Quantum Computing Seminar 5

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Entanglement

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- All of these are true but fails to distinguish the above non-entangled probabilistic state with an entanglement in a meaningful way.
- The power of entanglement lies in what we can do with it.

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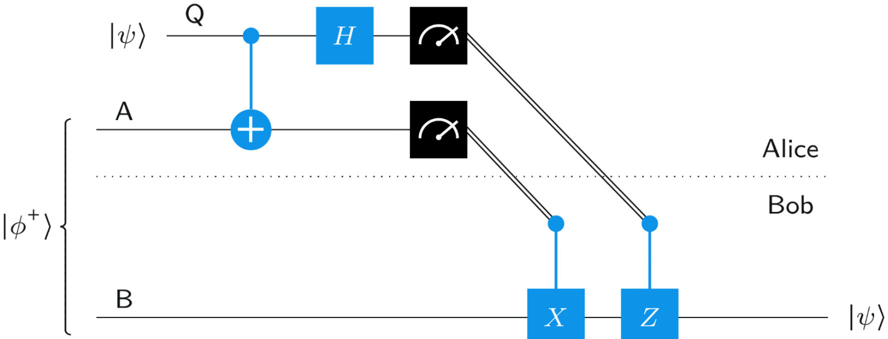
- They all consume a pair of entangled qubits $|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ as a resource, which we'll call an **e-bit**.
- All of these are either impossible if we have a pair of non-entangled probabilistic bits instead of an e-bit.

Teleportation Protocol

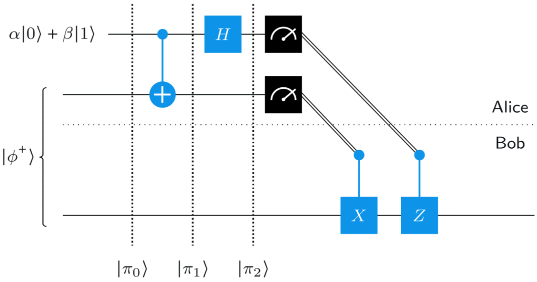
Teleportation Protocol

Preparation	Alice and Bob each have a qubit, together forming an e-bit
Objective	Alice sends Bob one qubit of quantum information
Communication	Alice sends Bob two bits of classical information

Protocol Overview



Teleportation Protocol



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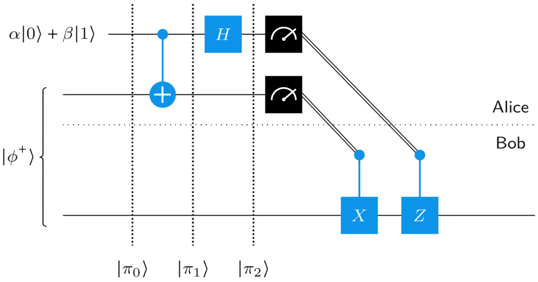

$$
\begin{aligned}
|\pi_0\rangle = |\phi^+\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) &= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|110\rangle + \beta|001\rangle + \beta|111\rangle) \\
&= \frac{1}{2}\Big(\alpha|0\rangle \otimes \big(|\phi^+\rangle + |\phi^-\rangle\big) + \alpha|1\rangle \otimes \big(|\psi^+\rangle - |\psi^-\rangle\big) + \beta|0\rangle \otimes \big(|\psi^+\rangle + |\psi^-\rangle\big) + \beta|1\rangle \otimes \big(|\phi^+\rangle - |\phi^-\rangle\big)\Big) \\
&= \frac{1}{2}\Big((\alpha|0\rangle + \beta|1\rangle) \otimes |\phi^+\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |\phi^-\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |\psi^+\rangle - (\alpha|1\rangle - \beta|0\rangle) \otimes |\psi^-\rangle\Big)
\end{aligned}
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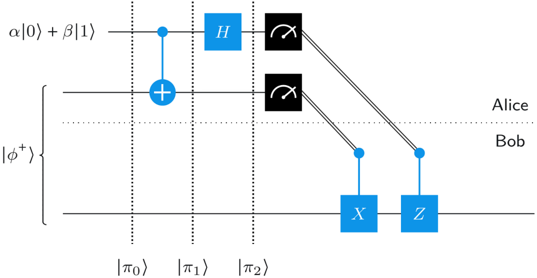
Teleportation Protocol



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\begin{aligned}
|\pi_2\rangle &= (I_2 \otimes I_2 \otimes H)\,(I_2 \otimes CX_{0,1})\,|\pi_0\rangle \\
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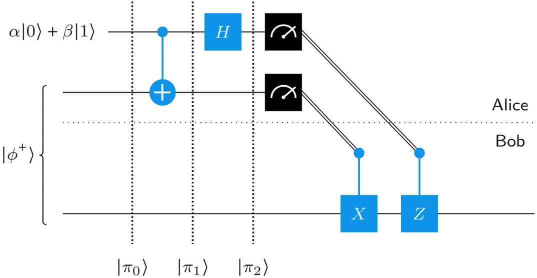

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There are 4 possible outcomes to Alice 's measurement.

