Quantum Computing Seminar 4

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Remark

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- Quantum information evolves deterministically when we apply any number of unitary operations.
- What is probabilistic is the way we interact with quantum information, i.e. the measurement.
- If we have information of the initial state and the unitary operations applied, we deterministically know the final state assuming there were no measurement.

Quantum Circuit

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Unitary operations and measurements are applied sequentially to the qubits and classical bits.

Quantum Circuit

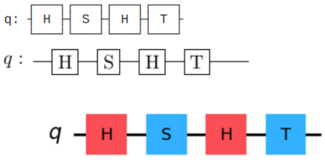
Default name of qubits are $q_0, \cdots, q_{n-1}$
(or q if there's only one qubit)

```
from qiskit import QuantumCircuit, QuantumRegister, ClassicalRegister
from qiskit.primitives import Sampler
from qiskit.visualization import plot_histogram

circuit = QuantumCircuit(1)

circuit.h(0)
circuit.s(0)
circuit.h(0)
circuit.t(0)

display(circuit.draw())
display(circuit.draw("latex"))
display(circuit.draw("mpl"))
```



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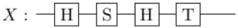
```

The name of qubits can be set explicitly

```python
X = QuantumRegister(1, "X")
circuit = QuantumCircuit(X)

circuit.h(X)
circuit.s(X)
circuit.h(X)
circuit.t(X)

display(circuit.draw())
display(circuit.draw("latex"))
display(circuit.draw("mpl"))
```

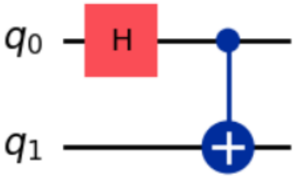


Quantum Circuit

```
circuit = QuantumCircuit(2)

circuit.h(0)
circuit.cx(0, 1)

circuit.draw("mpl")
```



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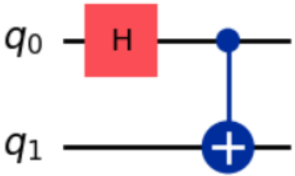

The first layer applies the Hadamard operation on q_0 while leaving q_1 untouched, which is the same as applying the identity operation.

$$I_2 \otimes H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

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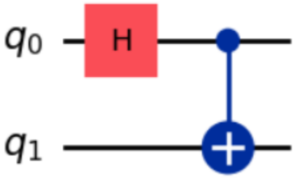

The second layer applies controlled-not operation on q_1 with q_0 as the control bit.

$$CX_{0,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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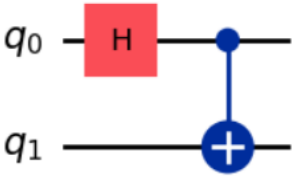

Therefore, the entire circuit represents the following matrix

$$U = CX_{0,1} \cdot (I_2 \otimes H) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}$$

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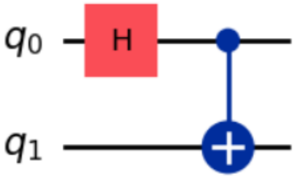


And the following relations completely characterizes the circuit

$U|00\rangle = |\phi^+\rangle$

$U|01\rangle = |\phi^-\rangle$

$U|10\rangle = |\psi^+\rangle$

$U|11\rangle = -|\psi^-\rangle$

where $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle$ are bell states

$|\phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$|\phi^-\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle$

$|\psi^+\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

$|\psi^-\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

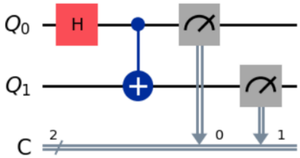
Quantum Circuit

We can measure by adding classical bits to the circuit; Classical bits are represented with double-lines

```
Q = QuantumRegister(2, "Q")
C = ClassicalRegister(2, "C")

circuit = QuantumCircuit(Q, C)
circuit.h(Q[0])
circuit.cx(Q[0], Q[1])
circuit.measure(Q[0], C[0])
circuit.measure(Q[1], C[1])

circuit.draw("mpl")
```



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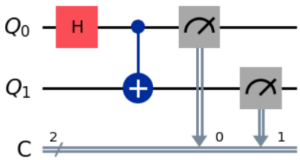
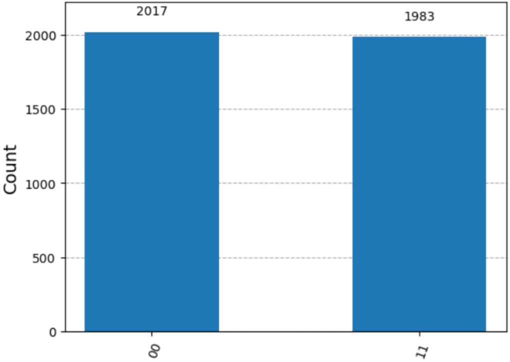


Qubits are initialized with $|0 \cdots 0\rangle$ by default

