Yong Zhou

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Preface

The concept of fractional derivative appeared for the first time in a famous correspondence between G.A. de L'Hospital and G.W. Leibniz, in 1695. Many mathematicians have further developed this area and we can mention the studies of L. Euler (1730), J.L. Lagrange (1772), P.S. Laplace (1812), J.B.J. Fourier (1822), N.H. Abel (1823), J. Liouville (1832), B. Riemann (1847), H.L. Greer (1859), H. Holmgren (1865), A.K. Grünwald (1867), A.V. Letnikov (1868), N.Ya. Sonin (1869), H. Laurent (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside (1892), S. Pincherle (1902), G.H. Hardy and J.E. Littlewood (1917), H. Weyl (1919), P. Lévy (1923), A. Marchaud (1927), H.T. Davis (1924), A. Zygmund (1935), E.R. Love (1938), A. Erdélyi (1939), H. Kober (1940), D.V. Widder (1941), M. Riesz (1949) and W. Feller (1952). In the past sixty years, fractional calculus had played a very important role in various fields such as physics, chemistry, mechanics, electricity, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc.

In the last decade, fractional calculus has been recognized as one of the best tools to describe long-memory processes. Such models are interesting for engineers and physicists but also for pure mathematicians. The most important among such models are those described by differential equations containing fractional-order derivatives. Their evolutions behave in a much more complex way than in the classical integer-order case and the study of the corresponding theory is a hugely demanding task. Although some results of qualitative analysis for fractional differential equations can be similarly obtained, many classical methods are hardly applicable directly to fractional differential equations. New theories and methods are thus required to be specifically developed, whose investigation becomes more challenging. Comparing with classical theory of differential equations, the researches on the theory of fractional differential equations are only on their initial stage of development.

This monograph is devoted to a rapidly developing area of the research for the qualitative theory of fractional differential equations. In particular, we are interested in the basic theory of fractional differential equations. Such basic theory should be the starting point for further research concerning the dynamics, control, numerical analysis and applications of fractional differential equations. The book is divided into six chapters. Chapter 1 introduces preliminary facts from fractional calculus, nonlinear analysis and semigroup theory. In Chapter 2, we present a unified framework to investigate the basic existence theory for discontinuous fractional functional differential equations with bounded delay, unbounded delay and infinite delay. Chapter 3 is devoted to the study of fractional differential equations in Banach spaces via measure of noncompactness method, topological degree method and Picard operator technique. In Chapter 4, we first present some techniques for the investigation of fractional evolution equations governed by C_0-semigroup, then we discuss fractional evolution equations with almost sectorial operators. In Chapter 5, by using critical point theory, we give a new approach to study boundary value problems of fractional differential equations. And in the last chapter, we present recent advances on theory for fractional partial differential equations including fractional Euler-Lagrange equations, time-fractional diffusion equations, fractional Hamiltonian systems and fractional Schrödinger equations.

The material in this monograph are based on the research work carried out by the author and other experts during the past four years. The book is self-contained and unified in presentation, and it provides the necessary background material required to go further into the subject and explore the rich research literature. Each chapter concludes with a section devoted to notes and bibliographical remarks and all abstract results are illustrated by examples. The tools used include many classical and modern nonlinear analysis methods. This book is useful for researchers and graduate students for research, seminars, and advanced graduate courses, in pure and applied mathematics, physics, mechanics, engineering, biology, and related disciplines.

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Yong Zhou
October 2013, Xiangtan, China

Contents

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we introduce some notations and basic facts on fractional calculus, nonlinear analysis and semigroup which are needed throughout this book.

1.2 Some Notations, Concepts and Lemmas

As usual $\mathbb{N}$ denotes the set of positive integer numbers and $\mathbb{N}_0$ the set of nonnegative integer numbers. $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ denotes the set of nonnegative reals and $\mathbb{R}^+$ the set of positive reals. Let $\mathbb{C}$ be the set of complex numbers.

We recall that a vector space X equipped with a norm $|\cdot|$ is called a normed vector space. A subset E of a normed vector space X is said to be bounded if there exists a number K such that $|x| \leq K$ for all $x \in E$. A subset E of a normed vector space X is called convex if for any $x, y \in E$, $ax + (1-a)y \in E$ for all $a \in [0,1]$.

A sequence $\{x_n\}$ in a normed vector space X is said to converge to the vector x in X if and only if the sequence $\{|x_n - x|\}$ converges to zero as $n \to \infty$. A sequence $\{x_n\}$ in a normed vector space X is called a Cauchy sequence if for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for all $n, m \geq N(\varepsilon)$, $|x_n - x_m| < \varepsilon$. Clearly a convergent sequence is also a Cauchy sequence, but the converse may not be true. A space X where every Cauchy sequence of elements of X converges to an element of X is called a complete space. A complete normed vector space is said to be a Banach space.

Let E be a subset of a Banach space X. A point $x \in X$ is said to be a limit point of E if there exists a sequence of vectors in E which converges to x. We say a subset E is closed if E contains all of its limit points. The union of E and its limit points is called the closure of E and will be denoted by $\bar{E}$. Let X, F be normed vector spaces, and E be a subset of X. An operator $\mathscr{T} : E \to F$ is continuous at a point $x \in E$ if and only if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $|\mathscr{T}x - \mathscr{T}y| < \varepsilon$ for all $y \in E$ with $|x-y| < \delta$. Further, $\mathscr{T}$ is continuous on E, or simply continuous, if it is continuous at all points of E.

We say that a subset E of a Banach space X is compact if every sequence of vectors in E contains a subsequence which converges to a vector in E. We say that E is relatively compact in X if every sequence of vectors in E contains a subsequence which converges to a vector in X, i.e., E is relatively compact in X if $\bar{E}$ is compact.

Let $J = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval of $\mathbb{R}$. We assume that X is a Banach space with the norm $|\cdot|$. Denote $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm

$$\|x\| = \sup_{t\in J} |x(t)|,$$

where $x \in C(J, X)$. $C^n(J, X)$ $(n \in \mathbb{N}_0)$ denotes the set of mappings having n times continuously differentiable on J, $AC(J, X)$ is the space of functions which are absolutely continuous on J and $AC^n(J, X)$ $(n \in \mathbb{N}_0)$ is the space of functions f such that $f \in C^{n-1}(J, X)$ and $f^{(n-1)} \in AC(J, X)$. In particular, $AC^1(J, X) = AC(J, X)$.

Let $1 \le p \le \infty$. $L^p(J, X)$ denotes the Banach space of all measurable functions $f : J \to X$. $L^p(J, X)$ is normed by

$$\|f\|_{L^pJ} = \begin{cases} \left(\displaystyle\int_J |f(t)|^p dt\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \displaystyle\inf_{\mu(\bar{J})=0} \{ \sup_{t\in J\setminus\bar{J}} |f(t)|\}, & p = \infty. \end{cases}$$

In particular, $L^1(J, X)$ is the Banach space of measurable functions $f : J \to X$ with the norm

$$\|f\|_{LJ} = \int_J |f(t)|dt,$$

and $L^\infty(J, X)$ is the Banach space of measurable functions $f : J \to X$ which are bounded, equipped with the norm

$$\|f\|_{L^\infty J} = \inf\{c > 0 : |f(t)| \le c, \text{ a.e. } t \in J\}.$$

Lemma 1.1 (Hölder inequality). Assume that $p, q \ge 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(J, X), g \in L^q(J, X)$, then for $1 \le p \le \infty$, $fg \in L^1(J, X)$ and

$$\|fg\|_{LJ} \le \|f\|_{L^pJ}\|g\|_{L^qJ}.$$

A family F in $C(J, X)$ is called uniformly bounded if there exists a positive constant K such that $|f(t)| \le K$ for all $t \in J$ and all $f \in F$. Further, F is called equicontinuous, if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ for all $t_1, t_2 \in J$ with $|t_1 - t_2| < \delta$ and all $f \in F$.

Lemma 1.2 (Arzela-Ascoli's theorem). If a family $F = \{f(t)\}$ in $C(J, \mathbb{R})$ is uniformly bounded and equicontinuous on J, then F has a uniformly convergent subsequence $\{f_n(t)\}_{n=1}^{\infty}$. If a family $F = \{f(t)\}$ in $C(J, X)$ is uniformly bounded and equicontinuous on J, and for any $t^* \in J$, $\{f(t^*)\}$ is relatively compact, then F has a uniformly convergent subsequence $\{f_n(t)\}_{n=1}^{\infty}$.

The Arzela-Ascoli's Theorem is the key to the following result: A subset F in $C(J,\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on J.

Lemma 1.3 (Lebesgue's dominated convergence theorem). Let E be a measurable set and let $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in E, and for every $n \in \mathbb{N}$, $|f_n(x)| \le g(x)$ a.e. in E, where g is integrable on E. Then

$$\lim_{n\to\infty}\int_E f_n(x)dx = \int_E f(x)dx.$$

Finally, we state the Bochner's theorem.

Lemma 1.4 (Bochner's theorem). A measurable function $f:(a,b)\to X$ is Bochner integrable if $|f|$ is Lebesgue integrable.

1.3 Fractional Calculus

The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (Re(z) > 0),$$

where $t^{z-1} = e^{(z-1)\log(t)}$. This integral is convergent for all complex $z \in \mathbb{C}$ $(Re(z) > 0)$.

For this function the reduction formula

$$\Gamma(z+1) = z\Gamma(z) \qquad (Re(z) > 0)$$

holds. In particular, if $z = n \in \mathbb{N}_0$, then

$$\Gamma(n+1) = n! \qquad (n \in \mathbb{N}_0)$$

with (as usual) $0! = 1$.

Let us consider some of the starting points for a discussion of fractional calculus. One development begins with a generalization of repeated integration. Thus if f is locally integrable on (c,∞), then the n-fold iterated integral is given by

$$\begin{aligned} {}_cD_t^{-n}f(t) &= \int_c^t ds_1\int_c^{s_1} ds_2\cdots\int_c^{s_{n-1}} f(s_n)ds_n \\ &= \frac{1}{(n-1)!}\int_c^t (t-s)^{n-1}f(s)ds \end{aligned}$$

for almost all t with $-\infty \le c < t < \infty$ and $n \in \mathbb{N}$. Writing $(n-1)! = \Gamma(n)$, an immediate generalization is the integral of f of fractional order $\alpha > 0$,

$${}_cD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_c^t (t-s)^{\alpha-1}f(s)ds \quad \text{(left hand)}$$

and similarly for $-\infty < t < d \le \infty$

$${}_tD_d^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^d (s-t)^{\alpha-1}f(s)ds \quad \text{(right hand)}$$

both being defined for suitable f.

A number of definitions for the fractional derivative has emerged over the years, we refer the reader to Diethelm, 2010; Hilfer, 2006; Kilbas, Srivastava and Trujillo, 2006; Miller and Ross, 1993; Podlubny, 1999. In this book, we restrict our attention to the use of the Riemann-Liouville and Caputo fractional derivatives. In this section, we introduce some basic definitions and properties of the fractional integrals and fractional derivatives which are used further in this book. The materials in this section are taken from Kilbas, Srivastava and Trujillo, 2006.

