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The Laplace and Poisson Equations in Schwarzschild's Space–Time

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The method of separation of variables is used to solve the Laplace equation in Schwarzschild's space–time. The solutions are given explicitly in series form and in terms of Legendre functions. Green's function is determined and remarks are made on the solution of Poisson's equation for a point source.

1. INTRODUCTION

A number of works have dealt recently with problems in Schwarzschild's space–time. Israel's theorem [1] presents Schwarzschild's solution as the only electrovac solution of Einstein's equations with closed, simply-connected equipotential surfaces and regular event horizon. The study of a collapsing star has indicated [2] that all multipole moments are radiated away, leaving behind a Schwarzschild black hole. Cohen and Wald [3] have studied the electrostatic field in a Schwarzschild background, with similar results.

In this paper a counterexample is reported. A test field obeying the Laplace equation is studied by solving exactly the appropriate differential equations in a Schwarzschild background. Using the method of separation of variables, we separate and examine in detail the radial dependence of the solution and its differences from its flat-space counterpart. Next, we express Green's function for Poisson's equation in a series form, and we find the coefficients. This enables us to write down the solution which is created by a pointlike source. Finally, conclusions are drawn concerning the behavior of the field when the source slowly approaches the horizon.

2. THE LAPLACE EQUATION

The Schwarzschild line element in the usual coordinates t, r, θ, φ is

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2\, dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{1}$$

where r_s is the *Schwarzschild radius*.

We consider a test field Ψ, namely, a field weak enough so that the geometry is unaffected by it. Let Ψ satisfy the wave equation

$$\Box\Psi \equiv g^{ij}\Psi_{;ij} = 4\pi f(t, \mathbf{r}), \tag{2}$$

where semicolon stands for covariant differentiation with respect to Schwarzschild's metric and Latin indices run from 0 to 3 corresponding to t, r, θ, φ, respectively. If f and Ψ are independent of time, Eq. (2) reduces to *Poisson's equation*

$$\triangle\Psi = -4\pi f(\mathbf{r}), \tag{3}$$

with

$$\triangle\Psi \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(1-\frac{r_s}{r}\right)\frac{\partial\Psi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}. \tag{4}$$

When

$$f(\mathbf{r}) = 0, \tag{5}$$

we get *Laplace's equation*

$$\triangle\Psi = 0. \tag{6}$$

We look for solutions of Eq. (6) of the form

$$\Psi = R(r)\, Y(\theta, \varphi). \tag{7}$$

Substituting into Eq. (6) we find the equation for spherical harmonics [4]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \ell(\ell+1)\, Y = 0, \tag{8}$$

and for R the equation

$$\frac{d}{dr}\left[r^2\left(1-\frac{r_s}{r}\right)\frac{dR}{dr}\right] - \ell(\ell+1)\, R = 0, \tag{9}$$

where ℓ has to be a nonnegative integer for $Y(\theta, \varphi)$ to be single-valued and convergent.

3. The Radial Equation

For $r_s = 0$, two independent solutions of Eq. (9) are r^ℓ and $r^{-\ell-1}$. When $r_s \neq 0$, we set $x = r/r_s$ into Eq. (9), which becomes

$$x(x-1)\frac{d^2R}{dx^2} + (2x-1)\frac{dR}{dx} - \ell(\ell+1)\, R = 0. \tag{10}$$

This is a special case of the *hypergeometric differential equation*. It has three regular singular points at $x = 0$, $x = 1$, and $x = \infty$. A solution can be found in the form of a power series of x. Since ℓ is an integer, the series terminates, and becomes, after replacing x,

$$R_\ell^{(i)}(r, r_s) = \sum_{n=0}^{\ell} (-1)^{\ell+n} \frac{(\ell+n)!\,(\ell!)^2\, r_s^{\ell}}{(\ell-n)!\,(n!)^2\,(2\ell)!} \left(\frac{r}{r_s}\right)^n. \tag{11}$$

In a similar way we have a solution which is a power series of x^{-1}:

$$R_\ell^{(e)}(r, r_s) = \sum_{n=0}^{\infty} \frac{[(\ell+n)!]^2\,(2\ell+1)!}{(\ell!)^2\, n!(2\ell+n+1)!\, r_s^{\ell+1}} \left(\frac{r_s}{r}\right)^{\ell+n+1}. \tag{12}$$

