

# MATH0024 PDEs Homework 2

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## 1. Fourier Transform Verification of the Fundamental Solution for the 3D Wave Equation

The goal is to prove that  $\langle \Phi_t, \hat{\varphi} \rangle = \langle \hat{\Phi}_t, \varphi \rangle$  for all Schwartz functions  $\varphi$  on  $\mathbb{R}^3$

By definition

$$\begin{aligned}\langle \Phi_t, \hat{\varphi} \rangle &= \left\langle \Phi_t, \int_{\mathbb{R}^3} \exp(-ix \cdot \xi) \varphi(\xi) d\xi \right\rangle \\ &= \frac{t}{4\pi} \int_{\|y\|=1} \int_{\mathbb{R}^3} \exp(-icty \cdot \xi) \varphi(\xi) d\xi dS_y \\ &= \int_{\mathbb{R}^3} \frac{t}{4\pi} \int_{\|y\|=1} \exp(-icty \cdot \xi) dS_y \varphi(\xi) d\xi\end{aligned}\tag{1}$$

$$\langle \Phi_t, \hat{\varphi} \rangle = \int_{\mathbb{R}^3} \frac{t}{4\pi} \int_{\|y\|=1} \exp(-icty_3 \|\xi\|) dS_y \varphi(\xi) d\xi\tag{2}$$

By replacing it with spherical coordinates the inner integral is

$$\begin{aligned}\int_{\|y\|=1} \exp(-icty_3 \|\xi\|) dS_y &= \int_0^{2\pi} \int_0^\pi \exp(-ict \cos(\theta) \|\xi\|) \sin(\theta) d\theta d\chi \\ &= 2\pi \int_0^\pi \exp(-ict \cos(\theta) \|\xi\|) \sin(\theta) d\theta\end{aligned}\tag{3}$$

By performing a small variable change  $p = \cos(\theta)$  and  $dp = -\sin(\theta) d\theta$

$$\begin{aligned}2\pi \int_{-1}^1 \exp(-ictp \|\xi\|) dp &= 2\pi \left[ \frac{\exp(-ictp \|\xi\|)}{-ict \|\xi\|} \right]_{-1}^1 \\ &= \frac{2\pi}{-ict \|\xi\|} (e^{-ict \|\xi\|} - e^{ict \|\xi\|}) \\ &= \frac{4\pi \sin(ct \|\xi\|)}{ct \|\xi\|}\end{aligned}\tag{4}$$

In summary

$$\begin{aligned}\langle \Phi_t, \hat{\varphi} \rangle &= \int_{\mathbb{R}^3} \frac{t}{4\pi} \cdot \frac{4\pi \sin(ct \|\xi\|)}{ct \|\xi\|} \varphi(\xi) d\xi \\ &= \int_{\mathbb{R}^3} \frac{\sin(ct \|\xi\|)}{c \|\xi\|} \varphi(\xi) d\xi = \langle \hat{\Phi}_t, \varphi \rangle\end{aligned}\tag{5}$$

## 2. Absolute Stability Analysis of the Forward Euler Method

This question will consider the forward euler method for the initial-value problem

$$\begin{cases} \frac{du}{dt}(t) = \lambda u(t) & \text{for } t > 0, \\ u(0) = 1 & \text{at } t = 0 \end{cases} \quad \lambda \in \mathbb{C} \quad (6)$$

The exact solution is given by

$$u(t) = e^{\lambda t} \quad (7)$$

The forward euler method is given by

$$u_{n+1} = (1 + \Delta t \cdot \lambda)u_n \quad (8)$$

**A time marching method is absolutely stable** for a specific timestep  $\Delta t$  if its application to this particular IVP leads for this timestep to a numerical solution with the same asymptotic behaviour.

$$u_n \rightarrow 0 \text{ as } n \rightarrow +\infty \text{ when } \operatorname{Re}(\lambda) < 0 \quad (9)$$

For the forward Euler method with initial condition,

$$|u_n| = |1 + \Delta t \cdot \lambda|^n \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (10)$$

This requires

$$|1 + \Delta t \cdot \lambda| < 1 \quad (11)$$

The region of absolute stability ( ie where the condition is respected) is a disk in the complex plane centered in  $(-1,0)$  with a radius of 1.

