TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

PROF. DR. ERIC SONNENDRÜCKER, DOMINIK BELL

Computational Plasma Physics, SS 2022
http://www-m16.ma.tum.de/Allgemeines/CompPlasmaPhys22
Exercise Sheet 4 (30. 05. 2022)

Explicit Maxwell solver with leap-frog scheme

This week we will be looking at the Maxwell equations:

$$-\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{1b}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1d}$$

1. Show that if you consider only (1a) and (1b) as the dynamical equations and take (1c), (1d) as initial conditions (i.e. satisfied at $t = 0$) that they remain true for all times. (Hint: Use the continuity equation $\frac{\mathrm{d}\rho}{\mathrm{d}t} + \nabla \cdot \mathbf{J} = 0$ and the fact that $c^2\epsilon_0\mu_0 = 1$).

In the following we will be considering equations (1) in vacuum (so $\mathbf{J} = 0$ and $\rho = 0$) and in natural units ($c = 1$):

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = 0 \tag{2a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{2b}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{2c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2d}$$

on the domain $[0, 2\pi]^3$ with periodic boundary conditions

$$\mathbf{E}(t, 0, y, z) = \mathbf{E}(t, 2\pi, y, z) \qquad \mathbf{B}(t, 0, y, z) = \mathbf{B}(t, 2\pi, y, z) \tag{3a}$$

$$\mathbf{E}(t, x, 0, z) = \mathbf{E}(t, x, 2\pi, z) \qquad \mathbf{B}(t, x, 0, z) = \mathbf{B}(t, x, 2\pi, z) \tag{3b}$$

$$\mathbf{E}(t, x, y, 0) = \mathbf{E}(t, x, y, 2\pi) \qquad \mathbf{B}(t, x, 0, z) = \mathbf{B}(t, x, 2\pi, z) \tag{3c}$$

As you showed above, it is enough to check (2c), (2d) for the initial conditions and then evolve the system using equations (2a) and (2b). In order to solve these equations explicitly we will split them into two steps; one where $\mathbf{B}$ is fixed $\mathbf{E}$ is updated, and one where $\mathbf{E}$ is fixed and $\mathbf{B}$ is updated. We will solve the equations for $\mathbf{E}$ at times $t = 0, \Delta t\ , 2\Delta t\ , \ldots$ and solve the equations for $\mathbf{B}$ at each half time-step $t = \Delta t\ /2, 3\Delta t\ /2, \ldots$. This is called a leap-frog scheme.

We use a short-cut notation indicating the position in space and time for the fields, e.g.

$$E_x^{n,i,j,k} = E_x(t^n, x_i, y_j, z_k) \tag{4}$$

To discretize the differential equations, we use first-order approximations for space and time, e.g.

$$\frac{\partial}{\partial x} E_z^{n,i,j,k} = \frac{1}{h_x}\left(E_z^{n,i,j,k} - E_z^{n,i-1,j,k}\right) \tag{5a}$$

$$\frac{\partial}{\partial t} E_x^{n,i,j,k} = \frac{1}{\Delta t}\left(E_x^{n+1,i,j,k} - E_x^{n,i,j,k}\right) \tag{5b}$$

2. Write down the fully discrete equations following from (2a) and (2b).
3. Implement update functions for **E** and **B**.
4. Create a code that starts with the analytical solution (6) at $t = 0$ and advances them in time via a leap-frog scheme using your update functions.

$$\mathbf{E}(\mathbf{x}, t) = \begin{pmatrix} \cos\left(x + y + z - \sqrt{3}t\right) \\ -2\cos\left(x + y + z - \sqrt{3}t\right) \\ \cos\left(x + y + z - \sqrt{3}t\right) \end{pmatrix} \quad \mathbf{B}(\mathbf{x}, t) = \begin{pmatrix} \sqrt{3}\cos\left(x + y + z - \sqrt{3}t\right) \\ 0 \\ -\sqrt{3}\cos\left(x + y + z - \sqrt{3}t\right) \end{pmatrix} \tag{6}$$

5. Run your code using the following parameters:
   - $\Omega = [0, 2\pi]^3$
   - $N_x = N_y = N_z = 8$
   - $\Delta t \ = 0.01$
6. Use the method of manufactured solutions: Plot your results and the analytical solution at different times. What do you notice? What does this say about the scheme?
7. *Optional*: What happens when you choose different parameters for Δt and N?