

2

Review of Fundamentals

2.1 Introduction

The operation of gas turbine engines and of rocket motors is governed by the laws of mechanics and thermodynamics. The field of mechanics includes the mechanics of both fluids and solids. However, since the process occurring in most propulsion devices involves a flowing fluid, our emphasis will be on fluid mechanics or, more specifically, gas dynamics.

Understanding and predicting the basic performance of gas turbine engines and rocket motors requires a closed set of governing equations (e.g., conservation of mass, energy, momentum, and entropy). For transparency, understanding, and capturing the basic physical phenomena, we model the gas as a *perfect gas* and the flow as *one-dimensional flow*, in which the fluid properties are constant across the flow and vary only in the direction of flow (axial direction).

This chapter is intended as a review of thermodynamics and one-dimensional gas dynamics that are the foundation of this textbook. We include new material from *Hypersonic Airbreathing Propulsion* by Heiser and Pratt¹³ on the graphical solution and description of one-dimensional gas dynamics in Section 2.8 to improve the reader's comprehension of this fundamental material.

This chapter concludes with a short section on chemical reactions to provide the fundamentals needed to understand reacting flows (e.g., combustion). With this material, we will predict the equilibrium state of reacting flows and the performance of liquid-fuel rocket motors. For further study, the student is directed to references such as Ref. 14.

2.2 Equations of State and Conservation of Mass

The state of a system is described by specifying the values of the properties of the system. Pressure P , temperature T , specific internal energy e , and density ρ or specific volume $v = 1/\rho$ are some basic thermodynamic properties. The specification of any two independent intensive properties will fix the thermodynamic state of a simple (meaning in absence of motion and force fields) system and, therefore, the values of all other thermodynamic properties of the system. The values of the other properties may be found through equations of state. The most common working fluid addressed in this book is a gas that is modeled using the perfect gas equations of state.

A function relating one dependent and two independent thermodynamic properties of a simple system of unit mass is called an *equation of state*. When the three properties are P , v , and T as in

$$f(P, v, T) = 0 \quad (2.1)$$

the equation is called the *thermal equation of state*. In general, we cannot write the functional relationship equation (2.1) in the form of an equation in which specified values of the two properties will allow us to determine the value of the third. Although humans may not know what the functional relation equation (2.1) is for a given system, one does exist and nature always knows what it is. When the solution set of Eq. (2.1) cannot be determined from relatively simple equations, tables that list the values of P , v , and T (elements of the solution set) satisfying the function may be prepared. This has been done for water (in all of its phases), air, and most common gases.

The functional relation between the internal energy e of a simple system of unit mass and any two independent properties for the set P , v ($=1/\rho$), T is called the *energy equation of state*. This equation can be written functionally as

$$e = e(T, v) \quad \text{or} \quad e = e(P, v) \quad \text{or} \quad e = e(P, T) \quad (2.2)$$

As with the thermal equation of state, we may not be able to write an analytical expression for any of the functional relations of Eq. (2.2). The important thing is that energy is a property; hence, the functional relations exist.

If the solution sets of the thermal and energy equations of state of a simple system of unit mass are known, all thermodynamic properties of the system can be found when any two of the three properties P , v , T are specified. From the solution set, we can form a tabulation of v and e against specified values of P and T for all states of the system. From these known values of P , T , v , and e , we can determine any other property of the simple system. For example, the value of the property *enthalpy* h is found for any state of the system by combining the tabulated values of P , v , and e for that state by

$$h \equiv e + Pv \quad (2.3)$$

Four other definitions are listed here for use in the later sections of this chapter: specific heat at constant volume c_v , specific heat at constant pressure c_p , ratio of specific heats γ , the speed of sound a , and the Mach number M :

$$c_v \equiv \left(\frac{\partial e}{\partial T} \right)_v \quad (2.4)$$

$$c_p \equiv \left(\frac{\partial h}{\partial T} \right)_p \quad (2.5)$$

$$\gamma \equiv c_p/c_v \quad (2.6)$$

$$a^2 \equiv g_c \left(\frac{\partial P}{\partial \rho} \right)_s = \gamma g_c \left(\frac{\partial P}{\partial \rho} \right)_T \quad (2.7)$$

$$M = V/a \quad (2.8)$$

2.2.1 Definition of Steady Flow

Consider the flow of fluid through the control volume σ shown in Fig. 2.1. If the properties of the fluid at any point i in the control volume do not vary with time, the flow is called *steady flow*. For such flows we may conclude that for any property R within σ

$$\frac{dR_\sigma}{dt} = 0 \quad \text{in steady flow} \quad (2.9)$$

2.2.2 Definition of One-Dimensional Flow

If the intensive stream properties at a permeable control surface section normal to the flow directions are uniform, the flow is called *one-dimensional*. Many flows in engineering may be treated as steady one-dimensional flows. The term *one-dimensional* is synonymous in this use with *uniform* and applies only at a control surface section. Thus the overall flow through a control volume may be in more than one dimension and still be uniform (one-dimensional flow) at permeable sections of the control surface normal to the flow direction. The flow in Fig. 2.2 is called one-dimensional flow because the intensive properties, such as velocity, density, and temperature, are uniform at sections 1 and 2.

2.2.3 Conservation of Mass Equation

The law of mass conservation for any control volume system σ is simply

$$\frac{dm_\sigma}{dt} = \dot{m}_{\text{in}} - \dot{m}_{\text{out}} \quad (2.10a)$$

where

$$\dot{m} = \int_A \rho V_\perp dA \quad (2.10b)$$

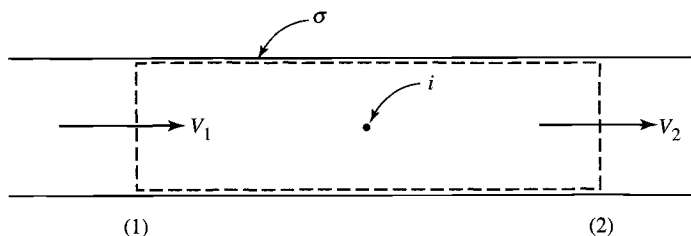


Fig. 2.1 Control volume for steady flow.

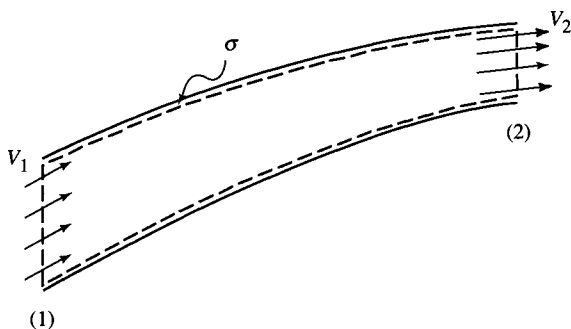


Fig. 2.2 One-dimensional flow through a convergent duct. The flow is uniform at sections 1 and 2, hence one-dimensional, even though the flow direction may vary elsewhere in the flow.

and V_{\perp} is the velocity component normal to area A . This equation is known as the *conservation of mass equation*. For steady flows through any control volume, Eq. (2.10a) simplifies to

$$\dot{m}_{\text{out}} = \dot{m}_{\text{in}} \quad (2.11)$$

If the flow is steady and one-dimensional through a control volume with a single inlet and exit such as shown in Fig. 2.2, then by Eq. (2.10b)

$$\dot{m} = \rho A V_{\perp} \quad (2.12a)$$

which is called the *one-dimensional mass flow equation*, and

$$\rho_1 A_1 V_{\perp 1} = \rho_2 A_2 V_{\perp 2} \quad (2.12b)$$

2.3 Steady Flow Energy Equation

We consider steady one-dimensional flow of a fluid through a control volume and surface σ (Fig. 2.3). Fluid crosses σ at the in and out stations only. A shaft-work interaction \dot{W}_x and heat interaction \dot{Q} occur at the boundary of σ . If the energy within the control volume does not change with time (steady flow), the first law of thermodynamics can be written as

$$\dot{Q} - \dot{W}_x = \dot{m} \left(h + \frac{V^2}{2g_c} + \frac{gz}{g_c} \right)_{\text{out}} - \dot{m} \left(h + \frac{V^2}{2g_c} + \frac{gz}{g_c} \right)_{\text{in}} \quad (2.13)$$

where the dimensions of \dot{Q} , \dot{W}_x , etc., are those of power or energy per unit time. Dividing by \dot{m} gives

$$q - w_x = \left(h + \frac{V^2}{2g_c} + \frac{gz}{g_c} \right)_{\text{out}} - \left(h + \frac{V^2}{2g_c} + \frac{gz}{g_c} \right)_{\text{in}} \quad (2.14)$$

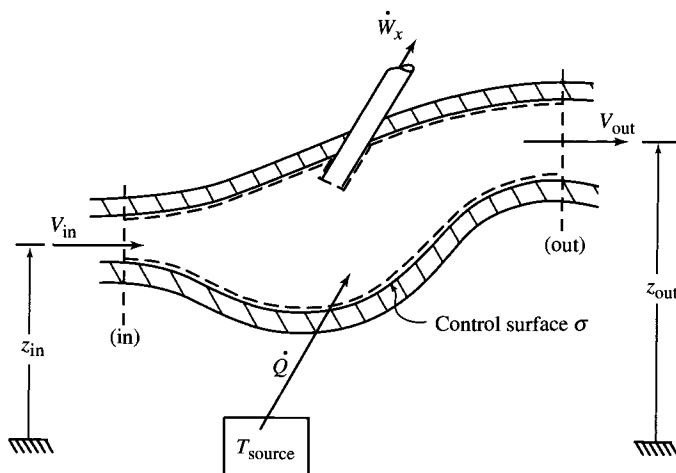


Fig. 2.3 Steady flow through control volume σ .

where q and w_x are the heat and shaft-work interactions per unit mass flow through σ . All of the terms in this equation have units of energy per unit mass.

Example 2.1

The first step in the application of the steady flow energy equation is a clear definition of a control surface σ . This is so because each term in the equation refers to a quantity at the boundary of a control volume. Thus, to use the equation, one needs only to examine the control surface and identify the applicable terms of the equation.

In the application of Eq. (2.14) to specific flow situations, many of the terms are zero or may be neglected. The following example will illustrate this point. The perfect gas equation (presented in detail later in this chapter) is used to model the relationship between change in enthalpy and temperature of the gas.

Consider a turbojet aircraft engine as shown in Fig. 2.4a. We divide the engine into the control volume regions:

σ_1 : inlet	σ_4 : turbine
σ_2 : compressor	σ_5 : nozzle
σ_3 : combustion chamber	

Let us apply the steady flow energy equation to each of these control volumes. In all cases, the potential energy change $(gz/g_c)_{\text{out}} - (gz/g_c)_{\text{in}}$ is zero and will be ignored.

It is advisable in using the steady flow energy equation to make two sketches of the applicable control surface σ , showing the heat and shaft work interactions (q , w_x) in one sketch and the fluxes of energy $[h, V^2/(2g_c)]$ in the second sketch. The term $[h + V^2/(2g_c)]_{\text{out}}$ is a flux per unit mass flow of internal energy

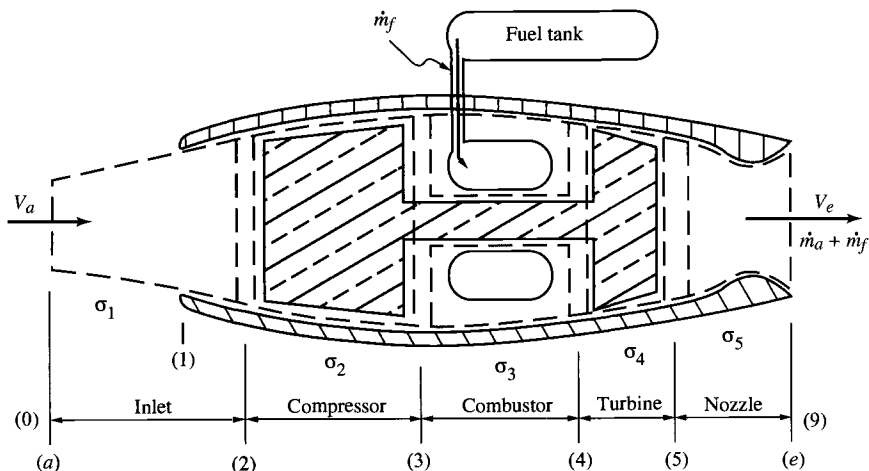


Fig. 2.4a Control volume for analyzing each component of a turbojet engine.

e , kinetic energy $V^2/(2g_c)$, and flow work Pv . We will use the expression *flux of energy* to include the flow work flux also.

a) Inlet and nozzle: σ_1 and σ_5 . There are no shaft work interactions at control surfaces σ_1 and σ_5 . Heat interactions are negligible and may be taken as zero. Therefore, the steady flow energy equation, as depicted in Fig. 2.4b, for the inlet or nozzle control surfaces gives the result

$$0 = \left(h + \frac{V^2}{2g_c} \right)_{\text{out}} - \left(h + \frac{V^2}{2g_c} \right)_{\text{in}}$$

Nozzle: numerical example Let the gases flowing through the nozzle control volume σ_5 be perfect with $c_p g_c = 6000 \text{ ft}/(\text{s}^2 \cdot ^\circ\text{R})$. Determine V_9 for $T_5 = 1800^\circ\text{R}$, $V_5 = 400 \text{ ft/s}$, and $T_9 = 1200^\circ\text{R}$.

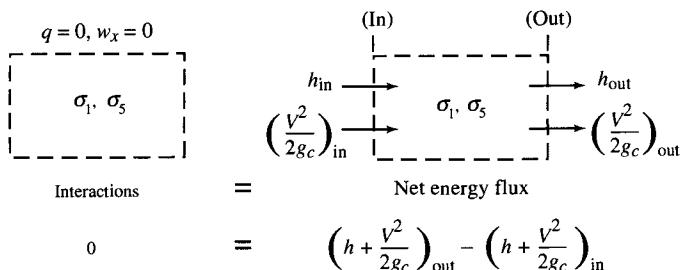


Fig. 2.4b Energy equation applied to control volumes σ_1 and σ_5 .

Solution: From the steady flow energy equation with 5 and 9 as the in and out stations, respectively, we have

$$h_9 + \frac{V_9^2}{2g_c} = h_5 + \frac{V_5^2}{2g_c}$$

and

$$V_9 = \sqrt{2g_c(h_5 - h_9) + V_5^2} = \sqrt{2c_p g_c(T_5 - T_9) + V_5^2}$$

or

$$= \sqrt{2(6000)(1800 - 1200) + 400^2}$$

so

$$= 2700 \text{ ft/s}$$

b) Compressor and turbine: σ_2 and σ_4 . The heat interactions at control surfaces σ_2 and σ_4 are negligibly small. Shaft work interactions are present because each control surface cuts a rotating shaft. The steady flow energy equation for the compressor or for the turbine is depicted in Fig. 2.4c and gives

$$-w_x = \left(h + \frac{V^2}{2g_c} \right)_{\text{out}} - \left(h + \frac{V^2}{2g_c} \right)_{\text{in}}$$

Compressor and turbine: numerical example For an equal mass flow through the compressor and turbine of 185 lb/s, determine the compressor power and the turbine exit temperature T_5 for the following conditions:

$$c_p g_c = 6000 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R})$$

Compressor

$$T_2 = 740^\circ\text{R}, T_3 = 1230^\circ\text{R}$$

$$V_2 = V_3$$

Turbine

$$T_4 = 2170^\circ\text{R}, T_5 = ?$$

$$V_4 = V_5$$

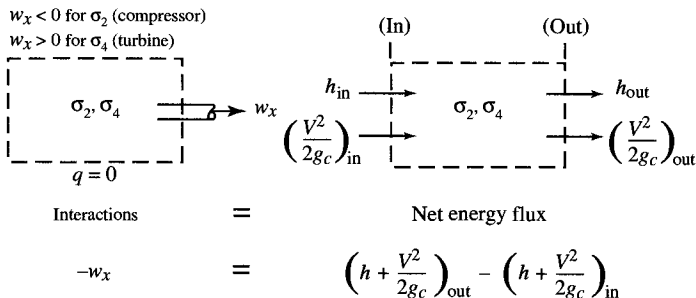


Fig. 2.4c Energy equation applied to control volumes σ_2 and σ_4 .

Solution: The compressor power $\dot{W}_c = (\dot{m}w_x)_{\sigma_2}$ is, with $V_2 = V_3$,

$$\begin{aligned}\dot{W}_c &= -\dot{m}(h_3 - h_2) = -\dot{m}c_p(T_3 - T_2) \\ &= -(185 \text{ lbm/s}) \frac{6000 (\text{ft/s})^2}{32.174 \text{ ft-lbf/(lbm-s}^2\text{)}} (1230 - 740) \\ &= -16.9 \times 10^6 \text{ ft-lbf/s} \times \frac{1 \text{ hp}}{550 \text{ ft-lbf/s}} \\ &= -30,700 \text{ hp}\end{aligned}$$

The minus sign means the compressor shaft is delivering energy to the air in σ_2 .

The turbine drives the compressor so that the turbine power $\dot{W}_t = (\dot{m}w_x)_{\sigma_4}$ is equal in magnitude to the compressor power. Thus $\dot{W}_t = -\dot{W}_c$, where, from the energy equation,

$$\dot{W}_t = \dot{m}(h_5 - h_4) \quad \text{and} \quad \dot{W}_c = -\dot{m}(h_3 - h_2)$$

Thus

$$\dot{m}c_p(T_5 - T_4) = -\dot{m}c_p(T_3 - T_2)$$

and

$$\begin{aligned}T_5 &= T_4 - (T_3 - T_2) \\ &= 2170^\circ\text{R} - (1230^\circ\text{R} - 740^\circ\text{R}) = 1680^\circ\text{R} = 1220^\circ\text{F}\end{aligned}$$

c) Combustion chamber: σ_3 . Let us assume that the fuel and air entering the combustion chamber mix physically in a mixing zone (Fig. 2.4d) to form what we will call *reactants* (denoted by subscript *R*). The reactants then enter a combustion zone where combustion occurs, forming *products* of combustion (subscript *P*) that leave the combustion chamber. We apply the steady flow energy equation to combustion zone σ_3 . Because the temperature in the combustion zone is higher than that of the immediate surroundings, there is a heat interaction between σ_3 and the surroundings that, per unit mass flow of reactants, is negligibly small ($q < 0$ but $q = 0$). Also the velocities of the products leaving and of the reactants entering the combustion zone are approximately equal. There is no shaft work interaction for σ_3 . Hence the steady flow energy equation, as depicted in Fig. 2.4d, reduces to

$$h_{R_3} = h_{P_4} \quad (2.15)$$

We must caution the reader about two points concerning this last equation. First, we cannot use the relation $c_p\Delta T$ for computing the enthalpy difference between

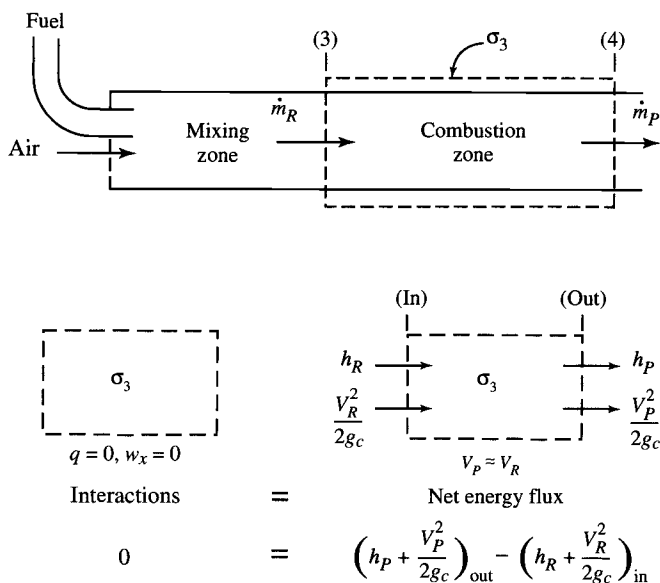


Fig. 2.4d Energy equation applied to control volumes σ_3 .

two states of a system when the chemical aggregation of the two states differs. Second, we must measure the enthalpy of each term in the equation relative to the same datum state. To place emphasis on the first point, we have introduced the additional subscripts R and P to indicate that the chemical aggregations of states 3 and 4 are different.

To emphasize the second point, we select as our common enthalpy datum a state d having the chemical aggregation of the products at a datum temperature T_d . Then, introducing the datum state enthalpy $(h_P)_d$ into the last equation, we have

$$h_{R_3} - h_{P_d} = h_{P_4} - h_{P_d} \quad (2.16)$$

Equation (2.16) can be used to determine the temperature of the products of combustion leaving an adiabatic combustor for given inlet conditions. If the combustor is not adiabatic, Eq. (2.16), adjusted to include the heat interaction term q on the left-hand side, is applicable. Let us treat the reactants and products as perfect gases and illustrate the use of Eq. (2.16) in determining the temperature of the gases at the exit of a turbojet combustion chamber via an example problem.

Combustion chamber: numerical example For the turbojet engine combustion chamber, 45 lbm of air enters with each 1 lbm of JP-8 (kerosene) fuel. Let us assume these reactants enter an adiabatic combustor at 1200°R. The heating value h_{PR} of JP-8 is 18,400 Btu/lbm of fuel at 298 K. [This is also called the

lower heating value (LHV) of the fuel.] Thus the heat released $(\Delta H)_{298\text{K}}$ by the fuel per 1 lbm of the products is 400 Btu/lbm (18,400/46) at 298 K. The following data are known:

$$c_{pP} = 0.267 \text{ Btu}/(\text{lbm}^\circ\text{R}) \quad \text{and} \quad c_{pR} = 0.240 \text{ Btu}/(\text{lbm}^\circ\text{R})$$

Determine the temperature of the products leaving the combustor.

Solution: A plot of the enthalpy equations of state for the reactants and the products is given in Fig. 2.5. In the plot, the vertical distance $h_R - h_P$ between the curves of h_R and h_P at a given temperature represents the enthalpy of combustion ΔH of the reactants at that temperature (this is sometimes called the *heat of combustion*). In our analysis, we know the enthalpy of combustion at $T_d = 298 \text{ K}$ (536.4°R).

States 3 and 4, depicted in Fig. 2.5, represent the states of the reactants entering and the products leaving the combustion chamber, respectively. The datum state d is arbitrarily selected to be products at temperature T_d . State d' is the reactants' state at the datum temperature T_d .

In terms of Fig. 2.5, the left-hand side of Eq. (2.16) is the vertical distance between states 3 and d , or

$$h_{R_3} - h_{P_d} = h_{R_3} - h_{R_{d'}} + h_{R_{d'}} - h_{P_d}$$

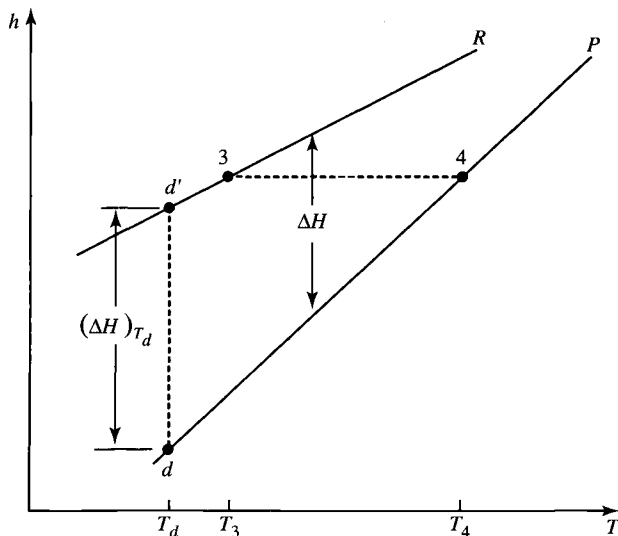


Fig. 2.5 Enthalpy vs temperature for reactants and products treated as perfect gases.

and since

$$\Delta h_R = c_{pR} \Delta T \quad \text{and} \quad h_{R_d} - h_{P_d} = (\Delta H)_{T_d}$$

then

$$h_{R_3} - h_{P_d} = c_{pR}(T_3 - T_d) + (\Delta H)_{T_d} \quad (\text{i})$$

Similarly, the right-hand side of Eq. (2.16) is

$$h_{P_4} - h_{P_d} = c_{pP}(T_4 - T_d) \quad (\text{ii})$$

Substituting Eqs. (i) and (ii) in Eq. (2.16), we get

$$c_{pR}(T_3 - T_d) + (\Delta H)_{T_d} = c_{pP}(T_4 - T_d) \quad (2.17)$$

We can solve this equation for T_4 , which is the temperature of the product gases leaving the combustion chamber. Solving Eq. (2.17) for T_4 , we get

$$\begin{aligned} T_4 &= \frac{c_{pR}(T_3 - T_d) + (\Delta H)_{T_d}}{c_{pP}} + T_d \\ &= \frac{0.240(1200 - 536.4) + 400}{0.267} + 536.4 \\ &= 2631^\circ\text{R} \quad (2171^\circ\text{F}) \end{aligned}$$

This is the so-called adiabatic flame temperature of the reactants for a 45:1 mixture ratio of air to fuel weight. For the analysis in portions of this book, we choose to sidestep the complex thermochemistry of the combustion process and model it as a simple heating process. The theory and application of thermochemistry to combustion in jet engines are covered in many textbooks, such as the classic text by Penner (see Ref. 14).

2.4 Steady Flow Entropy Equation

From the second law of thermodynamics, we have for steady state

$$\dot{S}_{\text{out}} - \dot{S}_{\text{in}} \geq \frac{\dot{Q}}{T_{\text{source}}} \quad (2.18)$$

where $\dot{S} = \dot{m}s$, \dot{Q} is the rate of heat interaction into the control volume σ , and T_{source} is the heat source temperature as shown in Fig. 2.3. For steady state and adiabatic flow through a control volume, this reduces to the statement that the entropy flux out is greater than or equal to the entropy flux in

$$\dot{S}_{\text{out}} \geq \dot{S}_{\text{in}}$$

For one outlet section (2) and one inlet section (1), this and the continuity condition [Eq. (2.11)] yield

$$s_2 \geq s_1 \quad (2.19)$$

2.5 Steady Flow Momentum Equation

Newton's second law of motion for a control volume σ is

$$\sum \mathbf{F}_\sigma = \frac{1}{g_c} \left(\frac{d\mathbf{M}_\sigma}{dt} + \dot{\mathbf{M}}_{\text{out}} - \dot{\mathbf{M}}_{\text{in}} \right) \quad (2.20)$$

where $\mathbf{M}_\sigma = \int_\sigma \mathbf{V} \, dm$ is the momentum of the mass within the control volume σ and $\dot{\mathbf{M}}_{\text{out}} = \int_{\text{out}} \mathbf{V} \, d\dot{m}$ is the momentum flux leaving the control volume. In words, Eq. (2.20) says that the net force acting on a fixed control volume σ is equal to the time rate of increase of momentum within σ plus the net flux of momentum from σ . This very important momentum equation is in fact a *vector* equation, which implies that it must be applied in a specified direction to solve for an unknown quantity.

Applying control volume equations to a steady flow problem gives useful results with only a knowledge of conditions at the control surface. Nothing needs to be known about the state of the fluid interior to the control volume. The following examples illustrate the use of the steady one-dimensional flow condition and the momentum equation. We suggest that the procedure of sketching a control surface (and showing the applicable fluxes through the surface and the applicable forces acting on the surface) be followed whenever a control volume equation is used. This situation is similar to the use of free body diagrams for the analysis of forces on solid bodies. We illustrate this procedure in the solutions that follow.

Example 2.2

Water ($\rho = 1000 \text{ kg/m}^3$) is flowing at a steady rate through a convergent duct as illustrated in Fig. 2.6a. For the data given in the figure, find the force of the fluid $F_{\sigma D}$ acting on the convergent duct D between stations 1 and 2.

