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硕士学位论文

论文题目　种群动态模型的马氏骨架方法

学科、专业　概率论与数理统计

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摘 要

本文采用马尔可夫骨架过程的方法来研究地震等强自然灾害发生的条件下单种群种群数量的变化。其核心内容借助于马尔可夫骨架过程(MSP)方法研究了种群动态学中单种群种群数量的瞬时分布。马尔可夫骨架过程是由侯振挺教授及其同事们于 1997 年首次提出的一类随机过程，它包含了许多已有的随机过程模型，如马尔可夫过程、半马尔可夫过程、逐段决定马氏过程、Doob 过程、再生过程、半再生过程等一系列经典的随机过程，具有重要的理论和应用价值。

在以往的种群数量研究中，生物种群数量的变化过程通常由微分方程或差分方程的解给出，即由一条连续的光滑曲线，或一条右连左极的阶梯曲线（离散时间的连续延拓）来描述,例如著名的具有密度制约的单种群 Logistic 模型。上述模型在局部或者相对较短的时期内可行。但当从整体来看或者相对较长的时期内，例如种群数量在特殊情况下发生了突变，它们就存在了明显的缺陷。而马氏骨架从概率论的角度解决了这个矛盾。在马氏骨架理论中，我们把发生突变的时刻当成是停时列的一个停时$\tau_n(n\ge1)$，再补充新的变量以寻求一个马尔可夫骨架过程，即种群数量发生变化的时刻$\tau_n(n\ge1)$为马尔可夫骨架过程在时间轴上的第n个间断点，然后研究单种群种群数量在任意时刻t的瞬时分布。例如，本文的模型就考虑了发生重大自然灾害的情况下种群数量的变化状况。

在本文中，先由两个引理给出了马尔可夫骨架过程的向后方程中

h和q的表达式，接着给出两个定理，分别研究了离散和连续两种条件下单种群种群数量瞬时分布，证明了单种群种群数量在时刻t的瞬时分布是某一非负线性方程的最小非负解。

关键字：马尔可夫骨架过程，单种群种群数量，最小非负解

ABSTRACT

In the thesis, we make use of the method of Markov skeleton processes in order to study the change of single-species population number under the disaster such as earthquakes. The clou recurs to the method of Markov skeleton processes to study the instantaneous distribution of single-species population number on population dynamics.Markov skeleton processes are a kind of comprehensive stochastic processes, which are firstly put forward by Prof.Hou Zhengting and his colleagues in 1997. The processes contain many exist classical stochastic processes models, such as Markov processes、semi-Markov processes、piecewise deterministic Markov processes、Doob processes、regenerative processes、semi-regenerative processes. They have important value in theory and application.

In most formal studies, the change of the population number is usually produced by differential equation or difference equation, namely, is discribed by a continuous smooth curve or a ladder curve (the continuous continuation of discrete time) which is right-continuous with left limits, such as the famous single-species Logistic model with density dependence. But when all comes to all or in a relatively long time, for example, when the population number are changed abruptly under the exceptive circumstances, there are distinct limitations in these models. However, we can put the axe in the helve by means of the method of Markov skeleton processes from the aspect of probability theory.In the theory of Markov skeleton processes,we can consider the time of break as a stopping time $\tau_n(n \ge 1)$ of the series.Then,in order to seek a Markov skeleton process,we supplement new variables.Thus,the time $\tau_n(n \ge 1)$ at which the population number changes is the discontinuity point of Markov skeleton processes at the time axis.At last,we study the instantaneous distribution of single-species population at random time t.For example, in the model of this thesis, when studying the change of the population number, we take the occurrence of the disaster into account.

In this thesis, firstly, we bring forward the expressions of h and q in the backward equation of Markov skeleton processes via Lemma 3.3.1 and Lemma 3.3.2. Sencondly, we put forward Theorem 3.3.1 and Theorem 3.3.2, studying the instantaneous distribution of single-species population under discrete state and continuous condition respectively, and proving that the instantaneous distribution of single-species population number at time t is the smallest non-negative solution of some non-negative linear equation .

Keywords: Markov skeleton processes, single-species population number, the smallest non-negative solution

目录

第一章 绪论

1.1 研究背景

长江上游的温带森林区，是全球34个生物多样性区域，是世界上为数不多的诺亚方舟之一。东邻成都盆地，西接青藏高原，喜玛拉雅山缓慢的造山运动中，中国西南山区的上升在急速和平缓中交替。青藏高原阻挡了东南季风，形成了西南多云潮湿的封闭环境，成为大熊猫、小熊猫、金丝猴、牛羚等多种濒危物种的天堂，中国50%的鸟类、哺乳动物，以及30%的高等植物生活在此。然而这个多样性区域又是和高风险区域相互重叠。

发生在2008年5月12号的汶川大地震，不仅对我国人民的生命财产安全造成破坏性影响，也对西南地区的野生种群的生存环境造成威胁。据四川省林业厅公布的初步调查结果，汶川大地震不仅造成野生动物大量死亡，而且形成了严重的生存环境隔离，山区已形成若干个“孤岛”，可能令丰富的生物多样性资源丧失。专家认为，地震可能加剧野生动物生存环境破碎化，而地震产生的山体滑坡、泥石流，会让已经割裂的动物栖息地更小更分散，对濒危物种将是很大的威胁。

1.2 研究现状

以上小节提出的问题就是种群生态学问题。其核心是种群动态研究。所谓种群动态（population dynamic）是指研究种群数量在时间上和空间上的变动规律及其变动原因（调节机制)。在数学生态的文献中，大多将描述生物种群数量变化的生态模型分为两类：1.生命长、世代重叠而且数量很大的种群，常常近似用连续过程来描述，通常表为微分方程，主要是用微分方程的理论和方法来研究。2.生命短、世代不重叠的种群，或者虽然生命长、世代重叠但数量比较少的种群，常用不连续过程来描述，通常表为差分方程，主要用差分方程的方法来研究。在上述两类模型中，通常把种群之间的影响以及环境对种群的影响归结到模型的参数中和差分方程来进行研究。因此种群数量的变化过程可由微分方程或差分方程的解给出，即由一条光滑曲线，或者一条右连左极的阶梯曲线来描述。从局部来看，或者说在一段相对较短的时期内，用上述模型来描述生物种群数量的发展和变化是可行的。但从整体看来，或者说在一段相对较长的时期内，用上述模型来

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描述生物种群的变化就存在明显的缺陷。例如，以上提到的汶川大地震的突然发生，无论是当地人口还是野生物数量都发生了突然下降。我们把类似于这种变化称为突变，它具有以下特征：其一这种突变发生的时刻是不确定的，或者说是随机的。其二，突变的强度一般来说也是不确定的，或者说是随机的。

由于以上原因，本文采用马尔可夫骨架过程的方法来研究地震等强自然灾害发生的条件下单种群种群数量的变化。我们把发生突变的时刻当成是停时列的一个停时$\tau_n(n\geq 1)$，再补充新的变量以寻求一个马尔可夫骨架过程，即种群数量发生变化的时刻$\tau_n(n\geq 1)$为马尔可夫骨架过程在时间轴上的第n个间断点，然后研究单种群种群数量在任意时刻t的瞬时分布。其核心内容借助于马尔可夫骨架过程(MSP)方法研究了种群动态学中单种群种群数量的瞬时分布，并证明单种群种群数量在时刻t的瞬时分布是某一非负线性方程的最小非负解。

1.3 论文的主要内容和结构

本文主要研究了种群动态学中单种群种群数量的瞬时分布。整个论文主要分为以下三部分：

第一章是绪论，主要介绍论文的选题背景和研究现状，以及本文的主要内容和结构；

第二章是基础知识，主要介绍了马尔可夫骨架过程的概念，其向前向后方程，正则性准则和有限维分布，其中部分定理和命题笔者按照自己的理解给出了证明。

第三章是马氏骨架过程方法在种群动态学中的应用，是本文的核心。先出两个引理给出了马尔可夫骨架过程的向后方程中h和q的表达式，接着给出两个定理，分别研究了离散和连续两种条件下单种群种群数量瞬时分布，证明了单种群种群数量在时刻t的瞬时分布是某一非负线性方程的最小非负解。

第二章 基础知识

2.1 马尔可夫骨架过程的概念

设(E,ε)是一可测空间，$X=\{X(t,\omega),0\le t<\tau\}$是定义在完备的概率空间$(\Omega,F,P)$上取值于$(E,\varepsilon)$寿命为$\tau$的随机过程，$\{F_t^X,t\ge 0\}$是由$X$生成的自然$\sigma-$代数流。

状态空间E中加入一个孤立点Δ，将其扩充到$\hat{E}=:E\cup\{\Delta\}$，对应过程X也扩展到$\hat{X}=:\{\hat{X}(t,\omega),0\le t<\tau\}$，其中

$$\hat{X}(t,\omega)=\begin{cases}X(t,\omega),0\le t<\tau(\omega)\\ \Delta,\tau(\omega)\le t<\infty\end{cases}$$

即过程在时间$\tau(\omega)$之后，被吸收在状态Δ，称之为坟墓点。假设Ω为定义在$IR^+=[0,\infty)$上取值于$\hat{E}$的所有右连续函数生成的空间。

令θ_t为推移算子：$(\theta_t\omega)_s=\omega_{t+s},(\omega_s)_{s\ge0}\in\Omega$.在不引起混淆的情况下，我们将不再区别$X$和$\hat{X}$。

定义 2.1.1 随机过程$X=\{X(t,\omega),0\le t<\tau\}$称为马尔可夫骨架过程，如果存在一停时列$\{\tau_n\}_{n\ge0}$，满足

（1）$\tau_0=0$且$\tau_n\uparrow\tau$，并对任意的$n\ge0,\tau_n<\infty\Rightarrow\tau_n<\tau_{n+1}$；

（2）对于一切$n=0,1,\cdots,$有$\tau_{n+1}=\tau_n+\theta_{\tau_n}\tau_1$；

（3）对每个τ_n和任意定义在$E^{[0,\infty)}$上的有界$\varepsilon^{[0,\infty)}-$可测函数，有

$E\left[f(X(\tau_n+\cdot))\middle|F_{\tau_n}^X\right]=E\left[f(X(\tau_n+\cdot))\middle|X(\tau_n)\right],P\text{-}a.s.$,（2.1）

有$\Omega_{\tau_n}=(\omega:\tau_n(\omega)<\infty)$，$F_{\tau_n}^X=\{A:\forall t\ge0,A\cap(\omega:\tau_n\le t)\in F_t^X\}$

是Ω_{τ_n}上的$\sigma-$代数。我们把$\{\tau_n\}_{n=0}^{\infty}$叫做马尔可夫骨架过程$X$的骨架时序列。

进而，如果在Ω_{τ_n}上

$$E[f(X(\tau_n)+\cdot))|F^X_{\tau_n}]=E[f(X(\tau_n+\cdot))|X(\tau_n)]=E_{X(\tau_n)}[f(X(\tau_n))]P\text{-}a.s. \qquad (2.1')$$

成立，则称X是时齐马尔可夫骨架过程,记为MSP。在这里$E_x(\cdot)$表示对应于$P(\cdot|X(0)=x)$的期望。

注 2.1.1 本文中，设E为 Polish 空间，是 Borel$\sigma-$代数,Ω是定义在$IR^+=[0,\infty)$取值E的右连续函数空间，取值于E的右连续随机过程$X=\{X(t,\omega),0\le t<\infty\}$。设$F_\infty=\vee_{t=0}^{\infty}F_t^X$。假设存在$(\Omega,F)$上的一族概率测度$P_x,x\in E$，满足$\forall A\in F_\infty,x\to P_x(A)$是$\varepsilon$一可测，$\forall x\in E$，$P_x(A)=P(A|X_0=x),\forall A\in F_\infty$。对任意$(E,\varepsilon)$上的概率测度$\mu$，我们定义$(\Omega,F_\infty)$上的概率测度$P_\mu$如下：

$\forall A\in F_\infty,P_\mu(A)=\int_E P_x(A)\mu(dx)$. 令$F_t^\mu$为$F_t^X$关于测度$P_\mu$的完备化，并且$F_t=\cap_{\mu\in P(E)}F_t^\mu,t\ge 0$，这里$P(E)$为$(E,\varepsilon)$上的概率测度的集合。

注 2.1.2 由于 Polish 空间可度量化，可视X是定义在度量空间上的右连续随机过程，所以X关于$\{F_t^X,t\ge 0\}$是循序可测的。故$X(\tau_n)$和$f(X(\tau_n+\cdot))$是可测的，这里f是$\left(E^{[0,\infty)},\varepsilon^{[0,\infty)}\right)$上的可测函数。设$X=\{X(t,\omega),0\le t<\infty\}$是一马尔可夫骨架过程。令$\eta_n=(\sigma_n,X(\tau_n),n\ge 0)$，这里$\sigma_0=0$, $\sigma_n=\tau_n-\tau_{n-1},n\ge 1$（约定：$\infty-\infty=0$），则$\eta_n,n\ge 0$是取值于可测空间$(IR^+\times E,B(IR^+)\times\varepsilon)$的随机变量序列。

命题 2.1.1 如果$\{X(t);t\ge 0\}$是以$\{\tau_n\}_{n=0}^{\infty}$为骨架时序列的马尔可夫骨架过程，则$\tau_{n+1}-\tau_n$是$\sigma(X(\tau_n+t);t\ge 0)$—可测的。

证明：由是一停时列可知它F_∞可测，从而存在一可测函数f和一序列$\{t_1,t_2,\cdots\}$，使得$\tau_1(\omega)=f(X(t_1,\omega),X(t_2,\omega),\cdots)$,故由$\tau_{n+1}-\tau_n=\theta_{\tau_n}\cdot\tau_1=f(X(\tau_n+t_1),X(\tau_n+t_2),\cdots)$得$\tau_{n+1}-\tau_n$是$\sigma(X(\tau_n+t);t\ge 0)$可测。

定义 2.1.2 设G为的一个子σ代数，(U,υ)为一可测空间，若ξ为在(U,υ)中取值的随机元。如果$\{k(\omega,A),\omega\in\Omega,A\in\upsilon\}$是$(U,\upsilon)$上的一簇概率测度，满足条件：(1)

$\forall A\in\upsilon, k(\cdot,A)$是$\Omega$上$G$可测函数;（2） $\forall A\in\upsilon, k(\cdot,A)$是$P(\xi^{-1}(A)|G)$的一个版本，

即$\forall B\in G,\ \int_B k(\omega,A)P(d\omega)=P(B\cap\xi^{-1}(A))$.

