Quantum Computing Seminar 5

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December 16, 2024

Entanglement

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- All of these are true but fails to distinguish the above non-entangled probabilistic state with an entanglement in a meaningful way.
- The power of entanglement lies in what we can do with it.

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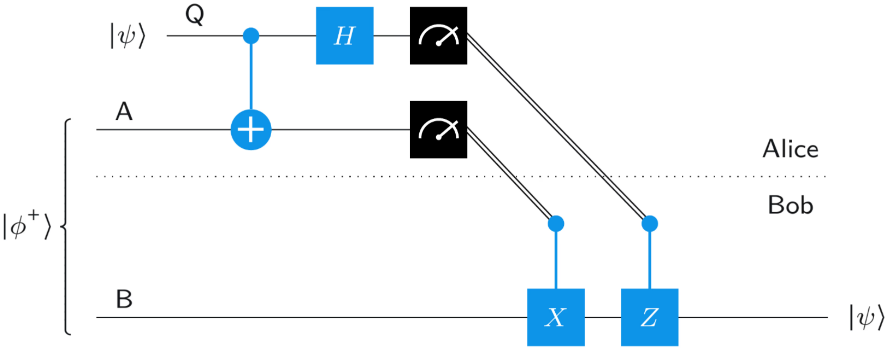
- They all consume a pair of entangled qubits $|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ as a resource, which we'll call an **e-bit**.
- All of these are either impossible if we have a pair of non-entangled probabilistic bits instead of an e-bit.

Teleportation Protocol

Teleportation Protocol

Preparation	Alice and Bob each have a qubit, together forming an e-bit
Objective	Alice sends Bob one qubit of quantum information
Communication	Alice sends Bob two bits of classical information

Protocol Overview



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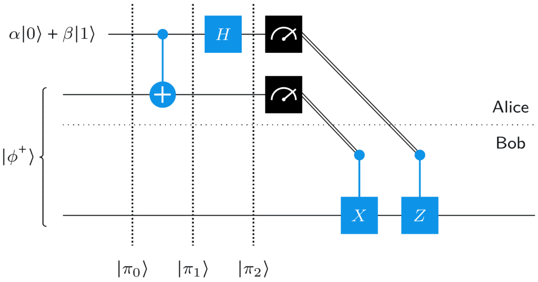

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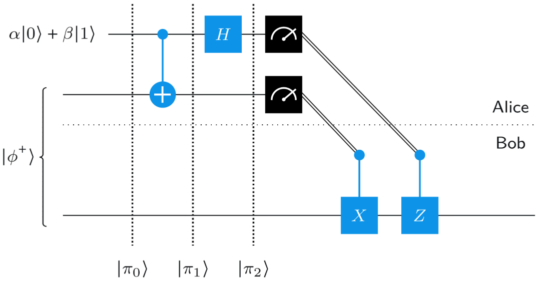

$$|\pi_0\rangle = |\phi^+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle)$$

$$= \frac{1}{2}\left(\alpha|0\rangle \otimes (|\phi^+\rangle + |\phi^-\rangle) + \alpha|1\rangle \otimes (|\psi^+\rangle - |\psi^-\rangle) + \beta|0\rangle \otimes (|\psi^+\rangle + |\psi^-\rangle) + \beta|1\rangle \otimes (|\phi^+\rangle - |\phi^-\rangle)\right)$$

$$= \frac{1}{2}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |\phi^+\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |\phi^-\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |\psi^+\rangle - (\alpha|1\rangle - \beta|0\rangle) \otimes |\psi^-\rangle\right)$$

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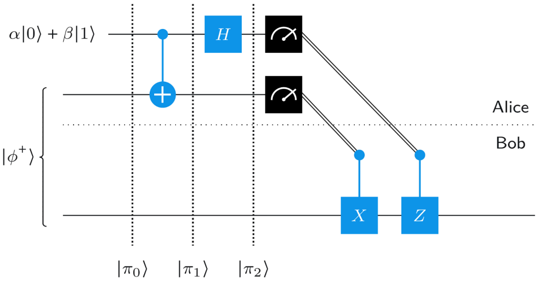

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\begin{aligned}
|\pi_2\rangle &= (I_2 \otimes I_2 \otimes H)\,(I_2 \otimes CX_{0,1})\,|\pi_0\rangle \\
&= \frac{1}{2}\left((\alpha\,|0\rangle + \beta\,|1\rangle) \otimes |00\rangle + (\alpha\,|0\rangle - \beta\,|1\rangle) \otimes |01\rangle + (\alpha\,|1\rangle + \beta\,|0\rangle) \otimes |10\rangle + (\alpha\,|1\rangle - \beta\,|0\rangle) \otimes |11\rangle\right)
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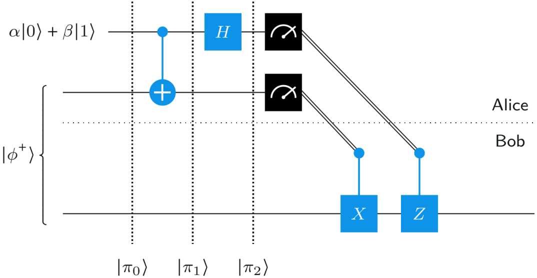

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There are 4 possible outcomes to Alice 's measurement.