The solutions of the radial equation can be related to Legendre functions, if we set

$$z = 1 - 2x. \tag{13}$$

Then Eq. (10) becomes

$$(1 - z^2)\frac{d^2R}{dz^2} - 2z\frac{dR}{dz} + \ell(\ell+1)\,R = 0, \tag{14}$$

which is the well known *Legendre equation*. The expressions (11) and (12) can be rewritten as

$$R_\ell^{(i)}(r, r_s) = (-1)^\ell \frac{(\ell!)^2\, r_s^{\ell}}{(2\ell)!} P_\ell\left(1 - \frac{2r}{r_s}\right) \tag{15}$$

and

$$R_\ell^{(e)}(r, r_s) = (-1)^{\ell+1} \frac{2(2\ell+1)!}{(\ell!)^2\, r_s^{\ell+1}} Q_\ell\left(1 - \frac{2r}{r_s}\right), \tag{16}$$

where $P_\ell(z)$ and $Q_\ell(z)$ are the *Legendre functions of the first and second kind*, respectively [5, 6]. The factors in Eqs. (11), (12), (15), and (16) have been chosen so that

$$\lim_{r_s \to 0^+} R_\ell^{(i)}(r, r_s) = r^\ell \tag{17}$$

and

$$\lim_{r_s \to 0^+} R_\ell^{(e)}(r, r_s) = r^{-\ell-1}. \tag{18}$$

We study now the behavior of $R_\ell^{(i)}$ and $R_\ell^{(e)}$ at the crucial points $r = 0$,

$r = r_s$, and $r = \infty$. Using the properties of $P_\ell(z)$ and $Q_\ell(z)$, we can prove that

$$\lim_{r\to 0^+} R_\ell^{(i)}(r, r_s) = (-1)^\ell \frac{(\ell!)^2}{(2\ell)!} r_s^{\ell}, \tag{19}$$

$$\lim_{r\to 0^+} \frac{R_\ell^{(e)}(r, r_s)}{\ln(r/r_s)} = (-1)^\ell \frac{(2\ell+1)!}{(\ell!)^2} r_s^{-\ell-1}, \tag{20}$$

$$\lim_{r\to r_s} R_\ell^{(i)}(r, r_s) = \frac{(\ell!)^2}{(2\ell)!} r_s^{\ell}, \tag{21}$$

$$\lim_{r\to r_s} \frac{R_\ell^{(e)}(r, r_s)}{\ln|1 - r/r_s|} = -\frac{(2\ell+1)!}{(\ell!)^2} r_s^{-\ell-1}, \tag{22}$$

$$\lim_{r\to+\infty} [r^{-\ell} R_\ell^{(i)}(r, r_s)] = 1, \tag{23}$$

and

$$\lim_{r\to+\infty} [r^{\ell+1} R_\ell^{(e)}(r, r_s)] = 1. \tag{24}$$

Note that the limits (21) and (22) do not depend on the way r approaches r_s. Also, the quantities $r^{-\ell}R_\ell^{(i)}$ and $r^{\ell+1}R_\ell^{(e)}$ have the same behavior in either case, $r_s \to 0^+$ or $r \to +\infty$.

Finally, we calculate the *Wronskian* of $R_\ell^{(i)}$ and $R_\ell^{(e)}$ to be

$$W[R_\ell^{(i)}, R_\ell^{(e)}] \equiv R_\ell^{(i)} \frac{dR_\ell^{(e)}}{dr} - R_\ell^{(e)} \frac{dR_\ell^{(i)}}{dr} = -\frac{2\ell+1}{r^2 - rr_s}. \tag{25}$$

Other properties of $R_\ell^{(i)}$ and $R_\ell^{(e)}$ (recurrence relations, Rodrigues' formula, generating functions, definite integrals) can be established, but will not be given here, since they are rather involved and will not be needed later.

