Flight Mechanics
AE321
LEC: TWF 16:00-17:00 L8

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Course Content

- Introduction to Flight, Aerodynamics forces, moments, elements of aircraft propulsion.

- Equations of Motion of a rigid body: translational and rotational dynamics

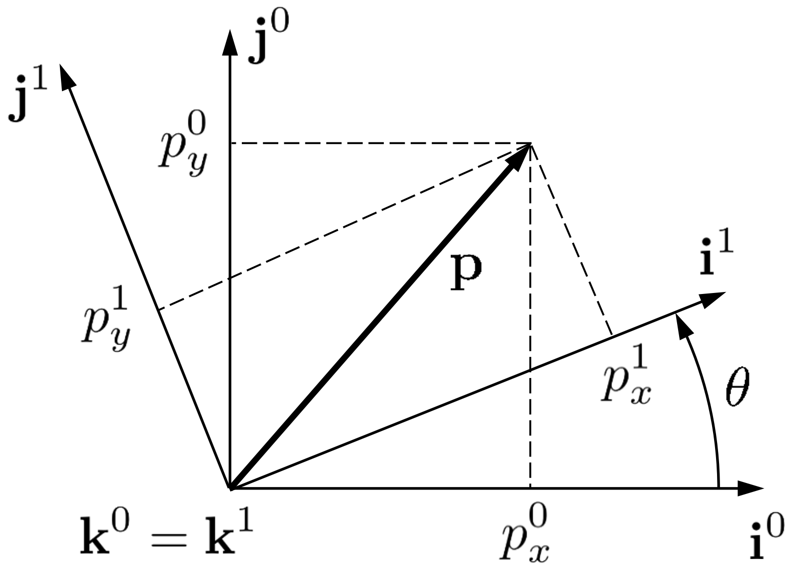
- Aircraft performance (Level flight, climbing/gliding flight, vertical and horizontal maneuvers, take-off and landing)

- Stability and control (Longitudinal static and dynamic stability, stick fixed and stick free analysis, lateral-directional static and dynamic stability).

Coordinated Frames

- Describe relative position and orientation of objects
  - Aircraft relative to direction of wind
  - Camera relative to aircraft
  - Aircraft relative to inertial frame
- Some things most easily calculated or described in certain reference frames
  - Newton’s law
  - Aircraft attitude
  - Aerodynamic forces/torques
  - Accelerometers, rate gyros
  - GPS
  - Mission requirements

Rotation of Reference Frame



$$\mathbf{p} = p_x^0\mathbf{i}^0 + p_y^0\mathbf{j}^0 + p_z^0\mathbf{k}^0$$

$$\mathbf{p} = p_x^1\mathbf{i}^1 + p_y^1\mathbf{j}^1 + p_z^1\mathbf{k}^1$$

$$p_x^1\mathbf{i}^1 + p_y^1\mathbf{j}^1 + p_z^1\mathbf{k}^1 = p_x^0\mathbf{i}^0 + p_y^0\mathbf{j}^0 + p_z^0\mathbf{k}^0$$

$$\mathbf{p}^1 \triangleq \begin{pmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \end{pmatrix} = \begin{pmatrix} \mathbf{i}^1\cdot\mathbf{i}^0 & \mathbf{i}^1\cdot\mathbf{j}^0 & \mathbf{i}^1\cdot\mathbf{k}^0 \\ \mathbf{j}^1\cdot\mathbf{i}^0 & \mathbf{j}^1\cdot\mathbf{j}^0 & \mathbf{j}^1\cdot\mathbf{k}^0 \\ \mathbf{k}^1\cdot\mathbf{i}^0 & \mathbf{k}^1\cdot\mathbf{j}^0 & \mathbf{k}^1\cdot\mathbf{k}^0 \end{pmatrix} \begin{pmatrix} p_x^0 \\ p_y^0 \\ p_z^0 \end{pmatrix}$$

$$\mathbf{p}^1 = \mathcal{R}_0^1\mathbf{p}^0 \quad \text{where} \quad \mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(rotation about $\mathbf{k}$ axis)

