Attitude Representation

- 3 Degrees of Freedom
- The most general representation is **Rotation matrices**

  - 9 elements

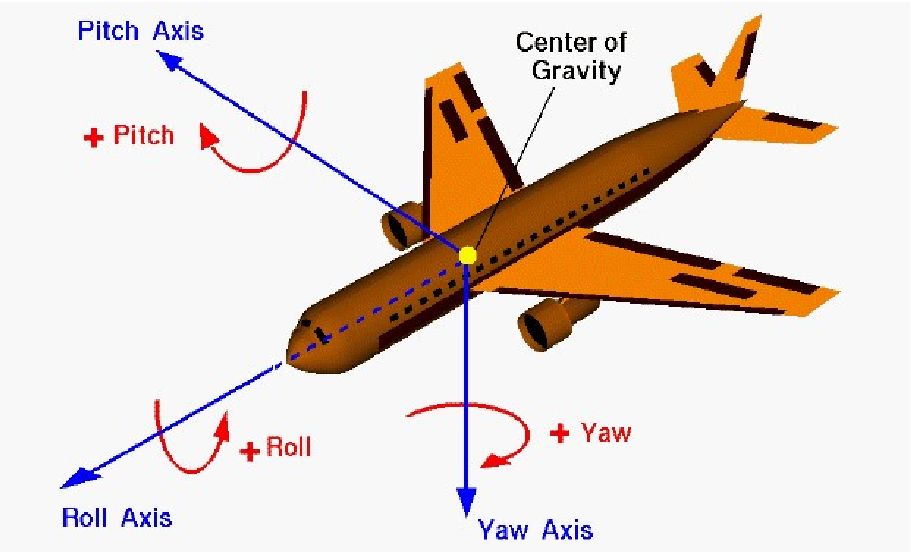
  - cumbersome to use
- Most commonly used representation: **Euler angles**

  -Intuitive **Physical interpretation**

  -Minimalistic representation :

  3 parameters for 3 DOF

  -But exhibit a phenomenon known as Gimble Lock



https://www.youtube.com/watch?v=zc8b2Jo7mno

What does it imply mathematically?

Body-axis angular rate vector (orthogonal)

$$\omega_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

But Euler angles do not form an orthogonal vector. The Euler rates are also not orthogonal.

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \neq \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Consider a 3-2-1 rotation

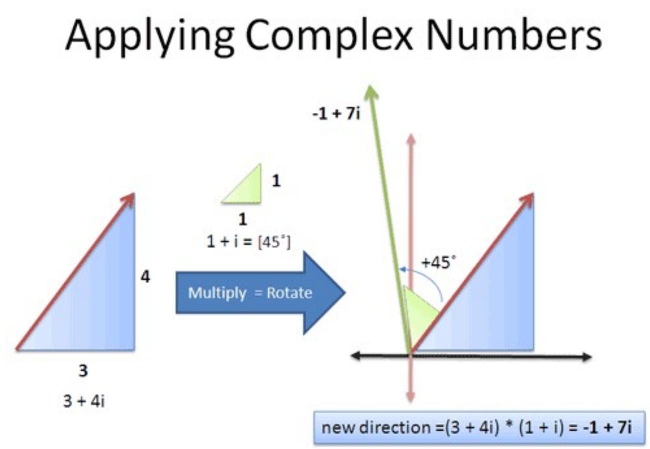
$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + R(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + R(\phi)R(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

At pitch 90^0 the matrix becomes singular.

Note: The singularity occurs in all Euler angle rotation sequences for the middle rotation

Quaternions



- 4 component extended complex number

$$\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$$

- Consists of scalar and vector part

$$q = (q_0, \vec{q})$$

- These are mathematical objects. *Can also be used to represent rotations*.

  - Recall: what does multiplying any complex number by $e^{i\theta}$ does? It rotates the vector by θ!

- Remove singularity at the cost of one more parameter. The main reason they started being used for satellites. Now used extensively for small Aerial vehicles, aerospace robotics, VTOLs, etc.
- Simpler to compose
- Some denote it as (w,x,y,z) with w being the scalar part.