```
result = StatevectorSampler().run([circuit], shots = 4000).result()[0]
plot_histogram(result.data.C.get_counts())
```



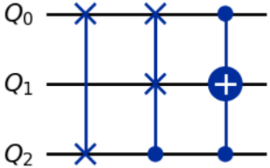
Quantum Circuit

Some more examples of gates (SWAP, CSWAP, CCX)

```
Q = QuantumRegister(3, "Q")

circuit = QuantumCircuit(Q)
circuit.swap(Q[0], Q[2])
circuit.cswap(Q[2], Q[0], Q[1])
circuit.ccx(Q[0], Q[2], Q[1])

circuit.draw("mpl")
```



Projective Measurement

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Definition (Orthogonal Projection)

A complex square matrix P is an **orthogonal projection** if

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1. $P^H = P$
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- Let $|u\rangle$ be a unit vector and $P = |u\rangle\langle u|$. Then P is an orthogonal projection.
- More generally, let $\{|u_0\rangle, \cdots, |u_{k-1}\rangle\}$ be an orthonormal set of vectors and let $P = \sum_{i=0}^{k-1} |u_i\rangle\langle u_i|$. Then P is an orthogonal projection.

Projective Measurement

Theorem

A complex square matrix P is an orthogonal projection if and only if there exists an orthonormal set of vectors $\{|u_0\rangle, \cdots, |u_{k-1}\rangle\}$ such that $P = \sum_{i=0}^{k-1} |u_i\rangle \langle u_i|$.

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Proof

- Since $P = P^2$, its eigenvalues are either 0 or 1.
- Since P is Hermitian, it has an orthonormal basis consisting of eigenvectors. Hence,

$$P = QDQ^{-1} = \sum_{i=0}^{n-1} \lambda_i Q |i\rangle \langle i| Q^{-1} = \sum_{i=0}^{n-1} \lambda_i (Q|i\rangle)(Q|i\rangle)^H$$

- Let $|u_i\rangle = Q|j\rangle$ for each j with $\lambda_j = 1$ and we're done.

Projective Measurement

Definition (Standard basis measurement and projective measurement)

The measurements we've discussed so far is called the **standard basis measurement**.
A **projective measurement** is a set $\{P_0, \cdots, P_{m-1}\}$ of orthogonal projection matrices such that $P_0 + \cdots + P_{m-1} = I_{2^n}$.

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Let X be a system with state $|u\rangle$.

- The projective measurement on X has probability $|P_i |u\rangle|^2$ of having outcome i.
- In such case, the state of X collapses to $\frac{1}{|P_i|u\rangle|} P_i |u\rangle$.

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- Verify that the probabilities sums up to 1.
- What happens when $m = 2^n$ and $P_i = |\mathrm{binary}_n(i)\rangle \langle \mathrm{binary}_n(i)|$?

Projective Measurement

Theorem

A set of 2^n by 2^n complex matrices $\{P_0, \cdots, P_{m-1}\}$ is a projective measurement if and only if there exists an orthonormal basis B and a partition

$$C_0 = \{|c_{0,0}\rangle, \cdots, |c_{0,n_0-1}\rangle\}, \cdots, C_{m-1} = \{|c_{m-1,0}\rangle, \cdots, |c_{m-1,n_{m-1}-1}\rangle\}$$

of B such that $P_i = \sum_{j=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}|$ for all $0 \leq i < m$.

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Proof of sufficiency

- $P_i^H = \sum_{j=0}^{n_i-1} (|c_{i,j}\rangle \langle c_{i,j}|)^H = \sum_{j=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}| = P_i$
- $P_i^2 = \sum_{j=0}^{n_i-1} \sum_{k=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}|c_{i,k}\rangle \langle c_{i,k}| = \sum_{j=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}| = P_i$
- For all vector $|u\rangle$, $\sum_{i=0}^{n-1} P_i |u\rangle = \sum_{i=0}^{n-1} \sum_{j=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}|u\rangle = |u\rangle$, hence $\sum_{i=0}^{n-1} P_i = I_{2^n}$

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of B such that $P_i = \sum_{j=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}|$ for all $0 \leq i < m$.

Proof of necessity

- We know that $P_i = \sum_{j=0}^{n_i-1} |c_{i,j}\rangle$ where $C_i = \{c_{i,0}, \cdots, c_{i,n_i-1}\}$ is an orthonormal basis of the eigenspace of P_i corresponding to the eigenvalue 1.
- Fix an i, then for each $|u\rangle \in C_i$,

$$1 = \langle u|u\rangle = \langle u| \sum_{j=0}^{m-1} P_j |u\rangle = 1 + \sum_{j=0, j\neq i}^{m-1} \sum_{k=0}^{n_j-1} |\langle c_{j,k}|u\rangle|^2$$

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of B such that $P_i = \sum_{j=0}^{n_i-1} |c_{i,j}\rangle \langle c_{i,j}|$ for all $0 \leq i < m$.

Proof of necessity

- Hence, $\langle c_{j,k}|u\rangle = 0$ for all $j \neq i$ and $0 \leq k < n_j$, and every eigenspaces corresponding to 1 of P_i are orthogonal to each other.
- Since the sum of ranks of P_i must sum up to 2^n, the union of C_i must form an orthonormal basis.

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Excercise

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Consider a system consisting of 4 qubits $q_0, \cdots, q_3$.

