenomenon where waves change direction when they move into a medium with a different wave speed. Snell's Law $n_1 \sin\theta_1 = n_2 \sin\theta_2$ predicts the direction θ (measured from the normal), provided the index of refraction n for medium 1 and medium 2. In typical media, n_1 and n_2 are positive numbers greater than or equal to one, and θ_1 and θ_2 are measured both counterclockwise from the normal, or both clockwise from the normal. However, in the figure for this problem, the path of the waves is to the left of the normal both before *and* after refraction. If θ_1 and θ_2 are the incident and refracted angles respectively, this implies that θ_2 has a negative value, but because n_{air} and θ_1 are positive, n_2 must be negative. Therefore, (A) is the only possible value in these five choices.

15. A – a *metamaterial* is a material composed of naturally existing materials that are arranged in a way that interacts with waves in a way unlike most natural materials. Because the metamaterial uses naturally occurring materials as building blocks in its internal geometry, it is in effect "the material's material," much like how protons, neutrons, and electrons are the "the atom's atoms." This gives rise to the *meta* in its name. The complex internal structure of a metamaterial will bump waves this way and that, absorbing and attenuating and reflecting, making waves zig and zag, as if they're falling down a pachinko board or a quincunx. While the polka dots in the figure may not do justice for visualizing this complex interaction, a joint team of researchers from Duke and Aalto University in Finland actually assembled a real metamaterial for sound, and you can find a photograph of their creation at the press release at `https://www.eurekalert.org/pub_releases/2019-03/du-rs030719.php`. You can also find a much more detailed and exciting overview of advances in metamaterials at `https://stories.duke.edu/beyond-materials-from-invisibility-cloaks-to-satellite-communications`.

Because this problem asks us to find the *best* example, we must consider why the special material doesn't match the characteristics of the incorrect choices. Smart materials are those that can respond to their environment. While it's possible for a material to be both a metamaterial and a smart material, never does this question explore the defining properties of smart materials, so we can eliminate (B). Aerogels are a type of solid gel that are extremely strong yet feel as light as air; they let light pass through, but not heat. The properties of aerogels are more relevant to light than sound, so (C) is not correct. In choice (D), birefringence would mean that the special material refracts most sinusoidal waves in more than one direction, or in other words, a single wave could have multiple indices of refraction in a birefringent material. Birefringence occurs because a medium is uneven, and behaves *anisotropically* (differently in different directions). Nothing here indicates the special material acts differently in different directions, nor does the sound get refracted in multiple directions. (The figure does depict two distinct paths, but the description states that the dashed line represents a typical medium, not the special material.) As a result, (D) is wrong. (E) is incorrect because condensed matter refers to all matter that is condensed, which typically means liquids and solids, but not gases. However, the special material isn't a good prototypical example of condensed matter, so it doesn't answer the question very well of which choice is best illustrated by the special material.

16. E – because there is air inside the space station, the inside string transfers some of its energy to air particles. As a result, the string inside the space station will come to a rest before the string outside the space station.

Wait, but if sound can't propagate in a vacuum, then why is (B) wrong? In fact, waves are already on a medium: the string. Because the waves are on the string, the vacuum of space does not prevent the waves from propagating on the string. Sure, nobody will be able to hear anything outside the space station—but the string keeps on wriggling just the same.

17. E – in a dispersive medium, the speed of a sinusoidal wave depends on its wavelength. If a wave packet is composed of sinusoidal waves of multiple wavelengths, then the packet will spread out since some waves in the group will speed ahead, while others will fall behind. As a result, statement I is possible because observer B may encounter the packet when it's more spread out, and so it takes a longer time for the entire group of waves to pass observer B. Statement II is always false because waves cannot have zero wavelength. Statement III could be true because two observer moving at different speeds could pass the wave packets at different rates.

18. A – Fourier's Theorem states that for any function continuous within an interval can be written as the sum of sine waves, where the sine waves have frequencies $f, 2f, 3f, 4f, \ldots, nf, \ldots$ for n up to infinity, and where each sine wave can have any amplitude. (The theorem also applies to some, but not all, non-continuous functions.) Therefore, the machine can create the wave depicted, even though the left side and right side are different as in (B), the positive and negative parts aren't rotationally symmetric as in (C), or a pointy cusp occurs as in (D). Because the wave has a finite amplitude in the graph, (E) is wrong as well.

19. D – if the previous question established that any continuous function can be written as the superposition of harmonic sine waves, then how come in this question, the answer is no? In fact, this wave can certainly exist—the critical piece of information, however, is that the wave cannot exist *within a pipe*. The world outside the pipe is at ambient pressure, by definition of ambient pressure, or the pressure of the surroundings. The two ends of the pipe ($x = 0$ and $x = L$) are both inside and outside the pipe, and because they are outside the pipe, they must be at ambient pressure by definition, so they must have zero pressure difference from ambient pressure. However, in the graph, Δp is nonzero at $x = 0$, so this pipe cannot exist. (B) is wrong for the reasons explained in the previous problem. (C) is incorrect because the graph is only a representation of the standing sound wave, but it isn't the wave itself. The graph looks like a transverse wave simply because in science, we like to graph things on perpendicular axes, but the graph doesn't imply the wave is actually transverse. (E) is irrelevant; just because the curve looks like a damping curve, the similarities are merely superficial. Damping occurs when a vibrating object returns closer and closer to its initial position over time; the graph in the problem plots pressure variation with position, and does not show time at all.

20. See **Appendix B** for the full solution of this problem.

21. See **Appendix C** for the full solution of this problem.

22. C – the piccolo is a type of transverse flute with extraordinarily high pitch range, and it is used in orchestral music. All transverse flutes consist of open pipes, so the machine can operate on a piccolo without a problem. Clarinets and bassoons are instruments with a reed attached to the end of a pipe. Because the reed closes off the end of the pipe, clarinets and bassoons consist of closed pipes, so (A) and (D) are incorrect. The ocarina is neither a closed pipe nor an open pipe, but is instead an irregularly shaped object, so (B) is incorrect.

23. A – the trombone is the only instrument with a slide, which allows the trombonist to continuously adjust the pitch. For the other brass instruments (trumpet, tuba, euphonium, and French horn), it is very difficult for players to intentionally play microtonal notes because the pitches are determined by valves which cannot be modified while playing.

