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Comparing the right-hand sides of Eqs. (A3) and (A4), it is seen that they are complex conjugates of each other. Hence, R_p is real. Q.E.D.

APPENDIX B: ABSENCE OF INFINITE RESONANCES

The condition for resonance requires that there exist real positive values of η and β for which the denominators of the coefficients c_p and d_p go to zero. The denominator for both coefficients is given by

$$H_p^{(1)}(\beta)[dR_p(k_0r)/d(k_0r)]_\beta - H_p^{(1)\prime}(\beta)R_p(\beta) = 0, \quad \text{(B1)}$$

where the prime represents the derivative with respect to the argument.

We proved in Appendix A that the function $R_p(r)$ is real. Hence, using the definition of the Hankel function,

$$H_p^{(1)}(\beta) = J_p(\beta) + iY_p(\beta), \quad \text{(B2)}$$

substituting into Eq. (B1), and setting the real and imaginary parts equal to zero yields

$$J_p(\beta)\left(\frac{dR_p(k_0r)}{d(k_0r)}\right)_\beta - J_p'(\beta)R_p(\beta) = 0, \quad \text{(B3)}$$

$$Y_p(\beta)\left(\frac{dR_p(k_0r)}{d(k_0r)}\right)_\beta - Y_p'(\beta)R_p(\beta) = 0. \quad \text{(B4)}$$

Eliminating

$$R_p(\beta)\Big/\left(\frac{dR_p(k_0r)}{d(k_0r)}\right)_\beta$$

from Eqs. (B3) and (B4) gives

$$J_p(\beta)Y_p'(\beta) - J_p'(\beta)Y_p(\beta) = 0 \quad \text{(B5)}$$

as the requirement for infinite resonance. However, the left-hand side of Eq. (B5) is the Wronskian[12] having the value of $2/\pi k_0 a$ which can never be zero for finite values of $k_0 a$.

[12] Reference 3, Chap. 9, p. 360, Formula 9.1.16.

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Scattering of Massless Scalar Waves by a Schwarzschild "Singularity"*

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This paper investigates the scattering and absorption of scalar waves satisfying the equation $\phi_{;\mu}^{;\mu} = 0$ in the Schwarzschild metric. This problem has been previously considered by Hildreth. We find, for a Schwarzschild mass m, the following cross sections in the zero-frequency limit for s-waves: σ(absorption) $= 0$, $d\sigma/d\Omega \simeq [c + \frac{1}{4}(2m)\ln(2m\omega)]^2$, where c is a constant of order m. These results disagree with the previous calculation. We exhibit a method of solution for the equation. Its limiting (Newtonian) form, with suitable identification of the coefficients, is the problem of Coulomb scattering in nonrelativistic quantum mechanics. By demanding coordinate conditions which for large l allow the usual Coulomb results in a partial-wave expansion, we are able to define a partial-wave cross section. The (summed) differential cross section for small frequencies inherits the logarithmic behavior of the s-wave part, which is the only contribution explicitly calculated. (The $l \neq 0$ contributions and the behavior of the cross sections for $\omega \neq 0$ are qualitatively indicated.) Cosmological considerations are given which cut off this divergence.

I. INTRODUCTION AND SUMMARY

At one time the Schwarschild "singularity" at[1] $r = 2m$ was dismissed as unphysical, since it does not occur in the gravitational fields of normal stars. But as unexpected observations have more recently made speculations about possible peculiar astrophysical objects interesting, there has been an increased appreciation of the work of Oppenheimer and Snyder[2] which showed, not only that one could tolerate the actual existence of the Schwarzschild $r = 2m$ surface, but that there were even reasonable physical processes by which it could be produced.

* Supported in part by NASA Grant NSG-436, and by the Aerospace Research Laboratory, Office of Aerospace Research, U.S. Air Force.
† National Science Foundation Pre-doctoral Fellow 1963–1967.
[1] We take $G = c = 1$.

[2] J. R. Oppenheimer and H. Snyder, Phys. Rev. **56**, 455 (1939).

(For some recent discussions see May and White[3] and Chap. 8 of Thorne's lectures.[4])

We refer to the remnant of such a process as an Oppenheimer–Snyder (OS) star, when some connotation of physical reality is not out of place, or as a Schwarzschild horizon, when it is more appropriate to suggest the mathematical idealizations we invoke in studying it. These idealizations include spherical symmetry, ignoring any angular momentum of the original matter, and the neglect of any matter expelled in the formation process, as well as of any orbiting debris with entrained magnetic fields, etc.

In this paper we illustrate by an example how an OS star is, like more normal final equilibrium states such as cooled white dwarfs and neutron stars, an object whose structure and whose response to various probes is independent of the detailed dynamical processes by which it is formed. In fact, we require no description whatsoever of the region $r < 2m$ beyond the assumption that all the matter responsible for the gravitational field has, some time in the past, fallen in beyond this $r = 2m$ Schwarzschild "horizon" leaving a spherically symmetric field. This assumption will be translated into a boundary condition at $r = 2m$.