4. Green's Function

If $f(\mathbf{r})$ in Eq. (3) is zero for large enough r, and

$$\Psi = \mathcal{O}(r^{-1}) \tag{26}$$

as $r \to +\infty$ then the solution of Eq. (3) poses a Dirichlet [7] problem. If $f(\mathbf{r})$ is different from zero only at a single point, then we have a "point source." For the special case [4]

$$f(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}'), \tag{27}$$

the solution of Eq. (3) is $G(\mathbf{r}, \mathbf{r}')$, namely, *Green's function* for the potential problem.

To find $G(\mathbf{r}, \mathbf{r}')$ we write

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{r^2}\, \delta(r - r') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi), \tag{28}$$

and

$$G(\mathbf{r}, \mathbf{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}(\theta', \varphi')\, R_\ell(r, r')\, Y_{\ell m}(\theta, \varphi), \tag{29}$$

where the possible dependence of G, $A_{\ell m}$, and R_ℓ or r_s has been suppressed. Substituting into Eq. (3), we find, using Eqs. (4) and (8)

$$A_{\ell m}(\theta', \varphi') = Y^*_{\ell m}(\theta', \varphi') \tag{30}$$

and

$$\frac{d}{dr}\left[r^2\left(1 - \frac{r_s}{r}\right)\frac{d}{dr} R_\ell(r, r')\right] - \ell(\ell + 1)\, R_\ell(r, r') = -\, 4\pi\delta(r - r'). \tag{31}$$

This is identical with Eq. (9), at any point $r \neq r'$; hence, R_ℓ is a linear combination of $R_\ell^{(i)}$ and $R_\ell^{(e)}$. Finiteness of $G(\mathbf{r}, \mathbf{r}')$ at $r = 0$ and $r = +\infty$, and symmetry under interchange of $\mathbf{r}$ and $\mathbf{r}'$ require that

$$R_\ell(r, r') = C_\ell R_\ell^{(i)}(r_<, r_s)\, R_\ell^{(e)}(r_>, r_s), \tag{32}$$

where $r_<$ ($r_>$) the smaller (larger) of r and r'. At $r = r'$ the function $R_\ell(r, r')$ is continuous. However, its first derivative is not. Integrating Eq. (31) from $r' - \epsilon$ to $r' + \epsilon$ ($\epsilon > 0$) and using the Wronskian of $R_\ell^{(i)}$ and $R_\ell^{(e)}$, we determine the value of C_l to be

$$C_\ell = \frac{4\pi}{2\ell + 1}. \tag{33}$$

Consequently, from Eq. (29) we have Green's function

$$G(\mathbf{r}, \mathbf{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell + 1}\, Y^*_{\ell m}(\theta', \varphi')\, R_\ell^{(i)}(r_<, r_s)\, R_\ell^{(e)}(r_>, r_s)\, Y_{lm}(\theta, \varphi) \tag{34}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{-\,8\pi}{r_s}\, Y^*_{\ell m}(\theta', \varphi')\, P_\ell\left(1 - \frac{2r_<}{r_s}\right) Q_\ell\left(1 - \frac{2r_>}{r_s}\right) Y_{\ell m}(\theta, \varphi), \tag{35}$$

where the last expression has been derived, using Eqs. (15) and (16).

5. The Point Source

We consider now the case where the only source is a point particle of strength q located at $\mathbf{r}'$ outside the Schwarzschild sphere ($r' > r_s$). The solution $\Psi(\mathbf{r}, \mathbf{r}', r_s)$ of Eq. (3) which behaves appropriately at $r = 0$ and $r = \infty$ is obviously

$$\Psi(\mathbf{r}, \mathbf{r}', r_s) = qG(\mathbf{r}, \mathbf{r}'), \tag{36}$$

since in this case

$$f(\mathbf{r}) = q\delta(\mathbf{r} - \mathbf{r}'). \tag{37}$$

We examine the behavior of the field at points $r \geqslant r_s$ as $r' \to r_s^+$. This process corresponds to a "slow fall" of the point source towards the Schwarzschild sphere, slow enough for our static considerations to hold. In a similar study, Cohen and Wald [3] have found that the electrostatic field remains finite at the horizon $r = r_s$ and approaches the spherically symmetric Reissner–Nordstrom solution for any observer outside the Schwarzschild sphere.

