International Journal of Modern Physics D
Vol. 25, No. 12 (2016) 1644015 (6 pages)
© World Scientific Publishing Company
DOI: 10.1142/S0218271816440156

Int. J. Mod. Phys. D Downloaded from www.worldscientific.com
by VANDERBILT UNIVERSITY on 09/11/16. For personal use only.

Black hole fusion made easy*

Roberto Emparan[†,‡,§] and Marina Martínez[‡,¶]

[†]*Institució Catalana de Recerca i Estudis Avançats (ICREA), Passeig Lluís Companys 23, E-08010 Barcelona, Spain*

[‡]*Departament de Física Fonamental, Institut de Ciències del Cosmos, Universitat de Barcelona, Martí i Franquès 1, E-08028 Barcelona, Spain*

[§]*emparan@ub.edu*

[¶]*marinamartinez@ub.edu*

Received 29 July 2016
Accepted 10 August 2016
Published 1 September 2016

The fusion of two black holes — a signature phenomenon of General Relativity — is usually regarded as a process so complex that nothing short of a supercomputer simulation can accurately capture it. In this essay, we explain how the event horizon of the merger can be found in an exact analytic way in the limit where one of the black holes is much smaller than the other. Remarkably, the ideas and techniques involved are elementary: the equivalence principle, null geodesics in the Schwarzschild solution, and the notion of event horizon itself. With these, one can identify features such as the line of caustics at which light rays enter the horizon, and find indications of universal critical behavior when the two black holes touch.

Keywords: Black hole; black hole merger; extreme mass ratio merger; event horizon; exact solutions.

PACS Number(s): 04.70.Bw, 04.20.Cv

The recent direct detection of gravitational waves is a momentous discovery,[1] but even more exciting is the identification of the event that triggered it over a billion years ago: the fusion of two black holes. One cannot help but marvel that the entire process is completely captured by such beautiful equations as $R_{\mu\nu} = 0$. Unfortunately, this bliss fizzles out when it comes to extracting from these equations the details of the phenomenon. It requires deploying a full arsenal of methods, culminating in sophisticated numerical analysis carried out by supercomputers running for days on end. The nonlinearity of Einstein's theory is at its most fiendish when

*This essay received an Honorable Mention in the 2016 Essay Competition of the Gravity Research Foundation.

the two horizons fuse into a single one. Surely any detailed information about the event horizon of a merger bearing any realistic resemblance to those occurring in the universe is unattainable without powerful machines.

Or so it would seem. In this essay, we argue that there is one limiting — but still realistic — instance in which the event horizon of the merger of a black hole binary becomes so simple that it can be described in an exact analytic way. Moreover, its construction involves only elementary techniques and ideas at the core of the theory, which have been well understood for many decades. They are

- the equivalence principle,
- the Schwarzschild solution and its null geodesics,
- the notion of event horizon.

The limit is that of extreme-mass ratio (EMR) in which one black hole is much smaller than the other. Given the findings of Ref. 1, it does seem possible that black hole binary mergers with mass ratios $\lesssim 1/30$ will be detected in ground-based observatories — and with much smaller ratios in space-based ones.

This limit is often taken as one where the size of the large black hole, M (in geometrized units where the gravitational constant is $G = 1$), is fixed while the small black hole is regarded as a pointlike object of size $m \to 0$. This is appropriate for extracting the gravitational waves from the collision (which have wavelengths $\sim M$), but not details on the scale of m, such as the geometry and evolution of the event horizon when the two black holes fuse with each other. For this, we must keep m fixed while $M \to \infty$.

Now consider the last moments before the merger, when the small black hole is at a distance $\ll M$ of the large one. The equivalence principle asserts that we can always place ourselves in the rest frame of the small black hole (where its center-of-mass is not moving), and that the curvature created by the large black hole (which is inversely proportional to its size M) can be neglected over distances $\ll M$. Therefore, this is a situation where all the length scales associated to the large black hole become extremely large, and the spacetime around the small black hole should be well approximated by the Schwarzschild geometry.[2] Although the curvature created by the large black hole vanishes in this limit, its horizon is still present: it becomes an infinite, Rindler-type, acceleration horizon that reaches asymptotic null infinity as a planar null surface. Therefore, in the EMR limit on scales much smaller than M, the event horizon of the black hole merger can be found by tracing in the Schwarzschild geometry a family of null geodesics that approach a planar horizon at a large distance from the small black hole. Given the enhanced symmetry of the limiting system, this can be done exactly.

