Problems in Classical Mechanics

1 Realistic Projectile Motion

1.1 Skydiver

Let's consider first the freefall of a skydiver. In the simplest case, we neglect air resistance, and the equation of motion is

$$\frac{d^2y}{dt^2} = -g \tag{1}$$

where $g = 9.81$ m/s^2. (This is a positive constant. We're assuming the downward direction - toward Earth - is the negative y direction, which is the reason for the negative sign in the equation.) Let's assume the initial height of the skydiver is 3000 m (about 9800 ft), that is $y(0) = y_0 = 3000$ and the skydiver starts from rest, with an initial velocity of $v_0 = 0$ m/s.

The first step is to make this second-order differential equation into two first-order differential equations. We do this by letting

$$v = \frac{dy}{dt} \tag{2}$$

so that

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -g \tag{3}$$

Let's plot $v(t)$ and $y(t)$ from $t = 0$ to $t = 15$ s using Euler's method with $h = 0.1$ s. This problem is solvable analytically of course, so we can compare to the true solution, $y(t) = y_0 - gt^2/2$ and $v(t) = -gt$.

So how do we solve these two simultaneous first-order differential equations? Writing it in our standard form, we have

$$\frac{dy}{dt} = f_1(y, v, t) \tag{4}$$

where

$$f_1(y, v, t) \equiv v \tag{5}$$

And

$$\frac{dv}{dt} = f_2(y, v, t) \tag{6}$$

where

$$f_2(y, v, t) \equiv -g \tag{7}$$

(a constant).

For a single step of size h, Euler's method gives us

$$\begin{aligned} y(t+h) &= y(t) + hf_1(y, v, t) \\ y(t+h) &= y(t) + hv(t) \end{aligned} \tag{8}$$

so that y at the future time $t+h$ is determined by y and v at the current time t.

$$\begin{aligned} v(t+h) &= v(t) + hf_2(y, v, t) \\ v(t+h) &= v(t) - hg \end{aligned} \tag{9}$$

so that v at the future time $t+h$ is determined by v at the current time t.

For repeated steps, this gives us

$$v_i = v_{i-1} - hg \tag{10}$$

for v with $v_0 = v(0) = 0$. For y, we have

$$y_i = y_{i-1} + hv_{i-1} \tag{11}$$

with $y_0 = y(0) = 3000$.

Example: Skydiver1

Now of course we know this solution isn't realistic, due to air resistance. In general the force due to air resistance can be characterized as

$$F_{\text{air}} \approx kv + Dv^2 \tag{12}$$

When v is very small, the first term dominates, and the constant k can be calculated for objects with simple shapes. For example, the first term is a good approximation for dust particles falling in air or a ball bearing falling in oil. For larger objects moving at larger velocities (falling raindrops, bicyclists, airplanes), the second term dominates. So for the case of a skydiver, we have

$$F_{\text{air}} \approx Dv^2 \tag{13}$$

The net force is given by

$$F_{\text{net}} = -mg + Dv^2, \tag{14}$$

with gravity in the negative y direction and the force of air resistance in the positive y direction. The equation of motion with air resistance becomes

$$\begin{aligned} m\frac{d^2y}{dt^2} &= -mg + Dv^2 \\ \frac{d^2y}{dt^2} &= -g + \frac{Dv^2}{m} \end{aligned} \tag{15}$$

What is the constant D? As an object moves through the air, the object must push the air in front of it out of the way. The mass of air moved is $m_{\text{air}} \sim \rho\mathcal{V}$, where ρ is the density of air and $\mathcal{V}$ is the volume of the air moved. The volume can be approximated by the cross-sectional area of the object, A, times vdt, the distance the object moves in time dt. Therefore $m_{\text{air}} \sim \rho Avdt$. The air is given a velocity of approximately v, so that the kinetic energy of the air is $E_{\text{air}} \sim m_{\text{air}}v^2/2$. E_{air} should be equal to the work done by the drag force, so that

$$\begin{aligned} E_{\text{air}} &\sim F_{\text{air}}dx \\ E_{\text{air}} &\sim F_{\text{air}}vdt \\ m_{\text{air}}v^2/2 &\sim F_{\text{air}}vdt \\ (\rho Avdt)v^2/2 &\sim F_{\text{air}}vdt \\ F_{\text{air}} &\sim \frac{1}{2}\rho Av^2 \\ F_{\text{air}} &\approx \frac{1}{2}C\rho Av^2 \end{aligned} \tag{16}$$

where C is a unitless constant called the drag coefficient that depends on the shape of the object. So we have

$$D = \frac{1}{2}C\rho A \tag{17}$$

Let's assume the mass of our skydiver is 80 kg and the cross-sectional area is $A \sim 1.0$ m^2. (Assume a rectangular skydiver, 180 cm or 6 ft tall by 60 cm or 2 ft wide, $1.8 \times 0.6 \approx 1$.) The density of air is 1.2 kg/m^3. We can assume $C \sim 0.7$. Therefore,

$$D = \frac{1}{2}(0.7)(1.2)(1.0) \sim 0.4 \text{ kg/m} \tag{18}$$

Note the units: D times velocity squared has units of (kg/m)(m^2/s^2) = kg m/s^2 = N, as it should.