Teleportation Protocol

Outcome 00

$$|\pi_2\rangle = \frac{1}{2}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle + (\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle\right)$$

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- **Probability** $\rightarrow$

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- **Probability** $\rightarrow 1/4$

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- **Bob 's Action** $\rightarrow$ Apply Z

Teleportation Protocol

Outcome 10

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- **Probability** $\rightarrow$

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Teleportation Protocol

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$$|\pi_2\rangle = \frac{1}{2}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle + (\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle\right)$$

- **Probability** $\rightarrow 1/4$
- **State of B** $\rightarrow \alpha|1\rangle - \beta|0\rangle$
- **Bob 's Action** $\rightarrow$

Teleportation Protocol

Outcome 10

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Teleportation Protocol

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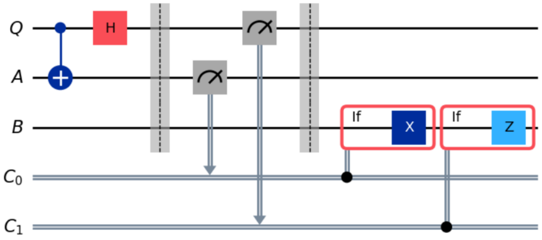
- Q) What if Q is entangled with some other qubits?
  - The state of Bob 's qubit at the end has the same entangled state as the initial state of Q.
  - It's not hard to repeat the same analysis to show this (left as an exercise), but it turns out that showing it for an independent qubit Q is enough, which will be discussed later.

Teleportation Protocol

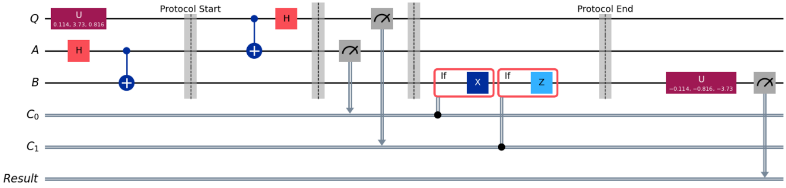
```python
from qiskit import QuantumCircuit, QuantumRegister, ClassicalRegister
from qiskit_aer import AerSimulator
from qiskit.visualization import plot_histogram
from qiskit.result import marginal_distribution
from qiskit.circuit.library import UGate
from numpy import pi, random
```

```python
Q = QuantumRegister(1, "Q")
A = QuantumRegister(1, "A")
B = QuantumRegister(1, "B")
C = ClassicalRegister(2, "C")

protocol = QuantumCircuit(Q, A, B, C)

# Alice's operations
protocol.cx(Q, A)
protocol.h(Q)
protocol.barrier()

# Alice measures and sends classical bits to Bob
protocol.measure(A, C[0])
protocol.measure(Q, C[1])
protocol.barrier()

# Bob uses the classical bits to conditionally apply gates
with protocol.if_test((C[0], 1)):
    protocol.x(B)
with protocol.if_test((C[1], 1)):
    protocol.z(B)

protocol.draw(output = "mpl", cregbundle = False)
```



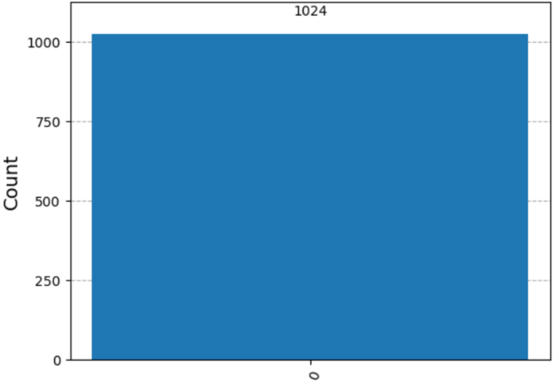
Teleportation Protocol

```
runner = QuantumCircuit(Q, A, B, C)

# Make ranodm gate
random_gate = UGate(
    theta=random.random() * 2 * pi,
    phi=random.random() * 2 * pi,
    lam=random.random() * 2 * pi,
)

# Randomly selected state on Q
runner.append(random_gate, Q)

# Entangle A and B
runner.h(A)
runner.cx(A, B)
runner.barrier(label = "Protocol Start")

# Run protocol
runner = runner.compose(protocol)
runner.barrier(label = "Protocol End")

# Check the result
runner.append(random_gate.inverse(), B)
result = ClassicalRegister(1, "Result")
runner.add_register(result)
runner.measure(B, result)

runner.draw(output = "mpl", cregbundle = False)
```



