MATH0024 PDEs Homework 2

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1. Fourier Transform Verification of the Fundamental Solution for the 3D Wave Equation

The goal is to prove that $\langle \Phi_t, \hat{\varphi} \rangle = \left\langle \hat{\Phi}_t, \varphi \right\rangle$ for all Schwartz functions φ on $\mathbb{R}^3$
By definition

$$\begin{aligned} \langle \Phi_t, \hat{\varphi} \rangle &= \left\langle \Phi_t, \int_{\mathbb{R}^3} \exp(-i\boldsymbol{x} \cdot \boldsymbol{\xi}) \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} \right\rangle \\ &= \frac{t}{4\pi} \int_{\|y\|=1} \int_{\mathbb{R}^3} \exp(-ict\boldsymbol{y} \cdot \boldsymbol{\xi}) \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} dS_y \\ &= \int_{\mathbb{R}^3} \frac{t}{4\pi} \int_{\|y\|=1} \exp(-ict\boldsymbol{y} \cdot \boldsymbol{\xi}) dS_y \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} \end{aligned} \tag{1}$$

$$\langle \Phi_t, \hat{\varphi} \rangle = \int_{\mathbb{R}^3} \frac{t}{4\pi} \int_{\|y\|=1} \exp(-icty_3 \|\boldsymbol{\xi}\|) dS_y \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} \tag{2}$$

By replacing it with spherical coordinates the inner integral is

$$\begin{aligned} \int_{\|y\|=1} \exp(-icty_3 \|\boldsymbol{\xi}\|) dS_y &= \int_0^{2\pi} \int_0^{\pi} \exp(-ict \cos(\theta) \|\boldsymbol{\xi}\|) \sin(\theta) d\theta d\chi \\ &= 2\pi \int_0^{\pi} \exp(-ict \cos(\theta) \|\boldsymbol{\xi}\|) \sin(\theta) d\theta \end{aligned} \tag{3}$$

By performing a small variable change $p = \cos(\theta)$ and $dp = -\sin(\theta) d\theta$

$$\begin{aligned} 2\pi \int_{-1}^{1} \exp(-ictp\|\boldsymbol{\xi}\|) dp &= 2\pi \left[\frac{\exp(-ictp\|\boldsymbol{\xi}\|)}{-ictp\|\boldsymbol{\xi}\|} \right]_{-1}^{1} \\ &= \frac{2\pi}{-ict\|\boldsymbol{\xi}\|} \left(e^{-ict\|\boldsymbol{\xi}\|} - e^{ict\|\boldsymbol{\xi}\|} \right) \\ &= \frac{4\pi \sin(ct\|\boldsymbol{\xi}\|)}{ct\|\boldsymbol{\xi}\|} \end{aligned} \tag{4}$$

> In summary
>
> $$\begin{aligned} \langle \Phi_t, \hat{\varphi} \rangle &= \int_{\mathbb{R}^3} \frac{t}{4\pi} \cdot \frac{4\pi \sin(ct\|\boldsymbol{\xi}\|)}{ct\|\boldsymbol{\xi}\|} \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} \\ &= \int_{\mathbb{R}^3} \frac{\sin(ct\|\boldsymbol{\xi}\|)}{c\|\boldsymbol{\xi}\|} \varphi(\boldsymbol{\xi}) d\boldsymbol{\xi} \qquad = \left\langle \hat{\Phi}_t, \varphi \right\rangle \end{aligned} \tag{5}$$

2. Absolute Stability Analysis of the Forward Euler Method

This question will consider the forward euler method for the initial-value problem

$$\begin{cases} \frac{du}{dt}(t) = \lambda u(t) & \text{for } t > 0, \qquad \lambda \in \mathbb{C} \\ u(0) = 1 & \text{at } t = 0 \end{cases} \tag{6}$$

The exact solution is given by

$$u(t) = e^{\lambda t} \tag{7}$$

The forward euler method is given by

$$u_{n+1} = (1 + \Delta t \cdot \lambda) u_n \tag{8}$$

A time marching method is absolutely stable for a specific timestep Δt if its application to this particular IVP leads for this timestep to a numerical solution with the same asymptotic behaviour.