则称$\{k(\omega,A),\omega\in\Omega,A\in\upsilon\}$为$\xi$ 关于G的混合条件分布。

定理 2.1.1 若(U,υ)为一个 Radon 可测空间，则对于任意的取值于(U,υ)的随机元及F的子σ代数G，存在关于G的混合条件分布。

定理 2.1.2 如果Y是取值于可测空间(V,ν)的随机元,ξ 是取值于(U,υ)的随机元,(U,υ)为一个 Radon 可测空间,则存在$V\times U$上的（混合）条件分布$K(y,A)$,使

（i） 对于固定的$y\in V, K(y,\cdot)$是(U,υ)上的概率测度;

(ii) 对于固定的$A\in\upsilon, K(\cdot,A)$是V上的ν可测函数;

（iii）对于固定的$A\in\upsilon$, $K(y,A)$是$P(\xi^{-1}(A)|\sigma(Y))$的一个版本。

注 2.1.3 由于 Polish 空间是 Radon 可测空间，故上述二定理对于取值于 Polish 空间的随机元亦成立。对于马尔可夫骨架过程$\{X_t,0\le t<\tau\}$来说，由于其轨道右连续，所以$X(\tau_n),n=1,2,\cdots$是取值于(E,ε)的随机变量，$(\tau_n,X(\tau_n))$是取值于 Polish 空间$IR^+\times E$的随机变量，于是由定理 2.1.2 知，关于随机变量

$X(0)$的条件分布$q^{(n)}(x,ds,dy)\equiv P(\tau_n\in ds,X_{\tau_n}\in dy|X(0)=x)$存在。对于任意的$x\in E,t\ge 0,q^{(n)}(x,[0,t),dy)$是$(E,\varepsilon)$上的测度，记作$q^{(n)}(x,t,dy)$。又由过程的齐次性及定理 2.1.2 易证

定理 2.1.3 $q^{(n)}(x,t,dy)$, $x\in E,t\ge 0,\ A\in\varepsilon$满足以下条件:

（i）对于固定的$A\in\varepsilon$, $q^{(n)}(\cdot,\cdot,A)$是$E\times IR^+$上的$\varepsilon\times B(IR^+)$可测的函数;

（ii）对于固定的$x\in E,t\ge 0,q^n(x,t,\cdot)$是$(E,\varepsilon)$上的有限测度;

（iii）对于任意的$t\ge 0$, $A\in\varepsilon$和$m\in Z^+=\{0,1,2,\cdots\}$

$$q^{(n)}(X(\tau_m),t,A)=P\{X(\tau_{m+n})\in A,\tau_{m+n}\le t|X(\tau_m)\},P-a.s.$$

其中$q^{(1)}(x,t,A)$简记作$q(x,t,A)$, $q^{(1)}(x,dt,dy)$简记作$q(x,dt,dy)$

命题 2.1.2 对于任意的$n\in N,t\geq 0,A\in\varepsilon,x\in E,$

$$q^{(n+1)}(x,t,A)=\int_E\int_0^t q^{(n)}(x,ds,dy)q(y,t-s,A)$$
$$=\int_E\int_0^t q(x,ds,dy)q^{(n)}(y,t-s,A)$$

证明： 由条件概率的基本性质

$$q^{(n+1)}(x,t,A)$$
$$=P(X(\tau_{n+1})\in A,\tau_{n+1}\leq t\,|\,X(0)=x)$$
$$=\int_E\int_0^t P(X(\tau_{n+1})\in A,\tau_{n+1}\leq t\,|\,X(\tau_n)=y,\tau_n=s,X(0)=x)$$
$$P(X(\tau_n)\in dy,\tau_n\in ds\,|\,X(0)=x)$$
$$=\int_E\int_0^t P(X(\tau_n+\theta_{\tau_n}\tau_1)\in A,\theta_{\tau_n}\tau_1\leq t-s\,|\,X(\tau_n)=y,\tau_n=s)$$
$$P(X(\tau_n)\in dy,\tau_n\in ds\,|\,X(0)=x)$$
$$=\int_E\int_0^t P(X(\tau_1)\in A,\tau_1\leq t-s\,|\,X(0)=y)P(X(\tau_n)\in dy,\tau_n\in ds\,|\,X(0)=x)$$
$$=\int_E\int_0^t q(y,t-s,A)q^{(n)}(x,ds,dy)$$

又有 $q^{(n+1)}(x,t,A)$

$$=P(X(\tau_{n+1})\in A,\tau_{n+1}\leq t\,|\,X(0)=x)$$
$$=\int_E\int_0^t P(X(\tau_{n+1})\in A,\tau_{n+1}\leq t\,|\,X(\tau_1)=y,\tau_1=s,X(0)=x)$$
$$P(X(\tau_1)\in dy,\tau_1\in ds\,|\,X(0)=x)$$
$$=\int_E\int_0^t P(X(\tau_{n+1})\in A,\tau_{n+1}\leq t\,|\,X(\tau_1)=y,\tau_1=s)q(x,ds,dy)$$
$$=\int_E\int_0^t P(X(\tau_n)\in A,\tau_n\leq t-s\,|\,X(0)=y)q(x,ds,dy)$$
$$=\int_E\int_0^t q^{(n)}(y,t-s,A)q(x,ds,dy)$$

所以定理得证。

2.2 向前向后方程

定义 2.2.1 称时齐的马尔可夫骨架过程$X=\{X(t,\omega),0\leq t<\infty\}$是正规的，如果存在$E\times R^{+}\times\varepsilon$上的函数$h(x,t,A)$，使得

（i）对固定的$x,t,h(x,t,\cdot)$是ε上的有限测度；

（ii）对固定的$A\in\varepsilon,h(\cdot,\cdot,A)$是$E\times R^{+}$上的$\varepsilon\times B(R^{+})$可测函数；

（iii）对于任意的$t\geq 0,A\in\varepsilon$

$$h(X(\tau_n),t,A)=P\{X(t)\in A,t<\tau_{n+1}-\tau_n\,|X(\tau_n)\}P-a.s.$$

$$h(x,t,A)=P\{X(t)\in A,t<\tau_{n+1}-\tau_n\,|X(\tau_n)=x\}$$

$$=P\{X(t)\in A,t<\tau_1\,|X(0)=x\} \tag{2.2}$$

令 $$p(x,t,A)=P\{X(t)\in A|X(0)=x\}$$

定理 2.2.1 设$\{X(t);t\geq 0\}$是以$\{\tau_n\}_{n=0}^{\infty}$为骨架时序列的正规马尔可夫骨架过程，则对于任意的$x\in E,t\geq 0,A\in\varepsilon$，有

$$p(x,t,A)=h(x,t,A)+\int_E\int_0\sum_{n=1}^{\infty}q^{(n)}(x,ds,dy)h(y,t-s,A) \tag{2.3}$$

从而$p(x,t,A)$是如下非负方程组的最小非负解：

$$p(x,t,A)=h(x,t,A)+\int_E\int_0 q(x,ds,dy)p(y,t-s,A) \tag{2.4}$$

对于任意$x\in E,t\geq 0,A\in\varepsilon$，其中$q^{(n)}(x,ds,dy),q(x,ds,dy)$为上节定义的$(E,\varepsilon)$上的测度族。

证明：对于任意的$x\in E,t\geq 0,A\in\varepsilon,n\in N$

$$P(X(t)\in A,\tau_n\leq t<\tau_{n+1}|X(0)=x)$$

$$=\int_E\int_0 P(X(t)\in A,\tau_n\leq t<\tau_{n+1}\,|X(\tau_n)=y,\tau_n=s,X(0)=x)$$

$$P(X(\tau_n)\in dy,\tau_n\in ds|X(0)=x)$$

$$=\int_E\int_0 P(X(t-s+\tau_n)\in A,t-s<\theta_{\tau_n}\tau_1\,|X(\tau_n)=y,\tau_n=s,X(0)=x)q^{(n)}(x,ds,dy)$$

由$\{X(t);t\geq 0\}$的齐次性及$\{X(t);t\geq 0\}$在τ_n处的马尔可夫性立得：

$$P(X(t-s+\tau_n)\in A,t-s<\theta_{\tau_n}\tau_1\,|X(\tau_n)=y,\tau_n=s,X(0)=x)$$

$$=P(X(t-s)\in A,t-s<\tau_1\,|X(0)=y)=h(y,t-s,A)$$

故$P(X(t)\in A,\tau_n\le t<\tau_{n+1}|X(0)=x)=\int_E\int_0^t q^{(n)}(x,ds,dy)h(y,t-s,A)$

从而$p(x,t,A)=P\{X(t)\in A|X(0)=x\}$

$$=P(X(t)\in A,t<\tau_1|X(0)=x)+\sum_{n=1}^{\infty}P(X(t)\in A,\tau_n\le t<\tau_{n+1}|X(0)=x)$$

$$=h(x,t,A)+\int_E\int_0^t\sum_{n=1}^{\infty}q^{(n)}(x,ds,dy)h(y,t-s,A)$$

即$\{p(x,t,A);t\ge 0,x\in E\}$满足方程（2.3）。由最小非负解理论可以知道$\{p(x,t,A)\}$是方程组（2.4）的最小非负解。方程组（2.4）称为正规马尔可夫骨架过程$\{X(t);t\ge 0\}$的向后方程组。$\{h(x,t,A)\}$记作H，$\{q(x,dt,dy)\}$记作Q。由定理 2.2.1 可知$X(t)$的一维分布由(H,Q)唯一决定。

定义 2.2.2 如果存在测度簇$\hat{Q}=\{\hat{q}(x,dt,dy)\}_{t\ge 0}$，使得对于任意的$A\in\varepsilon$, $x\in E,t\ge 0$,

$$\int_E\int_0^t q(x,ds,dy)h(y,t-s,A)=\int_E\int_0^t h(x,t-s,dy)\hat{q}(y,ds,A) \quad (2.5)$$

则称如下方程组：

$$p(x,t,A)=h(x,t,A)+\int_E\int_0^t p(x,t-s,dy)\hat{q}(y,ds,A) \quad A\in\varepsilon,x\in E,t\ge 0 \quad (2.6)$$

为马尔可夫骨架过程$\{X(t);t\ge 0\}$的向前方程组。

令
$$\hat{q}^{(1)}(x,ds,dy)=\hat{q}(x,ds,dy)$$
$$\hat{q}^{(2)}(x,ds,dy)=\int_E\int_0^t\hat{q}^{(1)}(x,ds,dz)\hat{q}(z,t-s,dy)$$
$$\cdots$$
$$\hat{q}^{(n)}(x,ds,dy)=\int_E\int_0^t\hat{q}^{(n-1)}(x,ds,dz)\hat{q}(z,t-s,dy)$$
$$\cdots$$

定理 2.2.2 如果存在满足等式（2.5）的$\hat{Q}=\{\hat{q}(x,dt,dy)\}_{t\ge 0}$，则马尔可夫骨架过程$\{X(t);t\ge 0\}$的转移概率$p(x,t,A)=P\{X(t)\in A|X(0)=x\}$也是向前方程组（2.6）的最小非负解。

证明： 下面用数学归纳法证明，若（2.5）成立，则对于任意的$n\in N$，有

$$\int_E\int_0^t q^{(n)}(x,ds,dy)h(y,t-s,A)=\int_E\int_0^t h(x,t-s,dy)\hat{q}^{(n)}(y,ds,A)$$

当$n=1$时，由（2.5）式已证；当$n=k$时，假设有下面式子成立：

$$\int_E\int_0^t q^{(k)}(x,ds,dy)h(y,t-s,A)=\int_E\int_0^t h(x,t-s,dy)\hat{q}^{(k)}(y,ds,A);$$

当$n=k+1$时，由以上假设及命题 2.1.2：

$$\begin{aligned}
&\int_E\int_0^t q^{(k+1)}(x,ds,dy)h(y,t-s,A)\\
&=\int_E\int_0^t[\int_E\int_0^s q^{(k)}(x,dw,dz)q(z,ds-dw,dy)]h(y,t-s,A)\\
&=\int_E\int_0^t q^{(k)}(x,dw,dz)[\int_E\int_0^t q(z,ds-dw,dy)h(y,t-s,A)]\\
&=\int_E\int_0^t q^{(k)}(x,dw,dz)[\int_E\int_0^t h(z,t-s,dy)\hat{q}(y,ds-dw,A)]\\
&=\int_E\int_0^t[\int_E\int_0^s q^{(k)}(x,dw,dz)h(z,t-s,dy)]\hat{q}(y,ds-dw,A)\\
&=\int_E\int_0^t[\int_E\int_0^s h(x,t-s,dz)\hat{q}^{(k)}(z,dw,dy)]\hat{q}(y,ds-dw,A)\\
&=\int_E\int_0^t h(x,t-s,dz)[\int_E\int_0^s \hat{q}^{(k)}(z,dw,dy)\hat{q}(y,ds-dw,A)]\\
&=\int_E\int_0^t h(x,t-s,dz)\hat{q}^{(k+1)}(z,ds,A)\\
&=\int_E\int_0^t h(x,t-s,dy)\hat{q}^{(k+1)}(y,ds,A)\ \text{即证}
\end{aligned}$$

由定理 2.2.1，对于任意的$A\in\varepsilon,\ x\in E,t\geq 0,$

$$\begin{aligned}
p(x,t,A)&=h(x,t,A)+\int_E\int_0^t\sum_{n=1}^{\infty}q^{(n)}(x,ds,dy)h(y,t-s,A)\\
&=h(x,t,A)+\int_E\int_0^t\sum_{n=1}^{\infty}h(x,t-s,dy)\hat{q}^{(n)}(y,ds,A)
\end{aligned}$$

故$p(x,t,A)$是（2.6）的最小非负解。定理证毕。

令 $h_\lambda(x,A)=\int_0^\infty e^{-\lambda t}h(x,t,A)dt\quad q_\lambda(x,A)=\int_0^\infty e^{-\lambda t}q(x,t,A)dt$

$$p_\lambda(x,A)=\int_0^\infty e^{-\lambda t}P(x,t,A)dt\quad {q_\lambda}^{(n)}(x,A)=\int_0^\infty e^{-\lambda t}q^{(n)}(x,t,A)dt$$

于是由定理 2.2.1 得：

定理 2.2.3 设$X=\{X(t);t\geq 0\}$是以$\{\tau_n\}_{n=0}^{\infty}$为骨架时序列的正规马尔可夫骨架过程，则对于任意的$x\in E,t\geq 0,A\in\varepsilon$，有

$$p_\lambda(x,A)=h_\lambda(x,A)+\int_E\sum_{n=1}^{\infty}q_\lambda{}^{(n)}(x,dy)h_\lambda(y,A) \tag{2.7}$$

从而$\{p_\lambda(x,A)\}$是如下方程的最小非负解：

$$p_\lambda(x,A)=h_\lambda(x,A)+\int_E q_\lambda(x,dy)p_\lambda(y,A) \tag{2.8}$$

方程（2.8）也称为X的向后方程。同样，我们也把与（2.6）等价的方程

$$p_\lambda(x,A)=h_\lambda(x,A)+\int_E p_\lambda(x,dy)\hat{q}_\lambda(y,A) \tag{2.9}$$

称为X的向前方程。其中$\hat{q}_\lambda(x,A)=\int_0^\infty e^{-\lambda t}\hat{q}(x,t,A)dt$

由定理 2.2.1 知正规马尔可夫骨架过程应是我们进一步研究的对象。下面我们给出马尔可夫骨架过程成为正规马尔可夫骨架过程的一个充分条件，这个条件就是过程轨道以概率 1 处处有左极限。在实际应用中，到目前为止，我们所遇到的马尔可夫骨架过程都具有这个性质。在今后的研究中，我们所说的马尔可夫骨架都是指正规马尔可夫骨架过程，不再一一申明。

令$D_E[0,\infty)=\{w:[0,\infty)\to E;w(\cdot)$是$[0,\infty)$上的右连左极的映射$\}$，周知，$D_E[0,\infty)$在 Skorohod 度量下是一个 Polish 空间。令$\{w(t)_{t\geq 0}\}$为$D_E[0,\infty)$上的坐标过程，$B(D_E[0,\infty))$表示$D_E[0,\infty)$上的 Borel 代数。显然$B(D_E[0,\infty))=\sigma\{w(t),t\in IR^+\}$。

引理 2.2.1 对于任意的$A\in\varepsilon$，$I_A(w(t))$是$IR^+\times D_E[0,\infty)$上的$B(IR^+)\times B(D_E[0,\infty))$可测函数。

定理 2.2.4 如果(E,ε)是 Polish 空间，$\{X(t);t\geq 0\}$是取值于(E,ε)且具有右连左极轨道的马尔可夫骨架过程，则$\{X(t);t\geq 0\}$是正规的。

证明 设$\{\tau_n\}_{n=0}^{\infty}$是$\{X(t);t\geq 0\}$的骨架过程时序列。由于的所有轨道是右连左极的，故可以将$\{X(t);t\geq 0\}$看成是取值于$D_E[0,\infty)$的随机元ξ。由定理 2.1.2 和注 2.1.3 存在ξ关于$(X(0),\tau_1)$的混合条件分布$K(x,s,dw)\equiv P(\xi\in dw|X(0)=x,\tau_1=s)$使得

(i) 对于固定的$x\in\varepsilon,s\geq 0,K(x,s,\cdot)$是$D_E[0,\infty)$上的概率测度；

(ii)对于固定的$C\in B(D_E[0,\infty)),K(\cdot,\cdot,C)$是$\varepsilon\times B(IR^+)$可测的；

(iii) $\forall C \in B(D_E[0,\infty)), K(X(0),\tau_1,C)$是$P(\xi \in C|X(0),\tau_1)$的一版本。

对于任意的$A \in \varepsilon$,由引理 2.2.1,$I_A(w(t))$是$IR^+ \times D_E[0,\infty)$上的$B(IR^+)\times B(D_E[0,\infty))$可测函数。

