Introduction to Proof and Problem Solving

James Sandefur

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Contents

Chapter 0

Preface

0.1 Notation

1. $\mathbb{R}$ is the set of all **real** numbers.
2. $\mathbb{Q}$ is the set of all **rational** numbers.
3. $\mathbb{Z}$ is the set of all **integers**.
4. $\mathbb{N}$ is the set of all **natural** numbers, $\{1,2,\ldots\}$.
5. $\mathbb{E}$ is the set of all **even** integers.
6. $\mathbb{O}$ is the set of all **odd** integers.
7. S^+ is the set of all elements of S that are **positive**. For example, $\mathbb{R}^+$ is the set of all positive real numbers. Similarly for $\mathbb{Q}^+$, $\mathbb{Z}^+$, $\mathbb{E}^+$, and $\mathbb{O}^+$. Note that using this definition, $\mathbb{N} = \mathbb{Z}^+$.
8. S_0^+ is the set of all elements of S that are **nonnegative**, that is, positive or zero.
9. S^- is the set of all elements of S that are **negative**. For example, $\mathbb{R}^-$ is the set of all negative real numbers.
10. S_0^- is the set of all elements of S that are **nonpositive**, that is, negative or zero.
11. $\exists$ means **there exists** or **there is**.
12. $\forall$ means **for every**.

13. s.t. means **such that**.
14. $\neg$ means **negation** of a statement.
15. $\vee$ is the **disjunction** of two statements, that is, $p \vee q$ means statement p or q or both are true.
16. $\wedge$ is the **conjunction** of two statements, that is, $p \wedge q$ means statements p and q are both true.
17. $\lceil x \rceil =$ the smallest integer $n \geq x$. So $\lceil 11.8 \rceil = 12$ and $\lceil 15 \rceil = 15$. This is called the **ceiling** function.
18. $n = m \bmod k$ means $n - m$ is divisible by k. Another way to say this is that there exists an integer j such that $n = m + kj$, as in $23 = 8 \bmod 5$ since $23 = 8 + 3(5)$. This can also be written as
$$n \underset{k}{=} m.$$
19. $x \in S$ denotes that x is an **element** of set S.
20. $T \subseteq S$ means set T is a **subset** of S, that is, if $x \in T$, then $x \in S$. In this case, we say that T is contained in S or possibly equal to S.
21. $T \subset S$ means $T \subseteq S$ but does not equal S, that is, $\exists\, x \in S$ s.t. $x \notin T$. The set T is called a **proper subset** of S.
22. Two sets are **equal**, $S = T$, if and only if $S \subseteq T$ and $T \subseteq S$. This definition is used quite often to show set equality!
23. $\emptyset$ is the **null set**. It is the set which contains no elements. This means that $\forall\, x \in \mathbb{U}$, the universe from which elements are taken, $x \notin \emptyset$.
24. $S \cup T = \{x : x \in S \text{ or } x \in T\}$ is the **union** of the sets S and T.
25. $S \cap T = \{x : x \in S \text{ and } x \in T\}$ is the **intersection** of the sets S and T.
26. We can generalize the definitions of **intersection** and **union** to a collection of sets. Let I be a set (often $I = \mathbb{Z}^+$ or $I = \mathbb{R}^+$). For each $i \in I$, let $S_i \subseteq \mathbb{U}$ where $\mathbb{U}$ represents the universe of elements being considered. Then
$$\bigcap_{i \in I} S_i = \{x \in \mathbb{U} : \forall\, i \in I,\, x \in S_i\}$$
$$\bigcup_{i \in I} S_i = \{x \in \mathbb{U} : \exists\, i \in I, \text{s.t. } x \in S_i\}.$$

For simplicity, we will often write $\bigcap S_i$ and $\bigcup S_i$ where the index is assumed.

27. S^c is the **complement** of set S. Letting $\mathbb{U}$ represent the universe of elements being considered, $x \in \mathbb{U}$ is in S^c if and only if $x \notin S$.

28. Sets S and T are **disjoint** if and only if $S \cap T = \emptyset$.

29. $S - T = \{x \in S \cap T^c\}$, that is, it is the set of all elements in S minus those elements which are also in T. For example, $\mathbb{R} - \{1\}$ is the set of all real numbers except 1.

30. We can write sets in a number of different ways.

    (a) One standard method is to just list the elements in the set, as in

    $$S = \{1, 2, 3, 4, 5\}.$$

    In such cases, we should describe the universe $\mathbb{U}$ of elements from which the elements of S are taken. The universe is sometimes implied from the context of the problem, often being $\mathbb{U} = \mathbb{R}$, $\mathbb{N}$, or $\mathbb{Z}$.

    (b) Another method for writing sets is

    $$T = \{j \in \mathbb{Z} : \ j = 3k \text{ for some } k \in \mathbb{N}\}.$$

    where ":" is read as *such that*. In this case, the universe is given as part of the definition, that is,

    $$T = \{x \in \mathbb{U} : \ x \text{ satisfies some condition}\}.$$

    (c) We often write sets as intervals, as in

    $$(a, b) = \{x \in \mathbb{R} : \ a < x < b\} \text{ and } [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

    A parenthesis before or after a number indicates that number *is not* included in the set. A bracket before or after a number indicates that number *is* included in the set. *Note that (a, b) can be used to denote the interval from a to b. It can also denote a point in $\mathbb{R}^2$. We must use the context to determine which it is.*

0.2 Assumptions

What can we assume we know? When writing proofs, many students have difficulty in knowing what they can assume is true and what they need to show. Below are some facts that we will assume we all know.

1. $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ are closed under addition and multiplication. For example, the sum of 2 integers is an integer and the product of 2 integers is an integer. We will actually show that the product of 2 rational numbers is rational.

2. Addition and multiplication are associative and commutative, and the distributive property holds.

3. If we multiply or divide both sides of an inequality by a positive number, the inequality sign remains the same, but if we multiply or divide both sides of an inequality by a negative number, the inequality switches.

4. For any $x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ such that $x \leq n$. One such n is $n = \lceil x \rceil$. Similarly, for any $x \in \mathbb{R}$, $\exists n \in \mathbb{Z}$ such that $x < n$. In this case, one such n would be
$$n = \lceil x \rceil + 1.$$
This means that no matter how large a number we pick, we can find an integer at least that large or larger. This can be used to show that, for any small positive number, ϵ, there exists a positive integer n such that $0 < 1/n < \epsilon$ by letting $n = \lceil 1/\epsilon \rceil + 1$.

5. Suppose $n, m \in \mathbb{Z}$ and p is a prime number which divides nm. Then p divides n or p divides m.

6. Division property when dividing n by m: For all $n, m \in \mathbb{Z}^+$, there exist unique $q \in \mathbb{Z}$ (called quotient) and $r \in \{0, 1, \ldots, m-1\}$ (called remainder) such that
$$n = qm + r.$$

7. Once we prove a statement, we can then assume that statement is true from then on. For example, we will prove that the sum of 2 even integers is even and the sum of 2 odd integers is even. After that, this can be assumed. Another obvious but useful fact we will prove is that if $n \in \mathbb{Z}$, then $n(n+1)$ is even. This just means that the product of two consecutive integers is even. When in doubt about what you can assume, ask your instructor.

0.3 Definitions

Definition 1.5 in Section 1.2: The **let-method**: To prove a statement of the form

$$r \equiv \{\text{for every } x \in S, \, p(x)\},$$

the proof should begin

Let x_0 represent an arbitrary value in S.

We call the value x_0 a **let-variable**.

Definition 1.8 in Section 1.2: Suppose we want to prove a statement of the form

$$r \equiv \{\text{for every } x \text{ in some set } S, \, p(x) \text{ is true}\}$$

is false. We assume statement r is true, then find one value, $x_0 \in S$ for which $p(x_0)$ is false. We call the value x_0 a **set-variable**.

Definition 1.13 in Section 1.3: An integer n is **even** if and only if there exists an integer k such that $n = 2k$. An integer n is **odd** if and only if there exists an integer k such that $n = 2k + 1$.

Definition 1.19 in Section 1.3: A set of objects S is called a **subset** of a set of objects T, denoted $S \subseteq T$, if and only if every element x that is in S is also in T, that is, for every $x \in S$, $x \in T$.

Definition 1.21 in Section 1.3: A set of objects S is **not a subset** of a set of objects T, denoted $S \nsubseteq T$, if and only if there exists $x \in S$ such that $x \notin T$.

Definition 1.23 in Section 1.3, Problem 2: Integer n **divides** integer m if and only if there exists an integer k such that $m = kn$.

Definition 1.31 in Section 1.4: A function f maps the set A **into** the set B if and only if for every $x \in A$, $f(x) \in B$.

Definition 1.32 in Section 1.4: A function f maps the set A **onto** the set B if and only if for every $y \in B$ there exists an $x \in A$ such that $f(x) = y$.

Definition 1.40 in Section 1.5: Suppose f is a function with domain $D \subseteq \mathbb{R}$. The function f is **bounded above** if and only if there exists $M \in \mathbb{R}$ such that for all $x \in D$, $f(x) \leq M$. In this case, M is an upper bound for the function.

Definition 1.43 in Section 1.5: A function f is said **not** to map set S onto the set T if and only if there exists $y \in T$ such that for every $x \in S$, $f(x) \neq y$.

Definition 2.2 in Section 2.1: The **negation** of a statement p is the statement that is true precisely when statement p is false and is denoted $\neg p$.

Definition 2.4 in Section 2.1: The function f is **not bounded above** if and only if for all $M \in \mathbb{R}$, there exists an $x \in D$ such that $f(x) > M$, that is

$$\{\langle \forall\, M \in \mathbb{R}\rangle, \langle \exists x \in D \text{ such that}\rangle \langle f(x) > M\rangle\}.$$

Definition 2.11 in Section 2.2: The **conjunction** of p and q is the statement that is true precisely when statements p and q are both true and is denoted

$$r \equiv \{p \wedge q\}.$$

Definition 2.12 in Section 2.2: The **disjunction** of p and q is the statement that is true when statement p is true or statement q is true or both statements p and q are true, and is denoted

$$s \equiv \{p \vee q\}.$$

Definition 2.29 in Section 2.4: A function f is **one-to-one** from its domain D into a set R if and only if for all $x_1,\ x_2 \in D$, such that $x_1 \neq x_2$, we have that $f(x_1) \neq f(x_2)$.

Definition 2.33 in Section 2.4: A function f is **not one-to-one** from its domain D into a set R if and only if

$$\{\exists\, x_1, x_2 \in \mathbb{R} \text{ such that } x_1 \neq x_2 \text{ and } f(x_1) = f(x_2)\}.$$

Definition 2.41 in Section 2.5: A number x is **rational** if and only if there exists integers n and m with $m \neq 0$ such that $x = n/m$.

Definition 2.42 in Section 2.5: A number is **irrational** if and only if for all integers n and $m \neq 0$, $x \neq n/m$.

Definition 4.2 in Section 4.1: Let f be a function whose domain contains an open interval about the point a, except possibly a itself. The **limit** of $f(x)$ as x approaches a equals L, written

$$\lim_{x \to a} f(x) = L$$

if and only if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

it Definition 4.17 in Section 4.3 A limit **does not equal** L, written as

$$\lim_{x \to a} f(x) \neq L$$

if and only if

$$\{\langle \exists\, \epsilon > 0 \rangle \text{s.t. } \langle \forall\, \delta > 0 \rangle,\, \exists\, x \text{ s.t.} \langle 0 < |x-a| < \delta \rangle \wedge \langle |f(x) - L| \geq \epsilon \rangle \}.$$

Definition 4.23 in Section 4.4: Suppose a function f is defined on an interval (a, b) and $c \in (a, b)$. The function is said to be **continuous** at $x = c$ if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Definition 4.28 in Section 4.4: If f is defined on an interval (a, b) and $x_0 \in (a, b)$, then f is differentiable at x_0 if and only if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. The limit is called the derivative of f at x_0 and is denoted $f'(x_0)$, that is,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Definition 5.4 in Section 5.1: We define the **difference** of two sets as

$$A - B = \{x : x \in A \cap B^c\}.$$

Definition 5.11 in Section 5.2: A set $R \subseteq S^2$ is called a **relation** on S. If $(a, b) \in R$, then we say a is related to b.

Definition 5.15 in Section 5.2: Consider a relation R on the set S.

- R is **symmetric** iff if $(a, b) \in R$ then $(b, a) \in R$. Thus, if a is related to b, then b is related to a.
- R is **asymmetric** iff if $(a, b) \in R$ then $(b, a) \notin R$. Thus, if a is related to b, then b is not related to a.
- R is **antisymmetric** iff if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$. Thus, if a is related to b, and b is related to a, then $a = b$.

Definition 5.25 in Section 5.2: Let R be a relation on the set S. R is **transitive** iff if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ meaning that if a is related to b and b is related to c, then a is related to c.

Definition 5.28 in Section 5.3: Consider a relation R on the set S.

- R is **reflexive** iff for all $a \in S$, $(a, a) \in R$, that is every a is related to itself.
- R is **irreflexive** iff for all $a \in S$, $(a, a) \notin R$, that is a is never related to itself.

Definition 5.37 in Section 5.3: A relation R on S is an **equivalence relation** if and only if R is Reflexive, Symmetric, and Transitive.

Definition 5.38 in Section 5.3: A relation R on set S is a **partial order** if and only if it is reflexive, antisymmetric and transitive.

Definition 5.39 in Section 5.3: A relation R on set S is a **strict partial order** if and only if it is asymmetric and transitive.(This means it is also irreflevice

Definition 6.3 in Section 6.1: We say the **cardinality** of sets S and T are the same if and only if there exists a one-to-one function f from S onto T. We then write

$$C(S) = C(T).$$

Definition 6.4 in Section 6.1: The set S has **cardinality** n if and only if there exists a one-to-one function f from S onto

$$\mathbb{Z}_n = \{i \in \mathbb{Z}^+ : i \leq n\} = \{1, 2, \cdots, n\},$$

in which case we write

$$C(S) = n$$

and say that set S is **countable**.

Definition 6.7 in Section 6.1: A set S is **countably infinite** or **denumerable** and its cardinality is denoted as

$$C(S) = \aleph_0$$

(read as 'aleph naught') if and only if

$$C(S) = C(\mathbb{Z}^+).$$

Definition 6.10 in Section 6.2: We say that if there is a one-to-one function f from S into T then the cardinality of S is less than or equal to T, and write

$$C(S) \preceq C(T).$$

If there is a one-to-one function f from S into T but there does not exist a one-to-one function from S onto T, then the cardinality of S is less than the cardinality of T and write

$$C(S) \prec C(T).$$

Definition 6.12 in Section 6.1: The cardinality of $I = (0,1)$ is the **continuum** and is denoted as

$$C(I) = c.$$

Definition 7.2 in Section 7.1: The set $S \subseteq \mathbb{R}$ is called an **open set** if and only if

$$p \equiv \{\forall\, x \in S,\, \exists\,(a,b) \subseteq S \text{ such that } x \in (a,b)\}.$$

Definition 7.4 in Section 7.1: A set $S \subseteq \mathbb{R}$ is **closed** if and only if S^c is open.

Definition 7.6 in Section 7.1: Let $S \subseteq \mathbb{R}$. The number $b \in \mathbb{R}$ is a **boundary value** for S if and only if

$$\{\forall\, x < b \text{ and } \forall\, y > b,\, (x,y) \cap S \neq \emptyset \text{ and } (x,y) \cap S^c \neq \emptyset\}.$$

We call the set

$$B_S = \{b \in \mathbb{R} : b \text{ is a boundary value for } S\}$$

the boundary of S.

Definition 7.10 in Section 7.1: The number $b \in \mathbb{R}$ is **not a boundary value** for S if and only if

$$\{\exists\,(x,y) \text{such that } \langle b \in (x,y) \subseteq S\rangle \vee \langle b \in (x,y) \subseteq S^c\rangle\}.$$

Definition 7.15 in Section 7.1: Given an arbitrary set S. If S has measure at most ϵ for every $\epsilon > 0$, then we say set S has **measure zero**.

Definition 7.13 in Section 7.1: c is an **interior value** of S if and only if there exists an interval (a,b) such that $c \in (a,b) \subseteq S$.

Definition 7.14 in Section 7.1: A value x is an **upper bound** for a set S if and only if $\forall\, a \in S,\, x \geq a$.

0.4 Video Lessons

To access the following videos, use the password 'Proof'.

1. https://vimeo.com/85102582: See Section 1.2. This 8:49 minute video is an introduction to implications.

2. https://vimeo.com/101854757 This 2:59 minute video shows two students doing a backwards proof in Problem 10 of Section 1.2.

3. https://vimeo.com/180908097 This 11:40 minute video introduces the concept of **let** variables. See Section 1.3.

4. https://vimeo.com/46317620 This 6:29 minute video shows two students trying to prove two sets are equal in Problem 9 in Section 1.3.

5. https://vimeo.com/45913468 This 4:28 minute video proves two sets are equal, $S = T$, Problem 11 in Section 1.3.

6. https://vimeo.com/6166001 This 5:41 minute video has two students trying to show two sets are equal in Problem 12 of Section 1.3.

7. https://vimeo.com/83721714 This 5:45 minute video gives a *for every, there exists* proof. See Section 1.4.

8. https://vimeo.com/44892151: This 5:02 minute video has students trying to prove or disprove a *for every, there exists, if then* statement. See Section 1.4, Problem 20.

9. https://vimeo.com/45875554: See Section 1.4, Problem 21. This 4:33 minute video has students trying to prove or disprove the statement

$$\{\forall\, a \in \mathbb{R}, \exists\, b \in \mathbb{R} \text{ s.t. if } x > a \text{ then } 2x - 5 > b\}$$

10. https://vimeo.com/83696588 This 6:48 minute video gives a *there exists/for every* proof. See Section 1.5.

11. https://vimeo.com/46575162 This 5:17 minute video shows two students solving Problem 3, Section 1.5 which is of the form, there exists a such that for all b, A is true.

12. https://vimeo.com/45866012: This 5:55 minute video has two students trying to show a function maps onto the integers before realizing it does not map onto, Problem 4 of Section 1.5.

13. https://vimeo.com/73971958: See Section 2.1. This 5:25 minute video goes through the process of negating the statement that a function is bounded above.

14. https://vimeo.com/85102409: See Section 2.1. This 3:36 minute video negates an implication.

15. https://vimeo.com/74398407: See Section 2.1. This 4:13 minute video Negates an implication.

16. https://vimeo.com/498727792: See Section 2.2. This 5:05 minute video proves a conjunction.

17. https://vimeo.com/498747664: See Section 2.2. This 3:51 minute video proves a disjunction.

18. https://vimeo.com/498772074: See Section 2.3. This 6:49 minute video gives proofs about properties of functions.

19. https://vimeo.com/26512295: See Section 2.3, Problem 8. This 3:40 minute video has students trying to find the error in their proof that the difference of two bounded above functions is bounded above.

20. https://vimeo.com/70952875: See Section 2.3, Problem 8. This 2:27 minute video has students trying to prove or disprove that the difference of two bounded above functions is bounded above.

21. https://vimeo.com/71067008: See Section 2.3, Problem 11. This 6:15 minute video has students trying to prove or disprove two functions are equal given the property that a third function is onto.

22. https://vimeo.com/74332781: See Section 2.4. This 9:42 minute video proves a statement using contraposition.

23. https://vimeo.com/74893416: See Section 2.4. This 7:04 minute video proves a complicated statement using contraposition, and shows several approaches to rewriting implications.

24. https://vimeo.com/45321903. This 7:17 minute video has students trying to find key idea behind why intersection of some two must contain third, Problem 14 in Section 2.4.

25. https://vimeo.com/5918679 This 4:59 minute video has students determining if f is one-to-one, then $g = h$ in Problem 16 in Section 2.4.

26. https://vimeo.com/87615555: See Section 2.5. In this 5:55 minute video, you can see a proof by contradiction that a certain cubic expression has only irrational roots.

27. https://vimeo.com/73398243: See Section 3.1. This 8:00 minute video develops conceptual insight into why a map made of n lines can be two-colored. It introduces induction.

28. https://vimeo.com/76630587: See Section 3.2. This approximately 10:27 minute video introduces proof by induction.

29. https://vimeo.com/76709853: See Section 3.2. This 8:53 minute video shows a simple example that brings home the importance of the base case in an induction proof.

30. https://vimeo.com/76713928: See Section 3.2, Problem 2. This 7:28 minute video gives a false induction proof.

31. https://vimeo.com/44885997: See Section 3.3, This 9:13 minute video gives an example of strong induction.

32. https://vimeo.com/44885997: See Section 3.3, Problem 5. In this 5:28 minute video, students try to determine what values of $n \in \mathbb{Z}_0^+$ is $2^{2n+1} + 1$ prime?

33. https://vimeo.com/87882345: See Section 3.4. In this 15:08 minute video, a formula is proven for the number of lines of blocks of length n using induction and two values for the base case.

34. https://vimeo.com/77532939: See Section 3.4. In this 10:14 minute video, the formula for binomial coefficients in Pascal's triangle is proven using induction on rows.

35. https://vimeo.com/77537399: See Section 4.1. This 7:12 minute video introduces the ϵ-δ definition of limit. This proof is similar to those done in Section 1.4.

36. https://vimeo.com/189383604: See Section 4.1. This 6:48 minute video introduces a technical handle to help with piecewise defined functions.

37. https://vimeo.com/399478447: See Section 4.2 This 5:26 minute video introduces a technique called the restriction method on piecewise defined functions.

38. https://vimeo.com/77748556: See Section 4.2. This 11:38 minute video uses the restriction method on a rational function to prove a limit.

39. https://vimeo.com/400028412 See Section 4.3. This 5:47 minute videos introduces the negation of a limit equaling L.

    https://vimeo.com/400028737 See Section 4.3. This 4:45 minute video gives ideas on selecting an appropriate ϵ_0 for showing a limit does not equal L.

40. https://vimeo.com/79835786: See Section 4.3. This 10:27 minute video shows that a limit does not exist. This 7:00 minute video shows how to prove a derivative is what we think using limits.

41. https://vimeo.com/499674749: See Section 5.1. This 9:35 minute video gives examples of set inclusion proofs.

42. https://vimeo.com/499675227: See Section 5.2. This 10:49 minute video gives an introduction to relations.

43. https://vimeo.com/499675608: See Section 5.3. This 7:00 minute video shows how to prove a relation is an equivalence relation. .

44. https://vimeo.com/499703835: See Section 5.3. This 7:10 minute video shows how to prove a relation is a partial order..

45. https://vimeo.com/499675985: See Section 6.1. This 9:25 minute video shows two infinite sets have the same cardinality.

46. https://vimeo.com/499676410: See Section 6.2. This 6:23 minute video gives insight into the difference between the sizes of set of integers and set of real numbers.

47. https://vimeo.com/79834502: See Section 6.2. This 14:45 minute video sketches a proof that $C(\mathbb{R}) = C(P(\mathbb{Z}^+)$.

48. https://vimeo.com/499676930: See Section 7.1. This 5:57 minute video discusses open sets.

0.5 Acknowledgements:

There are too many people to thank all of them for the preparation of this text. but I will mention a few.

Many years ago, Manya Raman started me on a project of videotaping students doing proofs. As part of this project, we enlisted Kay Somers, Connie Campbell, and Geoff Birky. On a sabbatical at Oxford University, I began working with Anne Watson, John Mason and Gabriel Stylianides. Again, through discussions while watching student-videos, I gained much insight into what works, which also led to several papers written by this group. More recently, I have been teaching sections of Proof in parallel with two other instructors, Michael Raney and Erblin Mehmetaj. We have had numerous discussion about this course and the material, and observed each others' classes. Finally, I would also like to thank all the mathematics education researchers in the area of student-proving. Their research has greatly impacted the writing of this material. Let me mention a few.

- Keith Weber and Lara Alcock wrote several papers on students tendency to work either syntactically or semantically.
- Guershon Harel has written about introducing induction using problems that are inherently recursive in nature as opposed to problems which can be proven using induction but for which induction does not explain.
- A paper by Paola Iannone, Matthe Inglis, Juan Pablo Mejia Ramos, Adrian Simpson and Keith Weber cast doubt on example construction being helpful to students, which led to work on how to get students to use examples productively.

Finally, I would like to thank Georgetown University, and more specifically, their Center for New Designs in Learning and Scholarship (CNDLS), for its support in this project over the years. Of particular support from CNDLS was Randall Bass who recruited me, and Susannah McGowen, who guided me in my initial attempts at videotaping my students.

Chapter 1

Introduction to Proving

1.1 Heuristic Arguments

Popular opinion is that mathematics is about computation. In reality, mathematics is more about thinking logically. One important aspect of this logical thinking is in the analysis of the veracity of complex statements.

We define a **statement** to be a sentence or several sentences that is either true or false. For easy reference, we will often identify a statement with one of the variables p, q, r, and s, as in

$$p \equiv \{\text{statement}\}$$

For example,

$$\begin{aligned} p &\equiv \{15 \text{ is a prime number}\} \text{ and} \\ q &\equiv \{21 \text{ is divisible by } 3\} \end{aligned}$$

Note that p is a false statement and q is a true statement. We use the symbol '$\equiv$' to emphasize that the variable **represents** the statement which follows.

Some care must be taken when defining statements. For example, we cannot determine whether the statement $s \equiv \{x > 1\}$ is true or false until we know more about x. Technically we should write this statement as

$$s(x) \equiv \{x > 1\}$$

In this case the statement $s(4)$ is true and the statement $s(0)$ is false. Statements such as $s(x)$ are often called **predicate functions**. Sometimes we will assume that a predicate function is true, for example, assume that statement $s(x)$ is true. In this case, we are assuming that $x > 1$.

Note that the sentence "Turn in all your homework" is not a statement since it is neither true nor false but is a command.

There are several goals in this text. First is to learn how to determine if a statement is true or not. This can come from developing **conceptual insight** into the problem, often by first considering some examples, then possibly through writing a **heuristic** argument, that is, an informal argument that attempts to explain the underlying reasons why a statement is true or false. Unfortunately, heuristic arguments frequently miss subtleties in the statement, resulting in an incorrect conclusion. For example, some students were asked if it was true that for every real number x, there exists a real number y such that $xy = 1$. They claimed this statement was true by letting $y = 1/x$, missing that this does not work if $x = 0$.

We have nothing against developing a heuristic argument for why a statement is true, and, in fact, encourage students to develop such arguments which are often the first step in proving a statement true or false. The second step to proving a statement is true or false is to convert our insight or heuristic argument into a logical argument that is perceived to be clear and complete by others. This often requires some technical handle which can consist of some algebraic manipulation and/or choosing an appropriate logical approach to the proof. The third step is to convert our logical argument into a well-written proof that others can easily read.

To emphasize the importance of the second and third steps, in the rest of this section, we give a variety of heuristic arguments for why a particular statement is true or false. Some of these arguments actually do give insight into the problem while others are flat-out wrong, reinforcing our point that care should be taken with heuristic arguments. The reader should try to evaluate these arguments, noting where an argument gives insight into a problem and where the argument is missing something important. Hopefully, these examples will convince the reader of the importance of completing the second and third steps in our proving process, which are, in short, to refine and review our arguments.

Example 1.1: What does $0.99\cdots$ equal?

We are all familiar with fractions that equal infinitely repeating decimals, such as

$$\frac{1}{3} = 0.333\cdots = 0.\overline{3}$$

In this example, we explore the decimal 0.999.... In particular, we wish to determine the truth of the claim that

$$0.999\cdots = 0.\overline{9} = 1$$

If $x = 0.999\cdots \neq y = 1$, how far apart are they?
Heuristic Argument 1 that they are equal: Consider the known result

$$\frac{1}{3} = 0.333\cdots = 0.\overline{3}.$$

Multiplying both sides by 3 gives $1 = 0.999\cdots$.
Heuristic Argument 2 that they are equal: Multiply both sides of $x = 0.\overline{9}$ by 10, giving

$$10x = 9.999\cdots = 9.\overline{9}.$$

Subtracting $x = 0.\overline{9}$ from $10x = 9.\overline{9}$ gives

$$10x - x = 9.\overline{9} - 0.\overline{9}$$

which simplifies to

$$9x = 9 \quad \text{or} \quad x = 1 = y$$

so the infinitely repeating decimal is actually an integer.
Heuristic Argument 3 that they are not equal: We argue that

$$1.000\cdots \neq 0.999\cdots$$

Since they are different decimal expansions, they cannot possibly be equal as decimal expansions are unique.
We have two arguments that $x = y$ and another argument that $x \neq y$. Which is correct? In the first argument, is multiplying an infinite decimal by 3 allowed? In the second argument, we were subtracting an infinite number of decimal places from an infinite number of decimals. Is that allowed? Since we multiplied by 10, shouldn't $0.99\cdots$

have more decimals places than $9.99\cdots$ and then we actually need to borrow one at the 'infinity' place?
In evaluating the third heuristic argument, if $0.999\cdots$ does not equal 1, then what is the decimal representation of the average of x and y,

$$\frac{x+y}{2} = \frac{1.999\cdots}{2} = ?$$

Would it be $0.999\cdots 5 = 0.\overline{9}5$? Have we ever seen a number like this? Could we write it as a fraction? What does it mean? Are x and y just two different numbers which are infinitely close to one another. What does 'infinitely close' even mean? The third argument used the 'fact' that every number has a unique decimal expansion. Just because our intuition tells us this is true doesn't mean it actually is true. We should be careful about assuming something we have not actually seen proven or even claimed.

Too often someone is positive that a statement is true or false but others are not convinced by their heuristic argument. We have seen students so convinced that $0.999\cdots \neq 1$ that they resorted to what we call *proof by intimidation*, that is, they started talking increasingly forcefully, possibly making some sarcastic comments questioning the intelligence of the audience. This is not allowed: an acceptable argument should stand on its own without the need to intimidate.

On the other hand, sometimes we have a proof that we know must be false. This means that there must be a mistake somewhere in the proof. We should not be lazy and just state that the proof is false, but should go back and find the error to make sure we do not make the same error on another proof. Following is what appears to be a convincing argument that the false statement

$$p \equiv \{90 \text{ degrees equals } 89 \text{ degrees}\}$$

is actually true.

We begin this false proof with Figure 1.1, in which rectangle ABCE is constructed with sides AB and CE of length 1. Line CD is constructed with angle $\angle$DCP$=$ 89 degrees and $\overline{CD} = 1$. We are going to try to show that $\angle$ABP$=$ $\angle$DCP.

Why is point D below line AE? Line AD is constructed and extended to

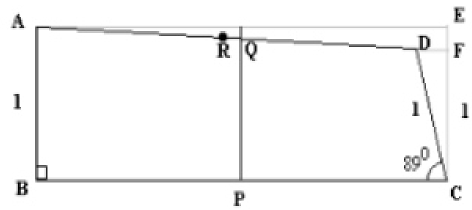

Figure 1.1: Initial figure with ∠ABP= 90 degrees and ∠DCP= 89 degrees.

point F on line EC. The perpendicular bisector PQ of line BC is constructed. PQ bisects line AF. This means that

$$\overline{AQ} = \overline{QF} > \overline{QD}$$

so the bisector R of line AD must lie to the left of Q.

In Figure 1.2, the perpendicular bisector PQ of BC is extended and the perpendicular bisector of AD is constructed. The two perpendicular bisectors are not parallel since AD is sloping downward, so must intersect at some point above line AD as seen in Figure 1.2. Call this point of intersection S.

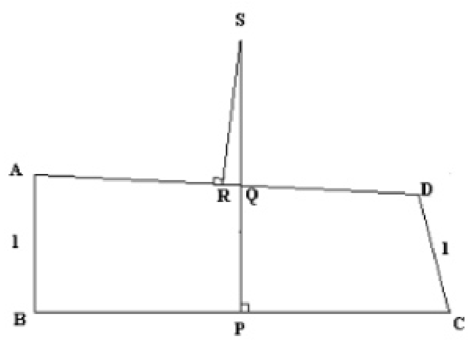

Figure 1.2: Perpendicular bisectors to AD and to BC meet above the figure.

In Figure 1.3 a, point S has been connected to points A, B, C, and D. Figure 1.3 b shows a part of Figure 1.3 a. Note that $\overline{SP} = \overline{SP}$ by identity and $\overline{BP} = \overline{CP}$ since P bisects BC. Also, ∠SPB= ∠SPC= 90 degrees. Thus △PBS≅ △PCS because of side-angle-side. This is just the simple result that the two triangles formed by a perpendicular bisector of a line are congruent. Thus, $\overline{BS} = \overline{CS}$ because they are corresponding sides of congruent triangles. Also, ∠SBP= ∠SCP because they are corresponding angles of congruent triangles. Figure 1.3 a is duplicated as Figure 1.4 with the results marked, that is, $\overline{BS} = \overline{CS}$ and ∠SBP= ∠SCP.

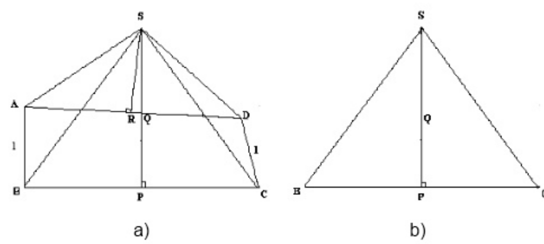

Figure 1.3: In Figure a), points have been connected to point where perpendicular bisectors meet. Figure b) displays a portion of Figure a) for simplicity.

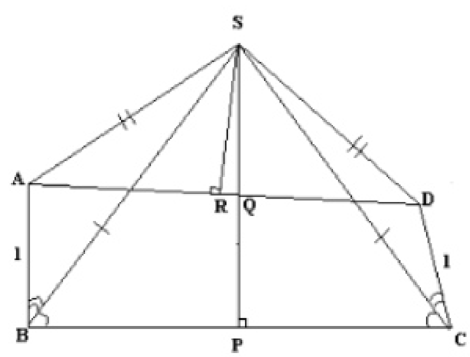

Figure 1.4: Equal lines and equal angles have been marked.

Similarly, △ASR≅ △DSR by side-angle-side (triangles formed by perpendicular bisector of AD) since $\overline{SR} = \overline{SR}$ by identity, $\overline{AR} = \overline{RD}$ because R is the bisector of AD, and ∠ARS= ∠DRS= 90 degrees. This means that $\overline{AS} = \overline{SD}$.

Note that $\overline{AB} = \overline{CD}$ since they are both constructed with length 1. This means △BAS≅ △CDS since corresponding sides are equal. From this, we know that ∠ABS= ∠DCS since they are corresponding angles of congruent triangles.

Finally, ∠ABP= ∠DCP by addition of equal angles. So the 90 degree angle ∠ABP equals the 89 degree angle ∠DCP.

Our detailed proof that 89 degrees equals 90 degrees appears to be accurate but is clearly wrong because we know the statement is wrong. For fun, the reader should try to find the mistake in this argument. As a hint, nowhere in our arguement did we use the fact that $\angle$DCP was 89 degrees. We only used the fact it was less than 90 degrees. Thus, you might try redrawing the figure as accurately as possible using different acute angles for $\angle$DCP.

This false proof reinforces our warning that, just like the students considering $xy = 1$ missed the case that $x = 0$ is a possibility, it is unfortunately all too easy to miss a subtle point in a proof. So we need to make sure we are careful and critically re-evaluate our own proofs once we have finished writing them.

Summary 1.1

When considering a statement, construct some examples to get an idea if the statement is true or not and to get some conceptual insight into why the statement is true or not.

1.1.1 Problems

1. Is $n^2 + n$ even for all $n \in \mathbb{Z}$? Check the validity of this claim by computing $n^2 + n$ for several different values of n. Give a heuristic argument supporting your conclusion. Answer 1

2. Is $n^2 + 63 \geq 16n$ for every $n \in \mathbb{Z}^+$? Check the validity of this claim by computing $n^2 + 63$ and $16n$ for several different positive integers, n. Give a heuristic argument supporting your conclusion.

3. What is wrong with the following argument that $3 = 1$?

   Proof? Let $x = 1$. Then
   $$x^2 = 1$$
   Adding x to both sides of the equation and subtracting 2 gives
   $$x^2 + x - 2 = x - 1$$
   Factoring gives
   $$(x-1)(x+2) = x - 1$$
   Dividing both sides of the equation by $x - 1$ gives
   $$x + 2 = 1$$

Substitution of 1 for x gives that $3 = 1$. Answer 2

4. Let's investigate the statement

$$r \equiv \{\text{There are infinite sets of different sizes}\}$$

Heuristic Argument 1: The set $\mathbb{E}^+ = \{2, 4, \cdots\}$ is smaller than the set $\mathbb{Z}^+ = \{1, 2, \cdots\}$ since $\mathbb{E}^+ \subseteq \mathbb{Z}^+$, but there are elements of $\mathbb{Z}^+$ that are not in $\mathbb{E}^+$, such as $1, 3, \cdots$.

Heuristic Argument 2: The set $\mathbb{E}^+$ is the same size as set $\mathbb{Z}^+$, since we can put the elements of $\mathbb{E}^+$ in a one-to-one correspondence with the elements of $\mathbb{Z}^+$;

$$\begin{array}{ccc} \mathbb{E}^+ & \leftrightarrow & \mathbb{Z}^+ \\ 2 & \leftrightarrow & 1 \\ 4 & \leftrightarrow & 2 \\ 6 & \leftrightarrow & 3 \\ 8 & \leftrightarrow & 4 \\ \vdots & \vdots & \vdots \\ 2n & \leftrightarrow & n \end{array}$$

Which claim do you believe and why? What do you think of the two arguments? Suppose we were comparing $\mathbb{E}^- = \{\cdots, -6, -4, -2\}$ to $\mathbb{Z}^+$ where $\mathbb{E}^- \cap \mathbb{Z}^+ = \emptyset$, so the argument of Claim 1 was not valid. In this case, would an argument similar to Claim 2 in which $-2n$ is paired with n for $n \in \mathbb{Z}^+$ be valid, indicating these sets were the same size? Would this mean that

$$\mathbb{E}_0^- = \{\cdots, -6, -4, -2, 0\}$$

was larger than $\mathbb{Z}^+$? What is your definition for one set being larger than another?

5. If the net in Figure 1.5 a) is cut out along the solid black lines, then folded along the dotted lines, it will form a cube. The same is true for the net in Figure 1.5 b). How many different nets of 6 squares, similar to those seen in Figure 1.5 can be drawn, which when cut out and folded, form a cube? Be sure you define what it means for two nets to be different. Carefully explain your answer. Answer 3

6. Find the fallacy in the following *proof* that $\pi = 3$. Identify as closely as possible where an incorrect conclusion is made.

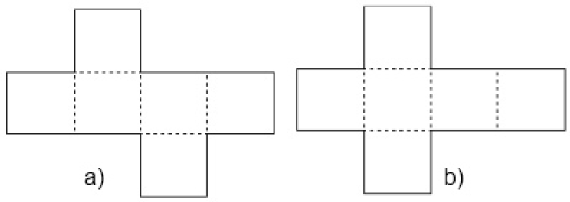

Figure 1.5: Nets that form squares.

Proof: Let $a = \frac{3+\pi}{2}$, then $3+\pi = 2a$. Multiplying both sides by $3-\pi$ gives

$$(3-\pi)(3+\pi) = 3^2 - \pi^2 = 6a - 2\pi a$$

and therefore

$$3^2 - 6a = \pi^2 - 2\pi a$$

Adding a^2 to both sides gives

$$3^2 - 6a + a^2 = \pi^2 - 2\pi a + a^2$$

which can be rewritten as

$$(3-a)^2 = (\pi - a)^2.$$

Taking square roots gives $3 - a = \pi - a$. Therefore, $3 = \pi$.

7. Find the fallacy in the following *proof* that $i = -\sqrt{3}$ where $i^2 = -1$. Identify as closely as possible where an incorrect conclusion is made.

   Proof: We start with the identity

$$\left(\frac{1}{2} + \sqrt{\frac{3}{4}}i\right)^3 = -1$$

   which implies that

$$\frac{1}{2} + \sqrt{\frac{3}{4}}i = -1.$$

   Hence

$$\frac{\sqrt{3}}{2}i = \sqrt{\frac{3}{4}}i = -\frac{3}{2}$$

   and therefore

$$i = -\frac{3}{\sqrt{3}} = -\sqrt{3} < 0.$$

   Answer 4

8. We tend to think that a function is continuous on an interval if we can 'draw' the function on that interval without lifting our pencil. Consider the function

$$f(x) = \begin{cases} 0 & x = \pm 1, \pm 1/2, \pm 1/3, \cdots \\ x & \text{otherwise} \end{cases} \tag{1.1}$$

A sketch of the graph of f is seen in Figure 1.6

Heuristic Argument 1: The function f is discontinuous at $x = 0$. The function is discontinuous at $x = 1$ since $f(1) = 0$ but for x close enough to 1, $f(x) = x \approx 1$. Similar arguments show that f is discontinuous at $x = \pm 1, \pm 1/2, \pm 1/3, \cdots$. This means that f 'jumps' for x-values arbitrarily close to 0, so f is discontinuous at $x = 0$.

Heuristic Argument 2: The function f is continuous at $x = 0$. If x is close to 0 then $f(x) = 0$ or $f(x) = x \approx 0$ both of which are close to $f(0) = 0$ which is the definition of continuous at $x = 0$.

Which argument do you think is the most correct and why?

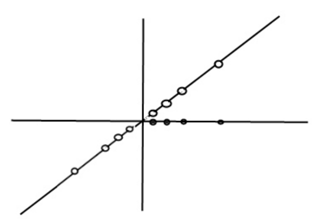

Figure 1.6: Graph of f from Problem 8.

9. Suppose we have 3 spinners, as in Figure 1.7. Assume for each spinner that when you spin the pointer, each of the 3 numbers is equally likely to occur. We play a 2-person game in which we each select a different one of these spinners, spin its pointer, and the person whose

pointer is on the highest number wins. Which is the best spinner to choose?

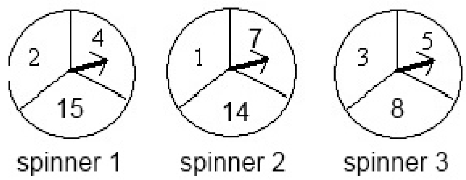

Figure 1.7: Three spinners.

Heuristic Argument 1 that Spinner 1 is best: The largest number is 15 and is on Spinner 1.

Heuristic Argument 2 that Spinner 2 is best: We take the average spin for each spinner and the spinner with the highest average must be better. For spinner 1, this would be

$$\frac{2+4+15}{3} = 7$$

You should check that the average of spinner 2 is $7\frac{1}{3}$ and spinner 3 is $5\frac{1}{3}$, so we have an argument that spinner 2 is the best.

Heuristic Argument 3 that Spinner 3 is best: Let's determine what fraction of the time spinner 1 will win against spinner 2 by making a list of the 9 equally likely results for the spins of the two spinners, which are

$$(2,1)\ (2,7)\ (2,14)\ (4,1)\ (4,7)\ (4,14)\ (15,1)\ (15,7)\ (15,14)$$

with the result for spinner 1 given first and the result for spinner 2 given second. Notice that spinner 1 wins 5 of them, so spinner 1 will win 5/9ths of the time playing against spinner 2, so spinner 1 is clearly superior to spinner 2, which we denote as

$$\text{spinner 1} > \text{spinner 2}$$

We now compare the results between spinner 1 and spinner 3, which are

$$(2,3)\ (2,5)\ (2,8)\ (4,3)\ (4,5)\ (4,8)\ (15,3)\ (15,5)\ (15,8)$$

with the result for spinner 1 given first and the result for spinner 3 given second. In this case, note that spinner 3 wins 5/9ths of the time when playing against spinner 1, so

$$\text{spinner } 3 > \text{spinner } 1$$

It is clear that since spinner 3 wins more than 50% of the time against spinner 1 and that spinner 1 wins more than 50% of the time against spinner 2, thus spinner 3 must win more than 50% of the time against spinner 2, and is thus the best spinner, that is,

$$\text{spinner } 3 > \text{spinner } 2$$

Given these heuristic arguments, which spinner do you believe is best and why? Answer 5

10. Alice, Bob, and Celeste are playing a tournament of Rock-Paper-Scissors. *Ask a friend if you don't know how this game is played.* Each player is playing unpredictably, that is, they are using a random mechanism to decide which of the three moves to use. Alice likes paper, so she uses this move 50% of the time and the other two moves 25% of the time. Bob likes rock, so he uses this move 50% of the time and the other two moves 25% of the time. Finally, Celeste likes scissors, so she uses this move 50% of the time and the other two moves 25% of the time.

    (a) If Alice plays many times against Bob, explain why in the long run Alice will win six times, Bob will win five times, and there will be five draws, out of 16 games.

    (b) Determine similarly how Bob will do against Celeste in the long run if many games are played.

    (c) From parts a) and b), we see that Alice tends to beat Bob and Bob tends to beat Celeste in the long run. Now determine how Alice will do against Celeste in the long run.

    (d) There is a paradox here (something that runs against our intuition). State what the paradox is. What could be its resolution?

1.1.2 Answers to selected problems

1. **Problem 1** $n^2 + n$ is always even. Make sure you check using negative values for n as well as positive. Factoring gives

$$n^2 + n = n(n+1)$$

This is product of two consecutive integers, so one of them must be even. This convincing argument gives the idea behind why this is true. It is not a proof. That will come later.

2. **Problem 3:** In the last step, both sides were divided by $x - 1$ which equals 0 since we were given that $x = 1$. We cannot divide both sides of an equation by 0.

3. **Problem 5:** Did you define two nets as the same if a rotation of one gives another? If a flip and rotation of one gives another?

4. **Problem 7**: In the last step, we took the cube root of both sides. Cube roots are not unique. For example,if you compute

$$\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$$

you will see it equals 1. So the numbers were different but their cubes were equal. Both terms are cube roots of -1.

5. **Problem 9**: There is no best spinner. Continuing Heuristic Argument 3, if we list the 9 equally likely possible results of spinner 2 versus spinner 3, we see that spinner 2 beats spinner 3, on average, 5/9ths of the time, that is

$$\text{spinner } 2 > \text{spinner } 3$$

Each spinner beats one of the other spinners and loses to the other spinner, so there is no best spinner. No matter which spinner we pick, our opponent can pick a spinner that beats ours more than 50% of the time. Note that we are getting contradictory results from our heuristic arguments. It should be clear that the three arguments, while seeming to be reasonable, missed an important aspect to the problem.

1.2 Implications and Let/Set Variables

Video Lesson 1.2

Watch the 8:49 minute video at
https://vimeo.com/85102582 Password:Proof
before reading this section. That video will introduce you to the method of using let-variables. To change the speed at which the video plays, click on the gear at the lower right of the video.

Consider the statement

$$r \equiv \left\{ \text{for every } x > -1, \frac{x}{x+1} < 1 \right\}. \tag{1.2}$$

This statement is called an **implication**. Another way to write it is

$$r \equiv \left\{ \text{if } x > -1, \text{ then } \frac{x}{x+1} < 1 \right\}.$$

A third way to write this statement is by letting

$$r \equiv \{p(x) \Rightarrow q(x)\}$$

where

$$p(x) \equiv \{x > -1\} \text{ and } q(x) \equiv \left\{ \frac{x}{x+1} < 1 \right\}.$$

This statement is read as '$p(x)$ implies $q(x)$', meaning that any x that satisfies statement $p(x)$ also satisfies $q(x)$.

While all three forms are acceptable, you should get into the habit of rewriting every 'if/then' or implication statement as a 'for every' statement, such as statement 1.2. This will help you avoid mistakes later on.

When given a statement, we often construct some examples to determine if the statement is reasonable or not. For example, we can substitute values such as $x = 2$, $x = 1,000,000$, $x = -0.9999$ and $x = 0$ into the expression

$$x/(x+1)$$

and observe the result is always less than 1, leading us to believe this statement is probably true.

Conceptual Insight: 1.3

As mentioned in the previous section, when looking at a statement, it is always a good idea to try to develop some **conceptual insight** into why a statement might be true or false, true in this case. While examples can help us see that the statement is reasonable, they often do not give much conceptual insight. Graphs are often a good means of developing conceptual insight. The graph of

$$y = \frac{x}{x+1}$$

seen in Figure 1.8 clearly has a vertical asymptote at $x = -1$ and a horizontal asymptote of $y = 1$, and remains below 1 when $x > -1$. This graph gives us good conceptual insight into why a statement is true, but it is not a proof.

How do we prove something is true for an infinite number of values, all the values greater than -1 in this case? The idea is to think of x in this expression as a placeholder for numbers greater than -1. In other words, we are claiming that if we substitute any number greater than -1 in for $\sqcup$ in the expression

$$\frac{\sqcup}{\sqcup + 1}$$

we will get a result less than 1. Since we have to show something is true **for every** $x > -1$, we let x_0 represent an arbitrary value greater than -1, that is, $x_0 > -1$. Note we did not actually choose a value but we treat x_0 as a known number greater than -1 and then will show that

$$\frac{x_0}{x_0 + 1} < 1. \tag{1.3}$$

Note that since we are treating x_0 as a fixed number greater than -1, then $x_0/(x_0 + 1)$ is also treated as a fixed number.

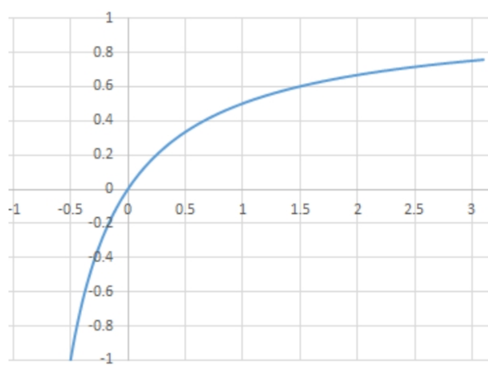

Figure 1.8: Graph of $y = x/(x + 1)$ has vertical asymptote at $x = -1$ and horizontal asymptote at $x = 1$.

We now convert our conceptual insight into a clear proof.

Claim 1.4: For every $x > -1$, $x/(x+1) < 1$.

Proof: We will prove that for every $x > -1$, $x/(x+1) < 1$. Let x_0 represent an arbitrary number, $x_0 > -1$. *(We are now assuming x_0 is a known number greater than -1.)* We know that

$$0 < 1.$$

Adding x_0 to both sides gives

$$x_0 < x_0 + 1.$$

Since $x_0 > -1$, *(using the property of the set from which x_0 is a representative)* $x_0 + 1 > 0$, so we can divide both sides of the inequality by $x_0 + 1$ without reversing the inequality. This gives

$$\frac{x_0}{x_0+1} < 1,$$

which is what we were to show. Thus, the statement is true.□

In the last step, we used the fact that $x_0 + 1 > 0$ to justify not reversing the inequality sign. So we didn't need to know the value of x_0, only the property $x_0 > -1$. We note that a proof is actually a template for proving the statement is true given any value in the given set. So to check a proof, just repeat the steps in the proof using actual values to see how it works.

Proof. Verification of Proof for an example: We rewrite the proof using an actual value for $x_0 > -1$, say $x_0 = -0.9$. We know that

$$0 < 1.$$

Adding $x_0 = -0.9$ to both sides gives

$$x_0 < x_0 + 1 \text{ or } -0.9 < 0.1.$$

Since $x_0 > -1$, $x_0 + 1 = 0.1 > 0$, so we can divide both sides of the inequality by $x_0 + 1 = 0.1$ without reversing the inequality. This gives

$$\frac{x_0}{x_0+1} < 1 \text{ or } \frac{-0.9}{0.1} = -9 < 1,$$

which is what we were to show. □

We repeat the proof again using another value for x_0.

> *Proof.* Verification of Proof for another example: Let's rewrite the proof using $x_0 = 1000 > -1$. We know that
>
> $$0 < 1.$$
>
> Adding $x_0 = 1000$ to both sides gives
>
> $$x_0 < x_0 + 1 \text{ or } 1000 < 1001.$$
>
> Since $x_0 > -1$, $x_0 + 1 = 1001 > 0$, so we can divide both sides of the inequality by $x_0 + 1 = 1001$ without reversing the inequality. This gives
>
> $$\frac{x_0}{x_0 + 1} < 1 \text{ or } \frac{1000}{1001} < 1,$$
>
> which is what we were to show. □

We checked our argument for two values, one just a little greater than -1 and another that was a lot greater than -1. Note that this argument would work for any value we used for $x_0 > -1$. So that you understand let-variables, you should get into the habit of repeating your proofs using actual values for let-variables.

You may wonder how we came up with the idea to start the proof with the statement $0 < 1$. Before writing the proof, we did some heuristic work. We began with the statement we were trying to prove,

$$\frac{x_0}{x_0 + 1} < 1,$$

and worked backwards. Since $x_0 > -1$, $x_0 + 1 > 0$, so we can multiply both sides of the inequality by $x_0 + 1$ without reversing the inequality sign. This yields

$$x_0 < x_0 + 1 \text{ or } 0 < 1$$

after subtracting x_0 from both sides. The actual proof resulted from reversing these steps. We call this a *backward/forward* approach since we worked backward from what we were to prove, $x/(x+1) < 1$, to a statement we know is true $0 < 1$, then reversed the steps to write the proof from what we know is true, $0 < 1$, toward what we are supposed to prove, $x/(x+1) < 1$. It would not be clear to someone reading our proof why we began with the statement $0 < 1$, but the reader will acknowledge it is true.

We call the combination of the backward/forward approach with the use of a let-variable, the **technical handle** we need to convert our insight, gained from the graph in this case, into an acceptable proof.

We summarize this process. Suppose we have a statement of the form

$$\{\text{for every } x \text{ in set } S,\ p(x) \text{ is true.}\}$$

First, we check the reasonableness of this statement by trying several numbers in set S to check if $p(x)$ seems to be true. If it seems like $p(x)$ is true for every $x \in S$, then we try what we call the 'let-method.'

Definition 1.5: Let-Variable

To prove a statement of the form

$$r \equiv \{\text{for every } \ x \in S,\ p(x)\},$$

the proof should begin

Let x_0 represent an arbitrary value in S.

We call the value x_0 a **let-variable**.

This method is used when we are trying to prove some statement for every value in an infinite set S and the subscripted variable is just used as a placeholder for all the values for which we have to prove the statement, all the values in S. To help identify variables that are treated as known fixed representatives of set S, we use subscripts.

Conceptual Insight: 1.6

We contrast the previous statement r and its proof with a similar one:

$$s \equiv \left\{\text{for every } \ x > -1, \frac{x}{x+1} < 0.9\right\}. \tag{1.4}$$

From the graph in Figure 1.8, we can see that it is not true that $x/(x+1) < 0.9$ for every $x > -1$. One approach to proving a statement is false is to assume the statement is true and arrive at a contradiction. Let us assume statement 1.4 is true. Since we have assumed that $x/(x+1) < 0.9$ is true for every $x > -1$, it must be true when $x = x_0 = 9$. In this case, we are using a particular value for x so we

set $x = x_0 = 9$. Substitution gives

$$\frac{x_0}{x_0 + 1} = 9/10 < 0.9,$$

which is a contradiction. Therefore statement 1.4 is false.

Claim 1.7: It is false that for every $x > -1$, $x/(x+1) < 0.9$.

Proof: We will prove it is false that for every $x > -1$, $x/(x+1) < 0.9$. Assume statement s is true. Set $x_0 = 9$. Then $x_0/(x_0+1) = 9/10 \not< 0.9$. This is a contradiction to our assumption, so the statement must be false. □

In arriving at our contradiction, we had to find **one** value for x that did not work. We picked 9, but any $x \geq 9$ would have worked. For example, we could have set $x_0 = 10$ or $x_0 = 100$, but we could not have used any value less than 9. Sometimes there is only one value that works but other times, there are numerous values that arrive at a contradiction and we only have to use one of them. Note the difference between proving a 'for-every' statement is true, in which case we verify it using a let-variable, and proving it is false, by assuming it is true and setting the variable equal to a value for which it does not work.

Definition 1.8: Set-Variable

Suppose we want to prove a statement of the form

$$r \equiv \{\text{for every } \ x \ \text{ in some set } S,\ p(x) \text{ is true}\}$$

is false. We assume statement r is true, then find one value, $x_0 \in S$ for which $p(x_0)$ is false. We call the value x_0 a **set-variable**.

To apply a set-variable, note the steps: first, assume statement r is true; next, set x_0 equal to the value that does not work; finally, verify $p(x_0)$ is false. This is called **proof by contradiction** and the value x_0 is called a **counterexample**.

Conceptual Insight: 1.9

Consider the implication that if $x > 10$, then $x^2 + 40 > 14x$ or

$$r \equiv \{p(x) \Rightarrow q(x)\}$$

where

$$p(x) \equiv \{x > 10\} \text{ and } q(x) \equiv \{x^2 + 40 > 14x\}.$$

We should usually try to rewrite an implication as a for-all statement, as in

$$r \equiv \{\text{for every } x > 10,\ x^2 + 40 > 14x\}.$$

We try some examples to get an idea if the statement is true. If we substitute values such as $x = 10.1$ or $x = 100$, which are greater than 10, into the inequality in statement $q(x)$, we will see that the inequality is satisfied. The examples indicate that the statement is probably true but they do not give us conceptual insight into why the statement is true. We graph the functions $f(x) = x^2 + 40$ and $g(x) = 14x$ as seen in Figure 1.9. From the graphs, it is apparent that $f(x) > g(x)$ for $x > 10$. This graph gives us the conceptual insight to understand why the implication is true.
Since the statement begins with a 'for every' in that 'for every real $x \in S = (10, \infty)$,' we use a let-variable and let x_0 represent an arbitrary real number greater than 10. What we have to show is that $x_0^2 + 40 > 14x_0$.
Assumed: We assume that $x_0 > 10$.
To Show: We want to show that $x_0^2 + 40 > 14x_0$
It is not clear at first how we go about this. When dealing with inequalities, it is often helpful to have all of the terms on the same side, that is, greater than or less than 0, so we will reframe what we have to show as that $x_0^2 - 14x_0 + 40 > 0$. We then try factoring the quadratic on the left of the inequality giving

$$(x_0 - 10)(x_0 - 4) > 0.$$

Since $x_0 > 10$, both factors are positive, so this inequality holds. This is the backward part of a backward/forward proof.

The forward proof would start from the fact that $x_0 > 10$.

Claim 1.10: For every $x > 10$, $x^2 + 40 > 14x$.

Let x_0 represent a number such that $x_0 > 10$. Note that the terms $x_0 - 10$ and $x_0 - 4$ are then both positive, so their product is positive, giving $x_0^2 - 14x_0 + 40 > 0$. This can now be written as $x_0^2 + 40 > 14x_0$, which is what we were to show.□

Let us now consider the similar problem

$$s = \{\text{for every } x > 9,\ x^2 + 40 > 14x\}.$$

We can see from Figure 1.9 that this statement is not true since it is not true for $9 < x \leq 10$. So to show this statement is not true, we assume it is true. We then set $x = x_0 = 10$, which is greater than 9, and arrive at the contradiction that

$$10^2 + 40 > 14(10).$$

So by contradiction, this statement is false.

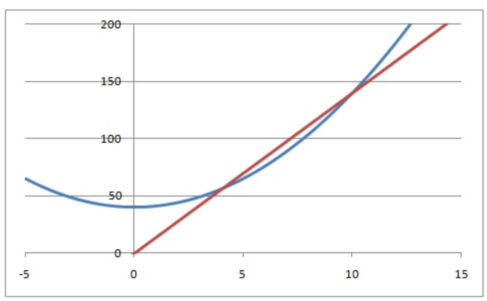

Figure 1.9: Graphs of $f(x) = x^2 + 40$ and $g(x) = 14x$.

Summary 1.2

When considering a statement, construct some examples and possibly draw some graphs to get an idea if the statement is true or not and to get some conceptual insight into why the statement is true or not. If you believe the statement is true and it begins with 'for every,' then you can construct a direct proof using a let-variable. You can prove it is false using contradiction by assuming it is true and then using a set-variable equal to some value for which it does not work. Remember that a statement of the form 'if $p(x)$ then $q(x)$' should be rewritten as 'for every $x \in S$, something is true,' then the let method should be used.

In the next several chapters, we will continue expanding on our proof techniques, including variations on direct proofs and proofs by contradiction.

Remark 1.11

The terms'for every' and 'for all', as in 'for every $x > 1$' and 'for all $x >$', mean the same thing and are used interchangeably in the rest of this text.

1.2.1 Problems

In the following problems, note when you use the let method and the backward/forward approach. Look for insight into each of the statements through the use of appropriate examples, graphs, and tables.

1. Consider the following proof that for every $x < -3$,

$$x^2 + x > 6.$$

   Proof: Let x_0 represent an arbitrary number, $x_0 < -3$. We then know that $x_0 - 2 < 0$ and $x_0 + 3 < 0$. This means that

$$(x_0 - 2)(x_0 + 3) > 0.$$

   Expanding this expression gives $x_0^2 + x_0 - 6 >$ or $x_0^2 + x_0 > 6$ after adding 6 to both sides, which is what we needed to prove. $\square$

   Verify this proof rewriting it using several different values for $x_0 < -3$. Answer 1

2. Consider the following proof that for every $x > 4$,

$$\frac{3x + 1}{x - 4} > 2.$$

   Proof: Let x_0 represent an arbitrary number, $x_0 > 4$. We then know that

$$9 > 0.$$

   Since $x_0 > 4 > 0$, $x_0 + 9 > 0$. Adding $2x_0$ to both sides of this inequality gives $3x_0 + 9 > 2x_0$. Subtracting 8 from both sides gives

$$3x_0 + 1 > 2x_0 - 8 = 2(x_0 - 4).$$

Since $x_0 > 4$, $x_0 - 4 > 0$, so we can divide both sides of this inequality by $x_0 - 4$ without reversing the inequality sign. This gives

$$\frac{3x_0 + 1}{x_0 - 4} > 2,$$

which is what we were to prove. □

Verify this proof rewriting it using several different values for $x_0 > 4$.

3. Consider the statement that if $x > -2$, then

$$\frac{x^2 - x}{x^2 + 2} < 1.$$

Rewrite this as a 'for every' statement. Then convert the following backwards heuristic proof into a correct forward proof using the let method.

Heuristic Proof: Since $x^2 + 2 > 0$, we can multiply both sides of $(x^2 - x)/(x^2 + 2) < 1$ by $x^2 + 2$ to get

$$x^2 - x < x^2 + 2.$$

Subtracting $x^2 + 2$ from both sides gives $-x - 2 = -(x+2) < 0$. Multiplying both sides of this inequality by -1 and reversing the inequality sign gives $x + 2 > 0$, which we know is true since $x > -2$. Answer 2

4. Consider the statement that if $x > 7$, then

$$\frac{x+1}{7-x} < \sin(x).$$

Rewrite this as a 'for every' statement. Then convert the following backwards heuristic proof into a correct forward proof using the let method.

Heuristic Proof: The graph of

$$y = \frac{x+1}{7-x}$$

always appears to be less than -1 when $x > 7$. Since $-1 \le \sin(x) \le 1$, if we can show that

$$\frac{x+1}{7-x} < -1,$$

it must be true that

$$\frac{x+1}{7-x} < \sin(x).$$

Since $x > 7$, $7 - x < 0$, so when we multiply both sides of $(x+1)/(7-x) < -1$ by $7 - x$, we must reverse the inequality, giving

$$x + 1 > x - 7,$$

which simplifies to $1 > -7$ which we know is true.

5. In this problem, we are going to prove or disprove two similar statements.

   (a) Rewrite the statement

   $$\{\text{if } 2 < x < 4, \text{ then } \frac{1}{x^2 - 6x + 8} \leq -1\}$$

   as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable. Answer 3

   (b) Rewrite the statement that if $2 < x < 4$, then

   $$\frac{1}{x^2 - 6x + 8} < -1$$

   as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable. Answer 4

6. In this problem, we are going to prove or disprove two similar statements. Recall that $-1 \leq \cos(\theta) \leq 1$.

   (a) Rewrite the statement that if $x > 5$, then

   $$x^2 - 7x + 11 + \cos(\pi x) > 0$$

   as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable.

   (b) Rewrite the statement that if $x \geq 5$, then

   $$x^2 - 7x + 11 + \cos(\pi x) > 0$$

   as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable.

7. In this problem, we are going to prove or disprove two similar statements.

   (a) Rewrite the statement that if n is an even integer ($n \in \mathbb{Z}$), then
   $$n^2 + 3n$$
   is divisible by 4 as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable. Answer 5

   (b) Rewrite the statement that if n is an even integer ($n \in \mathbb{Z}$), then
   $$n^2 + 12$$
   is divisible by 4 as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable. Answer 6

8. In this problem, we are going to prove or disprove two similar statements.

   (a) Rewrite the statement that if $x < -2$, then
   $$\frac{2x+5}{x-3} > 0$$
   as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable.

   (b) Rewrite the statement that if $x < 3$, then
   $$\frac{2x+5}{x-3} < 2$$
   as a 'for every' statement, then prove it using a let-variable and backwards/forwards approach, or disprove it using contradiction and a set-variable.

9. In this problem, we are going to prove or disprove two similar statements.

   (a) If x and y are real numbers, then
   $$|x+y| \leq |x| + |y| \tag{1.5}$$

This is called the Triangle Inequality. It is useful in many situations, so remember it. Rewrite this statement as a 'for every' statement, then prove it is true. Answer 7

(b) If x and y are real numbers, then

$$|x+y| = |x| + |y|$$

Rewrite this statement as a 'for every' statement, then prove it is false using contradiction. Answer 8

10. Consider the statement that if $n \geq 3$ then $n^3 > (n+1)^2$, where $n \in \mathbb{Z}$.

(a) Rewrite this statement as a 'for every' statement.

(b) In the video

https://vimeo.com/101854757 Password:Proof

two students do a backwards proof of this statement. Convert their proof into a well written proof.

(c) Prove or disprove that if $n \geq 2$ then $n^3 > (n+1)^2$, where $n \in \mathbb{Z}$.

1.2.2 Answers to Selected Problems

1. **Problem 1**: Let $x_0 = -5 < -3$. We then know that $x_0 - 2 = -7 < 0$ and $x_0 + 3 = -2 < 0$. This means that

$$(x_0 - 2)(x_0 + 3) = (-7)(-2) = 14 > 0.$$

Expanding this expression gives $x_0^2 + x_0 - 6 > 0$ or $x_0^2 + x_0 = 25 - 5 = 20 > 6$ after adding 6 to both sides, which is what we needed to prove.

2. **Problem 3**: For every $x > -2$,

$$\frac{x^2 - x}{x^2 + 2} < 1.$$

Let x_0 represent an arbitrary number greater than -2. Then $x_0 + 2 > 0$ or $-x_0 - 2 < 0$ after multiplying by -1 and reversing the inequality sign. Adding $x_0^2 + 2$ to both sides of this inequality gives

$$x_0^2 - x_0 < x_0^2 + 2.$$

Since $x_0^2 + 2 > 0$, we can divide it into both sides of the inequality giving

$$\frac{x_0^2 - x_0}{x_0^2 + 2} < 1,$$

which is what we were to prove. □

3. **Problem 5 a**: The statement can be rewritten as, for every $x \in (2,4)$,

$$\frac{1}{x^2 - 6x + 8} \leq -1.$$

Proof: Let x_0 represent an arbitrary value in the interval $(2,4)$. Clearly $(x_0 - 3)^2 \geq 0$. Expanding gives $x_0^2 - 6x_0 + 9 \geq 0$ or $1 \geq -x_0^2 + 6x_0 - 8$. Factoring gives

$$1 \geq -(x_0 - 2)(x_0 - 4).$$

Since $x_0 > 2$, $x_0 - 2 > 0$ so we can divide both sides of the inequality without reversing the inequality, giving

$$\frac{1}{x_0 - 2} \geq -(x_0 - 4).$$

Since $x_0 < 4$, then $x_0 - 4 < 0$ so we can divide both sides of the inequality by $x_0 - 4$ if we reverse the inequality sign, giving

$$\frac{1}{(x_0 - 2)(x_0 - 4)} \leq -1,$$

which becomes

$$\frac{1}{x_0^2 - 6x_0 + 8} \leq -1. \square$$

4. **Problem 5 b**: The statement can be rewritten as, for every $x \in (2,4)$,

$$\frac{1}{x^2 - 6x + 8} < -1.$$

Proof: Assume that this statement is true, that is, for every $x \in (2,4)$

$$\frac{1}{x^2 - 6x + 8} < -1.$$

Since this is true for every x between 2 and 4, it will be true for $x_0 = 3$. But substitution gives

$$\frac{1}{(3)^2 - 6(3) + 8} = \frac{1}{9 - 18 + 8} = -1 \not< -1,$$

which is a contradiction, so this statement is false. □

5. **Problem 7 a**: The statement is that for every even integer n, $n^2 + 3n$ is divisible by 4. We prove this statement is false by contradiction. We assume it is true and thn set $n_0 = 2$. Then $n_0^2 + 3n_0 = 10$, which is not divisible by 4, and hence is a contradiction. Therefore, this statement is false.

6. **Problem 7 b**: The statement is that for every even integer n, $n^2 + 12$ is divisible by 4. Let n_0 represent an arbitrary even integer. Since n_0 has a factor of 2, n_0^2 must have a factor of 4. Since $12 = 3 \times 4$ and n_0^2 has a factor of 4, both terms are divisible by 4 so their sum is divisible by 4.

7. **Problem 9 a**: for every $(x, y) \in \mathbb{R}^2$, $|x + y| \leq |x| + |y|$.

   Proof 1 using cases: Let x_0 and y_0 represent two arbitrary numbers in $\mathbb{R}$. Either $|x_0 + y_0| = x_0 + y_0$ or $|x_0 + y_0| = -x_0 - y_0$.

   Case 1: Assume $|x_0 + y_0| = x_0 + y_0$. Then

$$|x_0 + y_0| = x_0 + y_0 \leq |x_0| + |y_0|$$

   since $x_0 \leq |x_0|$ and $y_0 \leq |y_0|$.

   Case 2: Assume $|x_0 + y_0| = -(x_0 + y_0) = -x_0 - y_0$. Then

$$|x_0 + y_0| = -x_0 - y_0 \leq |x_0| + |y_0|$$

   since $-x_0 \leq |x_0|$ and $-y_0 \leq |y_0|$.□

   Proof 2: This is a clever alternate proof. Let x_0 and y_0 represent two arbitrary numbers in $\mathbb{R}$. Then

$$\begin{aligned} (x_0 + y_0)^2 &= x_0^2 + 2x_0 y_0 + y_0^2 \\ &\leq |x_0|^2 + 2|x_0||y_0| + |y_0|^2 \\ &= (|x_0| + |y_0|)^2. \end{aligned}$$

   Taking positive square roots of both sides gives

$$|x_0 + y_0| \leq ||x_0| + |y_0|| = |x_0| + |y_0|.\square$$

8. **Problem 9 b**: This statement is false. Assume it is true. Set $x_0 = 3$ and set $y_0 = -2$. Then $|x_0 + y_0| = 1$, which does not equal $|x_0| + |y_0| = 5$.

1.3 There-Exist-Variables

Video Lesson 1.12

Before reading this section, you should watch the 11:40 minute video https://vimeo.com/180908097 (password:Proof).
which introduces much of this material. The reading may then make more sense. To change the speed at which the video plays, click on the gear at the lower right of the video.

Like in the introductory video, suppose we want to show that for all odd integers n, the number

$$n^2 + 3n + 1$$

is an odd integer. (This is the same as the implication that if n is odd, then $n^2 + 3n + 1$ is odd.) If we let $n = 1$, we get that $n^2 + 3n + 1 = 5$; if we let $n = 7$, then $n^2 + 3n + 1 = 71$; and if we let $n = -5$, then $n^2 + 3n + 1 = 11$, all of which are odd. Thus, we have verified in a few cases that when n is odd, $n^2 + 3n + 1$ is also odd, but we have not shown it for all odd integers n.

We might give a heuristic argument that n^2 is odd, $3n$ is odd, and 1 is odd, and the sum of three odd integers is odd. This gives us the conceptual insight into why this statement is true, but we do not consider this a proof. To give a proof of this statement, we need to show something is true for all of the integers in the infinite set of odd integers, that is, we need to evaluate

$$\sqcup^2 + 3 \sqcup + 1$$

by substituting every possible odd integer into the space $\sqcup$. Since this is impossible, as discussed in the previous section, we use a let-variable. In particular, we let n_0 represent an arbitrary odd integer, using the subscript to denote that from this point onward, n_0 is treated as a known odd integer. Thus, we can do anything with n_0 that we could do with any other odd integer. We now substitute n_0 into the placeholder $\sqcup$ giving

$$n_0^2 + 3n_0 + 1, \tag{1.6}$$

which we now have to argue is an odd integer. We must ask; What can we assume when we assume an integer is odd? What has to be proven to show an integer is odd? To answer these questions, we need a definition of odd.

Definition 1.13: Even and Odd

An integer n is **even** if and only if there exists an integer k such that $n = 2k$. Similarly, an integer n is **odd** if and only if there exists an integer k such that $n = 2k + 1$.

We now know that an odd integer can be written as two times an integer plus 1. For example,

$$11 = 2(5) + 1,\ 113 = 2(56) + 1,\ -15 = 2(-8) + 1.$$

This means that when we assume n_0 is an odd integer, we can also assume there exists another integer, which we call k_0, such that

$$n_0 = 2k_0 + 1.$$

We use a subscript because if we 'know' n_0, then we know k_0, as in the previous 3 examples. Note that n_0 is just a placeholder for odd integers and k_0 is the placeholder for the integer we know exists to write n_0 as two times an integer plus 1. The variable k_0 is called a *there-exists-variable* because we have assumed it exists.

We can now substitute $n_0 = 2k_0 + 1$ into expression 1.6, to get

$$(2k_0 + 1)^2 + 3(2k_0 + 1) + 1 = 4k_0^2 + 10k_0 + 5 = 2(2k_0^2 + 5k_0 + 2) + 1.$$

This integer is clearly odd since it is written as $2j_0 + 1$ where j_0 is the integer

$$j_0 = 2k_0^2 + 5k_0 + 2.$$

Alternatively, since n_0 can be written as an even integer plus 1, instead of assuming that k_0 exists, we could have set

$$k_0 = \frac{n_0 - 1}{2},$$

which we know is an integer because n_0 is odd so $n_0 - 1$ is even. We note that written this way, it may not be as obvious to the reader that k_0 is an integer since it is written as a fraction.

The use of *there-exist-variables* is often combined with *let-variables*. For example, in showing $n_0^2 + 3n_0 + 1$ is odd, we knew there existed an integer k_0 such that $n_0 = 2k_0 + 1$ since we knew n_0 was odd.

1. Suppose we are given that *there exists* at least one solution to the equation $t^3 - t^2 + 3t = 7$. We can let t_0 represent one of those solutions.

2. Let n_0 represent an arbitrary even integer. We know *there exists* an integer k_0 such that $n_0 = 2k_0$.

3. Let i_0 represent an arbitrary compound integer greater than 3 (that is, not prime). Then *there exist* two integers, p_0 and q_0 (called factors of i_0), both greater than 1 such that $i_0 = p_0 q_0$.

4. Let x_0 represent an arbitrary number greater than 3. Then we can assume there exists a positive number a_0 such that
$$x_0 = a_0 + 3.$$

These are all examples of *there-exist-variables*. Again, we will usually represent these variables using subscripts to remind us they are fixed but have a certain property. Note that these variables may represent a unique number that we know exists or one of several numbers that exist: we just do not know its exact value. For example, since $f(t) = t^3 - t^2 + 3t - 7$ goes from negative infinity to positive infinity, it must have at least one root, but those roots may be difficult to find. We can still state 'let t_0 represent a root of this function.'

The phrase 'for every' indicates that we are going to use a let-variable. The phrase, 'there is' or 'there exists', $\exists$ for short, is a little more complicated. In some situations we know a certain number exists. For example, with n_0 representing an even integer, we know 'there exits' an integer k_0 such that $n_0 = 2k_0$. In some cases, to show a value exists, we actually have to find it. For example, we might want to show there exists an x satisfying the equation $x^2 - x = 6$. In this case, x is treated as an unknown we need to find, if it actually does exist. To show the statement is true, we set $x_0 = -2$ (or set $x_0 = 3$) and show it satisfies the equation, verifying $x_0^2 - x_0 = 6$. In the beginning example of this section, we wanted to show $n_0^2 + 3n_0 + 1$ was odd, which meant we had to show there exists an integer j such that $n_0^2 + 3n_0 + 1 = 2j + 1$. We found that integer by setting $j = j_0 = 2k_0^2 + 5k_0 + 2$. In this case, we do not use a subscript for j until we have actually found it and therefore know it exists.

As a word of caution, try to always quantify variables, giving the set it is from, and using 'for all' or 'there exists' whenever appropriate.

In the next two examples, we prove the obvious, that the square of an even integer and the sum of two even integers are both even. We do this to demonstrate the different ways let-variables and there-exist-variables can be used in the same problem when proving a 'for every' statement.

Conceptual Insight: 1.14

We prove that if n is even, then n^2 is even. We begin by letting n_0 represent an arbitrary even integer. We use the definition of even to write this in a usable form using the there-exists-variable $k_0 \in \mathbb{Z}$ such that $n_0 = 2k_0$.
We must now show there exists an integer j such that $n_0^2 = 2j$. Remember that j is now treated as an unknown we must find. By substitution, we have that

$$n_0^2 = 4k_0^2 = 2(2k_0^2) = 2j_0$$

where $j = j_0 = 2k_0^2$ so n_0^2 is even.
It is often helpful in doing a proof to carefully write down what is assumed and what must be shown:
Assumed: There exists an integer k_0 such that $n_0 = 2k_0$.
To Show: There exists an integer j such that $n_0^2 = 2j$.
Written this way, it is clear that our goal is to find the unknown (unsubscripted) variable j.

Claim 1.15: If n is an even integer, then n^2 is even.

Proof: We prove that if n is an even integer, then n^2 is even. Let n_0 represent an arbitrary even integer. By definition of even, there exists an integer k_0 such that $n_0 = 2k_0$. Set $j_0 = 2k_0^2$. Then

$$n_0^2 = (2k_0)^2 = 4k_0^2 = 2(2k_0^2) = 2j_0,$$

so n_0^2 is even. $\square$

We now consider an example with two let-variables and two there-exist-variables.

Conceptual Insight: 1.16

Let's show that the sum of two even integers is even, that is,

$$\forall\, n, m \in \mathbb{E},\ n + m \in \mathbb{E}.$$

We begin with two let-variables by letting n_1 and n_2 represent two arbitrary even integers. From the definition, we have two there-exist-variables in that there exist two integers k_1 and k_2 such that

$$n_1 = 2k_1 \quad \text{and} \quad n_2 = 2k_2.$$

The subscripts indicate that we consider the integers n_1, n_2, k_1 and k_2 as known numbers.
To show $n_1 + n_2 \in \mathbb{E}$ we must, from the definition, find an integer $j \in \mathbb{Z}$ such that

$$n_1 + n_2 = 2j.$$

Note that j is unknown and must be shown to exist. Substitution and factoring gives that

$$n_1 + n_2 = 2k_1 + 2k_2 = 2(k_1 + k_2).$$

We have now found an integer, so we set

$$j_0 = k_1 + k_2$$

so that $n_1 + n_2 = 2j_0$. Hence, the sum is even. We now use a subscript since j_0 is known, that is, is given in terms of two other 'known' variables.

- **Assumed:** There exist integers k_1 and k_2 such that $n_1 = 2k_1$ and $n_2 = 2k_2$. (Note n_1, n_2, k_1 and k_2 are treated as known integers.)
- **To Show:** Show there exists an integer j such that $n_1 + n_2 = 2j$. (Note that j must be found.)

With the assumed and to show written this way, it is much easier to construct a proof, and more importantly, to know when we are finished, that is, when we have found j in this case.

Claim 1.17: The sum of two even integers is even.

Proof: Let n_1 and n_2 represent two arbitrary even integers. From the definition, there exist two integers k_1 and k_2 such that

$$n_1 = 2k_1 \quad \text{and} \quad n_2 = 2k_2.$$

Substitution and factoring gives that

$$n_1 + n_2 = 2k_1 + 2k_2 = 2(k_1 + k_2) = 2j_0,$$

where

$$j_0 = k_1 + k_2.$$

Therefore, $n_1 + n_2$ is even.□

Remember that the proof is just a template for proving the result is true no matter what even integers we use. We show this by rewriting the proof 1.17 using a particular value for the let-variable.

> *Proof.* Verification of Proof for an example: Let n_1 and n_2 be two even integers, say $n_1 = 10$ and $n_2 = 28$. From the definition, there exist two integers $k_1 = 5$ and $k_2 = 14$ such that
>
> $$n_1 = 2k_1 \quad \text{and} \quad n_2 = 2k_2.$$
>
> Substitution and factoring gives that
>
> $$n_1 + n_2 = 10 + 28 = 2k_1 + 2k_2 = 2(k_1 + k_2) = 2(5 + 14) = 2j_0 = 2(19).$$
>
> Therefore, $n_1 + n_2$ is even. □

In the proof 1.17, we used the definition of 'even' in two different ways. We first used the definition to rewrite the 'let' integers n_1 and n_2 in terms of other 'there-exist' integers, k_1 and k_2. Then we used the definition again to determine precisely what we needed to do: find the unknown j in the equation $n_1 + n_2 = 2j$. This was what we call the **technical handle** needed to write a proof. It often helps to rewrite a problem in terms of what is given and what has to be shown in as usable form as possible which usually means in terms of definitions.

A there-exist-variable could have been used in the proof of Claim 1.10, for every $x > 10$, $x^2 + 40 > 14x$, instead of the factoring approach. The idea is

that once we have the let-variable, $x_0 > 10$, we know there exists a positive number a_0 such that
$$x_0 = a_0 + 10.$$
This is a type of technical handle that sometimes works. The proof is relatively easy, once we have found this algebraic technical handle. We give a model of how the proof could be written. Notice that it does not show the thinking behind the proof.

Claim 1.18: For all $x > 10$, $x^2 + 40 > 14x$.

Proof: We prove that for all $x > 10$, $x^2 + 40 > 14x$, or equivalently, $x^2 - 14x + 40 > 0$. We let x_0 represent an arbitrary number such that $x_0 > 10$. We now know there exists a positive number a_0 such that $x_0 = 10 + a_0$. Substitution of $x_0 = 10 + a_0$ into $x_0^2 - 14x_0 + 40$ gives, after simplification,
$$x_0^2 - 14x_0 + 40 = 6a_0 + a_0^2.$$
Since $a_0 \in \mathbb{R}^+$, $6a_0 + a_0^2 > 0$, so $x_0^2 - 14x_0 + 40 > 0$. This is equivalent to $x_0^2 + 40 > 14x_0$, which is what we wanted to prove. $\square$

Let's apply this approach to a different type of problem, set inclusion. Sets can be written in several different ways, as described in the Notation Section 0.1 of the Preface. For example, we might just list the elements in the set as in
$$S = \{1, 2, 3\},$$
which is just the set containing the numbers 1, 2, and 3, or we might have the set
$$T = \{x \in \mathbb{R} : x \geq 1\},$$
which is the set of all real numbers that are greater than or equal to 1. The symbol ':' should be read as 'such that.' When the set is an interval, we can write it as
$$T = [1, \infty).$$
When we use interval notation, remember that the parenthesis means the number is not included and the bracket means it is included. Since infinity is not a number, it is never included. We note that $S \subseteq T$ since every element in S is also in T.

Definition 1.19: Subset

A set of objects S is called a subset of a set of objects T, denoted $S \subseteq T$, if and only if every element x that is in S is also in T.

In short, to show that $S \subseteq T$, we must show for every $x \in S$, $x \in T$, or equivalently, that if $x \in S$, then $x \in T$.

Conceptual Insight: 1.20

Let's consider the following two sets,

$$S = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z} \text{ s.t. } n = 3k\} \text{ and } T = \{m \in \mathbb{Z} : \exists j \in \mathbb{Z} \text{ s.t. } m = 6j\}.$$

To get a sense of these sets, we construct Table 1.1.
Consider the integer $n_0 = 9$. There exists an integer $k_0 = 3$ such that $n_0 = 3k_0$. So $n_0 = 9 \in S$ as seen in the table. On the other hand, there does not exist an integer k such that $9 = 6k$, for in this case, $k = 3/2$ which is not an integer, so $9 \notin T$. Constructing tables is another method for developing conceptual insight into problems. In particular, Table 1.1 gives us the conceptual insight that S is just the set of all integers that are divisible by 3 and T is the set of all integers that are divisible by 6.
To show that $T \subseteq S$, we must show **every** integer in T is also in S. To do this, we use a let-variable. We *let* m_0 represent an arbitrary integer in T. In this case, we know *there exists* an integer j_0 such that $m_0 = 6j_0$. This is what we have assumed. What we need to show is that there is an integer k such that $m_0 = 3k$.

- **Assumed:** We assume that $m_0 \in T$, and that means there exists an integer j_0 such that $m_0 = 6j_0$. (Note the use of m_0 as our let-variable and j_0 as our there-exists-variable.)
- **To Show:** To show $m_0 \in S$, we must show there exists an integer k such that $m_0 = 3k$. (At this point, k is treated as an unknown that we have to find.)

Since $m_0 = 6j_0 = 3(2j_0)$, we can just set $k = k_0 = 2j_0$. This would

$S = \{\cdots,$	$-3,$	$0,$	$3,$	$6,$	$9,$	$\cdots\}$
$T = \{\cdots,$	$-6,$	$0,$	$6,$	$12,$	$18,$	$\cdots\}$

Table 1.1: Elements in sets S and T.

imply that $m_0 = 3k_0 \in S$, and hence, $T \subseteq S$.

For sets S and T in Example 1.20, the statement that $S \not\subseteq T$ is clearly true: to prove it is true, we set $n_0 = 15 = 3 \times 5 \in S$. Since $n_0 = 6(5/2)$, and $5/2$ is not an integer, then $n_0 \notin T$. Therefore, $S \not\subseteq T$. We have used the following definition.

Definition 1.21: Not Subset

A set of objects S is **not a subset** of a set of objects T, denoted $S \not\subseteq T$, if and only if there exists $x \in S$ such that $x \notin T$.

We now consider the two sets of points in the plane,

$$S = \{(s-2, 2s-1) \in \mathbb{R}^2 : s \in \mathbb{R}\},\ T = \{(-2t+3, -4t+9) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

For example, setting $s = 1$ shows that the point $(s-2, 2s-1) = (-1, 1) \in S$, and setting $t = 1$ shows the point $(-2t+3, -4t+9) = (1, 5) \in T$.

Table 1.2 is a table of points in S. We try to see if the first point $(-3, -3)$ is in T by setting $-3 = -2t + 3$. Sovling for t gives $t = 3$. Substitution of $t = 3$ gives $(-2t+3, -4t+9) = (-3, -3)$, so it is in T. We do the same for each point in Table 1.2 resulting in Table 1.3.

$s =$	-1	0	1	2	3
Points in S	$(-3, -3)$	$(-2, -1)$	$(-1, 1)$	$(0, 3)$	$(1, 5)$

Table 1.2: Points in set $S = \{(s-2, 2s-1) \in \mathbb{R}^2 : s \in \mathbb{R}\}$.

$t =$	3	2.5	2	1.5	1
Points in T	$(-3, -3)$	$(-2, -1)$	$(-1, 1)$	$(0, 3)$	$(1, 5)$

Table 1.3: Points in set $T = \{(-2t+3, -4t+9) \in \mathbb{R}^2 : t \in \mathbb{R}\}$.

From these tables, it appears $S \subseteq T$. We are going to prove $S \subseteq T$. To begin our heuristic proof, we first let p_0 represent an arbitrary point in S. Note that our let-variable is not a number, but is a point in the plane. We use the definition of S to state that there exists a value s_0 such that $p_0 = (s_0 - 2, 2s_0 - 1)$.

- **Assumed:** We assume that $p_0 \in S$. This means there exists a real number s_0 such that $p_0 = (s_0 - 2, 2s_0 - 1)$. (The point p_0 is our let-variable and the number s_0 is our there-exists-variable since we know it exists by the definition of what it means for $p_0 \in S$.)

- **To Show:** We want to show that $p_0 = (s_0 - 2, 2s_0 - 1) \in T$, that is, there exists a real number t such that $(-2t + 3, -4t + 9) = (s_0 - 2, 2s_0 - 1)$. Note that $-2t + 3$ and $-4t + 9$ are unknown since t is unknown and needs to be found, but $s_0 - 2$ and $2s_0 - 1$ are treated as known numbers.

