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On the radial wave equation in Schwarzschild's space–time

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The radial factor of a separable solution of the wave equation in Schwarzschild's space–time satisfies a second-order linear differential equation. This equation is studied in detail. The behavior of the solutions near the singular points (the origin, the horizon, and infinity) of the equation is analyzed. By an appropriate transformation two simpler differential equations are obtained corresponding to retarded and advanced solutions with characteristic asymptotic expansions. Their properties permit the expression of the general solution of the radial equation in terms of a single contour integral. Finally, through a "matching" technique, the behavior of a solution at the singular points is determined from its behavior at a single singular point.

1. INTRODUCTION

In curved space–times the detailed mathematical study of the wave equation must precede any systematic investigation of wave phenomena, exactly as in flat space–time. However, even in the simple space–time of Schwarzschild, separation of variables in the wave equation leads to a second order linear differential equation, *the radial wave equation*, which is not related to any known differential equation of mathematical physics. Expression of the solution in closed form is not possible. Even methods containing infinite steps[1,2] have not given satisfactory expressions and have raised unanswered questions of convergence. In fact, the solution of the radial wave equation has not gone essentially beyond the stage of writing down the differential equation.[1,3] In all physical problems, which lead to the radial wave equation[3,4,5] (or similar second order differential equations[6,7]), techniques of "effective potential" tailored to the specific requirements of the problem have been used.

In this paper we set and reach a limited objective, that is, the investigation of those properties of the solutions which are essential for the study of time-dependent wave phenomena around a Schwarzschild black hole. These essential properties of the solutions can be considered in two groups. The first group concerns the behavior of the solutions at the origin of the coordinate system and the horizon of the black hole. It is intimately related with the radiation of multipole moments[4,8] and the possibility of destruction of the black hole. The second group concerns the behavior of the solution at infinity, the retarded and advanced contributions to the wave solution, and is related to the observations of a distant observer.

In Sec. 2 we review briefly the radial wave equation in flat space–time. In Secs. 3 and 4 we consider the radial wave equation in Schwarzschild's space–time, and we study the behavior of the solution at the origin and the horizon (Sec. 3) and at infinity in terms of retarded and advanced solutions (Sec. 4). In Sec. 5 we derive certain linear relationships among the characteristic solutions of the differential equation. These relations enable us to find the behavior of a solution near a singular point from its behavior near another singular point.

2. THE RADIAL WAVE EQUATION IN FLAT SPACE–TIME

We present briefly the solution of the radial wave equation in flat space–time in a way which avoids the use of Bessel functions. The method of solution will indicate the generalization needed to derive the retarded and advanced solutions in Schwarzschild's space–time (see Sec. 4). Moreover, the formulas presented in this section will help in demonstrating the correspondence between the flat-space and the Schwarzschild-space solutions.

In flat space–time the metric tensor in spherical coordinates is

$$g_{\mu\nu} = \mathrm{diag}[c^2, -1, -r^2, -r^2 \sin^2\theta], \tag{1}$$

and the scalar wave equation[9]

$$\Box\Psi \equiv g^{\mu\nu}\Psi_{;\mu\nu} = 0 \tag{2}$$

is separable[10] (the semicolon denotes covariant differentiation).

If

$$\Psi = R(r)Y(\theta, \varphi)e^{-i\omega t}, \tag{3}$$

then $Y(\theta, \varphi)$ is a spherical harmonic and $R(r)$ satisfies the equation

$$r^2 \frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + [k^2r^2 - l(l+1)]R = 0, \tag{4}$$

where $k = \omega/c$.

A change of the dependent variable to $r^{1/2}R$ will give a Bessel equation[10] of fractional order. However, we can avoid the Bessel functions. If we set

$$R(r) = e^{\mp ix}F_{\pm}(x), \tag{5}$$

where $x = kr$, Eq. (4) reduces to

$$x^2\frac{d^2F_\pm}{dx^2} + 2(x \mp ix^2)\frac{dF_\pm}{dx} + [\mp 2ix - l(l+1)]F_\pm = 0. \tag{6}$$

This equation has an irregular singular point at $x = +\infty$, but we can obtain closed-form solutions (one F_+ and one F_-), which are polynomials of x^{-1} of degree $l+1$. In fact, we have[11]

$$F_{l\pm} = (\pm i)^{l+1}x^{-1}\sum_{n=0}^{l}\frac{(l+n)!}{(l-n)!n!}(\pm 2ix)^{-n}, \tag{7}$$

with F_{l-} corresponding to retarded waves and F_{l+} to advanced waves. Usually we consider the two linearly independent combinations[12]

$$j_l = \tfrac{1}{2}(e^{ix}F_{l-} + e^{-ix}F_{l+}) \tag{8}$$

and

$$h_l = e^{ix} F_{l-}, \tag{9}$$

which are finite at $r = 0$ and $r = +\infty$ respectively.