Teleportation Protocol

Outcome 00

$$|\pi_2\rangle = \frac{1}{2}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle + (\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle\right)$$

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- **Probability** $\rightarrow$

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- **Probability** $\rightarrow 1/4$

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- **Bob 's Action** $\rightarrow$ Apply Z

Teleportation Protocol

Outcome 10

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- **Probability** $\rightarrow$

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Teleportation Protocol

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$$|\pi_2\rangle = \frac{1}{2}\left((\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle + (\alpha|0\rangle - \beta|1\rangle) \otimes |01\rangle + (\alpha|1\rangle + \beta|0\rangle) \otimes |10\rangle + (\alpha|1\rangle - \beta|0\rangle) \otimes |11\rangle\right)$$

- **Probability** $\rightarrow 1/4$
- **State of B** $\rightarrow \alpha|1\rangle - \beta|0\rangle$
- **Bob 's Action** $\rightarrow$

Teleportation Protocol

Outcome 10

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Teleportation Protocol

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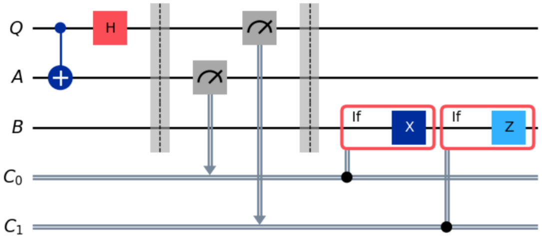
- Q) What if Q is entangled with some other qubits?
  - The state of Bob 's qubit at the end has the same entangled state as the initial state of Q.
  - It's not hard to repeat the same analysis to show this (left as an exercise), but it turns out that showing it for an independent qubit Q is enough, which will be discussed later.

Teleportation Protocol

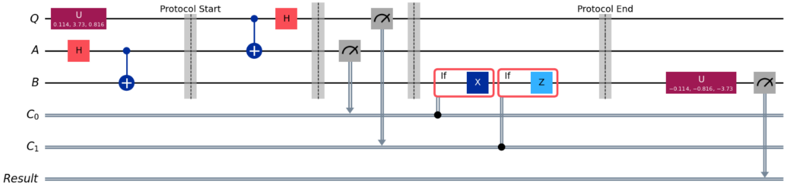
```python
from qiskit import QuantumCircuit, QuantumRegister, ClassicalRegister
from qiskit_aer import AerSimulator
from qiskit.visualization import plot_histogram
from qiskit.result import marginal_distribution
from qiskit.circuit.library import UGate
from numpy import pi, random
```

```python
Q = QuantumRegister(1, "Q")
A = QuantumRegister(1, "A")
B = QuantumRegister(1, "B")
C = ClassicalRegister(2, "C")

protocol = QuantumCircuit(Q, A, B, C)

# Alice's operations
protocol.cx(Q, A)
protocol.h(Q)
protocol.barrier()

# Alice measures and sends classical bits to Bob
protocol.measure(A, C[0])
protocol.measure(Q, C[1])
protocol.barrier()

# Bob uses the classical bits to conditionally apply gates
with protocol.if_test((C[0], 1)):
    protocol.x(B)
with protocol.if_test((C[1], 1)):
    protocol.z(B)

protocol.draw(output = "mpl", cregbundle = False)
```



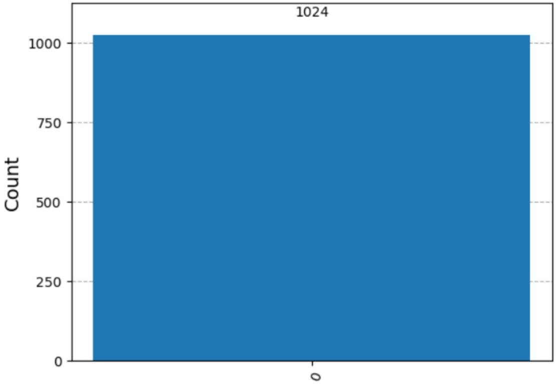
Teleportation Protocol

```
runner = QuantumCircuit(Q, A, B, C)

# Make ranodm gate
random_gate = UGate(
    theta=random.random() * 2 * pi,
    phi=random.random() * 2 * pi,
    lam=random.random() * 2 * pi,
)

# Randomly selected state on Q
runner.append(random_gate, Q)

# Entangle A and B
runner.h(A)
runner.cx(A, B)
runner.barrier(label = "Protocol Start")

# Run protocol
runner = runner.compose(protocol)
runner.barrier(label = "Protocol End")

# Check the result
runner.append(random_gate.inverse(), B)
result = ClassicalRegister(1, "Result")
runner.add_register(result)
runner.measure(B, result)

runner.draw(output = "mpl", cregbundle = False)
```



