Quantum Computing Seminar 3

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Multi Qubit System

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Reminder: endianness

Following the Qiskit's convention, bitstring will be indexed in decreasing order: $s = s_{N-1} \cdots s_0$, representing that the qubit q_i has collapsed into the value s_i.

Multi Qubit System

Reminder: Kronecker product of vectors

We denote $|abc\cdots\rangle := |a, b, c, \cdots\rangle := |a\rangle \otimes |b\rangle \otimes |c\rangle \cdots$ and $\langle abc\cdots| := \langle a, b, c, \cdots| := \langle a| \otimes \langle b| \otimes \langle c| \cdots$.

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- $\langle abc|abc\rangle = (\langle a| \otimes \langle b| \otimes \langle c|) \cdot (|a\rangle \otimes |b\rangle \otimes |c\rangle) = \langle a|a\rangle \otimes \langle b|b\rangle \otimes \langle c|c\rangle = 1 \otimes 1 \otimes 1 = 1$

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- Assume $b \neq d$.
  $\langle abc|adc\rangle = (\langle a| \otimes \langle b| \otimes \langle c|) \cdot (|a\rangle \otimes |d\rangle \otimes |c\rangle) = \langle a|a\rangle \otimes \langle b|d\rangle \otimes \langle c|c\rangle = 1 \otimes 0 \otimes 1 = 0$

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$\cdot$	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$\langle 00\|$	1	0	0	0
$\langle 01\|$	0	1	0	0
$\langle 10\|$	0	0	1	0
$\langle 11\|$	0	0	0	1

Multi Qubit System

Example

Consider the following state on a system of 2 qubits q_0 and q_1.

$$\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{4} - \frac{1}{4}i\right) = \begin{bmatrix} 1/4 \\ 2/4 \\ 3/4 \\ 1/4 - i/4 \end{bmatrix} = \frac{1}{4}|00\rangle + \frac{2}{4}|01\rangle + \frac{3}{4}|10\rangle + \left(\frac{1}{4} - \frac{1}{4}i\right)|11\rangle$$

The following table shows the probability of the system collapsing into each bitstring upon measurement.

Bitstring	00	01	10	11
Probability	1/16	4/16	9/16	2/16

Multi Qubit System

Reminder: Kronecker product of matrices

$$A \otimes B := \begin{bmatrix} A_{1,1} \cdot B & \cdots & A_{1,n} \cdot B \\ \vdots & \ddots & \vdots \\ A_{m,1} \cdot B & \cdots & A_{m,n} \cdot B \end{bmatrix}$$

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Reminder: Kronecker product of matrices

Let A, B be N by N and M by M matrices, and u and v be N and M dimensional complex vectors respectively.

$$(A \otimes B) \cdot (|u\rangle \otimes |v\rangle) = A \cdot |u\rangle \otimes B \cdot |v\rangle$$

Multi Qubit System

Definition (independence and entanglement)

Let X be a system, Y be a subsystem of X, and Z be the complement of Y, each of which are in the state $|u\rangle$, $|v\rangle$, and $|w\rangle$ respectively.

- Y and Z are **independent** if $|u\rangle = |v\rangle \otimes |w\rangle$.
- If no such Y and Z each of which having at least one qubit exists, X is said to be **entangled**.

Multi Qubit System

Example

Which of the following states are entangled?

1. $|a\rangle = 3/5\,|00\rangle + 4/5\,|11\rangle$
2. $|b\rangle = 1/2\,|00\rangle + \mathfrak{i}/2\,|01\rangle - 1/2\,|10\rangle - \mathfrak{i}/2\,|11\rangle$
3. $|c\rangle = 1\sqrt{2}\,|001\rangle + 1/2\,|010\rangle + 1/2\,|100\rangle$

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1. $|a\rangle$ is entangled.
   - If $|a\rangle = |u\rangle \otimes |v\rangle$, then $\langle 01|a\rangle = \langle 0|u\rangle \otimes \langle 1|v\rangle = 0$, which contradicts that $\langle 00|a\rangle = \langle 0|u\rangle \otimes \langle 0|v\rangle$ and $\langle 11|a\rangle = \langle 1|u\rangle \otimes \langle 1|v\rangle$.