Rotation of Reference Frame

Right-handed rotation about $\mathbf{j}$ axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

Right-handed rotation about $\mathbf{i}$ axis:

$$\mathcal{R}_0^1 \triangleq \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

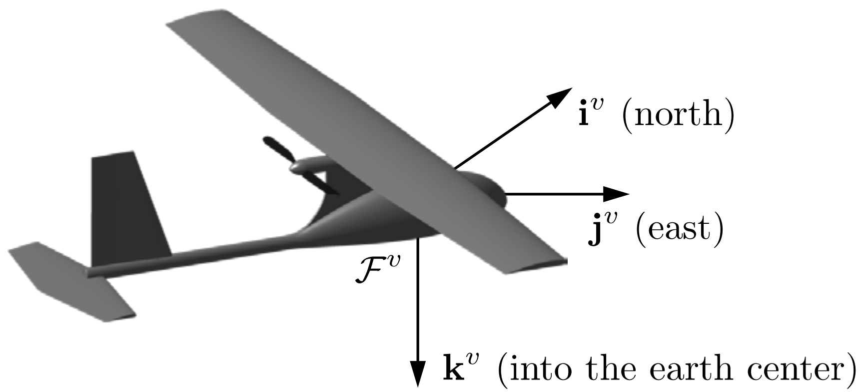
Orthonormal matrix properties:

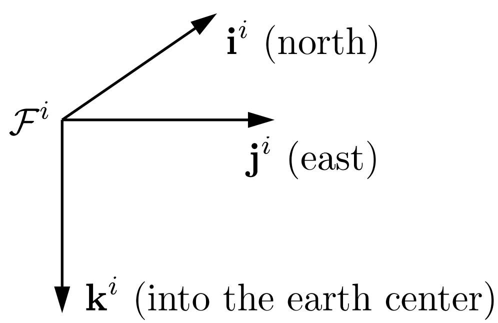
P.1. $(\mathcal{R}_a^b)^{-1} = (\mathcal{R}_a^b)^\top = \mathcal{R}_b^a$

P.2. $\mathcal{R}_b^c \mathcal{R}_a^b = \mathcal{R}_a^c$

P.3. $\det\big(\mathcal{R}_a^b\big) = 1$

Inertial Frame and Vehicle Frame





- Vehicle frame has same orientation as inertial frame
- Vehicle frame is fixed at cm of aircraft
- Inertial and vehicle frames are referred to as NED frames
- N→x, E→y, D→z

Euler Angles

- Need way to describe attitude of aircraft
- Common approach: Euler angles

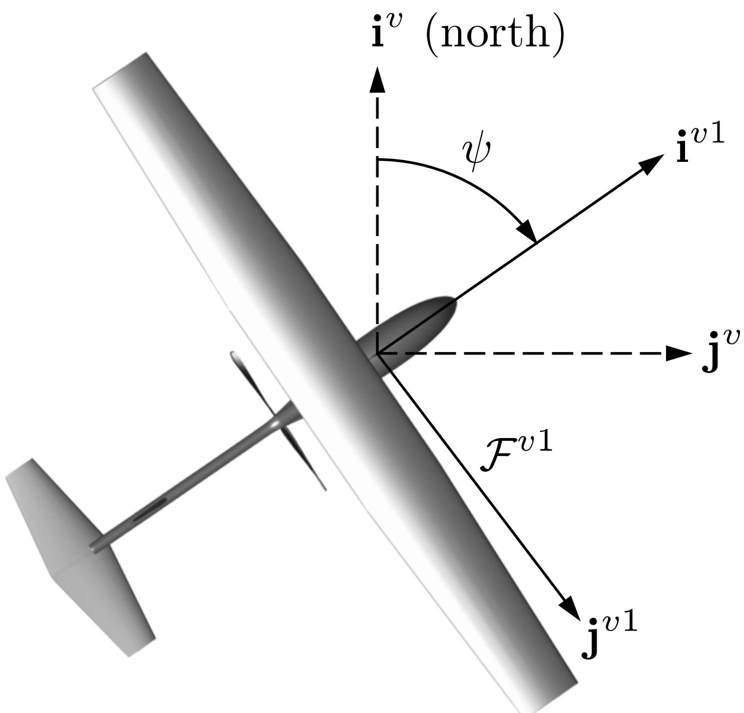
ψ: heading (yaw)

