TECHNISCHE UNIVERSITÄT MÜNCHEN
Zentrum Mathematik

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Computational Plasma Physics, SS 2022
http://www-m16.ma.tum.de/Allgemeines/CompPlasmaPhys22
Exercise Sheet 7 (27. 06. 2022)

Explicit Maxwell solver with Yee scheme

This week we will again solve the Maxwell equations in vacuum:

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = 0 \tag{1a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \tag{1b}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1d}$$

on the domain $[0, 2\pi]^3$ with periodic boundary conditions.

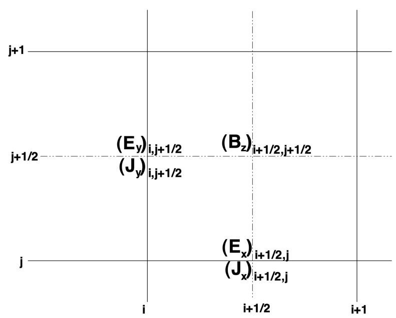
This time, we will implement a second order scheme in space, the Yee scheme, to complement the leap-frog scheme in time, and preserve energy.

To remind you, the leap-frog scheme solves the system in two steps; one where $\mathbf{B}$ is fixed $\mathbf{E}$ is updated, and one where $\mathbf{E}$ is fixed and $\mathbf{B}$ is updated. We will solve the equations for $\mathbf{E}$ at times $t = 0, \Delta t\ , 2\Delta t\ , \ldots$ and solve the equations for $\mathbf{B}$ at each half time-step $t = \Delta t\ /2, 3\Delta t\ /2, \ldots$ with the update rule

$$\frac{\partial}{\partial t} E_x^{n,i,j,k} = \frac{1}{\Delta t} \left(E_x^{n+1,i,j,k} - E_x^{n,i,j,k} \right) \tag{2}$$

using again the short-cut notation indicating the position in space and time for the fields, e.g.

$$E_x^{n,i,j,k} = E_x(t^n, x_i, y_j, z_k) \tag{3}$$



For the space discretization we implement the Yee scheme, which defines approximation for the field quantities at two slightly shifted grids: the electric field $\mathbf{E}$ is defined at the centre of the cell boundaries, while the magnetic field is defined in the centre of the cells. This is shown schematically in 2D, where only E_x, E_y, B_z exist, in the figure on the right.

So the components of $\mathbf{E}$ are shifted in the direction to which they correspond (e.g. E_x in the x-direction) and the components of $\mathbf{B}$ remain in the plane to which they correspond and are shifted in the other two directions (e.g. B_z shifted in x- and y-direction). The derivatives in space then read e.g.

$$\frac{\partial}{\partial x} E_z^{n,i,j,k+\frac{1}{2}} = \frac{1}{h_x}\left(E_z^{n,i,j,k+\frac{1}{2}} - E_z^{n,i-1,j,k+\frac{1}{2}}\right) \tag{4a}$$

$$\frac{\partial}{\partial y} B_x^{n,i,j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{h_y}\left(B_x^{n,i,j+\frac{1}{2},k+\frac{1}{2}} - B_x^{n,i-\frac{1}{2},j,k+\frac{1}{2}}\right) \tag{4b}$$

1. Write down the fully discrete equations following from (1a) and (1b) using the Yee scheme. *Hint:* Pay attention where both the left- and right-hand side are defined in space and time.

2. Implement update functions for **E** and **B**.

3. Create a code that starts with the analytical solution (5) at $t = 0$ and $t = \frac{\Delta t}{2}$, respectively, and advances them in time via a leap-frog scheme using your update functions.

$$\mathbf{E}(\mathbf{x}, t) = \begin{pmatrix} \cos\left(x+y+z-\sqrt{3}t\right) \\ -2\cos\left(x+y+z-\sqrt{3}t\right) \\ \cos\left(x+y+z-\sqrt{3}t\right) \end{pmatrix} \quad \mathbf{B}(\mathbf{x}, t) = \begin{pmatrix} \sqrt{3}\cos\left(x+y+z-\sqrt{3}t\right) \\ 0 \\ -\sqrt{3}\cos\left(x+y+z-\sqrt{3}t\right) \end{pmatrix} \tag{5}$$

4. Run your code using the following parameters:
   - $\Omega = [0, 2\pi]^3$
   - $N_x = N_y = N_z = 32$
   - $\Delta t \ = 0.005$
   - $T = 20$

5. Use the method of manufactured solutions: Plot your results and the analytical solution (5) at different times. Also plot the energy versus time. What do you notice?