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2. $|b\rangle$ is not entangled.
   - $|b\rangle = \left(1/\sqrt{2}\,|0\rangle - 1/\sqrt{2}\,|1\rangle\right) \otimes \left(1/\sqrt{2}\,|0\rangle + \mathrm{i}/\sqrt{2}\,|1\rangle\right)$

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3. $|c\rangle$ is entangled.

Multi Qubit System

Definition (Bell states)

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

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Multi Qubit System

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- Each state in the Bell states are entangled.

Multi Qubit System

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- Each state in the Bell states are entangled.
- They form an orthonormal basis.

Multi Qubit System

Definition (GHZ state and W state)

GHZ state

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

W state

$$\frac{1}{\sqrt{3}}|001\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{1}{\sqrt{3}}|100\rangle$$

Multi Qubit System

Definition (measurement)

Given a system X of N qubits $q_0, \cdots, q_{N-1}$ with state $|v\rangle$, **measuring** it yields the bitstring $s \in \{0, 1\}^N$ with probability $\mathcal{P}_s(|v\rangle) := |\langle s|v\rangle|^2$

Probability

Consider a set of balls, each of which are either white(W) or black(B), and either hard(H) or soft(S).

Number of balls	H	S
W	2	3
B	5	7

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Probability can be thought of as proportion of events.

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For example, the probability that a randomly chosen ball is black and soft is $\frac{7}{17}$.

Probability

Conditional probability re-defines domain based on a set of conditions. For example, if the condition is “the color is black”, then the distribution becomes

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For example,

$$\mathcal{P}\left(\text{Hardness} : \text{Soft} | \text{Color} : \text{Black}\right) = \frac{7}{12}$$

and

$$\mathcal{P}\left(\text{Color} : \text{White}, \text{Hardness} : \text{Soft} | \text{Color} : \text{Black}\right) = 0$$

Probability

For another example, if the condition is "white or soft", then the distribution becomes

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Probability	H	S
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B	0	$\frac{7}{12}$

For example,

$$\mathcal{P}\left(\text{Color} : \text{White} | \text{Color} : \text{White}, \text{Hardness} : \text{Soft}\right) = \frac{5}{12}$$

Multi Qubit System

Consider a system on two qubits q_0 and q_1 which are in the state $|v\rangle$. The outcome of the measurement follows the probability in the following table.

$q_1 \backslash q_0$	0	1
0	$\lvert\langle 00\vert v\rangle\rvert^2$	$\lvert\langle 01\vert v\rangle\rvert^2$
1	$\lvert\langle 10\vert v\rangle\rvert^2$	$\lvert\langle 11\vert v\rangle\rvert^2$

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From the table, we can see that the probability that measuring the whole system causes q_0 to collapse to 0 is $\mathcal{P}_{q_0=0}(|v\rangle) := |\langle 00|v\rangle|^2 + |\langle 10|v\rangle|^2$

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From the table, we can see that the probability that measuring the whole system causes q_0 to collapse to 0 is $\mathcal{P}_{q_0=0}(|v\rangle) := |\langle 00|v\rangle|^2 + |\langle 10|v\rangle|^2$
The conditional probability table with the condition "q_0 has collapsed to 0" is

$q_1 \backslash q_0$	0	1
0	$\frac{\lvert\langle 00\vert v\rangle\rvert^2}{\mathcal{P}_{q_0=0}(\lvert v\rangle)}$	0
1	$\frac{\lvert\langle 10\vert v\rangle\rvert^2}{\mathcal{P}_{q_0=0}(\lvert v\rangle)}$	0