Solution: We first select the control volume σ such that the force of interest is acting at the control surface. Because we want the force interaction between D and the flowing water, we choose a control surface coincident with the inner wall surface of D bounded by the permeable surfaces 1 and 2, as illustrated in Fig. 2.6a. By applying the steady one-dimensional continuity equation [Eq. (2.11)], as depicted in Fig. 2.6b, we find V_1 as follows:

$$\begin{aligned} \rho_1 A_1 V_1 &= \rho_2 A_2 V_2 \\ V_1 &= \frac{A_2}{A_1} V_2 \quad (\rho_1 = \rho_2) \\ &= 3 \text{ m/s} \end{aligned}$$

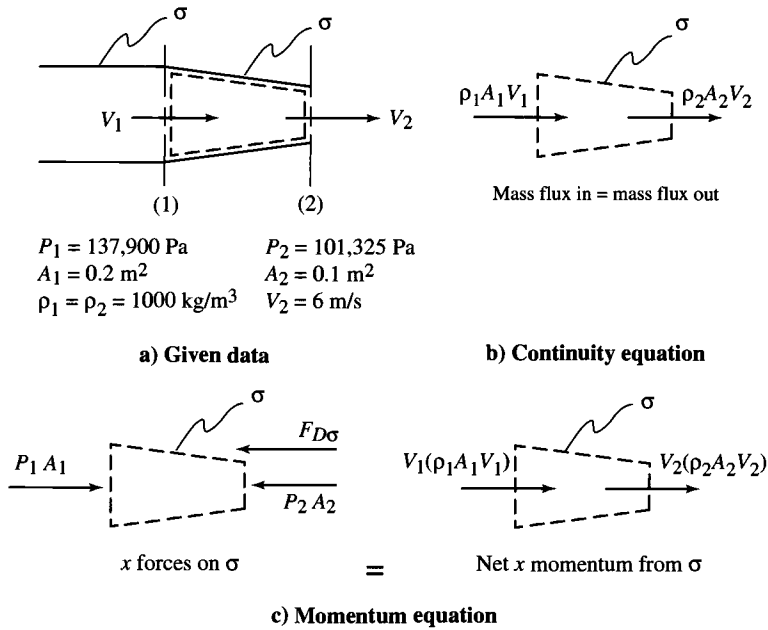


Fig. 2.6 Flow through a convergent duct.

With V_1 determined, we can apply the momentum Eq. (2.20) to σ and find the force of the duct walls on σ , denoted by $F_{D\sigma}$ ($F_{\sigma D} = -F_{D\sigma}$). By symmetry, $F_{D\sigma}$ is a horizontal force, and so the horizontal x components of forces and momentum fluxes will be considered. The x forces acting on σ are depicted in Fig. 2.6c along with the x momentum fluxes through σ .

From Fig. 2.6c, we have momentum:

$$P_1 A_1 - F_{D\sigma} - P_2 A_2 = \frac{1}{g_c} [(\rho_2 A_2 V_2) V_2 - (\rho_1 A_1 V_1) V_1]$$

And by Fig. 2.6b, continuity:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \dot{m}$$

Combining the continuity and momentum equations, we obtain

$$P_1 A_1 - F_{D\sigma} - P_2 A_2 = \frac{\dot{m}}{g_c} (V_2 - V_1)$$

or

$$F_{D\sigma} = P_1 A_1 - P_2 A_2 - \frac{\dot{m}}{g_c} (V_2 - V_1)$$

With $\dot{m} = \rho_1 A_1 V_1 = 1000 \text{ kg/m}^3 \times 0.2 \text{ m}^2 \times 3 \text{ m/s} = 600 \text{ kg/s}$, we have

$$F_{D\sigma} = 137,000 \text{ N/m}^2 \times 0.2 \text{ m}^2 - 101,325 \text{ N/m}^2 \times 0.1 \text{ m}^2 \\ - 600 \text{ kg/s} \times (6 - 3) \text{ m/s}$$

or

$$F_{D\sigma} = 27,580 \text{ N} - 10,132 \text{ N} - 1800 \text{ N}$$

and

$$= 15,648 \text{ N} \quad \text{acts to left in assumed position}$$

Finally, the force of the water on the duct is

$$F_{\sigma D} = -F_{D\sigma} = -15,648 \text{ N} \quad \text{acts to right}$$

Example 2.3

Figure 2.7 shows the steady flow conditions at sections 1 and 2 about an airfoil mounted in a wind tunnel where the frictional effects at the wall are negligible. Determine the section drag coefficient C_d of this airfoil.

Solution: Since the fluid is incompressible and the flow is steady, the continuity equation may be used to find the unknown velocity V_B as follows:

$$(\rho AV)_1 = (\rho AV)_2 \quad \text{that is, } \dot{m}_1 = \dot{m}_2$$

or

$$\rho_1 A_1 V_A = \rho_2 \left(\frac{2}{3} V_B + \frac{1}{3} V_C \right) A_2$$

but

$$\rho_1 = \rho_2 \quad \text{and} \quad A_1 = A_2$$

$$V_A = \frac{2}{3} V_B + \frac{1}{3} V_C$$

thus

$$V_B = \frac{3}{2} V_A - \frac{1}{2} V_C = 31.5 \text{ m/s}$$

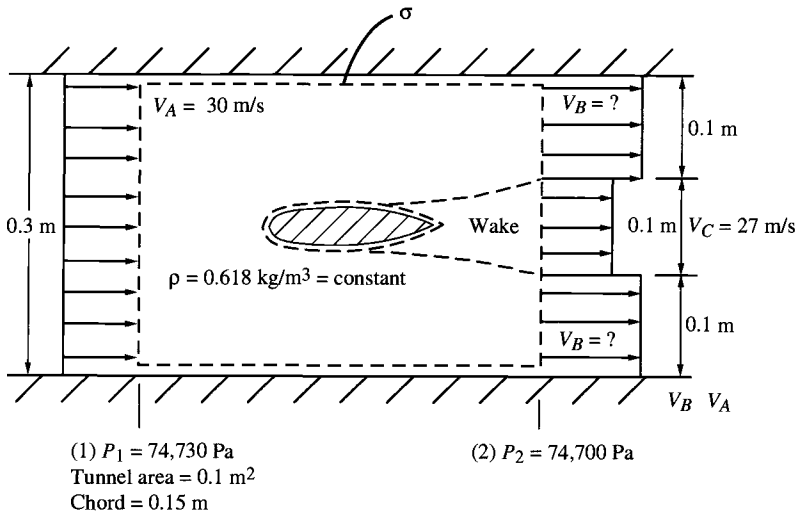


Fig. 2.7 Wind tunnel drag determination for an airfoil section.

The momentum equation may now be used to find the drag on the airfoil. This drag force will include both the skin friction and pressure drag. We sketch the control volume σ with the terms of the momentum equation as shown in Fig. 2.8.

Taking forces to right as positive, we have from the sketch

$$\begin{aligned}\sum F_{\sigma} &= P_1 A_1 - P_2 A_2 + F_{D\sigma} \\ \dot{M}_1 &= (\rho_1 A_1 V_A) V_A = \rho_1 A_1 V_A^2 \\ \dot{M}_2 &= \rho_2 \left(\frac{2}{3} A_2 \right) V_B^2 + \rho_2 \left(\frac{1}{3} A_2 \right) V_C^2 \\ &= \rho_2 A_2 \left(\frac{2}{3} V_B^2 + \frac{1}{3} V_C^2 \right)\end{aligned}$$

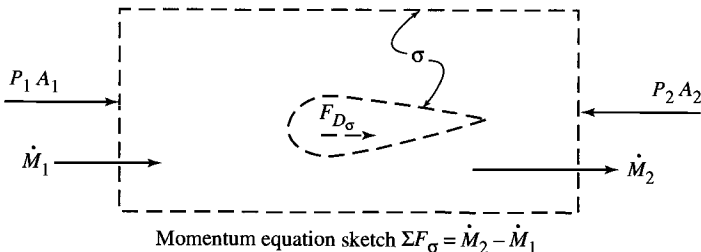


Fig. 2.8 Sketch for momentum equation for airfoil section.

For $\rho = \rho_1 = \rho_2$ and $A = A_1 = A_2$,

$$-F_{D\sigma} = (P_1 - P_2)A + \frac{\rho A}{g_c} \left(V_A^2 - \frac{2}{3} V_B^2 - \frac{1}{3} V_C^2 \right)$$

$$\begin{aligned} \text{or} \quad &= (74,730 - 74,700) 0.1 + 0.618 \times 0.1 \left[30^2 - \frac{2}{3} (31.5^2) - \frac{1}{3} (27^2) \right] \\ &= 3.0 \text{ N} - 0.278 \text{ N} \end{aligned}$$

$$-F_{D\sigma} = 2.722 \text{ N} \quad \therefore F_{D\sigma} \text{ acts to left}$$

$F_{\sigma D}$ = drag force for section

$$= -F_{D\sigma} \quad \text{and} \quad F_{\sigma D} \text{ acts to left}$$

$$F'_D = \frac{F_{\sigma D}}{b} = \frac{2.722 \text{ N}}{0.333 \text{ m}} = 8.174 \text{ N/m}$$

$$C_d = \frac{F'_D}{qc} \quad \text{and} \quad q = \frac{\rho V_\infty^2}{2g_c} \quad \text{where} \quad V_\infty = V_A$$

$$C_d = \frac{F'_D}{[(\rho V_\infty^2)/(2g_c)]c} = \frac{8.174}{(0.618 \times 30^2/2)0.15} = 0.196$$

Example 2.4

Figure 2.9 shows a test stand for determining the thrust of a liquid-fuel rocket. The propellants enter at section 1 at a mass flow rate of 15 kg/s, a velocity of 30 m/s, and a pressure of 0.7 MPa. The inlet pipe for the propellants is very flexible, and the force it exerts on the rocket is negligible. At the nozzle exit, section 2, the area is 0.064 m², and the pressure is 110 kPa. The force read by the scales is 2700 N, atmospheric pressure is 82.7 kPa, and the flow is steady.

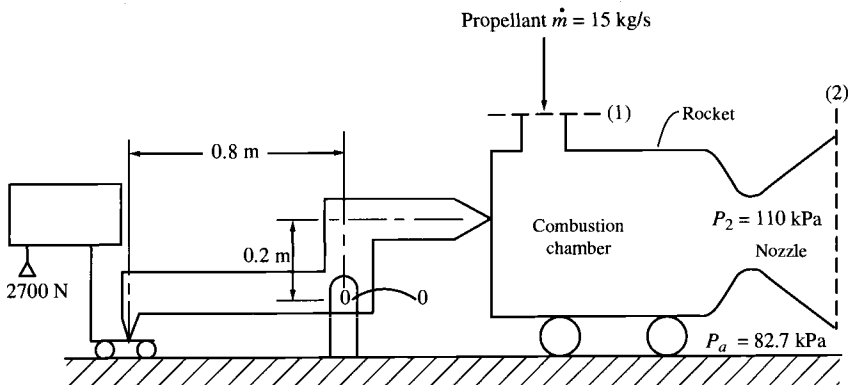


Fig. 2.9 Liquid-fuel rocket test setup.

Determine the exhaust velocity at section 2, assuming one-dimensional flow exists. Mechanical frictional effects may be neglected.

Solution: First, determine the force on the lever by the rocket to develop a 2700 N scale reading. This may be done by summing moments about the fulcrum point 0 (see Fig. 2.10a). Sum the horizontal forces on the rocket engine as shown in Fig. 2.10b. We note that the unbalanced pressure force on the exterior of the rocket engine is $P_a A_2$, and the interior forces (pressure and friction) are contained within the force $F_{C\sigma}$. Next, draw an internal volume σ around

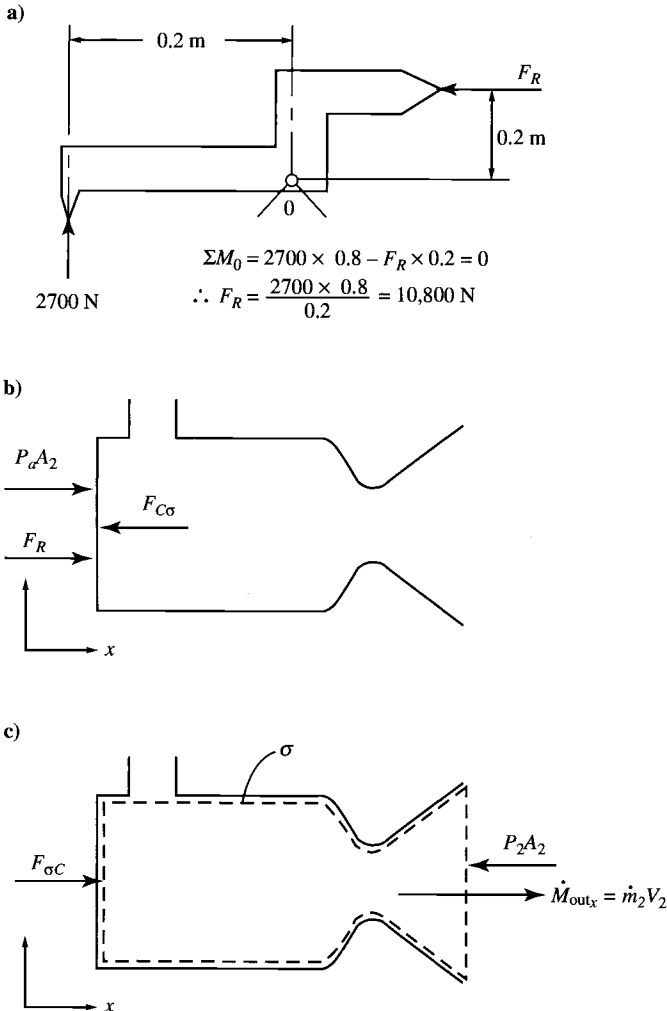


Fig. 2.10 Momentum equation sketch $\Sigma F_\sigma = (\dot{M}_2 - \dot{M}_1)/g_c$.

the fluid within the rocket engine as shown, and indicate the horizontal forces and momentum flux (see Fig. 2.10c).

Summing the forces on the rocket engine as shown in Fig. 2.10b, we obtain

$$F_R + P_a A_2 = F_{C\sigma}$$

Applying the momentum equation to the control volume σ shown in Fig. 2.10c, we obtain

$$F_{\sigma C} - P_2 A_2 = \frac{\dot{m}_2 V_2}{g_c}$$

Combining these two equations to remove $F_{\sigma C}$ gives

$$F_R - (P_2 - P_a)A_2 = \frac{\dot{m}_2 V_2}{g_c}$$

which is the same as Eq. (1.52). Because the flow is steady, the continuity equation yields $\dot{m}_1 = \dot{m}_2 = 15 \text{ kg/s}$. Therefore,

$$\begin{aligned} V_2 &= \frac{F_R - (P_2 - P_a)A_2}{\dot{m}_2/g_c} \\ &= \frac{10,800 - [(110 - 82.7) \times 10^3 \text{ N/m}^2](0.064 \text{ m}^2)}{15 \text{ kg/s}} \\ &= 603.5 \text{ m/s} \end{aligned}$$

2.6 Perfect Gas

2.6.1 General Characteristics

The thermodynamic equations of state for a perfect gas are

$$P = \rho RT \quad (2.21)$$

$$e = e(T) \quad (2.22)$$

where P is the thermodynamic pressure, ρ is the density, R is the gas constant, T is the thermodynamic temperature, and e is the internal energy per unit mass and a function of temperature only. The gas constant R is related to the universal gas constant \mathcal{R} and the molecular weight of the gas \mathcal{M} by

$$R = \frac{\mathcal{R}}{\mathcal{M}} \quad (2.23)$$

Values of the gas constant and molecular weight for typical gases are presented in Table 2.1 in several unit systems; $\mathcal{R} = 8.31434 \text{ kJ}/(\text{kmol} \cdot \text{K}) = 1.98718 \text{ Btu}/(\text{mol} \cdot ^\circ\text{R})$.

Table 2.1 Properties of ideal gases at 298.15 K (536.67°R)

Gas	Molecular weight	c_p , kJ/(kg · K)	c_p , Btu/(lbm · °R)	R , kJ/(kg · K)	R , (ft · lbm)/ (lbm · °R)	γ
Air	28.97	1.004	0.240	0.286	53.34	1.40
Argon	39.94	0.523	0.125	0.208	38.69	1.67
Carbon dioxide	44.01	0.845	0.202	0.189	35.1	1.29
Carbon monoxide	28.01	1.042	0.249	0.297	55.17	1.40
Hydrogen	2.016	14.32	3.42	4.124	766.5	1.40
Nitrogen	28.02	1.038	0.248	0.296	55.15	1.40
Oxygen	32.00	0.917	0.217	0.260	48.29	1.39
Sulfur dioxide	64.07	0.644	0.154	0.130	24.1	1.25
Water vapor	18.016	1.867	0.446	0.461	85.78	1.33

From the definition of enthalpy per unit mass h of a substance [Eq. (2.3)], this simplifies for a perfect gas to

$$h = e + RT \quad (2.24)$$

Equations (2.24) and (2.3) combined show that the enthalpy per unit mass is also only a function of temperature $h = h(T)$. Differentiating Eq. (2.24) gives

$$dh = de + R dT \quad (2.25)$$

The differentials dh and de in Eq. (2.25) are related to the specific heat at constant pressure and specific heat at constant volume [see definitions in Eqs. (2.4) and (2.5)], respectively, as follows:

$$dh = c_p dT$$

$$de = c_v dT$$

Note that both specific heats can be functions of temperature. These equations can be integrated from state 1 to state 2 to give

$$e_2 - e_1 = \int_{T_1}^{T_2} c_v dT \quad (2.26)$$

$$h_2 - h_1 = \int_{T_1}^{T_2} c_p dT \quad (2.27)$$

Substitution of the equations for dh and de into Eq. (2.25) gives the relationship between specific heats for a perfect gas

$$c_p = c_v + R \quad (2.28)$$

and γ is the ratio of the specific heat at constant pressure to the specific heat at constant volume, or, as in Eq. (2.6),

$$\gamma \equiv c_p/c_v$$

The following relationships result from using Eqs. (2.28) and (2.6):

$$\frac{R}{c_v} = \gamma - 1 \quad (2.29)$$

$$\frac{R}{c_p} = \frac{\gamma - 1}{\gamma} \quad (2.30)$$

The Gibbs equation relates the entropy per unit mass s to the other thermodynamic properties of a substance. It can be written as

$$ds = \frac{de + P d(1/\rho)}{T} = \frac{dh - (1/\rho) dP}{T} \quad (2.31)$$

For a perfect gas, the Gibbs equation can be written simply as

$$ds = c_v \frac{dT}{T} + R \frac{d(1/\rho)}{1/\rho} \quad (2.32)$$

$$ds = c_p \frac{dT}{T} - R \frac{dP}{P} \quad (2.33)$$

These equations can be integrated between states 1 and 2 to yield the following expressions for the change in entropy $s_2 - s_1$:

$$s_2 - s_1 = \int_{T_1}^{T_2} c_v \frac{dT}{T} + R \ell n \frac{\rho_1}{\rho_2} \quad (2.34)$$

$$s_2 - s_1 = \int_{T_1}^{T_2} c_p \frac{dT}{T} - R \ell n \frac{P_2}{P_1} \quad (2.35)$$

If the specific heats are known functions of temperature for a perfect gas, then Eqs. (2.26), (2.27), (2.34), and (2.35) can be integrated from a reference state and tabulated for further use in what are called *gas tables*.

The equation for the speed of sound in a perfect gas is easily obtained by use of Eqs. (2.7) and (2.21) to give

$$a = \sqrt{\gamma R g_c T} \quad (2.36)$$

2.6.2 Calorically Perfect Gas

A *calorically perfect gas* is a perfect gas with constant specific heats (c_p and c_v). In this case, the expressions for changes in internal energy e , enthalpy h , and

entropy s simplify to the following:

$$e_2 - e_1 = c_v(T_2 - T_1) \quad (2.37)$$

$$h_2 - h_1 = c_p(T_2 - T_1) \quad (2.38)$$

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} \quad (2.39)$$

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1} \quad (2.40)$$

Equations (2.39) and (2.40) can be rearranged to give the following equations for the temperature ratio T_2/T_1 :

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1} \right)^{R/c_v} \exp \frac{s_2 - s_1}{c_v}$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1} \right)^{R/c_p} \exp \frac{s_2 - s_1}{c_p}$$

From Eqs. (2.29) and (2.30), these expressions become

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma-1} \exp \frac{s_2 - s_1}{c_v} \quad (2.41)$$

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1} \right)^{(\gamma-1)/\gamma} \exp \frac{s_2 - s_1}{c_p} \quad (2.42)$$

2.6.3 Isentropic Process

For an isentropic process ($s_2 = s_1$), Eqs. (2.41), (2.42), and (2.21) yield the following equations:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1} \right)^{(\gamma-1)/\gamma} \quad (2.43)$$

$$\frac{T_2}{T_1} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma-1} \quad (2.44)$$

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma} \quad (2.45)$$

Note that Eqs. (2.43), (2.44), and (2.45) apply only to a calorically perfect gas undergoing an isentropic process.

Example 2.5

Air initially at 20°C and 1 atm is compressed reversibly and adiabatically to a final pressure of 15 atm. Find the final temperature.

Solution: Because the process is isentropic from initial to final state, Eq. (2.43) can be used to solve for the final temperature. The ratio of specific heats for air is 1.4:

$$\begin{aligned} T_2 &= T_1 \left(\frac{P_2}{P_1} \right)^{(\gamma-1)/\gamma} = (20 + 273.15) \left(\frac{15}{1} \right)^{0.4/1.4} \\ &= 293.15 \times 2.1678 = 635.49 \text{ K } (362.34^\circ\text{C}) \end{aligned}$$

Example 2.6

Air is expanded isentropically through a nozzle from $T_1 = 3000^\circ\text{R}$, $V_1 = 0$, and $P_1 = 10 \text{ atm}$ to $V_2 = 3000 \text{ ft/s}$. Find the exit temperature and pressure.

Solution: Application of the first law of thermodynamics to the nozzle gives the following for a calorically perfect gas:

$$c_p T_1 + \frac{V_1^2}{2g_c} = c_p T_2 + \frac{V_2^2}{2g_c}$$

This equation can be rearranged to give T_2 :

$$\begin{aligned} T_2 &= T_1 - \frac{V_2^2 - V_1^2}{2g_c c_p} = 3000 - \frac{3000^2}{2 \times 32.174 \times 0.240 \times 778.16} \\ &= 3000 - 748.9 = 2251.1^\circ\text{R} \end{aligned}$$

Solving Eq. (2.43) for P_2 gives

$$P_2 = P_1 \left(\frac{T_2}{T_1} \right)^{\gamma/(\gamma-1)} = 10 \left(\frac{2251.1}{3000} \right)^{3.5} = 3.66 \text{ atm}$$

2.6.4 Mollier Diagram for a Perfect Gas

The Mollier diagram is a thermodynamic state diagram with the coordinates of enthalpy and entropy s . Because the enthalpy of a perfect gas depends on temperature alone,

$$dh = c_p dT$$

temperature can replace enthalpy as the coordinate of a Mollier diagram for a perfect gas. When temperature T and entropy s are the coordinates of a Mollier diagram, we call it a T - s diagram. We can construct lines of constant pressure and density in the T - s diagram by using Eqs. (2.34) and (2.35). For a calorically

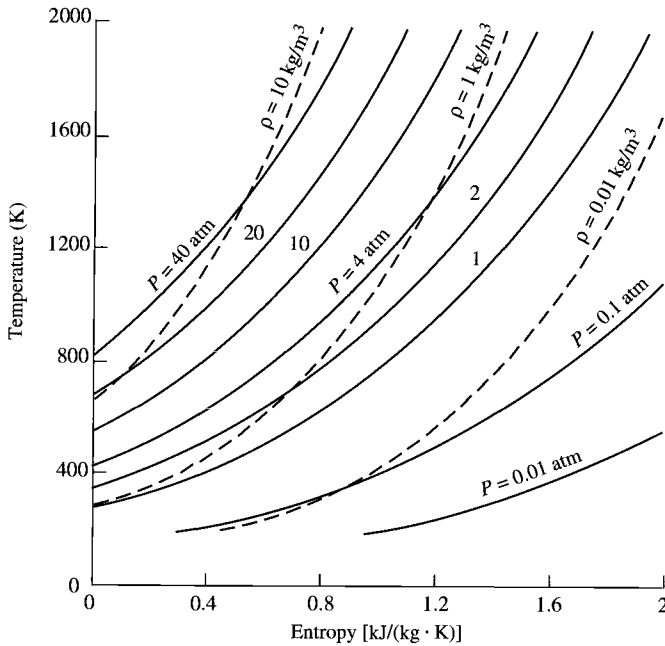


Fig. 2.11 A T - s diagram for air as a calorically perfect gas.

perfect gas, Eqs. (2.39) and (2.40) can be written between any state and the entropy reference state ($s = 0$) as

$$s = c_v \ln \frac{T}{T_{\text{ref}}} - R \ln \frac{\rho}{\rho_{\text{ref}}}$$

$$s = c_p \ln \frac{T}{T_{\text{ref}}} - R \ln \frac{P}{P_{\text{ref}}}$$

where T_{ref} , P_{ref} , and ρ_{ref} are the values of temperature, pressure, and density, respectively, when $s = 0$. Because the most common working fluid in gas turbine engines is air, Fig. 2.11 has been constructed for air by using the preceding equations with these data:

$$c_p = 1.004 \text{ kJ}/(\text{kg} \cdot \text{K}), \quad T_{\text{ref}} = 288.2 \text{ K} \quad \rho_{\text{ref}} = 1.225 \text{ kg}/\text{m}^3$$

$$R = 0.286 \text{ kJ}/(\text{kg} \cdot \text{K}), \quad P_{\text{ref}} = 1 \text{ atm} = 101,325 \text{ Pa}$$

2.6.5 Mixtures of Perfect Gases

We consider a mixture of perfect gases, each obeying the perfect gas equation:

$$PV = NRT$$

where N is the number of moles and \mathcal{R} is the universal gas constant. The mixture is idealized as independent perfect gases, each having the temperature T and occupying the volume V . The partial pressure of gas i is

$$P_i = N_i \mathcal{R} \frac{T}{V}$$

According to the Gibbs–Dalton law, the pressure of the gas mixture of n constituents is the sum of the partial pressures of each constituent:

$$P = \sum_{i=1}^n P_i \quad (2.46)$$

The total number of moles N of the gas is

$$N = \sum_{i=1}^n N_i \quad (2.47)$$

The ratio of the number of moles of constituent i to the total number of moles in the mixture is called the *mole fraction* χ_i . By using the preceding equations, the mole fraction of constituent i can be shown to equal the ratio of the partial pressure of constituent i to the pressure of the mixture:

$$\chi_i = \frac{N_i}{N} = \frac{P_i}{P} \quad (2.48)$$

The Gibbs–Dalton law also states that the internal energy, enthalpy, and entropy of a mixture are equal, respectively, to the sum of the internal energies, the enthalpies, and the entropies of the constituents when each alone occupies the volume of the mixture at the mixture temperature. Thus we can write the following for a mixture of n constituents.

Energy:

$$E = \sum_{i=1}^n E_i = \sum_{i=1}^n m_i e_i \quad (2.49)$$

Enthalpy:

$$H = \sum_{i=1}^n H_i = \sum_{i=1}^n m_i h_i \quad (2.50)$$

Entropy:

$$S = \sum_{i=1}^n S_i = \sum_{i=1}^n m_i s_i \quad (2.51)$$

where m_i is the mass of constituent i .

The specific heats of the mixture follow directly from the definitions of c_p and c_v and the preceding equations. For a mixture of n constituents, the specific heats are

$$c_p = \frac{\sum_{i=1}^n m_i c_{pi}}{m} \quad \text{and} \quad c_v = \frac{\sum_{i=1}^n m_i c_{vi}}{m} \quad (2.52)$$

where m is the total mass of the mixture.