In Sec. 4 a generalization of transformation (5) will result in equations similar to Eq. (6).

3. THE RADIAL WAVE EQUATION IN SCHWARZSCHILD'S SPACE-TIME

We consider now the wave equation (2) in Schwarzschild's space-time with metric tensor

$$g_{\mu\nu} = \mathrm{diag}\left[\left(1 - \frac{r_s}{r}\right) c^2, -\left(1 - \frac{r_s}{r}\right)^{-1}, -r^2, -r^2 \sin^2\theta\right], \tag{10}$$

where r_s is the *Schwarzschild radius* (a constant related to the mass M by the relation $r_s = 2GMc^{-2}$).

Assuming a solution[13] of Eq. (2) of the form (3), we have for $R(r)$ the *radial wave equation*[14]

$$x(x - x_s)^2 \frac{d^2R}{dx^2} + (x - x_s)(2x - x_s) \frac{dR}{dx} + [x^3 - l(l+1)(x - x_s)]R = 0, \tag{11}$$

where

$$x = kr, \quad x_s = kr_s. \tag{12}$$

Eq. (11) has two regular singular points[15,16] at $x = 0$ and $x = x_s$ and an irregular singular[17] point at $x = +\infty$.

A. Behavior near the origin

In the neighborhood of $x = 0$ we try a power series of x as a solution of Eq. (11). The indicial equation has a double root equal to zero and, consequently, two linearly independent solutions are[17]

$$\mathcal{R}_1(x) = \sum_{n=0}^{\infty} a_n x^n \tag{13}$$

and

$$\mathcal{R}_2(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) \ln x + \sum_{n=0}^{\infty} b_n x^n. \tag{14}$$

Substituting these expressions into Eq. (11), we find that a_n and b_n satisfy the recurrence relations

$$n^2 x_s^2 a_n + [l(l+1) - (n-1)(2n-1)]x_s a_{n-1} + [(n-1)(n-2) - l(l+1)]a_{n-2} + a_{n-4} = 0 \tag{15}$$

and

$$n^2 x_s^2 b_n + [l(l+1) - (n-1)(2n-1)]x_s b_{n-1} + [(n-1)(n-2) - l(l+1)]b_{n-2} + b_{n-4} + 2n x_s^2 a_n - (4n-3)x_s a_{n-1} + (2n-3)a_{n-2} = 0. \tag{16}$$

The coefficients a_0 and b_0 are not specified by Eqs. (15) and (16) and have to be chosen arbitrarily. We must choose $a_0 \neq 0$; otherwise $\mathcal{R}_1(x) \equiv 0$ (trivial solution.[18] Any arbitrary pair (a_0, b_0) with $a_0 \neq 0$ will give two linearly independent solutions. We choose $a_0 = b_0 = 1$, thus making $\mathcal{R}_1(x)$ and $\mathcal{R}_2(x)$ particular solutions of Eq. (11). The general solution of Eq. (11) is an arbitrary linear combination of $\mathcal{R}_1(x)$ and $\mathcal{R}_2(x)$.

The series in Eqs. (13) and (14) converge for $x < x_s$ ($x = kr > 0$). Hence, *there is one solution finite at the origin* [expression (13)], which can be regarded as "physically preferable."[19]