With these idealizations, the exterior to a collapsed star is the Schwarzschild solution, and so the star's mass can be measured by observing planetary orbits or by doing bending of light experiments. In the latter case, Darwin[5] has shown that we cannot expect to probe more closely than $3m$ by means of high-frequency light, as light rays from infinity which reach smaller radii are trapped and never return to $r = \infty$. But with this limitation, we cannot tell with certainty whether the mass we observe is an OS star or, for instance, a neutron star with radius $2.5m$. A probe which gets closer in would be desirable, and we should investigate longer wavelength scattering. This will replace the corpuscular picture of Einstein's deflection-of-light calculation by a wave packet picture. We would expect the maximum amount of detail to be found from wave packets whose typical wavelength is approximately m. Lower frequencies would lose resolution; higher frequencies would be trapped if they probed close to $r = 2\mathrm{m}$.

The problem that we actually discuss is a simplified version of that stated above. Instead of light, we calculate for a scalar field ϕ. And, although we state the problem for finite frequencies ω and for any angular momentum quantum number l, we will present a solution only for the $\omega \to 0$, s-wave limit. We seek cross sections for the scattering and absorption of scalar waves ϕ by a background Schwarzschild field.

The equation which determines the scattering is a generalization of the flat-space wave equation, namely

$$\phi_{;\mu}^{;\mu} = (-g)^{\frac{1}{2}} \frac{\partial}{\partial x^\mu}\left[(-g)^{\frac{1}{2}} g^{\mu\nu} \frac{\partial}{\partial x^\nu}\phi\right] = 0. \quad (1)$$

The metric entering this equation is taken to be the Schwarzschild metric, and it is its differing from the flat Lorentz metric which causes the scattering. By using the Schwarzschild metric, we have of course neglected any small gravitational field which the packet of ϕ waves itself might produce.

This same scattering problem has been considered previously by Hildreth[6] who thought of ϕ as an additional component of the gravitational field according to the Brans–Dicke[7] theory. The present restudy of this scattering problem obtains solutions of the basic differential equations somewhat more directly and leads to results which differ from Hildreth's

The radial equation is cast into the form of an effective potential equation by using the Regge–Wheeler[8] coordinate r^*. The boundary conditions at $r = 2m+$ are found. The difficulties of long-range forces, which are familiar from the problem of Coulomb scattering, are discussed. Although they have been extensively investigated in the literature, they pose the most difficult problems to be solved in the cross section calculation, particularly in terms of the coordinate invariance of the result. The separation into a "distorted plane wave" and outgoing waves is discussed. An ambiguity arises in the definition of the partial-wave scattering phase shifts. We fix the ambiguity in such a way that the usual "Coulomb" result is obtained for large l, so the large l partial cross sections contribute the usual large amount to small angle scattering (only).

We quote our results for the s wave $\omega \to 0$ limit:

$$\begin{aligned} &\sigma(\text{absorption}) = 0, \\ &d\sigma/d\Omega \simeq \left[c + \frac{2m}{3}\ln(2m\omega)\right]^2. \end{aligned} \quad (2)$$

Here c is a constant of order m.

[3] M. M. May and R. H. White, Phys. Rev. **141**, 1232 (1966).
[4] K. S. Thorne, in *Les Houches Summer School Proceedings, 1966* (Gordon and Breach Science Publishers, London, 1967).
[5] Sir Charles Darwin, Proc. Roy. Soc. **A249**, 180 (1958).
[6] W. W. Hildreth, Ph.D. thesis, Princeton University (1964).
[7] R. H. Dicke, Phys. Rev. **125**, 2163 (1962), and earlier references there.
[8] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

These disagree with Hildreth's results,[9] which are

$$\sigma(\text{absorption}) \simeq 12.5\,(2m)^2,$$

$$\frac{d\sigma}{d\Omega} - \frac{d\sigma}{d\Omega}\text{ newtonian} \simeq 1.96\,(2m)^2.$$

The logarithmic divergence in Eq. (2) is cut off by the requirement that the observer be outside the near field zone, which requires $\omega \gg 1/R_0$ where R_0 is, as a maximum, the Hubble radius. The cross section is then limited to $\sim(50\,m)^2$ for scatterers the mass of the sun. (In our $G = c = 1$ units, $m_{\text{sun}} = 1.5 \times 10^5$ cm.) However, this cross section is small enough so that it would contribute negligibly to the behavior of the scalar wave ϕ in a universe which contains many real stars whose radii are $\sim 10^{11}$ cm.