θ: elevation (pitch)

ϕ: bank (roll)

- Pro: Intuitive
- Con: Mathematical singularity
  - Quaternions are alternative for overcoming singularity

Vehicle-1 Frame

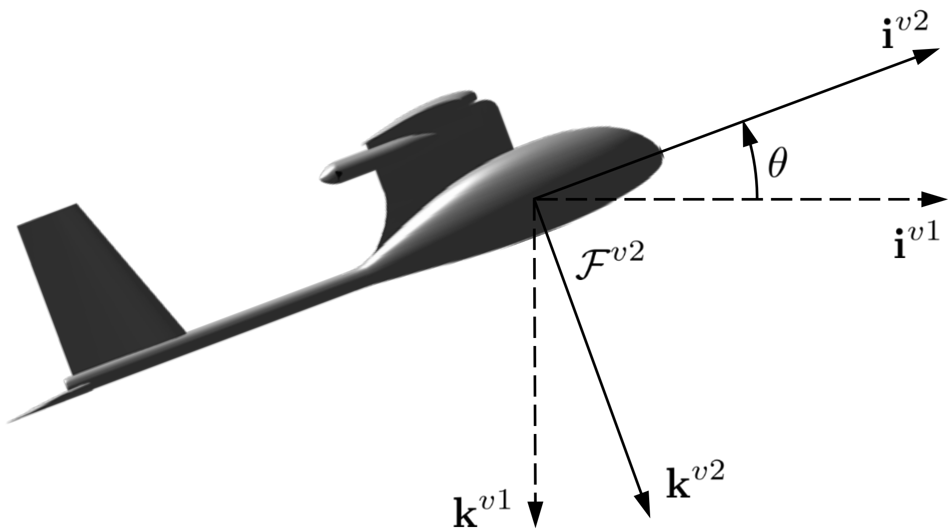


$$\mathcal{R}_v^{v1}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{p}^{v1} = \mathcal{R}_v^{v1}(\psi)\mathbf{p}^v$$