To find t, we solve

$$-2t + 3 = s_0 - 2,$$

giving $t = -0.5s_0 + 2.5$, and we solve $-4t + 9 = 2s_0 - 1$, resulting in the same value $t = -0.5s_0 + 2.5$. If the values were different, then p_0 would not be in T. This is our backward work. We are now ready for the proof.

Claim 1.22: $S \subseteq T$ where

$$S = \{(s - 2, 2s - 1) \in \mathbb{R}^2 : s \in \mathbb{R}\} \textbf{ and}$$

$$T = \{(-2t + 3, -4t + 9) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Proof: Let p_0 represent an arbitrary point in S (let method). This means there exists a real number s_0 such that $p_0 = (s_0 - 2, 2s_0 - 1)$. Set $t = t_0 = -0.5s_0 + 2.5$. (The subscript for t here indicates it is now a 'known' number.) This means that $(-2t_0 + 3, -4t_0 + 9) \in T$. Substitution means that

$$(-2t_0 + 3, -4t_0 + 9) = (-2(-0.5s_0 + 2.5) + 3, -4(-0.5s_0 + 2.5) + 9),$$

which after collecting terms and simplifying, gives

$$(-2t_0 + 3, -4t_0 + 9) = (s_0 - 2, 2s_0 - 1) = p_0 \in T,$$

which is what we needed to show. □

We now verify our proof by repeating the proof using actual values, so we can see how the steps fit together.

Proof. **Verification of Proof:** We will show that

$$S = \{(s - 2, 2s - 1) \in \mathbb{R}^2 : s \in \mathbb{R}\} \subseteq T = \{(-2t + 3, -4t + 9) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Let $p_0 = (1, 5)$ represent an arbitrary point in S (let-variable). This means there exists a real number (there-exists-variable) $s_0 = 3$ such that $p_0 = (1,5) = (s_0 - 2, 2s_0 - 1)$. Set $t = t_0 = -0.5s_0 + 2.5 = 1$. This means that $(-2t_0 + 3, -4t_0 + 9) = (-2(1) + 3, -4(1) + 9) = (1,5) = p_0 \in T$, which is what we needed to show. $\square$

To show two sets are equal, we must show each is a subset of the other, which requires two proofs. The reader is encouraged to write a similar proof to show that $T \subseteq S$ for this example. In this case, $S = T$.

Summary 1.3

- Recall from Section 1.1 that when considering a statement, construct some examples and possibly draw some graphs to get an idea if the statement is true or not and to get some conceptual insight into why the statement is true or not.
- From Section 1.2, if the statement begins with 'for every,' then use a let-variable to represent the variable using a subscript to indicate it is assumed as known. Remember that a statement of the form 'if $\cdots$, then $\cdots$' should be rewritten as a 'for every $\cdots$, something is true,' then use the let method.
- From this section we learned the following:
  - If we can assume something exists in terms of the let-variable, we can write it as a 'there-exists' variable using a subscript.
  - If we have to show something exists, we do not use subscripts and it is a variable that needs to be solved for.
  - When proving a statement, clearly write down what is assumed using let-variables and there-exists variables, and what has to be shown using a there-exists variable without a subscript, so we know what must be found.
  - If any of these items are confusing, refer back to Example 1.16 in which they are displayed.

We state Definition 1.23 which will be used in the following problems. This definition will be extremely useful in the rest of the text, as well.

Definition 1.23: Divides

Integer n **divides** integer m if and only if there exists an integer k such that $m = kn$.

Note that division is defined in terms of multiplication. If n divides m, we write $n|m$ which means that m/n is an integer. So we would say that 3 divides 6, or $3|6$ since $6 = 3 \times 2$. Problems are often easier if fractions are avoided.

1.3.1 Problems

In the following problems, note when you use the let method and the backward/forward approach. Look for insight into each of the statements through the use of appropriate examples, graphs, and tables.

1. Consider the statement

$$p \equiv \{5 \text{ divides the sum of any 5 consecutive integers}\}.$$

   Using Definition 1.23, work the following parts.

   (a) Test the validity of this statement by checking if it is true for several sets of 5 consecutive integers. Make sure you try a case in which all 5 of the integers are positive, all 5 of the integers are negative, and some of the integers are positive and some are negative. We want to make sure it is true under a variety of situations. Answer 1

   (b) Prove the statement is true using the let method, letting n_0 represent the first of the 5 integers. Answer 2

   (c) Explore sums of different numbers of consecutive integers. Develop your own conjectures about possible true statements. Can you prove any of them? Answer 3

2. Use Definition 1.23 in Problem 1 to prove each of the following statements.

   (a) Show that 5 divides 30.

   (b) Show that 1 divides every integer m.

   (c) Show that every integer n divides itself.

   (d) Suppose that n divides m and n divides k. Show n divides $m + k$.

   (e) Suppose that n divides m and that m divides j. Show that n divides j.

3. Recall the definitions of even and odd integers: an integer $n \in \mathbb{Z}$ is called **even**, denoted $n \in \mathbb{E}$, if and only if $\exists k \in \mathbb{Z}$ such that $n = 2k$ and is called **odd**, denoted $n \in \mathbb{O}$, if and only if $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. For each of the following, rewrite the statement using quantifiers, and then develop an informal proof from the definition and write it down. Finally give a formal proof, using the *let* proof method.

   (a) The product of two odd integers is odd. Answer 4

   (b) The sum and difference of two odd integers are even. Answer 5

   (c) If n is a positive integer and m is even, then $m^n + 7$ is odd. Answer 6

4. Consider the statement that if a and b are rational numbers, then $a + b$ is a rational number. (Recall that a rational number is just a number that can be written as the quotient of two integers.) This statement can be rewritten as, for every a and b in the set of rational numbers, $a + b$ is rational. Prove this statement using 2 let-variables and 4 there-exist-variables.

5. If a and b are rational numbers, then ab is a rational number. Prove this statement using 2 let-variables and 4 there-exist-variables. Answer 7

6. Let's consider the sets

$$S = \{n \in \mathbb{Z} : \exists k \in \mathbb{E} \text{ such that } n = 2k + 12\}$$

   and

$$T = \{m \in \mathbb{Z} : \exists j \in \mathbb{Z} \text{ such that } m = 4j\}.$$

   (a) Prove or disprove $T \subseteq S$.

   (b) Prove or disprove $S \subseteq T$.

7. Let's consider the sets

$$S = \{n \in \mathbb{Z} : \exists k \in \mathbb{Z} \text{ such that } n = 5k + 11\}$$

   and

$$T = \{m \in \mathbb{Z} : \exists j \in \mathbb{Z} \text{ such that } m = 10j + 1\}.$$

   (a) Prove or disprove $T \subseteq S$. Answer 8

(b) Prove or disprove $S \subseteq T$. Answer 9

8. Let $S = \{(-n+2, -4n+13) : n \in \mathbb{Z}\}$ and $T = \{(3m-1, 12m+1) : m \in \mathbb{Z}\}$. Remember that it helps to make a table of values for each set to get an idea what the sets look like.

   (a) Prove $T \subseteq S$ using let variables or prove it is not true using contradiction.

   (b) Prove $S \subseteq T$ using let variables or prove it is not true using contradiction.

9. Set S equal to the set of points in $\mathbb{R}^2$ defined by

$$S = \{(2n-2, 4n-1) : n \in \mathbb{Z}\}$$

   and T as the set of points in $\mathbb{R}^2$ defined by

$$T = \{(m, 2m+3) : m \in \mathbb{Z}\}.$$

   Remember that it helps to make a table of values for each set to get an idea what the sets look like.

   (a) In the video

   https://vimeo.com/46317620 Password:Proof

   two students believe they have proven that $T \subseteq S$. What do you think of their proof? What do you think about their approach to this problem? Answer 10

   (b) Prove $T \subseteq S$ using let variables or prove it is not true using contradiction. Answer 11

   (c) Prove $S \subseteq T$ using let variables or prove it is not true using contradiction. Answer 12

10. Set S equal to the set of points in $\mathbb{R}^2$ defined by

$$S = \left\{ \left(\frac{x+1}{x-2}, \frac{5x-1}{x-2} \right) : x \in \mathbb{R} - \{2\} \right\}.$$

   Similarly set T equal to the set of points in $\mathbb{R}^2$ defined by

$$T = \{(y+4, 3y+14) : y \in \mathbb{R}\}.$$

   In working this problem, it might help to again generate a table of values.

(a) Prove $S \subseteq T$ using let-variables or prove it is not true using contradiction.

(b) Prove $T \subseteq S$ using let-variables or prove it is not true using contradiction.

11. Let $S = \{(3x, 6x-1) : x \in \mathbb{R}\}$ and $T = \{(y+2, 2y+3) : y \in \mathbb{R}\}$. We are going to investigate if $S = T$. In the video

    https://vimeo.com/45913468 Password:Proof

    two students try to prove that $S \subseteq T$ and that $T \subseteq S$. Do they have the right idea? What problems do you find with their proof? Can you write an acceptable proof of these two statements?

    (a) Prove $S \subseteq T$.

    (b) Prove $T \subseteq S$. Answer 13

12. Let $S = \{(x, 2x-2) : x \in \mathbb{R}\}$ and $T = \{(t+1, 2t) : t \in \mathbb{R}\}$. In the video

    https://vimeo.com/6166001 Password:Proof

    two students try to prove that $S \subseteq T$ and that $T \subseteq S$. Do they have the right idea? What problems do you find with their proof? Can you write an acceptable proof of these two statements?

1.3.2 Answers to Selected Problems

1. **Problem 1 a)**: A check of several examples all seem to work, so this statement seems true.

2. **Problem 1 b)**: Proof: Let S represent an arbitrary set of 5 consecutive numbers and let n_0 represent the first of these integers. Then the 5 consecutive integers are n_0, n_0+1, n_0+2, n_0+3, n_0+4 and their sum is
$$s = 5n_0 + 10 = 5(n_0 + 2).$$
   Set $k_0 = n_0 + 2$. Then the sum of these 5 consecutive integers is
$$s = 5k_0,$$
   which by definition means the sum is divisible by 5.

3. **Problem 1 c)**: The sum of m consecutive integers when m is odd appears to be divisible by m, while the sum is only divisible by $m/2$ if m is even. Let m_0 represent an odd integer. Then there exists an integer k_0 such that $m_0 = 2k_0 + 1$. The easiest way to prove the result is to let the middle of the m_0 integers be represented by n_0. Then the sum is

$$s = (n_0 - k_0) + (n_0 - (k_0 - 1)) + \cdots + (n_0 + k_0) = (2k_0 + 1)n_0 = m_0 n_0.$$

4. **Problem 3 a):** Let n_1 and n_2 represent two odd integers. Then there exist integers k_1 and k_2 such that $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$. Then $n_1 n_2 = (2k_1 + 1)(2k_2 + 1) = 2(2k_1 k_2 + k_1 + k_2) + 1$ after multiplying out, collecting terms and factoring out a two. This is odd since it is of the form $2j_0 + 1$, where $j_0 = 2k_1 k_2 + k_1 + k_2$ is an integer.

5. **Problem 3 b):** Let n_1 and n_2 represent two odd integers. Then there exist integers k_1 and k_2 such that $n_1 = 2k_1 + 1$ and $n_2 = 2k_2 + 1$. Then $n_1 + n_2 = 2(k_1 + k_2 + 1)$ and $n_1 - n_2 = 2(k_1 - k_2)$, which are both two times an integer so are even.

6. **Problem 3 c):** This statement is really 'for all positive integers n and all even integers m, $m^n + 7$ is odd.' So we can use the let method. Let n_0 represent an arbitrary positive integer and m_0 represent an arbitrary even integer. Then there exists an integer k_0 such that $m_0 = 2k_0$. Then
$$m_0^{n_0} + 7 = (2k_0)^{n_0} + 7 = 2(2^{n_0-1}k_0^{n_0} + 3) + 1.$$
Since $n_0 > 0$, $n_0 - 1 \geq 0$, and so 2^{n_0-1} is an integer. Thus,
$$m_0^{n_0} + 7 = 2j_0 + 1,$$
is odd, since
$$j_0 = 2^{n_0-1}k_0^{n_0} + 3$$
is an integer.

7. **Problem 5:** This problem is really 'for all rational numbers a and b, ab is rational.' So we can use the let method. Let a_0 and b_0 represent two rational numbers. This means there exists 4 integers, n_0, m_0, j_0 and k_0, ($m_0 \neq 0$ and $k_0 \neq 0$) such that $a_0 = n_0/m_0$ and $b_0 = j_0/k_0$. Then $a_0 b_0 = (n_0 j_0)/(m_0 k_0)$, which is the ratio of two integers, the denominator being nonzero, and so $a_0 b_0$ is rational.

8. **Problem 7 a:** If we try some values for j, say $j = 1, 2, 3$, we get that the values $11, 21, 31 \in T$. We note that $11 = 5(0) + 11$, $21 = 5(2) + 11$, and $31 = 5(4) + 11$, so all three numbers are in S. This indicates that T may be a subset of S. To show this, let m_0 represent an arbitrary element of T. This means there exists j_0 such that $m_0 = 10j_0 + 1$. We can rewrite this as

$$m_0 = 10(j_0 - 1) + 11 = 5(2j_0 - 2) + 11 = 5k_0 + 11 \in S,$$

where $k_0 = 2j_0 - 2$ is an integer. Thus $T \subseteq S$.

9. **Problem 7 b:** We assume $S \subseteq T$. Set $n_0 = 5(1) + 11 = 16 \in S$. Since we have assumed $S \subseteq T$, there must exist an integer j such that $16 = 10j + 1$. But then $j = 3/2 \notin \mathbb{Z}$, so we have a contradiction and $S \not\subseteq T$.

10. **Problem 9 a:** The students should have constructed a set of points in S and a set of points in T. Had they done that, perhaps, they would have noticed there are points in T which are not in S. In trying to write their proof that $T \subseteq S$, they began correctly by letting $(m_0, 2m_0 + 3)$ be an arbitrary point in T. Their mistake, which they almost caught, was that they needed to do a backward proof, solving

$$2n - 2 = m_0$$

for n. This gives $n = (m_0 + 2)/2$, which is not always an integer. Therefore, we cannot always find an n, so there must be some points in T that are not in S. In fact, this approach gives us a method for finding a point in T that is not in S by picking a value for m_0 that makes n a non-integer. For example, if we set $m_0 = 1$, then $n_0 = 1.5$. This means $t_0 = (1, 5) \in T$, but $t_0 \notin S$. For t_0 to be in S, n_0 needs to be an integer, but $n_0 = 1.5 \notin \mathbb{Z}$.

11. **Problem 9 b:** We show $T \not\subseteq S$ by assuming it is, then getting a contradiction. By setting $m_0 = 1$, we get that $(m_0, 2m_0 + 3) = (1, 5) \in T$. Since we have assumed T is a subset of S, there must exist an integer n_0 such that $(2n_0 - 2, 4n_0 - 1) = (1, 5)$. In this case $n_0 = 3/2$ which contradicts the assumption n_0 is an integer. Therefore, $T \not\subseteq S$.

12. **Problem 9 c:** We show $S \subseteq T$ by letting s_0 represent an arbitrary point in S. This means there exists n_0 such that

$$s_0 = (2n_0 - 2, 4n_0 - 1).$$

Set $m_0 = 2n_0 - 2$ and substitute to get that $s_0 = (2n_0 - 2, 4n_0 - 1) = (m_0, 2m_0 + 3) \in T$, so $S \subseteq T$.

13. **Problem 11:** The proof that $S \subseteq T$ is pretty good, but a little hard to follow in that the student does a lot of the algebra and substitution in his head and does not explain it clearly to his partner. It would have been helpful in understanding the process if they had generated a set of points in set S using different values for x, then seen how they found corresponding values y that result in showing those points are also in T, as did the students in the video for Problem 12. Since the students showed $S \subseteq T$, we will show $T \subseteq S$. Let p_0 represent an arbitrary point in T. Then there exists $y_0 \in \mathbb{R}$ such that $p_0 = (y_0 + 2, 2y_0 + 3)$. Set $x_0 = (y_0 + 2)/3$. Show that the point

$$(3x_0, 6x_0 - 1),$$

which we know is in S, is actually equal to p_0 by substitution. Thus, for every $p_0 \in T$, we have $p_0 \in S$ so $T \subseteq S$.

1.4 For Every, There Exists

Video Lesson 1.24

See the 5:45 minute video
https://vimeo.com/83721714 (password:Proof)
before reading this section as it introduces you to some of the concepts of this section. To change the speed at which the video plays, click on the gear at the lower right of the video.

One common theme in statements is:

$$\left\{ \langle \forall\, a \in S \rangle, \langle \exists\, b \in T \text{ such that} \rangle \langle \text{claim is true} \rangle \right\}. \tag{1.7}$$

We use $\langle \cdots \rangle$ to group the separate parts of the statement. This grouping helps us unpack the meaning of the statement. We note that the phrase ‘such that’ is usually paired with $\exists$, which makes the statement read

for every *something*, there exists *something else* such that *something* is true

Statements of this form are just extensions of statements considered in the previous two sections which were of the form

for every *something*, *something* is true

Let's consider the statement

$$p \equiv \left\{ \langle \forall\, y \in \mathbb{R} - \{-1\} \rangle, \langle \exists\, x \in \mathbb{R} \text{ such that} \rangle \langle y = \frac{3+x}{2-x} \rangle \right\}. \tag{1.8}$$

Remark 1.25

We note that $y \in S - T$ means y is an element of the set S but it is not an element of the set T.

To get insight into Statement 1.8, we graph the function $y = (3+x)/(2-x)$, as seen in Figure 1.10.

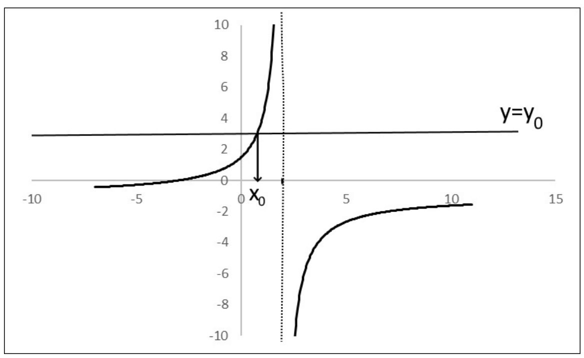

Figure 1.10: For every $y_0 \neq -1$, there exists an x_0 such that $(3+x_0)/(2-x_0) = y_0$.

Conceptual Insight: 1.26

Since p begins with 'for all' so we let y_0 represent an arbitrary real number other than -1. The next clause states that x exists. Note that every horizontal line $y = y_0$, $y_0 \neq -1$, intersects this graph at some point. The x-coordinate of this point of intersection is the x-value that is claimed to exist. To find this value, we work backwards, solving $y_0 = (3+x)/(2-x)$ for x, giving

$$x = x_0 = \frac{2y_0 - 3}{1 + y_0},$$

which exists when $y_0 \neq -1$. This was the backwards approach.

We now carefully write what is given and what we have to show, then give a model of how the proof could actually be written.

- **Assumed:** We assume that $y_0 \neq -1$. Note y_0 is treated as a known value.
- **To Show:** Find a value x such that $y_0 = (3+x)/(2-x)$. Note that x is treated as an unknown which must be found using backwards work.

Claim 1.27: Statement p 1.8 is true.

Proof: Let y_0 represent an arbitrary real number other than -1. Set

$$x_0 = \frac{2y_0 - 3}{1 + y_0},$$

which exists since $y_0 \neq -1$. We substitute, giving

$$\frac{3 + x_0}{2 - x_0} = \frac{3 + \frac{2y_0 - 3}{1 + y_0}}{2 - \frac{2y_0 - 3}{1 + y_0}}.$$

We multiply numerator and denominator by $1 + y_0 \neq 0$ getting

$$\frac{3(1 + y_0) + 2y_0 - 3}{2(1 + y_0) - (2y_0 - 3)} = \frac{5y_0}{5} = y_0,$$

which is what we wanted to show. So Statement 1.8 is true. □

Let's try verifying our proof using some specific value for y_0.

Proof. Verification of Proof for one example Let $y_0 = 3 \neq -1$. Set

$$x_0 = \frac{2y_0 - 3}{1 + y_0} = \frac{3}{4}.$$

We substitute, giving

$$\frac{3 + 3/4}{2 - 3/4}.$$

We multiply numerator and denominator by $1 + y_0 = 4$ to get

$$\frac{12 + 3}{8 - 3} = \frac{15}{5} = 3 = y_0,$$

which is what we wanted to show. □

A similar result happens no matter what number we pick for y_0.

Suppose we instead were considering the statement

$$p \equiv \left\{ \langle \forall\, y \in \mathbb{R} \rangle, \langle \exists\, x \in \mathbb{R} \text{ such that} \rangle \langle y = \frac{3+x}{2-x} \rangle \right\}. \tag{1.9}$$

Since the graph of $y = (3+x)/(2-x)$ has a horizontal asymptote at $y = -1$, as seen in Figure 1.10, we believe that Statement 1.9 is false in that it is not true that there is an x for every y. To prove this statement if false, we assume it is true. Since it is true for every y, then it must be true for $y = y_0 = -1$, that is, there exists an x_0 such that

$$-1 = \frac{3+x_0}{2-x_0}.$$

We use subscripts now since we have assumed x_0 exists. Multiplying by $2 - x_0$ gives

$$x_0 - 2 = 3 + x_0.$$

After subtracting x_0 from both sides, we get $-2 = 3$, which is a contradiction. So Statement 1.9 is not true since it is not true for every y.

Let's consider another example of a statement of Type 1.7, which arises when considering a function that is unbounded above.

Conceptual Insight: 1.28

We now develop conceptual insight into why the statement

$$p \equiv \{ \langle \forall\, M \geq 1 \rangle, \langle \exists\, x \in \mathbb{R} \text{ such that} \rangle \langle f(x) = x^2 + 1 > M \rangle \} \tag{1.10}$$

is true. From the graph of this function in Figure 1.11, it appears that if we draw a line $y = M$, $M \geq 1$, then for any $x > c$, $f(x) > M$. This is the **conceptual insight** that helps us understand why the statement is true. As previously mentioned, conceptual insight is often a graph or table of values that not only convinces us a statement is true or false, but gives us insight into **why** the statement is true.
The first phrase states that $\forall\, M \geq 1$ and this means that we must let M_0 represent an arbitrary number, $M_0 \geq 1$. From the graph, we see that if we find the point of intersection (c, M_0) of f and the line $y = M_0$ in the first quadrant, then for any $x > c$, $f(x) > M_0$. Thus, to show that x exists, we have to first find c, and then find an $x > c$.

We find the intersection point by solving

$$c^2 + 1 = M_0.$$

This gives $c = c_0 = \sqrt{M_0 - 1}$ which exists since $M_0 - 1 \geq 0$. This is what we call the **technical handle** that allows us to prove the statement. A technical handle is often an algebraic manipulation that transforms our conceptual insight into a concrete algebraic verification. We now show x exists by defining it. Since $M_0 > M_0 - 1 \geq 0$, set

$$x = x_0 = \sqrt{M_0} > \sqrt{M_0 - 1} = c_0.$$

(Note that we use a subscript since x_0 is now a 'known' value.) All that remains is to show that

$$f(x_0) > M_0.$$

This is easy to do in that, after squaring, we get

$$f(x_0) = x_0^2 + 1 = (\sqrt{M_0})^2 + 1 = M_0 + 1 > M_0.$$

- **Assumed:** We assume that $M_0 \geq 1$.
- **To Show:** Find an x-value such that $x^2 + 1 > M_0$.

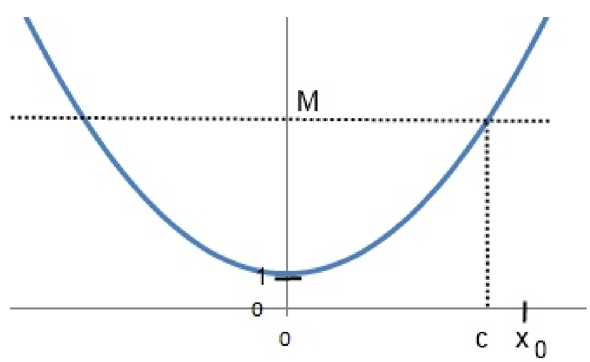

Figure 1.11: For every $M \geq 1$, if $x_0 > c$ then $f(x_0) = x_0^2 + 1 > M$.

Claim 1.29: For ever $M \geq 1$, there exists a real x such that $f(x) = x^2 + 1 > M$.

Proof: Let M_0 represent an arbitrary real number that is greater than or equal to 1. Set

$$x_0 = \sqrt{M_0},$$

so it exists. Then

$$f(x_0) = x_0^2 + 1 = (\sqrt{M_0})^2 + 1 = M_0 + 1 > M_0.$$

□

We now verify our proof for one value of M_0.

Proof. Verification of Proof for one example: Let $M_0 = 10 \geq 1$. Set

$$x_0 = \sqrt{M_0} = \sqrt{10},$$

so it exists. Then

$$f(x_0) = x_0^2 + 1 = (\sqrt{10})^2 + 1 = 10 + 1 > 10.$$

□

Note in the video lesson 1.24, the restriction that $M \geq 1$ is omitted, that is, it proves

$$p \equiv \{\langle \forall\, M \in \mathbb{R}\rangle, \langle \exists\, x \in \mathbb{R} \text{ such that}\rangle \langle f(x) = x^2 + 1 > M\rangle\}.$$

We note that any $x > \sqrt{M_0 - 1}$ would work. For example, $x = x_0 = \sqrt{M_0 - 1} + 1$ would work, but this value would be more difficult to work with since

$$x_0^2 = M_0 + 2\sqrt{M_0 - 1}.$$

When you have multiple choices for setting a variable, look for one that is easier to work with.

We compare the proof of statement 1.10 to the proof that statement

$$q \equiv \{\langle \forall\, M \geq 1\rangle, \langle \exists\, x \in \mathbb{R} \text{ such that}\rangle \langle f(x) = -x^2 + 1 > M\rangle\} \tag{1.11}$$

is false.

Claim 1.30: Statement 1.11 is false

Proof: From the graph in Figure 1.12, we know that $f(x) = -x^2 + 1$ will always be less that or equal to 1, so it is not true that for every M, there is an x such that $-x^2 + 1 > M$. Let's assume this statement is true and set $M_0 = 2$. Then we have assumed there exists an x_0 such that

$$-x_0^2 + 1 > 2$$

or that $-x_0^2 > 1$. This is clearly a contradiction since the left side is negative or 0 and the right side is positive. Therefore, q is false. $\square$

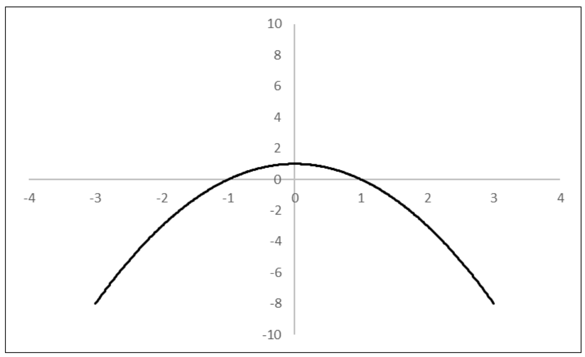

Figure 1.12: For any $M > 1$, there does not exists an x such that $-x^2 + 1 > M$.

We say f maps the set S **into** the set T if and only if for every $x \in S$, $f(x) \in T$. Mapping **onto** the set T intuitively means every element in T is the image of some element in S. We state this more formally with the following definitions.

Definition 1.31: Into

A function f maps the set S **into** the set T if and only if for every $x \in S$, $f(x) \in T$.

Definition 1.32: Onto

A function f maps the set S **onto** the set T if and only if for every $y \in T$, there exists an $x \in S$ such that $f(x) = y$.

Note that the definition of a function mapping into a set T is a statement of the form

'for every $\cdots$, something is true'

while the definition of a function mapping onto a set T is a statement of the form 1.7,

'for every $\cdots$, there exists $\cdots$ such that something is true.'

In Figure 1.13 is the sketch of a function f that

- maps set S_1 into and onto set T_1 (red),
- maps set S_2 into but not onto set T_2 because no $x \in S_2$ maps onto $y_2 \in T_2$ (green),
- does not map set S_3 into set T_3 since $f(x_4) \notin T_3$ (blue).

Also note that even though S_1 maps onto T_1, it may happen that several different x-values may map onto the same y-value, as is the case for x_1, x_2 and x_3 all mapping onto y_0.

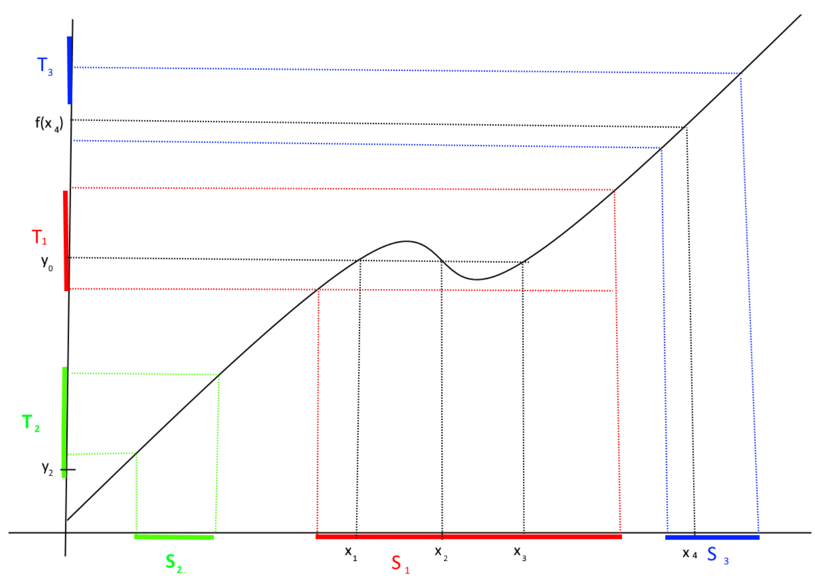

Figure 1.13: The function f maps sets S_1 **onto** the set T_1 and S_2 into but not onto T_2. Set S_3 does not map into T_3.

Let's see how we could show a function is onto some set T. Consider the function

$$f(x) = x^2 + 1$$

as a function from $S = \mathbb{R}$ into the set $T = [1, \infty)$, that is, all numbers greater than or equal to 1. We will show f is onto T.

- **Assumed:** We assume that $y_0 \geq 1$.
- **To Show:** Find a real number x such that $x^2 + 1 = y_0$.

Let y_0 represent an arbitrary number in T, that is, $y_0 \geq 1$. Working backwards, we want to find an x such that $f(x) = x^2 + 1 = y_0$. Solving for x, we get one or two possible solutions (one solution when $y_0 = 1$),

$$x = \pm\sqrt{y_0 - 1}$$

which exist since $y_0 - 1 \geq 0$. Set $x = x_0 = \sqrt{y_0 - 1}$. Then

$$f(x_0) = x_0^2 + 1 = (\sqrt{y_0 - 1})^2 + 1 = y_0 - 1 + 1 = y_0$$

and we are finished.

If $T = \mathbb{R}^+$, then f would map into T but would not be onto T. To show this, we assume f maps onto $\mathbb{R}^+$. We then set $y_0 = 0.5$ and have assumed there exists an x_0 such that $f(x_0) = x_0^2 + 1 = 0.5$. But this means $x_0^2 = -0.5$ which is a contradiction. When a function does not map onto a set, it is usually easy to prove once we develop conceptual insight.

Conceptual Insight: 1.33

Is the function f mapping $S = \mathbb{R}$ into $T = \mathbb{R}$ onto T?

$$f(x) = \begin{cases} 2x - 5 & x > 4 \\ x + 1 & x \leq 4 \end{cases}. \tag{1.12}$$

To get some conceptual insight, we graph this function in Figure 1.14. From this graph, it appears every horizontal line (corresponding to an arbitrary y-value) intersects the graph at least once to determine an x-value that maps onto it. There appear to be two cases: if $y_0 \leq 5$, then we use the line $x + 1$, while if $y_0 > 5$, we use $2x - 5$ to find x. We will solve $x + 1 = y_0$ for x when $y_0 \leq 5$ and solve $2x - 5 = y_0$ for x when $y_0 > 5$. This is our backward work. Note that for $3 < y \leq 5$ there are two choices for x: we will use the line $x + 1$ for these.

Let's now fill in the details.

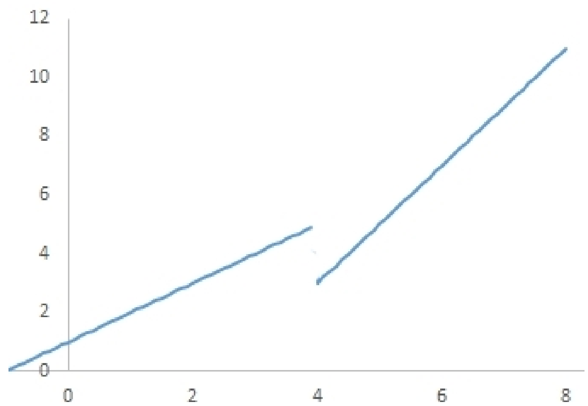

Figure 1.14: The function 1.12 with domain $S = \mathbb{R}$ apparently maps **onto** the set $T = \mathbb{R}$.

Claim 1.34: The function 1.12 maps its domain $S = \mathbb{R}$ onto the set $T = \mathbb{R}$.

Proof: We consider two cases.
Case 1: Suppose $y_0 \leq 5$. Then set $x = x_0 = y_0 - 1$. It is important to note that since $y_0 \leq 5$, $x_0 = y_0 - 1 \leq 4$. Since $x_0 \leq 4$, we have

$$f(x_0) = x_0 + 1 = (y_0 - 1) + 1 = y_0,$$

which is what we needed to show. (Note that we had to ensure that $x_0 \leq 4$ to evaluate $f(x_0)$ using $x_0 + 1$.)
Case 2: Suppose $y_0 > 5$. Set $x = x_0 = (y_0 + 5)/2 = y_0/2 + 2.5$. Again, it is important to note that since $y_0 > 5$, $x_0 = y_0/2 + 2.5 > 5/2 + 2.5 = 5 > 4$, so

$$f(x_0) = 2x_0 - 5 = 2(y_0 + 5)/2 - 5 = y_0 + 5 - 5 = y_0,$$

which is what we needed to show.
So for all $y \in T$, there exists an $x \in \mathbb{R}$ such that $f(x) = y$. Hence, f is onto T.

Proof. **Verification of Proof, Example 1:** Let $y_0 = 3 \leq 5$, and so it satisfies Case 1.
Case 1: Since $y_0 = 3 \leq 5$, we set $x = x_0 = y_0 - 1 = 2 \leq 4$. Then, since

$x_0 \leq 4$, we have

$$f(x_0) = x_0 + 1 = 2 + 1 = 3 = y_0,$$

which is what we needed to show. (Note that we had to ensure that $x_0 \leq 4$ to evaluate $f(x_0)$ using $x_0 + 1$. □

We now verify our proof for a second example.

Proof. **Verification of Proof, Example 2:** Let $y_0 = 7 > 5$ and so Case 2 is satisfied.
Case 2: Since $y_0 = 7 > 5$, we set $x = x_0 = (y_0 + 5)/2 = (7 + 5)/2 = 6$. Since $x_0 > 4$,

$$f(x_0) = 2x_0 - 5 = 2(6) - 5 = 7 = y_0,$$

which is what we needed to show. □

On the other hand, consider the function

$$f(x) = \begin{cases} 2x - 1 & x > 4 \\ x + 1 & x \leq 4 \end{cases}. \tag{1.13}$$

To get some conceptual insight, we graph this function in Figure 1.15. From this graph, it appears to be missing y-values greater than 5 and less than or equal to 7, so it is not onto $T = \mathbb{R}$. To prove this, we assume the function maps onto $\mathbb{R}$ and arrive at a contradiction.

Claim 1.35: The function 1.13 does not map onto $\mathbb{R}$.

Proof: Assume f maps onto $\mathbb{R}$. This means there exists an $x_0 \in \mathbb{R}$ which maps onto $y_0 = 6$.
Case 1: Suppose $x_0 > 4$. Then

$$f(x_0) = 2x_0 - 1 > 2(4) - 1 = 7.$$

So, $f(x_0) \neq y_0 = 6$, which is a contradiction.
Case 2: Suppose $x_0 \leq 4$. Then

$$f(x_0) = x_0 + 1 \leq 5 < 6 = y_0,$$

which is also a contradiction.
Therefore, there is no x such that $f(x) = y_0 = 6$ and we conclude that f does not map onto $\mathbb{R}$.

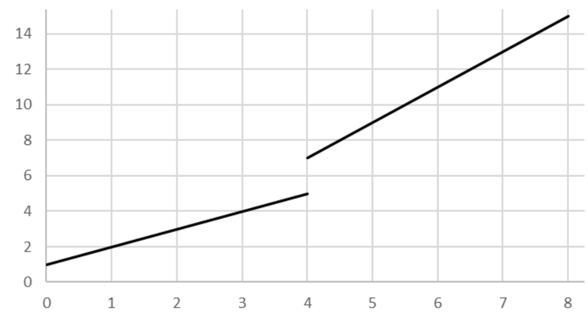

Figure 1.15: The function 1.13 with domain $\mathbb{R}$ apparently does not map **onto** the set $T = \mathbb{R}$.

We note that in many cases, it is easy to see from a graph that a function maps onto a specific set T, but it is difficult to prove it because it is difficult to solve for x in terms of an arbitrary $y_0 \in T$.

Consider the statement

$$p \equiv \{\forall a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. if } x > b, \text{ then } 3x + 5 > a\}. \tag{1.14}$$

Statements of this form seem complicated, but not if we just slowly break it down into parts, and remember that a statement of the form 'if $x > b$' really means 'for all $x > b$.' Doing this, we rewrite statement 1.14 as

$$p \equiv \{\forall a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. } \forall x > b, 3x + 5 > a\}.$$

To get a sense of what this statement means, we graph the function $y = 3x + 5$, as seen in Figure 1.16. The statement begins with 'for every a' and at then end of the statement, a is compared to y on $y = 3x + 5$, so we begin in the figure by plotting an arbitrary a on the y-axis. The next part is 'exists b' which comes from the a, so we go backwards, indicated by the arrows to find b on the x-axis. The next part is 'for every $x > b$' so we pick an arbitrary x to the right of b. Finally, we need to verify that $3x + 5 > a$, which we do graphically by following the arrows from x, up to the graph and over to the y-axis, noting the value is greater than a.

The proof of such a statement is just to algebraically do what we did graphically.

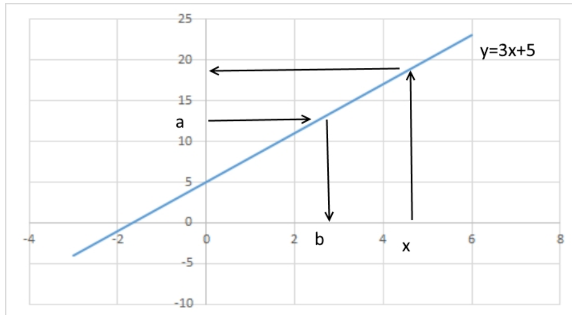

Figure 1.16: Graph of $y = 3x + 5$ indicating statement 1.14 is true.

Conceptual Insight: 1.36

We begin with the let method, let a_0 represent an arbitrary real number. There exists b means we have to find b such that for all $x > b$, $3x + 5 > a_0$. To find b, we solve $3b + 5 = a_0$ for b, giving

$$b_0 = \frac{a_0 - 5}{3}$$

(so we have found b). Note that (b_0, a_0) is a point on the line $y = 3x + 5$.

We now give the proof by reversing all our work, backward/forward approach.

Claim 1.37: Statement 1.14 is true.

Proof: Let a_0 represent an arbitrary real number. Set $b_0 = (a_0 - 5)/3$. Let x_0 represent an arbitrary number greater than b_0. Thus,

$$x_0 > b_0 = \frac{a_0 - 5}{3}.$$

Multiplying both sides by 3 (which is positive) and adding 5 to both sides gives

$$3x_0 + 5 > a_0,$$

which is what we needed to show. □

1.4.1 Problems

1. Consider the function f defined by $f(x) = x^5 + 7$. Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) > M$. Answer 1.

2. Consider the function f defined by $f(x) = -2x + 3$. Prove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) > M$.

3. Consider the function f defined by $f(x) = x^4 - 5$. Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) < M$. Answer 2.

4. Consider the function f defined by $f(x) = x^2 - 33$. Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) < M$.

5. Consider the function f defined by

$$f(x) = \begin{cases} -2x+1 & x > 0 \\ 3x - 1 & x \leq 0 \end{cases}.$$

   (a) Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) > M$. Answer 3.

   (b) Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) < M$. Answer 4.

6. Consider the function f defined by

$$f(x) = \begin{cases} -x+1 & x > 1 \\ 2x & x \leq 1 \end{cases}.$$

   (a) Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) > M$.

   (b) Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) < M$.

7. Consider the function f defined by

$$f(x) = \begin{cases} 2x+1 & x > 0 \\ 3x - 1 & x \leq 0 \end{cases}.$$

   Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) > M$. Answer 5.

8. Consider the function f defined by

$$f(x) = \begin{cases} -x+1 & x > 1 \\ -2x & x \leq 1 \end{cases}.$$

Prove or disprove that for all $M \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) > M$.

9. Consider the function f with domain $\mathbb{R}$ defined by $f(x) = x^2$.

   (a) Prove or disprove that f maps into $\mathbb{R}^+ = (0, \infty)$. Answer 6.

   (b) Prove or disprove that f maps onto $\mathbb{R}^+ = (0, \infty)$. Answer 7.

   (c) Prove or disprove that f maps into $\mathbb{R}_0^+ = [0, \infty)$. Answer 8.

   (d) Prove or disprove that f maps onto $\mathbb{R}_0^+ = [0, \infty)$. Answer 9.

10. Consider the function f with domain $[-2, 4] = \{x : -2 \leq x \leq 4\}$ defined by

$$f(x) = 3x - 5.$$

   (a) Prove or disprove that f maps into $(0, 15) = \{x : 0 < x < 15\}$.

   (b) Prove or disprove that f maps onto $(0, 15) = \{x : 0 < x < 15\}$.

   (c) Prove or disprove that f maps into $(-5, 18) = \{x : -5 < x < 18\}$.

   (d) Prove or disprove that f maps onto $(-5, 18) = \{x : -5 < x < 18\}$.

11. Consider the function f mapping $S = \mathbb{R}$ into the set $T = \mathbb{R}$ defined by $f(x) = x^3 + 4$. Prove or disprove that f maps onto T. Answer 10.

12. Consider the function f mapping $\mathbb{R} - \{1\}$ (all real numbers except 1) into the set $T = \mathbb{R}$ defined by

$$f(x) = \frac{2x+1}{x-1}.$$

Prove f does not map onto T.

13. Consider the function f mapping $S = \mathbb{R} - \{3\}$ (all real numbers except 3) into the set $T = \mathbb{R}$ defined by

$$f(x) = \frac{4x+5}{x-3}.$$

Prove f does not map onto T. Answer 11.

14. Show the function f mapping $S = \mathbb{R}$ into $T = \mathbb{R}$ is not onto T, where
$$f(x) = \begin{cases} x+3 & x > 2 \\ x+2 & x \leq 2 \end{cases}.$$

15. Show the function f mapping $S = \mathbb{R}$ into $T = \mathbb{R}$ is not onto T, where
$$f(x) = \begin{cases} 2x-1 & x > 1 \\ -x+7 & x \leq 1 \end{cases}.$$
Answer 12.

16. Prove or disprove that the function f mapping $S = \mathbb{R}$ into $T = \mathbb{R}$ is onto T, where
$$f(x) = \begin{cases} -2x+7 & x > 3 \\ -x+2 & x \leq 3 \end{cases}.$$

17. Prove or disprove that the function f mapping $S = \mathbb{R}$ into $T = \mathbb{R}$ is onto T, where
$$f(x) = \begin{cases} x^2 & x > 0 \\ 3x & x \leq 0 \end{cases}.$$
Answer 13.

18. Show the function f mapping $S = \mathbb{R}$ into $T = \mathbb{R}$ is onto T, where
$$f(x) = \begin{cases} x+3 & x \notin \mathbb{Z} \\ x-2 & x \in \mathbb{Z} \end{cases}.$$

19. Recall that an integer n is even (denoted $n \in \mathbb{E}$) if there exists an integer k such that $n = 2k$ and n is odd (denoted $n \in \mathbb{O}$) if there exists an integer k such that $n = 2k+1$. Show the function f mapping $S = \mathbb{Z}$ into $T = \mathbb{Z}$ is onto or find a $j \in T$ for which there does not exist an $n \in S = \mathbb{Z}$ such that $f(n) = j$, where
$$f(n) = \begin{cases} 0.5n+3 & n \in \mathbb{E} \\ 3n-1 & n \in \mathbb{O} \end{cases}.$$
Answer 14.

20. Watch the video at

    https://vimeo.com/44892151 Password: Proof

    in which two students try to prove or disprove the statement

    $$\left\{\forall\, a \in \mathbb{R}^+,\, \exists\, b \in \mathbb{R} \text{ s.t. if } x > b, \text{ then } \frac{2}{3x+5} < a\right\}.$$

    (a) To get conceptual insight into this problem, graph $y = 2/(3x + 5)$ and find the x-value when $y = a_0 > 0$. What does this have to do with the statement?

    (b) What do you think of their approach to this problem?

    (c) Convert their proof into a well-written proof.

21. Watch the video at

    https://vimeo.com/45875554 Password: Proof

    in which two students try to prove or disprove the statement

    $$\{\forall\, a \in \mathbb{R},\, \exists\, b \in \mathbb{R} \text{ s.t. if } x > a \text{ then } 2x - 5 > b\}.$$

    (a) To get conceptual insight into this problem, graph $y = 2x - 5$ and find the y-value when $x = a_0$. What does this have to do with the statement? Answer 15

    (b) What do you think of their approach to this problem? Answer 16

    (c) Convert their proof into a well-written proof. Answer 17

1.4.2 Answers to Selected Problems

1. **Problem 1:** Let M_0 represent an arbitrary real number and set $x_0 = (M_0 - 6)^{1/5}$. Then

   $$f(x_0) = x_0^5 + 7 = ((M_0 - 6)^{1/5})^5 + 7 = M_0 - 6 + 7 = M_0 + 1 > M_0.$$

2. **Problem 3:** This statement is false. Assume it is true and set $M_0 = -6$. We have assumed there exists an x_0 such that $f(x_0) = x_0^4 - 5 < -6$. This means $0 \leq x_0^4 < -1$, which is a contradiction. Therefore this statement is false.

3. **Problem 5 a:** From the graph of this function in Figure 1.17, it is clear the function never goes above 1, so this statement is false. We assume it is true and set $M_0 = 2$. From our assumption, there exists an x_0 such that $f(x_0) > 2$.

   Case 1: Suppose $x_0 > 0$. Then $f(x_0) = -2x_0 + 1 > 2$. After simplifying, $-2x_0 > 1$ or after dividing by -2 and reversing the inequality, $x_0 < -1/2$, which contradicts $x_0 > 0$.

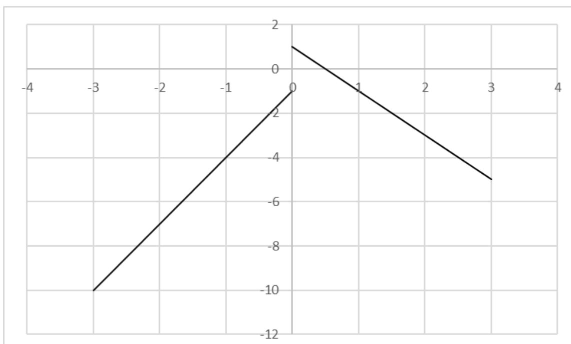

Figure 1.17: The graph of the function for Problem 5 with domain $\mathbb{R}$.

   Case 2: Suppose $x_0 \leq 0$. Then $f(x_0) = 3x_0 - 1 > 2$. After simplifying, $3x_0 > 3$ or after dividing by 3, $x_0 > 1$, which contradicts $x_0 \leq 0$.

4. **Problem 5 b:** From the graph of this function in Figure 1.17, it appears this statement is true. Let M_0 represent an arbitrary real number. If $M_0 < -1$, there are two expressions we could use, either line segment. We will use $-2x + 1$ although either is fine. There are two cases.

   Case 1: Suppose $M_0 \geq 1$. Set $x_0 = 1 > 0$ so $f(x_0) = -2x_0 + 1 = -1 < 1 \leq M_0$. Hence, $f(x_0) < M_0$.

   Case 2: Suppose $M_0 < 1$. We find when $f(x_0) = M_0$ by solving $M_0 = -2c + 1$. Since $M_0 < 1$, we get

$$c_0 = \frac{1 - M_0}{2} > 0.$$

   Any $x > c_0$ will work, so we set $x_0 = (2 - M_0)/2 > 0$. Then

$$f(x_0) = -2x_0 + 1 = -2(2 - M_0)/2 + 1 = -2 + M_0 + 1 = -1 + M_0 < M_0.$$

In both cases for M_0, there exists an x_0 such that $f(x_0) < M_0$, so this statement is true.

5. **Problem 7:** Let M_0 represent an arbitrary real number.

   Case 1: Suppose $M_0 > 1$. Finding the point of intersection of the function and the line $y = M_0$ results in $2c_0 + 1 = M_0$ or $c_0 = (M_0 - 1)/2 > 0$ since $M_0 > 1$. We set $x = x_0 = c_0 + 1 = (M_0 - 1)/2 + 1$ as seen in Figure 1.18 which gives conceptual insight into the problem. Then since $x_0 > 0$, $f(x_0) = 2x_0 + 1 = M_0 + 2 > M_0$.

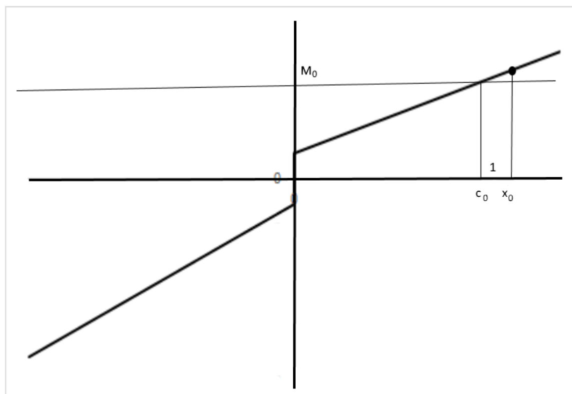

Figure 1.18: The graph of the function for Problem 7 with domain $\mathbb{R}$. Point giving y-value above M_0 is to right of point of intersection of function with $y = M_0$.

   Case 2: Suppose $M_0 \leq 1$. Figure 1.19 gives conceptual insight into this case in that any point on the right half of the graph has a y-value above M_0. For example, use $x_0 = 1 > 0$. Then $f(x_0) = f(1) = 3 > M_0$. Note that we did not need to use the branch $3x - 1$.

6. **Problem 9 a**: Set $x_0 = 0$. Then $f(x_0) = 0 \notin \mathbb{R}^+$, so the function is not into $\mathbb{R}^+$.

7. **Problem 9 b**: Let y_0 be an arbitrary element of $\mathbb{R}^+$ and set $x_0 = \sqrt{y_0}$. Then $f(x_0) = x_0^2 = y_0$, so this function does map onto $\mathbb{R}^+$.

8. **Problem 9 c**: Let x_0 be an arbitrary real number. Then $f(x_0) = x_0^2 \geq 0$, so $f(x_0) \in \mathbb{R}_0^+$. Hence, the function is into $\mathbb{R}_0^+$.

9. **Problem 9 d**: Let y_0 be an arbitrary element of $\mathbb{R}_0^+$ and set $x_0 = \sqrt{y_0}$. Then $f(x_0) = x_0^2 = y_0$, so this function does map onto $\mathbb{R}_0^+$.

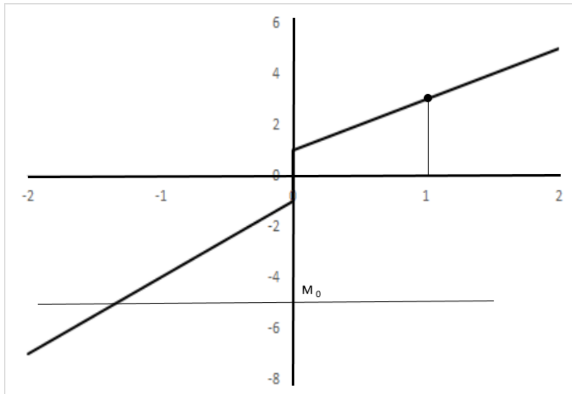

Figure 1.19: The graph of the function for Problem 5 with domain $\mathbb{R}$. Point with $x_0 = 1$ gives a y-value above M_0.

10. **Problem 11:** You should sketch a graph of this function to see that it appears to be onto $\mathbb{R}$. Let y_0 represent an arbitrary real number and set $x = x_0 = (y_0 - 4)^{1/3}$. Thus $f(x_0) = y_0$.

11. **Problem 13:** From looking at the graph of the function in Figure 1.20, there appears to be a horizontal asymptote at $y = 4$, so no x gives a value of $y = 4$. We use contradiction, assume the map is onto and set $y_0 = 4$. Let x_0 represent the value that results in $f(x_0) = 4$. Since

$$4 = \frac{4x_0 + 5}{x_0 - 3}$$

results in $-12 = 5$ which is a contradiction, $f(x_0) \neq 4$ and f does not map onto $\mathbb{R}$.

12. **Problem 15:** From looking at the graph of the function

$$f(x) = \begin{cases} 2x - 1 & x > 1 \\ -x + 7 & x \leq 1 \end{cases}$$

in Figure 1.21, it is clear that f is not onto $T = \mathbb{R}$. For example, no x gives a value of $y = 0$. We assume the function is onto and set $y_0 = 0$. We assume x_0 is the value that gives $f(x_0) = 0$.

Case 1: Suppose $x_0 \leq 1$. Then $f(x_0) = -x_0 + 7 \geq 6 \neq y_0 = 0$, which is a contradiction.

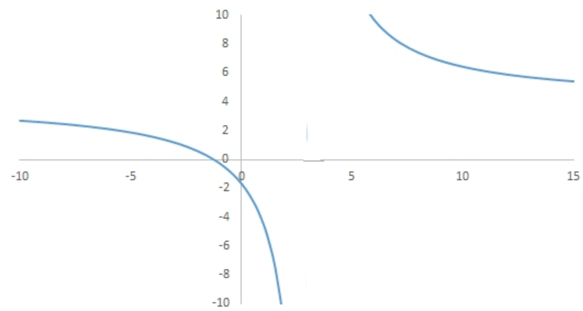

Figure 1.20: The graph of the function for Problem 13 with domain $\mathbb{R}$ apparently does not map **onto** the set $S = \mathbb{R}$.

Case 2: Suppose $x_0 > 1$. Then $f(x_0) = 2x_0 - 1 > 1 > y_0 = 0$. So in both cases, we have a contradiction and therefore there does not exists a real x such that $f(x) = 0$ so f is not onto T.

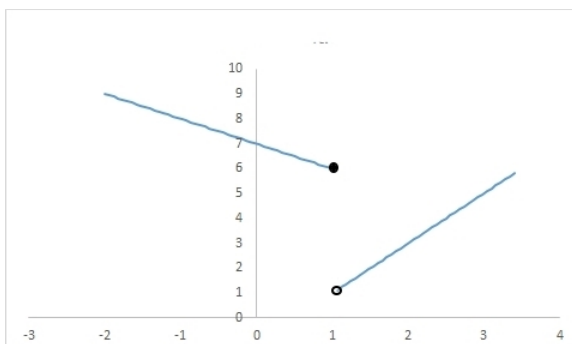

Figure 1.21: The graph of the function for Problem 15 with domain $\mathbb{R}$ apparently does not map onto the set $TS = \mathbb{R}$, leaving off all values 1 and lower.

13. **Problem 17:** From looking at the graph of the function in Figure 1.22, f appears to be onto. To prove this, we let y_0 represent an arbitrary real number, then consider two cases.

    Case 1: Assume $y_0 \leq 0$. Set $x_0 = y_0/3 \leq 0$.

    Case 2: Assume $y_0 > 0$. Set $x_0 = \sqrt{y_0} > 0$.

    In both cases, substitution yields that $f(x_0) = y_0$. Note that to substitute the x_0 values, we needed to know whether they were less than or equal to 0, or greater than 0.

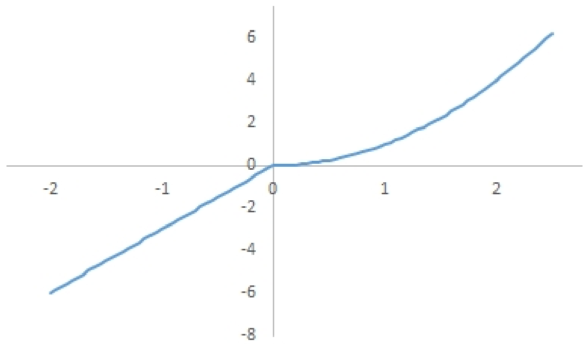

Figure 1.22: The graph of the function for Problem 17 with domain $S = \mathbb{R}$ apparently maps **onto** the set $T = \mathbb{R}$.

14. **Problem 19:** In first two rows of Table 1.4, we computed $f(n)$-values for odd integers n and last two rows we computed $f(n)$-values for even integers n. We can see that even integers seem to give us all integers as $f(n)$-values. This is our conceptual insight. We now let m_0 represent an arbitrary integer. We work backward, solving $0.5n+3 = m_0$ for n. This gives $n = n_0 = 2(m_0 - 3)$. Since $m_0 - 3$ is an integer, n_0 is an even integer, so

$$f(n_0) = 0.5(2(m_0-3)) + 3 = m_0 - 3 + 3 = m_0.$$

Hence, f maps onto $T = \mathbb{Z}$.

$n =$	-1	1	3	5	7	9
$f(n) =$	-4	2	8	14	20	26
$n =$	-2	0	2	4	6	8
$f(n) =$	2	3	4	5	6	7

Table 1.4: First two rows give $f(n)$-values for odd n and last two rows give values for even n for Problem 19:

.

15. **Problem 21 a:** We begin by plotting a on the x-axis, going to b on the y-axis, and then noticing for all $x > a$, $f(x) > b$, as seen in Figure 1.23.

16. **Problem 21 b:** The students began saying they just needed to solve $2x - 5 = b$, which would give them the backwards work they needed, but they wandered around before actually doing it.

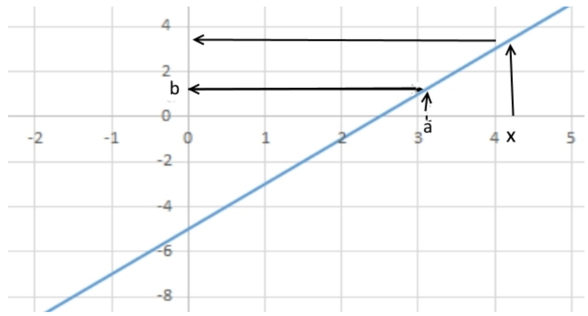

Figure 1.23: Graph of $y = 2x - 5$ indicating statement for Problem 21 is true.

17. **Problem 21 c: Proof** of the statement

$$p \equiv \{\forall a \in \mathbb{R}, \exists b \in \mathbb{R}\, \text{s.t.}\ \forall x > a, 2x - 5 > b\}.$$

Let a_0 represent an arbitrary real number. Set $b_0 = 2a_0 - 5$, which means $a_0 = (b_0 + 5)/2$. Let x_0 represent an arbitrary number greater than a_0. Thus,

$$x_0 > \frac{b_0 + 5}{2}.$$

Multiplying both sides by 2 (which is positive) and subtracting 5 to both sides gives

$$2x_0 - 5 > b_0,$$

which is what we needed to show.

1.5 There Exists, For Every

Video Lesson 1.38

Before reading this section, you should watch the 6:48 minute video https://vimeo.com/83696588 (password:Proof).
which introduces much of this material. The reading may then make more sense. To change the speed at which the video plays, click on the gear at the lower right of the video.

A second common theme is a statement of the form:

$$p \equiv \{\exists\, a \in S \ \text{s.t. claim is true}\}, \tag{1.15}$$

read as, 'there exists an a in the set S such that claim is true.' Such statements are often relatively easy to verify in that all we have to do is find a value that works. We note that sometimes there is only one value that works, sometimes there are several values that work and sometimes there are an infinite number of values that work. For example, we might be asked to show that

$$p \equiv \{\exists x \in \mathbb{R} \text{ such that } 2x + 5 = 9\}.$$

To prove this statement we only need to set $x_0 = 2$ and verify the equation holds. On the other hand, we might be proving

$$p \equiv \{\exists x \in \mathbb{R} \text{ such that } x^2 = 9\},$$

in which case, we can prove the statement is true by setting $x_0 = 3$ or setting $x_0 = -3$ and verifying the statement. For the statement

$$p \equiv \{\exists x \in \mathbb{R} \text{ such that } \sin(\pi x) = 0\},$$

we can set x_0 equal to any integer.

More importantly are statements of the form

$$p \equiv \{\big\langle \exists\, a \in S \text{ such that } \big\rangle \big\langle \forall\, b \in T \big\rangle \, \big\langle \text{claim is true} \big\rangle\}. \tag{1.16}$$

For example, consider the statement

$$p \equiv \{\exists\, y \in \mathbb{R}, \text{ such that } \forall x \in \mathbb{R},\ (y+2)(x-1) = 0\}. \tag{1.17}$$

If we set $y = y_0 = -2$, the expression equals 0 for every x-value, so the statement is true.

Assumed: From backwards work, we assume $y_0 = -2$.

To Show: Using let-variable, x_0, show $(y_0 + 2)(x_0 - 1) = 0$.

Claim 1.39: We prove statement 1.17.

Proof: Set $y = y_0 = -2$ (so it exists). Let x_0 represent an arbitrary real number. The expression $(y_0 + 2)(x_0 - 1)$ becomes

$$(0)(x_0 - 1) = 0$$

after substitution. Thus, for $y = y_0 = -2$ and for every $x \in \mathbb{R}$, the expression equals zero, which is what we were trying to prove. □

Note in Proof 1.39 that after the there-exists-variable y_0 was found, we had to use a let-variable, x_0 because we were showing something for every x.

One application of such a statement occurs in showing a function is bounded above.

Definition 1.40: Bounded above

Suppose f is a function with domain $D \subseteq \mathbb{R}$. The function f is **bounded above** if and only if there exists $M \in \mathbb{R}$ such that for all $x \in D$, $f(x) \leq M$. In this case, M is an upper bound for the function.

Intuitively, a function being bounded above just means that we can draw some horizontal line $y = M$ for which the graph of the function is never above the line.

Consider the function
$$f(x) = \frac{2}{1+x^2}$$
with domain $D = \mathbb{R}$. Its graph can be seen in Figure 1.24. Our intuition tells us this function is bounded above. Let's think about what this means in relation to the definition.

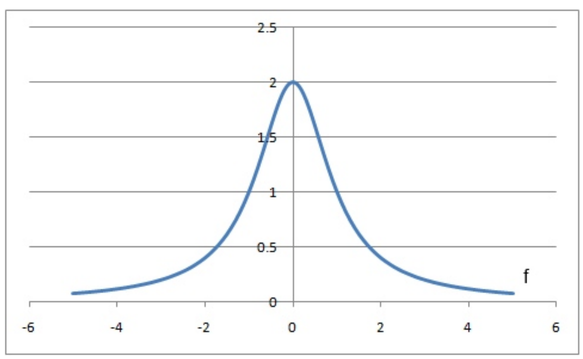

Figure 1.24: The function defined by $f(x) = 2/(1+x^2)$ is bounded above with upper bound M, $M \geq 2$.

Conceptual Insight: 1.41

It appears that if we draw a line $y = M$, $M > 2$, then the entire graph of the function will lie below that line, and if we draw the line $y = 2$, then the entire graph of the function lies on or below the line. This conceptual insight is based only on our intuition and a limited graph of the function. We now need a technical handle to write the proof. To show that $f(x) = 2/(1+x^2)$ is bounded above, we have to show a statement similar to Type 1.16 is true:

$$\{\langle \exists\, M \in \mathbb{R} \text{ such that } \rangle \langle \forall\, x \in D \rangle \langle f(x) \leq M \rangle\}.$$

The first phrase states that $\exists\, M \in \mathbb{R}$ which means that M is unknown and we need to find it to show it exists. From the graph of the function in Figure 1.24, f always appears to be on or below the line $y = 2$. While we can **set** M to be any value greater than or equal to 2, we will actually set it as 2. In the proof, we would write the there-exists-variable as

$$\text{Set } M = M_0 = 2.$$

We use the word 'set' to define a variable as equal to a particular value.

Since we have $M_0 = 2$, we have to show the following phrases are true for this M-value. The next phrase is easy, $\langle \forall\, x \in D \rangle$. Since this must be true for every real number x, we use a let-variable by letting x_0 represent an arbitrary number in the domain of f, that is, an arbitrary real number. We now treat x_0 as if it were a known value.

Now to the last phrase, we must show $f(x_0) \leq M_0$, which means showing

$$\frac{2}{1+x_0^2} \leq 2.$$

Working backwards, since $1 + x_0^2 > 0$, we can multiply both sides of the inequality by $1 + x_0^2$ giving

$$2 \leq 2(1 + x_0^2).$$

This is equivalent to

$$0 \leq 2x_0^2,$$

which we know is true. We can now prove our result by reversing our steps. The *backward/forward approach* gives us the technical handle we need to construct a proof.

Assumed: Assume that $M_0 \geq 2$.

To Show: We want to show that $x_0 \in \mathbb{R}$, $2/(x_0^2 + 1) \leq 2$.

Let's now convert our conceptual insight into a proof.

Claim 1.42: The function $f(x) = 2/(1+x^2)$ is bounded above.

Proof: Set $M_0 = 2$. Let x_0 represent an arbitrary real number. We know that for any real number, x_0, $0 \leq 2x_0^2$. Adding 2 to both sides gives $2 \leq 2 + 2x_0^2$ or

$$2 \leq 2(1+x_0^2).$$

Dividing both sides by the positive value, $1+x_0^2$, gives

$$f(x_0) = \frac{2}{1+x_0^2} \leq 2 = M_0,$$

which is what we needed to show. Since this was true for any arbitrary $x_0 \in \mathbb{R}$, we are finished; f is bounded above.□

Note that the insight gained from Figure 1.24 helped us guess a value for M that ended up working. Then we used a little algebraic manipulation combined with a let-variable and backward/forward approach to construct the proof that f is bounded above.

While in Section 1.4, we showed a function was not onto a set T using contradiction, we could also do this by using a statement of the form 1.16.

Definition 1.43: Not onto

A function f is said **not** to map set S onto the set T if and only if there exists $y \in T$ such that for every $x \in S$, $f(x) \neq y$.

Suppose we want to show that the function $f(x) = x^2$ from $S = \mathbb{R}$ into $T = \mathbb{R}$ does not map onto T. In this case, we begin by setting $y_0 = -1$ and then let x_0 represent an arbitrary element of the domain $\mathbb{R}$. Since $f(x_0) = x_0^2 \geq 0$, then $f(x_0) \neq y_0 = -1$, so f is not onto $T = \mathbb{R}$.

1.5.1 Problems

1. Prove that the function f mapping $\mathbb{R}$ into $\mathbb{R}$ is bounded above where

$$f(x) = \frac{x^2-2}{x^2+5}.$$

   Answer 1

2. Prove that the function f mapping $\mathbb{R}$ into $\mathbb{R}$ is bounded above where
$$f(x) = \frac{x+3}{x^2+16}.$$

3. Prove that the function f mapping $\mathbb{R}$ into $\mathbb{R}$ is bounded above where
$$f(x) = \begin{cases} 2x+1 & x \leq 3 \\ -x+5 & x > 3 \end{cases}.$$
Answer 2

4. Prove that the function f mapping $\mathbb{R}$ into $\mathbb{R}$ is bounded above where
$$f(x) = \begin{cases} 5x-3 & x \leq 0 \\ -x^2+5 & x > 0 \end{cases}.$$

5. Prove the statement that there exists a real number a such that for all $b > a$,
$$\frac{3b+1}{b+5} > 2.$$
Answer 3

6. Prove the statement that there exists a real number a such that for all $b < a$,
$$\frac{5b+6}{b+1} > 4.$$

7. Recall that an integer n is even (denoted $n \in \mathbb{E}$) if there exists an integer k such that $n = 2k$ and n is odd (denoted $n \in \mathbb{O}$) if there exists an integer k such that $n = 2k+1$. Show the function f mapping $S = \mathbb{Z}$ into $T = \mathbb{Z}$ is not onto $\mathbb{Z}$ where
$$f(n) = \begin{cases} 2n+1 & n \in \mathbb{E} \\ 4n-2 & n \in \mathbb{O} \end{cases}.$$
Answer 4

8. The following statements seem quite similar. Explain why you think each statement is true or false.
$$\begin{aligned} p &\equiv \{\forall\, x > 0, \exists y > 0 \text{ such that } y < x\} \text{ and} \\ q &\equiv \{\exists\, x > 0 \text{ such that } \forall y > 0,\, y < x\}. \end{aligned}$$

9. Work the following parts

   (a) Determine the truth of the statement

   $$p_1 \equiv \{\forall\, x \in \mathbb{R},\, \exists y \in \mathbb{R} \text{ such that } 2x + y = -2\}$$

   and explain your answer. Answer 5

   (b) Determine the truth of the statement

   $$q_1 \equiv \{\exists\, y \in \mathbb{R} \text{ such that } \forall\, x \in \mathbb{R},\, 2x + y = -2\}$$

   and explain your answer. Answer 6

   (c) Determine the truth of the statement

   $$p_2 \equiv \{\forall\, x \in \mathbb{R},\, \exists y \in \mathbb{R} \text{ such that } x + y^2 = 3\}$$

   and explain your answer. Answer 7

   (d) Determine the truth of the statement

   $$q_2 \equiv \{\exists\, y \in \mathbb{R} \text{ such that } \forall\, x \in \mathbb{R},\, x + y^2 = 3\}$$

   and explain your answer. Answer 8

   (e) Determine the truth of the statement

   $$p_3 \equiv \{\forall\, x \in \mathbb{R},\, \exists y \in \mathbb{R} \text{ such that } (x + y - 2)(y + 3) = 0\}$$

   and explain your answer. Answer 9

   (f) Determine the truth of the statement

   $$q_3 \equiv \{\exists\, y \in \mathbb{R} \text{ such that } \forall\, x \in \mathbb{R},\, (x + y - 2)(y + 3) = 0\}$$

   and explain your answer. Answer 10

   (g) Is it possible to find an equation for which

   $$p \equiv \{\forall\, x \in \mathbb{R},\, \exists y \in \mathbb{R} \text{ such that `equation' is satisfied}\}$$

   is false and

   $$q \equiv \{\exists\, y \in \mathbb{R} \text{ such that } \forall\, x \in \mathbb{R}, \text{ `equation' is satisfied}\}$$

   is true? Why or why not. Answer 11

10. Consider the following statements:

$$\begin{aligned}
p_1 &\equiv \{\forall x \in \mathbb{R}\ \exists y \in \mathbb{R} \text{ such that } && y^2 = x^3 + 2x + 1\},\\
p_2 &\equiv \{\forall y \in \mathbb{R}\ \exists x \in \mathbb{R} \text{ such that } && y^2 = x^3 + 2x + 1\},\\
p_3 &\equiv \{\exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R} && y^2 = x^3 + 2x + 1\} \text{ and}\\
p_4 &\equiv \{\exists y \in \mathbb{R} \text{ such that } \forall x \in \mathbb{R} && y^2 = x^3 + 2x + 1\}.
\end{aligned}$$

Which of the statements are true? Which of these statements are false? Are there any statements for which you cannot tell? If you think the statement is false, give a counterexample (a single one is enough, the simpler the better) or a very concise argument. If you think a statement is true, explain why (no formal proof is necessary).

1.5.2 Answers to selected problems

1. **Problem 1:** From the graph in Figure 1.25, we can use any $M \geq 1$. From the backwards work, set $M_0 = 1$ and let x_0 represent an arbitrary real number. Since $-2 \leq 5$, adding x^2 to both sides gives $x^2 - 2 \leq x^2 + 5$. Since $x^2 + 5 > 0$, we can divide both sides by it, giving

$$f(x) = \frac{x^2 - 2}{x^2 + 5} \leq 1 = M_0,$$

and the function is bounded above.

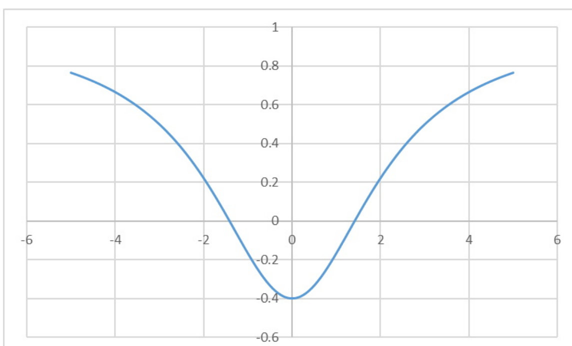

Figure 1.25: The graph of the function for Problem 1 indicates it is bounded above with a horizontal asymptote at $y = 1$.

2. **Problem 3:** From the graph in Figure 1.26, we can use any $M \geq 7$. From the backwards work, set $M_0 = 7$ and let x_0 represent an arbitrary real number.

Case 1: Suppose $x_0 \leq 3$. Then $f(x_0) = 2x_0 + 1 \leq 2(3) + 1 = M_0$.

Case 2: Suppose $x_0 > 3$. Then $f(x_0) = -x_0 + 5 < -3 + 5 = 2 < 7 = M_0$.

In both cases, $f(x_0) \leq M_0$ so the function is bounded above.

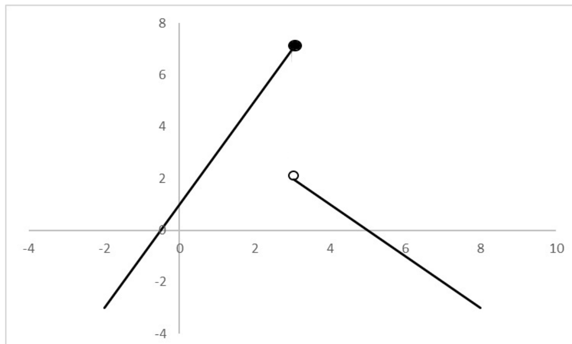

Figure 1.26: The graph of the function for Problem 3 indicates it is bounded above.

3. **Problem 5:** Set $a_0 = 9$ and let $b_0 > a_0 = 9$. Adding $2b_0$ to both sides gives

$$3b_0 > 2b_0 + 9.$$

Adding 1 to both sides gives

$$3b_0 + 1 > 2b_0 + 10 = 2(b_0 + 5).$$

Since $b_0 > 9$, $b_0 + 5 > 0$, and we can divide the inequality by it to get our result

$$\frac{3b_0 + 1}{b_0 + 5} > 2.$$

This proof was based on doing the backwards work, which was done by the students in the video

https://vimeo.com/46575162 Password:Proof

They give a reasonable proof of this statement, except that they 'Choose $b > 9$' instead of 'letting b_0 represent an arbitrary number greater than 9.

4. **Problem 7:** From looking at the graph of the function in Figure 1.27, it is clear that f is not onto $T = \mathbb{Z}$. For example, it appears there is no

integer n such that $f(n) = 4$. Thus, we set $y_0 = 4$ and let n_0 represent an arbitrary integer. There are two cases.

Case 1: Suppose n_0 is even. Then $f(n_0) = 2n_0 + 1$ is odd and cannot therefore equal 4.

Case 2: Suppose n_0 is odd. Then there exists an integer k_0 such that $n_0 = 2k_0 + 1$. Therefore $f(n_0) = 4n_0 - 2 = 4(2k_0 + 1) - 2 = 8k_0 + 2 = 2(4k_0 + 1)$ which cannot equal $4 = 2(2)$ for then $4k_0 + 1 = 2$ and $k_0 = 1/4$ which is not an integer. See the video at

https://vimeo.com/45866012 (Password:Proof)

to see how some students approached this problem, not knowing if the function was onto or not. In particular, you will notice that by not creating a table of values for $f(n)$ first, it took them longer to work this problem than it should have.

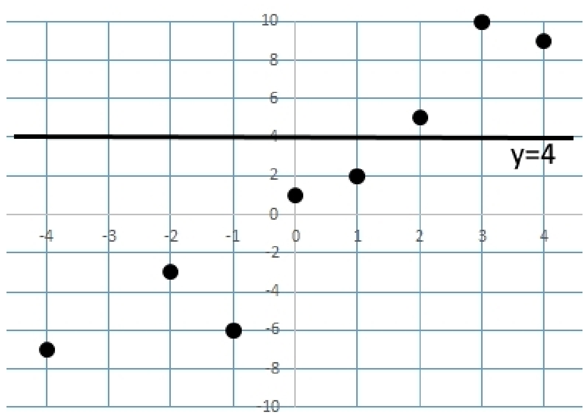

Figure 1.27: With the graph of the function for Problem 7 with domain $\mathbb{Z}$ being represented by the circles in this graph, it is apparent the function does not map onto the set $T = \mathbb{Z}$. For example, no point lies on the line $y = 4$.

5. **Problem 9 a)**: If we let x_0 represent an arbitrary real number, then $y_0 = -2x_0 - 2$ exists so statement p_1 is true.

6. **Problem 9 b)**: The statement q_1 is false for no matter what value we set as y, the equation is not satisfied for all x, but for only the one x that pairs with y, that is, $x = (-2 - y)/2$.

7. **Problem 9 c)**: If we set $x = 5$, then there does not exist a y such that $5 + y^2 = 3$. Hence, statement p_2 is false since it isn't true for all x.

8. **Problem 9 d)**: The statement q_2 is false; there cannot exist a single fixed y-value such that $x + y^2 = 3$ is true for every x since we already know that there does not exist a y such that $x + y^2 = 3$ is true when $x = 5$.

9. **Problem 9 e)**: Let x_0 be an arbitrary real number. Set $y = y_0 = -3$. Then $(x_0 + y_0 - 2)(y_0 + 3) = 0$, so p_3 is true.

10. **Problem 9 f)**: There exists a y, $y = y_0 = -3$ such that for all x, $(x - 2)(y_0 + 3) = 0$, so q_3 is true.

11. **Problem 9 g)**: We cannot have p false and q true. Suppose q is true for some y_0, that is,
$$E(x, y_0) = 0$$
for every x where $E(x, y)$ is some expression in terms of x and y. Then there exists some $y = y_0$, such that $E(x, y_0) = 0$ for every $x \in \mathbb{R}$. Then let x_0 represent an arbitrary fixed number in $\mathbb{R}$. Then we already know that $E(x_0, y_0) = 0$, so for any x-value, there exists a y-value so that the equation is satisfied, so p is also true.

Chapter 2

Complex Statements and their negations

2.1 Negating Statements

Video Lesson 2.1

This section will be easier to read if you first watch the 5:25 minute video
https://vimeo.com/73971958 (Password:Proof)
in which we go through the process of negating the statement that a function is bounded above, and the 3:36 minute video
https://vimeo.com/85102409 (Password:Proof)
in which an implication is proven false. To change the speed at which the video plays, click on the gear at the lower right of the video.

In previous sections, we were given a statement r, using quantifiers such as 'for every' and 'there exists,' and asked to prove the statement is true or false. To prove it was false, we used contradiction, assuming it was true and arriving at a contradiction. In this section, we are going to learn how to write $\neg r$, the negation of statement r. Then we develop conceptual insight into whether r or $\neg r$ is true. Once we have determined which statement we think is true, we will use our previously developed methods to prove our conjecture is true.

Definition 2.2: Negation

The **negation** of a statement r is the statement that is true precisely when statement r is false and is denoted $\neg r$.

Let's begin by considering how we negate statements. Consider these two standard types of statements, first that 'there exists an $x \in D$ such that $A(x)$ is true' written as

$$r \equiv \{\langle \exists\, x \in D \text{ s.t.}\rangle\ \langle A(x)\rangle\} \tag{2.1}$$

and second, 'for all $x \in D$, $B(x)$ is true' written as

$$s \equiv \{\langle \forall\, x \in D\rangle\ \langle B(x)\rangle\} \equiv \{\text{If } x \in D\text{, then } B(x)\}. \tag{2.2}$$

Examples of these statements might be

$$r_0 \equiv \{\text{there exists } x > 0 \text{ such that } x^2 - 1 > 0\}$$

and

$$s_0 \equiv \{\text{for all } x > 0,\, x^2 - 1 \leq 0\}.$$

The negation of r in Statement 2.1 is that there does not exist an $x \in D$ for which $A(x)$ is true. This must mean that for all $x \in D$, $A(x)$ must be false

$$\neg r \equiv \{\langle \forall\, x \in D, \rangle\ \langle \neg A(x)\rangle\} \equiv \{\text{If } x \in D \text{ then } \neg A(x)\}. \tag{2.3}$$

In other words, $\neg r \equiv s$ where $\neg A(x) = B(x)$. For example, $\neg r_0 \equiv s_0$.

The negation of s in Statement 2.2 is that it is not true that for every $x \in D$, $B(x)$ is true. This means there must exist an $x \in D$ such that $B(x)$ is false,

$$\neg s \equiv \{\langle \exists\, x \in D \text{ s.t.}\rangle\ \langle \neg B(x)\rangle\}. \tag{2.4}$$

This means that $\neg s \equiv r$ where $\neg B(x) = A(x)$. For example, $\neg s_0 \equiv r_0$.

Summary 2.1

$\neg r \equiv s$ and $\neg s \equiv r$ if $A(x) = \neg B(x)$ (and $B(x) = \neg A(x)$).

One of the most common uses of this process of negation is with implications involving predicate functions, that is, statements of the form

$$s \equiv \{\text{if } p(x)\text{, then } q(x)\},$$

which can be rewritten as

$$s \equiv \{\text{for all } x \text{ for which } p(x) \text{ is true, } q(x) \text{ is true}\}.$$

By the above, the negation of this statement is then

$$\neg s \equiv r \equiv \{\text{there exists } x \text{ such that } p(x) \text{ is true such that } q(x) \text{ is false}\}.$$

This means that the negation of an 'if/then' statement is a 'there exists' statement. For example, consider the simple implication,

$$s \equiv \{\text{if } x < 10, \text{ then } x > 3\},$$

which can be rewritten as

$$s \equiv \{\text{for all } x < 10,\ x > 3\}.$$

The negation of this statement is

$$\neg s \equiv \{\text{there exists } x < 10 \text{ such that } x \leq 3\}.$$

Implication s is clearly false since there exists $x_0 = 1$ which satisfies $x_0 = 1 < 10$ and $x_0 \leq 3$. Note that it was simple to prove this implication false. This seems counterintuitive until you think about what these statements actually mean.

Video Lesson 2.3

Watch the 4:13 minute video
https://vimeo.com/74398407 (Password:Proof)
which gives an additional discussion on how to prove an implication is false.

Remember this important and useful relation

$$\begin{aligned}\neg s &\equiv \neg\{\text{if } p(x) \text{ then } q(x)\} \\ &\equiv \neg\{\forall x \text{ s.t. } p(x), q(x)\} \\ &\equiv \{\exists\, x \text{ s.t. } p(x) \text{ s.t. } \neg q(x)\}.\end{aligned} \tag{2.5}$$

Let's use this equivalence to develop a definition for a function to not be bounded above. Recall that f is bounded above if and only if

$$p \equiv \{\langle \exists\, M \in \mathbb{R} \text{ such that}\rangle\ \langle \forall\, x \in D,\ f(x) \leq M\rangle\},$$

where D is the domain of f. This is of the form of Statement 2.1, so its negation is of the form of Statement 2.3

$$\{\text{not bounded above}\} \equiv \neg p \equiv \{\langle \forall M \in \mathbb{R} \rangle, \neg \langle \forall\, x \in D,\, f(x) \leq M \rangle\}.$$

We now apply Negation 2.4 to the second phrase giving

$$\begin{aligned} \neg\{\langle \forall\, x \in D \rangle\, \langle f(x) \leq M \rangle\} \quad &\equiv \quad \{\langle \exists\, x \in D \text{ such that} \rangle\, \neg \langle f(x) \leq M \rangle\} \\ &\equiv \quad \{\langle \exists\, x \in D \rangle \text{ s.t. } \langle f(x) > M \rangle\}. \end{aligned}$$

Putting it all together, we now have the definition for f not being bounded above.

Definition 2.4: Not bounded above

The function f is **not bounded above** if and only if for all $M \in \mathbb{R}$, there exists an $x \in D$ such that $f(x) > M$, that is

$$\{\langle \forall\, M \in \mathbb{R} \rangle, \langle \exists x \in D \text{ such that} \rangle \langle f(x) > M \rangle\}.$$

The ability to negate statements is crucial for understanding the rest of this text. For the most part, we will be writing a statement p and will ask if the statement is true or false. To answer this question, we need to write the statement p and the statement $\neg p$. We then need to develop conceptual insight into which of these statements is true. This often consists of constructing some examples (graphs, tables, etc.) to get insight into which we think is true. From there, we will try to construct a proof of our conjecture.

In the following, we use Definition 2.4 to show that a simple function is not bounded above. You can also rewatch video 1.24, mentioned in Section 1.3, in which we used this derived definition of not bounded above to show that the function $f(x) = x^2 + 1$ is not bounded above.

Suppose we want to show a function, such as the linear function

$$f(x) = 2x + 1$$

is not bounded above. The key idea is that some part of the graph of this function will lie above any horizontal line, $y = M$, that we draw, that is, for any line $y = M$, there exists an x-value for which the point $(x, f(x))$ is above the line $y = M$.

Conceptual Insight: 2.5

Let's use the definition to prove that the linear function

$$f(x) = 2x + 1$$

is not bounded above. Since we have to show something is true for every M, we begin with a let-variable and let M_0 represent an arbitrary number in $\mathbb{R}$. At this point, we treat M_0 as a known real number. The next step is to show that

$$\langle \exists\, x \in D = \mathbb{R} \text{ such that} \rangle \langle f(x) > M_0 \rangle$$

is true. At this point, we must find a value for the unknown x so that $f(x) > M_0$, that is,

$$2x + 1 > M_0.$$

Remembering that M_0 is now considered a known number, if we look at the inequality, we see that x is the only unknown, so we work backwards and solve the inequality for x getting

$$x > \frac{M_0 - 1}{2} = 0.5M_0 - 0.5.$$

We only have to find one x-value, so any value greater than $(M_0 - 1)/2$ will work, say set

$$x = x_0 = 0.5M_0,$$

completing our backward approach.

We now reverse our steps, that is, use a forward approach to prove our statement.

Claim 2.6: The function $f(x) = 2x + 1$ is not bounded above.

Proof: Let M_0 represent an arbitrary real number (must hold for all M). Set $x_0 = 0.5M_0$ showing x exists. We now check that the last statement is true,

$$f(x_0) = 2x_0 + 1 = 2(0.5M_0) + 1 = M_0 + 1 > M_0,$$

which is what we needed to show, so the function is not bounded above. □

We note that the backward/forward algebraic approach was what we call the 'technical handle' that we needed to construct an actual proof of our statement.

These techniques of negating statements can easily be applied, even when our statements get more complicated as in

$$p \equiv \{\forall a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. if } x > b, \text{ then } x\sin(\pi x) > a\}. \tag{2.6}$$

To get a sense of what this statement means, we graph the function $y = x\sin(\pi x)$, as seen in Figure 2.1.

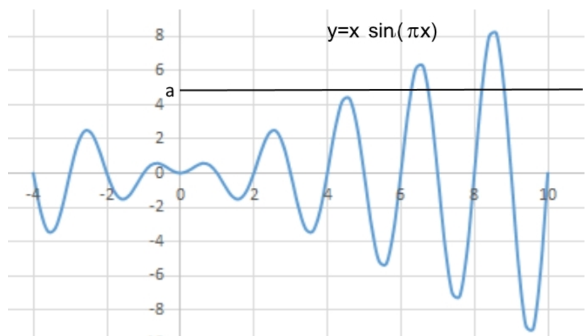

Figure 2.1: Graph of $y = x\sin(\pi x)$ indicating statement 2.6 is false.