Teleportation Protocol

```
result = AerSimulator().run(runner).result()
statistics = marginal_distribution(result.get_counts(), [2])
plot_histogram(statistics)
```

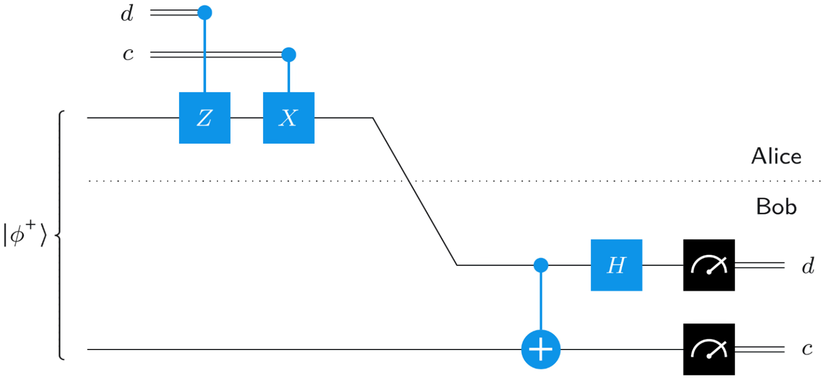


Superdense Coding Protocol

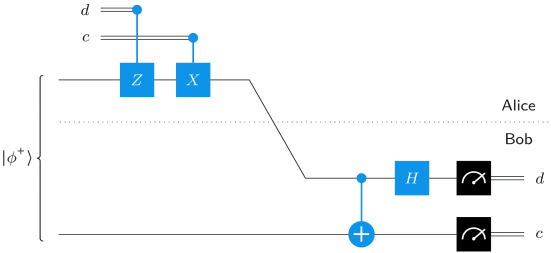
Superdense Coding Protocol

Preparation	Alice and Bob each have a qubit, together forming an e-bit
Objective	Alice sends Bob two bits of classical information
Communication	Alice sends Bob one qubit of quantum information

Protocol Overview

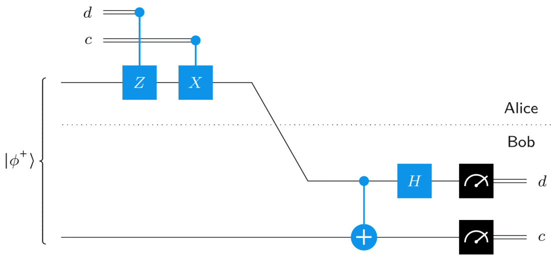


Superdense Coding Protocol



We only have 4 cases to analyze

Superdense Coding Protocol



We only have 4 cases to analyze

Input	**Initial state**	**State after Alice 's operation**	**Bob 's measurement**
00	$\|\phi^+\rangle$	$\|\phi^+\rangle$	00
01	$\|\phi^+\rangle$	$\|\phi^-\rangle$	01
10	$\|\phi^+\rangle$	$\|\psi^+\rangle$	10
11	$\|\phi^+\rangle$	$\|\psi^-\rangle$	11

Superdense Coding Protocol

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$$\begin{aligned}\mathcal{P}_{M=k}\left(|u\rangle\right) &= \left|\frac{1}{\sqrt{2}}\left(|0\rangle \otimes |b_k\rangle\langle b_k|c\rangle + (-1)^d\,|1\rangle \otimes |b_k\rangle\langle b_k|1-c\rangle\right)\right|^2 \\ &= \frac{1}{2}\left(|\langle b_k|c\rangle|^2 + |\langle b_k|1-c\rangle|^2\right) = \frac{1}{2}\end{aligned}$$

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- Therefore, Eve cannot gain any information.

Superdense Coding Protocol

Ultradense Coding

- Suppose there exists a protocol which can send $2 + \epsilon$ classical bits by sending 1 qubit for some positive real ϵ.

Superdense Coding Protocol

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- The probability that Bob guesses correctly at least once out of $N = 100$ attempt is $1 - \left(\frac{3}{4}\right)^{100} \approx 1 - 3.2072022 \cdot 10^{-13}$, and Bob almost surely can receive the message without physically receiving anything.

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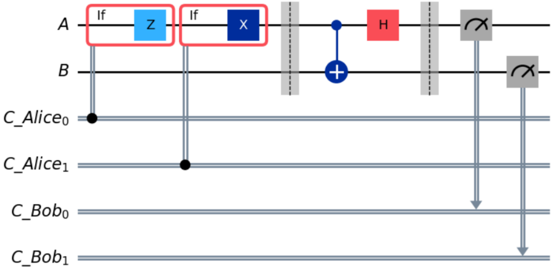
- The probability that Bob guesses correctly at least once out of $N = 100$ attempt is $1 - \left(\frac{3}{4}\right)^{100} \approx 1 - 3.2072022 \cdot 10^{-13}$, and Bob almost surely can receive the message without physically receiving anything.
- Alice and Bob now have achieved a faster-than-light communication.

