
Abstract Algebra HW3

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2. Find the order of each of the following elements.

Problem 2.1. $5 \in \mathbb{Z}_{12}$.

- The order of $5 \in \mathbb{Z}_{12}$ is $\frac{12}{\gcd(5,12)} = \frac{12}{1} = 12$.

Problem 2.2. $\sqrt{3} \in \mathbb{R}$.

- We will prove that the order is infinite via contradiction. Suppose the order of $\sqrt{3}$ is finite. This would mean there exists a positive integer k such that $k\sqrt{3} = 0$.
- Dividing both sides of this equality by $\sqrt{3}$, we get $k = 0$, which contradicts the statement that $k > 0$. This means $\sqrt{3}$ must be infinite.

Problem 2.3. $\sqrt{3} \in \mathbb{R}^*$.

- We will prove that the order is infinite, also via contradiction. Suppose the order of $\sqrt{3}$ in the group \mathbb{R}^* was finite. Then there exists a positive integer k such that $\sqrt{3}^k = 1$.
- Taking the natural logarithm of both sides, we get $k \ln \sqrt{3} = 0$. From this, we get $k = 0$, which is a contradiction.

Problem 2.4. $-i \in \mathbb{C}^*$.

- The order of $-i$ in the group of complex numbers under multiplication is 4.
- $(-i)^2 = -1$, and $(-i)^3 = i$. $(-i)^4 = 1$.

Problem 2.5. $72 \in \mathbb{Z}_{240}$.

- The order of $72 \in \mathbb{Z}_{240}$ is $\frac{240}{\gcd(240,72)}$.
- The gcd of 240 and 72 is 24:

$$\begin{aligned} 240 &= 72(3) + 24 \\ 72 &= 24(3) + 0. \end{aligned}$$

- So, the order of $72 \in \mathbb{Z}_{240}$ is $\frac{240}{24} = 10$.

Problem 2.6. $312 \in \mathbb{Z}_{471}$.

- The order of 312 in \mathbb{Z}_{471} is $\frac{471}{\gcd(471,312)}$.
- And the greatest common divisor of 471 and 312 is 1:

$$\begin{aligned} 471 &= 312(1) + 159 \\ 312 &= 159(2) - 66 \\ 159 &= 66(2) + 27 \\ 66 &= 27(2) + 12 \\ 27 &= 12(2) + 3 \\ 12 &= 3(4) + 0. \end{aligned}$$

- This means the order of $312 \in \mathbb{Z}_{471}$ is 471.

3. List all of the elements in each of the following subgroups.

Problem 3.1. The subgroup of \mathbb{Z} generated by 7.

- The elements in this subgroup are $\{\dots, -14, -7, 0, 7, 14, \dots\}$.
- This subgroup is infinite.

Problem 3.2. The subgroup of \mathbb{Z}_{24} generated by 15.

- The subgroup is $\{15, 6, 21, 12, 3, 18, 9, 0\}$.

Problem 3.3. All subgroups of \mathbb{Z}_{12} .

- The subgroups of \mathbb{Z}_{12} are:
 - $\langle 0 \rangle = \{0\}$
 - $\langle 1 \rangle = \{1, 2, 3, \dots, 11, 0\}$
 - $\langle 2 \rangle = \{2, 4, 6, 8, 10, 0\}$
 - $\langle 3 \rangle = \{3, 6, 9, 0\}$
 - $\langle 4 \rangle = \{4, 8, 0\}$
 - $\langle 6 \rangle = \{6, 0\}$

Problem 5. Find the order of every element in \mathbb{Z}_{18} .

- The order of any element $k \in \mathbb{Z}_{18}$ is given by $\frac{18}{\gcd(k, 18)}$.
- $\text{ord}(1) = 18$
- $\text{ord}(2) = 9$
- $\text{ord}(3) = 6$
- $\text{ord}(4) = 9$
- $\text{ord}(5) = 18$
- $\text{ord}(6) = 3$
- $\text{ord}(7) = 18$
- $\text{ord}(8) = 9$
- $\text{ord}(9) = 2$
- $\text{ord}(10) = 9$
- $\text{ord}(11) = 18$
- $\text{ord}(12) = 3$
- $\text{ord}(13) = 18$
- $\text{ord}(14) = 9$
- $\text{ord}(15) = 6$
- $\text{ord}(16) = 9$
- $\text{ord}(17) = 18$
- $\text{ord}(0) = 1$

Problem 26.