Individual 9

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Introduction to Proof and Problem Solving

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Problem 1. Show that

$$C((0,1]) = C((0,1))$$

by showing that the function

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n} \text{ for any } n \in \mathbb{Z}^+ \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{Z}^+ \end{cases}$$

is one-to-one from $(0,1]$ onto $(0,1)$. It might help to graph the function.

Proof. In order to prove these two sets have the same cardinality, we will prove that the function $f : (0,1] \to (0,1)$ is one-to-one and onto. We will first use a proof by contraposition to prove that f is one-to-one. Let x_1, x_2 be any two real numbers within $(0,1]$ such that $f(x_1) = f(x_2)$. Let's consider cases:

Case 1: Suppose $x_1, x_2 \neq 1/n$ for any $n \in \mathbb{Z}^+$. By definition, $f(x_1) = f(x_2)$, and it then follows that $x_1 = x_2$.

Case 2: Suppose there exist positive integers n_1, n_2 such that $x_1 = 1/n_1$ and $x_2 = 1/n_2$. Since $f(x_1) = f(x_2)$, it follows that

$$\frac{1}{n_1 + 1} = \frac{1}{n_2 + 1}.$$

This means that their reciprocals are also equivalent:

$$n_1 + 1 = n_2 + 1.$$

Subtracting 1 from both sides, we get $n_1 = n_2$. The reciprocal of both sides of this equation is $1/n_1 = 1/n_2$. From this it follows that $x_1 = x_2$.

Case 3: Without loss of generality, suppose $x_1 \neq 1/j$ for any $j \in \mathbb{Z}^+$ and there exists a positive integer n_2 such that $x_2 = 1/n_2$. Then $f(x_1) = x_1$ and $f(x_2) = 1/(n_2 + 1)$. Set $j_0 = n_2 + 1$. Since integers are closed under addition, j_0 is an integer. Then we get

$$x_1 = \frac{1}{j_0},$$

which is a contradiction. Since case 3 cannot occur and cases 1 and 2 hold true, we have proven that $f : (0,1] \to (0,1)$ is one-to-one.

We will now show that $f : (0,1] \to (0,1)$ is onto. Let y_0 be any real number such that $y_0 \in (0,1)$. We will consider cases again:

Case 1: Suppose there exists an $n_0 \in \mathbb{Z}^+$ such that $y_0 = 1/(n_0+1)$. Set $x_0 = 1/(n_0)$. Then

$$f(x_0) = \frac{1}{n+1}.$$

Case 2: Suppose there does not exist an integer $n \in \mathbb{Z}^+$ such that $y_0 = 1/n$. Set $x_0 = y_0$. Then

$$f(x_0) = x_0 = y_0.$$

Thus, we have proven that $f : (0,1] \to (0,1)$ maps $(0,1]$ onto $(0,1)$. Since we've proven that $f : (0,1] \to (0,1)$ is both one-to-one and onto, $C((0,1]) = C((0,1))$.

$\square$

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.