Superdense Coding Protocol

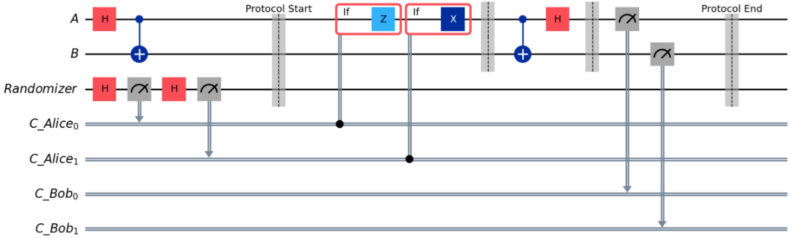
```python
from qiskit import QuantumCircuit, QuantumRegister, ClassicalRegister
from qiskit_aer.primitives import Sampler
from qiskit_aer import AerSimulator
from qiskit.visualization import plot_histogram
```

```python
A = QuantumRegister(1, "A")
B = QuantumRegister(1, "B")
C_Alice = ClassicalRegister(2, "C_Alice")
C_Bob = ClassicalRegister(2, "C_Bob")

protocol = QuantumCircuit(A, B, C_Alice, C_Bob)

# Alice modifies her qubit than send it to Bob
with protocol.if_test((C_Alice[0], 1)):
    protocol.z(A)
with protocol.if_test((C_Alice[1], 1)):
    protocol.x(A)
protocol.barrier()

# Bob's operation
protocol.cx(A, B)
protocol.h(A)
protocol.barrier()

# Bob's measurement
protocol.measure(A, C_Bob[0])
protocol.measure(B, C_Bob[1])

protocol.draw(output = "mpl", cregbundle = False)
```



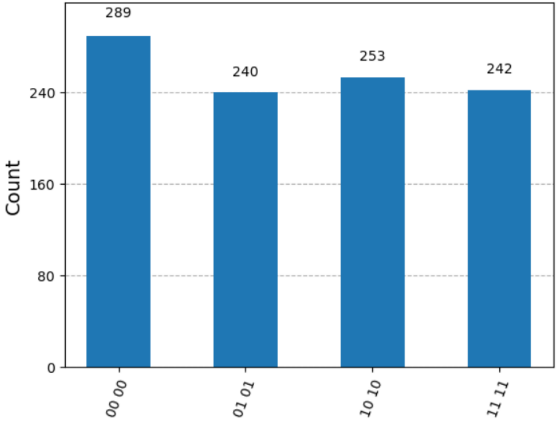
Superdense Coding Protocol

```
R = QuantumRegister(1, "Randomizer")
runner = QuantumCircuit(A, B, R, C_Alice, C_Bob)

# Make ranodm bits
runner.h(R)
runner.measure(R, C_Alice[0])
runner.h(R)
runner.measure(R, C_Alice[1])

# Entangle A and B
runner.h(A)
runner.cx(A, B)
runner.barrier(label = "Protocol Start")

# Run protocol
runner = runner.compose(protocol)
runner.barrier(label = "Protocol End")

# Check the result
runner.draw(output = "mpl", cregbundle = False)
```



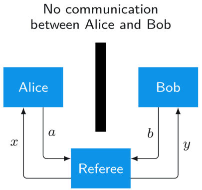
Superdense Coding Protocol

```
result = AerSimulator().run(runner).result()
statistics = result.get_counts()
display(plot_histogram(statistics))
```



CHSH Game

Game Description



(x,y)	win	lose
$(0,0)$	$a=b$	$a\neq b$
$(0,1)$	$a=b$	$a\neq b$
$(1,0)$	$a=b$	$a\neq b$
$(1,1)$	$a\neq b$	$a=b$

- It is a cooperative game where Alice and Bob work together to achieve a particular outcome.
- A referee uniformly and randomly choose a pair of integer x and y, each of which are either 0 or 1, and give x to Alice and y to Bob
- Alice and Bob each replies with an integer a and b, each of which are again 0 or 1. They win according to the table on the left.
- Alice and Bob can discuss their strategy beforehand, but they're not allowed to communicate after the game starts.

CHSH Game

Classical Strategy (Deterministic)

CHSH Game

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- Here, Alice and Bob 's responses are functions of x and y, i.e. $a = f(x)$ and $b = g(y)$ for some f and g.

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- Here, Alice and Bob 's responses are functions of x and y, i.e. $a = f(x)$ and $b = g(y)$ for some f and g.
- Q) Can they win on all inputs? No.

CHSH Game

Classical Strategy (Deterministic)

- Here, Alice and Bob 's responses are functions of x and y, i.e. $a = f(x)$ and $b = g(y)$ for some f and g.
- Q) Can they win on all inputs? No.
- However, $f(x) = g(x) = 0$ allows them to win on 3 out of 4 possible inputs. Thus, the best deterministic strategy has the winning probability of 0.75.

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- A probabilistic strategy can be regarded as choosing a deterministic strategy at random.

