Individual 8

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Introduction to Proof and Problem Solving

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Problem 1. Consider the function

$$f(x) = \begin{cases} x-2 & x \leq 4 \\ \frac{3}{2}x - 2 & x > 4 \end{cases}.$$

Show that $\lim_{x\to 4} f(x)$ does not exist.

Proof. We will prove $\lim_{x\to 4} f(x)$ does not exist using a proof by contradiction. Suppose that the limit did exist, that $\lim_{x\to 4} f(x) = L_0$ for some L_0. Then for every $\epsilon > 0$, there exists a real number $\delta > 0$ such that for all real numbers $x \in \mathbb{R}$ that satisfy the condition

$$0 < |x - 4| < \delta,$$

the following condition is also satisfied:

$$f(x) - L_0 < \epsilon.$$

Set $\epsilon_0 = b$. Let δ_0 be any real number greater than 0. Set $x_1 = 4 - \delta_0/2$ and $x_2 = 4 + \delta_0/2$. Then as demonstrated below, we know that $x_1 < \delta_0$ and $x_2 < \delta_0$:

$$0 < |x_1 - 4| = \left|4 - \frac{\delta_0}{2} - 4\right| = \left|-\frac{\delta_0}{2}\right| = \frac{\delta_0}{2} < \delta_0,$$

and

$$0 < |x_1 - 4| = \left|4 + \frac{\delta_0}{2} - 4\right| = \left|\frac{\delta_0}{2}\right| = \frac{\delta_0}{2} < \delta_0,$$

By the triangle inequality, we know that

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - L_0 + L_0 - f(x_2)| \\ &\leq |f(x_1) - L_0| + |L_0 - f(x_2)| \\ &= |f(x_1) - L_0| + |-(f(x_2) - L_0)| \end{aligned} \tag{1}$$

Since the limit exists, $|f(x_1) - L_0| < \epsilon_0$ and $|f(x_2) - L_0| < \epsilon_0$. Then, using the above inequality, we know that

$$
\begin{aligned}
|f(x_1) - f(x_2)| &< \epsilon_0 + \epsilon_0 \\
&= 2\epsilon_0 = .
\end{aligned}
$$

$\square$

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.