In the case of  $\lambda = -5$ , the absolute stability inequation gives us:

$$-1 \leq 1 - 5\Delta t \leq 1 \quad (12)$$

Where

- The upperbound is trivial for a positive  $\Delta t$
- the lower bound gives  $\Delta t \leq 0.4$

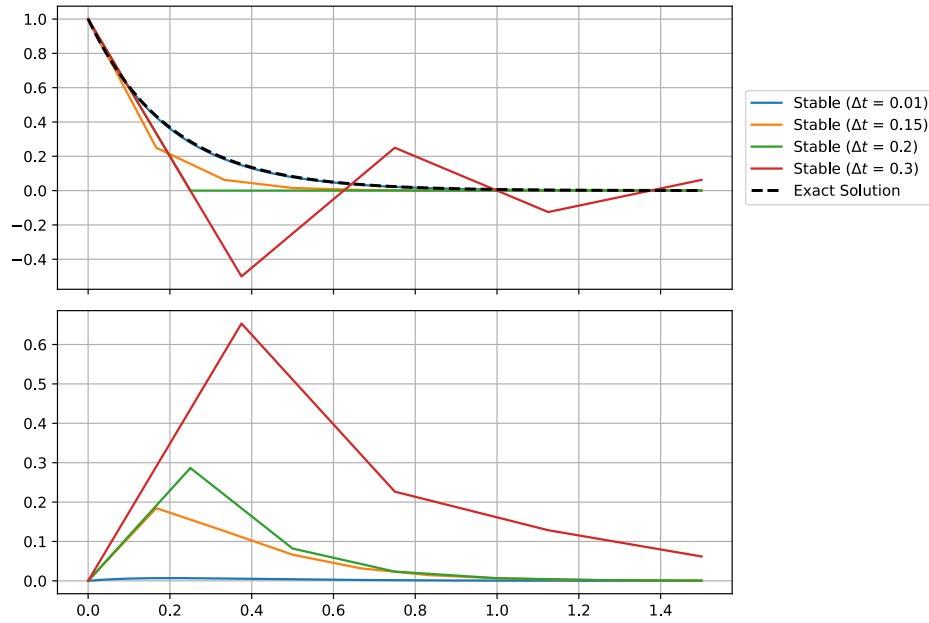


Figure 1: Representation of the forward Euler method and the representation of the absolute error associated to the method for stable  $\Delta t$