1.3.1 *Definitions*

Definition 1.5 (Left and right Riemann-Liouville fractional integrals). Let $J = [a, b]$ $(-\infty < a < b < \infty)$ be a finite interval of $\mathbb{R}$. The left and right Riemann-Liouville fractional integrals ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$ of order $\alpha \in \mathbb{R}^+$, are defined by

$$ {}_aD_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a, \quad \alpha > 0 \tag{1.1} $$

and

$$ {}_tD_b^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)ds, \quad t < b, \quad \alpha > 0, \tag{1.2} $$

respectively, provided the right-hand sides are pointwise defined on $[a, b]$. When $\alpha = n \in \mathbb{N}$, the definitions (1.1) and (1.2) coincide with the n-th integrals of the form

$$ {}_aD_t^{-n}f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)ds $$

and

$$ {}_tD_b^{-n}f(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} f(s)ds. $$

Definition 1.6 (Left and right Riemann-Liouville fractional derivatives). The left and right Riemann-Liouville fractional derivatives ${}_aD_t^{\alpha}f(t)$ and ${}_tD_b^{\alpha}f(t)$ of order $\alpha \in \mathbb{R}_+$, are defined by

$$ \begin{aligned} {}_aD_t^{\alpha}f(t) &= \frac{d^n}{dt^n} {}_aD_t^{-(n-\alpha)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(\int_a^t (t-s)^{n-\alpha-1} f(s)ds \right), \quad t > a \end{aligned} \tag{1.3} $$

and

$$ \begin{aligned} {}_tD_b^{\alpha}f(t) &= (-1)^n \frac{d^n}{dt^n} {}_tD_b^{-(n-\alpha)} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} (-1)^n \frac{d^n}{dt^n} \left(\int_t^b (s-t)^{n-\alpha-1} f(s)ds \right), \quad t < b, \end{aligned} \tag{1.4} $$

respectively, where $n = [\alpha] + 1$, $[\alpha]$ means the integer part of α. In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$ {}_aD_t^0 f(t) = {}_tD_b^0 f(t) = f(t), $$

$$ {}_aD_t^n f(t) = f^{(n)}(t) \quad \text{and} \quad {}_tD_b^n f(t) = (-1)^n f^{(n)}(t), $$

where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n. If $0 < \alpha < 1$, then

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\left(\int_a^t (t-s)^{-\alpha} f(s)ds \right), \quad t > a $$

and

$$ {}_tD_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}\left(\int_t^b (s-t)^{-\alpha} f(s)ds \right), \quad t < b. $$

Remark 1.7. If $f \in C([a,b], \mathbb{R}^N)$, it is obvious that Riemann-Liouville fractional integral of order $\alpha > 0$ exists on $[a,b]$. On the other hand, following Lemma 2.2 in Kilbas, Srivastava and Trujillo, 2006, we know that the Riemann-Liouville fractional derivative of order $\alpha \in [n-1, n)$ exists almost everywhere on $[a,b]$ if $f \in AC^n([a,b], \mathbb{R}^N)$.

The left and right Caputo fractional derivatives are defined via above Riemann-Liouville fractional derivatives.

Definition 1.8 (Left and right Caputo fractional derivatives). The left and right Caputo fractional derivatives ${}_a^CD_t^\alpha f(t)$ and ${}_t^CD_b^\alpha f(t)$ of order $\alpha \in \mathbb{R}_+$ are defined by

$$ {}_a^CD_t^\alpha f(t) = {}_aD_t^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k \right] \tag{1.5} $$

and

$$ {}_t^CD_b^\alpha f(t) = {}_tD_b^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!}(b-t)^k \right], \tag{1.6} $$

respectively, where

$$ n = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}_0; \ n = \alpha \text{ for } \alpha \in \mathbb{N}_0. \tag{1.7} $$

In particular, when $0 < \alpha < 1$, then

$$ {}_a^CD_t^\alpha f(t) = {}_aD_t^\alpha (f(t) - f(a)) $$

and

$$ {}_t^CD_b^\alpha f(t) = {}_tD_b^\alpha (f(t) - f(b)). $$

The Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected with each other by the following relations.

Property 1.9.

(i) If $\alpha \notin \mathbb{N}_0$ and $f(t)$ is a function for which the Caputo fractional derivatives ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ of order $\alpha \in \mathbb{R}^+$ exist together with the Riemann-Liouville fractional derivatives ${}_a D_t^\alpha f(t)$ and ${}_t D_b^\alpha f(t)$, then

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha} $$

and

$$ {}_t^C D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-t)^{k-\alpha}, $$

where $n = [\alpha] + 1$. In particular, when $0 < \alpha < 1$, we have

$$ {}_a^C D_t^\alpha f(t) = {}_a D_t^\alpha f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha} $$

and

$$ {}_t^C D_b^\alpha f(t) = {}_t D_b^\alpha f(t) - \frac{f(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}. $$

(ii) If $\alpha = n \in \mathbb{N}_0$ and the usual derivative $f^{(n)}(t)$ of order n exists, then ${}_a^C D_t^n f(t)$ and ${}_t^C D_b^n f(t)$ are represented by

$$ {}_a^C D_t^n f(t) = f^{(n)}(t) \quad \text{and} \quad {}_t^C D_b^n f(t) = (-1)^n f^{(n)}(t). \tag{1.8} $$

Property 1.10. Let $\alpha \in \mathbb{R}_+$ and let n be given by (1.7). If $f \in AC^n([a,b], \mathbb{R}^N)$, then the Caputo fractional derivatives ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ exist almost everywhere on $[a,b]$.

(i) If $\alpha \notin \mathbb{N}_0$, ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ are represented by

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds \right) \tag{1.9} $$

and

$$ {}_t^C D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) ds \right) \tag{1.10} $$

respectively, where $n = [\alpha] + 1$. In particular, when $0 < \alpha < 1$ and $f \in AC([a,b], \mathbb{R}^N)$,

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} f'(s) ds \right) \tag{1.11} $$

and

$$ {}_t^C D_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \left(\int_t^b (s-t)^{-\alpha} f'(s) ds \right). \tag{1.12} $$

(ii) If $\alpha = n \in \mathbb{N}_0$ then ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ are represented by (1.8). In particular,

$$ {}_a^C D_t^0 f(t) = {}_t^C D_b^0 f(t) = f(t). $$

Remark 1.11. If f is an abstract function with values in Banach space X, then integrals which appear in above definitions are taken in Bochner's sense.

The fractional integrals and derivatives, defined on a finite interval $[a, b]$ of $\mathbb{R}$, are naturally extended to whole axis $\mathbb{R}$.

Definition 1.12 (Left and right Liouville-Weyl fractional integrals on the real axis). The left and right Liouville-Weyl fractional integrals ${}_{-\infty}D_t^{-\alpha}f(t)$ and ${}_tD_{+\infty}^{-\alpha}f(t)$ of order $\alpha > 0$ on the whole axis $\mathbb{R}$ are defined by

$$ {}_{-\infty}D_t^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}f(s)ds \tag{1.13} $$

and

$$ {}_tD_{+\infty}^{-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(s-t)^{\alpha-1}f(s)ds, \tag{1.14} $$

respectively, where $t \in \mathbb{R}$ and $\alpha > 0$.

Definition 1.13 (Left and right Liouville-Weyl fractional derivatives on the real axis). The left and right Liouville-Weyl fractional derivatives ${}_{-\infty}D_t^{\alpha}f(t)$ and ${}_tD_{+\infty}^{\alpha}f(t)$ of order α on the whole axis $\mathbb{R}$ are defined by

$$ \begin{aligned} {}_{-\infty}D_t^{\alpha}f(t) &= \frac{d^n}{dt^n}({}_{-\infty}D_t^{-(n-\alpha)}f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left(\int_{-\infty}^{t}(t-s)^{n-\alpha-1}f(s)ds\right) \end{aligned} \tag{1.15} $$

and

$$ \begin{aligned} {}_tD_{+\infty}^{\alpha}f(t) &= (-1)^n\frac{d^n}{dt^n}({}_tD_{+\infty}^{-(n-\alpha)}f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}(-1)^n\frac{d^n}{dt^n}\left(\int_{t}^{\infty}(s-t)^{n-\alpha-1}f(s)ds\right), \end{aligned} \tag{1.16} $$

respectively, where $n = [\alpha] + 1$, $\alpha \geq 0$ and $t \in \mathbb{R}$.

In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$ {}_{-\infty}D_t^{0}f(t) = {}_tD_{+\infty}^{0}f(t) = f(t), $$

$$ {}_{-\infty}D_t^{n}f(t) = f^{(n)}(t) \quad \text{and} \quad {}_tD_{+\infty}^{n}f(t) = (-1)^n f^{(n)}(t), $$

where $f^{(n)}(t)$ is the usual derivative of $f(t)$ of order n. If $0 < \alpha < 1$ and $t \in \mathbb{R}$, then

$$ \begin{aligned} {}_{-\infty}D_t^{\alpha}f(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_{-\infty}^{t}(t-s)^{-\alpha}f(s)ds\right) \\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(t)-f(t-s)}{s^{\alpha+1}}ds \end{aligned} $$

and

$$ \begin{aligned} {}_tD_{+\infty}^{\alpha}f(t) &= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_{t}^{\infty}(s-t)^{-\alpha}f(s)ds\right) \\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(t)-f(t+s)}{s^{\alpha+1}}ds. \end{aligned} $$

Formulas (1.11) and (1.12) can be used for the definition of the Caputo fractional derivatives on the whole axis $\mathbb{R}$.

Definition 1.14 (Left and right Caputo fractional derivatives on the real axis). The left and right Caputo fractional derivatives ${}_{-\infty}^{\ C}\mathbf{D}_t^{\alpha} f(t)$ and ${}_t^C\mathbf{D}_{+\infty}^{\alpha} f(t)$ of order α (with $\alpha > 0$ and $\alpha \notin \mathbb{N}$) on the whole axis $\mathbb{R}$ are defined by

$$ {}_{-\infty}^{\ C}\mathbf{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\int_{-\infty}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds \right) \tag{1.17} $$

and

$$ {}_t^C\mathbf{D}_{+\infty}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\int_{t}^{\infty} (s-t)^{n-\alpha-1} f^{(n)}(s) ds \right), \tag{1.18} $$

respectively.

When $0 < \alpha < 1$, the relations (1.17) and (1.18) take the following forms

$$ {}_{-\infty}^{\ C}\mathbf{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\int_{-\infty}^{t} (t-s)^{-\alpha} f'(s) ds \right) $$

and

$$ {}_t^C\mathbf{D}_{+\infty}^{\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \left(\int_{t}^{\infty} (s-t)^{-\alpha} f'(s) ds \right). $$

Now we present the Fourier transform properties of the fractional integral and fractional differential operators.

Definition 1.15. The Fourier transform of a function $f(t)$ of real variable $t \in \mathbb{R}$ is defined by

$$ \hat{f}(w) = \int_{-\infty}^{\infty} e^{-it\cdot w} f(t) dt \qquad (w \in \mathbb{R}). $$

Let $f(t)$ be defined on $(-\infty, \infty)$ and $0 < \alpha < 1$. Then the Fourier transform of the Liouville-Weyl integral and differential operator satisfies

$$ \begin{aligned} \widehat{{}_{-\infty}D_t^{-\alpha} f(t)}(w) &= (iw)^{-\alpha} \hat{f}(w), \\ \widehat{{}_tD_{\infty}^{-\alpha} f(t)}(w) &= (-iw)^{-\alpha} \hat{f}(w), \\ \widehat{{}_{-\infty}D_t^{\alpha} f(t)}(w) &= (iw)^{\alpha} \hat{f}(w), \\ \widehat{{}_tD_{\infty}^{\alpha} f(t)}(w) &= (-iw)^{\alpha} \hat{f}(w). \end{aligned} $$

1.3.2 *Properties*

We present here some properties of the fractional integral and derivative operators that will be useful throughout this book.