24. A – the term "brass instrument" is a rather colloquial term, but it is generally agreed that the saxophone is not a brass instrument, because it produces sound from a reed, much like a woodwind.

25. C – sound localization is the ability of an organism to locate the origin of a sound. In echolocation, bats and whales create high-frequency sounds, which reflect from surrounding objects. Based on the reflection, the animal can determine the position of any objects, so this fits the description of sound localization. Some humans have been able to learn echolocation abilities as well, although they are much less refined compared to the abilities of bats.

26. A – Music boxes are instruments that play a sequence of notes when a listener rotates a handle. The rotation of the handle pulls a strip of paper or metal into the music box, although we only considered paper-reading music boxes in this problem. The music box consists of several tiny bars, and normally, these bars cannot be struck by pins on the opposite side of the music box because the piece of paper is in the way. However, when a hole appears, there is no paper to stop the pin and bar from striking each other. Each bar is tuned to a different note, so the location of each hole determines which bar gets struck by a pin. In this way, (A) is the only choice that accurately summarizes this process.

You didn't have to have a detailed knowledge about music boxes to be able to answer the question, as (A) is the only reasonable answer. If the rate at which the handle is turned determines the note, as in (B), a listener who turns slightly slower or slightly faster is prone to making notes flat or sharp. Humans aren't very good at doing things at a precise rate in the absence of a metronome or other time-tracking device, so playing so precisely seems like it would take a lot of skill and training to do—which doesn't seem to fit with the common (but accurate) notion that music boxes are instruments anybody can play. In (C), if all notes are playing by default, then the only way to include a rest in the music is to cut a hole in every location, which seems highly inefficient. In most compositions, only a few notes are playing in any instant, and this number is dwarfed in comparison to the dozens of notes that aren't playing at that moment in time. (D) is the least plausible out of all these choices, so it can be quickly eliminated by Occam's razor, which is a heuristic (mental shortcut) that goes along the lines of "simpler ideas are more likely to be true." Why is this choice unlikely? For starters, at least eight different types of hole punchers would be needed to create the eight notes of a major scale in this scenario. Next, because an octagon (8-gon), nonagon (9-gon), and decagon (10-gon) are all approximately circular, the music box would be filled with mistakes as a 10-gon could easily be mistaken for a 9-gon, or vice versa, thereby triggering the wrong note.

27. A – the clarinet is the only single-reed aerophone in this set of choices. (B), (C), and (D) all use double reeds, and (E) does not use any reeds.

28. D – an electric guitar, like an ordinary acoustic guitar, consists of six strings that are held under tension from end to end. So, an electric guitar doesn't need electricity to play—a pluck of the strings is all to set the guitar into vibration. An electric guitar is special, however, in that its body hosts a small device known as a pickup, which can send waves to an amplifier and a loudspeaker when plugged in. When plugged in, both the vibration of the string and the vibration of the loudspeaker account for the sound produced.

29. A – a pentatonic scale uses only five pitch classes. This eliminates (D) and (E). (C) isn't written in any apparent order—it goes up by a whole step, down by two whole steps, up by eleven half steps, then down by three whole steps—which makes it an unlikely choice. (The notes are wrong too.) Of the remaining two choices, only (A) has the correct notes. (B) is a pretty convincing choice, so don't feel bad for missing it. If you've ever played on only the black keys of a piano, you've played a pentatonic scale before. The pentatonic major scale is heard by playing ascending black keys starting with F♯. If you moved each black key down by 6 half steps (i.e. transposed to C), then the notes of the pentatonic major scale would be C, D, E, G, and A.

30. D – a half note is equal in duration to two quarter notes. From the problem, there are 120 quarter notes per minute, or two quarter notes per second, or one half note per second.

31. E – castanets, or clackers, are little instruments consisting of two plate-shaped objects that are hinged together. When shaken, the plates clap like hands, producing a clicking sound. This matches the definition of an idiophone, as it uses neither strings, membranes, an air column, nor electricity to produce sound.

32. C – the clef shown is known as the tenor clef, where the fourth bar line from the bottom (aka the second bar line from the top) is middle C. So, the note shown is a bar line below middle C, so it must be an A_3.

33. A

34. D

35. E

36. D – the correct sequence of notes is C – D – E – F – C – D – G – A – B – C. The first four measures are played normally up to *D.C. al coda*, which signals that the player should go to "da capo" (the beginning), and play to the "To coda" part. Therefore, the first six notes played are C, D, E, F, C, and D. Then, the player skips to the coda mark, which begins on the fifth measure. The player should play the next notes as G, A, B, and C. This gives (D) as the correct answer.

37. B – while the crazy overload of notes and accidentals on the staff may seem messy at first, we notice regular patterns exist within the notes. We can uncover the time signature through either the first or second measure. If we try the second measure, which is the easier of the two, we see that each sixteenth rest is followed immediately by a sixteenth note. Each pair of sixteenth rests and sixteenth notes (let's just call them "pairs") has a collective duration equal to an eight note. If this test hadn't been conducted virtually, you could have used your pen or pencil to circle each pair. You would have circled four such pairs, and the only notes left uncircled in the measure would be the eighth notes—four of them, to be exact. An eighth note and a pair have the same duration, so four of these durations plus another four is exactly one whole note, so we see the time signature must be $\begin{smallmatrix}4\\4\end{smallmatrix}$ time.

Alternatively, from the first measure, we find there are dotted dotted quarter notes. A dot to the right of a note means that you increase its duration by 50%, and a second dot to the right means that you increase its duration by another 25% of the *original* note, leading to a total increase in length of 75%. So each double dotted quarter note is 1.75 times the duration of a quarter note, which is, uh... aww rats, how many sixteenth notes does that equal? Fortunately, there are exactly two double dotted quarter notes, so together they must have a duration of 3.5 quarter notes. There are also exactly two sixteenth notes, and they must have a duration of one eighth note, or 0.5 quarter notes. So the total duration of the measure is 4 quarter notes, so only $\begin{smallmatrix}4\\4\end{smallmatrix}$ is a valid time signature.

38. A – the wedge-shaped mark on the final note is called an accent mark, or from time to time, a *marcato*. (Not everybody agrees that the accent mark should be called a *marcato*, however.) The accent mark means that a note should be played louder. The original piece didn't contain an accent mark at that position, but we added one here for the purpose of this problem. (C) can also be easily eliminated, as it is essentially a meaningless choice: on a piano, the pitch is fixed by the strings inside the piano, and there is no way to change the precision of each note without opening the piano and physically messing around with the string for that note.