Conceptual Insight: 2.7

Statement 2.6 begins with 'for every a' and at then end of the statement, a is compared to y on $y = x\sin(\pi x)$, so we begin in the figure by plotting an arbitrary a on the y-axis. The next part is 'exists b for which the function is above a for all x to the right of b.' This is clearly not true, since the function continues to oscillate below the line $y = a$ as x increases. Thus, statement 2.6 appears to be false.
To prove the implication is false, we need to negate statement 2.6. To do this, we first rewrite this statement using 'for every' instead of 'if' as follows

$$p \equiv \{\forall a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. } \forall x > b,\ x\sin(\pi x) > a\}.$$

We negate it, one step at a time.

$$\begin{aligned}\neg p &\equiv \{\exists a \in \mathbb{R} \text{ s.t. } \neg\langle \exists b \in \mathbb{R} \text{ s.t. } \forall x > b,\, x\sin(\pi x) > a\rangle\} \\ &\equiv \{\exists a \in \mathbb{R} \text{ s.t. } \forall b \in \mathbb{R},\, \neg\langle \forall x > b,\, x\sin(\pi x) > a\rangle\} \\ &\equiv \{\exists a \in \mathbb{R} \text{ s.t. } \forall b \in \mathbb{R},\, \exists x > b \text{ s.t. } \neg\langle x\sin(\pi x) > a\rangle\} \\ &\equiv \{\exists a \in \mathbb{R} \text{ s.t. } \forall b \in \mathbb{R},\, \exists x > b \text{ s.t. } x\sin(\pi x) \le a\}.\end{aligned}$$

The first clause in the statement is 'exists a.' We note that $\sin(\pi x) = 0$ whenever $x \in \mathbb{Z}$. Thus $x\sin(\pi x) < a$ for some x values and any positive a-value. Since we must show an a exists, we set $a_0 = 1$. The next clause states 'for every b' so we let b_0 represent an arbitrary real number. The next clause is 'exists $x > b_0$ so that the y-value is below $a_0 = 1$' so we have to find x. We know if $x \in \mathbb{Z}$, then $x\sin(\pi x) = 0 \le a_0 = 1$, so we have to find an integer greater than b_0. This is easy if we use the ceiling function defined in the Preface 17. Set

$$x_0 = \lceil b_0 \rceil + 1 > b_0.$$

Then $x_0 \sin(\pi x_0) = 0 \le 1$ and we are finished.

Remark 2.8

The function $f(x) = \lceil x \rceil$ is called the ceiling function and it gives the smallest integer that is greater than or equal to x. It is quite useful when we are looking for large integers instead of large numbers.

We now use our conceptual insight to construct a clear proof.

Claim 2.9: $\neg p \equiv \{\exists a \in \mathbb{R}$ **s.t.** $\forall b \in \mathbb{R},\, \exists x > b$ **s.t.** $x\sin(\pi x) \le a\}$.

Proof: Set $a_0 = 1$. Let b_0 represent an arbitrary real number. Set $x_0 = \lceil b_0 \rceil + 1 > b_0$. Then $x_0 \sin(\pi x_0) = 0 \le 1$, which is what we needed to show. □

2.1.1 Problems

1. Consider the two similar statements

$$p \equiv \{\text{if } x > 3, \text{ then } x^2 - x - 2 > 0\}$$

and

$$q \equiv \{\text{if } x > 3, \text{ then } x^2 - 3x - 4 > 0\}.$$

(a) Write the negation of p, then prove that either p is true or $\neg p$ is true. Answer 1

(b) Write the negation of q, then prove that either q is true or $\neg q$ is true. Answer 2

2. Consider the two similar statements

$$p \equiv \{\text{if } -1 < x < 2.5, \text{ then } x^2 + x - 6 < 0\}$$

and

$$p \equiv \{\text{if } -2.5 < x < 1, \text{ then } x^2 + x - 6 < 0\}.$$

(a) Write the negation of p, then prove that either p is true or $\neg p$ is true.

(b) Write the negation of q, then prove that either q is true or $\neg q$ is true.

3. Consider the statement

$$p \equiv \{\text{If } x > 2, \text{ then } x^2 - 3x + 2 > 0\}.$$

(a) Write the negation, $\neg p$, of this statement. Answer 3

(b) Prove p or $\neg p$. Answer 4

4. Consider the statement

$$p \equiv \{\text{If } -2 \leq x \leq 3, \text{ then } x^2 + x - 30 < 0\}.$$

(a) Write the negation, $\neg p$, of this statement.

(b) Prove p or $\neg p$.

5. Consider the statement

$$p \equiv \{\text{If } x > 2, \text{ then } x^2 - 2x - 3 > 0\}.$$

(a) Write the negation, $\neg p$, of this statement. Answer 5

(b) Prove p or $\neg p$. Answer 6

6. Consider the statement

$$p \equiv \left\{ \text{If } x > 1, \text{ then } \frac{2x-7}{x-1} < 1 \right\}.$$

   (a) Write the negation, $\neg p$, of this statement.

   (b) Prove p or $\neg p$.

7. Consider the statement

$$p \equiv \{\forall a > 0 \ \exists b > a \text{ s.t. } a + b > 3\}.$$

   (a) Write $\neg p$. Answer 7

   (b) Prove p or $\neg p$. Answer 8

8. Consider the statement

$$p \equiv \{\forall a > 0 \ \exists b > a \text{ s.t. } 2a + b > 1\}.$$

   (a) Write $\neg p$.

   (b) Prove p or $\neg p$.

9. Consider the two similar statements

$$p \equiv \left\{ \exists a > 0 \ \text{ s.t. } \forall b > a, \frac{3b+2}{b+5} > 4 \right\}$$

   and

$$q \equiv \left\{ \exists a > 0 \ \text{ s.t. } \forall b > a, \frac{3b+2}{b+5} > 2 \right\}.$$

   (a) Write $\neg p$. Answer 9

   (b) Prove p or $\neg p$. Answer 10

   (c) Write $\neg q$. Answer 11

   (d) Prove q or $\neg q$. Answer 12

10. Consider the two similar statements

$$p \equiv \left\{ \exists a < 0 \ \text{ s.t. } \forall b < a, \frac{3b-8}{b-1} < 5 \right\}$$

   and

$$q \equiv \left\{ \exists a < 0 \ \text{ s.t. } \forall b < a, \frac{3b-8}{b-1} < 2 \right\}.$$

(a) Write $\neg p$.

(b) Prove p or $\neg p$.

(c) Write $\neg q$.

(d) Prove q or $\neg q$.

11. Consider the two similar statements

$$p \equiv \{\forall a \in \mathbb{R}\ \exists b \in \mathbb{R} \text{ s.t. } \forall x > b,\ 2x - 7 > a\}$$

and

$$q \equiv \{\forall a \in \mathbb{R}\ \exists b \in \mathbb{R} \text{ s.t. } \forall x > b,\ 7 - 2x > a\}.$$

(a) Write $\neg p$. Answer 13

(b) Prove p or $\neg p$. Answer 14

(c) Write $\neg q$. Answer 15

(d) Prove q or $\neg q$. Answer 16

12. Consider the two similar statements

$$p \equiv \left\{\forall a \in \mathbb{R}\ \exists b \in \mathbb{R} \text{ s.t. } \forall x > b,\ x^2 + 1 > a\right\}$$

and

$$q \equiv \left\{\forall a > 0\ \exists b > a \text{ s.t. } \forall x > b,\ \frac{3x+8}{x+1} > a\right\}.$$

(a) Write $\neg p$.

(b) Prove p or $\neg p$.

(c) Write $\neg q$.

(d) Prove q or $\neg q$.

13. Let P_2 be the set of all quadratic polynomials with integer coefficients, leading coefficient of 1, and two distinct integer roots. For example $p(x) = x^2 - 4$ is in P_2. Consider the statement

$$q \equiv \{\forall n \in S \subseteq \mathbb{Z},\ \exists p \in P_2 \text{ and } m \in \mathbb{Z} \text{ s.t. } p(m) = n\}.$$

(a) What is $\neg q$, the negation of q? Answer 17

(b) What is the largest set $S = S_0 \subseteq \mathbb{Z}$ for which q is true? Answer 18

(c) Prove q is true for $S = S_0$. Answer 19

(d) Prove that for any $S \subseteq \mathbb{Z}$ larger than S_0 that $\neg q$ is true. Answer 20

(e) Let P be the set of all polynomials with integer coefficients and at least two distinct integer roots. What is the largest set S for which q' is true where

$$q' \equiv \{\forall n \in S \subseteq \mathbb{Z}, \exists p \in P \text{ and } m \in \mathbb{Z} \text{ s.t. } p(m) = n\}?$$

Answer 21

14. Let P_3 be the set of all cubic polynomials with integer coefficients, leading coefficient of 1, and three distinct integer roots. For example $p(x) = x^3 - 4x$ is in P_3. Consider the statement

$$q \equiv \{\forall n \in S \subseteq \mathbb{Z}, \exists p \in P_3 \text{ and } m \in \mathbb{Z} \text{ s.t. } p(m) = n\}.$$

(a) What is $\neg q$, the negation of q?

(b) What is the largest set $S = S_0 \subseteq \mathbb{Z}$ for which q is true?

(c) Prove q is true for $S = S_0$.

(d) Prove that for any $S \subseteq \mathbb{Z}$ larger than S_0 that $\neg q$ is true.

(e) Let P be the set of all polynomials with integer coefficients and at least three distinct integer roots. What is the largest set S for which q' is true where

$$q' \equiv \{\forall n \in S \subseteq \mathbb{Z}, \exists p \in P \text{ and } m \in \mathbb{Z} \text{ s.t. } p(m) = n\}?$$

Let P_3 be the set of all cubic polynomials with integer coefficients, leading coefficient of 1, and three distinct integer roots. For example $p(x) = x^3 - 4x$ is in P_3. For what $n \in \mathbb{Z}$ is it true that there exist $p \in P_3$ and $m \in \mathbb{Z}$ such that

$$p(m) = n?$$

Note that this is a two part proof. For those n for which the statement is true, you have to prove it is true. For those n for which this statement is not true, you have to write the negation of the statement and prove it is true.

15. Consider the statement

$$p \equiv \{\forall\, x \in \mathbb{R},\ \exists y \in \mathbb{R} \text{ s.t. } P(x,y) \text{ is true}\}.$$

   (a) Write the statement $\neg p$. Answer 22

   (b) Give an equation $P(x,y)$ for which p is true. Explain. Answer 23

   (c) Give an equation $P(x,y)$ for which $\neg p$ is true. Explain. Answer 24

16. Consider the statement

$$r \equiv \{\forall a \in S,\ \exists b \in T \text{ such that } p(x) \Rightarrow q(x)\},$$

where

$$p(x) \equiv \{x > b\}, \quad q(x) \equiv \{0 < \frac{1}{x} < 1\}.$$

In each of the following, is statement r true or false given the sets S and T. Explain.

   (a) $S = \{-0.6, 0, 1\}$, $T = \{-a, a+1, 2a\}$,

   (b) $S = \{-1, 0, 0.5\}$, $T = \{-a, a+1, 2a\}$.

   (c) Write the statement $\neg r$.

17. Consider the statement

$$p \equiv \{\forall\, a \in \mathbb{R},\ \exists\, b \in \mathbb{R} \text{ s.t. if } x > a \text{ then } -4x+1 < b\}.$$

   (a) Write the negation of this statement. Answer 25

   (b) Prove p or $\neg p$. Answer 26

18. Consider the statement

$$\{\forall\, a > 0,\ \exists\, b \in \mathbb{R} \text{ s.t. if } x > a, \text{ then } x^2+1 > b\}.$$

   (a) Write the negation of this statement.

   (b) Prove p or $\neg p$.

19. Try to prove or disprove the statement

$$p \equiv \{\forall\, a \in \mathbb{R},\ \exists\, b \in \mathbb{R} \text{ s.t. if } x > b, \text{ then } -2x+7 > a\}.$$

   (a) Write the negation of this statement. Answer 27

(b) Prove p or $\neg p$. Answer 28

20. Try to prove or disprove the statement

$$p \equiv \{\forall a \in \mathbb{R}, \exists b \in \mathbb{R} \text{ s.t. if } x > a, \text{ then } -5x + 7 > b\}.$$

(a) Write the negation of this statement.

(b) Prove p or $\neg p$.

2.1.2 Answers to selected problems

1. **Problem 1 a:** We have

$$\neg p \equiv \{\text{there exists } x > 3 \text{ such that } x^2 - x - 2 \leq 0\}.$$

We prove p. Let x_0 represent a number greater 3. Then $x_0 - 2 > 0$ and $x_0 + 1 > 0$ so their product $(x_0 - 2)(x_0 + 1) = x_0^2 - x_0 - 2$ is greater than 0, which is what we need to prove.

2. **Problem 1 b:** We have

$$\neg q \equiv \{\text{there exists } x > 3 \text{ such that } x^2 - 3x - 4 \leq 0\}.$$

We prove $\neg q$. Set $x_0 = 3.5$. Then $x^2 - 3x - 4 = -2.25 \leq 0$.

3. **Problem 3 a**: $\neg p \equiv \{\text{There exists } x > 2 \text{ such that } x^2 - 3x + 2 \leq 0\}$

4. **Problem 3 b**: We prove p. Let $x_0 > 2$. Then $x_0 - 2 > 0$ and $x_0 - 1 > 0$. Thus,

$$(x_0 - 2)(x_0 - 1) = x_0^2 - 3x_0 + 2 > 0,$$

being the product of two positive numbers.

5. **Problem 5 a**: $\neg p \equiv \{\text{There exists } x > 2 \text{ such that } x^2 - 2x - 3 \leq 0\}$

6. **Problem 5 b**: We prove $\neg p$. Set $x_0 = 2.5$. Then $2.5^2 - 2(2.5) - 3 = -1.75 \leq 0$.

7. **Problem 10 a**:

$$\neg p \equiv \{\exists a > 0, \text{ s.t. } \forall b > a, \ a + b \leq 3\}.$$

8. **Problem 10 a**: We prove p. Let a_0 represent a real number. Set $b_0 = |a_0| + 4$. Then

$$a_0 + b_0 = a_0 + |a_0| + 4 \geq 4,$$

since $a_0 + |a_0| \geq 0$. This means

$$a_0 + b_0 = a_0 + |a_0| + 4 \geq 4 > 3,$$

which is what we were to prove.

9. **Problem 9 a**:

$$\neg p \equiv \left\{ \forall a > 0,\ \exists b > a \text{ s.t. } \frac{3b+2}{b+5} \leq 4 \right\}.$$

10. **Problem 9 b**: We prove $\neg p$. Let a_0 represent an arbitrary number greater than 0. Set $b_0 = a_0 + 1$. Then $-18 \leq b_0$. Adding $3b_0 + 20$ to both sides gives $3b_0 + 2 \leq 4b_0 + 20 = 4(b_0 + 5)$. Since $b_0 > 0$, we can divide both sides by $b_0 + 5$ without reversing the inequality sign, giving

$$\frac{3b_0+2}{b_0+5} \leq 4,$$

which is what we needed to prove.

11. **Problem 9 c**:

$$\neg q \equiv \left\{ \forall a > 0,\ \exists b > a \text{ s.t. } \frac{3b+2}{b+5} \leq 2 \right\}.$$

12. **Problem 9 d**: We prove q. Set $a_0 = 8$ and let b_0 represent an arbitrary number greater than a_0. From backwards work, since $b_0 > 8$, adding $2b_0 + 2$ to both sides gives

$$3b_0 + 2 > 2b_0 + 10.$$

Since $b_0 > 0$, then $2b_0 + 5 > 0$ and we can divide both sides by $2b_0 + 5$ without reversing the inequality, giving our result,

$$\frac{3b+2}{b+5} > 2.$$

13. **Problem 11 a**:

$$\neg p \equiv \{\exists a \in \mathbb{R},\ \text{ s.t. } \forall b \in \mathbb{R}\ \exists x > b \text{ s.t. } 2x - 7 \leq a\}.$$

14. **Problem 11 b**: We prove p. Let a_0 represent an arbitrary number. Set
$$b_0 = \frac{a_0 + 7}{2}$$
and let x_0 represent a number greater than b_0. Then
$$x_0 > \frac{a_0 + 7}{2},$$
which simplifies to $2x_0 - 7 > a_0$ which is what we are to prove.

15. **Problem 11 c**:
$$\neg q \equiv \{\exists a \in \mathbb{R} \text{ s.t. } \forall b \in \mathbb{R}\ \exists x > b \text{ s.t. } 7 - 2x \leq a\}.$$

16. **Problem 11 d**: We prove $\neg q$. Since $7 - 2x$ goes to negative infinity as x goes to infinity, we can get less than any number we pick for a_0, so we set $a_0 = 1$. It has to hold for every b so we let b_0 be an arbitrary number. Doing backwards work on $7 - 2x \leq a$ gives
$$x \geq -0.5a + 3.5.$$
This means we need $x \geq -0.5a_0 + 3.5 = 3$, but we also need $x > b_0$. One simple solution is to set
$$x_0 = 3 + |b_0| > b_0.$$
This also means $x_0 \geq 3$ so $7 - 2x_0 \leq 1 = a_0$, which is what we need to show.

17. **Problem 13 a**:
$$\neg q \equiv \{\exists n \in S \subseteq \mathbb{Z} \text{ s.t. } \forall\, p \in P_2 \text{ and } \forall\, m \in \mathbb{Z},\ p(m) \neq n\}.$$

18. **Problem 13 b**: $S_0 = \mathbb{Z} - \{1\}$.

19. **Problem 13 c**: We prove that for all $n \in S_0 = \mathbb{Z} - \{1\}$, there exist $p \in P_2$ and $m \in \mathbb{Z}$ such that $p(m) = n$. Let n_0 be an integer such that $n_0 \neq 1$. Set
$$p_0(x) = x^2 + (n_0 - 1)x = x(x - 1 + n_0).$$
The leading coefficient of this polynomial is 1, the other coefficient, $n_0 - 1$, is an integer, and the two roots are $x = 0$ and $x = 1 - n_0$. Since $n_0 \neq 1$, the two roots are distinct. Set $m_0 = 1$. Then
$$p_0(m_0) = p_0(1) = 1(1 - 1 + n_0) = n_0,$$
so the statement is true.

20. **Problem 13 d**: The only subset of the integers that is larger than S_0 is the set of intgers, so for this part, we set $S_1 = \mathbb{Z}$. We prove the negation of q,

$$\neg q \equiv \{\exists n \in S_1 \subseteq \mathbb{Z} \text{ s.t. } \forall\, p \in P_2 \text{ and } \forall\, m \in \mathbb{Z},\ p(m) \neq n\}.$$

Set $n_0 = 1 \in S_1$, so it exists. We show that for all $p \in P_2$ and for all $m \in \mathbb{Z}$, $p(m) \neq 1$. Let p_0 be an element of P_2 and let m_0 be an integer. Since $p_0 \in P_2$, it has 2 distinct integer roots, say j_1 and j_2 where $j_1 \neq j_2$. Since the leading coefficient is 1, p_0 can be factored as

$$p_0(x) = (x - j_1)(x - j_2).$$

We then have that

$$p_0(m_0) = (m_0 - j_1)(m_0 - j_2).$$

We know $m_0 - j_1$ and $m_0 - j_2$ are distinct integers since j_1 and j_2 are distinct integers. For the product of these two integers to equal 1, they would both have to equal 1 or both would have to equal -1. But this contradicts the fact that the integers are distinct. Therefore

$$p_0(m_0) = (m_0 - j_1)(m_0 - j_2) \neq 1.$$

21. **Problem 13 e**: We show

$$q' \equiv \{\forall n \in S \subseteq \mathbb{Z}, \exists p \in P \text{ and } m \in \mathbb{Z} \text{ s.t. } p(m) = n\}$$

is true for $S = \mathbb{Z}$. From answer 19, we know q' is true for all integers n other than 1 since $P_2 \subset P$. Therefore, to show this statement is true, we only need to find a $p \in P$ and an $m \in \mathbb{Z}$ such that

$$p(m) = 1.$$

Set

$$p_1(x) = -x^2 + 1$$

which has the two roots, 1 and -1, so $p_1 \in P$. Set $m_0 = 0$. Then $p_1(0) = -0 + 1 = 1$ and we are finished.

22. **Problem 15 a**: The negation is

$$\neg p \equiv \{\exists\, x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, \neg P(x, y) \text{ is true}\}.$$

23. **Problem 15 b**: Let $P(x,y)$ be the statement that $2x+y=5$. Let x_0 be an arbitrary real number. Set $y_0 = 5-2x_0$. Then $2x_0+y_0=5$ so p is true.

24. **Problem 15 c**: Let $P(x,y)$ be the statement that $(x-1)(y+2)=5$. Then $\neg P(x,y)$ is the statement that $(x-1)(y+2)\neq 5$. Set $x_0=1$ and let y_0 be an arbitrary number. Then $\neg p$ is true.

25. **Problem 17 a:** We have

$$\neg p \equiv \{\exists\, a \in \mathbb{R} \text{ s.t.} \forall\, b \in \mathbb{R}, \exists x > a \text{ s.t. } -4x+1 \geq b\}.$$

26. **Problem 17 b:** We prove p. Let a_0 be an arbitrary real number. Set $b_0 = -4a_0+1$. Let x_0 be an arbitrary value greater than a_0. Since $x_0 > a_0$, multiplying both sides by -4 gives $-4x_0 < -4a_4$. Adding 1 to both sides gives

$$-4x_0+1 < -4a_0+1 = b_0,$$

which is what we wanted to prove.

27. **Problem 19 a:** We have

$$\neg p \equiv \{\exists\, a \in \mathbb{R} \text{ s.t.} \forall\, b \in \mathbb{R}, \exists x > b \text{ s.t. } -2x+7 \leq a\}.$$

28. **Problem 19 b:** From the graph, p is clearly false, so we will prove $\neg p$. Set $a=1$. Let b_0 be an arbitrary real number. (From the graph, we know that if $x\geq 3$, then $y=-2x+7\leq 1$, so we need $x_0\geq 3$. To do this, we set

$$x_0 = 3+|b_0|.$$

This means $x_0 > b_0$, which we needed to have. Since $x_0 \geq 3$, $-2x_0 \leq -6$ and $-2x_0+7\leq 1$ which we needed to show.

2.2 Conjunction and Disjunction

Video Lesson 2.10

This section will be easier to read if you first watch the 5:05 minute video
https://vimeo.com/498727792 (Password:Proof)
in which we prove a statement of the form $p(x)$ implies $q(x)$ or $r(x)$.
Also watch the 3:51 minute video
https://vimeo.com/498747664 (Password:Proof)
in which a conjunction is proven false. To change the speed at which the video plays, click on the gear at the lower right of the video.

In this section, we deal with compound statements, that is, statements involving 'and' and 'or.'

Definition 2.11: conjunction

The **conjunction** of p and q is the statement that is true precisely when statements p and q are both true and is denoted

$$r \equiv \{p \wedge q\}.$$

Consider the statement $r \equiv \{2 \text{ is even and prime}\}$. Then statement r is the conjunction of p and q,

$$r \equiv p \wedge q,$$

where $p \equiv \{2 \text{ is even}\}$ and $q \equiv \{2 \text{ is prime}\}$. Statement r is true since both p and q are true.

Definition 2.12: disjunction

The **disjunction** of p and q is the statement that is true when statement p is true or statement q is true or both statements p and q are true, and is denoted

$$s \equiv \{p \vee q\}.$$

Let $t \equiv \{6 \text{ is even}\}$, $u \equiv \{6 \text{ is prime}\}$, and $v \equiv \{6 \text{ is divisible by } 3\}$. Then

$$s_1 \equiv t \wedge u \equiv \{6 \text{ is even and prime}\}$$

is false since u is false,

$$s_2 \equiv t \vee u \equiv \{6 \text{ is even or prime}\}$$

is true since t is true, and

$$s_3 \equiv t \vee v \equiv \{6 \text{ is even or divisible by } 3\}$$

is true since t is true, but is also true since v is true. In everyday conversation, we usually use 'or' to mean one thing or the other, but not both, but in mathematics, 'or' includes the possibility that both are true.

Often our statements involve predicate functions. For example, let

$$p(x) \equiv \{x > 2\} \text{ and } q(x) \equiv \{x^2 > 16\}.$$

Let $r(x) = p(x) \vee q(x)$. In this case, $r(3)$ is true because $p(3)$ is true since $3 > 2$, $r(-5)$ is true because $q(-5)$ is true, $(-5)^2 > 16$, and $r(5)$ is true because both $p(5)$ and $q(5)$ are true. On the other hand, $r(1)$ is false because $1 < 2$ and $(1)^2 < 16$. In this case, the set of all values for which $r(x)$ is true is

$$S = \{x : x > 2\} \cup \{x : x < -4\}.$$

Consider the statement

$$r(x) \equiv \{p(x) \wedge q(x)\}.$$

The set of values for which r is true is all x-values for which $p(x)$ is true and $q(x)$ is true. Suppose there exists an x-value for which $\neg r(x)$ is true. Then, for this x, either $p(x)$ is not true or $q(x)$ is not true. Thus, $\neg r(x)$ is true for all x for which $\neg p(x) \vee \neg q(x)$, that is

$$\neg\{p(x) \wedge q(x)\} \equiv \{\neg p(x) \vee \neg q(x)\}. \tag{2.7}$$

When the statements are predicate functions, we can mentally picture conjunction, disjunction, and negation using Venn diagrams. In Figure 2.2 the points inside the p circle represent 'values' under which the predicate function p is true and points inside the q circle represent 'values' under which q is true. The shaded region in Figure 2.2 a) represents 'values' under which $p \wedge q$ is true and the shaded region in Figure 2.2 b) represents 'values' under which $\neg p \vee \neg q$ is true. From these figures, we have the conceptual insight into why relation 2.7 is true.

Similarly, consider the statement

$$s(x) \equiv \{p(x) \vee q(x)\}.$$

The set of values for which s is true is all x-values for which $p(x)$ is true or $q(x)$ is true. Suppose there exists an x-value for which $\neg s(x)$ is true. Then, for this x, $p(x)$ is not true and $q(x)$ is not true. Thus, $\neg s(x)$ is true for all x for which $\neg p(x) \wedge \neg q(x)$, that is

$$\neg\{p(x) \vee q(x)\} \equiv \{\neg p(x) \wedge \neg q(x)\}. \tag{2.8}$$

In Figure 2.3 a) the shaded points represent $p \vee q$ while the shaded points in Figure 2.3 b) represent $\neg p \wedge \neg q$. This again, gives us conceptual insight into why relation 2.8 is true.

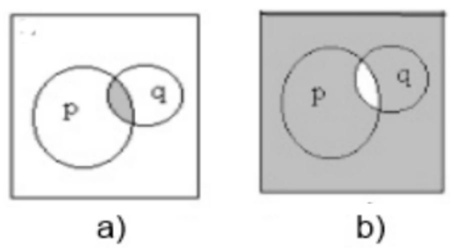

Figure 2.2: Venn diagram in which figure a) represents $p(x) \wedge q(x)$ while figure b) represents $\neg p(x) \vee \neg q(x)$.

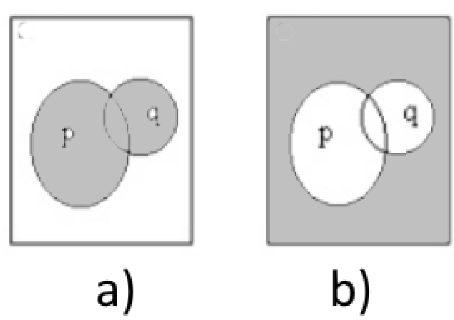

Figure 2.3: Venn diagram in which figure a) represents $p(x) \vee q(x)$ while figure b) represents $\neg p(x) \wedge \neg q(x)$.

Let's combine conjunction and disjunction with the use of quantifiers. Consider the statement

$$p \equiv \{\langle \forall\, n \in \mathbb{Z},\ \rangle\ \langle \exists m \in \mathbb{Z} \rangle \text{ such that } \langle 3n + 2m = 1 \text{ or } 3n + 2m = 2 \rangle\}. \tag{2.9}$$

Conceptual Insight: 2.13

We begin by constructing some examples to get conceptual insight. Since it says 'for all' n, we will pick some n-values. Let $n = 1$. We need to show there exists an integer m such that $3(1) + 2m = 1$ or $3(1) + 2m = 2$. We can solve the first equation, giving $m = -1$ which is an integer. The second equation gives $m = -1/2$ which is not an integer. But the statement is true for this value of n since there did exist an integer m such that $3n + 2m = 1$.
Let $n = 2$. We need to show there exists an integer m such that $3(2) + 2m = 1$ or $3(2) + 2m = 2$. We can solve the first equation, giving $m = -5/2$ which is not an integer, but solving the second equation gives $m = -2$ which is an integer. So the statement is true for this value of n since there did exist an integer m such that $3n + 2m = 2$.
If we continue, we see that when n is odd, we can find an integer m that satisfies the first equation and when n is even, we can find an integer m that satisfies the second equation.
Assumed: Assume $n_0 = 2k_0 + 1$ or $n_0 = 2k_0$.
To Show: Find m_0 such that $3n_0 + 2m_0 = 1$ (odd n_0) or $3n_0 + 2m_0 = 2$ (even n_0).

We now construct the actual proof.

Claim 2.14: $p \equiv \{\forall\, n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ **s.t.** $3n + 2m = 1$ **or** $3n + 2m = 2\}$.

Proof: Let n_0 represent an arbitrary integer.
Case 1: Suppose n_0 is odd. Then there exists an integer k_0 such that $n_0 = 2k_0 + 1$. Set $m_0 = -3k_0 - 1$. Then

$$3n_0 + 2m_0 = 3(2k_0 + 1) + 2(-3k_0 - 1) = 6k_0 + 3 - 6k_0 - 2 = 1,$$

so for this n-value, the statement is true, that is, there exists an m such that $3n + 2m = 1$.

Case 2: Suppose n_0 is even. Then there exists an integer k_0 such that $n_0 = 2k_0$. Set $m_0 = -3k_0 + 1$. Then

$$3n_0 + 2m_0 = 3(2k_0) + 2(-3k_0 + 1) = 6k_0 - 6k_0 + 2 = 2,$$

so for this n-value, the statement is true, that is, there exists an m such that $3n + 2m = 2$.
Thus, for any integer n, there exists an integer m such that either $3n + 2m = 1$ or $3n + 2m = 2$, and the statement is true. □

We now check our proof using two values for the let-variable, one for each case.

Proof. Verification of Proof, Case 1, n_0 odd: Let $n_0 = 5$, an odd integer. Then there exists an integer $k_0 = 2$ such that $n_0 = 2k_0 + 1$. Set $m_0 = -3k_0 - 1 = -3(2) - 1 = -7$. Then

$$3n_0 + 2m_0 = 3(5) + 2(-7) = 15 - 14 = 1.$$

So for this n-value, the statement is true, that is, there exists an m such that $3n + 2m = 1$. □

Proof. Verification of Proof, Case 2, n_0 even: Suppose $n_0 = 10$ which is even. Then there exists an integer $k_0 = 5$ such that $n_0 = 2k_0$. Set $m_0 = -3k_0 + 1 = -3(5) + 1 = -14$. Then

$$3n_0 + 2m_0 = 3(10) + 2(-14) = 30 - 28 = 2.$$

So for this n-value, the statement is true, that is, there exists an m such that $3n + 2m = 2$. □

Again, the purpose of verifying in these cases is to demonstrate how the let-variables are placeholders for any value in the set being considered in a 'for every' statement.

Let's now consider a statement that is similar to Statement 2.9, that is,

$$p \equiv \{\langle \forall\, n \in \mathbb{Z}\, \rangle\, \langle \exists m, k \in \mathbb{Z} \rangle \text{ such that } \langle 3n + 2m = 1 \text{ and } 3n + 2k = 2 \rangle\}. \tag{2.10}$$

Conceptual Insight: 2.15

Let's think about this statement. Suppose n is even. Then $3n+2m$ is even for every integer m, so no matter what m equals, $3n+2m$ cannot equal the odd integer 1. On the other hand, if n is odd, then $3n+2k$ is odd for every integer k, so can never equal 2. Therefore this statement does not appear to be true.

We believe $\neg p$ is true, so we need to write what it means. Using the previous process, the negation of this statement is then

$$\neg p \equiv \{\langle \exists\, n \in \mathbb{Z} \text{ s.t.}\rangle \, \langle \forall m, k \in \mathbb{Z}\rangle, \, \neg\langle 3n+2m = 1 \text{ and } 3n+2k = 2\rangle\}.$$

This last statement is of the form $\neg(A \wedge B)$ which by DeMorgan's rule 2.7 is equivalent to

$$\neg A \vee \neg B \equiv \{3n+2m \neq 1 \vee 3n+2k \neq 2\}.$$

Therefore, we have

$$\neg p \equiv \{\langle \exists\, n \in \mathbb{Z} \text{ s.t.}\rangle \, \langle \forall m, k \in \mathbb{Z}, \rangle \, \langle 3n+2m \neq 1 \text{ or } 3n+2k \neq 2\rangle\}.$$

If we begin with some examples, we could let $n = 1$. Are there integers m and k such that

$$3(1)+2m = 1 \text{ and } 3(1)+2k = 2?$$

We see that k cannot exist as an integer since $k = -1/2$, so statement $\neg p$ is true and statement p is false.

Claim 2.16: $\exists\, n \in \mathbb{Z}$ **s.t.** $\forall m, k \in \mathbb{Z}$, $3n+2m \neq 1$ **or** $3n+2k \neq 2\rangle\}$.

Proof: Set $n_0 = 1$. Let m_0 and k_0 represent arbitrary integers. Then

$$3n_0 + 2k_0 = 3 + 2k_0 = 2(k_0+1)+1$$

is odd and so cannot equal 2, and $\neg p$ is true. (Note that it did not matter if $3n_0 + 2m_0 = 1$ or not since we only had to show one of the statements held.) $\square$

We note that we could also prove $\neg p$ using contradiction, that is, we assume p is true, that is

$$p \equiv \{\langle \forall\, n \in \mathbb{Z}\,\rangle \, \langle \exists m, k \in \mathbb{Z}\rangle \text{ such that } \langle 3n+2m = 1 \text{ and } 3n+2k = 2\rangle\}.$$

Since we have assumed it is true for every integer n, it is true for $n_0 = 1$, that is, there are integers m_0 and k_0 such that

$$3(1) + 2m_0 = 1 \text{ and } 3(1) + 2k_0 = 2.$$

The second equation implies that $k_0 = -1/2 \notin \mathbb{Z}$ which is a contradiction.

2.2.1 Problems

1. Let $p(x) \equiv \{x > 3\}$ and let $q(x) \equiv \{x < 8\}$. For what values of x are each of the following statements true?
   (a) $p(x) \wedge q(x)$ Answer 1
   (b) $p(x) \vee q(x)$ Answer 2
   (c) $\neg p(x) \wedge q(x)$ Answer 3
   (d) $\neg(p(x) \vee q(x))$ Answer 4
2. Let $p \equiv \{n \text{ is an even integer}\}$ and let $q \equiv \{n \text{ is divible by } 3\}$.
   (a) For what values of n is $p \wedge q$ true?
   (b) For what values of n is $p \vee q$ true?
   (c) Verify property 2.7 by finding the set of n values for which $\neg(p \wedge q)$ is true and verifying it is the same set of values for which $\neg p \vee \neg q$ is true.
   (d) Verify property 2.8 by finding the set of n values for which $\neg(p \vee q)$ is true and verifying it is the same as the set of values for which $\neg p \wedge \neg q$ is true.
3. Consider the statement
   $$r \equiv \{\forall x \in \mathbb{R},\ p(x) \text{ is true or } q(x) \text{ is true}\}. \tag{2.11}$$
   (a) Give statements (equations or inequalities) $p(x)$ and $q(x)$ for which r is true. Explain. Answer 5
   (b) Write the statement $\neg r$. Answer 6
   (c) Give statements (equations or inequalities) $p(x)$ and $q(x)$ for which $\neg r$ is true. Explain. Answer 7
4. Consider the statement
   $$s \equiv \{\exists x \in \mathbb{R} \text{ s.t., } p(x) \text{ is true or } q(x) \text{ is true}\}. \tag{2.12}$$

(a) Give equations or inequalities $p(x)$ and $q(x)$ for which s is true. Explain.

(b) Give equations or inequalities $p(x)$ and $q(x)$ for which statement 2.12 is true but statement 2.11 is false. Explain.

5. Consider the statement

$$r \equiv \{\forall x \in \mathbb{R},\ p(x) \text{ is true or } \forall x \in \mathbb{R},\ q(x) \text{ is true}\}.$$

(a) Give equations or inequalities $p(x)$ and $q(x)$ for which r is true. Explain. Answer 8

(b) Write the statement $\neg r$. Answer 9

(c) Give equations or inequalities $p(x)$ and $q(x)$ for which $\neg r$ is true. Explain. Answer 10

6. Consider the statement

$$r \equiv \{\forall x \in \mathbb{R},\ p(x) \text{ is true and } q(x) \text{ is true}\}.$$

(a) Give equations or inequalities $p(x)$ and $q(x)$ for which r is true. Explain.

(b) Write the statement $\neg r$.

(c) Give equations or inequalities $p(x)$ and $q(x)$ for which $\neg r$ is true. Explain.

7. The number m is called the **absolute maximum** for the function f (with domain D) if and only if

$$\{\langle \exists a \in D \text{ s.t. } f(a) = m\rangle \text{ and } \langle \forall x \in D, \rangle\ \langle f(x) \leq m\rangle\}.$$

(a) Write the definition for m not being an absolute maximum for f. Answer 11

(b) Show that $m = 1$ is an absolute maximum for the function $f(x) = -x^2 + 1$. Answer 12

(c) Show that $m = 2$ is not an absolute maximum for the function $f(x) = -x^2 + 1$. Answer 13

(d) Show that $m = 0$ is not an absolute maximum for the function $f(x) = -x^2 + 1$. Answer 14

8. Consider the statement

$$\{\forall x \in \mathbb{R}, \text{ if } x^2 - 3x + 2 > 0, \text{ then } x > 2 \text{ or } x < 1\}.$$

   This statement is an implication of the form

$$\{\forall x \in \mathbb{R} \text{ s.t. } p(x) \text{ is true}, q(x) \text{ and } r(x) \text{ are true}\}.$$

   Write the negation of this statement. Then prove this statement or its negation.

9. A function f with domain D is even if and only if

$$\{\langle \forall x \in D, \rangle \langle -x \in D \rangle \wedge \langle f(x) = f(-x) \rangle\}.$$

   (a) Write what it means for f to not be even. Answer 15

   (b) Prove or disprove that the function $f(x) = \sqrt{1-x^2}$ with domain $D = [-1,1]$ is even. Answer 16

   (c) Prove or disprove that the function $f(x) = \sqrt{x}$ is even. Answer 17

   (d) Prove or disprove that the function $f(x) = x^3$ is even. Answer 18

10. Show the statement

$$p \equiv \{\langle \forall\, x \in \mathbb{R}, \rangle \langle \exists y, z \in \mathbb{R} \rangle \text{ such that } \langle 3x + 2y = 5 \text{ and } x - z = 4 \rangle\}$$

   is true.

11. Show the statement $p \equiv$

$$\{\forall\, n \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ s.t.}, 2n + 3m = 0 \text{ or } 2n + 3m = 1 \text{ or } 2n + 3m = 2\}$$

   is true. If you do some examples, you will see there are three cases. Answer 19

12. Write the negation of statement $p \equiv$

$$\{\forall\, n \in \mathbb{Z}, \exists m \in \mathbb{Z} \text{ s.t.}, 4n + 3m = 0 \text{ or } 4n + 3m = 1\}$$

   and prove p or $\neg p$ is true. Do some examples.

13. Consider the implication

$$s \equiv \{\text{For all } x \text{ s.t. } p(x) \wedge q(x), r(x)\} \equiv \{p(x) \wedge q(x) \Rightarrow r(x)\}.$$

(a) Draw a Venn diagram to indicate what it means for this statement to be true and write an explanation of your diagram. Answer 20

(b) Give an example of statements $p(x)$, $q(x)$, and $r(x)$ for which s is true. Answer 21

(c) Draw a Venn diagram indicating what it means for the implication s to be false. Write $\neg s$. Give an example of statements $p(x)$, $q(x)$ and $r(x)$ such that s is false. Answer 22

14. Consider the implication

$$s \equiv \{\text{For all } x \text{ s.t. } r(x), p(x) \wedge q(x)\} \equiv \{r(x) \Rightarrow p(x) \wedge q(x)\}.$$

(a) Draw a Venn diagram to indicate what it means for implication s to be true, and write an explanation of your diagram.

(b) Give an example of statements $p(x)$, $q(x)$, and $r(x)$ for which s is true.

(c) Draw a Venn diagram indicating what it means for the implication s to be false. Write $\neg s$. Give an example of statements $p(x)$, $q(x)$ and $r(x)$ such that s is false.

15. Consider the implication

$$s \equiv \{\text{For all } x \text{ s.t. } p(x) \vee q(x), r(x)\} \equiv \{p(x) \vee q(x) \Rightarrow r(x)\}.$$

(a) Draw a Venn diagram to indicate what it means for implication s to be true, and write an explanation of your diagram. Answer 23

(b) Give an example of statements $p(x)$, $q(x)$, and $r(x)$ for which s is true. Answer 24

(c) Draw a Venn diagram indicating what it means for the implication s to be false. Write $\neg s$. Give an example of statements $p(x)$, $q(x)$ and $r(x)$ such that s is false. Answer 25

16. Consider the implication

$$s \equiv \{\text{For all } x \text{ s.t. } r(x), p(x) \vee q(x)\} \equiv \{r(x) \Rightarrow p(x) \vee q(x)\}.$$

(a) Draw a Venn diagram to indicate what it means for implication s to be true, and write an explanation of your diagram.

(b) Give an example of statements $p(x)$, $q(x)$, and $r(x)$ for which s is true.

(c) Draw a Venn diagram indicating what it means for the implication s to be false. Write $\neg s$. Give an example of statements $p(x)$, $q(x)$ and $r(x)$ such that s is false.

2.2.2 Answers to selected problems

1. **Problem 1 a:** $p(x) \wedge q(x) \equiv \{3 < x < 8\}$;

2. **Problem 1 b:** $p(x) \vee q(x) \equiv \{x \in \mathbb{R}\}$;

3. **Problem 1 c:** $\neg p(x) \wedge q(x) \equiv \{x \leq 3\}$;

4. **Problem 1 d:** $\neg(p(x) \vee q(x)) \equiv \emptyset$

5. **Problem 3 a:** $p(x) \equiv \{x < 2\}$ and $q(x) \equiv \{x > 1\}$. One or the other or both are true for all x.

6. **Problem 3 b:**

$$\neg r \equiv \{\exists x \in \mathbb{R} \text{ s.t. } p(x) \text{ is false and } q(x) \text{ is false}\}.$$

7. **Problem 3 c:** $p(x) \equiv \{x < 2\}$ and $q(x) \equiv \{x > 3\}$. Then

$$\neg p(x) \wedge \neg q(x) \equiv \{x \geq 2 \wedge x \leq 3\}$$

and we need to show

$$\neg r \equiv \{\exists x \in \mathbb{R} \text{ s.t. } \{x \geq 2 \wedge x \leq 3\}\}.$$

One such x is $x_0 = 2.5$.

8. **Problem 5 a:** $p(x) \equiv \{x^2 + 1 > 0\}$ and $q(x) \equiv \{x > 1\}$. $p(x)$ is true for all x.

9. **Problem 5 b:**

$$\neg r \equiv \{\exists x \in \mathbb{R} \text{ s.t. } p(x) \text{ is false and } \exists\, x \in \mathbb{R} \text{ s.t. } q(x) \text{ is false}\}.$$

10. **Problem 5 c:** $p(x) \equiv \{x < 2\}$ and $q(x) \equiv \{x > 1\}$. $p(x)$ is false for $x = 2$ and $q(x)$ is false for $x = 1$.

11. **Problem 7 a:** The number m is not an absolute maximum for the function f (with domain D) if and only if

$$p \equiv \{\langle \forall x \in D,\, f(x) \neq m\rangle \text{ or } \langle \exists x \in D \text{ s.t. } f(x) > m\rangle\}.$$

12. **Problem 7 b:** $f(0) = 1$ and if x_0 represents an arbitrary real number, $f(x_0) = -x_0^2 + 1 \leq 1$ since $-x_0^2 \leq 0$.
13. **Problem 7 c:** Let a_0 represent an arbitrary real number. Since $-a_0^2 + 1 \leq 1 < 2$, $f(a_0) \neq 2$.
14. **Problem 7 d:** Since $f(0) = 1 > 0$, then 0 is not an absolute maximum.
15. **Problem 9 a:** A function f with domain D is not even if and only if

$$\{\langle \exists x \in D \text{ s.t.}\rangle \langle \langle -x \notin D\rangle \vee \langle f(x) \neq f(-x)\rangle\rangle\}.$$

16. **Problem 9 b:** Let x_0 represent a real number in $D = [-1, 1]$. Since $-1 \leq x_0 \leq 1$, multiplying by -1 and reversing the inequality signs gives $1 \geq -x_0 \geq -1$ so $-x_0 \in D$. In addition, $f(-x_0) = \sqrt{1 - (-x_0)^2} = \sqrt{1 - x_0^2} = f(x_0)$ so f is even.
17. **Problem 9 c:** The domain for f is $D = [0, \infty)$. Since $x_0 = 1 \in D$ but $-x_0 = -1 \notin D$, f is not even
18. **Problem 9 d:** Since $f(1) = 1 \neq f(-1) = -1$, f is not even.
19. **Problem 11:** Case 1 is when $n_0 = 3k_0$ in which $m_0 = -2k_0$ and $2n_0 + 3m_0 = 0$. Case 2 is when $n_0 = 3k_0 + 1$ in which $m_0 = -2k_0$ and $2n_0 + 3m_0 = 2$. Case 3 is when $n_0 = 3k_0 + 2$ in which $m_0 = -2k_0 - 1$ and $2n_0 + 3m_0 = 1$.
20. **Problem 13 a:** See Figure 2.4.
21. **Problem 13 b:** Let $p \equiv \{x < 1\}$, $q \equiv \{x > -0.5\}$, and $r \equiv \{x^2 < 1\}$.
22. **Problem 13 c:** See Figure 2.4.

$$\neg s \equiv \{\exists x \in D \text{ s.t. } p(x) \cap q(x) \cap \neg r(x)\}.$$

    Let $p \equiv \{x < 2\}$, $q \equiv \{x > -0.5\}$, and $r \equiv \{x^2 < 1\}$.

23. **Problem 15 a:** See Figure 2.5.
24. **Problem 15 b:** Let $p \equiv \{x < -1\}$, $q \equiv \{x > 1\}$, and $r \equiv \{1/x^2 < 1\}$.

Figure 2.4: Left figure is $s \equiv \{p \wedge q \Rightarrow r\}$ since $p \cap q \subseteq r$. Right figure is $\neg s$ since $p \cap q \not\subseteq r$.

25. **Problem 15 c:** See Figure 2.5.

$$\neg s \equiv \{\exists x \in D \text{ s.t. } (p(x) \cup q(x)) \cap \neg r(x)\}.$$

Let $p \equiv \{x < -0.5\}$, $q \equiv \{x > 1\}$, and $r \equiv \{1/x^2 < 1\}$.

Figure 2.5: Left figure is $s \equiv \{p \vee q \Rightarrow r\}$ since $p \cup q \subseteq r$. Right figure is $\neg s$ since $p \cup q \not\subseteq r$.

2.3 Comparing Functions

Video Lesson 2.17

This section will be easier to read if you first watch the 6:49 minute video
https://vimeo.com/498772074 (Password:Proof)
in which we prove statements about properties of functions. To change the speed at which the video plays, click on the gear at the lower right of the video.

Most of our statements have involved real numbers and expressions, such as

$$\{\text{for every } x \text{ satisfying } P(x), Q(x) \text{ is true}\},$$

and its negation

$$\{\text{there exists } x \text{ satifying } P(x) \text{ such that } \neg Q(x) \text{ is true}\}.$$

We now consider similar statements that involve functions, such as

$$\{\text{for every } f \text{ satisfying } P(f), Q(f) \text{ is true}\}.$$

Note that its negation would be

$$\{\text{there exists } f \text{ satifying } P(f) \text{ such that } \neg Q(f) \text{ is true}\}.$$

Statements involving functions are worked very similarly to those involving variables. In some sense, the function is now treated as the variable. For example, suppose we have the statement

$$\begin{aligned} s &\equiv \{\text{if } P(f), \text{then } Q(f)\} \\ &\equiv \{\forall\, f \text{ s.t. } P(f), Q(f) \text{ is true}\}. \end{aligned}$$

Then its negation would be

$$\begin{aligned} \neg s &\equiv \neg\{\text{if } P(f) \text{ then } Q(f)\} \\ &\equiv \neg\{\forall\, f \text{ s.t. } P(f), Q(f) \text{ is true}\} \\ &\equiv \{\exists\, f \text{ s.t. } P(f) \text{ s.t. } \neg Q(f)\}. \end{aligned}$$

A better way to write this last version of $\neg s$ is

$$\neg s \equiv \{\exists\, f \text{ s.t. } P(f) \text{ and } \neg Q(f)\}. \tag{2.13}$$

This is just an extension of statement 2.5.

We are now going to consider functions which satisfy different ones of the following properties (also considered predicate functions)

$$\begin{aligned} P_1(f) &\equiv \{\forall M \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } f(x) > M\} && (2.14) \\ P_2(f) &\equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\} && (2.15) \\ P_3(f) &\equiv \{\forall (M,K) \in \mathbb{R}^2, \exists x > K \text{ s.t. } f(x) > M\}, && (2.16) \\ P_4(f) &\equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, f(x) > M\}, && (2.17) \\ P_5(f) &\equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall K \in \mathbb{R}, \exists x > K \text{ s.t. } f(x) > M\}, && (2.18) \\ P_6(f) &\equiv \{\exists (M,K) \in \mathbb{R}^2 \text{ s.t. } \forall x > K, f(x) > M\}. && (2.19) \end{aligned}$$

With $i \neq j$, in the exercises, we will explore to see which of the statements

$$\{\forall\, f \text{ satisfying } P_i, f \text{ satisfies } P_j\}$$

are true. To do this, we also need to write down the negation 2.13 of this statement

$$\{\exists\, f \text{ satisfying } P_i \text{ s.t. } f \text{ does not satisfy } P_j\},$$

which can be rewritten as

$$\{\exists\, f \text{ satisfying } P_i \text{ s.t. } f \text{ satisfies } \neg P_j\}.$$

So to answer our questions about what statements are true, we need to write the negation of each of our properties. We do this for P_1 and P_2 and leave the rest for the reader. The negation of P_1 and P_2 are as follows:

$$\neg P_1(f) \equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, f(x) \le M\}, \quad (2.20)$$
$$\neg P_2(f) \equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall K \in \mathbb{R}, \exists x > K \text{ s.t. } f(x) \le M\}. \quad (2.21)$$

We note that P_1 is just the definition of being unbounded above and $\neg P_1$ is the definition of being bounded above.

To get some conceptual insight into what functions satisfy each of these properties and which do not, we sketch several different types of functions, especially those that are unbounded in different ways. Then we try to determine which predicate functions each of these satisfies. If we find that a sketch satisfies two of these properties, we try to sketch another function that satisfies only one of them. As an example, we sketch the three functions in Figure 2.6. You should try to determine which of the properties each of these functions satisfies. You should also try to come up with an algebraic formula for a function that behaves like each of the graphs in Figure 2.6.

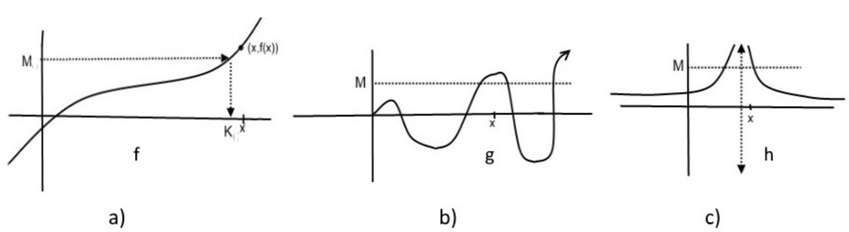

Figure 2.6: In figure a), f goes to infinity to right; in figure b), g oscillates to infinity; and in figure c), h has a vertical asymptote. A typical M is displayed for each.

Conceptual Insight: 2.18: Comparing P_1 and P_2

To help in this exploration, we will consider P_1 and P_2. Figure 2.6 a) shows a function f which satisfies both P_1 and P_2. To show f satisfies P_1, we pick an arbitrary M as shown in the graph. We go backwards from the y-axis to the x-axis to find K. To satisfy P_1, we have to find one x for which $f(x) > M$. Since this works not only for one $x > K$, it works for every $x > K$ so $P_1(f)$ is true for this f. Similarly, for P_2, the inequality must hold for all x larger than some value K, which it clearly does for f, so $P_2(f)$ is true. It would seem that P_2 is more restrictive than P_1 since the inequality $f(x) > M$ must hold, not just for one $x > K$, but for every $x > K$. This means

$$\{\forall f \text{ satisfying } P_2, f \text{ satisfies } P_1\}$$

is apparently true.
Figure 2.6 c) shows a function h which satisfies P_1 but not P_2. Again, for the given M in the figure, we can pick a $y_0 > M$ and go backwards from the y-axis to the x-axis to find an x_0 (actually there would be two possible values for x_0). In this case, $h(x_0) = y_0 > M$, so $P_1(h)$ is true. On the other hand, if we set $M = 1$, then no matter what value we pick for K, there will exist $x > K$ for which $h(x) \leq M = 1$ since there is a horizontal asymptote at $y = 0$: the inequality $h(x) > M = 1$ cannot hold for all $x > K$ no matter what value we pick for K. Thus, $\neg P_2(h)$ is true. This means

$$\{\exists f \text{ a function satisfying } P_1 \text{ which also satisfies } \neg P_2\}$$

is apparently true.

You should think about why

- function f in Figure 2.6 also satisfies properties P_3, P_5 and P_6
- function g in Figure 2.6 satisfies properties P_1, P_3 and P_5
- function h in Figure 2.6 also satisfies properties P_4, P_5 and P_6

Note that f, g and h all satisfy P_1 and P_5. See if you can find a function that satisfies one of these properties but not the other.

Let's see how we prove

$$\{\forall\, f \text{ satisfying } P_2,\, f \text{ satisfies } P_1\}.$$

We begin with the let method by assuming f_0 satisfies P_2 and try to show f_0 satisfies P_1.

- **Assume:** For all $M \in \mathbb{R}$, there exists $K \in \mathbb{R}$ such that $\forall x > K$, $f_0(x) > M$.
- **To Show:** For all $L \in \mathbb{R}$, there exists $t \in \mathbb{R}$ such that $f_0(t) > L$.

We note that the variables M, K, and x in each statement are independent of the variables for another statement. They are just placeholders. Thus, when writing the Assumed and To Show, we use different letters so as not to confuse ourselves.

Claim 2.19: Statement P_2 implies statement P_1

Proof: We are proving that if f satisfies

$$P_2(f) \equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\}$$

then f satisfies

$$P_1(f) \equiv \{\forall M \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } f(x) > M\}.$$

Let f_0 be an arbitrary function that satisfies P_2. To show f_0 satisfies P_1, we let L_0 represent an arbitrary number in $\mathbb{R}$ and must find a t such that $f_0(t) > L_0$. Since f_0 satisfies P_2, we know something is true for all M, so it is true for $M = M_0 = L_0$. Therefore, we set

$$M_0 = L_0.$$

We now know there exists K_0 such that for all $x > K_0$, $f(x) > M_0$. Therefore, we set $x_0 = K_0 + 1 > K_0$ and know that$f(x_0) > M_0 = L_0$. So if we set $t_0 = x_0$, then for L_0, there exists a $t_0 = x_0 = K_0 + 1$ such that $f(t_0) > L_0$. So, f_0 satisfies P_1. $\square$

We now show that P_1 does not imply P_2, that is, we prove

$$\{\exists\, f \text{ satisfying } P_1 \text{ s.t. } f \text{ satisfies } \neg P_2\}.$$

To construct this proof, we first find the right function f_0. We then prove f_0 satisfies 2.14, the conditions for P_1. Finally, we prove f_0 satisfies 2.21, the conditions for $\neg P_2$. In our discussion above, it appears that the function h whose graph is seen in Figure 2.6 c) will work. To prove that, we only need to come up with a formula for a function whose graph behaves like the graph in Figure 2.6 c) and show this function satisfies P_1 and $\neg P_2$. The graph gave us our conceptual insight, the formula is our technical handle to construct the proof.

Claim 2.20: There exists a function f that satisfies P_1 but which does not satisfy P_2.

Proof: We recall that

$$P_1 \equiv \{\forall M \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } f(x) > M\}$$

and

$$P_2(f) \equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\}.$$

Set

$$f_0(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Part 1: We first show that $P_1(f_0)$ is true. Let M_0 represent an arbitrary real number.

Case 1: Suppose $M_0 \leq 0$. Set $x_0 = 1$ so $f_0(1) = 1 > M_0$.

Case 2: Suppose $M_0 > 0$. Set

$$x_0 = \frac{1}{2M_0} \neq 0.$$

Then $f_0(x_0) = 2M_0 > M_0$, so $P_1(f_0)$ is true.

Part 2: To show $\neg P_2(f_0)$ is true, we need to show there exists an M. So set $M_0 = 1$. We now show something for all K, so let K_0 represent an arbitrary real number. We now have to find an $x > K_0$ such that $f_0(x) \leq 1$.

Conceptual Insight: 2.21

For f_0, any $x > 1$ will result in $f(x) < 1$. We have to make sure the x we use is also greater than K_0 but we don't know what K_0 is. We solve our problem using absolute values.

Claim 2.22: Proof of Claim 2.20 continued.

Proof continued: Set
$$x_0 = |K_0| + 2 > K_0.$$

Then
$$f(x_0) = f(|K_0| + 2) = \frac{1}{|K_0| + 2} \leq \frac{1}{2} \leq 1.$$

Therefore $\neg P_2(f_0)$ is true.

We have now shown there exists a function f_0 that satisfies P_1 and $\neg P_2$, so our statement is true. □

In the previous proof, we used the same letters for both parts, M, K, and x whereas we used different letters in the previous proof. In the first proof, we were considering both statements P_1 and P_2 at the same time, so needed to avoid confusion by changing the letters for one of the statements. For the second proof, we had two independent parts. In Part 1, we only considered P_1 and in Part 2, we only considered $\neg P_2$. Since we were only considering one of the statements at a time, there was no risk of confusion by using the same letters.

Whenever we have a new property, one of the things we do is determine how the property extends, that is, if f and g satisfy the property, what about $f + g$? Therefore, we will be considering statements similar to

$$p \equiv \{\text{if } f \text{ and } g \text{ are bounded above, then } f + g \text{ is bounded above}\}.$$

Since we recall that 'if' really means 'for every', we should rewrite this statement in the more usable form

$$p \equiv \{\forall \text{ functions } f \text{ and } g \text{ that are bounded above, } f + g \text{ is bounded above}\}.$$

As with statements involving variables, we also need to write the negation of such statements

$$\neg p \equiv \{\exists \text{ bounded above functions } f \text{ and } g \text{ s.t. } f + g \text{ is not bounded above}\}$$

and then, through examples and graphs, develop the conceptual insight into which of these statements is true.

We note that the term ‘bounded above’ is shorthand for

$$\{\exists\, M \text{ such that } \forall\, x \in D,\, f(x) \leq M\}.$$

To make sure we work these problems correctly, we could rewrite our statement using all the terminology, as in

$$\begin{aligned} p \quad \equiv \quad & \{\langle \forall\, f \text{ such that } \exists\, M \text{ such that } \forall\, x \in D,\, f(x) \leq M\rangle \\ & \text{and } \langle \forall\, g \text{ such that } \exists\, N \text{ such that } \forall\, x \in D,\, g(x) \leq N\rangle, \\ & \langle \exists\, L \text{ such that } \forall\, x \in D,\, f(x) + g(x) \leq L\rangle\}. \end{aligned}$$

In this case,

$$\begin{aligned} \neg p \quad \equiv \quad & \{\langle \exists\, f \text{ such that } \exists\, M \text{ such that } \forall\, x \in D,\, f(x) \leq M\rangle \\ & \text{and } \langle \exists\, g \text{ such that } \exists\, N \text{ such that } \forall\, x \in D,\, g(x) \leq N\rangle \text{ s.t.} \\ & \langle \forall\, L,\, \exists\, x \in D, \text{ such that } f(x) + g(x) > L\rangle\}. \end{aligned}$$

While p seems complicated, it is really just a statement of the form, if A and B, then C. Note that we use three different variables, M, N, and L, each one corresponding to a different function, f, g, and $f + g$, respectively. You can see the advantage to using the term ‘bounded above’ to replace all the conditions.

Conceptual Insight: 2.23

If you sketch different functions, you should conclude that p is likely to be true. To prove such a statement, we use the ‘let method’ and let f_0 and g_0 be two arbitrary bounded above functions and then show $h_0 = f_0 + g_0$ is bounded above.

- **Assumed:** There exists an M_0 such that for every $x \in D$, $f_0(x) \leq M_0$. Also, there exists an N_0 such that for every $x \in D$, $g_0(x) \leq N_0$.
- **To Show:** There exists an L such that for every $x \in D$, $h_0(x) = f_0(x) + g_0(x) \leq L$.

Remember that M_0 and N_0 are treated as known numbers and L is a number we need to find.

Claim 2.24: The sum of two bounded above functions is bounded above.

Proof: Let f_0 and g_0 be two functions with domain D which are bounded above. By the definition, we know there exist two real numbers M_0 and N_0 such that for all $x \in D$, $f_0(x) \leq M_0$ and $g_0(x) \leq N_0$. Set $L_0 = M_0 + N_0$ and let x_0 represent an arbitrary value in D. We are given that

$$f_0(x_0) \leq M_0 \text{ and } g_0(x_0) \leq N_0 .$$

Adding the two inequalities gives

$$h_0(x_0) = f_0(x_0) + g_0(x_0) \leq M_0 + N_0 = L_0.$$

Therefore, the sum of the two functions is bounded above. □

Now that we know that the sum of two functions, f and g, which are bounded above is bounded above, we might ask which of the functions $f - g$, $f \circ g$, fg, f/g are bounded above? This makes statements of the form

$$p \equiv \{\text{if } f \text{ and } g \text{ satisfy } P, \text{ then } f + g\ (f - g, fg, \text{etc.}) \text{ satisfies } P\}$$

particularly important. Remember to rewrite 'if/then' statements as 'for every' statements,

$$p \equiv \{\forall\ f \text{ and } g \text{ satisfying } P, f + g\ (f - g, fg, \text{etc.}) \text{ satisfies } P\}.$$

As another example, let's consider functions that map $\mathbb{R}$ into $\mathbb{R}$. Recall that a function f with domain D is **onto** set R if and only if for every $b \in R$ there is an $a \in D$ such that $f(a) = b$. We now want to prove or disprove that

$$p \equiv \{\text{if } f \text{ and } g \text{ both map onto } \mathbb{R}, \text{ then } h = f + g \text{ maps onto } \mathbb{R}\}.$$

This is a statement of the form

$$p \equiv \{\forall\ f \text{ and } g \text{ satisfying } \textit{onto} \text{ property}, h = f + g \text{ satisfies } \textit{onto} \text{ property}\}.$$

Just as we did with statements involving variables, the negation of this statement is

$$\neg p \equiv \{\exists\ f \text{ and } g \text{ that map } \textit{onto}\ \mathbb{R} \text{ s.t. } h = f + g \text{ does not map } \textit{onto}\ \mathbb{R}\}.$$

If you consider some examples, you should be able to determine that $\neg p$ is true. To prove this negation, we have to find two functions f and g such that

- **Assumed:** For every $y \in \mathbb{R}$, there exists an $x_1 \in \mathbb{R}$ such that $f(x_1) = y$ and there exists an $x_2 \in \mathbb{R}$ such that $g(x_2) = y$.
- **To Show:** Find a $y_0 \in \mathbb{R}$ such that for every $x \in \mathbb{R}$, $h(x) = f(x) + g(x) \neq y_0$.

Claim 2.25: There exists two functions that map $\mathbb{R}$ onto $\mathbb{R}$, but their sum does not map onto $\mathbb{R}$.

Proof: Set $f_0(x) = x$ and $g_0(x) = -x$, and $h_0 = f_0 + g_0$.

Part 1: We must prove that f_0 maps onto $\mathbb{R}$. Let y_0 be an arbitrary number in $\mathbb{R}$. Set $x_0 = y_0$. Then $f_0(x_0) = y_0$, so f_0 is onto $\mathbb{R}$.

Part 2: We must prove that g_0 maps onto $\mathbb{R}$. Let y_0 be an arbitrary number in $\mathbb{R}$. Set $x_0 = -y_0$. Then $g(x_0) = -x_0 = y_0$, so g_0 is onto $\mathbb{R}$.

Part 3: We must prove that h_0 does not map onto $\mathbb{R}$, that is, we must find a y such that for all x, $h_0(x) \neq y$. Set $y_0 = 1$ and let x_0 be an arbitrary number in $\mathbb{R}$. Then

$$h_0(x_0) = f_0(x_0) + g_0(x_0) = x_0 - x_0 = 0 \neq 1 = y_0.$$

Thus, h_0 is not onto. □

2.3.1 Problems

For simplicity for Problems 1 through 4 we repeat the conditions

$$
\begin{aligned}
P_1(f) &\equiv \{\forall M \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } f(x) > M\}, && 2.14\\
P_2(f) &\equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\}, && 2.15\\
P_3(f) &\equiv \{\forall (M,K) \in \mathbb{R}^2, \exists x > K \text{ s.t. } f(x) > M\}, && 2.16\\
P_4(f) &\equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, f(x) > M\}, && 2.17\\
P_5(f) &\equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall K \in \mathbb{R}, \exists x > K \text{ s.t. } f(x) > M\}, && 2.18\\
P_6(f) &\equiv \{\exists (M,K) \in \mathbb{R}^2 \text{ s.t. } \forall x > K, f(x) > M\}. && 2.19
\end{aligned}
$$

1. In this problem, we focus on Property $P_3(f)$ 2.16

(a) Prove or disprove

$$\{\forall f \text{ satisfying } P_1, f \text{ satisfies } P_3\}.$$

Answer 1

(b) Prove or disprove

$$\{\forall f \text{ satisfying } P_3, f \text{ satisfies } P_1\}.$$

Answer 2

(c) Prove or disprove

$$\{\forall f \text{ satisfying } P_2, f \text{ satisfies } P_3\}.$$

Answer 3

(d) Prove or disprove

$$\{\forall f \text{ satisfying } P_3, f \text{ satisfies } P_2\}.$$

Answer 4

2. Consider statement 2.17

$$P_4(f) \equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall x \in \mathbb{R}, f(x) > M\}.$$

(a) Prove or disprove

$$\{\forall f \text{ satisfying } P_1, f \text{ satisfies } P_4\}.$$

(b) Prove or disprove

$$\{\forall f \text{ satisfying } P_4, f \text{ satisfies } P_1\}.$$

(c) Prove or disprove

$$\{\forall f \text{ satisfying } P_2, f \text{ satisfies } P_4\}.$$

(d) Prove or disprove

$$\{\forall f \text{ satisfying } P_4, f \text{ satisfies } P_2\}.$$

(e) Prove or disprove

$$\{\forall f \text{ satisfying } P_3, f \text{ satisfies } P_4\}.$$

(f) Prove or disprove

$$\{\forall f \text{ satisfying } P_4, f \text{ satisfies } P_3\}.$$

3. Consider statement 2.18

$$P_5(f) \equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall K \in \mathbb{R}, \exists x > K \text{ s.t. } f(x) > M\}.$$

(a) Prove or disprove

$$\{\forall f \text{ satisfying } P_1, f \text{ satisfies } P_5\}.$$

Answer 5

(b) Prove or disprove

$$\{\forall f \text{ satisfying } P_5, f \text{ satisfies } P_1\}.$$

Answer 6

(c) Prove or disprove

$$\{\forall f \text{ satisfying } P_2, f \text{ satisfies } P_5\}.$$

(d) Prove or disprove

$$\{\forall f \text{ satisfying } P_5, f \text{ satisfies } P_2\}.$$

(e) Prove or disprove

$$\{\forall f \text{ satisfying } P_3, f \text{ satisfies } P_5\}.$$

Answer 7

(f) Prove or disprove

$$\{\forall f \text{ satisfying } P_5, f \text{ satisfies } P_3\}.$$

Answer 8

(g) Prove or disprove

$$\{\forall f \text{ satisfying } P_4, f \text{ satisfies } P_5\}.$$

(h) Prove or disprove

$$\{\forall f \text{ satisfying } P_5, f \text{ satisfies } P_4\}.$$

4. Consider statement 2.19

$$P_6(f) \equiv \{\exists (M, K) \in \mathbb{R}^2 \text{ such that } \forall x > K, f(x) > M\}.$$

(a) Prove or disprove

$$\{\forall f \text{ satisfying } P_1, f \text{ satisfies } P_6\}.$$

(b) Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_1\}.$$

(c) Prove or disprove

$$\{\forall f \text{ satisfying } P_2, f \text{ satisfies } P_6\}.$$

(d) Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_2\}.$$

(e) Prove or disprove

$$\{\forall f \text{ satisfying } P_3, f \text{ satisfies } P_6\}.$$

(f) Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_3\}.$$

(g) Prove or disprove

$$\{\forall f \text{ satisfying } P_4, f \text{ satisfies } P_6\}.$$

(h) Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_4\}.$$

(i) Prove or disprove

$$\{\forall f \text{ satisfying } P_5, f \text{ satisfies } P_6\}.$$

(j) Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_5\}.$$

5. Consider the function

$$f_1(x) = \begin{cases} x\cos(\pi x) & x \geq 0 \\ 0 & x < 0 \end{cases}, \tag{2.22}$$

whose graph is seen in Figure 2.7. This function satisfies statements P_1, P_3, and P_5 but these statements do not pick up all of the behavior of this function. Write a statement P_7 such that $P_7(f_1)$ is true but $P_7(g_i)$ is false for $i = 1, 2, 3,$ and 4, where

$$g_1(x) = x^2,\ g_2(x) = \cos(\pi x),\ g_3(x) = |x\cos(\pi x)|,\ g_4(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Answer 9

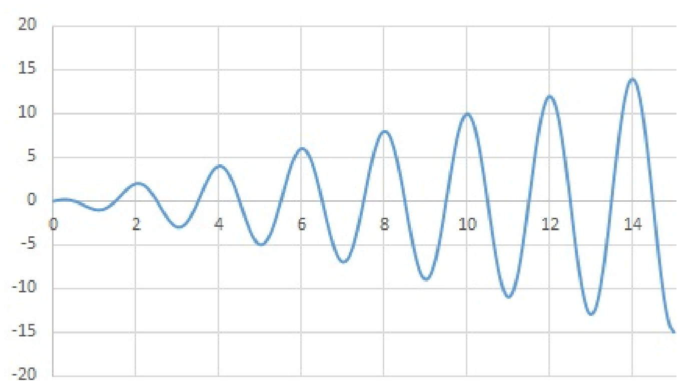

Figure 2.7: Sketch of function $f(x) = x\cos(\pi x)$ for $x \geq 0$.

6. Consider the function

$$g_4(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   from Problem 5 This function satisfies statements P_1, P_4, P_5 and P_6 but these statements do not pick up all of the behavior of g_4. Write a statement P_8 which is satisfied by g_4 but is not satisfied by the functions

$$g_1(x) = x^2,\ g_2(x) = \cos(\pi x),\ g_3(x) = |x\cos(\pi x)|,$$

   or the function 2.22, f_1.

7. Suppose function f_0, which maps $\mathbb{R}$ into $\mathbb{R}$, is bounded above and $c_0 > 0$. Prove or disprove the function $g_1 = c_0 f_0$ is bounded above. Answer 10

8. Consider the statement, 'if f and g are bounded above, then $f - g$ is bounded above'.

   (a) Write carefully what is given and what must be shown to prove this statement. Make some sketches so you get some idea of whether the statement is true or not.

   (b) Watch video at

   https://vimeo.com/26512295 Password:Proof

   One of the students shows that their proof is wrong. What is the error in their proof?

   (c) Watch video at

   https://vimeo.com/70952875 Password:Proof

   How do you like their approach to this problem? What could they do better?

   (d) Write a well-written proof that the negation of the statement is true. Make sure you actually prove that $f - g$ is not bounded above based on the definition of not bounded above.

   (e) Consider the statement, 'if f and g are bounded above, either $f - g$ or $g - f$ is bounded above. Prove or disprove this statement.

9. Prove or disprove that if function f with domain D which maps onto $\mathbb{R}$ and if $c > 0$, then the function $g = cf$ maps onto $\mathbb{R}$. (Make sure you rewrite using 'for every' in place of 'if'.) Answer 11

10. A function f with domain $\mathbb{R}$ mapping into $\mathbb{R}$ is increasing if whenever $a < b$, then $f(a) < f(b)$. In the following, f and g are assumed to have domain $D = \mathbb{R}$.

    (a) Prove or disprove that if f and g are increasing functions into $\mathbb{R}$, then $h = f + g$ is an increasing function into $\mathbb{R}$.

    (b) Prove or disprove that if f and g are increasing functions into $\mathbb{R}$, then $h = f - g$ is an increasing function into $\mathbb{R}$.

    (c) Prove or disprove that if f is an increasing function into $\mathbb{R}$, and $c > 0$ then $h = cf$ is an increasing function into $\mathbb{R}$.

    (d) Prove or disprove that if f and g are increasing functions into $\mathbb{R}$, then the composition of f and g, $h = f \circ g$ is an increasing function into $\mathbb{R}$.

    (e) Prove or disprove that if f and g are increasing functions into $\mathbb{R}$, then $h = fg$ is an increasing function into $\mathbb{R}$.

    (f) Prove or disprove that if f and g are increasing functions into $\mathbb{R}^+$, then $h = fg$ is an increasing function into $\mathbb{R}^+$.

11. Assume that A, B, and C are three sets. Assume the functions g and h map the set A into the set B and that the function f maps the set B into the set C. Also assume that

$$f \circ g = f \circ h.$$

    Consider the statement

$$p \equiv \{\text{if } f \text{ is onto, then } g = h\}.$$

    (a) Try to construct some appropriate functions (and corresponding sets A, B, and C) to convince your self whether this statement is true or false. To do this, write precisely what is given and what must be shown. Answer 12

    (b) Carefully write what is given in usable form, that is, what it means for f to be onto. Answer 13

(c) Carefully write what you must show in usable form, that is, what it means for $g = h$. Answer 14

(d) Develop an heuristic argument justifying your conclusion. Answer 15

(e) Watch the 6:15 minute video at

https://vimeo.com/71067008 (password:Proof)

where two students are working this same problem. Note their progression of examples. What do you think of their heuristic argument? Answer 16

(f) Prove or disprove the statement. Answer 17

12. Let S and T be sets and let $f : S \to T$ and $g : T \to S$ be functions such that
$$\forall x \in S,\ g(f(x)) = x\,.$$
(a) Prove or disprove that f must be onto?

(b) Prove or disprove that g must be onto?

2.3.2 Answers to selected problems

1. **Problem 1 a:** We disprove this statement by proving
$$\{\exists\, f \text{ satisfying } P_1 \text{ s.t. } f \text{ satisfies } \neg P_3\},$$
where
$$\neg P_3(f) \equiv \{\exists (M,K) \in \mathbb{R}^2, \text{ s.t. } \forall x > K,\ f(x) \le M\}.$$
We have already shown the function
$$f_0(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
satisfies P_1. To show $\neg P_3(f_0)$ is true, we first set $(M_0, K_0) = (1/2, 2)$. Let $x_0 > K_0 = 2$. Then, since $x_0 > 2$
$$f_0(x_0) = \frac{1}{x_0} \le \frac{1}{2},$$
which is what we needed to show. Thus P_1 does not imply P_3.

2. **Problem 1 b:** We prove

$$\{\forall f \text{ satisfying } P_3, f \text{ satisfies } P_1\}.$$

- **Assume** f_0 is an arbitrary function that satisfies

$$P_3(f_0) \equiv \{\forall (M, K) \in \mathbb{R}^2, \exists x > K \text{ s.t. } f(x) > M\}.$$

- **Show** f_0 satisfies

$$P_1(f_0) \equiv \{\forall L \in \mathbb{R}, \exists t \in \mathbb{R} \text{ s.t. } f(t) > L\}.$$

Let L_0 represent an arbitrary number in $\mathbb{R}$. We must find an t such that $f(t) > L_0$. Since P_3 is true, we can set $M_0 = L_0$ and $K_0 = 1$ (any value works for K_0). We know there exists $x_0 > K_0 = 1$ such that $f(x_0) > M_0 = L_0$ and we are finished with $t_0 = x_0$.

3. **Problem 1 c:** We show

$$\{\forall f \text{ satisfying } P_2, f \text{ satisfies } P_3\}.$$

We write the two statements using different letters for the variables.

- **Assume** f_0 is an arbitrary function that satisfies

$$P_2 \equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\}.$$

- **Show** f_0 satisfies

$$P_3 \equiv \{\forall (L, N) \in \mathbb{R}^2, \exists t > N \text{ s.t. } f(t) > L\}.$$

To prove $P_3(f_0)$, we must let L_0 and N_0 represent two arbitrary real numbers. We must find $t > N_0$ such that $f(t) > L_0$. Since $P_2(f_0)$ is true, we know for all M, there exists K such that for all $x > K$, $f(x) > M$. Set $M = M_0 = L_0$. We know K_0 exists. Now set $t_0 = |K_0| + |N_0| + 1$. Since $t_0 > K_0$, by P_2, $f(t_0) > M_0 = L_0$. Since $t_0 > N_0$, there exists a $t_0 > N_0$ such that $f(t_0) > L_0$, and $P_3(f_0)$ is true.

4. **Problem 1 d:** We show this statement isf false by proving

$$\{\exists f \text{ satisfying } P_3 \text{ s.t. } f \text{ satisfies } \neg P_2\},$$

where

$$\neg P_2(f) \equiv \{\exists M \in \mathbb{R} \text{ s.t. } \forall K \in \mathbb{R}, \exists x > K \text{ s.t. } f(x) \leq M\}.$$

Set $f_0(x) = x\cos(\pi x)$ whose graph is seen in Figure 2.7, for $x \geq 0$. Note that $\cos(\pi x) = 1$ if x is an even integer, $\cos(\pi x) = -1$ if x is an odd integer, and $\cos(\pi x) = 0$ if $x = n + 0.5$ where $n \in \mathbb{Z}$.

Part 1: To show f_0 satisfies P_3, let (M_0, K_0) represent a pair of real numbers. Set

$$x_0 = 2(\lceil |M_0| \rceil + \lceil |K_0| \rceil + 1).$$

Clearly $x_0 > K_0$. Also, x_0 is an even integer, so $\cos(\pi x_0) = 1$. This means

$$f_0(x_0) = x_0 > M_0,$$

so $P_3(f_0)$ is true.

Part 2: To show $\neg P_2(f_0)$ is true, we set $M_0 = 1$ and let $K_0 \in \mathbb{R}$. Set

$$x_0 = 2(\lceil |M_0| \rceil + \lceil |K_0| \rceil) + 0.5 > K_0.$$

This means that $x_0 = 2n_0 + 0.5$ for some $n_0 \in \mathbb{Z}$ so $\cos(\pi x_0) = \cos(\pi/2) = 0$ by periodicity and $f(x_0) = 0 < M_0 = 1$ so $\neg P_2(f_0)$ is true.

5. **Problem 3 a:** First, we need

$$\neg P_5(f) \equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R}, \text{ s.t. } \forall x > K, f(x) \leq M\}.$$

Set

$$f_0(x) = -x.$$

Then $P_1(f_0)$ is true but $P_5(f_0)$ is false. To see this, let M_0 represent an arbitrary real number and set $K_0 = -M_0$. For all $x > K_0$, $f_0(x) = -x < -K_0 = M_0$ so there does not exist an $x > K_0$ such that $f_0(x) > M_0$. Details are left to reader.

6. **Problem 3 b:** The function $f_0(x) = 1$ satisfies P_5 (with $M = 0$) but does not satisfy P_1.

7. **Problem 3 e:** We show $P_3(f) \Rightarrow P_5(f)$. We write the two statements using different letters for the variables.

- **Assume:** Let f_0 be an arbitrary function that satisfies P_3, that is for all $(M, K) \in \mathbb{R}^2$ there exists $x > K$ such that $f(x) > M$.
- **Show:** We must show f_0 satisfies P_5, that is, there exists an $L \in \mathbb{R}$ such that for all $N \in \mathbb{R}$, there exists $t > N$ such that $f(t) > L$.

We must show there exists an L, so set $L = L_0 = 1$. Then we must show for all N so we let N_0 represent an arbitrary real number and must now show there exists $t > N_0$ such that $f(t) > 1 = L_0$. Because $P_3(f_0)$ is true, we know something for all (M, K), so it must be true when $M = M_0 = 1$ and $K_0 = N_0$. This means that there exists $x = x_0 > K_0 = N_0$ such that $f(x_0) > 1 = L_0$. Set $t = t_0 = x_0$ and we have that $P_5(f_0)$ is satisfied.

8. **Problem 3 f:** The function $f_0(x) = 1$ satisfies P_5 but not P_3.

   To show f_0 satisfies P_5, set $M_0 = 0$. Let N_0 be an arbitrary real number and set $x_0 = N_0 + 1 > N_0$. Then
   $$f_0(x_0) = 1 > 0 = M_0.$$
   To show f_0 does not satisfy P_3, set $M_0 = 2$ and set $K_0 = 0$ (any number would work). Let $x_0 > K_0$. Then
   $$f_0(x_0) = 1 < 2.$$
   So f_0 satisfies $\neg P_3$.

9. **Problem 5:** One possibility is
   $$P_7(f) \equiv \{\forall M, K \in \mathbb{R}, \exists x_1, x_2 > K \text{ s. t. } f(x_1) < M \text{ and } f(x_2) > M\}.$$

10. **Problem 7:** We prove this statement is true. Let f_0 be a bounded above function and let c_0 represent an arbitrary positive number.

    Assumed: There exists an M_0 such that for every $x \in \mathbb{R}$, $f_0(x) < M_0$.

    To Show: There exists an N such that for every $x \in \mathbb{R}$, $cf_0(x) < N$.

    Note we use M_0 for the known number associated with f_0, and a different variable N for the unknown number associated with $c_0 f_0$.

    Claim: Suppose function f_0, which maps $\mathbb{R}$ into $\mathbb{R}$, is bounded above and $c_0 > 0$. The function $g_1 = c_0 f_0$ is bounded above.

Proof: Let f_0 be an arbitrary function that is bounded above and let c_0 represent an arbitrary positive number. We know there exists an $M_0 \in \mathbb{R}$ such that $f_0(x) \leq M_0$ for every $x \in \mathbb{R}$. Set $N_0 = c_0 M_0$. Let $x_0 \in \mathbb{R}$. We know

$$f_0(x_0) \leq M_0.$$

Multiplying both sides by c_0, which we know is positive, gives

$$c_0 f_0(x_0) \leq c_0 M_0 = N_0,$$

which is what we needed to prove. □

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11. **Problem 9:** Let f_0 be a function which maps D onto $\mathbb{R}$ and let c_0 be a non-zero number.

    Assumed: For every $y \in \mathbb{R}$, there exists an $x \in D$ such that $f_0(x) = y$.

    To Show: Let $y_0 \in \mathbb{R}$. Find $x \in D$ such that $g_0(x) = c_0 f_0(x) = y_0$.

    Claim: If f maps onto $\mathbb{R}$ and $c_0 \neq 0$, then $g = c_0 f$ maps onto $\mathbb{R}$.

    Proof: Let $y_0 \in \mathbb{R}$. We know that there exists an $x_0 \in D$ such that $f_0(x_0) = y_0/c_0$ since f_0 is onto $\mathbb{R}$. But then $g_0(x_0) = c_0 f(x_0) = c_0 y_0/c_0 = y_0$. □

    ---

12. **Problem 11 a:** We are given that for every $x \in A$, $f(g(x)) = f(h(x))$ and that for every $z \in C$, there exists a $y \in B$ such that $f(y) = z$. We must show for every $x \in A$, $g(x) = h(x)$. One example is, letting $A = \mathbb{R}$, $B = \mathbb{R}$ and $C = \mathbb{R}_0^+$, $g(x) = x$, $h(x) = -x$ and $f(x) = x^2$.
13. **Problem 11 b:** For every $z \in C$, there exists $y \in B$ such that $f(y) = z$.
14. **Problem 11 c:** For every $x \in A$, $g(x) = h(x)$.
15. **Problem 11 d:** From our example, the statement seems false. To show this, we must have an f which is onto C but there exists $x_0 \in A$ such that $g(x_0) \neq h(x_0)$ which is true for $x_0 = 1$.
16. **Problem 11 e:** The students did a good job of thinking through this statement. The figures were particularly helpful.

17. **Problem 11 f:** A simple proof that the statement is false is the following counterexample: we need to define sets A, B, and C and functions $g, h : A \to B$ and $f : B \to C$ where f is onto C, $(f \circ g)(x) = (f \circ h)(x)$ for every $x \in A$ and there exists $x \in A$ such that $g(x) \neq h(x)$.

    Set $A = \{a_0\}$, $B = \{b_1, b_2\}$ and $C = \{c_0\}$. (Note that a_0, b_1, b_2, and c_0 are not numbers, but just letter/elements of each set.) Define $g(a_0) = b_1$ and $h(a_0) = b_2$. Remember that $b_1 \neq b_2$. Define $f(y) = c_0$ for all $y \in B$. Then f maps onto C because for every element of C, that is, c_0, there exists an element of B (either b_1 or b_2) such that $f(y) = c_0$. Also, for every $a \in A$, $f \circ g(a) = f \circ h(a) = c_0$. Thus, the conditions are satisfied, but since there exists an element $a_0 \in A$ such that $g(a_0) \neq h(a_0)$, $g \neq h$.

2.4 Contraposition

Video Lesson 2.26

Watch the 9:42 minute video
https://vimeo.com/74332781 (Password:Proof)
This will make reading this section easier. To change the speed at which the video plays, click on the gear at the lower right of the video.

Recall from Section 2.1 that the implication s involving predicate functions

$$\begin{aligned} s &\equiv \{\text{if } p(x) \text{ then } q(x)\} \\ &\equiv \{\forall x \text{ s.t. } p(x), q(x)\ \} \end{aligned}$$

has as its negation 2.5

$$\begin{aligned} \neg s &\equiv \neg\{\text{if } p(x) \text{ then } q(x)\} \\ &\equiv \neg\{\forall x \text{ s.t. } p(x), q(x)\ \} \\ &\equiv \{\exists\, x \text{ s.t. } p(x) \text{ s.t. } \neg q(x)\}. \end{aligned} \tag{2.23}$$

For example, if

$$s \equiv \{\text{if } x > 3, \text{ then } x > 2\}$$

has as its negation

$$\neg s \equiv \{\text{there exists } x > 3 \text{ such that } x \leq 2\}. \tag{2.24}$$

Clearly, s is true and $\neg s$ is false.

Let's now consider the implication,

$$r \equiv \{\text{if} \neg q(x) \ \text{ then } \ \neg p(x)\},$$

which can be rewritten as

$$r \equiv \{\text{for all } x \text{ such that} \neg q(x), \ \neg p(x)\}. \tag{2.25}$$

In this case,

$$\begin{aligned} r &\equiv \{\text{if } x \not> 2, \text{ then } x \not> 3\} \\ &\equiv \{\text{For all } x \text{ s.t. } x \not> 2, x \not> 3\ \} \\ &\equiv \{\text{For all } x \text{ s.t. } x \leq 2, x \leq 3\ \}. \end{aligned}$$

We can write the negation of this statement as

$$\begin{aligned} \neg r &\equiv \{\text{There exists } x \text{ s.t. } \neg q(x) \text{ such that } p(x)\} \\ &\equiv \{\text{There exists } x \text{ s.t. } x \not> 2 \text{ s.t. } x > 3\ \} \\ &\equiv \{\text{There exists } x \text{ s.t. } x \leq 2 \text{ s.t. } x > 3\ \}. \end{aligned}$$

This is the same as statement 2.4, just written in a different order. In other words,

$$\neg\{\text{if} \neg q(x) \ \text{ then } \ \neg p(x)\} \equiv \neg\{\text{If } p(x) \text{ then } q(x)\} \equiv \{\exists\, x \ \text{ s.t. } \ \neg q(x) \wedge p(x)\}.$$

Since the negations of these two implications, r and s, are the same statement, then these two implications are equivalent.

$$\{p(x) \Rightarrow q(x)\} \equiv \{\neg q(x) \Rightarrow \neg p(x)\}.$$

For example,

$$r \equiv \{\text{if } x \not> 2, \text{ then } x \not> 3\} \equiv s \equiv \{\text{if } x > 3, \text{ then } x > 2\}.$$

The implication $\{\neg q(x) \Rightarrow \neg p(x)\}$ is called the **contrapositive** of the statment $\{p(x) \Rightarrow q(x)\}$. Another way to write this is

$$\{\forall \ x \text{ s.t. } \ p(x), q(x)\} \equiv \{\forall \ x \text{ s.t. } \ \neg q(x), \neg p(x)\}. \tag{2.26}$$

Thinking in terms of Venn diagrams, if we let P be the set of values for which $p(x)$ is true and let Q be the set of values for which $q(x)$ is true, then

$$\{P \subseteq Q\} \equiv \{Q^c \subseteq P^c\}.$$

This means that the statement that if x is inside the set where p is true, it is inside the set where q is true is equivalent to saying that if x is outside the set where q is true, then it is outside the set where p is true.