Our starting point, then, is the Schwarzschild solution with mass m and with radius $r_0 = 2m$,

$$ds^2 = -\left(1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right). \tag{1}$$

Int. J. Mod. Phys. D Downloaded from www.worldscientific.com
by VANDERBILT UNIVERSITY on 09/11/16. For personal use only.

It has a timelike Killing vector ∂_t, which defines the rest frame of the small black hole (i.e. the black hole has zero momentum relative to observers that move along orbits of ∂_t), and exact spherical symmetry. Then, it suffices to study null geodesics on the plane $\theta = \pi/2$, putting the collision axis along, say, the two segments $\phi = 0, \pi$. A textbook calculation gives the null geodesic equations as

$$\begin{aligned} t(r) &= \int \frac{r^3 dr}{(r - r_0)\sqrt{r(r^3 - q^2 r + q^2 r_0)}}, \\ \phi(r) &= -\int \frac{q\, dr}{\sqrt{r(r^3 - q^2 r + q^2 r_0)}}, \end{aligned} \tag{2}$$

with integration constants fixed by requiring that the null surface becomes a planar horizon at infinity — i.e. that the light rays arrive at $\mathcal{I}^+$ parallel to the collision axis, and all at the same retarded time. The geodesics on this event horizon are labeled by their impact parameter at infinity, q.

And that's it. The equations then produce — in terms of elliptic integrals[3] — the exact event horizon as a null surface ruled by null geodesics (Fig. 1). It is the expected 'pair-of-pants' diagram, with a hugely oversized (almost planar) leg that advances towards a thin leg — which is vertical, since our construction is made in the rest frame of the small black hole. The sequence of constant-t sections in Fig. 2 (drawn with *Mathematica*) neatly illustrates how the fusion proceeds.

The structure of the event horizon can now be analyzed in detail. For instance, we can clearly see in Fig. 1 the presence before the merger of a line of caustics — the 'seam' in the pants — where null generators enter to be part of the event

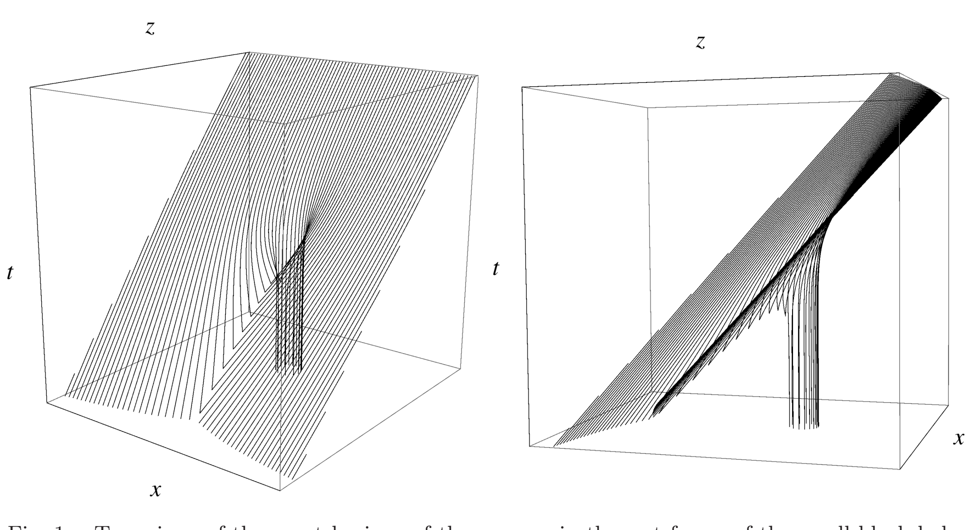

Fig. 1. Two views of the event horizon of the merger, in the rest frame of the small black hole. The large black hole advances towards it in the z-direction. Each curve is a null sgenerator of the hypersurface with a different asymptotic value $x \to q$ at future infinity. t is the Schwarzschild time and $r = \sqrt{x^2 + z^2}$ is the Schwarzschild area-radius. The full three-dimensional (3D) event horizon is obtained by rotating around the axis $x = 0$.

Int. J. Mod. Phys. D Downloaded from www.worldscientific.com by VANDERBILT UNIVERSITY on 09/11/16. For personal use only.