2.6.6 Gas Tables

In the case of a perfect gas with variable specific heats, the specific heat at constant pressure c_p is normally modeled by several terms of a power series in temperature T . This expression is used in conjunction with the general equations presented and the new equations that are developed next to generate a gas table for a particular gas (see Ref. 15).

For convenience, we define

$$h \equiv \int_{T_{\text{ref}}}^T c_p \, dT \quad (2.53)$$

$$\phi \equiv \int_{T_{\text{ref}}}^T c_p \frac{dT}{T} \quad (2.54)$$

$$P_r \equiv \exp \frac{\phi - \phi_{\text{ref}}}{R} \quad (2.55)$$

$$v_r \equiv \exp \left(-\frac{1}{R} \int_{T_{\text{ref}}}^T c_v \frac{dT}{T} \right) \quad (2.56)$$

where P_r and v_r are called the *reduced pressure* and *reduced volume*, respectively. Using the definition of ϕ from Eq. (2.54) in Eq. (2.35) gives

$$s_2 - s_1 = \phi_2 - \phi_1 - R \ln \frac{P_2}{P_1} \quad (2.57)$$

For an isentropic process between states 1 and 2, Eq. (2.57) reduces to

$$\phi_2 - \phi_1 = R \ln \frac{P_2}{P_1}$$

which can be rewritten as

$$\left(\frac{P_2}{P_1} \right)_{s=\text{const}} = \exp \frac{\phi_2 - \phi_1}{R} = \frac{\exp(\phi_2/R)}{\exp(\phi_1/R)}$$

Using Eq. (2.55), we can express this pressure ratio in terms of the reduced pressure P_r as

$$\left(\frac{P_2}{P_1}\right)_{s=\text{const}} = \frac{P_{r2}}{P_{r1}} \quad (2.58)$$

Likewise, it can be shown that

$$\left(\frac{v_2}{v_1}\right)_{s=\text{const}} = \frac{v_{r2}}{v_{r1}} \quad (2.59)$$

For a perfect gas, the properties h , P_r , e , v_r , and ϕ are functions of T , and these can be calculated by starting with a polynomial for c_p . Say we have the seventh-order polynomial

$$c_p = A_0 + A_1 T + A_2 T^2 + A_3 T^3 + A_4 T^4 + A_5 T^5 + A_6 T^6 + A_7 T^7 \quad (2.60)$$

The equations for h and ϕ as functions of temperature follow directly from using Eqs. (2.53) and (2.54):

$$h = h_{\text{ref}} + A_0 T + \frac{A_1}{2} T^2 + \frac{A_2}{3} T^3 + \frac{A_3}{4} T^4 + \frac{A_4}{5} T^5 + \frac{A_5}{6} T^6 + \frac{A_6}{7} T^7 + \frac{A_7}{8} T^8 \quad (2.61)$$

$$\phi = \phi_{\text{ref}} + A_0 \ln T + A_1 T + \frac{A_2}{2} T^2 + \frac{A_3}{3} T^3 + \frac{A_4}{4} T^4 + \frac{A_5}{5} T^5 + \frac{A_6}{6} T^6 + \frac{A_7}{7} T^7 \quad (2.62)$$

After we define reference values, the variations of P_r and v_r follow from Eqs. (2.55) and (2.56), and the preceding equations.

2.6.7 Comment for Appendix D

Typically, air flows through the inlet and compressor of the gas turbine engine whereas products of combustion flow through the engine components downstream of a combustion process. Most gas turbine engines use hydrocarbon fuels of composition $(\text{CH}_2)_n$. We can use the preceding equations to estimate the properties of these gases, given the ratio of the mass of fuel burned to the mass of air. For convenience, we use the fuel/air ratio f , defined as

$$f = \frac{\text{mass of fuel}}{\text{mass of air}} \quad (2.63)$$

The maximum value of f is 0.0676 for the hydrocarbon fuels of composition $(\text{CH}_2)_n$.

Given the values of c_p , h , and ϕ for air and the values of combustion products, the values of c_p , h , and ϕ for the mixture follow directly from the mixture

Table 2.2 Constants for air and combustion products used in Appendix D and program AFRPROP (from Ref. 16)

Air alone		Combustion products of air and $(\text{CH}_2)_n$ fuels	
A_0	2.5020051×10^{-1}	A_0	7.3816638×10^{-2}
A_1	$-5.1536879 \times 10^{-5}$	A_1	1.2258630×10^{-3}
A_2	6.5519486×10^{-8}	A_2	$-1.3771901 \times 10^{-6}$
A_3	$-6.7178376 \times 10^{-12}$	A_3	$9.9686793 \times 10^{-10}$
A_4	$-1.5128259 \times 10^{-14}$	A_4	$-4.2051104 \times 10^{-13}$
A_5	$7.6215767 \times 10^{-18}$	A_5	$1.0212913 \times 10^{-16}$
A_6	$-1.4526770 \times 10^{-21}$	A_6	$-1.3335668 \times 10^{-20}$
A_7	$1.0115540 \times 10^{-25}$	A_7	$7.2678710 \times 10^{-25}$
h_{ref}	$-1.7558886 \text{ Btu/lbm}$	h_{ref}	30.58153 Btu/lbm
ϕ_{ref}	$0.0454323 \text{ Btu}/(\text{lbm} \cdot ^\circ\text{R})$	ϕ_{ref}	$0.6483398 \text{ Btu}/(\text{lbm} \cdot ^\circ\text{R})$

equations [Eqs. (2.49)–(2.52)] and are given by

$$R = \frac{1.9857117 \text{ Btu}/(\text{lbm} \cdot ^\circ\text{R})}{28.97 - f \times 0.946186} \quad (2.64a)$$

$$c_p = \frac{c_{p \text{ air}} + f c_{p \text{ prod}}}{1 + f} \quad (2.64b)$$

$$h = \frac{h_{\text{air}} + f h_{\text{prod}}}{1 + f} \quad (2.64c)$$

$$\phi = \frac{\phi_{\text{air}} + f \phi_{\text{prod}}}{1 + f} \quad (2.64d)$$

Appendix D is a table of the properties h and P_r as functions of the temperature and fuel/air ratio f for air and combustion products [air with hydrocarbon fuels of composition $(\text{CH}_2)_n$] at low pressure (perfect gas). These data are based on the preceding equations and the constants given in Table 2.2, which are valid over the temperature range of 300–4000°R. These constants come from the gas turbine engine modeling work of Capt. John S. McKinney (U.S. Air Force) while assigned to the Air Force's Aero Propulsion Laboratory,¹⁶ and they continue to be widely used in the industry. Appendix D uses a reference value of 2 for P_r at 600°R and $f = 0$.

2.6.8 Comment for Computer Program AFRPROP

The computer program AFRPROP was written by using the preceding constants for air and products of combustion from air with $(\text{CH}_2)_n$. The program can calculate the four primary thermodynamic properties at a state (P , T , h , and s) given the fuel/air ratio f and two independent thermodynamic properties (say, P and h).

To show the use of the gas tables, we will resolve Examples 2.5 and 2.6, using the gas tables of Appendix D. These problems could also be solved by using the computer program AFPROP.

Example 2.7

Air initially at 20°C and 1 atm is compressed reversibly and adiabatically to a final pressure of 15 atm. Find the final temperature.

Solution: Since the process is isentropic from initial to final state, Eq. (2.58) can be used to solve for the final reduced pressure. From Appendix D at 20°C (293.15 K) and $f = 0$, $P_r = 1.2768$ and

$$\frac{P_{r2}}{P_{r1}} = \frac{P_2}{P_1} = 15$$

$$P_{r2} = 15 \times 1.2768 = 19.152$$

From Appendix D for $P_{r2} = 19.152$, the final temperature is 354.42°C (627.57 K). This is 7.9 K lower than the result obtained in Example 2.5 for air as a calorically perfect gas.

Example 2.8

Air is expanded isentropically through a nozzle from $T_1 = 3000^\circ\text{R}$, $V_1 = 0$, and $P_1 = 10$ atm to $V_2 = 3000$ ft/s. Find the exit temperature and pressure.

Solution: Application of the first law of thermodynamics to the nozzle gives the following for a calorically perfect gas:

$$h_1 + \frac{V_1^2}{2g_c} = h_2 + \frac{V_2^2}{2g_c}$$

From Appendix D at $f = 0$ and $T_1 = 3000^\circ\text{R}$, $h_1 = 790.46$ Btu/lbm and $P_{r1} = 938.6$. Solving the preceding equation for h_2 gives

$$h_2 = h_1 - \frac{V_2^2 - V_1^2}{2g_c} = 790.46 - \frac{3000^2}{2 \times 32.174 \times 778.16}$$

$$= 790.46 - 179.74 = 610.72 \text{ Btu/lbm}$$

For $h = 610.72$ Btu/lbm and $f = 0$, Appendix D gives $T_2 = 2377.7^\circ\text{R}$ and $P_{r2} = 352.6$. Using Eq. (2.58), we solve for the exit pressure

$$P_2 = P_1 \frac{P_{r2}}{P_{r1}} = 10 \left(\frac{352.6}{938.6} \right) = 3.757 \text{ atm}$$

These results for temperature and pressure at station 2 are higher by 126.6°R and 0.097 atm, respectively, than those obtained in Example 2.6 for air as a calorically perfect gas.

2.7 Compressible Flow Properties

For a simple compressible system, we learned that the state of a unit mass of gas is fixed by two independent intensive properties such as pressure and temperature. To fully describe the condition and thus fix the state of this same gas when it is in motion requires the specification of an additional property that will fix the speed of the gas. Thus three independent intensive properties are required to fully specify the state of a gas in motion.

At any given point in a compressible fluid flowfield, the thermodynamic state of the gas is fixed by specifying the velocity of the gas and any two independent thermodynamic properties such as pressure and temperature. However, we find that to specify the velocity directly is not always the most useful or the most convenient way to describe one-dimensional flow. There are other properties of the gas in motion that are dependent on the speed of the gas and that may be used in place of the speed to describe the state of the flowing gas. Some of these properties are the Mach number, total pressure, and total temperature. In this section we define these properties and describe briefly some of the characteristics of compressible flow.

2.7.1 Total Enthalpy and Total Temperature

The steady flow energy equation in the absence of gravity effects is

$$q - w_x = \left(h + \frac{V^2}{2g_c} \right)_{\text{out}} - \left(h + \frac{V^2}{2g_c} \right)_{\text{in}}$$

For a calorically perfect gas this becomes

$$q - w_x = c_p \left(T + \frac{V^2}{2g_c c_p} \right)_{\text{out}} - c_p \left(T + \frac{V^2}{2g_c c_p} \right)_{\text{in}}$$

The quantities $h + V^2/(2g_c)$ and $T + V^2/(2g_c c_p)$ in these equations are called the *stagnation* or *total enthalpy* h_t and the *stagnation* or *total temperature* T_t , respectively.

Total enthalpy:

$$h_t \equiv h + \frac{V^2}{2g_c} \quad (2.65)$$

Total temperature:

$$T_t \equiv T + \frac{V^2}{2g_c c_p} \quad (2.66)$$

The temperature T is sometimes called the *static temperature* to distinguish it from the total temperature T_t . When $V = 0$, the static and total temperatures are identical. From these definitions, it follows that, for a calorically perfect gas,

$$\Delta h_t = c_p \Delta T_t$$

Using these new definitions, we see that the steady flow energy equation in the absence of gravity effects becomes

$$q - w_x = h_{t\text{out}} - h_{t\text{in}} \quad (2.67)$$

or, for a calorically perfect gas,

$$q - w_x = c_p(T_{t\text{out}} - T_{t\text{in}}) \quad (2.68)$$

If $q - w_x = 0$, we see from Eqs. (2.67) and (2.68) that $h_{t\text{out}} = h_{t\text{in}}$ and that, for a calorically perfect gas, $T_{t\text{out}} = T_{t\text{in}}$.

Consider an airplane in flight at a velocity V_a . To an observer riding with the airplane, the airflow about the wing of the plane appears as in Fig. 2.12. We mark out a control volume σ as shown in the figure between a station far upstream from the wing and a station just adjacent to the wing's leading edge stagnation point, where the velocity of the airstream is reduced to a negligibly small magnitude.

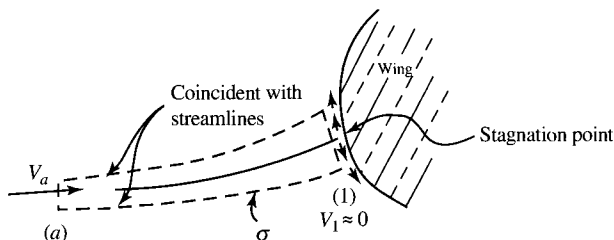
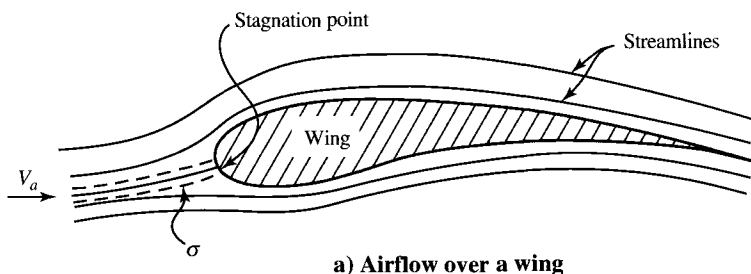


Fig. 2.12 Control volume σ with freestream inlet and stagnation exit conditions (reference system at rest relative to wing).

Applying the steady flow energy equation to the flow through σ of Fig. 2.12, we have

$$q - w_x = \left(h + \frac{V^2}{2g_c} \right)_1^0 - \left(h + \frac{V^2}{2g_c} \right)_a$$

or

$$0 = c_p T_1 - c_p \left(T + \frac{V^2}{2g_c c_p} \right)_a$$

From this equation, we find that the temperature of the air at the stagnation point of the wing is

$$T_1 = T_a + \frac{V_a^2}{2g_c c_p} = T_{ta}$$

Thus we see that the temperature, which the leading edge of the wing “feels,” is the total temperature T_{ta} .

At high flight speeds, the freestream total temperature T_{ta} is significantly different from the freestream ambient temperature T_a . This is illustrated in Fig. 2.13, where $T_{ta} - T_a$ is plotted against V_a by using the relation

$$(T_t - T)_a = \frac{V_a^2}{2g_c c_p} = \frac{V_a^2}{12,000} \approx \left(\frac{V_a}{110} \right)^2 {}^\circ\text{R}$$

with V_a expressed in ft/s. Because the speed of sound at 25,000 ft is 1000 ft/s, a Mach number scale for 25,000 ft is easily obtained by dividing the scale for V_a in Fig. 2.13 by 1000. (Mach number M equals V_a divided by the local speed of sound a .) Therefore, Mach number scales are also given on the graphs.

Referring to Fig. 2.13, we find that at a flight speed of 800 ft/s corresponding to a Mach number of 0.8 at 25,000-ft altitude, the stagnation points on an airplane experience a temperature that is about 50°R higher than ambient temperature. At 3300 ft/s ($M = 3.3$ at 25,000 ft), the total temperature is 900°R higher than ambient! It should be evident from these numbers that vehicles such as the X-15 airplane and reentry bodies experience high temperatures at their high flight speeds.

These high temperatures are produced as the kinetic energy of the air impinging on the surfaces of a vehicle is reduced and the enthalpy (hence, temperature) of the air is increased a like amount. This follows directly from the steady flow energy equation, which gives, with $q = w_x = 0$,

$$\left(h + \frac{V^2}{2g_c} \right)_{\text{in}} = \left(h + \frac{V^2}{2g_c} \right)_{\text{out}}$$

or

$$\Delta h = c_p \Delta T = -\Delta \left(\frac{V^2}{2g_c} \right)$$

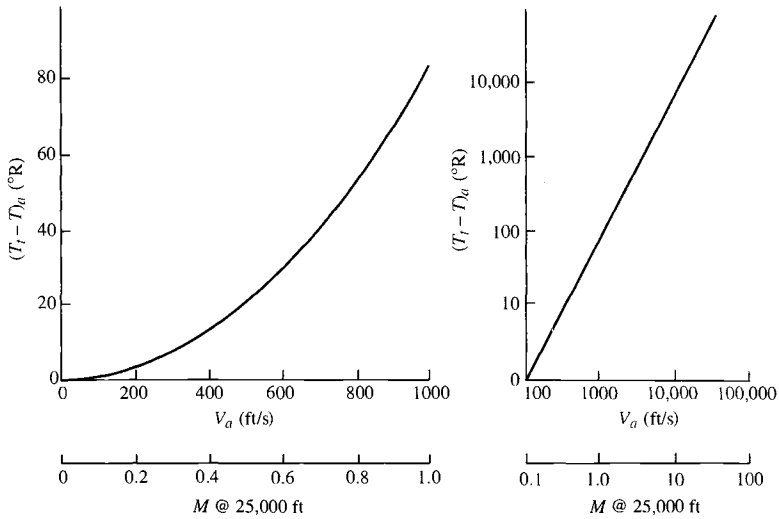


Fig. 2.13 Total temperature minus ambient temperature vs flight speed and vs flight Mach number at 25,000 ft [$g_c c_p$ is assumed constant at $6000 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R})$; therefore, these are approximate curves].

and

$$\Delta T = -\frac{\Delta V^2}{2g_c c_p}$$

Thus a decrease in the kinetic energy of air produces a rise in the air temperature and a consequent heat interaction between the air and the surfaces of an air vehicle. This heat interaction effect is referred to as *aerodynamic heating*.

Example 2.9

The gas in a rocket combustion chamber is at 120 psia and 1600°R (Fig. 2.14). The gas expands through an adiabatic frictionless (isentropic) nozzle to 15 psia. What are the temperature and velocity of the gas leaving the nozzle? Treat the gas as the calorically perfect gas air with $\gamma = 1.4$ and $g_c c_p = 6000 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R})$.

Solution: Locate the state of the combustion chamber gas entering control volume σ on a T - s diagram like Fig. 2.11 in the manner depicted in Fig. 2.15. Then, from the diagram, find s_1 . Because the process is isentropic, $s_2 = s_1$. The entropy at 2, along with the known value of P_2 , fixes the static state of 2. With 2 located in the T - s diagram, we can read T_2 from the temperature scale as 885°R , and we can verify this graphical solution for T_2 by using the isentropic relation (2.43) with $\gamma = 1.4$. Thus $T_2 = (1600^\circ\text{R})(15/120)^{0.286} = 885^\circ\text{R}$ (checks).

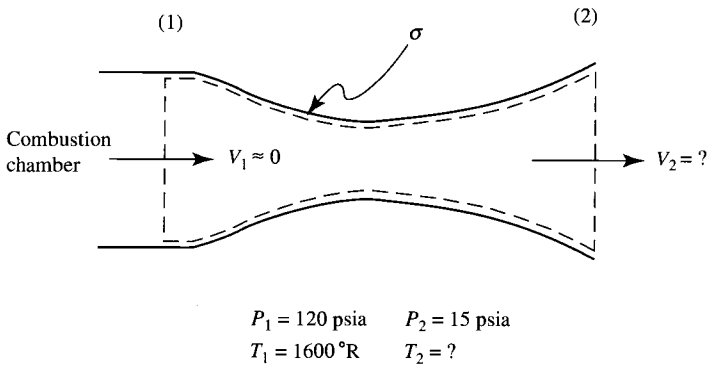


Fig. 2.14 Rocket exhaust nozzle.

If, in addition to P_2 and T_2 , the total temperature T_{t2} of the flowing gas at 2 is known, then the state of the gas at 2 is completely fixed. For with P_2 and T_2 specified, the values of all thermodynamic properties independent of speed (the *static* properties) are fixed, and the speed of the gas is determined by T_2 and T_{t2} .

From the steady flow energy equation (Fig. 2.15), we find that $T_{t2} = T_{t1} = T_1$ and, hence, V_2 from the relation

$$\frac{V_2^2}{2g_c c_p} = T_{t2} - T_2$$

We see from this equation that the vertical distance $T_{t2} - T_2$ in the T - s diagram is indicative of the speed of the gas at 2, which is $V_2^2 = 2(6000)(1600 - 885)\text{ft}^2/\text{s}^2$. Thus $V_2 = 2930 \text{ ft/s}$.

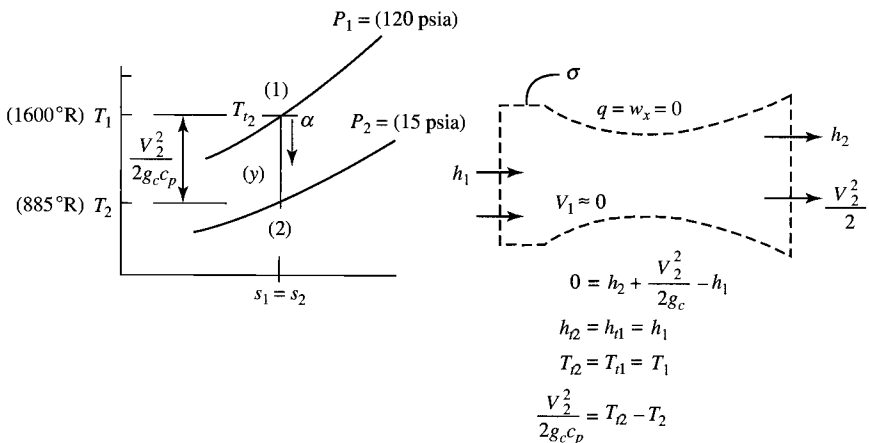


Fig. 2.15 Process plot for example rocket nozzle.

The series of states through which the gas progresses in the nozzle as it flows from the combustion chamber (nozzle inlet) to the nozzle exit is represented by path line α in the T - s diagram. The speed of the gas at any intermediate state y in the nozzle is represented by the vertical distance on the path line from 1 to the state in question. This follows from the relations

$$T_{ty} = T_1 \quad \text{and} \quad \frac{V_y^2}{2g_c c_p} = T_{ty} - T_y = T_1 - T_y$$

2.7.2 Stagnation or Total Pressure

In the adiabatic, no-shaft-work slowing of a flowing perfect gas to zero speed, the gas attains the same final stagnation temperature whether it is brought to rest through frictional effects (irreversible) or without them (reversible). This follows from the energy control volume equation applied to σ of Fig. 2.16 for a calorically perfect gas. Thus, from

$$q - w_x = c_p(T_y - T_1) + \frac{V_y^2 - V_1^2}{2g_c}$$

with $q = w_x = 0$ and $V_y = 0$, T_y becomes

$$T_y = T_0 = T_1 + \frac{V_1^2}{2g_c c_p}$$

Because the energy control volume equation is valid for frictional or frictionless flow, $T_y = T_0$ is constant and independent of the degree of friction between 1 and y as long as $q = w_x = V_y = 0$.

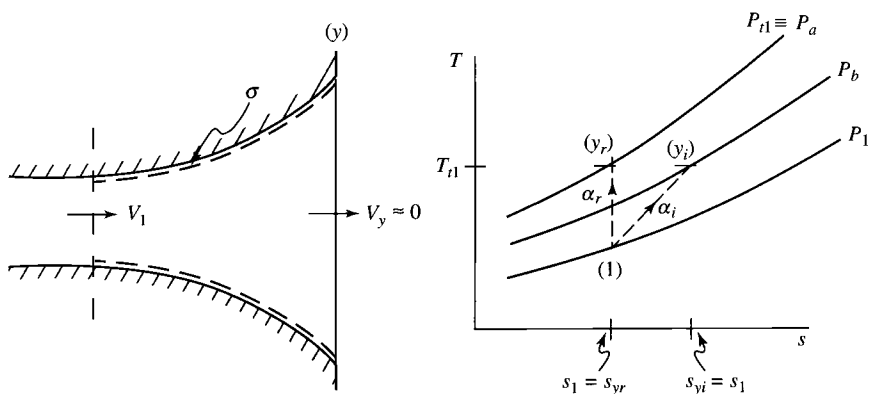


Fig. 2.16 Definition of total pressure.

Although the gas attains the same final temperature T_0 in reversible or irreversible processes, its final pressure will vary with the degree of irreversibility associated with the slowing down process. The entropy state and control volume equations for the flow through σ are

$$c_p \ln \frac{T_y}{T_1} - R \ln \frac{P_y}{P_1} = s_y - s_1 \geq 0 \quad (2.69)$$

Since $T_y = T_0 = \text{const}$ from Eq. (3.5), the final value of P_y depends on the entropy increase $s_y - s_1$, which in turn is a measure of the degree of irreversibility between 1 and y.

When the slowing down process between 1 and y is reversible, with $s_y - s_1 = 0$, the final pressure is defined as the total pressure P_t . The final state is called the *total state* t_1 of the static state 1. Using this definition of total pressure, we have, from Eq. (2.43),

$$P_t \equiv P \left(\frac{T_t}{T} \right)^{\gamma/(\gamma-1)} \quad (2.70)$$

These ideas are illustrated in the T - s diagram of Fig. 2.16. Let us imagine the flowing gas at station 1 to be brought to rest adiabatically with no shaft work by means of a duct diverging to an extremely large area at station y, where the flow velocity is zero. If the diverging duct is frictionless, then the slowing down process from 1 to y is isentropic with the path line α_r in the T - s diagram of the figure. If the diverging duct is frictional, then the slowing down process from 1 to y is irreversible and adiabatic ($s_{yi} > s_1$) to satisfy the entropy control volume equation for adiabatic flow and is shown as the path line α_i in the T - s diagram.

The total pressure of a flowing gas is defined as the pressure obtained when the gas is brought to rest isentropically. Thus the pressure corresponding to state y_r of the T - s diagram is the total pressure of the gas in state 1. The state point y_r is called the *total or stagnation state* t_1 of the static point 1.

The concepts of total pressure and total temperature are very useful, for these two properties along with the third property (static pressure) of a flowing gas are readily measured, and they fix the state of the flowing gas. We measure these three properties in flight with pitot-static and total temperature probes on modern high-speed airplanes, and these properties are used to determine speed and Mach number and to provide other data for many aircraft subsystems.

Consider a gas flowing in a duct in which P and T may change due to heat interaction and friction effects. The flow total state points t_1 and t_2 and the static state points 1 and 2, each of which corresponds to flow stations 1 and 2, are located in the T - s diagram of Fig. 2.17. By definition, the entropy of the total state at any given point in a gas flow has the same value as the entropy of the static state properties at that point. Therefore, $s_{t1} = s_1$ and $s_{t2} = s_2$.

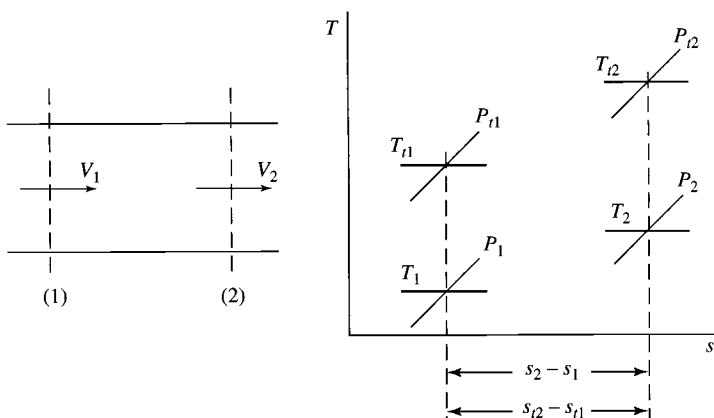


Fig. 2.17 Entropy change in terms of the stagnation properties T_t and P_t .

