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Computational Plasma Physics, SS 2022
http://www-m16.ma.tum.de/Allgemeines/CompPlasmaPhys22
Exercise Sheet 3 (23. 05. 2022)

1D Poisson solver with fast Fourier transform

Again, we want to solve the 1-dimensional Poisson equation

$$-\phi''(x) = \rho(x) \qquad\qquad x \in [a,b] \subset \mathbb{R}\,, \tag{1}$$

with periodic boundary conditions

$$\phi(a) = \phi(b) \tag{2}$$

As we have seen last week, this equation does not have a unique solution. In order to get a unique solution we will be imposing the 0-mean condition:

$$\int_a^b \phi(x)\,\mathrm{d}x = 0 \tag{3}$$

So far, we have explicitly solved the system by solving $A_h\Phi_h = \rho_h$. This week, we want to use the fact that the system matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix} \tag{4}$$

is circulant with entries $c_0 = 2/h^2$, $c_1 = -1/h^2$, $c_{N-1} = -1/h^2$. We have seen in the lecture that a circulant matrix C is always diagonalisable

$$C = P\Lambda P^* \tag{5}$$

where P is the matrix from the discrete Fourier transformation with entries

$$P_{jk} = \exp\left(-\frac{2i\pi jk}{N}\right) \tag{6}$$

and $P^* = \frac{1}{N}P^{-1}$ its complex conjugate.

This means that if Λ is invertible, we can simply compute the solution via

$$\Phi_h = P\Lambda^{-1}P^*\rho_h \tag{7}$$

1. What are the eigenvalues of A_h? Is Λ indeed invertible?
2. Discretize (3). How does this affect the 0-th Fourier mode of ϕ?
3. Now that you have determined the value of $\hat{\phi}_0$, implement the Poisson-solver based on FT, following this prescription:
   (a) Evaluate ρ at the internal grid points and apply the inverse Fourier transform P^* resulting in the vector $\hat{\rho}$.
   (b) Multiply $\hat{\rho}$ with Λ^{-1} and add $\hat{\phi}_0$ as the 0-th entry.
   (c) Apply the Fourier transform P to obtain your numerical solution Φ_h.
4. Test your code using the same manufactured solution as on Sheet 2. Plot your solutions and check the convergence rate.