Multi Qubit System

Now consider a separate state $|w\rangle$ on q_0 and q_1

$$
\begin{aligned}
|w\rangle &= \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}} |00\rangle \langle 00|v\rangle + \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}} |10\rangle \langle 10|v\rangle \\
&= \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}} \Bigg(\Big(|0\rangle \langle 0| + |1\rangle \langle 1| \Big) \otimes |0\rangle \langle 0| \Bigg) |v\rangle = \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}} \Big(I_2 \otimes |0\rangle \langle 0| \Big) |v\rangle
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&= \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}} \Bigg(\Big(|0\rangle \langle 0| + |1\rangle \langle 1| \Big) \otimes |0\rangle \langle 0| \Bigg) |v\rangle = \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}} \Big(I_2 \otimes |0\rangle \langle 0| \Big) |v\rangle
\end{aligned}
$$

The table for measuring $|w\rangle$ produces the identical table to the previous conditional probability table.

$q_1 \backslash q_0$	0	1
0	$\frac{\lvert\langle 00\rvert v\rangle\rvert^2}{\mathcal{P}_{q_0=0}(\lvert v\rangle)}$	0
1	$\frac{\lvert\langle 10\rvert v\rangle\rvert^2}{\mathcal{P}_{q_0=0}(\lvert v\rangle)}$	0

Multi Qubit System

This is not a coincidence. When we partially measure the qubit q_0 from the state $|v\rangle$, q_0 collapses to 0 with probability

$$\mathcal{P}_{q_0=0}\left(|v\rangle\right) := \left|\langle 00|v\rangle\right|^2 + \left|\langle 10|v\rangle\right|^2$$

which causes the whole system to collapse into the state

$$\mathcal{S}_{q_0=0}\left(|v\rangle\right) := \frac{1}{\sqrt{\mathcal{P}_{q_0=0}\left(|v\rangle\right)}}\left(I_2 \otimes |0\rangle\langle 0|\right)|v\rangle$$

Similarly, partially measuring q_0 from the state $|v\rangle$ causes q_0 to collapse to 1 with probability

$$\mathcal{P}_{q_0=1}\left(|v\rangle\right) := \left|\langle 01|v\rangle\right|^2 + \left|\langle 11|v\rangle\right|^2$$

which causes the whole system to collapse into the state

$$\mathcal{S}_{q_0=1}\left(|v\rangle\right) := \frac{1}{\sqrt{\mathcal{P}_{q_0=1}\left(|v\rangle\right)}}\left(I_2 \otimes |1\rangle\langle 1|\right)|v\rangle$$

Multi Qubit System

- Let $M \leq N$ and consider subsystems Y and Z of X consisting of M qubits $q_0, \cdots, q_{M-1}$ and $N-M$ qubits $q_M, \cdots, q_{N-1}$ respectively, where X is in the state $|v\rangle$.

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- Let $s \in \{0,1\}^M$, $t \in \{0,1\}^{N-M}$.

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- Let $s \in \{0, 1\}^M$, $t \in \{0, 1\}^{N-M}$.
- Probability that measuring X causes Y to collapse to s is

$$\sum_{u \in \{0,1\}^{N-M}} |\langle s, u|v\rangle|^2$$

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- Let $s \in \{0,1\}^M$, $t \in \{0,1\}^{N-M}$.
- Probability that measuring X causes Y to collapse to s is

$$\sum_{u \in \{0,1\}^{N-M}} |\langle s, u|v\rangle|^2$$

- Probability that measuring X causes Z to collapse to t, given that Y has collapsed to s, is

$$\frac{|\langle s, t|v\rangle|^2}{\sum_{u \in \{0,1\}^{N-M}} |\langle s, u|v\rangle|^2}$$

Multi Qubit System

We introduce partial measurement which respect these probabilities.