II. GENERAL CONSIDERATIONS FOR SCALAR SCATTERING

A. Radial Equation

With the usual Schwarzschild metric

$$ds^2 = dr^2\left(1 - \frac{2m}{r}\right)^{-1} + r^2\,d\Omega^2 - \left(1 - \frac{2m}{r}\right)dt^2$$

(where m is the mass of the scatterer), Eq. (1) is

$$r^{-2}\left(1 - \frac{2m}{r}\right)\frac{d}{dr}\left[r^2\left(1 - \frac{2m}{r}\right)\frac{dR}{dr}\right] - \left(1 - \frac{2m}{r}\right)\frac{l(l+1)}{r^2}R + \omega^2 R = 0, \quad (3)$$

where separation of variables has given

$$\phi = e^{-i\omega t}R(r)P_l(\theta). \quad (4)$$

If

$$R = u/r, \quad (5)$$

we find

$$\left(1 - \frac{2m}{r}\right)\frac{d}{dr}\left[\left(1 - \frac{2m}{r}\right)\frac{du}{dr}\right] + \left[\omega^2 - \left(1 - \frac{2m}{r}\right)\left(\frac{2m}{r^3} + \frac{l(l+1)}{r^2}\right)\right]u = 0. \quad (6)$$

Equation (6) is still not free from first derivatives $d\mu/dr$.

However, the substitution[8]

$$dr^* = dr\left(1 - \frac{2m}{r}\right)^{-1}, \quad (7)$$

$$r^* = r + 2m\ln(r/2m - 1) + \text{const}, \quad (8)$$

puts Eq. (6) in the form

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - \left(1 - \frac{2m}{r}\right)\left(\frac{2m}{r^3} + \frac{l(l+1)}{r^2}\right)\right]u = 0. \quad (9)$$

[9] W. W. Hildreth, Ref. 6, p. 86. His B is $\frac{1}{2}m$, and we have factored out $(2m)^2$.

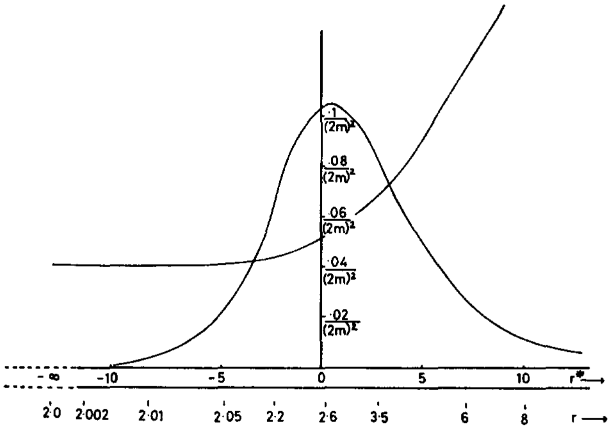

FIG. 1. The effective potential $V(r^*)$ plotted against a scale linear in r^*. (r and r^* are in units of m.) The constant in Eq. (8) has been set equal to zero. Also shown is the curve $u = r$ (not to scale), which is the solution to the s-wave radial equation for $\omega = 0$.

The problem has been reduced to a one-dimensional Schrödinger equation with independent variable r^*. This Schrödinger equation has an effective potential

$$V(r^*) = \left(1 - \frac{2m}{r}\right)\frac{2m}{r^3}, \quad (10)$$

with r considered a function of r^*.

The effective potential has been plotted in Fig. 1, taking the integration constant in r^* to be zero. Note that $V(r^*)$ has a rather low peak:

$$V_{\text{peak}} = 3\left(\frac{3}{16}\frac{1}{2m}\right)^2 \quad \text{at} \quad r = \tfrac{4}{3}(2m).$$

Furthermore, $V(r^*)$ is a finite range potential: it vanishes as $(r^*)^{-3}$ for $r^* \to +\infty$ and as $\exp(r^*/2m)$ for $r^* \to -\infty$ (corresponding to $r - 2m \to 0^+$). Therefore solutions of Eq. (9) can be obtained by familiar approximations.

B. Boundary Condition at $r = 2m$

We can write Eq. (9) in a dimensionless form by multiplying it by $(2m)^2$:

$$\left[\frac{d^2}{dy^{*2}} + q^2 - \left(1 - \frac{1}{y}\right)\frac{1}{y^3}\right]u = 0. \quad (11)$$

Here $y^* = r^*/2m$, $y = r/2m$, and $q = 2m\omega$. We set $l = 0$ since that is the case we investigate in Sec. III. (The boundary condition for $l \neq 0$ can be done in exactly the same way.) We set the integration constant in Eq. (8) equal to zero for convenience, and so

$$y^* = y + \ln(y - 1). \quad (12)$$

Now consider a well-defined pulse of scalar radiation of small but nonzero frequency, incident from $y^* = +\infty$, where $V(y^*) = 0$. This pulse will be partly reflected by the potential, so for $y^* \to +\infty$, we expect both ingoing and outgoing waves. Because the potential disappears so rapidly for $y^* \to -\infty$, there is no backscatter from even small negative values of y^*. The pulse will be entirely ingoing for $y^* \to -\infty$, i.e., at the Schwarzschild horizon $r = 2m+$. We then take as the condition at $y^* = -\infty$ $(r - 2m = +0)$ that the waves be pure ingoing.

If we suppose the source of the field to be an OS star, this boundary condition means that there can be no scalar radiation from the star.

Since at $t \to \infty$ the surface of an OS star will approach arbitrarily close to $r = 2m$, any radiation from its surface will be red-shifted, so that its intensity goes to zero.[10] We saw above that the active region for scattering of the ϕ waves is $r \gtrsim \frac{4}{3}(2m)$, well away from the infinite red-shift surface. So the OS star, when "completely" collapsed, becomes a cold sink for radiation and satisfies our idealized boundary condition.