39. A – as explained in the description, computers only store data in terms of discrete values. This is because the computer represents everything as a number (for most computers, a binary number), and each number represents one tick mark. As computer would need infinite memory to store arbitrarily large or arbitrarily many numbers, this is why they need to sample only some of the points on a wave. The rate at which the sound data is collected is called the *sampling rate*, so it's what determines how many samples are taken per unit of time. Therefore, (A) is the correct answer.

40. C – the vertical axis corresponds to the number of fixed values that the amplitude is allowed to have. This is known as the *bit depth*.

Short answer

41. We have it that $v = \lambda f = (0.0883\text{ m})(654\text{ Hz}) = 57.7\text{ m/s}$.

42. The observer hears tones of 342.5 Hz and 347.5 Hz. With correct significant figures, these are 343 Hz and 348 Hz. Students got credit even if they didn't use correct significant figures for this problem. When two tones f_1 and f_2 are very close (usually within 20 Hz of each other), they produce a single pitch that seems to switch between loud and soft. These alternations between loud and soft are called *beats*. When the individual tones have equal amplitudes, the resultant tone has frequency of $(f_1 + f_2)/2$, and the beats have a frequency of $|f_1 - f_2|$.

43. Intensity at a distance of r is given by $I = \dfrac{P}{4\pi r^2}$, where P is the power output of the source. Therefore,

$$\begin{aligned}
\frac{I_{3.00\text{ m}}}{I_{8.00\text{ m}}} &= \frac{1/(3.00\text{ m})^2}{1/(8.00\text{ m})^2} \\
I_{3.00\text{ m}} &= \left(7.125 \times 10^{-5}\text{ W/m}^2\right)\frac{(8.00\text{ m})^2}{(3.00\text{ m})^2} \\
&= 5.07 \times 10^{-4}\text{ W/m}^2
\end{aligned}$$

44. The fundamental frequency f of a cone of length L is $f = \dfrac{v}{2L} = \dfrac{343\text{ m/s}}{2(0.35\text{ m})} = 490\text{ Hz}$.

45. For an ideal gas at temperature $\mathcal{T}$ with heat capacity ratio γ and molar mass m, the speed of sound inside the gas is $v = \sqrt{\dfrac{\gamma R \mathcal{T}}{m}}$, where R is the ideal gas constant. Therefore, v is proportional to $\sqrt{\mathcal{T}}$. We can now find that

$$\begin{aligned}
\frac{v_{\text{final}}}{v_{\text{initial}}} &= \frac{\sqrt{\mathcal{T}_{\text{final}}}}{\sqrt{\mathcal{T}_{\text{initial}}}} \\
\frac{4.00 \times 10^3\text{ m/s}}{2.37 \times 10^3\text{ m/s}} &= \frac{\sqrt{\mathcal{T}_{\text{final}}}}{\sqrt{1.82 \times 10^2\text{ K}}} \\
\mathcal{T}_{\text{final}} &= 518\text{ K}.
\end{aligned}$$

It should be noted that sound speeds upwards of 2000 m/s are extremely unrealistic for ideal gases, and is more on par with the wave speed in very stiff solids. I honestly can't remember for the life of me why I didn't choose more realistic numbers.

46. By definition, there are 100 cents in a half step, and a half step has a frequency ratio of $2^{1/12}$. Because the frequency ratio of a quarter tone is the square root of the frequency ratio of a half step, a quarter tone is 50 cents. So, there are $\dfrac{50}{1.954} = 25.59$ schismas in a quarter tone.

47. The lowest string on a cello is C_2, so this is the lowest note on a cello.

48. The microphone receives 10^{10} times more power. (Students who only wrote "10^{10}" without further context still received full credit.) This is because a *decibel* (dB) is a unit such that a difference of 10 decibels corresponds to a ratio of 10 in another unit. The "SIL" refers to "sound intensity level," so if two sounds that differ by 10 dB SIL, one of the sounds must have 10 times more intensity than the other. In this problem the sounds are 100 dB apart, so their intensities must have a ratio of $\underbrace{10 \times 10 \times \cdots \times 10}_{\text{10 written 10 times}} = 10^{10}$.

49. Sonar stands for *sound navigation and ranging*. The "and" is not necessary for full credit.

50. The pitch is closest to G_5. Both the pitch class (letter) and note number must be present for full credit. This is because the string is split into three equal parts, which implies the third harmonic must have been excited. The third harmonic has a frequency three times the fundamental, and while its frequency isn't exactly equal to any note in equal temperament, it is exceedingly close to an octave and a half step, or G_5.

51. The frequency is greater.

52. The leading tone is 11 semitones higher.

53. The correct order is *lento*, *andante*, *andantino*, *allegro*, *presto*, *prestissimo*. We'd also like to express appreciation to the teams who pointed out that *andante* is slower than *adantino* in current usage, but that a now-obsolete historical usage held *adante* to be the faster tempo marker.

54. From left to right, the notes are *re mi fa sol mi do re*. One point is awarded for having at least four notes correct in the correct order. A second point is awarded for all seven notes correct. If seven notes are already present, there is no penalty for repeating the final *re* as the eight note in the sequence, even though the tie means that there are only seven notes.

55. From left to right, these chords are F♯ minor, $\mathrm{D}^{\mathrm{add2}}$, E major, and C♯ major.

One point is awarded for at least one correct chord for part (a). A second point is awarded for at least two correct chords for part (a). A third point is awarded for all four correct chords for part (a). We counted a chord as correct as long as we could tell what a team meant; for example, "f♯" is an acceptable substitute for "F♯ minor," and "C♯ over G♯" (i.e. in its second inversion) is an acceptable substitute for "C♯ major."

56. The key signature needs three sharps because it is in F♯ minor, forming a i-VI-VII-V progression.

57. For a *la*-based minor: *do la do mi la mi la do fa re mi fa*
For a *do*-based minor: *me do me sol do sol do me la fa sol le*

One point is awarded for at least four notes correct. A second point is awarded for at least eight notes correct. A third and final point is awarded for all notes correct.