Let's now see an example in which the contrapositive form of the implication is helpful.

Conceptual Insight: 2.27

We prove the statement, if n^2 is even, then n is even. We note that a direct proof of this statement is quite difficult using the 'let method.' Using the contrapositive form of the implication gives

$$\{\forall n \notin \mathbb{E},\, n^2 \notin \mathbb{E}\},$$

which just means that
- we **assume** n is odd and
- **show** n^2 is odd.

Claim 2.28: For all even n^2, n is even.

Proof: We prove this statement using contraposition, that is, we assume n is odd and show n^2 is odd. We now use the let method and let n_0 represent an arbitrary odd integer. Being odd means there exists $k_0 \in \mathbb{Z}$ such that $n_0 = 2k_0 + 1$. We show that n_0^2 is odd, that is, there exists an integer j such that $n_0^2 = 2j + 1$. Since $n_0 = 2k_0 + 1$, then

$$n_0^2 = (2k_0 + 1)^2 = 4k_0^2 + 4k_0 + 1 = 2j_0 + 1.$$

where $j_0 = 2k_0^2 + 2k_0$. □

Let's consider another concept from calculus, a function being one-to-one. This just means that different x-values get mapped to different y-values. Often a visual example of a one-to-one function is a curve drawn in the plane such that every horizontal line intersects the graph at most once, such as $f(x) = x^3$. If a function is not one-to-one, it just means that there is (at least) one pair of x-values that gets mapped to the same y-value. The graph of such a function is often something like a parabola which intersects some horizontal line twice (or more). Formally, we have the following.

Definition 2.29: One-to-one

A function f is **one-to-one** from its domain D into a set R if and only if for all $x_1, x_2 \in D$, such that $x_1 \neq x_2$, we have that $f(x_1) \neq f(x_2)$.

This definition is often awkward to use. For example, suppose we wanted to show that $f(x) = 2x + 3$ is one-to-one. We would begin by letting x_1 and x_2 represent two arbitrary values in D such that $x_1 \neq x_2$.

Assumed: $x_1 \neq x_2$

We would then have to show that

To Show: $f(x_1) = 2x_1 + 3 \neq 2x_2 + 3 = f(x_2)$.

One way to do this is to state that since $x_1 \neq x_2$, then $2x_1 \neq 2x_2$ since nonzero multiples of two unequal numbers would still not be equal. We could then state that $2x_1 + 3 \neq 2x_2 + 3$ since adding the same amount to unequal numbers results in unequal numbers. We now have that $f(x_1) \neq f(x_2)$. While being correct, it is awkward to argue about numbers being unequal.

One of the most useful uses of contraposition is in proving that a function is one-to-one. Let's see what happens if we write the contrapositive of the definition of one-to-one, that is, a function f is one-to-one if and only if for every x_1 and x_2 in the domain

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2. \tag{2.27}$$

Now let's again prove that $f(x) = 2x + 3$ is one-to-one. Let x_1 and x_2 represent two arbitrary numbers in the domain of f such that $f(x_1) = f(x_2)$. This means that $2x_1 + 3 = 2x_2 + 3$. Subtracting 3 from both sides gives $2x_1 = 2x_2$. Dividing both sides by 2 gives $x_1 = x_2$ which is what we needed to show. This is a much clearer and more satisfying proof.

Summary 2.2

The contrapositive approach to proving a function is one-to-one means:
Assumed: $f(x_1) = f(x_2)$
To Show: $x_1 = x_2$.

Let's show that the function

$$f(x) = \begin{cases} 2x - 3 & x \geq 5 \\ x + 1 & x < 5 \end{cases} \tag{2.28}$$

is one-to-one. We would first graph the function as in Figure 2.8 which would give us the conceptual insight into that it is actually one-to-one. (It is not onto as for all $x \in \mathbb{R}$, $f(x) \neq 6.5$.)

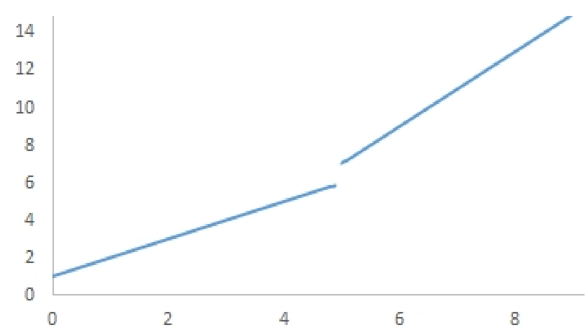

Figure 2.8: Graph of function 2.28 demonstrating it is one-to-one

Conceptual Insight: 2.30

We now proceed to prove the function is one-to-one using the contrapositive form of the definition, that is, we let x_1 and x_2 represent two arbitrary real numbers such that $f(x_1) = f(x_2)$. We now have to show $x_1 = x_2$. Since x_1 and x_2 are arbitrary numbers, we have to consider the cases, 1) both are greater than or equal to 5, 2) both are less than 5, or 3) one is greater than or equal to 5 and the other is less than 5. This third case is the important one: for the function to be one-to-one, we have to show this case cannot happen. Otherwise, there would be two non-equal x-values with the same y-value. This involves three cases.

Claim 2.31: Function 2.28,

$$f(x) = \begin{cases} 2x-3 & x \geq 5 \\ x+1 & x < 5 \end{cases}$$

is one-to-one.

Proof: We prove f is one-to-one using contraposition, that is, we assume x_1 and x_2 are two real numbers such that

$$f(x_1) = f(x_2).$$

Case 1: Assume $x_1 \geq 5$ and $x_2 \geq 5$. Then we have $f(x_1) = 2x_1 - 3 = 2x_2 - 3 = f(x_2)$. Adding 3 to both sides then dividing by 2 gives $x_1 = x_2$.

Case 2: Assume $x_1 < 5$ and $x_2 < 5$. Then we have $f(x_1) = x_1 + 1 = x_2 + 1 = f(x_2)$. Subtracting 1 from both sides gives $x_1 = x_2$.

Case 3: Assume one of the x-values is greater than or equal to 5 and the other is less than 5. Without loss of generality, assume $x_1 \geq 5$ and $x_2 < 5$. Then we have

$$f(x_1) = 2x_1 - 3 = x_2 + 1 = f(x_2).$$

Since $x_1 \geq 5$, then $f(x_1) = 2x_1 - 3 \geq 2(5) - 3 = 7$ and since $x_2 < 5$, then $f(x_2) = x_2 + 1 < 6$. Therefore, $f(x_1) > f(x_2)$ which contradicts the assumption that $f(x_1) = f(x_2)$, so this case cannot happen.

Thus, in each of the **possible** cases, $x_1 = x_2$ which is what we needed to show. This function is one-to-one.□

Remark 2.32

Case 3 is often the most important one when trying to determine if a function is one-to-one or not and should probably be done first. In particular, in Case 3, since $x_1 \geq 5 > x_2$, then $x_1 \neq x_2$. If this case cannot happen, the function is likely one-to-one (still have to check Cases 1 and 2). If this case can happen, then we have found a counterexample, the function is not one-to-one, and we do not need to consider Cases 1 and 2.

We note that in the Proof of Claim 2.31, we could have used four cases with Case 3 being $x_1 \geq 5$ and $x_2 < 5$, and Case 4 being $x_2 \geq 5$ and $x_1 < 5$. Because of symmetry, the work for both of these cases is the same with just the subscripts switched. In such cases, we can combine these two cases into one case as we did in this example, combining them into what we called Case 3. When we realize that the choice is arbitrary, we use the phrase **without loss of generality** or WLOG for short. This is quite useful. For example, suppose we let x_1 and x_2 be two arbitrary real numbers. Then we can say, WLOG assume $x_1 \leq x_2$. Or we could let n_1 and n_2 be two arbitrary integers numbers. We could consider three cases, both are even, both are odd, and one is even and one is odd. Then for the third case we can say, WLOG assume n_1 is even and n_2 is odd.

To show a function is not one-to-one, we have to show the negation of the definition of one-to-one. Since definition of one-to-one is of the form

$$r \equiv \{\forall x_1, x_2 \in \mathbb{R},\ x_1 \neq x_2,\ f(x_1) \neq f(x_2)\},$$

then the negation is defined as the following.

Definition 2.33: Not one-to-one

A function f is **not one-to-one** from its domain D into a set R if and only if

$$\{\exists\, x_1, x_2 \in \mathbb{R} \text{ such that } x_1 \neq x_2 \text{ and } f(x_1) = f(x_2)\}.$$

Consider the function

$$f(x) = \begin{cases} 2x - 5 & x \geq 5 \\ x + 1 & x < 5 \end{cases} \tag{2.29}$$

whose graph is seen in Figure 2.9. Clearly this function is not one-to-one as some horizontal lines intersect this function at two points.

Claim 2.34: Function 2.29 is not one-to-one.

Proof: Set $x_1 = 4 \neq x_2 = 5$. Then $f(x_1) = x_1 + 1 = 5 = 2x_2 - 5 = f(x_2)$.$\square$

We note that function 2.29 maps onto $\mathbb{R}$.

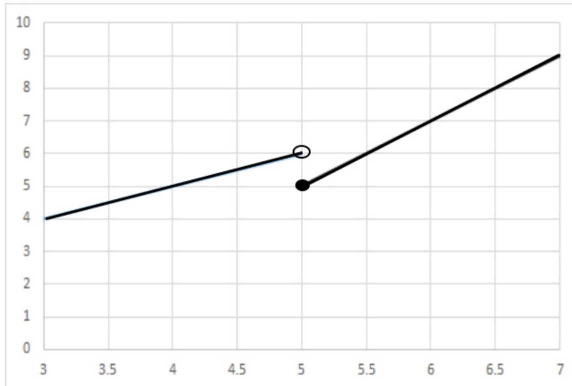

Figure 2.9: Graph of function 2.29 demonstrating it is not one-to-one

Video Lesson 2.35

At this point, you should watch the video
https://vimeo.com/74893416 password:Proof
which shows several approaches to rewriting a complicated implication.

2.4.1 Problems

1. Let $n, m \in \mathbb{Z}$. Consider the statement that if nm is even, then n is even or m is even. This statement as an implication of the form $\{p(n,m) \Rightarrow q(n) \vee r(m)\}$. Prove or disprove this implication using contraposition. Answer 1

2. Consider the statement

$$\{\text{if } x^2 - 4x - 5 > 0 \text{ then } x > 5 \text{ or } x < -1\}.$$

Prove this statement using contraposition.

3. Consider the statement

$$\{\text{if } x^2 - 8x + 12 < 0 \text{ then } x \in (2,6)\}.$$

Prove this statement using contraposition. Answer 2

4. Consider the function f from Section 1.5 Problem 12 mapping $\mathbb{R} -$

$\{1\}$ (all real numbers except 1) into the set $S = \mathbb{R}$ defined by

$$f(x) = \frac{2x+1}{x-1}.$$

Prove f is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

5. Consider the function f mapping $\mathbb{R}$ into the set $S = \mathbb{R}$ defined by $f(x) = x^3 + 4$. Prove f is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Answer 3

6. Show the function f mapping $\mathbb{R}$ into $S = \mathbb{R}$ is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, where

$$f(x) = \begin{cases} -2x+7 & x > 3 \\ -x+2 & x \leq 3 \end{cases}.$$

7. Consider the function f mapping $\mathbb{R} - \{3\}$ (all real numbers except 3) into the set $S = \mathbb{R}$ defined by

$$f(x) = \frac{4x+5}{x-3}.$$

Prove f is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Answer 4

8. Show the function f mapping $\mathbb{R}$ into $S = \mathbb{R}$ is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, where

$$f(x) = \begin{cases} x+3 & x > 2 \\ x+2 & x \leq 2 \end{cases}.$$

9. Show the function f mapping $\mathbb{R}$ into $S = \mathbb{R}$ is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, where

$$f(x) = \begin{cases} x^2 & x > 0 \\ 3x & x \leq 0 \end{cases}.$$

Answer 5

10. Show the function f mapping $\mathbb{R}$ into $S = \mathbb{R}$ is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, where

$$f(x) = \begin{cases} x+3 & x \notin \mathbb{Z} \\ x-2 & x \in \mathbb{Z} \end{cases}.$$

11. Show the function f mapping $\mathbb{R}$ into $S = \mathbb{R}$ is one-to-one or find two values x_1 and x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$, where

$$f(x) = \begin{cases} 2x-1 & x > 1 \\ -x+7 & x \leq 1 \end{cases}.$$

Answer 6

12. Show the function f mapping $\mathbb{Z}$ into $S = \mathbb{Z}$ is one-to-one or find two integers n_1 and n_2 such that $n_1 \neq n_2$ but $f(n_1) = f(n_2)$, where

$$f(n) = \begin{cases} 0.5n+3 & n \in \mathbb{E} \\ 3n-1 & n \in \mathbb{O} \end{cases}.$$

13. Show the function f mapping $\mathbb{Z}$ into $S = \mathbb{Z}$ is one-to-one or find two integers n_1 and n_2 such that $n_1 \neq n_2$ but $f(n_1) = f(n_2)$, where

$$f(n) = \begin{cases} 2n+1 & n \in \mathbb{E} \\ 4n-2 & n \in \mathbb{O} \end{cases}.$$

Answer 7

14. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ be three open intervals on the real line. Let

$$p \equiv \{A \cap B \cap C \neq \emptyset\},$$

meaning there is at least one real number x which is in all three intervals. Let

$$q \equiv \{\text{The union of some pair of these intervals must contain the third}\}.$$

Consider the implication $p \Rightarrow q$.

   (a) Draw some examples to convince yourself whether this statement is true or false.

(b) Develop an heuristic argument justifying your conclusion

(c) Watch the 7:17 minute video at

https://vimeo.com/45321903 (password:Proof)

where two students are working this same problem. Note their progression of examples. Which type of examples were easier to follow and why? What do you think of their heuristic argument?

15. We are going to investigate the sum of squares of two integers.

(a) For any integers n and m, if $n^2 + m^2$ is a multiple of 3, then n is a multiple of 3. Answer 8

(b) For any integers n and m, if $n^2 + m^2$ is a multiple of 5, then n is a multiple of 5. Answer 9

(c) For any integers n and m, if $n^2 + m^2$ is a multiple of 6, then n is a multiple of 6. Answer 10

(d) For any integers n and m, if $n^2 + m^2$ is a multiple of 7, then n is a multiple of 7. Answer 11

16. Assume that A, B, and C are three sets. Assume the functions g and h map the set A into the set B and that the function f maps the set B into the set C. Also assume that

$$f \circ g = f \circ h.$$

Consider the statement

$$p \equiv \{\text{if } f \text{ is one-to-one, then } g = h\}.$$

(a) Try to construct some appropriate functions (and corresponding sets A, B, and C) to convince your self whether this statement is true or false. To do this, write precisely what is given and what must be shown.

(b) Carefully write what is given in usable form, that is, what it means for f to be one-to-one.

(c) Carefully write what you must show in usable form, that is, what it means for $g = h$.

(d) Develop an heuristic argument justifying your conclusion

(e) Watch the 6:15 minute video at

https://vimeo.com/5918679 (password:Proof)

where two students are working this same problem. Note their progression of examples. What do you think of their heuristic argument?

(f) Prove or disprove the statement.

17. Let S and T be sets and let $f : S \to T$ and $g : T \to S$ be functions such that

$$\forall x \in S, g(f(x)) = x\,.$$

(a) Must f be one-to-one? Sketch an heuristic proof or give a counterexample. Answer 12

(b) Must g be one-to-one? Sketch an heuristic proof or give a counterexample. Answer 12

18. Given $j \in \mathbb{Z}^+ - \{1\}$. Suppose that for every $n, m \in \mathbb{Z}^+$, if j divides nm then j divides n or j divides m. Then j is prime. With

$$p(j) \equiv \{\forall\, n, m \in \mathbb{Z}^+, \langle\langle j \text{ divides } nm\rangle \Rightarrow \langle j \text{ divides } n\rangle \vee \langle j \text{ divides } m\rangle\rangle\}$$

and $q(j) \equiv \{j \text{ is prime}\}$, this can be written as an implication $p(j) \Rightarrow q(j)$ where statement $p(j)$ is an implication, that is, $p(j) \equiv \{r(j) \Rightarrow s(j)\}$.

(a) Pick a prime number for j. Pick different pairs of integers n and m such that j divides nm. In each case, does j divide n or m? (so the implication $p \equiv \{r \Rightarrow s\}$ is true)? Pick a composite number for j. Can you find integers n and m such that j divides nm but j does not divide n or m? (This part corresponds to Start-up Examples.)

(b) Write statement $\neg p$ in usable form. Writing $\neg p$ in usable form is one key to the problem.

(c) Write statement $\neg q$ in usable form. Writing 'composite' in a usable form is another key to the problem.

(d) Construct a proof by the Contrapositive Approach.

2.4.2 Answers to selected problems

1. **Problem 1:** This statement is of the form $\{p \Rightarrow q \vee r\}$ or

$$\{\forall n, m \text{ s.t. } p(n,m) \text{ is true, } q(n,m) \text{ or } r(n,m) \text{ is true}\},$$

where $p(n,m) \equiv \{nm \text{ is even}\}$, $q(n,m) \equiv \{n \text{ is even}\}$, and $r(n,m) \equiv \{m \text{ is even}\}$. This problem will be difficult to prove directly as knowing $nm = 2k$ will not help determining n or m is even. For example,

Assumed: $n_0 m_0 = 2k_0$

To Show: Either $n_0 = 2j$ for some j or $m_0 = 2j$ for some j.

We try contraposition, that is, we assume

$$\neg\{q(n) \vee r(m)\} \equiv \{\neg q(n) \wedge \neg r(m)\},$$

noting that $\neg q(n)$ means n is not even so is odd. Similarly, $\neg r(m)$ is that m is odd. We then prove $\neg p(n,m)$ which is that nm is not even, that is, nm is odd.

Assumed: $n_0 = 2i_0 + 1$ and $m_0 = 2k_0 + 1$

To Show: $n_0 m_0 = 2j + 1$ for some j.

We then let n_0 and m_0 represent two arbitrary odd integers. This means $\exists, i_0, k_0 \in \mathbb{Z}$ such that $n_0 = 2i_0 + 1$ and $m_0 = 2k_0 + 1$. Substitution in the equation gives that

$$(2i_0 + 1)(2k_0 + 1) = 2(2i_0 k_0 + i_0 + k_0) + 1 = 2j_0 + 1.$$

Therefore, $n_0 m_0$ is odd and our statement is proven using contraposition.

2. **Problem 3:** $p(x) \equiv \{x^2 - 8x + 12 < 0\}$ so $\neg p(x) \equiv \{x^2 - 8x + 12 \geq 0\}$. Also $q(x) \equiv \{x \in (2,6)\}$ so $\neg q(x) \equiv \{x \leq 2 \text{ or } x \geq 6\}$. We will show $\neg q(x) \Rightarrow \neg p(x)$. Let x_0 be an arbitrary value for which $\neg q(x)$ is true. There are two cases.

Case 1: $x_0 \leq 2$. Since $x^2 - 8x + 12 = (x-2)(x-6)$, we have that $x_0 - 2 \leq 0$ and $x_0 - 6 < 0$ so $x^2 - 8x + 12 = (x-2)(x-6) \geq 0$ which is what we needed to prove.

Case 2: $x_0 \geq 6$. Since $x^2 - 8x + 12 = (x-2)(x-6)$, we have that $x_0 - 2 > 0$ and $x_0 - 6 \geq 0$ so $x^2 - 8x + 12 = (x-2)(x-6) \geq 0$ which is what we needed to prove.

In both cases, the inequality is satisfied, so the statement is true by contraposition.

3. **Problem 5:** From the graph of this function, it appears to be one-to-one. We will prove it using contraposition. We let x_1 and x_2 represent two arbitrary real numbers such that $f(x_1) = f(x_2)$. This means that $x_1^3 + 4 = x_2^3 + 4$. Subtracting 4 from both sides gives $x_1^3 = x_2^3$. We take cube roots, which are unique, giving $x_1 = x_2$, so the function is one to one.

   If we wanted to be careful, we could factor $x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$. If $x_1^2 + x_1x_2 + x_2^2 = 0$, then by the quadratic formula

   $$x_1 = \frac{-x_2 \pm \sqrt{x_2^2 - 4x_2^2}}{2} = \frac{-x_2 \pm \sqrt{-3x_2^2}}{2},$$

   which has no real solutions, other than $x_1 = 0$ when $x_2 = 0$, but then $x_1 = x_2$. Since the second factor cannot equal 0, then the first factor, $x_1 - x_2$ must equal 0, meaning $x_1 = x_2$ which is what we needed to show. Therefore this function is one-to-one.

4. **Problem 7:** From the graph of this function in Figure 1.20, it appears to be one-to-one. We let x_1 and x_2 represent two arbitrary real numbers in the domain of f such that $f(x_1) = f(x_2)$. This means that

   $$\frac{4x_1 + 5}{x_1 - 3} = \frac{4x_2 + 5}{x_2 - 3}.$$

   Multiplying both sides by the nonzero $(x_1 - 3)(x_2 - 3)$ and expanding gives

   $$4x_1x_2 + 5x_2 - 12x_1 - 15 = 4x_1x_2 + 5x_1 - 12x_2 - 15.$$

   Canceling $4x_1x_2$ and -15 leaves

   $$5x_2 - 12x_1 = 5x_1 - 12x_2,$$

   which simplifies to $17x_2 = 17x_1$ or $x_1 = x_2$ which is what we needed to show.

5. **Problem 9:** From the graph of this function in Figure 1.22, it appears to be one-to-one. We let x_1 and x_2 represent two arbitrary real numbers in the domain of f such that $f(x_1) = f(x_2)$. We now have to show $x_1 = x_2$. Consider three cases.

Case 1: Assume $x_1 > 0$ and $x_2 > 0$. Then we have $f(x_1) = x_1^2 = x_2^2 = f(x_2)$. This simplifies to $(x_1 - x_2)(x_1 + x_2)$. Since both x-values are positive, $x_1 + x_2 > 0$ so $x_1 - x_2 = 0$ or $x_1 = x_2$ in this case.

Case 2: Assume $x_1 \leq 0$ and $x_2 \leq 0$. Then we have $f(x_1) = 3x_1 = 3x_2 = f(x_2)$ and dividing by 3 gives $x_1 = x_2$ in this case, also.

Case 3: Assume one of the values is greater than 0 and the other is less than or equal to 0. Without loss of generality, assume $x_1 > 0$ and $x_2 \leq 0$. Then we have $f(x_1) = x_1^2 = 3x_2 = f(x_2)$. But since $x_1^2 > 0$ and $3x_2 \leq 0$, we have a contradiction so this case cannot occur.

From the three cases, we have shown if $f(x_1) = f(x_2)$ then $x_1 = x_2$ and f is one-to-one.

6. **Problem 11:** From the graph of this function in Figure 1.21, f appears not to be one-to-one. To show this, we need to find $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. There are numerous choices. One possibility is to let $x_1 = 0$ so that $f(x_1) = -x_1 + 7 = 7$. We then find x_2 such that $f(x_2) = 2x_2 - 1 = 7$, but then we have to check that $x_2 > 1$. Solving the equation, we get $x_2 = 4$. Therefore we have found $x_1 \neq x_2$ such that $f(x_1) = f(x_2) = 7$ so this function is not one-to-one. We note that if we had tried to prove this function was one-to-one, we would have had no problem with Case 1 when both values were less than one or with Case 2 when both values were greater than 1. It is Case 3 where the problem would occur. Instead of getting a contradiction when $x_1 \leq 1$ and $x_2 > 1$, we are actually able to find two values that work, as we did here. This means **the third case is critical**.

7. **Problem 13:** From the graph of this function in Figure 1.27, f appears to be one-to-one, but it is not totally clear this is true. We will proceed to try to prove it is one-to-one and see what happens. We let n_1 and n_2 represent two arbitrary integers such that $f(n_1) = f(n_2)$. We now have to show $n_1 = n_2$. Consider three cases.

Case 1: Assume n_1 and n_2 are both even. Then we have $f(n_1) = 2n_1 + 1 = 2n_2 + 1 = f(n_2)$. This simplifies to $n_1 = n_2$.

Case 2: Assume n_1 and n_2 are both odd. Then we have $f(n_1) = 4n_1 - 2 = 4n_2 - 2 = f(n_2)$. This also simplifies to $n_1 = n_2$.

Case 3: Assume one of the values is even and the other is odd. Without loss of generality, assume n_1 is even and n_2 is odd. This means there exist integers k_1 and k_2 such that $n_1 = 2k_1$ and $n_2 = 2k_2 + 1$.

Then we have

$$f(n_1) = 2n_1 + 1 = 4k_1 + 1$$

and

$$f(n_2) = 4n_2 - 2 = 4(2k_2 + 1) - 2 = 8k_2 + 2.$$

This means $4k_1 + 1 = 8k_2 + 2$ or after dividing by 2, $2k_1 + 1/2 = 4k_2 + 1$ which is a contradiction since the right side is an integer and the left side is not an integer. So this case cannot happen.

From the three cases, we have shown if $f(x_1) = f(x_2)$ then $x_1 = x_2$ and f is one-to-one.

8. **Problem 15 a:** True. Whenever you square an integer, it is either a multiple 3 or one more than a multiple of 3. Use contraposition, that is, show that if n is not divisible by 3, then $n^2 + m^2$ is not divisible by 3. Let n_0 represent an integer which is not divisible by 3. There are two cases,

   Case 1: There exists an integer k_0 such that $n_0 = 3k_0 + 1$. There are three subcases, $m_0 = 3j_0$, $m_0 = 3j_0 + 1$ and $m_0 = 3j_0 + 2$. In each case, substitution into $n_0^2 + m_0^2$ leads to an integer that is not divisible by 3.

   Case 2: There exists an integer k_0 such that $n_0 = 3k_0 + 2$. There are three subcases, $m_0 = 3j_0$, $m_0 = 3j_0 + 1$ and $m_0 = 3j_0 + 2$. In each case, substitution into $n_0^2 + m_0^2$ leads to an integer that is not divisible by 3.

9. **Problem 15 b:** False: Let $n = 1$ and $m = 2$, then $n^2 + m^2 = 5$ which is a multiple of 5, but $n = 1$ is not a multiple of 5.

10. **Problem 15 c:** False: Let $n = 3$ and $m = 3$, then $n^2 + m^2 = 18$ which is a multiple of 6, but 3 is not a multiple of 6.

11. **Problem 15 d:** True: The proof is similar to part a but with many more cases. The reason it is true is that for every combination of remainders when n and m are divided by 7, that is the remainders are in the set $S = 1, 2, 3, 4, 5, 6$, we never get that $n^2 + m^2$ is a multiple of 7, that is, the square of the remainders never adds to 7.

12. **Problem 17 a:** This statement is true. Let x_1 and x_2 be two elements of set S and assume $f(x_1) = f(x_2)$. Since g is a function, we know that $g(f(x_1)) = g(f(x_2))$. But we are given that $g(f(x_1)) = x_1$ and $g(f(x_2)) = x_2$, so $x_1 = x_2$ and f is one-to-one.

13. **Problem 17 b:** This statement is false. As a simple example, let $S = \{1\}$ and $T = \mathbb{R}$. Let $f(x) = 1$ for all $x \in S$ and let $g(x) = 1$ for all $x \in T$. Therefore, for all $x \in S$, $f(g(x)) = x$ since $x = 1$. On the other hand g is not one-to-one since $1 \neq 2$ but $g(1) = g(2) = 1$.

2.5 Contradiction

Video Lesson 2.36

Watch the 5:55 minute video
https://vimeo.com/87615555 (Password:Proof)
where you can see a proof by contradiction that a certain cubic expression has only irrational roots. This will make the reading easier. To change the speed at which the video plays, click on the gear at the lower right of the video.

Let's consider the simple statement

$$q \equiv \{\langle \forall\, M \in \mathbb{R}\rangle\ \langle \exists\, x \in \mathbb{R}\rangle \text{ such that } \langle\, M < x\rangle\}.$$

This statement is clearly true in that for every $M \in \mathbb{R}$, there exists an $x = M + 1 \in \mathbb{R}$ such that $M < x$. The proof by the let method would look like the following.

Claim 2.37: For every real numbers M, there exists a real number x such hat $M < x$.

Proof: Let M_0 represent an arbitrary real number (let method). Set $x_0 = M_0 + 1$. Since $0 < 1$, adding M_0 to both sides gives that $M_0 < M_0 + 1 = x_0$.$\square$

Statement q can be rephrased as

$$q \equiv \{\text{There is no largest real number}\}.$$

An alternate approach to proving q is to show that the negation of q

$$\neg q \equiv \{\text{There exists a largest real number}\}$$

leads to a contradiction.

Claim 2.38: We prove q, that there is no largest number by showing $\neg q$ cannot be true.

Proof: Assume

$$\neg q \equiv \{\exists\, M_0 \in \mathbb{R} \text{ such that } \forall\, x \in \mathbb{R},\, M_0 \geq x\}.$$

Set $x = x_0 = M_0 + 1$. We know that $0 < 1$. Adding M_0 to both sides gives that $M_0 < M_0 + 1 = x_0$. But by $\neg q$, $M_0 \geq x_0$ which is a contradiction. Therefore, $\neg q$ is false and therefore q is true, there is no largest real number.□

This is just another example of the powerful proof technique, proof by contradiction, which we have been using frequently in earlier sections. This method gives an alternate method for proving implications of the form

$$\{p(x) \Rightarrow q(x)\} \equiv \{\forall x \text{ s.t. } p(x), q(x)\}.$$

As we already know, the negation of this statement is

$$\neg\{p(x) \Rightarrow q(x)\} \equiv \neg\{\forall x \text{ s.t. } p(x), q(x)\} \equiv \{\exists x \text{ s.t. } p(x) \wedge \neg q(x)\}.$$

This means

$$\neg\{\exists x \text{ s.t. } p(x) \wedge \neg q(x)\} \equiv \{\forall x \text{ s.t. } p(x), q(x)\}, \tag{2.30}$$

that is, if there does not exists an x such that $p(x)$ and $\neg q(x)$ are true, then the implication must be true. This just means that the statement $\{p(x) \wedge \neg q(x)\}$ cannot happen. The idea behind this approach is to assume that there does exist an x such that both $p(x)$ and $\neg q(x)$ are true, then arrive at a contradiction, that is, a statement that is clearly wrong. Using this form of an implication is called **proof by contradiction**.

Contradiction often works when we are stuck on a problem. The idea is that we can assume a lot, that for some x both statements $p(x)$ and $\neg q(x)$ are true. Then we only have to find something that must go wrong, so $p(x) \Rightarrow q(x)$ is true, or we actually find an x that works, so the implication is false.

We have already seen how this method works in simple cases, such as, if $p(x) \equiv \{x > 3\}$ then $q(x) \equiv \{x > 2\}$. We then assume there exists an x_0 such $p(x_0)$, $x_0 > 3$, and $\neg q(x_0)$, $x_0 \leq 2$. Combining these inequalities gives

$$3 < x_0 \leq 2$$

or $3 < 2$ which is a contradiction. Therefore our implication must be true.

As another simple example, let $p(x,y) \equiv \{x^2 + y^2 < -1\}$ and $q(x,y) \equiv \{x^2 + y^2 < 4\}$. Note that $p(x,y)$ is never true. Amazingly, the implication

$$r \equiv \{\forall (x,y) \in \mathbb{R}^2 \text{ such that } p(x,y), q(x,y)\}$$

is true, since there is no point (x,y) where $p(x,y)$ is true and $q(x,y)$ is false since $p(x,y)$ is never true.

The proof that r is true is to assume $\neg r$ is true, that is,

$$\neg r \equiv \{\exists (x,y) \in \mathbb{R}^2 \text{ such that } p(x,y) \text{ and } \neg q(x,y)\}.$$

Let (x_0, y_0) represent such a point. This means $x_0^2 + y_0^2 < -1$ and $x_0^2 + y_0^2 \geq 4$. But the first inequality means that

$$0 \leq x_0^2 + y_0^2 < -1,$$

which implies $0 < -1$ which is a contradiction. Therefore $\neg r$ cannot be true so r is true.

While it would seem silly to have a hypothesis that is never true, this situation actually came up in Section 2.4 when we assumed a Case 3 that couldn't happen in proving some functions were one-to-one. The implication where $p(x)$ is empty just means that case cannot happen.

Let's see how contradiction works on an example which is a little more complicated by showing

$$r \equiv \{\forall x \in \mathbb{R}, \text{ if } p(x) \equiv \{\forall i \in \mathbb{Z}^+,\ -\frac{1}{i} < x\} \text{ then } q(x) \equiv \{0 \leq x\}\}. \quad (2.31)$$

Conceptual Insight: 2.39

To get a sense of the problem, we try some examples. Suppose $x \geq 0$. Then p is true because $\forall i \in \mathbb{Z}^+$, $-1/i < 0 \leq x$. Likewise q is also true and there is no problem. So we need to consider examples in which $x < 0$, say $x = -0.001$. For p to be true, we must have that for all positive integers i,

$$-\frac{1}{i} < -0.001 = -\frac{1}{1000},$$

but this is clearly not true since we could set $i = 1001$, so $p(x)$ is not true for $x = -0.001$ so $q(x)$ does not need to be true (which it isn't). It appears that we could do something similar for any $x < 0$, but this is just a heuristic proof.

To prove the implication, we use proof by contradiction, that is, we assume

$$\neg r \equiv \{\exists x \in \mathbb{R} \text{ such that } p(x) \equiv \{\forall i \in \mathbb{Z}^+, -\frac{1}{i} < x\} \text{ and } \neg q(x) \equiv \{x < 0\}\} \tag{2.32}$$

and try to find a contradiction.

Claim 2.40: Statement 2.31 is true

Proof: We prove this statement using contradiction, that is, we assume statement 2.32 is true. Since we assumed at least one such an x exists, we let x_0 represent an arbitrary one of those values where

$$p(x_0) \wedge \neg q(x_0)$$

is true. Since x_0 satisfies $p(x_0)$, then we know that for every positive integer i,

$$-\frac{1}{i} < x_0$$

Since $\neg q(x_0)$, we know that $x_0 < 0$ which means we reverse the sign in the inequality when dividing by x_0, giving

$$-\frac{1}{x_0} > i$$

for every positive integer i. But since $-1/x_0 > 0$ is a known number, we know there exists an integer i larger than $-1/x_0$. Recall the **ceiling** function 17

$$f(x) = \lceil x \rceil = (\text{the smallest integer greater than or equal to } x).$$

Using this function, we set

$$i_0 = \left\lceil -\frac{1}{x_0} \right\rceil + 1 > \frac{-1}{x_0}.$$

Multiplying both sides by the negative number x_0 and dividing by the positive number i_0 gives

$$-\frac{1}{i_0} > x_0,$$

but one of our assumptions was that for every positive integer i,

$$-\frac{1}{i} < x_0.$$

This is a contradiction. So our assumption that for some x_0, $p(x_0) \wedge \neg q(x_0)$ was true led to a contradiction. Therefore

$$\neg\{\exists x \text{ such that } \; p(x) \wedge \neg q(x)\}$$

is true which is equivalent to our original implication.□

We now explore a difference between rational and irrational numbers and how to use this difference in a simple case. Later in the text, we explore other differences between rational and irrational numbers.

Definition 2.41: Rational

A number x is **rational** if and only if there exists integers n and $m \neq 0$ such that $x = n/m$.

A number is irrational if it is not rational, that is, the negation of the definition for rational.

Definition 2.42: Irrational

A number is **irrational** if and only if for all integers n and $m \neq 0$, $x \neq n/m$.

How do we show we cannot write a number as a fraction? Let's see one approach in which we show

$$p \equiv \{x = \sqrt{2} \text{ is irrational}\}.$$

Conceptual Insight: 2.43

First, we could try several values for n and m and show that $\sqrt{2} \neq n/m$ for those values.
Next, we would try to find values for n and m such that n^2 is about twice m^2, say $n = 7$ and $m = 5$. Since

$$2 \neq (7/5)^2 = 49/25, \text{ then } \sqrt{2} \neq 7/5.$$

While it is clear that $2 \neq 49/25$, one way to tell is to rewrite the equation as $50 \neq 49$ because one is even and one is odd. Whatever numbers we use for n and m, after rewriting the equation $\sqrt{2} = n/m$, we get

$$2m^2 = n^2.$$

This always results in an even integer on the left, so n must also be even. What if we let $n = 10$ and $m = 7$? Then we would have

$$2m^2 = n^2 \quad \text{or} \quad 2(49) = 100.$$

Other than the fact they clearly are not equal, we see if we divide both sides by 2, that the right side is even and the left side is odd. This must mean that in addition to n being even, m must also be even. But then the fraction is not reduced to lowest terms. This gives some conceptual insight into the proof.

- **Assumed:** $x = \sqrt{2}$.
- **To Show:** For every $n, m \in \mathbb{Z}$, $\sqrt{2} \neq n/m$.
- **Conceptual Insight:** From our start-up examples, it appears that the numerator must be even, which forces the denominator to be even, but the integers are supposed to be in lowest terms.
- **Technical Handle:** A direct proof would be difficult, since we would need to check every pair of integers. Contradiction seems a reasonable choice of methods since we are to show something is NOT true, $\sqrt{2} \neq n/m$. We now write $\neg p$ in usable form.

We now cycle back around, rewriting what we are assuming and what we have to show using contradiction.

- **Assumed:** $\neg p \equiv \{\exists\, n_0,\, m_0 \in \mathbb{Z}$ such that $\sqrt{2} = n_0/m_0$ and n_0 and m_0 are reduced to lowest terms $\}$.

- **To Show:** $\neg p$ is false, that is, something is wrong.
- **Technical Handle:** It is easier to deal with integers, so we square both sides, getting

$$2 = \frac{n_0^2}{m_0^2} \text{ or } 2m_0^2 = n_0^2.$$

This implies that n_0^2 is even. From Claim 2.28, this means n_0 is even, that is, for some $k_0 \in \mathbb{Z}$, $n_0 = 2k_0$. Substitution gives $2m_0^2 = 4k_0^2$ or

$$m_0^2 = 2k_0^2.$$

Since m_0^2 is even, then m_0 is even.
Remember that if we have a fraction, we can reduce it to lowest terms. Since n_0/m_0 was assumed to be reduced to lowest terms, n_0 and m_0 have no common factors. We have shown that n_0 and m_0 both have a factor of 2 which contradicts the lowest terms assumption.

Note it was the assumption that n_0 and m_0 have no common factors which led to the contradiction. When two integers n and m have no common factors, other than ± 1, then n and m are said to be **relatively prime**. We give a complete proof that $\sqrt{2}$ is irrational as a another model for how well written proofs might look.

Claim 2.44: The number $\sqrt{2}$ is irrational

Proof: To show $\sqrt{2}$ is irrational, we have to show that for every pair of integers n and m,

$$\sqrt{2} \neq n/m.$$

Instead, we prove this statement by assuming $\sqrt{2}$ is rational and finding a contradiction. Assume there are 2 integers n_0 and m_0 such that

$$\sqrt{2} = n_0/m_0,$$

and that n_0 and m_0 are relatively prime. Squaring both sides of $\sqrt{2} = n_0/m_0$ and multiplying by m_0^2 gives

$$2m_0^2 = n_0^2.$$

This means that n_0^2 is even, so n_0 is even.

Since n_0 is even, there exists a k_0 such that $n_0 = 2k_0$. Substitution gives

$$2m_0^2 = n_0^2 = (2k_0)^2 = 4k_0^2 \quad \text{or} \quad m_0^2 = 2k_0^2,$$

so m_0^2 is even. This means m_0 is even and $m_0 = 2i_0$ for some integer i_0. So n_0 and m_0 have a common factor, 2 which contradicts our assumption that n_0 and m_0 are relatively prime. Thus, $\neg p$ is false and p is true, $\sqrt{2}$ is irrational.□

2.5.1 Problems

1. The statement that if a is rational and b is irrational, then $a + b$ is irrational can be written as

   $$s \equiv \{\forall a \in \mathbb{Q} \text{ and } b \in \mathbb{Q}^c, a + b \in \mathbb{Q}^c\}.$$

   Write $\neg s$ and prove that s is true by getting a contradiction to statement $\neg s$. Answer 1

2. Consider the statement

   $$p \equiv \{\text{if } x^2 - 3x + 2 > 0 \text{ then } x > 2 \text{ or } x < 1\}.$$

   This implication is of the form

   $$\{\forall x \text{ s.t. } x^2 - 3x + 2 > 0, x > 2 \text{ or } x < 1\}.$$

   The negation is

   $$\neg p \equiv \{\exists x \text{ s.t. } x^2 - 3x + 2 > 0 \text{ and } \neg q\}.$$

   Write $\neg q$ and use this to arrive at a contradiction to $\neg p$.

3. Consider the statement

   $$p \equiv \{\text{if } x^2 - 2x - 3 > 0 \text{ then } x > 3 \text{ or } x < -1\}.$$

   This implication is of the form

   $$\{\forall x \text{ s.t. } x^2 - 2x - 3 > 0, x > 3 \text{ or } x < -1\}.$$

   The negation is

   $$\neg p \equiv \{\exists x \text{ s.t. } x^2 - 2x - 3 > 0 \text{ and } \neg q\}\}.$$

   Find $\neg q$ and use this to arrive at a contradiction to $\neg p$. Answer 2

4. Prove or disprove that $\sqrt[3]{2} \notin \mathbb{Q}$.
5. Let $a_0, b_0, d_0 \in \mathbb{Z}^+ - 1$. Let

$$p \equiv \{\exists\, n, m \in \mathbb{Z} \text{ such that } a_0 n + b_0 m = 1\}$$

and

$$q \equiv \{d_0 \text{ does not divide } a_0 \text{ or } d_0 \text{ does not divide } b_0\}.$$

Prove $p(a_0, b_0) \Rightarrow q(a_0, b_0)$. Answer 3

6. Consider the statement that if $0 < a < b < c < 1$, then $b - a < 0.5$ or $c - b < 0.5$. This is an implication of the form

$$\{p(a, b, c) \Rightarrow q(a, b) \vee r(b, c)\}.$$

Write the negation of this implication, then prove this statement is true using contradiction.

7. Consider the statement that if n^2 is divisible by 3, then n is divisible by 3, where $n \in \mathbb{Z}$. Prove this statement using contradiction. Answer 4
8. Consider the statement

$$\{\text{ if } x_1 \neq x_2 \text{ then } f(x_1) \neq f(x_2)\}.$$

Prove this statement is true using contradiction where $f(x) = 2x + 1$.

9. In this problem, you need to use the result of Problem 7 where you proved that if n^2 is divisible by 3, then n is divisible by 3, where $n \in \mathbb{Z}$.
   (a) Prove that $x = \sqrt{3}$ is irrational. Answer 5
   (b) Prove that $x = \sqrt{2} + \sqrt{3}$ is irrational. Hint: Using contradiction, square both sides of $\sqrt{2} = n/m - \sqrt{3}$. Answer 6
10. We are going to study roots of the equation

$$x^3 + x - k = 0 \tag{2.33}$$

for different integer values k.

   (a) Show that there exists a rational x_0 that satisfies equation 2.33 when $k = 2$.

(b) Show that if $k = 3$, then all roots x of equation 2.33 are irrational.

(c) Find several values for $k \in \mathbb{Z}$ such that equation 2.33 has a rational root.

(d) Does there exist $k \in \mathbb{O}$ such that this equation has a rational root? Why (prove your conjecture)?

11. Suppose $i, j \in \mathbb{Z}$. We consider the equation

$$ix^2 + jx + i = 0. \tag{2.34}$$

(a) Try finding integers i, j, and x such that $ix^2 + jx + i = 0$. Can you find a solution where $j, x \in \mathbb{E}$? How about a solution where $j \in \mathbb{E}$ and $x \in \mathbb{O}$? $j \in \mathbb{O}$ and $x \in \mathbb{E}$? $j, x \in \mathbb{O}$? Answer 7

(b) Prove or disprove the statement that if $j \in \mathbb{O}$, then $x \in \mathbb{E}$. Answer 8

2.5.2 Answers to selected problems

1. **Problem 1:** We write the negation as

$$\neg s \equiv \{\exists a \text{ and } b \text{ such that } a \in \mathbb{Q}, b \in \mathbb{Q}^c, \text{ and } a + b \in \mathbb{Q}\}.$$

Let a_0 and b_0 be the numbers that we assume exist. Since $a_0 \in \mathbb{Q}$, there exists integers n_0 and $m_0 \neq 0$ such that $a_0 = n_0/m_0$. Since $a_0 + b_0 \in \mathbb{Q}$, there exists integers i_0 and $j_0 \neq 0$ such that $a_0 + b_0 = i_0/j_0$. Since b_0 is irrational, there do not exist integers k andℓ such that $b_0 = k/\ell$. But, by substitution,

$$b_0 = (a_0 + b_0) - a_0 = i_0/j_0 - n_0/m_0 = (i_0 m_0 - n_0 j_0)/(j_0 m_0),$$

so b_0 does equal the ratio of two integers. This is a contradiction, so $\neg p$ is false and p is true.

2. **Problem 3:** The negation of $\{x > 3\} \vee \{x < -1\}$ is by DeMorgan's law

$$\neg\{x > 3\} \wedge \neg\{x < -1\} \equiv \{x \leq 3\} \wedge \{x \geq -1\} \equiv \{x \in [-1, 3]\}.$$

Then

$$\neg p \equiv \{\exists x \text{ s.t. } x^2 - 2x - 3 > 0 \text{ and } x \in [-1, 3]\}.$$

Suppose there is such an x_0. This means $x_0 \in [-1, 3]$ and $x_0^2 - 2x_0 - 3 = (x_0 - 3)(x_0 + 1) > 0$. Since $x_0 \in [-1, 3]$, then $x_0 - 3 \leq 0$ and $x_0 + 1 \geq 0$ so their product is $(x_0 - 3)(x_0 + 1) \leq 0$. This is a contradiction and the original statement is true.

3. **Problem 5:** We have

$$p \equiv \{\exists\, n, m \in \mathbb{Z} \text{ such that } a_0 n + b_0 m = 1\}$$

and

$$\neg q \equiv \{d_0 \text{ does divide } a_0 \text{ and } d_0 \text{ does divide } b_0\}.$$

We assume

$$\neg r \equiv \{\exists\, n, m \in \mathbb{Z} \text{ such that } a_0 n + b_0 m = 1 \text{ and } d_0 \text{ divides } a_0 \text{ and } b_0\}.$$

Let n_0 and m_0 be the two integers such that $a_0 n_0 + b_0 m_0 = 1$. Then there exists positive integers j_0 and k_0 such that $a_0 = j_0 d_0$ and $b_0 = k_0 d_0$. Substitution gives

$$a_0 n_0 + b_0 m_0 = j_0 d_0 n_0 + k_0 d_0 m_0 = 1,$$

or after factoring and dividing

$$d_0 = \frac{1}{j_0 n_0 + k_0 m_0} \le 1,$$

which contradicts that $d_0 > 1$.

4. **Problem 7:** We assume the negation of this statement

$$\{\exists n \in \mathbb{Z} \text{ such that } n^2 \text{ is divisible by 3 and } n \text{ is not divisible by 3}\}.$$

Let n_0 represent such a value. Then there exists an integer k_0 such that $n_0^2 = 3k_0$ and there exists an integer j_0 such that $n_0 = 3j_0 + 1$ or $n_0 = 3j_0 + 2$.

Case 1: Suppose there exists an integer j_0 such that $n_0 = 3j_0 + 1$. Then

$$n_0^2 = (3j_0 + 1)^2 = 9j_0^2 + 6j_0 + 1 = 3(3j_0^2 + 2j_0) + 1 = 3k_0.$$

Therefore, $3(3j_0^2 + 2j_0 - k_0) = 1$ so 1 is divisible by 3, which is a contradiction.

Case 2: Suppose there exists an integer j_0 such that $n_0 = 3j_0 + 2$. Then

$$n_0^2 = (3j_0 + 2)^2 = 9j_0^2 + 12j_0 + 4 = 3(3j_0^2 + 4j_0 + 1) + 1 = 3k_0.$$

Therefore, $3(3j_0^2 + 4j_0 + 1 - k_0) = 1$ so 1 is divisible by 3, which is a contradiction.

In both cases, we have a contradiction, so we have proven our original statement, if n^2 is divisible by 3, then n is divisible by 3

5. **Problem 9 a:** We need to show that

$$p \equiv \{\langle \forall\, n, m \in \mathbb{Z}\rangle\, \langle \sqrt{3} \neq \frac{n}{m}\rangle\}.$$

We will try contradiction, that is we assume

$$\neg p \equiv \{\langle \exists\, n_0, m_0 \in \mathbb{Z}\rangle \text{ such that } \langle \sqrt{3} = \frac{n_0}{m_0}\rangle\}.$$

We assume n_0 and m_0 have no common factors, since otherwise we could reduce the fraction n_0/m_0. With $\sqrt{3} = n_0/m_0$, we multiply by m_0 and square both sides, giving

$$3m_0^2 = n_0^2.$$

Since $3m_0^2$ is divisible by 3, then n_0^2 is divisible by 3 which means n_0 is divisible by 3, that is, there exists $k_0 \in \mathbb{Z}$ such that $n_= 3k_0$. This gives $3m_0^2 = 9k_0^2$ or $m_0^2 = 3k_0^2$. Thus, m_0^2 is divisible by 3, so m_0 is divisible by 3, that is, there exists an integer j_0 such that $m_0 = 3j_0$. Thus, m_0 and n_0 have a common factor, 3, which contradicts our assumption. Thus, $\neg p$ is false so p is true, $\sqrt{3}$ is irrational.

6. **Problem 9 b:** We prove that $\sqrt{2} + \sqrt{3}$ is irrational using contradiction, that is we assume

$$\neg p \equiv \{\langle \exists\, n_0, m_0 \in \mathbb{Z}\rangle \text{ such that } \langle \sqrt{2} + \sqrt{3} = \frac{n_0}{m_0}\rangle\}$$

and derive a contradiction.

Proof: Assume there exists $n_0, m_0 \in \mathbb{Z}$ such that $\sqrt{2} + \sqrt{3} = n_0/m_0$. We square both sides of

$$\sqrt{2} = n_0/m_0 - \sqrt{3},$$

giving

$$2 = \frac{n_0^2}{m_0^2} - \frac{2n_0\sqrt{3}}{m_0} + 3.$$

Solving this equation for $\sqrt{3}$ results in $\sqrt{3}$ equaling an integer over an integer. This means $\sqrt{3}$ is rational, which contradicts the result of the first part. Thus $\sqrt{2} + \sqrt{3}$ is irrational.$\square$

7. **Problem 11 a:** One approach would be to solve for j, giving

$$j = -ix - i/x,$$

so if x divides i, then j is an integer. A solution where both j and x are even is $i = 4$, $x = 2$ and $j = -10$. A solution where x is even and j is odd is $i = 6$, $x = 2$ and $j = -15$. A solution where x is odd and j is even is $i = 9$, $x = 3$ and $j = -30$. You will not be able to find a solution where both x and j are odd.

8. **Problem 11 b:** (Proof by contraposition) Assume i, j and x satisfy equation 2.34. The negation of the statement that if j is odd then x is even is

$$\neg p \equiv \{\exists\, x, j \in \mathbb{O} \text{ such that } ix^2 + jx + i = 0\}.$$

Assume $\neg p$. We assume there exists $i_0 \in \mathbb{Z}$, and j_0, $x_0 \in \mathbb{O}$ such that $ix^2 + jx + i = 0$. This means there exist integers n_0 and k_0 such that $x_0 = 2n_0 + 1$ and $j_0 = 2k_0 + 1$. Substitution gives

$$i_0 x_0^2 + j_0 x_0 + i_0 = i_0(2n_0 + 1)^2 + (2k_0 + 1)(2n_0 + 1) + i_0.$$

This simplifies to

$$2(2n_0^2 + 2n_0 + 1)i_0 + 2(2k_0 n_0 + n_0 + k_0) + 1,$$

which is an odd integer. Since 0 is an even integer, this leads to a contradiction, so p is true.

Chapter 3

Induction

3.1 Gaining Insight through Patterns

Video Lesson 3.1

This section will be easier to read if you first watch the approximately 8 minute video
https://vimeo.com/73398243 (Password:Proof)
in which we go through the process of gaining conceptual insight into why a particular statement is true. To change the speed at which the video plays, click on the gear at the lower right of the video.

In this section, we are going to gain insight into statements by developing heuristic arguments that are quite convincing to ourselves. In the following sections, we are going to learn how to turn these heuristic arguments into formal proofs that will be convincing to others. We will start with two examples.

Example 3.2

Let $s_1 = \sqrt{2} \approx 1.41421$,

$$s_2 = \sqrt{2 + s_1} = \sqrt{2 + \sqrt{2}} \approx 1.84776,$$

$$s_3 = \sqrt{2 + s_2} = \sqrt{2 + \sqrt{2 + \sqrt{2}}} \approx 1.96157,$$

and continuing, $s_4 \approx 1.99037$, $s_5 \approx 1.99759, \cdots$ using the pattern that

$$s_{n+1} = \sqrt{2 + s_n} \text{ for } n \in \mathbb{Z}^+.$$

If we keep computing, we see that s_n is always less than 2 but appears to be increasing toward 2. Let's see if we can show the first of these statements, that is, for all $n \geq 1$, $s_n < 2$.
We note that $s_1 < 2$ and $s_2 = \sqrt{2 + s_1}$. Since $s_1 < 2$, then $2 + s_1 < 4$ so its square root, $s_2 = \sqrt{2 + s_1} < 2$. It appears that we keep taking the square root of 2 plus a number which is less than 2. Since the total is less than 4, the square root is less than 2. For example, since we know $s_5 \approx 1.99759$, then $2 + s_5 \approx 3.99759 < 4$ so $s_6 = \sqrt{2 + s_5} < 2$. If we keep repeating this argument, we will continue getting $s_n < 2$ so $2 + s_n < 4$ so $s_{n+1} = \sqrt{2 + s_n} < 2$. Thus $s_n < 2$ for all $n \geq 1$.

The heuristic argument in Example 3.2, while not being a complete formal proof, does give us conceptual insight into the key idea why this statement is true, which is that we keep taking the square roots of numbers which are 2 plus a number less than 2.

A classic mathematics problem is to determine how many colors are needed to color a map so that no two countries with a common border are colored the same color. It is assumed that no country consists of two separate pieces. It is okay for two countries that meet at just a point to be colored the same color. For example, a chess board is two colorable since it can be colored with two colors, usually red and black. In 1976, Kenneth Appel and Wolfgang Haken proved that any planar map can be colored with at most 4 colors, with every pair of countries sharing a border being different colors. The proof of this result is far beyond this text, but the reader is encouraged to find maps that require four colors and to try to find one that requires five to gain insight into why four colors are sufficient.

We now consider a simpler problem. Suppose we have a rectangular map that is formed by drawing any number of straight lines across the rectangle. Each line must start on one side of the rectangle and end on another side, not necessarily the opposite side. How many colors are required to color this map? How does the number of colors depend on the number of lines? In Figure 3.1 are two such maps, one with 4 lines and one with 5 lines. Both have been colored with just two colors, which is clearly the smallest number of colors possible since neither can be colored with just one color.

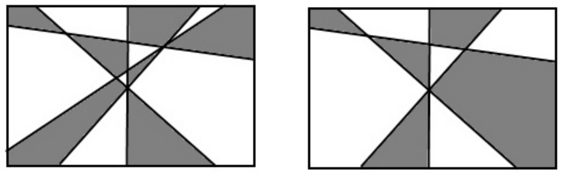

Figure 3.1: Two maps that have been 2-colored.

To try to understand this problem, we would normally begin drawing a number of maps and try coloring them, which the reader should do. In each case, you will find that it is easy to color the map with two colors, but you will probably not understand why it is always easy as the maps can be quite complicated. Many students at this point would stop generating examples and would try to prove the result without having any idea on how to go about it. This is the point where we should come back and be more systematic in our example construction. We could start with a map with just one line and color it. Then add a line to that map and recolor it. Then add another line and recolor it again, as seen in Figure 3.2. Some readers may notice a pattern in the coloring each time another line is added.

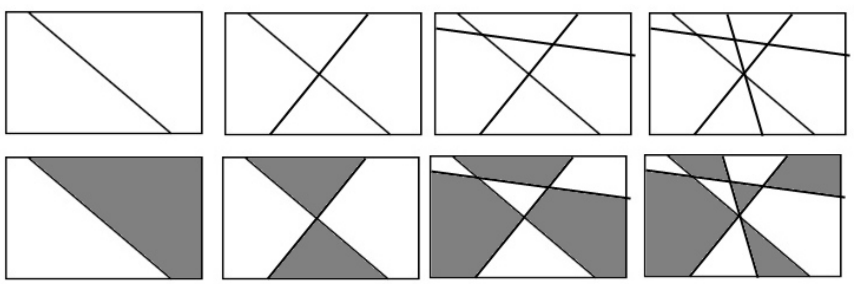

Figure 3.2: Four maps that have been 2-colored. Each map results from adding a line to the previous map, going left to right, then is recolored appropriately.

If we have not noticed a pattern, we continue our process by looking a little closer at what is happening. On the left of Figure 3.3 a) is a map with 4 lines that has been 2-colored and a fifth line is added without the map being recolored. On the right of Figure 3.3 a) is the same map with 5 lines that has been 2-colored. Again, we may or may not notice a pattern. If not, then in Figure 3.3 b) is the portion of the maps in Figure 3.3 a) below the 5th line. Here, we cannot avoid noticing that the coloring is identical. In

Figure 3.3 c) is the portion of each map of Figure 3.3 a), but above the fifth line. Here, we notice the coloring has been reversed. We now see, if we had not seen it earlier, that when another line is added, all we have to do is switch all the colors on one side of the line.

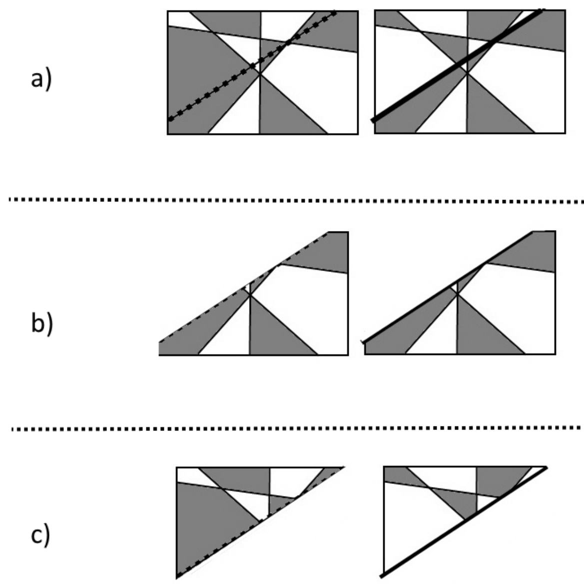

Figure 3.3: Left map in a) has four lines and is 2-colored. A fifth dotted line is added. Map on right of a) is recolored version of map on left with fifth line added. Figure b) gives portions of maps from a) below the fifth line added. Notice coloring is the same. Figure c) gives portions of maps from a) above the fifth line added. Notice coloring is the reverse.

Conceptual Insight: 3.3: Map coloring:

Let's think about why the process of changing the colors on one side of the last line added results in a 2-coloring. All the countries on the unchanged side of the line are properly colored since if two of them have a common border before the last line was added, then they are different colors and since nothing has changed on that side, they still have different colors. If two countries have a common border on the changed side of the line, then they had different colors before the last line was added and since the colors of both have change, then the colors are still different. What about two countries with a common border but on different sides of the last line. Then a segment of the last line must be the common border, meaning the last line divided one country into two countries. Since the country was all one color before adding the line, then the portion of the country on the changed side of the last line must have changed, so again, the countries are colored differently. If you had difficulty following this argument, rewatch video 3.1.

Summary 3.1

In these examples we were trying to prove a collection of statements which we will call $p(n)$, $n \geq 1$. In the first example, the statement was

$$p(n) \equiv \{s_n < 2\},$$

while in the second example, the statement was

$$p(n) \equiv \{\text{Every rectangular map consisting of } n \text{ lines can be two colored.}\}$$

We checked that the statement was true for the first several values of n. Then we developed a process to use the truth of the statement for one value of n to prove the statement for the next value. In the first example, we used the fact that if, for some n, $s_n < 2$, then $s_{n+1} < 2$ also. In the second example, we used that if we knew a map with n lines was 2-colorable, we could construct a 2-coloring on the map with an $n+1$ line added.

This approach is called **process pattern generalization** in that we have a process that generated a pattern going from one value to the next, and we generalized it for all values. This is not yet a proof, but it should be fairly convincing. In the Exercises, you should look for a process that allows you to go from one step to the next.

3.1.1 Problems

1. Suppose we have a chess board that is 2^n-by-2^n (2-by-2, 4-by-4, 8-by-8, $\cdots$), $n \geq 1$, with one corner piece removed. Can you cover your chess board with L-shaped pieces consisting of three squares without any of the L's overlapping? See Figure 3.4 which displays a 2-by-2 and 4-by-4 chessboard. Answer 1

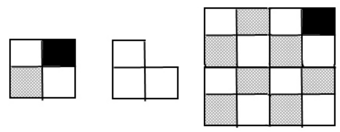

Figure 3.4: 2-by-2 and 4-by-4 chessboards with corner missing to be covered by copies 3-square piece in middle.

2. Find a formula for

$$\prod_{j=2}^{n}(1-\frac{1}{j}) = (1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n}),$$

$n \geq 2$ and show that your formula is correct.

3. Consider a 2-player game in which there is a pile of N beads. On each player's turn, that player can remove 1, 2, or 3 beads from the pile. The player who takes the last bead wins the game.

   (a) Suppose $N = 10$. What is the optimal strategy and which player should win the game? Answer 2

   (b) Suppose $N = 20$. What is the optimal strategy and which player should win the game? Answer 3

   (c) For what values of N can the second player always win? Write a heuristic argument explaining why your approach should work? Answer 4

   (d) For what values of N can the first player always win? What is the optimal strategy and explain your answer? Answer 5

4. Notice that each of the sums

$$1+3,\ 1+3+3^2+3^3,\ 1+3+3^2+3^3+3^4+3^5,\cdots$$

are divisible by 4.

   (a) Explain why the sum

$$1+3+3^2+\cdots+3^n$$

   is divisible by 4 when n is odd.

   (b) Use similar reasoning to show that the sum is never divisible by 4 if n is even.

5. Draw a square, which is one region. If we draw one line across the square, we have two regions. If we draw a second line across the region, intersecting the first line, we have four regions. If we draw a third line intersecting the other two lines, and no three lines intersecting at the same point, we have seven regions. Come up with a formula for computing the number of regions if n lines are drawn across the square, with every pair of lines intersecting once and no three lines intersecting at the same point. Explain your answer. Answer 6

6. The Power Set $P(S)$ of a set S is the set of all subsets of S. For example, if $S_0 = \emptyset$ then $P(S_0) = \{\emptyset\}$, which is the set that contains the null set as an element. Let
$$S_n = \{1, 2, \cdots, n\},$$
that is, S_n is the set that contains the first n integers, with S_0 being the null set or empty set. Let $C(S)$ be the number of elements in a set. Then $C(S_n) = n$. Note that $C(P(S_0)) = 1$ since $P(S_0)$ is the set that contains the empty set as its one element. Since
$$P(S_1) = \{\emptyset, \{1\}\} \text{ then } C(P(S_1)) = 2.$$
Also,
$$P(S_2) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\},$$
so $C(P(S_2)) = 4 = 2^2$. Note that elements of $P(S_2)$ are sets, that is, $\{1\} \in P(S_2)$ while $\{1\} \subseteq S_2$. Similarly, $\emptyset$ is a set containing no elements, but $\emptyset$ is an element of $P(S_2)$, that is, $\emptyset \in P(S_2)$.

   (a) Let $S_3 = \{1, 2, 3\}$. List the elements in $P(S_3)$ and find $C(P(S_3))$. Try to do this in an organized manner, listing all the sets not containing 3 and all the sets containing 3.

   (b) Let $S_4 = \{1, 2, 3, 4\}$. List the elements in $P(S_4)$ and find $C(P(S_4))$. Try to do this in an organized manner, listing all the sets not containing 4 and all the sets containing 4.

   (c) Explain why $C(P(S_{n+1})) = 2C(P(S_n))$. Use this to explain why $C(P(S_n)) = 2^n$.

7. A diagonal in a polygon is a line that connects any two non-adjacent vertices. A polygon is convex if and only if every diagonal lies entirely within the polygon. Develop a relationship between the number of diagonals in a convex polygon with n vertices and a convex

polygon with $n+1$ vertices, Letting d_n represent the number of diagonals in a convex polygon with n sides (called an n-gon), note in Figure 3.5 that $d_3 = 0$, $d_4 = 2$, and $d_5 = 5$. Argue that your relationship is correct. Note that this relationship gives that the number of diagonals of a convex polygon with n vertices is given by a simple finite sum. Answer 7

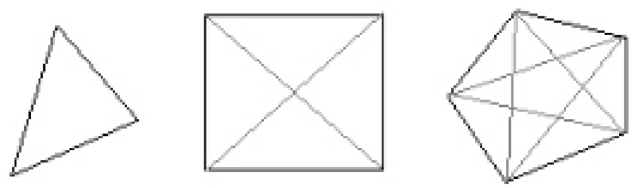

Figure 3.5: Number of diagonals for polygon with 3, 4, and 5 vertices is 0, 2, and 5, respectively.

8. Find a pattern for the remainder when 7^n is divided by 4, $n \geq 0$. Give an argument that your pattern is correct.
9. Consider the sum

$$\sum_{i=0}^{n}(-1)^{i+1}4^i = -1 + 4 - 16 + \cdots + (-1)^n 4^{n-1} + (-1)^{n+1}4^n.$$

   (a) Note that if n is odd, this sum is always divisible by 3. Explain why this is true by combining the last two numbers added. Answer 8

   (b) Note that if n is even, the argument used in the previous part still holds, but the sum is never divisible by three. Explain what is wrong. Answer 9

10. Consider the two person game in Problem 3 in which there is a pile of n beads and on each turn, a player picks up 1, 2, or 3 beads. Suppose the rules are changed in that the last person who picks up a bead loses. Determine the optimal strategy for playing this game. In particular, for what values of n can player 2 always win, if playing optimally? Give a heuristic argument in support of your conjecture.

3.1.2 Answers to selected problems

1. **Problem 1:** See the 8-by-8 chessboard in Figure 3.6. Can you continue

this process?

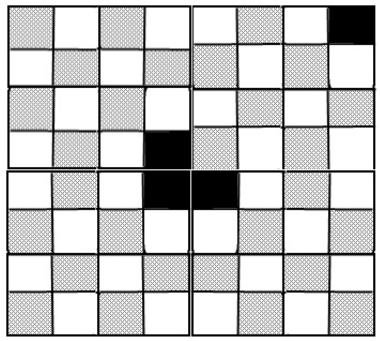

Figure 3.6: Can you use your covering of the 4-by-4 chessboard to cover this 8-by-8 chessboard? Only the upper right corner is actually missing.

2. **Problem 3 a:** The first player will win by removing 2 beads bringing the total to 8. Whatever the second player removes, the first player can remove an amount that brings the total to 4. Then, whatever the second player removes, the first player can remove the rest.

3. **Problem 3 b:** The second player will win. Whatever the first player removes, the second player can bring the total to 16, then to 12, then to 8, then to 4, and then, whatever the first player removes, the second player can remove the rest.

4. **Problem 3 c:** If the total number of beads is a multiple of 4, the second player can win. We have seen that the second player can win if the total is 4, 8, 12, 16, and 20. We can use this to show the second player wins if the total is 24, then 28, and so forth, working up to any multiple of 4.

5. **Problem 3 d:** The first player will win if the total is not a multiple of 4. If the total is $4n+1$, the first player removes 1 to get to a multiple of 4 and by the previous part, this player can now win. The same approach will work if the total is $4n+2$ or $4n+3$; The first player just removes the number of marbles needed to bring the total to a multiple of 4.

6. **Problem 5:** For n lines, the number of regions will be

$$1+1+2+3+\cdots+n.$$

This formula works for the first several lines. Suppose it works for n lines. If we add an $n+1$th line, it intersects n other lines. Each time it intersects a line, one region is split into two regions, for one additional region. Thus, we will add n new regions. We add one additional region as the line goes to the border, splitting one additional region into two. So for a total of $n+1$ new regions, so there will be

$$1+1+2+3+\cdots+n+(n+1)$$

regions.

7. **Problem 7:** Let n be the number of sides of a polygon. Clearly, $n \geq 3$. The values for numbers of diagonals in Figure 3.7 suggest the following pattern: each time a new vertex is added to a polygon with n sides to get an $n+1$ sided polygon, the number of diagonals increases by $n-1$. For example, if $n=5$, then we add a vertex to get an $n+1=6$ sided polygon and the number of diagonals increases by $n-1=4$.

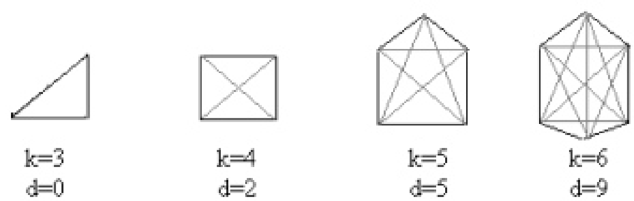

Figure 3.7: Polygons with 3, 4, 5 and 6 sides.

$$\begin{aligned} n=4 &\quad \text{yields} \quad d_4=2, \\ n=5 &\quad \text{yields} \quad d_5=2+3, \\ n=6 &\quad \text{yields} \quad d_6=2+3+4, \\ n=7 &\quad \text{yields} \quad d_7=2+3+4+5. \end{aligned}$$

This suggests that if we let d_n be the number of diagonals of an n-sided convex polygon, then

$$d_{n+1}=d_n+n-1.$$

By repeated addition, this would give that

$$d_n=2+3+\cdots+(n-2),$$

that is, the number of diagonals is the sum from 2 to the number of sides minus 2.

Let's see why this happens. Take an n-sided convex polygon and suppose it has d_n diagonals. (Figure 3.8 helps us understand the argument.) We add a new vertex called v (see Figure 3.8) between two other vertices, u and w. Now the line from u to w, which was a side of the n-sided polygon is now a diagonal of our $n+1$-sided polygon. Plus we pick up a diagonal for each of the other vertices connected to v. Since there are $n+1$ vertices, but u, v,and w are three of them, this is another $n+1-3 = n-2$ diagonals. Thus, the number of new diagonals is $1+(n-2) = n-1$ and we have our relation $d_{n+1} = d_n + n - 1$.

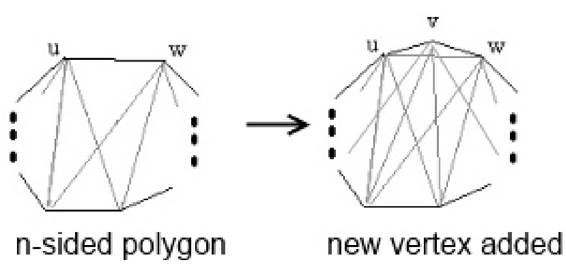

Figure 3.8: Creating $n+1$-sided polygon from n-sided polygon.

8. **Problem 9 a:** Note that the last two numbers added are

$$(-1)^n 4^{n-1} + (-1)^{n+1} 4^n = (-1)^n 4^{n-1}(1-4) = 3((-1)^{n+1} 4^{n-1}),$$

which is divisible by 3. So as we go from one odd sum to the next odd sum, we are adding two terms whose sum is divisible by 3. The first odd sum is $-1+4$ which is divisible by 3, so if we continue adding something divisible by 3 to something that is divisible by 3, it will still be divisible by 3.

9. **Problem 9 b:** The factoring of the last two terms added is still the same, but the first even sum is -1, not 3. So we continue adding pairs of terms that are divisible by 3 to sums which are not divisible by 3, meaning the new sum is not divisible by 3.

3.2 Introduction to Induction

Video Lesson 3.4

This section will be easier to read if you first watch the approximately 10:27 minute video
https://vimeo.com/76630587 (Password:Proof)
which introduces proof by induction. To change the speed at which the video plays, click on the gear at the lower right of the video.

In this section, we are going to learn how to formalize heuristic arguments of the form seen in the previous section. The method is called **proof by induction** and it adds one more proof technique to our repertoire of technical handles. This approach often works when we are trying to prove that a collection of statements $p_1, p_2, \cdots$ is true.

Let's consider a sequence of numbers, $a_1, a_2, \ldots$ Suppose we are given that $a_1 = 2$ and that each number in the sequence is twice the previous number, $a_{n+1} = 2a_n$ for $n = 1, \ldots$. In this case, we have enough information to find each number in the sequence. In particular, $a_2 = 4$, $a_3 = 8$, $\ldots$, $a_n = 2^n$, $\ldots$. The idea is that if we know a first value and we have a rule for finding each value from the previous value, then we can find any of the values. Even if we haven't computed a value, we can consider it known, because we have a method for finding it.

Induction is a proof technique that is similar to finding the values in the sequence a_n. In particular, we have a sequence of statements p_n, $n = 1, 2, \ldots$. If we know:

- The first statement p_1 is true.
- Each statement is true if the previous statement is true.

then we know that all of the statements must be true, because for any statement, p_n, we have a method for showing it is true, just as we had a method for finding the number a_n. We see this visually in Figure 3.9.

We saw this in the examples in Section 3.1. In Example 3.2 we gave an argument that all of the numbers s_n, $n \geq 1$ were less than 2. We noted the first one was less than two, and we wrote an argument that if one of these numbers was less than 2, then the next one must be less than 2 also. In the example 3.3, we gave an argument that all maps composed of n lines going

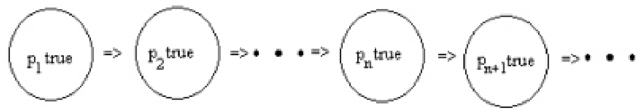

Figure 3.9: Visualization of Principle of Induction.

from one side to another, $n \geq 0$, could be 2-colored. We showed the graph with zero lines could be 2-colored and we gave a method for 2-coloring any map if we could 2-color a map with one fewer line.

The process is fairly simple, logically. For example, if we wanted to show that p_6 is true, we would first show that p_1 is true. We use this to show p_2 is true, from which it follows that statement p_3 is true, from which it follows that statement p_4 is true, from which it follows that statement p_5 is true from which it follows that statement p_6 is true.

Principle: 3.5: Induction:

Suppose we have a sequence of statements p_1, p_2, p_3, $\ldots$, and that we know:

- *Base case:* Statement p_1 is true.
- *Induction step:* For every $n \in \mathbb{Z}^+$, if statement p_n is true, then p_{n+1} must be true.

Then all of the statements are true.

To show the Induction step is true means we must prove the implication

$$p_n \Rightarrow p_{n+1}$$

for all positive integers n. This is done using a variation on the let method. We 'assume' there exists an arbitrary positive integer n_0 such that p_{n_0} (Assumed) is true. We then show p_{n_0+1} (To Show) is true. It is important to note that we are now treating n_0 as a known fixed number. We are not assuming p_n is true for all n, we are just assuming it is true for this one n_0-value.

A sum you may be familiar with is

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \text{ for } n = 1, 2, \ldots$$

when $n \geq 1$. We will now prove that this sum is correct using induction, that is, we will show the truth of

$$p_n \equiv \left\{ \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \right\}$$

for all $n \geq 1$.

Conceptual Insight: 3.6: Sum first n integers:

The conceptual insight behind why

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

is that, if we define

$$s_n = 1 + 2 + \cdots + (n-1) + n,$$

then we can also write s_n as

$$s_n = n + (n-1) + \cdots + 2 + 1.$$

By adding corresponding terms in the two equations, this gives that

$$2s_n = (n+1) + \cdots + (n+1) = n(n+1)$$

since $(n+1)$ is added n times. Then division by 2 gives our result. This is a heuristic that allows us to see why the statement is true. This is not satisfactory as a proof because of the use of '$\cdots$'. Induction is the technical handle that allows us to construct a more acceptable proof.

A proof by induction always consists of 2 parts.

Claim 3.7: For every $n \geq 1$,

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

Proof: We use induction to prove this claim.
Base case: We see that

$$p_1 \equiv \left\{ \sum_{j=1}^{1} j = \frac{1(1+1)}{2} \right\}$$

is true since each side of the equation equals 1.
Induction step: (Assumed) We assume there exists a positive integer n_0 such that p_{n_0} is true. In other words, we assume that

$$\sum_{j=1}^{n_0} j = \frac{n_0(n_0+1)}{2}.$$

(To show) We show that p_{n_0+1} is true, that is,

$$\sum_{j=1}^{n_0+1} j = \frac{(n_0+1)(n_0+2)}{2}$$

using a forward proof. We rewrite the sum as,

$$\sum_{j=1}^{n_0+1} j = \sum_{j=1}^{n_0} j + (n_0+1).$$

Since we have assumed that p_{n_0} is true, we can substitute

$$\frac{n_0(n_0+1)}{2} \text{ for } \sum_{j=1}^{n_0} j,$$

giving

$$\begin{aligned} \sum_{j=1}^{n_0} j + (n_0+1) &= \frac{n_0(n_0+1)}{2} + \frac{2(n_0+1)}{2}. \\ &= \frac{n_0^2 + 3n_0 + 2}{2} \\ &= \frac{(n_0+1)(n_0+2)}{2} \end{aligned}$$

Thus, we have shown

$$p_{n_0+1} \equiv \left\{ \sum_{j=1}^{n_0+1} j = \frac{(n_0+1)(n_0+2)}{2} \right\},$$

which is what we were to show. So if p_{n_0} is true, then p_{n_0+1} is true. By the Principle of Induction,

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

is true for all $n \in \mathbb{Z}^+$. □

Usually the base case, showing p_1 is true, is simple. To show the induction step, that p_{n+1} is true assuming p_n is true, we use the methods we have previously studied to prove the implication, $p_n \Rightarrow p_{n+1}$. Often the proof is a relatively simple application of the let method combined with the backward/forward approach, but sometimes we need to combine induction with contraposition or contradiction.

In Example 3.2 and the map coloring 3.3, the induction step not only proved that $p_n \Rightarrow p_{n+1}$, it gave insight into why the statements were true. In the sum problem 3.6, the heuristic argument that gave us insight into why the statement was true was quite different from the induction proof which was used to actually verify the statement. Induction proofs are generally a technical handle used to verify a statement: they do not always give conceptual insight into why the statement is true. Therefore, do not worry if you do an induction proof but are still not quite certain why it worked.

It is irrelevant that the first statement is numbered 1. Sometimes the sequence of statements begins at some point other than 1.

Principle: 3.8: General Induction:

Suppose we have a sequence of statements p_n, $n = k_0, k_0+1, \ldots$, and we know:

- *Base case:* Statement p_{k_0} is true and
- *Induction step:* For every $n \geq k_0$, if statement p_n is true, then

p_{n+1} must be true.
Then p_n is true for $n = k_0, k_0 + 1, \ldots$.

All that is really relevant is that we show 1) p_{k_0} is true for some $k_0 \in \mathbb{Z}$, and 2) if p_n is true then p_{n+1} is true, for $n \geq k_0$. Then p_n is true for $n \geq k_0$.

We now give a proof by induction with a different starting point, showing that
$$2^n \geq n^2 \text{ for } n \geq 4.$$

Conceptual Insight: 3.9

In this case, we have the sequence of statements $p_n \equiv \{2^n \geq n^2\}$ for $n \geq 4$.
Start-up Examples: The first step is to show that p_4 is true, that is $2^4 = 16 \geq 4^2 = 16$, which is clearly true. We note that the statement is not true when $n = 3$ since $2^3 < 3^2$. When we let $n = 5$, we get $2^5 = 32 > 5^2 = 25$. The statement seems to be true because exponentials, 2^n, grow faster than quadratics, n^2, which is a conceptual insight behind this statement.
We have already shown the base case is true. The induction step is to assume there exists an integer $n_0 \geq 4$ such that p_{n_0} is true.

Assumed true: $p_{n_0} \equiv \{2^{n_0} \geq n_0^2\}$, $n_0 \geq 4$

To Show: $p_{n_0+1} \equiv \{2^{n_0+1} \geq (n_0+1)^2\}$

Technical Handle: At this point, we will try a forward approach, taking the left term, 2^{n_0+1}, and seeing if we can work toward the right term, $(n_0+1)^2$. Using what is assumed results in
$$2^{n_0+1} = 2(2^{n_0}) \geq 2n_0^2.$$

What we want is
$$2^{n_0+1} \geq (n_0+1)^2.$$

If we can show that
$$2n_0^2 \overset{?}{\geq} (n_0+1)^2,$$

we are finished.

We now develop conceptual insight into why the expression

$$2n^2 \geq (n+1)^2$$

when $n \geq 4$.

Conceptual Insight: 3.10: $2n_0^2 \overset{?}{\geq} (n_0+1)^2$

Now we use a backwards approach in working with

$$2n_0^2 \overset{?}{\geq} n_0^2 + 2n_0 + 1 \quad \text{or} \quad n_0^2 - 2n_0 - 1 \overset{?}{\geq} 0.$$

Completing the square gives

$$n_0^2 - 2n_0 - 1 = (n_0 - 1)^2 - 2 \overset{?}{\geq} 0.$$

Since we know $n_0 \geq 4$, then

$$(n_0 - 1)^2 \geq 3^2 = 9,$$

so

$$n_0^2 - 2n_0 - 1 = (n_0 - 1)^2 - 2 \geq 7.$$

We can now piece this into a proof. To begin, we first prove the result 3.10 which we need for the induction proof.

Lemma 3.11: $n^2 - 2n - 1 \geq 0$ for $n \geq 4$.

Proof: Let $n_0 \geq 4$. *Then*

$$n_0^2 - 2n_0 - 1 = (n_0 - 1)^2 - 2 \geq 3^2 - 2 = 7 > 0.$$

□

Remark 3.12

An alternate proof of the Lemma would be to first find the roots of $n^2 - 2n - 1$ which are $n = 1 \pm \sqrt{2}$. Our knowledge of algebra is that the polynomial $f(n) = n^2 - 2n - 1$ will have the same sign for all n-values greater than the largest root, so $n^2 - 2n - 1 > 0$ for $n \geq 4 > 1 + \sqrt{2}$.

We are now ready to prove our main result.

Claim 3.13: If $n \geq 4$, $2^n \geq n^2$.

Proof: We use induction to prove our claim.

Base case: When $n = 4$, we get that $2^n = 16 \geq 16 = n^2$.

Induction step: We assume there exists an integer $n_0 \geq 4$ such that $2^{n_0} \geq n_0^2$. Then

$$2^{n_0+1} = 2(2^{n_0}) \geq 2n_0^2 = n_0^2 + n_0^2$$

by assumption. By Lemma 3.11,$n_0^2 \geq 2n_0 + 1$ so

$$2^{n_0+1} \geq 2n_0^2 = n_0^2 + n_0^2 \geq n_0^2 + 2n_0 + 1 = (n_0 + 1)^2,$$

which is what we needed to show.□

Let's again note the logic on which induction is based. We show some statement is true, say p_{k_0}. Since p_{k_0} is true, we can use that to show statement p_{k_0+1} is true. Now that statement p_{k_0+1} is true, we can use that to show p_{k_0+2} is true. For any $n_0 \geq k_0$, at some point p_{n_0} will be shown to be true and that can be used to show p_{n_0+1} is true. We can keep repeating the same general argument to show p_n is true for any $n \geq k_0$. So p_n must be true $\forall\, n \in \{k_0, k_0 + 1, \cdots\}$.

Summary 3.2

- When proving a statement, our backwards work often reveals a result that we need to finish our proof. We usually state this side result as a lemma and prove it before starting the proof of the main result, as we did in Proof 3.13.
- An induction proof consists of a Base Case and the Induction Step. We first prove the base case is true and then assume the statement is true for some value. Because the Base Case is usually easy to show, students often underestimate its value. **Without the Base Case, there is no induction proof.** The induction step is based on the fact that we know the statement is true for some value.

Video Lesson 3.14

See the 5:46 minute video
https://vimeo.com/76709853 (Password:Proof)
to see a simple example that brings home the importance of the base case.

3.2.1 Problems

1. Find the error in the following induction proof.

   Claim 3.15: All even positive integers are equal.

   Proof: To prove this claim, we will prove that the statements $p_n = \{2n = 2n + 2\}$ are true for all $n \in \mathbb{Z}^+$. Assume there exists a positive integer n_0 such that p_{n_0} is true, that is, $2n_0 = 2n_0 + 2$. Adding 2 to both sides gives that $2n_0 + 2 = 2n_0 + 4$ or $2(n_0 + 1) = 2(n_0 + 1) + 2$, so p_{n_0+1} is true. Therefore, by induction, all even positive integers are equal.$\square$

   Answer 1

2. Find the error in the following induction proof.

   Claim 3.16: All students at Georgetown have the same major.

   Proof: Define the statements
   $p_n \equiv \{$ All students in every set of n GU students have the same major $\}$.
   Base case: p_1 is true because any one student has the same major as him or herself.
   Induction step: Assume there exists a positive integer n_0 such that p_{n_0} is true where
   $p_{n_0} \equiv \{$ All students in every set of n_0 students have the same major $\}$.
   We are now going to show that p_{n_0+1} is true, that is
   $p_{n_0+1} \equiv \{$ All students in every set of $n_0 + 1$ students have the same major $\}$.

We begin using the let method, that is, we pick an arbitrary set

$$G = \{s_1, s_2, \cdots, s_{n_0}, s_{n_0+1}\}$$

of $n_0 + 1$ Georgetown students, where

s_i represents the ith student in the set.

All we need to do is show that all of these students have the same major. Take the subset

$$S = \{s_1, s_2, \cdots, s_{n_0}\}$$

of n_0 of these students. By the assumption, all n_0 of these students have the same major, as seen on left of Figure 3.10. This means that all students, s_i, $i \in \{1, \cdots, n_0\}$ have the same major as student s_2. There is one student left, student s_{n_0+1}. Now take another subset

$$T = \{s_2, s_3, \cdots, s_{n_0}, s_{n_0+1}\}$$

of G that has n_0 students which includes student s_{n_0+1}, but not student s_1 (see right of Figure 3.10). By the assumption, all of the students in set T have the same major. Thus, student $n_0 + 1$ has the same major as student s_2 who has the same major as the rest of the students, so s_{n_0+1} has the same major as all the other students. Thus, all students at Georgetown have the same major.□

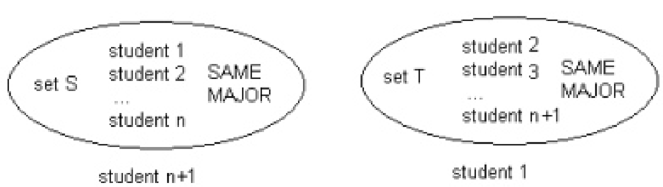

Figure 3.10: Students with same major in circle

Video Lesson 3.17

If you have trouble following this argument, see the 7:28 minute video
https://vimeo.com/76713928 (Password:Proof)
in which this argument is given.

3. Find the error in the following induction proof.

Claim 3.18: If $n \geq 4$, then n can be written as the product of two positive integers that are less than n.

