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Computational Plasma Physics, SS 2022
http://www-m16.ma.tum.de/Allgemeines/CompPlasmaPhys22

Exercise Sheet 1 (02. 05. 2022)

1D Poisson solver with finite differences

When the magnetic field is constant and the time evolution of the electric field is slow, we can replace the Maxwell equations by the Poisson equation. In this exercise we will solve the 1-dimensional Poisson equation which reads

$$-\phi''(x) = \rho(x) \qquad\qquad x \in [a,b] \subset \mathbb{R}\,, \tag{1}$$

with given $\rho \in C^0([a,b])$. We will consider (1) with Dirichlet boundary conditions

$$\phi(a) = \alpha \qquad\qquad \phi(b) = \beta \tag{2}$$

1. Show that the problem (1)-(2) has a unique solution. (Hint: Assume that both ϕ and ψ are solutions of (1), then verify that $\eta := \phi - \psi$ satisfies the Laplace equation.)

Now we want to use finite differences to approximate the solution of (1). We divide the domain $[a,b]$ into N equally large cells leading to $N+1$ grid points $x_i \in [a,b]$, $i = 0,\dots,N$, on which the discrete solution is defined.

2. Write a finite difference solver for (1)-(2) for arbitrary a, b, α, β, N and ρ by approximating the Laplacian at the grid points $i = 1,\dots,N-1$ with

$$\phi''(x_i) = \frac{1}{h^2}\left(\phi_{i+1} - 2\phi_i + \phi_{i-1}\right) + \mathcal{O}(h^2) \tag{3}$$

   where $\phi_i = \phi(x_i)$ and $h = \frac{b-a}{N}$ is the cell size (grid spacing). For different N, how does the system matrix look like? Check the eigenvalues of the system matrix; what does this say about our algorithm?

3. Set $a = 0$, $b = 2\pi$, $\rho(x) = 2\sin(x) + x\cos(x)$, $\alpha = 0$, $\beta = 2\pi$ and solve (1)-(2) for $N = 8, 16, 32, 64, 128, 256$. In each run, compute and save the errors with respect to the analytical solution $\phi(x) = x\cos(x)$ in the L^1-, the L^2- and the L^∞-norm. Plot the numerical and analytical solutions.

4. Plot the errors as a function of N. Does the solution converge as $N \to \infty$? If so, is the rate what you would expect considering (3)?