Individual 7

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Introduction to Proof and Problem Solving

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Lemma 1. For every integer $n > 0$, $n(n+1)$ is even. We will prove this via induction.

Base Case: Set $n = n_0 = 1$. Then $n_0(n_0 + 1) = 2$, which is clearly true.

Inductive Step: Assume there exists integers $n_0 > 0$ and k_0 such that

$$n_0(n_0 + 1) = 2k_0.$$

We want to show that there exists an integer m_0 such that

$$(n_0 + 1)(n_0 + 2) = 2m_0.$$

Set $m_0 = (k_0 + n_0 + 1)$. Since k_0 is an integer and integers are closed under addition, j_0 is also an integer. Unfactoring the left side of the statement above, we get $(n_0 + 1)(n_0 + 2) = n_0^2 + 3n_0 + 2$. When we substitute $2k + 1$ into this, we're left with $2k_0 + 2n_0 + 2$. When we rewrite this expression with the 2 factored, the equation becomes

$$2(k_0 + n_0 + 1).$$

We can rewrite this as $2m_0$, which is what we needed to show.

Problem 1. Use induction to prove that $(n^3 - n)(n + 2)$ is divisible by 12 for all $n \geq 1$.

Proof. We will prove problem 1 using induction.

Base Case: Set $n = n_0 = 1$. Then

$$(n^3 - n)(n + 2) = 0.$$

This is clearly true, since 0 can be written as 12(0), and is thus divisible by 12.

Inductive Step: Assume there exists integers $n \geq 1$ and k_0 such that

$$(n^3 - n)(n + 2) = 12k_0.$$

We want to show that this is true for $n+1$, meaning we want to show that there exists an integer j_0 such that

$$((n+1)^3 - (n+1))((n+1)+2) = 12j_0.$$

Set $j_0 =$ (something). When we unfactor the equation above, we get

$$n_0^4 + 6n_0^3 + 11n_0^2 + 6n_0 = 12j_0.$$

When we substitute in $12k_0$, we get

$$12k_0 + 4n_0^3 + 12n_0^2 + 8n_0 = 12j_0.$$

We can rewrite the statement like this:

$$12k_0 + 4n_0^3 + 4n_0^2 + 8n_0^2 + 8n_0 = 12j_0.$$

Utilizing grouping, we can factor out $4n_0(n_0+1)$, and are left with

$$12k_0 + 4n_0(n_0+1)(n_0+2) = 12j_0.$$

By lemma 1, $n_0(n_0+1)$ is even, and can be rewritten as $2m_0$, with m_0 being an integer. When we substitute this in, we get

$$12k_0 + 8m_0(n_0+2) = 12j_0.$$

□

Problem 2. Let r represent an arbitrary real number other than 0 and 1. Show that for $n \in \mathbb{Z}_0^+$

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

This is the formula for what is called the finite geometric series. This formula is quite important in many different fields of mathematics and should be committed to memory.

Proof. We will prove this using mathematical induction.

Base Case: Set $n_0 = 0$. Then

$$\sum_{i=0}^{n_0} r^i = r^0 = 1 = \frac{1 - r^{n_0+1}}{1 - r}.$$

This statement is clearly true.

Inductive Step: Assume there exists a nonnegative integer n_0 such that for every real number r aside from 0 and 1,

$$\sum_{i=0}^{n_0} r^i = \frac{1 - r^{n_0+1}}{1 - r}.$$

We want to show that this is true for $n + 1$, so we need to show that

$$\sum_{i=0}^{n_0} r^i + r^{n_0+1} = \sum_{i=0}^{n_0+1} r^i.$$

□

While working on this proof, I received no external assistance aside from advice from Professor Mehmetaj.