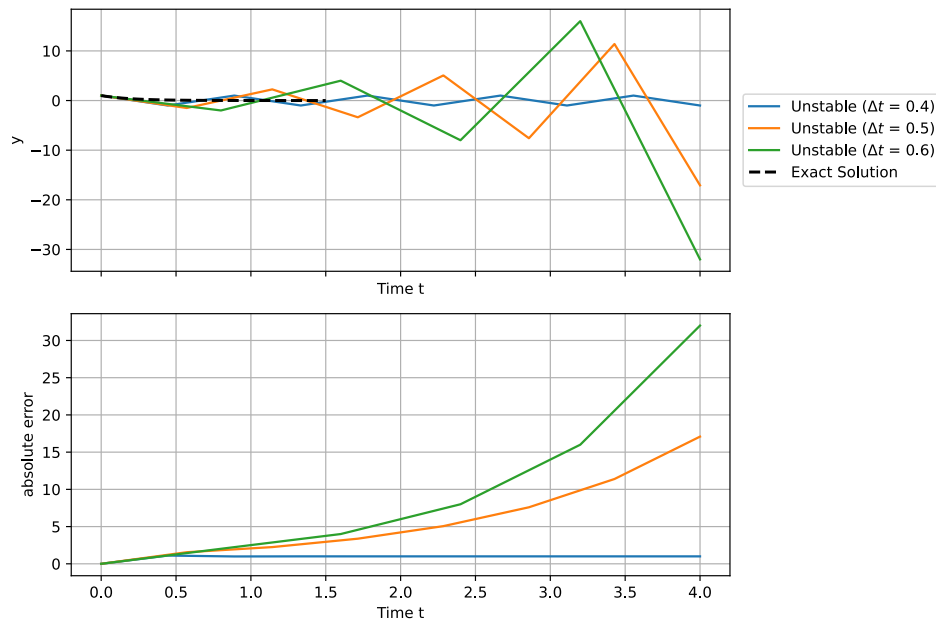


Figure 2: Representation of the forward Euler method and the representation of the absolute error associated to the method for unstable and stable marginally  $\Delta t$

The term  $|1 + \Delta t \cdot \lambda|$  can be considered as the amplification factor. In the case of small value, the error will decay. In the particular case of  $\Delta t = 0.4$ , the shema is marginally stable, the oscillation does not even growth or decay. After this limit value the amplification increase the amplitude of the oscillations. The absolute error explodes.

### 3. Spectral Analysis and Wave Propagation in Rectangular Waveguides

#### 3.1. Spectral solution

The spectral problem equation is

$$-\frac{\partial^2 \phi}{\partial x^2} = \lambda \phi \quad (13)$$

Three cases must be discussed based on the sign of  $\lambda$

- Case  $\lambda < 0$

Let  $\lambda = -\alpha^2$ , with  $\alpha > 0$

In this particular case the general solution is given by:

$$\varphi(x) = A \cosh(\alpha x) + B \sinh(\alpha x) \quad (14)$$

By applying boundary

$$\begin{cases} \varphi(-\frac{L}{2}) = A \cosh(-\alpha \frac{L}{2}) + B \sinh(-\alpha \frac{L}{2}) = 0 \\ \varphi(\frac{L}{2}) = A \cosh(\alpha \frac{L}{2}) + B \sinh(\alpha \frac{L}{2}) = 0 \end{cases} \quad (15)$$

Knowing that  $\sinh$  is a odd function and that  $\cosh$  is even:

$$\begin{cases} \varphi(-\frac{L}{2}) = A \cosh(\alpha \frac{L}{2}) - B \sinh(\alpha \frac{L}{2}) = 0 \\ \varphi(\frac{L}{2}) = A \cosh(\alpha \frac{L}{2}) + B \sinh(\alpha \frac{L}{2}) = 0 \end{cases} \quad (16)$$

By rearranging, this gives

$$\begin{cases} 2A \cosh(\alpha \frac{L}{2}) = 0 \\ 2B \sinh(\alpha \frac{L}{2}) = 0 \end{cases} \quad (17)$$

That concludes that  $A = B = 0$  and in extenso  $\varphi(x) = 0$

- Case  $\lambda = 0$

In this case the general solution is

$$\varphi(x) = Ax + b \quad (18)$$

By applying boundary condition;

$$\begin{cases} \varphi(-\frac{L}{2}) = -A\frac{L}{2} + B = 0 \\ \varphi(\frac{L}{2}) = A\frac{L}{2} + B = 0 \end{cases} \quad (19)$$

This situation also conclude that  $A = B = 0$  and then that  $\varphi(x) = 0$

- Case  $\lambda > 0$

In this last case, the general solution is given by

$$\phi(x) = A \sin(\sqrt{\lambda}x) + B \cos(\sqrt{\lambda}x) \quad (20)$$

With the help of the boundary condition

$$\begin{cases} \phi(-\frac{L}{2}) = 0 \\ \phi(\frac{L}{2}) = 0 \end{cases} \quad (21)$$

This show that  $B = 0$  and  $\lambda_m = \left(\frac{2m\pi^2}{L}\right)$  with  $m \in \mathbb{N}_0$

The eigenfunction is then

$$\phi_m(x) = \sin\left(\frac{2m\pi x}{L}\right) \quad (22)$$

By normalizing

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin^2\left(\frac{2m\pi x}{L}\right) dx = \frac{L}{2} \quad (23)$$