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- Since the average is equal or less than the maximum, it cannot perform better than a deterministic one.

CHSH Game

Classical Strategy (Probabilistic)

- A probabilistic strategy can be regarded as choosing a deterministic strategy at random.
- Since the average is equal or less than the maximum, it cannot perform better than a deterministic one.
- Therefore, the best probabilistic strategy has the winning probability of 0.75.

CHSH Game

Quantum Strategy

- Can they achieve a better winning probability than 0.75 if they had prepared a shared e-bit?

Quantum Strategy

- Can they achieve a better winning probability than 0.75 if they had prepared a shared e-bit?
- The answer is yes. We demonstrate one such strategy.

CHSH Game

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We define the following variables.

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We define the following variables.

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   The following formulae can easily be verified.

   - $\langle\psi_\alpha|\psi_\beta\rangle = \cos(\alpha - \beta)$
   - $\langle\psi_\alpha \otimes \psi_\beta|\phi^+\rangle = \frac{1}{\sqrt{2}}\cos(\alpha - \beta) = \frac{1}{\sqrt{2}}\langle\psi_\alpha|\psi_\beta\rangle$

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2. $$\begin{aligned} U_\theta &= |0\rangle\langle\psi_\theta| + |1\rangle\langle\psi_{\theta+\pi/2}| \\ &= \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \end{aligned}$$

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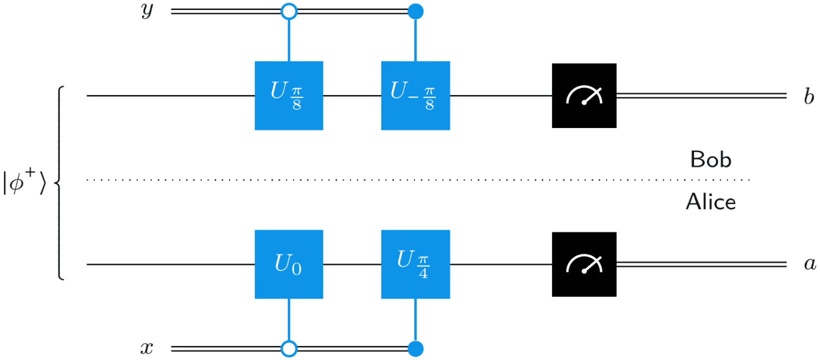
   It represents the clockwise rotation by θ.

Strategy Overview



CHSH Game

Alice and Bob are choosing angles α (which is 0 if $x = 0$ and $\pi/4$ otherwise) and β (which is $\pi/8$ if $y = 0$ and $-\pi/8$ otherwise) depending on x and y, applying U_α and U_β, then measuring their qubit.

$$
\begin{aligned}
&(U_\alpha \otimes U_\beta) |\phi^+\rangle \\
&= |00\rangle \langle \psi_\alpha \otimes \psi_\beta | \phi^+\rangle + |01\rangle \langle \psi_\alpha \otimes \psi_{\beta+\pi/2} | \phi^+\rangle + |10\rangle \langle \psi_{\alpha+\pi/2} \otimes \psi_\beta | \phi^+\rangle \\
&+ |11\rangle \langle \psi_{\alpha+\pi/2} \otimes \psi_{\beta+\pi/2} | \phi^+\rangle \\
&= \frac{1}{\sqrt{2}} \left(\langle \psi_\alpha | \psi_\beta \rangle |00\rangle + \langle \psi_\alpha | \psi_{\beta+\pi/2} \rangle |01\rangle + \langle \psi_{\alpha+\pi/2} | \psi_\beta \rangle |10\rangle + \langle \psi_{\alpha+\pi/2} | \psi_{\beta+\pi/2} \rangle |11\rangle \right)
\end{aligned}
$$

CHSH Game

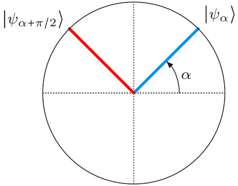
$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

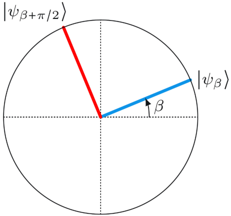
The probabilities of each outcome can be thought of as the following.

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

The probabilities of each outcome can be thought of as the following.
Alice and Bob first pick the following orthonormal basis, where blue represents 0 and red represents 1.





