Individual 4

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Introduction to Proof and Problem Solving

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Problem 12. (1) Consider the statement

$$p \equiv \{\forall M \in \mathbb{R}, \exists K \in \mathbb{R} \text{ s.t. } \forall x > K, f(x) > M\}.$$

(a) Write $\neg p$.

(b) Consider the function f from $\mathbb{R}$ into $\mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Does f satisfy p or $\neg p$? Prove your answer.

Solution. (a) The negation of p is

$$\neg p \equiv \{\exists M \in \mathbb{R} \text{ such that } \forall K \in \mathbb{R}, \exists x < K \text{ such that } f(x) \leq M\}$$

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Proof. We will prove that f satisfies $\neg p$. Set $M_0 = 1$. Let K_0 be any real number. Set $x_0 = -|K_0| - 1$. Since $|K_0| \geq 0$, then we know that $-|K_0| \leq 0$. Since $-1 < 0$, when we subtract 1 from the left hand side we get

$$\begin{aligned} -|K_0| - 1 &< 0 \\ x_0 &< 0 \end{aligned}$$

We know that $-1 < 0$. Since $x_0 < 0$, $x_0 \neq 0$ and we are allowed to divide both sides of the inequality by x_0.

$$\frac{1}{x_0} < 0.$$

Since $0 < 1$, we know that the inequality below must also be true:

$$\frac{1}{x_0} \leq 1.$$

Thus, f satisfies $\neg p$.

$\square$

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.