$$u_n \to 0 \text{ as } n \to +\infty \text{ when } \mathrm{Re}(\lambda) < 0 \tag{9}$$

For the forward Euler method with initial condition,

$$|u_n| = |1 + \Delta t \cdot \lambda|^n \to 0 \text{ as } n \to +\infty \tag{10}$$

This requires

$$|1 + \Delta t \cdot \lambda| < 1 \tag{11}$$

The region of absolute stability (ie where the condition is respected) is a disk in the complex plane centered in (-1,0) with a radius of 1.
In the case of $\lambda = -5$, the absolute stability inequation gives us:

$$-1 \leq 1 - 5\Delta t \leq 1 \tag{12}$$

Where
- The upperbound is trivial for a positive Δt
- the lower bound gives $\Delta t \leq 0.4$

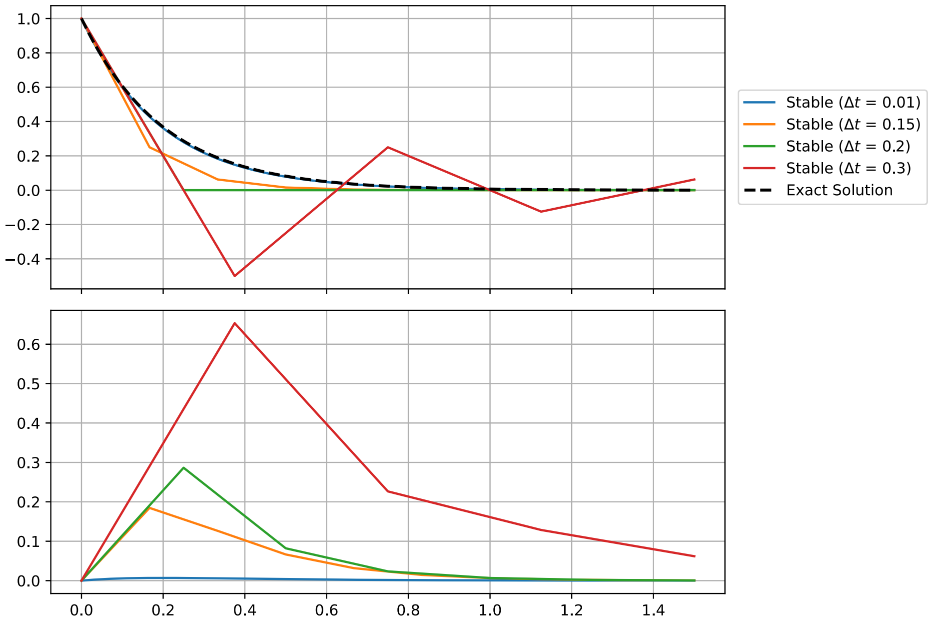

Figure 1: Representation of the forward Euler method and the representation of the absolute error associated to the method for stable Δt

