Science Olympiad
Astronomy C
BEARSO Invitational

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Section C Solutions

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Section C: Calculations

Points are shown for each question or sub-question, for a total of 84 points.

1. (27 points) For any body balanced between gravitational collapse and radiation pressure, there is an upper limit to its luminosity called the **Eddington luminosity** L_{Edd}. The Eddington luminosity of a star of mass M and opacity κ is given by

$$L_{Edd} = \frac{4\pi GMc}{\kappa}$$

(a) (4 points) This equation can be simplified into the form

$$L_{Edd} = k\left(\frac{M}{M_\odot}\right)\mathrm{L}_\odot$$

where $\frac{M}{M_\odot}$ is the star's mass in solar masses and k is a constant. Find k. Assume the star is composed of ionized hydrogen, whose opacity is given by $\kappa = \frac{\sigma_T}{m_p}$, where $\sigma_T = 6.652 \times 10^{-29}\,\mathrm{m}^2$ is the Thomson scattering cross-section for the electron, and $m_p = 1.673 \times 10^{-27}\,\mathrm{kg}$ is the mass of the proton.

(b) (2 points) Calculate the Eddington luminosity of the sun, in solar luminosities. Is this smaller or larger than the Sun's actual luminosity?

(c) (9 points) Luminous Blue Variables are stars that sometimes exceed their Eddington limit during their outburst states. η Carinae is a star famous for its "Great Eruption" in the 19th century, where it brightened up to magnitude -0.8, becoming the second brightest star in the night sky at the time. It has since dimmed back to an apparent magnitude of 4.3.

i. Assuming that η Carinae's current luminosity is $5.0 \times 10^6\,\mathrm{L}_\odot$, calculate its luminosity during the outburst state, in solar luminosities.

ii. Assuming that η Carinae had a mass of $150\,\mathrm{M}_\odot$ at the time of the outburst, what was its Eddington luminosity, in solar luminosities?

iii. There is a cloud of gas surrounding η Carinae that is so prominent we have named it the Homunculus Nebula. Connect this with the star's Great Eruption and the Eddington luminosity.

(d) (12 points) The following data is given for the quasar 152156.48+520238.5.

Object name (J2000)	$\log(L_{Bol})$ erg s^{-1}	$\log(M_{BH})$ $\mathrm{M}_\odot$	L_{Bol}/L_{Edd}
152156.48+520238.5	48.2	9.8	2.09

i. Calculate the quasar's bolometric luminosity, in solar luminosities.

ii. Calculate the quasar's Eddington luminosity, in solar luminosities.

iii. What is the calculated Eddington ratio (L_{Bol}/L_{Edd})? Is this consistent with the one given in the table?

iv. Not many objects can sustain a super-Eddington luminosity. Explain why, and why an active galactic nucleus might be able to. *Hint: what are AGNs powered by?*

Solution:

(a) By plugging in all the constants, we can find

$$L_{Edd} = \frac{4\pi \times 6.67 \times 10^{-11} \times 3 \times 10^{8}}{(6.652 \times 10^{-29})/(1.673 \times 10^{-27})} \times \left(\frac{M}{\text{kg}}\right) \text{W} = 6.32 \left(\frac{M}{\text{kg}}\right) \text{W}$$

While we did not write out the full dimensional analysis of these constants, we have used SI units for all of our constants, and thus we expect L_{Edd} to be in Watts if we use a M in kilograms. Next, we can use some dimensional analysis to convert kilograms to solar masses and Watts to solar luminosities:

$$\begin{aligned} L_{Edd} &= 6.32M \times \frac{\text{W}}{\text{kg}} \times \frac{1.989 \times 10^{30}\,\text{kg}}{1\,\text{M}_\odot} \times \frac{1\,\text{L}_\odot}{3.828 \times 10^{26}\,\text{W}} \\ &= 3.28 \times 10^{4} \left(\frac{M}{\text{M}_\odot}\right) \text{L}_\odot \end{aligned}$$

Thus, $k = 3.28 \times 10^4$.

Note that k is also just the Eddington luminosity of the sun in solar luminosities, since the sun has $M = 1\,\text{M}_\odot$, so it is also possible to find k by simply plugging in $M = 1\,\text{M}_\odot$ and converting L_{Edd} to solar luminosities. (This is probably a more intuitive way to solve this problem.)

(b) Plugging in $\frac{M}{\text{M}_\odot} = 1$ to our previous equation, we get $L_{Edd,\,\odot} = 3.28 \times 10^4\,\text{L}_\odot$. This is significantly greater than $1\,\text{L}_\odot$ as expected, since the Eddington luminosity is an upper limit on the luminosity of stars.