Property 1.16. If $\alpha \geq 0$ and $\beta > 0$, then

$$ {}_aD_t^{-\alpha}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1} \quad (\alpha > 0), \tag{1.19} $$

$$ {}_aD_t^{\alpha}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1} \quad (\alpha \geq 0) \tag{1.20} $$

and

$$ {}_tD_b^{-\alpha}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1} \quad (\alpha > 0), \tag{1.21} $$

$$ {}_tD_b^{\alpha}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1} \quad (\alpha \geq 0). \tag{1.22} $$

In particular, if $\beta = 1$ and $\alpha \geq 0$, then the Riemann-Liouville fractional derivatives of a constant are, in general, not equal to zero:

$$ {}_aD_t^{\alpha}1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad {}_tD_b^{\alpha}1 = \frac{(b-t)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{1.23} $$

On the other hand, for $j = 1, 2, ..., [\alpha] + 1$,

$$ {}_aD_t^{\alpha}(t-a)^{\alpha-j} = 0, \quad {}_tD_b^{\alpha}(b-t)^{\alpha-j} = 0. \tag{1.24} $$

The semigroup property of the fractional integral operators ${}_aD_t^{-\alpha}$ and ${}_tD_b^{-\alpha}$ are given by the following result.

Property 1.17. If $\alpha > 0$ and $\beta > 0$, then the equations

$$ {}_aD_t^{-\alpha}\left({}_aD_t^{-\beta}f(t)\right) = {}_aD_t^{-\alpha-\beta}f(t) \text{ and } {}_tD_b^{-\alpha}\left({}_tD_b^{-\beta}f(t)\right) = {}_tD_b^{-\alpha-\beta}f(t) \tag{1.25} $$

are satisfied at almost every point $t \in [a,b]$ for $f \in L^p([a,b],\mathbb{R}^N)$ $(1 \leq p < \infty)$. If $\alpha + \beta > 1$, then the relations in (1.25) hold at any point of $[a,b]$.

Property 1.18.

(i) If $\alpha > 0$ and $f \in L^p([a,b],\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, then the following equalities

$$ {}_aD_t^{\alpha}\left({}_aD_t^{-\alpha}f(t)\right) = f(t) \text{ and } {}_tD_b^{\alpha}\left({}_tD_b^{-\alpha}f(t)\right) = f(t) \quad (\alpha > 0) \tag{1.26} $$

hold almost everywhere on $[a,b]$.

(ii) If $\alpha > \beta > 0$, then, for $f \in L^p([a,b],\mathbb{R}^N)$ $(1 \leq p \leq \infty)$, the relations

$$ {}_aD_t^{\beta}\left({}_aD_t^{-\alpha}f(t)\right) = {}_aD_t^{-\alpha+\beta}f(t) \text{ and } {}_tD_b^{\beta}\left({}_tD_b^{-\alpha}f(t)\right) = {}_tD_b^{-\alpha+\beta}f(t) \tag{1.27} $$

hold almost everywhere on $[a,b]$.

In particular, when $\beta = k \in \mathbb{N}$ and $\alpha > k$, then

$$ {}_aD_t^{k}\left({}_aD_t^{-\alpha}f(t)\right) = {}_aD_t^{-\alpha+k}f(t) \text{ and } {}_aD_t^{k}\left({}_tD_b^{-\alpha}f(t)\right) = (-1)^k {}_tD_b^{-\alpha+k}f(t). \tag{1.28} $$

To present the next property, we use the spaces of functions ${}_aD_t^{-\alpha}(L^p)$ and ${}_tD_b^{-\alpha}(L^p)$ defined for $\alpha > 0$ and $1 \leq p \leq \infty$ by

$$ {}_aD_t^{-\alpha}(L^p) = \{f : \ f = {}_aD_t^{-\alpha}\varphi, \ \varphi \in L^p([a,b],\mathbb{R}^N)\} $$

and

$$ {}_tD_b^{-\alpha}(L^p) = \{f : \ f = {}_tD_b^{-\alpha}\phi, \ \phi \in L^p([a,b],\mathbb{R}^N)\}, $$

respectively. The composition of the fractional integration operator ${}_aD_t^{-\alpha}$ with the fractional differentiation operator ${}_aD_t^{\alpha}$ is given by the following result.

Property 1.19. Let $\alpha > 0$, $n = [\alpha] + 1$ and let $f_{n-\alpha}(t) = {}_aD_t^{-(n-\alpha)}f(t)$ be the fractional integral (1.1) of order $n - \alpha$.

(i) If $1 \le p \le \infty$ and $f \in {}_aD_t^{-\alpha}(L^p)$, then

$$ {}_aD_t^{-\alpha}\Big({}_aD_t^{\alpha} f(t)\Big) = f(t). \tag{1.29}$$

(ii) If $f \in L^1([a,b],\mathbb{R}^N)$ and $f_{n-\alpha} \in AC^n([a,b],\mathbb{R}^N)$, then the equality

$$ {}_aD_t^{-\alpha}\Big({}_aD_t^{\alpha} f(t)\Big) = f(t) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(t-a)^{\alpha-j}, \tag{1.30}$$

holds almost everywhere on $[a,b]$.

Property 1.20. Let $\alpha > 0$ and $n = [\alpha]+1$. Also let $g_{n-\alpha}(t) = {}_tD_b^{-(n-\alpha)}g(t)$ be the fractional integral (1.2) of order $n-\alpha$.

(i) If $1 \le p \le \infty$ and $g \in {}_tD_b^{-\alpha}(L^p)$, then

$$ {}_tD_b^{-\alpha}\Big({}_tD_b^{\alpha} g(t)\Big) = g(t). \tag{1.31}$$

(ii) If $g \in L^1([a,b],\mathbb{R}^N)$ and $g_{n-\alpha} \in AC^n([a,b],\mathbb{R}^N)$, then the equality

$$ {}_tD_b^{-\alpha}\Big({}_tD_b^{\alpha} g(t)\Big) = g(t) - \sum_{j=1}^{n} \frac{(-1)^{n-j} g_{n-\alpha}^{(n-j)}(a)}{\Gamma(\alpha-j+1)}(b-t)^{\alpha-j}, \tag{1.32}$$

holds almost everywhere on $[a,b]$.

In particular, if $0<\alpha<1$, then

$$ {}_tD_b^{-\alpha}\Big({}_tD_b^{\alpha} g(t)\Big) = g(t) - \frac{g_{1-\alpha}(a)}{\Gamma(\alpha)}(b-t)^{\alpha-1}, \tag{1.33}$$

where $g_{1-\alpha}(t) = {}_tD_b^{\alpha-1}g(t)$ while for $\alpha = n \in \mathbb{N}$, the following equality holds:

$$ {}_tD_b^{-n}\Big({}_tD_b^{n} g(t)\Big) = g(t) - \sum_{k=0}^{n-1} \frac{(-1)^{k} g_{n-\alpha}^{(k)}(a)}{k!}(b-t)^{k}. \tag{1.34}$$

Property 1.21. Let $\alpha > 0$ and let $y \in L^\infty([a,b],\mathbb{R}^N)$ or $y \in C([a,b],\mathbb{R}^N)$. Then

$$ {}_a^CD_t^{\alpha}\Big({}_aD_t^{-\alpha} y(t)\Big) = y(t) \ \text{ and } \ {}_t^CD_b^{\alpha}\Big({}_tD_b^{-\alpha} y(t)\Big) = y(t). \tag{1.35}$$

Property 1.22. Let $\alpha > 0$ and let n be given by (1.7). If $y \in AC^n([a,b],\mathbb{R}^N)$ or $y \in C^n([a,b],\mathbb{R}^N)$, then

$$ {}_aD_t^{-\alpha}\Big({}_a^CD_t^{\alpha} y(t)\Big) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^{k} \tag{1.36}$$

and

$$ {}_tD_b^{-\alpha}\Big({}_t^CD_b^{\alpha} y(t)\Big) = y(t) - \sum_{k=0}^{n-1} \frac{(-1)^k y^{(k)}(b)}{k!}(b-t)^{k}. \tag{1.37}$$

In particular, if $0<\alpha\le 1$ and $y \in AC([a,b],\mathbb{R}^N)$ or $y \in C([a,b],\mathbb{R}^N)$, then

$$ {}_aD_t^{-\alpha}\Big({}_a^CD_t^{\alpha} y(t)\Big) = y(t) - y(a) \ \text{ and } \ {}_tD_b^{-\alpha}\Big({}_t^CD_b^{\alpha} y(t)\Big) = y(t) - y(b). \tag{1.38}$$

On the other hand, we have the following property of fractional integration.

Property 1.23. Let $\alpha > 0$, $p \ge 1$, $q \ge 1$, and $\frac{1}{p}+\frac{1}{q} \le 1+\alpha$ ($p \ne 1$ and $q \ne 1$ in the case when $\frac{1}{p}+\frac{1}{q} = 1+\alpha$).

(i) If $\varphi \in L^p([a,b],\mathbb{R}^N)$ and $\psi \in L^q([a,b],\mathbb{R}^N)$, then

$$\int_a^b \varphi(t)\ {}_aD_t^{-\alpha}\psi(t)dt = \int_a^b \psi(t)\ {}_tD_b^{-\alpha}\varphi(t)dt. \tag{1.39}$$

(ii) If $f \in {}_tD_b^{-\alpha}(L^p)$ and $g \in {}_aD_t^{-\alpha}(L^q)$, then

$$\int_a^b f(t)\ {}_aD_t^{\alpha}g(t)dt = \int_a^b g(t)\ {}_tD_b^{\alpha}f(t)dt. \tag{1.40}$$

Then applying Property 1.9, we can derive the integration by parts formula for the left and right Riemann-Liouville fractional derivatives looks as follows.

Property 1.24.

$$\int_a^b {}_aD_t^{\alpha}f(t)\cdot g(t)dt = \int_a^b {}_tD_b^{\alpha}g(t)\cdot f(t)dt, \quad 0<\alpha\leq 1,$$

provided the boundary conditions

$$f(a)=f(b)=0, \quad f' \in L^\infty([a,b],\mathbb{R}^N),\ g\in L^1([a,b],\mathbb{R}^N),$$

or

$$g(a)=g(b)=0, \quad g' \in L^\infty([a,b],\mathbb{R}^N),\ f\in L^1([a,b],\mathbb{R}^N)$$

are fulfilled.

Remark 1.25. If f, g are abstract functions with values in Banach space X, then integrals which appear in above properties are taken in Bochner's sense.

1.4 Some Results from Nonlinear Analysis

1.4.1 *Sobolev Spaces*

We refer to Cazenave and Haraux, 1998, for the definitions and results given below.

Consider an open subset Ω of $\mathbb{R}^N$. $\mathcal{D}(\Omega)$ is the space of C^∞ (real-valued or complex valued) functions with compact support in Ω and $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. A distribution $T\in\mathcal{D}'(\Omega)$ is said to belong to $L^p(\Omega)$ $(1\leq p\leq\infty)$ if there exists a function $f\in L^p(\Omega)$ such that

$$\langle T,\varphi\rangle = \int_\Omega f(x)\varphi(x)dx,$$

for all $\varphi\in\mathcal{D}(\Omega)$. In that case, it is well known that f is unique. Let $m\in\mathbb{N}$ and let $p\in[1,\infty]$. Define

$$W^{m,p}(\Omega) = \{f\in L^p(\Omega); D^\alpha f\in L^p(\Omega) \ \text{ for all } \alpha\in\mathbb{N}^N \ \text{ such that } |\alpha|\leq m\}.$$

$W^{m,p}(\Omega)$ is a Banach space when equipped with the norm defined by

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha|\leq m}\|D^\alpha f\|_{L^p},$$

for all $f \in W^{m,p}(\Omega)$. For all $m,\ p$ as above, we denote by $W_0^{m,p}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$. If $p = 2$, one sets $W^{m,2}(\Omega) = H^m(\Omega)$, $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ and one equips $H^m(\Omega)$ with the following equivalent norm:

$$\|f\|_{H^m} = \left(\sum_{|\alpha|\le m} \|D^\alpha u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

Then $H^m(\Omega)$ is a Hilbert space with the scalar product

$$\langle u, v\rangle_{H^m} = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u \cdot D^\alpha v dx.$$

If Ω is bounded, there exists a constant $C(\Omega)$ such that

$$\|u\|_{L_2} \le C(\Omega)\|\nabla u\|_{L_2},$$

for all $u \in H_0^1(\Omega)$ (this is Poincaré's inequality). It may be more convenient to equip $H_0^1(\Omega)$ with the following scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v dx,$$

which defines an equivalent norm to $\|\cdot\|_{H^1}$ on the closed space $H_0^1(\Omega)$.

