Quantum Computing Seminar 8

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Integer Factorization with Quantum Computer

Problem (Integer factorization)

Input: N-bit integer A
Output: Factorization of A

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As we've discussed before, the best known classical algorithm has super-polynomial time complexity.
We'll look at **Shor's algorithm**, which solves integer factorization problem using $O(N^2 \cdot \log(N))$ gates.

Order Finding Problem

Order Finding Problem

Definition (Euler's totient function)

Euler's totient function $\phi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is defined as
$\phi(n) =$ (Number of integers $1 \leq a < n$ which is coprime to n)

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Theorem

Let n be a positive integer with factorization $n = p_1^{e_1} \cdots p_k^{e_k}$. Then
$\phi(n) = (p_1^{e_1} - p_1^{e_1 - 1}) \cdots (p_k^{e_k} - p_k^{e_k - 1})$.

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Definition (Multiplicative order)

Given a positive integer n and a positive integer a coprime to n, the **(multiplicative) order** of a modulo n is the minimum positive integer k with $a^k = 1 \mod n$.

Order Finding Problem

Problem (Order finding problem)

Input: Positive coprime integers a and n
Output: Order of a modulo n

Shor's Algorithm

Shor's Algorithm

Shor's algorithm consists of two parts.

1. Classical reduction of the integer factorization problem to the order finding problem
2. Quantum algorithm to solve order finding problem

Shor's Algorithm (Part 1)

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Algorithm	Factorize
Input	Positive integer n
Output	Factorization of n

1. If n is prime, we're done.
2. Otherwise, let $d = \text{FindANontrivialFactor(n)}$. We output the combination of $\text{Factorize}(d)$ and Factorize(n/d).

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Input	Positive composite integer n
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    3.4 If $\gcd(a^{\mathrm{FindOrder}_n(a)/2} - 1, n) > 1$, return $\gcd(a^{\mathrm{FindOrder}_n(a)/2} - 1, n)$.
    3.5 Otherwise, go to step 3.

Shor's Algorithm (Part 2)

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Reminder

For a non-negative integer n and a complex number r,

$$1 + r + \cdots + r^{n-1} = \begin{cases} n & \text{if } r = 1 \\ \frac{1-r^n}{1-r} & \text{if } r \neq 1 \end{cases}$$

Shor's Algorithm (Part 2)

Definition (Primitive root of unity)

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Theorem (Root-of-unity filter)

For a positive integer n and an integer k,

$$\omega_n^{0\cdot k} + \omega_n^{1\cdot k} + \cdots + \omega_n^{(n-1)\cdot k} = \begin{cases} n & \text{if } k = 0 \mod n \\ 0 & \text{if } k \neq 0 \mod n \end{cases}$$

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Proof

- If $k = 0 \mod n$, $\omega_n^k = 1$, so $\omega_n^{0\cdot k} + \omega_n^{1\cdot k} + \cdots + \omega_n^{(n-1)\cdot k} = 1 + \cdots + 1 = n$.

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- If $k \neq 0 \mod n$, $\omega_n^k \neq 1$, so $\omega_n^{0\cdot k} + \omega_n^{1\cdot k} + \cdots + \omega_n^{(n-1)\cdot k} = \frac{1-\omega_n^{n\cdot k}}{1-\omega_n^k} = 0$.

Shor's Algorithm (Part 2)

Definition (Root-of-unity basis)

For a positive integer n, let $|b_k\rangle = \frac{1}{\sqrt{n}}(\omega_n^{0\cdot k}, \cdots, \omega_n^{(n-1)\cdot k})$ for integers $0 \leq k < n$. Then the set $\{ \, | \, b_0\rangle, \cdots, \, | \, b_{n-1}\rangle \, \}$ is an orthonormal basis of $\mathbb{C}^n$, called the **root-of-unity basis**.

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Proof

$$\begin{aligned}\langle b_i | b_j \rangle &= \frac{1}{n}\left(\overline{\omega_n^{0\cdot i}} \cdot \omega_n^{0\cdot j} + \cdots + \overline{\omega_n^{(n-1)\cdot i}} \cdot \omega_n^{(n-1)\cdot j}\right) \\ &= \frac{1}{n}\left(\omega_n^{0\cdot(j-i)} + \cdots + \omega_n^{(n-1)\cdot(j-i)}\right) \\ &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}\end{aligned}$$

Shor's Algorithm (Part 2)

Definition (Discrete Fourier transform)

For a positive integer n, **discrete Fourier transform** DFT_n is the n by n matrix transforming the standard basis vector $|i\rangle$ to the root-of-unity basis vector $|b_i\rangle$.

$$DFT_n := \begin{bmatrix} | & & | \\ |b_0\rangle & \cdots & |b_{n-1}\rangle \\ | & & | \end{bmatrix} = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \omega_n^{i \cdot j} |i\rangle \langle j|$$

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Since $\{ b_0, \cdots, b_{n-1} \}$ is an orthonormal basis, DFT_n is a unitary matrix.

$$DFT_n^H = \begin{bmatrix} - & \langle b_0| & - \\ & \vdots & \\ - & \langle b_{n-1}| & - \end{bmatrix} = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \omega_n^{-i \cdot j} |i\rangle \langle j|$$

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Definition (Quantum Fourier transform)

For a non-negative integer m, **quantum Fourier transform** QFT_{2^m} is the unitary quantum gate on m-qubits, whose action corresponds to DFT_{2^m}.