(c) i. Because the distance to η Carinae should still be the same as the 19th century, the distance modulus remains unchanged as well: $m - M = m_0 - M_0$, where m_0 and M_0 are the apparent and absolute magnitudes of η Carinae today, while m and M correspond to its state during the Great Eruption.

We can use this in the magnitude equation to find its luminosity L that corresponds to the absolute magnitude M:

$$\frac{L}{L_0} = 10^{-0.4(M-M_0)} = 10^{-0.4(m-m_0)}$$

Plugging in $L_0 = 5.0 \times 10^6\,\text{L}_\odot$, $m_0 = 4.3$, and $m = -0.8$, we get

$$L_0 = 5.48 \times 10^{8}\,\text{L}_\odot$$

ii. We again use the equation derived in part (a).

$$\begin{aligned} L_{Edd} &= 3.28 \times 10^{4} \times 150\,\text{L}_\odot \\ &= 4.92 \times 10^{6}\,\text{L}_\odot \end{aligned}$$

η Carinae's luminosity during the Great Eruption was greater than its Eddington luminosity.

iii. During the Great Eruption, η Carinae greatly exceeded its Eddington luminosity. Outward radiation pressure was stronger than its own gravitational pull, and much of its outer layers were ejected. We can still see its ejected gas today surrounding η Carinae as the Homonculus Nebula.

(d) i. The table gives us $\log L_{Bol} = 48.2$, where L_{Bol} is in ergs per second. ($1\,\mathrm{J} = 10^7$ erg.) Then,

$$L_{Bol} = 10^{48.2}\ \mathrm{erg\,s^{-1}} = 1.585 \times 10^{48}\,\mathrm{erg\,s^{-1}} = 1.585 \times 10^{41}\,\mathrm{W}$$

Converting to solar luminosities, $L_{Bol} = 4.14 \times 10^{14}\,\mathrm{L_\odot}$.

ii. Again, we use our equation from part (a). The mass of this quasar is $10^{9.8}\ \mathrm{M_\odot} = 6.31 \times 10^9\,\mathrm{M_\odot}$. Then, $L_{Edd} = 3.28 \times 10^4 \times 6.31 \times 10^9\ \mathrm{L_\odot} = 2.07 \times 10^{14}\,\mathrm{L_\odot}$.

iii. $\frac{L_{Bol}}{L_{Edd}} = \frac{4.14 \times 10^{14}\,\mathrm{L_\odot}}{2.07 \times 10^{14}\,\mathrm{L_\odot}} = 2.00$.
This is reasonably consistent with the one given in the table. Our Eddington luminosity equation is an approximation, since it corresponds to a star of pure ionized hydrogen.

iv. Most objects with a super-Eddington luminosity would usually quickly lose mass until they return to a stable state. However, since quasars are powered by accretion, they are able to sustain constant mass loss and thus a super-Eddington luminosity. They constantly lose mass via their AGN jets but this mass is constantly supplied via infalling matter / accretion.

2. (31 points) We observe a star orbiting within a certain galaxy with redshift z = 0.00300 and angular radius 5.00′; the star is orbiting 2.50′ from the galaxy's center. Assume the galaxy is circular and that we are viewing it edge-on (i = 90°); assume also that the star's orbit is circular, and at the time we are viewing this star, its apparent separation from the galaxy center is maximum.

(a) (4 points) What is the distance to this galaxy, in Mpc? Use $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

(b) (3 points) What is the physical radius of this galaxy, in pc?

(c) (2 points) What is the physical orbital radius of the star, in pc?

(d) (4 points) After taking the spectrum of this star, we find that its H-alpha line ($\lambda_0 = 656.281\,\mathrm{nm}$) is centered at 658.416 nm. Find this star's recessional velocity with respect to Earth, in $\mathrm{km\,s^{-1}}$.

(e) (4 points) Find this star's orbital velocity with respect to the galaxy, in $\mathrm{km\,s^{-1}}$.

(f) (4 points) Find the galaxy's mass interior to the star's orbital radius, in $\mathrm{M_\odot}$.

(g) (4 points) Assuming the galaxy has a spherical uniform mass distribution, find its total mass, in $\mathrm{M_\odot}$.

(h) (3 points) There are an estimated 70 billion stars in this galaxy. Assuming that each star is sunlike on average, estimate this galaxy's mass to light ratio, in $\mathrm{M_\odot\,L_\odot{}^{-1}}$.

(i) (3 points) Compared to the composition of a typical galaxy, does this galaxy have more or less dark matter?