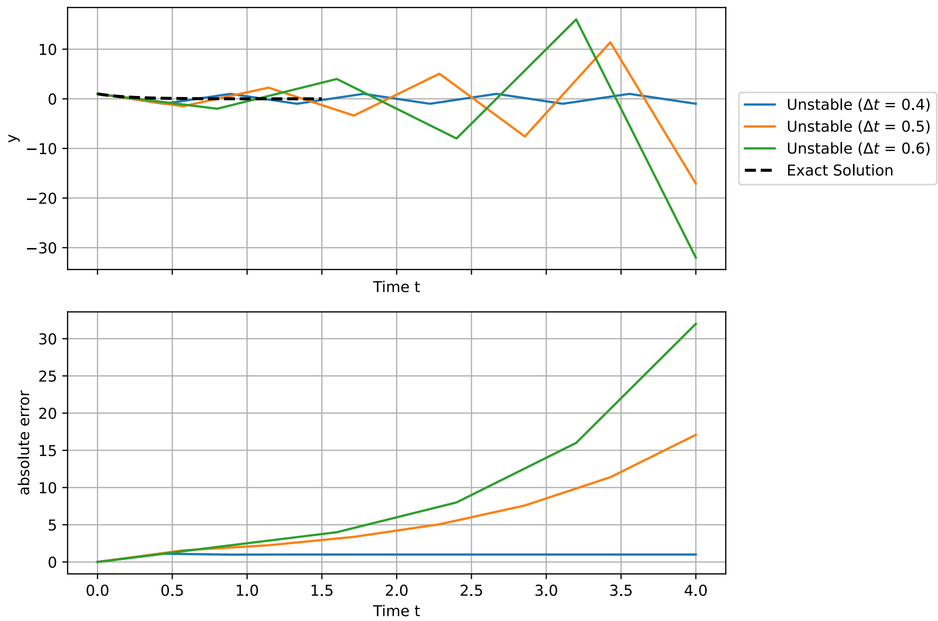

Figure 2: Representation of the forward Euler method and the representation of the absolute error associated to the method for unstable and stable marginally Δt

> The term $|1 + \Delta t \cdot \lambda|$ can be considered as the amplification factor. In the case of small value, the error will decay. In the particular case of $\Delta t = 0.4$, the shema is marginally stable, the oscillation does not even growth or decay. After this limit value the amplification increase the amplitude of the oscillations. The absolute error explodes.

3. Spectral Analysis and Wave Propagation in Rectangular Waveguides

4. Spectral solution

The spectral problem equation is

$$-\frac{\partial^2 \phi}{\partial x^2} = \lambda \phi \tag{13}$$

Three cases must be discussed based on the sign of λ

- Case $\lambda < 0$

Let $\lambda = -\alpha^2$, with $\alpha > 0$
In this particular case the general solution is given by:

$$\varphi(x) = A\cosh(\alpha x) + B\sinh(\alpha x) \tag{14}$$

By applying boundary

$$\begin{cases} \varphi\left(-\frac{L}{2}\right) = A\cosh\left(-\alpha\frac{L}{2}\right) + B\sinh\left(-\alpha\frac{L}{2}\right) = 0 \\ \varphi\left(\frac{L}{2}\right) = A\cosh\left(\alpha\frac{L}{2}\right) + B\sinh\left(\alpha\frac{L}{2}\right) = 0 \end{cases} \tag{15}$$

Knowing that sinh is a odd function and that cosh is even:

$$\begin{cases} \varphi\left(-\frac{L}{2}\right) = A\cosh\left(\alpha\frac{L}{2}\right) - B\sinh\left(\alpha\frac{L}{2}\right) = 0 \\ \varphi\left(\frac{L}{2}\right) = A\cosh\left(\alpha\frac{L}{2}\right) + B\sinh\left(\alpha\frac{L}{2}\right) = 0 \end{cases} \tag{16}$$

By rearranging, this gives

$$\begin{cases} 2A\cosh\left(\alpha\frac{L}{2}\right) = 0 \\ 2B\sinh\left(\alpha\frac{L}{2}\right) = 0 \end{cases} \tag{17}$$

That concludes that $A = B = 0$ and in extenso $\varphi(x) = 0$

- Case $\lambda = 0$

In this case the general solution is

$$\varphi(x) = Ax + b \tag{18}$$

By applying bondary condition;

$$\begin{cases} \varphi\left(-\frac{L}{2}\right) = -A\frac{L}{2} + B = 0 \\ \varphi\left(\frac{L}{2}\right) = A\frac{L}{2} + B = 0 \end{cases} \tag{19}$$

This situation also conclude that $A = B = 0$ and then that $\varphi(x) = 0$

- Case $\lambda > 0$

In this last case, the general solution is given by

$$\phi(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right) \tag{20}$$

With the help of the bondary condition

$$\begin{cases} \phi(-\frac{L}{2}) = 0 \\ \phi(\frac{L}{2}) = 0 \end{cases} \tag{21}$$