Teleportation Protocol

```
result = AerSimulator().run(runner).result()
statistics = marginal_distribution(result.get_counts(), [2])
plot_histogram(statistics)
```

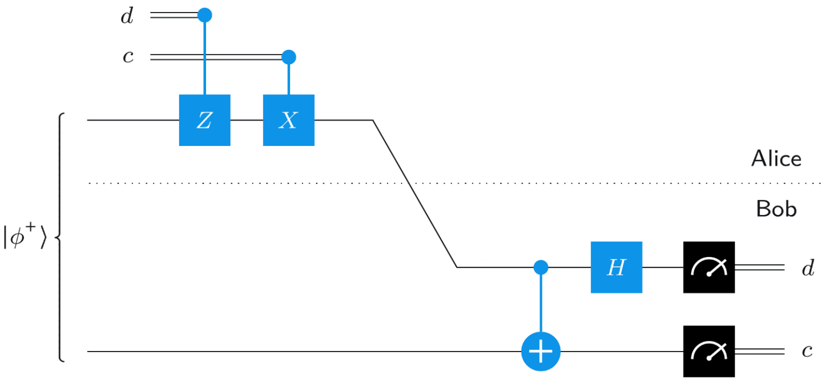


Superdense Coding Protocol

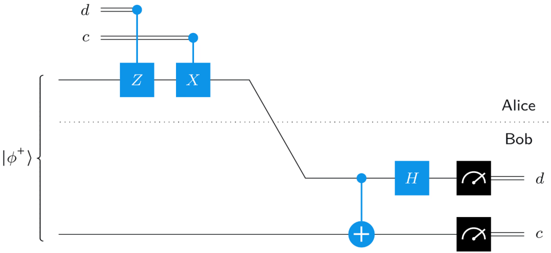
Superdense Coding Protocol

Preparation	Alice and Bob each have a qubit, together forming an e-bit
Objective	Alice sends Bob two bits of classical information
Communication	Alice sends Bob one qubit of quantum information

Protocol Overview

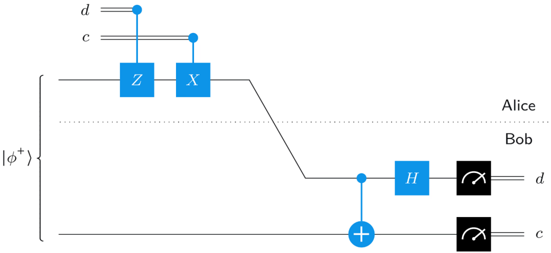


Superdense Coding Protocol



We only have 4 cases to analyze

Superdense Coding Protocol



We only have 4 cases to analyze

Input	Initial state	State after Alice 's operation	Bob 's measurement
00	$\|\phi^+\rangle$	$\|\phi^+\rangle$	00
01	$\|\phi^+\rangle$	$\|\phi^-\rangle$	01
10	$\|\phi^+\rangle$	$\|\psi^+\rangle$	10
11	$\|\phi^+\rangle$	$\|\psi^-\rangle$	11

Superdense Coding Protocol

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$$|u\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle \otimes |c\rangle + (-1)^{d}\,|1\rangle \otimes |1-c\rangle\Big)$$

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$$\begin{aligned}\mathcal{P}_{M=k}(|u\rangle) &= \left|\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |b_k\rangle \langle b_k|c\rangle + (-1)^d |1\rangle \otimes |b_k\rangle \langle b_k|1-c\rangle\right)\right|^2 \\ &= \frac{1}{2}\left(|\langle b_k|c\rangle|^2 + |\langle b_k|1-c\rangle|^2\right) = \frac{1}{2}\end{aligned}$$

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- Therefore, Eve cannot gain any information.

Superdense Coding Protocol

Ultradense Coding

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Superdense Coding Protocol

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- The probability that Bob guesses correctly at least once out of $N = 100$ attempt is $1-\left(\frac{3}{4}\right)^{100} \approx 1 - 3.2072022 \cdot 10^{-13}$, and Bob almost surely can receive the message without physically receiving anything.