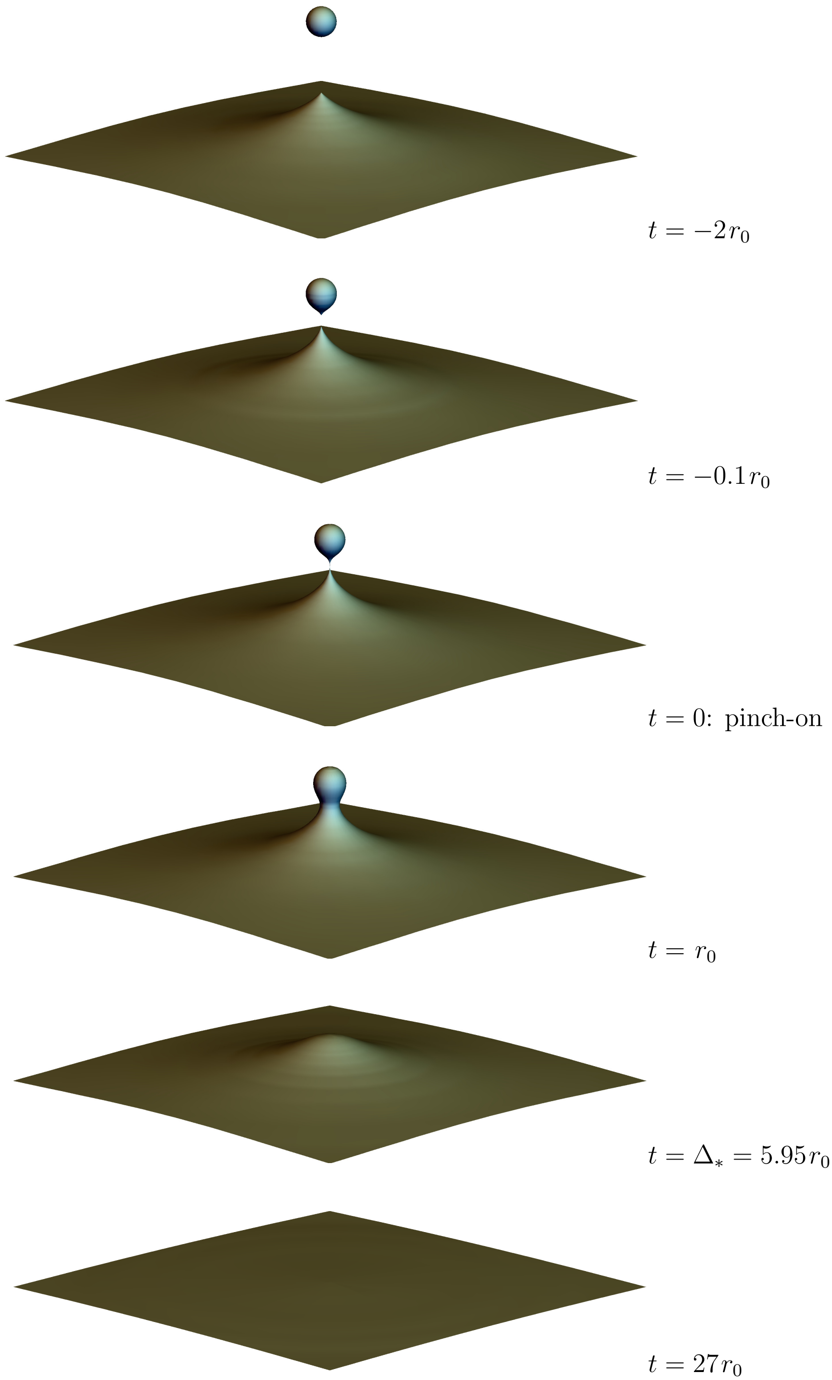

Fig. 2. Sequence of 3D constant-time slices of the exact event horizon. New null generators enter the horizon until pinch-on, creating cones at the caustic points.

Int. J. Mod. Phys. D Downloaded from www.worldscientific.com
by VANDERBILT UNIVERSITY on 09/11/16. For personal use only.

Int. J. Mod. Phys. D Downloaded from www.worldscientific.com by VANDERBILT UNIVERSITY on 09/11/16. For personal use only.

horizon. On spatial sections, this makes the horizon acquire a conical shape at the caustic.

