

# Individual 7

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Introduction to Proof and Problem Solving

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**Problem 1.** Use induction to prove that  $(n^3 - n)(n + 2)$  is divisible by 12 for all  $n \geq 1$ .

Before we prove this, we will need a lemma.

**Lemma 1.** For every integer greater than 0,  $n^3 + 3n^2 + 2n$  is divisible by 3.

*Proof.* We will prove this lemma via induction.

Base case: Set  $n_0 = 1$ . Then  $n_0^3 + 3n_0^2 + 2n_0 = 6 = 3(2)$ .

Inductive Step: Assume there exists an integer  $k_0$  and an integer  $n_0 \geq 1$  such that

$$n_0^3 + 3n_0^2 + 2n_0 = 3k_0.$$

We want to show that there exists an integer  $j$  such that

$$(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) = 3j.$$

Set  $j = j_0 = k_0 + 3n_0^2 + 3n_0 + 2$ . Then we know that

$$\begin{aligned}(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) &= n_0^3 + 3n_0^2 + 3n_0 + 1 + 3(n_0^2 + 2n_0 + 1) + 2n_0 + 2 \\ &= n_0^3 + 6n_0^2 + 11n_0 + 6.\end{aligned}$$

We can substitute in  $3k_0$ , and we get

$$\begin{aligned}(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) &= 3k_0 + 3n_0^2 + 9n_0 + 6 \\ &= 3(k_0 + n_0^2 + 3n_0 + 2) \\ &= 3j_0.\end{aligned}$$

Thus, we have proven what we needed to prove. □

*Proof.* We will prove problem 1 using induction.

Base Case: Set  $n = n_0 = 1$ . Then

$$(n_0^3 - n_0)(n_0 + 2) = 0 = 12(0).$$

Inductive Step: Assume there exists an integer  $n \geq 1$  and  $k_0$  such that

$$(n^3 - n)(n + 2) = 12k_0.$$

We want to show that there exists an integer  $j$  such that

$$((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) = 12j.$$

We need to prove that this is true for  $n_0 + 1$ , meaning we want to show that there exists an integer  $j_0$  such that

$$((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) = 12j_0.$$

Set  $j = j_0 = k_0 + m$ , where  $m$  is an integer. Unfactoring the left hand side, we get

$$((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) = n_0^4 + 6n_0^3 + 11n_0^2 + 6n_0.$$

When we substitute in  $12k_0$ , we get

$$\begin{aligned} ((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) &= 12k_0 + 4n_0^3 + 12n_0^2 + 8n_0 \\ &= 12k_0 + 4(n_0^3 + 3n_0^2 + 2n_0). \end{aligned}$$

By lemma 1, we know that  $n_0^3 + 3n_0^2 + 2n_0$  is divisible by 3. We can rewrite the statement like this:

$$\begin{aligned} ((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) &= 12k_0 + 4(3m) \\ &= 12k_0 + 12m \\ &= 12(k_0 + m) \\ &= 12j_0 \end{aligned}$$

Thus, we have proven what we needed to prove.

□

**Problem 2.** Let  $r$  represent an arbitrary real number other than 0 and 1. Show that for  $n \in \mathbb{Z}_0^+$

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}.$$

This is the formula for what is called the finite geometric series. This formula is quite important in many different fields of mathematics and should be committed to memory.

*Proof.* We will prove this using mathematical induction.

Base Case: Set  $n_0 = 0$ . Then

$$\sum_{i=0}^{n_0} r^i = r^0 = 1 = \frac{1 - r^{n_0+1}}{1 - r}.$$

This statement is clearly true.

Inductive Step: Assume there exists a nonnegative integer  $n_0$  such that for every real number  $r$  aside from 0 and 1,

$$\sum_{i=0}^{n_0} r^i = \frac{1 - r^{n_0+1}}{1 - r}.$$

We want to show that this is true for  $n_0 + 1$ , so we need to show that

$$\sum_{i=0}^{n_0} r^i + r^{n_0+1} = \sum_{i=0}^{n_0+1} r^i.$$

By our inductive hypothesis, we can rewrite the left side of the equation as

$$\frac{1 - r^{n_0+1}}{1 - r} + r^{n_0+1} = \sum_{i=0}^{n_0+1} r^i.$$

We can multiply  $r^{n_0+1}$  by  $(1 - r)/(1 - r)$  to get a common denominator:

$$\begin{aligned} \sum_{i=0}^{n_0+1} r^i &= \frac{1 - r^{n_0+1} + r^{n_0+1}(1 - r)}{1 - r} \\ &= \frac{1 - r^{n_0+1} + r^{n_0+1} - r^{n_0+2}}{1 - r} \\ &= \frac{1 - r^{n_0+2}}{1 - r}. \end{aligned}$$

Thus, we have proven this is true. □

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.