Representation	No. of Parameters
Rotation Matrix	9
Euler Angles	3
Quaternions	4

Quaternion Algebra

Have their own definition of operations

Illustration with a right hand rule:

$$
\begin{aligned}
i^2 &= j^2 = k^2 = ijk = -1 \\
ij &= k, ji = -k \\
jk &= i, kj = -i \\
ki &= j, ik = -j
\end{aligned}
$$

Can be used to define quaternion product

$$(q_0 + q_1 i + q_2 j + q_3 k)(p_0 + p_1 i + p_2 j + p_3 k)$$

$$
\begin{aligned}
q \circ p &= (p_0 q_0 - \vec{p} \cdot \vec{q}, q_0 \vec{p} + p_0 \vec{q} + \vec{q} \times \vec{p}) \\
&= \begin{pmatrix} p_0 q_0 - q_1 p_1 - q_2 p_2 - q_3 p_3 \\ q_1 p_0 + q_0 p_1 + q_2 p_3 - q_3 p_2 \\ q_2 p_0 + q_0 p_2 + q_3 p_1 - q_1 p_3 \\ q_3 p_0 + q_0 p_3 + q_1 p_2 - q_2 p_1 \end{pmatrix}
\end{aligned}
$$

Properties similar to complex numbers

- Norm: $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$
- Conjugate: $\bar{q} = (q_0, -\vec{q})$
- Inverse $q^{-1} = \frac{\bar{q}}{|q|}$
- Product is non-commutative $p \circ q \neq q \circ p$
- Product is associative $p \circ q \circ r = (p \circ q) \circ r = p \circ (q \circ r)$

Recall $(a + ib)^{-1} = \frac{a - ib}{\sqrt{(a^2 + b^2)}}$

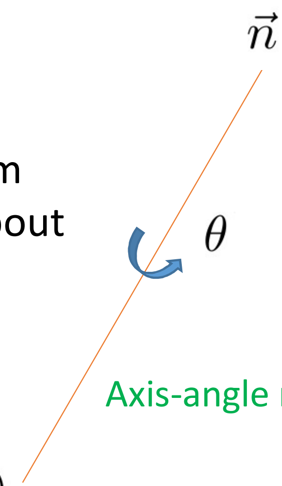
Rotation using Quaternions

4 parameters to represent 3 degrees of freedom → Must satisfy a constraint

Unit modulus quaternions: $|q| = 1$

Euler's Rotation Theorem

Any rotation or sequence of rotations of a rigid body or coordinate system about a fixed point is equivalent to a single rotation by a given angle θ about a fixed axis (called Euler axis) that runs through the fixed point.



Axis-angle representation

$$q = \left(cos\left(\frac{\theta}{2}\right), \vec{n}\sin\left(\frac{\theta}{2}\right) \right)$$

- Rotation operator: $x' = \bar{q} \circ x \circ q \qquad x = (0, \vec{x}) \qquad q = (q_0, \vec{q})$

- Exercise: $\vec{x}' = (1 - \cos(\theta))(\vec{n}.\vec{x})\vec{n} + \cos\theta\vec{x} + \sin\theta(\vec{x} \times \vec{n})$

Conventions

- In loose terms, Rotation is a *directional* and *relative* quantity with a magnitude (but remember rotation is not a vector!)
- Need to first set the rules: *Left or right handed?*
  Rotating frames (passive) or rotating vectors (active)?
  Direction of operation (in passive case)

Quaternion Type	Hamilton	JPL
1. Component order	$(q_0, \vec{q})$	$(\vec{q}, q_0)$
2. Algebra	ij=k(right handed)	ij=-k(left handed)
3. Default notation	Local to Global	Global to local
	$q = q_{GL}$	$q = q_{LG}$
	$x_G = q \circ x_L \circ q*$	$x_L = q \circ x_G \circ q*$

Conventions

- Hamilton Notation Order: ***if go towards left: local to global***
- Implications

  → Local perturbations are compounded to right (post-multiplied)

