Individual 9

Yu Fan Mei
Introduction to Proof and Problem Solving

December 6, 2024

Problem 1. Show that
$$C((0,1]) = C((0,1))$$
by showing that the function
$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n} \text{ for any } n \in \mathbb{Z}^+ \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{Z}^+ \end{cases}$$
is one-to-one from $(0,1]$ onto $(0,1)$. It might help to graph the function.

Proof. In order to prove these two sets have the same cardinality, we will prove that the function $f : (0,1] \to (0,1)$ is one-to-one and onto. We will first use a proof by contraposition to prove that f is one-to-one. Let x_1, x_2 be any two real numbers within $(0,1]$ such that $f(x_1) = f(x_2)$. Let's consider cases:

Case 1: Suppose $x_1, x_2 \neq 1/n$ for any $n \in \mathbb{Z}$. By definition, $f(x_1) = f(x_2)$, and it then follows that $x_1 = x_2$.

Case 2: Suppose there exists positive integers n_1, n_2 such that $x_1 = 1/n_1$ and $x_2 = 1/n_2$. Since $f(x_1) = f(x_2)$, it follows that
$$\frac{1}{n_1+1} = \frac{1}{n_2+1}.$$

This means that their reciprocals are also equivalent:
$$n_1 + 1 = n_2 + 1.$$

Subtracting 1 from both sides, we get $n_1 = n_2$. The reciprocal of both sides of this equation is $1/n_1 = 1/n_2$. From this it follows that $x_1 = x_2$.

Case 3: Without loss of generality, suppose $x_1 \neq 1/j$ for any $j \in \mathbb{Z}$ and there exists a positive integer n_2 such that $x_2 = 1/n_2$. Then $f(x_1) = x_1$ and $f(x_2) = 1/(n_2+1)$. Set $j_0 = n_2$

□

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.