ψ: heading

Vehicle-2 Frame

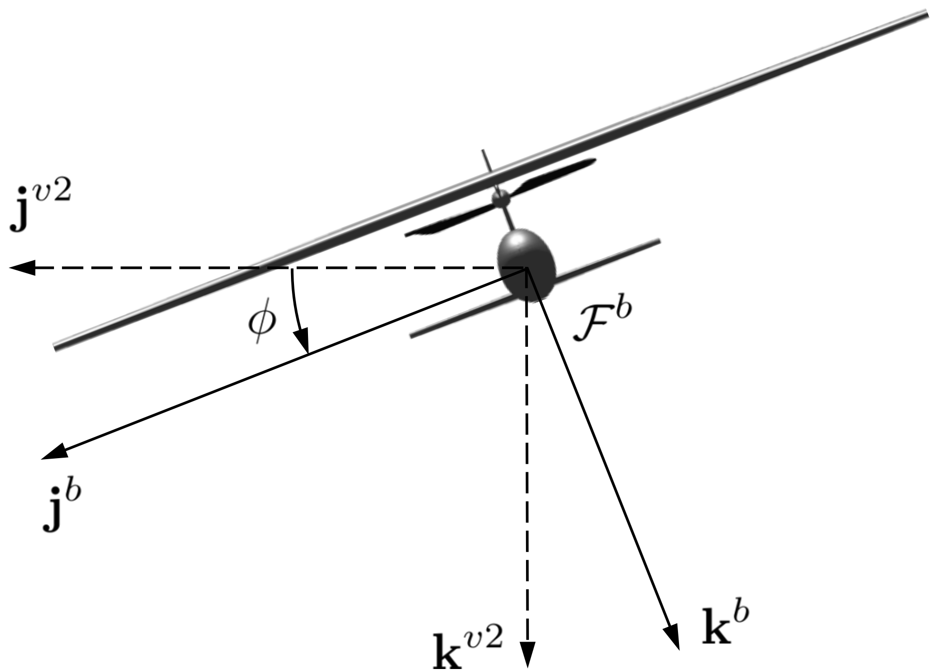


$$\mathcal{R}_{v1}^{v2}(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$\mathbf{p}^{v2} = \mathcal{R}_{v1}^{v2}(\theta)\mathbf{p}^{v1}$$

θ: elevation (pitch)

Body Frame



$$\mathcal{R}_{v2}^{b}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

$$\mathbf{p}^b = \mathcal{R}_{v2}^{b}(\phi)\mathbf{p}^{v2}$$

ϕ: bank (roll)

Inertial Frame to Body Frame Transformation

$$\begin{aligned}
\mathcal{R}_v^b(\phi,\theta,\psi) &= \mathcal{R}_{v2}^b(\phi)\mathcal{R}_{v1}^{v2}(\theta)\mathcal{R}_v^{v1}(\psi) \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} c_\theta c_\psi & c_\theta s_\psi & -s_\theta \\ s_\phi s_\theta c_\psi - c_\phi s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & s_\phi c_\theta \\ c_\phi s_\theta c_\psi + s_\phi s_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi & c_\phi c_\theta \end{pmatrix}
\end{aligned}$$

$$\mathbf{p}^b = \mathcal{R}_v^b(\theta)\mathbf{p}^v$$

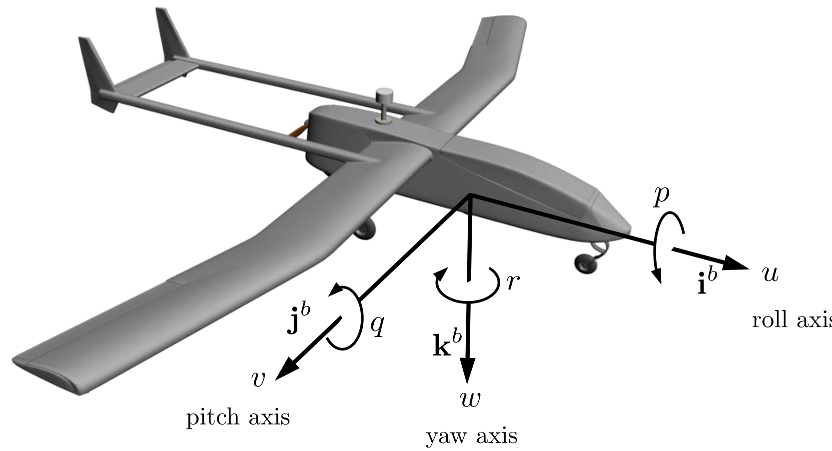
Translational Kinematics

$$\frac{d}{dt}\begin{pmatrix} p_n \\ p_e \\ p_d \end{pmatrix} = \mathcal{R}_b^v \begin{pmatrix} u \\ v \\ w \end{pmatrix} = (\mathcal{R}_v^b)^\top \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Rotational Kinematics

$$
\begin{aligned}
\begin{pmatrix} p \\ q \\ r \end{pmatrix} &= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}_{v2}^{b}(\phi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathcal{R}_{v2}^{b}(\phi)\mathcal{R}_{v1}^{v2}(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}
\end{aligned}
$$