The solution is then given by

$$\phi_m(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2m\pi x}{L}\right) \quad (24)$$

With the same analysis the y-direction problem gives with

$$\kappa_n = \frac{2n\pi^2}{H} \quad (25)$$

The normalized form is given by

$$\psi_n = \sqrt{\frac{2}{H}} \sin\left(\frac{2n\pi}{H}\right) \quad (26)$$

### 3.2. Derivation in the sense of the theory of distribution

For all test function  $\varphi \in \mathbb{R}$

$$\begin{aligned} \left\langle \frac{d}{dz} G_{mn}, \varphi \right\rangle &= - \left\langle G_{mn}, \frac{d}{dz} \varphi \right\rangle \\ &= - \int_{-\infty}^{+\infty} G_{mn}(z) \frac{d}{dz} \varphi dz = - \int_{-\infty}^0 B_{mn} e^{-ik_{f,mn}z} \frac{d}{dz} \varphi dz \\ &\quad - \int_0^{+\infty} B_{mn} e^{-ik_{f,mn}z} \frac{d}{dz} \varphi dz \end{aligned} \quad (27)$$

The integration by part returns

$$\begin{aligned} &- [B_{mn} e^{-ik_{f,mn}z}]_{-\infty}^0 + \int_{-\infty}^0 B_{mn} (-ik_{f,mn}) e^{-ik_{f,mn}z} \varphi dz \\ &- [B_{mn} e^{ik_{f,mn}z}]_0^{+\infty} + \int_0^{+\infty} B_{mn} (ik_{f,mn}) e^{ik_{f,mn}z} \varphi dz \end{aligned} \quad (28)$$

Since  $\varphi$  has the properties  $\lim_{x \rightarrow \pm\infty} \varphi(x) = 0$

$$\begin{aligned}
& -B_{mn}\varphi(0) + B_{mn}\varphi(0) + \int_{-\infty}^0 B_{mn}(-ik_{f,mn})e^{-ik_{f,mn}z}\varphi dz \\
& + \int_0^{+\infty} B_{mn}(ik_{f,mn})e^{ik_{f,mn}z}\varphi dz
\end{aligned} \tag{29}$$

The first order derivative is then

$$\left\langle \frac{d}{dz}G_{mn}, \varphi \right\rangle = \int_{-\infty}^0 B_{mn}(-ik_{f,mn})e^{-ik_{f,mn}z}\varphi dz + \int_0^{+\infty} B_{mn}(ik_{f,mn})e^{ik_{f,mn}z}\varphi dz \tag{30}$$

The second order derivative can be obtain in the same way

$$\begin{aligned}
\left\langle \frac{d^2}{dz^2}G_{mn}, \varphi \right\rangle &= \left\langle G_{mn}, \frac{d^2}{dz^2}\varphi \right\rangle \\
&= \int_{-\infty}^{+\infty} G_{mn}(z) \frac{d^2}{dz^2}\varphi dz \\
&= \int_{-\infty}^0 B_{mn}e^{-ik_{f,mn}z} \frac{d^2}{dz^2}\varphi dz + \int_0^{+\infty} B_{mn}e^{ik_{f,mn}z} \frac{d^2}{dz^2}\varphi dz
\end{aligned} \tag{31}$$

By decomposing the left integral (negative domain)

$$\begin{aligned}
\int_{-\infty}^0 B_{mn}e^{-ik_{f,mn}z} \frac{d^2}{dz^2}\varphi dz &= \left[ B_{mn}e^{-ik_{f,mn}z} \frac{d}{dz}\varphi \right]_{-\infty}^0 \\
&\quad - \int_{-\infty}^0 (-ik_{f,nm}B_{mn}e^{-ik_{f,mn}z}) \frac{d}{dz}\varphi dz \\
&= B_{mn} \frac{d}{dz}\varphi(0) + ik_{f,mn}B_{mn}\varphi(0) + \int_{-\infty}^0 (-k_{f,mn}^2)B_{mn}e^{-ik_{f,nm}z}\varphi dz
\end{aligned} \tag{32}$$