From the entropy equation of state of a perfect gas, the entropy change between 1 and 2 is

$$s_2 - s_1 = c_p \ln \frac{T_2}{T_1} - R \ln \frac{P_2}{P_1}$$

The entropy change between total state points t_1 and t_2 is

$$s_{t2} - s_{t1} = c_p \ln \frac{T_{t2}}{T_{t1}} - R \ln \frac{P_{t2}}{P_{t1}} \quad (2.71a)$$

Since $s_{t1} = s_1$ and $s_{t2} = s_2$, we have

$$s_{t2} - s_{t1} = s_2 - s_1$$

Therefore, the change of entropy between two states of a flowing gas can be determined by using total properties in place of static properties.

Equation (2.71a) indicates that in an adiabatic and no-shaft-work constant- T_t flow (such as exists in an airplane engine inlet and nozzle, or flow through a shock wave), we have

$$s_2 - s_1 = -R \ln \frac{P_{t2}}{P_{t1}} \quad (2.71b)$$

By virtue of this equation and the entropy control volume equation for adiabatic flow, $s_2 - s_1 \geq 0$. Thus, in a constant- T_t flow,

$$\frac{P_{t2}}{P_{t1}} \leq 1$$

Hence the total pressure of air passing through an engine inlet and nozzle or a shock wave cannot increase and must, in fact, decrease because of the irreversible effects of friction.

2.7.3 T/T_t and P/P_t as Functions of Mach Number

The speed of sound a in a perfect gas is given by

$$a = \sqrt{\gamma g_c R T}$$

Using this relation for the speed of sound and Eq. (2.8), we can write the Mach number as

$$M^2 = \frac{V^2}{\gamma g_c R T}$$

With the help of this expression for the Mach number, we can obtain many useful relations that give gas flow property ratios in terms of the flow Mach number alone. Two such relations for T/T_t and P/P_t are

$$\frac{T}{T_t} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1} \quad (2.72)$$

$$\frac{P}{P_t} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\gamma/(\gamma - 1)} \quad (2.73)$$

Equations (2.71) and (2.72) appear graphically in Fig. 2.18 and are given in the isentropic flow functions of the GASTAB program for user input γ value. These equations show that for each freestream Mach number (hence, for each flight Mach number of an airplane), the ratios P/P_t and T/T_t have unique values.

Both Fig. 2.18 and the corresponding equations provide Mach numbers and ambient temperatures for known values of P , P_t , and T_t . For example, we are given the following in-flight measurements:

$$P = 35.4 \text{ kPa} \quad T_t = 300 \text{ K} \quad P_t = 60.0 \text{ kPa}$$

From these data, $P/P_t = 0.59$. If we enter Fig. 2.18 with this value of P/P_t , we find $M = 0.9$ and $T/T_t = 0.86$. Then we obtain the ambient temperature by using $T = (T/T_t) T_t = 0.86(300) = 258 \text{ K}$.

Figure 2.18 shows that in a sonic ($M = 1.0$) stream of gas with $\gamma = 1.4$,

$$\frac{P}{P_t} = 0.528$$

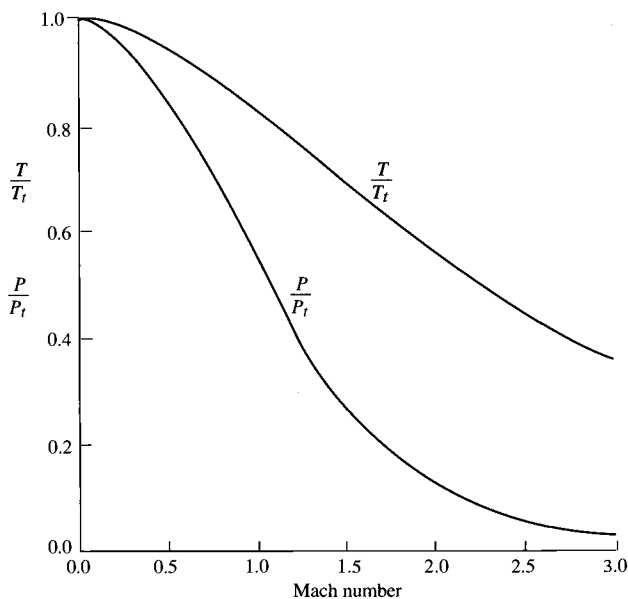


Fig. 2.18 P/P_t and T/T_t vs Mach number ($\gamma = 1.4$).

and

$$\frac{T}{T_t} = 0.833$$

and for supersonic flow,

$$\frac{P}{P_t} < 0.528$$

and

$$\frac{T}{T_t} < 0.833$$

Consider the one-dimensional steady flow of a gas in a duct with T_t and P_t constant at all stations along the duct. This means a total temperature probe will measure the same value of T_t at each duct station, and an isentropic total pressure probe will measure the same value of P_t at each station. The path line of α of the flow is a vertical line in the T - s diagram. The state points on the path line can be categorized as follows.

Subsonic:

$$T > 0.883 T_t \quad P > 0.528 P_t$$

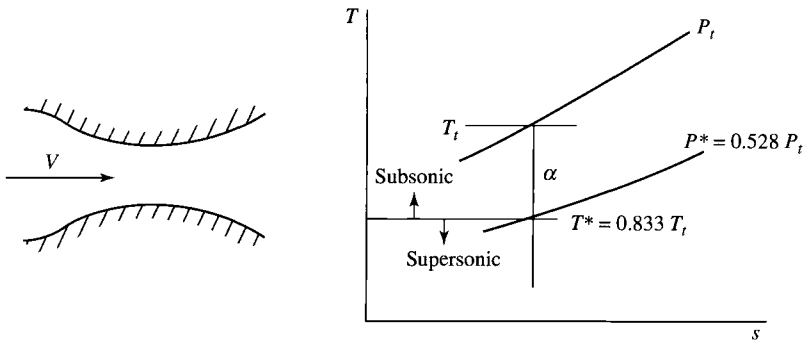


Fig. 2.19 Subsonic and supersonic state points in an isentropic flow.

Sonic:

$$T = 0.833 T_t \quad P = 0.528 P_t$$

Supersonic:

$$T < 0.833 T_t \quad P < 0.528 P_t$$

Figure 2.19 delineates the subsonic, sonic, and supersonic portions of path line α .

The thermodynamic properties at the state point where $M = 1$ on α are denoted by P^* , T^* , V^* , etc. (read as P star, etc.) and the state is called the “star state.” In addition, the cross-sectional flow area at the $M = 1$ point is indicated by A^* . The magnitude of A^* is determined by the relation

$$A^* = \frac{\dot{m}}{\rho^* V^*} \quad (2.74)$$

2.7.4 Mass Flow Parameter

The stream properties at any general station of a calorically perfect gas are related by Eqs. (2.21), (2.36), (2.66), (2.70), (2.72), (2.73), and the one-dimensional mass flow equation

$$\dot{m} = \rho A V$$

The mass flow per unit area at duct area A in the flow of a calorically perfect gas is given by the expression

$$\frac{\dot{m}}{A} = \rho V = \frac{PV}{RT} = \frac{V}{\sqrt{\gamma g_c RT}} \frac{P \sqrt{\gamma g_c}}{\sqrt{RT}} = M \sqrt{\frac{\gamma g_c}{R}} \frac{P}{P_t} \sqrt{\frac{T_t}{T}} \frac{P_t}{\sqrt{T_t}}$$

where the one-dimensional flow equation [Eq. (2.12a)], the perfect gas equation of state [Eq. (2.21)], the perfect gas speed of sound equation [Eq. (2.36)], and the

definition of Mach number have been used. Replacing the static/total property ratios with Eqs. (2.72) and (2.73) and rearranging, we obtain the grouping defined as the *mass flow parameter* (MFP):

$$\text{MFP} \equiv \frac{\dot{m} \sqrt{T_t}}{A P_t} \quad (2.75)$$

$$\text{MFP}(M) = \frac{\dot{m} \sqrt{T_t}}{A P_t} = \sqrt{\frac{\gamma g_c}{R}} M \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma + 1}{2(\gamma - 1)}} \quad (2.76)$$

Thus the MFP is a unique function of the Mach number in a calorically perfect gas. Values of the mass flow parameter are plotted in Fig. 2.20 for $\gamma = 1.4$ and $\gamma = 1.3$, and calculated in the GASTAB program for user input values of γ and mean molecular weight \mathcal{M} . From Fig. 2.20 and the GASTAB program, one sees that the maximum value of the mass flow parameter occurs when the Mach number is unity. Thus, for a given total temperature and pressure T_t and P_t , the maximum mass flow rate per area corresponds to a flow Mach number of 1.

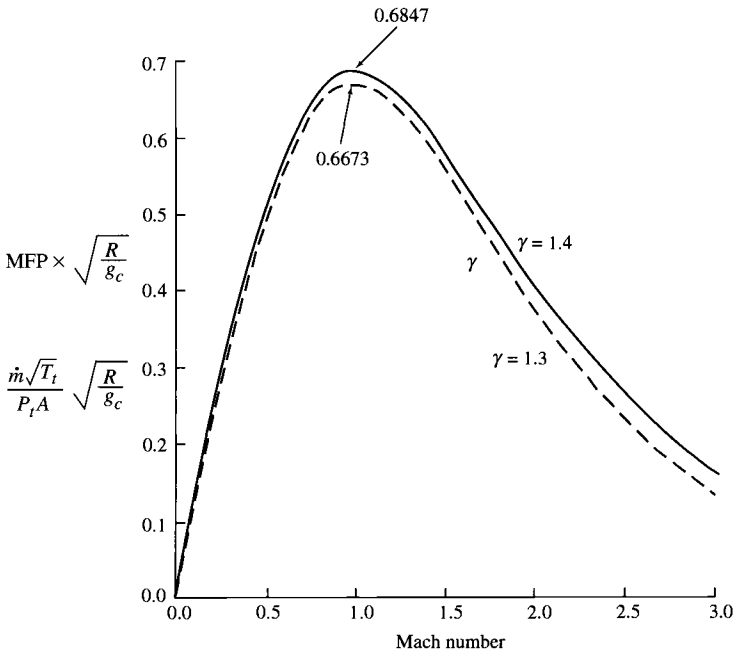


Fig. 2.20 Mass flow parameter vs Mach number ($\gamma = 1.4$ and $\gamma = 1.3$).

2.7.5 Isentropic Area Ratio A/A^*

The ratio of the one-dimensional flow area A at any flow station to the one-dimensional area for that same flow rate at sonic velocity A^* is a unique function of the Mach number M at the flow station and the ratio of specific heats γ . Consider the isentropic flow of a calorically perfect gas in an isentropic duct from ρ , M , P , T , and A to the sonic state where the properties are ρ^* , $M^* = 1$, P^* , T^* , and A^* . Because both states have the same mass flow, we write

$$\dot{m} = \rho AV = \rho^* A^* V^*$$

Rewriting gives

$$\frac{A}{A^*} = \frac{\rho^* V^*}{\rho V} = \frac{P^* T}{T^* P M} \frac{1}{a} = \frac{1}{M} \frac{P^*/P}{T^*/T} \sqrt{T^*/T} = \frac{1}{M} \frac{P^*/P}{\sqrt{T^*/T}} \quad (i)$$

However,

$$\frac{T}{T^*} = \frac{T/T_t}{T^*/T_t} = \left[\frac{2}{\gamma+1} \left(1 + \frac{\gamma-1}{2} M^2 \right) \right]^{-1} \quad (ii)$$

and

$$\frac{P}{P^*} = \frac{P/P_t}{P^*/P_t} = \left[\frac{2}{\gamma+1} \left(1 + \frac{\gamma-1}{2} M^2 \right) \right]^{-\gamma/(\gamma-1)} \quad (iii)$$

Substitution of Eqs. (ii) and (iii) into (i) gives

$$\frac{A}{A^*} = \frac{1}{M} = \left[\frac{2}{\gamma+1} \left(1 + \frac{\gamma-1}{2} M^2 \right) \right]^{(\gamma+1)/[2(\gamma-1)]} \quad (2.77)$$

and A/A^* , P/P_t , and T/T_t are plotted vs Mach number in Fig. 2.21 for $\gamma = 1.4$ and calculated in GASTAB for user input γ value.

Example 2.10

Air at a total temperature of 300 K and total pressure of 1 atm flows through a 2-m^2 duct at a Mach number of 0.5.

1) What must the area of the duct be for the flow to isentropically accelerate to Mach 1?

2) What is the mass flow rate of air through the duct?

Solution: From GASTAB for $M = 0.5$ and $\gamma = 1.4$, $A/A^* = 1.3398$ and $MFP = 0.0301563$.

1) The area of the duct at $M = 1$ is then

$$A^* = \frac{A}{A/A^*} = \frac{2}{1.3398} = 1.4928 \text{ m}^2$$

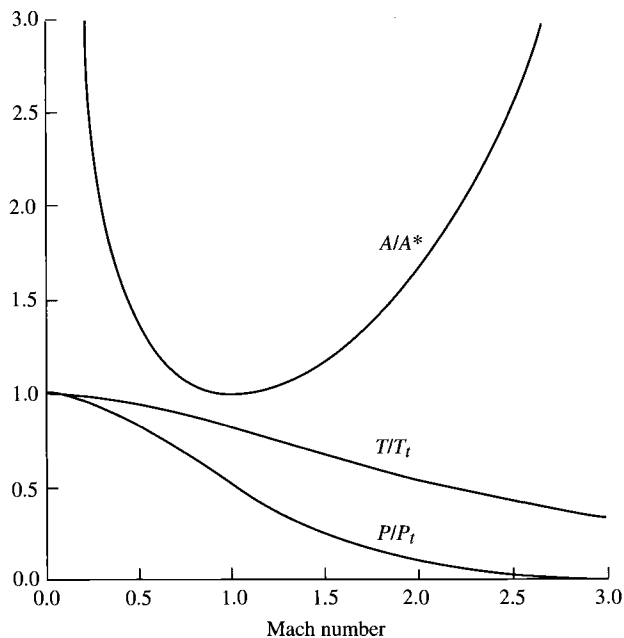


Fig. 2.21 A/A^* , P/P_t , and T/T_t vs Mach number ($\gamma = 1.4$).

2) The mass flow rate can be calculated by using the MFP as follows:

$$\dot{m} = \frac{P_t A}{\sqrt{T_t}} \text{MFP} = \frac{101,325 \times 2\text{N}}{\sqrt{300\text{K}}} 0.0301563 \text{ kg} \cdot \sqrt{\text{K}}/(\text{N} \cdot \text{s}) = 352.83 \text{ kg/s}$$

2.7.6 Impulse Function I

The impulse function I is defined by

$$I \equiv PA + \dot{m}V/g_c \quad (2.78)$$

Using one-dimensional mass flow equation ($\dot{m} = \rho AV$) and equations for a calorically perfect gas, Eq. (2.78) can be rewritten as

$$I = PA(1 + \gamma M^2) \quad (2.79)$$

From Eq. (2.20) for steady state, the streamwise axial force exerted on the fluid flowing through a control volume is $I_{\text{exit}} - I_{\text{entry}}$, while the reaction force exerted by the fluid on the control volume is $I_{\text{entry}} - I_{\text{exit}}$.

The impulse function makes possible almost unimaginable simplification of the evaluation of forces on aircraft engines, rocket engines, and their components. For example, although one could determine the net axial force exerted on the fluid flowing through any device by integrating the axial components of pressure and viscous forces over every infinitesimal element of internal wetted surface area, it is certain that no one ever has. Instead, the integrated result of the forces is obtained with ease and certainty by merely evaluating the change in impulse function across the device.

Compressible flow functions tabulate the impulse function as a ratio to that at the star state ($M = 1$) or I/I^* . We note that

$$\frac{I}{I^*} = \frac{P}{P^*} \frac{A}{A^*} \frac{1 + \gamma M^2}{1 + \gamma} \quad (2.80)$$

2.8 One-Dimensional Gas Dynamics—Finite Control Volume Analysis and the H - K Diagram

Much of the material in this section, through page 121, is reprinted from *Hypersonic Airbreathing Propulsion*, by Heiser and Pratt, Section 2.6,¹³ with permission of the publisher.* Minor adaptations have been made to the original text.

The six example cases that follow are based on the steady, one-dimensional flow of a calorically perfect gas as shown in Fig. 2.22. They are meant to show the breadth and power of one-dimensional analysis, while demonstrating several key features of hypersonic flows that play prominent roles in ramjets and scramjets and familiarizing the reader with applying the set of governing equations.

The reader should be aware that the example cases are also found in the subroutines of the GASTAB program (Gas Tables) included with this textbook.

Example Case 2.1: Constant Energy Flows

Consider a one-dimensional flow without energy interactions with the surroundings, i.e.,

$$w_x = q = 0$$

Under these conditions, the energy equation, Eq. (2.14), reduces to

$$h_i + \frac{V_i^2}{2g_c} = h_e + \frac{V_e^2}{2g_c} = h_t \quad (2.81)$$

which means that the *total enthalpy* h_t is a fixed property of constant energy flows, even if the gas is *not* calorically perfect. The total enthalpy is also equal

*Copyright © 1994 by the American Institute of Aeronautics and Astronautics, Inc.

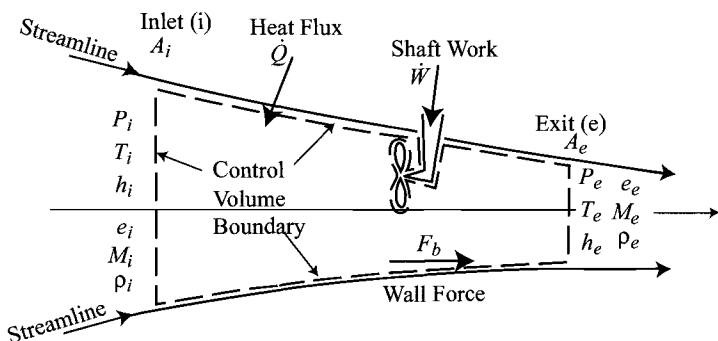


Fig. 2.22 Finite control volume for one-dimensional gas dynamics.

to the enthalpy that would exist locally if the flow were brought completely to rest without energy interactions, and is therefore also known as the *stagnation enthalpy*.

For a calorically perfect gas, Eqs. (2.38) and (2.81) can be combined to yield

$$c_p T_i + \frac{V_i^2}{2g_c} = c_p T_e + \frac{V_e^2}{2g_c} = c_p T_t \quad (2.82)$$

where T_t is the *total temperature*. This expression may be rearranged with the help of Eqs. (2.8) and (2.36) to show that

$$T_t = T_i \left(1 + \frac{\gamma - 1}{2} M_i^2 \right) = T_e \left(1 + \frac{\gamma - 1}{2} M_e^2 \right) \quad (2.83)$$

It is interesting and productive to portray Eq. (2.82) in terms of the *dimensionless static enthalpy* $c_p T / c_p T_{ti}$ and the *dimensionless kinetic energy* $V^2 / (2g_c c_p T_{ti})$, or

$$\frac{c_p T}{c_p T_{ti}} + \frac{V^2}{2g_c c_p T_{ti}} = 1 \quad (2.84)$$

which can be written simply as

$$H + K = 1 \quad (2.85)$$

where

$$H \equiv \frac{c_p T}{c_p T_{ti}} \quad \text{and} \quad K \equiv \frac{V^2}{2g_c c_p T_{ti}} \quad (2.86)$$

as shown in Fig. 2.23. The state of the flow *must* be somewhere on this straight line. This line will be referred to as a *constant total enthalpy line*.

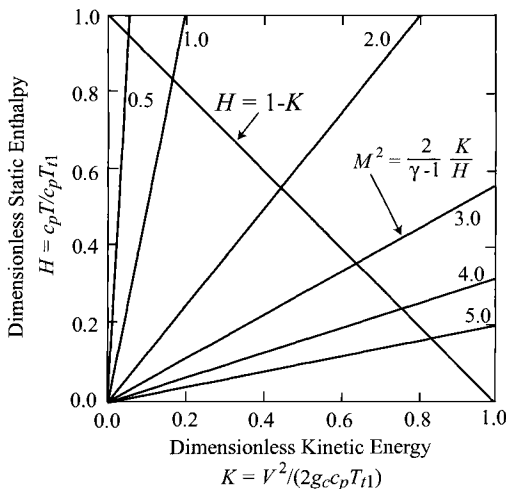


Fig. 2.23 Dimensionless energy diagram for the constant energy flow of a calorically perfect gas, showing the relationship between dimensionless static enthalpy H and the dimensionless kinetic energy K , Eq. (2.85). The straight lines emanating from the origin are lines of constant Mach number, Eq. (2.87), increasing from left to right.

From the definition of the Mach number, isolines of constant Mach number M are given by the straight line

$$\frac{\gamma - 1}{2} M^2 H - K = 0 \quad (2.87)$$

which is a ray passing through the coordinate origin (Ref. 17). Lines of constant Mach number are shown on Fig. 2.23 along with a line of constant enthalpy, Eq. (2.85).

Equation (2.71b) may now be combined and manipulated to show that

$$\frac{s_e - s_i}{R} = -\ell n \left(\frac{P_{ti}}{P_{te}} \right) \quad (2.88)$$

where the *total pressure* [Eq. (2.73)]

$$P_t = P \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}}$$

is a fixed property of constant energy *isentropic* flows of calorically perfect gases. Because the entropy can only increase for this type of flow, Eq. (2.88) shows that the total pressure can only decrease as the flow proceeds. Because γ for air is 1.40 or less, the exponent of this expression is at least 3.50, which means

that the dependence of the total pressure on flight Mach number is *remarkably* strong for a natural process.

Note should also be taken of the fact that, according to Eq. (2.88),

$$\pi = \frac{P_{te}}{P_{ti}} = e^{-(s_e - s_i)/R} \leq 1 \quad (2.89)$$

which means that the *total pressure ratio* π depends exponentially on the entropy increase. In other words, the total pressure ratio should be expected to decrease very rapidly as entropy, the natural indicator of the effects of dissipation or *flow losses*, increases.

Example Case 2.2: Impulse Function and Stream Thrust Function

Consider a one-dimensional flow for which all properties at the inlet and exit planes are known. In this situation Eq. (2.12) becomes

$$\dot{m} = \rho_i V_i A_i = \rho_e V_e A_e \quad (2.90)$$

and Eq. (2.20) becomes

$$F_b = (P_e A_e + \dot{m} V_e / g_c) - (P_i A_i + \dot{m} V_i / g_c) \quad (2.91)$$

so that the total force acting on the flow within the control volume can be directly determined.

In the regime where the fluid behaves as a calorically perfect gas, the axial momentum equation may be rearranged into several useful forms. For example, with the help of Eqs. (2.78) and (2.79), Eq. (2.91) becomes

$$F_b = P_e A_e (1 + \gamma M_e^2) - P_i A_i (1 + \gamma M_i^2) = I_e - I_i \quad (2.92)$$

where [Eq. (2.79)]

$$I = PA(1 + \gamma M^2)$$

Next, multiplying Eq. (2.91) by g_c and dividing through by Eq. (2.90), and using Eq. (2.21), we find that

$$\frac{F_b g_c}{\dot{m}} = V_e (1 + g_c R T_e / V_e^2) - V_i (1 + g_c R T_i / V_i^2) = S_{a_e} - S_{a_i} \quad (2.93)$$

where

$$S_a = \frac{I g_c}{\dot{m}} = V(1 + g_c R T / V^2) \quad (2.94)$$

The latter expression is known in compressible fluid mechanics as the *stream thrust function* (S_a originally standing for mass flow rate specific thrust of air). The stream thrust function is the parameter that most easily leads to the

determination of mass flow rate specific thrust, a quantity often used in performance evaluations to eliminate dependence on absolute size.

Finally, dividing Eq. (2.94) by $\sqrt{g_c c_p T_t}$ and rearranging, we find that the *dimensionless stream thrust function* for axial flow as

$$\Phi = \frac{Sa}{\sqrt{g_c c_p T_{ti}}} = \sqrt{\frac{1}{2} \left(\frac{2 g_c c_p T_{ti}}{V^2} \right)} \left\{ 2 \left(\frac{V^2}{2 g_c c_p T_{ti}} \right) + \frac{\gamma - 1}{\gamma} \left(\frac{c_p T}{c_p T_{ti}} \right) \right\} \quad (2.95)$$

or

$$\Phi = \frac{Sa}{\sqrt{g_c c_p T_{ti}}} = \sqrt{\frac{1}{2K}} \left\{ 2K + \frac{\gamma - 1}{\gamma} H \right\} \quad (2.96)$$

Note that Φ depends upon the same dimensionless quantities that are found in Eq. (2.84). Equation (2.95) is portrayed in Fig. 2.24 for typical values of Φ and γ , where it can be seen that the curve resembles an ellipse passing through the origin. For a given Φ , the state of the flow *must* be somewhere on this curved line. For the *special* case of constant area flow with heat interaction but without friction, this line is known in the classical literature as a *Rayleigh line*. However, the same relation applies to the *much* more *general* situations to be encountered here, and so the line will *not* be referred to by that name. The breaks at the right end of the line are meant to remind us that it may not be possible to find *mean* values of the gas constants valid over the entire range, although one may judiciously select the range for which they are suitably accurate.

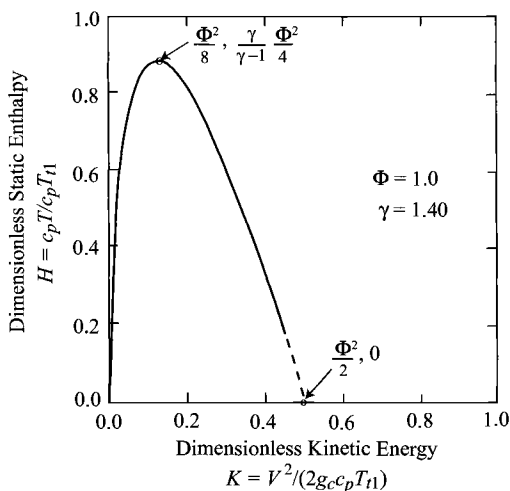


Fig. 2.24 Dimensionless energy diagram for the constant stream thrust flow of a calorically perfect gas, showing the relationship between dimensionless static enthalpy H and the dimensionless kinetic energy K , Eq. (2.95).

Example Case 2.3: Normal Shock Waves

Normal shock waves are discontinuities in one-dimensional, constant through-flow area, axial flows that are subject to the three constraints of constant mass flow, constant energy, and constant stream thrust. The upstream and downstream or *jump* conditions for normal shock waves can therefore be obtained for calorically perfect gases by the simultaneous solution of Eqs. (2.84) and (2.95) for constant values of $c_p T_{ii}$ and Φ . A typical *graphical* solution is portrayed in Fig. 2.25, which combines Figs. 2.23 and 2.24.