1.4.2 *Measure of Noncompactness*

We recall here some definitions and properties of measure of noncompactness.

Assume that X is a Banach space with the norm $|\cdot|$. The measure of noncompactness α is said to be:

(i) *Monotone* if for all bounded subsets B_1, B_2 of X, $B_1 \subseteq B_2$ implies $\alpha(B_1) \le \alpha(B_2)$;
(ii) *Nonsingular* if $\alpha(\{x\} \cup B) = \alpha(B)$ for every $x \in X$ and every nonempty subset $B \subseteq X$;
(iii) *Regular* $\alpha(B) = 0$ if and only if B is relatively compact in X.

One of the most important examples of measure of noncompactness is the Hausdorff measure of noncompactness α defined on each bounded subset B of X by

$$\alpha(B) = \inf\{\varepsilon > 0 :\ B \subset \bigcup_{j=1}^m B_\varepsilon(x_j) \text{ where } x_j \in X\}, \tag{1.41}$$

where $B_\varepsilon(x_j)$ is a ball of radius $\le \varepsilon$ centered at x_j, $j = 1, 2, ..., m$. Without confusion, Kuratowski measure of noncompactness α_1 defined on each bounded subset B of X by

$$\alpha_1(B) = \inf\{\varepsilon > 0 :\ B \subset \bigcup_{j=1}^m M_j \text{ and } \operatorname{diam}(M_j) \le \varepsilon\}, \tag{1.42}$$

where the diameter of M_j is defined by $\text{diam}(M_j) = \sup\{|x-y| : x, y \in M_j\}$, $j = 1, 2, ..., m$.

It is well known that the Hausdorff measure of noncompactness α and Kuratowski measure of noncompactness α_1 enjoy the above properties (i)-(iii) and other properties. We refer the reader to Banaŝ and Goebel, 1980; Deimling, 1985; Heinz, 1983; Lakshmikantham and Leela, 1969.

(iv) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$, where $B_1 + B_2 = \{x + y : x \in B_1,\ y \in B_2\}$;
(v) $\alpha(B_1 \cup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;
(vi) $\alpha(\lambda B) \leq |\lambda|\alpha(B)$ for any $\lambda \in \mathbb{R}$.

In particular, the relationship of Hausdorff measure of noncompactness α and Kuratowski measure of noncompactness α_1 is given by
(vii) $\alpha(B) \leq \alpha_1(B) \leq 2\alpha(B)$.

Let $J = [0, a], a \in \mathbb{R}^+$, For any $W \subset C(J, X)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : u \in W \right\},\ \text{for } t \in [0, a],$$

where $W(s) = \{u(s) \in X : u \in W\}$.

We present here some useful properties.

Property 1.26. If $W \subset C(J, X)$ is bounded and equicontinuous, then $\overline{co}W \subset C(J, X)$ is also bounded and equicontinuous.

Property 1.27. (Guo, Lakshmikantham and Liu, 1996) If $W \subset C(J, X)$ is bounded and equicontinuous, then $t \to \alpha(W(t))$ is continuous on J, and

$$\alpha(W) = \max_{t \in J} \alpha(W(t)), \quad \alpha\left(\int_0^t W(s)ds\right) \leq \int_0^t \alpha(W(s))ds,\ \text{for } t \in [0, a].$$

Property 1.28. (Mönch, 1980) Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X with $|u_n(t)| \leq \tilde{m}(t)$ for almost all $t \in J$ and every $n \geq 1$, where $\tilde{m} \in L(J, \mathbb{R}^+)$, then the function $\psi(t) = \alpha(\{u_n(t)\}_{n=1}^{\infty})$ belongs to $L(J, \mathbb{R}^+)$ and satisfies

$$\alpha\left(\left\{ \int_0^t u_n(s)ds : n \geq 1 \right\}\right) \leq 2\int_0^t \psi(s)ds.$$

Property 1.29. (Bothe, 1998) If W is bounded, then for each $\varepsilon > 0$, there is a sequence $\{u_n\}_{n=1}^{\infty} \subset W$, such that

$$\alpha(W) \leq 2\alpha(\{u_n\}_{n=1}^{\infty}) + \varepsilon.$$

1.4.3 *Topological Degree*

For a minute description of the following notions we refer the reader to Banaŝ and Goebel, 1980; Deimling, 1985; Heinz, 1983; Lakshmikantham and Leela, 1969.

Definition 1.30. Consider $\Omega \subset X$ and $\mathscr{F} : \Omega \to X$ a continuous bounded mapping. We say that $\mathscr{F}$ is α-Lipschitz if there exists $k \geq 0$ such that

$$\alpha(\mathscr{F}(B)) \leq k\alpha(B) \quad (\forall)\ B \subset \Omega \text{ bounded}.$$

If, in addition, $k < 1$, then we say that $\mathscr{F}$ is a strict α-contraction.

We say that $\mathscr{F}$ is α-condensing if

$$\alpha(\mathscr{F}(B)) < \alpha(B) \ \ (\forall)\ B \subset \Omega \text{ bounded with } \alpha(B) > 0.$$

In other words, $\alpha(\mathscr{F}(B)) \geq \alpha(B)$ implies $\alpha(B) = 0$. The class of all strict α-contractions $\mathscr{F} : \Omega \to X$ is denoted by $\mathcal{S}C_\alpha(\Omega)$ and the class of all α-condensing mappings $\mathscr{F} : \Omega \to X$ is denoted by $C_\alpha(\Omega)$.

We remark that $\mathcal{S}C_\alpha(\Omega) \subset C_\alpha(\Omega)$ and every $\mathscr{F} \in C_\alpha(\Omega)$ is α-Lipschitz with constant $k = 1$. We also recall that $\mathscr{F} : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that

$$|\mathscr{F}x - \mathscr{F}y| \leq k|x - y| \quad (\forall)\ x, y \in \Omega$$

and that $\mathscr{F}$ is a strict contraction if $k < 1$.

Next, we collect some properties of the applications defined above.

Property 1.31. If $\mathscr{F}, \mathscr{G} : \Omega \to X$ are α-Lipschitz mappings with constants k, k', respectively, then $\mathscr{F} + \mathscr{G} : \Omega \to X$ is α-Lipschitz with constant $k + k'$.

Property 1.32. If $\mathscr{F} : \Omega \to X$ is compact, then $\mathscr{F}$ is α-Lipschitz with constant $k = 0$.

Property 1.33. If $\mathscr{F} : \Omega \to X$ is Lipschitz with constant k, then $\mathscr{F}$ is α-Lipschitz with the same constant k.

The theorem below asserts the existence and the basic properties of the topological degree for α-condensing perturbations of the identity. For more details, see Isaia, 2006.

Let

$$\mathcal{T} = \left\{(I - \mathscr{F}, \Omega, y) : \Omega \subset X \text{ open and bounded}, \mathscr{F} \in C_\alpha(\overline{\Omega}), y \in X \setminus (I - \mathscr{F})(\partial\Omega)\right\}$$

be the family of the admissible triplets.

Theorem 1.34. There exists one degree function $D : \mathcal{T} \to \mathbb{N}_0$ which satisfies the properties:

(i) *Normalization* $D(I, \Omega, y) = 1$ for every $y \in \Omega$;

(ii) *Additivity on domain* For every disjoint, open sets $\Omega_1, \Omega_2 \subset \Omega$ and every y does not belong to $(I - \mathscr{F})(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$ we have

$$D(I - \mathscr{F}, \Omega, y) = D(I - \mathscr{F}, \Omega_1, y) + D(I - \mathscr{F}, \Omega_2, y);$$

(iii) *Invariance under homotopy* $D(I - H(t, \cdot), \Omega, y(t))$ is independent of $t \in [0, 1]$ for every continuous, bounded mapping $H : [0, 1] \times \overline{\Omega} \to X$ which satisfies

$$\alpha(H([0, 1] \times B)) < \alpha(B) \ \ (\forall)\ B \subset \overline{\Omega} \text{ with } \alpha(B) > 0$$

and every continuous function $y : [0, 1] \to x$ which satisfies

$$y(t) \neq x - H(t, x) \ \ (\forall)\ t \in [0, 1], \ (\forall)\ x \in \partial\Omega;$$

(iv) *Existence* $D(I - \mathscr{F}, \Omega, y) \neq 0$ implies $y \in (I - \mathscr{F})(\Omega)$;

(v) *Excision* $D(I-\mathscr{F},\Omega,y)=D(I-\mathscr{F},\Omega_1,y)$ for every open set $\Omega_1\subset\Omega$ and every y does not belong to $(I-\mathscr{F})(\overline{\Omega}\backslash\Omega_1)$.

Having in hand a degree function defined on $\mathcal{T}$, we collect the usability of the a priori estimate method by means of this degree.

Theorem 1.35. Let $\mathscr{F}: X\to X$ be α-condensing and

$$\mathcal{S}=\{x\in X:\ (\exists)\ \lambda\in[0,1]\ \text{ such that }\ x=\lambda\mathscr{F}x\}.$$

If $\mathcal{S}$ is a bounded set in X, so there exists $r>0$ such that $\mathcal{S}\subset B_r(0)$, then

$$D(I-\lambda\mathscr{F},B_r(0),0)=1\quad(\forall)\ \lambda\in[0,1].$$

Consequently, $\mathscr{F}$ has at least one fixed point and the set of the fixed points of $\mathscr{F}$ lies in $B_r(0)$.

1.4.4 *Picard Operator*

Let (X,d) be a metric space and $A:X\to X$ an operator. We shall use the following notations:

$P(X)=\{Y\subseteq X\mid Y\neq\varnothing\}$; $F_A=\{x\in X\mid A(x)=x\}$−the fixed point set of A;
$I(A)=\{Y\in P(X)\mid A(Y)\subseteq Y\}$;
$O_A(x)=\{x,A(x),A^2(x),...,A^n(x),...\}$−the A-orbit of $x\in X$;
$H:P(X)\times P(X)\to\mathbb{R}_+\cup\{+\infty\}$;
$H(Y,Z)=\max\{\sup\limits_{a\in Y}\inf\limits_{b\in Z}d(a,b),\ \sup\limits_{b\in Z}\inf\limits_{a\in Y}d(a,b)\}$−the Pompeiu-Hausdorff functional on $P(X)$.

Definition 1.36. (Rus, 1987) Let (X,d) be a metric space. An operator $A:X\to X$ is a Picard operator if there exists $x^*\in X$ such that $F_A=\{x^*\}$ and the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^* for all $x_0\in X$.

Definition 1.37. (Rus, 1993) Let (X,d) be a metric space. An operator $A:X\to X$ is a weakly Picard operator if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0\in X$ and its limit (which may depend on x_0) is a fixed point of A.

If A is a weakly Picard operator, then we consider the operator

$$A^\infty:X\to X,\quad A^\infty(x)=\lim_{n\to\infty}A^n(x).$$

The following results are useful in what follows.

Theorem 1.38. (Rus, 1979) Let (Y,d) be a complete metric space and $A,B:Y\to Y$ two operators. Suppose that:

(i) A is a contraction with contraction constant ρ and $F_A=\{x_A^*\}$;
(ii) B has fixed points and $x_B^*\in F_B$;
(iii) There exists $\eta>0$ such that $d(A(x),B(x))\le\eta$, for all $x\in Y$.

Then $d(x_A^*,x_B^*)\le\frac{\eta}{1-\rho}$.