B. Behavior near the horizon

We consider the solutions of Eq. (11) in the neighborhood of the other regular singular point $x = x_s$. Expanding in powers of $x - x_s$, we obtain an indicial equation with roots $\pm ix_s$. Consequently, two linearly independent solutions are

$$\mathcal{R}_3(x) = e^{ix_s \ln|x - x_s|} \sum_{n=0}^{\infty} c_n (x - x_s)^n \tag{17}$$

and

$$\mathcal{R}_4(x) = e^{-ix_s \ln|x - x_s|} \sum_{n=0}^{\infty} d_n (x - x_s)^n, \tag{18}$$

where

$$(n + 2ix_s) n x_s c_n + [(n+l)(n-l-1) + 2x_s^2 + (2n-1)ix_s]c_{n-1} + 3x_s c_{n-2} + c_{n-3} = 0 \tag{19}$$

and

$$(n - 2ix_s) n x_s d_n + [(n+l)(n-l-1) + 2x_s^2 - (2n-1)ix_s]d_{n-1} + 3x_s d_{n-2} + d_{n-3} = 0. \tag{20}$$

The coefficients c_0 and d_0 must be different than zero,[18] but are otherwise arbitrary. We choose $c_0 = d_0 = 1$. Thus, $\mathcal{R}_3(x)$ and $\mathcal{R}_4(x)$ are particular solutions of Eq. (11) and the general solution is an arbitrary linear combination of them.

From expressions (17) and (18) we derive two important properties of the solutions of Eq. (11). First, *every solution remains bounded on the horizon* $r = r_s$ (and, consequently, every solution is "physically acceptable"[19]). Second, *no solution goes to zero as* $r \to r_s$. The proof of these properties is simple, since near $x = x_s$ any solution behaves as $A(x - x_s)^{ix_s} + B(x - x_s)^{-ix_s}$, which remains absolutely smaller than $|A| + |B|$ and does not have a limit as $x \to x_s$.

The importance of these two properties is due to the fact[8] that they are intimately connected with the possibility of destruction of the black hole and the radiation of higher multipole moments during the fall of a small scalar particle into the black hole.

4. RETARDED AND ADVANCED SOLUTIONS AT INFINITY

We ask now for a generalization[20] of transformation (5), which will "separate" the retarded and advanced solutions of Eq. (11). Two remarks indicate the generalization. First expressions (17) and (18) indicate that a factor $e^{\pm ix_s \ln|x - x_s|}$ should be removed from R. Second, the retarded (advanced) solution would have been reached, if we had worked from the beginning in a retarded[21] (advanced) coordinate system. This means that a factor $e^{-i\omega u}$ would have been removed from Ψ instead of $e^{-i\omega t}$. Hence the new transformation is

$$R_{l\pm}(x, x_s) = e^{\mp i(x + x_s \ln|x - x_s|)} F_{l\pm}(x, x_s). \tag{21}$$

Obviously, for $x_s = 0$ we have again Eq. (5).

Replacing $R_{l\pm}$ in Eq. (11) we find two equations satisfied by F_{l+} and F_{l-}, respectively. At this point the consideration of complex values for the independent variable appears to be useful. If $F_l(z, x_s; \epsilon)$ is a solution of

$$z(z - x_s) \frac{d^2F_l}{dz^2} + (-\epsilon z^2 + 2z - x_s) \frac{dF_l}{dz} - [\epsilon z + l(l+1)]F_l = 0, \tag{22}$$

then

$$F_{l\pm}(x, x_s) = F_l(x, x_s; \pm 2i), \tag{23}$$

namely, Eq. (19) for $z = x$ gives the differential equations for F_{l+} ($\epsilon = 2i$) and F_{l-} ($\epsilon = -2i$). Note also that for $x_s = 0$ we rediscover Eq. (6).

The solutions of Eq. (22) present the following important property. *If $F_l(z, x_s; \epsilon)$ is a solution of Eq. (22), then*

$$e^{\epsilon[z+x_s \ln(z-x_s)]}F_l(z, x_s; -\epsilon) \tag{24}$$

is also a solution. This property can be proved easily by substituting expression (24) into Eq. (22).

We will express now the retarded and advanced solutions as contour integrals. According to the theory of contour integration[15,16] of ordinary linear differential equations, the integral

$$\int_C G_l(w) e^{zw} dw \tag{25}$$

will be a solution of Eq. (22), if $G_l(w)$ satisfies the equation[22]

$$w(w-\epsilon)\frac{d^2G}{dw^2} + (x_s w^2 + 2w - \epsilon)\frac{dG}{dw} + [x_s w - l(l+1)]G = 0. \tag{26}$$