By doing the same thing developpement for the right integral (positive domain)

$$\begin{aligned}
\int_0^{+\infty} B_{mn}e^{ik_{f,mn}z} \frac{d^2}{dz^2}\varphi dz &= -B_{mn} \frac{d}{dz}\varphi(0) + ik_{f,mn}B_{mn}\varphi(0) \\
&\quad + \int_0^{+\infty} (-k_{f,mn}^2)B_{mn}e^{ik_{f,mn}z}\varphi dz
\end{aligned} \tag{33}$$

By adding both contribution

The second order derivative is then

$$\left\langle \frac{d^2}{dz^2}G_{mn}, \varphi \right\rangle = -k_{f,mn}^2 \langle G_{mn}, \varphi \rangle + 2ik_{f,mn}B_{mn}\varphi(0) \tag{34}$$

### 3.3. Evaluation of $B_{mn}$

By substituting in the equation

$$-c^2 \left[ -k_{f,nm}^2 G_{mn} + 2ik_{f,nm}B_{mn}\delta(z) \right] + \left( -\omega_f^2 + c^2(\lambda_m + \kappa_n) \right) G_{mn} = \phi_m(0)\psi_n(0)\delta(z) \tag{35}$$

Knowing that  $k_{f,mn}^2 = \frac{\omega_f^2}{c^2} - (\lambda_m + \kappa_n)$

$$-2ic^2 k_{\{f,mn\}} B_{\{mn\}} \delta(z) = \phi_m(0) \psi_n(0) \delta(z) \quad (36)$$

The value found is:

$$B_{mn} = i \frac{\phi_m(0) \psi_n(0)}{2c^2 k_{f,mn}} \quad (37)$$

## 4. Von Neuman Stability Analysis of the Upwind Finite Difference Scheme

The upwind method is defined as

$$\begin{cases} u_j^{n+1} = u_j^n - c \frac{\Delta t}{h} (u_j^n - u_{j-1}^n) \\ u_j^0 = u_0(x_j) \end{cases} \quad (38)$$

Where  $\nu = \frac{c \Delta t}{h}$

By assuming a solution in the form:

$$u_j = \gamma^n e^{i\xi j h} \quad (39)$$

The schema can be rewritten as

$$\gamma^{n+1} e^{i\xi j h} = \gamma^n e^{i\xi j h} - \nu \gamma^n (e^{i\xi j h} - e^{i\xi (j-1) h}) \quad (40)$$

$$\gamma = 1 - \nu(1 - e^{-i\xi h}) \quad (41)$$

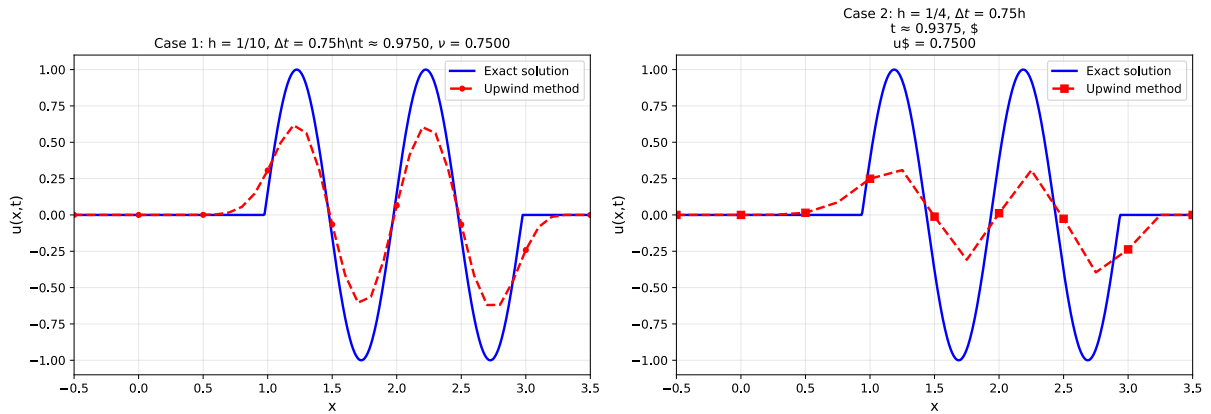


Figure 3: Comparison of exact and numerical solution for the transport equation at  $t \approx 1$  using the upwind finite difference method.