Theorem 1.39. (Rus and Mureşan, 2000) Let (X,d) be a complete metric space and $A,B:X\to X$ two orbitally continuous operators. Assume that:

(i) there exists $\rho \in [0, 1)$ such that

$$d(A^2(x), A(x)) \leq \rho d(x, A(x)),$$
$$d(B^2(x), B(x)) \leq \rho d(x, B(x))$$

for all $x \in X$;

(ii) there exists $\eta > 0$ such that $d(A(x), B(x)) \leq \eta$ for all $x \in X$.

Then $H(F_A, F_B) \leq \frac{\eta}{1-\rho}$, where H denotes the Pompeiu-Hausdorff functional.

Theorem 1.40. (Rus, 1993) Let (X, d) be a metric space. Then $A : X \to X$ is a weakly Picard operator if and only if there exists a partition $X = \bigcup_{\lambda \in \Lambda} X_\lambda$ of X such that

(i) $X_\lambda \in I(A)$;

(ii) $A\,|_{X_\lambda}: X_\lambda \to X_\lambda$ is a Picard operator, for all $\lambda \in \Lambda$.

1.4.5 *Fixed Point Theorems*

In this subsection, we present some fixed point theorems which will be used in the following chapters.

Theorem 1.41 (Banach contraction mapping principle). Let (X, d) be a complete metric space, and $\mathscr{T} : \Omega \to \Omega$ a contraction mapping:

$$d(\mathscr{T}x, \mathscr{T}y) \leq kd(x, y),$$

where $0 < k < 1$, for each $x, y \in \Omega$. Then, there exists a unique fixed point x of $\mathscr{T}$ in Ω: $\mathscr{T}x = x$.

Theorem 1.42 (Schauder's fixed point theorem). Let X be a Banach space and $\Omega \subset X$ a convex, closed and bounded set. If $\mathscr{T} : \Omega \to \Omega$ is a continuous operator such that $\mathscr{T}\Omega \subset X$, $\mathscr{T}\Omega$ is relatively compact, then $\mathscr{T}$ has at least one fixed point in Ω.

Theorem 1.43 (Schaefer's fixed point theorem). Let X be a Banach space and let $F : X \to X$ be a completely continuous mapping. Then either

(i) the equation $x = \lambda Fx$ has a solution for $\lambda = 1$, or

(ii) the set $\{x \in X : x = \lambda Fx \text{ for some } \lambda \in (0, 1)\}$ is unbounded.

Theorem 1.44 (Darbo-Sadovskii's fixed point theorem). If Ω is bounded closed and convex subset of a Banach space X, the continuous mapping $\mathscr{T} : \Omega \to \Omega$ is an α-contraction, then the mapping $\mathscr{T}$ has at least one fixed point in Ω.

Theorem 1.45 (Krasnoselskii's fixed point theorem). Let X be a Banach space, let Ω be a bounded closed convex subset of X and let $\mathscr{S}$, $\mathscr{T}$ be mappings of Ω into X such that $\mathscr{S}z + \mathscr{T}w \in \Omega$ for every pair $z, w \in \Omega$. If $\mathscr{S}$ is a contraction and $\mathscr{T}$ is completely continuous, then the equation $\mathscr{S}z + \mathscr{T}z = z$ has a solution on Ω.

Theorem 1.46 (O'Regan's fixed point theorem). Let U be an open set in a closed, convex set $\mathcal{C}$ of X. Assume $0 \in U$, $T(\overline{U})$ is bounded and $T : \overline{U} \to \mathcal{C}$ is given by $T = T_1 + T_2$ where $T_1 : \overline{U} \to X$ is completely continuous, and $T_2 : \overline{U} \to X$ is a nonlinear contraction. Then either

(i) T has a fixed point in $\overline{U}$, or
(ii) there is a point $x \in \partial U$ and $\lambda \in (0,1)$ with $x = \lambda T(x)$.

A non-empty closed set K in a Banach space X is called a cone if:

(i) $K + K \subseteq K$;
(ii) $\lambda K \subseteq K$ for $\lambda \in R, \lambda \geq 0$;
(iii) $\{-K\} \cap K = \{0\}$, where 0 is the zero element of X.

We introduce an order relation " $\leq$ " in X as follows. Let $z, y \in X$. Then $z \leq y$ if and only if $y - z \in K$. A cone K is called normal if the norm $\|\cdot\|_X$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|z\|_X \leq N\|y\|_X$ for all $z, y \in K$ with $z \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. Similarly, the cone K in X is called regular if every monotone increasing (resp. decreasing) order bounded sequence in X converges in norm.

For any $a, b \in X, a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{z \in X : \ a \leq z \leq b\}.$$

Let X and Y be two ordered Banach spaces. A mapping $\mathscr{T} : X \to Y$ is said to be nondecreasing or monotone increasing if $z \leq y$ implies $\mathscr{T}z \leq \mathscr{T}y$ for all $z, y \in [a, b]$.

Theorem 1.47 (Hybrid fixed point theorem). (Dhage, 2006) Let X be a Banach space and $A, B, C : X \to X$ be three monotone increasing operators such that

(i) A is a contraction with contraction constant $k < 1$;
(ii) B is completely continuous;
(iii) C is totally bounded;
(iv) there exist elements a and b in X such that $a \leq Aa + Ba + Ca$ and $b \geq Ab + Bb + Cb$ with $a \leq b$.

Further if the cone K in X is normal, then the operator equation $Az + Bz + Cz = z$ has a least and a greatest solution in $[a, b]$.

1.4.6 *Critical Point Theorems*

Let H be a real Banach space and $C^1(H, \mathbb{R}^N)$ denotes the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H.

We need to use the critical point theorems to consider the fractional boundary value problems and fractional Hamiltonian systems. For the reader's convenience, we state some necessary definitions and theorems and skip the proofs.

Definition 1.48. (Rabinowitz, 1986) Let $\psi \in C^1(H, \mathbb{R}^N)$. If any sequence

$\{u_k\} \subset H$ for which $\{\psi(u_k)\}$ is bounded and $\psi'(u_k) \to 0$ as $k \to \infty$ possesses a convergent subsequence, then we say ψ satisfies Palais-Smale condition (denoted by (PS) condition for short).

Definition 1.49. (Mawhin and Willem, 1989) Let H be a real Banach space, $\psi : H \to \mathbb{R}$ is differentiable and $c \in \mathbb{R}$. We say that ψ satisfies the $(\mathrm{PS})_{\mathrm{c}}$ condition if the existence of a sequence $\{u_k\}$ in H such that

$$\psi(u_k) \to c, \quad \psi'(u_k) \to 0$$

as $k \to \infty$, implies that c is a critical value of ψ.

Theorem 1.50. (Mawhin and Willem, 1989) Let H be a real reflexive Banach space. If the functional $\psi : H \to \mathbb{R}^N$ is weakly lower semi-continuous and coercive, i.e. $\lim_{|z|\to\infty} \psi(z) = +\infty$, then there exists $z_0 \in H$ such that $\psi(z_0) = \inf_{z\in H} \psi(z)$. Moreover, if ψ is also Fréchet differentiable on H, then $\psi'(z_0) = 0$.

Let B_r be the open ball in H with the radius r and centered at 0 and ∂B_r denote its boundary. Let us recall two critical point results due to Rabinowitz, 1986.

Theorem 1.51 (Mountain pass theorem). (Rabinowitz, 1986) Let H be a real Banach space and $I \in C^1(H, \mathbb{R})$ satisfy Palais-Smale condition. Suppose that I satisfies the following conditions:

(i) $I(0) = 0$,
(ii) there exist constants $\rho, \beta > 0$ such that $I|_{\partial B_\rho(0)} \geq \beta$,
(iii) there exist $e \in H \setminus \overline{B}_\rho(0)$ such that $I(e) \leq 0$.

Then I possesses a critical value $c \geq \beta$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ is an open ball in H of radius ρ centered at 0, and

$$\Gamma = \{g \in C([0,1], H) : \ g(0) = 0, \ g(1) = e\}.$$

Theorem 1.52. (Rabinowitz, 1986) Let H be a real Banach space and $I \in C^1(H, \mathbb{R})$ with I even. Suppose that I satisfies (PS) condition, (i), (ii) of Theorem 1.51 and the following condition:
(iii$'$) For each finite dimensional subspace $H' \subset H$, there is $r = r(H') > 0$ such that $I(u) \leq 0$ for $u \in H' \setminus B_r(0)$.
Then I possesses an unbounded sequence of critical values.

Remark 1.53. A deformation lemma can be proved with condition (C) replacing the usual (PS) condition, and it turns out that Theorem 1.52 holds true under condition (C). We say I satisfies condition (C), i.e., for every sequence $\{u_n\} \subset H$, $\{u_n\}$ has a convergent subsequence if $I(u_n)$ is bounded and $(1 + |u_n|)|I'(u_n)| \to 0$ as $n \to +\infty$.

Let X be a reflexive and separable Banach space, then there are $e_j \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\mathrm{span}\{e_j : j = 1, 2, ...\}} \quad \text{and} \quad X^* = \overline{\mathrm{span}\{e_j^* : j = 1, 2, ...\}},$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{1.43}$$

For convenience, we write

$$X_j := \operatorname{span}\{e_j\}, \quad Y_k := \bigoplus_{j=1}^{k} X_j, \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}. \tag{1.44}$$

And let

$$B_k := \{u \in Y_k : \ |u| \le \rho_k\}, \quad N_k := \{u \in Z_k : \ |u| = \gamma_k\}. \tag{1.45}$$

Theorem 1.54 (Fountain Theorem). (Bartsch, 1993) Suppose:
(H1) X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional, the subspace X_k, Y_k and Z_k are defined by (1.44).

If for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that

(H2) $a_k := \max\limits_{\substack{u \in Y_k \\ |u| = \rho_k}} \varphi(u) \le 0$;

(H3) $b_k := \inf\limits_{\substack{u \in Z_k \\ |u| = r_k}} \varphi(u) \to \infty$, as $k \to \infty$;

(H4) φ satisfies the $(\mathrm{PS})_{\mathrm{c}}$ condition for every $c > 0$.
Then φ has an unbounded sequence of critical values.

Theorem 1.55 (Dual Fountain Theorem). (Bartsch, 1993) Assume (H1) is satisfied, and there is a $k_0 > 0$ so as to for each $k \ge k_0$, there exist $\rho_k > r_k > 0$ such that

(H5) $d_k := \inf\limits_{\substack{u \in Z_k \\ |u| \le \rho_k}} \varphi(u) \to 0$, as $k \to \infty$;

(H6) $i_k := \max\limits_{\substack{u \in Y_k \\ |u| = r_k}} \varphi(u) < 0$;

(H7) $\inf\limits_{\substack{u \in Z_k \\ |u| = \rho_k}} \varphi(u) \ge 0$;

(H8) φ satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in [d_{k_0}, 0)$.
Then φ has a sequence of negative critical values converging to 0.

Remark 1.56. φ satisfies the $(\mathrm{PS})_c^*$ condition means that: if any sequence $\{u_{n_j}\} \subset X$ such that $n_j \to \infty, u_{n_j} \in Y_{n_j}, \varphi(u_{n_j}) \to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ contains a subsequence converging to a critical point of φ. It is obvious that if φ satisfies the $(\mathrm{PS})_c^*$ condition, then φ satisfies the $(\mathrm{PS})_c$ condition.