Multi Qubit System

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Definition (partial measurement)

Given are a system X of N qubits with state $|v\rangle$, subsystem Y and Z of X consisting of M and $N-M$ qubits respectively, and a bitstring $s \in \{0,1\}^M$.
The probability that the **partial measurement on Y** yields the bitstring s is

$$\mathcal{P}_{Y=s}(|v\rangle) := \sum_{u \in \{0,1\}^{N-M}} |\langle s, u|v\rangle|^2$$

which causes X to partially collapses into the state

$$\mathcal{S}_{Y=s}(|v\rangle) := \frac{1}{\sqrt{\mathcal{P}_{Y=s}(|v\rangle)}} \cdot \left(|s\rangle \langle s| \otimes I_{2^{N-M}} \right) \cdot |v\rangle$$

Multi Qubit System

Let $N = 2, M = 1$, and

$$|v\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{\mathfrak{i}}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

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which can be rewritten as

$$|v\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{6}}|1\rangle\right) + |1\rangle \otimes \left(\frac{\mathfrak{i}}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle\right)$$

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Measuring Y

- $\mathcal{P}_{Y=0}(|v\rangle) = 1/2 + 1/6 = 2/3$
- $\mathcal{S}_{Y=0}(|v\rangle) = \sqrt{1/\mathcal{P}_{Y=0}(|v\rangle)}\,|0\rangle \otimes \left(1/\sqrt{2}\cdot|0\rangle - 1/\sqrt{6}\,|1\rangle\right) = \sqrt{3}/2\,|00\rangle - 1/2\,|01\rangle$
- $\mathcal{P}_{Y=1}(|v\rangle) = 1/6 + 1/6 = 1/3$
- $\mathcal{S}_{Y=1}(|v\rangle) = \sqrt{1/\mathcal{P}_{Y=1}(|v\rangle)}\,|1\rangle \otimes \left(\mathrm{i}/\sqrt{6}\cdot|0\rangle + 1/\sqrt{6}\,|1\rangle\right) = \mathrm{i}/\sqrt{2}\,|10\rangle + 1/\sqrt{2}\,|11\rangle$

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this time, rewrite it as

$$|v\rangle = \left(\frac{1}{\sqrt{2}} |0\rangle + \frac{i}{\sqrt{6}} |1\rangle \right) \otimes |0\rangle + \left(-\frac{1}{\sqrt{6}} |0\rangle + \frac{1}{\sqrt{6}} |1\rangle \right) \otimes |1\rangle$$

Multi Qubit System

Let $N = 2, M = 1$, and

$$|v\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{6}}|01\rangle + \frac{i}{\sqrt{6}}|10\rangle + \frac{1}{\sqrt{6}}|11\rangle$$

this time, rewrite it as

$$|v\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{6}}|1\rangle\right) \otimes |0\rangle + \left(-\frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{6}}|1\rangle\right) \otimes |1\rangle$$

Measuring Z

- $\mathcal{P}_{Z=0}(|v\rangle) = 1/2 + 1/6 = 2/3$
- $\mathcal{S}_{Z=0}(|v\rangle) = \sqrt{1/\mathcal{P}_{Z=0}(|v\rangle)}\left(1/\sqrt{2} \cdot |0\rangle + i/\sqrt{6}\,|1\rangle\right) \otimes |0\rangle = \sqrt{3}/2\,|00\rangle + i/2\,|10\rangle$
- $\mathcal{P}_{Z=1}(|v\rangle) = 1/6 + 1/6 = 1/3$
- $\mathcal{S}_{Z=1}(|v\rangle) = \sqrt{1/\mathcal{P}_{Z=1}(|v\rangle)}\left(-1/\sqrt{6} \cdot |0\rangle + 1/\sqrt{6}\,|1\rangle\right) \otimes |1\rangle = -1/\sqrt{2}\,|01\rangle + 1/2\,|11\rangle$

Multi Qubit System

THEOREM

Let $|u\rangle$ and $|v\rangle$ be states with N and M qubits respectively. Then,

$$\mathcal{P}_s(|u\rangle) = \frac{\mathcal{P}_{s,t}(|u,v\rangle)}{\mathcal{P}_t(|v\rangle)}$$

for all $s \in \{0,1\}^N$ and $t \in \{0,1\}^M$.