令$\hat{h}(x,s,t,A)=\int_{D_E[0,\infty)} I_A(W(t))K(x,s,dw)$，显然$\hat{h}(x,s,t,A)$满足:

(i) 对于固定的$x,s,t,\hat{h}(x,s,t,A)$是(E,ε)上的概率测度;

(ii)对于固定的$A \in \varepsilon$,则$\hat{h}(x,s,t,A)$是$E\times IR^+ \times IR^+$上的$\varepsilon \times B(IR^+)\times B(IR^+)$可测的函数;

(iii)对于固定的$t \ge 0$, $A \in \varepsilon$, $\hat{h}(x,s,t,A)=P(X(t)\in A|X(0)=x,\tau_1=s)\ P-a.s.$

令$h(x,t,A)=\int^{\infty}\hat{h}(x,s,t,A)\hat{P}(x,ds)$,其中$\hat{P}(x,ds)$是$\tau_1$关于$X(0)$的混合条件分布$P(\tau_1 \in ds|X(0)=x)$。显然有:

(i) 对于$A \in \varepsilon, h(\cdot,\cdot,A)$是$E \times R^+$上的$\varepsilon \times B(R^+)$可测函数;

(ii) 对固定的$x \in E, t \ge 0, h(x,t,\cdot)$是$(E,\varepsilon)$上的有限测度;

(iii)并且由条件概率的性质，对于固定的$A \in \varepsilon$,$t \ge 0$,

$$\begin{aligned} h(x,t,A) &= \int^{\infty}\hat{h}(x,s,t,A)\hat{P}(x,ds) \\ &= \int^{\infty} P(X(t)\in A|X(0)=x,\tau_1=s)P(\tau_1 \in ds|X(0)=x) \\ &= \int^{\infty} P(X(t)\in A, t<\tau_1|X(0)=x) \end{aligned} \qquad P-a.s.$$

即$h(X(0),t,A)=\int^{\infty} P(X(t)\in A,t<\tau_1|X(0)=x) \qquad P-a.s.$

由$\{X(t);t\ge 0\}$在τ_n上的马尔可夫性及该过程的齐次性，立得

$$P(X(\tau_n+t)\in A,\tau_{n+1}-\tau_n>t|X(\tau_n)=x)$$

$$=P(X(t)\in A,\tau_1>t|X(0)=x)=h(x,t,A) \qquad P-a.s.$$

即$P(X(\tau_n+t)\in A,\tau_{n+1}-\tau_n>t|X(\tau_n)=x)=h(X(\tau_n),t,A) \qquad P-a.s.$

故$\{X(t);t\ge 0\}$是正规的马尔可夫骨架过程。

2.3 正则性准则

为了讨论方便，我们假定马尔可夫骨架过程$\{X(t);t\geq 0\}$是不中断的（或称为正则的）。实际上，上面得到的定理 2.2.1—2.2.4 对于中断马尔可夫骨架过程$\{X(t);0\leq t<\tau\}$也成立，这里τ称为$X(t)$的生命。在应用中常需要判定一个马尔可夫骨架过程是否是正则的，下面我们给出判定准则。

定义 2.3.1 过程$\{X(t);0\leq t<\tau\}$称为正则的，如果对于每个$x\in E$，有

$$P\{\tau=\infty|X(0)=x\}=1 \tag{2.10}$$

引理 2.3.1 $\{X(t)\}$为正则的充分必要条件是对于每个$x\in E$及$t\geq 0$，有

$$p(x,t,E)=1$$

或等价地，对于每个$x\in E$及$\lambda\geq 0$,有$\lambda p_{\lambda}(x,E)=1$.

证明 先证,对于每个$x\in E$及$t\geq 0$, $p(x,t,E)=1$与$\lambda p_{\lambda}(x,E)=1$等价

若$p(x,t,E)=1$, $\lambda p_{\lambda}(x,E)=\lambda\int_0^{\infty}e^{-\lambda t}p(x,t,E)dt=\lambda\int_0^{\infty}e^{-\lambda t}dt=1$

若$\lambda p_{\lambda}(x,E)=1$, $p_{\lambda}(x,E)=\int_0^{\infty}e^{-\lambda t}p(x,t,E)dt=1/\lambda$

由反查拉氏变换简表可得：$p(x,t,E)=1$

再证$P\{\tau=\infty|X(0)=x\}=1$与$p(x,t,E)=1$等价。

我们可以根据正则性定义的直观含义来理解上式两者的等价性：过程在 0 点从某一状态开始在有限时间内跳有限次后所处的状态仍在E中，即不会发生在有限时间内跳到无穷远处回不来了这种情况，即过程的寿命可以认为是无限的。证毕。

令 $B_E\triangleq\{f:f$是E上的有界 Borel 可测函数$\}$

引理 2.3.2 若$0\leq f\in B_E$,且对于某$\lambda>0$,存在$0\leq u(x)\in B_E$,使得

$$f(x)-\int_E q_{\lambda}(x,dy)f(y)=\int_E h_{\lambda}(x,dy)u(y)\geq 0 \tag{2.11}$$

则 $$f(x)\geq\int_E p_{\lambda}(x,dy)u(y),\forall x\in E. \tag{2.12}$$

进一步，若方程$$\begin{cases}g(x)=\int_E q_{\lambda}(x,dy)g(y),\forall x\in E\\ 0\leq g(x)\in B_E.\end{cases} \tag{2.13}$$

只有零解，则（2.12）成为等式 $f(x)=\int_E p_\lambda(x,dy)u(y),\forall x\in E.$ （2.14）

证明 由定理 2.2.3 知 $\{p_\lambda(x,A);x\in E\}$ 是非负方程

$$z(x)=\int_E q_\lambda(x,dy)z(y)+h_\lambda(x,A),x\in E \tag{2.15}$$

的最小非负解，从而有

$$p_\lambda(x,dy)=\int_E q_\lambda(x,dw)p_\lambda(w,dy)+h_\lambda(x,dy)$$

$$p_\lambda(x,dy)u(y)=\int_E q_\lambda(x,dw)p_\lambda(w,dy)u(y)+h_\lambda(x,dy)u(y)$$

$$\int_E p_\lambda(x,dy)u(y)=\int_E\int_E q_\lambda(x,dw)p_\lambda(w,dy)u(y)+\int_E h_\lambda(x,dy)u(y)$$

$$\int_E p_\lambda(x,dy)u(y)=\int_E q_\lambda(x,dw)[\int_E p_\lambda(w,dy)u(y)]+\int_E h_\lambda(x,dy)u(y)$$

令 $z(x)=\int_E p_\lambda(x,dy)u(y)$，则上式化为

$$z(x)=\int_E q_\lambda(x,dw)z(w)+\int_E h_\lambda(x,dy)u(y)$$

即证得 $\{\int_E p_\lambda(x,dy)u(y);x\in E\}$ 是非负方程

$$z(x)=\int_E q_\lambda(x,dy)z(y)+\int_E h_\lambda(x,dy)u(y),\quad x\in E \tag{2.16}$$

的最小非负解。而由定理的条件得

$$f(x)=\int_E q_\lambda(x,dy)f(y)+\int_E h_\lambda(x,dy)u(y),\ x\in E \tag{2.17}$$

即 $\{f(x);x\in E\}$ 也是（2.16）的非负解，所以有（2.12）成立。

将（2.16）式与（2.17）式相减，有 $f(x)-z(x)=\int_E q_\lambda(x,dy)(f(y)-z(y))$

当令 $g(x)\triangleq f(x)-z(x)=f(x)-\int_E p_\lambda(x,dy)u(y)$

即有 $$g(x)=\int_E q_\lambda(x,dy)g(y),\forall x\in E$$

由已知 $0\le f\in B_E,\lambda>0,0\le u(x)\in B_E$ 及 $g(x)$ 的定义有：$g(x)\in B_E$

又由（2.12）及 $g(x)$ 的定义有：$g(x)\ge 0$

即得到了（2.13）。由已知（2.13）只有零解，即 $g(x)=0$，即得（2.14）。

推论 2.3.1 如果状态空间 E 只有有限个点，且 $\forall x\in E$，

$$P(\tau>0|X(0)=x)>0 \text{ 则 } X \text{ 正则。}$$

2.4 有限维分布

令 $D^{(n)}=\{(t_1,t_2,\cdots,t_n);t_1,\cdots,t_n\in IR^+,0<t_1<t_2<\cdots<t_n\}$

定义 2.4.1 称时齐马尔可夫骨架过程 $\{X(t),t\geq 0\}$ 是严正规的，如果对于任意的 $n\in N$，存在 $E\times D^{(n)}\times\varepsilon^n$ 上的函数 $h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$，使得(i)对于固定的 $x,t_1,\cdots,t_n,h(x,t_1,\cdots,t_n,\cdot)$ 是 (E^n,ε^n) 上的有限测度；

(ii)对于固定的 $A_1,\cdots,A_n\in\varepsilon,h(\cdot,\cdots,A_1,\cdots,A_n)$ 是 $E\times D^{(n)}$ 上的 Borel 可测的函数；

(iii)对于任意的 $0<t_1<t_2<\cdots<t_n$，$A_1,\cdots,A_n\in\varepsilon$,

$$P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_n<\tau_1|X(0)=x)$$

$$=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)\qquad P-a.s.$$

对于任意的 $n\in N,(t_1,t_2,\cdots,t_n)\in D^{(n)},A_1,\cdots,A_n\in\varepsilon$，令

$$\eta=X(\tau_1)I_{A_1}(X(t_1))\cdots I_{A_n}(X(t_n))I_{[t_n,+\infty)}(\tau_1)$$

(ξ,η) 关于 $X(0)$ 的混合分布记作 $q(x,t_1,t_2,\cdots,t_n,A_1,\cdots,A_n,dt,dy)$，即

$$P(\xi\in dt,\eta\in dy|X(0))=q(X(0),t_1,\cdots,t_n,A_1,\cdots,A_n,dt,dy)\ P-a.s.$$

当 $n=0$ 时，即为 $q(x,dt,dy)$。

下面我们利用 $\bar{H}\equiv\{h(x,t_1,t_2,\cdots,t_n,A_1,\cdots,A_n);n\in N\}$ 和

$\bar{Q}\equiv\{q(x,t_1,t_2,\cdots,t_n,A_1,\cdots,A_n,dt,dy);n\in N\}$ 来确定马尔可夫骨架过程有限维分布。

对于任意的 $A\in\varepsilon$，$t_1<t_2<\cdots<t_n$，令

$$p(x,t_1,\cdots,t_n,A_1,\cdots,A_n)=P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n|X(0)=x)$$

定理 2.4.1 设 $X=\{X(t);t\geq 0\}$ 是严正规的马尔可夫骨架过程，X 的有限维分布由下列递推公式被 $(\bar{H},\bar{Q})$ 唯一决定:

$$p(x,t_1,A)=h(x,t_1,A)+\sum_{n=1}^{\infty}\int_E\int_0^t q^{(n)}(x,ds,dy)h(y,t-s,A)\qquad(2.18)$$

$$(\forall x\in E,\forall t\geq 0,A\in\varepsilon)$$

$$p(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$$

$$=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)+\sum_{i=1}^{n-1}\int_E\int_{t_i}^{t_{i+1}}h(x,t_1,\cdots,t_i,A_1,\cdots,A_i)$$

$$q(x,ds,dy)p(y,t_{i+1}-s,\cdots,t_n-s,A_{i+1},\cdots,A_n)$$

$$+\sum_{m=1}^{\infty}\int_E\int_0^{t_1}q^{(m)}(x,ds_1,dy_1)h(y_1,t_1-s_1,\cdots,t_n-s_1,A_1,\cdots,A_n)$$

$$+\sum_{m=1}^{\infty}\int_E\int_0^{t_1}q^{(m)}(x,ds_1,dy_1)\sum_{i=1}^{n-1}\int_E\int_{t_i-s_1}^{t_{i+1}-s_1}h(y_1,t_1-s_1,\cdots,t_i-s_1,A_1,\cdots,A_i)$$

$$\cdot q(y_1,ds_2,dy_2)p(y_2,t_{i+1}-s_1-s_2,\cdots,t_n-s_1-s_2,A_{i+1},\cdots,A_n)$$

$$(\forall n>1,x\in E,\forall(t_1,\cdots,t_n)\in D^{(n)},A_1\in\varepsilon,\cdots,A_n\in\varepsilon) \quad (2.19)$$

证明 (2.18)即是(2.19),故只须证明（2.19)。

对于任意的$\forall n>1,x\in E,\forall(t_1,\cdots,t_n)\in D^{(n)},A_1\in\varepsilon,\cdots,A_n\in\varepsilon$，易得

$$p(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$$

$$=P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_n<\tau_1|X(0)=x)$$

$$+\sum_{i=1}^{n-1}P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_i<\tau_1\le t_{i+1}|X(0)=x)$$

$$+\sum_{m=1}^{\infty}P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,\tau_m\le t_1<t_n<\tau_{m+1}|X(0)=x)$$

$$+\sum_{m=1}^{\infty}\sum_{i=1}^{n-1}P(X(t_1)\in A_1,\cdots X(t_n)\in A_n,\tau_m\le t_1<t_i<\tau_{m+1}\le t_{i+1}|X(0)=x)$$

由h的定义，第一项$=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$；

第二项中，由马氏性和齐次性得

$$P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_i<\tau_1\le t_{i+1}|X(0)=x)$$

$$=P(X(t_1)\in A_1,\cdots,X(t_i)\in A_i,X(t_{i+1})\in A_{i+1},\cdots,X(t_n)\in A_n,t_i<\tau_1,\tau_1\le t_{i+1}|X(0)=x)$$

$$=P(X(t_1)\in A_1,\cdots,X(t_i)\in A_i,t_i<\tau_1|X(0)=x)$$

$$\cdot P(X(t_{i+1})\in A_{i+1},\cdots,X(t_n)\in A_n,\tau_1\le t_{i+1}|X(0)=x)$$

$$=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$$

$$\cdot\int_E\int_{t_i}^{t_{i+1}} P(X(t_{i+1})\in A_{i+1},\cdots,X(t_n)\in A_n \,|\, X(0)=x,\tau_1=s,X(\tau_1)=y)$$

$$P(\tau_1\in ds, X(\tau_1)\in dy \,|\, X(0)=x)$$

$$=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$$

$$\cdot\int_E\int_0^{t} P(X(t_{i+1}-s)\in A_{i+1},\cdots,X(t_n-s)\in A_n \,|\, X(0)=y)q(x,ds,dy)$$

$$=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)\cdot\int_E\int_0^{t} p(y,t_{i+1}-s,\cdots,t_n-s,A_{i+1},\cdots,A_n)q(x,ds,dy)$$

$$=\int_E\int_{t_i}^{t_{i+1}} h(x,t_1,\cdots,t_i,A_1,\cdots,A_i)q(x,ds,dy)p(y,t_{i+1}-s,\cdots,t_n-s,A_{i+1},\cdots,A_n)$$

第三项中，由马氏性和齐次性得：

$$P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,\tau_m\le t_1<t_n<\tau_{m+1} \,|\, X(0)=x)$$

$$=\int_E\int_0^{t} P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_n<\tau_{m+1} \,|\, X(0)=x,\tau_m=s_1,X(\tau_m)=y_1)$$

$$\cdot P(\tau_m\in ds_1,X(\tau_m)\in dy_1 \,|\, X(0)=x)$$

$$=\int_E\int_0^{t} P(X(t_1-s_1)\in A_1,\cdots,X(t_n-s_1)\in A_n,t_n<\tau_1 \,|\, X(0)=y_1)$$

$$\cdot P(\tau_m\in ds_1,X(\tau_m)\in dy_1 \,|\, X(0)=x)$$

$$=\int_E\int_0^{t_1} h(y_1,t_1-s_1,\cdots,t_n-s_1,A_1,\cdots,A_n)q^{(m)}(x,ds_1,dy_1)$$