Now that we have a value for D, we can solve Equation 15:

$$v = \frac{dy}{dt} \tag{19}$$

The equation of motion above becomes two first-order differential equations:

$$\frac{dy}{dt} = v \equiv f_1(y, v, t) \tag{20}$$

$$\frac{dv}{dt} = -g + \frac{Dv^2}{m} \equiv f_2(y, v, t) \tag{21}$$

For a single step of size h, Euler's method gives us

$$\begin{aligned} y(t+h) &= y(t) + hf_1(y, v, t) \\ y(t+h) &= y(t) + hv(t) \end{aligned} \tag{22}$$

so that y at the future time $t+h$ is determined by y and v at the current time t.

$$\begin{aligned} v(t+h) &= v(t) + hf_2(y, v, t) \\ v(t+h) &= v(t) + h\left(-g + \frac{D[v(t)]^2}{m}\right) \\ v(t+h) &= v(t) - hg + \frac{hD[v(t)]^2}{m} \end{aligned} \tag{23}$$

so that v at the future time $t+h$ is determined by v at the current time t.

Using Euler's approximation in repeated steps:

$$v_i = v_{i-1} - hg + \frac{hDv_{i-1}^2}{m} \tag{24}$$

for v with $v_0 = v(0) = 0$. For y, we have

$$y_i = y_{i-1} + hv_{i-1} \tag{25}$$

with $y_0 = y(0) = 3000$.

Example: Skydiver2

The so-called "terminal velocity" is the constant speed that a falling object will reach when air resistance prevents further acceleration. Constant speed means zero acceleration, so

$$\begin{aligned} -g + \frac{Dv_t^2}{m} &= 0 \\ \frac{Dv_t^2}{m} &= g \\ v_t &= \sqrt{\frac{mg}{D}} \end{aligned} \tag{26}$$

Using the given values we have

$$\begin{aligned} v_t &= \sqrt{\frac{mg}{D}} \\ &= \sqrt{\frac{(80\text{ kg})(9.81\text{ m/s}^2)}{0.4\text{ kg/m}}} \\ &= 44\text{ m/s} = 99\text{ mph} \end{aligned} \tag{27}$$

Section 4, Exercise 1

1.2 The Trajectory of a Cannon Shell

Let's consider a projectile such as a shell shot by a large cannon. Ignoring air resistance, the equations of motion are

$$\frac{d^2x}{dt^2} = 0 \tag{28}$$

$$\frac{d^2y}{dt^2} = -g \tag{29}$$

where x is the horizontal position and y is the vertical position. We can produce four first-order differential equations from these two second-order equations:

$$\frac{dx}{dt} = v_x \tag{30}$$

$$\frac{dv_x}{dt} = 0 \tag{31}$$

$$\frac{dy}{dt} = v_y \tag{32}$$

$$\frac{dv_y}{dt} = -g \tag{33}$$

To add in the effects of air resistance, we have the vector force $\vec{F}_{\rm air} = Dv^2(-\hat{v})$, where $\hat{v}$ is a unit vector in the direction of the velocity of the projectile. (The force of air resistance will be directed exactly opposite to the motion.) The unit vector $\hat{v}$ is

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{v_x\ \hat{\rm x} + v_y\ \hat{\rm y}}{v} \tag{34}$$

where $v = \sqrt{v_x^2 + v_y^2}$. Therefore, the components of $\vec{F}_{\rm air}$ are

$$F_{{\rm air},x} = -Dv^2\frac{v_x}{v} = -Dvv_x \tag{35}$$

$$F_{{\rm air},y} = -Dv^2\frac{v_y}{v} = -Dvv_y \tag{36}$$

The equations of motion become

$$\frac{d^2x}{dt^2} = -\frac{Dvv_x}{m} \tag{37}$$

$$\frac{d^2y}{dt^2} = -g - \frac{Dvv_y}{m} \tag{38}$$

Transforming to first-order equations:

$$\frac{dx}{dt} = v_x \tag{39}$$
$$\frac{dv_x}{dt} = -\frac{Dvv_x}{m} \tag{40}$$
$$\frac{dy}{dt} = v_y \tag{41}$$
$$\frac{dv_y}{dt} = -g - \frac{Dvv_y}{m} \tag{42}$$

Applying Euler's approximation in repeated steps:

$$v_{xi} = v_{xi-1} - \frac{hDv_{i-1}v_{xi-1}}{m} \tag{43}$$
$$v_{yi} = v_{yi-1} - hg - \frac{hDv_{i-1}v_{yi-1}}{m} \tag{44}$$
$$x_i = x_{i-1} + hv_{xi-1} \tag{45}$$
$$y_i = y_{i-1} + hv_{yi-1} \tag{46}$$

where $v_i = \sqrt{v_{xi}^2 + v_{yi}^2}$.

We will assume the projectile is launched from the position $(x, y) = (0, 0)$ at time $t = 0$. We will consider different launch velocities and launch angles. A reasonable value for the constant D (based on a measurement) is $D/m \approx 4 \times 10^{-5}$ m^{-1}. Since D and m never appear separately, and only as the ratio D/m, we don't need the individual values of D and m. The simple formula we used in the previous section to estimate D (Equation 17) doesn't work very well in this case; we'll see why later.