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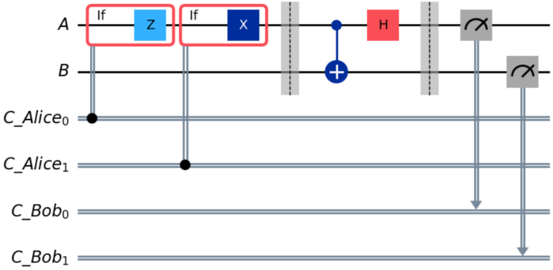
- The probability that Bob guesses correctly at least once out of $N = 100$ attempt is $1 - \left(\frac{3}{4}\right)^{100} \approx 1 - 3.2072022 \cdot 10^{-13}$, and Bob almost surely can receive the message without physically receiving anything.
- Alice and Bob now have achieved a faster-than-light communication.

Superdense Coding Protocol

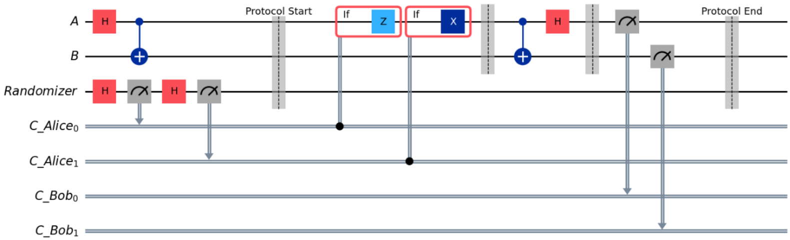
```python
from qiskit import QuantumCircuit, QuantumRegister, ClassicalRegister
from qiskit_aer.primitives import Sampler
from qiskit_aer import AerSimulator
from qiskit.visualization import plot_histogram
```

```python
A = QuantumRegister(1, "A")
B = QuantumRegister(1, "B")
C_Alice = ClassicalRegister(2, "C_Alice")
C_Bob = ClassicalRegister(2, "C_Bob")

protocol = QuantumCircuit(A, B, C_Alice, C_Bob)

# Alice modifies her qubit than send it to Bob
with protocol.if_test((C_Alice[0], 1)):
    protocol.z(A)
with protocol.if_test((C_Alice[1], 1)):
    protocol.x(A)
protocol.barrier()

# Bob's operation
protocol.cx(A, B)
protocol.h(A)
protocol.barrier()

# Bob's measurement
protocol.measure(A, C_Bob[0])
protocol.measure(B, C_Bob[1])

protocol.draw(output = "mpl", cregbundle = False)
```



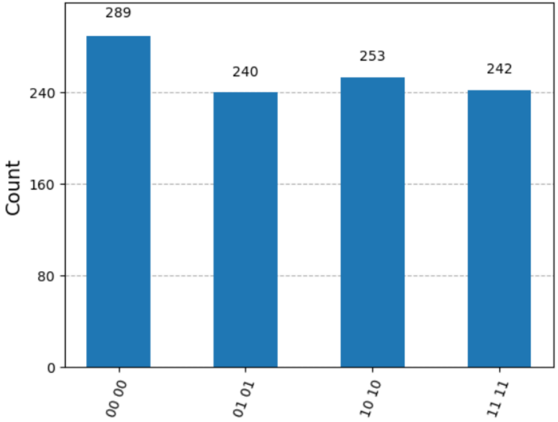
Superdense Coding Protocol

```
R = QuantumRegister(1, "Randomizer")
runner = QuantumCircuit(A, B, R, C_Alice, C_Bob)

# Make ranodm bits
runner.h(R)
runner.measure(R, C_Alice[0])
runner.h(R)
runner.measure(R, C_Alice[1])

# Entangle A and B
runner.h(A)
runner.cx(A, B)
runner.barrier(label = "Protocol Start")

# Run protocol
runner = runner.compose(protocol)
runner.barrier(label = "Protocol End")

# Check the result
runner.draw(output = "mpl", cregbundle = False)
```



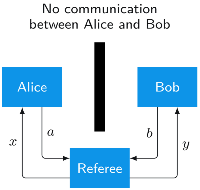
Superdense Coding Protocol

```
result = AerSimulator().run(runner).result()
statistics = result.get_counts()
display(plot_histogram(statistics))
```



CHSH Game

Game Description



(x,y)	win	lose
$(0,0)$	$a=b$	$a\neq b$
$(0,1)$	$a=b$	$a\neq b$
$(1,0)$	$a=b$	$a\neq b$
$(1,1)$	$a\neq b$	$a=b$

- It is a cooperative game where Alice and Bob work together to achieve a particular outcome.
- A referee uniformly and randomly choose a pair of integer x and y, each of which are either 0 or 1, and give x to Alice and y to Bob
- Alice and Bob each replies with an integer a and b, each of which are again 0 or 1. They win according to the table on the left.
- Alice and Bob can discuss their strategy beforehand, but they're not allowed to communicate after the game starts.

Classical Strategy (Deterministic)

CHSH Game

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- Here, Alice and Bob 's responses are functions of x and y, i.e. $a = f(x)$ and $b = g(y)$ for some f and g.