第四项中，由马氏性和齐次性和第二项的推导得

$$\sum_{i=1}^{n-1} P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,\tau_m\le t_1<t_i<\tau_{m+1}\le t_{i+1} \,|\, X(0)=x)$$

$$=\sum_{i=1}^{n-1}\int_E\int_0^{t_1} P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_i<\tau_{m+1}\le t_{i+1} \,|\, X(0)=x,$$

$$\tau_m=s_1,X(\tau_m)=y_1)\cdot P(\tau_m\in ds_1,X(\tau_m)\in dy_1 \,|\, X(0)=x)$$

$$=\sum_{i=1}^{n-1}\int_E\int_0^{t_1} P(X(t_1-s_1)\in A_1,\cdots,X(t_n-s_1)\in A_n,t_i<\tau_1\le t_{i+1} \,|\, X(0)=y_1)q^{(m)}(x,ds_1,dy_1)$$

$$=\int_E\int_0^{t_1} q^{(m)}(x,ds_1,dy_1)\sum_{i=1}^{n-1}\int_E\int_{t_i-s_1}^{t_{i+1}-s_1} h(y_1,t_1-s_1,\cdots,t_i-s_1,A_1,\cdots,A_i)q(y_1,ds_2,dy_2)$$

$$\cdot p(y_2,t_{i+1}-s_1-s_2,\cdots,t_n-s_1-s_2,A_{i+1},\cdots,A_n)$$

由上面几个式子即得（2.19）式。定理证毕。

下面我们给出了马尔可夫骨架过程成为严正规马尔可夫骨架过程的充分条件—过程的轨道以概率 1 处处有左极限。

引理 2.4.1 对于任意的 $n\in N, A_1\cdots A_n\in\varepsilon, \omega\mapsto I_{A_1}(w(t_1))\cdots I_{A_n}(w(t_n))$ 是 $D^{(n)}\times D_E[0,\infty)$ 上的 Borel 可测函数。

定理 2.4.2 设 $X=\{X(t);t\geq 0\}$ 是以 $\{\tau_n\}_{n=0}^{\infty}$ 为骨架时序列的马尔可夫骨架过程。如果 X 的轨道都是左极的，则 $\{X(t);t\geq 0\}$ 是严正规马尔可夫骨架过程。

证明 设 $\{\tau_n\}_{n=0}^{\infty}$ 是 $\{X(t);t\geq 0\}$ 的骨架过程时序列。由于的所有轨道是右连左极的，故可以将 $\{X(t);t\geq 0\}$ 看成是取值于 $D_E[0,\infty)$ 的随机元 ξ。由定理 2.1.2 和注 2.1.3 存在 ξ 关于 $(X(0),\tau_1)$ 的混合条件分布

$$K(x,s,dw)\equiv P(\xi\in dw|X(0)=x,\tau_1=s)\text{ 使得}$$

(i) 对于固定的 $x\in\varepsilon, s\geq 0, K(x,s,\cdot)$ 是 $D_E[0,\infty)$ 上的概率测度；

(ii)对于固定的 $C\in B(D_E[0,\infty)), K(\cdot,\cdot,C)$ 是 $\varepsilon\times B(IR^+)$ 可测的；

(iii) $\forall C\in B(D_E[0,\infty)), K(X(0),\tau_1,C)$ 是 $P(\xi\in C|X(0),\tau_1)$ 的一版本。

对于任意的 $n\in N, A_1\cdots A_n\in\varepsilon$, 由引理 2.4.1 $\omega\mapsto I_{A_1}(w(t_1))\cdots I_{A_n}(w(t_n))$ 是 $D^{(n)}\times D_E[0,\infty)$ 上的 Borel 可测函数。令

$$\hat{h}(x,s,t_1,\cdots,t_n,A_1,\cdots,A_n)=\int_{D_E[0,\infty)} I_{A_1}(w(t_1))\cdots I_{A_n}(w(t_n))K(x,s,dw)$$

显然 $\hat{h}(x,s,t_1,\cdots,t_n,A_1,\cdots,A_n)$ 满足以下条件：

(i)对于固定的 $x,s,t_1,\cdots,t_n,\hat{h}(x,s,t_1,\cdots,t_n,\cdot)$ 是 (E^n,ε^n) 上的概率测度；

(ii)对固定的 $A_1,\cdots,A_n\in\varepsilon,\hat{h}(\cdots,\cdots,A_1,\cdots,A_n)$ 是 $E\times IR^+\times D^{(n)}$ 上的 Borel 可测函数；

(iii)对于固定的 $0<t_1<t_2<\cdots<t_n, A_1,\cdots,A_n\in\varepsilon$,

$$\hat{h}(x,s,t_1,\cdots,t_n,A_1,\cdots,A_n)$$
$$=P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n|X(0)=x,\tau_1=s)P-a.s.$$

令 $h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)=\int_t^{\infty}\hat{h}(x,s,t_1,\cdots,t_n,A_1,\cdots,A_n)P(x,ds)$

其中 $P(x,ds)$ 是 τ_1 关于的混合条件分布 $P(\tau_1\in ds|X(0)=x)$。

显然，$h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)$ 满足以下条件：

(1)对于固定的 $A_1,\cdots,A_n,h(\cdot,\cdots,A_1,\cdots,A_n)$ 是 $E\times D^{(n)}$ 上 Borel 可测函数；

(2)对于固定的 $x,t_1<t_2<\cdots<t_n,h(x,t_1,\cdots,t_n,\cdot)$ 是 (E^n,ε^n) 上有限测度；

(3)对于固定的 $A_1,\cdots,A_n\in\varepsilon,t_1<t_2<\cdots<t_n$,

$$
\begin{aligned}
&h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)\\
&=\int_t^{\infty}\hat{h}(x,s,t_1,\cdots,t_n,A_1,\cdots,A_n)P(x,ds)\\
&=\int_t^{\infty}P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n|X(0)=x,\tau_1=s)P(\tau_1\in ds|X(0)=x)\\
&=P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_n<\tau_1|X(0)=x)\qquad P-a.s.
\end{aligned}
$$

由 $\{X(t);t\ge 0\}$ 在 τ_m 处的马尔可夫性及齐次性，可得

$$
\begin{aligned}
&P(X(\tau_m+t_1)\in A_1,\cdots,X(\tau_m+t_n)\in A_n,\tau_{m+1}-\tau_m>t_n|X(\tau_m)=x)\\
&=P(X(t_1)\in A_1,\cdots,X(t_n)\in A_n,t_n<\tau_1|X(0)=x)\\
&=h(x,t_1,\cdots,t_n,A_1,\cdots,A_n)\qquad \forall m\in N,t_1<t_2<\cdots<t_n
\end{aligned}
$$

因此，$\{X(t);t\ge 0\}$ 是严正规的马尔可夫骨架过程。

第三章 马氏骨架过程方法在种群动态学中的应用

3.1 引言

掌握自然种群动态的规律，便可以更好地计划渔捞量、毛皮兽猎取量，以及野生动物的经济利用等，并对农林业的害虫、害兽以及传播疾病的动物进行有效防治。在做这部分研究之前，首先给大家介绍几个生物学上的概念:1.种群(population):在特定时间内，由分布在同一区域的许多同种生物个体自然组成的生物系统;具有四个基本特征：数量特征、空间特征、遗传特征、系统特征;从系统的角度，通过研究种群内在的因子，以及生存环境内各种环境因子与种群数量变化的相互关系，从而揭示种群数量变化的机制与规律。种群的界限是人为划定的;2.种群生态学：是研究种群生物系统的规律的科学，研究种群内部各成员之间，种群（或其成员）与其他生物种群之间，以及种群与周围环境非生物因素的相互作用规律,核心是种群动态研究;3.种群动态（population dynamic)：研究种群数量在时间上和空间上的变动规律及其变动原因（调节机制)；数量特征是种群的最基本特征；种群是由多个个体所组成的，其数量大小受四个种群参数(出生率、死亡率、迁入率和迁出率)的影响，这些参数继而又受种群的年龄结构、性别比率、内分布格局和遗传组成的影响，从而形成种群动态。

3.2 模型描述

本文将利用马尔可夫骨架过程理论来研究种群动态学中的数学模型。我们所考虑的是某区域内的封闭种群数量，即只考虑该区域内某种群的出生和死亡，不考虑种群的迁移。现在假定所研究的种群动态学数学模型如下：

假设某区域内某种群在时刻0的种群数量为$i(i>0)$。

(1) 假定每次只有一个幼仔出生且新生幼仔生命力较强，相邻两个幼仔出生的时间间隔a_m独立同分布，其分布函数为$A(t)=P(a_m\le t)$。

(2) 非天灾人祸造成的死亡我们称之为自然死亡。对于自然死亡而言，假定该地区此种群的寿命是随机变量，且单个个体的寿命是独立同分布的，其共同分布函数为$B(t)=P(\xi\le t)$。

(3) 相邻两次发生自然灾害(如冰冻，泥石流，海啸，地震等)的时间间隔c_m独立同分布，分布函数为$C(t)=P(c_m\le t)$。并且我们假定一次自然灾害中死亡的此种群数量ς是一随机变量，且死亡k个个体的概率是$P_k(k=1,2,\cdots,i)$，那么设其期望值$E\varsigma=\sum_{k=1}^{i}kP_k=Q$，即在一次自然灾害中平均死亡的种群数量是$Q$。

在讨论此模型时，我们视Q为发生一次自然灾害死亡的种群数量。

(4) 相邻个体被捕食的时间间隔d_m独立同分布，分布函数$D(t)=P(d_m\le t)$。

用$L(t)$表示时刻t该区域内的所要研究的单种群种群数量，由假设可知，一般$L(t)$不是马尔可夫骨架过程，现在采纳[3]中的方法引进补充变量如下：

$\bar{\theta}(t)$表示到时刻t为止已有多长时间没有幼仔出生，

$\hat{\theta}_k(t)$表示到时刻t为止第k个个体已有多长时间未发生自然死亡，

$\tilde{\theta}(t)$表示到时刻t为止已有多长时间未发生自然灾害，

$\theta_k(t)$表示到时刻t为止第k个个体已有多长时间未被捕食，

以上变量中$k=1,2,\cdots,L(t)$。

则$\{L(t),\bar{\theta}(t),\hat{\theta}_1(t),\cdots,\hat{\theta}_{L(t)}(t),\tilde{\theta}(t),\theta_1(t),\cdots,\theta_{L(t)}(t)\}$为一个马尔可夫过程，以上假设中$a_m,\xi,c_m,d_m$均相互独立，$Q,m\in Z^+$。

3.3 单种群种群数量的分析

下面利用马尔可夫骨架过程的理论来讨论单种群种群数量的变化情况：

以B表示$IR^+=[0,\infty)$上所有的 Borel 集的全体。设$\tau_0\equiv 0,\tau_n(n\ge 1)$为$\{L(t),\bar{\theta}(t),\hat{\theta}_1(t),\cdots,\hat{\theta}_{L(t)}(t),\tilde{\theta}(t),\theta_1(t),\cdots,\theta_{L(t)}(t)\}$ $(t\ge 0)$在$[0,\infty)$上的第n个间断点，即在时刻$\tau_n(n\ge 1)$有一个幼仔出生；或者有l个个体发生自然死亡$(1\le l\le L(t))$；或者发生一次自然灾害死了Q个个体；或者有r个个体被捕食$(1\le r\le L(t))$；或者

以上之中的 2 个，3 个，4 个事件同时发生，则有 $\{L(t),\bar{\theta}(t),\hat{\theta}_1(t),\cdots,\hat{\theta}_{L(t)}(t),\tilde{\theta}(t),\theta_1(t),\cdots,\theta_{L(t)}(t)\}$ 是以 $\{\tau_n\}_{n=0}^{\infty}$ 为骨架时序列的马尔可夫骨架过程。

令
$$A_{\bar{\theta}}(t)=\frac{A(\bar{\theta}+t)-A(\bar{\theta})}{1-A(\bar{\theta})} \tag{3.1}$$

$$B_{\hat{\theta}_k}(t)=\frac{B(\hat{\theta}_k+t)-B(\hat{\theta}_k)}{1-B(\hat{\theta}_k)},k=1,2,\cdots,L(t) \tag{3.2}$$

$$C_{\tilde{\theta}}(t)=\frac{C(\tilde{\theta}+t)-C(\tilde{\theta})}{1-C(\tilde{\theta})} \tag{3.3}$$

$$D_{\theta_k}(t)=\frac{D(\theta_k+t)-D(\theta_k)}{1-D(\theta_k)},k=1,2,\cdots,L(t) \tag{3.4}$$

下面说明以上 4 个式子代表的含义，我们以(3.1)为例：

$$A_{\bar{\theta}}(t)=\frac{A(\bar{\theta}+t)-A(\bar{\theta})}{1-A(\bar{\theta})}=\frac{P(a_m\le\bar{\theta}+t)-P(a_m\le\bar{\theta})}{1-P(a_m\le\bar{\theta})}$$

$$=\frac{P(\bar{\theta}<a_m\le\bar{\theta}+t)}{P(a_m>\bar{\theta})}=P(a_m\le\bar{\theta}+t\,|\,a_m>\bar{\theta})$$

即表示在 $(\bar{\theta},\bar{\theta}+t)$ 时间段内有一个幼仔出生，故 $A_{\theta}(t)$ 是一个幼仔出生的剩余时间分布。用同样方法可得：(3.2)表示在 $(\hat{\theta}_k,\hat{\theta}_k+t)$ 时间段内第 k 个个体发生死亡，故 $B_{\hat{\theta}_k}(t)$ 是第 k 个个体发生自然死亡的剩余时间分布；(3.3)表示在 $(\tilde{\theta},\tilde{\theta}+t)$ 时间段内发生一次自然灾害，故 $C_{\tilde{\theta}}(t)$ 是发生一次自然灾害的剩余时间分布；(3.4)表示在 (θ_k,θ_k+t) 内第 k 个个体被捕食,故 $D_{\theta_k}(t)$ 是第 k 个个体被捕食的剩余时间分布。

显然有 $\tau_n\uparrow\infty(n\to\infty)$ 。对于 $i\gg Q>0,j>0;\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i\in R^+$ 以及 $\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j\in B(R^+)$，令

$$h(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$
$$=P\{L(t)=j,\bar{\theta}(t)\in\bar{A},\hat{\theta}_1(t)\in\hat{A}_1,\cdots,\hat{\theta}_j(t)\in\hat{A}_j,\tilde{\theta}(t)\in\tilde{A},\theta_1(t)\in A_1,\cdots,\theta_j(t)\in A_j,t<\tau_1$$
$$\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i\};\quad(3.5)$$

$$p(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$

$$=P\{L(t)=j,\bar{\theta}(t)\in\bar{A},\hat{\theta}_1(t)\in\hat{A}_1,\cdots,\hat{\theta}_j(t)\in\hat{A}_j,\tilde{\theta}(t)\in\tilde{A},\theta_1(t)\in A_1,\cdots,\theta_j(t)\in A_j$$

$$\left|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i\right\};\quad(3.6)$$

$$q(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,ds,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j)$$

$$=P\{\tau_1\in ds,L(\tau_1)=j,\bar{\theta}(\tau_1)\in\bar{A},\hat{\theta}_1(\tau_1)\in\hat{A}_1,\cdots,\hat{\theta}_j(\tau_1)\in\hat{A}_j,\tilde{\theta}(\tau_1)\in\tilde{A},\theta_1(\tau_1)\in A,\cdots,$$

$$\theta_j(\tau_1)\in A_j\left|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,\cdots,\right.$$

$$\theta_i(0)=\theta_i\};\quad(3.7)$$

特别地，对于 $\bar{a},\hat{a}_1,\cdots,\hat{a}_j,\tilde{a},a_1,\cdots,a_j\in IR^+$，

$$q(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,ds,j,\bar{a},\hat{a}_1,\cdots,\hat{a}_j,\tilde{a},a_1,\cdots,a_j)$$

$$\triangleq q(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,ds,j,\{\bar{a}\},\{\hat{a}_1\},\cdots,\{\hat{a}_j\},\{\tilde{a}\},\{a_1\},\cdots,\{a_j\})\quad(3.8)$$

由以上有：$h(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$ 表示 $t<\tau_1$ 的情况，即我们考察的时刻 t 在第一个间断点 τ_1 之前，在这样的时刻，种群数量未发生变化，无出生无死亡，没有自然灾害，也没有个体被捕食。则有