The probability of each outcome is $\frac{1}{2}\cos^2(\theta)$ where θ is the angle between corresponding vectors of Alice and Bob

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

And finally,

$$\mathcal{P}(a = b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_\beta\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle\right|^2 = \cos(\alpha-\beta)^2$$
$$\mathcal{P}(a \neq b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\right|^2 = \sin(\alpha-\beta)^2$$

CHSH Game

$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

And finally,

$$\mathcal{P}(a = b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_\beta\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle\right|^2 = \cos(\alpha-\beta)^2$$
$$\mathcal{P}(a \neq b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\right|^2 = \sin(\alpha-\beta)^2$$

Can be interpreted as $\cos(\lambda)^2$ and $\sin(\lambda)^2$ where λ is the angle between vectors of the same color for the first one, and

CHSH Game

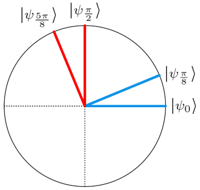
$$\frac{1}{\sqrt{2}}\left(\langle\psi_\alpha|\psi_\beta\rangle|00\rangle + \langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle|01\rangle + \langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle|10\rangle + \langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle|11\rangle\right)$$

And finally,

$$\mathcal{P}(a=b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_\beta\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_{\beta+\pi/2}\rangle\right|^2 = \cos(\alpha-\beta)^2$$
$$\mathcal{P}(a\neq b) = \frac{1}{2}\left|\langle\psi_\alpha|\psi_{\beta+\pi/2}\rangle\right|^2 + \frac{1}{2}\left|\langle\psi_{\alpha+\pi/2}|\psi_\beta\rangle\right|^2 = \sin(\alpha-\beta)^2$$

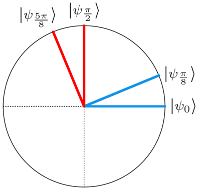
Can be interpreted as $\cos(\lambda)^2$ and $\sin(\lambda)^2$ where λ is the angle between vectors of the same color for the first one, and
Note that you can pick any such pair of vectors and the result will be the same.

Case $(x, y) = (0, 0)$


$|\psi_{\frac{5\pi}{8}}\rangle$
$|\psi_{\frac{\pi}{2}}\rangle$
$|\psi_{\frac{\pi}{8}}\rangle$
$|\psi_0\rangle$

CHSH Game

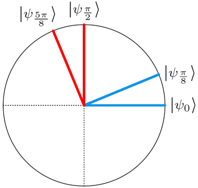
Case $(x, y) = (0, 0)$



- $\mathcal{P}\,(a = b) =$

CHSH Game

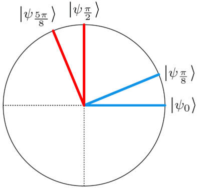
Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$

CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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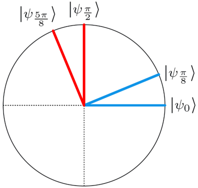
CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}\left(a = b\right) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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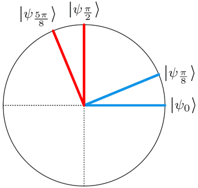
CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}\left(a = b\right) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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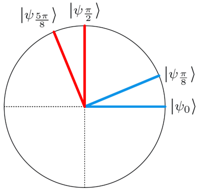
- (Winning Probability) $=$

CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
- $\mathcal{P}(a \neq b) = \sin\left(\frac{\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
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CHSH Game

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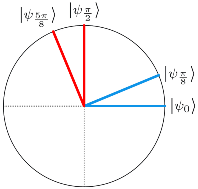

- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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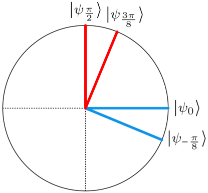
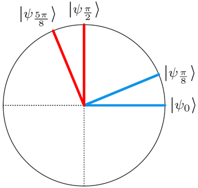
- (Winning Probability) $= \frac{2+\sqrt{2}}{4}$

Case $(x, y) = (0, 1)$



CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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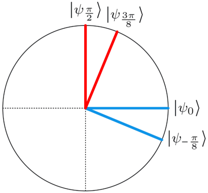
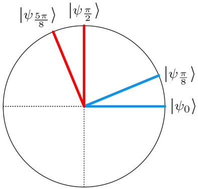


- $\mathcal{P}(a = b) =$

CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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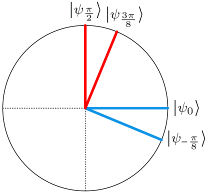
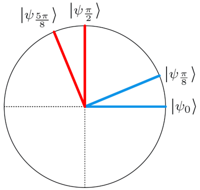


- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$

CHSH Game

Case $(x, y) = (0, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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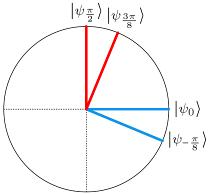

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CHSH Game

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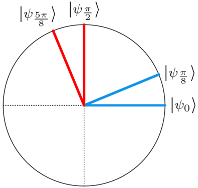

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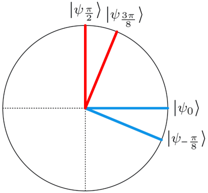

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CHSH Game

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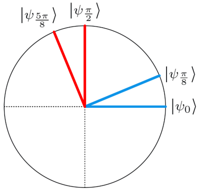

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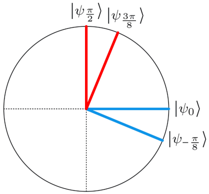

- $\mathcal{P}\left(a = b\right) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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CHSH Game