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Q) How would you perform the root-of-unity basis measurement on a given quantum state?

Implementing quantum Fourier transform

Shor's Algorithm (Part 2)

Implementing quantum Fourier transform

We recursively build QFT_{2^m}, starting from $QFT_{2^0} = I_1$. For an arbitrary integer $0 \leq x < 2^m$,

$$
\begin{aligned}
QFT_{2^m} |\text{binary}(x)\rangle &= \frac{1}{\sqrt{2^m}} \sum_{i=0}^{2^m-1} \omega_{2^m}^{x \cdot i} |\text{binary}(i)\rangle \\
&= \frac{1}{\sqrt{2^m}} \sum_{i=0}^{2^{m-1}-1} \left(\omega_{2^m}^{2 \cdot x \cdot i} |\text{binary}(2 \cdot i)\rangle + \omega_{2^m}^{2 \cdot x \cdot i + x} |\text{binary}(2 \cdot i + 1)\rangle \right) \\
&= \frac{1}{\sqrt{2^m}} \sum_{i=0}^{2^{m-1}-1} \omega_{2^{m-1}}^{x \cdot i} |\text{binary}(i)\rangle \otimes (|0\rangle + \omega_{2^m}^{x} |1\rangle) \\
&= QFT_{2^{m-1}} |\text{binary}(x - 2^{\lfloor \log_2(x) \rfloor})\rangle \otimes \frac{1}{\sqrt{2}} (|0\rangle + \omega_{2^m}^{x} |1\rangle)
\end{aligned}
$$

Implementing quantum Fourier transform

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\begin{aligned}
& QFT_{2^m} \left|\text{binary}(x)\right\rangle \\
& = QFT_{2^{m-1}} \left|\text{binary}(x - 2^{\lfloor \log_2(x) \rfloor})\right\rangle \otimes \frac{1}{\sqrt{2}} \left(\left|0\right\rangle + \omega_{2^m}^{x} \left|1\right\rangle\right) \\
& = \text{BITSWAPS}_{m-1\to 0} \left(\frac{1}{\sqrt{2}} \left(\left|0\right\rangle + \omega_{2^m}^{x} \left|1\right\rangle\right) \otimes QFT_{2^{m-1}} \left|\text{binary}(x - 2^{\lfloor \log_2(x) \rfloor})\right\rangle \right) \\
& = \text{BITSWAPS}_{m-1\to 0} \left(\left(I_2 \otimes QFT_{2^{m-1}}\right) \cdot \left(\frac{1}{\sqrt{2}} \left(\left|0\right\rangle + \omega_{2^m}^{x} \left|1\right\rangle\right) \otimes \left|\text{binary}(x - 2^{\lfloor \log_2(x) \rfloor})\right\rangle \right) \right)
\end{aligned}
$$

Implementing quantum Fourier transform

Reminder: action of H and CP

The action of the Hadamard gate H and the controlled phase gate CP is given by the following matrices

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, CP_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{\theta \mathrm{i}} \end{bmatrix}$$

Shor's Algorithm (Part 2)

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Therefore, for $x_0, \cdots, x_{m-1} \in \{0, 1\}$, $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, and $k \in \{0, \cdots, m-2\}$,

- $H_{m-1} |\overline{x_{m-1} \cdots x_0}\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_{m-1}} |1\rangle) \otimes |\overline{x_{m-2} \cdots x_0}\rangle$, and
- $CP_{\theta,k,m-1} ((\alpha |0\rangle + \beta |1\rangle) \otimes |\overline{x_{m-2} \cdots x_0}\rangle) = \left(\alpha |0\rangle + \beta e^{x_k \theta \mathrm{i}} |1\rangle\right) \otimes |\overline{x_{m-2} \cdots x_0}\rangle$

Shor's Algorithm (Part 2)

