Sailboat simulation and control

All constants, parametres and variables are given in following units or units derived from them:

m - metre for distance

s - second for time

kg - kilogram for mass

rad - radian for angle

1 Constants

4 constants are used in the simulation:

name	designation	value
air density	ρ_A	1.2
water density	ρ_W	1000
boat density	ρ_B	50
gravity	g	9.81

2 Parametres

2.1 Boat parametres

The boat is defined by 7 parametres entered by the user:

name	designation	min	max
length	L	8	12
width to length ratio	$\frac{W}{L}$	0.2	0.4
hull height to length ratio	$\frac{H}{L}$	0.05	0.15
sail area to length squared ratio	$\frac{A_S}{L^2}$	0.4	0.6
keel area to length squared ratio	$\frac{A_K}{L^2}$	0.02	0.06
rudder area to length squared ratio	$\frac{A_R}{L^2}$	0.004	0.01
ballast mass to upper hull mass	$\frac{M_B}{M_H}$	0.6	1.0

Using parametres initial parametres following boat attributes are calculated:

Attribute name	formula
Width	$W = \frac{W}{L} \cdot L$
Height	$H = \frac{H}{L} \cdot L$
Sail area	$A_S = \frac{A_S}{L^2} \cdot L^2$
Sail height	$H_S = 3\sqrt{\frac{S_A}{6}}$
Boom height	$H_B = \frac{H_S}{10}$
Keel area	$A_K = \frac{A_K}{L^2} \cdot L^2$
Keel length	$L_K = \sqrt{A_K}$
Keel depth	$D_K = \sqrt{A_K}$
Rudder area	$A_R = \frac{A_R}{L^2} \cdot L^2$
Rudder depth	$D_R = 2\sqrt{\frac{A_R}{2}}$
Ballast mass	$M_B = \frac{A_R}{A_S} \cdot A_S$
Hull depth	$D_H = \frac{H\rho_B + \frac{M_B}{LW}}{\rho_W}$
Hull mass	$M_H = LW(H + D_H)\rho_B$
Mass	$M = M_H + M_B$
Roll inertia	$I_R = M_H \left(\frac{W^2+H^2}{12} + \frac{H^2}{4} \right) + M_B \left(D_H + D_K\right)^2$
Yaw inertia	$I_Y = \frac{ML^2}{12}$

2.2 Wind paramtres

The wind has one parameter:

name	designation	min	max
speed	$\|v\|$	2	6

2.3 Target direction

The targets direction τ allowed values depend on parametres:

The maximum value is constant:

$$\tau_{max} = \pi$$

the minimum value is calculated from following code (the parametres are given as a list with the same order as they have been introduced in this document):

```
coefs = [0.8552868086961989, 0.007712865177737268, -0.6635550741181487,
-0.05841192614387274, -0.31648475784251945, -1.6655535903801222,
9.807621749941644, 0.07929555574104712, 0.06315510686158626,
0.0009738079842759349, 0.04545066981012345, 0.23370361838394102,
-0.00153894007075216, -0.17698848793072353, -2.6523817289024416,
-0.02158495576504988, -0.012685633340530873, 1.4272421376565094,
8.915130711749946, -0.9700459052289417, -6.881128775159292,
-13.401670280074645, 0.3479893882429269, -0.13957799128409407,
-1.6597110955923333, -3.2329323344279715, -14.059117892701058,
-23.95657534462534, 1.153037420385502, -0.4633601891168525,
0.648537362694002, 1.7286693533916218, -13.053751366389559,
-0.0012238530229256794, -0.02137821501069217, 33.81392492571629,
-20.0646238269576, 0.035802246950813695, -0.050679235086903925,
1783.841167909631, -1.5560822067854674, 0.9683803076029115,
0.016424440307737796, -0.013555068477408666, 0.026911175843521987]
BUFFER = 0.17453292519

arr = [1] + parametres
t_min = BUFFER
k = 0
for i in range(VALS):
    for j in range(i, VALS):
        t_min += coefs[k] * arr[i] * arr[j]
        k += 1
```

3 Dynamic variables

Outside of parametres set by the user at the beginning of the simulation, there are also variables whose values change over course of simulation.