C. Comparison Coulomb Problem

Equation (9) has permitted us to find boundary conditions for $r - 2m = 0+$; but since r^* differs by a logarithm from r, it is not a good coordinate to use to interpret results at $r \to \infty$. We can give a statement of the problem which makes the asymptotic properties more transparent by making a different substitution in (3):

$$R = \frac{\bar{u}}{r\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}}, \tag{13}$$

which gives

$$\frac{d^2\bar{u}}{dr^2} + \left[\frac{\omega^2}{\left(1 - \frac{2m}{r}\right)^2} + \left(\frac{m}{r}\right)^2 \frac{1}{r^2\left(1 - \frac{2m}{r}\right)} - \frac{l(l+1)}{r^2}\frac{1}{\left(1 - \frac{2m}{r}\right)}\right]\bar{u} = 0. \tag{14}$$

As $r \to \infty$, Eq. (14) becomes

$$\frac{d^2\bar{u}}{dr^2} + \left[\omega^2 + \frac{4m\omega^2}{r} + \frac{12m^2\omega^2}{r^2} - \frac{l(l+1)}{r^2} + O(r^{-3})\right]\bar{u} = 0. \tag{15}$$

There is an attractive Newtonian coupling between the "energy density" $4\omega^2$ and the mass of the OS star.[11]

For large l, we expect the wavefunction solving Eqs. (9) and (14) to have small amplitude for a radius less than some distance of closest approach $r_\perp \sim [l(l+1)]^{\frac{1}{2}}/\omega$. The solution depends essentially only on the region $r > r_\perp$. Thus, for any ω, there is an $l_{\min}(\omega)$ such that $l \geq l_{\min}$ means the term $12m^2\omega^2/r^2$ and the terms $O(r^{-3})$ are negligible in Eq. (14). So for large enough l, we need only consider the solutions to

$$\frac{d^2\bar{u}}{dr^2} + \left[\omega^2 + \frac{4m\omega^2}{r} - \frac{l(l+1)}{r^2}\right]\bar{u} = 0, \qquad R = \frac{\bar{u}}{r}, \tag{14'}$$

which are satisfactory approximations to those of Eq. (9) or Eq. (14). The second of Eq. (14') follows from the assumption that the neglected terms in the first Eq. (14') are small.

We call Eq. (14') the "comparison Coulomb problem." We should investigate solutions to Eq. (9) for $l < l_{\min}$; but for $l \geq l_{\min}$, we may take the solutions given by Eq. (14'). This means considering the standard nonrelativistic wave mechanics problem of scattering of (negative) electrons from a nucleus of charge $+Ze$ (and infinite mass) with the substitutions

$$\hbar^2/2m_e \to 1, \quad Ze^2 \to 4m\omega^2.$$

We may immediately compute the "Newtonian" scattering [given by Eq. (14')] by making these substitutions into the Coulomb formula. In particular, we find the Coulomb "phase shift," $-\Omega_l(\omega)$:

$$-\Omega_l = -\arg \Gamma(l + 1 + 2im\omega).$$

For small ω and $l = 0$

$$\Omega_l \simeq 2im\omega\psi(1).$$

$\psi(x)$ is the logarithmic derivative of $\Gamma(x)$; $\psi(1) = 0.577\cdots$ (Euler's constant). The $l = 0$, $\omega \to 0$ "partial cross section" is then

$$\frac{d\sigma}{d\Omega} \cong \omega^{-2}\,|2im\omega\psi(1)|^2 \cong \tfrac{1}{3}(2m)^2.$$

The summed cross section is given by the Rutherford formula[12,13]

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{\sin^4\theta/2}. \tag{16}$$

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[10] C. W. Misner, January 1965 Relativity Meeting, Stevens Institute of Technology, Hoboken, N.J. (unpublished).

[11] This same $1/r$ coupling would be found by reducing Eq. (1) to Schroedinger form—free from first derivatives—in any coordinate system r' which goes as $r' = r + \text{const} + O(1/r)$. The $1/r^2$ terms in Eq. (14) are not invariant under such coordinate changes, but it is shown in Secs. IIE and IIIB below that the cross sections are invariant.

[12] H. Bethe and E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Academic Press Inc., New York 1957), p. 31, Eq. (6.24).

[13] It is consistent to use the Rutherford formula for the geometrical optics case of deflection by the sun, where $2m \ll r_\perp$, since the dominant terms in the partial-wave expansion will come from the l values such that $l \sim r_\perp\omega$. Thus the term $l(l+1)/r^2$ completely dominates $12m^2\omega^2/r^2$ and $O(1/r^3)$ in Eq. (14), so Eq. (14') gives a correct result.

We note in passing that the Einstein deflection $\Delta\theta \simeq 4m/r_\perp$ gives a scattering cross section $d\sigma/d\Omega \simeq m^2(\theta/2)^{-4}$, in satisfactory agreement with Eq. (16) [we expect scalar and vector (light) waves to give the same answer in the geometrical optics limit] and justifying Eq. (14).