Proof: We prove this statement by induction.
Base case: $n = n_0 = 4 = 2 \times 2$.
Induction step: Assume there exists an integer $n_0 \geq 4$ such that $n_0 = k_0 k_1$ where $0 < k_0 < n_0$, $0 < k_1 < n_0$, and $k_0, k_1 \in \mathbb{Z}$. Then

$$n_0 + 1 = k_0 k_1 + 1 = (k_0 + 1)(k_1 + 1).$$

Since $0 < k_0 < n_0$, then $0 < k_0 + 1 < n_0 + 1$. Similarly $0 < k_1 + 1 < n_0 + 1$, so $n_0 + 1$ can be written as the product of 2 integers less than $n_0 + 1$.

Answer 2

4. Complete the following induction proof:

Claim 3.19: $2^n > n$ for $n \geq 0$.

Proof: We prove this statement by induction.
Base case: When $n = 0$, $2^0 = 1 > 0$.
Induction step: Assume there exists a nonnegative integer n_0 such that $2^{n_0} > n_0$. Show $2^{n_0+1} > n_0 + 1$.
By assumption

$$2^{n_0+1} = 2(2^{n_0}) > 2n_0.$$

We now have 3 cases: $n_0 = 0$, $n_0 = 1$, and $n_0 > 1$. $\cdots$

5. In Problem 5 from Section 3.1 we drew a square, which is one region. A line across the square results in two regions. A second line across the region, intersecting the first line, results in four regions. A third

line intersecting the other two lines with no three lines intersecting at the same point results in seven regions. In that problem, we came up with the formula that for n lines, there are

$$1+1+2+3+\cdots+n$$

regions. From Example 3.7, this equals

$$1+\frac{n(n+1)}{2}.$$

Prove that this formula is correct. Answer 3

6. Let r represent an arbitrary real number other than 0 and 1. Show that for $n \in \mathbb{Z}_0^+$
$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}.$$
This is the formula for what is called the **finite geometric series**. This formula is quite important in many different fields of mathematics and should be committed to memory.

7. Use induction to prove that $n^3 - n$ is divisible by 6 for $n \geq 0$. We have to show that
$$p_n \equiv \{\exists\, j \in \mathbb{Z} \text{ such that } n^3 - n = 6j\}$$
is true for $n \geq 0$. Answer 4

8. Use induction to prove that $7^n - 1$ is divisible by 6 for $n \geq 0$. Make sure you clearly write what is assumed and what needs to be proven.

9. Use induction to prove that $5^n - 1$ is divisible by 4 for $n \geq 0$. Make sure you clearly write what is assumed and what needs to be proven. Answer 5

10. Show that if $n \in \mathbb{Z}^+$, then
$$\frac{n^3}{3}+\frac{n^5}{5}+\frac{7n}{15}$$
is an integer.

11. Suppose we have a sequence of numbers such that $a_0 = 15$ and
$$a_{n+1} = 0.8a_n + 1$$

for $n = 0, 1, 2, \cdots$. Show that

$$a_n = 5 + 10(0.8^n),\ n \in \mathbb{Z}_0^+.$$

Answer 6

12. Show that 7 divides

$$3^{2n+1} + 2^{n+2}$$

for $n \in \mathbb{Z}_0^+$

13. Show that 9 divides

$$10^n + 3(4^{n+2}) + 5$$

when $n \geq 0$. Answer 7

14. Let $f_1(x) = 1/(2 - x)$ and let $f_{n+1}(x) = f_1 \circ f_n(x)$. Find a formula for $f_n(x)$ and show that this formula is correct.

15. Problem 3 from Section 3.1 considered a 2-player game in which there is a pile of n beads. On each player's turn, that player can remove 1, 2, or 3 beads from the pile. The player who takes the last bead wins the game. In that game, you developed a heuristic argument that if n is divisible by 4, then the second person can win the game. Give an induction proof for this problem. Answer 8

16. In Problem 4 from Section 3.1 you considered a sum of the form

$$1 + 3 + 3^2 + \cdots + 3^n.$$

   (a) Give an induction proof that this sum is divisible by 4 if n is odd.

   (b) Give an induction proof that this sum is never divisible by 4 if n is even.

17. Consider the sum

$$\sum_{i=0}^{n} (-1)^{i+1} 4^i = -1 + 4 - 16 + \cdots + (-1)^n 4^{n-1} + (-1)^{n+1} 4^n.$$

   (a) Note that if n is odd, this sum is always divisible by 3. Prove this is true using induction. Answer 9

   (b) Note that if n is even, the argument used in the previous part still holds, but the sum is never divisible by three. Explain what is wrong. Answer 10

18. For every $i \in \mathbb{Z}^+$, find a formula for a_i (in terms of i) where a_i is the remainder when 4^i is divided by 6. Prove your formula is correct.

19. Show that for $n \geq 2$
$$\sqrt{n} < \sum_{j=1}^{n} 1/\sqrt{j}.$$
Answer 11

3.2.2 Answers to selected problems

1. **Problem 1:** This is a classic example in which we have correctly shown the implication $p_{n_0} \Rightarrow p_{n_0+1}$, but in which p_{n_0} is false for all $n_0 \in \mathbb{Z}$. The base case was omitted; it was not shown that p_1 is true. In fact,
$$p_1 \equiv \{2 = 4\}$$
is false, so the implication step is irrelevant.

2. **Problem 3:** The proof is wrong because $k_0k_1 + 1 \neq (k_0+1)(k_1+1) = k_0k_1 + k_0 + k_1 + 1$.

3. **Problem 5:** We prove by induction that for n lines there are
$$1 + \frac{n(n+1)}{2}$$
regions, $n \geq 0$.

Base case: For $n_0 = 0$ there is one region which equals
$$1 + \frac{0(0+1)}{2}.$$
Induction step: Assume for some n_0 that there are
$$1 + \frac{n_0(n_0+1)}{2}$$
regions, $n_0 \geq 0$. We need to show that if we add an $n_0 + 1$th line, the number of regions will be
$$1 + \frac{(n_0+1)(n_0+2)}{2}.$$
When we add the $n_0 + 1$th line, it intersects the n_0 other lines. Each time it intersects a line, one region is split into two regions, for one

additional region. Thus, we will add n_0 new regions. We add one additional region as the line goes from the last line to the border, splitting one additional region into two, for a total of $n_0 + 1$ new regions. Thus, the number of regions will be

$$1 + \frac{n_0(n_0+1)}{2} + n_0 + 1 = 1 + \frac{(n_0+1)(n_0+2)}{2},$$

after getting a common denominator and collecting terms.

Thus, our result is proven using induction.

4. **Problem 7:** Let $p_n \equiv \{\exists\, j \in \mathbb{Z} \text{ such that } n^3 - n = 6j\}$.

   We prove $n^3 - n$ is divisible by 6 for all $n \in \mathbb{Z}_0^+$ by induction.

   Base case: Set $n = 0$. Then $n^3 - n = 0 - 0 = 6 \times 0$, so p_0 is true with $j = 0$.

   Induction step: **Assumption:** There exists an integer n_0 and an integer $j_0 \in \mathbb{Z}$ such that $n_0^3 - n_0 = 6j_0$.

   To show: Show that $\exists\, i \in \mathbb{Z}$ such that $(n_0+1)^3 - (n_0+1) = 6i$.

   Proof: Multiplying out gives that

$$\begin{aligned}(n_0+1)^3 - (n_0+1) &= n_0^3 + 3n_0^2 + 3n_0 + 1 - n_0 - 1 \\ &= n_0^3 - n_0 + 3n_0^2 + 3n_0 \\ &= 6j_0 + 3n_0(n_0+1).\end{aligned}$$

   We have shown earlier that for every integer n_0,

$$n_0(n_0+1) = 2k_0$$

   for some $k_0 \in \mathbb{Z}$. Substitution gives that

$$(n_0+1)^3 - (n_0+1) = 6j_0 + 6k_0 = 6i,$$

   where $i = j_0 + k_0$ so p_{n_0+1} is true.□

5. **Problem 9:** Let

$$p_n \equiv \{\exists\, j \in \mathbb{Z} \text{ such that } 5^n - 1 = 4j\}.$$

   We show $5^n - 1$ is divisible by 4 for all $n \in \mathbb{Z}_0^+$ by induction.

Base case: Set $n = 0$. Then $5^0 - 1 = 1 - 1 = 4 \times 0$, so p_0 is true with $j = 0$.

Induction step: **Assumption:** There exists a nonnegative integer n_0 and an integer $j_0 \in \mathbb{Z}$ such that

$$5^{n_0} - 1 = 4j_0.$$

Show: There exists $i \in \mathbb{Z}$ such that

$$5^{n_0+1} - 1 = 4i.$$

We can do this proof directly multiplying both sides of $5^{n_0} - 1 = 4j_0$ by 5 giving

$$5^{n_0+1} - 5 = 20j_0.$$

Adding 4 to both sides gives

$$5^{n_0+1} - 1 = 4(5j_0 + 1) = 4i,$$

with $i = 5j_0 + 1$. Since i exists, p_{n_0+1} is true.□

6. **Problem 11:** We are given that $a_0 = 15$ and

$$a_{n+1} = 0.8a_n + 1$$

for $n = 0, 1, 2, \cdots$. We are to show that

$$p_n \equiv \{a_n = 5 + 10(0.8^n)\}$$

is true, $n \in \mathbb{Z}_0^+$.

Proof: By induction.

Base case: We know that $a_0 = 15$. By the formula, $a_0 = 5 + 10(0.8^0) = 15$, so p_0 is true.

Induction step: **Assuption:** There exists an integer n_0 such that

$$a_{n_0} = 5 + 10(0.8^{n_0}).$$

We also know that

$$a_{n_0+1} = 0.8a_{n_0} + 1.$$

Show that

$$a_{n_0+1} = 5 + 10(0.8^{n_0+1}).$$

By substitution, we get that

$$a_{n_0+1} = 0.8a_{n_0} + 1 = 0.8(5 + 10(0.8^{n_0})) + 1 = 10(0.8^{n_0+1}) + 5,$$

after simplification. So p_{n_0+1} is true.□

The difficulty in this problem would be to come up with the formula $a_n = 5 + 10(0.8^n)$ in the first place.

7. **Problem 13:** Let

$$p_n \equiv \{\exists\, j \in \mathbb{Z} \text{ such that } 10^n + 3(4^{n+2}) + 5 = 9j\}.$$

Base case: Set $n = 0$. Then $10^0 + 3(4^2) + 5 = 9 \times 6$, so p_0 is true with $j = 6$.

Induction step: **Assume** there exists a nonnegative integer n_0 and an integer $j_0 \in \mathbb{Z}$ such that

$$10^{n_0} + 3(4^{n_0+2}) + 5 = 9j_0.$$

Show that there exists $i \in \mathbb{Z}$ such that

$$10^{n_0+1} + 3(4^{n_0+3}) + 5 = 9i.$$

Using a backward approach, we begin with the left side of the equation, getting

$$10^{n_0+1} + 3(4^{n_0+3}) + 5 = (10)10^{n_0} + 12(4^{n_0+2}) + 5.$$

We factor out $10^{n_0} + 3(4^{n_0+2}) + 5$ from the right term so we can use our assumption. We are left with $9(10^{n_0} + 4^{n_0+2})$. Combining this, we get

$$\begin{aligned} 10^{n_0+1} + 3(4^{n_0+3}) + 5 &= 10^{n_0} + 3(4^{n_0+2}) + 5 + 9(10^{n_0} + 4^{n_0+2}) \\ &= 9j_0 + 9(10^{n_0} + 4^{n_0+2}) \\ &= 9i, \end{aligned}$$

where $i = j_0 + 10^{n_0} + 4^{n_0+2}$. Since i exists, p_{n_0+1} is true.□

8. **Problem 15:** To use induction, we could assume N is divisible by 4 and player 2 wins, then show player 2 wins for $4N + 4$, inducting by skipping 4, not 1. It is better to assume $N = 4n$ for some integer $n \geq 1$. Then the base case is $n_0 = 1$ giving 4 beads.

Base case: Suppose $n_0 = 1$ giving 4 beads. Player 1 takes k_0 beads, $1 \leq k_0 \leq 3$. Player 2 takes $j_0 = 4 - k_0$ beads. In this case, $1 \leq j_0 \leq 3$ and $j_0 + k_0 = 4$ so player 2 wins.

Induction step: Suppose for some $n_0 \geq 1$ that player 2 wins if there are $4n_0$ beads. Suppose there are $4(n_0+1) = 4n_0 + 4$ beads. Player 1 takes k_0 beads, $1 \leq k_0 \leq 3$. Player 2 takes $j_0 = 4 - k_0$ beads. In this case, $1 \leq j_0 \leq 3$ and $j_0 + k_0 = 4$, so there are $4n_0$ beads left and it is player 1's turn. By assumption, player 2 wins.

9. **Problem 17 a:** We are to prove that when n is odd,

$$\sum_{i=0}^{n}(-1)^{i+1}4^i = -1 + 4 - 16 + \cdots + (-1)^n 4^{n-1} + (-1)^{n+1}4^n$$

is divisible by 3. Instead of proving for odd n, we use the definition of odd, that is, we prove when $n = 2k+1$, $k \geq 0$, that

$$\sum_{i=0}^{2k+1}(-1)^{i+1}4^i = -1 + 4 - 16 + \cdots + (-1)^{2k+1}4^{2k} + (-1)^{2k+2}4^{2k+1}$$

is divisible by 3.

Base case: Let $k = 0$. Then the sum is

$$-1 + 4 = 2,$$

which is divisible by 3.

Induction Step: Assume for some $k_0 \geq 0$ there exists an integer j_0 such that

$$\sum_{i=0}^{2k_0+1}(-1)^{i+1}4^i = 3j_0.$$

Then for $k_0 + 1$, the sum is

$$\begin{aligned}
\sum_{i=0}^{2(k_0+1)+1}(-1)^{i+1}4^i &= \sum_{i=0}^{2k_0+3}(-1)^{i+1}4^i \\
&= \sum_{i=0}^{2k_0+1}(-1)^{i+1}4^i + \sum_{i=2k_0+2}^{2k_0+3}(-1)^{i+1}4^i.
\end{aligned}$$

We note that

$$\begin{aligned}
\sum_{i=2k_0+2}^{2k_0+3}(-1)^{i+1}4^i &= (-1)^{2k_0+3}4^{2k_0+2}+(-1)^{2k_0+4}4^{2k_0+3} \\
&= (-1)4^{2k_0+2}+4^{2k_0+3} \\
&= 4^{2k_0+2}(-1+4).
\end{aligned}$$

Thus

$$\sum_{i=0}^{2(k_0+1)+1}(-1)^{i+1}4^i = 3(j_0+4^{2k_0+2}),$$

which is divisible by 3.

10. **Problem 17 b:** Since n is even, we can let $n = 2k$ and try to prove for all $k \geq 0$, that the sum

$$\sum_{i=0}^{2k_0}(-1)^{i+1}4^i$$

is divisible by 3. But then the base case is

$$\sum_{i=0}^{0}(-1)^{i+1}4^i = -1,$$

which is not divisible by 3. If you note that $-1 = -(3)+1$, then we say that -1 has a remainder of 2 when divided by 3. We could then use induction to show that for any k, this sum has a remainder of 2 when divided by 3, since the last two terms added are divisible by 3, when added together and factored.

11. **Problem 19:** Let $p_n \equiv \{\sqrt{n} < \sum_{j=1}^{n} 1/\sqrt{j}\}$.

Conceptual Insight For the base case, we set $n = 2$. We try a backward proof. Is

$$\sqrt{2} < 1 + \frac{1}{\sqrt{2}}?$$

Multiplying both sides by $\sqrt{2}$ gives $2 < \sqrt{2}+1$ which can be rewritten as $1 < \sqrt{2}$ or after squaring both sides, $1 < 2$, which is true. The actual proof is to reverse these steps which shows that

$$\sqrt{2} < 1 + \frac{1}{\sqrt{2}},$$

which is the base case.

Induction step: **Assume** there exists an integer $n_0 \geq 2$ such that

$$\sqrt{n_0} < \sum_{j=1}^{n_0} 1/\sqrt{j}.$$

We must **show** that

$$\sqrt{n_0+1} < \sum_{j=1}^{n_0+1} 1/\sqrt{j}.$$

We use a direct approach.

$$\sum_{j=1}^{n_0+1} 1/\sqrt{j} = \sum_{j=1}^{n_0} 1/\sqrt{j} + \frac{1}{\sqrt{n_0+1}} > \sqrt{n_0} + \frac{1}{\sqrt{n_0+1}}$$

by assumption. We are finished if we can show that

$$\sqrt{n_0} + \frac{1}{\sqrt{n_0+1}} > \sqrt{n_0+1}$$

when $n_0 \geq 2$.

Lemma: For every $n \in \mathbb{Z}^+$, $n \geq 2$

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} > \sqrt{n+1}.$$

We will prove the Lemma using a backward approach. Multiplying both sides of the inequality in the lemma by $\sqrt{n+1}$ gives

$$\sqrt{n^2+n} + 1 > n + 1.$$

Subtracting 1 from both sides then squaring gives

$$n^2 + n > n^2,$$

which is true, so the lemma appears to be true, which implies the claim is true. (Note that the lemma appears to be true for $n = 1$ also.)

Following is a proof of the induction step written in a more polished form, now that we have the idea.

Proof of Lemma: Let n_0 represent an arbitrary integer such that $n_0 > 1$ which means that $n_0^2 + n_0 > n_0^2$. Taking positive square roots gives

$$\sqrt{n_0^2 + n_0} > n_0.$$

Adding 1 to both sides, then dividing by $\sqrt{n_0+1}$ gives

$$\sqrt{n_0} + \frac{1}{\sqrt{n_0+1}} > \sqrt{n_0+1}$$

for $n_0 \geq 2.\square$

Claim: $\sqrt{n} < \sum_{j=1}^{n} 1/\sqrt{j}$ for all $n \in \{2, 3, \cdots\}$.

Base case: We begin with the fact that $1 < 2$. Taking positive roots then adding 1 to both sides gives $2 < \sqrt{2} + 1$. Dividing both sides by $\sqrt{2}$ gives

$$\sqrt{2} < 1 + \frac{1}{\sqrt{2}} = \sum_{j=1}^{2} 1/\sqrt{j}.$$

Induction step: Assume there exists an integer $n_0 \geq 2$ such that $\sqrt{n_0} < \sum_{j=1}^{n_0} 1/\sqrt{j}$. Then

$$\sum_{j=1}^{n_0+1} \frac{1}{\sqrt{j}} = \sum_{j=1}^{n_0} \frac{1}{\sqrt{j}} + \frac{1}{\sqrt{n_0+1}} > \sqrt{n_0} + \frac{1}{\sqrt{n_0+1}} > \sqrt{n_0+1}$$

by Lemma.$\square$

3.3 Strong Induction

Video Lesson 3.20

This section will be easier to read if you first watch the 9:13 minute video
https://vimeo.com/499710548 (Password:Proof)
which gives an example of strong induction. To change the speed at which the video plays, click on the gear at the lower right of the video.

If we look at the logic of induction a little deeper, we see that it actually shows more. We show p_1 is true and use it to show p_2 is true. It is now known that p_1 and p_2 are true. In most induction proofs, we only need to use p_2 to show p_3 is true. On the other hand, if we need to, we can use the truth of both p_1 and p_2 to show p_3 is true since we know they are both true. Once we find that p_3 is true, we can use the fact that we know p_1, p_2, and p_3 are true to show p_4 is true. Sometimes it is necessary to use all of this information. There is no difference in the logic except that we are using ALL of what we know, not just the previous case. When we assume all of the previous statements, it is called **strong induction** even though the logic is in reality, the same.

Principle: 3.21: Strong Induction

Suppose we have a sequence of statements $p_1, p_2, p_3, \ldots$, and that we know:

- *Base case:* Statement p_1 is true.
- *Induction Step:* For every $nin\mathbb{Z}^+$ that if $p_1, p_2, \cdots, p_n$ are true, then p_{n+1} is true.

Then all of the statements are true.

The logic behind strong induction is the same as for standard induction and we could use strong induction anytime that we have used standard induction, but good practice is to only use the version we need. Normally, as we go through the thinking process looking for conceptual insight and a technical handle, we get an idea of which to use.

Let's consider the statement that every integer greater than 1 has a prime factor, a fact we all instinctively know. In this case, the statements are

$$p_n \equiv \{\text{integer } n \text{ has a prime factor}\}$$

for $n = 2, 3, \cdots$. We now go through our proof process.

Conceptual Insight: 3.22

We know statement p_2 is true because 2 is a prime factor of 2. p_3 is true because 3 is a prime factor of 3. Note we did not use previous information. To show p_4 is true, we note that 4 is not prime but can be factored into 2×2 and we know that 2 has a prime factor from p_2, so we needed to know, not that the previous statement p_3 was true, but that the statement before that was true. Standard induction is not going to work.

We must show a sequence of statements is true, so this indicates induction might help. In the examples, we did not use the previous information when n was prime, and we used more than the previous statement is true when we showed 4 had a prime factor; We did not use the fact that 3 had a prime factor, but that 2 had a prime factor. To use induction, in this case we will prove the implication

$$p_2 \wedge p_3 \wedge \cdots \wedge p_n \Rightarrow p_{n+1}.$$

- **Assumed:** For some $n_0 \geq 2$, k has a prime factor $\forall k \in \{2, \cdots, n_0\}$
- **To Show:** $n_0 + 1$ has a prime factor.

Conceptual Insight: Either $n_0 + 1$ is prime, as was 3, or $n_0 + 1$ factors into smaller integers, one of which must have a prime factor.

We are now prepared to prove that every integer $n > 1$ has a prime factor using Strong Induction.

Claim 3.23: Every integer greater than 1 has a prime factor.

Proof: We use (strong) induction to prove our claim.

Base case: We know p_2 is true since 2 is a prime factor of 2.

Induction step: Assume there exists an integer $n_0 \geq 2$ such that p_2 through p_{n_0} are true, that is,

$$p_k \equiv \{k \text{ has a prime factor }\}$$

for all $k \in \{2, \cdots, n_0\}$. Consider $n_0 + 1$. There are 2 cases, $n_0 + 1$ is prime and $n_0 + 1$ is composite.

Case 1: Suppose $n_0 + 1$ is prime. Then it has a prime factor, itself.

Case 2: Suppose $n_0 + 1$ is composite. Then there exist two integers a_0 and b_0 where $2 \leq a_0, b_0 \leq n_0$ such that

$$n_0 + 1 = a_0 b_0.$$

By our assumption, we know a_0 has a prime factor, that is,

$$a_0 = p_0 c_0,$$

where p_0 is prime and $c_0 \in \mathbb{Z}^+$. Substitution gives that

$$n_0 + 1 = p_0 c_0 b_0,$$

where p_0 is prime and $c_0 b_0 \in \mathbb{Z}^+$, so $n_0 + 1$ has a prime factor, p_0.□

3.3.1 Problems

1. Suppose we have a collection of rods of all integer lengths. Let a_n be the number of different lines of rods of length n that can be made, with order being important. We have that $a_1 = 1$, being just one rod of length 1. Also, $a_2 = 2$, being 11 and 2 (two 1-rods or one 2-rod); $a_3 = 4$ being 111, 12, 21, 3. Make a conjecture on a formula for

$$a_n$$

and prove this conjecture. Did you use standard induction or strong induction. Answer 1

2. The Fibonacci sequence is the sequence of numbers

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \cdots$$

in which each number is the sum of the previous two numbers, i.e., if $f_0 = 0$ and $f_1 = 1$, then $f_{n+2} = f_{n+1} + f_n$.

   (a) Is f_{3n} always even, $n \geq 0$? Prove or disprove. If you proved it true, did you use strong or standard induction?

   (b) Is f_{3n+1} always odd, $n \geq 0$? Prove or disprove. If you proved it true, did you use strong or standard induction?

   (c) Is f_{3n+2} always odd, $n \geq 0$. See if you can prove your conjecture without using induction.

3. The Well Ordering Principle or WOP states that if S is a nonempty subset of $\mathbb{Z}_0^+$, then S has a smallest value. This important fact seems obviously true.

   (a) Show that WOP is true by letting $p_n \equiv$ {every set of nonnegative integers that contains the integer n has a smallest integer}. In the induction step, assume there is an n_0 for which p_j is true for every $j \leq n_0$. Answer 2

(b) Show that every nonnegative integer has an interesting property. (Hint: Use contradiction by letting $S = \{n \in \mathbb{Z}_0^+$: such that n does not have an interesting property$\}$ and use WOP.) Answer 3

4. Consider a 2-player game in which there are two piles of beads. The first pile has n beads and the second pile has l beads. On each player's turn, that player may remove any number of beads from one pile. The player that removes the last bead wins.

   (a)) Set $n = 5$ and $l = 3$. Does the first or second player have a strategy that can force a win? What is that strategy? Write a convincing argument that it works.

   (b) Complete this claim and prove that it is true using induction. Suppose that there are n beads in each pile. Then player ?? can always win. (In proving the claim, the optimal strategy for that player must be given.)

   (c) Complete this claim and prove that it is true using part b). Suppose that $n > l$. Then player ?? can always win. (In proving the claim, the optimal strategy for that player must be given.)

   (d) Suppose the rules are changed such that the player who removes the last bead loses. Suppose there are n beads in both piles, $n > 1$. Determine who wins this game if both players play optimally and prove your claim.

5. For what values of $n \in \mathbb{Z}_0^+$ is

$$2^{2n+1} + 1$$

   prime? Answer 4

6. For what values of $n \in \mathbb{O}^+$ is

$$\sum_{j=0}^{n} 2^j$$

   prime?

7. Show that if $0 \leq a \leq 1$ and $n \in \mathbb{Z}^+$, then $(1-a)^n \geq 1 - na$. For what values of n and a does equality hold? Answer 5

8. Show that for $n \in \mathbb{Z}_0^+$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

9. In this problem, we explore factoring the difference of powers

   (a) Show that $x - y$ is a factor of $x^2 - y^2$. Answer 6

   (b) Show that $x - y$ is a factor of $x^3 - y^3$. Answer 7

   (c) What does it mean to say that $x - y$ is a factor of $x^n - y^n$? Answer 8

   (d) Show that $x - y$ is a factor of $x^n - y^n$ for $n \in \mathbb{Z}^+$. Answer 9

   (e) Give a formula for the other factor of $x^n - y^n$ when $x - y$ is factored out. Answer 10

10. Recall the triangle inequality

$$|a+b| \leq |a|+|b|,$$

which was proven previously.

   (a) Show that

$$|a+b+c| \leq |a|+|b|+|c|$$

   by using the triangle inequality twice.

   (b) Show that

$$|a+b+c+d| \leq |a|+|b|+|c|+|d|$$

   by using the triangle inequality repeatedly.

   (c) Use induction to show the generalized triangle inequality, that for $n \geq 2$

$$\left|\sum_{j=1}^{n} a_j\right| \leq \sum_{j=1}^{n} |a_j|.$$

11. In this problem, we are going to explore 6^i mod 7 where $i \in \mathbb{Z}_0^+$. (Note that $n = m$ mod k means $n - m$ is divisible by k. See definition of mod 18 in Notation Section 0.1 of Preface.)

   (a) Show that

$$6^i \underset{7}{=} 1$$

   if i is even. Carefully write what must be shown and what is assumed in the induction step. Answer 11

(b) Make a conjecture about 6^i mod 7 when i is odd. Prove your result using part a). Answer 12

12. Again, consider the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \cdots$$

in which each number is the sum of the previous two numbers, i.e., if $f_0 = 0$ and $f_1 = 1$, then $f_{n+2} = f_{n+1} + f_n$. suppose k_0 is a positive integer. If k_0 is a Fibonacci number then it can be written as a sum of Fibonacci numbers, specifically, itself. Let's consider some integers which are not Fibonacci numbers. Notice that

$$\begin{aligned} 4 = 3+1 &= f_3 + f_1 \\ 6 = 5+1 &= f_5 + f_1 \\ 7 = 5+2 &= f_5 + f_3 \\ 9 = 8+1 &= f_6 + f_1 \\ 10 = 8+2 &= f_6 + f_3 \\ 11 = 8+3 &= f_6 + f_4 \\ 12 = 8+3+1 &= f_6 + f_4 + f_1 \\ &. \end{aligned} \tag{3.1}$$

Note that each positive integer can apparently be written as the sum of non-consecutive Fibonacci numbers. We note that there were cases we could have used f_1 or f_2 since they both equal 1, but for 4, we had to use f_1 to keep them non-consecutive. Prove that for every positive integer k, that k can be written as the sum of non-consecutive Fibonacci numbers.

13. Guess a formula for the n-th derivative

$$\frac{d^n}{dx^n}\left(x^2 e^x\right)$$

and prove your result. Did you use standard or strong induction. Answer 13

14. In Figure 3.11 is a 5-by-5 square covered with 8 smaller squares, four which are 1-by-1, three which are 2-by-2, and one which is 3-by3. For what numbers n can a square be covered completely with n squares, none of which overlap and all of which have sides of integer lengths?

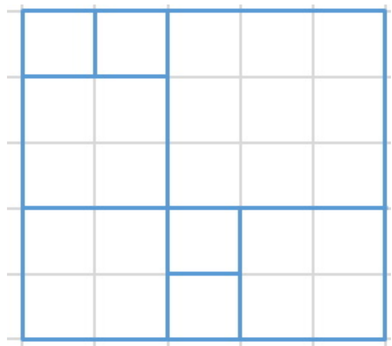

Figure 3.11: In this figure, a 5 by 5 square is covered by 8 smaller squares.

3.3.2 Answers to selected problems

1. **Problem 1:** *Base case:* $a_1 = 2^{1-1} = 1$ which is just a line containing a 1 rod.

   Strong Induction step: Assume for some $n_0 \geq 1$ that $a_k = 2^{k-1}$ for $k \in \{1, 2, \cdots, n_0\}$. We need to show $a_{n_0+1} = 2^{n_0}$. We put down the first rod which we assume is of length k, $1 \leq k \leq n_0 + 1$.

   Case 1: If $1 \leq k \leq n_0$, then there is a length of $n_0 + 1 - k$ left to place which can be done in a_{n_0+1-k} ways. Since k goes from 1 to n_0, then $n_0 + 1 - k$ goes n_0 to 1, that is, the sum of all of these cases is

   $$a_1 + a_2 + \cdots + a_{n_0},$$

   which by our assumption, equals

   $$\sum_{j=1}^{n_0} a_j = \sum_{j=1}^{n_0} 2^{j-1} = \sum_{j=0}^{n_0-1} 2^j.$$

   We note that by Problem 6,

   $$\sum_{j=0}^{n_0-1} 2^j = \frac{1 - 2^{n_0}}{1 - 2} = 2^{n_0} - 1.$$

   Case 2: If $k = n_0 + 1$, there are no more rods to place, so this can be done in just one way.

   We sum the two cases, giving

   $$a_{n_0+1} = 2^{n_0} - 1 + 1 = 2^{n_0}.$$

2. **Problem 3 a:** Define statements

$$p_n \equiv \{\forall\, S \subseteq \mathbb{Z}_0^+,\ \text{ if } n \in S \text{ then } S \text{ has a smallest integer}\}.$$

We will show p_n is true $\forall\, n \in \mathbb{Z}_0^+$ using induction.

Base case: To show p_0 is true, we must show that $\forall\, S \subseteq \mathbb{Z}_0^+$, if $0 \in S$, then S has a smallest element. Let S_0 represent an arbitrary subset of $\mathbb{Z}_0^+$ such that $0 \in S_0$ (let method). We will show 0 is the smallest element in S_0. Let i_0 represent an arbitrary integer in S_0. We know that $i_0 \in \mathbb{Z}_0^+$ and that $0 \leq i\, \forall\, i \in \mathbb{Z}_0^+$. Thus, S_0 has a smallest element, 0.

Induction step: Assume there is an $n_0 \in \mathbb{Z}_0^+$ such that p_j is true for all $0 \leq j \leq n_0$. We now show p_{n_0+1} is true. Let $S_0 \subseteq \mathbb{Z}_0^+$ be such that $n_0 + 1 \in S_0$ (let method). We must now show that $S_)$ has a smallest element.

Case 1: There exists $i_0 \in S_0$ such that $0 \leq i_0 \leq n_0$. Since we know that p_{i_0} is true, then S_0 has a smallest element.

Case 2: $i \notin S_0$ for $0 \leq i \leq n_0$. Let j_0 represent an arbitrary integer in S_0. Since $j_0 \notin \{0, 1, \cdots, n_0\}$, then $j_0 \geq n_0 + 1$. Thus $n_0 + 1$ is the smallest element in S_0 since it is less than or equal to all other elements in S_0.

In both case, S_0 has a smallest element, so p_{n_0+1} is true.$\square$

We now show WOP is true. Let S_0 be a nonempty subset of $\mathbb{Z}_0^+$. Since $S_)$ is nonempty, there exists $j_0 \in S_0$. We know $j_0 \geq 0$. Since we know p_{j_0} is true, then S_0 contains a smallest integer.

3. **Problem 3 b:** We will use proof by contradiction. Since we want to show every nonnegative integer has an interesting property, we will assume some nonnegative integers do not have interesting properties, that is, we assume

$$S = \{n \in \mathbb{Z}_0^+ : n \text{ has no interesting properties}\} \neq \emptyset.$$

Clearly, $S \subseteq \mathbb{Z}_0^+$. We know that $0 \notin S$ since 0 is the additive identity, which is pretty interesting. $1 \notin S$ since 1 is the multiplicative identity, which is also an interesting property. $2 \notin S$ since 2 is the only even prime number, $3 \notin S$ since 3 is the first odd prime number, and $4 \notin S$ since 4 is the first composite number; these are all interesting properties.

We have assumed $S \neq \emptyset$. By the well ordering principle, there exists $j_0 \in S$ such that $j_0 \leq i$ for all $i \in S$, that is, j_0 is the smallest nonnegative integer with no interesting properties. But that is an interesting property of j_0, so $j_0 \notin S$, which is a contradiction. Thus, all nonnegative integers have interesting properties.$\square$

4. **Problem 5:** The only value is $n = 0$. For all $n \in \mathbb{Z}_0^+$, $2^{2n+1} + 1$ is divisible by 3, so is not prime unless it equals 3, which it does when $n = 0$. One approach to the proof is to show that $2^{2n+1} + 1$ is divisible by 3 for all nonnegative n using induction on n. The base case is when $n = 0$ in which case $2^{2n+1} + 1 = 3$ which is divisible by 3. Then assume there exist a nonnegative integer n_0 and an integer k_0 for which $2^{2n_0+1} + 1 = 3k_0$. Show $2^{2(n_0+1)+1} + 1$ is divisible by 3. To see how to approach such problems see

   https://vimeo.com/44885997 (Password:Proof)

   in which two students attempt this problem.

5. **Problem 7:** We will prove this by induction. Let $a_0 \in [0, 1]$.

   Base case: Set $n = 1$. Then $(1 - a_0)^1 = 1 - a_0 = 1 - 1(a_0)$, so statement is true when $n = 1$. Equality holds for all values of $a \in [0, 1]$.

   Induction step: Assume there exists a positive integer n_0 such that $(1 - a_0)^{n_0} \geq 1 - n_0 a_0$. Show $(1 - a_0)^{n_0+1} \geq 1 - (n_0 + 1)a_0$.

   We have that

   $$(1-a_0)^{n_0+1} \geq (1-a_0)(1-n_0a_0) = 1-(n_0+1)a_0 + n_0a_0^2 \geq 1-(n_0+1)a_0,$$

   with equality holding only when $a_0 = 0$.$\square$

   From the proof we see that equality holds when $n = 1$ or when $a = 0$. Otherwise, $(1 - a)^n > 1 - na$.

6. **Problem 9 a:** $x^2 - y^2 = (x - y)(x + y)$

7. **Problem 9 b:** $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ which can be checked by multiplying the factors together. The factor can be found using long division.

8. **Problem 9 c:** $(x - y)$ is a factor of $x^n - y^n$ if there exists a polynomial in terms of x and y, $p(x, y)$, such that for all $x, y \in \mathbb{R}$,

   $$x^n - y^n = p(x, y)(x - y).$$

9. **Problem 9 d:** We prove that

$$\{\langle \exists\, p(x,y)\rangle \text{ such that } \langle \forall\, x,y \in \mathbb{R}\rangle\, \langle x^n - y^n = p(x,y)(x-y)\rangle\}$$

for $n \geq 1$ using induction.

Base case: Set $n = 1$. We must find $p_1(x,y)$ such that

$$x^1 - y^1 = p_1(x,y)(x-y)\,\forall\, x,y \in \mathbb{R}.$$

But this is true when $p_1(x,y) = 1$.

Induction step: Assume there exists a positive integer n_0 and there exists a polynomial $p_{n_0}(x,y)$ such that

$$x^{n_0} - y^{n_0} = p_{n_0}(x,y)(x-y)\,\forall\, x,y \in \mathbb{R}.$$

We must find a polynomial $p_{n_0+1}(x,y)$ such that

$$x^{n_0+1} - y^{n_0+1} = p_{n_0+1}(x,y)(x-y)\,\forall\, x,y \in \mathbb{R}.$$

We now do a backward/forward approach, although it is not entirely clear what to do next. We know we must use $x^{n_0} - y^{n_0} = p_{n_0}(x,y)(x-y)$, but how? Maybe a substitution, rewriting

$$x^{n_0} = y^{n_0} + p_{n_0}(x,y)(x-y).$$

Then

$$\begin{aligned} x^{n_0+1} - y^{n_0+1} = xx^{n_0} - yy^{n_0} &= x[y^{n_0} + p_{n_0}(x,y)(x-y)] - yy^{n_0}, \\ &= xy^{n_0} + p_{n_0}(x,y)x(x-y) - yy^{n_0} \\ &= (x-y)y^{n_0} + p_{n_0}(x,y)x(x-y) \\ &= [y^{n_0} + p_{n_0}(x,y)x](x-y) \end{aligned}$$

so our statement is true with

$$p_{n_0+1}(x,y) = y^{n_0} + p_{n_0}(x,y)x. \square.$$

10. **Problem 9 e:** Let $p_n(x,y)$ represent the other factor of $x^n - y^n$, that is,

$$x^n - y^n = (x-y)p_n(x,y).$$

Our goal is to find a formula for $p_n(x,y)$. From the base case,

$$p_1(x,y) = 1.$$

From the induction step,

$$p_{n+1}(x,y) = y^n + p_n(x,y)x.$$

Applying this formula gives

$$\begin{array}{ll} p_2(x,y) & = y + p_1(x,y)x = y + x \\ p_3(x,y) & = y^2 + p_2(x,y)x = y^2 + xy + x^2 \\ p_4(x,y) & = y^3 + p_3(x,y)x = y^3 + y^2x + yx^2 + x^3 \\ p_5(x,y) & = y^4 + y^3x + y^2x^2 + yx^3 + x^4 \end{array}.$$

We then conjecture that

$$p_n(x,y) = \sum_{i=0}^{n-1} y^{n-1-i}x^i.$$

A proof that this formula works is by induction.

Base case: We already know $p_1(x,y) = 1$. This agrees with the formula

$$p_n(x,y) = \sum_{i=0}^{0} y^{0-i}x^i = 1.$$

Induction step: Assume there exists a positive integer n_0 such that

$$p_{n_0}(x,y) = \sum_{i=0}^{n_0-1} y^{n_0-1-i}x^i.$$

By induction step in part d), we know that

$$p_{n_0+1}(x,y) = y^{n_0} + x\sum_{i=0}^{n_0-1} y^{n_0-1-i}x^i.$$

But

$$x\sum_{i=0}^{n_0-1} y^{n_0-1-i}x^i = \sum_{i=0}^{n_0-1} y^{n_0-1-i}x^{i+1} = \sum_{i=1}^{n_0} y^{n_0-i}x^i.$$

Combining, this means that

$$p_{n_0+1}(x,y) = y^{n_0} + x\sum_{i=0}^{n_0-1} y^{n_0-1-i}x^i = \sum_{i=0}^{n_0} y^{n_0-i}x^i.$$

which is what we needed to show.□

This proof shows how an induction proof can sometimes be used to actually find the formula, by writing it out for the first several cases.

11. **Problem 11 a:** Set $i = 2n$, $n \geq 0$. Show there exists $j \in \mathbb{Z}$ such that

$$6^{2n} = 7j + 1.$$

Base case: When $n = 0$, $6^0 = 7(0) + 1$.

Induction step: Assume there exists a nonnegative integer n_0 and an integer $j_0 \in \mathbb{Z}$ such that

$$6^{2n_0} = 7j_0 + 1.$$

Then

$$6^{2(n_0+1)} = 6^{2n_0}6^2 = (7j_0+1)36 = 7(36j_0)+36 = 7(36j_0+5)+1 = 7i+1,$$

where $i = 36j_0 + 5$.

12. **Problem 11 b:** $6^1 = 7(0) + 6$ and $6^3 = 7(30) + 6$, so conjecture is that for all $n \in \mathbb{Z}_0^+$ there exists $j \in \mathbb{Z}$ such that

$$6^{2n+1} = 7j + 6.$$

Let n_0 represent an arbitrary nonnegative integer. From part a), we know that there exists $i_0 \in \mathbb{Z}$ such that

$$6^{2n_0} = 7i_0 + 1.$$

Multiplying both sides by 6 gives

$$6^{2n_0+1} = 7(6i_0) + 6$$

and we are done. This result is a corollary to part a).

13. **Problem 13:** After taking a few derivatives and observing a pattern, and remembering that

$$1 + \cdots + (n-1) = \frac{(n-1)n}{2},$$

we come up with the formula

$$f^{(n)}(x) = ((n-1)n + 2nx + x^2)e^x.$$

Base case: $f^{(0)}(x) = f(x) = x^2e^x = ((0-1)0 + 2(0)x + x^2)e^x$

Induction step: Assume for some n_0 that

$$f^{(n_0)}(x) = ((n_0-1)n_0 + 2n_0x + x^2)e^x.$$

We must show

$$f^{(n_0+1)}(x) = ((n_0)(n_0+1) + 2(n_0+1)x + x^2)e^x.$$

We use the product rule to differentiate $f^{(n_0)}(x)$ giving

$$f^{(n_0+1)}(x) = (2n_0 + 2x)e^x + ((n_0-1)n_0 + 2n_0x + x^2)e^x.$$

Factoring and collecting like terms gives

$$f^{(n_0+1)}(x) = ((n_0-1)n_0 + 2n_0 + (2n_0+2)x + x^2)e^x.$$

Since

$$(n_0-1)n_0 + 2n_0 = n_0^2 + n_0 = n_0(n_0+1),$$

substitution gives what we want

$$f^{(n_0+1)}(x) = ((n_0)(n_0+1) + 2(n_0+1)x + x^2)e^x.$$

By induction, our conjecture is proven.

3.4 Variations on Induction and Binomial Coefficients

Sometimes we need to know more than that the previous statement is true, but we don't need to know all of the statements are true. In induction proofs, most mathematicians assume the minimum needed to complete the proof, even though all of the previous statements could be assumed true. We now consider an example of induction in which 2 values are assumed true.

Video Lesson 3.24

Watching the 15:08 minute video at
https://vimeo.com/87882345 (Password:Proof)
will make reading this next example easier to follow. To change the speed at which the video plays, click on the gear at the lower right of the video.

Suppose we have three types of blocks, blue ones that are one unit long denoted by $\boxed{\text{b}}$, yellow ones that are 2 units long denoted by $\boxed{\text{y y}}$, and red ones that are 2 units long denoted by $\boxed{\text{r r}}$. We place blocks in a row from left to right so that the total length is n units.

Claim 3.25: For $n \geq 1$, there are

$$a_n = \frac{1}{3}(-1)^n + \frac{2}{3}(2)^n \tag{3.2}$$

different ways to make a line of blocks that is n units long from left to right, where order matters.

In Figure 3.12 a), we see that there is $a_1 = 1$ way to make a line 1-unit long, b, $a_2 = 3$ ways to make a line that is 2-units long, bb, yy, rr, and $a_3 = 5$ ways to make a line that is 3-units long, bbb, byy, yyb, brr, rrb, since r's and y's must be in pairs.

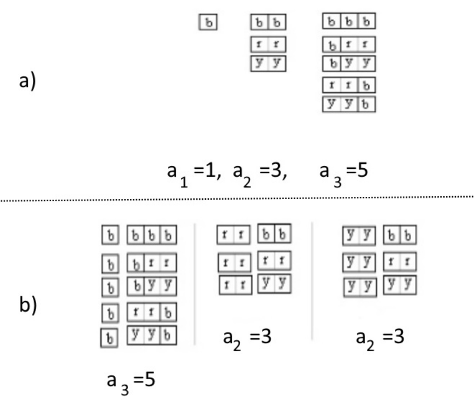

Figure 3.12: Figure a) shows lines of length 1, 2, and 3. Figure b) shows lines of length 4, constructed from lines of length 1, 2, and 3.

Substitution of $n = 1$, $n = 2$, and $n = 3$ into equation 3.2 gives the values 1, 3 and 5, respectively, so the formula works in these cases, but it is not clear why.

Conceptual Insight: 3.26

Let's develop some conceptual insight into what is happening. To make a line that is 4-units long, we would place down the first block, either $\boxed{\text{b}}$, $\boxed{\text{yy}}$, or $\boxed{\text{rr}}$. If we place $\boxed{\text{b}}$ first, we must now add blocks with total length 3 and there are $a_3 = 5$ ways to do this, as can be seen in Figure 3.12 b). If we place $\boxed{\text{y y}}$ or $\boxed{\text{r r}}$, we must add blocks with total length of 2 units and there are $a_2 = 3$ ways to do this in each case. So

$$a_4 = a_3 + 2a_2 = 11.$$

If we were making a line of length $n+2$ and we placed the first block down, we would then need to add lines of total length $n+1$ if the first block was $\boxed{\text{b}}$ or lines of total length n if the first block was $\boxed{\text{yy}}$ or $\boxed{\text{rr}}$. So this tells us that

$$a_{n+2} = a_{n+1} + 2a_n, \tag{3.3}$$

so we can determine the value for a_{n+2} if we know the 2 previous values. This is a conceptual insight for this problem.

Equation 3.3 is a recursive equation meaning that each value is found recursively from the previous two values. This, combined with induction, are technical handles for constructing a proof even though neither explains why formula 3.2 works.

Since we are trying to show something is true about a_n for integer values of n, induction seems appropriate. We seem to need to use the previous two cases, depending on whether a block of length 1 or 2 is placed down first. We note that because of equation 3.3, if we have a formula for a_n and a_{n+1}, then using substitution, we can find a formula for a_{n+2}. So for the induction step, we will assume the formula works for n_0 and $n_0 + 1$ and then prove it works for $n_0 + 2$, that is,

- **Assumed:** Assume there exists a positive integer n_0 such that there are

  $$a_{n_0} = \frac{1}{3}(-1)^{n_0} + \frac{2}{3}(2)^{n_0}$$

  different lines of blocks of total length n_0 units and that there are

  $$a_{n_0+1} = \frac{1}{3}(-1)^{n_0+1} + \frac{2}{3}(2)^{n_0+1}$$

  lines of total length $(n_0 + 1)$ units. (When $n_0 = 1$, we know the formula works for n_0 and $n_0 + 1$ because of the base case, a_1 and a_2. This is why the base case consisted of showing the formula worked for the first 2 values.)

- **Show:** We must now show that a_{n_0+2}, the number of lines of length $n_0 + 2$, is
$$a_{n_0+2} = \frac{1}{3}(-1)^{n_0+2} + \frac{2}{3}(2)^{n_0+2}\}.$$
- **Base case:** So to get started, we need to know that the formula works, not for just $n = 1$, but for $n = 1$ and $n + 1 = 2$. So our base case will consist of showing the first two steps are true.

Claim 3.27: The number of different possible arrangements of n blocks is
$$a_n = \frac{1}{3}(-1)^n + \frac{2}{3}(2)^n \textbf{ for } n \geq 1.$$

Proof: We prove our claim using induction.
Base case: There is 1 line of blocks of length 1 unit and there are 3 lines of blocks with lengths of 2 units. This agrees with formula 3.2 with $n = 1$ and $n = 2$. This is the base case. In the induction step, we need to assume we know 2 previous values, so for the base case, we need to verify the formula for 2 cases, not 1.
Proof of Induction Step: Assume there exists a positive integer n_0 such that there are
$$\begin{aligned} a_{n_0} &= \frac{1}{3}(-1)^{n_0} + \frac{2}{3}(2)^{n_0} \\ a_{n_0+1} &= \frac{1}{3}(-1)^{n_0+1} + \frac{2}{3}(2)^{n_0+1} \end{aligned}$$
ways of making lines of length n_0 and $n_0 + 1$ units, respectively. We must show there are
$$a_{n_0+2} = \frac{1}{3}(-1)^{n_0+2} + \frac{2}{3}(2)^{n_0+2}$$
ways of making lines of length $n_0 + 2$ units.
The set of a_{n_0+2} lines of length $n_0 + 2$ is the union of 3 disjoint sets, the set of lines that begin with $\boxed{\text{b}}$, the set of lines that begin with $\boxed{\text{yy}}$, and the set of lines that begin with $\boxed{\text{rr}}$. See Figure 3.12 b) where $n_0 + 2 = 4$ to help understand this.

- *Set 1:* Consider lines that begin with $\boxed{\text{b}}$. There is a length of $n_0 + 1$ units that must be completed and this can be done in
$$a_{n_0+1} = \frac{1}{3}(-1)^{n_0+1} + \frac{2}{3}(2)^{n_0+1} = -\frac{1}{3}(-1)^{n_0} + \frac{4}{3}(2)^{n_0}$$

ways, by what is assumed and by simplifying the exponents.

- *Sets 2 and 3:* Similarly, lines that begin with $\boxed{\text{yy}}$ or $\boxed{\text{rr}}$ have an n_0-unit length to be completed and this can be done in

$$a_{n_0} = \frac{1}{3}(-1)^{n_0} + \frac{2}{3}(2)^{n_0}$$

ways in each case, by assumption.

Adding the number of elements in each set, we have that

$$a_{n_0+2} = a_{n_0+1} + 2a_{n_0} = -\frac{1}{3}(-1)^{n_0} + \frac{4}{3}(2)^{n_0} + 2\left(\frac{1}{3}(-1)^{n_0} + \frac{2}{3}(2)^{n_0}\right),$$

which simplifies to

$$a_{n_0+2} = \frac{1}{3}(-1)^{n_0} + \frac{8}{3}(2)^{n_0} = \frac{1}{3}(-1)^{n_0+2} + \frac{2}{3}(2)^{n_0+2}.$$

So formula 3.2 holds for a_{n_0+2} also.□

While we were able to prove formula 3.2 is correct, the induction argument did not shed any light on where the formula came from. Sometimes, we have to explore patterns until we have a conjectured result or formula, then use induction as a technical handle to prove the formula is correct.

Video Lesson 3.28

We now consider Pascal's triangle, the beginning of which is seen in Figure 3.13. You may find it easier to follow the reading if you first watch the 10:14 minute video
https://vimeo.com/77532939 (Password:Proof)
in which this example is carefully discussed. To change the speed at which the video plays, click on the gear at the lower right of the video.

As you may recall, Pascal's triangle consists of 1's down the sides. Each number in the middle of the triangle is derived by adding the numbers above and to each side. Let n represent the row of Pascal's triangle. The top row is numbered 0. Let i represent the ith number from the left in a row, starting with 0. We denote

$$\text{the number in row } n\text{, position } i \text{ as } \binom{n}{i}$$

as seen in Figure 3.14. Comparing Figure 3.13 to Figure 3.14, we see that, for example, $\binom{4}{2} = 6$.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Figure 3.13: Pascal's triangle.

$$\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

Figure 3.14: Pascal's triangle symbolically

In summary, each number in Pascal's triangle is given as

$$\binom{n}{i} = \begin{cases} 1 & i = 0 \\ 1 & i = n \\ \binom{n-1}{i-1} + \binom{n-1}{i} & 0 < i < n \end{cases}, \tag{3.4}$$

where $n \in \mathbb{Z}_0^+$ and $0 \leq i \leq n$. We will use induction to show that

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}, \tag{3.5}$$

where $n \in \mathbb{Z}_0^+$, $0 \le i \le n$, $0! = 1$ and

$$i! = i(i-1)(i-2)\cdots(3)(2)(1),$$

$i > 0$. The numbers $\binom{n}{i}$ are called **binomial coefficients**.

Conceptual Insight: 3.29

There seems to be 2 variables instead of 1. A logical approach might be to fix the row, say row n and show the statement is true for all i, $0 \le i \le n$. So we will do induction on the rows, n, starting at $n = 0$.

We now give the proof, but notice that some thinking is omitted. To show that

$$\binom{n}{i} = \frac{n!}{i!(n-i)!},$$

we will use induction.

Claim 3.30: We will show that for each $n \in \mathbb{Z}_0^+$ and each $i \in \{0, \cdots, n\}$, Formula 3.5 holds for binomial coefficients.

Proof: We prove our claim using induction.
Base case: Set $n = 0$. We know that $\binom{0}{0} = 1$ from equation 3.4. We also know that

$$\frac{0!}{0!(0-0)!} = 1,$$

since $0! = 1$ by definition. Thus,

$$\frac{0!}{0!(0-0)!} = \binom{0}{0},$$

that is, the entire $n = 0$ row of Pascal's Triangle satisfies equation 3.5.
Induction step: **Assumption:** Assume there exists a nonnegative integer n_0 such that

$$\binom{n_0}{i} = \frac{n_0!}{i!(n_0-i)!}\ \forall\, i \in \{0, \cdots, n_0\}.$$

(Note we have a variation on induction in that we suppose for some fixed n_0, the entire n_0th row of Pascal's Triangle satisfies equation 3.5.)

To Show: We must show that the entire $n_0 + 1$ row of Pascal's Triangle satisfies equation 3.5, that is,

$$\binom{n_0+1}{i} = \frac{(n_0+1)!}{i!(n_0+1-i)!} \, \forall \, i \in \{0, \cdots, n_0, n_0+1\}.$$

To do this, we consider 3 cases.
Case 1: Set $i = 0$. Then by definition of Pascal's triangle 3.4 we have that

$$\binom{n_0+1}{i} = \binom{n_0+1}{0} = 1.$$

By substitution of $i = 0$, we get

$$\frac{(n_0+1)!}{i!(n_0+1-i)!} = \frac{(n_0+1)!}{0!(n_0+1-0)!} = \frac{(n_0+1)!}{(n_0+1)!} = 1.$$

So equation 3.5 is true for $n = n_0 + 1$ and $i = 0$, that is, the first number in row $n_0 + 1$ satisfies equation 3.5.
Case 2: Let's consider the last number in row $n_0 + 1$, that is, $i = n_0 + 1$. Then by definition $\binom{n_0+1}{i} = \binom{n_0+1}{n_0+1} = 1$. By substitution,

$$\frac{(n_0+1)!}{i!(n_0+1-i)!} = \frac{(n_0+1)!}{(n_0+1)!(n_0+1-n_0-1)!} = \frac{(n_0+1)!}{(n_0+1)!0!} = 1,$$

so the claim is true in this case.
Case 3: Remembering that $n_0 + 1$ is fixed, we now consider an arbitrary value of row $n_0 + 1$, other than the first or last value and use the let method, that is, let i_0 represent an arbitrary integer such that

$$i = i_0 \in \{1, \cdots, n_0\}.$$

We must show

$$\binom{n_0+1}{i_0} = \frac{(n_0+1)!}{i_0!(n_0+1-i_0)!}.$$

We know by definition 3.4 that

$$\binom{n_0+1}{i_0} = \binom{n_0}{i_0-1} + \binom{n_0}{i_0}.$$

By substitution using our assumption that row n_0 satisfies equation 3.5,

$$\binom{n_0}{i_0-1} + \binom{n_0}{i_0} = \frac{n_0!}{(i_0-1)!(n_0-i_0+1)!} + \frac{n_0!}{i_0!(n_0-i_0)!}.$$

We now get a common denominator on the right side of this equation by multiplying and dividing the first fraction by i_0 and the second fraction by $(n_0 - i_0 + 1)$, giving

$$\begin{aligned}\binom{n_0+1}{i_0} &= \frac{n_0!}{(i_0-1)!(n_0-i_0+1)!}\left(\frac{i_0}{i_0}\right) + \frac{n_0!}{i_0!(n_0-i_0)!}\left(\frac{n_0-i_0+1}{n_0-i_0+1}\right),\\ &= \frac{i_0(n_0!)}{i_0!(n_0-i_0+1)!} + \frac{(n_0-i_0+1)(n_0!)}{i_0!(n_0-i_0+1)!}\\ &= \frac{(i_0+n_0-i_0+1)(n_0!)}{i_0!(n_0-i_0+1)!}\\ &= \frac{(n_0+1)!}{i_0!(n_0+1-i_0)!}\end{aligned}$$

which is what we wanted to show.
For each $i \in \{0, \cdots, n_0+1\}$, the equation 3.5 holds therefore, by induction

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

for all $n \in \mathbb{Z}_0^+$ and $i \in \{0, 1, \cdots, n\}\square$

The proof of equation 3.5 seems complicated, but it is really not difficult if you check the algebra, step by step. It is just a matter of getting a common denominator, adding two fractions, then simplifying.

3.4.1 Problems

1. Suppose that $a_1 = 1$ and $a_2 = 5$ and that $a_{n+2} = 5a_{n+1} - 6a_n$. The claim is that $a_n = 3^n - 2^n$, $n \in \mathbb{Z}^+$.

   (a) Show the claim is true for $n = 1$, 2, and 3. Answer 1

   (b) What is the base case for this claim? Show the base case is true. Answer 2

   (c) Prove the claim is true by completing the induction step. Answer 3

2. Suppose we have three types of blocks, blue and yellow ones that are one unit long denoted by $\boxed{\text{b}}$ and $\boxed{\text{y}}$ and red ones that are 2 units long denoted by $\boxed{\text{r r}}$. We place blocks in a row from left to right so

that the total length is n units. We let a_n represent the number of ways of placing blocks of total length n. Then $a_1 = 2$ and $a_2 = 5$.

(a) Find a_3 and a_4.

(b) Develop a recursive equation similar to equation 3.3 for a_{n+2} and double-check that it gives the correct values for a_3 and a_4.

(c) Use induction to show that for all $k \geq 1$

$$a_k = (0.4 + 0.25\sqrt{2})(1 + \sqrt{2})^k + (0.4 - 0.25\sqrt{2})(1 - \sqrt{2})^k.$$

3. Let $\binom{n}{i}$ be the binomial coefficient as defined in equation 3.4. Use induction and the definition to prove that

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

for all $n \geq 0$. Do not use the explicit formula 3.5. (This formula follows very easily from the fact that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i},$$

by setting $x = y = 1$. Do not use this argument either.) Answer 4

4. The Fibonacci sequence is the sequence of numbers

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ \cdots$$

in which each number is the sum of the previous two numbers, i.e., if $f_0 = 0$ and $f_1 = 1$, then $f_{n+2} = f_{n+1} + f_n$. Use induction to show that $f_n \leq 2^n$ for $n \geq 0$.

3.4.2 Answers to selected problems

1. **Problem 1 a:** We have that

$$\begin{aligned} a_1 = 1 &= 3^1 - 2^1, \\ a_2 = 5 &= 3^2 - 2^2. \end{aligned}$$

Also, by definition

$$a_3 = 5a_2 - 6a_1 = 5(5) - 6(1) = 19,$$

but by formula

$$a_3 = 3^3 - 2^3 = 19.$$

2. **Problem 1 b:** Base case is showing the formula works for a_1 and a_2 which has been done.

3. **Problem 1 c:** *Induction step:* We assume there exists an integer $n_0 \geq 1$ such that the formula works for a_{n_0} and a_{n_0+1}, that is

$$a_{n_0} = 3^{n_0} - 2^{n_0} \text{ and } a_{n_0+1} = 3^{n_0+1} - 2^{n_0+1}.$$

We also know that

$$a_{n_0+2} = 5a_{n_0+1} - 6a_{n_0}.$$

We show that

$$a_{n_0+2} = 3^{n_0+2} - 2^{n_0+2}.$$

Proof of induction step: By substitution, we get that

$$a_{n_0+2} = 5a_{n_0+1} - 6a_{n_0} = 5(3^{n_0+1} - 2^{n_0+1}) - 6(3^{n_0} - 2^{n_0}).$$

Collecting like powers and simplifying gives what we needed to show.

4. Problem 3 We prove this by induction. We let p_n represent the sum of all the binomial coefficients in the nth row of Pascal's triangle, $n \geq 0$. We easily note that $p_0 = 2^0 = 1$, $p_1 = 2 = 2^1$, and $p_2 = 4 = 2^2$. We actually consider p_1 as our base case, which we will see in the induction step.

Induction step: Assume for some $n_0 \geq 1$ that $p_{n_0} = 2^{n_0}$. We now compute p_{n_0+1} which is the sum of all of the binomial coefficients on the $n_0 + 1$ row of Pascal's triangle. We note that for $\binom{n_0+1}{i}$, $1 \leq i \leq n_0$ that this number is the sum of two binomial coefficients from the n_0 row. This means

$$\sum_{i=1}^{n_0} \binom{n_0+1}{i}$$

is the sum of every binomial coefficient from row n_0 added twice except the 1's at each end of the row are added only once. Since we have assumed the sum of the binomial coefficients from the n_0 row is 2^{n_0} then

$$\sum_{i=1}^{n_0} \binom{n_0+1}{i} = 2\left(2^{n_0}\right) - 2.$$

Now we add the first and last 1 on the $n_0 + 1$ row to get that

$$p_{n_0+1} = 2^{n_0+1}.$$

Chapter 4

Concepts of Calculus

4.1 Introduction to Limits

Video Lesson 4.1

This section will be easier to read if you first watch the approximately 7:12 minute video
https://vimeo.com/77537399 (Password:Proof)
which introduces the definition of 'limit.' These proofs are similar to those done in Section 1.4, so a review of that section might help in reading this section. To change the speed at which the video plays, click on the gear at the lower right of the video.

Calculus is based on the concept of a limit. When students first learn calculus, limits are usually introduced in an intuitive manner. Even if students learn the definition of a limit, they tend to understand it only at a superficial level. In this section, we are going to concentrate on developing a deeper understanding of limit. We begin with the formal definition of limit, discuss what that means, and then see what it means in a particular example.

Definition 4.2: Limit

Let f be a function whose domain contains an open interval about the point a, except possibly a itself. The **limit** of $f(x)$ as x approaches a equals L, written

$$\lim_{x \to a} f(x) = L$$

if and only if $\forall\, \epsilon > 0$, $\exists\, \delta > 0$, such that

$$\text{if } 0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

We know that whenever we have an if/then statement, it is usually better to rewrite it as a for-every statement, so we will rewrite this definition as

$$\{\forall\, \epsilon > 0,\ \exists\, \delta > 0 \text{ s.t. } \forall\, x \text{ s.t. } 0 < |x - a| < \delta,\ |f(x) - L| < \epsilon\}.$$

The definition of limit is therefore a statement of the form

$$r \equiv \{< \forall \epsilon >< \exists \delta \text{ s. t. } >< \forall\, x \text{ s.t. } p(x), q(x) >\}.$$

The statement begins 'for every ϵ' so we let ϵ_0 represent an arbitrary positive number, which is how close we want $f(x)$ to be to L. The range of acceptable y-values is then $(L - \epsilon_0, L + \epsilon_0)$, as seen on the vertical axis in Figure 4.1 a.

The next phrase is 'there exists δ' so we must find a δ, which we do by using the backward/forward approach; we go to the x-axis as seen from arrows in Figure 4.1 a, then we set δ equal to any number so that the interval

$$(a - \delta, a + \delta)$$

is between the vertical lines, as seen in Figure 4.1 b.

The next phrase is that for every

$$\forall\, x \in (a - \delta, a + \delta),$$

so we now let x_0 be an arbitrary value in $(a - \delta, a + \delta)$, and verify that

$$f(x) \in (L - \epsilon_0, L + \epsilon_0),$$

also by the arrows in Figure 4.1 b. This is the forward part of the proof.

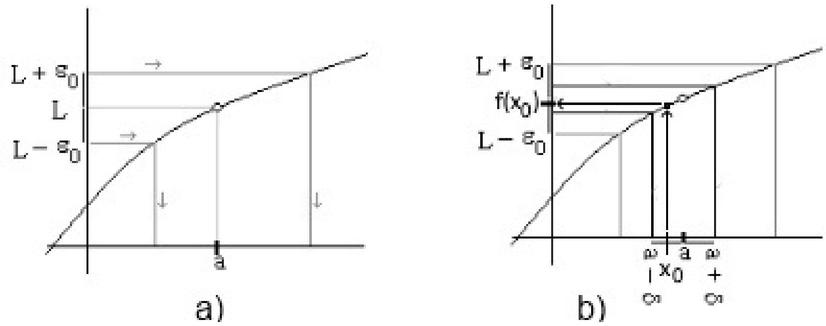

Figure 4.1: a) Pick ϵ_0, go from y-axis to x-axis. b) Find δ, go back to y-axis.

This is a classic example of a backward/forward approach combined with the let method. The key idea behind this definition is that $f(x)$ will be arbitrarily close (ϵ_0) to L if x is close enough (δ) to a.

We will use this definition to show that

$$\lim_{x \to 2} f(x) = 11,$$

where $f(x) = 3x + 5$, even though it can easily be seen from the graph in Figure 4.2. It is intuitively clear that as x gets close to 2 that $f(x)$ gets close to 11 which is $f(2)$. Let's think about what this mean.

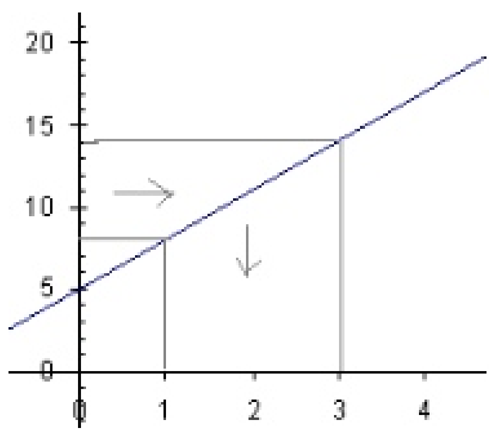

Figure 4.2: Graph of $f(x) = 3x + 5$ near $x = 2$. If $\epsilon_0 = 3$, the goal is for $f(x) \in (11 - 3, 11 + 3) = (8, 14)$, which is true when $x \in (1, 3)$.

We know that ϵ represents how close we want to be to 11. For example, set

$\epsilon_0 = 3$. Then

$$(L - \epsilon_0, L + \epsilon_0) = (11 - 3, 11 + 3) = (8, 14).$$

Graphically, as seen in Figure 4.2, we want to find x-values that ensure the y-values are between 8 and 14. Going from the y-axis to the x-axis, as in Figure 4.2, shows that if $1 < x < 3$, then the y-values will be in the correct range. In this case, we can set $\delta = 1$ or any number less than 1.

We have to find δ, so set $\delta = 1$. Then

$$(a - \delta, a + \delta) = (2 - 1, 2 + 1) = (1, 3).$$

We now have to show the implication is true, that is, we must show that

$$\forall\, x \in (1, 3)\ (x \neq 2),\ f(x) \in (8, 14).$$

We do this by letting x_0 represent a number such that $0 < |x_0 - 2| < 1$. Substitution gives that

$$|f(x_0) - 11| = |3x_0 + 5 - 11| = |3x_0 - 6| = 3|x_0 - 2| < 3,$$

since $|x_0 - 2| < 1$.

Conceptual Insight: 4.3

To prove our limit we have to do the following.

- **Assumed:** Let ϵ_0 represent an arbitrary positive number.
- **To Show:** Find δ so that $x \in (2 - \delta, 2 + \delta)$ and $x \neq 2$ implies $f(x) \in (11 - \epsilon_0, 11 + \epsilon_0)$. Alternatively, we can write this as

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 11| < \epsilon_0.$$

Using backward/forward, we start at the end

$$q(x) \equiv \{|f(x) - 11| < \epsilon_0\}$$

and work backward to find δ.

$$|f(x) - 11| = |3x + 5 - 11| = |3x - 6| = 3|x - 2| < \epsilon_0$$

or

$$|x - 2| < \frac{\epsilon_0}{3} = \delta$$

which is $p(x)$. So setting $\delta = \delta_0 = \epsilon_0/3$ will work.

We now reverse our steps to construct our proof using Definition 4.2.

Claim 4.4: $\lim_{x \to 2} 3x + 5 = 11$

Proof: From the definition, we have to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 2| < \delta$, $|(3x + 5) - 11| < \epsilon$. Therefore, we let ϵ_0 represent an arbitrary positive number and set

$$\delta_0 = \frac{\epsilon_0}{3}.$$

Let x_0 represent an arbitrary number such that

$$0 < |x_0 - 2| < \frac{\epsilon_0}{3}.$$

By substitution

$$|f(x_0) - 11| = |3x_0 + 5 - 11| = |3x_0 - 6| = 3|x_0 - 2| < 3\left(\frac{\epsilon_0}{3}\right).$$

We have thus shown that

$$|f(x_0) - 11| < \epsilon_0,$$

which is what we needed to show.□

Note in Proof 4.4 that the forward proof hides the thinking that went into finding δ.

Summary 4.1

Outline for limit proofs

- $\langle \forall\ \epsilon > 0 \rangle$ *(let method)* Let ϵ_0 represent an arbitrary positive number.
- $\langle \exists\ \delta > 0 \rangle$ *(Technical handle into constructing limit proofs is use of backward proof to solve for unknown δ in terms of ϵ_0.)* Start with $|f(x) - L| < \epsilon_0$ and work backward to $0 < |x - a| < \delta$, to set δ equal to a number given in terms of ϵ_0 as seen in Figure 4.1.
- *(Forward proof)* Use δ_0 found in backward proof and let method (let x_0) to verify that

$$\forall\, x_0 \text{ satisfying } 0 < |x_0 - a| < \delta_0, \text{ that } |f(x_0) - L| < \epsilon_0.$$

As the previous explanation demonstrates, the idea behind limits is not difficult, and neither is the method for proving limits exist for simple functions.

Suppose we have the function

$$f(x) = \frac{3x^2 - x - 10}{x - 2}$$

and we are to show that

$$\lim_{x \to 2} f(x) = 11.$$

The fact that 2 is not in the domain does not present a problem. Since $0 < |x - 2|$, then $x \neq 2$ and so

$$f(x) = \frac{3x^2 - x - 10}{x - 2} = \frac{(3x+5)(x-2)}{x-2} = 3x + 5.$$

From here, the proof that $\lim_{x \to 2} f(x) = 11$ is the same as in Proof 4.4.

The conceptual insight in limits is that whenever we choose an open interval on the y-axis about L and go backward to the x-axis, we can find an open interval about a as seen visually in Figures 4.1 a and 4.1 b. But for most limit problems, we need some additional technical handles to actually find a symmetric interval about a. Figure 4.3 shows graphically how an appropriate δ can be determined when the function is nonlinear. We 'pick' ϵ_0 and, using the backward approach (arrows to right and down) find an interval on the x-axis that results in the $f(x)$-values being within ϵ_0 of L. This interval is not usually symmetric about $x = a$. We then must find a symmetric interval about $x = a$ that is within this interval. In Figure 4.3, the largest possible value was chosen for δ, which was the distance from a to the closest endpoint, which was the endpoint to the left of a on the x-axis. The forward proof consists of showing that this interval (arrows up and to left) results in y-values within the correct interval on the y-axis. We will need to develop some technical handle to help us do this algebraically.

Video Lesson 4.5: T

e 6:48 minute video
https://vimeo.com/189383604 (Password:Proof)
introduces the technical handle used in the next example to prove a limit exits. You should watch this video before reading the example. To change the speed at which the video plays, click on the gear at the lower right of the video.

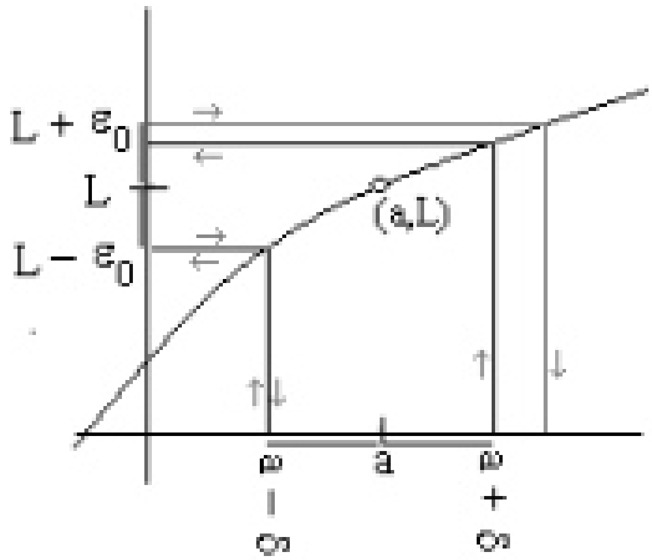

Figure 4.3: Finding δ when f is nonlinear.

Consider the function

$$f(x) = \begin{cases} 2x+3 & x > 5 \\ 7 & x = 5 \\ 3x-2 & x < 5 \end{cases}. \tag{4.1}$$

whose graph is seen in Figure 4.4.

We want to show that

$$\lim_{x \to 5} f(x) = 13.$$

We begin by picking an actual value, say $\epsilon = 3$, to help us understand how the proof proceeds. For this value of ϵ, we want

$$|f(x) - 13| < 3 \text{ or } 10 < f(x) < 16.$$

As seen in Figure 4.4, if $4 < x < 6.5$ ($x \neq 5$), then $|f(x) - 13| < 3$ and we are finished. But this x-interval is not symmetric about $x = 5$. Since the endpoint of this interval which is closest to 5 is 4, which is one unit away, we set

$$\delta_0 = 1.$$

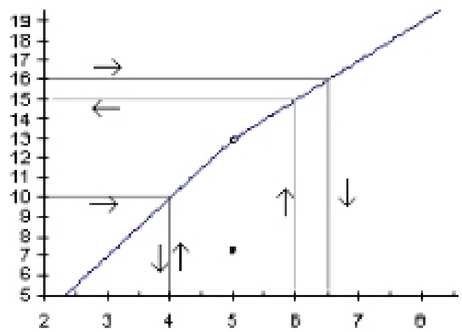

Figure 4.4: Graph of function 4.1 near $x = 5$, along with graphic approach to finding that $\delta = 1$ when $\epsilon = 3$.

Now if $0 < |x - 5| < 1$, then $4 < x < 6$ and $x \neq 5$ which results in

$$10 < f(x) < 15,$$

so $|f(x) - 13| < 3$. Notice how the geometry of the graph helped in determining the optimal value for δ. Also, note that since $f(5) = 7$, which is more than 3 units from 13, it was important that $0 < |x - 5|$.

Conceptual Insight: 4.6

Let's see how, for an arbitrary ϵ, we can find δ algebraically, using a backward approach.
Assumed: Let ϵ_0 represent an arbitrary positive number.
To Show: We must find $\delta > 0$ so that

$$\text{if } 0 < |x - 5| < \delta, \text{ then } |f(x) - 13| < \epsilon_0.$$

We gain insight by using a backward approach. We want

$$|f(x) - 13| < \epsilon_0.$$

To substitute, we need to know if $x < 5$ or $x > 5$.
Case 1: Suppose $x > 5$. We want

$$|f(x) - 13| = |2x + 3 - 13| = 2(x - 5)| < \epsilon_0$$

or

$$0 < |x - 5| < \frac{\epsilon_0}{2}.$$

Any δ such that

$$0 < \delta \leq \epsilon_0/2$$

will work.

Case 2: Suppose $x < 5$. Then

$$|f(x) - 13| = |3x - 2 - 13| = 3|x - 5| < \epsilon_0$$

or

$$0 < |x - 5| < \frac{\epsilon_0}{3}.$$

We need

$$0 < \delta \leq \epsilon_0/3,$$

also. So we can just set

$$\delta_0 = \min\{\epsilon_0/3, \epsilon_0/2\} = \epsilon_0/3.$$

We now reverse our backwards work to construct a proof based on Definition 4.2.

Claim 4.7: $\lim_{x\to 5} f(x) = 13$

Proof: We need to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 5| < \delta$, $|f(x) - 11| < \epsilon$ where

$$f(x) = \begin{cases} 2x + 3 & x > 5 \\ 7 & x = 5 \\ 3x - 2 & x < 5 \end{cases}.$$

Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \frac{\epsilon_0}{3}$$

and let x_0 represent an arbitrary number such that $0 < |x_0 - 5| < \delta_0 = \epsilon_0/3$. We must show that

$$|f(x_0) - 13| < \epsilon_0.$$

We have two cases.

Case 1: Suppose

$$5 < x_0 < 5 + \frac{\epsilon_0}{3}$$

or

$$0 < x_0 - 5 < \frac{\epsilon_0}{3}.$$

Then

$$|f(x_0) - 13| = |2x_0 + 3 - 13| = 2|x_0 - 5| < 2\left(\frac{\epsilon_0}{3}\right) < \epsilon_0.$$

Case 2: Suppose

$$5 - \frac{\epsilon_0}{3} < x_0 < 5,$$

which means that

$$0 < |x_0 - 5| < \frac{\epsilon_0}{3}.$$

Then

$$|f(x_0) - 13| = |3x_0 - 2 - 13| = 3|x_0 - 5| < 3\left(\frac{\epsilon_0}{3}\right) = \epsilon_0.$$

In both cases we have our desired result.□

Proof 4.7 was not too difficult, just requiring us to use two cases, depending on whether $x_0 > a$ or $x_0 < a$. In each case, we get a possible δ then set δ_0 equal to the smaller of the two values. There are a number of additional technical handles we may need when considering more complicated functions. We will consider some these in the next section.

4.1.1 Problems

1. Consider the function $f(x) = 5 - 4x$.

   (a) Show that
   $$\lim_{x\to 3} f(x) = -7.$$
   Answer1

   (b) Show that
   $$\lim_{x\to c} f(x) = 5 - 4c.$$
   Answer2

2. Consider the function $f(x) = 6x + 3$.

   (a) Show that
   $$\lim_{x\to -1} f(x) = -3.$$

(b) Show that
$$\lim_{x \to c} f(x) = 6c + 3.$$

3. Consider the function
$$f(x) = \begin{cases} \frac{1}{5x} & x \leq 2 \\ 2x - 4 & x > 2 \end{cases}.$$
What is wrong with the following proof that
$$\lim_{x \to 2} f(x) = 0?$$
Proof: Set $\epsilon = 0.2$. If $\delta = 0.1$ then
$$|f(x) - 0| < 0.2 \text{ when } |x - 2| < 0.1.$$
This is because $|x - 2| < 0.1$ means that $1.9 < x < 2.1$. If $1.9 < x < 2.1$ then
$$\frac{1}{9.5} > \frac{1}{5x} > \frac{1}{10.5}$$
and $-0.2 < 2x - 4 < 0.1$, so whichever function is used, $|f(x) - 0| < 0.2$. $\square$Answer3

4. Consider the function
$$f(x) = \sin(1/x).$$
What is wrong with the following proof that
$$\lim_{x \to 0} f(x) = 0?$$
Proof: Let ϵ_0 represent an arbitrary positive number. Let δ_0 represent an arbitrary positive number. There exist an $n \in \mathbb{Z}^+$ such that $1/n < \delta_0$. Therefore $1/2\pi n < \delta_0$. Set $x_0 = 1/2\pi n$. Note that $f(x_0) = 0$. Therefore for any $\epsilon > 0$ and any $\delta > 0$, there exists an x_0 such that
$$0 < |x_0 - 0| < \delta \text{ and } |f(x_0) - 0| < \epsilon. \square$$

5. Consider the function
$$f(x) = \begin{cases} x + 1 & x \leq 2 \\ 2x - 1 & x > 2 \end{cases}.$$
Show that
$$\lim_{x \to 2} f(x) = 3.$$
Answer4

6. Consider the function

$$f(x) = \begin{cases} 3x+2 & x \le 1 \\ -2x+7 & x > 1 \end{cases}.$$

   Show that $\lim_{x\to 1} f(x) = 5$

7. Consider the function

$$f(x) = \begin{cases} x-1 & x \le 2 \\ 2x-1 & x > 2 \end{cases}.$$

   (a) Find δ so that if $|x-4| < \delta$ then $|f(x)-7| < 0.2$. Answer5

   (b) Find δ so that if $|x-4| < \delta$ then $|f(x)-7| < 5$. Answer6

   (c) Find δ so that if $|x-4| < \delta$ then $|f(x)-7| < \epsilon_0$. Answer7

8. Consider the function

$$f(x) = \begin{cases} 3x & x \text{ rational} \\ -2x & x \text{ irrational} \end{cases}.$$

   You should make a 'sketch' of the function and use the sketch to help in working each part of this problem.

   (a) Set $\epsilon = 0.1$. Find δ such that if x is rational and $|x-0| < \delta$, then

$$|f(x)-0| < \epsilon = 0.1.$$

   (b) Set $\epsilon = 0.1$. Find δ such that if x is irrational and $|x-0| < \delta$, then

$$|f(x)-0| < \epsilon = 0.1.$$

   (c) Show that

$$\lim_{x\to 0} f(x) = 0.$$

9. Consider the function

$$f(x) = \begin{cases} 2x\,\sin(\frac{1}{x}) & x \ne 0 \\ 0 & x = 0 \end{cases}.$$

   Show that

$$\lim_{x\to 0} f(x) = 0.$$

   Answer8

10. Consider a sequence of real numbers s_n, $n \in \mathbb{R}^+$. We say that

$$\lim_{n\to\infty} s_n = L$$

if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that if $n > N$, then $|s_n - L| < \epsilon$. Show that if

$$s_n = \frac{n+2}{n+3} \text{ then } \lim_{n\to\infty} s_n = 1.$$

11. Consider a sequence of real numbers s_n, $n \in \mathbb{R}^+$. We say that

$$\lim_{n\to\infty} s_n = L$$

if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that if $n > N$, then $|s_n - L| < \epsilon$. Show that if

$$s_n = \frac{5n+9}{n+1} \text{ then } \lim_{n\to\infty} s_n = 5.$$

Answer 9

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and assume that for some a, $L \in \mathbb{R}$

$$\lim_{x\to a} f(x) = L.$$

Let $C \in \mathbb{R}$, and define the new function g as $g(x) = Cf(x)$. Prove that

$$\lim_{x\to a} g(x) = CL.$$

4.1.2 Answers to selected problems

1. **Problem 1 a:** Let ϵ_0 represent an arbitrary positive number. Set $\delta_0 = \epsilon_0/4$ (found by backward approach). Let x_0 represent a number such that $|x_0 - 3| < \delta_0 = \epsilon_0/4$. Then

$$\begin{aligned} f(x_0) - (-7)| &= |5 - 4x_0 + 7| \\ &= |12 - 4x_0| \\ &= 4|3 - x_0| \\ &= 4|x_0 - 3| \\ &< 4\left(\frac{\epsilon_0}{4}\right) \\ &= \epsilon_0. \end{aligned}$$

This could also be done backwards as

$$\epsilon_0 > 4|x_0 - 3| = |4x_0 - 12| = |-4x_0 + 12| = |5 - 4x_0 - (-7)| = |f(x_0) - (-7)|.$$

2. **Problem 1 b:** Forward Proof which is result of reversing the backward approach: Let ϵ_0 represent an arbitrary positive number. Set $\delta_0 = \epsilon_0/4$ (found by backward approach). Let x_0 represent a number such that $|x_0 - c| < \delta_0 = \epsilon_0/4$. Then

$$|f(x_0) - (5 - 4c)| = |5 - 4x_0 - 5 + 4c| = |-4x_0 + 4c| = 4|x_0 - c| < 4(\epsilon_0/4) = \epsilon_0.$$

3. **Problem 3:** A particular value was chosen for ϵ. In fact, $\lim_{x\to 2} f(x)$ does not exist. Since $1/(5x)$ approaches 0.1 as x approaches 2, if we picked $\epsilon_0 < 0.1$, we would not be able to find δ_0. We will consider limits not existing later in this chapter.

4. **Problem 5:** *Forward Proof:* Let ϵ_0 represent an arbitrary positive number. Set $\delta_0 = \epsilon_0/2$. Now let x_0 represent a number such that

$$0 < |x_0 - 2| < \delta_0 = \epsilon_0/2.$$

Case 1: Suppose that $x_0 < 2$. Then

$$|f(x_0) - 3| = |x_0 + 1 - 3| = |x_0 - 2| < \delta_0 = \frac{\epsilon_0}{2} < \epsilon_0,$$

which is what we wanted to show.

Case 2: Suppose instead that $x_0 > 2$. Then

$$|f(x_0) - 3| = |2x_0 - 1 - 3| = 2|x_0 - 4| < 2\delta_0 = 2\left(\frac{\epsilon_0}{2}\right) = \epsilon_0,$$

which is what we wanted to show.

In either case, if $0 < |x - 2| < \delta_0$, then $|f(x) - 3| < \epsilon_0$ which is what we needed to show.$\square$

5. **Problem 7 a:** We set $\delta_0 = 0.1$ so that $|x_0 - 4| < \delta_0$ implies that $3.9 < x_0 < 4.1$ so $x_0 > 2$. Now

$$|f(x_0) - 7| = |2x_0 - 1 - 7| = 2|x_0 - 4| < 0.2,$$

since $|x_0 - 4| < 0.1$.

6. **Problem 7 b:** We set $\delta_0 = 2$ so that $|x_0 - 4| < \delta_0$ implies that $2 < x_0 < 6$. Now

$$|f(x_0) - 7| = |2x_0 - 1 - 7| = 2|x_0 - 4| < 4 < 5,$$

since $|x_0 - 4| < 2$.

7. **Problem 7 c:** Set

$$\delta_0 = \min\{2, \epsilon_0/2\}.$$

8. **Problem 9:** Let ϵ_0 represent an arbitrary positive number. We need to find $\delta_0 > 0$ so that if $0 < |x| < \delta_0$, then

$$|f(x) - 0| = |f(x)| = |2x \sin\left(\frac{1}{x}\right)| < \epsilon_0.$$

Since $|\sin(\frac{1}{x})| \leq 1$ for all x, then

$$\left|2x \sin\left(\frac{1}{x}\right)\right| \leq 2|x|,$$

which we want to be less than ϵ_0 where $x \neq 0$. So we can set $\delta_0 = \epsilon_0/2$.

Proof: Let ϵ_0 represent an arbitrary positive number. Set $\delta_0 = \epsilon_0/2$. Let x_0 represent an arbitrary number such that $0 < |x_0| < \delta_0$. Then

$$|f(x_0) - 0| = \left|2x_0 \sin\left(\frac{1}{x_0}\right)\right| \leq 2|x_0| < 2\delta_0 = \epsilon_0. \square$$

9. **Problem 11:** Let ϵ_0 represent an arbitrary positive number. From backwards work, we set

$$N_0 = \frac{4}{\epsilon_0} - 1.$$

Let n_0 represent a positive integer greater than N_0. Then

$$n_0 + 1 > \frac{4}{\epsilon_0}.$$

Dividing by the positive number $n_0 + 1$ and multiplying by the positive number ϵ_0 givess

$$\frac{4}{n_0 + 1} < \epsilon_0.$$

We now have

$$|s_{n_0} - 5| = \left|\frac{5n_0 + 9}{n_0 + 1} - 5\right| = \left|\frac{5n_0 + 9 - 5n_0 - 5}{n_0 + 1}\right| = \left|\frac{4}{n_0 + 1}\right| < \epsilon.$$

4.2 The Restriction Method

Video Lesson 4.8

This section will be easier to read if you first watch the approximately 5:26 minute video
https://vimeo.com/399478447 (Password:Proof)
which introduces a technique called the restriction method. This is a technical handle allowing us to prove limits exist using the definition. To change the speed at which the video plays, click on the gear at the lower right of the video.

Suppose we have a function with a discontinuity near $x = a$ such as

$$f(x) = \begin{cases} 2x+3 & x \geq 5 \\ 3x-6 & x < 5 \end{cases},$$

which has a discontinuity at $x = 5$ and we want to show

$$\lim_{x \to 6} f(x) = 15. \tag{4.2}$$

We begin as before, letting $\epsilon_0 > 0$ and work backward

$$|f(x) - 15| = |2x + 3 - 15| = 2|x - 6| < \epsilon_0,$$

leading to $|x - 6| < \epsilon_0/2$ so apparently

$$\delta_0 = \epsilon_0/2.$$

The case $x > 6$ is similar to Proof 4.7, but the case $x < 6$ can cause a problem because of the discontinuity at $x = 5$.