Implementing quantum Fourier transform

Let $x = 2^0 x_0 + \cdots + 2^{m-1} x_{m-1}$ for binary integers $x_0, \cdots, x_{m-1}$. Then

$$
\begin{aligned}
&\frac{1}{\sqrt{2}} \left(|0\rangle + \omega_{2^m}^{x} |1\rangle \right) \otimes |\overline{x_{m-2} \cdots x_0}\rangle \\
&= \frac{1}{\sqrt{2}} \left(|0\rangle + \omega_{2^m}^{2^{m-1} x_{m-1}} \cdots \omega_{2^m}^{2^0 x_0} |1\rangle \right) \otimes |\overline{x_{m-2} \cdots x_0}\rangle \\
&= \frac{1}{\sqrt{2}} \left(|0\rangle + \omega_{2^1}^{x_{m-1}} \cdots \omega_{2^m}^{x_0} |1\rangle \right) \otimes |\overline{x_{m-2} \cdots x_0}\rangle \\
&= CP_{\frac{\pi}{2^{m-1}},0,m-1} \left(\frac{1}{\sqrt{2}} \left(|0\rangle + \omega_{2^1}^{x_{m-1}} \cdots \omega_{2^{m-1}}^{x_1} |1\rangle \right) \otimes |\overline{x_{m-2} \cdots x_0}\rangle \right) \\
&\vdots \\
&= CP_{\frac{\pi}{2^{m-1}},0,m-1} \cdots CP_{\frac{\pi}{2},m-2,m-1} \left(\frac{1}{\sqrt{2}} \left(|0\rangle + \omega_{2^1}^{x_{m-1}} |1\rangle \right) \otimes |\overline{x_{m-2} \cdots x_0}\rangle \right)
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Therefore,

$$
\begin{aligned}
&QFT_{2^m}|\text{binary}(x)\rangle \\
&= \text{BITSWAPS}_{m-1\to 0}\cdot\left(I_2 \otimes QFT_{2^{m-1}}\right)\cdot CP_{\frac{\pi}{2^{m-1}},0,m-1}\cdots CP_{\frac{\pi}{2},m-2,m-1}\cdot H_{m-1}|\text{binary}(x)\rangle
\end{aligned}
$$

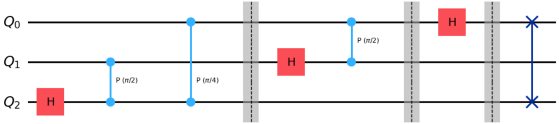
Shor's Algorithm (Part 2)

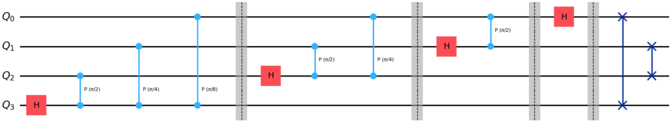
Implementing quantum Fourier transform

```
from qiskit import QuantumRegister, QuantumCircuit
from numpy import pi

def QFT(m):
    assert m >= 0
    q = QuantumRegister(m, "Q")
    C = QuantumCircuit(q)
    for i in reversed(range(m)):
        C.h(i)
        for j in reversed(range(i)):
            C.cp(pi / 2**(i-j), j, i)
        C.barrier()
    i, j = 0, m - 1
    while i < j:
        C.swap(i, j)
        i, j = i + 1, j - 1
    return C

display(QFT(3).draw(output = "mpl"))
display(QFT(4).draw(output = "mpl"))
```





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Note fast Fourier transform, the best classical algorithm currently known for computing DFT, takes $O(m \cdot 2^m)$ time, assuming all the underlying field operation takes $O(1)$ time.

Shor's Algorithm (Part 2)

Problem (Phase estimation problem)

Input: A unitary quantum circuit for an operation U on n qubits, along with one of its unit eigenvector $|\psi\rangle$

Output: An approximation to the number $\theta \in [0, 1)$ satisfying $U|\psi\rangle = e^{2\pi\theta \mathrm{i}}|\psi\rangle$

Shor's Algorithm (Part 2)

Special Case

- We first solve with the assumption that $\theta = y/2^m$ for some integer $0 \le y < 2^m$.

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1. Initial state is $|\psi\rangle \otimes |\overline{0}^m\rangle$. Here, the lower m indices will act as the control qubits.
2. We first uniformize the coefficients of the control qubits:

$$\left(I_n \otimes H^{\otimes m}\right)\left(|\psi\rangle \otimes |\overline{0}^m\rangle\right) = \frac{1}{\sqrt{2^m}} \sum_{x=0}^{2^m-1} |\psi\rangle \otimes |\text{binary}(x)\rangle$$

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3. By applying U^k on $|\psi\rangle$ with control qubit k for each $0 \leq k < m$, we obtain the following state

$$\frac{1}{\sqrt{2^m}} \sum_{x=0}^{2^m-1} U^x |\psi\rangle \otimes |\text{binary}(x)\rangle = |\psi\rangle \otimes \frac{1}{\sqrt{2^m}} \sum_{x=0}^{2^m-1} e^{2\pi\theta x \mathrm{i}} |\text{binary}(x)\rangle$$