We have used the terms "phase shift" and "partial cross section" in the conventional sense, but it should be emphasized that for long-range forces, like the Newtonian gravitational force, these quantities are to a large extent really conventional, although the summed (over l) cross sections are well defined.

For short-range forces in ordinary quantum mechanics, the separation into partial waves and the separation into "plane incident" and scattered outgoing spherical waves is unambiguous. The boundary condition (usually square integrability at the origin) is also easily handled. Although the boundary condition in the ϕ-scattering problem was readily found in Sec. IIB above, the separation into "plane" and "scattered" waves and the application of the method of partial waves are more difficult. The difficulty arises partly because of the long-range force which distorts "plane" waves, even asymptotically, but also because of a need in relativity for invariance under coordinate transformation. These problems are dealt with in the next sections.

D. Form for the Asymptotic "Plane Wave"

A solution which satisfies Eq. (9), if we ignore terms in the equation which are $O(r^{*-3}\ln r^*)$ or smaller, is $r^* j_l(\omega r^*)$ where j_l is a spherical Bessel function with asymptotic behavior $j_l(\rho) \to \rho^{-1}\sin(\rho - \frac{1}{2}l\pi)$.

It also solves Eq. (14′), if *all* $O(r^{-2})$ terms are ignored. These solutions are significantly different from the solutions for short-range potentials, because the term $\ln r$ appears in the argument of j_l. In fact, it is well known[14] that for the comparison Coulomb Eq. (14′), a partial-wave solution has the asymptotic form

$$R_l(\text{Coulomb}) = \frac{\bar{u}}{r} \to r^{-1}\sin[\omega r + 2m\omega\ln(2\omega r) - \Omega_l(\omega) - l\pi/2]. \tag{17}$$

If we were presented with only the partial-wave expansion of this Coulomb solution, it would be impossible to apply straightforward partial-wave scattering theory because of the logarithmic term in Eq. (17), which apparently precludes an unambiguous asymptotic phase-shift determination. But the logarithm can be ignored, and it is the $\Omega_l(\omega)$ which determines the scattering. We can see this by noting the identity[14,15]

$$\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta) = 2\delta(1-\cos\theta).$$

We use this identity in the definition of the scattering amplitude $f(\theta)$:

$$f(\theta) = (2i\omega)^{-1}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta), \tag{18}$$

to obtain, for $\theta \neq 0$:

$$f(\theta \neq 0) = (2i\omega)^{-1}\sum_{l=0}^{\infty}(2l+1)e^{2i\delta_l}P_l(\cos\theta). \tag{19}$$

Note that this holds only for the *summed* scattering amplitude. But in Eq. (19), it is clear that if δ_l is real, then additive terms in δ_l which are independent of l in the asymptotic region—even if they involve $\ln r$—change the value of $f(\theta \neq 0)$ by only a phase. They do not affect $|f(\theta \neq 0)|^2$, and we have, for instance,

$$\frac{d\sigma}{d\Omega}(\theta \neq 0) = f^*(\theta \neq 0)f(\theta \neq 0),$$
$$= (2\omega)^{-2}\,|\sum(2l+1)e^{-2i\Omega_l(\omega)}P_l(\cos\theta)|^2.$$

the scattering cross section for the equivalent Coulomb problem, Eq. (14′).

We are dealing with a potential $\sim r^{-1}$ at large distances. [It is explicitly $1/r$ in Eq. (14) and the long-range character is included in the definition of r^*, and so appears when solutions of Eq. (9) are expressed as a function of r.] Both the forward-scattering amplitude and the total-scattering cross section (which diverge for r^{-n}, $n \leqslant 3$ and $n \leqslant 2$, respectively,[14]) are infinite. The only relevant quantity remaining to calculate or measure is the differential-scattering cross section. We thus lose nothing by modifying the forward behavior of $f(\theta)$, as was done in going from Eq. (18) to Eq. (19) above.

Ambiguities still exist in the definition of each particular phase shift, since we must decide which zero point to take for δ_l. (We must clearly take the *same* reference for every δ_l.) For short-range forces the phase shifts are also required to give the correct total cross section, which effectively fixes them so that $\lim_{l\to\infty}\delta_l = 0$, but this normalization does not hold for Coulomb quantum scattering.

[14] E. Landau and L. Lifschitz, *Quantum Mechanics, Non Relativistic Theory* (Pergamon Press, Inc., London 1958), para. 105, 106, and 112.

[15] This equation is true (the left side converges uniformly) except for $\cos\theta = -1$, where the convergence is only "in the mean." We point out that this lack of convergence in the backward direction is shared by the expansion for a plane wave, $\exp(ikr\cos\theta) = \sum i^l(2l+1)j_l(kr)P_l(\cos\theta)$.