3.1 Boat variables

The boat has following 8 variables:

name	designation	starting value
position	$\vec{x}$	$[0 \quad 0]$
velocity	$\vec{v}$	$[0 \quad 0]$
yaw	α	$-\frac{\pi}{2}$
yaw speed	$\dot{\alpha}$	0
roll	β	0
roll speed	$\dot{\beta}$	0
rudder position	γ	0
sail position	δ	$\frac{\pi}{4}$

3.2 Wind variables

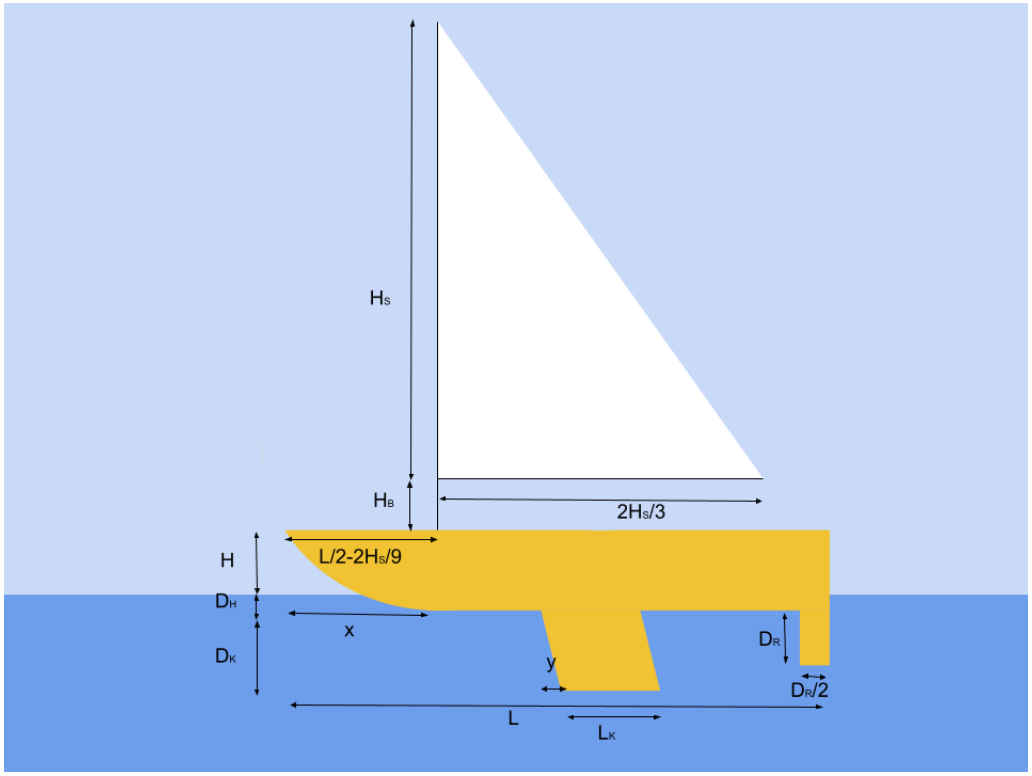
The wind has only one variable: wind velocity ($\vec{v_w}$).

$$\vec{v_w} = \begin{bmatrix} -\left|v\right| & 0 \end{bmatrix}$$

4 Visual representation

2 images of the boat should be available to user when setting parametres: side view and front view. Additionally top view should be available during simulation.