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PROOF)

$$\mathcal{P}_{s,t}(|u,v\rangle) = |\langle s,t|u,v\rangle|^2 = |\langle s|u\rangle \otimes \langle t|v\rangle|^2 = |\langle s|u\rangle|^2 \cdot |\langle t|v\rangle|^2 = \mathcal{P}_s(|u\rangle) \cdot \mathcal{P}_t(|v\rangle)$$

Qiskit Examples: Multi Qubit System

```
from qiskit.quantum_info import Statevector, Operator
from numpy import sqrt

zero_ket, one_ket = Statevector.from_label("0"), Statevector.from_label("1")
zero_ket.tensor(one_ket).draw("latex")
```

$$|01\rangle$$

```
minus_ket = Statevector.from_label("-")
u_ket = Statevector([1 / sqrt(2), 1j / sqrt(2)])
v_ket = minus_ket.tensor(u_ket)

v_ket.draw("latex")
```

$$\frac{1}{2}|00\rangle + \frac{i}{2}|01\rangle - \frac{1}{2}|10\rangle - \frac{i}{2}|11\rangle$$

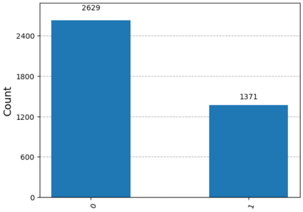
```
v_ket = Statevector([1 / sqrt(2), -1 / sqrt(6), 1j / sqrt(6), 1 / sqrt(6)])
v_ket.draw("latex")
```

$$\frac{\sqrt{2}}{2}|00\rangle - \frac{\sqrt{6}}{6}|01\rangle + \frac{\sqrt{6}i}{6}|10\rangle + \frac{\sqrt{6}}{6}|11\rangle$$

Qiskit Examples: Multi Qubit System

```python
from qiskit.visualization import plot_histogram

plot_histogram(v_ket.sample_counts(4000, [0]))
```




```python
result, next_ket = v_ket.measure([0])

print(f"{result = }")
next_ket.draw("latex")
```

```
result = '0'
```

$$\frac{\sqrt{3}}{2}|00\rangle + \frac{i}{2}|10\rangle$$

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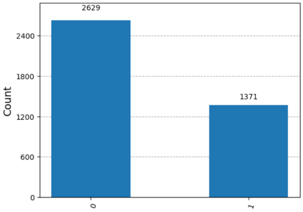


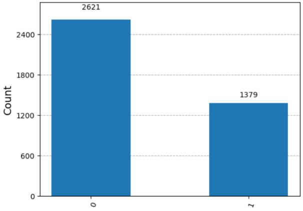

```
result, next_ket = v_ket.measure([0])

print(f"{result = }")
next_ket.draw("latex")
```

```
result = '0'
```

$$\frac{\sqrt{3}}{2}|00\rangle + \frac{i}{2}|10\rangle$$

```
plot_histogram(v_ket.sample_counts(4000, [1]))
```




```
result, next_ket = v_ket.measure([1])

print(f"{result = }")
next_ket.draw("latex")
```

```
result = '1'
```

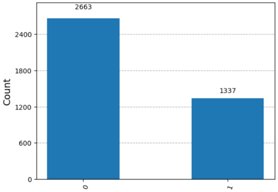
$$\frac{\sqrt{2}i}{2}|10\rangle + \frac{\sqrt{2}}{2}|11\rangle$$

Qiskit Examples: Multi Qubit System

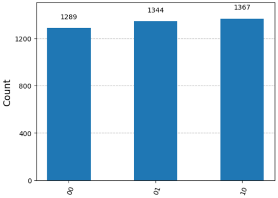
```
w_ket = Statevector([0, 1, 1, 0, 1, 0, 0, 0] / sqrt(3))
w_ket.draw("latex")
```

$$\frac{\sqrt{3}}{3}|001\rangle + \frac{\sqrt{3}}{3}|010\rangle + \frac{\sqrt{3}}{3}|100\rangle$$

```
plot_histogram(w_ket.sample_counts(4000, [1]))
```




```
plot_histogram(w_ket.sample_counts(4000, [0, 2]))
```



Unitary Operation

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Operation on joint system

Let $X_1, \cdots, X_n$ be systems which together form a joint system X.