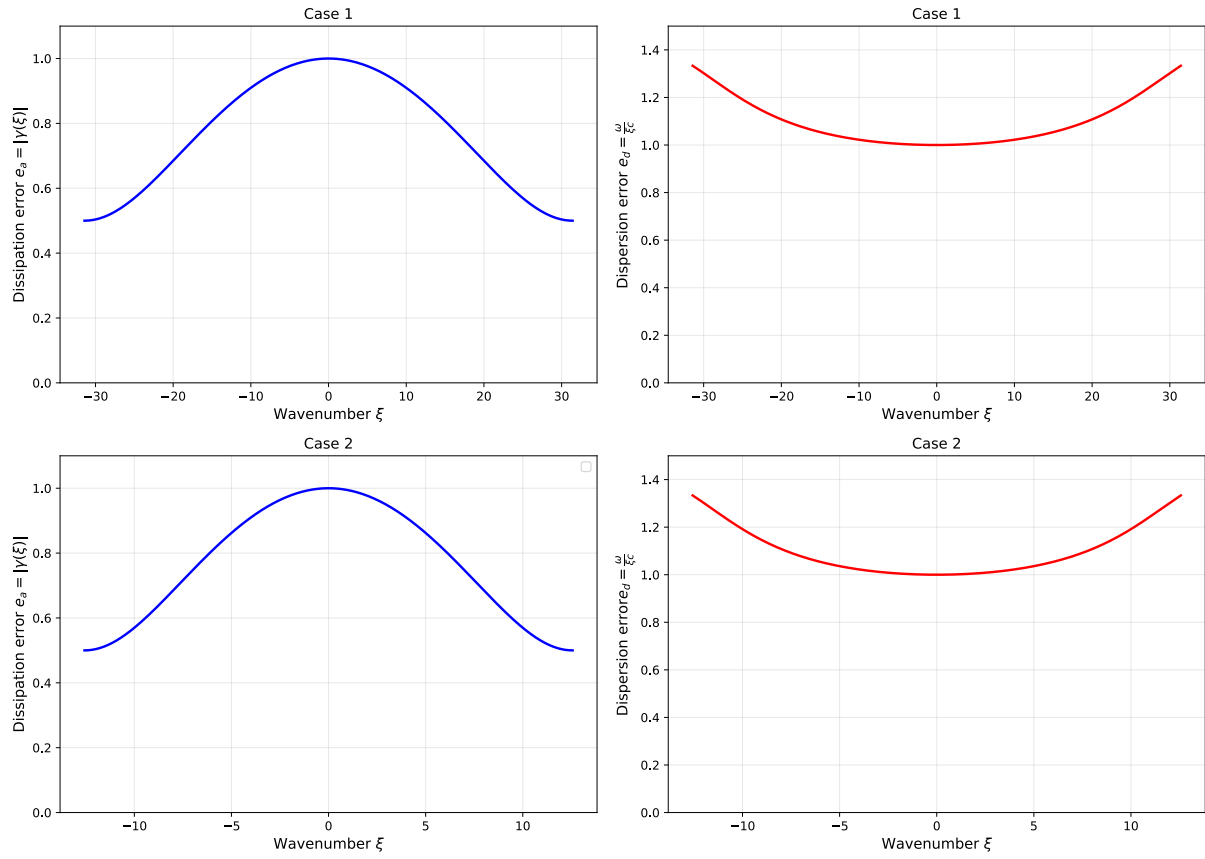


Figure 4: Von Neumann analysis of the upwind method, with dissipation  $e_a$  and dispersion  $e_d$  errors.

#### 4.1. Use of AI

AI was used to summarise the instructions for question 3.