This show that $B = 0$ and $\lambda_m = \left(\frac{2m\pi}{L}^2\right)$ with $m \in \mathbb{N}_0$
The eigenfunction is then

$$\phi_m(x) = \sin\left(\frac{2m\pi x}{L}\right) \tag{22}$$

By normalizing

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \sin^2\left(\frac{2m\pi x}{L}\right) dx = \frac{L}{2} \tag{23}$$

> The solution is then given by
>
> $$\phi_m(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2m\pi x}{L}\right) \tag{24}$$

With the same analysis the y-direction problem gives with

$$\kappa_n = \frac{2n\pi}{H}^2 \tag{25}$$

> The normalized form is given by
>
> $$\psi_n = \sqrt{\frac{2}{H}} \sin\left(\frac{2n\pi}{H}\right) \tag{26}$$

4.1. Derivation in the sense of the theory of distribution

For all test function $\varphi \in \mathbb{R}$

$$\begin{aligned} \left\langle \frac{d}{dz} G_{mn}, \varphi \right\rangle &= -\left\langle G_{mn}, \frac{d}{dz}\varphi \right\rangle \\ &= -\int_{-\infty}^{+\infty} G_{mn}(z) \frac{d}{dz}\varphi dz = -\int_{-\infty}^{0} B_{mn} e^{-ik_{f,mn}z} \frac{d}{dz}\varphi dz \\ &\quad - \int_{0}^{+\infty} B_{mn} e^{-ik_{f,mn}z} \frac{d}{dz}\varphi dz \end{aligned} \tag{27}$$

The integration by part returns

$$\begin{aligned} -\left[B_{mn} e^{-ik_{f,mn}z}\right]_{-\infty}^{0} + \int_{-\infty}^{0} B_{mn}(-ik_{f,mn}) e^{-ik_{f,mn}z} \varphi dz \\ -\left[B_{mn} e^{ik_{f,mn}z}\right]_{0}^{+\infty} + \int_{0}^{+\infty} B_{mn}(ik_{f,mn}) e^{ik_{f,mn}z} \varphi dz \end{aligned} \tag{28}$$

Since φ has the properties $\lim_{x\to\pm\infty} \varphi(x) = 0$

$$
\begin{aligned}
-B_{mn}\varphi(0) + B_{mn}\varphi(0) + \int_{-\infty}^{0} B_{mn}(-ik_{f,mn})e^{-ik_{f,mn}z}\varphi dz \\
+ \int_{0}^{+\infty} B_{mn}(ik_{f,mn})e^{ik_{f,mn}z}\varphi dz
\end{aligned}
\tag{29}
$$

> The first order derivative is then
>
> $$\left\langle \frac{d}{dz}G_{mn}, \varphi \right\rangle = \int_{-\infty}^{0} B_{mn}(-ik_{f,mn})e^{-ik_{f,mn}z}\varphi dz + \int_{0}^{+\infty} B_{mn}(ik_{f,mn})e^{ik_{f,mn}z}\varphi dz \tag{30}$$

The second order derivative can be obtain in the same way

$$
\begin{aligned}
\left\langle \frac{d^2}{dz^2}G_{mn}, \varphi \right\rangle &= \left\langle G_{mn}, \frac{d^2}{dz^2}\varphi \right\rangle \\
&= \int_{-\infty}^{+\infty} G_{mn}(z)\frac{d^2}{dz^2}\varphi dz \\
&= \int_{-\infty}^{0} B_{mn}e^{-ik_{f,mn}z}\frac{d^2}{dz^2}\varphi dz + \int_{0}^{+\infty} B_{mn}e^{ik_{f,mn}z}\frac{d^2}{dz^2}\varphi dz
\end{aligned}
\tag{31}
$$

By decomposing the left integral (negative domain)

$$
\begin{aligned}
\int_{-\infty}^{0} B_{mn}e^{-ik_{f,mn}z}\frac{d^2}{dz^2}\varphi dz &= \left[B_{mn}e^{-ik_{f,mn}z}\frac{d}{dz}\varphi\right]_{-\infty}^{0} \\
&\quad - \int_{-\infty}^{0} \left(-ik_{f,nm}B_{mn}e^{-ik_{f,mn}z}\right)\frac{d}{dz}\varphi dz \\
= B_{mn}\frac{d}{dz}\varphi(0) + ik_{f,mn}B_{mn}\varphi(0) &+ \int_{-\infty}^{0} \left(-k_{f,mn}^2\right)B_{mn}e^{-ik_{f,nm}z}\varphi dz
\end{aligned}
\tag{32}
$$