The contour C consists of a straight line parallel to the real axis from $\mathrm{Re}\, w = -\infty$ to $w = 0$ (or $w = \epsilon$), a circle around $w = 0$ (or $w = \epsilon$) described positively, and a straight line also parallel to the real axis from $w = 0$ (or $w = \epsilon$) to $\mathrm{Re}\, w = -\infty$. At $w = 0$ the indicial equation of Eq. (26) has a double root equal to zero; hence of the two solutions only the one containing $\ln w$ (the nonanalytic at $w = 0$) will contribute to the integral. Specifically, let $G_l(w, x_s; \epsilon)$ be the solution of Eq. (26), which near $w = 0$ is given by

$$G_l(w, x_s; \epsilon) = \sum_{n=0}^{\infty} g_n w^n, \tag{27}$$

with $g_0 = 1$ and

$$n^2 \epsilon g_n + (l+n)(l-n+1)g_{n-1} - (n-1)x_s g_{n-2} = 0. \tag{28}$$

Then a solution of Eq. (22) is

$$F_l(z, x_s; \epsilon) = \frac{i}{2\pi}\left(\frac{\epsilon}{2}\right)^{l+1} \int_C G_l(w, x_s; \epsilon) \ln w e^{zw} dw, \tag{29}$$

with C surrounding the negative real axis $\mathrm{Re}\, w < 0$. Its asymptotic expansion[23] for $\mathrm{Re}\, z > 0$ is

$$F_l(z, x_s; \epsilon) \sim \left(\frac{\epsilon}{2}\right)^{l+1} \sum_{n=0}^{\infty} \tau_n z^{-(n+1)}, \tag{30}$$

with $\tau_0 = 1$ and[18]

$$n\epsilon\tau_n - (l+n)(l-n+1)\tau_{n-1} - (n-1)^2 x_s \tau_{n-2} = 0. \tag{31}$$

Equations (27) and (28) have been normalized so that when $x_s = 0$, G_l becomes equal[24] to $P_l(1 - 2w/\epsilon)$ and F_l equal to F_{l+} and F_{l-} of Eq. (7) for $\epsilon = 2i$ and $\epsilon = -2i$, respectively.

From F_l we determine a second solution F_l' of Eq. (22) using expression (24). *F_l and F_l' are linearly independent* (for $\epsilon \neq 0$), since they are independent in the special case $x_s = 0$.

The solution F_l' is related to the second contour-integral solution given by the integral (25), say F_l'' when C starts and ends at $\mathrm{Re}\, w = -\infty$ surrounding the point $w = \epsilon$. In fact, *F_l' gives that integral* (up to a constant factor). The proof of this statement consists of two steps. First, we show that F_l' and F_l'' have the same asymptotic expansion (up to a factor). Secondly, we argue that F_l' and F_l'' cannot be linearly independent because in that case every solution would have the same asymptotic expansion, which is not correct for F_l. Consequently, F_l' and F_l'' are proportional to each other.

Reviewing the results of the present section, we see that two linearly independent solutions of Eq. (11) have been determined in terms of the contour integral (29). These solutions are

$$\mathcal{R}_5 = R_{l-} \quad \text{and} \quad \mathcal{R}_6 = R_{l+}. \tag{32}$$

They are defined by Eqs. (21), (23), and (29), and have asymptotic expansions given by Eqs. (30) and (31). The notation R_{l-} and R_{l+} has been adopted to indicate the retarded and advanced character of the solutions, while the notation $\mathcal{R}_5$ and $\mathcal{R}_6$ has been adopted to show the association of the asymptotic expansions with the third singular point at $x = +\infty$.

5. GLOBAL PROPERTIES OF THE SOLUTIONS

In the mathematical formulation of a physical problem the differential equations obeyed by the field are supplemented by a set of boundary conditions. In our case, in addition to Eq. (2), Ψ will have to satisfy some conditions containing Ψ and/or its derivatives evaluated on some surfaces, most probably[19] $r = 0$, $r = r_s$, and $r = +\infty$. Consequently, we must know how a particular solution of Eq. (11) behaves over all space–time. In fact, it will suffice to know the behavior of a particular solution at the singular points of Eq. (11), since every solution is analytic at the regular points. To put it differently, we have to know to what linear combination of $\mathcal{R}_3$ and $\mathcal{R}_4$ (or $\mathcal{R}_5$ and $\mathcal{R}_6$) a given linear combination of $\mathcal{R}_1$ and $\mathcal{R}_2$ corresponds.