Given a launch velocity v_0, $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$, where θ is the launch angle.

Example: Cannonball trajectory 1

Now we are going to discuss the effect that variable air density, projectile shape and speed, and wind will have on our projectile. But first...

Section 4, Exercise 2

The next bit of reality we want to incorporate into our calculation is the fact that the density of air varies as a function of altitude. Given the high altitudes (up to several kilometers) the cannon shell attains, this variation can affect the trajectory. The air is less dense at higher altitudes, making the drag force less at higher altitudes.

A simple approximation is to treat the atmosphere as an ideal gas at constant temperature. In that case, the pressure P depends on altitute according to

$$P(y) = P(0)e^{-Mgy/k_BT} \tag{47}$$

where M is the average mass of an air molecule, y is the height from sea level, k_B is Boltzmann's constant, and T is the temperature. According to the ideal gas law, the product of pressure and volume is constant if temperature is constant:

$$PV = nRT \equiv \alpha \tag{48}$$

where n is the number of moles, R is the gas constant, and V is the volume. Density is mass over volume, $\rho = m/V$, so we have

$$\begin{aligned} PV &= \alpha \\ \frac{PV}{m} &= \frac{\alpha}{m} \equiv \beta \\ \frac{P}{\rho} &= \beta \\ P &= \beta\rho \end{aligned} \tag{49}$$

i.e. pressure is proportional to the density. Therefore, density will have the same functional form as pressure:

$$\begin{aligned} \rho(y) &= \rho_0 e^{-Mgy/k_BT} \\ \frac{\rho(y)}{\rho_0} &= e^{-Mgy/k_BT} \end{aligned} \tag{50}$$

where ρ_0 is the density at sea level ($y = 0$). Let

$$\kappa = \frac{Mg}{k_BT} = \frac{(5.6 \times 10^{-26}\ \text{kg})(9.81\ \text{m/s}^2)}{(1.38 \times 10^{-23}\ \text{m}^2\text{kg}/(\text{s}^2K))(300\ \text{K})} \sim 10^{-4}\ \text{m}^{-1} \tag{51}$$

so that

$$\frac{\rho(y)}{\rho_0} = e^{-\kappa y} \tag{52}$$

Now recall that the drag force is

$$F_{\text{air}} = Dv^2 \tag{53}$$

The exact calculation of D is complicated in this case, but in Equation 17, we found it should be proportional to the air density. Therefore, it should change with altitude as the air density. Let $f_d(y)$ be the drag force "constant" (which is no longer constant). If we assume the value of D we used in the calculation above is the value at sea level ($f_D(y = 0) = D$), then

$$\begin{aligned} f_D(y) &= \frac{D}{\rho_0}\rho \\ f_D(y) &= De^{-\kappa y} \end{aligned} \tag{54}$$

We can take this effect into account by replacing D with $f_D = De^{-\kappa y}$ in our calculations.

Our Euler's method equations become:

$$v_{xi} = v_{xi-1} - \frac{hf_D(y_{i-1})v_{i-1}v_{xi-1}}{m} \tag{55}$$

$$v_{yi} = v_{yi-1} - hg - \frac{hf_D(y_{i-1})v_{i-1}v_{yi-1}}{m} \tag{56}$$

$$x_i = x_{i-1} + hv_{xi-1} \tag{57}$$

$$y_i = y_{i-1} + hv_{yi-1} \tag{58}$$

Example: Cannonball Trajectory 2

We've just seen an example of how the constant of proportionality for the drag force (which we've called D) can vary with altitude. We worked out a qualitative idea of what D should depend on, $D = \frac{1}{2}C\rho A$, but the drag coefficient C is not necessarily constant. For example, see the plot in Figure 1, which is obtained from wind tunnel measurements with baseballs. The drag coefficient depends strongly on the speed of the ball. At low speeds, the air flow around the ball is smooth. But at high speeds, the air flow around the ball is turbulent, and the ball is able to move through the air with less resistance. Note that the speed at which the transition happens is right in the region of the typical speed for a baseball pitch, so this effect needs to be taken into account to realistically model a baseball pitch. The figure also shows the same curve for "rough" and "smooth" balls, which affects the speed dependence.

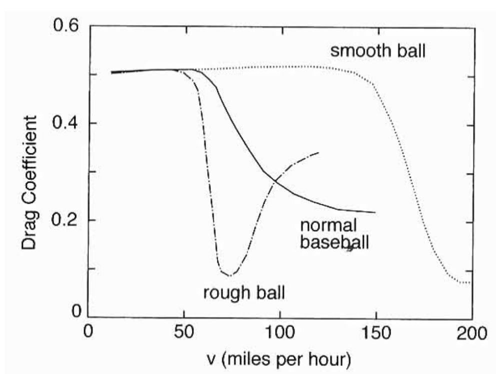

Figure 1: Variation of the drag coefficient C with velocity for normal, rough, and smooth baseballs. Copied from Giordano and Nakanishi.

Figure 2 shows another example of a measured drag coefficient curve.

We won't incorporate this effect into our model today, just note that it could be an important effect to keep in mind, depending on the desired accuracy of your model.