Inverting gives

$$
\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}
$$

State Equations

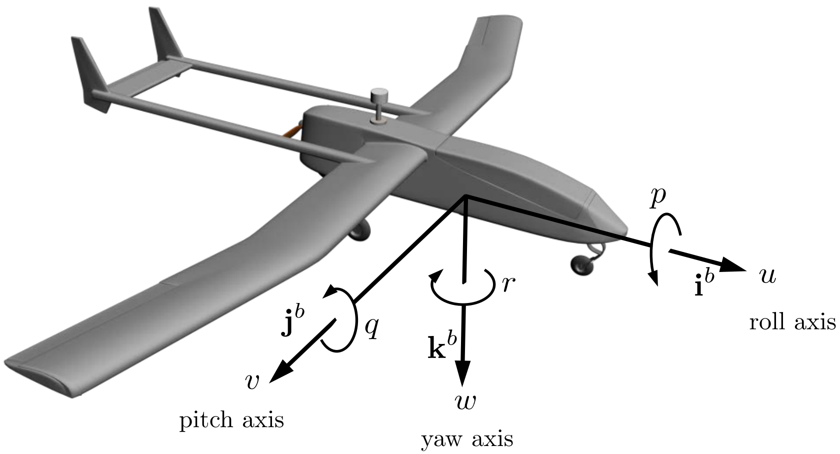
Six of the 12 state equations for the UAV come from the kinematic equations relating positions and velocities:

$$\begin{pmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

The remaining six equations will come from applying Newton's 2nd law to the translational and rotational motion of the aircraft.

Translational Dynamics



Newton's 2nd Law:

$$\mathrm{m}\frac{d\mathbf{V}_g}{dt_i} = \mathbf{f}$$

What is $\mathbf{V}_g$?

- $\mathbf{f}$ is the sum of all external forces
- m is the mass of the aircraft
- Time derivative taken wrt inertial frame

$$\frac{d\mathbf{V}_g}{dt_i} = \frac{d\mathbf{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{V}_g$$

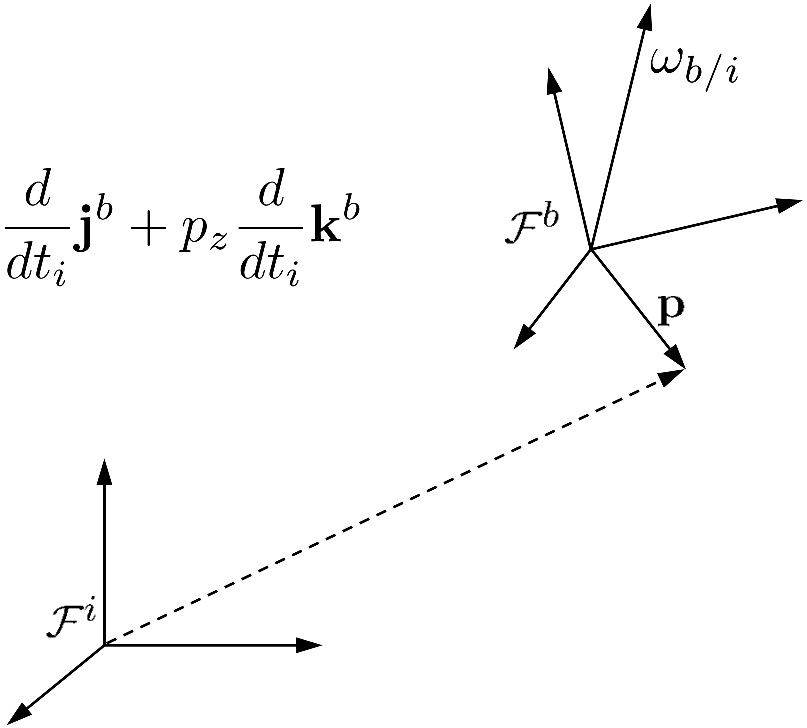
$$\mathrm{m}\left(\frac{d\mathbf{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{V}_g\right) = \mathbf{f}$$