By doing the same thing developpement for the right integral (positive domain)

$$
\begin{aligned}
\int_{0}^{+\infty} B_{mn}e^{ik_{f,mn}z}\frac{d^2}{dz^2}\varphi dz = -B_{mn}\frac{d}{dz}\varphi(0) + ik_{f,mn}B_{mn}\varphi(0) \\
+ \int_{0}^{+\infty} \left(-k_{f,mn}^2\right)B_{mn}e^{ik_{f,mn}z}\varphi dz
\end{aligned}
\tag{33}
$$

By adding both contribution

> The second order derivative is then
>
> $$\left\langle \frac{d^2}{dz^2}G_{mn}, \varphi \right\rangle = -k_{f,mn}^2\langle G_{mn}, \varphi\rangle + 2ik_{f,mn}B_{mn}\varphi(0) \tag{34}$$

4.2. Evaluation of B_{mn}

By substituing in the equation

$$-c^2\left[-k_{f,nm}^2G_{mn} + 2ik_{f,nm}B_{mn}\delta(z)\right] + \left(-\omega_f^2 + c^2(\lambda_m + \kappa_n)\right)G_{mn} = \phi_m(0)\psi_n(0)\delta(z) \tag{35}$$

Knowing that $k_{f,mn}^2 = \frac{\omega_f^2}{c^2} - (\lambda_m + \kappa_n)$

$$-2ic^2 k_{\{f,mn\}} B_{\{mn\}} \delta(z) = \phi_m(0)\psi_n(0)\delta(z) \tag{36}$$

> The value found is:
>
> $$B_{mn} = i\frac{\phi_m(0)\psi_n(0)}{2c^2 k_{f,mn}} \tag{37}$$

5. Von Neuman Stability Analysis of the Upwind Finite Difference Scheme

The upwind method is defined as

$$\begin{cases} u_j^{n+1} = u_j^n - c\frac{\Delta t}{h}\left(u_j^n - u_{j-1}^n\right) \\ u_j^0 = u_0(x_j) \end{cases} \tag{38}$$

Where $\nu = \frac{c\Delta t}{h}$
By assuming a solution in the form:

$$u_j = \gamma^n e^{i\xi jh} \tag{39}$$

The schema can be rewritten as

$$\gamma^{n+1} e^{i\xi jh} = \gamma^n e^{i\xi jh} - \nu\gamma^{n\left(e^{i\xi jh} - e^{i\xi(j-1)h}\right)} \tag{40}$$

$$\gamma = 1 - \nu\left(1 - e^{-i\xi h}\right) \tag{41}$$

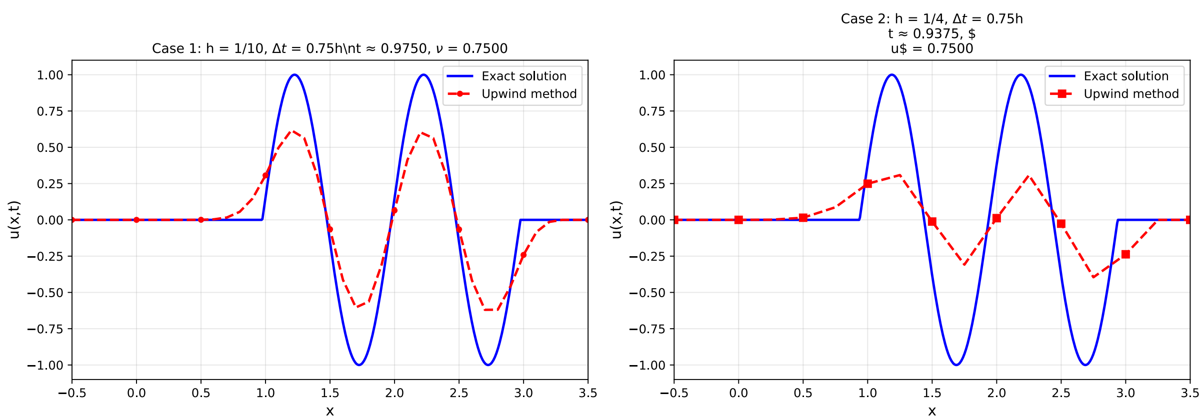

Figure 3: Comparaison of exact and numerical solution for the transport equation at t≃1using the upwind finite difference method.