Of great interest is the instant — recall that there is a preferred time-slicing, set by the rest frame of the small black hole — when the two black holes touch each other, i.e. the 'crotch' of the pants. The gravitational attraction of the large black hole, present as an acceleration effect, pulls on the small horizon deforming it into a tear-drop shape. At the pinch-on instant, it is stretched from its initial radius r_0 to a cusp at $r = r_*$ (measured in Schwarzschild area-radius). The exact solution gives this value as a root of a transcendental equation, easily obtained with high precision as

$$r_* = 1.76031\, r_0. \tag{3}$$

Other parameters of the merger can also be found. For instance, its duration Δ_* can be characterized by the time t elapsed between the pinch-on instant and the moment when the post-merger horizon reaches the distance r_* in the antipodal direction. We find

$$\Delta_* = 5.94676\, r_0\,. \tag{4}$$

Figure 2 shows that by this time the two horizons have noticeably fused with each other.

The geometry of the event horizon exhibits critical behavior in the instants before and after pinch-on at $t = 0$. Right before the merger, when the caustic cone on the large horizon closes off as $t \to 0^-$, we find that

$$\text{aperture angle of caustic cone} \propto |t|^{1/2}. \tag{5}$$

We obtain the same behavior in a five-dimensional (5D) merger, which suggests that it may be valid in all $D \geq 4$. A natural question, still open, is whether this exponent is universal, e.g. when charge or rotation are introduced.

Immediately after pinch-on, a thin throat forms connecting the two horizons. It then grows as

$$\text{width of throat} \propto t, \tag{6}$$

until at a time $\sim 0.25\, r_0$, when the growth begins to slow down. This behavior is also present in five dimensions. It may indeed be universal: linear growth has also been found in a rather different model of a merger.[4] This might actually be expected since the post-merger surface is smooth.

What is most remarkable about this simple construction is that it captures the evolution of the horizon in a realistic (e.g. astrophysical) black hole collision, with the only approximations that

(a) the spin of the small black hole is negligible, and
(b) the mass ratio m/M is small enough.

Within the validity of these assumptions, the construction is completely general. In particular, we do not have to find the horizon a new if there is a relative transverse velocity between the two black holes, as e.g. when a small black hole falls into a large rotating one along a noncorotating trajectory. The equivalence principle reduces this to the case above.[3]

Issue (a) is dealt with by replacing (1) with the Kerr metric. The lower degree of symmetry makes the calculations harder, but still very much simpler than when m/M is finite, since we do not need to obtain a new solution to Einstein's equations, only to find an appropriate family of geodesics in the Kerr geometry. Black hole uniqueness then implies that, by appropriately aligning the asymptotic null plane, and up to corrections for nonzero m/M, this construction *accurately captures the event horizon of any fusion between a small and a large black hole in the universe.*

Issue (b) is closely related to an aspect that is conspicuously absent from our description of the merger: gravitational wave radiation. In the limit $M \to \infty$, the radiation zone is pushed infinitely far away. Nevertheless, gravitational waves will reappear when corrections for small but finite m/M are computed. Indeed, our construction gives the near-zone solution of the merger to leading order in m/M. One can then match it to the far-zone construction of the EMR event horizon in,[5] to obtain the effect of gravitational wave emission on the horizon. This can be carried out iteratively to higher orders in m/M. It is not inconceivable that the sensitivity of future detectors will require these corrections. Our construction is the first step in the description of their event horizons.

The equivalence principle — Einstein's first and firmest intuition about gravity — lies at the heart of General Relativity. A notion so deep is not easily exhausted even after a century of use. In a beautiful sleight of hand, it has allowed us to accurately capture a phenomenon that involves two black holes, using a geometry that seemingly would contain only one.

Int. J. Mod. Phys. D Downloaded from www.worldscientific.com
by VANDERBILT UNIVERSITY on 09/11/16. For personal use only.

References

1. LIGO Scientific and Virgo Collab. (B. P. Abbott *et al.*), *Phys. Rev. Lett.* **116** (2016) 061102, arXiv:1602.03837 [gr-qc].
2. K. Schwarzschild, *Sitzungsber. Preuss. Akad. Wiss. Berlin* (*Math. Phys.*) (1916) 189, arXiv:physics/9905030.
3. R. Emparan and M. Martínez, *Class. Quant. Grav.* **33** (2016) 155003, arXiv:1603.00712 [gr-qc].
4. S. A. Caveny and R. A. Matzner, *Phys. Rev. D* **68** (2003) 104003, arXiv:gr-qc/0303109.
5. R. Hamerly and Y. Chen, *Phys. Rev. D* **84** (2011) 124015, arXiv:1007.5387 [gr-qc].