Let X be a nonempty set and $\Phi, \tilde{\Psi} : X \to \mathbb{R}$ be two functionals. For $r, r_1, r_2, r_3 \in \mathbb{R}$ with $r_1 < \sup_X \Phi$, $r_2 > \inf_X \Phi$, $r_2 > r_1$, and $r_3 > 0$, we define

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \tilde{\Psi}(u) - \tilde{\Psi}(u)}{r - \Phi(u)}, \tag{1.46}$$

$$\beta(r_1, r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\tilde{\Psi}(v) - \tilde{\Psi}(u)}{\Phi(v) - \Phi(u)}, \tag{1.47}$$

$$\gamma(r_2, r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \tilde{\Psi}(u)}{r_3}, \tag{1.48}$$

$$\alpha(r_1, r_2, r_3) := \max \{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}. \tag{1.49}$$

Lemma 1.57. (Averna and Bonanno, 2009; Bonanno and Candito, 2008) Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive, and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\tilde{\Psi} : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that

(i) $\inf_X \Phi = \Phi(0) = \tilde{\Psi}(0) = 0$;
(ii) for every u_1, u_2 satisfying $\tilde{\Psi}(u_1) \geq 0$ and $\tilde{\Psi}(u_2) \geq 0$, one has

$$\inf_{t\in[0,1]} \tilde{\Psi}(tu_1 + (1-t)u_2) \geq 0.$$

Assume further that there exist three positive constants r_1, r_2 and r_3, with $r_1 < r_2$, such that

(iii) $\alpha(r_1, r_2, r_3)) < \beta(r_1, r_2)$.

Then, for each $\lambda \in \big(1/\beta(r_1, r_2), 1/\alpha(r_1, r_2, r_3)\big)$, the functional $\Phi - \lambda\tilde{\Psi}$ has three distinct critical points u_1, u_2 and u_3 such that $u_1 \in \Phi^{-1}(-\infty, r_1)$, $u_2 \in \Phi^{-1}[r_1, r_2)$ and $u_3 \in \Phi^{-1}(-\infty, r_2 + r_3)$.

1.5 Semigroups

1.5.1 *C_0-Semigroup*

Let X be a Banach space and $B(X)$ be the Banach space of linear bounded operators.

Definition 1.58. A semigroup is a one parameter family $\{\mathscr{T}(t) : t \geq 0\} \subset B(X)$ satisfying the conditions:

(i) $\mathscr{T}(t)\mathscr{T}(s) = \mathscr{T}(t + s)$, for $t, s \geq 0$;
(ii) $\mathscr{T}(0) = I$.

Here I denotes the identity operator in X.

Definition 1.59. A semigroup $\{\mathscr{T}(t)\}_{t\geq0}$ is uniformly continuous if

$$\lim_{t\to0+} \|\mathscr{T}(t) - \mathscr{T}(0)\|_{B(X)} = 0,$$

that is if

$$\lim_{|t-s|\to0} \|\mathscr{T}(t) - \mathscr{T}(s)\|_{B(X)} = 0.$$

Definition 1.60. We say that the semigroup $\{\mathscr{T}(t)\}_{t\geq 0}$ is strongly continuous (or a C_0 semigroup) if the map $t \to \mathscr{T}(t)x$ is strongly continuous, for each $x \in X$, i.e.

$$\lim_{t\to 0+} \mathscr{T}(t)x = x, \quad \forall x \in X.$$

Definition 1.61. Let $\mathscr{T}(t)$ be a C_0-semigroup defined on X. The infinitesimal generator A of $\mathscr{T}(t)$ is the linear operator defined by

$$A(x) = \lim_{t\to 0+} \frac{\mathscr{T}(t)x - x}{t}, \quad \text{for} \quad x \in D(A),$$

where $D(A) = \{x \in X : \lim_{t\to 0+} \frac{\mathscr{T}(t)x-x}{t}$ exists in $X\}$.

1.5.2 *Almost Sectorial Operators*

We firstly introduce some special functions and classes of functions which will be used in the following, for more details, we refer to Markus, 2006; Periago and Straub, 2002.

Let S^0_μ with $0 < \mu < \pi$ be the open sector

$$\{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \mu\}$$

and S_μ be its closure, that is

$$S_\mu = \{z \in \mathbb{C}\backslash\{0\} : \ |\arg z| \leq \mu\} \cup \{0\}.$$

We state the concept of almost sectorial operators as follows.

Definition 1.62. (Periago and Straub, 2002) Let $-1 < p < 0$ and $0 < \omega < \pi/2$. By $\Theta^p_\omega(X)$ we denote the family of all linear closed operators $A : D(A) \subset X \to X$ which satisfy:

(i) $\sigma(A) \subset S_\omega$;
(ii) for every $\omega < \mu < \pi$ there exists a constant C_μ such that

$$\|R(z; A)\|_{B(X)} \leq C_\mu |z|^p, \quad \text{for all } z \in \mathbb{C} \setminus S_\mu, \tag{1.50}$$

where $R(z; A) = (zI - A)^{-1}$, $z \in \rho(A)$, which are bounded linear operators the resolvent of A. A linear operator A will be called an almost sectorial operator on X if $A \in \Theta^p_\omega(X)$.

Remark 1.63. Let $A \in \Theta^p_\omega(X)$. Then the definition implies that $0 \in \rho(A)$.

We denote the semigroup associated with A by $\{Q(t)\}_{t\geq 0}$. For $t \in S^0_{\frac{\pi}{2}-\omega}$

$$Q(t) = e^{-tz}(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e^{-tz} R(z; A) dz,$$

where the integral contour $\Gamma_\theta = \{\mathbb{R}^+ e^{i\theta}\} \cup \{\mathbb{R}^+ e^{-i\theta}\}$ is oriented counter-clockwise and $\omega < \theta < \mu < \pi/2 - |\arg t|$, forms an analytic semigroup of growth order $1 + p$.

Remark 1.64. From Periago and Straub, 2002, note that if $A \in \Theta^p_\omega(X)$, then A generates a semigroup $Q(t)$ with a singular behavior at $t = 0$ in a sense, called semigroup of growth $1+p$. Moreover, the semigroup $Q(t)$ is analytic in an open sector of the complex plane $\mathbb{C}$, but the strong continuity fails at $t=0$ for data which are not sufficiently smooth.

Property 1.65. (Periago and Straub, 2002) Let $A \in \Theta_\omega^p(X)$ with $-1<p<0$ and $0<\omega<\frac{\pi}{2}$. Then the following properties remain true.

(i) $Q(t)$ is analytic in $S^0_{\frac{\pi}{2}-\omega}$ and $\dfrac{d^n}{dt^n}Q(t)=(-A)^nQ(t)$ $(t\in S^0_{\frac{\pi}{2}-\omega})$;

(ii) the functional equation $Q(s+t)=Q(s)Q(t)$ for all $s,t\in S^0_{\frac{\pi}{2}-\omega}$ holds;

(iii) there is a constant $C_0=C_0(p)>0$ such that $\|Q(t)\|_{B(X)}\le C_0t^{-p-1}(t>0)$;

(iv) if $\beta>1+p$, then $D(A^\beta)\subset\Sigma_Q=\{x\in X:\ \lim_{t\to0+}Q(t)x=x\}$;

(v) $R(\lambda,A)=\int_0^\infty e^{-\lambda t}Q(t)dt$ for every $\lambda\in\mathbb{C}$ with $Re(\lambda)>0$.

Chapter 2

Fractional Functional Differential Equations

2.1 Introduction

The main objective of this chapter is to present a unified framework to investigate the basic existence theory for a variety of fractional functional differential equations with applications. As far as we know, many complex processes in nature and technology are described by functional differential equations which are dominant nowadays because the functional components in equations allow one to consider prehistory or after-effect influence. Various classes of functional differential equations are of fundamental importance in many problems arising in bionomics, epidemiology, electronics, theory of neural networks, automatic control, etc. Quite long ago delay differential equations had shown their efficiency in the study of the behavior of real populations. One can show that even though the delay terms occurring in the equations are unbounded, the domain of the initial data (past history or memory) may be finite or infinite. Consequently, those two cases need to be discussed independently. Moreover, one can consider functional differential equations so that the delay terms also occur in the derivative of the unknown solution. Since the general formulation of such a problem is difficult to state, a special kind of equations called neutral functional differential equations has been introduced.

On the other hand, fractional calculus is one of the best tools to characterize long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviors, power laws, allometric scaling laws, and so on. So the corresponding mathematical models are fractional differential equations. Their evolutions behave in a much more complicated way so to study the corresponding dynamics is much more difficult. Although the existence theorems for the fractional differential equations can be similarly obtained, not all the classical theory of differential equation can be directly applied to the fractional differential equations. Hence, a somewhat theoretical frame needs to be established.

In Section 2.2, we discuss existence and uniqueness of solutions and existence of extremal solutions of initial value problem for the fractional neutral differential equations with bounded delay. Section 2.3 is devoted to study of basic existence theory for fractional p-type neutral differential equations with unbounded delay but finite memory. In Section 2.4, we present a unified treatment of fundamental existence theory of fractional neutral differential equations with infinite memory. In Section 2.5, we consider a fractional order iterative functional differential equation with parameter. Some theorems to prove existence of the iterative series solutions are presented under some nature conditions.

2.2 Neutral Equations with Bounded Delay

2.2.1 *Introduction*

Let $I_0 = [-\tau, 0],\ \tau > 0,\ t_0 \geq 0$ and $I = [t_0, t_0 + \sigma],\ \sigma > 0$ be two closed and bounded intervals in $\mathbb{R}$. Denote $J = [t_0 - \tau, t_0 + \sigma]$.

Let $\mathcal{C} = C(I_0, \mathbb{R}^n)$ be the space of continuous functions on I_0. For any element $\varphi \in \mathcal{C}$, define the norm

$$\|\varphi\|_* = \sup_{\theta \in I_0} |\varphi(\theta)|.$$

If $z \in C(J, \mathbb{R}^n)$, then for any $t \in I$ define $z_t \in \mathcal{C}$ by

$$z_t(\theta) = z(t + \theta), \quad \theta \in [-\tau, 0].$$

Consider the initial value problems (IVP for short) of fractional neutral functional differential equations with bounded delay of the form

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}(x(t) - k(t, x_t)) = F(t, x_t), \quad \text{a.e.} \ \ t \in (t_0, t_0 + a], \\ x_{t_0} = \varphi, \end{cases} \tag{2.1}$$

where ${}_{t_0}^{C}D_t^{\alpha}$ is Caputo fractional derivative of order $0 < \alpha < 1$, $F : I \times \mathcal{C} \to \mathbb{R}^n$ is a given function satisfying some assumptions that will be specified later, and $\varphi \in \mathcal{C}$.

In Subsection 2.2.2, we establish the existence and uniqueness theorems of IVP (2.1). In Subsection 2.2.3, we discuss the existence of extremal solutions for IVP (2.1). We firstly give the definitions of $L^{\frac{1}{\beta}}$-Carathéodory, $L^{\frac{1}{\gamma}}$-Chandrabhan and $L^{\frac{1}{\delta}}$-Lipschitz, where β, γ, δ are some given numbers. Next, we apply the hybrid fixed point theorem to prove the existence results of extremal solutions for IVP (2.1) under $L^{\frac{1}{\beta}}$-Carathéodory, $L^{\frac{1}{\gamma}}$-Chandrabhan and $L^{\frac{1}{\delta}}$-Lipschitz conditions. We do not require the continuity of the nonlinearities involved in the equation (2.1). In the end, we will present an example to illustrate our main results.

2.2.2 *Existence and Uniqueness*

Let $A(\sigma, \gamma) = \{x \in C([t_0 - \tau, t_0 + \sigma], \mathbb{R}^n) : \ x_{t_0} = \varphi, \ \sup_{t_0 \leq t \leq t_0 + \sigma} |x(t) - \varphi(0)| \leq \gamma\}$, where σ, γ are positive constants.