Special Case

4. Now substitute $\theta = y/2^m$.

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Shor's Algorithm (Part 2)

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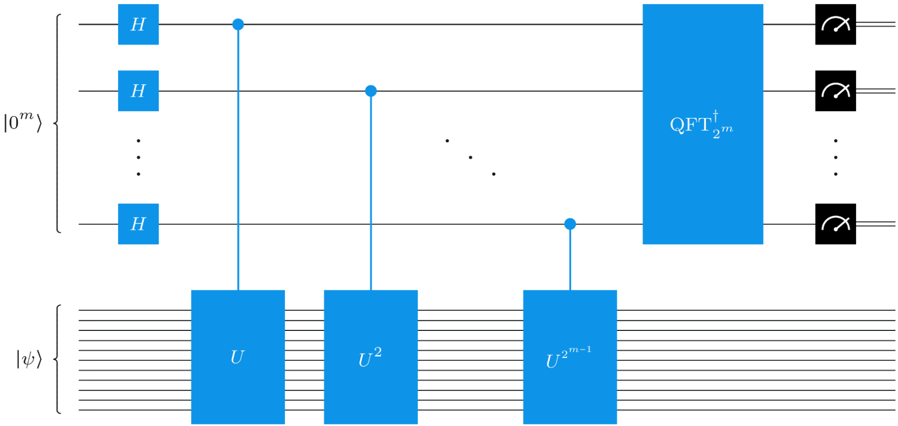
5. We have obtained the desired state. For $s \in \{0, 1\}^m$, the probability that the root-of-unity basis measurement yields the bitstring s is

$$\left| |b_s\rangle \langle b_s | b_y\rangle \right|^2 = \begin{cases} 1 & \text{if } s = y \\ 0 & \text{if } s \neq y \end{cases}$$

Special Case



Shor's Algorithm (Part 2)

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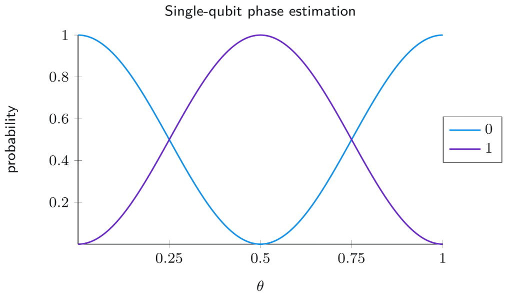

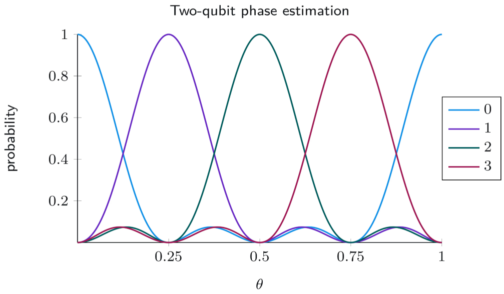

Shor's Algorithm (Part 2)

Solving order-finding problem with phase estimation problem

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Solving order-finding problem with phase estimation problem

- Due to the structure of U, we get that the following vector is a unit eigenvector of U with eigenvalue $\omega_r = e^{2\pi \mathrm{i}/r}$.

$$|\psi_1\rangle = \frac{\omega_r^{-0}\left|a^0 \mod n\right\rangle + \omega_r^{-1}\left|a^1 \mod n\right\rangle + \cdots + \omega_r^{-(r-1)}\left|a^{r-1} \mod n\right\rangle}{\sqrt{r}}$$

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- We run phase estimation on U and $|\psi_1\rangle$ with accuracy $m = 2N$, and we obtain a number k such that $\left|\frac{1}{r} - \frac{k}{2^{2N}}\right| \leq \frac{1}{2^{2N+1}}$, and $r = \lfloor \frac{2^{2N}}{k} + \frac{1}{2} \rfloor$.

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- There is one issue: we don't know r so we can't construct $|\psi_1\rangle$.

Shor's Algorithm (Part 2)

Solving order-finding problem with phase estimation problem

- Note that the following vector is a unit eigenvector of U with eigenvalue ω_r^k for all integer $0 \leq k < r$.

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- It turns out that we can uniquely recover the pair $k/\gcd(k,r), r/\gcd(k,r)$ with this given constraint in polynomial time, using continued fraction algorithm.

Solving order-finding problem with phase estimation problem

- We run this procedure few times, and take the least common multiple of all denominator as the order r.

List of experiments (from wikipedia)

- In 2001, a group at IBM demonstrated Shor's algorithm by factoring 15 using an NMR implementation of a quantum computer with seven qubits.
- In 2012, factorization of 21 was achieved.
- In 2019, an attempt was made to factor the number 35 using Shor's algorithm on an IBM Q System One, but the algorithm failed because of accumulating errors.

The End