We fix the constant in δ_l by demanding that the $\delta_l(\omega)$ approach $-\Omega_l(\omega)$, the phase shifts for the equivalent Coulomb problem (14′), for large l. (For fixed ω these are the partial-wave components which always stay far from the scattering center.) We do this by taking the integration constant in Eq. (8) to be $2m \ln (4m\omega)$, so

$$r^* = r + 2m \ln [2\omega(r - 2m)]. \tag{20}$$

As $r \to \infty$, $r^* \to r_c + O(m/r)$, where

$$r_c = r + 2m \ln (2\omega r)$$

is the combination appearing in the Coulomb partial-wave terms, Eq. (17). Thus, by the argument in Sec. IIC above, the phase shifts determined by Eq. (9), $\delta_l(\omega)$, tend to $-\Omega_l(\omega)$ as $l \to \infty$.

The method to use is (14′) and the known Coulomb phase shifts for the large l partial waves, and to apply Eq. (9) with the constant fixed by Eq. (20) to find the few small l phase shifts which will be significantly different, because of the effective potential $V(r^*)$, from the Coulomb ones. [We actually compute only one partial wave from Eq. (9), the $l = 0$, $\omega \to 0$ case.]

It is easy to show that this choice of constant (20) and this type of partial-wave manipulation is equivalent to splitting the total solution ϕ into

$$\begin{aligned}\phi_{\text{“plane”}} &= \frac{r^*}{r} e^{i\omega r^* \cos\theta} \\ &\to (\omega r)^{-1} \sum i^l (2l+1) P_l(\cos\theta) \\ &\qquad \times \sin(\omega r^* - l\pi/2), \qquad (21\text{a})\end{aligned}$$

$$\begin{aligned}\phi_{\text{“out”}} &= \phi - \phi_{\text{“plane”}} \\ &\to (2i\omega)^{-1} r^{-1} e^{i\omega r^*} \\ &\qquad \times \sum (2l+1)(e^{2i\delta_l} - 1) P_l(\cos\theta), \qquad (21\text{b})\end{aligned}$$

where the $\delta_l(\omega)$ are obtained by the usual partial-wave manipulation of the radial equation, treating r^* as an ordinary radial variable at infinity. Note that each term in the sums of Eq. (21) is an asymptotic solution to Eq. (9) and Eq. (14).[16]

E. Coordinate Invariance of the Scattering Cross Section

The total (incident + outgoing) solution to Eq. (9) or (14) for each partial wave has an asymptotic form like

$$u_l \to \sin(\omega r^* - l\pi/2 + \delta_l). \tag{22}$$

Since adding a constant to r^* amounts to adding a constant to each of the phase shifts, the summed differential cross section is independent of the choice of the constant in r^*. Similarly, coordinate changes $r' = r + \text{const} + O(r^{-1})$ add only an l-independent constant to the phase shift in the asymptotic region, and so the summed differential cross section is invariant under them. Also, the asymptotic value ($|\omega|$ in the z direction) of the current given by the "plane" wave part Eq. (21a) of the total solution is invariant both under $r' = r + \text{const} + O(r^{-1})$ and under additive constants in r^*, as can be verified by direct calculation.

We emphasize that the constant in Eq. (20) is irrelevant for determining a summed-scattering cross section, but the choice of (20) allows us to compute only a few phase shifts and then rely on tabulated Coulomb results for the higher l values.

III. CALCULATION OF CROSS SECTIONS

A. Exact solution for $\omega = 0$, $l = 0$

We write Eq. (9) for $l = 0$, $\omega = 0$:

$$\left(1 - \frac{2m}{r}\right)\left[\frac{d}{dr}\left(1 - \frac{2m}{r}\right)\frac{d}{dr} - \frac{2m}{r^3}\right]u = 0.$$

This equation has the solution $u = r$, as may be verified by direct substitution. We make use of this solution in the following sections as a starting point for our discussion of small ω behavior. It is especially useful since, as shown below, in the limit $\omega = 0$ this solution is exactly the scattering solution that we require (i.e., it solves the equation and fits the boundary condition.)

B. Calculation at Low Energy

It is shown in the Appendix that the scattering length approximation is valid for the $l = 0$, $\omega \to 0$ case which we now consider.

The ingoing wave condition at $y^* = -\infty$ corresponds (in the scattering length approximation) to

$$\begin{aligned} u &= 1 - iqy^* + O(q^2 y^{*2}), \\ \frac{du}{dy^*} &= -iq + O(q^2 y^{*2}), \end{aligned} \tag{23}$$

for $y^* \to -\infty$ with $qy^* \ll 1$. [The potential falls off exponentially in this region, so this ingoing condition is well satisfied for y^* even slightly to the inside of the peak of $V(y^*)$.] As $q \to 0$, these become

$$u \simeq 1, \quad du/dy^* \simeq 0.$$

The $q = 0$ wavefunction, $u(y^*) = y$, fits smoothly onto the ingoing wave Eq. (23) with errors of order q (see Appendix). We thus take ingoing free waves in the region inside the peak in $V(y^*)$, match them smoothly to this solution $u = y$, which is valid in the region where the potential dominates, and then match (by a scattering-length approximation) to a combination of

[16] An objection might be raised because the "plane wave," Eq. (21a), does not have a divergenceless current, even in the asymptotic region where $r^* \to r_c$. This characteristic is shared, however, by the "plane" part of the total Coulomb solution ~~(see~~ Ref. 14, p. 419). The current has the numerical value $|\omega|$ in the $z = r\cos\theta$ direction at spatial infinity.

ingoing and outgoing free waves on the outside of the peak in $V(y^*)$. In the $q = 0$ limit, we may approximate the free waves outside by a straight line, just as we approximated those on the inside by a straight line (of zero slope).

