Quantum Computing Seminar 7

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Computational Cost

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- Q) Is the number of gates a good way to define the computational cost?

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Elementary gates (boolean circuit)

We choose AND, OR, NOT, $FANOUT$ as elementary gates for boolean circuit, which forms a universal gate set for deterministic computation.

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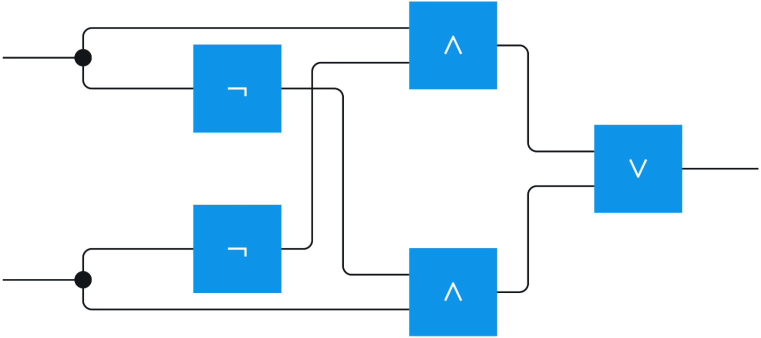
- We'll mostly associate circuit size with computational cost, despite the fact that circuit depth is more realistic estimation of runtime, for simplicity.

Computational Cost

Exercise: XOR circuit

What is the size and depth of the following circuit which computes the XOR of two bits?



Computational Cost

Problem (Integer addition)

Input: N-bit integers A and B
Output: $(N+1)$-bit integer $A+B$

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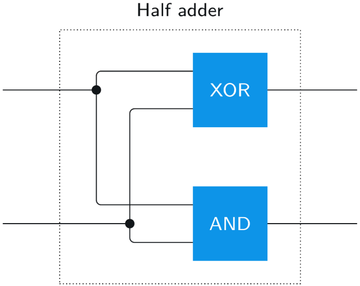

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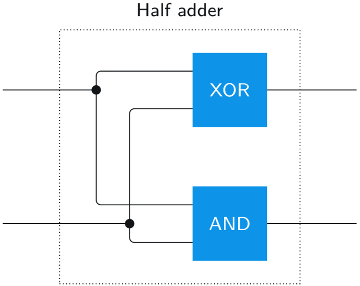
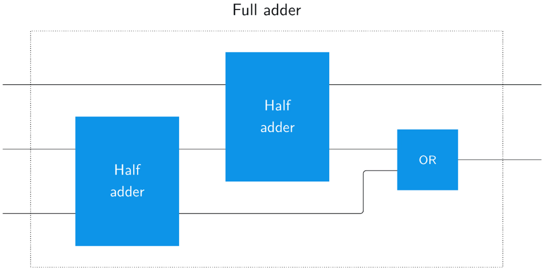


Other bits can be added with the following full adder.



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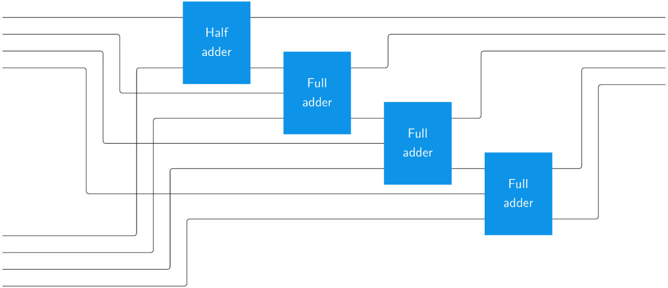
For example, following is the full circuit for $N = 4$.



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Cost: Same as integer multiplication

Computational Cost

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Input: N-bit integers A and B
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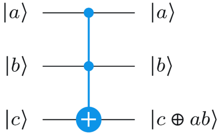
Cost: The state-of-art factorization algorithm (general number field sieve) is conjectured to take $O(e^{1.9 \cdot N^{1/3} \log(N)^{2/3}})$

Simulation of Boolean Circuit

Boolean circuit can be simulated using the Toffoli (CCX) gate



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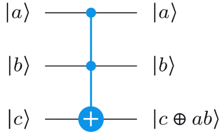
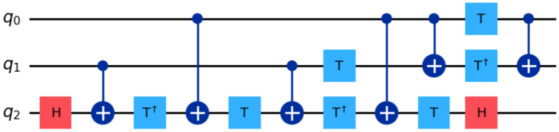


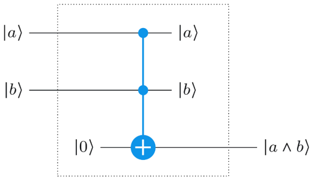
which can be implemented with the following circuit.



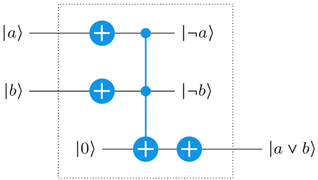
Simulation of Boolean Circuit

Given a boolean circuit C describing a function $f : \{0, 1\}^n \to \{0, 1\}^m$, we replace every NOT gate with an X gate, and the other elementary gates with the following quantum circuits.