58. Normal points can be awarded for a discussion of any of these reasons:

- The amount of energy released by sound is so small that it may be impossible to detected.
- If either of the balls contain hollow cavities, part of the released sound may go inside the ball, and Lyle won't be able to hear it.
- Teams cannot earn points for saying only that the collision is inelastic. Because the presence of sound requires inelastic collisions, a statement like that doesn't add anything to their response. However, teams *can* earn points for explaining that not all energy is converted into sound. For example, some energy is converted into heat or into deformation of each ball.

- Any other valid scientific reason, including practical reasons why the procedure is infeasible.

Tiebreaker points should be awarded as needed. No rubric is provided for tiebreaker points so that any necessary criteria for breaking ties may be considered. Typically, responses that demonstrate greater depth were awarded with more tiebreaker points.

59. First, we determine how many half steps above A_4 this note is. Because one half step corresponds to a ratio of frequencies of $\sqrt[12]{2}$, we find the number of half steps above A_4 is

$$\begin{aligned}\log_{\sqrt[12]{2}}\left(\frac{2020^{2021}}{440}\right) &= \frac{\ln\left(\frac{2020^{2021}}{440}\right)}{\ln\sqrt[12]{2}} \\ &= \frac{\ln\left(2020^{2021}\right)-\ln 440}{\ln\sqrt[12]{2}} \\ &= \frac{2021\ln 2020-\ln 440}{\ln\sqrt[12]{2}} \\ &= 266184.97 \text{ half steps.}\end{aligned}$$

There are 12 half steps in an octave. First, let's ignore the fractional part of 0.97, and focus on 266184. We find that $\frac{266184}{12} = 22182$ octaves, so the note is at least 22182 octaves above A_4. We bring back in the fractional part, so a note of frequency 2020^{2021} Hz is 22182 octaves and 0.97 half steps above A_4. 0.97 is almost equal to one, so the nearest note must be one half step above A, or A♯ or B♭. We see that this note must be $A\sharp_{22186}$ or $B\flat_{22186}$.

We awarded one normal point for a correct pitch class, and another point for a correct note number. Both of these points may be earned independently. Tiebreaker points should be awarded as needed. No rubric is provided for tiebreaker points so that any necessary criteria for breaking ties may be considered.

60. One normal point may be earned by any of these:

- The quantum particle in a box model is directly analogous to sound waves in a pipe closed at both ends. Particles in a box can only occur at a number of discrete energy levels, much like standing waves in a pipe only occurring at certain harmonics. Similarly, atomic orbitals are directly analogous to Chladni plates, and Schrödinger derived his equations for one-electron atomic orbitals by using the same math as that used for Chladni plates, and changing a few things to make it work for quantum mechanics. (McBride 2008, *OYS*)

- In 1878, physicist Lord Rayleigh published a theorem for the emission of sound from a small absorptive 3D room. Twenty-seven years later, in 1905, Rayleigh made a fateful decision to re-prove and republish the theorem along with fellow physicist James Jeans, with one crucial exception: they would examine light instead of sound. Called the Rayleigh–Jeans law, both physicists realized their derivation was lacking—in fact, it was quite terrible—at describing light emitted by an absorptive room. Funny enough, Max Planck had derived the correct law five years earlier, so at first glance, the Rayleigh–Jeans derivation was needless. However, what the law did do was to lead many prominent physicists to abandon classical physics in favor of quantum, and it traces its roots back to acoustics. (Beyer 1999, *Sounds of our Times*; Kox 2013, *Archive for History of Exact Sciences*, 67)

- Julian Schwinger, a physicist with many contributions to quantum electrodynamics, drew on his knowledge of sound in his research on radio waveguides. While radio waveguides are generally considered to be a non-quantum area of study, Schwinger drew on a mathematical technique known as the *variational technique* first developed for use in the acoustics of vibrating plates. A few years later, the same mathematical technique would guide his research on quantum physics of atomic nuclei. (Beyer 1999; Mehra, Kimball, & Rembiesa 1999, *Foundations of Physics*, 29(6), DOI: 10.1023/A:1018825429390)

  Schwinger also used the variational technique to find an exact solution to end correction in pipes with infinitely thin walls (or "unflanged" pipes, as he called them). In this way, the variational technique not only originated from acoustics, but acoustics also further sharpened the technique thanks to Schwinger's forays into end corrections, before it was ultimately used in quantum theory. It should be clear, though, that the influence of end corrections was small, as many other areas of research also influenced the development of the variational technique. (Mehra, Kimball, & Rembiesa 1999)

- In addition to the examples above where acoustics shaped the foundations of quantum mechanics, any legitimate modern influence of acoustics on quantum theory may earn points.

Tiebreaker points should be awarded as needed. No rubric is provided for tiebreaker points so that any necessary criteria for breaking ties may be considered, including how funny the response was.

61. Tiebreaker points should be awarded as needed. No rubric is provided for tiebreaker points so that any necessary criteria for breaking ties may be considered. Because this is a silly tiebreaker, we used it only as a tiebreaker of last resort, and we awarded tiebreaker points with much greater hesitation than we did for the other three tiebreakers.

Free response

The weight of each point on the free-response question was multiplied by two when calculating test score.

62. **10 points**

(a) For identifying the temperature difference, density, or both as the reason for reflection — 1 point

For stating that a change in impedance occurs — 1 point

The *impedance* of a medium depends on its density and the speed of waves inside the medium. The intuition for impedance is a bit difficult, but you can think of it as a number that tracks the "differentness" between media. Two media with very different impedances are very "different," while two media with very close impedances are more similar. Impedance matters here because waves reflect wherever impedance changes, and the more different the impedances, the more reflection. For example, when you yell at a wall, your yell reflects back at you, because the impedance of the wall is significantly higher than the impedance of air. In contrast, when you yell at air, air has the same impedance as air, so none of the yell reflects back (unless, of course, it hits something, like a wall, or a different patch of air with a different impedance).

Using the information provided, because temperature rises extremely rapidly in the stratosphere (much more steeply than the troposphere), the impedance difference causes reflection there.