Solution:

(a) We can get the recessional velocity from the redshift:

$$v = cz = 0.00300 \times 3 \times 10^5\,\mathrm{km\,s^{-1}} = 900\,\mathrm{km\,s^{-1}}$$

. Then, we use Hubble's law

$$d = \frac{v}{H_0} = \frac{900\,\mathrm{km\,s^{-1}}}{H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}} = 12.9\,\mathrm{Mpc}$$

(b) For small angles, we can use the small angle approximation $\tan\alpha \approx \alpha = \frac{r}{d}$, where r is the physical radius and d is the distance. If r and d are the same units, then α must be in radians. However, 1 parsec is defined as the distance at which 1 arcsecond subtends 1 AU, and so if we use α in arcseconds, we can have r in AU and d in parsecs. Either method will yield the same answer.

Here, we have $\alpha = 5.00' = 300"$ and $d = 1.29 \times 10^7\,\mathrm{pc}$, which yields $r = 3.76 \times 10^9\,\mathrm{AU}$. There are 206265 AU in a parsec (and 206265 arcseconds in a radian, due to the way the parsec is defined), and thus $r = 18\,700\,\mathrm{pc}$.

(c) The star is orbiting at an angular radius of 2.50', while the angular radius of the galaxy is 5.00'. Thus, the star is orbiting at half the physical radius of the galaxy, which is at 9350 pc.

(d) The redshift of the star $z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{658.416 - 656.281}{656.281} = 0.00325$. The recessional velocity of the star is then $v = cz = 0.00325 \times 3 \times 10^5\,\mathrm{km\,s^{-1}} = 976\,\mathrm{km\,s^{-1}}$.

(e) The star's recessional velocity is $975.9\,\mathrm{km\,s^{-1}}$, while the galaxy's recessional velocity is $900\,\mathrm{km\,s^{-1}}$. Thus, the star is moving at $75.9\,\mathrm{km\,s^{-1}}$ with respect to the galaxy. This is the star's orbital velocity.

(f) The equation for a circular orbit is

$$v = \sqrt{\frac{GM}{r}}$$

where M is the mass interior to the star's orbit, since for a spherical mass distribution only the mass interior to the orbit has any net gravitational effect. From the previous parts, we have $v = 75.9 \times 10^3$ m s^{-1} and $r = 9350\,\text{pc} = 9350 \times 3.086 \times 10^16$ m. Evaluating $M = \frac{v^2 r}{G}$ with these numbers, we have $M = 2.49 \times 10^{40}\,\text{kg} = 1.25 \times 10^{10}\,\text{M}_\odot$.

(g) Let M_* and V_* be the mass and volume of the galaxy interior to the orbital radius of the star, $r_* = 9350\,\text{pc}$. Since we are assuming the galaxy has a uniform mass distribution, $\frac{M}{M_*} = \frac{V}{V_*} = \frac{r^3}{r_*^3}$ (the volume of a sphere is proportional to r^3). From earlier, we know that $r/r_* = 2$. Thus, the total mass of the galaxy $M = 2^3 M_* = 8 \times 1.25 \times 10^{10}\,\text{M}_\odot = 1 \times 10^{11}\,\text{M}_\odot$.

(h) The luminosity of this galaxy is $7 \times 10^{10}\,\text{L}_\odot$ if we assume each star is on average $1\,\text{L}_\odot$. The mass to light ratio is then $\frac{1 \times 10^{11}\,\text{M}_\odot}{7 \times 10^{10}\,\text{M}_\odot} = 1.4\frac{\text{M}_\odot}{\text{L}_\odot}$.

(i) A mass to light ratio of $1.4\frac{\text{M}_\odot}{\text{L}_\odot}$ implies that the majority of the galaxy is visible or non-dark matter; most galaxies have a mass to light ratio between 2-10 $\frac{\text{M}_\odot}{\text{L}_\odot}$ (i.e. a majority of their matter is dark matter). Thus, this galaxy has less dark matter than usual.

3. (8 points) Gravitational lensing is a well-known phenomenon of general relativity that astrophysicists use for weighing galaxies, detecting exoplanets, and more. The angle of deflection of light for a massive body is given by

$$\theta = \frac{2R_s}{r}$$

where r is the impact parameter (the distance of closest approach from the light ray to the center of the lensing object) and R_s is the object's Schwarzschild radius.

(a) (4 points) Find the maximum angle of deflection for a photon passing near the sun, in arcseconds.

(b) (4 points) Find the maximum angle of deflection for a photon passing near Jupiter, in arcseconds.