In principle, we face the general problem of finding the analytic continuation of a given solution of a differential equation.[25] However, here we are interested in practical answers, which can be used in numerical computations. In what follows we will limit ourselves to the real axis $z = \mathrm{Re}\, z = x$.

The matching of the solutions can be attained through the use of some linear relationships among $\mathcal{R}_i$ ($i = 1, 2, 3, 4, 5, 6$). If

$$W[\mathcal{R}_i, \mathcal{R}_j] = \mathcal{R}_i \frac{d\mathcal{R}_j}{dx} - \mathcal{R}_j \frac{d\mathcal{R}_i}{dx} \tag{33}$$

is the *Wronskian* of any two of the six solutions given by Eqs. (13), (14), (17), (18), and (32), then a constant K_{ij} exists such that

$$W[\mathcal{R}_i, \mathcal{R}_j] = K_{ij}/x(x - x_s), \tag{34}$$

as it can be proved easily from Eq. (11). Moreover, the identities

$$K_{ij}\mathcal{R}_k + K_{jk}\mathcal{R}_i + K_{ki}\mathcal{R}_j = 0 \tag{35}$$

and

$$K_{ij}K_{kl} + K_{ik}K_{lj} + K_{il}K_{jk} = 0 \tag{36}$$

are direct consequencies of Eq. (34). Equation (35) is obviously the key in relating the solutions among themselves. It is enough to find the fifteen K_{ij} ($K_{ij} = -K_{ji}$), although they are not all independent.

We start by evaluating the simplest of them, namely K_{12}, K_{34}, K_{56}, using the definition (34) and the series expansions for $\mathcal{R}_i$. Since K_{ij} is a constant, it can be evaluated at any point, where the respective series converge. However, the expressions become simpler when we consider the limit of K_{ij} as x goes to one of the singular points $0, x_s, +\infty$. We find

$$K_{12} = -x_s, \quad K_{34} = -2ix_s^2, \quad K_{56} = -2i. \tag{37}$$

Note that K_{56} can be evaluated only by taking the limit as $x \to +\infty$ because $\mathcal{R}_5$ and $\mathcal{R}_6$ have asymptotic expansions only.

We come now to the evaluation of $K_{13}, K_{14}, K_{23}, K_{24}$. The expansions of the needed $\mathcal{R}_i$ converge for $0 < x < x_s$ and, consequently, we can take the limits as x goes to x_s from below. We find

$$K_{A3} = ix_s^2 \lim_{x\to x_s-} \left[e^{ix_s \ln|x-x_s|} \left(\mathcal{R}_A + i\, \frac{x-x_s}{x_s} \frac{d\mathcal{R}_A}{dx} \right) \right], \tag{38}$$

where $A = 1, 2$. K_{14} and K_{24} are found to be the complex conjugates of K_{13} and K_{23}, respectively. Note that the coefficients c_n and d_n of the expansions for $\mathcal{R}_3$ and $\mathcal{R}_4$ do not appear in expression (38).

The evaluation of $K_{35}, K_{36}, K_{45}, K_{46}$ requires a more elaborate scheme of matching. The series in Eqs. (17) and (18) converge for $0 < x < 2x_s$ only, and the point $x = +\infty$ lies far from the circle of convergence. Hence, we have to reexpress $\mathcal{R}_3$ and $\mathcal{R}_4$ so that the expansion of the solution around $x = x_s$ will converge up to $x = +\infty$. Using the transformation $y = x^{-1}$, we rewrite Eq. (11) as

$$y^4(y-y_s)^2 \frac{d^2R}{dy^2} + y^4(y-y_s)\frac{dR}{dy} + [y_s^2 + y_s l(l+1)(y-y_s)y^2]R = 0. \tag{39}$$