Now let's consider the impact of wind on our calculation. This means that the air itself will have a non-zero velocity $\vec{v}_{\text{wind}}$. Before, we were assuming the air had zero velocity.

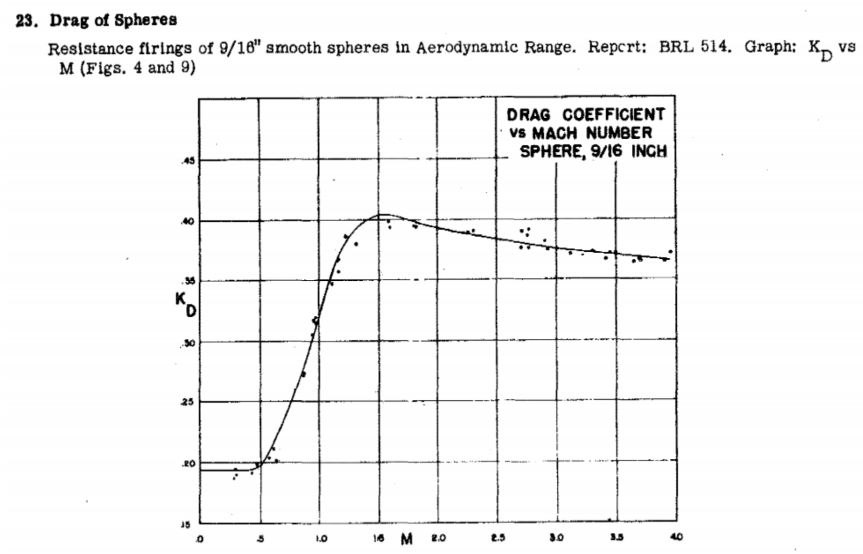

Figure 2: Variation of the drag coefficient with speed for 9/16" smooth spheres from measurements by the Ballistic Research Laboratories (BRL) in the 1940's. Speed is given in units of "Mach number"; a Mach number of 1 is the speed of sound, ~340 m/s. I copied from `https://apps.dtic.mil/dtic/tr/fulltext/u2/800469.pdf` in 2020, but it is no longer available at this link. Interesting tidbit: Astronomer Edwin Hubble served as Chief of the External Ballistics Branch of the BRL during World War II.

Let's recall (non-relativistic) velocity addition in one dimension. If $v_{\rm AB}$ is the velocity of object A with respect to object B, and $v_{\rm BC}$ is the velocity of object B with respect to object C, then the velocity of object A with respect to object C is $v_{\rm AC} = v_{\rm AB} + v_{\rm BC}$. In our case, object A is the projectile, object B is the air, and object C is the cannon. The velocity that matters for air resistance is $v_{\rm AB}$, the velocity of the projectile with respect to the air, but the trajectory that we've been calculating is in the rest frame of the cannon, i.e. our $v(t)$ is $v_{\rm AC}$, the velocity of the projectile with respect to the cannon. We've been assuming that v_{BC}, the velocity of the air with respect to the cannon, is zero. Therefore the velocity of the projectile with respect to the air is the same as the velocity of the projectile with respect to the cannon, $v_{\rm AC} = v_{\rm AB}$.

With wind, $v_{\rm BC} \neq 0$, so we have to distinguish between $v_{\rm AC}$ and $v_{\rm AB}$ in our calculation. When calculating x and y we need $v_{\rm AC}$ as before. But in the drag force term, we need $v_{\rm AB} = v_{\rm AC} - v_{\rm BC}$, i.e. projectile speed minus the wind speed.

Let's assume the wind is purely in the horizontal (x) direction, so that the velocity of the wind (in the rest frame of the cannon) is $\vec{v}_{\rm wind} = v_{\rm wind}\ \hat{\rm x}$. In the drag term, we need to use

$$\vec{v}_{\rm proj-air} = \vec{v} - \vec{v}_{\rm wind} \tag{59}$$
$$v_{\rm proj-air,x} = v_x - v_{\rm wind} \tag{60}$$
$$v_{\rm proj-air,y} = v_y \tag{61}$$

Note that a "tail wind" is in the same direction as the motion of our projectile (positive x), and therefore corresponds to a positive value of v_{wind}. A "head wind" is in the opposite direction as the motion of our projectile, and therefore corresponds to a negative value of v_{wind}.

Our Euler's method equation become:

$$
\begin{aligned}
v_{xi} &= v_{xi-1} - \frac{hf_D(y_{i-1})v_{\text{proj-air},i-1}v_{\text{proj-air},x,i-1}}{m} \\
&= v_{xi-1} - \frac{hf_D(y_{i-1})(v_{xi-1} - v_{\text{wind}})\sqrt{(v_{xi-1} - v_{\text{wind}})^2 + v_{yi-1}^2}}{m} && (62) \\
v_{yi} &= v_{yi-1} - hg - \frac{hf_D(y_{i-1})v_{\text{proj-air},i-1}v_{yi-1}}{m} \\
&= v_{yi-1} - hg - \frac{hf_D(y_{i-1})v_{yi-1}\sqrt{(v_{xi-1} - v_{\text{wind}})^2 + v_{yi-1}^2}}{m} && (63) \\
x_i &= x_{i-1} + hv_{xi-1} && (64) \\
y_i &= y_{i-1} + hv_{yi-1} && (65)
\end{aligned}
$$

A tail wind (positive v_{wind}) will reduce drag; a head wind (negative v_{wind}) will increase drag. In the case of a tail wind, the air is pushing the projectile and creating less resistance, while a head wind is creating more resistance.