(b) i. For a correct calculation — 1 point

Distance traveled $= 2d = (343 \text{ m/s})(6.80 \times 10^2 \text{ s}) = 2.33 \times 10^5$ m or 2.33×10^2 km

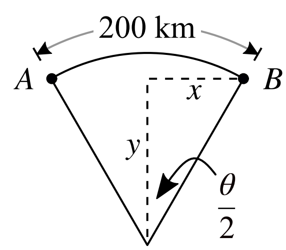
ii. For a calculation consistent with part (b)(i) — 1 point

Method 1 (preferred method): because the distance between the cities (200 km) is much smaller than the radius of the Earth, the Earth is very nearly flat at the scale between both cities. Now, we can form a right triangle, with the path of length d as hypotenuse, and one leg will have a length equal to half the distance from city A to B. Applying the Pythagorean theorem yields

$$\sqrt{\left(\frac{2.33 \times 10^2 \text{ km}}{2}\right)^2 - \left(\frac{2.0 \times 10^2 \text{ km}}{2}\right)^2} = 6.0 \times 10^1 \text{ km or } 6.0 \times 10^4 \text{ m}$$

which is correct to two significant figures.

Method 2: unlike Method 1, if we know the actual radius R of the Earth in meters, we can determine the angle θ subtended between cities A and B on Earth's surface, as well as the true shortest distance (the chord distance) between A and B; we'll let x be half this distance, as illustrated below.



By trigonometry, $x = \cos(\theta/2)$ and $y = \sin(\theta/2)$. The distance from the right angle vertex of the triangle to the reflection point is $\sqrt{d^2 - x^2}$ by the Pythagorean Theorem; the distance from the vertex of the triangle to the surface of the Earth is $R - y$, so the distance from the surface of the Earth to the reflection point is the difference between these, or $\sqrt{d^2 - x^2} - (R - y)$. Taking $R = 6.4 \times 10^3$ km and $\theta = \frac{200 \text{ km}}{R} = 0.031$ radians, the final answer is still 60. km to two significant figures. This is also why Method 2, despite producing the only truly accurate distance, isn't all that great. If we didn't round to two significant figures, the value for Method 2 would be only 6 m bigger. When we're estimating something as large as the stratosphere, if we're off by six meters (about 0.0001% error), the approximation turn out to be no big deal.

(c) For explaining that the speed of sound decreases with altitude in the troposphere — 1 point

Example: "sound is slower in cold air, so as the sound travels higher, it takes longer to travel any distance, which means the calculation is an overestimate."

(d) i. For explaining how refraction causes sound to propagate from city B to city A through the stratosphere — 2 points

Example: "because the stratosphere warms up with altitude and because sound moves faster in warmer air, the sound bends downwards in the stratosphere."

As you've probably guessed from the rest of this solution, the hypothesis from the early 20th century is punctured with weaknesses. Reflection exists, but it's not enough to account for the entire effect where A hears sound from B, but nobody else located between them can do the same. So how does refraction provide a better explanation?

We met our friend Snell's Law in problem 14, and in the context of the atmosphere, Snell's Law implies that waves refract downward in direction when the faster wave speed medium is above the slower wave speed medium. This situation matches the stratosphere well, because the temperature keeps rising and rising with height, and therefore, the speed of sound rises as well. (Despite popular belief, the density of the stratosphere negligibly impacts speed of sound, because it's canceled out by the change in pressure with altitude. Now on the other hand, the molar mass of the gas does matter, but the types of molecules don't change—mostly just temperature.) So, sound waves in the stratosphere will keep refracting downwards and downwards, until they're headed back towards Earth's surface, and towards city A. If you previously competed in Optics, the effect here occurs on the same principle as a desert mirage, although atmospheric refraction has several other factors that make it slightly more complicated (such as the temperature profile of the stratosphere almost mirroring that of the troposphere, which means the paths of sound rays is slightly different from that of light).

ii. For any scientifically reasonable explanation for a possible error in the calculations in part (b), other than their explanation in (c) — 1 point

Example: "because of refraction in part (d)(i), sound doesn't travel in a straight line, but the diagram assumes straight line travel."

Example: "wind or turbulence can change the effective propagation speed of a wave. Over large distances of hundreds of kilometers, it's hard to believe that wind didn't have any non-negligible effects."

Example: "there is no guarantee that the path shown in the figure is the only path taken by sound. It could reflect at other points on the atmosphere, or it could bounce off the ground."

While the prompt in part (c) specifically asks teams to address the speed of sound, their answer to part (d)(ii) may be anything as long as it isn't identical in substance to their response to part (c). A team could address how the speed of sound affects the calculation in *one* way as part of (c), and then address how the speed of sound affects the calculation in a *different* way as part (d)(ii), and still receive full credit.

(e) For explaining that the microphones triangulate sound — 2 points

Example: "the sound reaches all three microphones at different times. This time lag can be used to estimate how far sound traveled to each microphone, and at what angle."

Intensity-based answers received only one point. Methods that measure intensity indeed can determine the direction of a sound. However, over long distances, and without using certain microphones (those with directional sensitivity), intensity-based methods are not ideal for determining the direction of a sound. Moreover, students must explain how intensity works, and can't just drop "volume of the sound" to get points.

Remarks:

- I fabricated the time data in part (b)(i), so a time of 6.80×10^2 seconds (about 10 minutes) may not be realistic. I had no idea what a realistic time value would have been, so I decided to make the answers realistic instead. So, the height of 60 km where the sound *refracts* (not reflects) in the stratosphere is reasonable, but not the assumption that it takes 10 minutes.

- Some data from real scientists: Yang (2016) gives that the point where sound reflects is typically at an altitude of 50 km, and states that 200 km is in the ballpark of the distance along Earth's surface where refracted sound becomes audible. Even so, these ballparks may not always be reasonable either, given that Earth's atmosphere differs from place to place, and from season to season. Moreover, atmospheric turbulence can waylay the effects of refraction (Gabrielson 2006).

- This question was hard. The part about acoustic impedance was especially difficult, and it's okay if you're learning it for the first time. You shouldn't feel bad if you found this problem difficult—I've been writing, rewriting, and reworking the question since July 2019, and I *still* found it difficult. I wrote so that every subpart of this question was answerable with only the background information plus your Sounds of Music knowledge. Sure, it was nowhere near easy, but it was possible nevertheless to get a perfect score with nothing but acoustics. And this is one of the most amazing things about science: you never know where your knowledge will take you. Knowing musical physics helps you to understand the atmosphere. Knowing how mirages of light happen helps to illuminate how mirages of sound happen. Every small morsel of scientific knowledge could be part of a long trail of bread crumbs leading you to something entirely different. Your adventure through science may cross borders over many fields and disciplines, including fields outside of science, but whatever you learn, however hard it is, exciting travels await you—it just might not be the journey you had planned.