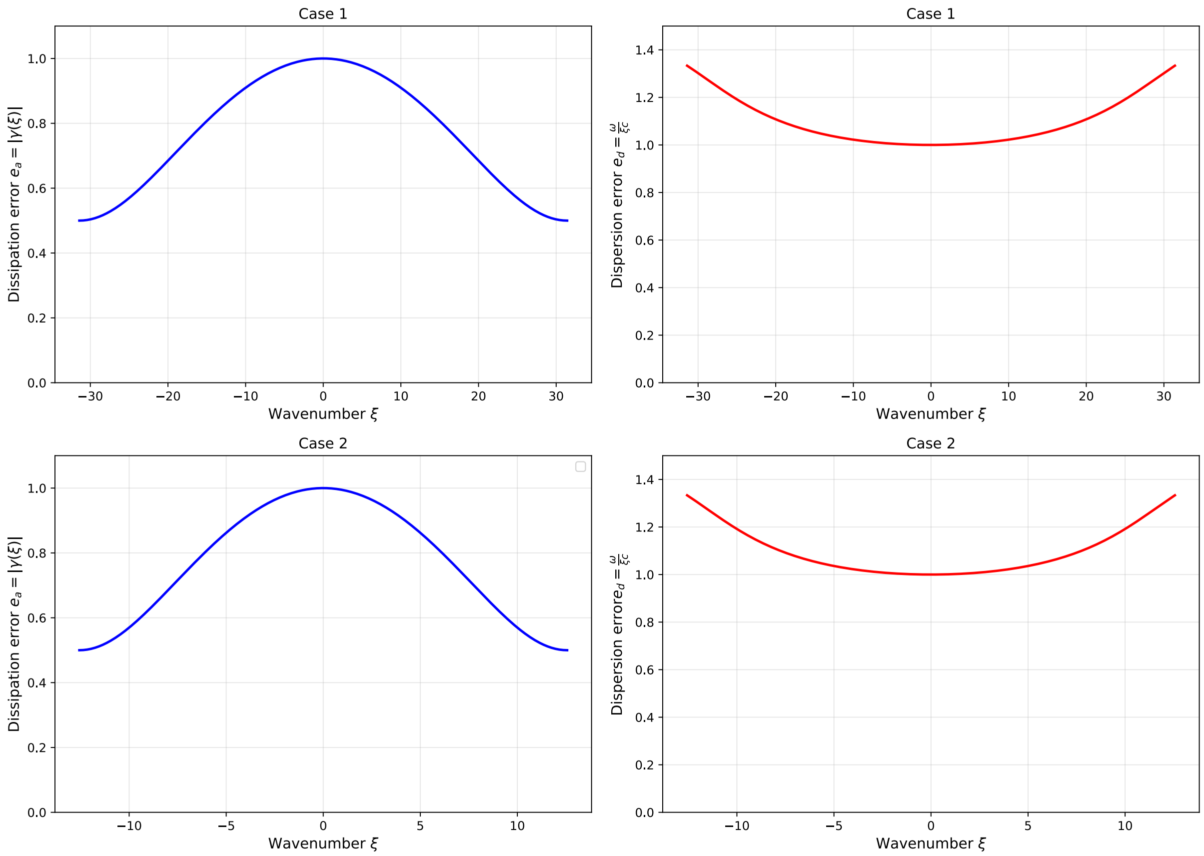

Figure 4: Von Neumann analysis of the upwind method, with dissipation e_a and dispersion e_d errors.

5.1. Use of AI

AI was used to summarise the instructions for question 3.