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Zentrum Mathematik

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Computational Plasma Physics, SS 2022
http://www-m16.ma.tum.de/Allgemeines/CompPlasmaPhys22
Exercise Sheet 2 (16. 05. 2022)

1D Poisson solver with FD for mixed and periodic BC

We revisit the 1-dimensional Poisson equation

$$-\phi''(x) = \rho(x) \qquad\qquad x \in [a,b] \subset \mathbb{R}\,, \tag{1}$$

with given $\rho \in C^0([a,b])$.

1. Mixed boundary conditions.

We will solve (1) with mixed boundary conditions (also called Neumann BC):

$$\phi(a) = \alpha \qquad\qquad \phi'(b) = \gamma\,. \tag{2}$$

1. Does (1)-(2) possess a unique solution? (Hint: Look at the exercise from last week.)

In order to apply finite differences we now also need a finite differences approximation of the derivative at the boundary. First we will consider

$$\phi'(x_N) = \frac{1}{h}\left(\phi_N - \phi_{N-1}\right) \tag{3}$$

2. Write a finite difference solver with the boundary conditions (2). Use the same parameters as in exercise sheet 1, replacing β with $\gamma = 1$. Again, check the eigenvalues of the system matrix and the convergence rate, and plot your results. (Hint: You can reuse most of the code from last week.)

As you can see, the approximation of the derivative has affected our convergence order. Our aim now, is to find a second order approximation for the derivative at the boundary. For that, take the ansatz

$$\phi'(x_N) = \frac{1}{h}\left(c_1\phi_N + c_2\phi_{N-1} + c_3\phi_{N-2}\right) \tag{4}$$

3. With the use of the Taylor expansion, find the coefficients c_1, c_2, c_3 for which this approximation is of order 2.

4. Implement the alternative, second order derivative in your code. Check the convergence rate and plot your results.

2. Periodic boundary conditions.

Consider now a periodic solution with period $L = b - a$

$$\phi(x + L) = \phi(x) \qquad \forall x \in \mathbb{R} \tag{5}$$

i.e. periodic boundary conditions

$$\phi(a) = \phi(b) \tag{6}$$

It is sufficient to compute the solution in $[a, b)$ and there are now N unknowns.

1. Does this type of boundary condition bring a solvability condition onto ρ? Does the problem (1),(5) have a unique solution?
2. Discretize the problem (1), (5). (Hint: $\phi_N = \phi_0$ and $\phi_{-1} = \phi_{N-1}$). Check the eigenvalues of the system matrix A. Does this correspond to your answer in the previous question?
3. Impose $\phi(a) = 0$ on the system matrix and check the eigenvalues again.
4. Set $a = 0$, $b = 2\pi$ and take $\rho = 4\sin(2x)$ for which the analytical solution reads $\phi(x) = \sin(2x)$. Plot your results and check the convergence rate.