

# Individual 7

Yu Fan Mei

Introduction to Proof and Problem Solving

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**Problem 1.** Use induction to prove that  $(n^3 - n)(n + 2)$  is divisible by 12 for all  $n \geq 1$ .

Before we prove this, we will need a lemma.

**Lemma 1.** For every integer greater than 0,  $n^3 + 3n^2 + 2n$  is divisible by 3.

*Proof.* We will prove this lemma via induction.

Base case: Set  $n_0 = 1$ . Then  $n_0^3 + 3n_0^2 + 2n_0 = 6 = 3(2)$ .

Inductive Step: Assume there exists an integer  $k_0$  and an integer  $n_0 \geq 1$  such that

$$n_0^3 + 3n_0^2 + 2n_0 = 3k_0.$$

We want to show that there exists an integer  $j$  such that

$$(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) = 3j.$$

Set  $j = j_0 = k_0 + 3n_0^2 + 3n_0 + 2$ . Then we know that

$$\begin{aligned}(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) &= n_0^3 + 3n_0^2 + 3n_0 + 1 + 3(n_0^2 + 2n_0 + 1) + 2n_0 + 2 \\ &= n_0^3 + 6n_0^2 + 11n_0 + 6.\end{aligned}$$

We can substitute in  $3k_0$ , and we get

$$\begin{aligned}(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) &= 3k_0 + 3n_0^2 + 9n_0 + 6 \\ &= 3(k_0 + n_0^2 + 3n_0 + 2) \\ &= 3j_0.\end{aligned}$$

Thus, we have proven what we needed to prove. □

*Proof.* We will prove problem 1 using induction.

Base Case: Set  $n = n_0 = 1$ . Then

$$(n^3 - n)(n + 2) = 0.$$

This is clearly true, since 0 can be written as  $12(0)$ , and is thus divisible by 12.

Inductive Step: Assume there exists integers  $n \geq 1$  and  $k_0$  such that

$$(n^3 - n)(n + 2) = 12k_0.$$

We want to show that this is true for  $n + 1$ , meaning we want to show that there exists an integer  $j_0$  such that

$$((n + 1)^3 - (n + 1))((n + 1) + 2) = 12j_0.$$

Set  $j_0 = (\text{something})$ . When we unfactor the equation above, we get

$$n_0^4 + 6n_0^3 + 11n_0^2 + 6n_0 = 12j_0.$$

When we substitute in  $12k_0$ , we get

$$12k_0 + 4n_0^3 + 12n_0^2 + 8n_0 = 12j_0.$$

We can rewrite the statement like this:

$$12k_0 + 4n_0^3 + 4n_0^2 + 8n_0^2 + 8n_0 = 12j_0.$$

Utilizing grouping, we can factor out  $4n_0(n_0 + 1)$ , and are left with

$$12k_0 + 4n_0(n_0 + 1)(n_0 + 2) = 12j_0.$$

By lemma 1,  $n_0(n_0 + 1)$  is even, and can be rewritten as  $2m_0$ , with  $m_0$  being an integer. When we substitute this in, we get

$$12k_0 + 8m_0(n_0 + 2) = 12j_0.$$

□

**Problem 2.** Let  $r$  represent an arbitrary real number other than 0 and 1. Show that for  $n \in \mathbb{Z}_0^+$

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}.$$

This is the formula for what is called the finite geometric series. This formula is quite important in many different fields of mathematics and should be committed to memory.

*Proof.* We will prove this using mathematical induction.

Base Case: Set  $n_0 = 0$ . Then

$$\sum_{i=0}^{n_0} r^i = r^0 = 1 = \frac{1 - r^{n_0+1}}{1 - r}.$$

This statement is clearly true.

Inductive Step: Assume there exists a nonnegative integer  $n_0$  such that for every real number  $r$  aside from 0 and 1,

$$\sum_{i=0}^{n_0} r^i = \frac{1 - r^{n_0+1}}{1 - r}.$$

We want to show that this is true for  $n + 1$ , so we need to show that

$$\sum_{i=0}^{n_0} r^i + r^{n_0+1} = \sum_{i=0}^{n_0+1} r^i.$$

□

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.