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- Let U_i be a unitary operation on X_i for each i. Operating U_i on X_i simultaneously is equivalent to operating $U_1 \otimes \cdots \otimes U_n$ on X.

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- Let U_i be a unitary operation on X_i for each i. Operating U_i on X_i simultaneously is equivalent to operating $U_1 \otimes \cdots \otimes U_n$ on X.
- In particular, operating U on X_1 is equivalent to operating $U \otimes I \otimes \cdots \otimes I$ on X.

Unitary Operation

Swap operation

Let X and Y be system of N qubits. SWAP is the unitary operation on (X, Y) defined as the following.

$$\mathrm{SWAP} := \sum_{s,t \in \{0,1\}^N} |t\rangle\langle s| \otimes |s\rangle\langle t|$$

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- Note that the above completely characterizes SWAP. We'll use these interchangeably.

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- For any $s, t \in \{0,1\}^N$, $\mathrm{SWAP}\,|s, t\rangle = |t, s\rangle$.
- Note that the above completely characterizes SWAP. We'll use these interchangeably.
- In particular, when $N = 1$,

$$\mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Unitary Operation

Controlled operation

Let X be a system of single qubit and Y be a system of N qubits. For any operation U on Y, we define controlled operation CU on (X, Y) as the following.

$$CU = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U$$

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- For any $s \in \{0, 1\}^N$, $CU|0, s\rangle = |0\rangle \otimes |s\rangle$ and $CU|1, s\rangle = |1\rangle \otimes U|s\rangle$
- Think of it as an IF-statement on the controlled qubit.

Unitary Operation

Controlled not

$$CX = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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- For all $s, t \in \{ 0, 1 \}$, $CX |s, t\rangle = |s, s \oplus t\rangle$

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- For all $s, t \in \{0, 1\}$, $CX|s, t\rangle = |s, s \oplus t\rangle$

Controlled phase flip

$$CZ = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Unitary Operation

Controlled swap

$$CSWAP = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Controlled swap

$$
\mathrm{CSWAP} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes \mathrm{SWAP} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

- For all $s, t \in \{0, 1\}$, $\mathrm{CSWAP}\,|0, s, t\rangle = |0, s, t\rangle$ and $\mathrm{CSWAP}\,|1, s, t\rangle = |1, t, s\rangle$.

Unitary Operation

Controlled controlled not

$$CCX = (|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I + |11\rangle\langle 11| \otimes X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

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- For all $s, t, u \in \{0, 1\}$, $CCX\,|s, t, u\rangle = |s, t, (s \cdot t) \oplus u\rangle$.

Qiskit Examples: Unitary Operation

Exercise

Construct an operator U on 4 qubits which maps $|0000\rangle$ to a state with uniform probability.

$$\mathcal{P}_{0000}\left(U\left|0000\right\rangle\right) = \mathcal{P}_{0001}\left(U\left|0000\right\rangle\right) = \cdots = \mathcal{P}_{1111}\left(U\left|0000\right\rangle\right)$$

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- Hadamard operator on a single qubit maps $|0\rangle$ to a state with uniform probability.