Differentiation of a Vector

$$\mathbf{p} = p_x \mathbf{i}^b + p_y \mathbf{j}^b + p_z \mathbf{k}^b$$

$$\frac{d}{dt_i}\mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b + p_x \frac{d}{dt_i}\mathbf{i}^b + p_y \frac{d}{dt_i}\mathbf{j}^b + p_z \frac{d}{dt_i}\mathbf{k}^b$$

$$\frac{d}{dt_b}\mathbf{p} = \dot{p}_x \mathbf{i}^b + \dot{p}_y \mathbf{j}^b + \dot{p}_z \mathbf{k}^b$$



$$\dot{\mathbf{i}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{i}^b$$

$$\dot{\mathbf{j}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{j}^b$$

$$\dot{\mathbf{k}}^b = \boldsymbol{\omega}_{b/i} \times \mathbf{k}^b$$

$$\begin{aligned} p_x \dot{\mathbf{i}}^b + p_y \dot{\mathbf{j}}^b + p_z \dot{\mathbf{k}}^b &= p_x(\boldsymbol{\omega}_{b/i} \times \mathbf{i}^b) + p_y(\boldsymbol{\omega}_{b/i} \times \mathbf{j}^b) + p_z(\boldsymbol{\omega}_{b/i} \times \mathbf{k}^b) \\ &= \boldsymbol{\omega}_{b/i} \times \mathbf{p} \end{aligned}$$

$$\frac{d}{dt_i}\mathbf{p} = \frac{d}{dt_b}\mathbf{p} + \boldsymbol{\omega}_{b/i} \times \mathbf{p}$$

Translational Dynamics

$$m\left(\frac{d\mathbf{V}_g}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{V}_g\right) = \mathbf{f}$$

can be expressed in body frame as

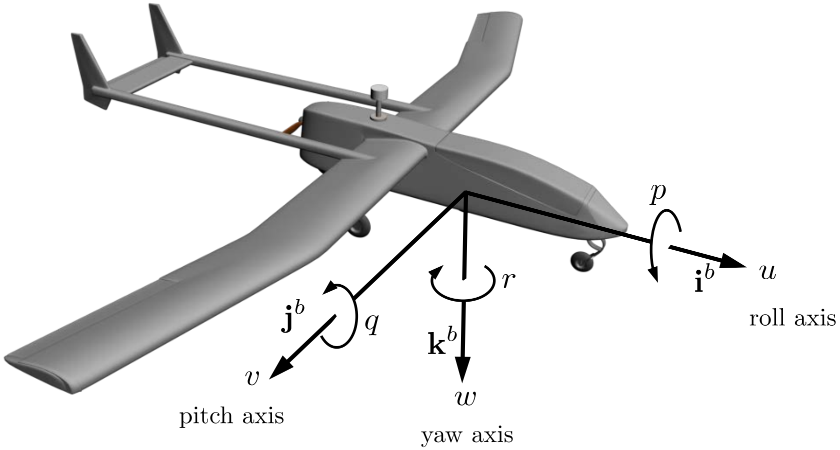
$$m\left(\frac{d\mathbf{V}_g^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{V}_g^b\right) = \mathbf{f}^b$$

where $\mathbf{V}_g^b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ $\quad \boldsymbol{\omega}_{b/i}^b = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ $\quad \boldsymbol{f}^b = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$

Since $\dfrac{d\mathbf{V}_g^b}{dt_b} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix}$ we have that

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} rv - qw \\ pw - ru \\ qu - pv \end{pmatrix} + \frac{1}{m}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Rotational Dynamics



Newton's 2nd Law:

$$\frac{d\mathbf{h}}{dt_i} = \mathbf{m}$$

- $\mathbf{h}$ is the angular momentum vector
- $\mathbf{m}$ is the sum of all external moments
- Time derivative taken wrt inertial frame

$$\frac{d\mathbf{h}}{dt_i} = \frac{d\mathbf{h}}{dt_b} + \boldsymbol{\omega}_{b/i} \times \mathbf{h} = \mathbf{m}$$

Expressed in the body frame,

$$\frac{d\mathbf{h}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{h}^b = \mathbf{m}^b$$

Rotational Dynamics

$$\frac{d\mathbf{h}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \mathbf{h}^b = \mathbf{m}^b$$

Because $\mathbf{J}$ is unchanging in the body frame, $\frac{d\mathbf{J}}{dt_b} = 0$ and

$$\mathbf{J}\frac{d\boldsymbol{\omega}_{b/i}^b}{dt_b} + \boldsymbol{\omega}_{b/i}^b \times \left(\mathbf{J}\boldsymbol{\omega}_{b/i}^b\right) = \mathbf{m}^b$$