We make the outside match at some matching point. The wavefunction must have continuous amplitude and slope.

$$y_0 \equiv \text{(Solution for potential region at match)}$$
$$= \text{(solution for free region at match)} \equiv A(y_0^* + b). \tag{24}$$

A and b are constants to be determined; the subscript zero means "at the joining point." For $y^* \gg 1$, $V(y^*) \sim y^{*-3}$, and the joining point will be the point where $V(y_0) \simeq h^{-3}q^2$ for some constant h of order unity. We may write

$$y_0 \equiv hy_{+TP} \simeq hq^{-\frac{2}{3}}. \tag{25}$$

Here y_{+TP} is the outside turning point:

$$q^2 = y_{+TP}^{-3}(1 - y_{+TP}^{-1}).$$

Matching slopes, we have

$$A \equiv \left.\frac{dy}{dy^*}\right|_0 = 1 - y_0^{-1}, \tag{26}$$

$$b = y_0(1 - y_0^{-1})^{-1} - y_0^*. \tag{27}$$

The wavefunction u may be written (for large y_0)

$$u \simeq (1 - y_0^{-1})(y^* + y_0 + 1 - y_0^*) \tag{24a}$$

and is clearly independent of additive constants in y^*.

The explicit form for y^* which was chosen in Eq. (20) is

$$y^* = y + \ln [2q(y - 1)].$$

Substituting this into Eq. (27), we find (in the limit $q \to 0$ so $y_0 \gg 1$)

$$b \simeq 1 - \ln [2q(y_0 - 1)].$$

From Eq. (25) above, we have

$$b \simeq 1 - \ln [2q(hq^{-\frac{2}{3}})] = 1 - \ln 2h - \tfrac{1}{3} \ln q. \tag{28}$$

We note that Eq. (28) contains a term proportional to $\ln q$. This term cannot be subtracted out of the scattering length b, as constants in y^* can be, because it is a specifically small l term. As mentioned in Sec. IIID below, the higher angular momentum components are kept from feeling the effective potential $V(y^*)$ by the centrifugal terms $l(l + 1)/r$. Only the small l solutions can feel the y^{*-3} potential which gives the logarithmic shift. [In the "starred" coordinate, the wavefunction in the potential region is suffering a logarithmic phase shift compared to the "free" wave $\sim \sin qy^*$. The difference between the inner solution and the comparison free wave increases as the turning point moves out ($y_{+TP} \simeq hq^{-\frac{2}{3}}$) so the value of this logarithmic shift increases.]

Since we argue that only the $l = 0$ term will be significantly different from the Coulomb result, we conclude there must be a logarithmic term in the summed differential cross section.

The absorption cross section is particularly easy to determine. On the inside of the potential bump, the wavefunction u has unit amplitude and its imaginary part $-i \sin qy^*$ is approximately zero. The wavefunction on the outside, however, is

$$u \simeq y^* \simeq q^{-1} \sin qy^*.$$

We see that the ratio of the amplitudes, outside to in, is q^{-1}; so if the incident wave (outside) were normalized, we would have no flux on the inside of the potential barrier as $q \to 0$. We may immediately conclude that *there is no absorption in the s wave, $\omega \to 0$ limit.* From the way this result was obtained, it is clearly coordinate invariant.

An observer who is monitoring extremely long wavelength scattering may, in fact, be within the effective potential region. This observer will see the solution $u = r$. This corresponds to $R = r/r = \text{const}$, i.e., $\phi(S \text{ wave}) = \text{constant}$ in space. (Of course there will be the slow time variation also.) The observer then sees no reason to prefer the origin over any other point. He would say that the scattering cross section is zero, since otherwise the location of the massive scatterer would have been distinguished by the behavior of the field.

As an example, if the orbit of the earth ($\sim 1.5 \times 10^8$ km; $y_E = r_E/2m. \sim 0.5 \times 10^8$) is within the region of the effective potential due to the sun, we must have

$$y_E^{\frac{3}{2}} \sim q^{-1} \sim \text{wavelength}/2m. \sim 0.3 \times 10^{12},$$

so

$$\text{wavelength} \sim 10^{12}\text{ km},$$
$$\text{period} \sim 3 \times 10^6\text{ sec},$$
$$\sim \text{month}.$$

Periods of this order or longer are of cosmological interest.

C. Cutoff for the Divergent Zero-Energy Cross Section

The last paragraphs of Sec. IIIB have shown (a) that an observer in the free wave zone would note a logarithmically diverging scattering length, but (b) an observer within the potential-dominated region would not be able to see any scattering at all.