As $r' \to r_s^+$ an observer at $\mathbf{r}$ ($r > r_s$) measures a scalar field

$$\Psi(\mathbf{r}, \mathbf{r}', r_s)\,|_{r'=r_s} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^{\ell+1}\, 8\pi q r_s^{-1} Y_{\ell m}^*(\theta', \varphi') Q_\ell\left(1 - \frac{2r}{r_s}\right) Y_{\ell m}(\theta, \varphi), \tag{38}$$

which obviously has nonzero multipoles contrary to the electrostatic case.

On the other hand, the field at $r = r_s$ due to a point particle at $\mathbf{r}'$ ($r' > r_s$) is given by the right side of Eq. (38) after replacement of r by r'.

Taking again the limit we find

$$\lim_{r' \to r_s^+} \left[\frac{\Psi(\mathbf{r}, \mathbf{r}', r_s)}{\ln(r'/r_s - 1)}\right]_{r=r_s} = -4\pi q r_s^{-1} \delta(\varphi - \varphi')\, \delta(\cos\theta - \cos\theta'). \tag{39}$$

Hence, as $r' \to r_s^+$, Ψ blows up with an additional factor $\ln(r'/r_s - 1)$. This factor is a direct result of the Schwarzschild metric and raises the possibility of destruction for the horizon.

It is possible that an appropriate choice of the source will change this "singular" behavior on the horizon. Let us assume [8] that in Eq. (2)

$$f(t, \mathbf{r}) = q_0 c^{-1} (u^0)^{-1}\, \delta(\mathbf{r} - \mathbf{r}'), \tag{40}$$

where $u^0 = dt/dS$ is the time-component of the four-velocity of the particle, and q_0 is a constant. Then for a particle with r, θ, φ constant

$$u^0 = \left(1 - \frac{r_s}{r}\right)^{-1/2} c^{-1}, \tag{41}$$

and in Eq. (3) we have

$$f(\mathbf{r}) = q_0 \left(1 - \frac{r_s}{r'}\right)^{1/2} \delta(\mathbf{r} - \mathbf{r}'). \tag{42}$$

For this source the field at $\mathbf{r}$ is

$$\Psi(\mathbf{r}, \mathbf{r}', r_s) = -8\pi q_0 \cdot r_s^{-1} \left(1 - \frac{r_s}{r'}\right)^{1/2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^{*}(\theta', \varphi') P_\ell \left(1 - \frac{2r_<}{r_s}\right) \times Q_\ell \left(1 - \frac{2r_>}{r_s}\right) Y_{\ell m}(\theta, \varphi). \tag{43}$$

From this expression it is obvious that when the particle "falls" on the horizon ($r' \to r_s^+$), the field at an arbitrary point $\mathbf{r}$ ($r > r_s$) goes to zero as $(1 - r_s/r')^{1/2}$. Even the monopole term does not survive, contrary to the electrostatic case. Finally, the field at $r = r_s$ due to a "falling" particle does not become infinite. Instead of Eq. (39), we have

$$\lim_{r' \to r_s} \left[\frac{\Psi(\mathbf{r}, \mathbf{r}', r_s)}{(1 - r_s/r')^{1/2} \ln(r'/r_s - 1)}\right]_{r=r_s} = -4\pi q_0 r_s^{-1} \delta(\varphi - \varphi')\, \delta(\cos\theta - \cos\theta'). \tag{44}$$

Consequently, as the point source "falls" on the horizon, the strength of the field at $r = r_s$ goes to zero as $(1 - r_s/r')^{1/2} \ln(r'/r_s - 1)$.

Summarizing, we can say that the behavior of the scalar field generated by a point scalar particle in Schwarzschild's space–time is different from that of a point charged particle. Depending on the source, we have the following two cases for "slow fall." The multipoles are not radiated and infinite stresses appear on the horizon (if Eq. (37) is assumed), or all multipoles are radiated and no stresses appear on the horizon (if Eq. (42) is assumed).

A word of caution also is needed. Since our considerations are purely static, remarks and statements made on time-dependent phenomena should be taken only as good indications of what really happens. These phenomena will be discussed rigorously in the next step, namely, the solution of the wave equation in Schwarzschild's space–time.

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