Rearranging we get

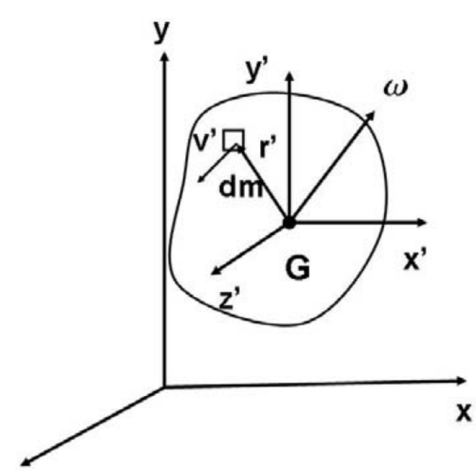
$$\dot{\boldsymbol{\omega}}_{b/i}^b = \mathbf{J}^{-1}\left[-\boldsymbol{\omega}_{b/i}^b \times \left(\mathbf{J}\boldsymbol{\omega}_{b/i}^b\right) + \mathbf{m}^b\right]$$

where $\dot{\boldsymbol{\omega}}_{b/i}^b = \frac{d\boldsymbol{\omega}_{b/i}^b}{dt_b} = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix}$

Angular Momentum

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{r}'_i)) = \sum_{i=1}^{n} m_i r'^2_i \boldsymbol{\omega}$$

$$= \int_m \boldsymbol{r}' \times \boldsymbol{v}' \, dm$$



Angular Momentum

$$\boldsymbol{H}_G = \int_m \boldsymbol{r}' \times (\boldsymbol{\omega} \times \boldsymbol{r}')\, dm = \int_m [(\boldsymbol{r}' \cdot \boldsymbol{r}')\boldsymbol{\omega} - (\boldsymbol{r}' \cdot \boldsymbol{\omega})\boldsymbol{r}']\, dm$$

$$\boldsymbol{r}' = x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}$$

$$\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$$

$$\begin{aligned}
\boldsymbol{H}_G \;&=\; \left(\omega_x \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')x'\ dm\right)\boldsymbol{i} \\
&+\; \left(\omega_y \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')y'\ dm\right)\boldsymbol{j} \\
&+\; \left(\omega_z \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')z'\ dm\right)\boldsymbol{k}
\end{aligned}$$

Inertia Components

$$
\begin{aligned}
\boldsymbol{H}_G &= \left(\; I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \right) \boldsymbol{i} \\
&+ \left(-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \right) \boldsymbol{j} \\
&+ \left(-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \right) \boldsymbol{k}
\end{aligned}
$$

$$
\left.
\begin{aligned}
I_{xx} &= \int_m (y'^2 + z'^2)\, dm \\
I_{yy} &= \int_m (x'^2 + z'^2)\, dm \\
I_{zz} &= \int_m (x'^2 + y'^2)\, dm
\end{aligned}
\right\} \text{Moments of inertia}
$$

Products of Inertia

$$I_{xy} = I_{yx} = \int_m x'y' \, dm$$

$$I_{xz} = I_{zx} = \int_m x'z' \, dm$$

$$I_{yz} = I_{zy} = \int_m y'z' \, dm$$

The Tensor of Inertia

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}}_{\text{Inertia matrix (J)}} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Inertia matrix

$$\mathbf{J} = \begin{pmatrix} \int (y^2 + z^2)\, d\mathsf{m} & -\int xy\, d\mathsf{m} & -\int xz\, d\mathsf{m} \\ -\int xy\, d\mathsf{m} & \int (x^2 + z^2)\, d\mathsf{m} & -\int yz\, d\mathsf{m} \\ -\int xz\, d\mathsf{m} & -\int yz\, d\mathsf{m} & \int (x^2 + y^2)\, d\mathsf{m} \end{pmatrix}$$

