

# Individual 9

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Introduction to Proof and Problem Solving

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**Problem 1.** Show that

$$C((0, 1]) = C((0, 1))$$

by showing that the function

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n} \text{ for any } n \in \mathbb{Z}^+ \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{Z}^+ \end{cases}$$

is one-to-one from  $(0, 1]$  onto  $(0, 1)$ . It might help to graph the function.

*Proof.* In order to prove these two sets have the same cardinality, we will prove that the function  $f : (0, 1] \rightarrow (0, 1)$  is one-to-one and onto. We will first use a proof by contraposition to prove that  $f$  is one-to-one. Let  $x_1, x_2$  be any two real numbers within  $(0, 1]$  such that  $f(x_1) = f(x_2)$ . Let's consider cases:

Case 1: Suppose  $x_1, x_2 \neq 1/n$  for any  $n \in \mathbb{Z}^+$ . By definition,  $f(x_1) = x_1$  and  $f(x_2) = x_2$ , and it then follows that  $x_1 = x_2$ .

Case 2: Suppose there exists positive integers  $n_1, n_2$  such that  $x_1 = 1/n_1$  and  $x_2 = 1/n_2$ . Since  $f(x_1) = f(x_2)$ , it follows that

$$\frac{1}{n_1 + 1} = \frac{1}{n_2 + 1}.$$

This means that their reciprocals are also equivalent:

$$n_1 + 1 = n_2 + 1.$$

Subtracting 1 from both sides, we get  $n_1 = n_2$ . The reciprocal of both sides of this equation is  $1/n_1 = 1/n_2$ . From this it follows that  $x_1 = x_2$ .

Case 3: Without loss of generality, suppose  $x_1 \neq 1/j$  for any  $j \in \mathbb{Z}^+$  and there exists a positive integer  $n_2$  such that  $x_2 = 1/n_2$ . Then  $f(x_1) = x_1$  and  $f(x_2) = 1/(n_2 + 1)$ . Set  $j_0 = n_2 + 1$ . Since integers are closed under addition,  $j_0$  is an integer. Then we get

$$x_1 = \frac{1}{j_0},$$

which is a contradiction. Since case 3 cannot occur and cases 1 and 2 hold true, we have proven that  $f : (0, 1] \rightarrow (0, 1)$  is one-to-one.

We will now show that  $f : (0, 1] \rightarrow (0, 1)$  is onto. Let  $y_0$  be any real number such that  $y_0 \in (0, 1)$ . We will consider cases again:

Case 1: there exists an  $n \in \mathbb{Z}$  such that  $y_0 = 1/n_0$ .

□

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.