Let's consider a wind speed of $v_{\text{wind}} = \pm 4.5$ m/s ($\approx \pm 10$ mph) and a launch angle of 45°.

Example: Cannonball Trajectory 3

2 Oscillatory Motion

2.1 Nonlinear pendulum

Consider the simple pendulum shown in Figure 3. A particle of mass m is connected by a massless string of length ℓ to a rigid support, making the angle θ with the vertical. The force along the direction of motion is given by $-mg\sin\theta$ (minus sign because the motion is to the right, while the force component is to the left), so the equation of motion is

$$
\begin{aligned}
m\frac{d^2(\ell\theta)}{dt^2} &= -mg\sin\theta \\
\frac{d^2\theta}{dt^2} &= -\frac{g}{\ell}\sin\theta \approx -\frac{g}{\ell}\theta && (66)
\end{aligned}
$$

where $\ell\theta$ is the distance traveled by the pendulum along its circular path. For small angles, $\sin\theta \approx \theta$, and this replacement yields the equation of motion for simple harmonic motion. Let's study this case first.

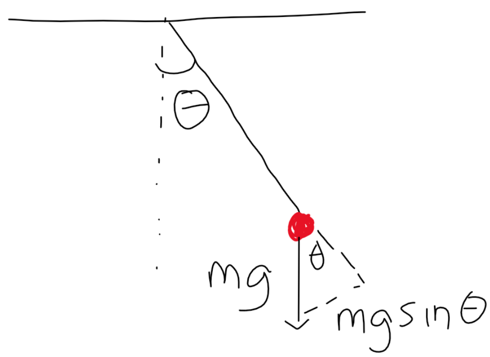

Figure 3:

As usual, we turn this into two first-order differential equations with the substitution

$$\frac{d\theta}{dt} = \omega \tag{67}$$

so that

$$\frac{d\omega}{dt} = -\frac{g}{\ell}\theta \tag{68}$$

Euler's method for one step is given by

$$\theta(t+h) = \theta(t) + h\omega(t) \tag{69}$$

$$\omega(t+h) = \omega(t) + h\left(-\frac{g}{\ell}\theta(t)\right) \tag{70}$$

This leads to these equations for Euler's method in repeated steps:

$$\theta_i = \theta_{i-1} + h\omega_{i-1} \tag{71}$$

$$\omega_i = \omega_{i-1} - \frac{hg}{\ell}\theta_{i-1} \tag{72}$$

We'll assume an initial angle of $\theta_0 = 0.2$, an initial angular velocity of $\omega_0 = 0$, and a pendulum length of $\ell = 1$ m.

Of course, we can easily verify that the solution to the equation of motion is

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \tag{73}$$

given the initial conditions above. $k = \sqrt{g/\ell}$ is called the "natural" frequency of the pendulum, the frequency of the pendulum swing if gravity is the only force acting on it.

Now let's use Euler's method to find the angle as a function of time, for $t = 0$ to $t = 10$ s, with a time step of $h = 0.04$ s:

$$\theta_i = \theta_{i-1} + h\omega_{i-1} \tag{74}$$

$$\omega_i = \omega_{i-1} - \frac{hg}{\ell}\theta_{i-1} \tag{75}$$

Example: Pendulum with Euler 1

Euler's method gives a solution with the feature we mentioned previously for oscillatory systems: the amplitude grows with time. This means the energy of the system grows with time, and is therefore not conserved.

Section 4, Exercise 3

Let's consider another way to program Euler's method in this case. Recall that we wrote a system of simultaneous first-order equations like this:

$$\frac{d\vec{r}}{dt} = \vec{f}(\vec{r}, t) \tag{76}$$

and Euler's method for one step is written as

$$\vec{r}(t + h) = \vec{r}(t) + h\vec{f}(\vec{r}, t) \tag{77}$$

For this problem,

$$\vec{r} = (\theta, \omega) \tag{78}$$

$$\frac{d\vec{r}}{dt} = \left(\frac{d\theta}{dt}, \frac{d\omega}{dt}\right)$$

$$\vec{f}(\vec{r}, t) = (\omega, -\frac{g}{\ell}\theta)$$

$$(f_\theta, f_\omega) = (\omega, -\frac{g}{\ell}\theta) \tag{79}$$

This leads to these equations for Euler's method in repeated steps:

$$\vec{r}_i = \vec{r}_{i-1} + h\vec{f}(\vec{r}_{i-1}, t_{i-1}) \tag{80}$$

Let's use the vector equations to program Euler's method.

