Abstract Algebra HW3

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2. Find the order of each of the following elements.

Problem 2.1. $5 \in \mathbb{Z}_{12}$.

- The order of $5 \in \mathbb{Z}_{12}$ is $\frac{12}{\texttt{gcd}(5,12)} = \frac{12}{1} = 12$.

Problem 2.2. $\sqrt{3} \in \mathbb{R}$.

- We will prove that the order is infinite via contradiction. Suppose the order of $\sqrt{3}$ is finite. This would mean there exists a positive integer k such that $k\sqrt{3} = 0$.
- Dividing both sides of this equality by $\sqrt{3}$, we get $k = 0$, which contradicts the statement that $k > 0$. This means $\sqrt{3}$ must be infinite.

Problem 2.3. $\sqrt{3} \in \mathbb{R}*$.

- We will prove that the order is infinite, also via contradiction. Suppose the order of $\sqrt{3}$ in the group $\mathbb{R}*$ was finite. Then there exists a positive integer k such that $\sqrt{3}^k = 1$.
- Taking the natural logarithm of both sides, we get $k \ln \sqrt{3} = 0$. From this, we get $k = 0$, which is a contradiction.

Problem 2.4. $-i \in \mathbb{C}*$.

- The order of $-i$ in the group of complex numbers under multiplication is 4.
- $(-i)^2 = -1$, and $(-i)^3 = i$. $(-i)^4 = 1$.

Problem 2.5. $72 \in \mathbb{Z}_{240}$.

- The order of $72 \in \mathbb{Z}_{240}$ is $\frac{240}{\texttt{gcd}(240,72)}$.
- The gcd of 240 and 72 is 24:

$$\begin{aligned} 240 &= 72(3) + 24 \\ 72 &= 24(3) + 0. \end{aligned}$$

- So, the order of $72 \in \mathbb{Z}_{240}$ is $\frac{240}{24} = 10$.

Problem 2.6. $312 \in \mathbb{Z}_{471}$.

- The order of 312 in $\mathbb{Z}_{471}$ is $\frac{471}{\texttt{gcd}(471,312)}$.
- And the greatest common divisor of 471 and 312 is 1:

$$\begin{aligned} 471 &= 312(1) + 59 \\ 312 &= 59(5) + 17 \\ 59 &= 17(3) + 8 \\ 17 &= 8(2) + 1 \\ 8 &= 1(8) + 0. \end{aligned}$$

- This means the order of $312 \in \mathbb{Z}_{471}$ is 471.

3. List all of the elements in each of the following subgroups.

Problem 3.1. The subgroup of $\mathbb{Z}$ generated by 7.

- The elements in this subgroup are $\{..., -14, -7, 0, 7, 14, ...\}$.
- This subgroup is infinite.

Problem 3.2. The subgroup of $\mathbb{Z}_{24}$ generated by 15.

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