There is actually no problem if our x-interval avoids the discontinuity. The technical handle is to initially **restrict** $\delta \leq 1$, which is small enough so that we avoid the discontinuity. Thus, we set

$$\delta_0 = \min\{1, \epsilon_0/2\}.$$

If $0 < \epsilon_0 \leq 2$ then $\delta_0 = \epsilon_0/2 \leq 1$ which would keep us away from the discontinuity as in Figure 4.5 a. If $\epsilon_0 > 2$ then $\delta_0 = 1$ as seen in Figure 4.5 b. We again avoid the discontinuity and are well within the y-lines.

We have done all of this graphically, but we need to make sure we include the algebra when writing the proof, as we did in Proof 4.7. We give the details for completeness.

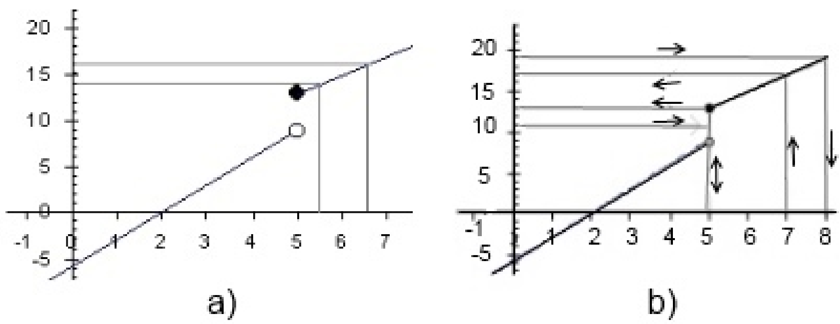

Figure 4.5: In Graph a, $\epsilon_0 = 1$ results in $\delta_0 = 0.5$. In Graph b, $\epsilon_0 = 4$ results in $\delta_0 = 1$.

Claim 4.9: $\lim_{x\to 6} f(x) = 15$ **where**

$$f(x) = \begin{cases} 2x+3 & x \geq 5 \\ 3x-6 & x < 5 \end{cases}.$$

Proof: Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \min\{1, \epsilon_0/2\}.$$

Now let x_0 represent a number such that

$$0 < |x_0 - 6| < \delta_0,$$

which means that

$$0 < |x_0 - 6| < \delta_0 \leq 1 \text{ and } 0 < |x_0 - 6| < \delta_0 \leq \epsilon_0/2.$$

The first inequality means that $5 < x_0 < 7$, so we know that

$$f(x_0) = 2x_0 + 3.$$

This gives

$$|f(x_0) - 15| = |2x_0 + 3 - 15| = 2|x_0 - 6| < 2\delta_0.$$

Since

$$|x_0 - 6| < \delta_0 = \min\{1, \epsilon_0/2\} \leq \epsilon_0/2,$$

then

$$|f(x_0) - 15| = 2|x_0 - 6| < 2\delta_0 \leq 2(\epsilon_0/2) = \epsilon_0,$$

which is what we needed to show.□

We now consider more complicated functions in which there is a different twist on initially restricting δ to show a limit exists.

Video Lesson 4.10

Before reading this example, watch the 11:38 minute video
https://vimeo.com/77748556 (Password:Proof)
in which a similar problem is worked. Notice the important difference in the example in the video and the following example. To change the speed at which the video plays, click on the gear at the lower right of the video.

We wish to prove that

$$\lim_{x \to 2} x^2 = 4.$$

Suppose $\epsilon_0 = 1$, that is, we want

$$|f(x) - 4| < 1 \text{ or } 3 < x^2 < 5,$$

which is true if $\sqrt{3} \approx 1.732 < x < \sqrt{5} \approx 2.236$. As seen in Figure 4.6, the interval is not symmetric about $x = 2$ since $|\sqrt{3} - 2| \approx 0.268$ and $|\sqrt{5} - 2| = 0.236$, but we can set $\delta_0 = \sqrt{5} - 2 \approx 0.236$, the smaller of the values. Now if

$$0 < |x - 2| < 0.236 = \delta_0,$$

then $3.112 < x^2 < 5$ so

$$|f(x) - 4| < 1.$$

How do we go about finding a symmetric interval about 2, algebraically? We at least know the closer end-point appears to be to the right of 2.

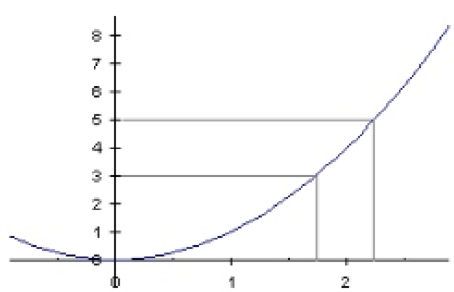

Figure 4.6: Graphically finding x-range when $3 < f(x) < 5$.

Let ϵ_0 represent an arbitrary positive number. Our goal is for

$$|x^2 - 4| = |x-2||x+2| < \epsilon_0,$$

which is true if

$$|x-2| < \frac{\epsilon_0}{|x+2|}.$$

While we would like to set

$$\delta_0 = \epsilon_0 / |x+2|,$$

if we look at the definition of a limit existing, we see that we must find δ_0 first, then pick an arbitrary x_0 in the interval $(2-\delta_0, 2+\delta_0)$, so δ_0 can be defined in terms of ϵ_0, but not in terms of x.

Remark 4.11

Note that δ_0 can only be defined in terms of numbers that are already assumed to be know. That x does not have a subscript yet means it is not known, so we cannot define δ_0 in terms of x.

Conceptual Insight: 4.12

Here is our **technical handle**. First, we will restrict δ so that x is in some preliminary interval, say

$$x \in (2-c_1, 2+c_1) = I_0.$$

Then we will find some number, $0 < N_{\epsilon_0}$, which is given only in terms of ϵ_0 and c_1 such that

$$N_{\epsilon_0} \leq \min\left\{\frac{\epsilon_0}{|x+2|}\right\} \tag{4.3}$$

for $x \in I_0$. We can then set $\delta_0 \leq N_{\epsilon_0}$ and assume

$$|x-2| < \delta_0 \leq N_{\epsilon_0}. \tag{4.4}$$

Combining inequalities 4.3 and 4.4 we have

$$|x-2| < \frac{\epsilon_0}{|x+2|}.$$

From here, we can reverse our steps, arriving at $|x^2-4| < \epsilon_0$.

Technical Handle: To see how this works, we begin by making a preliminary restriction $\delta \leq c_1 = 1$, which means

$$|x-2| < 1.$$

The choice of 1 was somewhat arbitrary. This means that $-1 < x - 2 < 1$ or $1 < x < 3$, so

$$3 < |x+2| < 5.$$

As mentioned in inequality 4.3, we find a number N_{ϵ_0} such that

$$0 < N_{\epsilon_0} \leq \min\left\{\frac{\epsilon_0}{|x+2|}\right\},$$

for $1 < x < 3$. This is easy since

$$\frac{1}{5} < \frac{1}{|x+2|},$$

since $1 < x < 3$. So

$$\frac{\epsilon_0}{5} < \frac{\epsilon_0}{|x+2|}$$

when $1 < x < 3$. This gives us our second restriction,

$$\delta \leq \frac{\epsilon_0}{5}.$$

Since we now know

$$|x-2| < \delta \leq \frac{\epsilon_0}{5} < \frac{\epsilon_0}{|x+2|},$$

we have $|x-2| < \epsilon_0/|x+2|$ which can be simplified to

$$|f(x) - 4| = |x^2 - 4| < \epsilon_0.$$

Summary 4.2

$$|f(x) - 4| < \epsilon_0 \quad \text{if} \quad |x-2| < 1 \quad \text{and} \quad |x-2| < \epsilon_0/5.$$

We now have two restrictions on δ, that it be less than or equal to 1 and less than or equal to $\epsilon_0/5$. We therefore set

$$\delta_0 = \min\{1, \epsilon_0/5\}.$$

The technical handle was to make an initial restriction, such as

$$|x-2|<1.$$

Remark 4.13

There was nothing special about this choice. We could have begun by restricting $\delta \leq 1/2$ which means we are assuming $|x-2|<0.5$ so that $1.5<x<2.5$ and therefore

$$\frac{\epsilon_0}{4.5}<\frac{\epsilon_0}{|x+2|},$$

so the second condition would be

$$|x-2|<\epsilon_0/4.5$$

and we would begin by setting

$$\delta_0=\min\{0.5, \epsilon_0/4.5\}.$$

We need to choose an initial bound to get a bound for $|x+2|$ so we can eliminate all x-values except for $|x-2|$.

Let's see how we use this approach to prove our limit.

Claim 4.14: $\lim_{x\to 2} x^2=4$.

Proof: Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0=\min\{1, \epsilon_0/5\},$$

so δ exists.
Let x_0 represent an arbitrary number such that

$$0<|x_0-2|<\delta_0.$$

Then

$$|x_0-2|<\frac{\epsilon_0}{5} \quad \text{and} \quad |x_0-2|<1,$$

since $\delta_0 \leq \epsilon_0/5$ and $\delta_0 \leq 1$.
The inequality $|x_0-2|<1$ implies that $1<x_0<3$ so that

$$3<|x_0+2|<5.$$

Multiplying both sides of $|x_0+2| < 5$ by ϵ_0 and dividing by $5|x_0+2|$, all of which are positive, gives

$$\frac{\epsilon_0}{5} < \frac{\epsilon_0}{|x_0+2|}.$$

Combining this with $|x_0 - 2| < \frac{\epsilon_0}{5}$ gives

$$0 < |x_0 - 2| < \frac{\epsilon_0}{5} < \frac{\epsilon_0}{|x_0+2|}.$$

Multiplying both sides by $|x_0+2|$ (which is positive), gives $|x_0^2 - 4| < \epsilon_0$ or

$$|f(x_0) - 4| < \epsilon_0,$$

which is what we needed to show.$\square$

To emphasize that the initial restriction is somewhat arbitrary, we construct an equally valid proof using a different choice for δ_0.

Claim 4.15: $\lim_{x\to 2} x^2 = 4$.

Alternate Proof: Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \min\{0.5, \epsilon_0/4.5\},$$

so δ exists.
Let x_0 represent an arbitrary number such that

$$0 < |x_0 - 2| < \delta_0.$$

Then

$$|x_0 - 2| < \frac{\epsilon_0}{4.5} \quad \text{and} \quad |x_0 - 2| < 0.5,$$

since $\delta_0 \le \epsilon_0/4.5$ and $\delta_0 \le 0.5$.
The inequality $|x_0 - 2| < 0.5$ implies that $1.5 < x_0 < 2.5$ so that

$$3.5 < |x_0 + 2| < 4.5.$$

Multiplying both sides of $|x_0 + 2| < 4.5$ by ϵ_0 and dividing by $4.5|x_0+2|$, all of which are positive, gives

$$\frac{\epsilon_0}{4.5} < \frac{\epsilon_0}{|x_0+2|}.$$

Combining this with $|x_0 - 2| < \frac{\epsilon_0}{4.5}$ gives

$$0 < |x_0 - 2| < \frac{\epsilon_0}{4.5} < \frac{\epsilon_0}{|x_0 + 2|}.$$

Multiplying both sides by $|x_0 + 2|$ (which is positive), gives $|x_0^2 - 4| < \epsilon_0$ or

$$|f(x_0) - 4| < \epsilon_0,$$

which is what we needed to show.□

You might ask what are our limits on the initial restriction. In this case, since we have

$$\frac{\epsilon_0}{|x_0 + 2|},$$

we do not want to worry about division by zero, so we want to keep x_0 away from -2. We note that if $|x_0 - 2| < 4$ then

$$-2 < x_0 < 6.$$

Thus, any initial interval

$$0 < |x_0 - 2| < c_1$$

will work as long as $0 < c_1 < 4$: staying away from zeros of the right side function is advisable. (We note that for this example, $c_1 = 4$ would also work but for other examples, this could cause a problem.)

Summary 4.3

Usually when the function is not linear, as in the previous example, there are technical problems to finding a value for δ in terms of only ϵ_0. Often, these problems can be overcome using the following steps:

1. Impose an initial restriction on δ, say $\delta \leq c_1$ for some real number c_1.
2. Rewrite $|f(x) - L| < \epsilon_0$ as $|x - a| < |g(x)|\epsilon_0$ if possible.
3. Use the initial restriction that $a - c_1 < x < a + c_1$ to find the minimum m of $|g(x)|$ for x in this interval. (If the minimum of $|g(x)|$ is zero, choose a smaller initial estimate, c_1.
4. Set
$$\delta_0 = \min\{c_1, m\epsilon_0\}.$$

5. Let x_0 be such that $0 < |x_0 - a| < \delta_0$. You can now show that $|f(x_0) - L| < \epsilon_0$ by first using step 3, then step 4.

We note that when proving a limit

$$\lim_{x \to a_0} f(x) = L,$$

the technical difficulties in finding a symmetric interval about $x = a_0$ actually have little to do with the concept of limit, but are artifacts of the definition. This means we can get a good conceptual insight into the limit definition, but not be able to actually prove certain limits exist because we lack the technical proficiency needed.

4.2.1 Problems

1. Consider the function

$$f(x) = \begin{cases} \frac{1}{5x} & x \leq 2 \\ 2x - 4 & x > 2 \end{cases}.$$

Complete the following proof that $\lim_{x \to 2.1} f(x) = 0.2$. *Thinking:* If we set $\delta \leq$???, then $f(x) = 2x - 4$ when $|x - 2.1| < \delta$. We want

$$|f(x) - 0.2| = |2x - 4 - 0.2| = 2|x - 2.1| < \epsilon,$$

which it will be if $\delta \leq$???.

Proof: Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = ???.$$

Let x_0 be such that $|x_0 - 2.1| < \delta_0$. Then $|f(x_0) - 0.2| = \cdots$. Answer 1

2. Consider the function

$$f(x) = \begin{cases} 3x + 2 & x \leq 1 \\ -2x + 7 & x > 1 \end{cases}.$$

Show that $\lim_{x \to 1.5} f(x) = 4$.

3. Consider the function

$$f(x) = \begin{cases} x + 1 & x \leq 2 \\ 2x - 1 & x > 2 \end{cases}.$$

Show that $\lim_{x \to 3} f(x) = 5$. Answer 2

4. Consider the function
$$f(x) = \begin{cases} x \sin(2/x) & x \leq 2 \\ 3x + 2 & x > 2 \end{cases}.$$
Show that $\lim_{x \to 3} f(x) = 11$.

5. Consider the function
$$f(x) = \begin{cases} x - 1 & x \leq 2 \\ 2x - 1 & x > 2 \end{cases}.$$
Show that
$$\lim_{x \to 4} f(x) = 7.$$
Answer 3

6. Consider the function
$$f(x) = \begin{cases} x - 1 & x \leq 2 \\ 2x - 1 & x > 2 \end{cases}.$$
Show that
$$\lim_{x \to 1} f(x) = 0.$$

7. Consider the function
$$f(x) = 2x^2 + 1.$$
   (a) Show
   $$\lim_{x \to 3} f(x) = 19.$$
   Answer 4

   (b) Show
   $$\lim_{x \to -3} f(x) = 19.$$
   Answer 5

   (c) Show
   $$\lim_{x \to 0.3} f(x) = 1.18.$$
   Answer 6

(d) Show
$$\lim_{x\to 0} f(x) = 1.$$
Answer 7

8. Consider the function
$$f(x) = x^2 + 2x - 1.$$
(a) Show
$$\lim_{x\to 1} f(x) = 2.$$
(b) Show
$$\lim_{x\to 0} f(x) = -1.$$
(c) Show
$$\lim_{x\to -1} f(x) = -2.$$

9. Consider the function
$$f(x) = \frac{2x+4}{x-1}.$$
(a) Show
$$\lim_{x\to 3} f(x) = 5.$$
Answer 8

(b) Show
$$\lim_{x\to 0.5} f(x) = -10.$$
Answer 9

10. Consider the function
$$f(x) = \frac{10}{x-3}.$$
(a) Show
$$\lim_{x\to 5} f(x) = 5.$$
(b) Show
$$\lim_{x\to 4} f(x) = 10.$$

11. Suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in (-1, 1)$ and that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = L.$$

Show that

$$\lim_{x\to 0} g(x) = L.$$

Answer 10

4.2.2 Answers to selected problems

1. **Problem 1:** *Thinking:* If we initially restrict $\delta \leq 0.1$, then $f(x) = 2x - 4$ when $|x - 2.1| < \delta$. We want

$$|f(x) - 0.2| = |2x - 4 - 0.2| = 2|x - 2.1| < \epsilon_0,$$

which it will be if $\delta \leq \epsilon_0/2$.

Proof: Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \min(0.1, \epsilon_0/2).$$

Let x_0 represent a number such that $|x_0 - 2.1| < \delta_0 \leq 0.1$. Then $2 < x_0 < 2.2$ so $|f(x_0) - 0.2| = |2x_0 - 4 - 0.2| = 2|x_0 - 2.1| < 2\delta_0 \leq 2\epsilon_0/2 = \epsilon_0$ which is what we wanted to show.$\square$

2. **Problem 3:** *Forward Proof:* Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \min\{1, \epsilon_0/2\}.$$

Now let x_0 represent an arbitrary number such that $0 < |x_0 - 3| < \delta_0$. This means

$$|x_0 - 3| < 1 \text{ and } |x_0 - 3| < \epsilon_0/2.$$

The first inequality means that $2 < x_0$, so

$$|f(x_0) - 5| = |2x_0 - 1 - 5| = 2|x_0 - 3|.$$

The second inequality means that

$$|f(x_0) - 5| = 2|x_0 - 3| < 2\delta_0 \leq 2(\epsilon_0/2) = \epsilon_0,$$

which is what we wanted to show.$\square$

3. **Problem 5:** *Forward Proof:* Let ϵ_0 represent an arbitrary positive number. Set
$$\delta_0 = \min\{2, \epsilon_0/2\}.$$
Now let x_0 represent an arbitrary number such that $0 < |x_0 - 4| < \delta_0$. This means
$$|x_0 - 4| < 2 \text{ and } |x_0 - 4| < \epsilon_0/2.$$
The first inequality means that $x_0 > 2$, so
$$|f(x_0) - 7| = |2x_0 - 1 - 7| = 2|x_0 - 4|.$$
The second inequality means that
$$|f(x_0) - 7| = 2|x_0 - 4| < 2\delta_0 \leq \epsilon_0,$$
which is what we wanted to show.□

4. **Problem 7 a:** Consider the function
$$f(x) = 2x^2 + 1.$$
We need to make
$$|2x^2 + 1 - 19| = 2|x^2 - 9| = 2|x - 3||x + 3| < \epsilon_0$$
by making x close enough to 3. Rewritten, we need to find a number N so that
$$0 < |x - 3| < N < \frac{\epsilon_0}{2|x + 3|}.$$
We make a preliminary estimate by making
$$0 < |x - 3| < \delta \leq 1,$$
so that $2 < x < 4$. This means
$$\frac{\epsilon_0}{14} < \frac{\epsilon_0}{2|x + 3|} < \frac{\epsilon_0}{10}.$$
So
$$N = \frac{\epsilon_0}{14}$$
and we set
$$\delta_0 = \min\{1, \epsilon_0/14\}.$$

We now construct the proof. Let ϵ_0 represent an arbitrary positive number and set

$$\delta_0 = \min\{1, \epsilon_0/14\}.$$

Let x_0 represent an arbitrary number such that

$$0 < |x_0 - 3| < \delta_0.$$

Since $\delta_0 \leq 1$, then $2 < x_0 < 4$ so

$$\frac{\epsilon_0}{14} < \frac{\epsilon_0}{2|x_0 + 3|}.$$

This gives that

$$0 < |x_0 - 3| < \delta_0 \leq \frac{\epsilon_0}{14} < \frac{\epsilon_0}{2|x_0 + 3|}.$$

Multiplying both sides by $2|x_0 + 3|$ gives

$$|2(x_0 + 3)(x_0 - 3)| = |2x_0^2 - 18| = |f(x_0) - 19| < \epsilon_0.$$

5. **Problem 7 b:** We need to make

$$|2x^2 + 1 - 19| = 2|x^2 - 9| = 2|x - 3||x + 3| < \epsilon_0$$

by making x close enough to -3. Rewritten, we need to find a number N so that

$$0 < |x + 3| < N < \frac{\epsilon_0}{2|x - 3|}.$$

We make a preliminary estimate by making

$$0 < |x + 3| < \delta \leq 1,$$

so that $-4 < x < -2$. This means

$$\frac{\epsilon_0}{14} < \frac{\epsilon_0}{2|x - 3|} < \frac{\epsilon_0}{10}.$$

So again

$$N = \frac{\epsilon_0}{14}$$

and we set

$$\delta_0 = \min\{1, \epsilon_0/14\}.$$

We now construct the proof. Let ϵ_0 represent an arbitrary positive number and set
$$\delta_0 = \min\{1, \epsilon_0/14\}.$$
Let x_0 represent an arbitrary number such that
$$0 < |x_0 + 3| < \delta_0.$$
Since $\delta_0 \leq 1$, then $-4 < x_0 < -2$ so
$$\frac{\epsilon_0}{14} < \frac{\epsilon_0}{2|x_0 - 3|}.$$
This gives that
$$0 < |x_0 + 3| < \delta_0 \leq \frac{\epsilon_0}{14} < \frac{\epsilon_0}{2|x_0 - 3|}.$$
Multiplying both sides by $2|x_0 - 3|$ gives
$$|2(x_0 + 3)(x_0 - 3)| = |2x_0^2 - 18| = |f(x_0) - 19| < \epsilon_0.$$

6. **Problem 7 c:** We need to make
$$|2x^2 + 1 - 1.18| = 2|x^2 - 0.09| = 2|x - 0.3||x + 0.3| < \epsilon_0$$
by making x close enough to 0.3. Rewritten, we need to find a number N so that
$$0 < |x - 0.3| < N < \frac{\epsilon_0}{2|x + 0.3|}.$$
We cannot make a preliminary estimate of
$$0 < |x - 0.3| < \delta \leq 1,$$
since then $-0.7 < x < 1.3$ and then
$$\frac{\epsilon_0}{2|x + 0.3|}$$
doesn't exist when $x = -0.3$. Our preliminary estimate has to be chosen with some thought. In this case, we let our preliminary estimate be
$$0 < |x - 0.3| < \delta \leq 0.2,$$

so that $0.1 < x < 0.5$. This means

$$\frac{\epsilon_0}{1.6} < \frac{\epsilon_0}{2|x+0.3|} < \frac{\epsilon_0}{0.8}.$$

So

$$N = \frac{\epsilon_0}{1.6}$$

and we set

$$\delta_0 = \min\{0.2, \epsilon_0/1.6\}.$$

We now construct the proof. Let ϵ_0 represent an arbitrary positive number and set

$$\delta_0 = \min\{0.2, \epsilon_0/1.6\}.$$

Let x_0 represent an arbitrary number such that

$$0 < |x_0 - 0.3| < \delta_0.$$

Since $\delta_0 \leq 0.2$, then $0.1 < x_0 < 0.5$ so

$$\frac{\epsilon_0}{1.6} < \frac{\epsilon_0}{2|x_0+0.3|}.$$

This gives that

$$0 < |x_0 - 0.3| < \delta_0 \leq \frac{\epsilon_0}{1.6} < \frac{\epsilon_0}{2|x_0+0.3|}.$$

Multiplying both sides by $2|x_0 + 0.3|$ gives

$$|2(x_0+0.3)(x_0-0.3)| = |2x_0^2 - 0.18| = |f(x_0) - 1.18| < \epsilon_0.$$

7. **Problem 7 d:** We need to make

$$|2x^2 + 1 - 1| = 2|x^2| < \epsilon_0$$

by making x close enough to 0. This problem is easy. Dividing by 2 and taking square roots gives

$$|x| < \frac{\sqrt{\epsilon_0}}{\sqrt{2}}.$$

Set

$$\delta = \delta_0 = \frac{\sqrt{\epsilon_0}}{\sqrt{2}}$$

and let x_0 be such that

$$|x_0| < delta_0 = \frac{\sqrt{\epsilon_0}}{\sqrt{2}}.$$

Squaring both sides and multiplying by 2 gives

$$|f(x_0) - 1| = |2x_0^2 + 1 - 1| = 2|x_0^2| < \epsilon_0.$$

8. **Problem 9 a:** We need to make

$$\left|\frac{2x+4}{x-1} - 5\right| = \left|\frac{-3x+9}{x-1}\right| = \frac{3|x-3|}{|x-1|} < \epsilon_0$$

by making x close enough to 3. Rewritten, we need to find a number N so that

$$0 < |x-3| < N < \frac{\epsilon_0|x-1|}{3}.$$

We make a preliminary estimate by making

$$0 < |x-3| < \delta \leq 1,$$

so that $2 < x < 4$. This means

$$\frac{\epsilon_0}{3} < \frac{\epsilon_0|x-1|}{3} < \frac{3\epsilon_0}{3} = \epsilon_0.$$

So

$$N = \frac{\epsilon_0}{3}$$

and we set

$$\delta_0 = \min\{1, \epsilon_0/3\}.$$

We now construct the proof. Let ϵ_0 represent an arbitrary positive number and set

$$\delta_0 = \min\{1, \epsilon_0/3\}.$$

Let x_0 represent an arbitrary number such that

$$0 < |x_0 - 3| < \delta_0.$$

Since $\delta_0 \leq 1$, then $2 < x_0 < 4$ so

$$\frac{\epsilon_0}{3} < \frac{\epsilon_0|x_0-1|}{3}.$$

This gives that

$$0 < |x_0 - 3| < \delta_0 \leq \frac{\epsilon_0}{3} < \frac{\epsilon_0 |x_0 - 1|}{3}.$$

Multiplying both sides by $3/|x_0 - 1|$ gives

$$\left|\frac{3(x_0-3)}{x_0-1}\right| = \left|\frac{-3x_0+9}{x_0-1}\right| = \left|\frac{2x_) + 4}{x_0 - 1} - 5\right| < \epsilon_0.$$

9. **Problem 9 b:** We need to make

$$\left|\frac{2x+4}{x-1} + 10\right| = \left|\frac{12(x-0.5)}{x-1}\right| < \epsilon_0$$

by making x close enough to 0.5. Rewritten, we need to find a number N so that

$$0 < |x - 0.5| < N < \frac{\epsilon_0 |x-1|}{12}.$$

We cannot make the preliminary estimate

$$0 < |x - 0.5| < \delta \leq 1,$$

since then $-0.5 < x < 1.5$. In this case, if $x = 1$, then

$$0 < N < \frac{\epsilon_0 |x-1|}{12} = 0$$

and we cannot find a positive N. We need to keep the possible x-values away from 1 with our preliminary estimate. We make the preliminary estimate

$$0 < |x - 0.5| < \delta \leq 0.2$$

say, so that $0.3 < x < 0.7$. This means

$$\frac{0.3\epsilon_0}{12} < \frac{\epsilon_0 |x-1|}{12} < \frac{0.7\epsilon_0}{12}.$$

So

$$N = \frac{0.3\epsilon_0}{12} = \frac{\epsilon_0}{40}$$

and we set

$$\delta_0 = \min\{0.2, \epsilon_0/40\}.$$

We now construct the proof. Let ϵ_0 represent an arbitrary positive number and set

$$\delta_0 = \min\{0.2, \epsilon_0/40\}.$$

Let x_0 represent an arbitrary number such that

$$0 < |x_0 - 0.5| < \delta_0.$$

Since $\delta_0 \leq 0.2$, then $0.3 < x_0 < 0.7$ so

$$\epsilon_0/40 < \frac{\epsilon_0|x_0 - 1|}{12}.$$

This gives that

$$0 < |x_0 - 0.5| < \delta_0 \leq \epsilon_0/40 < \frac{\epsilon_0|x_0 - 1|}{12}.$$

Multiplying both sides by $12/|x_0 - 1|$ gives

$$\left|\frac{12(x_0 - 3)}{x_0 - 1}\right| = \left|\frac{12x_0 - 6}{x_0 - 1}\right| = \left|\frac{2x_0 + 4}{x_0 - 1} + 10\right| = |f(x_0) - (-10)| < \epsilon_0.$$

10. **Problem 11:** We are given that for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } 0 < |x - 0| < \delta \text{ then } |f(x) - L| < \epsilon$$

and similarly for $h(x)$. We are also given that

$$f(x) \leq g(x) \leq h(x)$$

for all $x \in (-1, 1)$. We must show that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - 0| < \delta$ then $|g(x) - L| < \epsilon$.

We begin by letting ϵ_0 represent an arbitrary positive number. We know that

$$f(x) - L \leq g(x) - L \leq h(x) - L$$

for all $x \in (-1, 1)$. So we begin by assuming $\delta \leq 1$ so that if $|x| < \delta$ then $x \in (-1, 1)$. We know there exists $\delta_1 > 0$ so that

$$\text{if } 0 < |x - 0| < \delta_1 \text{ then } |f(x) - L| < \epsilon_0$$

and there exists $\delta_2 > 0$ so that

$$\text{if } 0 < |x - 0| < \delta_2 \text{ then } |h(x) - L| < \epsilon_0.$$

If $0 < |x| < \delta_1$, then

$$-\epsilon_0 < f(x) - L < \epsilon_0.$$

Similarly, if $0 < |x| < \delta_2$, then

$$-\epsilon_0 < h(x) - L < \epsilon_0.$$

Now set

$$\delta_0 = \min\{1, \delta_1, \delta_2\}$$

and let x_0 represent an arbitrary number such that $0 < |x_0| < \delta_0$. Since $|x_0| < \delta_0 \leq 1$, then

$$f(x_0) - L \leq g(x_0) - L \leq h(x_0) - L.$$

Since $|x_0| < \delta_0 \leq \delta_1$, then

$$-\epsilon_0 < f(x_0) - L \leq g(x_0) - L.$$

Since $|x_0| < \delta_0 \leq \delta_2$, then

$$g(x_0) - L \leq h(x_0) - L < \epsilon_0.$$

This means if $0 < |x| < \delta_0$, then $|g(x) - L| < \epsilon_0$.

4.3 Limits do not exist

Video Lesson 4.16

This section will be easier to read if you first watch the 6:16 minute video
https://vimeo.com/400028412 (Password:Proof)
and then watch the 4:45 minute video
https://vimeo.com/400028737 (Password:Proof)
which negate the definition of 'limit equaling L' and show how to use this negation. To change the speed at which the video plays, click on the gear at the lower right of the video.

We now consider the difficulty of showing a limit does not exist. We know that

$$\lim_{x \to a} f(x) = L$$

means

$$p \equiv \Big\{\langle\forall\,\epsilon > 0\rangle,\ \langle\exists\,\delta > 0\rangle \text{ s.t. } \langle\forall\,x\ \text{ s.t. }\ 0 < |x-a| < \delta,\ \text{ that } |f(x) - L| < \epsilon\rangle\Big\}.$$

Thus,

$$\lim_{x\to a} f(x) \neq L$$

means

$$\begin{aligned}\neg p \quad &\equiv \quad \Big\{\langle\exists\,\epsilon > 0 \text{ s.t. } \forall\,\delta > 0\rangle, \neg\langle\forall\,x \text{ s.t. } 0 < |x-a| < \delta,\ |f(x) - L| < \epsilon\rangle\Big\} \\ &\equiv \quad \Big\{\langle\exists\,\epsilon > 0 \text{ s.t. } \forall\,\delta > 0\rangle, \langle\exists\,x \text{ s.t. } 0 < |x-a| < \delta \text{ and } |f(x) - L| \geq \epsilon\rangle\Big\}.\end{aligned}$$

Definition 4.17: Limit does not equal L

A limit **does not equal** L, written as

$$\lim_{x\to a} f(x) \neq L$$

if and only if

$$\{\langle\exists\,\epsilon > 0\rangle \text{ s.t. } \langle\forall\,\delta > 0\rangle, \exists\,x \text{ s.t.}\langle 0 < |x-a| < \delta\rangle \wedge \langle|f(x) - L| \geq \epsilon\rangle\}.$$

Again consider the function

$$f(x) = \begin{cases} 2x+3 & x \geq 5 \\ 3x-6 & x < 5 \end{cases}, \tag{4.5}$$

seen in Figure 4.7. We will show that

$$\lim_{x\to 5} f(x) \neq 13.$$

Conceptual Insight: 4.18

From Figure 4.7, we see that at $x = 5$, the y-values ‘jump’ by 4 units. Again going backward, this means that a symmetric open interval about 13 with $\epsilon \leq 4$ would not result in an open interval about 5 on the x-axis. In Figure 4.7, we see that if $\epsilon = 3$, then the x-interval which maps into $(10, 16)$ is the half-open interval $[5, 6.5)$. This means

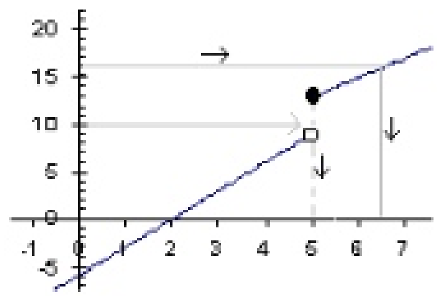

Figure 4.7: Graphic approach to finding an ϵ such that $|f(x) - 13| > \epsilon$.

for any $x < 5$, $f(x) < 9$ so $f(x) - 13 < 9 - 13 = -4$ so

$$|f(x) - 13| > |9 - 13| = 4 > 3 = \epsilon.$$

Therefore, we think that any x that satisfies

$$5 - \delta_0 < x < 5$$

should work.

The steps to showing the limit does not equal 13 are:

- Show ϵ exists by finding one. (From our discussion and Figure 4.7, any value for ϵ_0 that is less than or equal to 4 will work.)
- Let δ_0 represent an arbitrary positive number. (We must show something for every δ so we use the let method.)
- Find an x so that $q(x) \equiv \{0 < |x - a| < \delta_0\}$ and $\neg r(x) \equiv \{|f(x) - L| \geq \epsilon_0\}$.

Claim 4.19: $\lim_{x \to 5} f(x) \neq 13$ **for function 4.5**

Proof: Set $\epsilon_0 = 4$, so it exists. Let δ_0 represent an arbitrary positive number and set

$$x_0 = 5 - \delta_0/2,$$

so it exists and clearly satisfies

$$0 < |x_0 - 5| = \delta_0/2 < \delta_0.$$

Then

$$\begin{aligned} |f(x_0) - 13| &= |f(5 - \delta_0/2) - 13| \\ &= |3(5 - \delta_0/2) - 6 - 13| \\ &= |-3\delta_0/2 - 4|, \end{aligned}$$

since $x_0 < 5$. Continuing,

$$\begin{aligned} |f(x_0) - 13| &= |-3\delta_0/2 - 4| \\ &= 3\delta_0/2 + 4 \\ &> 4, \end{aligned}$$

since $\delta_0 > 0$. So the limit is not 13.□

We showed that

$$\lim_{x \to 5} f(x) \neq 13$$

for function 4.5. Looking at the graph in Figure 4.7, it is clear that this limit does not only not equal 13, this limit does not equal any number. This means this limit does not exist. How would we show this limit does not exist? One approach would be to show the limit cannot equal L for any $L \geq 11$ and then show the limit cannot equal L for any $L < 11$. (We picked 11 because it was halfway between the two values the function approaches as x approaches 5, that is, 13 and 9.) The approach is similar to that in which we showed the limit was not 13.

Claim 4.20: For function 4.5, the limit $\lim_{x \to 5} f(x)$ does not exist.

Proof: We show

$$\lim_{x \to 5} f(x) \neq L$$

for any real L. We do this in two cases.
Case 1: $\lim_{x \to 5} f(x) \neq L$ for any $L \geq 11$. Let $L_0 \geq 11$. Set $\epsilon_0 = 2$ (the distance between 11 and 9), so it exists. Let δ_0 represent an arbitrary positive number and set

$$x_0 = 5 - \delta_0/2,$$

so it exists and clearly satisfies

$$0 < |x_0 - 5| = \delta_0/2 < \delta_0.$$

Then

$$\begin{aligned}|f(x_0) - L_0| &= |f(5 - \delta_0/2) - L_0| \\ &= |3(5 - \delta_0/2) - 6 - L_0| \\ &= | - 3\delta_0/2 + 9 - L_0|,\end{aligned}$$

since $x_0 < 5$. Because $-3\delta_0/2 < 0$ and $9 - L_0 \leq 9 - 11 = -2$, we then have

$$|f(x_0) - L_0| = 3\delta_0/2 + L_0 - 9 \geq 3\delta_0/2 + 2 > 2.$$

So the limit is not L_0.

Case 2: $\lim_{x\to 5} f(x) \neq L$ for any $L < 11$. Let $L_0 < 11$. Set $\epsilon_0 = 2$ (the distance between 11 and 13), so it exists. Let δ_0 represent an arbitrary positive number and set

$$x_0 = 5 + \delta_0/2,$$

so it exists and clearly satisfies

$$0 < |x_0 - 5| = \delta_0/2 < \delta_0.$$

Then, since $x_0 > 5$ and $13 - L_0 > 2$,

$$\begin{aligned}|f(x_0) - L_0| &= |f(5 + \delta_0/2) - L_0| \\ &= |2(5 + \delta_0/2) + 3 - L_0| \\ &= |\delta_0 + 13 - L_0| \\ &= \delta_0 + 13 - L_0 \\ &> 2.\end{aligned}$$

So the limit is not $L_0 < 11$.

Since the limit cannot equal any number $L \geq 11$ and cannot equal any $L < 11$, the limit does not exists. □

Video Lesson 4.21

There is a clever use of the triangle inequality that allows us to show the limit cannot equal any number L in one computation, not the two cases we used here. This approach will be explored in Problems 3 e and 4 e. You can also see this method worked in detail in the 10:27 minute video at
https://vimeo.com/79835786 Password:Proof To change the speed at which the video plays, click on the gear at the lower right of the video.

4.3.1 Problems

1. Consider the function

$$f(x) = 3x - 1.$$

Show that

$$\lim_{x \to 2} f(x) \neq 8.$$

Answer 1

2. Consider the function

$$f(x) = -2x + 9.$$

Show that

$$\lim_{x \to 4} f(x) \neq 2.$$

3. Consider the function

$$f(x) = \begin{cases} x - 1 & x \leq 2 \\ 2x - 1 & x > 2 \end{cases}$$

as seen in Figure 4.8.

(a) Find x such that

$$0 < |x - 2| < 0.2 \text{ and } |f(x) - 1| > 2.$$

Answer 2

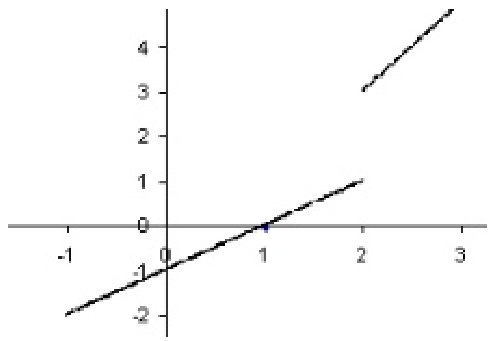

Figure 4.8: Graph of Function for Problem 3.

(b) Show that

$$\lim_{x\to 2} f(x) \neq 1$$

by finding x such that

$$0 < |x-2| < \delta_0 \text{ and } |f(x)-1| > 2 = \epsilon_0,$$

where $\delta_0 > 0$ represents a known fixed positive number. Answer 3

(c) In this part, you will show that if

$$\lim_{x\to 2} f(x) = L,$$

then $L > 2$. (We chose 2 since it is halfway between $x-1=1$, and $(2x-1)=3$, both when $x=2$.) Answer 4

(d) In this part, you will show that

$$\lim_{x\to 2} f(x)$$

does not exist by showing that if $L > 2$, then L is not a limit. Combined with the previous part, this shows the limit does not exist. Answer 5

(e) In this part, you will show that

$$\lim_{x\to 2} f(x) \text{ does not exist}$$

in one step instead of two by using contradiction. Assume that

$$\lim_{x\to 2} f(x) = L.$$

Set $\epsilon = 1$, half the gap in the jump of the function at $x = 2$. Since we have assumed the limit is L, we assume there exists $\delta_0 > 0$ such that

$$\text{if } |x-2| < \delta_0 \text{ then } |f(x) - L| < 1.$$

Compute

$$|f(2 - \delta/2) - f(2 + \delta/2)|$$

using the definition of f, then get a bound on this expression using our assumptions and the triangle inequality to get a contradiction. Answer 6

4. Consider the function

$$f(x) = \begin{cases} -2x + 11 & x \leq 3 \\ -3x + 13 & x > 3 \end{cases}.$$

(a) Find x such that

$$0 < |x-3| < 0.4 \text{ and } |f(x) - 4.5| > 0.5.$$

(b) Show that

$$\lim_{x \to 3} f(x) \neq 4.5$$

by finding x such that

$$0 < |x-3| < \delta_0 \text{ and } |f(x) - 4.5| > 0.5,$$

where $\delta_0 > 0$ is assumed to be a known fixed number.

(c) In this part, show that if

$$\lim_{x \to 3} f(x) = L,$$

then $L > 4.5$ by showing that if $L \leq 4.5$ then

$$\lim_{x \to 3} f(x) \neq L.$$

(d) In this part, show that

$$\lim_{x \to 3} f(x) \neq L$$

if $L > 4.5$, meaning the limit does not exist.

(e) In this part, you will show that

$$\lim_{x\to 3} f(x) \text{ does not exist}$$

in one step instead of two by using contradiction. Assume that

$$\lim_{x\to 3} f(x) = L.$$

Set ϵ equal to an appropriate number. Since we have assumed the limit is L, we assume there exists $\delta_0 > 0$ such that

$$\text{if } |x-3| < \delta \text{ then } |f(x) - L| < \epsilon.$$

Compute

$$|f(3-\delta/2) - f(3+\delta/2)|$$

using the definition of f, then get a bound on this expression using our assumptions and the triangle inequality to get a contradiction.

5. For each of the following parts, show the claim is true or give a counterexample.

(a) Suppose that

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

both exist. Then

$$\lim_{x\to a}[f(x) + g(x)]$$

exists. Answer 7

(b) Suppose that

$$\lim_{x\to a}[f(x) + g(x)]$$

exists. Then

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

both exist. Answer 8

6. Consider the function

$$f(x) = \cos(\pi/x).$$

We are going to show that

$$\lim_{x\to 0} f(x) \text{ does not exist}$$

by showing that for any $L \in \mathbb{R}$

$$\lim_{x \to 0} f(x) \neq L.$$

We first need to explore some properties of this function through some sequence of x-values which converge to 0. Set $x_j = 1/(2j)$. Since

$$\lim_{j \to \infty} x_j = 0$$

and $f(x_j) = 1$ for all $j \in \mathbb{Z}^+$, then for all $\delta > 0$, there exists x_j such that

$$|x_j - 0| < \delta \text{ and } f(x_j) = 1.$$

Find a sequence of numbers x'_j such that

$$\lim_{j \to \infty} x'_j = 0$$

and $f(x'_j) = -1$. Then show that for all $\delta > 0$, there exists x_j, x'_k such that $|x_j| < \delta$ and $|x'_k| < \delta$, but

$$|f(x) - f(x')| = 2.$$

Assume that

$$\lim_{x \to 0} f(x) = L.$$

Set $\epsilon = 1$. This means that there exists $\delta > 0$ such that for all x if $0 < |x| < \delta$, then

$$|f(x) - L| < 1.$$

Show that this implies that for all x, x' such that $0 < |x|,|x'| < \delta$, that

$$|f(x) - f(x')| < 2.$$

Why does this lead to a contradiction?

7. Consider a sequence of real numbers s_n, $n \in \mathbb{N}$. We say that

$$\lim_{n \to \infty} s_n = L$$

if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{R}$ such that if $n > N$, then $|s_n - L| < \epsilon$.

(a) Write what it means for

$$\lim_{n\to\infty} s_n \neq L.$$

Answer 9

(b) Show that if $s_n = n^2$ then for every real number L,

$$\lim_{n\to\infty} s_n \neq L.$$

Answer 10

8. Consider the function $f(x) = 1/x$.

   (a) Write the statement:

   $$\lim_{x\to 0} \frac{1}{x} \neq L$$

   using quantifiers.

   (b) Show that $\lim_{x\to 0} 1/x$ does not equal any non negative number L.

   (c) Show that $\lim_{x\to 0} 1/x$ does not equal any negative number L. (This shows that this limit does not exist.)

9. Consider the following statement about a function $f : \mathbb{R} \to \mathbb{R}$ and numbers a, $L \in \mathbb{R}$.

   $$\forall\, n \in \mathbb{Z}^+,\ \exists\, x_n \in \mathbb{R}\,\text{such that}\ \left\{0 < |x_n - a| < \frac{1}{n} \wedge |f(x_n) - L| < \frac{1}{n}\right\}.$$

   (a) Give a proof that if $\lim_{x\to a} f(x)$ exists, it must equal L.

   (b) Given an example where the above statement is true but $\lim_{x\to a} f(x)$ does not exist.

4.3.2 Answers to selected problems

1. **Problem 1:** When $f(x) = 3x - 1$, we intuitively know that

   $$\lim_{x\to 2} f(x) = 5,$$

   not 8. Conceptually, when x is close to 2, $f(x)$ will be close to 5, which is 3 units away from 8. So if we pick $\epsilon \leq 3$, then there will be some x-values close to 2 whose y-values are at least ϵ away from 8.

We now construct the proof. Set $\epsilon_0 = 3$ and let δ_0 represent an arbitrary positive number. Since f is a line with positive slope, we know that for any $x < 2$, $f(x) < 5$ so will be more than 3 units away from 8. We set

$$x_0 = 2 - \delta_0/2.$$

We now have that

$$0 < |x_0 - 2| = \delta_0/2 < \delta_0$$

and that

$$|f(x_0) - 8| = \left|3\left(2 - \frac{\delta_0}{2}\right) - 1 - 8\right| = \left|-\frac{3\delta_0}{2} - 3\right| = \frac{3\delta_0}{2} + 3 > 3 = \epsilon_0,$$

so the limit is not 8. Note we have still not shown that the limit is 5.

2. **Problem 3 a:** Set $x_0 = 2.1$. Then

   $|x_0 - 2| = 0.1 < 0.2$ and $|f(2.1) - 1| = |(2(2.1) - 1) - 1| = 2.2 > 2.$

   (Any value $2 < x < 2.2$ will work.)

3. **Problem 3 b:** To show that

   $$\lim_{x\to 2} f(x) \neq 1,$$

   we must show that

   $$\{\exists\, \epsilon > 0 \text{ s.t. } \forall\, \delta > 0\, \exists\, x \text{ s.t. } 0 < |x - 2| < \delta \wedge |f(x) - 1| \geq \epsilon\}.$$

   The following proof is based on the graph in Figure 4.8. Note that if $x \leq 2$ then $f(x) \leq 1$, but if $x > 2$, then $f(x) > 3$. Since $3 - 1 = 2$, set $\epsilon_0 = 2$ so ϵ exists. Now let δ_0 represent an arbitrary positive number.

   Set $x_0 = 2 + \delta_0/2$. We have that

   $$|x_0 - 2| = \delta_0/2 < \delta_0.$$

   Since $x_0 > 2$, then

   $$f(x_0) = f(2 + \delta_0/2) = 2(2 + \delta_0/2) - 1 = 3 + \delta_0.$$

   This means that

   $$|f(x_0) - 1| = |3 + \delta_0 - 1| = 2 + \delta_0 > 2,$$

   since $\delta_0 > 0$. So we have shown that for some ϵ and all δ, there is an x such that $|f(x) - 1| > \epsilon$ and $0 < |x - 2| < \delta$. This shows $\lim_{x\to 2} f(x) \neq 1.\square$

4. **Problem 3 c:** Let $L_0 \leq 2$. This is a proof that

$$\lim_{x\to 2} f(x) \neq L_0 \leq 2.$$

Set $\epsilon_0 = 1$ and let $\delta_0 > 0$. Set $x_0 = 2 + \delta_0/2$. Then $0 < |x_0 - 2| = \delta_0/2 < \delta_0$, and

$$|f(x_0) - L_0| = |2(2+\delta_0/2) - 1 - L_0| = 3 + \delta_0 - L_0 \geq 1 + \delta_0 > 1 = \epsilon_0,$$

since $3 - L_0 \geq 1$ because $L_0 \leq 2$. This shows that L cannot be less than or equal to 2.

5. **Problem 3 d:** Let $L_0 > 2$. This is a proof that

$$\lim_{x\to 2} f(x) \neq L_0 > 2.$$

Set $\epsilon_0 = 1$ and let $\delta_0 > 0$. Set $x_0 = 2 - \delta_0/2$. Then $0 < |x_0 - 2| = \delta_0/2 < \delta_0$, and

$$|f(x_0) - L_0| = |(2-\delta_0/2) - 1 - L_0| = |1 - \delta_0/2 - L_0| = L_0 - 1 + \delta_0/2,$$

since $L_0 > 2$. This also means $L_0 - 1 > 1$ so

$$|f(x_0) - L_0| = L_0 - 1 + \delta_0/2 > 1 + \delta_0/2 > 1 = \epsilon_0.$$

This shows that L cannot be greater than 2. Combined with the previous part, L cannot exist so the limit does not exist.

6. **Problem 3 e:** This is a proof by contradiction. From the graph in Figure 4.8, it is clear that for any δ,

$$f(2+\delta/2) - f(2-\delta/2) > 2,$$

since

$$f(2+\delta/2) > 3 \text{ and } f(2-\delta/2) < 1.$$

Suppose that

$$\lim_{x\to 2} f(x) = L_0$$

for some $L_0 \in \mathbb{R}$. Set $\epsilon_0 = 1$, half the gap in the jump. We have assumed the limit exists which means δ_0 exists. This means that for every x satisfying

$$|x - 2| < \delta_0,$$

that
$$|f(x) - L_0| < 1.$$
In other words, if $x_0 = 2 - \delta_0/2$, then
$$|f(2 - \delta_0/2) - L_0| < 1.$$
Similarly,
$$|f(2 + \delta_0/2) - L_0| < 1.$$
Combining gives
$$\begin{aligned} 2 &> |f(2-\delta_0/2) - L_0| + |L_0 - f(2+\delta_0/2)| \\ &\geq |[f(2-\delta_0/2) - L_0] + [L_0 - f(2+\delta_0/2)]| \end{aligned}$$
by the triangle inequality. Continuing
$$2 > |f(2-\delta_0/2) - f(2+\delta_0/2)| = |(2-\delta_0/2-1) - (4+\delta_0-1)| = 2+3\delta_0/2 > 2.$$
This is a contradiction. So $\lim_{x\to 2} f(x) \neq L$ for any $L \in \mathbb{R}$.

7. **Problem 5 a:** Suppose
$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M.$$
We will show
$$\lim_{x\to a}[f(x) + g(x)] = L + M.$$
We must show that for all $\epsilon > 0$, there exists $\delta > 0$ such that
$$\text{if } |x - a| < \delta, \text{ then } |f(x) + g(x) - L - M| < \epsilon.$$
The proof is based on the triangle inequality.

Let ϵ_0 represent an arbitrary positive number. We are given that for $\epsilon_0/2 > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that for all x satisfying
$$|x - a| < \delta_1,\ |f(x) - L| < \epsilon_0/2$$
and for all x satisfying
$$|x - a| < \delta_2,\ |g(x) - M| < \epsilon_0/2.$$
Set
$$\delta_0 = \min\{\delta_1, \delta_2\}.$$

Let x_0 represent an arbitrary number such that

$$|x_0 - a| < \delta_0.$$

Then $|x_0 - a| < \delta_1$ and $|x_0 - a| < \delta_2$. By the triangle inequality, this means that

$$\epsilon_0 > |f(x_0) - L| + |g(x_0) - M| \geq |f(x_0) + g(x_0) - L - M|,$$

which is what we needed to show.

8. **Problem 5 b:** This statement is not true. Set

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases}.$$

Then $f(x) + g(x) = 0$. In this case,

$$\lim_{x \to 0} [f(x) + g(x)] = 0,$$

but $\lim_{x \to 0} f(x)$ does not exist and $\lim_{x \to 0} g(x)$ does not exist.

9. **Problem 7 a:** there exists an $\epsilon > 0$ such that for every $N \in \mathbb{R}$, there exists an $n > N$ such that $|s_n - L| \geq \epsilon$.

10. **Problem 7 b:** Let L_0 be a real number. Set $\epsilon_0 = 1$ (any positive number works). Let N_0 be a real number.

Backwards work: We now have to find an n such that $n > N_0$ and that

$$|s_n - L_0| = |n^2 - L_0| \geq \epsilon_0 = 1.$$

Since n^2 gets larger and larger as n gets larger, the idea is to find an n such that

$$n^2 \geq L_0 + 1 \text{ and } n > N_0.$$

To make sure n is an integer, we use the ceiling function and set

$$n_0 = \left\lceil \sqrt{|L_0| + 1 + N_0^2} \right\rceil.$$

Then $n_0 > N_0$ and

$$s_{n_0} = n_0^2 \geq |L_0| + 1 + |N_0|^2 \geq L_0 + 1.$$

Therefore

$$s_{n_0} - L_0 = |s_{n_0} - L_0| \geq 1 = \epsilon_0$$

and we are finished.

4.4 Continuity and Derivatives

Video Lesson 4.22

This section will be easier to read if you first watch the 7:00 minute video
https://vimeo.com/499307805 (Password:Proof)
which shows how to use limits to prove a derivative is what we think it is. To change the speed at which the video plays, click on the gear at the lower right of the video.

In this section, we use the definition of limit to study continuity and differentiability.

Definition 4.23: Continuous

Suppose a function f is defined on an interval (a, b) and $c \in (a, b)$. The function is said to be **continuous** at $x = c$ if and only if

$$\lim_{x \to c} f(x) = f(c).$$

We showed in Proof 4.4 that

$$\lim_{x \to 2} 3x + 5 = 11.$$

For the function $f(x) = 3x + 5$, substitution gives that

$$f(2) = 11.$$

Thus,

$$\lim_{x \to 2} 3x + 5 = f(2),$$

so f is continuous at $x = 2$. On the other hand, the function

$$f(x) = \frac{3x^2 - x - 10}{x - 2}$$

is not defined at $x = 2$, so

$$\lim_{x \to 2} f(x) = 11 \neq f(2),$$

since $f(2)$ doesn't exist. This function is not continuous at $x = 2$. If we redefine this function as

$$f(x) = \begin{cases} \frac{3x^2-x-10}{x-2} = 3x+5 & x \neq 2 \\ 10 & x = 2 \end{cases},$$

then $f(2)$ exists, but the function is still not continuous at $x = 2$ since

$$\lim_{x\to 2} f(x) = 11 \neq 10 = f(2).$$

If instead, we redefine this function as

$$f(x) = \begin{cases} \frac{3x^2-x-10}{x-2} & x \neq 2 \\ 11 & x = 2 \end{cases},$$

then it would be continuous at $x = 2$ since

$$\lim_{x\to 2} f(x) = 11 = f(2).$$

Suppose we wish to prove that $f(x) = x^2$ is continuous at x_0 for all $x_0 \in \mathbb{R}$. To do this, we need to show that

$$\lim_{x\to x_0} x^2 = f(x_0) = x_0^2.$$

Assumed: Let ϵ_0 represent an arbitrary positive number.

To Show: We need to find $\delta > 0$ such that

$$\text{if } 0 < |x - x_0| < \delta, \text{ then } |f(x) - f(x_0)| < \epsilon_0.$$

We will break this into 3 cases. $x_0 > 0$, $x_0 < 0$, $x_0 = 0$.

Case 1: $x_0 > 0$.

Conceptual Insight: 4.24

Using the backward/forward approach, we start at the end,

$$|f(x) - x_0^2| = |x^2 - x_0^2| = |x - x_0||x + x_0| < \epsilon_0.$$

After dividing by $|x+x_0|$, we want to find a number N such that

$$|x-x_0| < N < \frac{\epsilon_0}{|x+x_0|}.$$

The number N can be written in terms of ϵ_0 and x_0, two already defined numbers, but cannot depend on x which at the moment is unknown. To do this requires picking an initial interval for x. We will have to use care to make sure $|x+x_0| \neq 0$.
While there are several choices we could make, we assume $\delta \leq |x_0|/2$. In this case, $|x-x_0| < |x_0|/2$ so

$$x_0/2 < x < 3x_0/2.$$

Then

$$3|x_0|/2 < |x+x_0| < 5|x_0|/2.$$

We now make the denominator of $\epsilon_0/|x+x_0|$ as large as possible,

$$N = \frac{\epsilon_0}{5|x_0|/2} = \frac{2\epsilon_0}{5|x_0|} < \frac{\epsilon_0}{|x+x_0|}.$$

Now we can also set

$$\delta \leq \frac{2\epsilon_0}{5|x_0|}.$$

We now construct a proof that

$$\lim_{x\to x_0} x^2 = x_0^2,$$

but we do it in three cases.

Claim 4.25: Case 1: $\lim_{x\to x_0} x^2 = x_0^2$ **for** $x_0 > 0$**.**

Proof of Case 1: Let $x_0 > 0$ represent an arbitrary number in $\mathbb{R}^+$. Show

$$\lim_{x\to x_0} x^2 = f(x_0) = x_0^2.$$

To do this, let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \min\left\{\frac{|x_0|}{2}, \frac{2\epsilon_0}{5|x_0|}\right\}.$$

Let x_1 represent an arbitrary positive number such that

$$0 < |x_1 - x_0| < \delta_0.$$

Since $|x_1 - x_0| < |x_0|/2$, then

$$0 < x_0/2 < x_1 < 3x_0/2.$$

Since $x_0 > 0$, adding x_0 to all sides gives

$$0 < 3x_0/2 < x_1 + x_0 < 5x_0/2.$$

Since $x_0 > 0$, we have

$$|x_1 + x_0| < 5|x_0|/2.$$

This means that

$$\frac{\epsilon_0}{|x_1 + x_0|} > \frac{2\epsilon_0}{5|x_0|}.$$

Since we also know that

$$|x_1 - x_0| < \frac{2\epsilon_0}{5|x_0|},$$

we then combine inequalities to get

$$|x_1 - x_0| < \frac{\epsilon_0}{|x_1 + x_0|}.$$

Multiplying both sides by $|x_1 + x_0|$, which we know is positive, we get

$$|f(x_1) - x_0^2| = |x_1^2 - x_0^2| < \epsilon_0.$$

Thus,

$$\lim_{x \to x_0} x^2 = f(x_0) = x_0^2$$

and so f is continuous at $x = x_0$ if $x_0 > 0$.□

The next

Claim 4.26: Case 2: $\lim_{x \to x_0} x^2 = x_0^2$ **for** $x_0 < 0$**.**

Proof of Case 2: Let $x_0 < 0$ represent an arbitrary number in $\mathbb{R}^-$, and show

$$\lim_{x \to x_0} x^2 = f(x_0) = x_0^2.$$

To do this, let ϵ_0 represent an arbitrary positive number. Again, set

$$\delta_0 = \min\left\{\frac{|x_0|}{2}, \frac{2\epsilon_0}{5|x_0|}\right\}.$$

Let x_1 represent an arbitrary positive number such that

$$0 < |x_1 - x_0| < \delta_0.$$

Since $|x_1 - x_0| < |x_0|/2$, and $x_0 < 0$ then

$$0 > x_0/2 > x_1 > 3x_0/2.$$

Adding x_0 gives

$$3x_0/2 > x_1 + x_0 > 5x_0/2.$$

Since $x_0 < 0$, all terms are negative so taking absolute values reverses the inequality, giving that

$$|x_1 + x_0| < 5|x_0|/2.$$

This means that

$$\frac{\epsilon_0}{|x_1 + x_0|} > \frac{2\epsilon_0}{5|x_0|}.$$

Since we also know that

$$|x_1 - x_0| < \frac{2\epsilon_0}{5|x_0|},$$

we then combine inequalities to get

$$|x_1 - x_0| < \frac{\epsilon_0}{|x_1 + x_0|}.$$

Multiplying both sides by $|x_1 + x_0|$, which we know is positive, we get

$$|f(x_1) - x_0^2| = |x_1^2 - x_0^2| < \epsilon_0.$$

Thus,

$$\lim_{x \to x_0} x^2 = f(x_0) = x_0^2$$

and so f is continuous at $x = x_0 < 0$. $\square$

We now prove the third case, which is the easiest of all.

Claim 4.27: Case 3: $\lim_{x\to x_0} x^2 = x_0^2$ **for** $x_0 = 0$**.**

Proof of Case 3: Set $x_0 = 0$. Show

$$\lim_{x\to 0} x^2 = f(0) = 0.$$

Let ϵ_0 represent an arbitrary positive number. Set

$$\delta_0 = \sqrt{\epsilon_0}.$$

Let x_1 represent an arbitrary number such that

$$0 < |x_1 - x_0| = |x_1 - 0| = |x_1| < \delta_0 = \sqrt{\epsilon_0}.$$

Squaring both sides gives

$$|x_1|^2 = |x_1^2 - 0^2| = |f(x_1) - f(0)| < \epsilon_0.\square$$

So in each case

$$\lim_{x\to x_0} x^2 = f(x_0)$$

and f is continuous at every x value.

We now consider the definition of the derivative.

Definition 4.28: Derivative

If f is defined on an interval (a, b) and $x_0 \in (a, b)$, then f is differentiable at x_0 if and only if

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. The limit is called the derivative of f at x_0 and is denoted $f'(x_0)$, that is,

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0).$$

Outline for using definition of derivative:

- *Let method:* Let ϵ_0 represent an arbitrary positive number.
- *Backwards approach:* Start with

$$\left|\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)\right|<\epsilon_0$$

  and work backwards to $0<|h-0|=|h|<\delta$ to find δ in terms of ϵ_0.
- *Technical handle:* If there is a problem finding δ in terms of ϵ_0, make an initial bound for δ, say $\delta<c$ for some appropriate c.
- *Forward proof:* Using δ found previously, verify that

$$\langle 0<|h|<\delta\rangle \Rightarrow \left\langle\left|\frac{f(x_0+h)-f(x_0)}{h}-f'(x_0)\right|<\epsilon_0\right\rangle.$$

Let's use these steps to show that $f(x)=x^3$ is differentiable at x_0 and that its derivative is

$$f'(x_0)=3x_0^2,$$

that is, show that

$$\lim_{h\to 0}\frac{(x_0+h)^3-x_0^3}{h}=3x_0^2.$$

As in Example 4.25, we consider three cases, $x_0>0$, $x_0<0$, and $x_0=0$.

We skip the start-up examples which would consist of trying particular values for ϵ_0 and finding δ for those values.

Assumed: Let x_0 represent an arbitrary positive number. To show the limit is as claimed, we let ϵ_0 represent an arbitrary positive number.

To Show: We must find $\delta>0$ such that

$$\text{if } 0<|h|<\delta \text{ then } \left|\frac{(x_0+h)^3-x_0^3}{h}-3x_0^2\right|<\epsilon_0.$$

Conceptual Insight: 4.29

We work backwards from

$$\left|\frac{(x_0+h)^3-x_0^3}{h}-3x_0^2\right|<\epsilon_0 \tag{4.6}$$

to find δ, which can be given in terms of the defined numbers ϵ_0 and

x_0. Simplification of inequality 4.6 gives

$$|3x_0h + h^2| = |h||3x_0 + h| < \epsilon_0.$$

Remember that x_0 and ϵ_0 are fixed constants so δ can depend on them, but not on h which can vary. Therefore, we cannot set δ equal to $\epsilon_0/|3x_0 + h|$.
Technical Handle: We need to pick an initial estimate for δ making sure that if $|h| < \delta$, then $3x_0 + h \neq 0$. One such delta would be

$$\delta \leq x_0.$$

Thus, if $|h| < \delta \leq x_0$, we know

$$0 < 2x_0 < 3x_0 + h < 4x_0.$$

Now we have

$$\frac{\epsilon_0}{|3x_0 + h|} \geq \frac{\epsilon_0}{4|x_0|},$$

so we want

$$|h| < \delta \leq \frac{\epsilon_0}{4|x_0|},$$

noting that the denominator cannot be equal to zero. We now combine the inequalities to get

$$0 < |h| < \frac{\epsilon_0}{4|x_0|} \leq \frac{\epsilon_0}{|3x_0 + h|},$$

which can now be simplified to the desired inequality $|3x_0h + h^2| = |h||3x_0 + h| < \epsilon_0$. Note that $3x_0 + h \neq 0$ since $|h| < |x_0|$. We now have two conditions on δ, so we set

$$\delta_0 = \min\{|x_0|, \epsilon_0/(4|x_0|)\}. \tag{4.7}$$

We now reverse our steps to prove the derivative is as claimed.

Claim 4.30: If $f(x) = x^3$, then $f'(x) = 3x^2$.

Proof: We complete our proof using three cases.

Case 1: Let $x_0 > 0$. Let $\epsilon_0 > 0$. Set

$$\delta_0 = \min\{|x_0|, \epsilon_0/(4|x_0|)\} = \min\{x_0, \epsilon_0/(4x_0)\}.$$

Let h_0 be such that

$$0 < |h_0 - 0| = |h_0| < \delta_0.$$

This means $-x_0 < |h_0| < x_0$ so

$$0 < |3x_0 + h_0| < 4x_0.$$

We now have that

$$\begin{aligned}
& \left|\frac{f(x_0+h_0) - f(x_0)}{h_0} - 3x_0^2\right| \\
= & \left|\frac{(x_0+h_0)^3 - x_0^3}{h_0} - 3x_0^2\right| \\
= & \left|\frac{(x_0^3 + 3x_0^2h_0 + 3x_0h_0^2 + h_0^3) - x_0^3}{h_0} - \frac{3x_0^2h_0}{h_0}\right| \\
= & \left|\frac{3x_0^2h_0 + 3x_0h_0^2 + h_0^3 - 3x_0^2h_0}{h_0}\right| \\
= & \left|\frac{3x_0h_0^2 + h_0^3}{h_0}\right| \\
= & \left|3x_0h_0 + h_0^2\right| \\
= & |3x_0 + h_0||h_0|.
\end{aligned}$$

Since we know that

$$|h_0| < \delta_0 \leq |x_0|,$$

then $|3x_0 + h_0| < |4x_0|$, giving that

$$|3x_0 + h_0||h_0| < |4x_0||h_0|.$$

Since

$$|h_0| < \delta_0 \leq \frac{\epsilon_0}{(4|x_0|)},$$

we have

$$|4x_0||h_0| < \epsilon_0.$$

Combining all this algebraic work gives that

$$\left|\frac{f(x_0+h_0)-f(x_0)}{h_0}-3x_0^2\right|<\epsilon_0.$$

Case 2: Let $x_0 < 0$. The proof is precisely the same as for Case 1, except absolute values must be kept all the way through. You use the same δ_0 as defined in Equation 4.7. We leave this to the reader, just follow the steps of Case 1, but be careful.

Case 3: Set $x_0 = 0$ and let $\epsilon_0 > 0$. In this case, we must find a δ so that for $0 < |h| < \delta$.

$$\left|\frac{(0+h)^3-0^3}{h}-3(0)^2\right|<\epsilon_0.$$

This simplifies to

$$|h^2| < \epsilon_0,$$

which will be true if

$$0 < |h| < \sqrt{\epsilon_0} = \delta_0.$$

We again leave the details to the reader. □

4.4.1 Problems

1. Recall the function

$$f(x)=\begin{cases}x+1 & x\le 2\\ 2x-1 & x>2\end{cases}$$

from Problem 5 in Section 4.1.

(a) Show that f is continuous at $x=2$. Answer 1

(b) Show $f'(3)=2$ Answer 2

2. Recall the function

$$f(x)=\begin{cases}3x & x \text{ rational}\\ -2x & x \text{ irrational}\end{cases}$$

from Section 4.1, Problem 8. Show that this function is continuous at $x = 0$ and discontinuous at $x = 0.5$. You should make a 'sketch' of the function and use the sketch to help in working each part of this problem.

3. Consider the function

$$f(x) = \begin{cases} x^2 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}.$$

   (a) Make a 'sketch' of the function and use the sketch to help in working each of the following parts of this problem. Answer 3

   (b) Set $\epsilon = 0.1$. Find $\delta > 0$ such that

$$\text{if } 0 < |h| < \delta \text{ then } \left| \frac{f(h) - f(0)}{h} - 0 \right| < 0.1.$$

   Answer 4

   (c) Show that f is differentiable at $x = 0$ and that $f'(0) = 0$. (Note that this function is discontinuous when $x \neq 0$, so it is differentiable at exactly one point. This problem could have been stated as 'show that there exists a unique x for which $f(x)$ is differentiable.' Answer 5

4. Show $f'(0) = 0$ for the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

5. Recall the function

$$f(x) = \begin{cases} 2x \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

from Problem 9, Section 4.1. In that problem, we showed that

$$\lim_{x \to 0} f(x) = 0,$$

so this function is continuous at $x = 0$. Show $f'(0)$ does not exist. Answer 6

6. Consider a sequence of real numbers s_n, $n \in \mathbb{Z}_0^+$. We say that
$$\lim_{n\to\infty} s_n = L$$
if and only if
$$\forall\, \epsilon > 0,\ \exists N \in \mathbb{R} \text{ such that if } n > N, \text{ then } |s_n - L| < \epsilon.$$
   (a) Letting
   $$s_n = \sum_{i=0}^{n} 0.8^i,$$
   find a value N such that if $n > N$, then
   $$|s_n - 5| < 0.1.$$
   Recall from Section 3.2, Problem 6 that the finite geometric series satisfies the equation
   $$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r},\ r \neq 0,\ r \neq 1.$$
   (b) Show that
   $$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=0}^{n} 0.8^i = 5.$$
   (c) Show that
   $$\sum_{i=0}^{\infty} r^i = \lim_{n\to\infty} \sum_{i=0}^{n} r^i = \frac{1}{1-r} \text{ if } 0 < |r| < 1.$$

7. We know that if $f(x) = x$, then $f'(x) = 1$. Prove that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$ for $n \in \mathbb{Z}^+$. (Hint: You may use the product rule.) Answer 7

4.4.2 Answers to selected problems

1. **Problem 1 a:** We have already shown in Problem 5 in Section 4.1 that
$$\lim_{x\to 2} f(x) = 3.$$
Substitution of $x = 2$ gives that $f(2) = 2 + 1 = 3$, so f is continuous at $x = 2$.

2. **Problem 1 b:** We must show that
$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{f(3+h)-5}{h}=2.$$
This means we must find δ so that when $|h-0|<\delta$ then
$$|\frac{f(3+h)-5}{h}-2|<\epsilon.$$

Proof: Let ϵ_0 represent an arbitrary positive number. Set $\delta_0=1$. Let h_0 represent an arbitrary number such that
$$0<|h_0-0|<\delta_0=1.$$
Then
$$\left|\frac{f(3+h_0)-f(3)}{h_0}-2\right|=\left|\frac{2(3+h_0)-1-5}{h_0}-2\right|=|0|<\epsilon_0,$$
so
$$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=2=f'(3),$$
which is what we were to show.□

3. **Problem 3 a:** Figure 4.9 gives a loose sketch of this function. Figure 4.9 a) indicates why the function is continuous at $x=0$ while Figure 4.9 b) indicates why the function is not continuous at $x=1$.

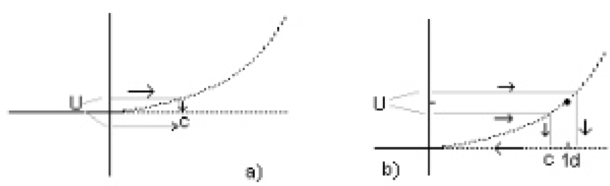

Figure 4.9: In a), for any value $x<c$, $f(x)\in U$ which is a small interval about 0. In b), for any irrational $x\in(c,d)$, $f(x)=0\notin U$.

4. **Problem 3 b:** We want
$$\left|\frac{f(h)-f(0)}{h}-0\right|=\left|\frac{f(h)}{h}\right|<0.1.$$
If h is irrational,
$$\left|\frac{f(h)}{h}\right|=0$$

no matter how large h is. If h is rational, then

$$\left|\frac{f(h)}{h}\right| = |h|.$$

This will be less than 0.1 if $0 < |h| < 0.1$, so $\delta = 0.1$.

5. **Problem 3 c:** Let ϵ_0 represent an arbitrary positive number. We have to find $\delta > 0$ such that if $0 < |h| < \delta$, then

$$\left|\frac{f(h) - f(0)}{h} - 0\right| = \left|\frac{f(h)}{h}\right| < \epsilon_0.$$

If h is irrational, then

$$\left|\frac{f(h)}{h}\right| = \left|\frac{0}{h}\right| = 0 < \epsilon_0$$

for any h. If h is rational, then

$$\left|\frac{f(h)}{h}\right| = \left|\frac{h^2}{h}\right| = |h| < \epsilon_0.$$

So the statement will be true if we set $\delta_0 = \epsilon_0$.

6. **Problem 5b:** We need to show that

$$\lim_{h\to 0} \frac{f(h+0) - f(0)}{h} = \lim_{h\to 0} 2\sin\left(\frac{1}{h}\right)$$

does not exist. The idea is that for small values of h

$$2\sin\left(\frac{1}{h}\right)$$

will range between -2 and 2, so doesn't go to one value. We will show the limit doesn't exist using contradiction.

Assume

$$\lim_{h\to 0} 2\sin(1/h) = L.$$

Set $\epsilon = 0.5$. We have assumed that

$$\exists\, \delta_0 > 0 \text{ such that if } 0 < |h| < \delta_0 \text{ then } |2\sin(1/h) - L| < 0.5.$$

Set
$$j_0 = \left\lceil \frac{1}{\delta_0} \right\rceil + 1,$$
so j_0 is an integer and
$$0 < \frac{1}{j_0} < \delta_0,$$
so
$$0 < \frac{1}{2\pi j_0} < \delta_0 \text{ and } 0 < \frac{1}{2\pi j_0 + \pi/2} < \delta_0.$$
Set
$$h_0 = \frac{1}{2\pi j_0} \text{ and } h_1 = \frac{1}{2\pi j_0 + +\pi/2}.$$
We have $0 < |h_0|, |h_1| < \delta_0$. By the triangle inequality and the assumed limit property
$$\begin{aligned} |2\sin(1/h_0) - 2\sin(1/h_1)| &= |2\sin(1/h_0) - L + L - 2\sin(1/h_1)| \\ &\leq |2\sin(1/h_0) - L| + |L - 2\sin(1/h_1)| \\ &< 0.5 + 0.5 = 1. \end{aligned}$$
But
$$|2\sin(1/h_0) - 2\sin(1/h_1)| = |2\sin(2\pi j_0) - 2\sin(2\pi j_0 + \pi/2)| = |0 - 2| = 2.$$
This is a contradiction, so the limit doesn't exist.

7. **Problem 7:** We use induction.

   Basis case: Show that when $n = 1$, the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$, that is, the derivative of $f(x) = x$ is $f'(x) = x^0 = 1$. We know this is true (fairly easy to show using the definition of derivative).

   Induction Step: Assume for some $n_0 \geq 1$ that the derivative of
   $$f(x) = x^{n_0} \text{ is } f'(x) = n_0 x^{n_0 - 1}.$$
   We must show that the derivative of $f(x) = x^{n_0+1}$ is $f'(x) = (n_0 + 1)x_0^n$.

   Rewriting
   $$f(x) = x^{n_0+1} = (x)x^{n_0},$$
   we have by the product rule that
   $$f'(x) = (1)x^{n_0} + x(n_0 x^{n_0-1}) = x^{n_0} + n_0 x^{n_0} = (n_0 + 1)x^{n_0}.$$

Chapter 5

Sets and Relations

5.1 Set Inclusion

Video Lesson 5.1

This section will be easier to read if you first watch the 9:35 minute video
https://vimeo.com/499674749 (Password:Proof)
which gives an example of set inclusion proofs. To change the speed at which the video plays, click on the gear at the lower right of the video.

In this section, we return to our study of sets. As we know, a common problem for sets is to show that one set, say S, is contained in another set, T, written as

$$S \subseteq T.$$

The technical handle for showing set containment is the let method, we 'let' s_0 represent an arbitrary element in the set S and then show that $s_0 \in T$. If we want to show two sets, S and T, are equal, we have to prove

$$S \subseteq T \quad \text{and} \quad T \subseteq S.$$

Thus, we have a two part proof. Let's see how this works for a particular example.

Conceptual Insight: 5.2

The shaded region of Figure 5.1 a) gives the Venn diagram for $(S \cup T)^c$ while the shaded regions for Figures 5.1 b) and c) give the diagrams for S^c and T^c, respectively. It appears from the figures that the intersection of the sets in b) and c) equals the set in a),

$$(S \cup T)^c = S^c \cap T^c. \tag{5.1}$$

This is one of DeMorgan's laws. We could develop some insight by constructing several examples, but Figure 5.1 is quite convincing, so we will skip that step, coming back to it only if we have difficulty constructing the proof.

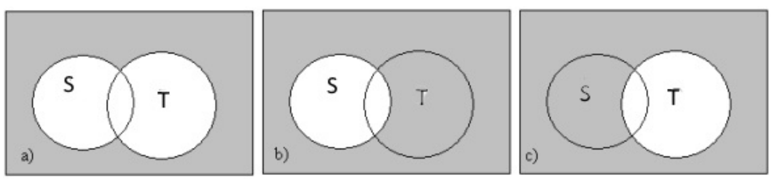

Figure 5.1: Visual display that $(S \cup T)^c = S^c \cap T^c$

Claim 5.3: For arbitrary sets S and T, $(S \cup T)^c = S^c \cap T^c$.

Proof: Let S and T represent two arbitrary sets (we use S and T instead of S_0 and T_0 for ease of reading). These sets are are now treated as fixed and known. This proof consists of doing two parts, showing $(S \cup T)^c \subseteq S^c \cap T^c$ and showing $S^c \cap T^c \subseteq (S \cup T)^c$. We continue using the let method.

Part 1: Let x_0 represent an arbitrary element in the complement of the union of S and T, that is,

$$x_0 \in (S \cup T)^c.$$

We must show

$$x_0 \in S^c \cap T^c,$$

that is, $x_0 \in S^c$ and $x_0 \in T^c$. We have clearly stated what is assumed and what we have to show.

By definition of complement, $x_0 \notin S \cup T$ which by the definition of union means that $x_0 \notin S$ and $x_0 \notin T$. Since $x_0 \notin S$, then $x_0 \in S^c$. Similarly, since $x_0 \notin T$, then $x_0 \in T^c$. By definition of intersection, $x_0 \in S^c \cap T^c$, so
$$(S \cup T)^c \subseteq S^c \cap T^c.$$

Part 2: To show $S^c \cap T^c \subseteq (S \cup T)^c$, let
$$x_0 \in S^c \cap T^c$$
and show
$$x_0 \in (S \cup T)^c.$$
Since $x_0 \in S^c \cap T^c$, then $x_0 \in S^c$ so $x_0 \notin S$. Similarly, $x_0 \in T^c$ so $x_0 \notin T$. By the definition of union, $x_0 \notin S \cup T$. By definition of complement, $x_0 \in (S \cup T)^c$, so
$$S^c \cap T^c \subseteq (S \cup T)^c.$$

Since $(S \cup T)^c \supseteq S^c \cap T^c$ and $(S \cup T)^c \subseteq S^c \cap T^c$, then
$$(S \cup T)^c = S^c \cap T^c. \square$$

What is different about the set inclusion argument in Proof 5.3 is that it is more of a written proof then algebraic manipulation. This will be a continuing theme. We now show how we can use definitions to find and prove properties about sets.

Definition 5.4: Difference of sets

We define the **difference** of two sets as
$$S - T = \{x : x \in S \cap T^c\}.$$

Intuitively, $S - T$ is the set S with all elements of T removed, as seen in Figure 5.2. Our intuition suggests that a distributive law for sets, such as
$$S \cap (T - V) = (S \cap T) - (S \cap V) \tag{5.2}$$
might hold since it is similar to the distributive law for numbers, $a(b - c) =$

$(ab) - (ac)$. In Figure 5.3 we can see that removing $(S \cap V)$ in part b) from $(S \cap T)$ in part a) gives $S \cap (T - V)$ in part c). You should construct actual sets S, T and V, then take $(S \cap V)$ from $(S \cap T)$ to see that the same shaded region results as in Figure 5.3 c), so this result is apparently true.

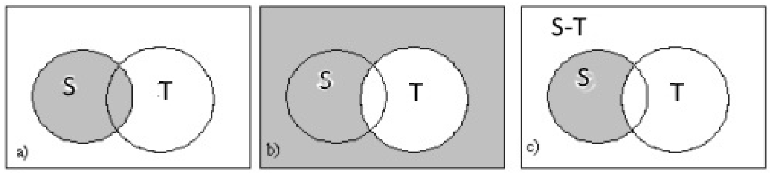

Figure 5.2: Intersection of shaded regions in a) and b) gives $S - T$, shaded region in c).

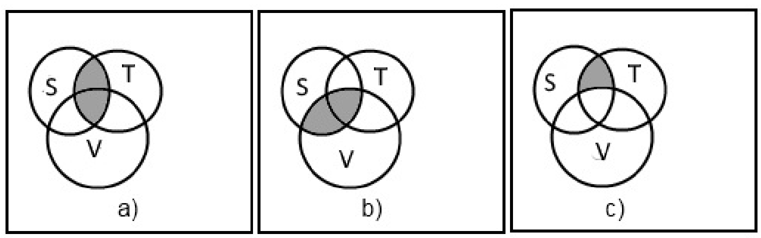

Figure 5.3: Intersection $(S \cap T)$, shaded a), minus $(S \cap V)$, shaded b), gives $S \cap (T - V)$, shaded c).

Conceptual Insight: 5.5

To show
$$S \cap (T - V) \subseteq (S \cap T) - (S \cap V),$$
we let x_0 represent an arbitrary element in $S \cap (T - V)$. We convert this into usable form:

- **Usable form of given**: $x_0 \in S$, $x_0 \in T$, and $x_0 \in V^c$

We must show that

$$x_0 \in (S \cap T) - (S \cap V) = \text{ (by definition) } (S \cap T) \cap (S \cap V)^c.$$

- **Usable form of what must be shown:** $x_0 \in S \cap T$ and $x_0 \notin (S \cap V)$.

Claim 5.6: For arbitrary sets S, T, and V,

$$S \cap (T - V) = (S \cap T) - (S \cap V).$$

Proof: We use the definition for difference of sets to show equation 5.2 holds. As usual for set equality, we do two parts.

Part 1: Let x_0 represent an arbitrary element in $S \cap (T - V)$. Then

$$x_0 \in S \cap (T \cap V^c),$$

which means $x_0 \in S$, $x_0 \in T$, and $x_0 \in V^c$
We must show that

$$x_0 \in (S \cap T) - (S \cap V) = \text{(by definition) } (S \cap T) \cap (S \cap V)^c,$$

that is, $x_0 \in S \cap T$ and $x_0 \notin (S \cap V)$.
Since $x_0 \in S$ and $x_0 \in T$ so

$$x_0 \in S \cap T.$$

Since $x_0 \in V^c$, then $x_0 \notin V$, so by definition of intersection, $x_0 \notin (S \cap V)$. Therefore $x_0 \in (S \cap T) \cap (S \cap V)^c$ so

$$S \cap (T - V) \subseteq (S \cap T) - (S \cap V)$$

and we are finished with the first part.
Part 2: See if you can reverse the previous steps to show

$$(S \cap T) - (S \cap V) \subseteq S \cap (T - V),$$

which would complete the proof that $S \cap (T - V) = (S \cap T) - (S \cap V)$.□

The proof consisted of using the definition for $S - T$ combined with thinking about the meaning at each step.

We now examine the intersection of sets in more detail. We know that the intersection of two open intervals is open, for example

$$(0, 7) \cap (2, 11) = (2, 7).$$

We can easily see that the intersection of three open intervals is open or the null set (which is considered to be open) by taking intersection one step at a time, as in

$$(-10,7) \cap (0,10) \cap (5,15) = (0,7) \cap (5,15) = (5,7).$$

In fact, we might think that the intersection of any number of open intervals would be an open interval, if we consider the empty set as an open interval. This is not actually true; the intersection of an infinite number of open intervals might in fact be a closed interval. Let's see how this can happen.

For $i \in \mathbb{Z}^+$, let

$$S_i = \left\{ x \in \mathbb{R} : x \in \left(-\frac{1}{i}, 1\right) \right\},$$

so $S_1 = (-1,1)$, $S_2 = (-0.5,1)$, $\cdots$, $S_{10} = (-0.1,1)$, $\cdots$. In fact, if we take intersection one step at a time, we get that $S_1 \cap S_2 = S_2$, $S_1 \cap S_2 \cap S_3 = S_3$

$$\bigcap_{i=1}^{n} S_i = S_n.$$

Figure 5.4 gives an idea of what

$$S = \bigcap_{i \in \mathbb{Z}^+} S_i$$

might look like.

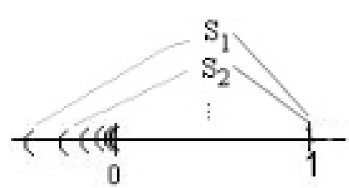

Figure 5.4: The sets $S_i = (-1/i, 1)$

This discussion is comparable to the startup examples of our proof process, leading to the conjecture

$$S = \bigcap_{i \in \mathbb{Z}^+} S_i = [0,1).$$

We are given the intervals S_i as our assumed and we have to show

$$[0,1) \subseteq \bigcap_{i \in \mathbb{Z}^+} S_i \text{ and } \bigcap_{i \in \mathbb{Z}^+} S_i \subseteq [0,1).$$

Claim 5.7:

$$S = \bigcap_{i \in \mathbb{Z}^+} S_i = [0,1).$$

Proof: Our proof requires two parts.
Part 1: We prove $[0,1) \subseteq S$. Let x_0 represent an arbitrary element in $[0,1)$ which means that $0 \leq x_0 < 1$. We must show that $\forall\, i \in \mathbb{Z}^+$

$$x_0 \in S_i = \left(-\frac{1}{i}, 1\right).$$

Let i_0 represent an arbitrary positive integer. Then

$$-\frac{1}{i_0} < 0 \leq x_0 < 1,$$

so $x_0 \in S_{i_0}$. Since i_0 was arbitrary, $x_0 \in S_i \,\forall\, i \in \mathbb{Z}^+$, so

$$x_0 \in \bigcap_{i \in \mathbb{Z}^+} S_i \text{ and } [0,1) \subseteq \bigcap_{i \in \mathbb{Z}^+} S_i.$$

Part 2: We prove $S \subseteq [0,1)$. Let x_0 represent an arbitrary element in

$$S = \bigcap_{i \in \mathbb{Z}^+} S_i.$$

We are given that

$$\forall\, i \in \mathbb{Z}^+,\, x_0 \in (-\frac{1}{i}, 1).$$

We must show $0 \leq x_0 < 1$.
Our given means that

$$\forall\, i \in \mathbb{Z}^+,\, -\frac{1}{i} < x_0 < 1.$$

You might review Proof 2.40 in Section 2.5, in which contradiction was used to show that

$$\text{if } -\frac{1}{i} < x \,\forall\, i \in \mathbb{Z}^+ \text{ then } 0 \leq x.$$

This means $x_0 \geq 0$. Since we also know $x_0 < 1$, we are done,

$$S = \bigcap_{i \in \mathbb{Z}^+} S_i = [0,1)$$

and the intersection of this infinite number of open intervals is not open.□

Remark 5.8

We 'cheated' somewhat. The main result we needed to complete this proof was done in Proof 2.40. In reality, we might have been trying to prove

$$\bigcap_{i \in \mathbb{Z}^+} S_i = [0, 1)$$

and realized we needed to show that if

$$-\frac{1}{i} < x \,\forall\, i \in \mathbb{Z}^+,$$

then $0 \leq x$. We would have then put our original problem aside and tried to prove this result. In presenting the whole proof, we normally state and prove these secondary results beforehand, as we did in Proofe 2.40, calling that result a Lemma. Then in the proof of the main result, we can just refer to the lemma. This makes the proof easier to follow.

Remark 5.9: I

fact, if

$$T_i = \left\{ x \in \mathbb{R} : x \in \left(-\frac{1}{i}, 1 + \frac{1}{i} \right) \right\},$$

then

$$T = \bigcap_{i \in \mathbb{Z}^+} T_i = [0, 1]$$

and the intersection of this infinite number of open intervals is actually closed.