Example: Pendulum with Euler 2

Now that we see we can get identical results using the vector equations, let's solve the same problem, but using the fourth-order Runge-Kutta method. The RK4 method for one step is written:

$$\vec{k}_1 = h\vec{f}(\vec{r}, t) \tag{81}$$
$$\vec{k}_2 = h\vec{f}(\vec{r} + \vec{k}_1/2, t + h/2) \tag{82}$$
$$\vec{k}_3 = h\vec{f}(\vec{r} + \vec{k}_2/2, t + h/2) \tag{83}$$
$$\vec{k}_4 = h\vec{f}(\vec{r} + \vec{k}_3, t + h) \tag{84}$$
$$\vec{r}(t + h) = \vec{r}(t) + \frac{1}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4) \tag{85}$$

Let's apply this to the pendulum.

Example: Pendulum with RK4

Using RK4 solves the problem of the increasing amplitude, so we will continue to use it. Now we're ready to introduce some other effects into the calculation. First, let's no longer assume the small-angle approximation, so that

$$\vec{r} = (\theta, \omega) \tag{86}$$
$$\frac{d\vec{r}}{dt} = \left(\frac{d\theta}{dt}, \frac{d\omega}{dt}\right)$$
$$\vec{f}(\vec{r}, t) = (\omega, -\frac{g}{\ell}\sin\theta)$$
$$(f_\theta, f_\omega) = (\omega, -\frac{g}{\ell}\sin\theta) \tag{87}$$

In this case, it is called the "non-linear" pendulum due to the $\sin\theta$ in the equation.

Example: Non-linear Pendulum

Now we consider the effect of friction, which is typically modeled as being proportional to the velocity:

$$F_f = \beta\frac{d(\ell\theta)}{dt} \tag{88}$$

where β is the damping coefficient. The equation of motion becomes

$$m\frac{d^2(\ell\theta)}{dt^2} = -mg\sin\theta - \beta\ell\frac{d\theta}{dt}$$
$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta - \frac{\beta}{m}\frac{d\theta}{dt} \tag{89}$$

This damping force also gets a negative sign, since it's directed opposite to the positive direction of θ.

Our problem can now be set up this way:

$$\vec{r} = (\theta, \omega) \tag{90}$$
$$\frac{d\vec{r}}{dt} = \left(\frac{d\theta}{dt}, \frac{d\omega}{dt}\right)$$
$$\vec{f}(\vec{r}, t) = (\omega, -\frac{g}{\ell}\sin\theta - \frac{\beta}{m}\omega)$$
$$(f_\theta, f_\omega) = (\omega, -\frac{g}{\ell}\sin\theta - \frac{\beta}{m}\omega) \tag{91}$$

Let's look at the results for various values of β/m.

Example: Damped Non-linear Pendulum

Small values of β result in an oscillation that is called "underdamped," where the system oscillates, but with a decreasing amplitude. Large values of β result in an oscillation that is called "overdamped," where the system exponentially decays back to equilibrium without oscillating. The decay time increases as β increases. There is a transition value which leads to "critical damping," where the system returns to equilibrium as quickly as possible without oscillating. In our example, $\beta/m = 5\ \text{s}^{-1}$ is close to the critical damping value.

Section 4, Exercise 4

So far we have assumed that once the pendulum is released from rest, and the only forces acting on it are gravity and the damping force. Now let's include a driving force, a force that is applied to the pendulum to keep it moving. We will assume the driving force is sinusoidal in time:

$$F_d = F_0 \sin(k_d t) \tag{92}$$

The equation of motion becomes:

$$m\frac{d^2(\ell\theta)}{dt^2} = F_0\sin(k_d t) - mg\sin\theta - \beta\ell\frac{d\theta}{dt}$$
$$\frac{d^2\theta}{dt^2} = \frac{F_0}{\ell m}\sin(k_d t) - \frac{g}{\ell}\sin\theta - \frac{\beta}{m}\frac{d\theta}{dt} \tag{93}$$

We adjust the definition of the vector function $\vec{f}$ to include the driving force:

$$\vec{r} = (\theta, \omega) \tag{94}$$
$$\frac{d\vec{r}}{dt} = \left(\frac{d\theta}{dt}, \frac{d\omega}{dt}\right)$$
$$\vec{f}(\vec{r}, t) = (\omega, \frac{F_0}{\ell m}\sin(k_d t) - \frac{g}{\ell}\sin\theta - \frac{\beta}{m}\omega)$$
$$(f_\theta, f_\omega) = (\omega, \frac{F_0}{\ell m}\sin(k_d t) - \frac{g}{\ell}\sin\theta - \frac{\beta}{m}\omega) \tag{95}$$

First, let's consider a driven, damped pendulum in the small angle approximation again. (We'll let $\sin\theta \to \theta$ in our function $\vec{f}$.) We'll consider different values of k_d, the "driving" frequency of the pendulum, relative to the natural frequency k.

In both examples, $\ell = 1$ m, $F_0/(\ell m) = 0.2$ s^{-2}, $\theta_0 = 0.2$, and $\omega_0 = 0$. The natural frequency of the oscillation (the frequency with no driving force) is $k = \sqrt{g/\ell} \approx 3.13$ s^{-1}.

Example: Driven Pendulum

In the first plot, we let $\beta/m = 1$ s^{-1} and $k_d = 2$ s^{-1}. For the first couple of cycles, we see a damped oscillation at the natural frequency, and then the driving force takes over, and we see a steady-state oscillation at the driving frequency.