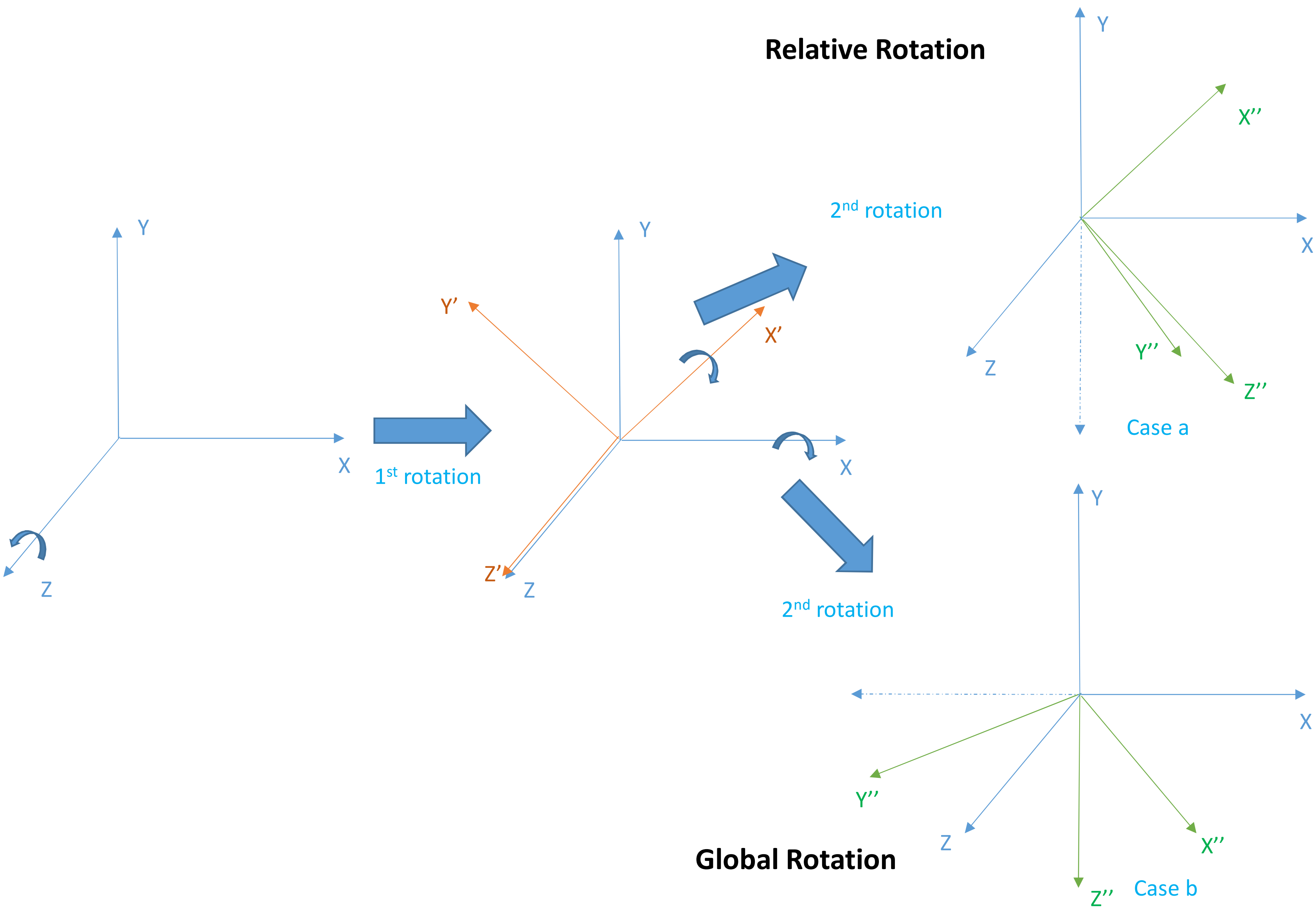
  → Global perturbations are compounded to left (pre-multiplied)
- Suppose 1st rotation is given by q^1 and 2nd by q^2

  if 2nd rotation is defined relatively: $q^a = q^1 \circ q^2$

  if 2nd rotation is defined globally: $q^b = q^2 \circ q^1$
- **Similar to Rotation Matrices**! - Recall: for a 321 rotation sequence, rotation matrix for conversion from local to Earth frame is: $R = R(\psi)R(\theta)R(\phi)$

  What if rotations were defined always with respect to original axis? $R = R(\phi)R(\theta)R(\psi)$

  (check for yourself!)