Solution: The angle of deflection is inversely proportional to the impact parameter, so we must minimize r in order to maximize θ. The closest distance a photon can be from the center of the celestial body is at its radius (or else the photon will be absorbed).

Let R be the radius of the object. The Schwarzschild radius is given by $R_s = 2GM/c^2$, where M is the mass of the object. Thus, we have

$$\theta_{max} = \frac{4GM}{Rc^2}$$

This angle is in radians, so we would multiply this answer by 206265 to convert to arcseconds.

(a)

$$\theta_{max,\ \odot} = \frac{4\mathrm{GM}_\odot}{\mathrm{R}_\odot} = \frac{4 \times 6.67 \times 10^{-11} \times 1.989 \times 10^{30}}{6.957 \times 10^8 \times (3 \times 10^8)^2} = 4.23 \times 10^{-6}\,\mathrm{rad} = 1.75"$$

(b)

$$\theta_{max,\ J} = \frac{4GM_J}{R_J} = \frac{4 \times 6.67 \times 10^{-11} \times 1.898 \times 10^{27}}{7.1492 \times 10^7 \times (3 \times 10^8)^2} = 7.88 \times 10^{-8}\,\mathrm{rad} = 0.0163"$$

4. (18 points) The **Friedmann equation** is a fundamental equation of modern cosmology and describes the expansion of an isotropic, homogeneous universe (which is a good approximation on large scales). While this equation is accurate in the context of general relativity, some of its components can also be understood with classical Newtonian gravity.

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3}$$

Here, H is the Hubble constant, which describes the universe's expansion rate; a is the scale factor, which describes the universe's size; ρ_m is the matter density, ρ_r is the relativistic particle (photons, neutrinos, etc) mass density, k is a coefficient that describes the universe's curvature (closed, flat, or open), and Λ is the cosmological constant (dark energy). These four components govern the expansion of the universe.

(a) (5 points) Derive an expression for the critical density ρ_c of the universe, which is the density required of a flat ($k = 0$) universe. *Hint: Matter density, relativistic particle density, and dark energy all contribute to the universe's density, so you can rewrite the Friedmann equation as*

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

(b) (4 points) Compute the critical density for the universe today, in $\mathrm{kg\,m^{-3}}$. Use $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

(c) (4 points) Using part the answer to part (b), estimate the baryonic mass density of the universe. Give your answer in hydrogen atoms per cubic meter.

(d) (5 points) The actual density of the universe (as measured by instruments like WMAP) is quite close to the critical density. A universe that is initially even just slightly closed or open will rapidly diverge from flatness; the fact that the universe today is flat (or nearly flat) seems like a remarkable coincidence. How do cosmologists explain this, besides using the anthropic principle?

Solution:

(a) Setting the curvature constant k to 0 and solving for ρ, we get

$$\rho_c = \frac{3H^2}{8\pi G}$$

(b) İn order to use the critical density equation from above, we must convert $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ to SI units ($\mathrm{s^{-1}}$).

$$70\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \times \frac{10^3\ \mathrm{m}}{1\,\mathrm{km}} \times \frac{1\,\mathrm{Mpc}}{3.086 \times 10^{22}\,\mathrm{m}} = 2.27 \times 10^{-18}\,\mathrm{s^{-1}}$$

Evaluating the critical density equation in SI units, we get

$$\rho_c = \frac{3 \times (2.27 \times 10^{-18})^2}{8\pi \times 6.67 \times 10^{-11}} = 9.21 \times 10^{-27}\,\mathrm{kg\,m^{-3}}$$

(c) The universe today is flat or nearly flat, so the total density (including baryonic mass, radiation, dark matter, and dark energy) is equal to the critical density. Baryonic (i.e. "ordinary") mass makes up about 4.6% of the universe's mass-energy. Thus, $\rho_b = 0.046 \times 9.21 \times 10^{-27}\,\mathrm{kg\,m^{-3}} = 4.23 \times 10^{-28}\,\mathrm{kg\,m^{-3}}$.

The mass of a hydrogen atom is $m_H \approx m_p = 1.673 \times 10^{-27}\,\mathrm{kg}$. The baryonic mass density of the universe in hydrogen atoms per meter cubed is then

$$\rho_b = 4.23 \times 10^{-28}\,\mathrm{kg\,m^{-3}} \times \frac{1\text{ Hydrogen atom}}{1.673 \times 10^{-27}\,\mathrm{kg}} = 0.25\text{ Hydrogen atoms m}^{-3}$$

(d) The remarkable flatness of the universe is one of the strongest supporting evidence for the **theory of inflation**. A period of strong inflation during the early universe would have flattened out the universe to $k = 1$ regardless of what k the universe began with.