Individual 8

Yu Fan Mei
Introduction to Proof and Problem Solving

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Problem 1. Consider the function

$$f(x) = \begin{cases} x - 2 & x \leq 4 \\ \frac{3}{2}x - 2 & x > 4 \end{cases}.$$

Show that $\lim_{x \to 4} f(x)$ does not exist.

Proof. We will prove $\lim_{x \to 4} f(x)$ does not exist using a proof by contradiction. Suppose that the limit did exist, that $\lim_{x \to 4} f(x) = L_0$ for some L_0. Then for every $\epsilon > 0$, there exists a real number $\delta > 0$ such that for all real numbers $x \in \mathbb{R}$ that satisfy the condition

$$0 < |x - 4| < \delta,$$

the following condition is also satisfied:

$$f(x) - L_0 < \epsilon.$$

Set $\epsilon_0 = 3$. Let δ_0 be any real number greater than 0. Set $x_1 = 4 - \delta_0/2$ and $x_2 = 4 + 2\delta_0/3$. Then as demonstrated below, we know that $x_1 < \delta_0$ and $x_2 < \delta_0$:

$$0 < |x_1 - 4| = \left|4 - \frac{\delta_0}{2} - 4\right| = \left|-\frac{\delta_0}{2}\right| = \frac{\delta_0}{2} < \delta_0,$$

and

$$0 < |x_2 - 4| = \left|4 + \frac{2\delta_0}{3} - 4\right| = \left|\frac{2\delta_0}{3}\right| = \frac{2\delta_0}{3} < \delta_0,$$

By the triangle inequality, we know that

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - L_0 + L_0 - f(x_2)| \\ &\leq |f(x_1) - L_0| + |L_0 - f(x_2)| \\ &= |f(x_1) - L_0| + |-(f(x_2) - L_0)| \end{aligned} \tag{1}$$

Since the limit exists, $|f(x_1) - L_0| < \epsilon_0$ and $|f(x_2) - L_0| < \epsilon_0$. Then, using the above inequality, we know that

$$\begin{aligned} |f(x_1) - f(x_2)| &< \epsilon_0 + \epsilon_0 \\ &= 2\epsilon_0 = 6. \end{aligned}$$

But since $x_1 \leq 4$ and $x_2 > 4$,

$$\begin{aligned} |f(x_1) - f(x_2)| &= \left| x_1 - 2 - \left(\frac{3}{2} x_2 - 2 \right) \right| \\ &= \left| \left(4 - \frac{\delta_0}{2} \right) - 2 - \frac{3}{2} \left(4 + \frac{2\delta_0}{3} \right) - 2 \right| \\ &= \left| -\frac{3\delta_0}{2} - 6 \right| \\ &= \left| -\left(\frac{3\delta_0}{2} + 6 \right) \right| \\ &= \frac{3\delta_0}{2} + 6 \\ &> 6. \end{aligned} \tag{2}$$

Combining (1) and (2), we get $6 < |f(x_1) - f(x_2)| < 6$, which is a contradiction. Thus, we have proven that $\lim_{x \to 4} f(x)$ does not exist.

□

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.