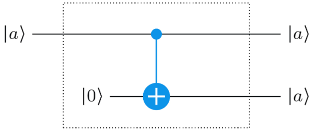
AND gate



OR gate

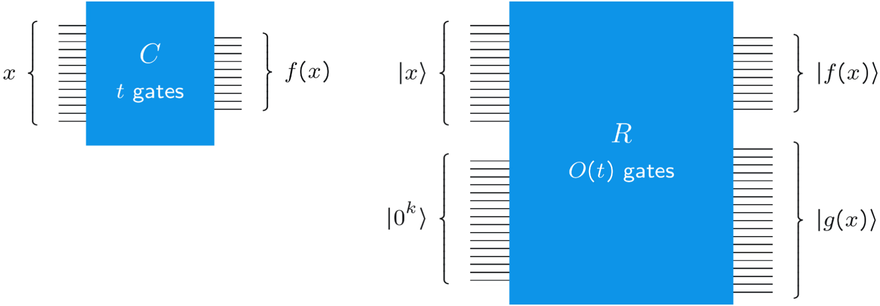


FANOUT gate



Simulation of Boolean Circuit

We obtain the quantum circuit R.



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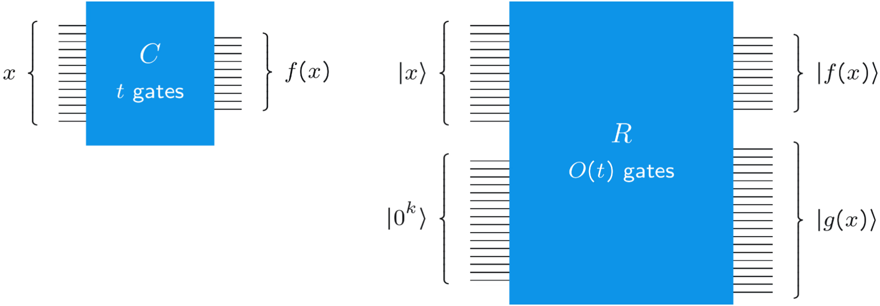

Here, k is the number of additional workspace qubits used, and $g : \{0, 1\}^n \to \{0, 1\}^{n+k-m}$ describes the garbage qubits at the end.

Simulation of Boolean Circuit

However, the garbage bits may be entangled with the output qubits, which is problematic if we want to do additional computation on output qubits.

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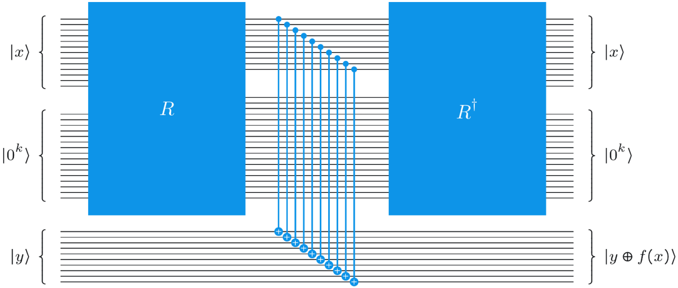
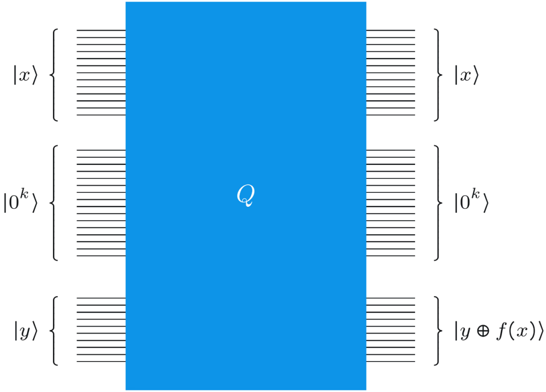
To ensure that the garbage qubits are independent from the rest, we construct the circuit as the following.



Simulation of Boolean Circuit

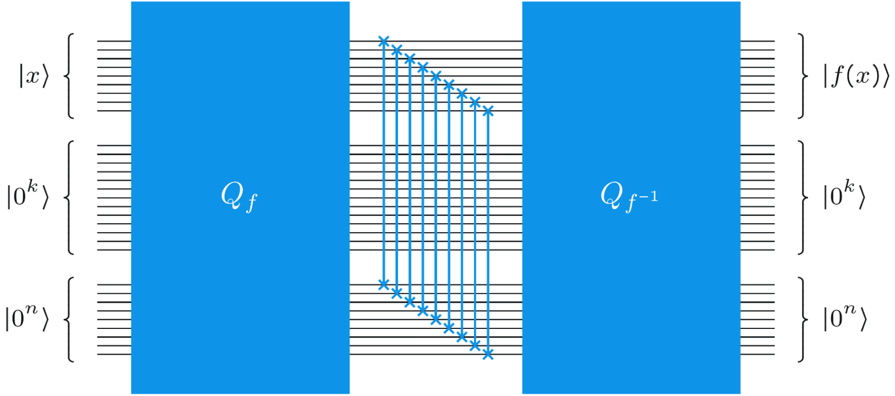
We now box everything and call it the query gate Q.



Note that if C has t gates, Q has $O(t)$ gates.

Simulation of Boolean Circuit

Extra If f is bijective, we can implement a circuit representing the gate U with $U|x\rangle = |f(x)\rangle$ as follows.



The End