5.1.1 Problems

1. With S and T being sets, prove that $(S \cap T)^c = S^c \cup T^c$. (This is the other De Morgan's Law.) Answer 1
2. Letting S_i be a set $\forall\, i \in \mathbb{Z}^+$, prove that

$$\left(\bigcup_{i \in \mathbb{Z}^+} S_i \right)^c = \bigcap_{i \in \mathbb{Z}^+} S_i^c.$$

3. Letting S_i be a set $\forall\, i \in \mathbb{Z}^+$, prove that

$$\left(\bigcap_{i\in\mathbb{Z}^+} S_i\right)^c = \bigcup_{i\in\mathbb{Z}^+} S_i^c.$$

   Answer 2

4. Prove or disprove that if $S \cup V \subseteq T \cup V$, then $S \subseteq T$.

5. For each of the following statements, prove it is true or give a counterexample. It might help to sketch a figure for each side of the equation first.

   (a) If $(S - T) \cap (S - V) = \emptyset$ then $T \cap V = \emptyset$ Answer 3

   (b) $S - (T - V) = (S \cup V) - T$. Answer 4

   (c) $[S - (T - V)] \cap V = S \cap V$. Answer 5

   (d) $S \cup (T - V) = [(S \cup T) - V] \cup (S \cap V)$.

6. Define $S\Delta T = (S - T) \cup (T - S)$. Are each of the following claims true? Why? The Venn diagram in Figure 5.5 gives a visualization of $S\Delta T$.

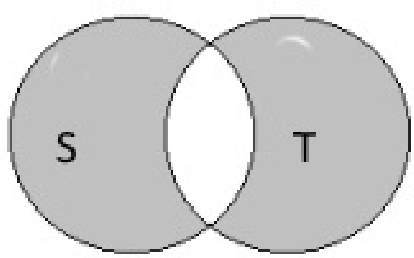

Figure 5.5: Venn diagram of $S\Delta T$.

   (a) $S\Delta T = (S \cup T) - (S \cap T)$.

   (b) $(S - T)\Delta(S - V) = (S \cap T)\Delta(S \cap V)$

7. For every $x \in \mathbb{R}^+$, define the set $S_x = (-x, x)$. Remember that a value a is in the intersection of a collection of sets if a is in every one of the sets.

   (a) Show that $0 \in S_1 \cap S_{0.5} \cap S_{0.01}$. Answer 6

(b) Show that $0.2 \notin S_1 \cap S_{0.5} \cap S_{0.01}$. Answer 7

(c) Let $a = 0.1$. Show that

$$a \notin \bigcap_{x \in \mathbb{R}^+} S_x.$$

Answer 8

(d) Show that

$$0 \in \bigcap_{x \in \mathbb{R}^+} S_x.$$

Answer 9

(e) Let $a > 0$. Show that

$$a \notin \bigcap_{x \in \mathbb{R}^+} S_x.$$

Answer 10

8. Let f be a real valued function with domain $\mathcal{D} \subseteq \mathbb{R}$. Let $Y \subseteq \mathbb{R}$. Define the set

$$f^{-1}(Y) = \{x \in \mathcal{D} : \exists y \in Y \text{ such that } f(x) = y\} \subseteq \mathcal{D}.$$

Intuitively, $f^{-1}(Y)$ is the set of all x-values which get mapped into the set Y. Figure 5.6 gives a visualization of $f^{-1}(Y)$. You should construct several different figures to develop good insight into $f^{-1}(Y)$. These should include cases in which f is not one-to-one and Y is not a subset of the range of f

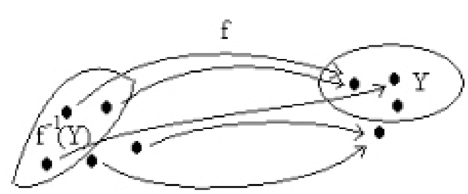

Figure 5.6: Visualization of $f^{-1}(Y)$

(a) Determine if $f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$ is always true. Explain your answer.

(b) Determine if $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$ is always true. Explain your answer.

(c) Prove or disprove that if $f^{-1}(S) \neq \emptyset$ for every $S \subseteq Y$, $S \neq \emptyset$, then f is onto Y.

9. Let $f : \mathcal{D} \to \mathbb{R}$ where $\mathcal{D} \subseteq \mathbb{R}$. Define the set

$$f(X) = \{y \in \mathbb{R} : \exists\, x \in X \text{ such that } f(x) = y\}.$$

Intuitively, $f(X)$ is the set of all y-values mapped onto from the set $X \subseteq \mathcal{D}$.

(a) Prove or disprove $f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$. Answer 11

(b) Prove or disprove $f(X_1 \cap X_2) = f(X_1) \cap f(X_2)$. Answer 12

10. If $X \subseteq \mathcal{D}$, is it true that $f(X^c) = f(X)^c$? Give a proof or a counterexample. (See Problem 9 for the definition of $f(X)$.)

11. Let $X \subseteq \mathcal{D}$. (See Problem 8 for the definition of $f^{-1}(Y)$ and Problem 9 for the definition of $f(X)$.)

(a) Prove or disprove that $X \subseteq f^{-1}(f(X))$. Answer 13

(b) Prove that $X = f^{-1}(f(X))$ or give an example in which

$$X \neq f^{-1}(f(X)).$$

Answer 14

12. If $f^{-1}(f(X)) = X$ for all $X \subseteq \mathcal{D}$, then prove f is one-to-one or find a counterexample. (See Problem 8 for the definition of $f^{-1}(Y)$ and Problem 9 for the definition of $f(X)$.)

13. Let $Y \subseteq \mathbb{R}$. (See Problem 8 for the definition of $f^{-1}(Y)$ and Problem 9 for the definition of $f(X)$.)

(a) Prove that $f(f^{-1}(Y)) \subseteq Y$ or give a counterexample. Answer 15

(b) Prove that $Y \subseteq f(f^{-1}(Y))$ or give a counterexample. Answer 16

14. Conjecture what the set

$$S = \bigcup_{i=2}^{\infty} \left[\frac{1}{i}, 1 - \frac{1}{i}\right]$$

equals and prove your result. Note that we are taking the infinite union of closed intervals.

15. Suppose

$$(a_1, b_1) \cap (a_2, b_2) \neq \emptyset.$$

Prove that there exists an interval (c, d) such that

$$(a_1, b_1) \cup (a_2, b_2) = (c, d).$$

Answer 17

5.1.2 Answers to selected problems

1. **Problem 1:**

   Proof of Part 1: Let x_0 represent an arbitrary element in $(S \cap T)^c$ This means $x_0 \notin S \cap T$. This means $x_0 \notin S$ or $x_0 \notin T$. WLOG, assume $x_0 \notin S$. This means $x_0 \in S^c$ so $x_0 \in S^c \cup T^c$. Thus, $(S \cap T)^c \subseteq S^c \cup T^c$.

   Proof of Part 2: We now let x_0 represent an arbitrary element in $S^c \cup T^c$. WLOG, assume $x_0 \in S^c$. This means $x_0 \notin S$, so $x_0 \notin S \cap T$. Thus, $x_0 \in (S \cap T)^c$, so $(S \cap T)^c \supseteq S^c \cup T^c$.

   This means $(S \cap T)^c = S^c \cup T^c$. $\square$

2. **Problem 3:**

   Proof of Part 1: Let x_0 represent an arbitrary element in

   $$\left(\bigcap_{i \in \mathbb{Z}^+} S_i \right)^c.$$

   This means that

   $$x_0 \notin \bigcap_{i \in \mathbb{Z}^+} S_i,$$

   that is, there exists $i_0 \in \mathbb{Z}^+$ such that $x_0 \notin S_{i_0}$ so $x_0 \in S_{i_0}^c$. This means that

   $$x_0 \in \bigcup_{i \in \mathbb{Z}^+} S_i^c,$$

   so

   $$\left(\bigcap_{i \in \mathbb{Z}^+} S_i \right)^c \subseteq \bigcup_{i \in \mathbb{Z}^+} S_i^c.$$

   Proof of Part 2: This direction is similar, so the sets are equal.

3. **Problem 5 a:** false: See sketch in Figure 5.7. All that needs to be true for $(S - T) \cap (S - V) = \emptyset$ is for S to be contained in $T \cup V$. For a counterexample, let $\mathcal{U} = \{1,2,3\}$, $S = \{1,2\}$, $T = \{1,3\}$, $V = \{2,3\}$. Then $S - T = \{2\}$, $S - V = \{1\}$ and $(S - T) \cap (S - V) = \emptyset$. But $T \cap V = \{3\}$.

4. **Problem 5 b:** False. Venn diagrams help in constructing a counterexample.

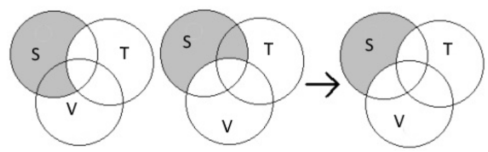

Figure 5.7: $(S - T) \cap (S - V)$

5. **Problem 5 c:** True:

   Proof of Part 1: To show $[S - (T - V)] \cap V \subseteq S \cap V$, let x_0 represent an arbitrary element in $[S - (T - V)] \cap V$. This means that $x_0 \in [S - (T - V)] \subseteq S$ and also $x_0 \in V$, so $x_0 \in S \cap V$.

   Proof of Part 2: To show $S \cap V \subseteq [S - (T - V)] \cap V$, Let x_0 represent an arbitrary element in $S \cap V$. Since $x_0 \in V$, all we need to show is that $x_0 \in S - (T - V) = S \cap (T - V)^c$. Since we already know that $x_0 \in S$, we only need to show that $x_0 \in (T - V)^c = (T \cap V^c)^c$. By DeMorgan's law, $(T \cap V^c)^c = T^c \cup (V^c)^c = T^c \cup V$. But we know $x_0 \in V$, so $x_0 \in T^c \cup V$. Part d) is similar.

6. **Problem 7 a:** $0 \in (-1, 1)$ and $0 \in (-0.5, 0.5)$ and $0 \in (-0.01, 0.01)$.

7. **Problem 7 b:** $0.2 \notin (-0.01, 0.01)$.

8. **Problem 7 c:** Set $x_0 = 0.01$. Then $a = 0.1 \notin S_{x_0} = (-0.01, 0.01)$.

9. **Problem 7 d:** We must show $0 \in (-x, x)$ for every $x \in \mathbb{R}^+$. Let x_0 represent an arbitrary number greater than 0. Then $0 \in (-x_0, x_0)$, so true.

10. **Problem 7 e:** Let a_0 represent an arbitrary number greater than 0. Then $a_0 \notin S_{a_0/2} = (-a_0/2, a_0/2)$ so false.

11. **Problem 9 a:** This is true. Let y_0 represent an arbitrary element in $f(X_1 \cap X_2)$. This means $\exists\, x_0 \in X_1 \cap X_2$ such that $y_0 = f(x_0)$. Since

$x_0 \in X_1$, then $y_0 \in f(X_1)$. Similarly, since $x_0 \in X_2$, then $y_0 \in f(X_2)$ so $y_0 \in f(X_1 \cap X_2)$.

12. **Problem 9 b:** This is false. Set $f(x) = x^2$, $X_1 = \{2\}$ and $X_2 = \{-2\}$. Then $f(X_1) = \{4\}$ and $f(X_2) = \{4\}$ so $f(X_1) \cap f(X_2) = \{4\}$. But $X_1 \cap X_2 = \emptyset$ so $f(X_1 \cap X_2) = \emptyset \neq \{4\}$.

13. **Problem 11 a:** Let x_0 represent an arbitrary element in X. Then $\exists\, y_0 \in f(X)$ such that $f(x_0) = y_0$. Since $y_0 \in f(X)$ and $f(x_0) = y_0$, then $x_0 \in f^{-1}(f(X))$ so $X \subseteq f^{-1}(f(X))$.

14. **Problem 11 b:** Set $X = \{2\}$ and $f(x) = x^2$. Then $f(X) = 4$ and $f^{-1}(f(X)) = \{-2, 2\}$, so $X \subset f^{-1}(f(X))$.

15. **Problem 13 a:** Let Y represent an arbitrary nonempty subset of $\mathbb{R}$. Let y_0 represent an arbitrary element in $f(f^{-1}(Y))$. Therefore,
$$\exists\, x_0 \in f^{-1}(Y) \text{ such that } f(x_0) = y_0.$$
But $x_0 \in f^{-1}(Y)$ means
$$\exists\, y_1 \in T \text{ such that } f(x_0) = y_1.$$
Since f is a function, $y_0 = y_1$ and $y_0 \in Y$, so $f(f^{-1}(Y)) \subseteq Y$.

16. **Problem 13 b:** Set $Y = \{-4, 4\}$ and $f(x) = x^2$. Then $f^{-1}(Y) = \{-2, 2\}$ since $f(2) = 4$ and $f(-2) = 4$. Then $f(f^{-1}(Y)) = f(\{-2, 2\}) = \{4\} \subset Y$. The key idea is not letting YT be a subset of the range of f.

17. **Problem 15:** If $(a_1, b_1) \subseteq (a_2, b_2)$ then $(c, d) = (a_2, b_2)$ and we are finished. Similarly if $(a_2, b_2) \subseteq (a_1, b_1)$. Let's assume that neither interval is contained in the other. If $a_1 = a_2$, then one of the intervals is contained in the other, depending on which is larger, b_1 or b_2. Therefore, we can assume $a_1 \neq a_2$. Without loss of generality, assume that $a_1 < a_2$. Similarly $b_2 > b_1$, otherwise (a_2, b_2) is contained in (a_1, b_1). Since the intersection is non-empty, there exists $x_0 \in (a_1, b_1) \cap (a_2, b_2)$. This means that
$$a_1 < a_2 < x_0 < b_1 < b_2.$$
Set $c = a_1$ and $d = b_2$.

We show $(a_1, b_1) \cup (a_2, b_2) = (c, d)$.

Part 1: We show $(a_1, b_1) \cup (a_2, b_2) \subseteq (c, d)$. Let x_1 represent an arbitrary element in $(a_1, b_1) \cup (a_2, b_2)$. WLOG, assume $x_1 \in (a_1, b_1)$. Then

$c = a_1 < x_1 < b_1 < b_2 = d$ so $x_1 \in (c, d)$. Therefore $(a_1, b_1) \cup (a_2, b_2) \subseteq (c, d)$.

Part 2: Now we let x_1 represent an arbitrary element in (c, d). Suppose $x_1 \leq x_0$. Then

$$c = a_1 < x_1 \leq x_0 < b_1,$$

so $x_1 \in (a_1, b_1) \subseteq (a_1, b_1) \cup (a_2, b_2)$. Similarly, if $x_1 \geq x_0$, then $x_1 \in (a_2, b_2)$. Therefore $(c, d) \subseteq (a_1, b_1) \cup (a_2, b_2)$.

Therefore,

$$(c, d) = (a_1, b_1) \cup (a_2, b_2).$$

5.2 Proofs about Relations

Video Lesson 5.10

This section will be easier to read if you first watch the 10:49 minute video
https://vimeo.com/499675227(Password:Proof)
which gives an introduction to relations. To change the speed at which the video plays, click on the gear at the lower right of the video.

Let S represent a set of objects. Then

$$S^2 = \{(x, y) : x, y \in S\},$$

which is called the Cartesian product of S with itself. When $S = \mathbb{R}$, then $S^2 = \mathbb{R}^2$ is just the Euclidean plane. In the following, we are going to study subsets, R of points in S^2, $R \subseteq S^2$.

Definition 5.11: Relation

A set $R \subseteq S^2$ is called a **relation** on S. If $(a, b) \in R$, then we say a is related to b.

One use of relations we are all familiar with is, with f representing some function from $D \subseteq \mathbb{R}$ into $\mathbb{R}$, the set

$$R = \{(x, f(x)) : x \in \mathbb{R}\}.$$

In this case, the relation R is called the graph of f. For example, R could just represent all the points on the graph of $f(x) = x^2$ in the plane, that is, all points of the form (x, x^2). In this case R is called a function-relation. An example of a relation R that is not a function-relation is the set of all points on the graph of the circle given by $x^2 + y^2 = 1$.

As another example, S could be a set of people and R could represent the set (a, b) where a and b are pairs of people under some rule. For example, we may group a set of students into study-groups. Then we would consider $(a, b) \in R$ if and only if a is in the same study group as b; Or we could pair males and females together for a dance. Then $(a, b) \in R$ if a and b are dance partners and a is the male and b is the female; Or $(a, b) \in R$ if a is older than b. In each case, we say that a (the first coordinate) is related to b (the second coordinate). For study groups, both (a, b) and (b, a) are in R, but for dance partners and age listing, the order in which we listed the students mattered.

Relation: 5.12: L, Less Than:

We can write the relation 'less than' on $\mathbb{R}$ as the set $L \subset \mathbb{R}^2$ defined as

$$L = \{(x, y) \in \mathbb{R}^2 : x < y\}.$$

Note that L represents the set of points in which the first value is less than the second value, and is visualized as the set of points in the shaded region of Figure 5.8.
So we would write

$$(1, 2) \in L,\ (2, 1) \notin L.$$

Using our notation, we can write $(1, 2) \in L$, but usually write $1 < 2$ instead of using L.

Let's consider a more complicated relation.

Relation: 5.13: I, Intersect:

Consider a nonempty Universe of objects $\mathcal{U}$. The power set of $\mathcal{U}$, denoted $P(\mathcal{U})$, is the set of all subsets of $\mathcal{U}$. Let $S \in P(\mathcal{U})$, meaning that $S \subseteq \mathcal{U}$. We define the relation 'intersects', I, on $P(\mathcal{U})$ as

$$(S, T) \in I \text{ iff } S \cap T \neq \emptyset$$

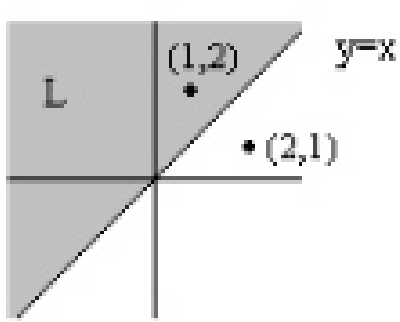

Figure 5.8: Set of points $(x, y) \in L$, that is, $x < y$.

Relation: 5.14: $\mathcal{S}$, Subset:

Consider a nonempty Universe of objects $\mathcal{U}$. We define the relation 'subset', $\mathcal{S}$, on $P(\mathcal{U})$ as

$$(S, T) \in \mathcal{S} \text{ iff } S \subseteq T.$$

We cannot 'visualize' the relations I and $\mathcal{S}$ since in both cases (S, T) is a pair of sets, not a point in the plane.

Definition 5.15: Symmetric

The relation R on the set S is **symmetric** iff if $(a, b) \in R$, then $(b, a) \in R$. Thus, if a is related to b, then b is related to a.

Definition 5.16: Asymmetric

The relation R on the set S is **asymmetric** iff if $(a, b) \in R$ then $(b, a) \notin R$. Thus, if a is related to b, then b is not related to a.

Definition 5.17: Antisymmetric

The relation R on the set S is **antisymmetric** iff if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$. Thus, if a is related to b, and b is related to a, then $a = b$.

Symmetric, asymmetric, and antisymmetric are all implications. To show a relation R is symmetric, we must show

$$\text{for all } (a,b) \in R, (b,a) \in R.$$

Thus, we begin with a 'let-variable', letting (a_0, b_0) represent an arbitrary element of the set R. Note that our let-variable is not a number but a point in $R \subseteq S^2$. We must now show that the point $(b_0, a_0) \in R$. To show a relation is not symmetric, we only need find one (a,b) such that $(a,b) \in R$ and $(b,a) \notin R$. Thus we have to show there exists $(a,b) \in R$ such that $(b,a) \notin R$.

Let's define the simple relation A on the set $S = \{1,2,3,4,5,6\}$ by

$$A = \{(1,2),(2,1),(1,4),(2,3),(5,5),(6,4)\}. \tag{5.3}$$

To show a relation is not symmetric, we must only find one pair that does not satisfy this implication. For example, $(1,4) \in A$, but $(4,1) \notin A$ so 1 is related to 4, but 4 is not related to 1. Thus, relation A is not symmetric since $(1,4) \in A$ but $(4,1) \notin A$.

Relation A is not asymmetric since $(1,2) \in A$ and $(2,1) \in A$. It is not antisymmetric since $(1,2) \in A$ and $(2,1) \in A$ but $1 \neq 2$.

Relation: 5.18: G, groups:

Suppose students in a math class are assigned to groups, with one group being Bob, Sue, and Al, and a second group being Mary, Ellen, and George. In this case, S would represent all students in the class and the relation G could be

$$G = \{(s,t) : \text{ student } s \text{ is in same group as student } t\}.$$

In this case,

$$G = \{\text{(Bob,Bob),(Bob,Sue),(Sue,Bob),(Mary,George)}, \cdots\}.$$

We can also write particular pairs as

$$\text{(Bob,Sue)} \in G$$

meaning Bob is in the same group as Sue, or

$$\text{(Bob,Mary)} \notin G$$

meaning Bob is not in the same group as Mary.

Suppose we wanted to show relation G is symmetric. To do this, we use let-variables by letting s_0 and t_0 represent two arbitrary students such that $(s_0, t_0) \in G$.

- **Assumed:** We assume $(s_0, t_0) \in G$, that is s_0 is in same group as t_0.
- **To Show:** Show $(t_0, s_0) \in G$, that is, t_0 is in same group as s_0.

Claim 5.19: Relation G is symmetric.

Proof: Let $(s_0, t_0) \in G$, that is, student s_0 is in the same group as student t_0. By the definition of groups, t_0 is in the same group as s_0 so $(t_0, s_0) \in G$. Therefore G is symmetric.□

Recall the relation 5.13 I in which $(S, T) \in I$ if and only if $S \cap T \neq \emptyset$. Like the group relation, this relation I is clearly symmetric.

Claim 5.20: Relation 5.13 I is symmetric.

Proof: Let S_0 and T_0 be two sets such that $(S_0, T_0) \in I$. This means $S_0 \cap T_0 \neq \emptyset$, so

$$\exists\, x_0 \in \mathcal{U} \text{ such that } x_0 \in S_0 \cap T_0.$$

Therefore,

$$x_0 \in T_0 \cap S_0,$$

so $T_0 \cap S_0 \neq \emptyset$ and $(T_0, S_0) \in I$. Therefore, I is symmetric.□

To show a relation is asymmetric, we must prove the implication

$$r \equiv \{(a, b) \in R \Rightarrow (b, a) \notin R\} \equiv \{\text{for all } (a, b) \in R, (b, a) \notin R\}.$$

Recall the relation 5.23 L in which $(x, y) \in L$ if and only if $x < y$. We will show this relation is asymmetric.

- **Assumed:** Let $(x_0, y_0) \in L$, that is, $x_0 < y_0$.
- **To Show:** Show $(y_0, x_0) \notin L$, that is, $y_0 \not< x_0$.

Claim 5.21: Relation 5.23 L is asymmetric.

Proof: We use let-variables by letting (x_0, y_0) be an arbitrary point in L. Then $x_0 < y_0$ so $(y_0, x_0) \notin L$ since $y_0 \nless x_0$. Thus, L is asymmetric.□

To show relation 5.18 G is not asymmetric, we only need to find one pair (s_1, s_2) such that $(s_1, s_2) \in G$ and $(s_2, s_1) \in G$. Note that (Bob,Sue)$\in G$ and (Sue,Bob)$\in G$, so G is not asymmetric.

Antisymmetric is a property that is similar to asymmetric, except that it allows (x, x) to be in the relation. To show a relation is antisymmetric, we must prove the implication

$$\{(a, b) \in R \wedge (b, a) \in R \Rightarrow a = b\}, \tag{5.4}$$

which can be rewritten as

$$\{\text{For all } a, b \in S \text{ such that } (a, b) \in R \text{ and } (b, a) \in R, a = b\}.$$

Recall the relation 5.14 $\mathcal{S}$ in which $(S, T) \in \mathcal{S}$ if and only if $S \subseteq T$. We will show this relation is antisymmetric.

- **Assumed:** We assume $(S_0, T_0) \in \mathcal{S}$ and $(T_0, S_0) \in \mathcal{S}$, that is $S_0 \subseteq T_0$ and $T_0 \subseteq S_0$.
- **To Show:** $S_0 = T_0$.

Claim 5.22: Relation 5.14 $\mathcal{S}$ is antisymmetric.

Proof: Let S_0 and T_0 be two sets such that $(S_0, T_0) \in \mathcal{S}$ and $(T_0, S_0) \in \mathcal{S}$. This means $S_0 \subseteq T_0$ and $T_0 \subseteq S_0$, so by definition of equality of sets, $S_0 = T_0$. This relation is antisymmetric.□

If a relation R is asymmetric, then it is antisymmetric, since the assumption is never true, and an implication $p \Rightarrow q$ is true if p is false. Thus, L is both asymmetric and antisymmetric.

Relation: 5.23: LE, less than or equal:

Similar to 'less than' is the relation 'less than or equal' on $\mathbb{R}$. We say that $(x, y) \in LE$ if and only if x is less than or equal to y, $x \leq y$, and

write
$$LE = \{(x, y) \in \mathbb{R}^2 : x \leq y\}.$$
Set LE is visualized as the shaded region together with the line $y = x$ of Figure 5.8. Note that as sets, $L \subset LE$.

The relation LE is antisymmetric, but not asymmetric. It is not asymmetric because if $x = y = 2$, then $(x, y) \in LE$ and $(y, x) \in LE$. To show LE is antisymmetric, we let x_0 and y_0 represent an arbitrary pair of numbers such that $(x_0, y_0) \in LE$,and $(y_0, x_0) \in LE$.

- **Assumed:** We assume $x_0 \leq y_0$ and $y_0 \leq x_0$.
- **To Show:** $x_0 = y_0$.

Claim 5.24: Relation 5.23 LE is antisymmetric.

Proof: Let x_0 and y_0 be two real numbers such that $(s_0, y_0) \in LE$ and $(y_0, x_0) = inLE$. The first assumption implies
$$x_0 - y_0 \leq 0$$
and the second implies
$$0 \leq x_0 - y_0,$$
which means that $x_0 - y_0 = 0$, or
$$x_0 = y_0.$$
This is what we needed to prove.□

Definition 5.25: Transitive

Let R be a relation on the set S. R is **transitive** iff if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ meaning that if a is related to b and b is related to c, then a is related to c.

In short, R is transitive if and only if
$$p \equiv \{\forall\, a, b, c \in S \text{ s.t. } (a, b), (bc) \in R, (a, c) \in R\}. \tag{5.5}$$

This means that the negation of a relation being transitive is, R is not transitive if and only if

$$\neg p \equiv \{\exists\, a, b, c \in S \text{ s.t. } (a,b), (bc) \in R \text{ s.t. } (a,c) \notin R\},$$

or in short

$$\neg p \equiv \{\exists\, a, b, c \in S \text{ s.t. } (a,b) \in R,\ (bc) \in R \text{ and } (a,c) \notin R\}.$$

Claim 5.26: Relation 5.18 G is transitive.

Proof: Let s_0, t_0, u_0 represent three arbitrary math students such that

$$(s_0, t_0) \in G \text{ and } (t_0, u_0) \in G.$$

This means s_0 is in the same group as t_0 and t_0 is in the same group as u_0. Therefore, by the definition of how groups are formed, s_0 and u_0 are in the same group as t_0 so are in the same group,

$$(s_0, u_0) \in G.$$

Therefore, the relation is transitive.□

The relation 5.12 L is transitive. Let x_0, y_0, z_0 represent three arbitrary numbers such that $(x_0, y_0) \in L$ and $(y_0, z_0) \in L$.

- **Assumed:** We assume $x_0 < y_0$ and $y_0 < z_0$.
- **To Show:** $x_0 < z_0$.

We know that

$$x_0 < y_0 \text{ and } y_0 < z_0, \text{ so } x_0 < z_0$$

and $(x_0, z_0) \in L$, so L is transitive.

The relation 5.13 I is not transitive as long as $\mathcal{U}$ contains at least 2 elements.

Claim 5.27: Suppose $1, 2 \in \mathcal{U}$. Then relation 5.13 I is not transitive.

Proof: Set $S_0 = \{1\}$, $T_0 = \{1, 2\}$, and $V_0 = \{2\}$. Then

$$(S_0, T_0) \in I,\ (T_0, V_0) \in I, \text{ but } (S_0, V_0) \notin I,$$

so I is not transitive.□

In Problem 9 of Section 1.1, we had three spinners and defined the relation 'better than' $(S_i, S_j) \in R$, meaning spinner S_i is better than spinner S_j, if the person with spinner S_i has a greater than 50% probability of winning against the person with spinner S_j. We noted in that example that (S_1, S_2) and (S_2, S_3). You would think that 'better than' would be transitive, but it is not since $(S_1, S_3) \notin R$ because spinner S_3 wins against S_1 more than half the time.

Our last set of properties are probably the easiest to prove or disprove.

Definition 5.28: Reflexive

Relation R on the set S is **reflexive** iff for all $a \in S$, $(a, a) \in R$, that is every a is related to itself.

Definition 5.29: Irreflexive

Relation R on the set S is **irreflexive** iff for all $a \in S$, $(a, a) \notin R$, that is a is never related to itself.

Note that if we have a relation R on the set S, then

- R is reflexive if and only if for every $x \in S$, $(x, x) \in R$.
- R is not reflexive if there exists $x \in S$ such that $(x, x) \notin R$.
- R is irreflexive if and only if for every $x \in S$, $(x, x) \notin R$.
- R is not irreflexive if and only if there exists $x \in S$ such that $(x, x) \in R$.

Thus, not reflexive and irreflexive are different. Similarly, not irreflexive is not the same as reflexive. Usually, it is nearly trivial to show that a relation is reflexive.

Relation: 5.30: Z_n, mod n:

Let's consider the following relation: letting n be a fixed positive integer, we define the relation Z_n on $\mathbb{Z}$ as

$$(i, j) \in Z_n \text{ iff } (i - j) \text{ is divisible by } n. \tag{5.6}$$

If we let $n = 24$, then $(i, j) \in Z_{24}$ iff $i - j$ is divisible by 24. Some relations we get are

$$(1, 25), (1, 49), (1, -23) \cdots \in Z_{24}.$$

This relation is just relating integers if they represent the same time on a 24-hour clock.

Let n be a fixed positive integer. To show the relation Z_n is reflexive, we let $i_0 \in \mathbb{Z}$ and note that $i_0 - i_0 = 0 \times n$. Thus, $(i_0, i_0) \in Z_n$, so Z_n is reflexive.

In addition to being reflexive, Z_n is also symmetric and transitive. To see symmetric, note that if $i_0 - j_0$ is divisible by n then so is $j_0 - i_0$ since it is just the negative of $i_0 - j_0$. If $i_0 - j_0$ is divisible by n and $j_0 - k_0$ is divisible by n, then their sum, which is $i_0 - k_0$, is also divisible by n, so Z_n is transitive.

Summary 5.1

- To show a relation is **symmetric**, you assume (a, b) satisfies the relation, and only need to show (b, a) satisfies the relation.
- To show a relation is **transitive**, you can assume a lot, that both (a, b) and (b, c) satisfy the relation, and only need to show (a, c) satisfies the relation.
- To show a relation is **reflexive**, you just show $(a, a) \in R$ for every $a \in S$.

These proofs are usually relatively easy if you clearly write down what you are assuming, using let-variables, and what you need to show.

5.2.1 Exercises

1. See the definition for divides 1.23 in Section 1.3.

   Relation: 5.31: D, divides:

   Define the relation 'divides', D, on $\mathbb{Z}$ as

   $$(i, j) \in D \text{ if and only if } i \text{ divides } j\}.$$

   For the following parts, explain your answers, making sure you understand the relation and why it does or does not satisfy different properties.

   (a) Prove or disprove that D is symmetric. Answer 1

   (b) Prove or disprove that D is asymmetric. Answer 2

   (c) Prove or disprove that D is antisymmetric. Answer 3

(d) Prove or disprove that D is transitive. Answer 4

(e) Prove or disprove that D is reflexive. Answer 5

(f) Prove or disprove that D is irreflexive. Answer 6

2. Let S be the set of students in math class.

Relation: 5.32: H, height:

Define the relation H on S by $(x, y) \in H$ if student x is within 1 inch of the height of student y.

For the following parts, explain your answers, making sure you understand the relation and why it does or does not satisfy different properties.

(a) Prove or disprove that H is symmetric.

(b) Prove or disprove that H is asymmetric.

(c) Prove or disprove that H is antisymmetric.

(d) Prove or disprove that H is transitive.

(e) Prove or disprove that H is reflexive.

(f) Prove or disprove that H is irreflexive.

3. The following relation is just relation 5.30 where $n = 2$. It has such importance in applications that we define it separately.

Relation: 5.33: P, parity:

Define the relation 'parity' on $\mathbb{Z}$ as $(i, j) \in P$ iff $i - j$ is even. If $(i, j) \in P$, then i and j are said to have the same parity.

In this case, $(6, 18) \in P$ and $(-3, 15) \in P$, but $(4, 11) \notin P$. For the following parts, explain your answers, making sure you understand the relation and why it does or does not satisfy different properties.

(a) Prove or disprove that parity is symmetric. Answer 7

(b) Prove or disprove parity is asymmetric. Answer 8

(c) Prove or disprove that parity is transitive. Answer 9

(d) Prove or disprove that parity is reflexive. Answer 9

(e) Prove or disprove that parity is irreflexive. Answer 10

4. Which of the properties, symmetric, asymmetric, antisymmetric, transitive, reflexive, and irreflexive does F satisfy, where F is defined as:

Relation: 5.34: F, **?:**

Let $S = \{(n, m) : n, m \in \mathbb{Z}^+\}$. Define the relation F on S by

$$((i, j), (k, l)) \in F \text{ if and only if } il = jk.$$

Explain your answers. F is a relation you are familiar with. Why is it important? What name might you give it?

5. Define the relation M on $\mathbb{R}$ by $(x, y) \in M$ if and only if $|x| + |y| = 6$. Which of the properties, symmetric, asymmetric, antisymmetric, transitive, reflexive, and irreflexive does M satisfy? Explain your answers. Make sure you understand the relation and why it does or does not satisfy different properties. Answer 11

6. Assume $\mathcal{U} = \{1, 2, 3, 4\}$. Let N be the relation on $P(\mathcal{U})$ such that $(S, T) \in N$ iff the number of elements in S equals the number of elements in T. Which of the properties, symmetric, asymmetric, antisymmetric, transitive, reflexive, and irreflexive does N satisfy? Explain your answers.

7. Which of the properties, symmetric, asymmetric, antisymmetric, transitive, reflexive, and irreflexive does M satisfy?

Relation: 5.35: M, **modulus:**

Define the relation M on the set of complex numbers $\mathbb{C}$ by

$$((x + yi), (a + bi)) \in M \text{ if and only if } x^2 + y^2 = a^2 + b^2.$$

Explain your answers. The number

$$|x + iy| = \sqrt{x^2 + y^2}$$

is called the **modulus** of the complex number $x + iy$. Answer 12

8. Let $x, y \in \mathbb{R}^+$.

(a) Define the relation A such that $(x, y) \in A$ if and only if there exists $n \in \mathbb{Z}_0^+$ such that $x = 2^n y$. Which of the properties, symmetric, asymmetric, antisymmetric, transitive, reflexive, and irreflexive does A satisfy? Explain your answers.

(b) Define the relation B such that $(x, y) \in B$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$. Which of the properties, symmetric, asymmetric, antisymmetric, transitive, reflexive, and irreflexive does B satisfy? Explain your answers.

9. We define the relation C on $\mathbb{R}$ by, $(x, y) \in C$ if and only if $x^2 + y^2 = 1$.

(a) Is C symmetric? Answer 13

(b) Is C transitive? Answer 14

(c) Is C reflexive? Answer 15

5.2.2 Answers to selected problems

1. **Problem 1a:** We have that $(3, 6) \in D$ since $6 = 3(2)$, but $(6, 3) \notin D$ since $3 \neq 6k$ for any $k \in \mathbb{Z}$, so D is not symmetric.

2. **Problem 1b:** Set $x_0 = 1$ and $y_0 = 1$. Then $(x_0, y_0) \in D$ and $(y_0, x_0) \in D$ so D is not asymmetric.

3. **Problem 1c:** After a little thinking, we should come up with a counterexample, such as

$$(2, -2) \in D \text{ and } (-2, 2) \in D,$$

but $2 \neq -2$, so D is not antisymmetric.

4. **Problem 1d:** The relation D is transitive. Let a_0, b_0 and c_0 represent three integers such that $(a_0, b_0) \in I$ and $(b_0, c_0) \in I$. This means that there exists integers k_1 and k_2 such that

$$b_0 = k_1 a_0 \text{ and } c_0 = k_2 b_0,$$

so

$$c_0 = (k_1 k_2) a_0 = k_3 a_0,$$

where $k_3 = k_1 k_2$, so $a_0 | c_0$ and $(a_0, c_0) \in D$.

5. **Problem 1e:** Let $i_0 \in \mathbb{Z}$ Since $i_0 = 1 \times i_0$, then i_0 divides i_0 so $(i_0, i_0) \in D$. Thus, D is reflexive.

6. **Problem 1f:** Set $i_0 = 1$. Since 1 divides 1, $(1,1) \in D$ so D is not irreflexive.

7. **Problem 3 a:** Parity is symmetric: Let i_0 and j_0 represent two integers such that $(i_0, j_0) \in P$. Then there exists an integer k_0 such that $i_0 - j_0 = 2k_0$. This means $j_0 - i_0 = -2k_0 = 2(-k_0)$ and $(j_0, i_0) \in P$.

8. **Problem 3 b:** Parity is not antisymmetric because $(2,4) \in P$ and $(4,2) \in P$.

9. **Problem 3 c:** Parity is transitive: Let i_0, j_0, and k_0 represent three integers such that $(i_0, j_0) \in P$ and $(j_0, k_0) \in P$. The usable form of this assumption is that there exist integers n_0 and m_0 such that

$$i_0 - j_0 = 2n_0 \text{ and } j_0 - k_0 = 2m_0.$$

Adding the equations gives that

$$i_0 - k_0 = 2(n_0 + m_0),$$

so $(i_0, k_0) \in P$.

item **Problem 3d:** Let $i_0 \in \mathbb{Z}$ Since $i_0 - i_0 = 0(2)$, it is even so $(i_0, i_0) \in P$. Thus, parity is reflexive.

10. **Problem 3e:** Set $i_0 = 1$. Since $1 - 1 = 0$ is even, $(1,1) \in P$ so parity is not irreflexive.

11. **Problem 5:**

Part 1: The relation M is symmetric. Let $(x_0, y_0) \in M$. Thus, $|x_0| + |y_0| = 6 = |y_0| + |x_0|$, so $(y_0, x_0) \in M$.

Part 2: The relation M is not asymmetric because $(1,5) \in M$ and $(5,1) \in M)$.

Part 3: The relation M is not antisymmetric because $(1,5) \in M$ and $(5,1) \in M)$ but $1 \neq 5$.

Part 4: Since $(-5,1) \in M$ and $(1,5) \in M$, but $(-5,5) \notin M$, it is not transitive.

Part 5: Set $x_0 = 4$. Since $|x_0| + |x_0| = 8 \neq 6$, $(x_0, x_0) \notin M$ so M is not reflexive.

Part 6: Set $x_0 = 3$. Since $|x_0| + |x_0| = 6$, $(x_0, x_0) \in M$ so M is not irreflexive. Note that not reflexive is different from irreflexive.

12. **Problem 7:**

 Part 1: Let $((x_0 + y_0 i), (a_0 + b_0 i)) \in M$. This means $x_0^2 + y_0^2 = a_0^2 + b_0^2$, so $a_0^2 + b_0^2 = x_0^2 + y_0^2$ and $((a_0 + b_0 i), (x_0 + y_0 i)) \in M$. Thus, M is symmetric.

 Part 2: The relation M is not asymmetric because $((3 + 2i), (2 + 3i)) \in M$ and $((2 + 3i), (3 + 2i)) \in M$. We note that a simpler proof that M is not asymmetric is to let $x_0 + y_0 i = a_0 + b_0 i = 3 + 2i$. Then $((x_0 + y_0 i), (a_0 + b_0 i)) \in M$ and $((a_0 + b_0 i), (x_0 + y_0 i)) \in M$.

 Part 3: The relation M is not antisymmetric because $((3 + 2i), (2 + 3i)) \in M$ and $((2 + 3i), (3 + 2i)) \in M$ but $3 + 2i \neq 2 + 3i$.

 Part 4: Let $((x_0 + y_0 i), (a_0 + b_0 i)) \in M$ and $((a_0 + b_0 i), (c_0 + d_0 i)) \in M$. Then

 $$x_0^2 + y_0^2 = a_0^2 + b_0^2 \text{ and } a_0^2 + b_0^2 = c_0^2 + d_0^2,$$

 so

 $$x_0^2 + y_0^2 c_0^2 + d_0^2$$

 and $((x_0 + y_0 i), (c_0 + d_0 i)) \in M$.

 Part 5: Let $x_0 + y_0 i$ be an arbitrary complex number. Since $x_0^2 + y_0^2 = x_0^2 + y_0^2$, $((x_0 + y_0 i), (x_0 + y_0 i)) \in M$, so M is reflexive.

 Part 6: Set $x_0 + y_0 = 1 + 2i$. Since $1^2 + 2^1 = 1^2 + 2^2$, $(1 + 2i, 1 + 2i) \in M$ so M is not irreflexive.

13. **Problem 9 a:** The relation C is symmetric because if $x_0^2 + y_0^2 = 1$, then $y_0^2 + x_0^2 = 1$.

14. **Problem 9 b:** The relation C is not transitive. Clearly $(1, 0) \in C$ and $(0, -1) \in C$, but $(1, -1) \notin C$ since $1^2 + (-1)^2 = 2 \neq 1$.

15. **Problem 9 c:** The relation C is not reflexive. Since $3^2 + 3^2 \neq 1$, $(3, 3) \notin C$.

5.3 Equivalence and Order Relations

Video Lesson 5.36

This section will be easier to read if you first watch the 7:00 minute video
https://vimeo.com/499675608(Password:Proof)
which shows how to prove a relation is an equivalence relation. Also watch the 7:10 minute video
https://vimeo.com/499703835(Password:Proof)
which shows how to prove a relation is a partial order. To change the speed at which the video plays, click on the gear at the lower right of the video.

When considering sets and relations from totally different areas of mathematics, we can note similarities through properties relations share, and differences through properties that are different. In this section, we consider two types of relations, equivalence relations and order relations.

Equivalence relations are relations that are similar to 'equality.' For example, we consider the numbers 2/3, 4/6, 6/9 and 10/15 as equal since they reduce to the same number. We note that 'equality' satisfies the properties

- reflexive ($a = a$)
- symmetric (if $a = b$, then $b = a$) and
- transitive (if $a = b$ and $b = c$, then $a = c$.

Consider the relation 5.18 G in which two students are related if they are in the same study-group. This relation is also reflexive, symmetric, and transitive, so behaves like 'equality' in some sense. Specifically, in this relation, we can consider two students equal, or at least equivalent, if they are in the same group: if the professor wants to meet with a representative of each group, any one of these group members could attend the meeting. The properties relations possess give structure to the sets they are defined on. When these three properties, reflexive, symmetric, and transitive, are satisfied, it means that in some sense, all related elements can be considered equivalent.

In relation 5.18 G, we say that all students in the same group are in some sense the same or equivalent and would call each group an **equivalence**

class. In the relation 5.30 Z_n, all integers with the same remainder, after dividing by n are related and form one equivalence class. Note that while G is about people and Z_n is about integers, there is something similar about these two relations.

Order relations come in two forms, those relations that behave like 'less than or equal' and those that behave like 'less than.' We note that the relation '$\leq$' is

- reflexive (the number a is less than or equal to itself)
- antisymmetric (if $a \leq b$, and $b \leq a$ then $a = b$) and
- transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$.

On the other hand, the relation '$<$' is

- irreflexive (the number $a \not< a$)
- asymmetric (if $a < b$, then $b \not< a$) and
- transitive (if $a < b$ and $b < c$, then $a < c$.

We actually note that if a relation is asymmetric, then it must also be irreflexive, so we could have omitted the irreflexive property for '$<$.'

Consider the subset relation5.14 $\mathcal{S}$ in which $(S, T) \in \mathcal{S}$ if and only if $S \subseteq T$. This relation is reflexive, antisymmetric, and transitive, so in some sense, relation $\mathcal{S}$ behaves like '$\leq$.' A similar relation to $\mathcal{S}$ is the relation we call $\mathcal{T}$ in which $(S, T) \in \mathcal{T}$ if and only if $S \subset T$. Remember that $S \subset T$ means S is a subset of T, but does not equal T. It is easy to prove this relation is irreflexive, asymmetric, and transitive, so behaves like $<$. Such relations as $\mathcal{S}$ and $\mathcal{T}$ are considered 'order' relations. This discussion leads to the following definitions.

Definition 5.37: Equivalence relation:

A relation R on S is an **equivalence relation** if and only if R is reflexive, symmetric, and transitive.

Definition 5.38: Partial order:

A relation R on set S is a **partial order** if and only if it is reflexive, antisymmetric and transitive.

Definition 5.39: Strict partial order:

A relation R on set S is a **strict partial order** if and only if it is asymmetric and transitive. (This means it is also irreflevice.)

If $(a, b) \in R$ where R is an equivalence relation, we consider a and b as being, if not equal, at least equivalent, and often write

$$a \sim b$$

instead of $(a, b) \in R$. In other words, in some sense, the set is partitioned into groups, with every object in each group being considered equivalent to the others in that group. This is typical for an equivalence relation.

The proof that a relation is an equivalence relation is a three part proof, that is, we must show each of the three properties holds. This means going back to the definition of each property and showing that the definition is satisfied. To show a relation is not an equivalence relation, we must find a counterexample that shows that one of the properties fails to hold in at least one case.

If $(a, b) \in R$ where R is a partial order, we consider a as being less than or equal to b, and often write

$$a \preceq b$$

instead of $(a, b) \in R$. Similarly to equivalence relation, the proof that a relation is a partial order is also a three part proof: we must show the relation is reflexive, antisymmetric and transitive. To show a relation is not a partial order, we must only find one counterexample to one of the properties; reflexive, antisymmetric, transitive.

Finally, to show a relation R is a strict partial order, we must only show it is asymmetric and transitive. In such a case, instead of writing $(a, b) \in R$, we often write

$$a \prec b.$$

Let's consider some examples.

Claim 5.40: Let n_0 represent an arbitrary positive integer. The relation 5.6 Z_{n_0} is an equivalence relation.

Proof: Recall that

$$Z_{n_0} = \{(i, j) \in \mathbb{Z}^2 : (i - j) \text{ is divisible by } n_0\}$$

We construct this proof in three parts.
Part 1, Reflexive: Let i_0 represent an arbitrary integer. We must show that $(i_0, i_0) \in Z_{n_0}$, that is, show that $(i_0 - i_0)$ is divisible by n_0. Since

$$(i_0 - i_0) = 0(n_0),$$

then $(i_0 - i_0)$ is divisible by n_0 and $(i_0, i_0) \in Z_{n_0}$, so this relation is reflexive.
Part 2, Symmetry: Let i_0 and j_0 represent arbitrary integers such that $(i_0, j_0) \in Z_{n_0}$. This means that there exists $k_0 \in \mathbb{Z}$ such that

$$i_0 - j_0 = k_0 n_0.$$

To show that $(j_0, i_0) \in Z_{n_0}$, set $k_1 = -k_0$ and note that

$$j_0 - i_0 = k_1\, n_0 = -k_0 n_0,$$

so Z_{n_0} is symmetric.
Part 3, Transitive: Let i_0, j_0 and l_0 represent arbitrary integers such that $(i_0, j_0) \in Z_{n_0}$ and $(j_0, l_0) \in Z_{n_0}$. We must show $(i_0, l_0) \in Z_{n_0}$. We are given that there exists $k_0, k_1 \in \mathbb{Z}$ such that

$$i_0 - j_0 = k_0 n_0 \text{ and } j_0 - l_0 = k_1 n_0.$$

Adding equations gives that

$$i_0 - l_0 = (k_0 + k_1) n_0 = k_2 n_0,$$

with $k_2 = k_0 + k_1 \in \mathbb{Z}$, and so

$$(i_0, l_0) \in Z_{n_0}.$$

Thus, Z_{n_0} is transitive.
From the three parts, Z_{n_0} is an equivalence relation. □

Now that we have seen how to prove a relation is an equivalence relation, let's see how to show one is not an equivalence relation

Relation: 5.41: C, not relatively prime

Consider the relation C on $S = \{2, 3, ...\} = \mathbb{Z}^+ - \{1\}$ where

$$(i, j) \in C \text{ iff } \exists k \geq 2 \text{ s.t. } k \text{ divides } i \text{ and } k \text{ divides } j.$$

In other words, i and j are related if they have a common divisor, other than 1. Pairs of integers that are not related by C are called **relatively prime**. We do not write $i \sim j$ since we do not know yet if this is an equivalence relation.

Let's do some thinking by checking the properties. Clearly, for all $i \geq 2$, $(i, i) \in C$ since i and i have the same divisor, i, which is greater than 1 so the relation is reflexive: no problem there. Suppose $(i, j) \in C$. Therefore, there exists $k \geq 2$ such that k divides i and k divides j, so k divides j and k divides i and $(j, i) \in C$. This relation is symmetric.

The problem occurs when we consider if relation C is transitive.

Claim 5.42: The relation 5.41 C is not an equivalence relation.

Proof: We only need to show that C is not transitive. To do this, we find $i, j, n \in \{2, 3, \cdots\}$ such that

$$(i, j) \in C, \ (j, n) \in C, \ (i, n) \notin C.$$

Set $i_0 = 2$, $j_0 = 6$, and $n_0 = 9$. We know that $(2, 6) \in C$ since $k_1 = 2$ divides i_0 and j_0, and that $(6, 9) \in C$ since $k_2 = 3$ divides j_0 and n_0, but $(2, 9) \notin C$ since there is no integer $k \geq 2$ which divides 2 and 9. So C is not transitive so is not an equivalence relation. □

The proof that C is not an equivalence relation consists only of showing C is not transitive, but we often need to consider all three properties before we find one that fails.

As noted previously, an equivalence relation R on a set S partitions the set S into mutually disjoint subsets or **equivalence classes**. For example, the equivalence relation 5.30 Z_5 partitions the integers into 5 equivalence classes,

$$\begin{aligned}
S_0 &= \{\ldots, -10, -5, 0, 5, 10, \ldots\} \\
S_1 &= \{\ldots, -9, -4, 1, 6, 11, \ldots\} \\
S_2 &= \{\ldots, -3, 2, 7, \ldots\} \\
S_3 &= \{\ldots, -2, 3, 8, \ldots\} \\
S_4 &= \{\ldots, -1, 4, 9, \ldots\}.
\end{aligned}$$

Any pair of integers in the same set S_i, $i = 0, 1, 2, 3, 4$ are considered equivalent. For example, $-9, 11 \in S_1$ means

$$(-9, 11) \in Z_5,$$

since $-9 - (11) = -20 = -4(5)$. Another way to say this would be that two integers are in the same equivalence class if they result in the same remainder when divided by 5. We consider 1 as the remainder when -9 is divided by 5, since $-9 = -2(5) + 1$: remainders should always be nonnegative and less than the divisor.

If $(i, j) \in Z_n$, we often write

$$i \underset{n}{=} j,$$

read as $i = j$ mod (or modulo) n. This is the sense in which the integers are equivalent. Note that Z_2 is the same relation as the parity relation 5.33 P in Problem 3 in Section 5.2, that is, all even integers are equivalent and all odd integers are equivalent.

Sometimes, we consider each equivalence class or set as a single object. For example, we consider the sets S_0, S_1, S_2, S_3, S_4 from Z_5 as individual objects. We can in some sense add these equivalence classes or sets. Suppose $i_0 \in S_2$ and $j_0 \in S_4$. Then there exists integers k_0 and k_1 such that

$$i_0 = 5k_0 + 2 \text{ and } j_0 = 5k_1 + 4.$$

Thus,

$$i_0 + j_0 = 5(k_0 + k_1 + 1) + 1,$$

so $i_0 + j_0 \in S_1$. This means if we add any element of S_2 to any element of S_4 we always get an element in S_1. Because of this, addition of the equivalence classes, $\oplus$, is well-defined as in

$$S_2 \oplus S_4 = S_1.$$

Using this notation for relation $P = Z_2$, we have a complete addition table for the two equivalence classes S_0 and S_1 in that

$$\begin{aligned} S_0 \oplus S_0 &= S_0 \\ S_0 \oplus S_1 = S_1 \oplus S_0 &= S_1 \\ S_1 \oplus S_1 &= S_0. \end{aligned}$$

This is similar for the equivalence classes of relation 5.30 Z_n for any $n \geq 2$.

In summary, an equivalence relation partitions a set $\mathcal{U}$ into nonempty disjoint subsets called equivalence classes, $A_, B, \cdots$ such that,

- every element $x \in \mathcal{U}$ is in exactly one subset and
- the intersection of any two subsets is empty, $A \cap B = \emptyset$ unless $A = B$.

As for the equivalence classes for Z_n, we can often impose additional structure on the equivalence classes such as addition or multiplication.

We now consider whether a relation is a partial order. Suppose we have a directed graph (also called a digraph) G consisting of 'points' or vertices, and 'arrows' or directed edges going from one vertex to another, such as seen in Figure 5.9.

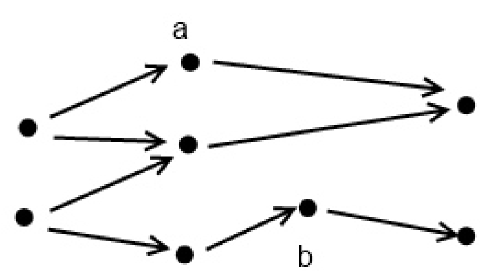

Figure 5.9: Digraph with partial order, where relation $\mathcal{P}$ is defined by vertex u is related to vertex v if there is a path from u to v. Vertices a and b are not related.

Relation: 5.43: $\mathcal{P}$, path

We define the path relation $\mathcal{P}$ on the vertices of a graph G by saying vertex u is related to vertex v, written $(u, v) \in \mathcal{P}$, if you can get from u to v by going along the edges in the directions of the arrows.

No matter what a graph G looks like, relation $\mathcal{P}$ is always reflexive since you can always get from any vertex to itself by not moving. It must also be transitive since if you can get from u to v and from v to w, then you can piece the two paths together to get from u to w. Let's see why relation $\mathcal{P}$ on the directed graph G in Figure 5.9 is antisymmetric. Note that the arrows are going left to right. Assume $(u_0, v_0) \in \mathcal{P}$ and $(v_0, u_0) \in \mathcal{P}$. This means u_0 is on or to the left of v_0 and v_0 is on or to the left of u_0. Thus u_0 is on v_0 so $u_0 = v_0$ so the relation $\mathcal{P}$ is antisymmetric and is a partial order in that vertex u being related to v means that u precedes or equals v. In some sense, we can consider u to be 'less than or equal' to v.

Partial order $\mathcal{P}$ on digraphs has some differences from the relation 'less than or equal' on the real numbers in that not all vertices may be related or comparable. For example, vertex a in Figure 5.9 is not related to vertex b nor is b related to a. Unlike real numbers, this means we cannot consider vertex a to be larger than b nor can we consider vertex b to be larger than a.

Suppose we modify relation $\mathcal{P}$ to the relation $\mathcal{Q}$ in which vertex u is related to vertex v if and only if there is a path with a least one edge from u to v. This relation will be a strict partial order on the digraph in Figure 5.9. This relation is clearly asymmetric in that if u is related to v, then u is to the left of v in the graph, so there does not exist a path from v to u, so v is not related to u. Also, $\mathcal{Q}$ is transitive since if you can get from u to v and from v to w, then you can put the two paths together to get from u to w. Often strict partial orders are just variations of partial orders, just as $\leq$ is a slight modification of $<$.

The path relation $\mathcal{P}$ on the directed graph G in Figure 5.10 is not antisymmetric since a is related to b and b is related to a, but $a \neq b$. So this relation is not a partial order on the vertices of G. Similarly, the path relation $\mathcal{Q}$ on the directed graph G in Figure 5.10 is not asymmetric since a is related to b and b is related to a. Thus, relation $\mathcal{Q}$ on the directed graph G in Figure 5.10 is not a strict partial order. Note that this graph has a cycle

$$a \to c \to b \to d \to a,$$

that is, a path with positive length and with the same starting and ending vertex.

A question we might have is, when is the relation $\mathcal{P}$ on the vertices of a directed graph G defined by the existence of a path from one vertex to the other a partial order? Clearly this relation is always reflexive and transitive. The question is whether it is antisymmetric or not.

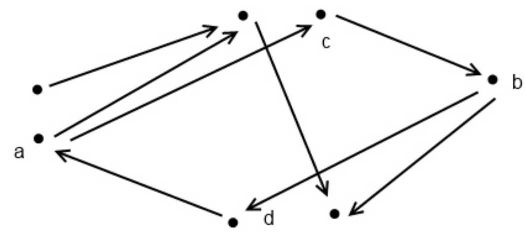

Figure 5.10: Digraph in which $\mathcal{P}$ is not a partial order and $\mathcal{Q}$ is not a strict partial order.

Claim 5.44: Suppose a directed graph G a cycle. Then relation $\mathcal{P}$ is not a partial order.

Proof: Suppose G has a cycle, that is, a path of the form

$$a \to b \to c \ldots d \to a.$$

Then vertex a is related to b and b is related to a, so $(a, b) \in \mathcal{P}$ and $(b, a) \in \mathcal{P}$ but $a \neq b$. The relation $\mathcal{P}$ is not antisymmetric and is not a partial order.

We also note that if G has a cycle, then $\mathcal{Q}$ is not a strict partial order.

Claim 5.45: Suppose relation $\mathcal{P}$ on a directed graph G is not a partial order. Then there is a cycle in G.

Proof: Since we know $\mathcal{P}$ is reflexive and transitive, it must not be antisymmetric. That means there exists vertices a_0 and b_0 such that $a_0 \neq b_0$ but $(a_0, b_0) \in \mathcal{P}$ and $(b_0, a_0) \in \mathcal{P}$. This means there is a path from a_0 to b_0,

$$a_0 \to a_1 \to \cdots \to b_0.$$

It also means there is a path from b_0 to a_0,

$$b_0 \to b_1 \to \cdots \to a_0.$$

Putting these two paths together gives us a cycle

$$a_0 \to a_1 \to \cdots \to b_0 \to b_1 \to \cdots \to a_0,$$

so there is a cycle in the graph.

Claims 5.44 and 5.45 mean that path relation $\mathcal{P}$ is a partial order if and only if there does not exist a cycle in the directed graph. Similarly, relation $\mathcal{Q}$ is a strict partial order if and only if there does not exist a cycle in the directed graph.

5.3.1 Problems

1. Let S be the set of all students that complete a given college class. Which relations are equivalence relations? Explain concisely why/why not.

   (a) Two students are related if their total scores in the class differ by less than 1%. Answer 1

   (b) Two students are related if they get the same course grade. Answer 2

   (c) Two students are related if they collaborate legally on the final course project. Answer 3

2. Consider the equivalence relation on $\mathbb{Z} \times \mathbb{Z}$ that is defined by

   $$(i, j) \sim (k, \ell) \text{ if and only if } i^2 + j^2 = k^2 + \ell^2.$$

   (You don't have to show that this is an equivalence relation.) For each of the parts, give an informal proof of your claim.

   (a) Find an equivalence class for this relation that has exactly 4 elements.

   (b) Find another equivalence class for this relation that has exactly 8 elements.

   (c) Find yet another equivalence class that has more than eight elements.

3. Assume $\mathcal{U} = \mathbb{R}$. Let Q be the relation in which two sets are related, $(S, T) \in Q$ iff $S \subseteq T$. Is Q an equivalence relation? Explain your answer. Answer 4

4. Consider the set of functions,

   $C^1 = \{f : f$ is a function with domain $\mathbb{R}$ and f' exists and is continuous on $\mathbb{R}\}$.

Define the relation D on C^1 as $(f, g) \in D$ if and only if $f' = g'$. Is D an equivalence relation. If so, how can you tell if two functions are in the same equivalence class for D?

5. Consider the relation M_n on $\mathbb{Z}_0^+$ such that $(i, j) \in M_n$ if and only if $i + j$ is divisible by n.

   (a) Is M_2 an equivalence relation? If so, what are the equivalence classes? Answer 5

   (b) Is M_3 an equivalence relation? If so, what are the equivalence classes? Answer 6

   (c) For what values $n \in \{2, 3, \cdots\}$ is M_n an equivalence relation? Give proof. Answer 7

6. Define the relation F on $\mathbb{R}^2$ by

$$((x, y), (s, t)) \in F \text{ iff } x^2 - y = s^2 - t.$$

   (Hint: Make sure you work enough examples to understand what pairs of points are related and why.) Show F is an equivalence relation or give a counterexample to one of the properties of an equivalence relation. If F is an equivalence relation, describe the equivalence classes for F.

7. Suppose we put the four integers 1, 2, 3, and 4 on the four corners of a square. The set S consists of all such possible squares, four of which are given in Figure 5.11. We denote a square by the 4 numbers on the corners, listed clockwise, starting with the upper left corner. So square a) would be denoted (1234) while square b) would be denoted (3412).

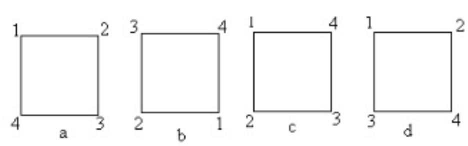

Figure 5.11: Squares with numbered corners.

   (a) How many different such squares are there? Answer 8

   (b) Define the relation R that two squares s_1 and s_2 are related if and only if square s_1 can be rotated to match square s_2. In Figure 5.11

we have that

$$\begin{aligned}((1234),(3412)) &\in R\\ ((1234),(1432)) &\notin R\\ ((1234),(1243)) &\notin R.\end{aligned}$$

Prove or disprove that R is an equivalence relation. If R is an equivalence relation, how many equivalence classes are there? List all elements of each equivalence class. Answer 9

(c) Define the relation T that two squares s_1 and s_2 are related if and only if square s_1 can be rotated or flipped (horizontally, vertically, or about either diagonal) to match square s_2. In Figure 5.11 we have that

$$\begin{aligned}((1234),(3412)) &\in R\\ ((1234),(1432)) &\in R\\ ((1234),(1243)) &\notin R.\end{aligned}$$

Prove or disprove that T is an equivalence relation. If T is an equivalence relation, how many equivalence classes are there? List all elements of each equivalence class. Answer 10

8. Let $x, y \in \mathbb{R}^+$.

   (a) Define the relation A such that $(x, y) \in A$ if and only if there exists $n \in \mathbb{Z}_0^+$ such that $x = 2^n y$. Is A an equivalence relation? Explain.

   (b) Define the relation A such that $(x, y) \in A$ if and only if there exists $n \in \mathbb{Z}_0^+$ such that $x = 2^n y$. Is A a partial order? Explain.

   (c) Define the relation B such that $(x, y) \in B$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$. Is B an equivalence relation? Explain.

   (d) Define the relation B such that $(x, y) \in B$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$. Is B a partial order? Explain.

9. Let A and B be arbitrary 2-by-2 matrices.

   (a) Define the relation M such that $(A, B) \in M$ if and only if there exists a 2-by-2 matrix X such that $AX = B$. Is M an equivalence relation? Explain. Answer 11

(b) Define the relation T such that $(A, B) \in T$ if and only if there exists an invertible 2-by-2 matrix X such that $AX = B$. Is T an equivalence relation? Explain. Answer 12

10. Consider the following relation R on $\mathbb{R}$:

$$(x, y) \in R \quad \text{iff} \quad x - y \in \mathbb{Z}.$$

Prove that this is an equivalence relation and describe the equivalence class that contains $x_0 = 1/7$.

11. Assume $\mathcal{U} = \mathbb{R}$. Let Q be the relation in which two sets are related, $(S, T) \in Q$ iff $S \subseteq T$. At the beginning of this section, we discussed why Q a partial order. Write a proof of this claim. Answer 13

12. Recall from Problem 6 that the relation F on $\mathbb{R}^2$ defined by

$$((x, y), (s, t)) \in F \text{ iff } x^2 - y = s^2 - t$$

is an equivalence relation.

(a) Let L_1 be a relation on the equivalence classes of F defined as follows: choose two equivalence classes C and D. Let $(x, y) \in C$ and let $(s, t) \in D$. Then C is related to D if and only if

$$x^2 - y \leq s^2 - t.$$

Prove or disprove that L_1 is a partial order?

(b) Let L_2 be a relation on the equivalence classes of F defined as follows: choose two equivalence classes C and D. Let $(x, y) \in C$ and let $(s, t) \in D$. Then C is related to D if and only if

$$x^2 - y < s^2 - t.$$

Prove or disprove that L_2 is a partial order?

13. Assume $\mathcal{U} = \{1, 2, 3, 4\}$. Let E be the relation on $P(\mathcal{U})$ such that $(S, T) \in E$ iff the number of elements in S is less than or equal to the number of elements in T. Is E a partial order? If not, modify the definition of E to make it a partial order. Explain. Answer 14

14. We define a relation A on $\mathbb{R}^2$ as

$$((a, b), (c, d)) \in A \text{ if } a \leq c \text{ and } b \leq d.$$

(a) Is A a partial order of S? Explain.

(b) If A is a partial order, can you find two points which are not related, thus showing a difference between a partial order and '$\leq$' on $\mathbb{R}$, in which every pair of real numbers is related one way or another.

15. We define a relation B on $\mathbb{R}^2$ as

$$((a,b),(c,d)) \in A \text{ if } a \leq c \text{ or } b \leq d.$$

(a) Is B a partial order of S? Explain. Answer 15

(b) If B is a partial order, can you find two points which are not related, thus showing a difference between a partial order and '$\leq$' on $\mathbb{R}$, in which every pair of real numbers is related one way or another. Answer 16

16. Define relation B on $\mathbb{R}^2$ as

$$((a,b),(c,d)) \in B \text{ if } a < c, \text{ or if both } a = c \text{ and } b \leq d.$$

(a) Is B a partial order? Explain.

(b) Can you modify relation B so that it is a strict partial order?

17. Let f and g represent two functions from the set of all functions with domain $\mathbb{R}$.

(a) Define the relation G such that f is related to g if and only if

$$f(x) \leq g(x)$$

for all $x \in \mathbb{R}$. Is G a partial order? Can you find two functions which are not related? Explain. Answer 17

(b) Define the relation H such that f is related to g if and only if

$$f(x) \leq g(x)$$

for some $x \in \mathbb{R}$. Is H a partial order? Can you find two functions that are not related? Explain. Answer 18

18. Let sets S and T be elements of $P(\mathbb{R})$.

(a) We define the relation R_1 by saying set S is related to set T, denoted $(S,T) \in R_1$, if and only if for every $x \in S$ and for every $y \in T$, $x \leq y$. Is R_1 a partial order?

(b) We define the relation R_2 by saying set S is related to set T, denoted $(S,T) \in R_2$, if and only if for every $x \in S$ and for every $y \in T$, $x < y$. Is R_2 a strict partial order?

19. Let sets S and T be elements of $P(\mathbb{R})$.

   (a) We define the relation R_3 by saying set S is related to set T, denoted $(S,T) \in R_3$, if and only if for every $x \in S$ there exists $y \in T$, such that $x \leq y$. Is relation R_3 a partial order? Answer 19

   (b) We define the relation R_4 by saying set S is related to set T, denoted $(S,T) \in R_4$, if and only if for every $x \in S$ there exists $y \in T$, such that $x < y$. Is relation R_4 a strict partial order? Answer 20

   (c) Let n be a fixed positive integer. We define the relation R_5 on $P(\mathbb{Z}_n) = P(\{1, 2, \cdots, n\})$ by saying set S is related to set T, denoted $(S,T) \in R_5$, if and only if for every $x \in S$ there exists $y \in T$, such that $x < y$. Is relation R_5 a strict partial order? Answer 21

20. Let sets S and T be elements of $P(\mathbb{R})$.

   (a) We define the relation R_6 by saying set S is related to set T, denoted $(S,T) \in R_6$, if and only if there exists $x \in S$ and there exists $y \in T$ such that, $x \leq y$. Is relation R_6 a partial order?

   (b) We define the relation R_7 by saying set S is related to set T, denoted $(S,T) \in R_7$, if and only if there exists $x \in S$ and there exists $y \in T$ such that, $x < y$. Is relation R_7 a strict partial order?

5.3.2 Answers to selected problems

1. **Problem 1 a:** Not an equivalence relation as it is not transitive. If student A has a score of 98.1, student B has a score of 98.9 and student C has a score of 99.5, then A is related to B and B is related to C, but A is not related to C.

2. **Problem 1 b:** Yes, it is reflexive, symmetric and transitive.

3. **Problem 1 c:** It depends on the rules for collaboration. If students are assigned groups and must collaborate with the students in their group, it is an equivalence relation. If the rules allow you to collaborate with whomever you like, then A could collaborate with B and B

with C, but A not with C, so it is not transitive.

4. **Problem 3:** Let $S = \{1\}$ and $T = \{1, 2\}$. Then $(S, T) \in Q$ and $(T, S) \notin Q$, so Q is not symmetric so is not an equivalence relation.

5. **Problem 5 a:** M_2 is an equivalence relation.

   Part 1: To show M_2 is reflexive, let i_0 represent an arbitrary integer in $\mathbb{Z}_0^+$. Since $i_0 + i_0 = 2i_0$ is divisible by 2, then $(i_0, i_0) \in M_2$ so it is reflexive.

   Part 2: To show M_2 is symmetric, let i_0 and j_0 represent arbitrary nonnegative integers such that $(i_0, j_0) \in M_2$. This means that there exists an integer k_0 such that $(i_0 + j_0) = 2k_0$. Thus, $(j_0 + i_0) = 2k_0$ so $(j_0, i_0) \in M_2$, and M_2 is symmetric.

   Part 3: To show M_2 is transitive, let i_0, j_0 and l_0 represent arbitrary nonnegative integers such that $(i_0, j_0) \in M_2$ and $(j_0, l_0) \in M_2$. This means there exists integers k_1 and k_2 such that $(i_0 + j_0) = 2k_1$ and $(j_0 + l_0) = 2k_2$. Adding gives $i_0 + 2j_0 + l_0 = 2(k_1 + k_2)$ or

   $$i_0 + l_0 = 2(k_1 + k_2 - j_0),$$

   so $(i_0, l_0) \in M_2$ and it is transitive.

6. **Problem 5 b:** $(1, 2) \in M_3$ since $1 + 2 = 3(1)$. Similarly, $(2, 4) \in M_3$ since $2 + 4 = 2(3)$. But $(1, 4) \notin M_3$ since $1 + 4$ is not divisible by 3, so M_3 is not transitive and is not an equivalence relation. Alternatively, set $2 + 2$ is not divisible by 3, so $(2, 2) \notin M_3$, so M_3 is not reflexive and is not an equivalence relation.

7. **Problem 5 c:** $n = 2$ is the only value where M_n is an equivalence relation. We have already shown M_2 is an equivalence relation. Let n_0 represent an arbitrary integer in the set $\{3, 4, \cdots\}$. $(1, 1) \notin M_n$ since $1 + 1 = 2 < n$, so $1 + 1$ is not divisible by n. Therefore, M_n is not reflexive and therefore not an equivalence relation.

8. **Problem 7 a:** There are $4! = 24$ different squares.

9. **Problem 7 b:** R is reflexive since a square rotated by 0 degrees gives the same square back, so $(s, s) \in R$ for every such square s. If s_1 is related to s_2 by a rotation of d degrees, then s_2 is related to s_1 by a rotation by $-d$ degrees, and R is symmetric. If s_1 is related to s_2 by a rotation of d degrees and s_2 is related to s_3 by a rotation of f degrees,

then s_1 is related to s_3 by a rotation of $d + f$ degrees, so R transitive. There are 6 equivalence classes. One equivalence class is

$$E_1 = \{(1234),\ (4123),\ (3412),\ \text{and}\ (4123)\}.$$

You should be able to find the other 5. Each contains four elements or squares.

10. **Problem 7 c:** The same argument works as in part b for reflexive and symmetric. Transitive is more difficult. For example,

$$((1234), (4123)) \in T$$

through a 90 degree clockwise rotation. Also

$$((4123), (4321)) \in T$$

by flipping the square about the diagonal that goes from upper left to lower right. The problem is to find one transformation that moves the square (1234) into the square (4321). The transformation is flipping about a horizontal line.

Suppose

$$(s_1, s_2) \in T \text{ and } (s_2, s_3) \in T.$$

This means there are two transformations t and u such that

$$t(s_1) = s_2 \text{ and } u(s_2) = s_3.$$

We now have to find one transformation v such that $v = u \circ t$. A simple, but time consuming check shows this is true for every transformation t and u. There are 3 equivalence classes, each containing 8 squares. One of them is

$$F_1 = \{(1234),\ (4123),\ (3412),\ (4123),\ (1432),\ (3214),\ (2143),\ (4321)\}.$$

11. **Problem 9 a:** It is reflexive, since X can be the identity matrix. M is transitive since if $AX_1 = B$ and $BX_2 = C$ then $A(X_1X_2) = C$. M is not symmetric since if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

then $AX = B$, so $(A, B) \in M$, but there does not exist a matrix Y such that $BY = A$ since $BY = B$, the 2-by-2 zero matrix, for all 2-by-2 matrices Y. M is not an equivalence relation since it isn't symmetric.

12. **Problem 9 b:** It is reflexive and transitive for the same reasons as part a). T is symmetric since if $AX = B$, then $A = BX^{-1}$. T is an equivalence relation since it is reflexive, symmetric, and transitive.

13. **Problem 11** We must show Q is reflexive, antisymmetric, and transitive.

    Reflexive: Let $S \subseteq \mathbb{R}$. Then every element in S is in S so
    $$S \subseteq S$$
    and $(S, S) \in Q$. Relation Q is reflexive.

    Antisymmetric: Assume $(S, T) \in Q$ and $(T, S) \in Q$. This means
    $$S \subseteq T \text{ and } T \subseteq S.$$
    From work with sets, we know this is the definition for $S = T$, so Q is antisymmetric.

    Transitive: To see that it is transitive, suppose $(S, T) \in Q$ and $(T, U) \in Q$, that is, $S \subseteq T$ and $T \subseteq U$. Let x_0 represent an arbitrary element in S. Since $S \subseteq T$, then $x_0 \in T$. Since $T \subseteq U$, then $x_0 \in U$. So for all $x \in S$, $x \in U$. Thus, $S \subseteq U$, so $(S, U) \in Q$ and the relation is transitive.

14. **Problem 13:** Relation E is not partial order. Let $S = 1$ and $T = 2$. Then $(S, T) \in E$ and $(T, S) \in E$ since the number of elements in each set is the same, 1. But $S \neq T$. So the relation is not antisymmetric. We could first define equivalence classes, by stating $(S, T) \in R$ if the number of elements in S and T are the same. Then we could define the relation E on the equivalence classes as $(C_S, C_T) \in E$ if the number of elements in S is less than or equal to the number of elements in T.

15. **Problem 15 a:** This relation is not a partial order. $((2, 5), (4, 3)) \in B$ since $2 < 4$ and $((4, 3), (2, 5)) \in B$ since $3 < 5$, but $(2, 5) \neq (4, 3)$, so the relation is not antisymmetric.

16. **Problem 15 b:** Since the relation is not a partial order, this part does not apply.

17. **Problem 17 a:** Clearly G is reflexive. Is it antisymmetric? Let f_0 and g_0 represent two functions such that $(f_0, g_0) \in G$ and $(g_0, f_0) \in G$. Then for every $x \in \mathbb{R}$,
    $$f_0(x) \leq g_0(x) \text{ and } g_0(x) \leq f_0(x).$$

This means $f_0(x) = g_0(x)$ for every x so $f_0 = g_0$ and G is antisymmetric. Is it transitive? Let f_0, g_0 and h_0 represent three functions such that $(f_0, g_0) \in G$ and $(g_0, h_0) \in G$. Then for every $x \in \mathbb{R}$,

$$f_0(x) \leq g_0(x) \leq h_0(x),$$

so $f_0(x) \leq h_0(x)$ for every $x \in \mathbb{R}$ and $(f_0, h_0) \in G$, so G is transitive and is a partial order. If $f(x) = x$ and $g(x) = -x$, then f is not related to g since $f(1) = 1 > g(1) = -1$ and g is not related to f since $g(-1) = 1 > f(-1) = -1$.

18. **Problem 17 b:** It is not a partial order because it is not antisymmetric. Set $f(x) = x$ and $g(x) = x^2$. Then

$$f(-1) = -1 < g(-1) = 1.$$

Thus, $(f, g) \in H$. But $f(0.5) = 0.5 > g(0.5) = 0.25$ so $(g, f) \in H$ but $f \neq g$.

19. **Problem 19 a:** It is not a partial order. Set $S = \{1, 3\}$ and $T = \{1.2.3\}$. We have $(S, T) \in R_3$ since for every $s \in S$, there exists a $t \in T$ ($t = 3$ in each case) such that $s \leq t$. Similarly, $(T, S) \in R_3$ since for every $t \in T$, there exists an $s \in S$ such that $t \leq s$. But $S \neq T$, so the relation is not antisymmetric. It is reflexive and transitive.

20. **Problem 19 b:** It is not a strict partial order. Set $S = \mathbb{E}$ and $T = \mathbb{O}$. We have $(S, T) \in R_4$ since for every $s \in S$, there exists a $t = s + 1 \in T$ such that $s < t$. Similarly, $(T, S) \in R_4$ since for every $t \in T$, there exists an $s = t + 1 \in S$ such that $t < s$. The relation is not asymmetric.

21. **Problem 19 c:** It is a strict partial order.

    Part 1: We prove R_5 is asymmetric. Let S and T be two arbitrary subsets of $\mathbb{Z}_n$ such that $(S, T) \in R_5$. Suppose S has at k elements, $k \leq n$, which we denote as $s_1, s_2, \cdots, s_k$. For every i, $1 \leq i \leq k$, there exists $t_i \in T$ such that $s_i < t_i$. Set

$$t_0 = \max\{t_1, \cdots, t_k\}.$$

    Then for every $i \in \{1, \cdots, k\}$, we have

$$s_i < t_i \leq t_0 \in T.$$

    Therefore, there does not exist an $s \in S$ such that $t_0 < s$, so $(T, S) \notin R_5$ and the relation is asymmetric.

Part 2: We prove R_5 is transitive. Suppose we have three sets, S, T, and U such that $(S,T) \in R_5$ and $(T,U) \in R_5$. Let s_0 be an arbitrary element in S. Since $(S,T) \in R_5$, then there exists a $t_0 \in T$ such that $s_0 < t_0$. Since $(T,U) \in R_5$, there exists $u_0 \in U$ such that $t_0 < u_0$. Therefore, $s_0 < u_0$ and $(S,U) \in R_5$. The relation is transitive and is therefore a strict partial order.

Chapter 6

Cardinality (Sizes of Sets)

6.1 Equivalence Classes of Sets

Video Lesson 6.1

This section will be easier to read if you first watch the 9:25 minute video
https://vimeo.com/499675985(Password:Proof)
that shows two infinite sets have the same cardinality. To change the speed at which the video plays, click on the gear at the lower right of the video.

In this Section, we define what we will show is an equivalence relation $\mathcal{C}$ on sets.

Relation: 6.2: Equal cardinality

Consider a universe of elements, $\mathcal{U}$. We define the relation $\mathcal{C}$ on $P(\mathcal{U})$, the power set of $\mathcal{U}$, by

$$(S, T) \in \mathcal{C}$$

if and only if there exists a one-to-one function f from S onto T.

If two sets are related by $\mathcal{C}$, we say they have the same cardinality, and write $C(S) = C(T)$. This study is a continuation of an investigation mentioned in

Problem 4, Section 1.1, concerning the comparison of sizes of infinite sets.

Definition 6.3: Cardinality

We say the **cardinality** of sets S and T are the same if and only if there exists a one-to-one function f from S onto T. We then write

$$C(S) = C(T).$$

When we say two sets have the same cardinality, we are really saying that, in some sense, the sets are the same size. This particularly makes sense when S is a finite set. Then S is said to have size or cardinality n if it contains precisely n elements.

Definition 6.4: Cardinalty n

The set S has **cardinality** n if and only if there exists a one-to-one function f from S onto

$$\mathbb{Z}_n = \{i \in \mathbb{Z}^+ : i \leq n\} = \{1, 2, \cdots, n\},$$

in which case we write
$$C(S) = n$$
and say that set S is **countable**.

The one-to-one correspondence with the elements in $\mathbb{Z}_n$ means we can count the elements in S: for $i \in \{1, 2, ..., n\}$, let s_i be the element in S such that $f(s_i) = i$. Then
$$S = \{s_1, \cdots, s_n\}.$$
For example, if we have the set

$$S = \{\text{apple, orange, banana}\},$$

then we can define the one-to-one function f onto $\mathbb{Z}_3$ as

$$f(x) = \begin{cases} 1 & x = \text{apple} \\ 2 & x = \text{orange} \\ 3 & x = \text{banana} \end{cases}$$

so that
$$s_1\text{=apple, } s_2 = \text{orange, and } s_3 = \text{banana}.$$

The function f is clearly not unique since there are 6 different ways we could have assigned the 3 elements of S.

To show that relation $\mathcal{C}$ is an equivalence relation, we must show it is reflexive, symmetric, and transition.

Claim 6.5: Consider a universe of elements, $\mathcal{U}$. The relation 6.2 on $P(\mathcal{U})$ is an equivalence relation.

Proof: This is a three part proof in which we show $\mathcal{C}$ is reflexive, symmetric and transitive.
Proof that $\mathcal{C}$ is reflexive: Let S_0 be a set. Then the identity function is a one-to-one function from S_0 onto itself, so $C(S_0) = C(S_0)$ and $(S_0, S_0) \in \mathcal{C}$. The relation is reflexive.
Proof that $\mathcal{C}$ is symmetric: Let S_0 and T_0 be two sets for which there is a one-to-one function f from S_0 onto T_0. Then f^{-1} is a one-to-one function from T_0 onto S_0 so $C(T_0) = C(S_0)$ and $(T_0, S_0) \in \mathcal{C}$. This relation is symmetric.
Proof that $\mathcal{C}$ is transitive: Suppose we have three sets, S_0, T_0 and U_0. Assume that f is a one-to-one function from S_0 onto T_0 so $(S_0, T_0) \in \mathcal{C}$. Also assume that g is a one-to-one function from T_0 onto U_0, so $(T_0, U_0) \in \mathcal{C}$. We will show $g \circ f$ is one-to-one from S_0 onto U_0 so $(S_0, U_0) \in \mathcal{C}$ and the relationship is transitive.
To show $g \circ f$ is one-to-one, we assume that x_1, $x_2 \in S_0$ such that $g(f(x_1)) = g(f(x_2))$. Since g is one-to-one, then $f(x_1) = f(x_2)$. Now, since f is one-to-one, $x_1 = x_2$, meaning $g \circ f$ is one-to-one. To show $g \circ f$ maps onto U_0, we let $z_0 \in U_0$. Since g maps T_0 onto U_0, there exists $y_0 \in T_0$ such that $g(y_0) = z_0$. Since f maps S_0 onto T_0, there exists an $x_0 \in S_0$ such that $f(x_0) = y_0$. Thus, $g(f(x_0)) = z_0$ and $g \circ f$ maps S_0 onto U_0. This means $C(S_0) = C(U_0)$ and $\mathcal{C}$ is transitive, and is therefore an equivalence relation.

Since $\mathcal{C}$ is an equivalence relation, then it means sets form equivalence classes. For each positive integer n, we have the equivalence class of all sets equivalent to $\mathbb{Z}_n$. Our next question is, what sets are equivalent to $\mathbb{Z}^+$?

Consider the sets

$$\mathbb{Z}^+ = \{1, 2, ...\} \text{ and } \mathbb{E}^+ = \{2, 4, 6, ...\} = \{n : \exists k \in \mathbb{Z}^+ \text{ such that } n = 2k\}.$$

These sets appear not to have the same number of elements since

$$\mathbb{E}^+ \subset \mathbb{Z}^+.$$

We might even think that $\mathbb{E}^+$ is 'half' the size of $\mathbb{Z}^+$ in some sense. On the other hand, the function

$$f(n) = n/2 \tag{6.1}$$

is one-to-one from $\mathbb{E}^+$ onto $\mathbb{Z}^+$.

Conceptual Insight: 6.6

To prove $C(\mathbb{E}^+) = C(\mathbb{Z}^+)$, we must show f is one-to-one and onto. To show it is one-to-one, we let a_0 and b_0 represent two arbitrary elements in $\mathbb{E}^+$ and assume that

$$f(a_0) = a_0/2 = b_0/2 = f(b_0).$$

Multiplying by 2 shows that $a_0 = b_0$, so the function 6.1 is one-to-one.
The function f is onto because if we let k_0 represent an arbitrary value in $\mathbb{Z}^+$ and set $a_0 = 2k_0 \in \mathbb{E}^+$, then

$$f(a_0) = f(2k_0) = k_0.$$

Thus,

$$C(\mathbb{E}^+) = C(\mathbb{Z}^+),$$

so $\mathbb{E}^+$ and $\mathbb{Z}^+$ are the same size or cardinality by the definition. This means that in some sense there are as many even positive integers as there are positive integers.

We leave writing the formula proof to the reader.

We have the positive integers to describe the sizes of finite, non-empty sets. We now define a symbol to represent the size of $\mathbb{Z}^+$ and sets which have the same cardinality as $\mathbb{Z}^+$, such as $\mathbb{E}^+$.

Definition 6.7: Countably infinite

A set S is **countably infinite** or **denumerable** and its cardinality is denoted as
$$C(S) = \aleph_0$$
(read as 'aleph naught') if and only if
$$C(S) = C(\mathbb{Z}^+).$$

As heuristic proof 6.6 shows, $C(\mathbb{E}^+) = \aleph_0$ even though $\mathbb{E}^+ \subset \mathbb{Z}^+$.

Amazingly, the set
$$\mathbb{Z}^+ \times \mathbb{Z}^+ = \{(i, j) : i, j \in \mathbb{Z}^+\}$$
is countably infinite. This seems to be counter to intuition since if we have a finite set S where $C(S) = n$, then
$$C(S \times S) = n^2 = C(S)^2.$$

Conceptual Insight: 6.8

To show that
$$C(\mathbb{Z}^+ \times \mathbb{Z}^+) = C(\mathbb{Z}^+) = \aleph_0,$$
we need to define a one-to-one function f from $\mathbb{Z}^+ \times \mathbb{Z}^+$ onto $\mathbb{Z}^+$. We will define this function f descriptively, which demonstrates the conceptual insight. In the appendix to this section, we will develop a formula for this function, which requires a technical handle. It is perfectly acceptable to define a function using a process, but for this function, it is interesting to see how the algebraic form of the function can be found and shown to be one-to-one and onto. Figure 6.1 describes the process of assigning a unique positive integer to each point in $\mathbb{Z}^+ \times \mathbb{Z}^+$.
From Figure 6.1, we see that
$$f((1,1)) = 1,\ f((2,1)) = 2,\ f((1,2)) = 3,\ f((3,1)) = 4,\ ...,\ f((1,4)) = 10,\ \cdots.$$
It is clear from this figure that f is one-to-one and onto. It is one-to-one since each point (i, j) is assigned a different integer k. It is onto since each integer k will be assigned to some point (i, j). See Appendix 6.1.3 to show this algebraically.

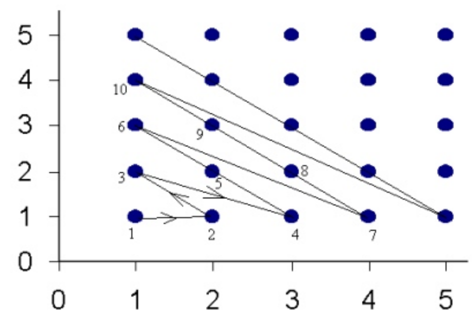

Figure 6.1: Visualization of one-to-one function from $\mathbb{Z}^+ \times \mathbb{Z}^+$ onto $\mathbb{Z}^+$.