In the second plot, we let $\beta/m = 0.1$ s^{-1} and $k_d = 1.01k$, 1% larger than the natural frequency. When the driving frequency is close to the natural frequency, we see "resonance". The steady state solution (after enough time passes) is an oscillation at the driving frequency with an amplitude

$$A = \frac{F_0/(\ell m)}{\sqrt{(k^2 - k_d^2)^2 + ((\beta/m)k_d)^2}} \tag{96}$$

(This amplitude is indicated by the red lines in the plot.) When $k_d \approx k$ and friction (β/m) is small, the amplitude can become large.

Now let's look at some results for a driven and damped non-linear pendulum. We'll assume $g/\ell = 1$ so that the natural frequency $k = 1$ s^{-1}, $\beta/m = 0.5$ s^{-1}, $k_d = 2/3$ s^{-1}, and initial condition $\omega_0 = 0$. We'll consider different values of $F_0/(\ell m)$ and the initial condition θ_0.

Example: Driven Damped Non-linear Pendulum 1

When the driving force is zero, we have a damped oscillation that decays to zero after a couple of periods (we studied this behavior above). When the driving force is small, the curve has an unusal shape at the beginning because we have an initial oscillation (due to the non-zero starting position of the pendulum) that exhibits damped behavior. But then the driving force takes over, and the pendulum falls into a steady state oscillation at the driving frequency k_d (much like in the case of the driven linear pendulum we studied above). When the driving force is larger, the behavior is very different.

Let's talk about the scale. When $\theta = 2\pi$, this means the pendulum has made one full revolution. If θ continues to increase past 2π, this means it continues to rotate in the same direction, with $\theta = 4\pi$ being two full revolutions, $\theta = 6\pi$ being three full revolutions, and so on. When θ has a local minimum or maximum, these are the points at which the pendulum slows down and then changes directions (begins rotating the other way).

For the larger values of the driving force, we don't see a periodic structure; the pattern never repeats. This is an example of *chaotic* behavior. A system can obey deterministic laws but still exhibit behavior that is unpredictable due to extreme sensitivity to the initial conditions. The deterministic nature does not make the system predictable. Sometimes it is also called *deterministic chaos*.

Let's see another example, where we keep the value of $F_0/(\ell m)$ constant, but use different values of θ_0.

Example: Driven Damped Non-linear Pendulum 2

So we see that very slight changes in the initial value of θ creates a large difference.

Let's look at the difference in the angular positions of the two pendulums with slightly different starting conditions, in the case of both small values and large values of the driving force.

Example: Pendulum Difference Plot

First consider the difference plot for a small value of the driving force. The dips in this plot occur when the two pendulums cross paths. Ignoring the dips, we see that the difference in the angular position of the two pendulums decreases over time at the small value of the driving force. The motion of the two pendulums becomes more similar over time. This system is predictable, because we could predict the position of the $\theta_0 = 0.201$ pendulum with high accuracy based on the position of the $\theta = 0.200$ pendulum, since the motion of the two pendulums converges to the same solution over time.

However, for the large value of the driving force, we see the difference in the angular position of the two pendulums generally increases over time; the motion of the two pendulums diverges. The dashed line shows the general trend of the increasing difference (if you did this for different starting values, keeping $\Delta\theta(t = 0)$ fixed, and averaged the results, the result would be similar to the dashed line). Note that the y-axis on these plots is a log scale, so a line on this plot represents an exponential function, so that $\Delta\theta \sim e^{\lambda t}$. This means that the trajectories diverge exponentially for very slight differences in initial conditions. This is characteristic of chaos. The behavior at high driving force is deterministic, but unpredictable since one could not know the initial conditions to infinite precision.

The parameter λ is known as the "Lyapunov" exponent. It is positive for chaotic systems and negative for non-chaotic systems. The transition to chaos occurs when $\lambda = 0$.

In chaos theory, it is standard to look at the system in terms of a "phase-space" plot. In this case, that means plotting ω vs θ. It is useful now to restrict θ to be in the interval $-\pi \to \pi$; since θ is periodic, $\theta = 3\pi$ is the same physical location as $\theta = \pi$, etc. We can accomplish this in our program by adding 2π if θ becomes less than $-\pi$, and subtracting 2π if θ becomes larger than π. We will consider different values of $F_0/(\ell m)$ and θ_0.

Example: Phase Space Plot

When the driving force is small, we have the strange shape in the middle as the oscillation due to the initial displacement of the pendulum damps, and then we settle into a steady state. For a small driving force, a slight change in the initial condition does not visibly alter the phase space plot. When the driving force is large, we get chaotic behavior. A slight change in the initial condition does visibly change the phase space plot. However, the phase-space plot does exhibit some clear (though complicated) features. Despite the fact that the θ vs time plot looks random, we can find patterns in the phase space plot. Chaotic systems generally show structure in their phase space plots.

