Group 4

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Introduction to Proof and Problem Solving

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Problem 4. Consider the following statements

$$P_1(f) \equiv \{\forall M \in \mathbb{R}, \exists x \in \mathbb{R} \text{ such that } f(x) > M\}$$

$$P_6(f) \equiv \left\{\exists (M, K) \in \mathbb{R}^2 \text{ such that } \forall x > K, f(x) > M\right\}.$$

(a) Prove or disprove

$$\{\forall f \text{ satisfying } P_1, f \text{ satisfies } P_6\}. \tag{1}$$

(b) Prove or disprove

$$\{\forall f \text{ satisfying } P_6, f \text{ satisfies } P_1\}. \tag{2}$$

Proof. (a) We will disprove the statement that for every function f satisfying P_1, f also satisfies P_6 by proving that its negation is true. The negation is:

$$\{\exists f \text{ satisfying } P_1 \text{ such that } f \text{ satisfies } \neg P_6\}.$$

Set $f_0 = -x_0$. We will first show that f_0 satisfies P_1. Let M_0 be any real number. Set $x_0 = -M_0 - 1$. Then,

$$f_0(x_0) = -(-M_0 - 1).$$

Distributing the negative, we get

$$f_0(x_0) = M_0 + 1.$$

We know that $1 > 0$, so adding M_0 to both sides, we get

$$M_0 + 1 > M_0.$$

Then,

$$f_0(x_0) > M_0.$$

Thus, f_0 satisfies P_1. Now we will prove f_0 satisfies $\neg P_6$. The negation of P_6 is

$$\neg P_6 \equiv \{\forall (M, K) \in \mathbb{R}^2, \exists x > K \text{ such that } f(x) \leq M\}.$$

Let (M_0, K_0) be any ordered pair in $\mathbb{R}^2$. Set $x_0 = |K_0| + |M_0| + 1$ Since $|M_0|$ is always nonnegative, we know that

$$|M_0| + 1 > 0.$$

Then $x_0 > K_0$. Plugging x_0 into f_0, we get

$$f_0(x_0) = -(|K_0| + |M_0| + 1).$$

Distributing the negative, we get

$$f_0(x_0) = -|K_0| - |M_0| - 1.$$

We know $|M_0| \geq M_0$. Multiplying both sides by -1, we get

$$-|M_0| \leq M_0.$$

Since we know $1 > 0$, the inequality still holds true when we subtract 1 from the left hand side of the inequality. Since $|K_0| \geq 0$, we can apply the same principle here and subtract $|K_0|$ from the left hand side. Then, we get

$$-|K_0| - |M_0| - 1 \leq M_0.$$

Since we know $f_0(x_0) = -|K_0| - |M_0| - 1$, we have

$$f_0(x_0) \leq M_0.$$

Thus, we have proven that there exists a f_0 that satisfies $\neg P_6$ and P_1, proving the negation of (1) to be true.

$\square$

Proof. (b) We will disprove the statement that for every function f satisfying P_6, f satisfies P_1 by proving that the negation is true. The negation of this statement is

$$\{\exists f \text{ satisfying } P_6 \text{ such that } f \text{ satisfies } \neg P_1\}.$$

Set $f_0(x) = 3$. We will first show that f_0 satisfies P_6. Set $(M_0, K_0) = (2, 0)$. Let x_0 be any real number greater than K_0. We know that $x_0 > 0$. Then,

$$f_0(x_0) = 3 > 2.$$

Next, we will show that f_0 satisfies $\neg P_1$. Set $M_0 = 4$. Let x_0 be any real number. Then,

$$f_0(x_0) = 3 \leq 4.$$

Thus, f_0 satisfies $\neg P_1$. Since f_0 satisfies both P_6 and $\neg P_1$, we disproved (2) by proving that the negation of (2) is true.

$\square$

Problem 10. A function f with domain $\mathbb{R}$ mapping into $\mathbb{R}$ is increasing if whenever $a < b$, then $f(a) < f(b)$. In the following, f and g are assumed to have domain $D = \mathbb{R}$.

(d) Prove or disprove that if f and g are increasing functions into $\mathbb{R}$, then the composition of f and g, $h = f \circ g$ is an increasing function into $\mathbb{R}$.

(e) Prove or disprove that if f and g are increasing functions into $\mathbb{R}$, then $h = fg$ is an increasing function into $\mathbb{R}$.

Proof. (d) We will prove this statement is true. Let g_0 be any increasing function into $\mathbb{R}$. By definition, for all $a_0 \in \mathbb{R}$ and $b_0 \in \mathbb{R}$ where $a_0 < b_0$, $g_0(a_0) < g_0(b_0)$. Let f_0 be any increasing function into $\mathbb{R}$. By definition, for all c_0 and $d_0 \in \mathbb{R}$, where $c_0 < d_0$, $f_0(c_0) < f_0(d_0)$. Set $m_0 = g_0(a_0)$ and set $n_0 = g_0(b_0)$. By definition, $m_0 < n_0$. Then, knowing that f_0 is an increasing function,

$$f_0(m_0) < f_0(n_0).$$

Plugging $g_0(a_0)$ and $g_0(b_0)$ back into the equation we get that,

$$f_0(g_0(a_0)) < f_0(g_0(b_0))$$

when $g_0(a_0) < g_0(b_0)$. As this is the definition of an increasing function into $\mathbb{R}$, we know that $h_0 = f_0 \circ g_0$ must be an increasing function into $\mathbb{R}$.

□

Proof. (e) We will disprove that if f and g are increasing functions into $\mathbb{R}$, then $h = fg$ is an increasing function into $\mathbb{R}$. Set $f_0(x_0) = x_0$ and set $g_0(x_0) = x_0$. First we will prove that these two functions are increasing functions by showing that for every real number a and b where $b > a$, $f_0(b) > f_0(a)$ and $g_0(b) > g_0(a)$.

Let a_0 be any real number. Let b_0 be any real number greater than a_0. Plugging a_0 into f_0, we get $f_0(a_0) = a_0$. Plugging b_0 into f_0, we get $f_0(b_0) = b_0$. But we know that

$$b_0 > a_0$$
$$f_0(b_0) > f_0(a_0).$$

Thus, f_0 is an increasing function. Plugging a_0 into g_0, we get $g_0(a_0) = a_0$. Plugging b_0 into g_0, we get $g_0(b_0) = b_0$. We know that

$$b_0 > a_0$$
$$g_0(b_0) > g_0(a_0).$$

Thus, g_0 is also an increasing function. Set $h_0(x) = f_0(x)g_0(x)$. Then $h_0(x) = x^2$. Suppose $h_0(x)$ is an increasing function. That would mean for every real number a and b where $b > a$, $h_0(b) > h_0(a)$. This statement should then hold true for $a_0 = -2$ and $b_0 = -1$, which is also greater than a_0. But substitution gives us $h_0(a_0) = 4$ and $h_0(b_0) = 1$. Since 1 is not greater than 4, we have proven that the original statement is false.

□

While working on this assignment, we did not receive any outside help except from Professor Mehmetaj.