Rotational Dynamics

If the aircraft is symmetric about the $\mathbf{i}^b$-$\mathbf{k}^b$ plane, then $J_{xy} = J_{yz} = 0$ and

$$\mathbf{J} = \begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix}$$

This symmetry assumption helps simplify the analysis. The inverse of $\mathbf{J}$ becomes

$$\mathbf{J}^{-1} = \frac{\mathrm{adj}(\mathbf{J})}{\det(\mathbf{J})} = \frac{\begin{pmatrix} J_y J_z & 0 & J_y J_{xz} \\ 0 & J_x J_z - J_{xz}^2 & 0 \\ J_{xz} J_y & 0 & J_x J_y \end{pmatrix}}{J_x J_y J_z - J_{xz}^2 J_y}$$

$$= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix} \qquad \Gamma \stackrel{\triangle}{=} J_x J_z - J_{xz}^2$$

Rotational Dynamics

$$\dot{\boldsymbol{\omega}}^b_{b/i} = \mathbf{J}^{-1}\left[-\boldsymbol{\omega}^b_{b/i} \times \left(\mathbf{J}\boldsymbol{\omega}^b_{b/i}\right) + \mathbf{m}^b\right]$$

Define $\mathbf{m}^b \triangleq \begin{pmatrix} l \\ m \\ n \end{pmatrix}$

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} &= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix}\left[\begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix}\begin{pmatrix} J_x & 0 & -J_{xz} \\ 0 & J_y & 0 \\ -J_{xz} & 0 & J_z \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix}\right] \\
&= \begin{pmatrix} \frac{J_z}{\Gamma} & 0 & \frac{J_{xz}}{\Gamma} \\ 0 & \frac{1}{J_y} & 0 \\ \frac{J_{xz}}{\Gamma} & 0 & \frac{J_x}{\Gamma} \end{pmatrix}\left[\begin{pmatrix} J_{xz}pq + (J_y - J_z)qr \\ J_{xz}(r^2 - p^2) + (J_z - J_x)pr \\ (J_x - J_y)pq - J_{xz}qr \end{pmatrix} + \begin{pmatrix} l \\ m \\ n \end{pmatrix}\right] \\
&= \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr + \Gamma_3 l + \Gamma_4 n \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) + \frac{1}{J_y} m \\ \Gamma_7 pq - \Gamma_1 qr + \Gamma_4 l + \Gamma_8 n \end{pmatrix}
\end{aligned}$$

Γ's are functions of moments and products of inertia

Gravity Force

$$\mathbf{f}_g^v = \begin{pmatrix} 0 \\ 0 \\ \mathsf{m}g \end{pmatrix}$$

expressed in vehicle frame

$$\mathbf{f}_g^b = \mathcal{R}_v^b \begin{pmatrix} 0 \\ 0 \\ \mathsf{m}g \end{pmatrix}$$

$$= \begin{pmatrix} -\mathsf{m}g\sin\theta \\ \mathsf{m}g\cos\theta\sin\phi \\ \mathsf{m}g\cos\theta\cos\phi \end{pmatrix}$$

expressed in body frame

Equation of Motion Summary

System of 12 first-order ODE's

Kinematics:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}_I = \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}_B$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Kinetics:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}_B = \begin{bmatrix} 0 \\ 0 \\ -T/M \end{bmatrix} + \begin{bmatrix} -g\sin\theta \\ g\cos\theta\sin\phi \\ g\cos\theta\cos\phi \end{bmatrix} + \begin{bmatrix} rv - qw \\ pw - ur \\ qu - pv \end{bmatrix}$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \frac{I_{yy}-I_{zz}}{I_{xx}}qr \\ \frac{I_{zz}-I_{xx}}{I_{yy}}pr \\ \frac{I_{xx}-I_{yy}}{I_{zz}}pq \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{xx}}l \\ \frac{1}{I_{yy}}m \\ \frac{1}{I_{zz}}n \end{bmatrix}$$

Here l, m, n corresponds to rolling pitching and yawing moments. Also T is the thrust produced by motors and M is the mass of quadrotor.