Individual 4

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Introduction to Proof and Problem Solving

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Problem 12. (1) Consider the statement

$$p \equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\}.$$

(a) Write $\neg p$.

(b) Consider the function f from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Does f satisfy p or $\neg p$? Prove your answer.

Solution. (a) The negation of p is

$$\neg p \equiv \{\exists M \in \mathbb{R} \text{ such that } \forall K \in \mathbb{R}, \exists x > K \text{ such that } f(x) \leq M\}.$$

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Proof. We will prove that f satisfies $\neg p$. Set $M_0 = 2$. Let K_0 be any real number. Set $x_0 = |K_0| + 3$. Since $|K_0| \geq 0$, we know that $|K_0| + 2 > 0$. Adding 1 to both sides, we get $|K_0| + 3 > 1$, or $x_0 > 1$.

We know that $2 > 1$. Multiplying both sides by x_0, we get $2x_0 > x_0$. But since we know $x_0 > 1$, we can also notice that $2x_0 > 1$. Dividing both sides by x_0, we get

$$2 > \frac{1}{x_0}.$$

Since $x_0 > 1$, $x \neq 0$. Then $f(x_0) = 1/x_0$. From the inequality above, we can observe that $f(x_0) < 2$. From this equation, we can see that $f(x_0) \leq M_0$ is also true. Thus, we have proven that f satisfies the negation of p. □

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.