Before stating and proving the main results, we introduce the following hypotheses:

(H1) $F(t, \varphi)$ is measurable with respect to t on I;
(H2) $F(t, \varphi)$ is continuous with respect to φ on $C(I_0, \mathbb{R}^n)$;

(H3) there exist $\alpha_1 \in (0,\alpha)$ and a real-valued function $m(t) \in L^{\frac{1}{\alpha_1}} I$ such that for any $x \in A(\sigma,\gamma)$, $|F(t,x_t)| \le m(t)$, for $t \in I_0$;
(H4) for any $x \in A(\sigma,\gamma)$, $k(t,x_t) = k_1(t,x_t) + k_2(t,x_t)$;
(H5) k_1 is continuous and for any $x', x'' \in A(\sigma,\gamma),\ t \in I$

$$|k_1(t,x'_t) - k_1(t,x''_t)| \le l\|x' - x''\|, \quad \text{where} \quad l \in (0,1);$$

(H6) k_2 is completely continuous and for any bounded set Λ in $A(\sigma,\gamma)$, the set $\{t \to k_2(t,x_t) : x \in \Lambda\}$ is equicontinuous in $C(I,\mathbb{R}^n)$.

Lemma 2.1. If there exist $\sigma \in (0,a)$ and $\gamma \in (0,\infty)$ such that (H1)-(H3) are satisfied, then for $t \in (t_0, t_0+\sigma]$, IVP (2.1) is equivalent to the following equation

$$\begin{cases} x(t) = \varphi(0) - k(t_0,\varphi) + k(t,x_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-s)^{\alpha-1} F(s,x_s)ds, \quad t \in I_0, \\ x_{t_0} = \varphi. \end{cases} \tag{2.2}$$

Proof. First, it is easy to obtain that $F(t,x_t)$ is Lebesgue measurable on I according to conditions (H1) and (H2). A direct calculation gives that $(t-s)^{\alpha-1} \in L^{\frac{1}{1-\alpha_1}}([t_0,t],\mathbb{R})$, for $t \in I$. In the light of Hölder inequality and (H3), we obtain that $(t-s)^{\alpha-1}F(s,x_s)$ is Lebesgue integrable with respect to $s \in [t_0,t]$ for all $t \in I_0$ and $x \in A(\sigma,\gamma)$, and

$$\int_{t_0}^{t} |(t-s)^{\alpha-1}F(s,x_s)|ds \le \|(t-s)^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}}[t_0,t]} \|m\|_{L^{\frac{1}{\alpha_1}} I}. \tag{2.3}$$

According to Definitions 1.5 and 1.8, it is easy to see that if x is a solution of the IVP (2.1), then x is a solution of the equation (2.2).

On the other hand, if (2.2) is satisfied, then for every $t \in (t_0, t_0+\sigma]$, we have

$$\begin{aligned} {}_{t_0}^{C}D_t^{\alpha}(x(t) - k(t,x_t)) &= {}_{t_0}^{C}D_t^{\alpha}\Big(\varphi(0) - k(t_0,\varphi) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F(s,x_s)ds\Big) \\ &= {}_{t_0}^{C}D_t^{\alpha}\Big(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F(s,x_s)ds\Big) \\ &= {}_{t_0}^{C}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}F(t,x_t)\Big) \\ &= {}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}F(t,x_t)\Big) - [{}_{t_0}D_t^{-\alpha}F(t,x_t)]_{t=t_0}\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= F(t,x_t) - [{}_{t_0}D_t^{-\alpha}F(t,x_t)]_{t=t_0}\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned}$$

According to (2.3), we know that $[{}_{t_0}D_t^{-\alpha}F(t,x_t)]_{t=t_0} = 0$, which means that ${}_{t_0}^{C}D_t^{\alpha}(x(t) - k(t,x_t)) = F(t,x_t),\ t \in (t_0,t_0+\sigma]$, and this completes the proof. □

Theorem 2.2. Assume that there exist $\sigma \in (0,a)$ and $\gamma \in (0,\infty)$ such that (H1)-(H6) are satisfied. Then the IVP (2.1) has at least one solution on $[t_0, t_0+\eta]$ for some positive number η.

Proof. According to (H4), the equation (2.2) is equivalent to the following equation

$$\begin{cases} x(t) = \varphi(0) - k_1(t_0,\varphi) - k_2(t_0,\varphi) + k_1(t,x_t) + k_2(t,x_t) \\ \qquad\qquad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}F(s,x_s)ds, \quad t \in I, \\ x_{t_0} = \varphi. \end{cases}$$

Let $\widetilde{\varphi} \in A(\sigma,\gamma)$ be defined as $\widetilde{\varphi}_{t_0} = \varphi$, $\widetilde{\varphi}(t_0+t) = \varphi(0)$ for all $t \in [0,\sigma]$. If x is a solution of the IVP (2.1), let $x(t_0+t) = \widetilde{\varphi}(t_0+t) + y(t)$, $t \in [-\tau,\sigma]$, then we have $x_{t_0+t} = \widetilde{\varphi}_{t_0+t} + y_t$, $t \in [0,\sigma]$. Thus y satisfies the equation

$$\begin{aligned} y(t) = &-k_1(t_0,\varphi) - k_2(t_0,\varphi) + k_1(t_0+t, y_t + \widetilde{\varphi}_{t_0+t}) + k_2(t_0+t, y_t + \widetilde{\varphi}_{t_0+t}) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(t_0+s, y_s+\widetilde{\varphi}_{t_0+s})ds, \quad t \in [0,\sigma]. \end{aligned} \tag{2.4}$$

Since k_1, k_2 are continuous and x_t is continuous in t, there exists $\sigma' > 0$, when $0 < t < \sigma'$,

$$|k_1(t_0+t, y_t+\widetilde{\varphi}_{t_0+t}) - k_1(t_0,\varphi)| < \frac{\gamma}{3}, \tag{2.5}$$

and

$$|k_2(t_0+t, y_t+\widetilde{\varphi}_{t_0+t}) - k_2(t_0,\varphi)| < \frac{\gamma}{3}. \tag{2.6}$$

Choose

$$\eta = \min\left\{\sigma, \sigma', \left(\frac{\gamma\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}{3M}\right)^{\frac{1}{(1+\beta)(1-\alpha_1)}}\right\} \tag{2.7}$$

where $\beta = \frac{\alpha-1}{1-\alpha_1} \in (-1,0)$ and $M = \|m\|_{L^{\frac{1}{\alpha_1}}I}$.

Define $E(\eta,\gamma)$ as follows

$$E(\eta,\gamma) = \{y \in C([-\tau,\eta],\mathbb{R}^n) : \ y(s) = 0 \ \text{ for } s \in [-\tau,0] \ \text{ and } \|y\| \le \gamma\}.$$

Then $E(\eta,\gamma)$ is a closed bounded and convex subset of $C([-\tau,\sigma],\mathbb{R}^n)$. On $E(\eta,\gamma)$ we define the operators S and U as follows

$$(Sy)(t) = \begin{cases} 0, & t \in [-\tau,0], \\ -k_1(t_0,\varphi) + k_1(t_0+t, y_t+\widetilde{\varphi}_{t_0+t}), & t \in [0,\eta], \end{cases}$$

$$(Uy)(t) = \begin{cases} 0, & t \in [-\tau,0], \\ -k_2(t_0,\varphi) + k_2(t_0+t, y_t+\widetilde{\varphi}_{t_0+t}) & \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(t_0+s, y_s+\widetilde{\varphi}_{t_0+s})ds, & t \in [0,\eta]. \end{cases}$$

It is easy to see that the operator equation

$$y = Sy + Uy \tag{2.8}$$

has a solution $y \in E(\eta,\gamma)$ if and only if y is a solution of the equation (2.4). Thus $x(t_0+t) = y(t) + \widetilde{\varphi}(t_0+t)$ is a solution of the equation (2.1) on $[0,\eta]$. Therefore, the existence of a solution of the IVP (2.1) is equivalent that (2.8) has a fixed point in $E(\eta,\gamma)$.

Now we show that $S+U$ has a fixed point in $E(\eta,\gamma)$. The proof is divided into three steps.

Step I. $Sz+Uy\in E(\eta,\gamma)$ for every pair $z,y\in E(\eta,\gamma)$.

In fact, for every pair $z,y\in E(\eta,\gamma)$, $Sz+Uy\in C([-\tau,\eta],\mathbb{R}^n)$. Also, it is obvious that $(Sz+Uy)(t)=0$, $t\in[-\tau,0]$.

Moreover, for $t\in[0,\eta]$, by (2.5)-(2.7) and the condition (H3), we have

$$
\begin{aligned}
&|(Sz)(t)+(Uy)(t)|\\
\le{}& |-k_1(t_0,\varphi)+k_1(t_0+t,z_t+\widetilde{\varphi}_{t_0+t})|+|-k_2(t_0,\varphi)+k_2(t_0+t,y_t+\widetilde{\varphi}_{t_0+t})|\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t|(t-s)^{\alpha-1}F(t_0+s,y_s+\widetilde{\varphi}_{t_0+s})|ds\\
\le{}& \frac{2\gamma}{3}+\frac{1}{\Gamma(\alpha)}\left(\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_{t_0}^{t_0+t}(m(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\
\le{}& \frac{2\gamma}{3}+\frac{1}{\Gamma(\alpha)}\left(\int_0^t(t-s)^{\frac{\alpha-1}{1-\alpha_1}}ds\right)^{1-\alpha_1}\left(\int_{t_0}^{t_0+\sigma}(m(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1}\\
\le{}& \frac{2\gamma}{3}+\frac{M\eta^{(1+\beta)(1-\alpha_1)}}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}\\
\le{}& \gamma.
\end{aligned}
$$

Therefore,

$$
\|Sz+Uy\|=\sup_{t\in[0,\eta]}|(Sz)(t)+(Uy)(t)|\le\gamma,
$$

which means that $Sz+Uy\in E(\eta,\gamma)$ for any $z,y\in E(\eta,\gamma)$.

Step II. S is a contraction on $E(\eta,\gamma)$.

For any $y',y''\in E(\eta,\gamma)$, $y'_t+\widetilde{\varphi}_{t_0+t},y''_t+\widetilde{\varphi}_{t_0+t}\in A(\delta,\gamma)$. So by (H5), we get that

$$
\begin{aligned}
|(Sy')(t)-(Sy'')(t)|&=|k_1(t_0+t,y'_t+\widetilde{\varphi}_{t_0+t})-k_1(t_0+t,y''_t+\widetilde{\varphi}_{t_0+t})|\\
&\le l\|y'-y''\|,
\end{aligned}
$$

which implies that

$$
\|Sy'-Sy''\|\le l\|y'-y''\|.
$$

In view of $0<l<1$, S is a contraction on $E(\eta,\gamma)$.

Step III. Now we show that U is a completely continuous operator.

Let

$$
(U_1y)(t)=\begin{cases}0, & t\in[-\tau,0],\\ -k_2(t_0,\varphi)+k_2(t_0+t,y_t+\widetilde{\varphi}_{t_0+t}), & t\in[0,\eta]\end{cases}
$$

and

$$
(U_2y)(t)=\begin{cases}0, & t\in[-\tau,0],\\ \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}F(t_0+s,y_s+\widetilde{\varphi}_{t_0+s})ds, & t\in[0,\eta].\end{cases}
$$

Clearly, $U = U_1 + U_2$.

Since k_2 is completely continuous, U_1 is continuous and $\{U_1 y : \ y \in E(\eta, \gamma)\}$ is uniformly bounded. From the condition that the set $\{t \to k_2(t, x_t) : x \in \Lambda\}$ is equicontinuous for any bounded set Λ in $A(\sigma, \gamma)$, we can conclude that U_1 is a completely continuous operator.

On the other hand, for any $t \in [0, \eta]$, we have

$$\begin{aligned}
|(U_2 y)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(t_0 + s, y_s + \widetilde{\varphi}_{t_0+s})| ds \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \left(\int_{t_0}^{t_0+t} (m(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\
&\leq \frac{M \eta^{(1+\beta)(1-\alpha_1)}}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}}.
\end{aligned}$$

Hence, $\{U_2 y : \ y \in E(\eta, \gamma)\}$ is uniformly bounded.

