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## Abstract Algebra HW3

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2. Find the order of each of the following elements.

**Problem 2.1.**  $5 \in \mathbb{Z}_{12}$ .

- The order of  $5 \in \mathbb{Z}_{12}$  is  $\frac{12}{\gcd(5, 12)} = \frac{12}{1} = 12$ .

**Problem 2.2.**  $\sqrt{3} \in \mathbb{R}$ .

- We will prove that the order is infinite via contradiction. Suppose the order of  $\sqrt{3}$  is finite. This would mean there exists a positive integer  $k$  such that  $k\sqrt{3} = 0$ .
- Dividing both sides of this equality by  $\sqrt{3}$ , we get  $k = 0$ , which contradicts the statement that  $k > 0$ . This means  $\sqrt{3}$  must be infinite.

**Problem 2.3.**  $\sqrt{3} \in \mathbb{R}^*$ .

- We will prove that the order is infinite, also via contradiction. Suppose the order of  $\sqrt{3}$  in the group  $\mathbb{R}^*$  was finite. Then there exists a positive integer  $k$  such that  $\sqrt{3}^k = 1$ .
- Taking the natural logarithm of both sides, we get  $k \ln \sqrt{3} = 0$ . From this, we get  $k = 0$ , which is a contradiction.

**Problem 2.4.**  $-i \in \mathbb{C}^*$ .

- The order of  $-i$  in the group of complex numbers under multiplication is 4.
- $(-i)^2 = -1$ , and  $(-i)^3 = i$ .  $(-i)^4 = 1$ .

**Problem 2.5.**  $72 \in \mathbb{Z}_{240}$ .

- The order of  $72 \in \mathbb{Z}_{240}$  is  $\frac{240}{\gcd(240, 72)}$ .
- The gcd of 240 and 72 is 24:

$$\begin{aligned} 240 &= 72(3) + 24 \\ 72 &= 24(3) + 0. \end{aligned}$$

- So, the order of  $72 \in \mathbb{Z}_{240}$  is  $\frac{240}{24} = 10$ .

**Problem 2.6.**  $312 \in \mathbb{Z}_{471}$ .

- The order of  $312$  in  $\mathbb{Z}_{471}$  is  $\frac{471}{\gcd(471, 312)}$ .
- And the greatest common divisor of 471 and 312 is 1:

$$\begin{aligned} 471 &= 312(1) + 59 \\ 312 &= 59(5) + 17 \\ 59 &= 17(3) + 8 \\ 17 &= 8(2) + 1 \\ 8 &= 1(8) + 0. \end{aligned}$$

- This means the order of  $312 \in \mathbb{Z}_{471}$  is 471.

**3.** List all of the elements in each of the following subgroups.

**Problem 3.1.** The subgroup of  $\mathbb{Z}$  generated by 7.

- The elements in this subgroup are  $\{\dots, -14, -7, 0, 7, 14, \dots\}$ .
- This subgroup is infinite.

**Problem 3.2.** The subgroup of  $\mathbb{Z}_{24}$  generated by 15.

- The subgroup is  $\{15, 6, 21, 12, 3, 18, 9, 0\}$ .

**Problem 3.3.** All subgroups of  $\mathbb{Z}_{12}$ .

- The subgroups of  $\mathbb{Z}_{12}$  are:
  - $\langle 0 \rangle = \{0\}$
  - $\langle 1 \rangle = \{1, 2, 3, \dots, 11, 0\}$
  - $\langle 2 \rangle = \{2, 4, 6, 8, 10, 0\}$
  - $\langle 3 \rangle = \{3, 6, 9, 0\}$
  - $\langle 4 \rangle = \{4, 8, 0\}$
  - $\langle 6 \rangle = \{6, 0\}$

**Problem 5.** Find the order of every element in  $\mathbb{Z}_{18}$ .