We have a scattering cross section

$$\frac{d\sigma}{d\Omega} = b^2 \simeq \left[2am + \frac{2m}{3} \ln (2\omega m)\right]^2,$$

where a is a constant of order unity. We see that as

$\omega \to 0$, the scattering length becomes infinite. However, the size of the universe gives a cutoff for this infinity, and the cutoff is rather small.

Consider the case of an OS star of one solar mass. Since $\omega_{\min} \sim 1/R_H$ where $R_H =$ "Radius of the universe" $\sim 10^{28}$ cm (the Hubble radius),

$$b \simeq 2m[a + \tfrac{1}{3}\ln(10^5 \times 10^{-28})],$$
$$\simeq 2m[a - 23] \sim 50 \text{ km}.$$

In this case there is nothing like an infinity. The longest scattering length occurs for $m \sim R_H$, in which case the scattering length $b \sim R_H$, also. The scattering length decreases monotonically for decreasing mass of the scatterers.

D. Higher Energy and Angular Momentum

The results given above hold only for the S-wave case in the zero-frequency limit. If S waves with ω^2 greater than

$$V_{\text{peak}} = 3\left(\frac{3}{16}\cdot\frac{1}{2m}\right)^2 = \frac{0.105}{(2m)^2}$$

are incident, they will be little affected by the bump, and so the absorption cross section would go up and the scattering becomes more "Newtonian" for s waves.

Since the effective potential is a close-in effect ($r_{\text{peak}} = \frac{4}{3}2m$), the higher angular momenta will be prevented by the repulsive centrifugal potential from feeling it. Recall the definition of a "distance of closest approach," $r_\perp = [l(l+1)]^{\frac{1}{2}}/\omega$. For any frequency low enough to feel the potential ($\omega \lesssim \frac{1}{2}m$), we have $r_\perp \geq 2m[l(l+1)]^{\frac{1}{2}}$. For $l = 1$ this is already outside, though near, the peak of the effective repulsive potential. Lower-frequency waves (for $l \neq 0$) will remain farther away and won't be affected by the effective potential, while higher-frequency waves, though they can get in to the peak of the effective potential, are not much affected by it. Thus we conclude, as Hildreth did by a similar argument, that the higher angular momentum components suffer essentially Newtonian scattering. However, there is a range ($\omega \sim \frac{1}{2}m$) where the $l = 1$ waves are somewhat affected by the non-Newtonian aspects of the scattering.

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APPENDIX: MATCHING BETWEEN FREE AND POTENTIAL REGIONS

Because $V(y^*) \sim \exp(y^*)$ for $y^* \to -\infty$, the potential falls off very quickly inside the peak, and so the linear approximation can be used ($e^{-iqy^*} \simeq 1 - iqy^*$). The turning point y^*_{-TP}, where the potential becomes negligible, occurs at a negative y^* which is much smaller than a wavelength ($\sim q^{-1}$).

Similarly, on the outside of the potential bump, the wavelength is $\sim q^{-1}$, while the turning point goes as $q^{-\frac{2}{3}}$. Thus the linear approximation through the turning point region is valid at both sides of the potential region.

Now consider the match at the inside of the potential. We assume the matching is done exactly at y^*_{-TP}, and note that the interior wave has $u = y$ and

$$y_{-TP} \simeq 1 + q^2,$$
$$\left.\frac{dy}{dy^*}\right|_{-TP} = (1 - y^{-1}_{-TP}) \simeq q^2.$$

The ingoing free wave has $u = \exp(-iqy^*)$:

$$u(y^*_{-TP}) \simeq 1 - iqy^*_{-TP},$$
$$\left.\frac{du}{dy^*}\right|_{-TP} \equiv u'_{-TP} \simeq -iq.$$

For sufficiently small q, the errors in joining these functions become small with q. Thus the solution $u = y$ is the correct continuation, in the potential region, of the ingoing wave determined by the boundary condition. We can see that errors in the match described here do not propagate and accumulate to cause large errors in the wavefunction for $y^* \to +\infty$. For we can write the relevant integral equations:

$$u'(x^*) = \int_{y^*-TP}^{x^*} Vu\, dy^* - \int_{y^*-TP}^{x^*} uq^2\, dy^* + u'(y^*_{-TP}), \tag{A1}$$

$$u(x^*) = \int_{y^*-TP}^{x^*}\left[u'(y^*_{-TP}) + \int_{y^*-TP}^{z^*}(V - q^2)u\, dy^*\right]dz^* + u(y^*_{-TP}). \tag{A2}$$

Consider (A1). Since $u(y^*)$ is of order unity near the potential peak, errors of order q in $u(y^*)$ will lead only to errors of order q in the first integral. The kinetic term will be of order q^2, so the second integral goes to zero as q^2. The error in u' at the inside matching point is also of order q, and so the last term $u(y^*_{-TP})$ is of that order (since it is of at least that smallness in the two solutions matched at y^*_{-TP}). Thus errors of order q in u can make only errors of order q in u'. Consideration of (A2) in the same way then shows that the system is stable; errors of order q in u' induce only errors of order q in u.