1. Show that the partial measurement of qubits q_0 and q_3 is a projective measurement.
2. Find the corresponding partition of an orthonormal basis given by the previous theorem.

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Implementation of projective measurement

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- It turns out we can implement projective measurement with unitary operations, standard basis measurement, and an extra workspace system.

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- Let $\{P_0, \cdots, P_{m-1}\}$ be a projective measurement on a system X with n qubits $q_0, \cdots, q_{n-1}$. We introduce a new system Y with a single special qubit which can have value of $|0\rangle, \cdots, |m-1\rangle$ upon measurement, which is initially on the state $|0\rangle$.

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- Let

$$U = \begin{bmatrix} P_0 & 0 & \cdots & 0 \\ P_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{m-1} & 0 & \cdots & 0 \end{bmatrix}$$

  be a matrix acting on the combined system (Y, X)
- U is not unitary since it has zero determinant due to having zero column. However,

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$$
\begin{aligned}
\langle C_i | C_j \rangle &= \left(\sum_{k=0}^{m-1} |k\rangle \otimes P_k \, |\mathrm{binary}_n(i)\rangle \right)^H \left(\sum_{l=0}^{m-1} |l\rangle \otimes P_k \, |\mathrm{binary}_n(j)\rangle \right) \\
&= \sum_{k=0}^{m-1} \sum_{l=0}^{m-1} \langle k|l\rangle \, \langle \mathrm{binary}_n(i) | P_k P_l | \mathrm{binary}_n(j)\rangle = \sum_{k=0}^{m-1} \langle \mathrm{binary}_n(i) | P_k | \mathrm{binary}_n(j)\rangle \\
&= \langle \mathrm{binary}_n(i) | \mathrm{binary}_n(j)\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

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- Suppose X is initially in the state $|u\rangle$, hence (Y, X) is on the state $|0, u\rangle$. Applying U changes the state of (Y, X) into

$$U|0, u\rangle = \sum_{k=0}^{m-1} |k\rangle \otimes P_k |u\rangle$$

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- Performing the standard basis measurement on Y has probability $|P_i |u\rangle|^2$ of having outcome i, in which case the state of (Y, X) collapses to

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- We now discard Y and has obtained the projective measurement on X.

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- The implementation of projective measurement hints at what a measurement is: if we perform a specific unitary process to a combined system, we obtain what we've known as a measurement on one of the subsystem.

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Limitation #1: Global phases are irrelevant

Let $|u\rangle$ and $|v\rangle$ be quantum states with $|v\rangle = e^{i\theta} |u\rangle$ for some real number θ. These states are said to **differ by a global phase**.
Then $|u\rangle$ and $|v\rangle$ have identical probability distribution of measurement results in any sequence of (projective) measurements and unitary operations.

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Proof

- If the first operation is an unitary operation U, the state after it is $U|u\rangle$ and $e^{i\theta}U|u\rangle$, which differs by a global phase.

Limitation on Quantum Information

Limitation #1: Global phases are irrelevant

Let $|u\rangle$ and $|v\rangle$ be quantum states with $|v\rangle = e^{i\theta}|u\rangle$ for some real number θ. These states are said to **differ by a global phase**.
Then $|u\rangle$ and $|v\rangle$ have identical probability distribution of measurement results in any sequence of (projective) measurements and unitary operations.

Proof

- If the first operation is an unitary operation U, the state after it is $U|u\rangle$ and $e^{i\theta}U|u\rangle$, which differs by a global phase.
- If the first operation is a projective measurement $\{P_0, \cdots, P_{m-1}\}$, the probability of the outcome being i is $|P_i|u\rangle|$ and $\left|e^{i\theta}\right| \cdot |P_i|u\rangle| = |P_i|u\rangle|$, which are identical. The state after the outcome i is $\frac{1}{|P|u\rangle|}P|u\rangle$ and $\frac{e^{i\theta}}{|P|u\rangle|}P|u\rangle$ which differs by a global phase.

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- Use induction on the length of the sequence and we're done.

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Note

global phase vs local phase

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global phase vs local phase

- Let $|u\rangle$ be either $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ or $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$, which differs by a local phase. They can be distinguished by measuring $H|u\rangle$ as mentioned before.

