Individual 7

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Introduction to Proof and Problem Solving

October 29, 2024

Problem 1. Use induction to prove that $(n^3 - n)(n + 2)$ is divisible by 12 for all $n \geq 1$.

Before we prove this, we will need a lemma.

Lemma 1. For every integer greater than 0, $n^3 + 3n^2 + 2n$ is divisible by 3.

Proof. We will prove this lemma via induction.

Base case: Set $n_0 = 1$. Then $n_0^3 + 3n_0^2 + 2n_0 = 6 = 3(2)$.

Inductive Step: Assume there exist integers k_0 and $n_0 \geq 1$ such that

$$n_0^3 + 3n_0^2 + 2n_0 = 3k_0.$$

We want to show that there exists an integer j such that

$$(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) = 3j.$$

Set $j = j_0 = k_0 + 3n_0^2 + 3n_0 + 2$. Then we know that

$$\begin{aligned}(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) &= n_0^3 + 3n_0^2 + 3n_0 + 1 + 3(n_0^2 + 2n_0 + 1) + 2n_0 + 2 \\ &= n_0^3 + 6n_0^2 + 11n_0 + 6.\end{aligned}$$

We can substitute in $3k_0$, and we get

$$\begin{aligned}(n_0 + 1)^3 + 3(n_0 + 1)^2 + 2(n_0 + 1) &= 3k_0 + 3n_0^2 + 9n_0 + 6 \\ &= 3(k_0 + n_0^2 + 3n_0 + 2) \\ &= 3j_0.\end{aligned}$$

Thus, we have proven that this lemma is true. □

Now, we will prove problem 1 using induction.

Proof. Base Case: Set $n = n_0 = 1$. Then

$$(n_0^3 - n_0)(n_0 + 2) = 0 = 12(0).$$

Inductive Step: Assume there exists an integer $n \geq 1$ and k_0 such that

$$(n^3 - n)(n + 2) = 12k_0.$$

We want to show that there exists an integer j such that

$$((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) = 12j.$$

We need to prove that this is true for $n_0 + 1$, meaning we want to show that there exists an integer j_0 such that

$$((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) = 12j_0.$$

Set $j = j_0 = k_0 + m$, where m is an integer. Unfactoring the left hand side, we get

$$((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) = n_0^4 + 6n_0^3 + 11n_0^2 + 6n_0.$$

When we substitute in $12k_0$, we get

$$\begin{aligned} ((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) &= 12k_0 + 4n_0^3 + 12n_0^2 + 8n_0 \\ &= 12k_0 + 4(n_0^3 + 3n_0^2 + 2n_0). \end{aligned}$$

By lemma 1, we know that $n_0^3 + 3n_0^2 + 2n_0$ is divisible by 3. We can rewrite the statement like this:

$$\begin{aligned} ((n_0 + 1)^3 - (n_0 + 1))((n_0 + 1) + 2) &= 12k_0 + 4(3m) \\ &= 12k_0 + 12m \\ &= 12(k_0 + m) \\ &= 12j_0 \end{aligned}$$

Thus, we have proven what we needed to prove.

$\square$

Problem 2. Let r represent an arbitrary real number other than 0 and 1. Show that for $n \in \mathbb{Z}_0^+$

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

This is the formula for what is called the finite geometric series. This formula is quite important in many different fields of mathematics and should be committed to memory.

Proof. We will prove this using mathematical induction.

Base Case: Set $n_0 = 0$. Then

$$\sum_{i=0}^{n_0} r^i = r^0 = 1 = \frac{1 - r^{n_0+1}}{1 - r}.$$

This statement is clearly true.

Inductive Step: Assume there exists a nonnegative integer n_0 such that for every real number r aside from 0 and 1,

$$\sum_{i=0}^{n_0} r^i = \frac{1 - r^{n_0+1}}{1 - r}.$$

We want to show that this is true for $n_0 + 1$, so we need to show that

$$\sum_{i=0}^{n_0} r^i + r^{n_0+1} = \sum_{i=0}^{n_0+1} r^i.$$

By our inductive hypothesis, we can rewrite the left side of the equation as

$$\frac{1 - r^{n_0+1}}{1 - r} + r^{n_0+1} = \sum_{i=0}^{n_0+1} r^i.$$

We can multiply r^{n_0+1} by $(1 - r)/(1 - r)$ to get a common denominator:

$$\begin{aligned}
\sum_{i=0}^{n_0+1} r^i &= \frac{1 - r^{n_0+1} + r^{n_0+1}(1 - r)}{1 - r} \\
&= \frac{1 - r^{n_0+1} + r^{n_0+1} - r^{n_0+2}}{1 - r} \\
&= \frac{1 - r^{n_0+2}}{1 - r}.
\end{aligned}$$

Thus, we have proven this is true.

□

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.