- I'm not an expert in earth science, and neither is my co-writer. I've consulted the literature on earth science to deepen my understanding as much as I could, but it's still possible I made a mistake! For this question, I consulted the following sources:

Attenborough, Keith. 2014. Sound propagation in the atmosphere. In Rossing, Thomas (ed.), *Springer Handbook of Acoustics*. New York: Springer-Verlag. See section 4.8 for a discussion of atmospheric refraction.

Calvert, James B. 2008. `http://mysite.du.edu/~jcalvert/waves/barisal.htm`. This was the source that sparked my interest in writing this question.

Gabrielson, Thomas B. 2006. Refraction of sound in the atmosphere. *Acoustics Today*.

Kallistratova, Margarita A. 2002. Acoustic waves in the turbulent atmosphere: A review. *Journal of Atmospheric and Oceanic Technology*, 19(8), 1139-1150. It should be noted that the author uses the word "reflection" loosely, as a way to refer to the *refraction*-induced direction change.

Yang, Xunren. 2016. *Atmospheric Acoustics*. Walter de Gruyter. See chapter 3.3.3 for details on abnormal sound propagation.

Possible points on the test:

$$\begin{aligned}99 = &\text{ multiple choice (40)} + \text{short answer, including tiebreakers (27)}\\ &+ \text{three free points per tiebreaker (12)}\\ &+ \text{free response } (2\times 10)\end{aligned}$$

Ties were broken by the number of tiebreaker points earned by each team. When ties could not be broken by tiebreaker points, we broke ties by first question correct in reverse order starting on question 22.

Share your thoughts!

How was this test? Did anything confuse you? Did you uncover any ~~typoos~~ typos or errors? Stuck on a problem? If you'd like to discuss, we'd appreciate it if you could shoot us an email!

- `yl708@duke.edu` is the corresponding author for questions 30-38 and 50-57.
- `qedgary@ad.unc.edu` is the corresponding author for questions 1-29, 39-49, and 58-62, as well as for images, typos, and formatting.

(Please type these email addresses instead of copying-and-pasting; they've been scrambled to fight automated spammers.) You're also welcome to discuss questions with us even if you received this test through a test exchange, such as the Scioly.org test exchange.

You can also anonymously give your feedback at the link below.

`https://scioly.web.unc.edu/rate-my-tests-gz839918`

Errata

The original version of this test and key included several errors. In question 29, answer choices (D) and (E) were identical. In question 53, "time signatures" was incorrectly used instead of "tempo markings." In question 58, the meaning of "larger" was not clearly synonymous with "louder," which led to confusion for some teams. Some errors also occurred when transcribing questions to Scilympiad; for example, the question mark was missing in question 5, but not the original PDF version of this test. Part (a) of question 62 initially did not ask why it was "reasonable" to expect reflection; part (a) also initially said "reflects in," which confused some teams, who interpreted that as "reflects into." This version of the answer key has corrected these errors, as well as various minor grammatical errors.

Appendix A: Solution to question 1.

Note: we offer appendices to provide further details and insight, and to help to maximize appreciation for science. Most of the knowledge presented here isn't essential to Sounds of Music, so don't let the length of the appendices frighten you! Read on for the joy of learning—not for the sake of winning. Or, if you find it endlessly boring, treat it like a good bedtime story, and tell us how we can improve in the future.

The correct answer is choice (C).

This question presents five statements, and four of them are largely meaningless, dressed in the highbrow imprecise vocabulary of pseudoscience and quackery. We want to know, which one is most meaningful to a scientist?

- (A) isn't very meaningful, since the word "irrational" in equally tempered music has little scientific connection to the word "irrational" as a description of human thought. Also, this statement comes off as rather silly: the first word "irrational" is mathematical, but the second psychological. By analogy, humans have been working with imaginary numbers for centuries, yet no mathematician has poofed out of existence for working on imaginary numbers.
- (B) is worse than (A), because the hypothesis is so poorly defined that we have little way of testing it. What is bad energy? Why is it bad? Energy is a physical feature of the world, and being a quantity measured in joules, it has no ulterior motives, so it'd be very rare that scientists would speak of energy as "bad."
- (D) is vague as well. How exactly is the body its own musical instrument? Do these natural frequencies refer to singing, where the vibration is in the air of our lungs and vocal tract? Or do they refer to the vibrations of our body's organs (as opposed to the air trapped inside them)? And what kind of balance is being improved—physical balance, emotional balance, nutritional balance? Of course, we *could* interpret this hypothesis literally, so that "balance" means our vestibular sense (our body's ability to resist collapsing under gravity), but the vague nature-oriented wording of this question makes us wonder whether a literal interpretation is appropriate. Maybe things would be different a hundred years ago, but in modern usage, the word "balance" is so often applied by advertisers and influencers to praise every kind of health product, to the point where "balance" no longer has anything to do with "sense of balance." (For instance, a "balanced skin treatment, as seen on TV" is likely irrelevant to your sense of balance. Instead, the word "balance" is there to appeal to consumers' desire for what seems natural and holistic.)
- In choice (E), we don't know how old is old enough to be "ancient," and it's unclear what makes a melody "tried and true." This raises several questions on the true meaning of the hypothesis. If an ancient melody was lost but rediscovered by archaeologists, does that mean it's worse compared to a melody that actually survived through the ages without being lost? Since melodies get passed down over time, many modern pieces make use of ancient melodies, so isn't modern music also "tried and true"? The choice also makes the fallacy known as an *appeal to tradition* by asserting that older must be better. While a fallacy doesn't necessarily make a statement wrong, it does make the supposed "therapeutic benefits" highly dubious.

(C) is the best answer because it uses word with precise meanings, like "high-intensity," "infrasound," "correlated," and "anxiety," where there can be no confusion on what this sentence signifies. While a well-defined meaning doesn't inherently prove a hypothesis true, it does make it plausible—and it also happens to correctly answer the multiple-choice question.