引 理 3.3.1

$$h(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$

$$=\delta_{ij}I_A(\bar{\theta}+t)[\prod_{k=1}^{i}I_{\hat{A}_k}(\hat{\theta}+t)]I_{\tilde{A}}(\tilde{\theta}+t)[\prod_{x=1}^{i}I_{A_x}(\theta_x+t)]$$

$$\times(1-A_{\theta}(t))[\prod_{y=1}^{i}(1-B_{\hat{\theta}_y}(t))](1-C_{\tilde{\theta}}(t))[\prod_{z=1}^{i}(1-D_{\theta_z}(t))]\qquad i,j>0$$

$$=\begin{cases}I_A(\bar{\theta}+t)[\prod_{k=1}^{i}I_{\hat{A}_k}(\hat{\theta}+t)]I_{\tilde{A}}(\tilde{\theta}+t)[\prod_{x=1}^{i}I_{A_x}(\theta_x+t)] \\ \times(1-A_{\bar{\theta}}(t))[\prod_{y=1}^{i}(1-B_{\hat{\theta}_y}(t))](1-C_{\tilde{\theta}}(t))[\prod_{z=1}^{i}(1-D_{\theta_z}(t))], j=i>0; \\ 0, \qquad\qquad j\neq i>0.\end{cases}\quad(3.9)$$

引理3.3.2 令集合 $\Delta_k=\{(n_1,\cdots,n_i)\big|\{n_1,\cdots,n_i\}=\{1,\cdots,i\},n_1<\cdots<n_k$ 且 $n_{k+1}<\cdots<n_i\}$，

$k=1,\cdots,i$ 则

$$q(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,ds,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j)=$$

$$①\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-C_{\tilde{\theta}}(s))dA_{\bar{\theta}}(s),\qquad j=i+1$$

$$②(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{k=1}^{i}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))$$

$$j=i$$

$$③(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{k=1}^{i}dB_{\hat{\theta}_k}(s)\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))$$

$$+(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{x=1}^{i}dD_{\theta_x}(s)\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))$$

$$(1-C_{\tilde{\theta}}(s))[\sum_{k=1}^{i}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))][\sum_{\substack{x=1\\ x\ne k}}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))]$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\substack{n_1<n_2,\\ n_3<\cdots<n_i\\ \{n_1,\cdots,n_i\}=\{1,\cdots,i\}}}[\prod_{k=1}^{2}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=3}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{\substack{n_1<n_2,\\ n_3<\cdots<n_i\\ \{n_1,\cdots,n_i\}=\{1,\cdots,i\}}}[\prod_{x=1}^{2}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=3}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$j=i-1$$

（注：③中第四式和第五式中“$\sum$ ”下的集合即是Δ_2，以下像这种集合均用Δ_k来表示，其中k为一变量。）

$$④(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_l}[\prod_{k=1}^{l}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=l+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]$$

$$+(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\sum_{e=1}^{l-1}\{[\sum_{\Delta_e}\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))][\sum_{\Delta_{l-e}}\prod_{x=1}^{l-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l-e+1}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{\Delta_l}[\prod_{x=1}^{l}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+1}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{l+1}}\prod_{k=1}^{l+1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=l+2}^{i}(1-B_{\hat{\theta}_{n_m}}(s))$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\theta}(s))\sum_{z=1}^{l}\{[\sum_{\Delta_z}\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))][\sum_{\Delta_{l+1-z}}\prod_{x=1}^{l+1-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2-z}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\theta}(s)-A_{\theta}(s-))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))[\sum_{\Delta_{l+1}}\prod_{x=1}^{l+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$j=i-l(l\in Z^{+}\text{ 且 }2\le l\le Q-2)$$

$$⑤\ (1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{Q-1}}[\prod_{k=1}^{Q-1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]$$

$$+(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))\sum_{e=1}^{Q-2}\{[\sum_{\Delta_e}\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))][\sum_{\Delta_{Q-1-e}}\prod_{x=1}^{Q-1-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q-e}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{\Delta_{Q-1}}[\prod_{x=1}^{Q-1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_Q}\prod_{k=1}^{Q}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\theta}(s))\sum_{z=1}^{Q-1}\{[\sum_{\Delta_z}\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]][\sum_{\Delta_{Q-z}}\prod_{x=1}^{Q-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1-z}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\theta}(s)-A_{\theta}(s-))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))[\sum_{\Delta_Q}\prod_{x=1}^{Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(A_{\theta}(s)-A_{\theta}(s-))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s)),\qquad j=i-(Q-1)$$

$$⑥\ (1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_Q}[\prod_{k=1}^{Q}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]$$

$$+(1-A_{\theta}(s))(1-C_{\theta}(s))\sum_{e=1}^{Q-1}\{[\sum_{\Delta_e}\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))][\sum_{\Delta_{Q-e}}\prod_{x=1}^{Q-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1-e}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{\Delta_Q}[\prod_{x=1}^{Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(A_{\theta}(s)-A_{\theta}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{Q+1}}\prod_{k=1}^{Q+1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+2}^{i}(1-B_{\hat{\theta}_{n_m}}(s))$$

$$+(A_{\theta}(s)-A_{\bar{\theta}}(s-))(1-C_{\theta}(s))\sum_{z=1}^{Q}\{[\sum_{\Delta_z}\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))][\sum_{\Delta_{Q+1-z}}\prod_{x=1}^{Q+1-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+2-z}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))[\sum_{\Delta_{Q+1}}\prod_{x=1}^{Q+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+2}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(1-A_{\bar\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))dC_{\tilde\theta}(s)$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\sum_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-)(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{k=1}^{i}(B_{\hat\theta_k}(s)-B_{\hat\theta_k}(s-))\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat\theta_m}(s))$$

$$j=i-Q$$

$$⑦(1-A_{\bar\theta}(s))(1-C_{\tilde\theta}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{Q+l}}[\prod_{k=1}^{Q+l}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=Q+l+1}^{i}(1-B_{\hat\theta_{n_m}}(s))]$$

$$+(1-A_{\bar\theta}(s))(1-C_{\tilde\theta}(s))\sum_{e=1}^{Q+l-1}\{[\sum_{\Delta_e}\prod_{k=1}^{e}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{Q+l-e}}\prod_{x=1}^{Q+l-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+1-e}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar\theta}(s))(1-C_{\tilde\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\sum_{\Delta_{Q+l}}[\prod_{x=1}^{Q+l}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+1}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(1-C_{\tilde\theta}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{Q+l+1}}\prod_{k=1}^{Q+l+1}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=Q+l+2}^{i}(1-B_{\hat\theta_{n_m}}(s))$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(1-C_{\tilde\theta}(s))\sum_{z=1}^{Q+l}\{[\sum_{\Delta_z}\prod_{k=1}^{z}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=z+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{Q+l+1-z}}\prod_{x=1}^{Q+l+1-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+2-z}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(1-C_{\tilde\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))[\sum_{\Delta_{Q+l+1}}\prod_{x=1}^{Q+l+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+2}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(1-A_{\bar\theta}(s))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_l}[\prod_{k=1}^{l}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=l+1}^{i}(1-B_{\hat\theta_{n_m}}(s))]$$

$$+(1-A_{\bar\theta}(s))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\sum_{r=1}^{l-1}\{[\sum_{\Delta_r}\prod_{k=1}^{r}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=r+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{l-r}}\prod_{x=1}^{l-r}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l-r+1}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar\theta}(s))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))[\sum_{\Delta_l}\prod_{x=1}^{l}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+1}^{i}(1-D_{\theta_{n_u}}(s))]$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-)(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{l+1}}\prod_{k=1}^{l+1}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=l+2}^{i}(1-B_{\hat\theta_{n_m}}(s))$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\sum_{v=1}^{l}\{[\sum_{\Delta_v}\prod_{k=1}^{v}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=v+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{l+1-v}}\prod_{x=1}^{l+1-v}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2-v}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\sum_{\Delta_{l+1}}\prod_{x=1}^{l+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2}^{i}(1-D_{\theta_{n_u}}(s))$$

$$j=i-Q-l(l\in Z^{+}\text{且}1\le l\le i-Q-2)$$

$$⑧(1-A_{\bar\theta}(s))(1-C_{\bar\theta}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{i-1}}[\prod_{k=1}^{i-1}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))(1-B_{\hat\theta_{n_i}}(s))]$$

$$+(1-A_{\bar\theta}(s))(1-C_{\bar\theta}(s))\sum_{e=1}^{i-2}\{[\sum_{\Delta_e}\prod_{k=1}^{e}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{i-1-e}}\prod_{x=1}^{i-1-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-e}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar\theta}(s))(1-C_{\bar\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\sum_{\Delta_{i-1}}\prod_{x=1}^{i-1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))(1-D_{\theta_{n_i}}(s))$$

$$+(A_{\bar\theta}(s)-A_{\theta}(s-))(1-C_{\bar\theta}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\prod_{k=1}^{i}(B_{\hat\theta_k}(s)-B_{\hat\theta_k}(s-))$$

$$+(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(1-C_{\bar\theta}(s))\sum_{z=1}^{i-1}\{[\sum_{\Delta_z}\prod_{k=1}^{z}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=z+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{i-z}}\prod_{x=1}^{i-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i+1-z}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\theta}(s)-A_{\theta}(s-))(1-C_{\bar\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\prod_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))$$

$$+(C_{\bar\theta}(s)-C_{\bar\theta}(s-))(1-A_{\bar\theta}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{i-1-Q}}\prod_{k=1}^{i-1-Q}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=i-Q}^{i}(1-B_{\hat\theta_{n_m}}(s))$$

$$+(1-A_{\theta}(s))(C_{\bar\theta}(s)-C_{\bar\theta}(s-))\sum_{r=1}^{i-2-Q}\{[\sum_{\Delta_r}\prod_{k=1}^{r}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=r+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{i-1-Q-r}}\prod_{x=1}^{i-1-Q-r}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q-r}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar\theta}(s))(C_{\bar\theta}(s)-C_{\bar\theta}(s-))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\sum_{\Delta_{i-1-Q}}\prod_{x=1}^{i-1-Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q}^{i}(1-D_{\theta_{n_u}}(s))$$

$$+(A_{\theta}(s)-A_{\theta}(s-)(C_{\bar\theta}(s)-C_{\bar\theta}(s-))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{i-Q}}\prod_{k=1}^{i-Q}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=i-Q+1}^{i}(1-B_{\hat\theta_{n_m}}(s))$$

$$+(A_{\bar\theta}(s)-A_{\theta}(s-))(C_{\theta}(s)-C_{\theta}(s-))\sum_{v=1}^{i-1-Q}\{[\sum_{\Delta_v}\prod_{k=1}^{v}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=v+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{i-Q-v}}\prod_{x=1}^{i-Q-v}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q+1-v}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(A_{\theta}(s)-A_{\bar\theta}(s-)(C_{\bar\theta}(s)-C_{\bar\theta}(s-))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\sum_{\Delta_{i-Q}}\prod_{x=1}^{i-Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q+1}^{i}(1-D_{\theta_{n_u}}(s))$$

$$j=i-Q-(i-Q-1),\text{即}\ j=1$$

$$⑨(1-A_{\bar\theta}(s))(1-C_{\bar\theta}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\prod_{k=1}^{i}(B_{\hat\theta_k}(s)-B_{\hat\theta_k}(s-))$$

$$+(1-A_{\bar\theta}(s))(1-C_{\bar\theta}(s))\sum_{e=1}^{i-1}\{[\sum_{\Delta_e}\prod_{k=1}^{e}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat\theta_{n_m}}(s))][\sum_{\Delta_{i-e}}\prod_{x=1}^{i-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-e+1}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(1-A_{\bar\theta}(s))(1-C_{\bar\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\prod_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))$$

$$+(1-A_{\bar{\theta}}(s))(C_{\bar{\theta}}(s)-C_{\bar{\theta}}(s-))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\sum_{\Delta_{i-Q}}\prod_{k=1}^{i-Q}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=i-Q+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))$$

$$+(1-A_{\bar{\theta}}(s))(C_{\theta}(s)-C_{\theta}(s-))\sum_{r=1}^{i-Q-1}\{[\sum_{\Delta_r}\prod_{k=1}^{r}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=r+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))][\sum_{\Delta_{i-Q-r}}\prod_{x=1}^{i-Q-r}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q-r+1}^{i}(1-D_{\theta_{n_u}}(s))]\}$$

$$+(C_{\bar{\theta}}(s)-C_{\bar{\theta}}(s-))(1-A_{\theta}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\sum_{\Delta_{i-Q}}\prod_{x=1}^{i-Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q+1}^{i}(1-D_{\theta_{n_u}}(s))$$

$j=0$

⑩ 0，　　　　其它

(3.10)

注 ① $j=i+1$：表示出生 1 个，未发生自然死亡，未发生自然灾害，且未发生被捕食的情形；

② $j=i$：或者表示出生 1 个，自然死亡 1 个，未发生自然灾害，未发生被捕食；或者出生 1 个，被捕食 1 个，未发生自然死亡和自然灾害的情形；

③ $j=i-1$：或者表示未出生，自然死亡 1 个，未发生自然灾害和被捕食；或者被捕食 1 个，未出生和自然死亡，未发生自然灾害；或者出生 1 个，自然死亡 1 个，被捕食 1 个，未发生自然灾害；或者出生 1 个，自然死亡 2 个，未发生自然灾害和被捕食；或者出生 1 个，被捕食 2 个，未发生自然死亡和灾害的情形；

④ $j=i-l$ ($l\in Z^{+}$ 且 $2\le l\le Q-2$)：或者表示未出生，自然死亡 l 个，未发生自然灾害和被捕食；或者未出生，自然死亡 e 个，被捕食 $l-e$ 个,未发生自然灾害 ($1\le e\le l-1$)；或者被捕食 l 个，未出生和自然死亡，未发生自然灾害；或者出生 1 个，自然死亡 $l+1$ 个，未发生自然灾害和被捕食；或者出生 1 个，自然死亡 z 个，被捕食 $l+1-z$ 个，未发生自然灾害 ($1\le z\le l$)；或者出生 1 个，被捕食 $l+1$ 个，未发生自然死亡和灾害的情形；

⑤ $j=i-(Q-1)$：或者表示未出生，自然死亡 $Q-1$ 个，未发生自然灾害和被捕食；或者未出生，自然死亡 e 个，被捕食 $Q-1-e$ 个,未发生自然灾害 ($1\le e\le Q-2$)；或者被捕食 $Q-1$ 个，未出生和自然死亡，未发生自然灾害；或者出生 1 个，自然死亡 Q 个，未发生自然灾害和被捕食；或者出生 1 个，自然死亡

z个，被捕食$Q-z$个，未发生自然灾害$(1\le z\le Q-1)$；或者出生1个，被捕食Q个，未发生自然死亡和灾害；或者出生1个，未发生自然死亡和被捕食，发生1次自然灾害的情形；

⑥$j=i-Q$：或者表示未出生，自然死亡Q个，未发生自然灾害和被捕食；或者未出生，自然死亡e个，被捕食$Q-e$个,未发生自然灾害$(1\le e\le Q-1)$；或者被捕食Q个，未出生和自然死亡，未发生自然灾害；或者出生1个，自然死亡$Q+1$个，未发生自然灾害和被捕食；或者出生1个，自然死亡z个，被捕食$Q+1-z$个，未发生自然灾害$(1\le z\le Q)$；或者出生1个，被捕食$Q+1$个，未发生自然死亡和灾害；未出生和自然死亡，未被捕食，发生自然灾害1次；出生1个，未发生自然死亡，被捕食1个，发生1次自然灾害；出生1个，自然死亡1个，未被捕食，发生1次自然灾害的情形；