4.1 Side view



The suggested values for x, y and z are:

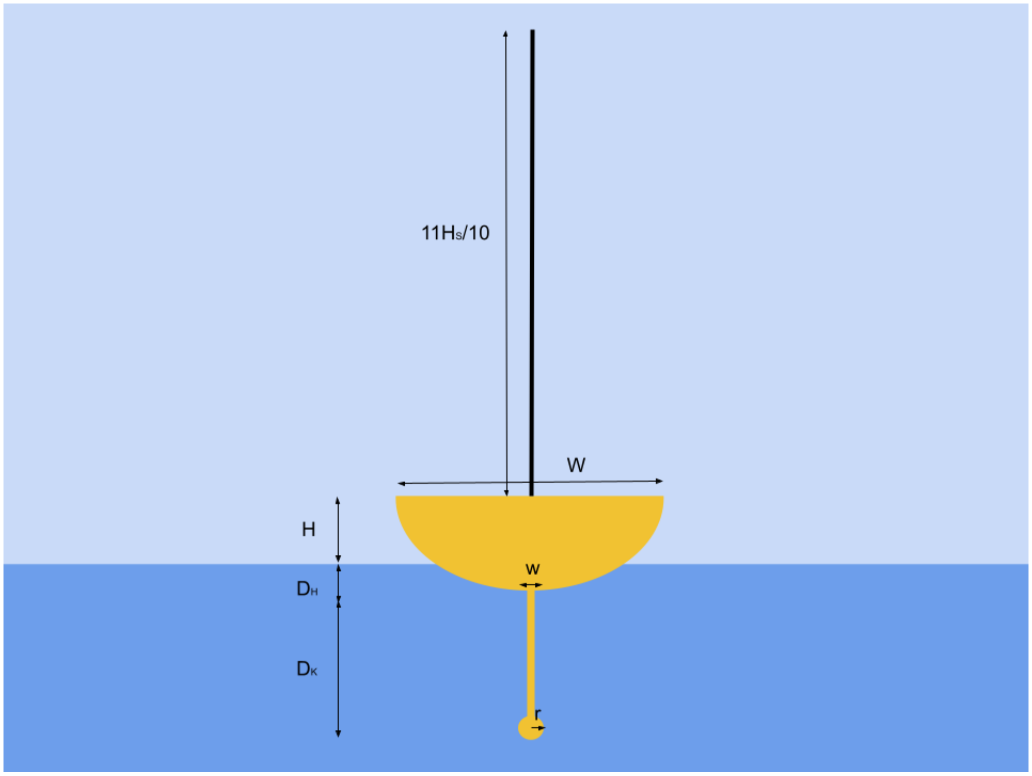
$$x = \frac{3\,(H + D_H)}{2}$$

$$y = \frac{L_K}{2}$$

$$z = \sqrt{\frac{A_R}{2}}$$

However, these values may be tweaked for aesthetic reasons.

4.2 Front view



The recommended values for w and r are:

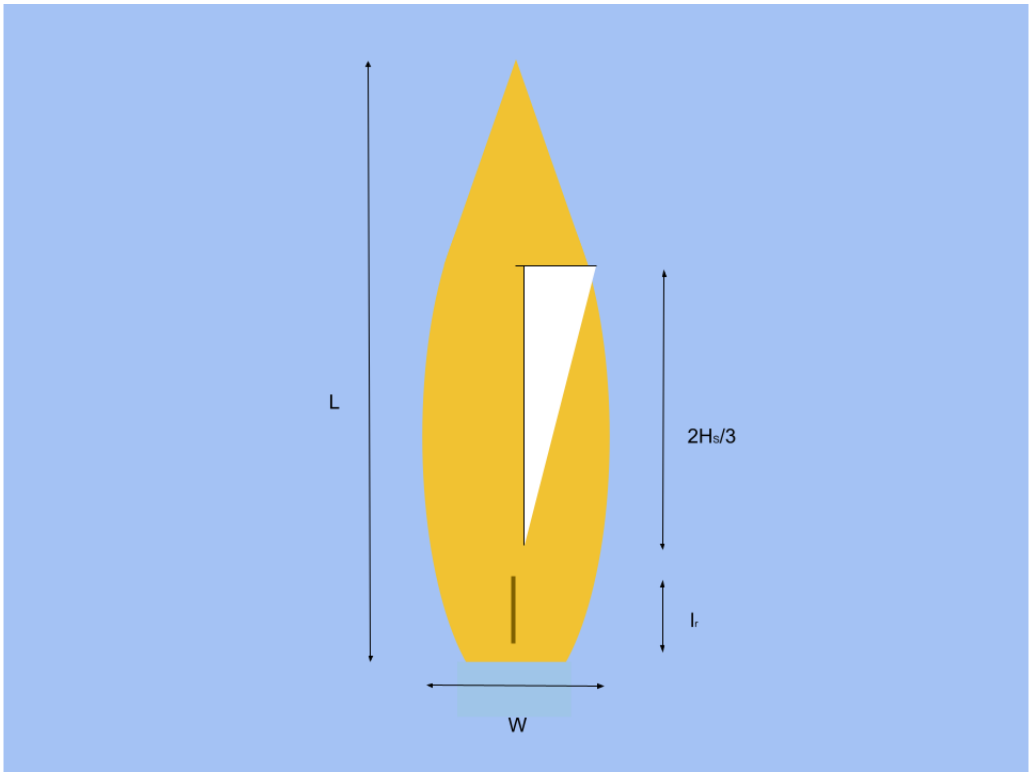
$$w = \frac{W}{30}$$

$$r = \sqrt{\frac{M_B}{\pi \rho L_K}} + \frac{w}{2}$$

$$\rho = 1.1 \cdot 10^4$$

The value of w may be tweaked for aesthetic reasons but it should remain very small. The shape of keel may be a subject to a change too. Changes to r require prior calculations (currently ballast is assumed to be lead cylinder with height equal to keel length and radius r encased by the same thickness of laminate as the rest of keel)

4.3 Top view



It is suggested that $l_r = 1$ but that may be changed for aesthetic reasons.

The boat in top view should be able to change its position, roll (angular position around axis from bow to stern), yaw (angular position around axis perpendicular to water surface), sail position (rotation in yaw axis) and rudder position (rotation in yaw axis).

5 Differential equations

The movement of the boat can be defined by a system of differential equations (Newtons notation used with differentiation over time).

First we need to define basic concepts of rotation matrix:

$$R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

And the concept of argument function:

$$Arg(\begin{bmatrix} x & y \end{bmatrix}) = \arccos\frac{x}{\sqrt{x^2+y^2}} \operatorname{sgn} y$$

And relative velocities:
Wind velocity relative to hull:

$$\vec{v_{WH}} = (\vec{v_W} - \vec{v})R(-\alpha)$$

Wind velocity relative to sail:

$$\vec{v_{WS}} = \left(\vec{v_{WH}} - \begin{bmatrix}0 & \frac{H_S+3H_B+3H}{3}\dot{\beta}\end{bmatrix}\right) R(-\delta)$$

Water velocity relative to hull:

$$\vec{v_{RH}} = -\vec{v}R(-\alpha)$$

Water velocity relative to keel:

$$\vec{v_{RK}} = \vec{v_{RH}} + \begin{bmatrix}0 & \dot{\beta}\frac{D_K+2D_H}{2}\end{bmatrix}$$

Water velocity relative to rudder:

$$\vec{v_{RR}} = \left(\vec{v_{RH}} + \begin{bmatrix}0 & \dot{\beta}\frac{D_R+2D_H}{2} + \frac{\dot{\alpha}}{2}L\end{bmatrix}\right) R(-\gamma)$$

When calculating forces acting on the boat we will use 2 empirical functions.

The first one is for the lift coefficient with the argument being angle between the wind and the sail:

$$f_1(\phi) = \frac{1}{25}\left(\frac{\phi(\phi-\pi)}{\phi - \frac{25}{24}\pi}\right)^5$$

The second one is for wave-making drag with the argument being ratio of boats speed to its length:

$$f_2(x) = 2 \cdot 10^4 \cdot \exp\left(-\frac{1}{x^2} - 12x\right)$$

To calculate magnitudes of most of the forces acting on the boat a drag or lift equation will be used with ρ being density of medium, v velocity of object, C - coefficient and A cross sectional area.