Straightforward mathematical manipulations of Eqs. (2.84) and (2.95) yield the more familiar closed-form, *algebraic* solutions for the jump conditions of normal shock waves. For example, the two dimensionless kinetic energy solutions are

$$K = \frac{V^2}{2g_c c_p T_{ii}} = \left\{ \frac{\gamma \Phi}{(\gamma + 1)\sqrt{2}} \pm \sqrt{\frac{1}{2} \left(\frac{\gamma \Phi}{\gamma + 1} \right) - \left(\frac{\gamma - 1}{\gamma + 1} \right)} \right\}^2 \quad (2.97)$$

and the respective dimensionless static enthalpy solutions are [Eq. (2.85)]

$$\frac{c_p T}{c_p T_{ii}} = 1 - \frac{V^2}{2g_c c_p T_{ii}} \quad \text{or} \quad H = 1 - K$$

To prove that these two solutions correspond to supersonic and subsonic flow, it is first necessary to observe that straight lines emanating from the origin of Fig. 2.25 are lines of constant Mach number, larger slopes belonging to lower Mach numbers. Next, it can be seen from the geometry of this diagram that there is only one value of Φ for which the constant stream thrust function line is tangent to the constant energy line *at point c* and for which there is one solution (rather than two or zero). This condition arises when the discriminant of Eq. (2.97) is zero, and consequently

$$\Phi_c = \sqrt{2 \left(1 - \frac{1}{\gamma^2} \right)} \quad (2.98)$$

($\Phi_c = 0.990$ for $\gamma = 1.40$), and

$$K_c = \frac{V_c^2}{2g_c c_p T_{ii}} = \frac{\gamma - 1}{\gamma + 1} \quad (2.99)$$

{ $K_c = V^2/(2g_c c_p T_{ii}) = 0.167$ for $\gamma = 1.40$ }. Using Eqs. (2.83) and (2.99) reveals that at point *c*

$$M_c = 1 \quad (2.100)$$

Finally, returning to the geometry of Fig. 2.25, it follows that all values of Φ that have two solutions must have one that is subsonic and one that is supersonic. The

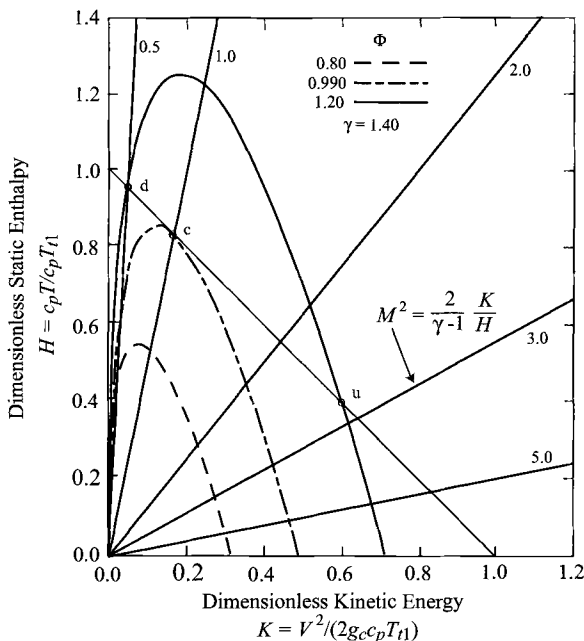


Fig. 2.25 Graphical solution for the jump condition of normal shock waves based on Eqs. (2.84) and (2.95). The straight lines emanating from the origin are lines of constant Mach number, increasing from left to right as indicated by their labels.

normal shock therefore takes the flow from the upstream, supersonic condition (point *u*) to the downstream, subsonic condition (point *d*).

Before moving on, it is worth noting in anticipation of coming topics that the graphical presentation of Fig. 2.25 has the special virtue that constant values of such primal physical quantities as static enthalpy, static temperature, kinetic energy, velocity, Mach number, and functions of Mach number only all appear as straight lines.

The GASTAB software is included to help solve problems. As an example, consider air ($\gamma = 1.4$) flowing at Mach 2.0 enters a normal shock. From normal shock tab of GASTAB, the downstream Mach number is 0.5774, the static pressure ratio across the normal shock is 4.5, and the total pressure ratio across the normal shock is 0.7209. Other example problems are included within the Supporting Material.

Example Case 2.4: Constant Area Heating and Thermal Choking

Consider a frictionless, one-dimensional flow of constant area for which there is only a total heat interaction q with the surroundings, i.e.,

$$F_{bx} = w_x = 0 \quad A_i = A_e \quad \dot{m}_i = \dot{m}_e = \dot{m}$$

Under these circumstances, which correspond to classical Rayleigh flow, Eq. (2.12) becomes

$$\rho_i V_i = \rho_e V_e \quad (2.101)$$

and Eq. (2.91) becomes

$$P_i + \rho_i V_i^2 / g_c = P_e + \rho_e V_e^2 / g_c \quad (2.102)$$

which means that the stream thrust function Sa is constant. Taking the working fluid to be calorically perfect, we must therefore reach the conclusion that

$$\Phi = \frac{Sa}{\sqrt{g_c c_p T_{ti}}} = \Phi_i = \text{const} \quad (2.103)$$

where Φ_i (the *inlet* dimensionless stream thrust function) as given by Eq. (2.95) and $c_p T_{ti}$ (the *inlet* total enthalpy) may be evaluated from the known inlet conditions.

Equation (2.14) becomes

$$c_p T_i + \frac{V_i^2}{2g_c} + q = c_p T_{ti} \left(1 + \frac{q}{c_p T_{ti}} \right) = c_p T_e + \frac{V_e^2}{2g_c} = c_p T_{te} \quad (2.104)$$

or

$$\frac{T_{ti}}{T_{te}} = \frac{c_p T_e}{c_p T_{ti}} + \frac{V_e^2}{2g_c c_p T_{ti}} = 1 + \frac{q}{c_p T_{ti}} \equiv \tau_e \geq 1 \quad (2.105a)$$

or

$$\frac{T_{ti}}{T_{te}} = H + K = 1 + \frac{q}{c_p T_{ti}} \equiv \tau_e \geq 1 \quad (2.105b)$$

where τ_e is the *total temperature ratio* and the primary indicator of *dimensionless heating*. Equations (2.103) and (2.105b) may be combined to find the *algebraic* solution for the two possible exit conditions, namely

$$K_e = \frac{V_e^2}{2g_c c_p T_{ti}} = \left\{ \frac{\gamma \Phi_i}{(\gamma + 1)\sqrt{2}} \pm \sqrt{\frac{1}{2} \left(\frac{\gamma \Phi_i}{\gamma + 1} \right)^2 - \tau_e \left(\frac{\gamma - 1}{\gamma + 1} \right)} \right\}^2 \quad (2.106)$$

and

$$\frac{c_p T_e}{c_p T_{ti}} = \tau_e - \frac{V_e^2}{2g_c c_p T_{ti}} \quad (2.107a)$$

or

$$H_e = \tau_e - K_e \quad (2.107b)$$

Equations (2.103) and (2.105b) can also be solved *graphically*, as shown in Fig. 2.26, where the positive root of Eq. (2.106) belongs to supersonic inlet conditions and the negative root to subsonic inlet conditions. Because τ_e may be treated as an independent variable, Fig. 2.26 reveals that heating the flow drives it toward an exit Mach number of 1 regardless of whether it was initially subsonic or supersonic. Thus, in the typical example of Fig. 2.25, a τ_e of 1.20 reduces the supersonic branch Mach number from an inlet value of 2.74 to an exit value of 1.89, and increases the subsonic branch Mach number from 0.493 to 0.598.

The remarks about the tangent point solution made in the previous example case also apply here, in particular that point *c* is the sonic condition, at which the discriminant of Eq. (2.106) must be zero, or

$$\tau_c = \frac{T_r}{T_{ti}} = \Phi_i^2 / \{2(1 - 1/\gamma^2)\} \quad (2.108)$$

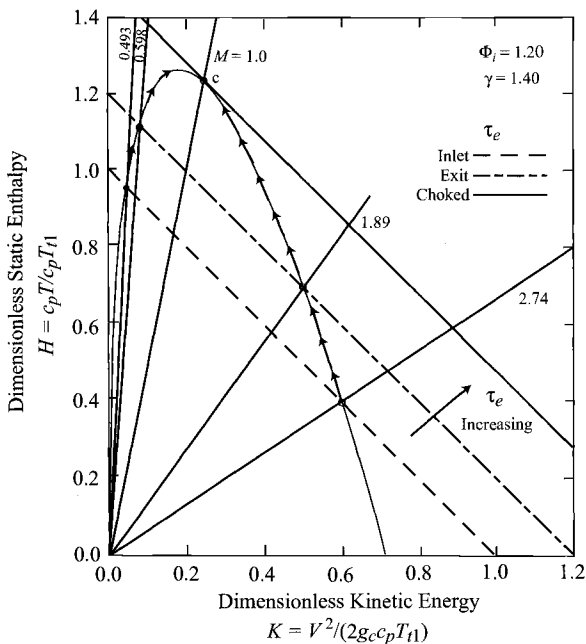


Fig. 2.26 Graphical solution for the frictionless, constant area heating based on Eqs. (2.103) and (2.104). The straight lines emanating from the origin are lines of constant Mach number, increasing from left to right as indicated by their labels.

($\tau_c = 1.47$ for $\Phi_i = 1.20$ and $\gamma = 1.40$), and

$$K_c = \frac{V_c^2}{2 g_c c_p T_{ti}} = \frac{1}{2} \left(\frac{\gamma \Phi_i}{\gamma + 1} \right)^2 \quad (2.109)$$

{ $K_c = V_c^2 / (2 g_c c_p T_{ti})$ } $i = 0.245$ for $\Phi_i = 1.20$ and $\gamma = 1.40$).

Most importantly, Fig. 2.26 clearly demonstrates that for any inlet condition there is a maximum amount of energy that can be added before the exit Mach number reaches 1. There are *no* solutions to the governing equations for $\tau_e > \tau_c$. Once that point has been reached, it is physically impossible to add any more energy without forcing the upstream conditions to change, and the flow is said to be *thermally choked*. The conditions necessary for thermal choking can and do occur in practice, especially when the inlet Mach number is not far from 1, and should not be considered merely as an intellectual curiosity.

Finally, it is also possible to show that the entropy always increases as the flow is heated. This is best done by substituting the preceding results into the Gibbs equation and rearranging to show that

$$\frac{s_e - s_i}{c_p} = \ln \left\{ \left(\frac{M_e}{M_i} \right)^2 \left(\frac{1 + \gamma M_i^2}{1 + \gamma M_e^2} \right)^{\frac{\gamma+1}{\gamma}} \right\} \quad (2.110)$$

Since the grouping

$$\frac{M^2}{(1 + \gamma M^2)^{\frac{\gamma+1}{\gamma}}}$$

has a maximum when M is 1, and since heating always drives M toward 1, it follows that heating is always accompanied by increased entropy.

The GASTAB software is included to help solve problems. As an example, consider air ($\gamma = 1.4$) flowing at Mach 0.5 in a constant area duct heated from 300 to 400 K. From Rayleigh flow tab of GASTAB, the maximum total temperature of the flow ($T_{r^*} = T_i \tau_c$) is 433.9 K ($= 300/0.6914$). For the exit temperature of 400 K ($T_i/T_{r^*} = 400/433.9 = 0.9219$), the exit Mach number is 0.7197 and the total pressure ratio is $0.9314[P_{t2}/P_{t1} = (P_{t2}/P_{r^*})/(P_{t1}/P_{r^*}) = 1.0377/1.1141]$. Other example problems are included within the Supporting Material.

Example Case 2.5: Constant Area Friction and Frictional Choking

Consider a one-dimensional flow of constant area without energy interactions and frictional force F_f with the surroundings, i.e.,

$$q = w_x = 0 \quad A_i = A_e$$

Under these circumstances, which correspond to classical Fanno line flow, Eq. (2.12b) becomes [Eq. (2.101)]

$$\rho_i V_i = \rho_e V_e$$

the energy equation, Eq. (2.14), reduces to [Eq. (2.81)]

$$h_i + \frac{V_i^2}{2g_c} = h_e + \frac{V_e^2}{2g_c} = h_t$$

and Eq. (2.20) becomes

$$-F_f = A(P_i + \rho_i V_i^2/g_c) - A(P_e + \rho_e V_e^2/g_c) = I_i - I_e \quad (2.111)$$

Taking the working fluid to be calorically perfect, Eq. (2.81) becomes [Eq. (2.82)]

$$c_p T_i + \frac{V_i^2}{2g_c} = c_p T_e + \frac{V_e^2}{2g_c} = c_p T_t$$

which can be written simply as [Eq. (2.85)]

$$H + K = 1$$

where [Eq. (2.86)]

$$H \equiv \frac{c_p T}{c_p T_{ti}} \quad \text{and} \quad K \equiv \frac{V^2}{2g_c c_p T_{ti}}$$

and Eq. (2.111) becomes

$$\frac{F_f g_c}{\dot{m}} = V_i(1 + g_c R T_i / V_i^2) - V_e(1 + g_c R T_e / V_e^2) = S a_i - S a_e \quad (2.112)$$

where [Eq. (2.94)]

$$S a = \frac{I g_c}{\dot{m}} = V(1 + g_c R T / V^2)$$

Using the dimensionless stream thrust function Φ [Eq. (2.96)], Eq. (2.112) becomes

$$\frac{F_f g_c / \dot{m}}{\sqrt{g_c c_p T_{ti}}} = \Phi_i - \Phi_e \quad (2.113)$$

which means that the change in stream thrust function is directly proportional to the frictional force F_f .

Graphical solution of Eqs. (2.85) and (2.103) is presented in Fig. 2.27. Consider flow entering at $M_i = 0.5$ and $\Phi_i = 1.20$ (point *d* in Fig. 2.27). If the frictional force decreases the dimensionless stream thrust function to $\Phi_e = 0.99$, then the exit flow chokes with $M_e = 1$ (point *c* in Fig. 2.27). There is no solution

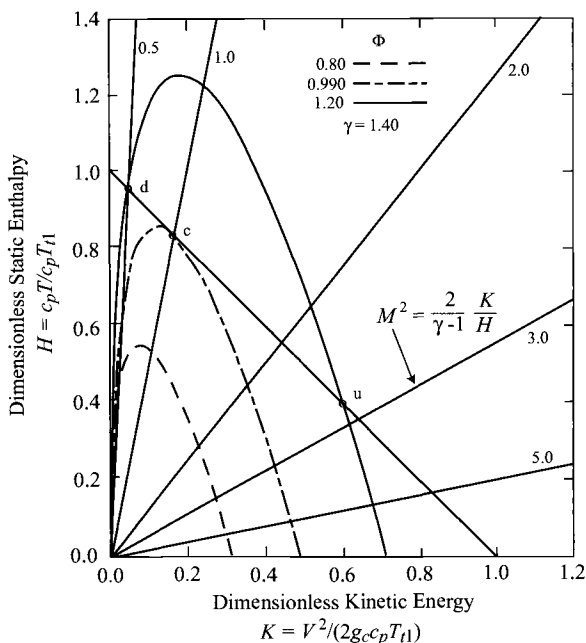


Fig. 2.27 Graphical solution for the frictionless, constant area frictional flow based on Eqs. (2.85) and (2.113). The straight lines emanating from the origin are lines of constant Mach number, increasing from left to right as indicated by their labels.

for further decreases of Φ with the same inlet conditions. Similarly, flow entering at $M_i = 2.74$ and $\Phi_i = 1.20$ (point u in Fig. 2.27) will choke when Φ is decreased to 0.99.

H-K diagram. The great utility of graphically displaying flow processes in terms of dimensionless static enthalpy vs dimensionless kinetic energy has been amply demonstrated by the elementary example cases already considered. This method of presentation will prove even more valuable in explaining and illustrating the more complex internal flow behavior of airbreathing engines. To ease communications, this diagram will hereinafter simply be called the *H-K diagram* (H for dimensionless static enthalpy and K for dimensionless kinetic energy).

Please note that the *H-K diagram*, for all its other virtues, is *not a state diagram* because only one axis is an intensive thermodynamic property. In other words, there is no necessary relationship between a point on the *H-K diagram* and the other intensive thermodynamic properties of the fluid, such as static pressure or static entropy. Nevertheless, the *H-K diagram* will provide more than enough information to reveal the things we really need to know about the flow. Also, under some frequently encountered conditions, such as one-dimensional flow with known \dot{m} , A , and T_{ti} , the *H-K diagram* is a state diagram.

An especially useful generalization for moving about the H - K diagram, based on the assemblage of preceding example cases, is that heating, friction, and area decrease *all* act separately and together to drive the Mach number toward 1. They may therefore in an intuitive sense be said to *block*, *constrict*, *obstruct*, *restrict*, or *occlude* the flow. Similarly, cooling, reverse friction (i.e., any streamwise force), and area increase could be said to *unblock*, *enlarge*, *relieve*, *open*, or *free* the flow. You will find these mental images helpful in what follows.

Figure 2.28 shows the H - K diagram with representative constant-property isolines. The H - K diagram will be used in this text to improve understanding of the flow through airbreathing engines.

The H - K diagram often provides a convenient means for visualizing, comprehending, and even analyzing the operation of propulsion devices, as illustrated in the following for the scramjet engine.¹⁷ It is important to note that the dimensionless stream thrust function at a given Mach number is not some arbitrary value but is specified by Eq. (2.114) as rearranged into the Mach number form:

$$\Phi = \sqrt{\frac{\tau_e(\gamma - 1)M^2}{1 + \frac{\gamma - 1}{2}M^2}} \left(1 + \frac{1}{\gamma M^2}\right) \quad (2.114)$$

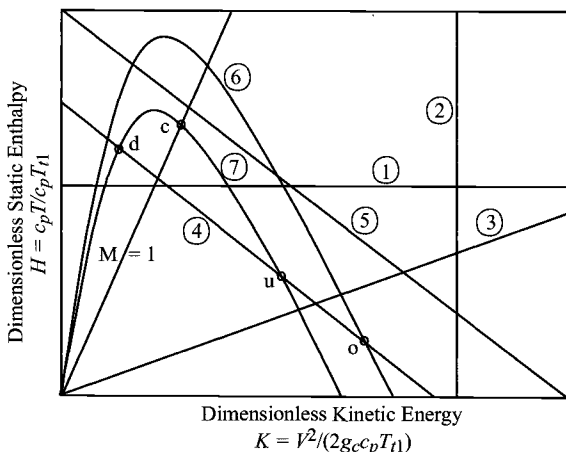


Fig. 2.28 The H - K diagram, depicting representative constant-property isolines. Key: 0 = freestream reference state. Point c = choked condition at constant impulse. Points u and d denote end states of normal shock. Circled numbers denote isolines of constant property as follows: 1) static enthalpy, static temperature; 2) kinetic energy, velocity, pressure (for frictionless heating or cooling only); 3) Mach number; 4) total enthalpy, total temperature (adiabat), $\tau \equiv T_i/T_0 = 1$; 5) post-heat release adiabat, $t > 1$; 6) impulse function/stream thrust, area (for frictionless flow with heating or cooling only), case $I = I_0$; 7) impulse function, case $\Phi > \Phi_0$.

Example Case 2.6: Scramjet

The basic concept of the scramjet is that the flow must remain supersonic throughout to avoid the high static temperatures that reduce performance by causing chemical dissociation of the combustion products, as well as the high static pressures that cause structural problems. Consequently, in practical terms, when the freestream Mach number exceeds 5–6, the flow is designed to enter the combustor at supersonic speeds, and the device is known as the supersonic combustion ramjet, or scramjet.

Figure 2.29 shows the H - K diagram for air with $\gamma = 1.40$ being processed by a scramjet that is powering a vehicle at a freestream Mach number of 10.0, where Eq. (2.114) has been used to show that $\Phi_0 = 1.390$. The air is first decelerated and compressed from the freestream condition (point 0) to the burner entry condition (point 1) by means of a combination of isentropic compression and oblique shock waves. The purposes of this compression are to provide a large enough static temperature ratio T_1/T_0 for satisfactory thermodynamic cycle efficiency (usually in the range of 6–8, and 6.50 for this example) and to produce high enough values of P_1 and T_1 to support complete and stable combustion in the burner. Even when these criteria have been met, the burner entry $M_1 = 3.340$ remains supersonic, as Fig. 2.29 clearly reveals.

The air is then heated in a combustion process that releases the chemical energy of the fuel. The heating is represented in this type of analysis by an increasing total temperature, in this example case by a factor of 1.40. The precise path of this process depends on the philosophy of the burner design, and two of many possible different types are depicted in Fig. 2.29. The first,

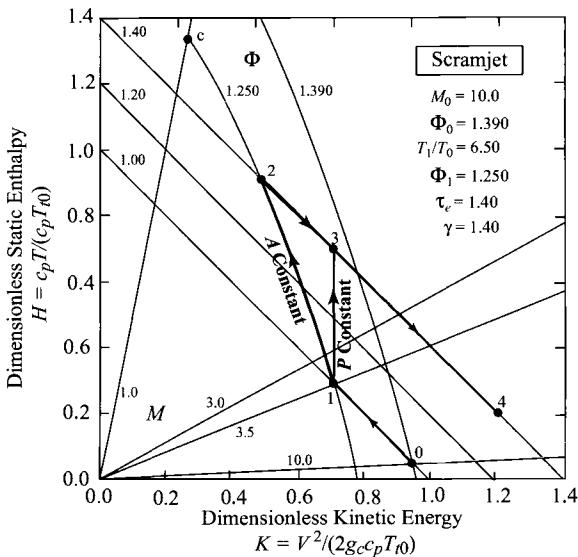


Fig. 2.29 The H - K diagram for an example scramjet.

joining point 1 to point 2, is frictionless, constant area heating, which is a Rayleigh line having $\Phi_1 = 1.250$. The second, joining point 1 to point 3, is frictionless, constant pressure heating, which is found in Problem 2.49d also to be a line of constant velocity. There is clearly no danger of reaching point c and thermal choking for either combustor in this scenario.

The heated air is then accelerated and expanded from a burner exit condition such as point 2 or 3 to the freestream static pressure at point 4. Because there are total pressure losses in the scramjet, the Mach number at point 4 can never be quite as large as the freestream Mach number, but it can be large enough that the kinetic energy and velocity at point 4 exceed that of point 0, which means that the scramjet produces net thrust. As a corollary, the total pressure losses and therefore the precise location of point 4 also depend on the type of burner design. Nevertheless, the H - K diagram makes it clear that the potential thermodynamic performance is greater for constant area heating than for constant velocity heating because each increment of heat is added at a higher temperature in the former case.

The readers are encouraged to expand their familiarity and facility with the H - K diagram by either explaining the impact of other heating processes (e.g., constant temperature or constant Mach number) on the potential thermodynamic performance of the scramjet, or by constructing the H - K diagram for the ramjet, for which the combustor Mach number is very subsonic.

2.9 Nozzle Design and Nozzle Operating Characteristics

Figure 2.30 shows flow through a nozzle from a subsonic flow (state 1) through the sonic conditions (state $*$) to supersonic flow (state 2). The flow is isentropic, and both the total temperature and total pressure are constant. For analysis, we consider accelerating a gas through a nozzle with mass flow rate \dot{m}_c . Let the flow originate in a large storage chamber (Fig. 2.31) at chamber pressure

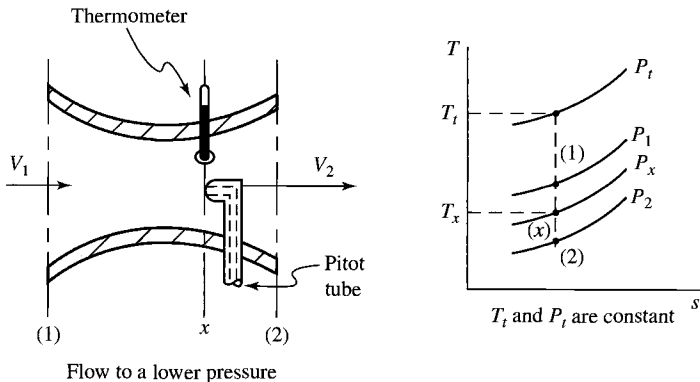


Fig. 2.30 Simple nozzle flow.

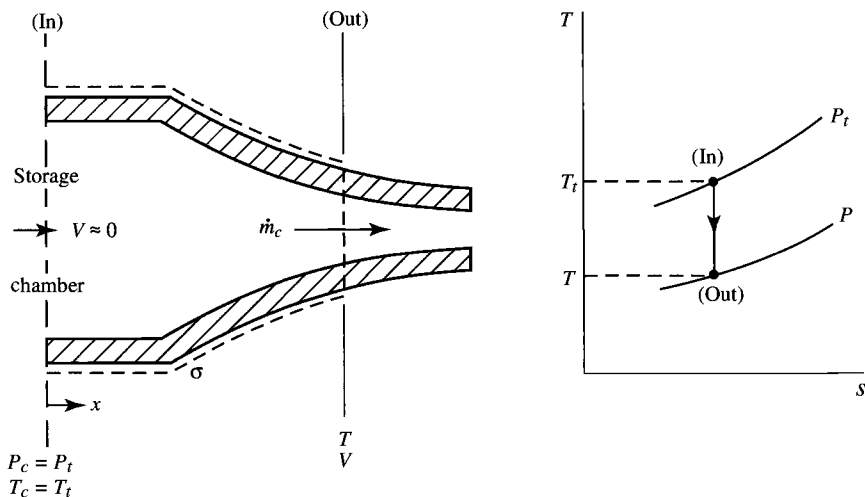


Fig. 2.31 Control volume for simple nozzle flow.

$P_c = P_t$, and chamber temperature $T_c = T_t$. The stream properties at any station in the flow are related by the following equations:

$$P_c = P_t = P(T_t/T)^{\gamma/(\gamma-1)} \quad (i)$$

$$T_c = T_t = T + V^2/(2 g_c c_p) \quad (ii)$$

$$\dot{m}_c = \rho AV = PAV/(RT) \quad (iii)$$

For given chamber gas conditions P_c , T_c , R , γ , and known \dot{m}_c , there are four variables P , T , V , A in these three equations. We may select one variable as independent and find each of the remaining three in terms of this one. Practical problems generally fall into two categories:

1) Nozzle design. We wish to pass a given mass flow with minimum frictional losses between two regions of different pressure (storage chamber at P_c and exhaust region at P_a) with, say, some assumed variation in pressure between the two regions.

2) Nozzle operating characteristics. Given a nozzle, what mass rates of flow and pressure distribution will prevail through the nozzle for various nozzle pressure ratios ($P_n = P_c/P_a$)?

In case 1, our independent, or known, variable is pressure P , which is a function of position x . In case 2, our independent, or known, variable is area A , which is a function of position x . We will consider each case in turn.

2.9.1 Nozzle Design

We shall illustrate the design of a nozzle by example.

Example 2.11

Assume we wish to expand gases at 28 lbm/s from a high-altitude second-stage rocket combustion chamber to an ambient pressure of 0.618 psia (≈ 70 kft altitude). Pertinent data simulating the Agena rocket engine are as follows:

$$P_c = 206 \text{ psia}, \quad c_p g_c = 6000 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R}), \quad T_c = 5000^\circ\text{R}$$

$$\text{Nozzle length} = 30 \text{ in.}, \quad \dot{m}_c = 28 \text{ lbm/s}$$

$$R g_c = 1715 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R}), \quad \text{Exit pressure} = 0.618 \text{ psia}$$

To solve the problem, we must determine the flow area, gas temperature, velocity, and Mach number (A , T , V , and M) at each nozzle station. Assume all sections of the nozzle are circular, and assume simple area flow. Figure 2.32 provides the length of, and pressure in, the nozzle to be designed.

Solution: For the given conditions, the constants of Eqs. (i) through (iii) are known. Thus

$$P_t = P_c = 206 \text{ psia}, \quad T_t = T_c = 5000^\circ\text{R}, \quad \dot{m}_c = 28 \text{ lbm/s}$$

$$R g_c = 1715 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R}), \quad c_p g_c = 6000 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R}), \quad \gamma = 1.4$$

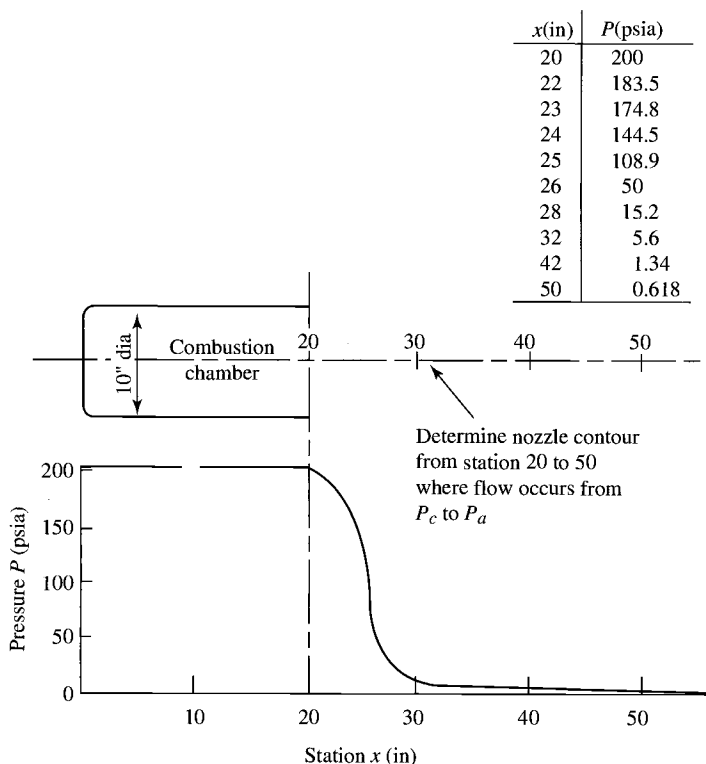


Fig. 2.32 Nozzle pressure distribution.