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- On the other hand, if $|u\rangle$ is either $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$ or $-|-\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ which differs by a global phase, they cannot be distinguished no matter what.

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Limitation #2: States cannot be copied (no cloning theorem)

Let X and Y be states with n qubits each, where Y is on $|v\rangle$. There is no unitary operation U on (X, Y) such that for all state $|u\rangle$ of X, $U(|u\rangle \otimes |v\rangle) = |u\rangle \otimes |u\rangle$.

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Adding each side and multiplying them by $\frac{1}{\sqrt{2}}$ yields

$$U\left(\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |v\rangle\right) = \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$$

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However,

$$U\left(\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |v\rangle\right) = \frac{1}{2}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$$

Which is a contradiction.

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For example, let $U = CX_{1,0}$ and Y is on $|0\rangle$. Then $U|00\rangle = |00\rangle$ and $U|10\rangle = |11\rangle$.

However, it cannot clone $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle)$ as seen before.

Limitation on Quantum Information

Limitation #3: Non-orthogonal states cannot be perfectly distinguished

Two states $|u\rangle$ and $|v\rangle$ over the system X with qubits $q_0, \cdots, q_{n-1}$ are orthogonal if and only if there exists a system Y which is on the all-zero state, a unitary operation U on (Y, X), and a projective measurement $M = \{M_0, M_1\}$ such that

$$\mathcal{P}_{M=0}\left(U\,|0\cdots0, u\rangle\right) = \mathcal{P}_{M=1}\left(U\,|0\cdots0, v\rangle\right) = 1$$

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Proof of sufficiency

- Let $A = \{\left|a_0\right\rangle, \cdots, \left|a_{n_a-1}\right\rangle\}$ and $B = \{\left|b_0\right\rangle, \cdots, \left|b_{n_b-1}\right\rangle\}$ the corresponding orthonormal sets for M_0 and M_1.

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- The condition implies that $U\left|0\cdots 0, u\right\rangle = \sum_{i=0}^{n_a-1} c_i \left|a_i\right\rangle$ and $U\left|0\cdots 0, v\right\rangle = \sum_{j=0}^{n_b-1} d_j \left|b_j\right\rangle$ for some scalars c_i and d_j.

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- The condition implies that $U\,|0\cdots 0, u\rangle = \sum_{i=0}^{n_a-1} c_i\,|a_i\rangle$ and $U\,|0\cdots 0, v\rangle = \sum_{j=0}^{n_b-1} d_j\,|b_j\rangle$ for some scalars c_i and d_j.
- These are equivalent to $|0\cdots 0, u\rangle = \sum_{i=0}^{n_a-1} c_i U^H\,|a_i\rangle$ and $|0\cdots 0, v\rangle = \sum_{j=0}^{n_b-1} d_j U^H\,|b_j\rangle$.

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Proof of sufficiency

- We take their inner product to obtain the orthogonality.

$$\text{LHS} = \langle 0\cdots 0, u|0\cdots 0, v\rangle = \langle 0\cdots 0|0\cdots 0\rangle\,\langle u|v\rangle = \langle u|v\rangle$$

$$\text{RHS} = \sum_{i=0}^{n_a-1}\sum_{j=0}^{n_b-1} c_i d_j \,\langle a_i|UU^H|b_j\rangle = \sum_{i=0}^{n_a-1}\sum_{j=0}^{n_b-1} c_i d_j\,\langle a_i|b_j\rangle = 0$$

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Proof of necessity

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Proof of necessity

- Set $Y = \emptyset$, $U = I_{2^n}$, and $M = \{|u\rangle\langle u|, I_{2^n} - |u\rangle\langle u|\}$.

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- $\mathcal{P}_{M=0}\left(\left|u\right\rangle\right) = \left|\left|u\right\rangle\left\langle u|u\right\rangle\right|^2 = \left|\left|u\right\rangle\right|^2 = 1$

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- $\mathcal{P}_{M=0}(|u\rangle) = \left||u\rangle\langle u|u\rangle\right|^2 = \left||u\rangle\right|^2 = 1$
- $\mathcal{P}_{M=1}(|v\rangle) = \left||v\rangle - |u\rangle\langle u|v\rangle\right|^2 = \left||v\rangle\right|^2 = 1$

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- When two states differs by a global phase, they cannot be distinguished
- When two states are orthogonal, they can be perfectly distinguished
- For every pair of states in-between them, you cannot perfectly distinguish them, but you can perform some unitary operation so that the probability distribution of some measurement differs. The probability can be maximized by the **Helstorm measurement**. Please refer to the **Wikipedia** for more detail.

The End