CHSH Game

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- Here, Alice and Bob 's responses are functions of x and y, i.e. $a = f(x)$ and $b = g(y)$ for some f and g.
- Q) Can they win on all inputs? No.

CHSH Game

Classical Strategy (Deterministic)

- Here, Alice and Bob 's responses are functions of x and y, i.e. $a = f(x)$ and $b = g(y)$ for some f and g.
- Q) Can they win on all inputs? No.
- However, $f(x) = g(x) = 0$ allows them to win on 3 out of 4 possible inputs. Thus, the best deterministic strategy has the winning probability of 0.75.

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Classical Strategy (Probabilistic)

- A probabilistic strategy can be regarded as choosing a deterministic strategy at random.
- Since the average is equal or less than the maximum, it cannot perform better than a deterministic one.
- Therefore, the best probabilistic strategy has the winning probability of 0.75.

Quantum Strategy

- Can they achieve a better winning probability than 0.75 if they had prepared a shared e-bit?

Quantum Strategy

- Can they achieve a better winning probability than 0.75 if they had prepared a shared e-bit?
- The answer is yes. We demonstrate one such strategy.

CHSH Game

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CHSH Game

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1. $$|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$

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   The following formulae can easily be verified.
   - $\langle\psi_\alpha|\psi_\beta\rangle = \cos(\alpha-\beta)$
   - $\langle\psi_\alpha\otimes\psi_\beta|\phi^+\rangle = \frac{1}{\sqrt{2}}\cos(\alpha-\beta) = \frac{1}{\sqrt{2}}\langle\psi_\alpha|\psi_\beta\rangle$

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2. $$\begin{aligned} U_\theta &= |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \end{aligned}$$

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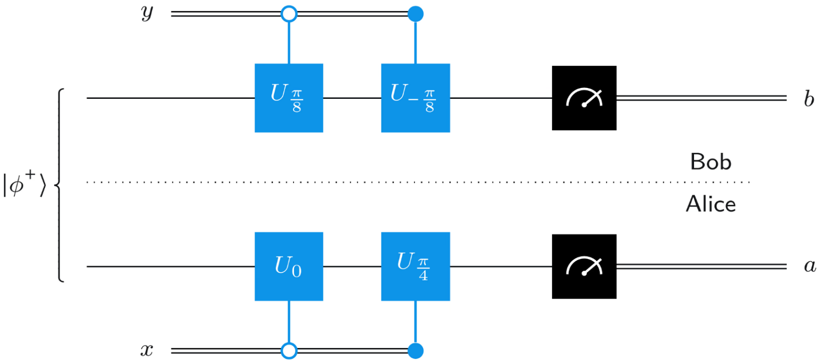
   It represents the clockwise rotation by θ.

Strategy Overview



CHSH Game

Alice and Bob are choosing angles α (which is 0 if $x = 0$ and $\pi/4$ otherwise) and β (which is $\pi/8$ if $y = 0$ and $-\pi/8$ otherwise) depending on x and y, applying U_α and U_β, then measuring their qubit.

$$
\begin{aligned}
&(U_\alpha \otimes U_\beta) |\phi^+\rangle \\
&= |00\rangle \langle \psi_\alpha \otimes \psi_\beta | \phi^+\rangle + |01\rangle \langle \psi_\alpha \otimes \psi_{\beta+\pi/2} | \phi^+\rangle + |10\rangle \langle \psi_{\alpha+\pi/2} \otimes \psi_\beta | \phi^+\rangle \\
&+ |11\rangle \langle \psi_{\alpha+\pi/2} \otimes \psi_{\beta+\pi/2} | \phi^+\rangle \\
&= \frac{1}{\sqrt{2}} \left(\langle \psi_\alpha | \psi_\beta \rangle |00\rangle + \langle \psi_\alpha | \psi_{\beta+\pi/2} \rangle |01\rangle + \langle \psi_{\alpha+\pi/2} | \psi_\beta \rangle |10\rangle + \langle \psi_{\alpha+\pi/2} | \psi_{\beta+\pi/2} \rangle |11\rangle \right)
\end{aligned}
$$

CHSH Game

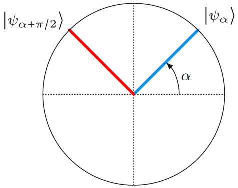
$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle\,|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\,|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\,|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle\,|11\rangle\right)$$

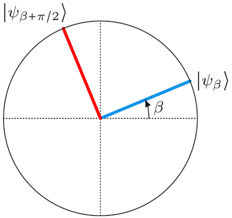
The probabilities of each outcome can be thought of as the following.