Rewrite Eqs. (i) through (iii) as

$$T = T_t(P/P_t)^{(\gamma-1)/\gamma} \quad (\text{iv})$$

$$V = \sqrt{2g_c c_p (T_t - T)} \quad (\text{v})$$

$$A = \dot{m}RT/(PV) \quad (\text{vi})$$

With P known (Fig. 2.32), use Eq. (iv) to find T at any given station. Equation (v) will then give V . With T and V determined, and \dot{m} , R , and P known, Eq. (vi) gives the nozzle area A at the selected station. In this manner, the nozzle area and gas properties can be found at all stations. The Mach number follows from

$$M = V/\sqrt{\gamma R g_c T} \quad (\text{vii})$$

The results of the computation outlined are plotted in Fig. 2.33. The curves and nozzle contour in this figure illustrate that in order to *decrease* P and *increase* V in a simple area flow:

- A converging nozzle contour is required in subsonic flow.*
- A diverging nozzle contour is required in a supersonic flow.*

Thus we find that the typical shape of a nozzle that is accelerating a gas from rest to supersonic speeds is convergent-divergent (C-D). At the design operating point of a *supersonic* C-D nozzle, the flow is subsonic up to the throat, sonic at the throat, and supersonic after the throat. The exit plane pressure P_e equals the exhaust region pressure P_a . We shall denote the nozzle pressure ratio for the

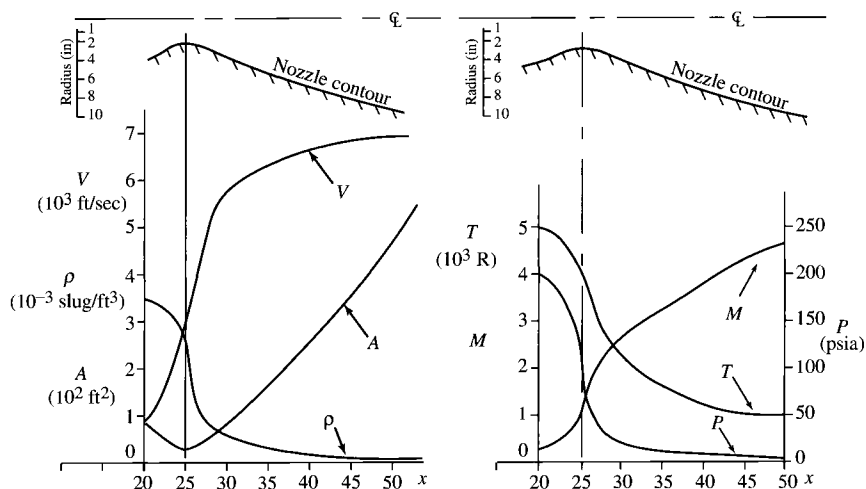


Fig. 2.33 Nozzle flow properties vs station for air.

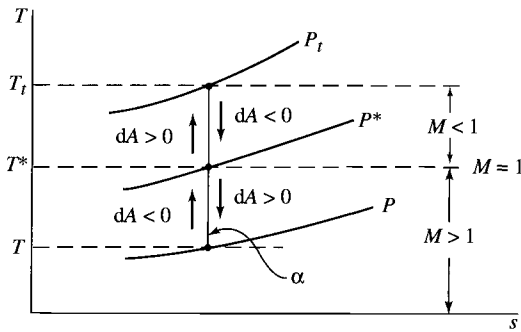


Fig. 2.34 Isentropes line α .

design point as $P_{\bar{n}}$, where

$$P_{\bar{n}} = (P_n)_{\text{design}} = \left(\frac{P_c}{P_a} \right)_{\text{design}}$$

The path line of an isentropic flow is called an *isentropes*. The area variation required to progress along an isentropes in a given direction is shown in the T - s diagram of Fig. 2.34. To progress *downward* along the isentropes requires a converging area ($dA < 0$) in subsonic flow and diverging area ($dA > 0$) in a supersonic stream. To progress *upward* along the isentropes requires a converging area ($dA < 0$) in supersonic flow and diverging area ($dA > 0$) in subsonic flow. This is why the engine intakes on the Concorde and various other supersonic aircraft converge from the inlet entrance to a throat and then diverge to the compressor face. This design reduces the speed of the air entering the compressor. Since P , T , ρ , and $V^2/(2g_c c_p)$ can be displayed in the T - s diagram, the isentropes line α properly interpreted summarizes most of the characteristics of isentropic flow.

The stream area/velocity variations discussed can be explained on the basis of the continuity equation by examining how the gas density varies with velocity in an isentropic flow. Area, velocity, and density are related as follows by the one-dimensional steady flow continuity equation:

$$A = \frac{\dot{m}}{\rho V} \quad (\dot{m} = \text{const})$$

By reference to a T - s diagram, with lines of constant density thereon, we know that in an isentropic flow, ρ decreases as V increases. In subsonic flow, V increases faster than ρ decreases so that, according to the preceding equation, A must decrease. In supersonic flow, ρ decreases more rapidly than V increases, and therefore A must increase to satisfy the continuity equation.

Last, we have the important result that $M = 1$ in the throat of a nozzle accelerating a gas. When $M = 1$ at the throat, the nozzle reaches maximum possible mass flow for the given chamber pressure and temperature, and the nozzle is

said to be *choked*, and M will equal 1 only at the nozzle throat. In a decelerating diffuser flow, on the other hand, the throat Mach number may be less than, equal to, or greater than 1.

2.9.2 Nozzle Operating Characteristics for Isentropic Flow

Having designed a nozzle for a specific operating condition, we now examine its off-design operating characteristics. We wish to answer the following question: Given a nozzle, what are the possible isentropic pressure distributions and mass flow rates through the nozzle?

A simple way to investigate this question is to deal with a single equation that contains all the restrictions placed on the flow by the perfect gas state equations and the control volume equations. The governing equations may be combined into a single equation. We have

Mass:

$$\dot{m} = \rho AV = \frac{PAV}{RT} \quad (\text{viii})$$

Energy:

$$T_t = T + \frac{V^2}{2g_c c_p} \quad (\text{ix})$$

Entropy:

$$P = P_t (T/T_t)^{\gamma/(\gamma-1)} \quad (\text{x})$$

Equation (viii) can be written as

$$\frac{\dot{m}}{A} = \frac{PV}{RT} \quad (\text{xi})$$

wherein

$$P = P_t \left(\frac{P}{P_t} \right) \quad T = T_t \left(\frac{T}{T_t} \right)$$

and

$$V = \sqrt{2g_c c_p (T_t - T)} = \left\{ 2g_c c_p T_t \left[1 - \left(\frac{P}{P_t} \right)^{(\gamma-1)/\gamma} \right] \right\}^{1/2}$$

Substituting these expressions for P , T , and V in Eq. (xi) and simplifying, we obtain a single equation representing the simultaneous solution of

Eqs. (viii), (ix), and (x):

$$\frac{\dot{m}}{A} = \frac{P_t}{\sqrt{T_t}} \sqrt{\frac{2g_c}{R} \frac{\gamma}{\gamma-1} \left[\left(\frac{P}{P_t} \right)^{2/\gamma} - \left(\frac{P}{P_t} \right)^{(\gamma+1)/\gamma} \right]} \quad (2.115)$$

If Eq. (2.115) is satisfied at every station of the flow through a nozzle, it follows that the conditions imposed on the flow by the thermal state equation and the mass, energy, and entropy control volume equations are satisfied. With P_c and T_c known in any given nozzle flow, we may effect a graphical solution of Eq. (2.115) by plotting \dot{m}/A vs P/P_t . In a physical flow, P/P_t may vary from 1.0 in a storage chamber ($P = P_c = P_t$) to 0 in a vacuum ($P = 0$). A graph of \dot{m}/A vs P/P_t is given in Fig. 2.35 for $P_c = 206$ psia and $T_c = 5000^\circ\text{R}$. Because there is a unique value of M for each P/P_t , we show a Mach number scale along with the P/P_t axis. Notice that M increases as P/P_t decreases from left to right in the figure.

We note that for a given value of \dot{m}/A , there are two possible values of P/P_t in Fig. 2.35. In a particular problem, we can determine which value of P/P_t is applicable by examining the physical aspects of the flow.

Assume the nozzle depicted in Fig. 2.36 is discharging air isentropically from a storage chamber with $P_c = 206$ psia and $T_c = 5000^\circ\text{R}$. Let us plot the nozzle

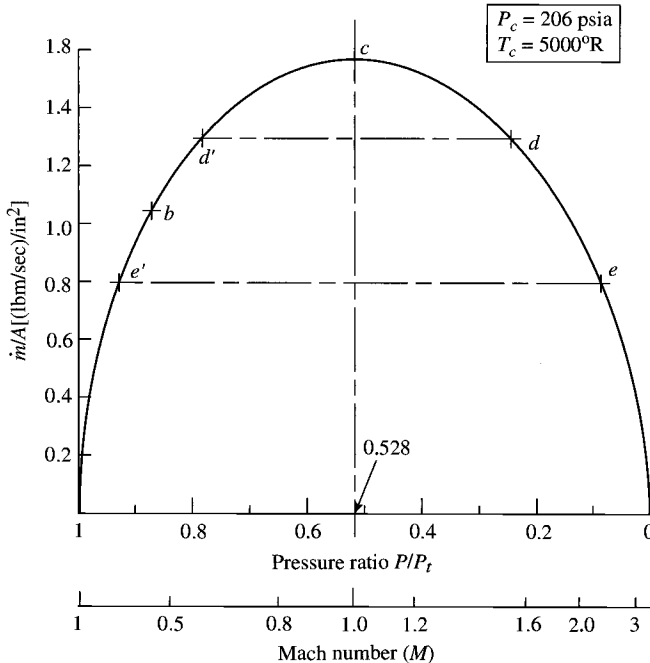


Fig. 2.35 \dot{m}/A vs P/P_t and Mach number for air.

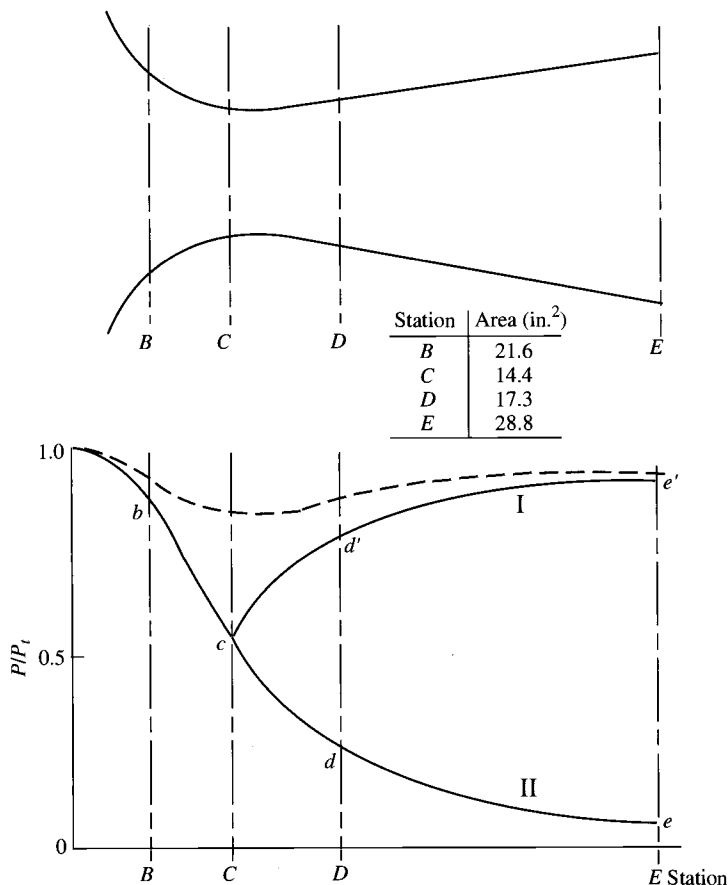


Fig. 2.36 Nozzle pressure distribution.

pressure distributions for various nozzle mass flows. We shall determine the pressure distribution for maximum mass flow first.

With the chamber pressure and temperature known, the \dot{m}/A vs P/P_t plot of Fig. 2.35 is made. Then, because for maximum mass flow $M = 1$ at the nozzle throat, \dot{m}_{\max} is determined by the relation

$$\begin{aligned}\dot{m}_{\max} &= \left(\frac{\dot{m}}{A} \right)_{M=1} A_{\text{throat}} \\ &= [1.55(\text{lbm/s})/\text{in.}^2](14.4 \text{ in.}^2) = 22.3 \text{ lbm/s}\end{aligned}$$

With \dot{m} and the areas at nozzle stations B, C, D, and E of Fig. 2.36 known, we determine \dot{m}/A at these stations. With these values of \dot{m}/A , we locate state

points b , c , d , and e , as shown in Fig. 2.35, and read values of P/P_t corresponding to stations B , C , D , and E .

Beginning at the storage chamber, $P/P_t = 1$. Then as A decreases and \dot{m}/A increases, P/P_t decreases from 1.0 to band c at the throat, as indicated in Figs. 2.35 and 2.36. After passing through the nozzle throat, A increases and \dot{m}/A decreases. Now there may physically exist either value of the ratio P/P_t , corresponding to a given \dot{m}/A with a continuous variation in pressure through the nozzle being maintained. Thus at section D , the pressure may be that corresponding to d or d' in Figs. 2.35 and 2.36. Whichever value exists depends on the nozzle pressure ratio $P_n = P_c/P_a$. The isentropic nozzle pressure distributions for maximum mass flow are the solid lines labeled I and II in the graph of Fig. 2.36. The dashed line represents a nozzle pressure distribution for a mass flow less than maximum and, hence, subsonic flow through the nozzle.

The nozzle pressure distribution corresponding to flow in I of Fig. 2.36 can be produced by nozzle pressure ratios other than the design value $P_{\bar{n}}$. However, the nozzle exit plane pressure P_e and the exhaust region pressure P_a are equal only for the design nozzle pressure ratio $P_{\bar{n}}$. At the off-design pressure ratios producing flow I, P_e remains the same, as given by

$$P_e = P_c/P_{\bar{n}}$$

but is either greater than or less than the exhaust region pressure P_a .

When $P_e > P_a$, the nozzle is said to be *underexpanded*. Under these conditions, the gas in the nozzle has not expanded down to the exhaust region pressure. Similarly, when $P_e < P_a$, the nozzle is said to be *overexpanded* because the gas in the nozzle has expanded to a value below the exhaust region pressure.

We see from Fig. 2.36 that there are no solutions of the equation

$$\frac{\dot{m}}{A} = f\left(\frac{P}{P_t}\right)$$

which gives $P_e = P_a$ for exhaust region P_a between $P_{e'}$ and P_e . Physically, it is possible to have a discharge region pressure in this range. What happens when such an exhaust region pressure exists? To answer this question, let us discuss the operating characteristics of a nozzle that might be used as a high-speed wind tunnel.

2.9.3 Nozzle Flow and Shock Waves

Figure 2.37 shows a wind-tunnel nozzle that we shall use for purposes of discussion. The tunnel operates between an air storage chamber maintained at P_c and T_c and an evacuated receiver. The pressure of the receiver P_a increases as air flows from the storage chamber through the tunnel into the receiver. In this way, the nozzle pressure ratio $P_n = P_c/P_a$ decreases from a very high value (due to a low P_a initially) to a value of 1 when the receiver pressure becomes equal to the storage chamber pressure and flow ceases. A rocket engine nozzle

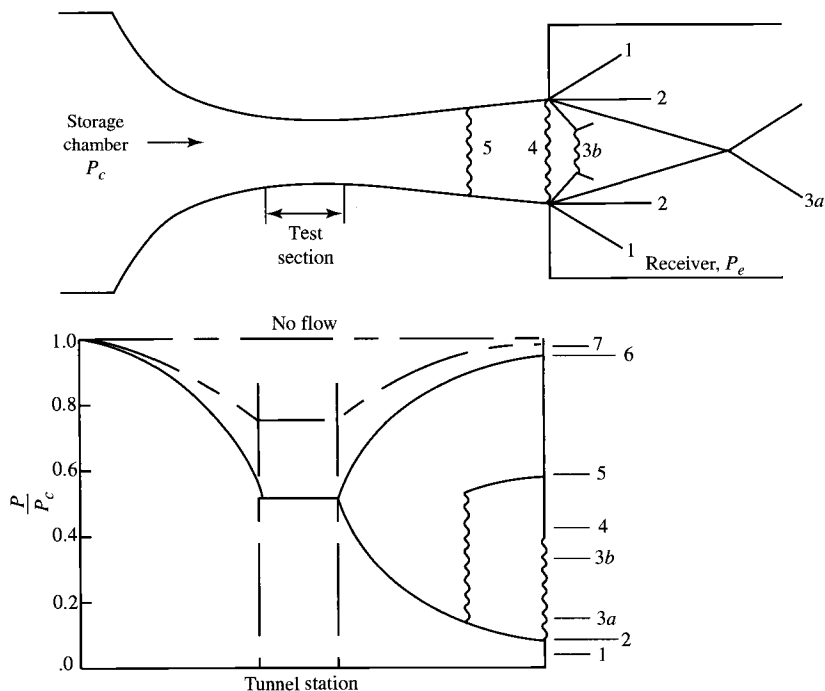


Fig. 2.37 Nozzle flow with shock waves.

descending through the atmosphere would experience a similar decrease in pressure ratio as P_a increases and P_c remains constant.

During the operation of the tunnel, the air flowing into the evacuated receiver raises the pressure in the nozzle exhaust region P_a and decreases the nozzle pressure ratio P_n . As a result, seven distinct nozzle pressure ratio operating conditions are present. They are depicted in Figs. 2.37 and 2.38 and tabulated in Table 2.3. The coordinates of the operating diagram in Fig. 2.38 are the nozzle pressure ratio and nozzle area ratio ϵ , where ϵ is the ratio of the nozzle exit area A_e to the nozzle throat area A_t . If we assume the tunnel of Fig. 2.37 has an area ratio of $\epsilon = 2$, then the operating points of the tunnel all lie along the horizontal line $\epsilon = 2$ in the nozzle operating diagram of Fig. 2.38. The following conditions are possible:

1) *Underexpanded*, $P_n > P_{\bar{n}}$. The pressure in the evacuated receiver is less than the nozzle exit plane pressure, so that $P_e > P_a$. The nozzle is operating underexpanded with $P_n > P_{\bar{n}}$. The flow *inside* the nozzle is the same as that corresponding to the design point pressure ratio $P_{\bar{n}}$. The flow *outside* the nozzle *does not* correspond to that for $P_{\bar{n}}$ since $P_e \neq P_a$. The transition of the nozzle exit plane pressure from P_e to the lower receiver pressure P_a occurs in the exhaust region, as depicted in the underexpanded portion of the nozzle operating diagram (Fig. 2.38).

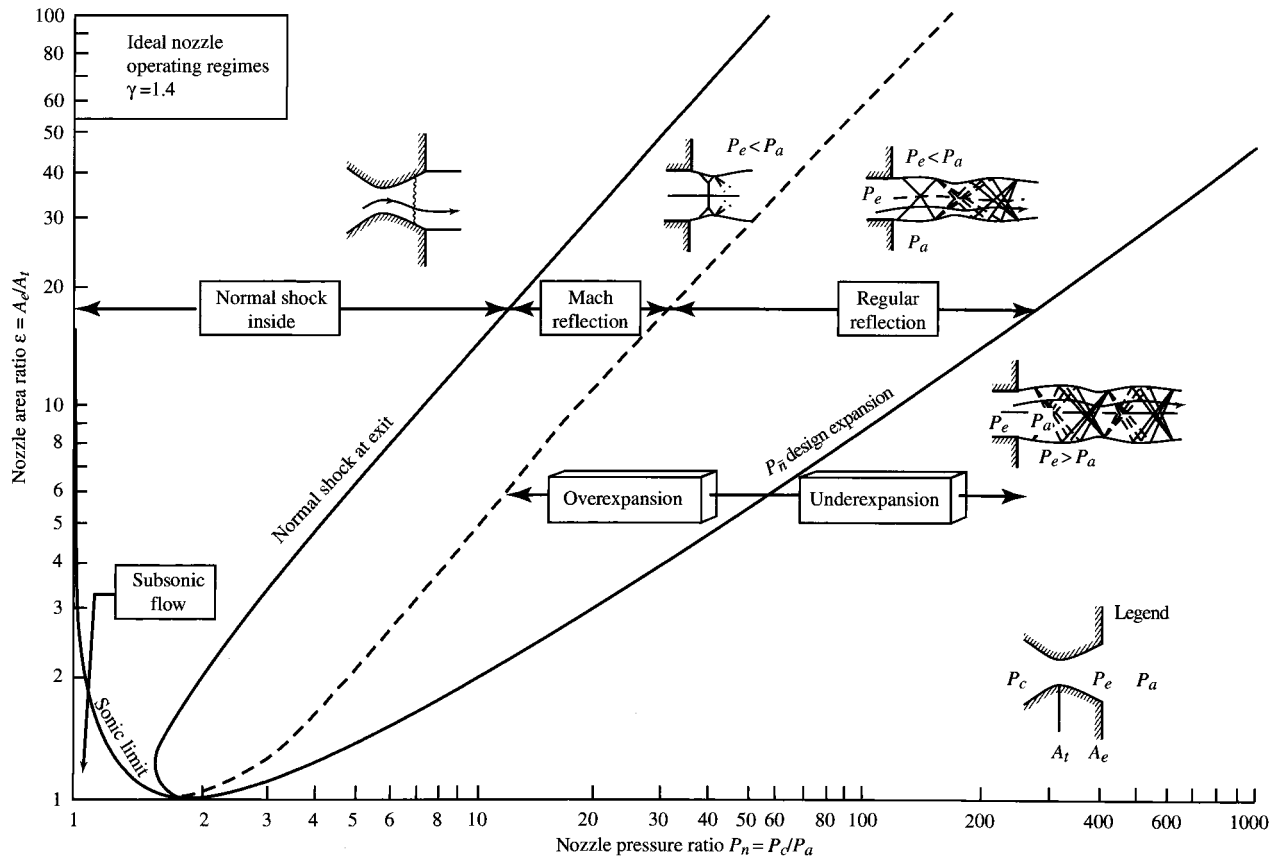


Fig. 2.38 Nozzle operating diagram.

Table 2.3 Nozzle operating points

Operand point	Exit section pressure P_e , $P_e = P_c/P_{\bar{n}}$	Nozzle pressure ratio $P_n = P_c/P_a$	Mass flow rate
1) Underexpanded	$P_e > P_a$	$P_n > P_{\bar{n}}$	Maximum
2) Design	$P_e = P_a$	$P_n = P_{\bar{n}}$	Maximum
3) Overexpanded	$P_e < P_a$	$P_n < P_{\bar{n}}$	Maximum
a) Regular reflection			
b) Mach reflection			
4) Normal shock at exit	$(P_e)_x < (P_e)_y = P_a$	$P_n < P_{\bar{n}}$	Maximum
5) Normal shock in divergent section	$P_e = P_a$	$P_n > P_{\bar{n}}$	Maximum
6) Sonic at throat subsonic elsewhere	$P_e = P_a$	$P_n < P_{\bar{n}}$	Maximum
7) Subsonic everywhere	$P_e = P_a$	$P_n < P_{\bar{n}}$	Less than maximum

2) *Design expansion*, $P_n = P_{\bar{n}}$. The pressure in the receiver has been increased to the nozzle exit plane pressure $P_e = P_a$ by the air flowing into the receiver. This is the nozzle design operating point with $P_n = P_{\bar{n}}$. No pressure disturbances occur in the jet issuing from the nozzle. This operating point is on the design expansion line of the nozzle operating diagram.

3) *Overexpanded*, $P_n < P_{\bar{n}}$. The pressure in the receiver is greater than the nozzle exit plane pressure $P_e < P_a$. The nozzle is operating overexpanded with $P_n < P_{\bar{n}}$. The transition from P_e to the higher receiver pressure is produced by either an oblique shock wave system (regular reflection 3a) or a combined oblique-normal shock wave system (Mach reflection pattern 3b). This nozzle operating condition lies between the design expansion line and normal-shock-at-exit line of the nozzle operating diagram.

4) *Normal shock at exit*. The pressure in the receiver has increased to a value that has moved the normal shock wave of condition 3b into the nozzle exit plane. The pressure of the gas entering the normal shock is $(P_e)_x$, and the pressure leaving is $(P_e)_y = P_a$. The loci of this operating condition are on the normal-shock-at-exit line of the nozzle operating diagram (Fig. 2.38).

5) *Normal shock inside*. The receiver pressure has increased to a value that has caused the normal shock to move into the diverging portion of the nozzle. Flow in the nozzle preceding the shock is unaffected. The flow through the shock is irreversible at constant T_t such that the total pressure decreases across the shock wave. The flow downstream of the shock is subsonic. This operating condition lies between the normal-shock-at-exit and sonic limit lines of the nozzle operating diagram.

6) *Sonic limit*. The receiver pressure has reached a value that produces isentropic shock-free flow throughout the nozzle with sonic throat conditions and

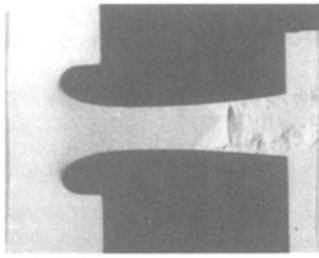
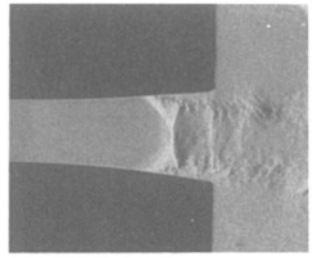
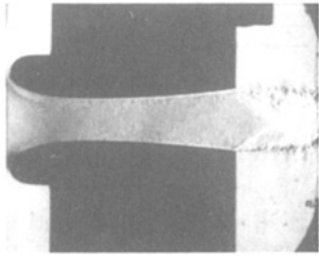
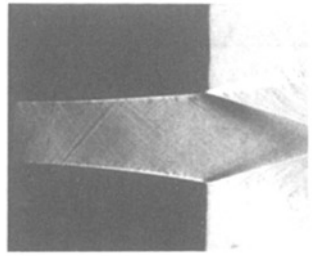
a) $P_n=1.5$, normal shock, insideb) $P_n=2.5$, Mach reflection, overexpansionc) $P_n=4.5$, regular reflection, overexpansiond) $P_n=8.0$, underexpansion

Fig. 2.39 Spark Schlieren photographs of nozzle exhaust flow patterns for an area ratio of 1.5. (Department of Aero-Mechanical Engineering, AFIT, WPAFB, Ohio.)

subsonic flow elsewhere. The nozzle pressure ratios corresponding to this operating condition lie on the sonic limit line of the nozzle operating diagram. This nozzle flow corresponds to flow I of Fig. 2.36.

7) *Subsonic flow.* The receiver pressure has risen to a value producing subsonic flow throughout with a reduced mass flow. This operating condition is bounded by the sonic limit and the $P_n = 1$ (no-flow) lines of the nozzle operating diagram.

