Group 6

Axel Abrica, Yu Fan Mei, Abby Porter
Introduction to Proof and Problem Solving

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Problem 1. For what values of $n \in \mathbb{O}^+$ is

$$\sum_{j=0}^{n} 2^j$$

prime?

Proof. The only values of $n \in \mathbb{O}^+$ for which the sum is prime is $n = 1$. We will prove this by showing that the sum is divisible by 3 for all $n \in \mathbb{O}^+$. That is, we will prove for all $n \in \mathbb{O}^+$, there exists a $k \in \mathbb{Z}$ such that

$$\sum_{j=0}^{n} 2^j = 3k.$$

Base Case: Set $n_0 = 1$. Then,

$$\sum_{j=0}^{1} 2^j = 2^0 + 2^1 = 1 + 2 = 3 = 3k_0,$$

where $k_0 = 1$.
Induction Step: We assume there exists an odd positive integer $n_0 \geq 1$ and an integer k_0 such that

$$\sum_{j=0}^{n_0} 2^j = 3k_0.$$

We want to show that there exists an integer l such that

$$\sum_{j=0}^{n_0+2} 2^j = 3l.$$

Set $l_0 = k_0 + 2^{n_0+1}$. Since integers are closed under addition and multiplication, $l_0 \in \mathbb{Z}$.

Rewriting the sum, we get

$$\begin{aligned}\sum_{j=0}^{n_0+2} 2^j &= \sum_{j=0}^{n_0} 2^j + 2^{n_0+1} + 2^{n_0+2} \\ &= 3k_0 + 2^{n_0} \cdot 2 + 2^{n_0} \cdot 2^2 \\ &= 3k_0 + 2^{n_0}(2 + 2^2) \\ &= 3k_0 + 2^{n_0}(6) \\ &= 3k_0 + 3(2^{n_0} \cdot 2) \\ &= 3(k_0 + 2^{n_0+1}) \\ &= 3l_0.\end{aligned}$$

So, for all $n_0 \in \mathbb{O}^+$, the sum is divisible by 3. However, the sum only equals 3 when $n_0 = 1$, and 3 is the only value divisible by 3 that is prime. Thus, for all other odd $n_0 > 1$, the sum is divisible by 3, but not prime. □

Problem 2. For each natural number n, the nth Fibonacci number f_n is given by

$$f_1 = 1, \;\; f_2 = 1, \;\text{ and }\;\; f_{n+2} = f_{n+1} + f_n \text{ for all } n \geq 1.$$

Let α be the positive solution and β the negative solution to the equation $x^2 = x + 1$. (The values are $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.) Show for all $n \in \mathbb{N}$ that

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

Proof. We start by checking the two base cases. Set $n_0 = 1$. Then

$$\begin{aligned}f_1 &= \frac{\alpha^1 - \beta^1}{\alpha - \beta} \\ &= \frac{\alpha - \beta}{\alpha - \beta} \\ &= 1.\end{aligned}$$

This matches $f_1 = 1$ so the base case holds for $n_0 = 1$. We now set $n_0 = 2$. Then we get

$$f_2 = \frac{\alpha^2 - \beta^2}{\alpha - \beta}.$$

We know α and β are roots of the equation $x^2 = x + 1$. Using the identity $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, we get

$$\begin{aligned}\frac{\alpha^2 - \beta^2}{\alpha - \beta} &= \frac{(\alpha + 1) - (\beta + 1)}{\alpha - \beta} \\ &= \frac{\alpha - \beta}{\alpha - \beta} \\ &= 1.\end{aligned}$$

This matches $f_2 = 1$, so the base case holds for $n_0 = 2$. We now move on to the induction step. Assume there exists an integer $n_0 \geq 1$ such that the formula holds true for n_0 and $n_0 + 1$. That is, we assume

$$f_{n_0} = \frac{\alpha^{n_0} - \beta^{n_0}}{\alpha - \beta}$$

and

$$f_{n_0+1} = \frac{\alpha^{n_0+1} - \beta^{n_0+1}}{\alpha - \beta}.$$

We need to show that the formula holds for $n_0 + 2$. That is, we need to prove

$$f_{n_0+2} = \frac{\alpha^{n_0+2} - \beta^{n_0+2}}{\alpha - \beta}.$$

Using the recurrence relation $f_{n+2} = f_{n+1} + f_n$, we have that

$$f_{n_0+2} = f_{n_0} + f_{n_0+1}.$$

From the induction step, we can substitute the expressions for f_{n_0} and f_{n_0+1} into this equation to get that

$$\begin{aligned} f_{n_0+2} &= f_{n_0} + f_{n_0+1} \\ &= \frac{\alpha^{n_0} - \beta^{n_0}}{\alpha - \beta} + \frac{\alpha^{n_0+1} - \beta^{n_0+1}}{\alpha - \beta} \\ &= \frac{\alpha^{n_0} + \alpha^{n_0+1} - (\beta^{n_0} + \beta^{n_0+1})}{\alpha - \beta}. \end{aligned}$$

We know α and β are roots of the equation $x^2 = x + 1$. Using the identity $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$, we can express α^{n_0+2} and β^{n_0+2} in terms of previous powers. We get that

$$\begin{aligned} \alpha^{n_0+2} &= \alpha^2 \cdot \alpha^{n_0} \\ &= (\alpha + 1) \cdot \alpha^{n_0} \\ &= \alpha^{n_0+1} + \alpha^{n_0}. \end{aligned}$$

Similarly, we get that

$$\beta^{n_0+2} = \beta^{n_0+1} + \beta^{n_0}.$$

Therefore,

$$\begin{aligned} \alpha^{n_0+2} - \beta^{n_0+2} &= (\alpha^{n_0+1} + \alpha^{n_0}) - (\beta^{n_0+1} + \beta^{n_0}) \\ &= \alpha^{n_0+1} - \beta^{n_0+1} + \alpha^{n_0} - \beta^{n_0}. \end{aligned}$$

We now substitute this back into our expression for f_{n_0+2} to get:

$$f_{n_0+2} = \frac{\alpha^{n_0+2} - \beta^{n_0+2}}{\alpha - \beta},$$

which is what we needed to show. Thus by induction, the formula f_n holds for all $n \in \mathbb{N}$. $\square$