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

The probabilities of each outcome can be thought of as the following.
Alice and Bob first pick the following orthonormal basis, where blue represents 0 and red represents 1.





The probability of each outcome is $\frac{1}{2}\cos^2(\theta)$ where θ is the angle between corresponding vectors of Alice and Bob

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle+\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle+\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle+\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

And finally,

$$\mathcal{P}(a=b)=\frac{1}{2}\left|\langle\psi_\alpha|\psi_\beta\rangle\right|^2+\frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle\right|^2=\cos(\alpha-\beta)^2$$
$$\mathcal{P}(a\neq b)=\frac{1}{2}\left|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\right|^2+\frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\right|^2=\sin(\alpha-\beta)^2$$

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

And finally,

$$\mathcal{P}(a=b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_\beta\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle\right|^2 = \cos(\alpha-\beta)^2$$
$$\mathcal{P}(a\neq b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\right|^2 = \sin(\alpha-\beta)^2$$

Can be interpreted as $\cos(\lambda)^2$ and $\sin(\lambda)^2$ where λ is the angle between vectors of the same color.

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle+\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle+\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle+\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

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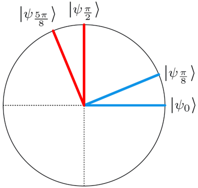
$$\mathcal{P}(a\neq b)=\frac{1}{2}\left|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\right|^2+\frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\right|^2=\sin(\alpha-\beta)^2$$

Can be interpreted as $\cos(\lambda)^2$ and $\sin(\lambda)^2$ where λ is the angle between vectors of the same color.

Note that you can pick any such pair of vectors and the result will be the same.

CHSH Game

Case $(x, y) = (0, 0)$



CHSH Game

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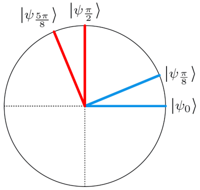
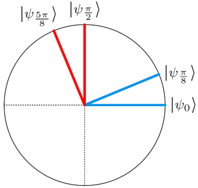


- $\mathcal{P}(a = b) =$

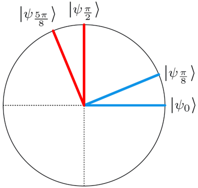
Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$

CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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CHSH Game

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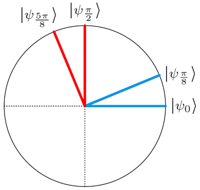

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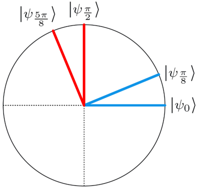

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CHSH Game

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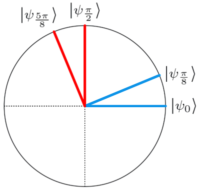

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CHSH Game

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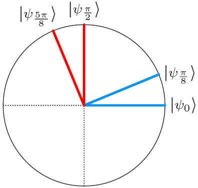
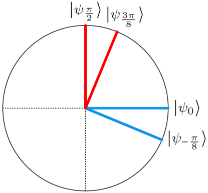


Case $(x, y) = (0, 1)$



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CHSH Game

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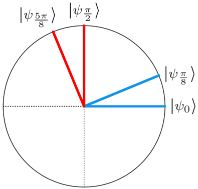

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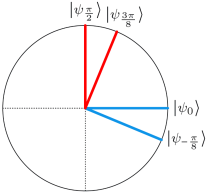

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CHSH Game

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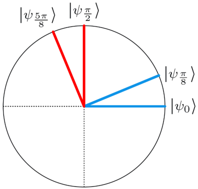

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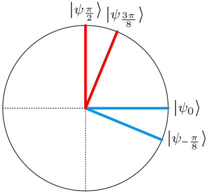

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CHSH Game

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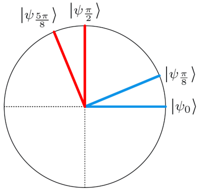

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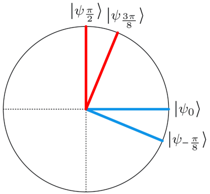

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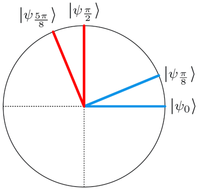

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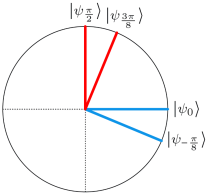

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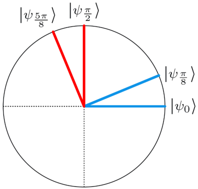

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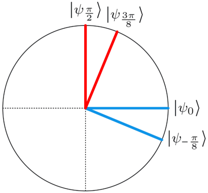

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CHSH Game

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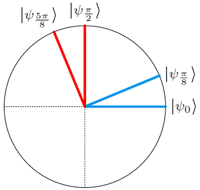