Of these five choices, only one of them was meaningful. However, all science derives from evidence, and just because a statement is meaningful, reasonable, clear, and precise, there holds no guarantee it must be true. So which statements are true? Well, perhaps surprisingly, it turns out that *none of them* are true. Every hypothesis was false, even though only one of them is precise enough for a scientist to understand at a glance without additional information.

Choice (A): this claim is false by Loosen (1995), who conducted a single-blind experiment that asked pianists, violinists, and non-musicians to listen to scales with scales with rational and irrational ratios between note frequencies. Pianos are usually tuned to irrational ratios between frequencies, while violins are not. The pianists said they liked scales with irrational ratios better; the violinists said the opposite; and the non-musicians didn't sway one way or the other. This illustrates that preference for 12-TET versus Pythagorean tuning is acquired, and not intrinsic to the human brain.[Note 1]

Choice (B): Choice (B) is so hopelessly empty of substance that it's impossible to prove it either right or wrong. Therefore, we assume it is false by default, much like we'd automatically assume the statement "a teapot is floating somewhere in outer space between the orbit of Earth and Mars" as false—it's simply not testable.

Choice (C): Choice (C) is false despite being the right answer choice: infrasound from wind turbines has been linked to increased anxiety, although one double-blind study suggests that this may be the result of a perceptual set, where if you think you'll feel anxious, you'll become anxious around wind turbines (Crichton et al. 2013). So, infrasound either raises anxiety, or it has no effect, but most likely, it doesn't help it.

Choice (D): Although it's hard to tell what this hypothesis even means, we can try to see whether the literal interpretation (where "balance" refers to the human sense of balance) stands strong against scientific scrutiny. The body's resonant frequencies, as well as the resonant frequencies of vital organs, are somewhere less than 20 Hz. The exact numbers have stirred up a substantial debate within the scientific literature, but few scientists give numbers greater than 20 Hz. For a discussion, see Oborne (1983); Brownjohn & Zheng (2001); and the widely accepted international standard for human body resonances, ISO 2631.

You may be wondering, who cares? Why would scientists want to argue over the exact normal mode frequencies of the average human? Well, compared to other frequencies, resonant frequencies force the body to vibrate at much, much greater amplitudes—so much, in fact, that it could lead to injury or death. Yikes. (That's why scientists care so much about the exact numbers.) Needless to say, getting injured probably doesn't improve balance in any sense of the term.

Choice (E): For choice (E), I have searched the scientific literature, and I've found no known studies on a systematic correlation between the age of music and its therapeutic effects. The

[1] Another source that debunks this myth, while simultaneously casting doubt on a few facts that turn out to be false, is Parncutt & Hair (2018). These researchers summarize inadequacies of a key assumption: humans perceive pitch based on ratios alone. It's an assumption with a certain idealized elegance, where the immense sophistication of human perception is distilled into manageable mathematical measures. This assumption pervades music theory—it's used on this Sounds of Music test too—but it merely approximates the complex reality of pitch perception. For instance, when left to themselves, people seem to prefer octaves *slightly* larger than a frequency ratio of 2-to-1, for reasons that are still not fully known to science. Getting spooky yet? One of the leading hypotheses is that the preference is acquired, and not innate. But whether that's right remains a mystery. Who knows what we'll find out next?

closest thing I got was one study in series, which reported Mozart's music was better at reducing epileptic seizures compared to "old time pop piano tunes" (Hughes et al. 1998). The study seems believable, as it was replicated successfully by other scientists (Grylls et al. 2018). However, two of the authors of the original study (Hughes & Fino 2000) specifically attributed these benefits to the properties of the waves in the music (instead of, you know, proclaiming that the musty age of the music miraculously made patients better, instead of the actual sound waves). And this is hardly evidence to support the idea that ancient music is better, since we can't call Mozart "ancient" (b. 1756) without saying the same of the first chemical refrigerator (invented 1755), the United States (1776), the Watt steam engine (1765), and the smallpox vaccine (1798). Therefore, for such a ridiculous claim made without evidence, it seems safe to conclude this hypothesis lacks merit as well.

By the way, while we're on the topic of Mozart, it's worth mentioning a related claim from pop culture, about intelligence rather than therapeutic benefits. These claims go that listening to Mozart makes people smarter. In reality, these claims are most likely false. Mozart's music fails to beat a placebo in making listeners perform better on intelligence tests (Pietschnig et al. 2010). So no, your choice of music doesn't affect intelligence, even though a small number of studies suggest Mozart may possibly lower the frequency of seizures. (Listening to Mozart is not a substitute for medical advice from a professional, though!)

Too long, didn't read? In summary, there are lots of claims that *A* causes *B*, and doing *C* will boost your health. Some claims may seem reasonable, but still be wrong. Fortunately, you can eliminate plenty of claims simply because they are too vague to be true.

Works consulted for this problem

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Hughes, John R., Yaman Daaboul, John J. Fino, & Gordon L. Shaw. 1998. The "Mozart effect" on epileptiform activity. *Clinical EEG and Neuroscience*, 29(3).

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Loosen, Franz. 1995. The effect of musical experience on the conception of accurate tuning. *Music Perception*, 12(3).

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Parncutt, Richard & Graham Hair. 2018. A psychocultural theory of musical interval: Bye bye Pythagoras. *Music Perception*, 35(4)

Pietschnig, Jakob, Martin Voracek, & Anton K. Formann. 2010. Mozart effect-Shmozart effect: A meta-analysis. *Intelligence*, 38(3)

Appendix B: Solution to question 20.

The correct answer is choice (A).

If a sound wave has intensity I and root-mean-square pressure amplitude p_{rms}, then the intensity is proportional to p_{rms}^2. In this problem, if the nth harmonic has root-mean-square pressure amplitude $p_n = p_1/n$, then the intensity of the nth harmonic is proportional to $\left(\frac{p_n}{n}\right)^2$. Because $I_1 \propto p_1^2$, the intensity of all the harmonics combined is the infinite series

$$I_{\text{total}} = \frac{I_1}{1^2} + \frac{I_1}{2^2} + \frac{I_1}{3^2} + \frac{I_1}{4^2} + \frac{I_1}{5^2} + \frac{I_1}{6^2} + \cdots .$$

The sum of the reciprocals of all perfect squares is a well-known problem in mathematics, and surprisingly, the value of the sum is actually $\pi^2/6$. This series can be evaluated using only algebra and geometry (as seen in the link), but a more straightforward proof is possible with Taylor series. No calculus is needed to use Taylor series, though a small amount of calculus is helpful to derive them, and if you want to explore how, the book *Calculus in a Week* offers an excellent overview.