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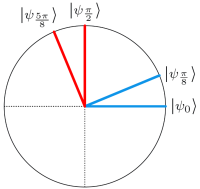

- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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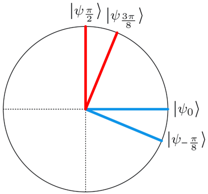
Case $(x, y) = (0, 1)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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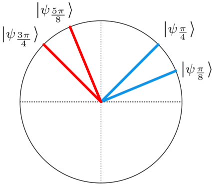
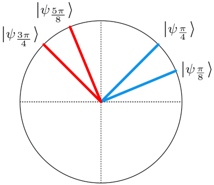
CHSH Game

Case $(x, y) = (1, 0)$



CHSH Game

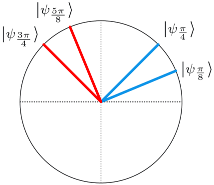
Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) =$

CHSH Game

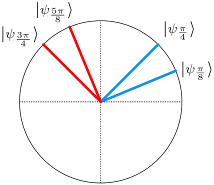
<u>**Case** $(x, y) = (1, 0)$</u>



- $\mathcal{P}\left(a = b\right) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$

CHSH Game

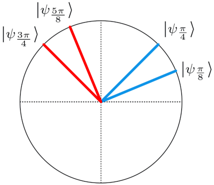
Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
- $\mathcal{P}(a \neq b) =$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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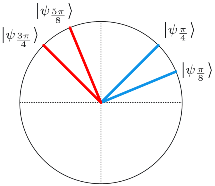
CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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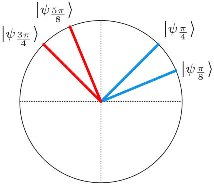
- (Winning Probability) $=$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}\left(a = b\right) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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CHSH Game

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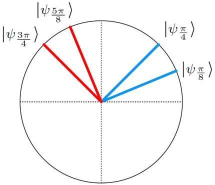

- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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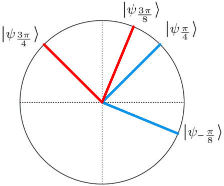
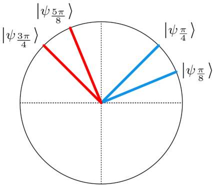
Case $(x, y) = (1, 1)$



CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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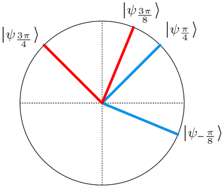
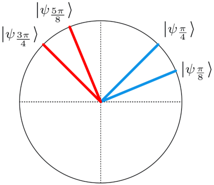


- $\mathcal{P}(a = b) =$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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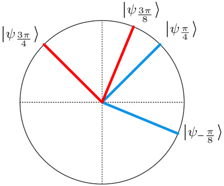
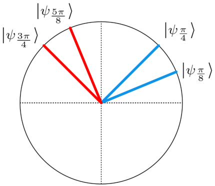
Case $(x, y) = (1, 1)$



- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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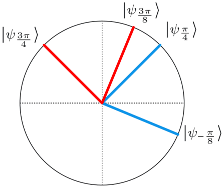

- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
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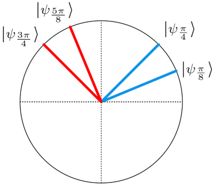
CHSH Game

Case $(x, y) = (1, 0)$



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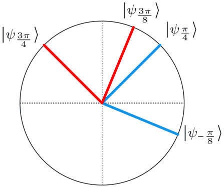
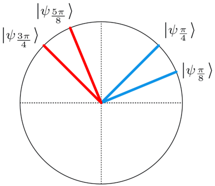


- $\mathcal{P}(a = b) = \cos\left(\frac{3\pi}{8}\right)^2 = \frac{2-\sqrt{2}}{4}$
- $\mathcal{P}(a \neq b) = \sin\left(\frac{3\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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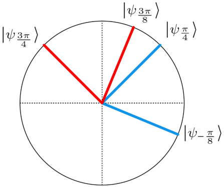

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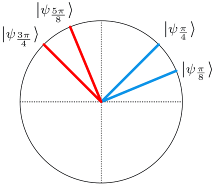
- (Winning Probability) $=$

CHSH Game

Case $(x, y) = (1, 0)$



- $\mathcal{P}(a = b) = \cos\left(\frac{\pi}{8}\right)^2 = \frac{2+\sqrt{2}}{4}$
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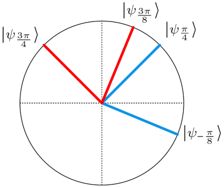
- $(\text{Winning Probability}) = \frac{2+\sqrt{2}}{4}$

Case $(x, y) = (1, 1)$



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- $(\text{Winning Probability}) = \frac{2+\sqrt{2}}{4}$

CHSH Game

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- CHSH game acts as a straightforward proof that entanglement is real, by providing a way to "observe" the entanglement.
- 2022 Nobel Prize in Physics was awarded to Alain Aspect, John Clauser, and Anton Zeilinger, for observing entanglement through Bell tests on entangled photons.