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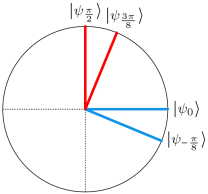

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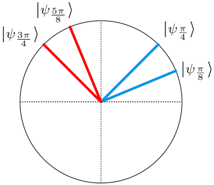
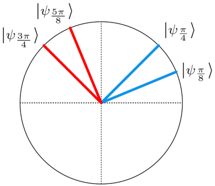
CHSH Game

Case $(x, y) = (1, 0)$



CHSH Game

Case $(x, y) = (1, 0)$



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CHSH Game

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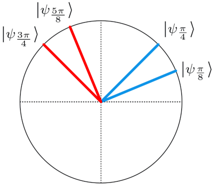

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CHSH Game

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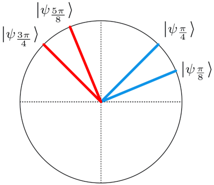

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CHSH Game

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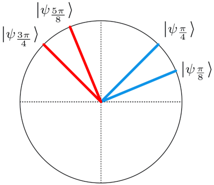

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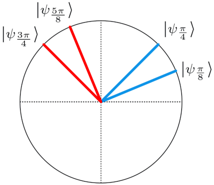
CHSH Game

Case $(x, y) = (1, 0)$



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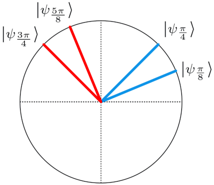
CHSH Game

Case $(x, y) = (1, 0)$



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CHSH Game

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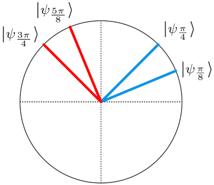
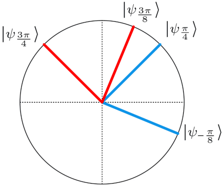


Case $(x, y) = (1, 1)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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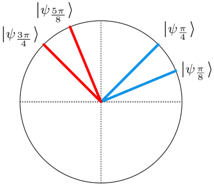
CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}\left(a = b\right) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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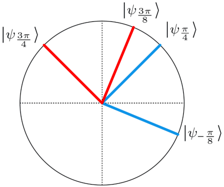
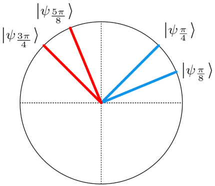


- $\mathcal{P}\left(a = b\right) =$

CHSH Game

Case $(x, y) = (1, 0)$



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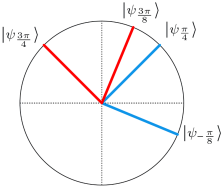
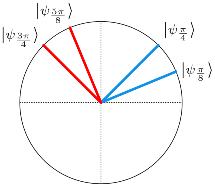


- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$

CHSH Game

<u>**Case** $(x, y) = (1, 0)$</u>



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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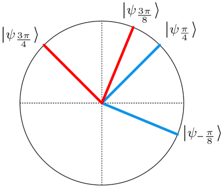

- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
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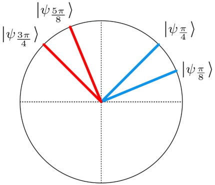
CHSH Game

Case $(x, y) = (1, 0)$



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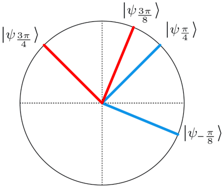

- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
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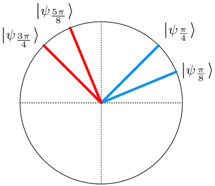
CHSH Game

Case $(x, y) = (1, 0)$



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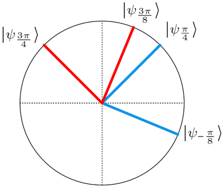
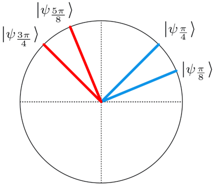
Case $(x, y) = (1, 1)$



- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
- $\mathcal{P}(a \neq b) = \sin\left(\frac{3\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
- (Winning Probability) $=$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
- $\mathcal{P}(a \neq b) = \sin\left(\frac{\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
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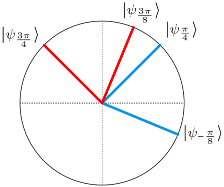

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- (Winning Probability) $= \frac{2+\sqrt{2}}{4}$

CHSH Game

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- CHSH game acts as a straightforward proof that entanglement is real, by providing a way to "observe" the entanglement.
- 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser, and Anton Zeilinger, for observing entanglement through Bell tests on entangled photons.

