Group 4

Helen Freedman, Pruthvi Jasty, Yu Fan Mei
Introduction to Proof and Problem Solving

October 2, 2024

Problem 4a.
$$P_1(f) \equiv \{\forall M \in \mathbb{R}, \exists x \in \mathbb{R} \text{ such that } f(x) > M\}$$

$$P_6(f) \equiv \left\{\exists (M, K) \in \mathbb{R}^2, \text{ such that } \forall x > K, f(x) > M\right\}$$

Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_1\}.$$

Example 1. It is fairly obvious that P_1 means the function is unbounded from above. To better understand the problem, we'll try to find functions that satisfy P_6.

Let $f_0(x) = 2$. Set $M = 1$ and $K = 1$. Let x be any real number larger than K. We can observe that $f_0(x) = 2 > M$ for all x in this case.

Let $f_1(x) = x$. Set $M = 0$ and $K = 1$. Let x be any real number larger than K. We can see that $K > M$. We know that $f_1(x) = x > K$, so $f_1(x) > M$ holds for positive linear and other increasing functions.

Proof. We will disprove the statement. The negation of the statement is as follows:

$$\{\exists f \text{ that satisfies } P_6 \text{ and satisfies } \neg P_1\}$$

The negation of P_1 is:

$$\neg P_1 \equiv \{\exists M \in \mathbb{R} \text{ such that } \forall x \in \mathbb{R}, f(x) \leq M\}$$

Set $f_0 = \sin(x)$. Set $(M_0, K_0) = (-2, 0)$. Let x_0 be any real number greater than K. We know that $-1 \leq \sin(n) \leq 1$ for any real number n, so $f_0(x_0) = \sin(x_0) \geq -1$.

We know that $-1 > -2$, so $f_0(x) > -2 = M$. Thus, $f_0(x)$ satisfies P_6.

Set $(M_0, K_0) = (2, 0)$. Let x_0 be any real number. Since $-1 \leq \sin(n) \leq 1$ for any real number n, so $f_0(x_0) = \sin(x_0) \leq 1$.

We know that $2 > 1$, so $f_0(x) < 2 = M$. Thus, $f_0(x)$ satisfies $\neg P_1$.

□

While working on this proof, we received no external assistance aside from advice from Professor Mehmetaj.