⑦$j=i-Q-l(l\in Z^{+}$且$1\le l\le i-Q-2)$：或者表示未出生，自然死亡$Q+l$个，未发生自然灾害和被捕食；或者未出生，自然死亡e个，被捕食$Q+l-e$个,未发生自然灾害$(1\le e\le Q+l-1)$；或者被捕食$Q+l$个，未出生和自然死亡，未发生自然灾害；或者出生1个，自然死亡$Q+l+1$个，未发生自然灾害和被捕食；或者出生1个，自然死亡z个，被捕食$Q+l+1-z$个，未发生自然灾害$(1\le z\le Q+l)$；或者出生1个，被捕食$Q+l+1$个，未发生自然死亡和灾害；或者未出生，自然死亡l个，未被捕食，发生1次自然灾害；或者未出生，自然死亡r个，被捕食$l-r$个，发生1次自然灾害$(1\le r\le l-1)$；或者未出生和自然死亡，被捕食l个，发生1次自然灾害；或者出生1个，自然死亡$l+1$个，未被捕食，发生1次自然灾害；或者出生1个，自然死亡v个，被捕食$l+1-v$个，发生1次自然灾害$(1\le v\le l)$；或者出生1个，未发生自然死亡，被捕食$l+1$个，发生1次自然灾害的情形；

⑧ $j=i-Q-(i-Q-1)$,即$j=1$：或者表示未出生，自然死亡$i-1$个，未发生自然灾害和被捕食；或者未出生，自然死亡e个，被捕食$i-1-e$个,未发生自然灾

害$(1\le e\le i-2)$；或者被捕食$i-1$个，未出生和自然死亡，未发生自然灾害；或者出生1个，自然死亡i个，未发生自然灾害和被捕食；或者出生1个，自然死亡z个，被捕食$i-z$个，未发生自然灾害$(1\le z\le i-1)$；或者出生1个，被捕食i个，未发生自然死亡和灾害；或者未出生，自然死亡$i-1-Q$个，未被捕食，发生1次自然灾害；或者未出生，自然死亡r个，被捕食$i-1-Q-r$个，发生1次自然灾害$(1\le r\le i-2-Q)$；或者未出生和自然死亡，被捕食$i-1-Q$个，发生1次自然灾害；或者出生1个，自然死亡$i-Q$个，未被捕食，发生1次自然灾害；或者出生1个，自然死亡v个，被捕食$i-Q-v$个，发生1次自然灾害$(1\le v\le i-1-Q)$；或者出生1个，未发生自然死亡，被捕食$i-Q$个，发生1次自然灾害的情形；

⑨$j=0$：或者表示未出生，自然死亡i个，未发生自然灾害和被捕食；或者未出生，自然死亡e个，被捕食$i-e$个,未发生自然灾害$(1\le e\le i-1)$；或者被捕食i个，未出生和自然死亡，未发生自然灾害；或者未出生，自然死亡$i-Q$个，未被捕食，发生1次自然灾害；或者未出生，自然死亡r个，被捕食$i-Q-r$个，发生1次自然灾害$(1\le r\le i-Q-1)$；或者未出生和自然死亡，被捕食$i-Q$个，发生1次自然灾害的情形。

证明：由$q(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,ds,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j)$的定义及条件概率的意义容易证明下面式子成立

$$
\begin{aligned}
&q(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,ds,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j)\\
&=P\{\tau_1\in ds,L(\tau_1)=j,\bar{\theta}(\tau_1)\in\bar{A},\hat{\theta}_1(\tau_1)\in\hat{A}_1,\cdots,\hat{\theta}_j(\tau_1)\in\hat{A}_j,\tilde{\theta}(\tau_1)\in\tilde{A},\theta_1(\tau_1)\in A,\cdots,\\
&\theta_j(\tau_1)\in A_j\,\big|\,L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i\}\\
&=P\{\tau_1\in ds,L(\tau_1)=j\,\big|\,L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,\\
&\cdots,\theta_i(0)=\theta_i\}P\{\bar{\theta}(\tau_1)\in\bar{A},\hat{\theta}_1(\tau_1)\in\hat{A}_1,\cdots,\hat{\theta}_j(\tau_1)\in\hat{A}_j,\tilde{\theta}(\tau_1)\in\tilde{A},\theta_1(\tau_1)\in A,\cdots,\theta_j(\tau_1)\in A_j
\end{aligned}
$$

$$\left| L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1\cdots,\theta_i(0)=\theta_i,\tau_1=s,\right.$$

$$L(\tau_1)=j\}, \tag{3.11}$$

(1)$P\{\tau_1\in ds,L(\tau_1)=i+1\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,$

$\cdots,\theta_i(0)=\theta_i\}=$①式

并且$P\{\bar{\theta}(\tau_1)=0,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\hat{\theta}_{i+1}(\tau_1)=0,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,$

$\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s,\theta_{i+1}(\tau_1)=0\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,$

$$\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1\cdots,\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=i+1\}=1 \tag{3.12}$$

(2)$P\{\tau_1\in ds,L(\tau_1)=i\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,$

$\cdots,\theta_i(0)=\theta_i\}=$②式 并且

$P\{[\bigcup\limits_{k=1}^{i}(\bar{\theta}(\tau_1)=0,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_{k-1}(\tau_1)=\hat{\theta}_{k-1}+s,\hat{\theta}_k(\tau_1)=0,\hat{\theta}_{k+1}(\tau_1)=\hat{\theta}_{k+1}+s,\cdots,$

$\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup\limits_{x=1}^{i}(\bar{\theta}(\tau_1)=0,$

$\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_{x-1}(\tau_1)=\theta_{x-1}+s,$

$\theta_x(\tau_1)=0,\theta_{x+1}(\tau_1)=\theta_{x+1}+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,$

$$\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1\cdots,\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=i\}=1 \tag{3.13}$$

(3)$P\{\tau_1\in ds,L(\tau_1)=i-1\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,$

$\cdots,\theta_i(0)=\theta_i\}=$ ③ 式 并 且

$P\{[\bigcup\limits_{k=1}^{i}(\bar{\theta}(\tau_1)=\bar{\theta}+s,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_{k-1}(\tau_1)=\hat{\theta}_{k-1}+s,\hat{\theta}_k(\tau_1)=0,\hat{\theta}_{k+1}(\tau_1)=\hat{\theta}_{k+1}+s,$

$\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup\limits_{x=1}^{i}(\bar{\theta}(\tau_1)=\bar{\theta}+s,$

$\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_{x-1}(\tau_1)=\theta_{x-1}+s,$

$$\theta_x(\tau_1)=0,\theta_{x+1}(\tau_1)=\theta_{x+1}+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{k=1}^{i}\bigcup_{\substack{x=1\\x\neq k}}^{i}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,$$

$$\cdots,\hat\theta_{k-1}(\tau_1)=\hat\theta_{k-1}+s,\hat\theta_k(\tau_1)=0,\hat\theta_{k+1}(\tau_1)=\hat\theta_{k+1}+s,\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,$$

$$\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_{x-1}(\tau_1)=\theta_{x-1}+s,\theta_x(\tau_1)=0,\theta_{x+1}(\tau_1)=\theta_{x+1}+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup$$

$$[\bigcup_{\Delta_2}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\hat\theta_{n_2}(\tau_1)=0,\hat\theta_{n_3}(\tau_1)=\hat\theta_{n_3}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,$$

$$\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{\Delta_2}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,$$

$$\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\theta_{n_2}(\tau_1)=0,\theta_{n_3}(\tau_1)=\theta_{n_3}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\Big|L(0)=i,$$

$$\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=i\}=1 \tag{3.14}$$

(4)$P\{\tau_1\in ds,L(\tau_1)=i-l\Big|L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,$

$\cdots,\theta_i(0)=\theta_i\}=$④式,并且

$$P\{[\bigcup_{\Delta_l}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_l}(\tau_1)=0,\hat\theta_{n_{l+1}}(\tau_1)=\hat\theta_{n_{l+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,$$

$$\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{e=1}^{l-1}\bigcup_{\Delta_e}\bigcup_{\Delta_{l-e}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\hat\theta_{n_e}(\tau_1)=0,\hat\theta_{n_{e+1}}(\tau_1)=\hat\theta_{n_{e+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{l-e}}(\tau_1)=0,$$

$$\theta_{n_{l-e+1}}(\tau_1)=\theta_{n_{l-e+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_l}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,$$

$$\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_l}(\tau_1)=0,\theta_{n_{l+1}}(\tau_1)=\theta_{n_{l+1}}+s,\cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{l+1}}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_{l+1}}(\tau_1)=0,\hat\theta_{n_{l+2}}(\tau_1)=\hat\theta_{n_{l+2}}+s,\cdots,$$

$$\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{z=1}^{l+1}\bigcup_{\Delta_z}\bigcup_{\Delta_{l+1-z}}(\bar\theta(\tau_1)=0,$$

$$\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_z}(\tau_1)=0,\hat\theta_{n_{z+1}}(\tau_1)=\hat\theta_{n_{z+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,$$

$$\cdots,\theta_{n_{l+1-z}}(\tau_1)=0,\theta_{n_{l+2-z}}(\tau_1)=\theta_{n_{l+2-z}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{l+1}}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,$$

$$\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{l+1}}(\tau_1)=0,\theta_{n_{l+2}}(\tau_1)=\theta_{n_{l+2}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]$$

$$\Big|L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i,\tau_1=s,$$

$$L(\tau_1)=i-l\}=1 \tag{3.15}$$

(5)$P\{\tau_1\in ds, L(\tau_1)=i-(Q-1)\big| L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,$

$\cdots,\theta_i(0)=\theta_i\}=$⑤式，并且

$$P\{[\bigcup_{\Delta_{Q-1}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_{Q-1}}(\tau_1)=0,\hat\theta_{n_Q}(\tau_1)=\hat\theta_{n_Q}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,$$

$$\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{e=1}^{Q-2}\bigcup_{\Delta_e}\bigcup_{\Delta_{Q-1-e}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,$$

$$\cdots,\hat\theta_{n_e}(\tau_1)=0,\hat\theta_{n_{e+1}}(\tau_1)=\hat\theta_{n_{e+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\theta_{n_{Q-1-e}}(\tau_1)=0,\theta_{n_{Q-e}}(\tau_1)=\theta_{n_{Q-e}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{Q-1}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_1(\tau_1)=\hat\theta_1+s,$$

$$\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{Q-1}}(\tau_1)=0,\theta_{n_Q}(\tau_1)=\theta_{n_Q}+s,\cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_Q}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_Q}(\tau_1)=0,\hat\theta_{n_{Q+1}}(\tau_1)=\hat\theta_{n_{Q+1}}+s,\cdots,$$

$$\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{z=1}^{Q-1}\bigcup_{\Delta_z}\bigcup_{\Delta_{Q-z}}(\bar\theta(\tau_1)=0,$$

$$\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_z}(\tau_1)=0,\hat\theta_{n_{z+1}}(\tau_1)=\hat\theta_{n_{z+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,$$

$$\cdots,\theta_{n_{Q-z}}(\tau_1)=0,\theta_{n_{Q-z+1}}(\tau_1)=\theta_{n_{Q-z+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_Q}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,$$

$$\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_Q}(\tau_1)=0,\theta_{n_{Q+1}}(\tau_1)=\theta_{n_{Q+1}}+s,\cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,$$

$$\cdots,\theta_i(\tau_1)=\theta_i+s)]\big|L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,\cdots,$$

$$\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=i-(Q-1)\}=1 \tag{3.16}$$

(6)$P\{\tau_1\in ds,L(\tau_1)=i-Q\big|L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,$

$\cdots,\theta_i(0)=\theta_i\}=$⑥式，并且

$$P\{[\bigcup_{\Delta_Q}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_Q}(\tau_1)=0,\hat\theta_{n_{Q+1}}(\tau_1)=\hat\theta_{n_{Q+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,$$

$$\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{e=1}^{Q-1}\bigcup_{\Delta_e}\bigcup_{\Delta_{Q-e}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,$$

$$\cdots,\hat\theta_{n_e}(\tau_1)=0,\hat\theta_{n_{e+1}}(\tau_1)=\hat\theta_{n_{e+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\theta_{n_{Q-e}}(\tau_1)=0,\theta_{n_{Q-e+1}}(\tau_1)=\theta_{n_{Q-e+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_Q}(\bar{\theta}(\tau_1)=\bar{\theta}+s,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,$$

$$\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_Q}(\tau_1)=0,\theta_{n_{Q+1}}(\tau_1)=\theta_{n_{Q+1}}+s,\cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{Q+1}}(\bar{\theta}(\tau_1)=0,\hat{\theta}_{n_1}(\tau_1)=0,\cdots,\hat{\theta}_{n_{Q+1}}(\tau_1)=0,\hat{\theta}_{n_{Q+2}}(\tau_1)=\hat{\theta}_{n_{Q+2}}+s,\cdots,$$

$$\hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_1(\tau_1)=\theta_1+s,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{z=1}^{Q}\bigcup_{\Delta_z}\bigcup_{\Delta_{Q+1-z}}(\bar{\theta}(\tau_1)=0,$$

$$\hat{\theta}_{n_1}(\tau_1)=0,\cdots,\hat{\theta}_{n_z}(\tau_1)=0,\hat{\theta}_{n_{z+1}}(\tau_1)=\hat{\theta}_{n_{z+1}}+s,\cdots,\hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_{n_1}(\tau_1)=0,$$

$$\cdots,\theta_{n_{Q+1-z}}(\tau_1)=0,\theta_{n_{Q+2-z}}(\tau_1)=\theta_{n_{Q+2-z}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{Q+1}}(\bar{\theta}(\tau_1)=0,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,$$

$$\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{Q+1}}(\tau_1)=0,\theta_{n_{Q+2}}(\tau_1)=\theta_{n_{Q+2}}+s,\cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup(\bar{\theta}(\tau_1)=\bar{\theta}+s,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,$$

$$\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{x=1}^{i}(\bar{\theta}(\tau_1)=0,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,$$

$$\cdots,\theta_{x-1}(\tau_1)=\theta_{x-1}+s,\theta_x(\tau_1)=0,\theta_{x+1}(\tau_1)=\theta_{x+1}+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{k=1}^{i}(\bar{\theta}(\tau_1)=0,$$

$$\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,\hat{\theta}_{k-1}(\tau_1)=\hat{\theta}_{k-1}+s,\hat{\theta}_k(\tau_1)=0,\hat{\theta}_{k+1}(\tau_1)=\hat{\theta}_{k+1}+s,\cdots,\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,$$

$$\tilde{\theta}(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,$$

$$\tilde{\theta}(0)=\tilde{\theta},\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=i-Q\}=1 \quad (3.17)$$

$(7)P\{\tau_1\in ds,L(\tau_1)=i-Q-l\big|L(0)=i,\bar{\theta}(0)=\bar{\theta},\hat{\theta}_1(0)=\hat{\theta}_1,\cdots,\hat{\theta}_i(0)=\hat{\theta}_i,\tilde{\theta}(0)=\tilde{\theta},$