Its solutions $\mathcal{R}'_3$ and $\mathcal{R}'_4$ in the neighborhood of the regular singular point $y = y_s = x_s^{-1}$ are (after resubstitution of y and y_s with x^{-1} and x_s^{-1})

$$\mathcal{R}'_3(x) = \exp[ix_s \ln|(x_s^2/x) - x_s|] \cdot \mathcal{R}''_3(x), \tag{40}$$

$$\mathcal{R}''_3(x) = \sum_{n=0}^{\infty} c'_n \left(\frac{1}{x} - \frac{1}{x_s} \right)^n, \tag{41}$$

with $\mathcal{R}'_4(x)$ and $\mathcal{R}''_4(x)$ given by the complex conjugates of expressions (40) and (41), respectively. The coefficients c'_n are related by the recurrence relation ($c'_0 = 1$)

$$\begin{aligned} &y_s^4(n+2ix_s)nc'_n + [4(n-1+ix_s)^2 + l(l+1)]y_s^3 c'_{n-1} \\ &\quad + [3(n-2+ix_s)^2 + l(l+1)]2y_s^2 c'_{n-2} \\ &\quad + [4(n-3+ix_s)^2 + l(l+1)]y_s c'_{n-3} \\ &\quad + (n-4+ix_s)^2 c'_{n-4} = 0. \end{aligned} \tag{42}$$

An easy calculation shows that the Wronskians of $\mathcal{R}_3, \mathcal{R}'_3$ and of $\mathcal{R}_4, \mathcal{R}'_4$ are zero and, consequently, $\mathcal{R}'_3$ and $\mathcal{R}'_4$ are proportional to $\mathcal{R}_3$ and $\mathcal{R}_4$, respectively. In fact, in Eqs. (40) and (41) we have normalized $\mathcal{R}'_3$ and $\mathcal{R}'_4$ so that (when $c_0 = c'_0 = 1$)

$$\mathcal{R}_3 = \mathcal{R}'_3, \quad \mathcal{R}_4 = \mathcal{R}'_4. \tag{43}$$

However, the expressions for $\mathcal{R}''_3$ and $\mathcal{R}''_4$ [Eq. (41) and its complex conjugate] converge for $0 < y < 2y_s$ or $x > x_s/2$, namely for x up to $+\infty$. Hence, we can take the limits of $K_{35}, K_{45}, K_{36}, K_{46}$ as $x \to +\infty$ and use the asymptotic expansions for $\mathcal{R}_5$ and $\mathcal{R}_6$. The result is

$$K_{3B} = (\mp i)^l e^{ix_s \ln x_s} \lim_{x\to+\infty} \left[x e^{\pm i(x + x_s \ln|x-x_s|)} \left(\mathcal{R}''_3 \pm i\, \frac{d\mathcal{R}''_3}{dx} \right) \right], \tag{44}$$

where the upper sign is to be taken when $B = 5$ and the lower sign when $B = 6$. K_{45} and K_{46} are the complex conjugates of K_{36} and K_{35}, respectively.

An attempt to calculate $K_{15}, K_{25}, K_{16}, K_{26}$ along the same lines as above results in highly complicated formulas, because no simple transformation[26] exists, which will bring $x = +\infty$ on the circumference of the circle of convergence of the solution around $x = 0$. However, $K_{15}, K_{16}, K_{25}, K_{26}$ can be calculated indirectly from the relation (36) in terms of the remaining eleven K_{ij}, for which formulas have already been given.

In the numerical evaluation of K_{ij}, the formulas (38) and (44) can be simplified by choosing appropriately the values through which x goes to the limit. If, for example, we set

$$x_\nu = x_s - \exp(-2\pi\nu/x_s), \quad x'_\nu = x_s + \exp(2\pi\nu/x_s), \tag{45}$$

where ν is a positive integer, then

$$K_{A3} = ix_s^2 \lim_{\nu\to+\infty} \left[\mathcal{R}_A + i\, \frac{x-x_s}{x_s} \frac{d\mathcal{R}_A}{dx} \right]_{x=x_\nu} \quad (A = 1, 2) \tag{46}$$

and

$$K_{3B} = (\mp i)^l e^{ix_s \ln x_s} \lim_{\nu\to+\infty} \left[x e^{\pm ix} \left(\mathcal{R}''_3 \pm i\, \frac{d\mathcal{R}''_3}{dx} \right) \right]_{x=x'_\nu} \tag{47}$$

(the upper sign if $B = 5$, the lower if $B = 6$). A different choice of x'_ν can eliminate completely the factor $e^{\pm ix}$ in Eq. (47).