$$F = \frac{\rho}{2}v^2CA$$

In case of the lift force, the coefficient is calculated from function f_1 and the direction of the force is perpendicular to the sail forming acute angle between lift force vector and wind velocity vector:

$$\vec{F_L} = \left|\frac{\rho_A}{2}\|\vec{v_{WS}}\|^2 A_S \frac{\vec{v_{WS}}}{\|\vec{v_{WS}}\|}\begin{bmatrix}0\\1\end{bmatrix}\cos\beta f_1\left(\left|Arg\left(\vec{v_{WS}}\right)\right|\right)\right| \begin{bmatrix}1 & 0\end{bmatrix} R\left(\delta + \alpha + \frac{\pi}{2}\operatorname{sgn}\left(\vec{v_{WS}}\begin{bmatrix}0\\1\end{bmatrix}\right)\right)$$

For sail drag the coefficient is 2 and the direction is the same as to wind velocity:

$$\vec{F_{DS}} = \left|\rho_A \|\vec{v_{WS}}\|^2 A_S \frac{\vec{v_{WS}}}{\|\vec{v_{WS}}\|} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos\beta\right| \frac{\vec{v_W} - \vec{v}}{\|\vec{v_W} - \vec{v}\|}$$

For upper hull drag the coefficient is 0.1 along the axis from bow to stern and 1 along the perpendicular axis:

$$\vec{F_{DUH}} = \frac{\rho_A}{2}\left(\begin{bmatrix} 1 & 0 \end{bmatrix} \left|\vec{v_{WH}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right| \vec{v_{WH}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} WH \cdot 0.1 + \begin{bmatrix} 0 & 1 \end{bmatrix} \left|\vec{v_{WH}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right| \vec{v_{WH}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} LH\right) R(\alpha)$$

The situation is similar for the lower hull with one notable exception: wave-making resistance increases the drag coefficient.

$$\vec{F_{DLH}} = \frac{\rho_W}{2}\left(\begin{bmatrix} 1 & 0 \end{bmatrix} \left|\vec{v_{RH}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right| \vec{v_{RH}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} WD_H \cdot 0.1 \left(1 + f_2\left(\frac{\left|\vec{v} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right|}{L}\right)\right) + \begin{bmatrix} 0 & 1 \end{bmatrix} \left|\vec{v_{RH}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right| \vec{v_{RH}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} LD_H \left(1 + f_2\left(\frac{\left|\vec{v} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right|}{L}\right)\right)\right) R(\alpha)$$

Both keel ($\vec{F_{DK}}$) and rudder ($\vec{F_{DR}}$) have drag coefficient of 2, but only along one axis. The drag along the other axis is negligibly small:

$$\vec{F_{DK}} = \rho_W \begin{bmatrix} 0 & 1 \end{bmatrix} \left|\vec{v_{RK}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right| \vec{v_{RK}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} A_K \cos\beta R(\alpha)$$

$$\vec{F_{DR}} = \rho_W \begin{bmatrix} 0 & 1 \end{bmatrix} \left|\vec{v_{RR}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right| \vec{v_{RR}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} A_R \cos\beta R(\gamma + \alpha)$$

Knowing all the forces acting on boat it is possible to calculate changes of position (including angular position):

$$\ddot{\vec{x}} = \dot{\vec{v}} = \frac{\vec{F_L} + \vec{F_{DS}} + \vec{F_{DUH}} + \vec{F_{DLH}} + \vec{F_{DK}} + \vec{F_{DR}}}{M}$$

$$\ddot{\alpha} = I_\alpha^{-1}\left(-\vec{F_{DR}} \frac{L}{2} R(-\alpha) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \frac{\dot{\alpha}^2 A_K \rho_W}{4}\right)$$

$$\ddot{\beta} = I_\beta^{-1}\left(\left(\frac{H_S + 3H_B + 3H}{3}\left(\vec{F_L} + \vec{F_{DS}}\right) - \frac{(2D_H + D_K)\vec{F_{DK}} + (2D_H + D_R)\vec{F_{DR}}}{2}\right) R(-\alpha) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - M_B g (D_H + D_K) \sin\beta\right)$$