$\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i\}=$⑦式，并且

$$P\{[\bigcup_{\Delta_{Q+l}}(\bar{\theta}(\tau_1)=\bar{\theta}+s,\hat{\theta}_{n_1}(\tau_1)=0,\cdots,\hat{\theta}_{n_{Q+l}}(\tau_1)=0,\hat{\theta}_{n_{Q+l+1}}(\tau_1)=\hat{\theta}_{n_{Q+l+1}}+s,\cdots,\hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s,$$

$$\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{e=1}^{Q+l-1}\bigcup_{\Delta_e}\bigcup_{\Delta_{Q-l-e}}(\bar{\theta}(\tau_1)=\bar{\theta}+s,\hat{\theta}_{n_1}(\tau_1)=0,\cdots,$$

$$\hat{\theta}_{n_e}(\tau_1)=0,\hat{\theta}_{n_{e+1}}(\tau_1)=\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{Q+l-e}}(\tau_1)=0,$$

$$\theta_{n_{Q+l-e+1}}(\tau_1)=\theta_{n_{Q+l-e+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{Q+l}}(\bar{\theta}(\tau_1)=\bar{\theta}+s,\hat{\theta}_1(\tau_1)=\hat{\theta}_1+s,\cdots,$$

$$\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s,\tilde{\theta}(\tau_1)=\tilde{\theta}+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{Q+l}}(\tau_1)=0,\theta_{n_{Q+l+1}}(\tau_1)=\theta_{n_{Q+l+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]$$

$$\bigcup[\bigcup_{\Delta_{Q+l+1}}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_{Q+l+1}}(\tau_1)=0,\hat\theta_{n_{Q+l+2}}(\tau_1)=\hat\theta_{n_{Q+l+2}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,$$

$$\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{z=1}^{Q+l}\bigcup_{\Delta_z}\bigcup_{\Delta_{Q+l+1-z}}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\hat\theta_{n_z}(\tau_1)=0,\hat\theta_{n_{z+1}}(\tau_1)=\hat\theta_{n_{z+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\theta_{n_{Q+l+1-z}}(\tau_1)=0,\theta_{n_{Q+l+2-z}}(\tau_1)=\theta_{n_{Q+l+2-z}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{Q+l+1}}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,$$

$$\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{Q+l+1}}(\tau_1)=0,\theta_{n_{Q+l+2}}(\tau_1)=\theta_{n_{Q+l+2}}+s,\cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_l}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_l}(\tau_1)=0,\hat\theta_{n_{l+1}}(\tau_1)=\hat\theta_{n_{l+1}}+s,\cdots,$$

$$\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{r=1}^{l-1}\bigcup_{\Delta_r}\bigcup_{\Delta_{l-r}}(\bar\theta(\tau_1)=\bar\theta+s,$$

$$\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_r}(\tau_1)=0,\hat\theta_{n_{r+1}}(\tau_1)=\hat\theta_{n_{r+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\theta_{n_{l-r}}(\tau_1)=0,\theta_{n_{l-r+1}}(\tau_1)=\theta_{n_{l-r+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_l}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,$$

$$\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_l}(\tau_1)=0,\theta_{n_{l+1}}(\tau_1)=\theta_{n_{l+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]$$

$$\bigcup[\bigcup_{\Delta_{l+1}}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_{l+1}}(\tau_1)=0,\hat\theta_{n_{l+2}}(\tau_1)=\hat\theta_{n_{l+2}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,$$

$$\tilde\theta(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{v=1}^{l}\bigcup_{\Delta_v}\bigcup_{\Delta_{l+1-v}}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,$$

$$\hat\theta_{n_v}(\tau_1)=0,\hat\theta_{n_{v+1}}(\tau_1)=\hat\theta_{n_{v+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{l+1-z}}(\tau_1)=0,$$

$$\theta_{n_{l+2-z}}(\tau_1)=\theta_{n_{l+2-z}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{l+1}}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,$$

$$\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{l+1}}(\tau_1)=0,\theta_{n_{l+2}}(\tau_1)=\theta_{n_{l+2}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\Big|L(0)=i,\bar\theta(0)=\bar\theta,$$

$$\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,\cdots,\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=i-Q-l\}=1 \tag{3.18}$$

(8) $P\{\tau_1\in ds,L(\tau_1)=1\Big|L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,\cdots,$

$\theta_i(0)=\theta_i\}=$⑧式，并且

$$P\{[\bigcup_{\Delta_{i-1}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_{i-1}}(\tau_1)=0,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,$$

$$
\begin{aligned}
&\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{e=1}^{i-2}\bigcup_{\Delta_e}\bigcup_{\Delta_{i-1-e}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_e}(\tau_1)=0,\\
&\hat\theta_{n_{e+1}}(\tau_1)=\hat\theta_{n_{e+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{i-1-e}}(\tau_1)=0,\\
&\theta_{n_{i-e}}(\tau_1)=\theta_{n_{i-e}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{i-1}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\\
&\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{i-1}}(\tau_1)=0,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)\bigcup(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=0,\\
&\cdots,\hat\theta_i(\tau_1)=0,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{z=1}^{i-1}\bigcup_{\Delta_z}\bigcup_{\Delta_{i-z}}(\bar\theta(\tau_1)=0,\\
&\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_z}(\tau_1)=0,\hat\theta_{n_{z+1}}(\tau_1)=\hat\theta_{n_{z+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_{n_1}(\tau_1)=0,\\
&\cdots,\theta_{n_{i-z}}(\tau_1)=0,\theta_{n_{i-z+1}}(\tau_1)=\theta_{n_{i-z+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\\
&\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=\tilde\theta+s,\theta_1(\tau_1)=0,\cdots,\theta_i(\tau_1)=0)\bigcup[\bigcup_{\Delta_{i-1-Q}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\\
&\hat\theta_{n_{i-1-Q}}(\tau_1)=10,\hat\theta_{n_{i-Q}}(\tau_1)=\hat\theta_{n_{i-Q}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\theta_1(\tau_1)=\theta_1+s,\cdots,\\
&\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{r=1}^{i-Q-2}\bigcup_{\Delta_r}\bigcup_{\Delta_{i-Q-1-r}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_r}(\tau_1)=0,\hat\theta_{n_{r+1}}(\tau_1)=\\
&\hat\theta_{n_{r+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{i-Q-1-r}}(\tau_1)=0,\theta_{n_{i-Q-r}}(\tau_1)=\theta_{n_{i-Q-r}}+s,\\
&\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{i-Q-1}}(\bar\theta(\tau_1)=\bar\theta+s,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=0,\\
&\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{i-Q-1}}(\tau_1)=0,\theta_{n_{i-Q}}(\tau_1)=\theta_{n_{i-Q}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{i-Q}}(\bar\theta(\tau_1)=0,\\
&\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_{i-Q}}(\tau_1)=0,\hat\theta_{n_{i-Q+1}}(\tau_1)=\hat\theta_{n_{i-Q+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\\
&\theta_1(\tau_1)=\theta_1+s,\cdots,\theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{v=1}^{i-Q-1}\bigcup_{\Delta_v}\bigcup_{\Delta_{i-Q-v}}(\bar\theta(\tau_1)=0,\hat\theta_{n_1}(\tau_1)=0,\cdots,\hat\theta_{n_v}(\tau_1)=0,\\
&\hat\theta_{n_{v+1}}(\tau_1)=\hat\theta_{n_{v+1}}+s,\cdots,\hat\theta_{n_i}(\tau_1)=\hat\theta_{n_i}+s,\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{i-Q-v}}(\tau_1)=0,\\
&\theta_{n_{i-Q-v+1}}(\tau_1)=\theta_{n_{i-Q-v+1}}+s,\cdots,\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{i-Q}}(\bar\theta(\tau_1)=0,\hat\theta_1(\tau_1)=\hat\theta_1+s,\cdots,\\
&\hat\theta_i(\tau_1)=\hat\theta_i+s,\tilde\theta(\tau_1)=0,\theta_{n_1}(\tau_1)=0,\cdots,\theta_{n_{i-Q}}(\tau_1)=0,\theta_{n_{i-Q+1}}(\tau_1)=\theta_{n_{i-Q+1}}+s,\cdots,\\
&\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\Big|L(0)=i,\bar\theta(0)=\bar\theta,\hat\theta_1(0)=\hat\theta_1,\cdots,\hat\theta_i(0)=\hat\theta_i,\tilde\theta(0)=\tilde\theta,\theta_1(0)=\theta_1,\cdots,\\
&\theta_i(0)=\theta_i,\tau_1=s,L(\tau_1)=1\}=1
\end{aligned}
\tag{3.19}
$$

$(9)P\{\tau_1 \in ds, L(\tau_1)=0 \big| L(0)=i, \bar{\theta}(0)=\bar{\theta}, \hat{\theta}_1(0)=\hat{\theta}_1, \cdots, \hat{\theta}_i(0)=\hat{\theta}_i, \tilde{\theta}(0)=\tilde{\theta}, \theta_1(0)=\theta_1, \cdots,$

$\theta_i(0)=\theta_i\}=$⑨式,并且

$$P\{(\bar{\theta}(\tau_1)=\bar{\theta}+s, \hat{\theta}_1(\tau_1)=0, \cdots, \hat{\theta}_i(\tau_1)=0, \tilde{\theta}(\tau_1)=\tilde{\theta}+s, \theta_1(\tau_1)=\theta_1+s, \cdots, \theta_i(\tau_1)=\theta_i+s)$$

$$\bigcup[\bigcup_{e=1}^{i-1}\bigcup_{\Delta_e}\bigcup_{\Delta_{i-e}}(\bar{\theta}(\tau_1)=\bar{\theta}+s, \hat{\theta}_{n_1}(\tau_1)=0, \cdots, \hat{\theta}_{n_e}(\tau_1)=0, \hat{\theta}_{n_{e+1}}(\tau_1)=\hat{\theta}_{n_{e+1}}+s, \cdots, \hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s,$$

$$\tilde{\theta}(\tau_1)=\tilde{\theta}+s, \theta_{n_1}(\tau_1)=0, \cdots, \theta_{n_{i-e}}(\tau_1)=0, \theta_{n_{i-e+1}}(\tau_1)=\theta_{n_{i-e+1}}+s, \cdots, \theta_{n_i}(\tau_1)=\theta_{n_i}+s)]$$

$$\bigcup(\bar{\theta}(\tau_1)=\bar{\theta}+s, \hat{\theta}_1(\tau_1)=\hat{\theta}_1+s, \cdots, \hat{\theta}_i(\tau_1)=\hat{\theta}_i+s, \tilde{\theta}(\tau_1)=\tilde{\theta}+s, \theta_1(\tau_1)=0, \cdots, \theta_i(\tau_1)=0)$$

$$\bigcup[\bigcup_{\Delta_{i-Q}}(\bar{\theta}(\tau_1)=\bar{\theta}+s, \hat{\theta}_{n_1}(\tau_1)=0, \cdots, \hat{\theta}_{n_{i-Q}}(\tau_1)=0, \hat{\theta}_{n_{i-Q+1}}(\tau_1)=\hat{\theta}_{n_{i-Q+1}}+s, \cdots, \hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s,$$

$$\tilde{\theta}(\tau_1)=0, \theta_1(\tau_1)=\theta_1+s, \cdots, \theta_i(\tau_1)=\theta_i+s)]\bigcup[\bigcup_{r=1}^{i-Q-1}\bigcup_{\Delta_r}\bigcup_{\Delta_{i-Q-r}}(\bar{\theta}(\tau_1)=\bar{\theta}+s, \hat{\theta}_{n_1}(\tau_1)=0, \cdots,$$

$$\hat{\theta}_{n_r}(\tau_1)=0, \hat{\theta}_{n_{r+1}}(\tau_1)=\hat{\theta}_{n_{r+1}}+s, \cdots, \hat{\theta}_{n_i}(\tau_1)=\hat{\theta}_{n_i}+s, \tilde{\theta}(\tau_1)=0, \theta_{n_1}(\tau_1)=0, \cdots, \theta_{n_{i-Q-r}}(\tau_1)=0,$$

$$\theta_{n_{i-Q-r+1}}(\tau_1)=\theta_{n_{i-Q-r+1}}+s, \cdots, \theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\bigcup[\bigcup_{\Delta_{i-Q}}(\bar{\theta}(\tau_1)=\bar{\theta}+s, \hat{\theta}_1(\tau_1)=\hat{\theta}_1+s, \cdots,$$

$$\hat{\theta}_i(\tau_1)=\hat{\theta}_i+s, \tilde{\theta}(\tau_1)=0, \theta_{n_1}(\tau_1)=0, \cdots, \theta_{n_{i-Q}}(\tau_1)=0, \theta_{n_{i-Q+1}}(\tau_1)=\theta_{n_{i-Q+1}}+s, \cdots,$$

$$\theta_{n_i}(\tau_1)=\theta_{n_i}+s)]\big| L(0)=i, \bar{\theta}(0)=\bar{\theta}, \hat{\theta}_1(0)=\hat{\theta}_1, \cdots, \hat{\theta}_i(0)=\hat{\theta}_i, \tilde{\theta}(0)=\tilde{\theta}, \theta_1(0)=\theta_1, \cdots,$$

$$\theta_i(0)=\theta_i, \tau_1=s, L(\tau_1)=0\}=1 \tag{3.20}$$

于是由(3.11)至(3.20)可以得到(3.9)、(3.10)，从而证得引理 3.3.1、引理 3.3.2。

3.4 离散条件下的单种群种群数量研究

定理 3.3.1

$p(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$是下列非负线性方程的最小非负解，也是唯一有界解：

$$p(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$

$$=h(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$

$$+\int_0^t\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-C_{\tilde{\theta}}(s))dA_{\bar{\theta}}(s)p(i+1,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,$$

$$\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{k=1}^{i}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))$$

$$p(i,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_{k-1}+s,\hat{\theta}_{k+1}+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{x=1}^{i}[(D_{\theta_x}(s)-D_{\theta_x}(s-))\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))]$$

$$p(i,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_{x-1}+s,\theta_{x+1}+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{k=1}^{i}\int_0^t\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))dB_{\hat{\theta}_k}(s)p(i-1,\bar{\theta}+s,\hat{\theta}_1+s,$$

$$\cdots,\hat{\theta}_{k-1}+s,\hat{\theta}_{k+1}+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{x=1}^{i}\int_0^t\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))dD_{\theta_x}(s)p(i-1,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,$$

$$\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_{x-1}+s,\theta_{x+1}+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{k=1}^{i}\sum_{\substack{x=1\\ x\ne k}}^{i}[(D_{\theta_x}(s)-D_{\theta_x}(s-))\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))$$

$$(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))]p(i-1,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_{k-1}+s,\hat{\theta}_{k+1}+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,$$

$$\theta_{x-1}+s,\theta_{x+1}+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_2}[\prod_{k=1}^{2}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=3}^{i}(1-B_{n_m}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))$$

$$p(i-1,0,\hat{\theta}_{n_3}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_2}[\prod_{x=1}^{2}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=3}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(A_{\theta}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))$$

$$p(i-1,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_3}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=2}^{Q-2}\sum_{\Delta_l}[\prod_{k=1}^{l}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=l+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(i-l,\bar{\theta}+s,\hat{\theta}_{n_{l+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=2}^{Q-2}\sum_{e=1}^{l-1}\{\sum_{\Delta_e}[\sum_{\Delta_{l-e}}\prod_{x=1}^{l-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l-e+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))p(i-l,\bar{\theta}+s,\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{l-e+1}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=2}^{Q-2}\sum_{\Delta_l}[\prod_{x=1}^{l}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(i-l,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{l+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=2}^{Q-2}\sum_{\Delta_{l+1}}[\prod_{k=1}^{l+1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=l+2}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\theta}(s)-A_{\theta}(s-))$$

$$(1-C_{\tilde{\theta}}(s))p(i-l,0,\hat{\theta}_{n_{l+2}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=2}^{Q-2}\sum_{z=1}^{l}\{\sum_{\Delta_z}[\sum_{\Delta_{l+1-z}}\prod_{x=1}^{l+1-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2-z}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(A_{\theta}(s)-A_{\theta}(s-))(1-C_{\tilde{\theta}}(s))p(i-l,0,\hat{\theta}_{n_{z+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,$$

$$\theta_{n_{l+2-z}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=2}^{Q-2}\sum_{\Delta_{l+1}}[\prod_{x=1}^{l+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(A_{\theta}(s)-A_{\theta}(s-))$$

$$(1-C_{\tilde{\theta}}(s))p(i-l,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{l+2}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,$$

$$A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_{Q-1}}[\prod_{k=1}^{Q-1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(i-(Q-1),\bar{\theta}+s,\hat{\theta}_{n_Q}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{e=1}^{Q-2}\{\sum_{\Delta_e}[\sum_{\Delta_{Q-1-e}}\prod_{x=1}^{Q-1-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q-e}^{i}(1-D_{\theta_{n_u}}(s))](1-A_{\bar{\theta}}(s))\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(1-C_{\tilde{\theta}}(s))p(i-(Q-1),\bar{\theta}+s,\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{Q-e}}+s,\cdots,\theta_{n_i}+s,$$

$$j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_{Q-1}}[\prod_{x=1}^{Q-1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(i-(Q-1),\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_Q}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_Q}[\prod_{k=1}^{Q}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))$$

$$p(i-(Q-1),0,\hat{\theta}_{n_{Q+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{z=1}^{Q-1}\{\sum_{\Delta_z}[\sum_{\Delta_{Q-z}}\prod_{x=1}^{Q-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1-z}^{i}(1-D_{\theta_{n_u}}(s))](1-C_{\tilde{\theta}}(s))\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))p(i-(Q-1),0,\hat{\theta}_{n_{z+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{Q+1-z}}+s,\cdots,$$

$$\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_Q}[\prod_{x=1}^{Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-C_{\tilde{\theta}}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))$$

$$p(i-(Q-1),0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{Q+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))$$