First, declare the CHSH game runner

```python
from qiskit import QuantumCircuit
from qiskit_aer.primitives import SamplerV2
from numpy import pi
from numpy.random import randint

"""Plays the CHSH game
Args:
    strategy (callable): A function that takes two bits (as `int`s) and
        returns two bits (also as `int`s). The strategy must follow the
        rules of the CHSH game.
Returns:
    int: 1 for a win, 0 for a loss.
"""
def CHSH_game(strategy):
    # Referee chooses x and y uniformly at random
    x, y = randint(0, 2), randint(0, 2)

    # Alice and Bob chooses a and b according to their strategy
    a, b = strategy(x, y)

    return 1 if a ^ b == x & y else 0

def CHSH_game_runner(strategy, NUM_GAMES):
    WINS = 0
    for _ in range(NUM_GAMES):
        WINS += CHSH_game(strategy)
    print(f"Won {WINS} out of {NUM_GAMES} games, with winrate {WINS/NUM_GAMES}")
```

CHSH Game

An optimal classical strategy has the winning probabilitiy of 0.75.

```python
def classical_strategy(x, y):
    """An optimal classical strategy for the CHSH game
    Args:
        x (int): Alice's bit (must be 0 or 1)
        y (int): Bob's bit (must be 0 or 1)
    Returns:
        (int, int): Alice and Bob's answer bits (respectively)
    """
    # Alice's answer
    if x == 0:
        a = 0
    elif x == 1:
        a = 1

    # Bob's answer
    if y == 0:
        b = 1
    elif y == 1:
        b = 0

    return a, b

CHSH_game_runner(strategy = classical_strategy, NUM_GAMES = 10000)
```

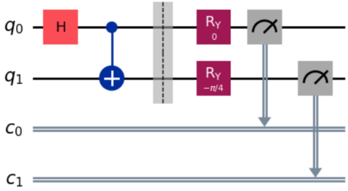
```
Won 7539 out of 10000 games, with winrate 0.7539
```

Declare the CHSH circuit builder

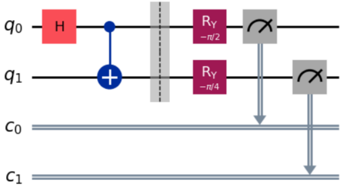
```python
def build_CHSH_circuit(x, y):
    """Creates a `QuantumCircuit` that implements the best CHSH strategy.
    Args:
        x (int): Alice's bit (must be 0 or 1)
        y (int): Bob's bit (must be 0 or 1)
    Returns:
        QuantumCircuit: Circuit that, when run, returns Alice and Bob's
            answer bits.
    """
    qc = QuantumCircuit(2, 2)
    qc.h(0)
    qc.cx(0, 1)
    qc.barrier()

    # Alice
    if x == 0:
        qc.ry(0, 0)
    else:
        qc.ry(-pi / 2, 0)
    qc.measure(0, 0)

    # Bob
    if y == 0:
        qc.ry(-pi / 4, 1)
    else:
        qc.ry(pi / 4, 1)
    qc.measure(1, 1)

    return qc

for x in range(2):
    for y in range(2):
        print(f"Circuit for (x, y) = ({x}, {y})")
        display(build_CHSH_circuit(x, y).draw(output = "mpl", cregbundle = False))
```

CHSH Game

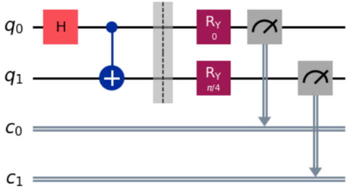
Circuit for (x, y) = (0, 0)



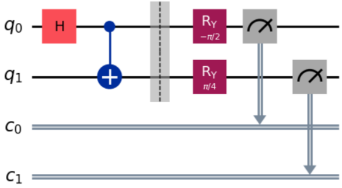
Circuit for (x, y) = (1, 0)



Circuit for (x, y) = (0, 1)



Circuit for (x, y) = (1, 1)



CHSH Game

An optimal quantum strategy has the winning probability of $\frac{2+\sqrt{2}}{2} \approx 0.85355$.

```python
sampler = SamplerV2()

def quantum_strategy(x, y):
    """Carry out the best strategy for the CHSH game.
    Args:
        x (int): Alice's bit (must be 0 or 1)
        y (int): Bob's bit (must be 0 or 1)
    Returns:
        (int, int): Alice and Bob's answer bits (respectively)
    """
    # `shots=1` runs the circuit once
    result = sampler.run([build_CHSH_circuit(x, y)], shots=1).result()[0].data.c.get_counts()
    bits = list(result.keys())[0]
    a, b = int(bits[0]), int(bits[1])
    return a, b

CHSH_game_runner(strategy = quantum_strategy, NUM_GAMES = 10000)
```

```
Won 8601 out of 10000 games, with winrate 0.8601
```

The End