Group 9

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Introduction to Proof and Problem Solving

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Problem 6. Define the relation F on $\mathbb{R}^2$ by

$$((x,y),(s,t)) \in F \iff x^2 - y = s^2 - t.$$

(Hint: Make sure you work enough examples to understand what pairs of points are related and why.)
Show F is an equivalence relation or give a counterexample to one of the properties of an equivalence relation. If F is an equivalence relation, describe the equivalence classes for F.

Proof. We will prove that F is an equivalence relation by proving that it is reflexive, symmetric, and transitive. We will first show that F is reflexive.
Let (x_0, y_0) be any point in $\mathbb{R}^2$. Then it follows that

$$x_0^2 - y_0^2 = x_0^2 - y_0^2.$$

Then $(x_0, y_0)F(x_0, y_0)$ holds true, and F is reflexive. Next, we must prove that F is symmetric.
Let (x_0, y_0) and (s_0, t_0) be any points in $\mathbb{R}^2$ such that $(x_0, y_0)F(s_0, t_0)$. By definition, $x_0^2 - y_0 = s_0^2 - t_0$. Then we know that

$$s_0^2 - t_0 = x_0^2 - y_0.$$

This shows that $(s_0, t_0)F(x_0, y_0)$, and F is symmetric. Finally, we must prove that F is transitive.

Let (x_0, y_0), (s_0, t_0), and (a_0, b_0) be any points in $\mathbb{R}^2$ such that $(x_0, y_0)F(s_0, t_0)$ and $(s_0, t_0)F(a_0, b_0)$. By definition, $x_0^2 - y_0 = s_0^2 - t_0$ and $s_0^2 - t_0 = a_0^2 - b_0$. Then it follows that

$$x_0^2 - y_0 = a_0^2 - b_0.$$

By the definition of the relation, this means that $(x_0, y_0)F(a_0, b_0)$, and F is transitive. Since F is reflexive, symmetric, and transitive, F is an equivalence relation.

Since F is an equivalence relation, we know that there exists an equivalence class $(m_0, n_0)/F$, where $(m_0, n_0) \in \mathbb{R}^2$ which consists of all the points $(x, y) \in \mathbb{R}^2$ such that $((m_0, n_0), (x, y) \in F$. From the definition of F, we know $m_0^2 - n_0 = x^2 - y$ and rearranging the equation we find $y = x^2 - m_0^2 + n_0$. Therefore we can define the equivalence class of F as $(m_0, n_0)/F = \{(x, x^2 - m_0^2 + n_0) | x \in \mathbb{R}\}$.

$\square$

Problem 8a. Let $x, y \in \mathbb{R}^+$.
Define the relation A such that $(x, y) \in A$ if and only if there exists $n \in \mathbb{Z}_0^+$ (the set of non-negative integers) such that $x = 2^n y$. Is A an equivalence relation? Explain.

Proof. We will prove that A is not an equivalence relation by proving that it is not symmetric through contradiction. Suppose that A is a symmetric relation. By definition, all points (x_0, y_0) that satisfy the relation A have a symmetric pair (y_0, x_0) also satisfies the relation A. Set $x_0 = 6$ and $y_0 = 3$. Then (x_0, y_0) satisfies the relation, as shown below:

$$6 = 2^1(3).$$

Since we assume that A is symmetric, this means that $(y_0, x_0) \in A$. Then there exists an $n_0 \in \mathbb{Z}_0^+$ such that

$$y_0 = 2^{n_0} x_0.$$

However, we observe that $3 = 2^{n_0} 6$. Dividing both sides by 6, we get

$$\frac{1}{2} = 2^{n_0}.$$

From this it follows that $n_0 = -1$. This is a contradiction, since n_0 cannot be negative. Thus, A is not symmetric, and thus is not an equivalence relation.

$\square$

Problem 8c. Let $x, y \in \mathbb{R}^+$.

Define the relation B such that $(x, y) \in B$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$. Is B an equivalence relation?

Proof. We will prove that B is an equivalence relation by showing that it is reflexive, symmetric, and transitive. We start with proving reflexivity. Let x_0 be any positive real number. We can observe that

$$x_0 = 2^0 x_0.$$

So $x_0 B x_0$ is clearly true, and B is reflexive. We will now prove that B is symmetric. Let $x_0, y_0 \in \mathbb{R}^+$ such that $x_0 B y_0$. By definition, we know that there exists an integer n_0 such that $x_0 = 2^{n_0} y$. Dividing both sides by 2^{n_0}, we get

$$y_0 = \frac{x_0}{2^{n_0}} = 2^{-n_0} x_0.$$

Thus, $y_0 B x_0$ holds true, and B is symmetric. Finally, we need to prove that B is transitive. Let x_0, y_0, z_0 be any positive real numbers such that $(x_0, y_0) \in B$ and $(y_0, z_0) \in B$. By definition, we know there exist integers n_0, j_0 such that $x_0 = 2^{n_0} y_0$ and $y_0 = 2^{j_0} z_0$. Then it follows that

$$x_0 = 2^{(n_0 + j_0)} z_0.$$

Since integers are closed under addition, $n_0 + j_0$ is an integer. This means that $x_0 B z_0$, and thus, B is transitive. Since we have proven that B is reflexive, symmetric, and transitive, B must be an equivalence relation.

$\square$

While working on this proof, we received no external assistance aside from advice from Professor Mehmetaj.