$$p(i-(Q-1),0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_Q}\prod_{k=1}^{Q}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(i-Q,\bar{\theta}+s,\hat{\theta}_{n_{Q+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{e=1}^{Q-1}\{\sum_{\Delta_e}[\sum_{\Delta_{Q-e}}\prod_{x=1}^{Q-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1-e}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))p(i-Q,\bar{\theta}+s,\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{Q+1-e}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_Q}[\prod_{x=1}^{Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(i-Q,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{Q+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_{Q+1}}[\prod_{k=1}^{Q+1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+2}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\theta}(s)-A_{\theta}(s-))(1-C_{\tilde{\theta}}(s))$$

$$p(i-Q,0,\hat{\theta}_{n_{Q+2}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{z=1}^{Q}\{\sum_{\Delta_z}[\sum_{\Delta_{Q+1-z}}\prod_{x=1}^{Q+1-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+2-z}^{i}(1-D_{\theta_{n_u}}(s))](1-C_{\tilde{\theta}}(s))(A_{\theta}(s)-A_{\theta}(s-))$$

$$\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}p(i-Q,0,\hat{\theta}_{n_{z+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{Q+2-z}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{\Delta_{Q+1}}[\prod_{x=1}^{Q+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+2}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(A_{\theta}(s)-A_{\theta}(s-))$$

$$(1-C_{\tilde{\theta}}(s))p(i-Q,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{Q+2}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\int_0^t(1-A_{\bar{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))dC_{\tilde{\theta}}(s)p(i-Q,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,$$

$$\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$
+\sum_{s\le t}[\sum_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))
$$

$$
p(i-Q,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_1+s,\cdots,\theta_{x-1}+s,\theta_{x+1}+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},
$$

$$
A_1,\cdots,A_j,t-s)
$$

$$
+\sum_{s\le t}[\sum_{k=1}^{i}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\bar{\theta}}(s)-A_{\theta}(s-))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))
$$

$$
p(i-Q,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_{k-1}+s,\hat{\theta}_{k+1}+s,\cdots,\hat{\theta}_i+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,
$$

$$
\cdots,A_j,t-s)
$$

$$
+\sum_{s\le t}\sum_{l=1}^{i-Q-2}\sum_{\Delta_{Q+l}}\prod_{k=1}^{Q+l}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+l+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))
$$

$$
p(i-Q-l,\bar{\theta}+s,\hat{\theta}_{n_{Q+l+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,
$$

$$
A_j,t-s)
$$

$$
+\sum_{s\le t}\sum_{l=1}^{i-Q-2}\sum_{e=1}^{Q+l-1}\{\sum_{\Delta_e}[\sum_{\Delta_{Q+l-e}}\prod_{x=1}^{Q+l-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+1-e}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))
$$

$$
\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))p(i-Q-l,\bar{\theta}+s,\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{Q+l+1-e}}+s,
$$

$$
\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)
$$

$$
+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_{Q+l}}\prod_{x=1}^{Q+l}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))
$$

$$
p(i-Q-l,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{Q+l+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)
$$

$$
+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_{Q+l+1}}\prod_{k=1}^{Q+l+1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=Q+l+2}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))
$$

$$
(1-C_{\tilde{\theta}}(s))p(i-Q-l,0,\hat{\theta}_{n_{Q+l+2}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},
$$

$$
A_1,\cdots,A_j,t-s)
$$

$$
+\sum_{s\le t}\sum_{l=1}^{i-Q-2}\sum_{z=1}^{Q+l}\{\sum_{\Delta_z}[\sum_{\Delta_{Q+l+1-z}}\prod_{x=1}^{Q+l+1-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+2-z}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))
$$

$$\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(A_{\tilde{\theta}}(s)-A_{\tilde{\theta}}(s-))(1-C_{\bar{\theta}}(s))p(i-Q-l,0,\hat{\theta}_{n_{z+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,$$

$$\theta_{n_{Q+l+2-z}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_{Q+l+1}}\prod_{x=1}^{Q+l+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=Q+l+2}^{i}(1-D_{\theta_{n_u}}(s))](1-C_{\bar{\theta}}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))$$

$$(A_{\tilde{\theta}}(s)-A_{\theta}(s-))p(i-Q-l,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_{Q+l+2}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,$$

$$\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_l}\prod_{k=1}^{l}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=l+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\theta}(s))$$

$$(C_{\bar{\theta}}(s)-C_{\bar{\theta}}(s-))p(i-Q-l,\bar{\theta}+s,\hat{\theta}_{n_{l+1}}+s,\cdots,\hat{\theta}_{n_i}+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,$$

$$\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}\sum_{r=1}^{l-1}\{\sum_{\Delta_r}[\sum_{\Delta_{l-r}}\prod_{x=1}^{l-r}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l-r+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{r}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=r+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(1-A_{\theta}(s))(C_{\bar{\theta}}(s)-C_{\bar{\theta}}(s-))p(i-Q-l,\bar{\theta}+s,\hat{\theta}_{n_{r+1}}+s,\cdots,\hat{\theta}_{n_i}+s,0,$$

$$\theta_{n_{l-r+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_l}\prod_{x=1}^{l}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(C_{\bar{\theta}}(s)-C_{\bar{\theta}}(s-))$$

$$(1-A_{\tilde{\theta}}(s))p(i-Q-l,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_{n_{l+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_{l+1}}\prod_{k=1}^{l+1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=l+2}^{i}(1-B_{\hat{\theta}_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(A_{\tilde{\theta}}(s)-A_{\tilde{\theta}}(s-)$$

$$(C_{\bar{\theta}}(s)-C_{\bar{\theta}}(s-))p(i-Q-l,0,\hat{\theta}_{n_{l+2}}+s,\cdots,\hat{\theta}_{n_i}+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,$$

$$\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}\sum_{v=1}^{l}\{\sum_{\Delta_v}[\sum_{\Delta_{l+1-v}}\prod_{x=1}^{l+1-v}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2-v}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{v}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=v+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))p(i-Q-l,0,\hat{\theta}_{n_{v+1}}+s,\cdots,\hat{\theta}_{n_i}+s,$$

$$0,\theta_{n_{l+2-v}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{l=1}^{i-Q-2}[\sum_{\Delta_{l+1}}\prod_{x=1}^{l+1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=l+2}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))$$

$$(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))p(i-Q-l,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_{n_{l+2}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-1}}\prod_{k=1}^{i-1}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))(1-B_{\hat{\theta}_{n_i}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(1,\bar{\theta}+s,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{e=1}^{i-2}\{\sum_{\Delta_e}[\sum_{\Delta_{i-1-e}}\prod_{x=1}^{i-1-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-e}^{i}(1-D_{\theta_{n_u}}(s))](1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))$$

$$\prod_{k=1}^{e}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}p(1,\bar{\theta}+s,\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{i-e}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-1}}\prod_{x=1}^{i-1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))(1-D_{\theta_{n_i}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))$$

$$p(1,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\prod_{k=1}^{i}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))$$

$$p(1,0,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{z=1}^{i-1}\{\sum_{\Delta_z}[\sum_{\Delta_{i-z}}\prod_{x=1}^{i-z}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i+1-z}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{m=z+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))(1-C_{\tilde{\theta}}(s))$$

$$\prod_{k=1}^{z}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\}(A_{\bar{\theta}}(s)-A_{\bar{\theta}}(s-))p(1,0,\hat{\theta}_{n_{z+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{i+1-z}}+s,\cdots,$$

$$\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}(A_{\bar\theta}(s)-A_{\bar\theta}(s-))(1-C_{\tilde\theta}(s))\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))\prod_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))$$

$$p(1,0,\hat\theta_1+s,\cdots,\hat\theta_i+s,\tilde\theta+s,j,\bar A,\hat A_1,\cdots,\hat A_j,\tilde A,A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-1-Q}}\prod_{k=1}^{i-1-Q}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=i-Q}^{i}(1-B_{\hat\theta_{n_m}}(s))]\prod_{x=1}^{i}(1-D_{\theta_x}(s))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))$$

$$(1-A_{\bar\theta}(s))p(1,\bar\theta+s,\hat\theta_{n_{i-Q}}+s,\cdots,\hat\theta_{n_i}+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar A,\hat A_1,\cdots,\hat A_j,\tilde A,A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{r=1}^{i-2-Q}\{\sum_{\Delta_r}[\sum_{\Delta_{i-1-Q-r}}\prod_{x=1}^{i-1-Q-r}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q-r}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{r}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))$$

$$\prod_{m=r+1}^{i}(1-B_{\hat\theta_{n_m}}(s))\}(1-A_{\bar\theta}(s))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))p(1,\bar\theta+s,\hat\theta_{n_{r+1}}+s,\cdots,\hat\theta_{n_i}+s,0,\theta_{n_{i-Q-r}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar A,\hat A_1,\cdots,\hat A_j,\tilde A,A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-Q-1}}\prod_{x=1}^{i-Q-1}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))$$

$$(1-A_{\bar\theta}(s))p(1,\bar\theta+s,\hat\theta_1+s,\cdots,\hat\theta_i+s,0,\theta_{n_{i-Q}}+s,\cdots,\theta_{n_i}+s,j,\bar A,\hat A_1,\cdots,\hat A_j,\tilde A,$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-Q}}\prod_{k=1}^{i-Q}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=i-Q+1}^{i}(1-B_{\hat\theta_{n_m}}(s))](A_{\bar\theta}(s)-A_{\bar\theta}(s-))(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))$$

$$\prod_{x=1}^{i}(1-D_{\theta_x}(s))p(1,0,\hat\theta_{n_{i-Q+1}}+s,\cdots,\hat\theta_{n_i}+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar A,\hat A_1,\cdots,\hat A_j,\tilde A,$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{v=1}^{i-1-Q}\{\sum_{\Delta_v}[\sum_{\Delta_{i-Q-v}}\prod_{x=1}^{i-Q-v}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q+1-v}^{i}(1-D_{\theta_{n_u}}(s))](A_{\bar\theta}(s)-A_{\bar\theta}(s-))$$

$$\prod_{k=1}^{v}(B_{\hat\theta_{n_k}}(s)-B_{\hat\theta_{n_k}}(s-))\prod_{m=v+1}^{i}(1-B_{\hat\theta_{n_m}}(s))\}(C_{\tilde\theta}(s)-C_{\tilde\theta}(s-))p(1,0,\hat\theta_{n_{v+1}}+s,\cdots,\hat\theta_{n_i}+s,$$

$$0,\theta_{n_{i-Q+1-v}}+s,\cdots,\theta_{n_i}+s,j,\bar A,\hat A_1,\cdots,\hat A_j,\tilde A,A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-Q}}\prod_{x=1}^{i-Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat\theta_k}(s))(A_{\bar\theta}(s)-A_{\bar\theta}(s-))$$

$$(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))p(1,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_{n_{i-Q+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))\prod_{k=1}^{i}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))p(0,\bar{\theta}+s,\tilde{\theta}+s,\theta_1+s,$$

$$\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{e=1}^{i-1}\{\sum_{\Delta_e}[\sum_{\Delta_{i-e}}\prod_{x=1}^{i-e}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-e+1}^{i}(1-D_{\theta_{n_u}}(s))](1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))$$

$$\prod_{k=1}^{e}(B_{\hat{\theta}_k}(s)-B_{\hat{\theta}_k}(s-))\prod_{m=e+1}^{i}(1-B_{\hat{\theta}_m}(s))\}p(0,\bar{\theta}+s,\hat{\theta}_{n_{e+1}}+s,\cdots,\hat{\theta}_{n_i}+s,\tilde{\theta}+s,\theta_{n_{i-e+1}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(D_{\theta_x}(s)-D_{\theta_x}(s-))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))p(0,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,$$

$$\hat{\theta}_i+s,\tilde{\theta}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-Q}}\prod_{k=1}^{i-Q}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))\prod_{m=i-Q+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))](1-A_{\theta}(s))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))\prod_{x=1}^{i}(1-D_{\theta_x}(s))$$

$$p(0,\bar{\theta}+s,\hat{\theta}_{n_{i-Q+1}}+s,\cdots,\hat{\theta}_{n_i}+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}\sum_{r=1}^{i-Q-1}\{\sum_{\Delta_r}[\sum_{\Delta_{i-Q-r}}\prod_{x=1}^{i-Q-r}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q-r+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{r}(B_{\hat{\theta}_{n_k}}(s)-B_{\hat{\theta}_{n_k}}(s-))$$

$$\prod_{m=r+1}^{i}(1-B_{\hat{\theta}_{n_m}}(s))\}(1-A_{\theta}(s))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))p(0,\bar{\theta}+s,\hat{\theta}_{n_{r+1}}+s,\cdots,\hat{\theta}_{n_i}+s,0,\theta_{n_{i-Q-r+1}}+s,$$

$$\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}[\sum_{\Delta_{i-Q}}\prod_{x=1}^{i-Q}(D_{\theta_{n_x}}(s)-D_{\theta_{n_x}}(s-))\prod_{u=i-Q+1}^{i}(1-D_{\theta_{n_u}}(s))]\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))$$

$$(1-A_{\bar{\theta}}(s))p(0,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_{n_{i-Q+1}}+s,\cdots,\theta_{n_i}+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},$$

$$A_1,\cdots,A_j,t-s)$$

证明：由 $\{L(t),\bar{\theta}(t),\hat{\theta}_1(t),\cdots,\hat{\theta}_{L(t)}(t),\tilde{\theta}(t),\theta_1(t),\cdots,\theta_{L(t)}(t)\}$ 是以 $\{\tau_n\}_{n=0}^{\infty}$ 为骨架时序列

的马尔可夫骨架过程，以及定理 2.2.1 即证得本定理成立。

3.5 连续条件下的单种群种群数量研究

定理 3.3.2 若$B(t),D(t)$为连续函数，则$\{p(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)\}$是下列非负线性方程的最小非负解，也是其唯一有界解：

$$p(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$

$$=h(i,\bar{\theta},\hat{\theta}_1,\cdots,\hat{\theta}_i,\tilde{\theta},\theta_1,\cdots,\theta_i,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t)$$

$$+\int_0^t\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-C_{\tilde{\theta}}(s))dA_{\bar{\theta}}(s)p(i+1,0,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,$$

$$\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{k=1}^{i}\int_0^t\prod_{\substack{1\le m\le i\\ m\ne k}}(1-B_{\hat{\theta}_m}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))(1-A_{\bar{\theta}}(s))(1-C_{\tilde{\theta}}(s))dB_{\hat{\theta}_k}(s)p(i-1,\bar{\theta}+s,\hat{\theta}_1+s,$$

$$\cdots,\hat{\theta}_{k-1}+s,\hat{\theta}_{k+1}+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{x=1}^{i}\int_0^t\prod_{\substack{1\le n\le i\\ n\ne x}}(1-D_{\theta_n}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))(1-A_{\theta}(s))(1-C_{\tilde{\theta}}(s))dD_{\theta_x}(s)p(i-1,\bar{\theta}+s,\hat{\theta}_1+s,$$

$$\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,\theta_1+s,\cdots,\theta_{x-1}+s,\theta_{x+1}+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\sum_{s\le t}(A_{\theta}(s)-A_{\theta}(s-))(C_{\tilde{\theta}}(s)-C_{\tilde{\theta}}(s-))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))p(i-(Q-1),0,$$

$$\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,0,\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

$$+\int_0^t(1-A_{\theta}(s))\prod_{k=1}^{i}(1-B_{\hat{\theta}_k}(s))\prod_{x=1}^{i}(1-D_{\theta_x}(s))dC_{\tilde{\theta}}(s)p(i-Q,\bar{\theta}+s,\hat{\theta}_1+s,\cdots,\hat{\theta}_i+s,\tilde{\theta}+s,$$

$$\theta_1+s,\cdots,\theta_i+s,j,\bar{A},\hat{A}_1,\cdots,\hat{A}_j,\tilde{A},A_1,\cdots,A_j,t-s)$$

证明：由定理 3.3.1 可得此定理。

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攻读硕士期间主要研究成果

[1] 张希娜，赵清贵，李占光，徐娟.马尔可夫骨架过程在种群动态学中的应用.数学理论与应用，2008，28（3)：77-80

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[3]《The probability of ruin in a markov-modulated risk model》（第三作者)《工程学报》已接收