CHSH Game

First, declare the CHSH game runner

```python
from qiskit import QuantumCircuit
from qiskit_aer.primitives import SamplerV2
from numpy import pi
from numpy.random import randint

"""Plays the CHSH game
Args:
    strategy (callable): A function that takes two bits (as `int`s) and
        returns two bits (also as `int`s). The strategy must follow the
        rules of the CHSH game.
Returns:
    int: 1 for a win, 0 for a loss.
"""
def CHSH_game(strategy):
    # Referee chooses x and y uniformly at random
    x, y = randint(0, 2), randint(0, 2)

    # Alice and Bob chooses a and b according to their strategy
    a, b = strategy(x, y)

    return 1 if a ^ b == x & y else 0

def CHSH_game_runner(strategy, NUM_GAMES):
    WINS = 0
    for _ in range(NUM_GAMES):
        WINS += CHSH_game(strategy)
    print(f"Won {WINS} out of {NUM_GAMES} games, with winrate {WINS/NUM_GAMES}")
```

CHSH Game

An optimal classical strategy has the winning probabilitiy of 0.75.

```python
def classical_strategy(x, y):
    """An optimal classical strategy for the CHSH game
    Args:
        x (int): Alice's bit (must be 0 or 1)
        y (int): Bob's bit (must be 0 or 1)
    Returns:
        (int, int): Alice and Bob's answer bits (respectively)
    """
    # Alice's answer
    if x == 0:
        a = 0
    elif x == 1:
        a = 1

    # Bob's answer
    if y == 0:
        b = 1
    elif y == 1:
        b = 0

    return a, b

CHSH_game_runner(strategy = classical_strategy, NUM_GAMES = 10000)
```

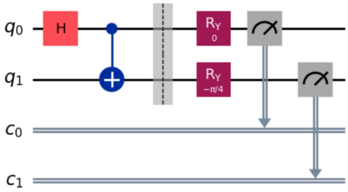
```
Won 7539 out of 10000 games, with winrate 0.7539
```

Declare the CHSH circuit builder

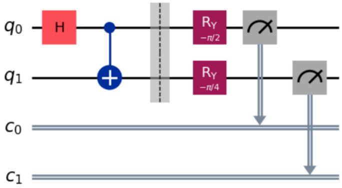
```python
def build_CHSH_circuit(x, y):
    """Creates a `QuantumCircuit` that implements the best CHSH strategy.
    Args:
        x (int): Alice's bit (must be 0 or 1)
        y (int): Bob's bit (must be 0 or 1)
    Returns:
        QuantumCircuit: Circuit that, when run, returns Alice and Bob's
            answer bits.
    """
    qc = QuantumCircuit(2, 2)
    qc.h(0)
    qc.cx(0, 1)
    qc.barrier()

    # Alice
    if x == 0:
        qc.ry(0, 0)
    else:
        qc.ry(-pi / 2, 0)
    qc.measure(0, 0)

    # Bob
    if y == 0:
        qc.ry(-pi / 4, 1)
    else:
        qc.ry(pi / 4, 1)
    qc.measure(1, 1)

    return qc

for x in range(2):
    for y in range(2):
        print(f"Circuit for (x, y) = ({x}, {y})")
        display(build_CHSH_circuit(x, y).draw(output = "mpl", cregbundle = False))
```

CHSH Game

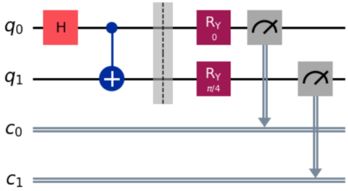
Circuit for (x, y) = (0, 0)



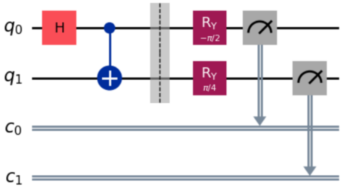
Circuit for (x, y) = (1, 0)



Circuit for (x, y) = (0, 1)



Circuit for (x, y) = (1, 1)



CHSH Game

An optimal quantum strategy has the winning probability of $\frac{2+\sqrt{2}}{2} \approx 0.85355$.

```
sampler = SamplerV2()

def quantum_strategy(x, y):
    """Carry out the best strategy for the CHSH game.
    Args:
        x (int): Alice's bit (must be 0 or 1)
        y (int): Bob's bit (must be 0 or 1)
    Returns:
        (int, int): Alice and Bob's answer bits (respectively)
    """
    # `shots=1` runs the circuit once
    result = sampler.run([build_CHSH_circuit(x, y)], shots=1).result()[0].data.c.get_counts()
    bits = list(result.keys())[0]
    a, b = int(bits[0]), int(bits[1])
    return a, b

CHSH_game_runner(strategy = quantum_strategy, NUM_GAMES = 10000)
```

```
Won 8601 out of 10000 games, with winrate 0.8601
```

The End