We now discuss an even more amazing result, that $C(\mathbb{Z}^+) = C(Q^+)$, that is, the number of positive rational numbers is the same as the number of integers. Instead of finding a one-to-one function from $\mathbb{Q}^+$ onto $\mathbb{Z}^+$, we find a one-to-one function from $\mathbb{Z}^+$ onto Q^+, which is equivalent. The key idea is seen in Figure 6.1. Thinking of the x-coordinate as the numerator and the y-coordinate as the denominator, the points in this figure correspond to rational numbers. The problem is that this function is not one-to-one as $f(1) = 1/1 = f(5) = 2/2$. To correct this, we just skip numbers which are already mapped onto, as seen in Figure 6.2. Following this diagram, we see that

$$f(1) = 1/1 = 1,\ f(2) = 2/1 = 2,\ f(3) = 1/2,\ f(4) = 3/1 = 3, \text{ and } f(5) = 1/3.$$

Note the point $(2,2)$ is skipped since the number $2/2 = 1/1$ which was already mapped onto.

While we do not have an algebraic formula for our function, it is well defined in that, for any integer n, we can find the rational number it maps onto, and for any rational number, we can go through the process to determine the integer which maps onto it. It is clearly onto since every positive rational number r can be written as a fraction reduced to lowest terms $r = n/m$ and there is an integer that maps onto the point (n,m), which can be found by going through this process.

At this point, we might ask if there are infinities that do not have the same size cardinality as $\mathbb{Z}^+$. This will be discussed in the next section.

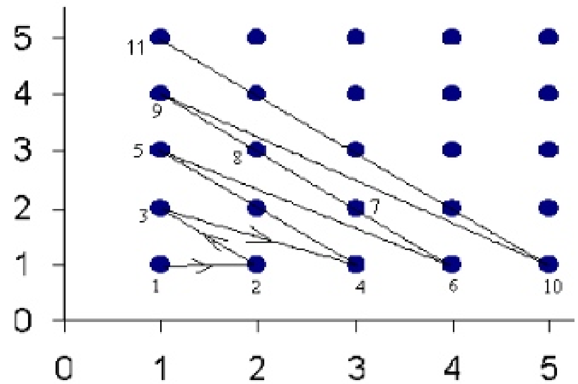

Figure 6.2: Graphic description of function f from $\mathbb{Z}^+$ onto $\mathbb{Q}^+$.

6.1.1 Exercises

1. Let S be the set of points in $\mathbb{R}^2$ defined by

$$S = \{(2t, t-1) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Define the function $f : S \to \mathbb{R}$ by

$$f((2t, t-1)) = t + 2.$$

We are going to investigate if this function shows $C(S) = C(\mathbb{R})$.

(a) Prove or disprove f is one-to-one. Answer 1

(b) Prove or disprove f is onto. Answer 2

(c) Does this prove the cardinalities are the same, different, or is it inconclusive? Answer 3

2. Let S be the set of points in $\mathbb{R}^2$ defined by

$$S = \{(t^2, 2t+1) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Define the function $f : S \to \mathbb{R}$ by

$$f((t^2, 2t+1)) = t.$$

We are going to investigate if this function shows $C(S) = C(\mathbb{R})$.

(a) Prove or disprove f is one-to-one.

(b) Prove or disprove f is onto.

(c) Does this prove the cardinalities are the same, different, or is it inconclusive?

3. Let S be the set of points in $\mathbb{R}^2$ defined by

$$S = \{(t+1, 3t-1) \in \mathbb{R}^2 : t \in \mathbb{R}\}.$$

Define the function $f : S \to \mathbb{R}_0^+$ by

$$f((t+1, 3t-1)) = (t+2)^2.$$

We are going to investigate if this function shows $C(S) = C(\mathbb{R}_0^+)$.

(a) Prove or disprove f is one-to-one. Answer 4

(b) Prove or disprove f is onto. Answer 5

(c) Does this prove the cardinalities are the same, different, or is it inconclusive? Answer 6

4. Let S be the set of points in $\mathbb{R}^2$ defined by

$$S = \{(1/t, t+1) \in \mathbb{R}^2 : t \in \mathbb{R} - \{0\}\}.$$

Define the function $f : S \to \mathbb{R}^+$ by

$$f((1/t, t+1)) = (t+2)/t.$$

We are going to investigate if this function shows $C(S) = C(\mathbb{R}^+)$.

(a) Prove or disprove f is one-to-one.

(b) Prove or disprove f is onto.

(c) Does this prove the cardinalities are the same, different, or is it inconclusive?

5. Show that $\mathbb{Z}_0^+$ and $\mathbb{Z}^+$ have the same cardinality by finding a one-to-one function from $\mathbb{Z}_0^+$ onto $\mathbb{Z}^+$. Show that your function is one-to-one and onto. Answer 7

6. Show that the sets $\mathbb{Z}$ and $\mathbb{Z}^+$ have the same cardinality by finding a one-to-one function from $\mathbb{Z}$ onto $\mathbb{Z}^+$. Show that your function is one-to-one and onto. (Hint: Try a conditionally defined function.)

7. (Galileo's paradox) Let P be the set of perfect squares, that is,

$$P = \{1, 4, 9, 16, 25 \cdots\}.$$

Show that $C(P) = C(\mathbb{Z}^+)$. Answer 8

8. Find a one-to-one function from $\mathbb{Z} \times \mathbb{Z}$ onto $\mathbb{Z}^+ \times \mathbb{Z}^+$. Combining this result with heuristic proof 6.8,

$$C(\mathbb{Z} \times \mathbb{Z}) = \aleph_0.$$

9. Show that

$$C(\mathbb{R}_0^+) = C(\mathbb{R}^+)$$

by showing that the function

$$f(x) = \begin{cases} x & x \notin \mathbb{Z}_0^+ \\ x + 1 & x \in \mathbb{Z}_0^+ \end{cases}$$

is one-to-one from $\mathbb{R}_0^+$ onto $\mathbb{R}^+$. It might help to graph the function. Answer 9

10. Show that

$$C((0, 1]) = C((0, 1))$$

by showing that the function

$$f(x) = \begin{cases} x & x \neq 1/n \text{ for any } n \in \mathbb{Z}^+ \\ \frac{1}{n+1} & x = \frac{1}{n} \text{ for some } n \in \mathbb{Z}^+ \end{cases}$$

is one-to-one from $(0, 1]$ onto $(0, 1)$. It might help to graph the function.

6.1.2 Answers to selected problems

1. **Problem 1 a:** This function is one-to-one. Assume there exists t_1 and t_2 such that

$$f((2t_1, t_1 - 1)) = t_1 + 2 = t_2 + 2 = f((2t_2, t_2 - 1)).$$

This means $t_1 = t_2$ so

$$(2t_1, t_1 - 1) = (2t_2, t_2 - 1).$$

2. **Problem 1 b:** This function is onto. Let a_0 represent an arbitrary value in $\mathbb{R}$. Set $t_0 = a_0 - 2$. Then
$$f((2t_0, t_0 - 1)) = t_0 + 2 = (a_0 - 2) + 2 = a_0.$$

3. **Problem 1 c:** Yes, this shows $C(S) = C(\mathbb{R})$.

4. **Problem 3 a:** Setting $t_0 = 0$, we get
$$f((t_0 + 1, 3t_0 - 1)) = f((1, -1)) = (0 + 2)^2 = 4.$$
Setting $t_1 = -4$ gives
$$f((t_1 + 1, 3t_1 - 1)) = f((-3, -13)) = (-4 + 2)^2 = 4.$$
Thus, $t_0 \neq t_1$ but $f(t_0) = f(t_1)$. This function is not one-to-one.

5. **Problem 3 b:** This function is onto. Let a_0 represent an arbitrary value in $\mathbb{R}_0^+$. Set $t_0 = \sqrt{a_0} - 2$. Then
$$f((t_0 + 1, 3t_0 - 1)) = (t_0 + 2)^2 = ((\sqrt{a_0} - 2) + 2)^2 = (\sqrt{a_0})^2 = a_0.$$

6. **Problem 3 c:** While this function is not one-to-one from S onto $\mathbb{R}_0^+$, there still may be some other function g which is one-to-one from S onto $\mathbb{R}_0^+$. Thus, we have an inconclusive result.

7. **Problem 5:** Define $f(n) = n + 1$ for all $n \in \mathbb{Z}_0^+$. To show f is one-to-one, let i_0 and j_0 represent two values in $\mathbb{Z}_0^+$ such that $f(i_0) = f(j_0)$. Then $f(i_0) = i_0 + 1 = f(j_0) = j_0 + 1$. Subtracting 1 from both sides gives $i_0 = j_0$. So f is one-to-one. To show f is onto, let n_0 represent an arbitrary value in $\mathbb{Z}^+$. Set $m_0 = n_0 - 1$. Since $n_0 \geq 1$, $m_0 = n_0 - 1 \geq 0$, so $m_0 \in \mathbb{Z}_0^+$. Since $f(m_0) = m_0 + 1 = n_0 - 1 + 1 = n_0$, then f is onto.

8. **Problem 7:** Let j_0 represent an arbitrary element in P. Then there exists $i_0 \in \mathbb{Z}^+$ such that $j_0 = i_0^2$. Define $f(j_0) = i_0$. This function is well defined, because there is only one positive integer i_0 such that $j_0 = i_0^2$. This function maps P onto $\mathbb{Z}^+$. To show this, let i_0 represent an arbitrary value in $\mathbb{Z}^+$. Then $i_0^2 \in P$ and $f(i_0^2) = i_0$. To show f is one-to-one, assume there exist $j_0, k_0 \in P$ such that $f(j_0) = f(k_0)$. Then there exists $i_0, n_0 \in \mathbb{Z}^+$ such that $j_0 = i_0^2$ and $k_0 = n_0^2$. Since $f(j_0) = i_0$ and $f(k_0) = n_0$ then $i_0 = n_0$ so $j_0 = i_0^2 = n_0^2 = k_0$ and f is one-to-one.

9. **Problem 9:** You need to show that f is one-to-one and onto. One way to show f is one-to-one is to let x_1 and x_2 represent two arbitrary values in $\mathbb{R}_0^+$ such that $f(x_1) = f(x_2)$. There are 3 cases, neither is an integer, both are integers, one is an integer. Suppose neither is an integer. Then $f(x_1) = x_1 = f(x_2) = x_2$. A similar argument holds when they are both integers. Suppose one is an integer and one is not an integer, say $x_1 \in \mathbb{Z}_0^+$ and $x_2 \notin \mathbb{Z}_0^+$. Then $f(x_1) = x_1 + 1$ is also an integer but $f(x_2) = x_2$ is not an integer so $f(x_1) \neq f(x_2)$. This contradicts the assumption that $f(x_1) = f(x_2)$, so this case cannot happen.

   To show f is onto, let y_0 represent an arbitrary value in $\mathbb{R}^+$. If $y_0 \notin \mathbb{Z}^+$, then set $x_0 = y_0$ and $f(x_0) = y_0$. If $y_0 \in \mathbb{Z}^+$, set

$$x_0 = y_0 - 1 \in \mathbb{Z}_0^+ \subset \mathbb{R}_0^+$$

   and $f(x_0) = x_0 + 1 = y_0 - 1 + 1 = y_0$. So for every $y \in \mathbb{R}^+$, there exists $x \in \mathbb{R}_0^+$ such that $f(x) = y$ and f is onto. Therefore

$$C(\mathbb{R}_0^+) = C(\mathbb{R}^+).$$

6.1.3 Appendix: Posiive integers in the plane

In this subsection, we construct a one-to-one algebraic function that maps $\mathbb{Z}^+ \times \mathbb{Z}^+$ onto $\mathbb{Z}^+$. In Figure 6.1, note that (i, j) is on the $n - 1$-diagonal, where

$$i + j = n.$$

For example, the points $(3, 1)$, $(2, 2)$ and $(1, 3)$ are the three points on the third diagonal, since

$$3 + 1 - 1 = 2 + 2 - 1 = 1 + 3 - 1 = 3.$$

Similarly,the first diagonal is $(1, 1)$ and the second diagonal is $(2, 1)$ to $(1, 2)$. In addition,

- the number of points on the kth diagonal is k,
- the first point on the kth diagonal is $(k, 1)$,
- the jth point on the kth diagonal is $(k - j + 1, j)$, $j = 1, 2, \cdots, k$.

Suppose we want to determine

$$f((i, j))$$

where $i + j = n$, so (i, j) is on the $n - 1$th diagonal. We have then already assigned numbers for the first $n - 2$ diagonals, that is, we have assigned 1 integer to the first diagonal, 2 integers to the second, and so forth. Thus we have assigned

$$\sum_{k=1}^{n-2} k$$

integers. We have previously learned that the sum of the first m integers is

$$1 + 2 + \cdots + m = \sum_{i=1}^{m} i = \frac{m(m+1)}{2}.$$

Applying that to our sum gives

$$\sum_{k=1}^{n-2} k = \frac{(n-1)(n-2)}{2}.$$

Since the point (i, j) is the jth point on the $n - 1$ diagonal, then it will be assigned the value

$$f((i,j)) = \frac{(n-1)(n-2)}{2} + j = \frac{(i+j-1)(i+j-2)}{2} + j \tag{6.2}$$

since $i + j = n$. For example, to find $f((2, 3))$ we note that $2 + 3 = 5 = n$ and $j = 3$. Applying the formula gives

$$f((2,3) = \frac{(5-1)(5-2)}{2} + 3 = 9,$$

which agrees with Figure 6.1. You should check that this function gives the correct values for the points in the fifth diagonal of Figure 6.1.

We outline how to show function 6.2 is one-to-one from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$. To do that, assume

$$f((i_0, j_0)) = f((i_1, j_1))$$

and show $i_0 = i_1$ and $j_0 = j_1$. Begin by assuming $i_0 + j_0 = n_0$ and $i_1 + j_1 = n_1$, and consider the two cases, $n_0 = n_1$ and $n_0 > n_1$. For the second case, show $j_0 \geq n_0$ which is a contradiction. This requires some algebraic manipulation and thinking.

We now outline how to show function 6.2 from $\mathbb{Z}^+ \times \mathbb{Z}^+$ to $\mathbb{Z}^+$ is onto. Begin by letting k_0 represent an arbitrary integer in $\mathbb{Z}^+$. We must now find (i, j) such that $f((i, j)) = k_0$. To do this, consider the set

$$S = \{n \in \mathbb{Z}^+ : \frac{(n-1)(n-2)}{2} < k_0\}.$$

Let n_0 be the largest integer in set S. Use n_0 to define $(i_0, j_0) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ such that $f((i_0, j_0)) = k_0$. You are not finished until you show that $1 \leq j_0 \leq n_0 - 1$. By the way j_0 is defined, it is clear that $j_0 \geq 1$. Assume that $j_0 > n_0 - 1$ and get a contradiction to the fact that n_0 is the largest integer in S.

6.2 Order Relations and Infinities

Video Lesson 6.9

This section will be easier to read if you first watch the 6:23 minute video
https://vimeo.com/499676410(Password:Proof)
which gives insight into the difference between the sizes of set of integers and set of real numbers. To change the speed at which the video plays, click on the gear at the lower right of the video.

We now have that sets satisfy an equivalence relation, so we can consider equivalence classes of sets. For example, all sets that are equivalent to $\mathbb{Z}_n$ can be considered the same, in some sense. We can also consider an ordering of the equivalence classes, such as

$$S \prec T$$

if $S \in \mathbb{Z}_{n_1}$ and $T \in \mathbb{Z}_{n_2}$ where $n_1 < n_2$. But what about sets of infinite size? Since $C(\mathbb{Z}^+) = C(\mathbb{Z}^+ \times \mathbb{Z}^+) = C(\mathbb{Q}^+)$, does this mean all infinite sets are the same 'size'? To begin answering this question, we are going to discuss what it would mean to have infinite sets of different 'sizes.'

Definition 6.10: $\preceq$ and $\prec$

We say that if there is a one-to-one function f from S into T, then the cardinality of S is **less than or equal** to T, and write

$$C(S) \preceq C(T).$$

If there is a one-to-one function f from S into T but there does not exist a one-to-one function from S onto T, then the cardinality of S is **less than** the cardinality of T and we write

$$C(S) \prec C(T).$$

Intuitively, '$\prec$' implies that set S is the same size as a subset of T, the subset mapped onto by the function f, but there are too many elements in T to match all of them with elements of S. This definition leads to many questions. We already know the sets $\mathbb{Z}^+$, $\mathbb{E}^+$, and $\mathbb{Z}^+ \times \mathbb{Z}^+$ are in the same equivalence class and are therefore countable but infinite. Are there actually sets T that are larger than $\mathbb{Z}^+$? If so, are there sets even larger than T? How would we even show a set was larger?

To answer one of these questions, we show that the set

$$I = (0,1) = \{x : 0 < x < 1\}$$

is uncountable, a result first shown by Georg Cantor in 1874. To show I is uncountable we first have to show that there exists a one-to-one function from $\mathbb{Z}^+$ into I. This is easy. One function is

$$f(n) = \frac{1}{n+1},$$

which is a one-to-one function that maps onto the set

$$\{1/2, 1/3, 1/4, \cdots\} \subset I.$$

How do we show there does not exist a one-to-one function from $\mathbb{Z}^+$ onto I? The 'obvious' method is to assume that such a function exists and arrive at a contradiction.

Cantor published three different proofs that $C(Z^+) \prec C(I)$ over about 15 years. The conceptual insight behind the proof presented below is based on his classic third proof, published in 1891, known as the Cantor **diagonalization** proof.

Claim 6.11: $C(Z^+) \prec C(I)$**:**

Proof: We prove this claim using contradiction. Assume a one-to-one function f exists from $\mathbb{Z}^+$ onto I. This means that for all $i \in \mathbb{Z}^+$, $f(i) = s_i \in (0,1)$, where

$$s_i = 0.a_{i1}a_{i2}a_{i3}a_{i4}a_{i5}\cdots$$

in decimal form. It also means that the list of numbers s_1, s_2, $\cdots$ includes every number between 0 and 1. This is summarized in Table 6.1.

We will now arrive at a contradiction by finding a number y between

0 and 1 that is not in this list of numbers, $L = \{s_i : i \in \mathbb{Z}^+\}$. To do this, for every $i \in \mathbb{Z}^+$, we make the ith digit of y's decimal expansion different from the ith digit in the decimal expansion of s_i (the bold digit), as seen in equation 6.3.

$$y = 0.\neg a_{11}\neg a_{22}\neg a_{33}\neg a_{44}\neg a_{55}\cdots = 0.y_1y_2y_3y_4y_5\cdots. \tag{6.3}$$

Clearly, y is not in the list of numbers mapped onto by f. To see this, we let s_i represent an arbitrary element in the set L, that is an arbitrary number mapped onto by f, say

$$s_i = 0.a_{i1}a_{i2}\cdots a_{i\,i-1}\boldsymbol{a_{ii}}a_{i\,i+1}\cdots.$$

Note that $s_i \neq y$ since the ith digit in s_i is a_{ii} while the ith digit in y is $\neg a_{ii} = y_i$. Since s_i was an arbitrary number in the set L, then no number in the set L equals y. Since L is the range of f, then f did not map onto I, and the equivalence class containing $\mathbb{Z}^+$ is smaller than the equivalence class containing the interval I, that is,

$$C(\mathbb{Z}^+) \prec C(I).$$

$f(1) =$	$s_1 =$	$0.\boldsymbol{a_{11}}a_{12}a_{13}a_{14}a_{15}\cdots$
$f(2) =$	$s_2 =$	$0.a_{21}\boldsymbol{a_{22}}a_{23}a_{24}a_{25}\cdots$
$f(3) =$	$s_3 =$	$0.a_{31}a_{32}\boldsymbol{a_{33}}a_{34}a_{35}\cdots$
$f(4) =$	$s_4 =$	$0.a_{41}a_{42}a_{43}\boldsymbol{a_{44}}a_{45}\cdots$
$\vdots$	$\vdots$	$\vdots$

Table 6.1: Finding a number that is not in the range of f.

While the idea seems simple, this result generated quite a bit of controversy among mathematicians at the time.

There is a technical detail that must be consider to turn this ad hoc argument into a proof. We know that the decimal expansion for a number is not unique since $0.99\cdots = 1.00\cdots$. Similarly, any terminating decimal can be written two ways. For example,

$$0.1399\cdots = 0.1400\cdots = 0.14,$$

so it is possible we changed the decimal expansion to an equivalent form of the decimal expansion of one of the numbers in the list. This nonuniqueness is caused by repeating 9's and 0's, so a simple solution is if $a_{ii} \neq 2$, then make $y_i = 2$ and if $a_{ii} = 2$, then make $y_i = 8$. You can find a proof in *Cantor's Other Proofs that* $\mathbb{R}$ *is Uncountable* by John Franks, published in October 2010 issue of Mathematics Magazine. This nice article includes two other clever proofs that Cantor gave of this result. The idea is to prove that a certain subset of I is not countable from which it easily follows that I is not countable.

Definition 6.12: Continuum

The cardinality of $I = (0, 1)$ is the **continuum** and is denoted as

$$C(I) = c.$$

It is not difficult to show

$$C(I) = C(\mathbb{R}).$$

All we need is a one-to-one function from I onto $\mathbb{R}$. Any function that looks like the function in Figure 6.3 shows that these two sets have the same cardinality. So the cardinality of the real numbers is the continuum and is uncountable while the cardinality of the integers is $\aleph_0$ and is countable.

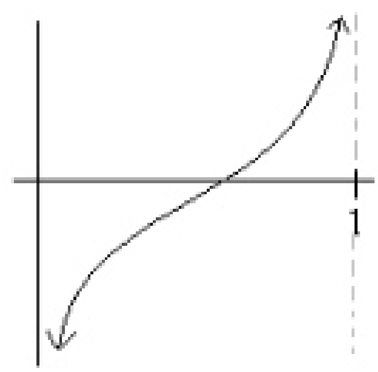

Figure 6.3: Graph of 1-1 function f mapping $(0, 1)$ onto $\mathbb{R}$. Note vertical asymptotes at 0 and 1.

We now have two equivalence classes of sets that are of infinite sizes, the class that contains $\mathbb{Z}^+$, which we designate as size $C(\mathbb{Z}^+) = \aleph_0$, and the class that contains the larger set $\mathbb{R}$, which we designate as size $C(\mathbb{R}) = c$. There are several issues remaining. Is there a set whose cardinality is larger than $\mathbb{R}$? How many different sizes of infinity are there? Is $\mathbb{R}$ the second largest infinity? If not, what is? These, and several other questions will be discussed later in this section.

We now group sets according to the equivalence class they are in, remembering that two sets are in the same equivalence class if and only if there is a one-to-one function from one onto the other. It is these equivalence classes that we will try to order according to size. Let C_S represent the equivalence class of all sets of the same cardinality as S. This means $C_S = C_T$ if and only if there is a one-to-one function f from S onto T.

Recall the relation '$\preceq$', defined by

$$C_S \preceq C_T$$

if and only if there exist a one-to-one function f from S into T. We note that this relation is well-defined. Suppose f is a one-to-one map from S_0 into T_0. Also, suppose S_0 and S_1 are both representatives of C_S, and T_0 and T_1 are both representatives of C_T. Then there is a function g which is one-to-one from S_0 onto S_1 and h which is one-to-one from T_0 onto T_1. Then hfg^{-1} is a one-to-one function from S_1 into T_1. Therefore, if there is a one-to-one function from S into T, then no matter which representatives we select from C_S and C_T, there is a one-to-one function from the first into the second.

We now ask if this relation is a partial order, that is, is it reflexive, antisymmetric, and transitive? Clearly the relation $\preceq$ is reflexive as the identity function is a one-to-one function from S into S. This relation is also clearly transitive since if f is one-to-one from S into T and g is one-to-one from T into U, then $g \circ f$ is a one-to-one function from S into U.

Summary 6.1

To determine if the relation $\preceq$ is a partial order on equivalence classes of sets, we have to show that it is antisymmetric meaning we have to show that if there is a one-to-one function f from S into T and a one-to-one function g from T into S, that S and T are from the same equivalence class, that is, there exists a one-to-one function h from S onto T, or in short, we have to show that

$$\text{if } C_S \preceq C_T \text{ and } C_T \preceq C_S \text{ then } C_S = C_T.$$

We note that if we can prove this statement, then we can also prove the relation $\prec$ is a strict partial order, that is, it is asymmetric and transitive.

There are times for which there is clearly a one-to-one function f from S into T and a one-to-one function g from T into S, but it is not obvious that there is a one-to-one function from S onto T. The proof of the famous Cantor-Schroder-Bernstein Theorem (CSB Theorem) not only shows that if f is a one-to-one map from S into T and g is a one-to-one map from T into S, then not only does there exist a one-to-one function h from S onto T, it shows how to construct such a function. This proves that $C_S = C_T$ and so $\preceq$ is a partial order and $\prec$ is a strict partial order. This is amazing since little is known about S, T, f, or g. We will proceed with a proof of the CSB Theorem for a specific example, which outlines the approach of the proof for the general case.

Let

$$S = (0,1) = \{x : 0 < x < 1\} \text{ and } T = (0,1] = \{x : 0 < x \leq 1\}.$$

Define the one-to-one function f from S into T as

$$f(x) = x$$

and the one-to-one function g from T into S as

$$g(x) = x/2.$$

This means

$$C_S \preceq C_T \text{ and } C_T \preceq C_S$$

If there is no one-to-one function h from S onto T, then this would mean

$$C_S \preceq C_T, \;\; C_T \preceq C_S \text{ and } C_S \neq C_T$$

and the relation $\preceq$ would not be antisymmetric and so not be a partial order. This would mean, in some sense, that $(0,1)$ is smaller than $(0,1]$ and $(0,1]$ is smaller than $(0,1)$. This does not make sense. Fortunately, there is a one-to-one function from $(0,1)$ onto $(0,1]$, so

$$C_{(0,1)} = C_{(0,1]}.$$

Let's see how we could construct such a function from the functions f and g^{-1}.

With $f(x) = x$ mapping $S = (0,1)$ into $T = (0,1]$ and $g(y) = y/2$ mapping $T = (0,1]$ into $S = (0,1)$, we have

$$\begin{aligned} f(S) &= (0,1) \subset T = (0,1] \\ g(T) &= (0,1/2] \subset S = (0,1). \end{aligned}$$

This also means that

$$g^{-1}(x) = 2x$$

is a one-to-one function from $(0, 1/2]$ onto $T = (0, 1]$. See Figure 6.4 for a visualization of f, g and g^{-1}.

Figure 6.4: Visualization of $f : S \to T$ and $g : T \to S$.

We now use f and g^{-1} to construct a function h which is 1-1 from S onto T. Note that f maps onto T except it does not map onto $y = 1$, which we designate as 'hole 1' in Figure 6.5 a.

We proceed by constructing a sequence of functions $h_n(x)$ which equal $f(x)$ except at a 'few' points, at which it equals $g^{-1}(x)$. Let

$$h_1(x) = \begin{cases} x & x \notin \{\frac{1}{2}\} \\ g^{-1}(x) = 2x & x \in \{\frac{1}{2}\} \end{cases},$$

a graphic representation of which is seen in Figure 6.5 b. Note that $h_1(1/2) = g^{-1}(1/2) = 1$, so the function h_1 corrects one problem, mapping $x = 1/2$ onto the hole at $y = 1 \in T$, but creates another problem, as now no x maps onto hole 2, $y = 1/2 \in T$. For easy reference, we let $S_1 = \{1/2\}$. Note that

$$S_1 = g(T - f(S)).$$

In particular, $T - f(S) = \{1\}$ and $g(\{1\}) = \{1/2\}$. See Figure 6.5 e.

We now define

$$h_2(x) = \begin{cases} f(x) = x & x \notin \{\frac{1}{2}, \frac{1}{4}\} \\ g^{-1}(x) = 2x & x \in \{\frac{1}{2}, \frac{1}{4}\} \end{cases}.$$

A graphic representation of h_2 can be seen in Figure 6.5 c. This means

$$h_2(1/2) = g^{-1}(1/2) = 1 \text{ and } h_2(1/4) = g^{-1}(1/4) = 1/2,$$

so h_2 corrects the problem that h_1 had in not mapping onto $1/2$ but creates another problem, not mapping any x onto hole 3, $y = 1/4 \in T$. Let $S_2 = \{1/4\}$. Note that

$$S_2 = g(f(S_1)),$$

as seen in Figure 6.5 e. In particular, $S_1 = \{1/2\}$, therefore $f(S_1) = f(\{1/2\}) = \{1/2\}$ and $g(\{1/2\}) = \{1/4\}$.

Define

$$h_3(x) = \begin{cases} f(x) = x & x \notin \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\} \\ g^{-1}(x) = 2x & x \in \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\} \end{cases},$$

which can been seen in Figure 6.5 d. This means $h_3(1/2) = g^{-1}(1/2) = 1$, $h_3(1/4) = g^{-1}(1/4) = 1/2$ and $h_3(1/8) = g^{-1}(1/8) = 1/4$. Therefore, h_3 corrects the problem that h_2 had in not mapping onto $1/4$ but creates another problem, not mapping any x onto hole 4, $y = 1/8 \in T$. Since $S_2 = \{1/4\}$, therefore $f(S_2) = f(\{1/4\}) = \{1/4\}$ and $g(\{1/4\}) = \{1/8\}$. Hence, we let

$$S_3 = g(f(S_2)) = \{1/8\},$$

again seen in Figure 6.5 e.

We continue correcting one point at a time giving the functions

$$h_n(x) = \begin{cases} f(x) = x & x \in S - S_1 \cup S_2 \cup \cdots \cup S_n \\ g^{-1}(x) = 2x & x \in S_1 \cup S_2 \cup \cdots \cup S_n, \end{cases}$$

where

$$\bigcup_{i=1}^{n} S_i = \left\{\frac{1}{2}, \frac{1}{4}, \cdots \frac{1}{2^n}\right\}.$$

We are just defining $h_n(x) = f(x)$ for all x in S except for the set

$$\left\{\frac{1}{2}, \frac{1}{4}, \cdots \frac{1}{2^n}\right\}.$$

On that set, we define $h_n(x) = g^{-1}(x) = 2x$ as seen in Figure 6.5 d.

For each n, h_n still has a problem, not mapping any x onto $y = 1/2^n \in T$. The idea is to take a union. In particular, we let

$$S_\infty = \bigcup_{i=1}^{\infty} S_i = \bigcup_{i=1}^{\infty} \left\{\frac{1}{2^i}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \cdots\right\}.$$

We now define our function

$$h(x) = \begin{cases} f(x) = x & x \in S - S_\infty \\ g^{-1}(x) = 2x & x \in S_\infty \end{cases}.$$

See Figure 6.5 d for a visual representation of h. An alternate visualization is seen in Figure 6.6.

Conceptual Insight: 6.13

Let's review the conceptual insight behind the proof of CSB, seen in Figure 6.5. The function f maps S into T (yellow region), leaving a 'hole', labeled hole 1 in T, as seen in Figure 6.5 a. As seen in this figure, the set S_1 is the image of hole 1 under the function g. We now redefine the mapping of S_1 using g^{-1} instead of f to map it, thus,

$$g^{-1}(S_1) = \text{hole } 1.$$

Since S_1 is now not being mapped by f, the set $f(S_1)$ is now hole 2 in T, again seen in Figure 6.5 e. We now consider the set $S_2 = g(\text{hole } 2) = g(f(S_1))$, so $g^{-1}(S_2)$ =hole 2 and leaving hole 3. We keep repeating this process, recursively filling in the new holes using

$$S_{n+1} = g(f(S_n))$$

and then defining

$$h(S_{n+1}) = g^{-1}(S_{n+1}).$$

If the number of sets were finite, this would be a problem and we would run out of sets to map, but because we have an infinite number of sets, we can take an infinite union of the S_n's to get S_∞ and map it using g^{-1} and map $S - S_\infty$ using f as seen in Figure 6.7. Miraculously, all the holes are filled.

We now show this function is one-to-one from S onto T.

Claim 6.14: The function 6.4, is one-to-one from S onto T.

Proof: This is a two part proof. We first show

$$h(x) = \begin{cases} f(x) = x & x \in S - S_\infty \\ g^{-1}(x) = 2x & x \in S_\infty \end{cases} \tag{6.4}$$

is one-to-one. We then show h is onto.

Part 1: We prove h defined by equation 6.4 is one-to-one from S onto T by first showing h is 1-1.

We let x_1 and x_2 represent two values in S such that $h(x_1) = h(x_2)$ and show $x_1 = x_2$. We consider three cases.

Case 1: Assume $x_1, x_2 \in S - S_\infty$. Then $h(x_1) = f(x_1) = x_1$ and

$h(x_2) = f(x_2) = x_2$, so $x_1 = h(x_1) = h(x_2) = x_2$ and we are finished.
Case 2: Assume $x_1, x_2 \in S_\infty$. Then $h(x_1) = g^{-1}(x_1) = 2x_1 = h(x_2) = g^{-1}(x_2) = 2x_2$. Dividing by 2 gives $x_1 = x_2$.
Case 3: WLOG assume $x_1 \in S_\infty$ and $x_2 \in S - S_\infty$ so that

$$h(x_1) = g^{-1}(x_1) = 2x_1 \text{ and } h(x_2) = f(x_2) = x_2.$$

Since $x_1 \in S_\infty$, then there exists $n_1 \in \mathbb{Z}^+$ such that $x_1 = 1/2^{n_1}$. Since $x_2 \in S - S_\infty$, then for all $n \in \mathbb{Z}^+$

$$x_2 \neq \frac{1}{2^n}.$$

Subcase 1: Assume $n_1 = 1$ so $x_1 = 1/2$. Then

$$h(x_1) = g^{-1}(x_1) = g^{-1}(1/2) = 1$$

but

$$h(x_2) = f(x_2) = x_2 < 1$$

since $x_2 \in S = (0, 1)$. So $h(x_1) \neq h(x_2)$ and we have a contradiction to the assumption that $h(x_1) = h(x_2)$: this subcase cannot happen.
Subcase 2: Assume $n_1 > 1$. Then

$$h(x_1) = g^{-1}(x_1) = g^{-1}(1/2^{n_1}) = \frac{1}{2^{n_1 - 1}}.$$

Since

$$h(x_2) = f(x_2) = x_2 \notin S_\infty$$

then for all $n \in \mathbb{Z}^+$,

$$x_2 \neq \frac{1}{2^n}$$

and in particular

$$h(x_2) = x_2 \neq h(x_1) = \frac{1}{2^{n_1 - 1}}.$$

Again, this subcase cannot happen, so Case 3 cannot happen.
In each of the possible cases, $h(x_1) = h(x_2)$ implies $x_1 = x_2$ so h is one-to-one.
Part 2: We must show h is onto. To do this, let y_0 represent an arbitrary value in T. We must find an $x \in S$ such that $h(x) = y_0$. We consider two cases.

Case 1: Assume there exists $n_0 \in \mathbb{Z}_0^+$ such that

$$y_0 = \frac{1}{2^{n_0}},$$

that is, $y_0 \in \{1, 1/2, 1/4, \cdots\}$. Set

$$x_0 = (1/2)y_0 = \frac{1}{2^{n_0+1}} \in \{1/2, 1/4, \cdots\} = S_\infty.$$

Then by definition,

$$h(x_0) = g^{-1}(x_0) = 2x_0 = \frac{1}{2^{n_0}} = y_0.$$

Case 2: Assume for all $n \in \mathbb{Z}_0^+$,

$$y_0 \neq \frac{1}{2^n}.$$

Set $x_0 = y_0 \in S - S_\infty$. Then $f(x_0) = x_0 = y_0$.
Thus, for every $y \in T$, we found an $x \in S$ such that $h(x) = y$ so h is onto. $\square$

While there is one more point in T than in S, $C_T = C_S$ since the function h is a one-to-one, onto function. We were, in essence, able to fill in all the 'holes'. We just map the first of these numbers, 1/2, over to 1, and then map each of the remaining numbers over to where the previous number was using g^{-1}, that is, reversing the arrows in Figure 6.6. Since there were an infinite number of them, all the holes are filled in.

We have now seen the idea behind the construction of a one-to-one function h from S onto T in a particular case. See Appendix 6.2.3 for a sketch of the proof of the CSB Theorem for the case of arbitrary sets S and T.

We now know that if $C_S \preceq C_T$ and $C_T \preceq C_S$ then $C_S = C_T$ and $\preceq$ is antisymmetric and therefore our relation $\preceq$ is a partial order on equivalence classes of sets. Similarly, $\prec$ is asymmetric and transitive, so is a strict partial order on equivalence classes of sets. For example,

$$C_{\mathbb{Z}^+} \prec C_{\mathbb{R}},$$

so we know there are at least two different equivalence classes of sets of infinite size. How many cardinalities can there be? Does it make sense that there are an infinite number of sizes of infinity? Is there a largest equivalence class, that is, a largest possible infinity? Are cardinalities like the real numbers in that between every two cardinalities, there is another set whose cardinality is between them? Or are cardinalities like the integers in that there are consecutive cardinalities? Power sets will help us answer some of these questions.

Suppose we have a finite set S such that

$$C(S) = n.$$

It is not difficult to show that

$$C(S \times S) = n^2.$$

In Section 3.3, Problem 6, we determined

$$C(P(S)) = 2^{C(S)} = 2^n.$$

We might expect that

$$\begin{aligned} C(\mathbb{Z}^+ \times \mathbb{Z}^+) &= \aleph_0 \times \aleph_0 \\ C(P(\mathbb{Z}^+)) &= 2^{\aleph_0}, \end{aligned}$$

whatever that means. We showed earlier, though, that

$$C(\mathbb{Z}^+ \times \mathbb{Z}^+) = C(\mathbb{Z}^+) = \aleph_0.$$

In the Exercises, you will show that

$$C_{\mathbb{Z}^+} \prec C_{P(\mathbb{Z}^+)}$$

meaning, in some sense, that $\aleph_0 \prec 2^{\aleph_0}$. The proof is very similar to the proof that $C_{\mathbb{Z}^+} \prec C_I$. This leads us to wonder if

$$\begin{aligned} C_{\mathbb{R}} &\prec C_{P(\mathbb{Z}^+)}, \\ C_{P(\mathbb{Z}^+)} &\prec C_{\mathbb{R}}, \\ C_{\mathbb{R}} &= C_{P(\mathbb{Z}^+)}, \end{aligned}$$

or maybe they are not related because there is no one-to-one function from either one of them into the other. In fact, $C_{\mathbb{R}} = C_{P(\mathbb{Z}^+)}$. Think about what it would take to show that the set of all subsets of $\mathbb{Z}^+$ is the same size as $\mathbb{R}$ in the sense that we can find a one-to-one function from one onto the other. It happens that this is possible but is beyond the scope of this text.

Video Lesson 6.15

For a sketch of this proof, see the 14:45 minute video at
https://vimeo.com/79834502 Password:Proof
To change the speed at which the video plays, click on the gear at the lower right of the video.

Similarly, we can show that

$$C_{P(\mathbb{Z}^+)} \prec C_{P(P(\mathbb{Z}^+))}$$

meaning that

$$\aleph_0 \prec 2^{\aleph_0} \prec 2^{2^{\aleph_0}}.$$

Repeating this process (as you will show in the Exercises using induction), there are infinite sets S_1, S_2, S_3, ... such that

$$C_{S_1} \prec C_{S_2} \prec \ldots,$$

that is, there are an infinite number of cardinalities.

You will also show that for any set S, there exists a set T such that

$$C_S \prec C_T,$$

so there is no largest infinity; no matter what the cardinality of a set, there is always a set of higher cardinality.

Another question we might ask is, is there a set S such that

$$\aleph_0 = C(\mathbb{Z}^+) \prec C(S) \prec C(\mathbb{R}) = C(P(\mathbb{Z}^+)) = c.$$

Define $\aleph_1$ (aleph one) as the next largest cardinality after $\aleph_0$ and let S be a set such that

$$C(S) = \aleph_1.$$

We know $\aleph_0 \prec 2^{\aleph_0}$. Is

$$\aleph_1 \prec c \text{ or is } \aleph_1 = c?$$

The continuum hypothesis is that $\aleph_1 = C(\mathbb{R}) = c$. The truth of the continuum hypothesis is unknown.

6.2.1 Exercises

1. Let $A = [0,1]$ and $B = [0,2)$. Define
$$f : A \to B \text{ as } f(x) = x.$$
Clearly f is one-to-one, but is not onto. Let
$$g : B \to A \text{ by } g(y) = y/2.$$
Again, g is one-to-one, but not onto. Define
$$S_1 = g(B - f(A))$$
and
$$S_{n+1} = g(f(S_n))$$
for $n = 1, 2, \ldots$

   (a) Find S_1, S_2, and S_3. Answer 1

   (b) Define
   $$S_\infty = \bigcup_{i \in \mathbb{Z}^+} S_i.$$
   What is S_∞? What is $A - S_\infty$? What is $g^{-1}(S_\infty)$? Is $g^{-1}(S_\infty) \subseteq B$? Answer 2

   (c) Define $h : A \to B$ as
   $$h(x) = \begin{cases} f(x) = x & x \in A - S_\infty \\ g^{-1}(x) = 2x & x \in S_\infty \end{cases}.$$
   What is $h(1/3)$? What is $h(1/8)$? Draw a figure, similar to Figure 6.6 that graphically shows that h is one-to-one and onto. Answer 3

   (d) Prove h is one-to-one from A into B. Answer 4

   (e) Prove h maps A onto B. Answer 5

2. Consider the 2 sets
$$E_0^+ = \{0, 2, 4, \ldots\} \text{ and } O^+ = \{1, 3, 5, \ldots\}.$$
Let $f : E_0^+ \to O^+$ be defined as
$$f(x) = x + 3$$

and $g : O^+ \to E_0^+$ be defined as

$$g(y) = y + 1.$$

The functions f and g are both one-to-one. This means that

$$C(E_0^+) \preceq C(O^+) \text{ and } C(O^+) \preceq C(E_0^+).$$

Clearly f and g are not onto. To show $C(E_0^+) = C(O^+)$, we have to find a one-to-one function

$$h : E_0^+ \overset{\text{onto}}{\to} O^+.$$

We know this would be easy, but the purpose of this problem is to practice the process given in Proof 6.14. Let

$$S_1 = g(O^+ - \text{Rng}(f))$$

and

$$S_{n+1} = g(f(S_n))$$

for $n = 1, 2, \ldots$

(a) Find S_1, S_2, S_3 and S_4.

(b) Define

$$S_\infty = \bigcup_{i \in \mathbb{Z}^+} S_i.$$

What is S_∞, in usable form? If $i \in S_\infty$, does $g^{-1}(i)$ exist? Why?

(c) Define $h : E_0^+ \to O^+$ as

$$h(i) = \begin{cases} f(i) & i \in E_0^+ - S_\infty \\ g^{-1}(i) & i \in S_\infty \end{cases}.$$

Find $h(i)$ for every $i \in E_0^+$.

(d) Prove h one-to-one.

(e) Prove h onto?

3. Suppose your are appointed as a student/TA for this class with the instructions that you will work this problem for those students, and only those students, who do not work this problem themselves. Who will work this problem for you? Answer 6

4. We are now going to compare $C(S)$ and $C(P(S))$ for any set S.

   (a) Show $C(S) \preceq C(P(S))$ by defining a one-to-one function f from S into $P(S)$. This is nearly trivial.

   (b) Let $S = \{1,2,3,4\}$. We are now going to define a function f from S into $P(S)$. Set $f(1) = \{2,3\}$. Define $f(2)$, $f(3)$, and $f(4)$. We define the set $T \subseteq S$ as

   $$T = \{x \in S : x \notin f(x)\}.$$

   What is the set T? Is there an element in $x \in S$ such that $f(x) = T$? Try defining a function f from S to $P(S)$, then constructing T, where T is actually in the range of f. Why can't you do this?

   (c) Now consider an arbitrary set S and let f_0 represent an arbitrary one-to-one function from S into $P(S)$. Set $T = \{x \in S : x \notin f(x)\} \subseteq S$. Show that for every $x \in S$, $T \neq f(x)$. Note: If we assume f is onto $P(S)$, this result gives a contradiction, meaning there does not exist a one-to-one function from S onto $P(S)$, so $C(S) \prec C(P(S))$. (This problem shows that there is no largest infinity: Whatever the cardinality of a set, its power set has a greater cardinality.)

   (d) Begin with set $S_0 = \mathbb{N}$. From the previous part, $C(S_0) \prec C(S_1)$ where $S_1 = P(S_0)$. Use induction to show that there exist an infinite number of different 'sizes' of infinity.

6.2.2 Answers to selected problems

1. **Problem 1 a:** $S_1 = (0.5,1)$, $S_2 = (0.25,0.5)$, $S_3 = (1/8,1/4)$.

2. **Problem 1 b:**

$$\begin{aligned} S_\infty &= (0,1) - \{1/2^n : n \in \mathbb{Z}^+\} \\ A - S_\infty &= \{0\} \cup \{1/2^n : n \in \mathbb{Z}_0^+\} \\ g^{-1}(S_\infty) &= (0,2) - \{1/2^n : n \in \mathbb{Z}_0^+\} \subseteq B. \end{aligned}$$

3. **Problem 1 c:**

$$h(x) = \begin{cases} f(x) = x & x = 1/2^n,\ n \geq 0 \text{ or } x = 0 \\ g^{-1}(x) = 2x & x \in (0,1],\ x \neq 1/2^n,\ n \geq 0 \end{cases}. \tag{6.5}$$

So $h(1/3) = 2/3$ and $h(1/8) = 1/8$. See Figure 6.8. Note points of form $1/2^n$ and 0 are mapped onto themselves. All other points are doubled.

4. **Problem 1 d:** We prove h defined by equation 6.5 is one-to-one from A into B.

   We let x_1 and x_2 represent two values in A such that $h(x_1) = h(x_2)$ and show $x_1 = x_2$. We consider three cases.

   Case 1: Assume $x_1, x_2 \in A - S_\infty$. Then $h(x_1) = f(x_1) = x_1$ and $h(x_2) = f(x_2) = x_2$, so $x_1 = h(x_1) = h(x_2) = x_2$ and we are finished.

   Case 2: Assume $x_1, x_2 \in S_\infty$. Then $h(x_1) = g^{-1}(x_1) = 2x_1 = h(x_2) = g^{-1}(x_2) = 2x_2$. Dividing by 2 gives $x_1 = x_2$.

   Case 3: WLOG assume $x_1 \in S_\infty$ and $x_2 \in A - S_\infty$ so that

   $$h(x_1) = g^{-1}(x_1) = 2x_1 \text{ and } h(x_2) = f(x_2) = x_2.$$

   Since

   $$x_2 \in A - S_\infty = \{0\} \cup \{1/2^n : n \in \mathbb{Z}_0^+\},$$

   we have two subcases.

   Subcase 1: Assume $x_2 = 0$. Then

   $$h(x_1) = g^{-1}(x_1) = 2x_1 > 0$$

   but

   $$h(x_2) = f(x_2) = 0.$$

   So $h(x_1) \neq h(x_2)$ and we have a contradiction to the assumption that $h(x_1) = h(x_2)$: this subcase cannot happen.

   Subcase 2: Assume there exists $n_2 \in \mathbb{Z}_0^+$ such that $x_2 = 1/2^{n_1}$. Then

   $$h(x_1) = 2x_1 = h(x_2) = x_2 = \frac{1}{2^{n_1}}.$$

   This means

   $$x_1 = \frac{1}{2^{n_1+1}},$$

   which implies $x_1 \in A - S_\infty$. This is a contradiction since $x_1 \in S_\infty$. Again, this subcase cannot happen, so Case 3 cannot happen.

   In each of the possible cases, $h(x_1) = h(x_2)$ implies $x_1 = x_2$ so h is one-to-one.

5. **Problem 1 e:** We must show h maps A onto B. To do this, let y_0 represent an arbitrary value in B. We must find an $x \in A$ such that $h(x) = y_0$. We consider three cases.

   Case 1: Assume $y_0 = 0$. Set $x_0 = 0 \in A - S_\infty$. Then $h(x_0) = f(x_0) = x_0 = 0 = y_0$.

   Case 2: Assume there exists $n_0 \in \mathbb{Z}_0^+$ such that
   $$y_0 = \frac{1}{2^{n_0}},$$
   that is, $y_0 \in \{1, 1/2, 1/4, \cdots\}$. Set
   $$x_0 = y_0 \in A - S_\infty.$$
   Then by definition,
   $$h(x_0) = f(x_0) = x_0 = y_0.$$

   Case 3: Assume $y_0 \in S_\infty$. This means that for all $n \in \mathbb{Z}_0^+$,
   $$y_0 \neq \frac{1}{2^n},$$
   and $y_0 \neq 0$. Set $x_0 = y_0/2$.

   - Since $y_0 \neq 0$, then $x_0 \neq 0$.
   - Since $y_0 \neq 1/2^n$ for any $n \in \mathbb{Z}_0^+$ then $x_0 \neq 1/2^m$ for any $m \in \mathbb{Z}^+$.

   Therefore $x_0 \in S_\infty$ and $h(x_0) = 2x_0 = y_0$. Thus, for every $y \in B$, we found an $x \in A$ such that $h(x) = y$ so h is onto. □

6. **Problem 3:** Suppose you work this problem yourself. Then from the instructions, you cannot work this problem for yourself. But if you don't work this problem for yourself, then you must, by the instructions, work this problem for yourself. You won't be able to follow instructions. This is a variation on Bertrand Russell's classic paradox.

6.2.3 Appendix: Outline of Proof of CSB Theorem

Let
$$S_1 = g(T - \text{Rng}(f)).$$

Since in Proof 6.14, $T = (0, 1]$ and $\text{Rng}(f) = (0, 1)$, then

$$\text{T} - \text{Rng}(f) = \{1\},$$

so

$$S_1 = \{1/2\}.$$

Conceptually, S_1 is just the elements in S that were used by way of the function g^{-1} to map onto the elements in T that were not originally mapped onto, 1 in this case Proof 6.14. We now let

$$S_{n+1} = g(s(S_n)).$$

For Proof 6.14

$$S_2 = g(f(S_1)) = \{1/4\}.$$

Note that for Proof 6.14, S_2 contains the elements that were used to map onto $f(S_1) = 1/2$. Similarly,

$$S_3 = g(f(S_2)) = \{1/8\}$$

is the point that maps onto $f(S_2)$. For Proof 6.14

$$S_n = \{1/2^n\}.$$

We define

$$S_\infty = \bigcup_{i=1}^{\infty} S_i.$$

- From the way the sets are defined, for every $x \in S_n$, for any n, there exists a $y \in T$ such that $g(y) = x$.
- If $x \in S_\infty$, then $x \in S_n$ for some $n \in \mathbb{Z}^+$. Since $x \in S_n$, then $x = g(y)$ for some $y \in T$. This means that

  $$S_\infty \subseteq \text{Rng}(g),$$

  so $x \in S_\infty$ implies $g^{-1}(x)$ exists and is in T.
- We now define h from S onto T as

$$h(x) = \begin{cases} f(x) & x \in S - S_\infty \\ g^{-1}(x) & x \in S_\infty \end{cases}. \tag{6.6}$$

This is identical to Equation 6.4 for Proof 6.14.

The general proof that h is one-to-one and onto is similar to the proof h defined by Equation 6.4 was one-to-one and onto, so we leave the details to the reader.

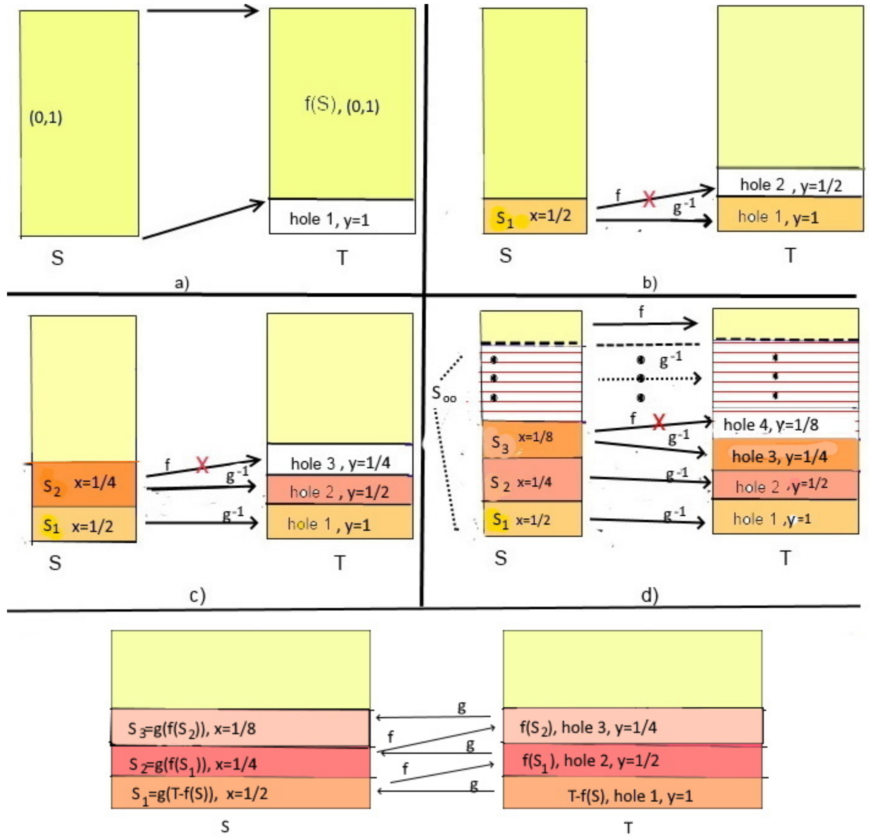

Figure 6.5: Idea behind proof of CSB Theorem. Arrows to right represent f and g^{-1}. Figure a represents the initial mapping of S into T by f, leaving hole 1 at $y = 1$. In Figures b through d, holes are successively filled in using g^{-1}. Figure e demonstrates how the sets S_i are constructed.

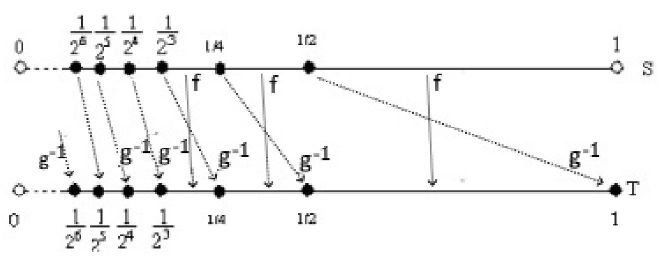

Figure 6.6: $h(x) = f(x) = x$ except at designated points, for which g^{-1} is used. Scale of interval has been distorted to make graph easier to see.

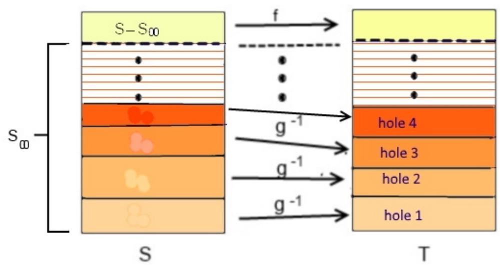

Figure 6.7: One-to-one function h mapping S onto T is given by f on $S - S_\infty$ and g^{-1} on S_∞.

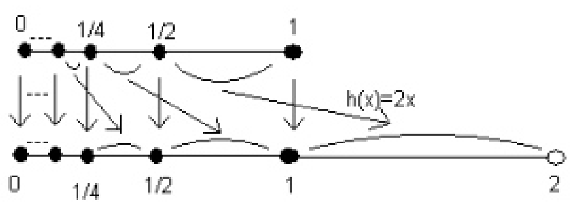

Figure 6.8: One-to-one function f from $[0,1]$ onto $[0,2)$.

Chapter 7

Open and Closed Sets

7.1 Proofs about the number line

Video Lesson 7.1

This section will be easier to read if you first watch the 5:57 minute video
https://vimeo.com/499676930(Password:Proof)
This video discusses what it means to be an open set. To change the speed at which the video plays, click on the gear at the lower right of the video.

In Section 6.2, we discovered that $C(\mathbb{Q}^+) \prec C(\mathbb{R}^+)$. This may conflict with our intuition since we work with rational numbers all the time, but only occasionally deal with irrational numbers. In this section, we extend our study of sets to the structure of sets on the real number line. Our goal in this study is to understand how much larger the set of irrational numbers is than the set of rational numbers. We will need some preparation for this.

Recall that an interval of the form

$$(a, b) = \{x : a < x < b\}$$

is called an open interval and an interval of the form

$$[a, b] = \{x : a \leq x \leq b\}$$

is called a closed interval. We now generalize the concept of open and closed to arbitrary sets on the real number line.

Definition 7.2: Open set

The set $S \subseteq \mathbb{R}$ is called an **open set** if and only if

$$p \equiv \{\forall\, x \in S,\ \exists\,(a,b) \subseteq S \text{ such that } x \in (a,b)\}.$$

This means if x is an element of S, then x is contained in an open interval that is contained in S. This intuitively means that an open set is, in some sense, made up of open intervals. We must remember, though, that when showing a set is open, we must use the definition, not our intuition. For example, let

$$S = \bigcup_{n \in \mathbb{Z}^+} I_n,$$

where

$$I_n = \left(n - \frac{1}{n}, n + \frac{1}{n}\right) = \{x : n - 1/n < x < n + 1/n\},$$

which is visualized in Figure 7.1. Since S is the union of an infinite number of open intervals, let's see if we can show that S is an open set.

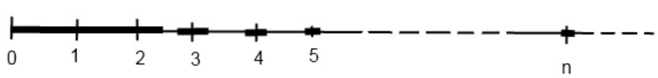

Figure 7.1: The open set which is the union of heavy lines.

Claim 7.3: The set

$$S = \bigcup_{n \in \mathbb{Z}^+} \left(n - \frac{1}{n}, n + \frac{1}{n}\right)$$

is an open set.

Proof: We are given the set S and use the let method, letting x_0 represent an arbitrary element in S. We have to find an open interval in S that contains x_0. Since we know that $x_0 \in S$, then there exists some $n_0 \in \mathbb{Z}^+$ such that

$$x_0 \in I_{n_0} = \left(n_0 - \frac{1}{n_0}, n_0 + \frac{1}{n_0}\right).$$

But this is an open interval contained in S. Therefore S is open.

We now generalize the concept of closed to sets.

Definition 7.4: Closed set

A set $S \subseteq \mathbb{R}$ is **closed** if and only if S^c is open.

The set

$$T = (-\infty, 0] \cup [1, 2] \cup [3, \infty)$$

is closed since $T^c = (0, 1) \cup (2, 3)$ is open. Note that some sets, such as $(0, 1]$, are neither open nor closed, so showing a set is not open does not prove it is closed. Since closed is defined through the complement, to show S is closed, we must show that S^c is open, that is

$$p \equiv \{\langle \forall x \in S^c \rangle \langle \exists\, (a, b) \subseteq S^c \rangle \text{ such that } \langle x \in (a, b) \rangle\}.$$

Claim 7.5: The set $S = [0, 1]$ is closed.

Proof: To show S is closed, we must show that $S^c = (-\infty, 0) \cup (1, \infty)$ is open. Let x_0 represent an arbitrary element in S^c. There are two cases.
Case 1: Assume $x_0 < 0$. Then $x_0 \in (x_0 - 1, 0) \subseteq S^c$.
Case 2: Assume $x_0 > 1$. Then $x_0 \in (1, x_0 + 1) \subseteq S^c$.
Thus, S^c is open and S is closed.

The boundary of an interval, open, closed or neither, would just be its end-points. For example, if we have an interval

$$I = (a, b) = \{x \in \mathbb{R} : a < x < b\},$$

the end points of the interval,

$$B = \{a, b\},$$

is the boundary set of I and I^c. If $x \in B$, then we call x a boundary value. Note that B is also the boundary set of the closed interval

$$J = [a, b] = \{x : a \leq x \leq b\}$$

and its complement J^c. While we have an intuitive idea of what the boundary of a set should be, it is not as clear what the boundary is when the sets become more complicated than intervals. What would be the boundary between a finite collection of points, say

$$T = \{1, 2, 3\}$$

and its complement? We might guess that the boundary is just T itself. What would be the boundary for an infinite set of points, such as

$$S = \{x : x = \frac{1}{n}, n \in \mathbb{Z}^+\}$$

and its complement. The set S is 'visualized' in Figure 7.2.

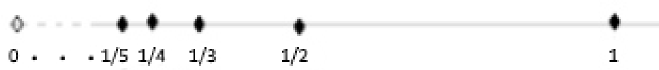

Figure 7.2: The set S of points approaching 0.

Each of the points in this set would seem to be a boundary value since they are isolated from one-another, but since the points seem to 'approach' 0, would $b = 0$ also be considered a boundary value? What do we even mean by a boundary value?

Intuitively, a boundary value seems to separate the set S from the set S^c. In some sense, it consists of values which are 'arbitrarily' close to values in both S and S^c. This leads to the following definition.

Definition 7.6: Boundary value

Let $S \subseteq \mathbb{R}$. The number $b \in \mathbb{R}$ is a **boundary value** for S if and only if

$$\{\forall\, x < b \text{ and } \forall\, y > b,\ (x, y) \cap S \neq \emptyset \text{ and } (x, y) \cap S^c \neq \emptyset\}.$$

We call the set

$$B_S = \{b \in \mathbb{R} : b \text{ is a boundary value for } S\}$$

the boundary of S.

To short, b is a boundary value if and only if

$$q \equiv \{\langle \forall\, (x,y) \text{ such that } b \in (x,y)\rangle,\ \langle S^c \cap (x,y) \neq \emptyset\rangle \wedge \langle S \cap (x,y) \neq \emptyset\rangle\},$$

that is, every interval containing b contains points in both S and S^c.

Suppose $S = (0,1)$. If we construct interval $I = (x,y)$ about 0, it should be clear that I contains points in S and S^c. We would think we could find these values as in Figure 7.3. The problem is that if $y > 2$, then $y/2 \notin S$. The proof that 0 is a boundary value requires care.

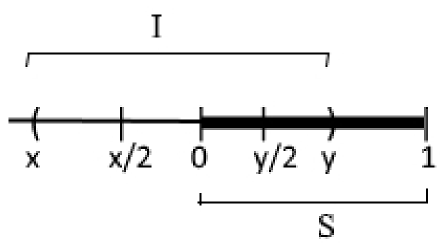

Figure 7.3: Interval $(0,1)$. Arbitrary interval (x,y) about 0 contains value $y/2$ in S and value $x/2$ in S^c.

To show, by the definition, that $b = 0$ is a boundary value $S = (0,1)$ we must show that every interval containing 0 also contains values in the set S and values not in the set. We let $I_0 = (x_0, y_0)$ represent an arbitrary interval such that

$$x_0 < 0 < y_0.$$

It is clear that $x_0/2 \in S^c \cap I$ so this part is easy. The difficult part is finding a value in $S \cap I$ since $y_0/2$ may not be in S.

Conceptual Insight: 7.7

We know that there are always values in $(0, y_0)$ that are in S, but there also may be values in $(0, y_0)$ that are in S^c so here is where we have to be careful that we choose appropriately. If $y_0 \leq 1$ we could just set $c_0 = y_0/2$, but if $y_0 > 1$, then $y_0/2$ may not be in S, but $c_0 = 0.5$ is a value in set S and (x_0, y_0). We could write a proof that involves two cases, but we can also solve this problem with the technical handle of setting

$$c_0 = \min\{0.5, y_0/2\}.$$

Since we 'know' y_0, then the value for c_0 is well-defined since it is the minimum of two known numbers. This technique of conditionally defining a value has been used often.

We have the values $x_0/2$ and c_0 which are both in I, one being in S^c and one in S, so 0 is a boundary value. Similarly, we can show 1 is a boundary value.

Claim 7.8: The number 0 is a boundary value for the set $S = (0, 1)$.

Proof: Let (x_0, y_0) be an arbitrary interval such that $0 \in (x_0, y_0)$. This means $x_0 < 0$ and $y_0 > 0$. This proof consists of two parts, showing $(x_0, y_0) \cap S^c \neq \emptyset$ and showing $(x_0, y_0) \cap S \neq \emptyset$.
Part 1: Since $0 \in (x_0, y_0)$, then $x_0 < 0$. This means

$$x_0/2 \in (x_0, y_0) \text{ and } x_0/2 \in S^c = (-\infty, 0] \cup [1, \infty),$$

so

$$(x_0, y_0) \cap S^c \neq \emptyset.$$

Part 2: Set $c_0 = \min\{0.5, y_0/2\}$. We consider two cases.
Case 1: Suppose $c_0 = 0.5$. This means $0.5 \leq y_0/2$ and so $1 \leq y_0$. Clearly, $c_0 = 0.5 \in (0, 1) = S$. Since $x_0 < 0$ then $c_0 \in (x_0, y_0)$. In this case,

$$(x_0, y_0) \cap S \neq \emptyset.$$

Case 2: Suppose $c_0 = y_0/2$. This means $y_0/2 \leq 0.5$ and so $y_0 \leq 1$. This means that both $c_0 \in (0, 1) = S$ and $c_0 \in (x_0, y_0)$. In this case,

$$(x_0, y_0) \cap S \neq \emptyset.$$

In both cases, $(x_0, y_0) \cap S \neq \emptyset$. Combined with Part 1 in which we showed $(x_0, y_0) \cap S^c \neq \emptyset$ gives that 0 is a boundary value for $S = (0, 1)$.

Remark 7.9

We must remark that Part 1 could also have been proven by just using 0, which is in both S^c and (x_0, y_0).

A similar argument to Proof 7.8 shows that 1 is also a boundary value for

S.

Now let's discuss what it means to not be a boundary value. Consider $0.8 \in S = (0,1)$. The interval $(0.7, 0.9)$ contains 0.8, but does not contain any points in S^c, so 0.8 is not a boundary point. Note that there are some intervals about 0.8 that contain points in both S and S^c, such as $(0.7, 1.1)$, but not every interval does this. It should be clear that b is not a boundary value if there exists an interval containing b that is contained entirely in S or S^c. that is,

Definition 7.10: Not a boundary value

The number $b \in \mathbb{R}$ is **not a boundary value** for S if and only if

$$\{\exists\, (x,y) \text{such that } \langle b \in (x,y) \subseteq S \rangle \vee \langle b \in (x,y) \subseteq S^c \rangle\}.$$

To show b is a boundary value, we must show something for every interval (x,y) containing b. To show b is not a boundary value, we must find one interval (x,y) that satisfies a certain condition. Again, notice the careful use of $\exists$ and $\forall$.

Conceptual Insight: 7.11

Let's now consider what is not a boundary value for $S = (0,1)$. Consider $b = -1$. We must find an interval (x,y) that contains -1 and which is entirely within S or S^c. Since b is in S^c, then we must have

$$(x,y) \subseteq S^c = \{x : x \leq 0 \text{ or } x \geq 1\}.$$

This is easy, set $(x,y) = (-2,0)$. There are lots of other equally valid intervals. Suppose instead that $b = 0.5$. Then $b \in S$ so we need to find x and y so that $0.5 \in (x,y) \subseteq S = (0,1)$. This again is easy. Just set $(x,y) = (0,1)$.
Let's try one more extreme value, say $b = 1.0001$. Since $b \in S^c$, then we need $(x,y) \subseteq S^c$ to show b is not a boundary point. For (x,y) to be in S^c we need $x \geq 1$ and for $b \in (x,y)$ we need $x < 1.0001$. There are lots of choices, but we can set $(x,y) = (1,2)$.

It appears that all values other than 0 and 1 are not boundary values for S. To show this,

- **Assumed**: we assume b_0 does not equal 0 or 1 and

- **Show**: we find (x, y) such that
$$\langle b_0 \in (x, y) \subseteq S \rangle \vee \langle b \in (x, y) \subseteq S^c \rangle.$$

Claim 7.12: If $b \neq 0$ and $b \neq 1$, then b is not a boundary value of $S = (0, 1)$.

Proof: Let b_0 be an arbitrary real number other than 0 or 1. As we saw in the examples, how we choose (x, y) depends on whether $b_0 \in S$ or $b_0 \in S^c$. There seem to be 3 natural cases,

$$b_0 < 0, \; 0 < b_0 < 1, \; b_0 > 1.$$

Case 1: Suppose $b_0 < 0$. Set $(x_0, y_0) = (b_0 - 1, 0)$. Since $b_0 - 1 < 0$ and $y_0 = 0$,

$$b_0 \in (x_0, y_0) = (b_0 - 1, 0) \subseteq S^c = (-\infty, 0] \cup [1, \infty).$$

Case 2: Suppose $0 < b_0 < 1$. Set $(x_0, y_0) = (0, 1)$. Clearly,

$$b_0 \in (x_0, y_0) = (0, 1) \subseteq S = (0, 1).$$

Case 3: Suppose $b_0 > 1$. Set $(x_0, y_0) = (1, b_0 + 1)$. Since $b_0 > 1$,

$$b_0 \in (x_0, y_0) = (1, b_0 + 1) \subseteq S^c = (-\infty, 0] \cup [1, \infty).$$

In all three cases, there existed an interval containing b_0 that was entirely in S or entirely in S^c. Thus, b_0 is not a boundary value.

Related to sets and boundaries are the concepts of interior points of a set and upper bounds of a set.

Definition 7.13: Interior value

The value c is an **interior value** of S if and only if there exists an interval (a, b) such that $c \in (a, b) \subseteq S$.

For the set $S = (0, 1)$, all the points in S are interior points. For the set $S^c = (-\infty, 0] \cup [1, \infty)$, all the points in $(-\infty, 0) \cup (1, \infty)$. Generally, all the points on the line are either interior points for S, interior points for S^c, or boundary points for both S and S^c. The proof of this is not difficult, but is left to the reader.

Definition 7.14: Upper bound

A value x is an **upper bound** for a set S if and only if $\forall\, a \in S$, $x \geq a$.

Let $U(S)$ be the set of all upper bounds for the set S. For example, if $S = (0,1) \cup [2,3] \cup \{4\}$, then $U(S) = [4,\infty)$, but if $T = (0,1)$, then $U(T) = [1,\infty)$. Note that $U(S)$ contains a value in S but $U(T)$ does not contain any values in T.

You will reconsider interior points and upper bounds in the problems.

An important question in probability is in determining the size of sets, in terms of length. For example, we would consider both the open set $(0,1)$ and the closed set $[0,1]$ of length one, but what about more complicated sets? This is where the concept of open sets is helpful. We define the length of an open interval (a,b) to be $|b-a|$. Suppose we have a set S. If we can enclose the set S within a collection of open intervals whose total length is L, then we will say the 'measure' of set S is less than or equal to L. Let's consider an example.

Suppose $S = \{1,2,3\}$. Let ϵ_0 represent a fixed positive number. Define the three open intervals

$$U_j = \left(j - \frac{\epsilon_0}{6}, j + \frac{\epsilon_0}{6}\right),\ j = 1,2,3$$

and the open set

$$U = \sum_{j=1}^{3} U_j.$$

Since the length of each of these open intervals is

$$\left|1 + \frac{\epsilon_0}{6} - (1 - \frac{\epsilon_0}{6})\right| = \frac{\epsilon_0}{3},$$

then the total size of U is less than or equal to ϵ_0 (less than because the intervals may overlap). We have now enclosed the set S in an open set U, $S \subset U$ of total measure at most ϵ_0, meaning that the measure of S is at most ϵ_0. Note that since ϵ_0 was an arbitrary positive number, the measure of S must be less than or equal to every positive number. Thus, S is considered to have measure zero.

Suppose we have a countably infinite set of open intervals

$$U_j = (a_j, b_j),\ j = 1,2,\cdots$$

where
$$\sum_{j=1}^{\infty} |b_j - a_j| \leq \epsilon.$$
Suppose
$$U = \bigcup_{j=1}^{\infty} U_j.$$
In the problems, you will show that U is an open set. We then state the measure of this open set is less than or equal to ϵ. We do not say 'equals' because the intervals could overlap. If some set $S \subseteq U$ then we say S has measure at most ϵ, also.

Definition 7.15: Measure zero

Given an arbitrary set S. If S has measure at most ϵ for every $\epsilon > 0$, then we say set S has **measure zero**.

Suppose we are interested in the size of the set $\mathbb{Z}^+$. We then construct the open intervals

$$\begin{aligned} U_1 &= (1-1/4, 1+1/4) = (0.75, 1.25), \\ U_2 &= (2-1/8, 2+1/8), \cdots \\ U_j &= (j-1/2^{j+1}, j+1/2^{j+1}), \cdots. \end{aligned}$$

Note that the length of U_1 is $1/2$, the length of U_2 is $1/4$, $\cdots$, the length of U_j is $1/2^j$, and so forth. Also note that
$$\mathbb{Z}^+ \subset U = \bigcup_{j=1}^{\infty} U_j.$$
You should recall the geometric series
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{j=1}^{\infty} \frac{1}{2^j} = 1,$$
so the measure of $\mathbb{Z}^+$ is at most 1.

We now use a technical handle you should find useful in a variety of situations. Let ϵ represent an arbitrary positive number. We redefine our open intervals as
$$U_j = \left(j - \frac{\epsilon}{2^{j+1}}, j + \frac{\epsilon}{2^{j+1}}\right),\ j = 1, 2, \cdots.$$

Note that $j \in U_j$ so $\mathbb{Z}^+ \subset U$ and that the length of U_j is $\epsilon/2^j$. This means that total length of all of these intervals is

$$\frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots = \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} == \epsilon \sum_{j=1}^{\infty} \frac{1}{2^j} = \epsilon.$$

Since ϵ was arbitrary and we can enclose $\mathbb{Z}^+$ in open intervals whose combined length is ϵ, then $\mathbb{Z}^+$ has measure zero. This may not be surprising since $\mathbb{Z}^+$ is a set of isolated points, although an infinite number of them.

We are now prepared to compare $\mathbb{R}$ to $\mathbb{Q}$. Recall from Section 6.1 that $C(\mathbb{Q}^+) = C(\mathbb{Z}^+)$, so the rational numbers behave somewhat like the integers. Precisely, it means that there is a one-to-one function from $\mathbb{Z}^+$ onto $\mathbb{Q}^+$, such that

$$f(1) = q_1,\, f(2) = q_2,\, \cdots$$

and for every positive rational number q, there exists an integer j such that $f(j) = q_j = q$. We can then do for the rational numbers what we did for the integers, construct intervals of decreasing size around each of them. In particular, we will construct the intervals

$$U_j = \left(q_j - \frac{\epsilon}{2^{j+1}}, q_j + \frac{\epsilon}{2^{j+1}}\right),\; j = 1, 2, \cdots .$$

Note that for every positive integer j, U_j contains q_j and the length of U_j is $\epsilon/2^j$. Summing up, we have

$$\mathbb{Q}^+ \subset U = \sum_{j=1}^{\infty} U_j,$$

where the total length of the intervals U_j is

$$\frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots = \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} == \epsilon \sum_{j=1}^{\infty} \frac{1}{2^j} = \epsilon,$$

as before.

This means that the set of positive rational numbers has measure zero. What this means is that essentially all the length on the real number line comes from the irrational numbers. All of the rational numbers in the interval $(0, \infty)$ put together has length zero and all the irrationals have infinite length. This gives a sense of how many more real numbers there are than rational numbers.

7.1.1 Exercises

1. Show $\emptyset$ is open. Answer 1
2. Show $\mathbb{R}$ is open.
3. Show $\mathbb{R}$ is closed. Answer 2
4. Show $\emptyset$ is closed.
5. For $n \in \mathbb{Z}^+$, let
$$J_n = [\frac{1}{n}, 1 - \frac{1}{n}]$$
and let
$$T = \bigcup_{n \geq 3} J_n.$$
Show T is open. (You could make a sketch of the sets J_n and use them to conjecture that $T = (0, 1)$ and then prove your conjecture. Instead, you should practice using the definition of an open set. To do this, carefully write what it is you assume and what it is you have to show.) Answer 3
6. For $n \in \mathbb{Z}^+$, let $K_n = (0, 1 + 1/n)$ and let
$$W = \bigcap_{n \in \mathbb{Z}^+} K_n.$$
Show W is not open. (What has to be shown to show W is not open? Carefully construct $\neg p$.)
7. Let S_n be an open set $\forall\, n \in \mathbb{Z}^+$. Show that
$$S = \bigcup_{n \in \mathbb{Z}^+} S_n$$
is open. Answer 4
8. Show that S is closed where
$$S = \{x : x = \frac{1}{n} \,\forall\, n \in \mathbb{Z}^+\} \cup \{0\}.$$
See Figure 7.4 for a visualization of set S. To work this problem, you need to let x_0 represent an arbitrary element in S^c, then find an open interval containing x_0 which is contained in S^c. There are three cases for where x_0 lies.

Figure 7.4: Set S is union of infinite number of points.

9. Let S_n be a closed set $\forall\, n \in \mathbb{Z}^+$. Show that

$$S = \bigcap_{n \in \mathbb{Z}^+} S_n$$

is closed. Answer 5

10. Let

$$S = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \right\}.$$

    (a) Show that if

    $$\frac{1}{n+1} < x < \frac{1}{n}$$

    for some positive integer n, then x is not a boundary value for the set S.

    (b) Show that $x = 1/4$ is a boundary value for the set S.

    (c) Show that $x = 0$ is a boundary value for the set S. What is the conceptual insight behind 0 being a boundary value for S? What is a technical handle needed to complete this proof?

11. What is $B(\mathbb{Q})$, the boundary set for the rational numbers $\mathbb{Q}$? Answer 6

12. In this problem we think about the relationship between open sets, closed sets and the boundary of a set. We note that if $T = (0, 1]$, then $T - B(T) = (0, 1)$ is open and $T \cup B(T) = [0, 1]$ is closed, where $B(T) = \{x : x \text{ is a boundary value for } T\}$.

    (a) Let S be a subset of the real numbers. Show that $S - B(S)$ is open. (What must you show to prove this statement? What method of proof is most likely to work? This problem requires you to carefully consider the definition of open.)

    (b) Show that $S \cup B(S)$ is closed. (This set is called the closure of S and is the smallest closed set containing S. What do you know about points in $(S \cup B(S))^c$?

13. Consider the Definition 7.13. Let $S = [1, 10] \subseteq \mathbb{R}$

(a) Prove or disprove that $c_1 = 3$ is an interior point of S. Answer 7

(b) Prove or disprove that $c_2 = 1$ is an interior point of S. Answer 8

14. Let $S \subseteq \mathbb{R}$ be a set of real numbers. Use the definition of an interior point from Problem 13.

    (a) Prove or find a counterexample: Let $x \in S$. If there exists a number $\epsilon > 0$ such that $[x - \epsilon, x + \epsilon] \subseteq S$, then x is an interior point of S.

    (b) Prove or find a counterexample: Let $x \in S$ be an interior point. Then there exists a number $\epsilon > 0$ such that $[x - \epsilon, x + \epsilon] \subseteq S$.

15. Use the definition of an interior point from Problem 13 on each part of this problem.

    (a) Determine which values $x \in [0, 1] = S$ are interior values and which are not interior values? Prove you claim. Answer 9

    (b) Let $S = \mathbb{R} - \{1, 2\} = \{x \in \mathbb{R} : x \neq 1, x \neq 2\}$. Which values of S are interior values and which are not interior values? Prove you claim. Answer 10

16. Let S_1, $S_2 \subseteq \mathbb{R}$ be sets of real numbers. Using the definition of an interior point from Problem 13, prove or disprove that if x is an interior point of S_1 and of S_2, then x is an interior point of $S_1 \cap S_2$. It will help to draw a picture of S_1, S_2 and of the requisite intervals get an insight.

17. Recall from Problem 13 the definition of an interior value. Which values
$$x \in S = \left\{ x : x \neq \frac{1}{n} \, \forall n \in \mathbb{Z}^+ \right\}$$
are interior values and which are not interior values? Prove you claim. Answer 11

18. Recall from Problem 13 the definition of an interior value. Show that if S is open, then every value is an interior value.

19. Consider the definition of an upper bound 7.14.

    (a) Find $U(S)$ where $S = \{1 - 1/n : n \in \mathbb{Z}^+\}$. Answer 12

    (b) Find $U(\mathbb{Z}^+)$. Answer 13

    (c) Find $U(\emptyset)$. Answer 14

20. Recall from Problem 19 the definition of upper bound. Show $U(S)$ is closed $\forall\, S \subseteq \mathbb{R}$. (Remember to use definitions.)

21. Recall from Problem 19 the definition of upper bound. Show that for any set of real numbers S, $B(U(S))$, the boundary of the set of upper bounds of S, consists of at most, one value. If this value exists, what name might you give it? Answer 15

22. Let $S \subseteq \mathbb{R}$ and let $U(S)$ be its set of upper bounds (see Problem 19). Prove or come up with a counterexample to the following statement:

$$\{\text{If } a \in S \cap U(S), \text{ then } a \text{ is a boundary point of } S.\}$$

23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. The function f is called an **open** function if and only if for any $B \subseteq \mathbb{R}$, if B is open, then $f^{-1}(B)$ is open.

    (a) Let $f(x) = ax + b$ where a and b are two fixed real numbers. Prove or disprove that f is an open function.

    (b) Find a function f which is not open.

7.1.2 Answers to selected problems

1. **Problem 1:** We must show that for every $x \in \emptyset$ there is an interval containing x that is also in $\emptyset$. But since there are no x-values in $\emptyset$, this is true automatically.

2. **Problem 3:** We must show $\mathbb{R}$ is closed which means we must show $\mathbb{R}^c = \emptyset$ is open, which we did in Problem 1.

3. **Problem 5:** Let x_0 represent an arbitrary element in S. This means for some n_0,

$$x_0 \in J_{n_0} = \left[\frac{1}{n_0}, 1 - \frac{1}{n_0}\right] \subset \left(\frac{1}{n_0+1}, 1 - \frac{1}{n_0+1}\right) \subset J_{n_0+1} \subset S.$$

4. **Problem 7:** Let x_0 represent an arbitrary element in S. Then there exists n_0 such that $x_0 \in S_{n_0}$. Since S_{n_0} is open, there exists an interval $(a, b) \subseteq S_{n_0}$ which contains x_0. Since $S_{n_0} \subseteq S$, then $x \in (a, b)$ and $(a, b) \subseteq S$, so S is open.

5. **Problem 9:** We must show

$$S^c = \left(\bigcap_{n \in \mathbb{Z}^+} S_n\right)^c$$

is open. Using DeMorgan's laws,

$$\left(\bigcap_{n\in\mathbb{Z}^+} S_n\right)^c = \left(\bigcup_{n\in\mathbb{Z}^+} S_n^c\right).$$

Since S_n^c is open for every n, by Problem 7, S^c is open so S is closed.

6. **Problem 11:** $B(\mathbb{Q}) = \mathbb{R}$. Let x_0 represent an arbitrary element in $\mathbb{R}$ and let (a_0, b_0) represent an arbitrary interval that contains x_0. Every interval contains rational numbers so $(a_0, b_0) \cap \mathbb{Q} \neq \emptyset$. Similarly every interval contains irrational numbers so $(a_0, b_0) \cap \mathbb{Q}^c \neq \emptyset$ and $x_0 \in B(\mathbb{Q})$.

7. **Problem 13 a:** We show $c_1 = 3$ is an interior value. Set $(a, b) = (2, 4) \subseteq [1, 10]$. Since $c_1 = 3 \in (2, 4)$, by the definition, $c_1 = 3$ is an interior value of $[1, 10]$.

8. **Problem 13:b** We show $c_2 = 1$ is not an interior value. Let (a_0, b_0) represent an arbitrary interval contained in $[1, 10]$. By the definition of open interval, for every $x \leq a_0$, x is not in (a_0, b_0). Since $(a_0, b_0) \subseteq [1, 10]$, $a_0 \geq 1$. This means $c_2 = 1$ is not an element of (a_0, b_0). Since (a_0, b_0) is an arbitrary open interval in $[1, 10]$, then $c_2 = 1$ is not in any open interval contained in $[1, 10]$ so is not an interior value of that interval. (An alternate method to prove this is to let (a_0, b_0) be an arbitrary interval containing $c_2 = 1$. This means $a_0 < 1$ so the value $(a_0 + 1)/2 \in (a_0, b_0)$ but $(a_0 + 1)/2 \notin [1, 10]$ so there is no open interval containing $c_2 = 1$ that is containe in $[1, 10]$ so c_2 is not an interior value.

9. **Problem 15 a:** $I = (0, 1)$ is the set of interior values for S. Let $x_0 \in I$. Then $x_0 \in (0, 1) \subseteq [0, 1]$ and $(0, 1)$ is an open interval, so x_0 is an interior value. If $x_0 < 0$ or $x_0 > 1$, then any interval containing x_0 is not in S since $x_0 \notin S$, so x_0 is not an interior value. Let $x_0 = 0$. We prove x_0 is not an interior value by contradiction. Assume $0 \in (a_0, b_0) \subseteq [0, 1]$. Since $0 \in (a_0, b_0)$, then $a_0 < 0$. Therefore $a_0/2 \in (a_0, b_0)$, but $a_0/2 \notin [0, 1]$, so $(a_0, b_0) \not\subseteq S$ which is a contradiction. So 0 is not an interior value. Similar proof shows 1 is not an interior value.

10. **Problem 15 b:** The set of interior points of S consists of all real numbers except 1 and 2. Let x_0 represent an arbitrary element in S. Case 1: If $x_0 < 1$, then the interval is $(x_0 - 1, 1)$. Case 2: if $1 < x_0 < 2$, then we use the interval $(1, 2)$. Case 3: if $x_0 > 2$, then we use the interval

$(2, x_0 + 1)$. Any interval containing 1 or 2 is not contained in S since 1 and 2 are not in S.

11. **Problem 17:** It is easy to prove $1/n$ is not an interior point for any positive integer n since any interval that contains $1/n$ contains a point not in S, namely $1/n$, itself. It is relatively straight forward to show that any other nonzero number x_0 is an interior point by considering three cases. For example, suppose that for some positive integer n_0, $1/(n_0 + 1) < x_0 < 1/n_0$. Then this is an interval that contains x_0 and is also in S. If $x_0 < 0$, then an interval we can use is $(x_0 - 1, 0)$. Suppose that $x = 0$ and that there exists an interval (a_0, b_0) such that $0 \in (a_0, b_0) \subseteq S$. Then $b_0 > 0$. Set $n = \lceil 1/b_0 \rceil + 1$ so $n > 1/b_0$. This means $0 < 1/n < b_0$ and $1/n \notin S$ which is a contradiction. So 0 is not an interior point.

12. **Problem 19 a:** It is clear that if $x \geq 1$, then $x \in U(S)$ since $x \geq 1 > 1 - 1/n$ for all $n \in \mathbb{Z}^+$. Let x_0 represent an arbitrary number less than 1. Set
$$n_0 = \left\lceil \frac{1}{1 - x_0} \right\rceil + 1.$$
Then $1 - 1/n_0 > x_0$ and since $1 - 1/n_0 \in S$. Thus, $x_0 \notin U(S)$ so $U(S) = \{x : x \geq 1\}$.

13. **Problem 19 b:** Let x_0 represent an arbitrary element in $\mathbb{R}$. Set $n_0 = \lceil x_0 \rceil + 1$. Since $n_0 \in \mathbb{Z}^+$ and $x_0 < n_0$, then $x_0 \notin U(\mathbb{Z}^+)$ so $U(\mathbb{Z}^+) = \emptyset$.

14. **Problem 19 c:** Let x_0 represent an arbitrary element in $\mathbb{R}$. Then for every $s \in \emptyset$, $x_0 > s$ since there are no such s-values. Thus, $U(\emptyset) = \mathbb{R}$.

15. **Problem 21:** We will construct a proof by contradiction by assuming $B(U(S))$ contains more than one value, say b_1 and b_2 and arrive at a contradiction. A WLOG, assume $b_1 < b_2$. Since $b_2 \in B(U(S))$, every open interval containing b_2 contains values in $U(S)$ and $U(S)^c$. Therefore the interval
$$\left(\frac{b_1 + b_2}{2}, b_2 + 1\right)$$
contains a value that is not in $U(S)$, say x_0. Since $x_0 \notin U(S)$, then there exist $s_0 \in S$ such that $s_0 > x_0$.

Since $b_1 \in B(U(S))$, every open interval containing b_1 contains values in $U(S)$ and $U(S)^c$. Therefore the interval

$$\left(b_1 - 1, \frac{b_1 + b_2}{2}\right)$$

contains a value that is in $U(S)$, say u_0. Since $u_0 \in U(S)$, then $u_0 \geq s$ for all $s \in S$.

We now have that

$$u_0 < \frac{b_1 + b_2}{2} < x_0 < s_0,$$

which contradicts the fact that $u_0 \in U(S)$. Therefore there cannot contain two values in $B(U(S))$. In fact, $B(U(S))$ has either one value or no values (when $U(S) = \emptyset$ or $U(S) = \mathbb{R}$). If $B(U(S))$ consists of one value, a good name for this one value is the **least upper bound**.