To be honest, if it weren't for the fact that the fame of this infinite series has outpaced its proof, I would have deemed this problem far too difficult for inclusion on a Science Olympiad test. The value of this infinite series is akin to general knowledge, even though not quite common knowledge. Much like the blue sky or green plants, it's a fact that many scientists know, but explaining why blue light scatters or why plants are green is much harder. (Ironically, the question of why plants use green pigments remains an unsolved problem in science, while the exact value of the sum of reciprocals of all squares was first solved almost three centuries ago.)

Does the thought of memorizing famous infinite series leave you uninspired? We could, as an alternative, estimate the root-mean-square amplitude *with only a typical graphing calculator*. How? We define a function on our calculator $f(x)$ that equals the sum of the harmonics from one to n. (Let's use the first 50 harmonics for simplicity.) This might look something like

$$\texttt{f(x)=}\ \sum_{\texttt{n=1}}^{\texttt{50}} \texttt{(1/n)*sin(}2\pi\texttt{nx)}.$$

This function represents what the pressure is like on the x-axis inside the pipe. Now, because $I \propto p_{\text{rms}}^2$, we can sample points along our function f to calculate the root-mean-square of its y-values, which we'll call y_{rms}. In our calculator's built-in statistics package, we can automatically fill up a spreadsheet column with cells $f(0)$, $f(0.01)$, $f(0.02)$, $f(0.03)$, and so forth until we reach $f(1)$. (You don't have to use intervals of 0.01, but smaller is better.) Here, we can square all values, add them together, then divide by the total number of all values, and this will be y_{rms}^2, which follows from the definition of root-mean-square. Alternatively, we can select our column from $f(0)$ to $f(1)$, then run `one-variable stats` $\rightarrow$ `Population standard deviation`. On some calculators, it may be labeled with a sigma (σ) or as `StDev-p`. This will give you y_{rms}. No matter which method we used, our desired answer is $2y_{\text{rms}}^2 I_1$. (Why did a factor of two mysteriously drop in front of y_{rms}^2? I'm leaving this as a puzzle for the reader, although I'd be happy to share the answer by email if requested.) Using our example data from above, we find $2y_{\text{rms}}^2 = 1.62473273362 \approx \pi^2/6$. While a rough approximation, we can eliminate every choice except for (A) and (E) because all the other choices are smaller than our approximation, and you know that the true answer must be larger because we didn't sum up infinitely many harmonics, but only the first 50.

Appendix C: Solution to question 21.

The correct answer is choice (E).

In contrast to the previous problem, which we visited in **Appendix B**, we're already told the *intensity* of each harmonic. So, the total intensity in this case is the harmonic series

$$I_{\text{total}} = \frac{I_1}{1} + \frac{I_1}{2} + \frac{I_1}{3} + \frac{I_1}{4} + \frac{I_1}{5} + \frac{I_1}{6} + \frac{I_1}{7} + \frac{I_1}{8} + \cdots$$

which turns out to be an infinite sum. Because the universe doesn't have an infinite amount of energy, this scenario is impossible, so (E) is the correct answer.

It seems to defy our expectations that this series of harmonics would have infinite intensity. Why should this sum be infinite? The problem tells us that each harmonic has n times less intensity than the first, so eventually, for very large n, shouldn't we expect to be adding waves of basically, well, zero intensity? After all, if we were filling up a jug of water, and we poured in half a teaspoon of water, then one-third a teaspoon, then one-fourth, and never stopped, we sure wouldn't imagine we'd eventually need to drain every ocean, and still need more water to carry on! Hmmm… how should we proceed? To prove our setup is impossible, we define infinite series A and B where

$$\begin{array}{ccccccccccccccccccc}
A & = & \frac{1}{1} & + & \frac{1}{2} & + & \frac{1}{3} & + & \frac{1}{4} & + & \frac{1}{5} & + & \frac{1}{6} & + & \frac{1}{7} & + & \frac{1}{8} & + & \cdots \\
 & & \Big\downarrow{\scriptstyle\text{equals}} & & \Big\downarrow{\scriptstyle\text{equals}} & & \Big\downarrow{\scriptstyle\text{greater than}} & & \Big\downarrow{\scriptstyle\text{equals}} & & \Big\downarrow{\scriptstyle\text{greater than}} & & \Big\downarrow{\scriptstyle\text{greater than}} & & \Big\downarrow{\scriptstyle\text{greater than}} & & \Big\downarrow{\scriptstyle\text{equals}} & & \\
B & = & \frac{1}{1} & + & \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{8} & + & \frac{1}{8} & + & \frac{1}{8} & + & \cdots
\end{array}$$

We notice that the infinite series A must be greater than B, because the nth term of A is either greater than or equal to the nth term of B, but never is any term in A smaller than its corresponding term in B. However, this entangles any attempt to extract a finite value from A, because

$$\begin{aligned}
B &= \frac{1}{1} + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty
\end{aligned}$$

but if B is infinite, and A is greater than B, then surely A must add up to infinity as well![Note 2]

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[2] Although you don't need to know any of the following for Sounds of Music, reality doesn't quite match our mathematical deductions. For sufficiently high harmonics, the wavelength λ of the harmonic would be so small, that λ would almost equal the average length between each molecule of air, which scientists call the *mean free path* of air molecules. The mean free path of air is on the order of magnitude of 10^{-7} m, so a one-meter pipe wouldn't be able to play any harmonics higher than $n \sim 10^7$, which means $I_{\text{total}} \approx 16.7\, I_1$. (The tilde symbol or "$\sim$" refers to order of magnitude. So, "$X \sim Y$" means that "if I rounded X and Y to the nearest power of ten, then they would both round to the same power of ten.") This still means (E) is correct, as $I_{\text{total}} \approx 16.7\, I_1$ is far too big for any of the choices (A) through (D). If we wanted I_{total} to be 100 dB greater than I_1, we would need $n > 10^{4000000000}$, which is larger than the number of protons in the universe to the millionth power. I'm squeezing this information into a footnote just to be perfectly clear: you'll probably never need this paragraph ever again for Sounds of Music (not that you needed it in the first place to answer this question). But it does show that science is weird—and cool!