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- Hadamard operator on a single qubit maps $|0\rangle$ to a state with uniform probability.
- Applying it on all qubits achieves our goal.

$$U = H \otimes H \otimes H \otimes H$$

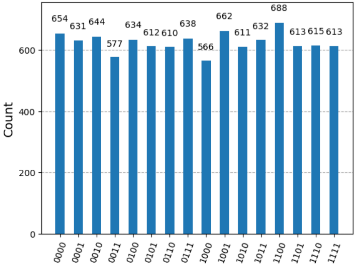
Qiskit Examples: Unitary Operation

```
H = Operator([[1/sqrt(2), 1/sqrt(2)], [1/sqrt(2), -1/sqrt(2)]])
H4 = H.tensor(H).tensor(H).tensor(H)

zero_ket = Statevector.from_label("0")
zero4_ket = zero_ket.tensor(zero_ket).tensor(zero_ket).tensor(zero_ket)
```

```
v_ket = zero4_ket.evolve(H4)

display(zero4_ket.draw("latex"))
display(v_ket.draw("latex"))
plot_histogram(v_ket.sample_counts(10000))
```

$$|0000\rangle$$

$$\frac{1}{4}|0000\rangle + \frac{1}{4}|0001\rangle + \frac{1}{4}|0010\rangle + \frac{1}{4}|0011\rangle + \frac{1}{4}|0100\rangle + \frac{1}{4}|0101\rangle + \ldots + \frac{1}{4}|1011\rangle + \frac{1}{4}|1100\rangle + \frac{1}{4}|1101\rangle + \frac{1}{4}|1110\rangle + \frac{1}{4}|1111\rangle$$



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Construct an operation U acting on 3 qubits as the following.

$$U\left|0ab\right\rangle = \left|0ba\right\rangle$$

$$U\left|1ab\right\rangle = \left|1ab\right\rangle$$

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We flip the first qubit, apply CSWAP, then flip the first qubit back.

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$$U = (X \otimes I \otimes I) \cdot \mathrm{CSWAP} \cdot (X \otimes I \otimes I)$$

Qiskit Examples: Unitary Operation

```
v_ket = Statevector(np.array([1, -1, sqrt(2), -sqrt(2), sqrt(3), sqrt(5), -sqrt(5), sqrt(6)]) / 5)
v_ket.draw("latex")
```

$$\frac{1}{5}|000\rangle - \frac{1}{5}|001\rangle + \frac{\sqrt{2}}{5}|010\rangle - \frac{\sqrt{2}}{5}|011\rangle + \frac{\sqrt{3}}{5}|100\rangle + \frac{\sqrt{5}}{5}|101\rangle - \frac{\sqrt{5}}{5}|110\rangle + \frac{\sqrt{6}}{5}|111\rangle$$

```
I = Operator([[1, 0], [0, 1]])
X = Operator([[0, 1], [1, 0]])
X0 = X.tensor(I).tensor(I)
CSWAP = Operator([
    [1, 0, 0, 0, 0, 0, 0, 0],
    [0, 1, 0, 0, 0, 0, 0, 0],
    [0, 0, 1, 0, 0, 0, 0, 0],
    [0, 0, 0, 1, 0, 0, 0, 0],
    [0, 0, 0, 0, 1, 0, 0, 0],
    [0, 0, 0, 0, 0, 0, 1, 0],
    [0, 0, 0, 0, 0, 1, 0, 0],
    [0, 0, 0, 0, 0, 0, 0, 1]
])

display(v_ket.evolve(CSWAP).draw("latex"))
display(v_ket.evolve(X0).evolve(CSWAP).evolve(X0).draw("latex"))
```

$$\frac{1}{5}|000\rangle - \frac{1}{5}|001\rangle + \frac{\sqrt{2}}{5}|010\rangle - \frac{\sqrt{2}}{5}|011\rangle + \frac{\sqrt{3}}{5}|100\rangle - \frac{\sqrt{5}}{5}|101\rangle + \frac{\sqrt{5}}{5}|110\rangle + \frac{\sqrt{6}}{5}|111\rangle$$

$$\frac{1}{5}|000\rangle + \frac{\sqrt{2}}{5}|001\rangle - \frac{1}{5}|010\rangle - \frac{\sqrt{2}}{5}|011\rangle + \frac{\sqrt{3}}{5}|100\rangle + \frac{\sqrt{5}}{5}|101\rangle - \frac{\sqrt{5}}{5}|110\rangle + \frac{\sqrt{6}}{5}|111\rangle$$

The End