Group 7

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Introduction to Proof and Problem Solving

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Problem 1. Show that $\lim_{x\to 1} 8/(x^2+1) = 4$.

Proof. Let ϵ_0 be any real number greater than 0. Set $\delta_0 = \min\{1, \epsilon_0/12\}$. Let x_0 be any real number such that $0 < |x_0 - 1| < \delta$.

Then we know that the following statements are true:

$$0 < |x_0 - 1| < 1 \tag{1}$$

and

$$0 < |x_0 - 1| < \epsilon_0/12. \tag{2}$$

From (1), we get that

$$-1 < x_0 - 1 < 1.$$

Adding 2 to all sides, we get

$$1 < x_0 + 1 < 3.$$

Since $x_0 + 1 > 1$, we know that $|x_0 + 1| < 3$. Then we can begin to solve for $f(x_0) - 4$:

$$\begin{aligned}
|f(x_0) - 4| = |\frac{8}{x_0^2+1} - 4| &= |\frac{8}{x_0^2+1} - \frac{4(x_0^2+1)}{x_0^2+1}| \\
&= |\frac{4-4x_0^2}{x_0^2+1}| \\
&= |\frac{-4(x_0^2-1)}{x_0^2+1}| \\
&= |\frac{-4(x_0-1)(x_0+1)}{x_0^2+1}| \\
&= |-4||x_0-1||x_0+1||\frac{1}{x_0^2+1}| \\
&= 4|x_0-1||x_0+1||\frac{1}{x_0^2+1}|.
\end{aligned}$$

Since we know that $|x_0 + 1| < 3$, we can substitute in 3 and see that

$$|f(x_0) - 4| < 12|x_0 - 1||\frac{1}{x_0^2 + 1}|.$$

Now we will examine $|1/(x_0^2 + 1)|$. If we take a look at (1) and add 1 to all sides, we get

$$0 < x_0 < 2.$$

Since $x > 0$, all numbers are nonnegative, and when we square all sides of the inequality, we get

$$0 < x_0^2 < 4.$$

Adding 1 to all sides, we get

$$1 < x_0^2 + 1 < 5.$$

When we take the reciprocal, we get

$$1/5 < 1/(x_0^2 + 1) < 1.$$

From this, we know that $|1/(x_0^2 + 1)| < 1$.

Then we know that

$$\begin{aligned} |f(x_0) - 4| &< 12|x_0 - 1| \\ &< 12\delta \\ &< 12(\epsilon/12) = \epsilon. \end{aligned}$$

Thus, $\lim_{x\to 1} 8/(x^2 + 1) = 4$.

□

While working on this proof, we received no external assistance aside from advice from Professor Mehmetaj.