Figure 2.39 is a picture of several nozzle exhaust flow patterns for a conical nozzle with an area ratio of $\epsilon = 1.5$. For $P_n = 1.5$, the normal shock is located well within the nozzle. As P_n is increased to 2.5, the flow is overexpanded in the Mach reflection regime, as evidenced by the clearly defined normal shock internal to the flow. Further increase in P_n causes this normal shock in the Mach reflection pattern to move farther out and diminish in size. It disappears when the regular reflection pattern is obtained. For $P_n = 4.5$, the nozzle is overexpanded in the regular reflection regime. The photograph with $P_n = 8.0$ corresponds to an underexpanded operating point with the exhaust gas expanding down to the ambient pressure in the jet plume.

2.9.4 Nozzle Characteristics of Some Operational Engines

The operating point of a nozzle is determined by the nozzle pressure ratio P_n and area ratio ϵ . These ratios are presented in Table 2.4 along with other data for

Table 2.4 Some nozzle characteristics of rocket and turbojet engines

Engine	Chamber pressure P_c , psia	Area ratio ϵ	P_a	$(P_a)_{\text{operate}}$
Saturn				
F-1 without extension	965	10	140	Varies with altitude P_a
F-1 with extension	965	16	275	
J-2	763	27.5	610	
Atlas				
Booster	703	8	100	Varies with altitude P_a
Sustainer	543	25	528	
Subsonic airbreathing turbofan (40,000 ft)	8–14	1.0	1.9	3–5
Supersonic airbreathing turbofan (40,000 ft)	8–56	1–2	1.9–8	3–20

some operational nozzles. The data in Table 2.4 permit us to locate the operating points of the Saturn and Atlas engines at a given ϵ in the nozzle operating diagram (Fig. 2.38). In the Saturn F-1 engine, $\epsilon = 16$ and $P_c = 965$ psia. Thus at 46,000 ft with $P_a = 2$ psia, we have $P_n = 965/2 = 482$ and a nozzle operating point of (482, 16). This operating point (assuming $\gamma = 1.4$) indicates that the F-1 nozzle is operating above the nozzle design point (underexpanded) at 46,000 ft. At sea level for the F-1, P_n is about 65, and the operating point of the engine nozzle (assuming $\gamma = 1.4$) is in the overexpanded regular reflection region. The turbojet engines of high-performance, airbreathing, subsonic aircraft generally use convergent nozzles ($\epsilon = 1.0$) and operate with nozzle pressure ratios greater than the design value of 1.9 (assuming $\gamma = 1.4$). These nozzles therefore operate in the underexpanded operating regime. Turbojet/turbofan engines of supersonic aircraft, however, have converging-diverging nozzles.

2.10 One-Dimensional Gas Dynamics—Differential Control Volume Analysis

The steady one-dimensional flow of a chemically inert perfect gas with constant specific heats is conveniently described and governed by the following definitions and physical laws.

2.10.1 Definitions

Perfect gas:

$$P = \rho RT \quad (\text{i})$$

Mach number:

$$M^2 = V^2/(\gamma R g_c T) \quad (\text{ii})$$

Total temperature:

$$T_t = T \left(1 + \frac{\gamma - 1}{2} M^2 \right) \quad (\text{iii})$$

Total pressure:

$$P_t = P \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{\gamma/(\gamma-1)} \quad (\text{iv})$$

2.10.2 Physical Laws

For one-dimensional flow through a control volume having a single inlet and exit sections 1 and 2, respectively, we have

One-dimensional mass flow:

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad (\text{v})$$

Momentum:

$$F_{\text{frict}} = -\Delta(PA + \rho A V^2 / g_c) \quad (\text{vi})$$

Energy equation (no shaft work):

$$q = c_p(T_{t2} - T_{t1}) \quad (\text{vii})$$

Entropy equation (adiabatic flow):

$$s_2 \geq s_1 \quad (\text{viii})$$

where F_{frict} is the frictional force of a solid control surface boundary on the flowing gas, and A is the flow cross-sectional area normal to the velocity V . The frictional force acting on a length L can be written as

$$F_{\text{frict}} = f \{ \rho V^2 / (2 g_c) \} \text{ wetted perimeter} \times L$$

or

$$F_{\text{frict}} = f \left(\frac{\rho V^2}{2 g_c} \right) \left(\frac{4A}{D} \right) L \quad (2.116)$$

where

f = frictional coefficient

D = hydraulic diameter = $\frac{4 \times A}{\text{wetted perimeter}}$

A = flow cross-sectional area

Now consider the differential element of duct with length dx , as shown in Fig. 2.40. The independent variables are the area change, total temperature change, and frictional force. The dependent variables are P , T , ρ , V , M^2 , and P_t . The application of Eqs. (i–vi) to flow in Fig. 2.40, having the presence of the simultaneous effects of area change, heat interaction, and friction, results in the following set of equations for the infinitesimal element dx .

Perfect gas:

$$-\frac{dP}{P} + \frac{d\rho}{\rho} + \frac{dT}{T} = 0 \quad (2.117a)$$

Total temperature:

$$\frac{dT}{T} + \frac{[(\gamma - 1)/2]M^2}{1 + [(\gamma - 1)/2]M^2} \frac{dM^2}{M^2} = \frac{dT_t}{T_t} \quad (2.117b)$$

One-dimensional mass flow:

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0 \quad (2.117c)$$

Total pressure:

$$\frac{dP}{P} + \frac{(\gamma/2)M^2}{1 + [(\gamma - 1)/2]M^2} \frac{dM^2}{M^2} = \frac{dP_t}{P_t} \quad (2.117d)$$

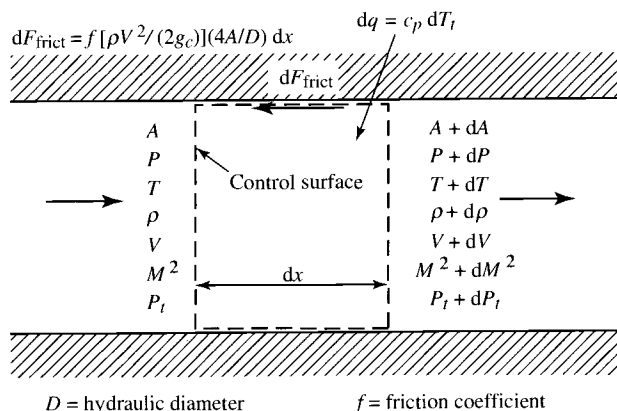


Fig. 2.40 Independent and dependent variables for one-dimensional flow.

Momentum:

$$\frac{dP}{P} + \gamma M^2 \frac{dV}{V} + 2\gamma M^2 f \frac{dx}{D} = 0 \quad (2.117e)$$

Mach number:

$$2 \frac{dV}{V} - \frac{dT}{T} = \frac{dM^2}{M^2} \quad (2.117f)$$

With the exception of the momentum equation, these equations are obtained by taking the derivative of the natural log of Eqs. (i–v).

In these equations, heat interaction effects are measured in terms of the total temperature change according to Eq. (vii). The entropy condition of Eq. (viii) is also applicable if $dT_t = 0$. If dT_t is not zero, then the entropy requirement is $ds = dq/T$. The six dependent variables M^2 , V , P , ρ , T , and P_t in the preceding set of six linear algebraic equations may be expressed in terms of the three independent variables A , T_t , and $4f \, dx/D$. The solution is given in Table 2.5. Note that D is the hydraulic diameter ($= 4 \times \text{flow cross-sectional area/wetted perimeter}$).

General conclusions can be made relative to the variation of the stream properties of the flow with each of the independent variables by the relations of Table 2.5. As an example, the relationship given for dV/V at the bottom of the table indicates that, in a constant-area adiabatic flow, friction will increase the stream velocity in subsonic flow and will decrease the velocity in supersonic flow. Similar reasoning may be applied to determine the manner in which any dependent property varies with a single independent variable.

Example 2.12

Consider the one-dimensional flow of a perfect gas in a channel of circular cross section. The flow is adiabatic, and we wish to design the duct so that its area varies with x such that the velocity remains constant.

1) Show that in such a flow, the temperature must be a constant, hence the Mach number must be a constant.

2) Show that the total pressure varies inversely with the cross-sectional area.

3) Show that $2 \, dA/C = \gamma M^2 f \, dx$, where C is the circumference.

4) For a constant f and a circular channel of diameter D , show that the diameter must vary in accordance with the following equation to keep the velocity constant:

$$D = D_1 + \gamma M^2 f x$$

Solution: This problem requires that we apply the relationships of Table 2.5 and other fundamental relationships.

1) Because the flow is adiabatic, the total temperature is constant. From the definition of the total temperature, we have $T_t = T + V^2/(2c_p g_c)$. Since T_t and

Table 2.5 Influence coefficients for steady one-dimensional flow

Dependent	Independent		
	$\frac{dA}{A}$	$\frac{dR}{T_i}$	$\frac{4f dx}{D}$
$\frac{dM^2}{M^2}$	$-\frac{2\left(1 + \frac{\gamma-1}{2}M^2\right)}{1-M^2}$	$\frac{(1 + \gamma M^2)\left(1 + \frac{\gamma-1}{2}M^2\right)}{1-M^2}$	$\frac{\gamma M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}{1-M^2}$
$\frac{dV}{V}$	$-\frac{1}{1-M^2}$	$\frac{1 + \frac{\gamma-1}{2}M^2}{1-M^2}$	$\frac{\gamma M^2}{2(1-M^2)}$
$\frac{dP}{P}$	$\frac{\gamma M^2}{1-M^2}$	$-\frac{\gamma M^2\left(1 + \frac{\gamma-1}{2}M^2\right)}{1-M^2}$	$-\frac{\gamma M^2[1 + (\gamma-1)M^2]}{2(1-M^2)}$
$\frac{dp}{p}$	$\frac{M^2}{1-M^2}$	$-\frac{\left(1 + \frac{\gamma-1}{2}M^2\right)}{1-M^2}$	$\frac{-\gamma M^2}{2(1-M^2)}$
$\frac{dT}{T}$	$\frac{(\gamma-1)M^2}{1-M^2}$	$\frac{(1 - \gamma M^2)\left(1 + \frac{\gamma-1}{2}M^2\right)}{1-M^2}$	$-\frac{\gamma(\gamma-1)M^4}{2(1-M^2)}$
$\frac{dP_t}{P_t}$	0	$\frac{-\gamma M^2}{2}$	$\frac{-\gamma M^2}{2}$

This table is read:

$$\frac{dV}{V} = \left(-\frac{1}{1-M^2}\right) \frac{dA}{A} + \frac{1 + \frac{\gamma-1}{2}M^2}{1-M^2} \frac{dT_i}{T_i} + \frac{\gamma M^2}{2(1-M^2)} \frac{4f dx}{D}$$

V are constant in the duct, the preceding equation requires that the static temperature remain constant. With the static temperature of the gas constant, the speed of sound will be constant ($a = \sqrt{\gamma R g_c T}$). With constant velocity and speed of sound, the Mach number will be constant.

2) Application of the continuity equation to the constant-velocity flow gives $A_i/A = \rho/\rho_i$. For a perfect gas with constant static temperature, we have $\rho/\rho_i = P/P_i$ and from Eq. (iv) for constant-Mach flow, we get $P_t/P_{ti} = P/P_i$. Thus $P_t/P_{ti} = A_i/A$.

3) To obtain this relationship, we need to get a relationship between two independent properties to keep the dependent property of velocity constant. Table 2.5 gives the basic relationships between dependent and independent properties for one-dimensional flow. We are interested in the case where velocity is constant,

and thus we write the equation listed as an example at the bottom of Table 2.5 for the case where both dV and dT_t are zero:

$$0 = \left(-\frac{1}{1-M^2} \right) \frac{dA}{A} + 0 + \frac{\gamma M^2}{2(1-M^2)} \frac{4f dx}{D}$$

The area of the duct is equal to the circumference C times one quarter of the diameter D . Thus $A = CD/4$, and the preceding relationship reduces to

$$2 dA/C = \gamma M^2 f dx$$

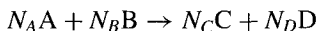
4) For a circular cross section, $A = \pi D^2/4$ and $C = \pi D$. Thus $2dA/C = dD$. Substitution of this relationship into the preceding equation and integration give the desired result (at $x = 0$, $D = D_i$):

$$D = D_i + \gamma M^2 f x$$

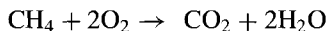
2.11 Chemical Reactions

2.11.1 General Characteristics

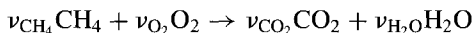
A chemical reaction from reactants A and B to products C and D is generally represented by



where N_A and N_B are the number of moles of reactants A and B, respectively. Likewise N_C and N_D are the number of moles of products C and D. An example reaction between one mole of methane (CH_4) and two moles of oxygen (O_2) can be written as



and represents *complete combustion* between the fuel, methane, and oxidizer, oxygen. Combustion is complete when all of the carbon in the fuel burns to CO_2 and the hydrogen burns to H_2O . The combustion is incomplete when there is any unburned fuel or compounds such as carbon, hydrogen, CO, or OH. This *theoretical process* of complete combustion is very useful in our analysis because it represents the number of moles of each reactant that are needed for complete combustion to the number of moles of each product. The theoretical or complete combustion process is also referred to as a stoichiometric combustion process and is represented as



where ν_{CH_4} and ν_{O_2} are the number of moles of reactants CH_4 and O_2 , and ν_{CO_2} and $\nu_{\text{H}_2\text{O}}$ are the number of moles of products CO_2 and H_2O . The ν_i in the

preceding equation are called the *stoichiometric coefficients* for the theoretical reaction. Thus the stoichiometric coefficients for the preceding reaction of methane and oxygen are

$$\nu_{\text{CH}_4} = 1 \quad \nu_{\text{O}_2} = 2 \quad \nu_{\text{CO}_2} = 1 \quad \nu_{\text{H}_2\text{O}} = 2$$

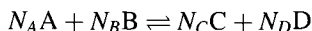
The stoichiometric coefficients give the number of moles of each constituent required for complete combustion. Note that the units of both the mole number N_i and the stoichiometric coefficients ν_i can also be regarded as moles *per unit mass* of the gas mixture.

The *actual process* between reactants normally has different amounts of each reactant than that of the theoretical (stoichiometric) process. For example, consider one mole of methane reacting with three moles of oxygen. The stoichiometric coefficients of the theoretical reaction indicates that only two moles of oxygen are required for each mole of methane. Thus one mole of oxygen will not react and remains on the products side of the chemical equation as shown:



2.11.2 Chemical Equilibrium

The chemical reactions represented so far were shown as complete reactions. In reality, the reactions go both forward and backward and the actual process is denoted as



Chemical equilibrium is reached when the forward rate of reaction of reactants A and B equals the backward rate of reaction of products C and D. It can be shown that chemical equilibrium corresponds to the minimum value of the Gibbs function ($G = H - TS$) for the entire gas mixture as shown in Fig. 2.41. Chemical equilibrium also corresponds to maximum entropy for an adiabatic system.

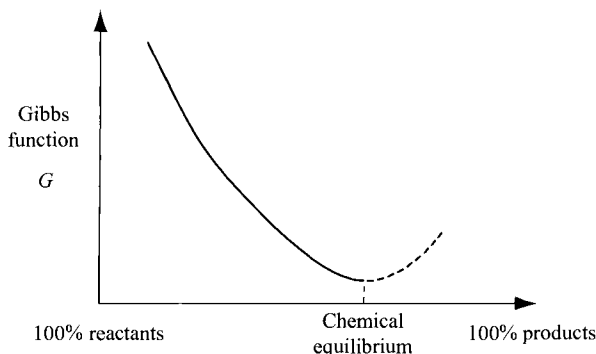


Fig. 2.41 Chemical equilibrium.

Chemical equilibrium for a stoichiometric (theoretical) reaction can be represented as



For solution of equilibrium problems, we define the *equilibrium constant* of a perfect gas as

$$K_P \equiv \frac{P_v^{v_C} P_D^{v_D}}{P_A^{v_A} P_B^{v_B}} \quad (2.119)$$

where P_A , P_B , P_C , and P_D are the partial pressures of components A, B, C, and D; v_A , v_B , v_C , and v_D are the stoichiometric coefficients shown in Eq. (2.118). For a perfect gas, K_P is only a function of temperature.

From Eq. (2.46), the partial pressure of constituent i can be written as

$$P_i = \frac{N_i}{N_{\text{total}}} P \quad (2.120)$$

where N_{total} is the total number of moles present in the reaction chamber. Using Eq. (2.120), Eq. (2.119) for the equilibrium constant can be rewritten as

$$K_P = \frac{N_C^{v_C} N_D^{v_D}}{N_A^{v_A} N_B^{v_B}} \left(\frac{P}{N_{\text{total}}} \right)^{\Delta v} \quad (2.121)$$

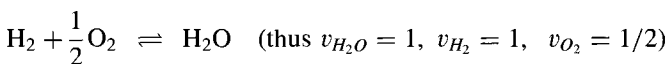
where $\Delta v = v_C + v_D - v_A - v_B$. Equation (2.121) can be used to verify equilibrium results from complex computer programs as will be done later in this section.

Example 2.13

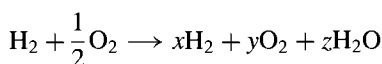
One mole of hydrogen and one-half mole of oxygen are in equilibrium at a temperature of 2000 K and a pressure of 10 atm. Determine the number of moles of H_2O , H_2 , and O_2 in the products (neglect other products such as OH, H, O, etc.).

Solution: We start by writing the chemical reaction equation for both the stoichiometric reaction and the actual reaction being modeled.

Stoichiometric:



Actual:



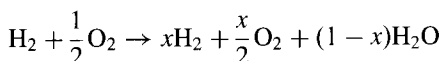
Because there are the three unknowns (x , y , and z), three equations are required. Two equations come from the atom balance of each element (H and O) and the third from the chemical equilibrium coefficient (K_p):

Atom balance:

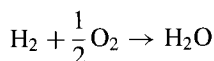
$$\text{H} : 2 = 2x + 2z \quad \text{or } z = 1 - x$$

$$\text{O} : 1 = 2y + z \quad \text{or } y = x/2$$

Now the actual reaction equation can be written with just one unknown:



From the JANNAF tables in the Supporting Material, $\log K_p = 3.540$ at 2000 K for the reaction



Thus

$$K_p = 10^{3.540} = 3467.4 = K_p = \frac{N_C^{v_{\text{H}_2\text{O}}}}{N_A^{v_{\text{H}_2}} N_B^{v_{\text{O}_2}}} \left(\frac{P}{N_{\text{total}}} \right)^{\Delta v}$$

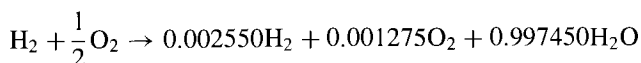
where $N_A = x$, $N_B = x/2$, $N_C = 1 - x$, $\Delta v = -1/2$, and $N_{\text{total}} = 1 + x/2$. Thus

$$K_p = \frac{(1-x)}{x^1 \sqrt{x/2}} \left(\frac{10}{1+x/2} \right)^{-1/2} = 3467.4$$

or

$$\frac{(1-x)\sqrt{1+x/2}}{x^{3/2}} = 3467.4\sqrt{10/2} = 7753.3$$

Solving for x gives $x = 0.002550$, and the reaction can be written as



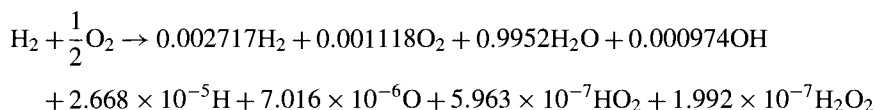
Thus there are 0.002550 moles of hydrogen, 0.001275 moles of oxygen, and 0.997450 moles of water.

2.11.3 EQL Software

The included EQL software program by David T. Pratt uses the NASA Glenn thermochemical data and the Gordon-McBride equilibrium algorithm (see Ref. 18). This equilibrium calculation is based on finding the mixture composition that minimizes the Gibbs function, which allows rapid solution of reactions resulting in many products.

Example 2.14

The EQL software program gives the following result for one mole of hydrogen and one-half mole of oxygen at a temperature of 2000 K and a pressure of 10 atm:



The 0.0022 moles less of H_2O are produced in this equilibrium mixture than Example 2.13, and these molecules show up in increases in the number of moles of H_2 and O_2 .

The equilibrium constant can be determined from these data and Eq. (2.121):

$$K_P = \frac{N_C^{v_{\text{H}_2\text{O}}}}{N_A^{v_{\text{H}_2}} N_B^{v_{\text{O}_2}}} \left(\frac{P}{N_{\text{total}}} \right)^{\Delta v} = \frac{0.9952}{0.00271 \times \sqrt{0.001118}} \left(\frac{10}{0.9990} \right)^{-1/2} = 3471$$

or $\log K_P = 3.540$, which agrees with the JANNAF tables.

2.11.4 Enthalpy of Chemical Component, Enthalpy of Formation, Heat of Reaction, and Adiabatic Flame Temperature

For reacting systems, the enthalpy of each component must be written in a form that has the same reference state. The *enthalpy of chemical components* (products or reactants) can be calculated using Eq. (2.50) with \bar{h} for a component written as

$$\bar{h} = \bar{h}_f^\circ + (\bar{h} - \bar{h}^\circ) = \bar{h}_f^\circ + \int_{T_d}^T \bar{c}_p \, dT' \quad (2.122)$$

where \bar{h}_f° is the *enthalpy of formation* (also called the heat of formation) at the reference state (datum) of 25°C (298 K), 1 atm ($\bar{h} - \bar{h}^\circ$) is the enthalpy change due to the temperature change from the reference state (T_d), and \bar{c}_p is the specific heat at constant pressure per mole. Typical values of \bar{h}_f° are given in Table 2.6 and values ($\bar{h} - \bar{h}^\circ$) for typical gases are given in the JANNAF tables of the Supporting Material.

Table 2.6 Enthalpy of formation \bar{h}_f° for some reactants and product gases at datum temperature 536°R/298 K (Refs. 12, 13)

Gas	Btu/lbmol	kJ/kgmol
Methane, CH ₄	-32,192	-74,883
Ethane, C ₂ H ₆	-36,413	-84,701
Hexane, C ₆ H ₁₄	-71,784	-166,978
Octane, C ₈ H ₁₈	-89,600	-208,421
Jet-A, C ₁₂ H ₂₃	-152,981	-355,853 ^a
Carbon monoxide, CO	-47,520	-110,537
Carbon dioxide, CO ₂	-169,181	-393,538
Atomic hydrogen, H	93,717	217,997
Hydrogen, H ₂	0	0
Water vapor, H ₂ O	-103,966	-241,838
Atomic oxygen, O	107,139	249,218
Oxygen, O ₂	0	0
Hydroxyl, OH	16,967	39,467
Atomic nitrogen, N	203,200	472,668
Nitrogen, N ₂	0	0
Nitrous oxide, N ₂ O	35,275	82,053
Nitric oxide, NO	38,817	90,293
Nitrogen dioxide, NO ₂	14,228	33,096

^aFor heating value $h_{PR} = 18,400 \text{ Btu/lbm} = 42,800 \text{ kJ/kg}$.

The enthalpy of both products and reactants in a reaction can be plotted vs temperature as shown in Fig. 2.41. A reaction typically causes a change in temperature. The *heat of reaction* ΔH is defined as the positive heat transfer to the products that is required to bring them back to the original temperature of the reactants. For ideal (perfect) gases, the heat of reaction ΔH at the standard reference temperature can be calculated using

$$\Delta H = H_P - H_R = \sum_{i=1}^{n_P} N_i \Delta \bar{h}_{f_i}^\circ - \sum_{j=1}^{n_R} N_j \Delta \bar{h}_{f_j}^\circ \quad (2.123)$$

where N_i and N_j are the number of moles of the products and reactants, respectively, and $\Delta \bar{h}_f^\circ$ is the heat of formation per mole of species i and j , ΔH is the vertical difference between the enthalpy line of the reactants and that of the products in Fig. 2.42.

Consider the adiabatic flow through a combustion chamber like that of Fig. 2.4d. The reactants enter at a temperature denoted by 1 in Fig. 2.42 and the products leave at a temperature denoted by 2. Neglecting changes in kinetic energy, the energy balance for this process yields

$$H_1 = H_2 \quad \text{or} \quad H_P = H_R \quad (2.124)$$

The temperature at state 2 is called the *adiabatic flame temperature*. This temperature can be calculated by determining the equilibrium state with the

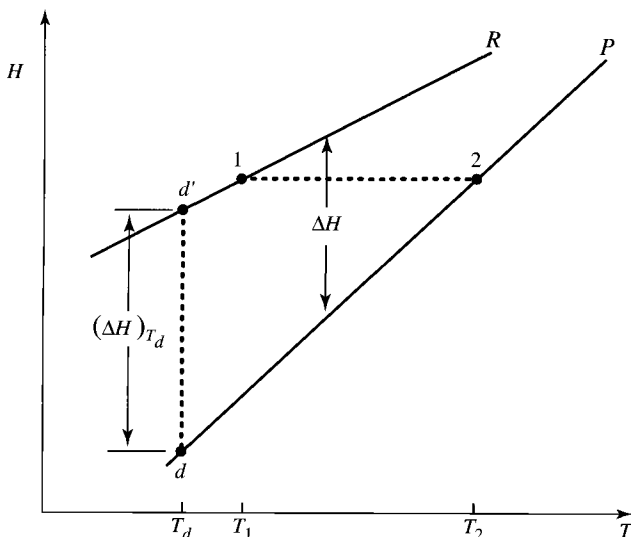


Fig. 2.42 Enthalpy states for a reaction.

same enthalpy H and pressure P as state 1. To facilitate calculation, we rewrite Eq. (2.124) in terms of the enthalpy difference $(\bar{h} - \bar{h}^\circ)$ and the heat of reaction at the datum (reference) state $[\Delta H]_{T_d}$. First we write

$$H_2 - [H_P]_{T_d} = H_1 - [H_R]_{T_d} + [\Delta H]_{T_d}$$

Since

$$H_2 - [H_P]_{T_d} = \sum_{j=1}^{n_P} N_j (\bar{h} - \bar{h}^\circ)_j$$

and

$$H_1 - [H_R]_{T_d} = \sum_{i=1}^{n_R} N_i (\bar{h} - \bar{h}^\circ)_i$$

then

$$H_2 - [H_P]_{T_d} = \sum_{j=1}^{n_P} N_j (\bar{h} - \bar{h}^\circ)_j = \sum_{i=1}^{n_R} N_i (\bar{h} - \bar{h}^\circ)_i + [\Delta H]_{T_d} \quad (2.125)$$

The following steps are used to calculate the adiabatic flame temperature:

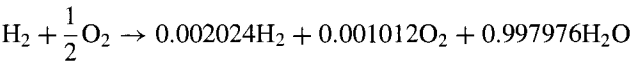
- 1) Assume a final temperature T_2 .
- 2) Calculate the mole fraction of the products for the resulting K_p at T_2 .
- 3) Calculate $\sum_{i=1}^{n_R} N_i (\bar{h} - \bar{h}^\circ)_i + [\Delta H]_{T_d}$.
- 4) Calculate $\sum_{j=1}^{n_P} N_j (\bar{h} - \bar{h}^\circ)_j$. If its value is greater than that of step 3, reduce the value of T_2 and perform steps 2–4 again.

This calculation by hand is tedious and has been programmed for rapid calculation using computers. The EQL software calculates the adiabatic flame temperature for given reactants and inlet temperature T and pressure P as shown in Example 2.16.

Example 2.15

Calculate the heat of reaction for the reaction of Example 2.13.

