Group 9

Sean Clavadetscher and Yu Fan Mei
Introduction to Proof and Problem Solving

December 9, 2024

Problem 6. Define the relation F on $\mathbb{R}^2$ by

$$((x, y), (s, t)) \in F \iff x^2 - y = s^2 - t.$$

(Hint: Make sure you work enough examples to understand what pairs of points are related and why.)

Show F is an equivalence relation or give a counterexample to one of the properties of an equivalence relation. If F is an equivalence relation, describe the equivalence classes for F.

Proof. We will prove that F is an equivalence relation by proving that it is reflexive, symmetric, and transitive. We will first show that F is reflexive:

Let (x_0, y_0) be any point in $\mathbb{R}^2$. Then it follows that

$$x_0^2 - y_0^2 = x_0^2 - y_0^2.$$

Then $(x_0, y_0)F(x_0, y_0)$ holds true, and F is reflexive. Next, we must prove that F is symmetric:

Let (x_0, y_0) and (s_0, t_0) be any points in $\mathbb{R}^2$ such that $(x_0, y_0)F(s_0, t_0)$. By definition, $x_0^2 - y_0 = s_0^2 - t_0$. Then we know that

$$s_0^2 - t_0 = x_0^2 - y_0.$$

This shows that $(s_0, t_0)F(x_0, y_0)$, and F is symmetric. Finally, we must prove that F is transitive:

Let (x_0, y_0), (s_0, t_0), and (a_0, b_0) be any points in $\mathbb{R}^2$ such that $(x_0, y_0)F(s_0, t_0)$ and $(s_0, t_0)F(a_0, b_0)$. By definition, $x_0^2 - y_0 = s_0^2 - t_0$ and $s_0^2 - t_0 = a_0^2 - b_0$. Then it follows that

$$x_0^2 - y_0 = a_0^2 - b_0.$$

By the definition of the relation, this means that $(x_0, y_0)F(a_0, b_0)$, and F is transitive. Since F is reflexive, symmetric, and transitive, F is an equivalence relation.

$\square$

Problem 8. Let $x, y \in \mathbb{R}^+$.
Define the relation A such that $(x, y) \in A$ if and only if there exists $n \in \mathbb{Z}_0^+$ (the set of non-negative integers) such that $x = 2^n y$. Is A an equivalence relation? Explain.

Proof. We will first prove that

$\square$

While working on this proof, we received no external assistance aside from advice from Professor Mehmetaj.