The next step in solving the complete boundary value problem will be the selection of two solutions $R_l^{(i)}$ and $R_l^{(e)}$ to represent the field in the "near zone" and the "far zone", respectively. $R_l^{(i)}$ and $R_l^{(e)}$ will be called "interior" and "exterior" and will play the roles of j_l and h_l of Eqs. (8) and (9). The interior and exterior solutions will be linear combinations of R_{l+} and R_{l-} and in view of the results of the present section can be expressed as linear combinations of $\mathcal{R}_1$ and $\mathcal{R}_2$ or $\mathcal{R}_3$ and $\mathcal{R}_4$. However, their selection has to be done after the exact formulation of the physical problem we wish to solve, since the boundary conditions will determine the appropriate $R_l^{(i)}$ and $R_l^{(e)}$.

Some final remarks should be added here. Contrary to the flat-space case, in a curved space the electromagnetic 4-potential does not satisfy the same wave equation [Eq. (2)] as the scalar field Ψ. Consequently, the radial factor of the electromagnetic potential will satisfy a radial wave equation[19] different from Eq. (11).

The study of the solutions of this new radial equation can be accomplished[27] along the lines of this paper. Beyond that, the methods of this paper can be used in studying radial equations which are derived from equations similar to Eq. (2) (as the Klein–Gordon equation) in spherically symmetric spaces. These spaces can satisfy the Einstein or similar equations, e.g., in the Brans–Dicke and Weyl theories.[28] However, in these more general cases the study of the static field[28] should be completed before going to time-dependent situations.

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[12]The functions j_l and h_l are normalized as in Refs. 10 and 11.

[13]Without the factor $e^{-i\omega t}$ Eq. (3) represents a static field. This case has been studied in detail by S. Persides, J. Math. Anal. Appl. (to appear).

[14]This equation has been given by many authors, i.e., Refs. 1 and 3. However, it has never been studied by the methods used in classical textbooks on the subject (as Refs. 15 and 16).

[15]E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956).

[16]A. R. Forsyth, *Theory of Differential Equations* (Dover, New York, 1959), Vol. IV.

[17]A concise study of regular, irregular points and asymptotic expansions can be found in V. I. Smirnov, *A Course of Higher Mathematics* (Pergamon Press, London, 1964), Vol. III.

[18]By definition all coefficients a_n, b_n, c_n, d_n, g_n, τ_n, and c'_n in the recurrence relations (15), (16), (19), (20), (28), (30), and (42) are zero, if $n < 0$.

[19]S. Persides, *Proceedings of the First European Astronomical Meeting, Athens, 1972* (to be published).

[20]An alternative transformation used in the literature to study disturbances in Schwarzschild's space-time is a change of the independent variable to $r^* = r + r_s \ln(r/r_s - 1)$. However, this transformation is not convenient, because the coefficients of the resulting equation are not rational functions of r^* (see, i.e., Refs. 3,4, and 5).

[21]In a retarded coordinate system, u, r, θ, φ, the only nonzero components of the metric tensor are $g_{00} = (1 - r_s/r)c^2$, $g_{01} = g_{10} = c$, $g_{22} = -r^2$, $g_{33} = -r^2 \sin^2\theta$. The transformation is $u = t - rc^{-1} - r_s c^{-1} \ln(r/r_s - 1)$ with r, θ, φ unchanged.

[22]If $F_l(z, x_s; \epsilon)$ is a solution of Eq. (22), then $F_l(w, \epsilon; -x_s)$ is a solution of Eq. (26). From this property an integral equation can be obtained for F or G.

[23]A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956). See also Ref. 17.

[24]P_l is the Legendre polynomial normalized so that $P_l(1) = 1$.

[25]See, for example, Ref. 15, p. 286 and Ref. 17, p. 363.

[26]In fact we must find a transformation $y = f(x)$ which will map $x = x_s$ to $y = y_s$ and $x = +\infty$ to $y = y_\infty$ so that (a) *no* other singular point lies *inside* the circle $(y_s, |y_s - y_\infty|)$ and (b) there is a large positive number x_0 such that all finite $x > x_0$ are mapped *inside* the circle $(y_s, |y_s - y_\infty|)$. It can be proved that a (complex) bilinear transformation cannot satisfy these conditions. A more involved transformation, such as a complex Schwarz–Christoffel transformation or a highly nonlinear transformation, can be found to satisfy the above conditions.

[27]S. Persides and B. Xanthopoulos, to be published.

[28]N. Cherry, Nuovo Cimento B **4**, 144 (1971).