Relative Rotation
Global Rotation
1st rotation
2nd rotation
2nd rotation
Case a
Case b
X
Y
Z
X'
Y'
Z'
X''
Y''
Z''

Derivative

By First Principles

$$\dot{q} = \lim_{\triangle t \to 0} \frac{q(t+\triangle t) - q(t)}{\triangle t}$$

If the change from previous attitude to current attitude is defined locally, the change in attitude $\triangle q_L$
is post-multiplied.

$$= \lim_{\triangle t \to 0} \frac{q \circ \triangle q_L - q(t)}{\triangle t}$$

For small angles

$q = \left(cos\left(\frac{\theta}{2}\right), \vec{n}\sin\left(\frac{\theta}{2}\right)\right)$

$$= \lim_{\triangle t \to 0} \frac{q \circ \left(\begin{bmatrix} 1 \\ \vec{n}\triangle\theta_L/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}{\triangle t}$$

$$= \lim_{\triangle t \to 0} \frac{q \circ \left(\begin{bmatrix} 0 \\ \vec{n}\triangle\theta_L/2 \end{bmatrix} \right)}{\triangle t} = \frac{1}{2} q \circ \begin{bmatrix} 0 \\ \vec{w}' \end{bmatrix}$$

$$\dot{q} = \frac{1}{2} q \circ w'$$

$$\dot{q} = \frac{1}{2} w \circ q$$

$$w' = 2\bar{q} \circ \dot{q}$$

$$w = 2\dot{q} \circ \bar{q}$$

Equivalent formulas

Quaternions vs Euler Angles

- No singularity vs Gimbal lock
- Computationally less expensive: no trigonometric function evaluation
- No discontinuity in representation like Euler angles

➢ Less intuitive
➢ Dual Covering $(q_0, \vec{q}) = (-q_0, -\vec{q})$
➢ Unit modulus constraint

Rotation matrix

$$x = q \circ x' \circ \bar{q}$$

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2 * (q_1 q_2 + q_0 q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Rigid Body Dynamics

$$\begin{aligned} \ddot{\mathbf{r}} &= \frac{1}{m}\mathbf{q} \circ \mathbf{F}^b \circ \mathbf{q}^* - \mathbf{g} \\ \mathbf{J}\dot{\vec{\omega}} &= \vec{M}^b - \vec{\omega} \times \mathbf{J}\vec{\omega} \\ \dot{\mathbf{q}} &= \frac{1}{2}\mathbf{q} \circ \mathbf{w} \end{aligned}$$

Conversion between Quaternions and Euler angles

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = R_\phi R_\theta R_\psi \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\theta \\ \sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi & \sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quaternions to Euler angles:

$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2 * (q_1 q_2 + q_0 q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi - \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\theta \\ \sin\phi\sin\psi - \cos\phi\sin\theta\cos\psi & \sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

$$\phi = \tan^{-1}\left(\frac{-2(q_2 q_3 - q_0 q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}\right)$$

$$\theta = \sin^{-1}\left(2(q_0 q_2 + q_1 q_3)\right)$$

$$\psi = \tan^{-1}\left(\frac{-2(q_1 q_2 - q_0 q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}\right)$$

Exercise: Try yourself!

Euler angles to Quaternions:

Quaternions corresponding to the three rotations are given by

$$q_\phi = \begin{bmatrix} \cos\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) \\ 0 \\ 0 \end{bmatrix}, \quad q_\theta = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) \\ 0 \\ \sin\left(\frac{\theta}{2}\right) \\ 0 \end{bmatrix}, \quad q_\psi = \begin{bmatrix} \cos\left(\frac{\psi}{2}\right) \\ 0 \\ 0 \\ \sin\left(\frac{\psi}{2}\right) \end{bmatrix}$$

Since the rotations are relative, we post-multiply the rotations.

For a 1-2-3 Euler rotation:

$$\begin{aligned} q &= q_\phi \circ q_\theta \circ q_\psi \\ &= \begin{bmatrix} \cos(\phi/2)\cos(\theta/2)\cos(\psi/2) - \sin(\phi/2)\sin(\theta/2)\sin(\psi/2) \\ \cos(\phi/2)\sin(\theta/2)\sin(\psi/2) + \sin(\phi/2)\cos(\theta/2)\cos(\psi/2) \\ \cos(\phi/2)\cos(\psi/2)\sin(\theta/2) - \sin(\phi/2)\cos(\theta/2)\sin(\psi/2) \\ \cos(\phi/2)\cos(\theta/2)\sin(\psi/2) + \cos(\psi/2)\sin(\theta/2)\sin(\phi/2) \end{bmatrix} \end{aligned}$$