You can find more structure if you sample the points of phase space plot in time with the driving force. The angular frequency (in rad/s) of the driving force is given by k_d. This means the frequency (in Hz or 1/s) is $k_d/(2\pi)$, and the period is $T = (2\pi/k_d)$ seconds. We will only plot points on the phase space plot if $t_{\text{sample}} = nT$ where n is an integer. (In practice, I did this by calculating all the points up to 15,000 seconds, but only plotting the points if the time was within 0.01 second of a value of t_{sample}. I also restricted the time to be greater than 30 s, since we saw in the previous examples that it takes almost that long for the system to settle into a steady state in the case of small driving force.) This curve is known as an attractor.

Example: Attractor

For the case of small driving force, where after 30 seconds, the system settles into the steady state, the attractor is simply a point. Consider the elliptical phase space plot. It takes one period of the driving force to fill in the entire ellipse. Therefore, if you sample the ellipse only once every period, you get the same combination of θ, ω values each time. For the larger value of the driving force, where we see chaotic behavior, notice that the points on the attractor are the same for both initial conditions. While the θ vs time plot looked random, and even the full phase space plot was different for the slightly different initial conditions, the attractor is the same! Even though we cannot predict the behavior of $\theta(t)$, we can predict that the system will be "attracted" to this curve (or surface if in more than 2 dimensions), regardless of initial conditions. In this chaotic case, this surface is known as a chaotic attractor, or more often a strange attractor.

3 Lorenz Model

Now we turn to a classic problem in chaos theory, the Lorenz equations. They were first studied by Edward Lorenz, while studying a simplfied model of atmospheric convection. (You can find his 1963 paper here: `https://journals.ametsoc.org/view/journals/atsc/20/2/1520-0469_1963_020_0130_dnf_2_0_co_2.xml`.) The equations represent the properties of a fluid layer when the top and bottom of the layer are held at different temperatures:

$$\frac{dx}{dt} = \sigma(y - x) \tag{97}$$

$$\frac{dy}{dt} = rx - y - xz \tag{98}$$

$$\frac{dz}{dt} = xy - bz \tag{99}$$

where σ, r, and b are constants. x is proportional to the intensity of the convective motion, y is proportional to the temperature difference between the ascending and descending currents (warm fluid rises, cold fluid falls). z is proportional to the distortion of the vertical temperature profile from a linear profile.

σ is called the Prandtl number, which is the ratio of the mometum diffusivity (or kinematic viscosity) to the thermal diffusivity. Kinematic viscosity is the viscosity over the density of the material, where viscosity is something like the internal friction between adjacent layers of fluid. The Prandtl number gives a measure of whether heat conduction is more or less significant than convection in a fluid. (Note that air is considered a fluid.)

r is the Rayleigh number, which describes the behavior of fluids when the mass density of the fluid is non-uniform; it is also related to conduction and convection in a fluid. When the number is below a critical value, there is no flow and heat transfer is purely by conduction. When the number is above the critical value, heat is transferred by convection. It is a measure of the temperature difference between the top and bottom surfaces of the fluid.

Finally, b is a parameter related to the physical size of the layer.

Setting up what we need to write the program:

$$\vec{r} = (x, y, z) \tag{100}$$

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\vec{f}(\vec{r}, t) = (\sigma(y - x), rx - y - xz, xy - bz)$$

$$(f_x, f_y, f_z) = (\sigma(y - x), rx - y - xz, xy - bz) \tag{101}$$

Customary values of σ and b are $\sigma = 10$ and $b = 8/3$. We'll consider different values of r. Note that $x = y = z = 0$ is a solution to the equations, and represents a steady state of no convection. Therefore we must consider initial conditions where at least one function is nonzero at $t = 0$.

Example: Lorenz Model

The value of r plays a similar role to the amplitude of the driving force in the pendulum example. At small values, there is an initial oscillation that decays to zero. These solutions represent steady convective motion in the fluid, the analog of nonchaotic motion of the pendulum. With a larger value of r, we see the chaotic behavior - a time dependence that doesn't repeat in a pattern.

The equations of fluid mechanics are called the Navier-Stokes equations and, like most equations in Physics, are difficult to solve analytically for any situation of practical interest. Lorenz developed his equations as a greatly simplified version of the Navier-Stokes equations, which he then solved computationally. Given the greatly increased computational power in the modern era, modern atmospheric weather models are much more complex and realistic. That being said, the simple model provided by Lorenz indicates that chaos is a property of virtually all fluid systems. This means that weather prediction is exceedingly complicated and very sensitive to the initial conditions. (Think about this the next time a hurricane is headed into the Gulf, and you wonder why there's such a large uncertainty on its path!) Lorenz gave a talk with the title, "Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?" This is the so-called "butterfly effect" that you have no doubt heard of in popular culture. (Newman, our textbook's author, notes that Ray Bradbury wrote a short story in 1952 where a time traveler's destruction of a butterfly changes the coures of history, so one could argue it was his idea, but Lorenz coined the term.) People also note that the phase space plot for solutions to the Lorenz model resembles a butterfly.

An attractor plot can be constructed for the Lorenz model. Since there's no characteristic frequency in the Lorenz model (like the driving force in the pendulum problem), the attractor must be constructed in a different way. The attractor can be seen by making a plot of, for example, z vs. y, when $x = 0$, i.e. selecting points in time $x = 0$ and plotting the values of z and y at those times.

Lorenz Attractor

Section 4, Exercise 5