Solution: We have



at 2000 K. From the JANNAF tables in the Supporting Materials, at 2000 K we have $\Delta \bar{h}_{f\text{H}_2}^\circ = 0$, $\Delta \bar{h}_{f\text{O}_2}^\circ = 0$, and $\Delta \bar{h}_{f\text{H}_2\text{O}}^\circ = -60.150$ kcal/mole. Eq. (2.78) gives

$$\begin{aligned} \Delta H &= H_P - H_R \\ &= \sum_{i=1}^{n_P} N_i \Delta \bar{h}_{fi}^\circ - \sum_{j=1}^{n_R} N_j \Delta \bar{h}_{fj}^\circ = 0.997976 \times -60.150 = -60.028 \text{ kcal} \end{aligned}$$

Thus 60.028 kcal (251.28 kJ) of energy must be removed during the reaction to keep the temperature at 2000 K.

Example 2.16

Using the EQL software, determine the adiabatic flame temperature for one mole of oxygen and one-half mole of methane with the reactants at a temperature of 500 K and pressure of 800 kPa.

Solution: We first enter the reactants and inlet pressure and temperature into the opening screen of EQL. Next we select the Equilibrium Processes Tab, which displays the numerous possible combustion processes. We select ‘Adiabatic Flame Temperature’ and perform the calculations. The resulting adiabatic flame temperature is 3349.85 K, and the resulting principal products are listed below:

Compound	Mole fraction
H ₂ O	0.4093
CO	0.1540
CO ₂	0.1190
OH	0.0989
O ₂	0.0792
H ₂	0.0681
H	0.0382
O	0.0330

Problems

- 2.1** Consider Fig. P2.1. A stream of air with velocity of 500 ft/s and density of 0.07 lbm/ft^3 strikes a stationary plate and is deflected 90 deg. Select an appropriate control volume and determine the force F_P necessary to hold the plate stationary. Assume that atmospheric pressure surrounds the jet and that the initial jet diameter is 1.0 in.

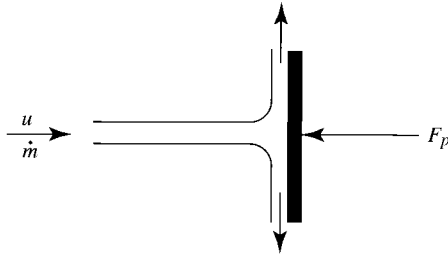


Fig. P2.1

- 2.2** Consider Fig. P2.1. An airstream with density of 1.25 kg/m^3 and velocity of 200 m/s strikes a stationary plate and is deflected 90 deg. Select an appropriate control volume and determine the force F_P necessary to hold the plate stationary. Assume that atmospheric pressure surrounds the jet and that the initial jet diameter is 1.0 cm.

- 2.3** Consider the flow shown in Fig. P2.2 of an incompressible fluid. The fluid enters (station 1) a constant-area circular pipe of radius r_0 with uniform velocity V_1 and pressure P_1 . The fluid leaves (station 2) with the parabolic velocity profile V_2 given by

$$V_2 = V_{\max} \left[1 - \left(\frac{r}{r_0} \right)^2 \right]$$

and uniform pressure P_2 . Using the conservation of mass and momentum equations, show that the force F necessary to hold the pipe in place can be

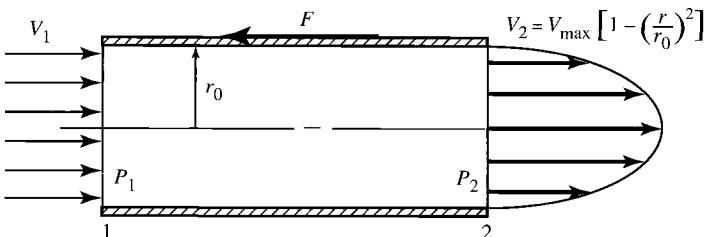


Fig. P2.2

expressed as

$$F = \pi r_0^2 \left(P_1 - P_2 + \frac{\rho V_1^2}{3g_c} \right)$$

- 2.4** Consider the flow of an incompressible fluid through a two-dimensional cascade as shown in Fig. P2.3. The airfoils are spaced at a distance s and have unit depth into the page. Application of the conservation of mass requires $V_i \cos \beta_i = V_e \cos \beta_e$.

(a) From the tangential momentum equation, show that

$$F_\theta = \frac{\dot{m}}{g_c} (V_i \sin \beta_i - V_e \sin \beta_e)$$

(b) From the axial momentum equation, show that

$$F_z = s(P_e - P_i)$$

(c) Show that the axial force can be written as

$$F_z = s \left[\frac{\rho}{2g_c} (V_i^2 \sin^2 \beta_i - V_e^2 \sin^2 \beta_e) - (P_{ii} - P_{ie}) \right]$$

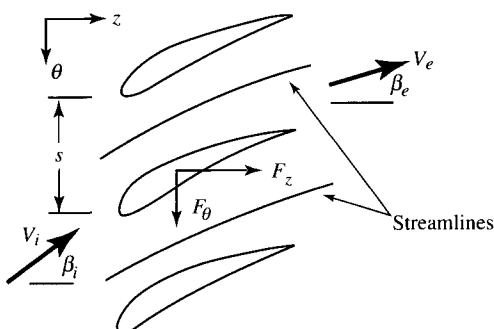


Fig. P2.3

- 2.5** When a freejet is deflected by a blade surface, a change of momentum occurs and a force is exerted on the blade. If the blade is allowed to move at a velocity, power may be derived from the moving blade. This is the basic principle of the impulse turbine. The jet of Fig. P2.4, which is initially horizontal, is deflected by a fixed blade. Assuming the same pressure surrounds the jet, show that the horizontal (F_x) and vertical (F_y) by the fluid on the blade are given by

$$F_x = \frac{\dot{m}(u_1 - u_2 \cos \beta)}{g_c} \quad \text{and} \quad F_y = \frac{\dot{m}u_2 \sin \beta}{g_c}$$

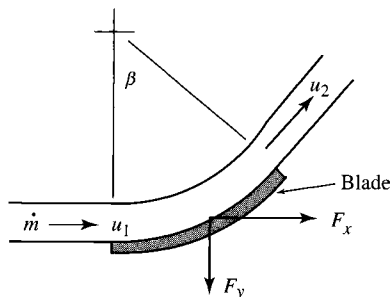


Fig. P2.4

Calculate the force F_y for a mass flow rate of 100 lbm/s, $u_1 = u_2 = 2000$ ft/s, and $\beta = 60$ deg.

- 2.6 One method of reducing an aircraft's landing distance is through the use of thrust reversers. Consider the turbofan engine in Fig. P2.5 with thrust reverser of the bypass airstream. It is given that 1500 lbm/s of air at 60°F and 14.7 psia enters the engine at a velocity of 450 ft/s and that 1250 lbm/s of bypass air leaves the engine at 60 deg to the horizontal, velocity of 890 ft/s, and pressure of 14.7 psia. The remaining 250 lbm/s leaves the engine core at a velocity of 1200 ft/s and pressure of 14.7 psia. Determine the force on the strut F_x . Assume the outside of the engine sees a pressure of 14.7 psia.

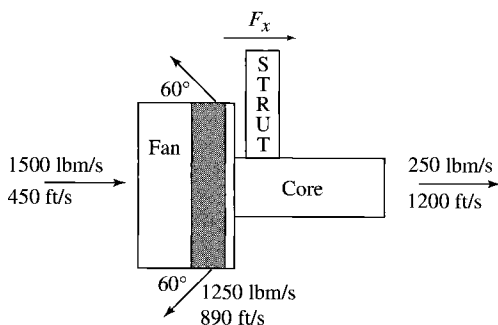


Fig. P2.5

- 2.7 Consider Fig. P2.6. Air with a density of 0.027 lbm/ft³ enters a diffuser at a velocity of 2470 ft/s and a static pressure of 4 psia. The air leaves the diffuser at a velocity of 300 ft/s and a static pressure of 66 psia. The entrance area of the diffuser is 1.5 ft², and its exit area is 1.7 ft². Determine the magnitude and direction of the strut force necessary to hold the diffuser stationary when this diffuser is operated in an atmospheric pressure of 4 psia.

- 2.8** Consider Fig. P2.6. It is given that 50 kg/s of air enters a diffuser at a velocity of 750 m/s and a static pressure of 20 kPa. The air leaves the diffuser at a velocity of 90 m/s and a static pressure of 330 kPa. The entrance area of the diffuser is 0.25 m^2 , and its exit area is 0.28 m^2 . Determine the magnitude and direction of the strut force necessary to hold the diffuser stationary when this diffuser is operated in an atmospheric pressure of 20 kPa.

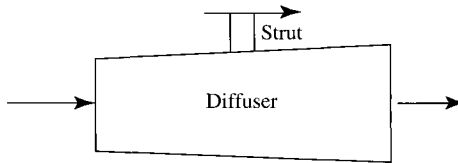


Fig. P2.6

- 2.9** Consider Fig. P2.7. It is given that 100 lbm/s of air enters a nozzle at a velocity of 600 ft/s and a static pressure of 70 psia. The air leaves the nozzle at a velocity of 4000 ft/s and static pressure of 2 psia. The entrance area of the nozzle is 14.5 ft^2 , and its exit area is 30 ft^2 . Determine the magnitude and direction of the strut force necessary to hold the nozzle stationary when this nozzle is operated in an atmospheric pressure of 4 psia.

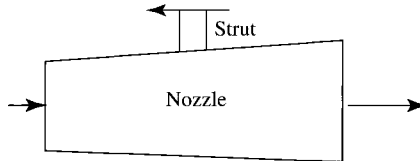


Fig. P2.7

- 2.10** Consider Fig. P2.7. Air with a density of 0.98 kg/m^3 enters a nozzle at a velocity of 180 m/s and a static pressure of 350 kPa. The air leaves the nozzle at a velocity of 1200 m/s and a static pressure of 10 kPa. The entrance area of the nozzle is 1.0 m^2 , and its exit area is 2.07 m^2 . Determine the magnitude and direction of the strut force necessary to hold the nozzle stationary when this nozzle is operated in an atmospheric pressure of 10 kPa.
- 2.11** For a calorically perfect gas, show that $P + \rho V^2/g_c$ can be written as $P(1 + \gamma M^2)$. Note that the Mach number M is defined as the velocity V divided by the speed of sound a .

- 2.12** Air at 1400 K, 8 atm, and 0.3 Mach expands isentropically through a nozzle to 1 atm. Assuming a calorically perfect gas, find the exit temperature and the inlet and exit areas for a mass flow rate of 100 kg/s.
- 2.13** It is given that 250 lbm/s of air at 2000°F, 10 atm, and 0.2 Mach expands isentropically through a nozzle to 1 atm. Assuming a calorically perfect gas, find the exit temperature and the inlet and exit areas.
- 2.14** Air at 518.7°R is isentropically compressed from 1 to 10 atm. Assuming a calorically perfect gas, determine the exit temperature and the compressor's input power for a mass flow rate of 150 lbm/s.
- 2.15** It is given that 50 kg/s of air at 288.2 K is isentropically compressed from 1 to 12 atm. Assuming a calorically perfect gas, determine the exit temperature and the compressor's input power.
- 2.16** Air at -55°F , 4 psia, and $M = 2.5$ enters an isentropic diffuser with an inlet area of 1.5 ft^2 and leaves at $M = 0.2$. Assuming a calorically perfect gas, determine:
- (a) The mass flow rate of the entering air
 - (b) The pressure and temperature of the leaving air
 - (c) The exit area and magnitude and direction of the force on the diffuser (assume outside of diffuser sees 4 psia)
- 2.17** Air at 225 K, 28 kPa, and $M = 2.0$ enters an isentropic diffuser with an inlet area of 0.2 m^2 and leaves at $M = 0.2$. Assuming a calorically perfect gas, determine:
- (a) The mass flow rate of the entering air
 - (b) The pressure and temperature of the leaving air
 - (c) The exit area and magnitude and direction of the force on the diffuser (assume outside of diffuser sees 28 kPa)
- 2.18** Air at 1800°F, 40 psia, and $M = 0.4$ enters an isentropic nozzle with an inlet area of 1.45 ft^2 and leaves at 10 psia. Assuming a calorically perfect gas, determine:
- (a) The velocity and mass flow rate of the entering air
 - (b) The temperature and Mach number of the leaving air
 - (c) The exit area and magnitude and direction of the force on the nozzle (assume outside of nozzle sees 10 psia)
- 2.19** Air at 1500 K, 300 kPa, and $M = 0.3$ enters an isentropic nozzle with an inlet area of 0.5 m^2 and leaves at 75 kPa. Assuming a calorically perfect gas, determine:
- (a) The velocity and mass flow rate of the entering air
 - (b) The temperature and Mach number of the leaving air
 - (c) The exit area and magnitude and direction of the force on the nozzle (assume outside of nozzle sees 75 kPa)

- 2.20** It is given that 100 lb/s of air enters a steady flow compressor at 1 atm and 68°F. It leaves at 20 atm and 800°F. If the process is adiabatic, find the input power, specific volume at exit, and change in entropy. Is the process reversible? (Assume a calorically perfect gas.)
- 2.21** It is given that 50 kg/s of air enters a steady flow compressor at 1 atm and 20°C. It leaves at 20 atm and 427°C. If the process is adiabatic, find the input power, specific volume at exit, and change in entropy. Is the process reversible? (Assume a calorically perfect gas.)
- 2.22** It is given that 200 lb/s of air enters a steady flow turbine at 20 atm and 3400°R. It leaves at 10 atm. For a turbine efficiency of 85%, determine the exit temperature, output power, and change in entropy. (Assume a calorically perfect gas.)
- 2.23** It is given that 80 kg/s of air enters a steady flow turbine at 30 atm and 2000 K. It leaves at 15 atm. For a turbine efficiency of 85%, determine the exit temperature, output power, and change in entropy. (Assume a calorically perfect gas.)
- 2.24** Rework Problem 2.13 for variable specific heats, using Appendix D or the program AFPROP. Compare your results to Problem 2.13.
- 2.25** Rework Problem 2.15 for variable specific heats, using Appendix D or the program AFPROP. Compare your results to Problem 2.15.
- 2.26** Rework Problem 2.16 for variable specific heats, using Appendix D or the program AFPROP. Compare your results to Problem 2.16.
- 2.27** Rework Problem 2.19 for variable specific heats, using Appendix D or the program AFPROP. Compare your results to Problem 2.19.
- 2.28** Rework Problem 2.20 for variable specific heats, using Appendix D or the program AFPROP. Compare your results to Problem 2.20.
- 2.29** Rework Problem 2.23 for variable specific heats, using Appendix D or the program AFPROP. Compare your results to Problem 2.23.
- 2.30** It is given that 100 lbm/s of air at total pressure of 100 psia, total temperature of 40°F, and static pressure of 20 psia flows through a duct. Find the static temperature, Mach number, velocity (ft/s), and flow area (ft²).
- 2.31** Products of combustion ($\gamma = 1.3$) at a static pressure of 2.0 MPa, static temperature of 2000 K, and Mach number of 0.05 are accelerated in an isentropic nozzle to a Mach number of 1.3. Find the downstream static pressure and static temperature. If the mass flow rate is 100 kg/s and the gas constant R is 286 J/(kg · K), use the mass flow parameter (MFP) and find the flow areas for $M = 0.5$ and $M = 1.3$.

- 2.32** Data for the JT9D high-bypass-ratio turbofan engine are listed in Appendix B. If the gas flow through the turbines (from station 4 to 5) is 251 lbm/s with the total properties listed, what amount of power (kW and hp) is removed from the gas by the turbines? Assume the gas is calorically perfect with $\gamma = 1.31$ and $Rg_c = 1716 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R})$.
- 2.33** At launch, the space shuttle main engine (SSME) has 1030 lbm/s of gas leaving the combustion chamber at $P_t = 3000 \text{ psia}$ and $T_t = 7350^\circ\text{R}$. The exit area of the SSME nozzle is 77 times the throat area. If the flow through the nozzle is considered to be reversible and adiabatic (isentropic) with $Rg_c = 3800 \text{ ft}^2/(\text{s}^2 \cdot ^\circ\text{R})$ and $\gamma = 1.25$, find the area of the nozzle throat (in^2) and the exit Mach number. *Hint:* Use the mass flow parameter to get the throat area and Eq. (3.14) to get the exit Mach number.
- 2.34** The experimental evaluation of a gas turbine engine's performance requires the accurate measurement of the inlet air mass flow rate into the engine. A bell-mouth engine inlet (shown schematically in Fig. P2.8) can be used for this purpose in the static test of an engine. The freestream velocity V_0 is assumed to be zero, and the flow through the bell mouth is assumed to be adiabatic and reversible. See Fig. P2.8.
- Measurements are made of the freestream pressure P_{t0} and the static pressure at station 2 (P_2), and the exit diameter of the inlet D_2 .
- (a) For the bell-mouth inlet, show that the Mach number at station 2 is given by

$$M_2 = \sqrt{\frac{2}{\gamma - 1} \left[\left(\frac{P_{t0}}{P_{t0} - \Delta P} \right)^{(\gamma-1)/\gamma} - 1 \right]}$$

and the inlet mass flow rate is given by

$$\dot{m} = \frac{P_{t0}}{\sqrt{T_{t0}}} \frac{\pi D_2^2}{4} \sqrt{\frac{2g_c}{R} \frac{\gamma}{\gamma - 1} \left[\left(\frac{P_{t0} - \Delta P}{P_{t0}} \right)^{2/\gamma} - \left(\frac{P_{t0} - \Delta P}{P_{t0}} \right)^{(\gamma+1)/\gamma} \right]}$$

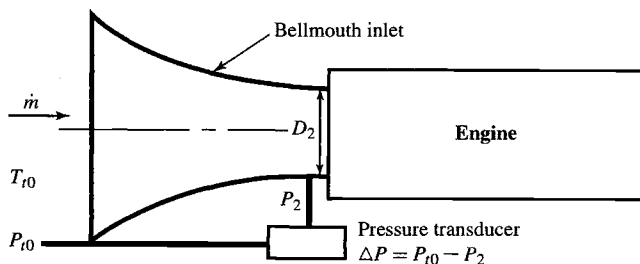


Fig. P2.8

- (b) For the following measured data, determine the inlet air mass flow rate, Mach number M_2 , static temperature T_2 , and velocity V_2 :

$$T_{t0} = 27^\circ\text{C} \quad P_{t0} = 77.80 \text{ kPa}$$

$$\Delta P = 3.66 \text{ kPa} \quad D_2 = 0.332 \text{ m}$$

- 2.35** An ideal ramjet (see Fig. P2.9) is operated at 50,000-ft altitude with a flight Mach number of 3. The diffuser and nozzle are assumed to be isentropic, and the combustion is to be modeled as an ideal heat interaction at constant Mach number with constant total pressure. The cross-sectional area and Mach number for certain engine stations are given in Table P2.1. The total temperature leaving the combustor T_{r4} is 4000°R. Assume ambient pressure surrounding the engine flow passage.
- Determine the mass flow rate of air through the engine (lbm/s).
 - Complete the table with flow areas, static pressures, static temperatures, and velocities.
 - Find the thrust (magnitude and direction) of the diffuser, combustor, and nozzle.
 - Find the thrust (magnitude and direction) of the ramjet.

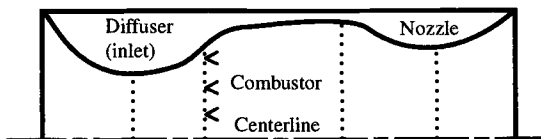


Fig. P2.9

- 2.36** If the flow enters the diffuser of Problem 2.7 at -55°F , is the process isentropic?
- 2.37** If the flow enters the nozzle of Problem 2.10 at 1250 K, is the process isentropic?

Table P2.1

Station	1	2	3	4	5	6
Area (ft ²)	4.235					
Mach	3	1	0.15	0.15	1	3
P (psia)						
T (°R)						
V (ft/s)						

- 2.38** Air at 20 kPa, 260 K, and Mach 3 passes through a normal shock. Determine:
- Total temperature and pressure upstream of the shock
 - Total temperature and pressure downstream of the shock
 - Static temperature and pressure downstream of the shock
- 2.39** Air upstream of a normal shock has the following properties: $P_t = 100$ psia, $T_t = 100^\circ\text{F}$, and $M = 2$. Find the upstream static temperature, static pressure, and velocity (ft/s). Find the downstream total temperature, Mach number, total pressure, static temperature, static pressure, and velocity (ft/s).
- 2.40** If the diffuser of the ideal ramjet in Problem 2.35 has a normal shock in front of the inlet, determine:
- The total and static pressures and the static temperature downstream of the shock
 - The mass flow rate through the engine (lbm/s) (assume choked flow at diffuser throat)
 - The thrust of the engine
- 2.41** Air at a total pressure of 1.4 MPa, total temperature of 350 K, and Mach number of 0.5 is accelerated isentropically in a nozzle (see Fig. P2.10) to a Mach number of 3 (station x), passes through a normal shock (x to y), and then flows isentropically to the exit. Given a nozzle throat area of 0.05 m^2 and the exit area of 0.5 m^2 :
- Find the area at the shock
 - Find the static pressure and static temperature upstream of the shock (station x)
 - Find the Mach number and the total and static pressures and temperatures downstream of the shock (station y)
 - Find the Mach number, static pressure, and static temperature at the exit

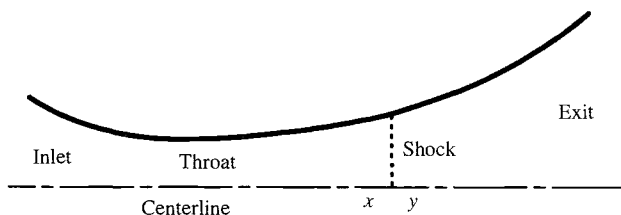


Fig. P2.10

- 2.42** Air flows through an isentropic nozzle with inlet conditions of $T_t = 2000^\circ\text{R}$ and $P_t = 100$ psia. The throat area is 2 ft^2 , and the exit area is 10.32 ft^2 . If the flow is choked at the throat, find:
- Mass flow rate through the nozzle

- (b) Mach number, static temperature, and static pressure at exit without a shock
- (c) Mach number, static temperature, and static pressure at exit with a shock in the divergent section where the flow area is 4.06 ft^2 (see Fig. P2.10)

- 2.43** Air at a total pressure of 400 psia, total temperature of 500°R , and subsonic Mach number is accelerated isentropically in a nozzle (see sketch in Fig. P2.10) to a supersonic Mach number (station x), passes through a normal shock (x to y), and then flows isentropically to the exit. Given a nozzle inlet area of 3 ft^2 , throat area of 1 ft^2 , and exit area of 6 ft^2 :
- (a) Find the Mach number at the inlet
 - (b) Find the Mach number upstream of the shock (station x)
 - (c) Find the static pressure and static temperature upstream of the shock (station x)
 - (d) Find the Mach number and the total and static pressures and temperatures downstream of the shock (station y)
 - (e) Find the Mach number, static pressure, and static temperature at the exit
- 2.44** A 20-deg wedge (see Fig. P2.11) is to be used in a wind tunnel using air with test conditions of $M = 3$, $T_t = 500^\circ\text{R}$, and $P_t = 100 \text{ psia}$. Determine the angle of the oblique shocks and the downstream total and static properties (pressure and temperature). Use the GASTAB program.

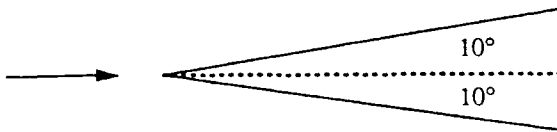


Fig. P2.11

- 2.45** A 15-deg ramp is used on a supersonic inlet at $M = 3.5$ and an altitude of 20 km. Use the GASTAB program.
- (a) Determine the angle of oblique shock and flow properties (Mach number, total and static temperature, total and static pressure) upstream and downstream of the shock
 - (b) At what Mach number does the shock detach from the ramp?
- 2.46** Draw an H - K diagram for the J-79 afterburning turbojet engine of Fig. 1.6. Assume c_p is constant and T_{t0} is 518.7°R .
- 2.47** Draw an H - K diagram for the F-100 afterburning turbofan engine of Table B.4. Assume c_p is constant and T_{t0} is 518.7°R .
- 2.48** Show that dP/P is a constant for Example 2.12.

- (a) Using the design conditions, complete Table P2.2
- (b) Make a plot of the nozzle contour (see Fig. 2.33)
- (c) Calculate the nozzle design pressure ratio $P_{\bar{n}}$ and the nozzle area ratio ε
- (d) Using the altitude table, determine the design altitude for this nozzle ($P_c = P_a$)
- (e) Determine the thrust of a rocket motor using this nozzle at its design altitude

2.51 A perfect gas enters a constant-area heater at a Mach number of 0.3, total pressure of 600 kPa, and total temperature of 500 K. A heat interaction of 500 kJ/kg into the gas occurs. Using the GASTAB software, determine the Mach number, total pressure, and total temperature after the heat interaction for the following gases:

- (a) $\gamma = 1.4$ and $c_p = 1.004 \text{ kJ}/(\text{kg} \cdot \text{K})$
- (b) $\gamma = 1.325$ and $c_p = 1.171 \text{ kJ}/(\text{kg} \cdot \text{K})$

2.52 A perfect gas enters a constant-area heater at a Mach number of 0.5, total pressure of 200 psia, and total temperature of 1000°R . A heat interaction of 100 Btu/lbm into the gas occurs. Using the GASTAB software, determine the Mach number, total pressure, and total temperature after the heat interaction for the following gases:

- (a) $\gamma = 1.4$ and $c_p = 0.24 \text{ Btu}/(\text{lbm} \cdot ^\circ\text{R})$
- (b) $\gamma = 1.325$ and $c_p = 0.28 \text{ Btu}/(\text{lbm} \cdot ^\circ\text{R})$

2.53 A convergent-only nozzle is to be used on an afterburning gas turbine engine as shown in Fig. P2.12. Model the afterburner (station 6 to station 7) as a constant-area duct ($A_6 = A_7$) with simple heat interaction q_{in} into the air. The flow through the nozzle (station 7 to station 8) is isentropic. The exit area of the nozzle A_8 is varied with the afterburner setting T_{i7} to keep sonic ($M = 1$) flow at station 8 and the inlet conditions (mass flow rate, P_{t6} , and T_{t6}) constant at station 6.

- (a) Using the mass flow parameter, show that the area ratio A_8/A_6 is given by

$$\frac{A_8}{A_6} = \frac{P_{t6}}{P_{t7}} \sqrt{\frac{T_{t7}}{T_{t6}}} \frac{\text{MFP}(M_6)}{\text{MFP}(M=1)} = \frac{P_{t6}}{P_{t7}} \sqrt{\frac{T_{t7}}{T_{t6}}} \left(\frac{A^*}{A} \right)_{M_6}$$

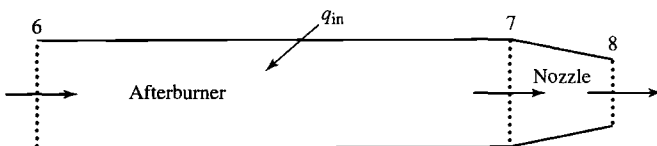


Fig. P2.12

(b) For $M_6 = 0.4$, determine the area ratio A_8/A_6 for the following values of T_{i7}/T_{i6} : 1.0, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, and $T_{i7} = T_i^*$

- 2.54** One mole of hydrogen and one mole of oxygen are in equilibrium at a temperature of 1500 K and a pressure of 5 atm. Determine the number of moles of H_2O , H_2 , and O_2 in the products (neglect other products such as OH , H , O , etc.). Compare your results with that of the EQL program.
- 2.55** One mole of hydrogen and 0.45 moles of oxygen are in equilibrium at a temperature of 2200 K and a pressure of 20 atm. Determine the equilibrium constant from the results of the EQL program and compare to the JANNAF tables.
- 2.56** Calculate the heat of reaction for the reaction of Problem 2.54.
- 2.57** Using the EQL software, determine the adiabatic flame temperature for one mole of hydrogen and one-half mole of oxygen with the reactants at a temperature of 300 K and a pressure of 10 atm.