Now, we will prove that $\{U_2 y : \ y \in E(\eta, \gamma)\}$ is equicontinuous. For any $0 \leq t_1 < t_2 \leq \eta$ and $y \in E(\eta, \gamma)$, we get that

$$\begin{aligned}
&|(U_2 y)(t_2) - (U_2 y)(t_1)| \\
&= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] F(t_0 + s, y_s + \widetilde{\varphi}_{t_0+s}) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} F(t_0 + s, y_s + \widetilde{\varphi}_{t_0+s}) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}] |F(t_0 + s, y_s + \widetilde{\varphi}_{t_0+s})| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |F(t_0 + s, y_s + \widetilde{\varphi}_{t_0+s})| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \\
&\quad + \frac{M}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} [(t_2 - s)^{\alpha-1}]^{\frac{1}{1-\alpha_1}} ds \right)^{1-\alpha_1} \\
&\leq \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_1 - s)^{\beta} - (t_2 - s)^{\beta} \right) ds \right)^{1-\alpha_1} + \frac{M}{\Gamma(\alpha)} \left(\int_{t_1}^{t_2} (t_2 - s)^{\beta} ds \right)^{1-\alpha_1} \\
&\leq \frac{M}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} \left(t_1^{1+\beta} - t_2^{1+\beta} + (t_2 - t_1)^{1+\beta} \right)^{1-\alpha_1} \\
&\quad + \frac{M}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} (t_2 - t_1)^{(1+\beta)(1-\alpha_1)} \\
&\leq \frac{2M}{\Gamma(\alpha)(1+\beta)^{1-\alpha_1}} (t_2 - t_1)^{(1+\beta)(1-\alpha_1)},
\end{aligned}$$

which means that $\{U_2 y : y \in E(\eta, \gamma)\}$ is equicontinuous. Moreover, it is clear that U_2 is continuous. So U_2 is a completely continuous operator. Then $U = U_1 + U_2$ is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\eta,\gamma)$, and hence the IVP (2.1) has a solution $x(t)=\varphi(0)+y(t-t_0)$ for all $t\in[t_0,t_0+\eta]$. □

In the case where $k_1\equiv 0$, we get the following result.

Corollary 2.3. Assume that there exist $\sigma\in(0,a)$ and $\gamma\in(0,\infty)$ such that (H1)-(H3) hold and

(H5)′ k is continuous and for any $x',x''\in A(\sigma,\gamma),\ t\in I$

$$|k(t,x'_t)-k(t,x''_t)|\le l\|x'-x''\|,\quad \text{where}\ \ l\in(0,1).$$

Then IVP (2.1) has at least one solution on $[t_0,t_0+\eta]$ for some positive number η.

In the case where $k_2\equiv 0$, we have the following result.

Corollary 2.4. Assume that there exist $\sigma\in(0,a)$ and $\gamma\in(0,\infty)$ such that (H1)-(H3) hold and

(H6)′ k is completely continuous and for any bounded set Λ in $A(\sigma,\gamma)$, the set $\{t\to k(t,x_t): x\in\Lambda\}$ is equicontinuous on $C(I,\mathbb{R}^n)$.

Then IVP (2.1) has at least one solution on $[t_0,t_0+\eta]$ for some positive number η.

2.2.3 *Extremal Solutions*

Define the order relation " $\le$ " by the cone K in $C(J,\mathbb{R}^n)$, given by

$$K=\{z\in C(J,\mathbb{R}^n)\mid z(t)\ge 0\ \ \text{for all}\ \ t\in J\}.$$

Clearly, the cone K is normal in $C(J,\mathbb{R}^n)$. Note that the order relation " $\le$ " in $C(J,\mathbb{R}^n)$ also induces the order relation in the space $\mathcal{C}$ which we also denote by " $\le$ " itself when there is no confusion.

We give the following definitions in the sequel.

Definition 2.5. A mapping $f: I\times\mathcal{C}\to\mathbb{R}^n$ is called $L^{\frac{1}{\delta}}$-Lipschitz if

(i) $t\mapsto f(t,z)$ is Lebesgue measurable for each $z\in\mathcal{C}$;

(ii) there exist a constant $\delta\in[0,\alpha)$ and a function $l\in L^{\frac{1}{\delta}}(I,\mathbb{R}_+)$ such that

$$|f(t,z)-f(t,y)|\le l(t)\|z-y\|_*,\quad \text{a.e.}\ \ t\in I$$

for all $z,y\in\mathcal{C}$.

Definition 2.6. A mapping $g: I\times\mathcal{C}\to\mathbb{R}^n$ is said to be Carathéodory if

(i) $t\mapsto g(t,z)$ is Lebesgue measurable for each $z\in\mathcal{C}$;

(ii) $z\mapsto g(t,z)$ is continuous almost everywhere for $t\in I$.

Furthermore, a Carathéodory function $g(t,z)$ is called $L^{\frac{1}{\beta}}$-Carathéodory if

(iii) for each real number $r>0$, there exist a constant $\beta\in[0,\alpha)$ and a function $m_r\in L^{\frac{1}{\beta}}(I,\mathbb{R}_+)$ such that

$$|g(t,z)|\le m_r(t),\quad \text{a.e.}\ \ t\in I$$

for all $z\in\mathcal{C}$ with $\|z\|_*\le r$.

Definition 2.7. A mapping $h : I \times \mathcal{C} \to \mathbb{R}^n$ is said to be Chandrabhan if

(i) $t \mapsto h(t, z)$ is Lebesgue measurable for each $z \in \mathcal{C}$;
(ii) $z \mapsto h(t, z)$ is nondecreasing almost everywhere for $t \in I$.
Furthermore, a Chandrabhan function $h(t, z)$ is called $L^{\frac{1}{\gamma}}$-Chandrabhan if
(iii) for each real number $r > 0$, there exist a constant $\gamma \in [0, \alpha)$ and a function $w_r \in L^{\frac{1}{\gamma}}(I, \mathbb{R}_+)$ such that

$$|h(t, z)| \leq w_r(t), \quad \text{a.e. } \ t \in I$$

for all $z \in \mathcal{C}$ with $\|z\|_* \leq r$.

Definition 2.8. A function $x \in C(J, \mathbb{R}^n)$ is called a solution of IVP (2.1) on J if

(i) the function $[x(t) - k(t, x_t)]$ is absolutely continuous on I;
(ii) $x_{t_0} = \varphi$, and
(iii) x satisfies the equation in (2.1).

Definition 2.9. A function $a \in C(J, \mathbb{R}^n)$ is called a lower solution of IVP (2.1) on J if the function $[a(t) - k(t, a_t)]$ is absolutely continuous on I, and

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}(a(t) - k(t, a_t)) \leq F(t, a_t), \quad \text{a.e. } \ t \in (t_0, t_0 + \sigma] \\ a_{t_0} \leq \varphi. \end{cases}$$

Again, a function $b \in C(J, \mathbb{R}^n)$ is called an upper solution of IVP (2.1) on J if the function $[b(t) - k(t, b_t)]$ is absolutely continuous on I, and $[b(t) - k(t, b_t)]$ is absolutely continuous on I, and

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}(b(t) - k(t, b_t)) \geq F(t, b_t), \quad \text{a.e. } \ t \in (t_0, t_0 + \sigma] \\ b_{t_0} \geq \varphi. \end{cases}$$

Finally, a function $x \in C(J, \mathbb{R}^n)$ is a solution of IVP (2.1) on J if it is a lower as well as an upper solution of IVP (2.1) on J.

Definition 2.10. A solution x_M of IVP (2.1) is said to be maximal if for any other solution x to IVP (2.1), one has $x(t) \leq x_M(t)$ for all $t \in J$. Again, a solution x_m of IVP (2.1) is said to be minimal if $x_m(t) \leq x(t)$ for all $t \in J$, where x is any solution for IVP (2.1) on J.

We need the following hypotheses in the sequel.

(F1) $F(t, z_t) = f(t, z_t) + g(t, z_t) + h(t, z_t)$, where $f, g, h : I \times \mathcal{C} \to \mathbb{R}^n$;
(F2) IVP (2.1) has a lower solution a and an upper solution b with $a \leq b$;
(k0) $k(t, z)$ is continuous with respect to t on I for any $z \in \mathcal{C}$;
(k1) $|k(t, z) - k(t, y)| \leq k_0 \|z - y\|_*$, for $z, y \in \mathcal{C}$, $t \in I$, where $k_0 > 0$;
(k2) $k(t, z)$ is nondecreasing with respect to z for any $z \in \mathcal{C}$ and almost all $t \in I$;
(f1) f is $L^{\frac{1}{\delta}}$-Lipschitz, and there exists $\eta \in [0, \alpha)$ such that $|f(t, 0)| \in L^{\frac{1}{\eta}}(I, \mathbb{R}_+)$;
(f2) $f(t, z)$ is nondecreasing with respect to z for any $z \in \mathcal{C}$ and almost all $t \in I$;
(g1) g is $L^{\frac{1}{\beta}}$-Carathéodory;
(g2) $g(t, z)$ is nondecreasing with respect to z for any $z \in \mathcal{C}$ and almost all $t \in I$;
(h1) h is $L^{\frac{1}{\gamma}}$-Chandrabhan.

For any positive constant r, let $B_r = \{z \in C(J, \mathbb{R}^n) : \|z\| \leq r\}$. Set

$$q_1 = \frac{\alpha - 1}{1 - \delta} \in (-1, 0), \qquad L = \|l\|_{L^{\frac{1}{\delta}} I}$$

and

$$q_2 = \frac{\alpha - 1}{1 - \beta} \in (-1, 0), \qquad M_r = \|m_r\|_{L^{\frac{1}{\beta}} I}.$$

In order to prove our main results, we need the following lemma.

Lemma 2.11. Assume that the hypotheses (F1), (f1), (g1) and (h1) hold. $x \in C(J, \mathbb{R}^n)$ is a solution for IVP (2.1) on J if and only if x satisfies the following relation

$$\begin{cases} x(t) = \varphi(0) + k(t, x_t) - k(t_0, \varphi) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t - s)^{\alpha - 1} F(s, x_s) ds, & \text{for } \ t \in I, \\ x(t_0 + \theta) = \varphi(\theta), & \text{for } \ \theta \in I_0. \end{cases} \tag{2.9}$$

Proof. For any positive constant r and $x \in B_r$, since x_t is continuous in t, according to (g_1) and Definition 2.6 (i)-(ii), $g(t, x_t)$ is a measurable function on I. Direct calculation gives that $(t - s)^{\alpha - 1} \in L^{\frac{1}{1-\beta}}[t_0, t]$, for $t \in I$ and $\beta \in [0, \alpha)$. By using Lemma 1.1 (Hölder inequality) and Definition 2.6 (iii), for $t \in I$, we obtain that

$$\begin{aligned} \int_{t_0}^{t} |(t - s)^{\alpha - 1} g(s, x_s)| ds &\leq \left(\int_{t_0}^{t} (t - s)^{\frac{\alpha - 1}{1 - \beta}} ds \right)^{1 - \beta} \|m_r\|_{L^{\frac{1}{\beta}}[t_0, t]} \\ &= \left(\int_{t_0}^{t} (t - s)^{q_2} ds \right)^{1 - \beta} \|m_r\|_{L^{\frac{1}{\beta}}[t_0, t]} \\ &\leq \frac{M_r}{(1 + q_2)^{1 - \beta}} \ \sigma^{(1 + q_2)(1 - \beta)}, \end{aligned} \tag{2.10}$$

which means that $(t - s)^{\alpha - 1} g(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in I$ and $x \in B_r$.

According to (f1), for $t \in I$ and $x \in B_r$, we get that

$$|f(t, x_t)| \leq l(t)\|x_t\|_* + |f(t, 0)| \leq l(t) r + |f(t, 0)|. \tag{2.11}$$

Using the similar argument and noting that (f1) and (h1), we can get that $(t - s)^{\alpha - 1} f(s, x_s)$ and $(t - s)^{\alpha - 1} h(s, x_s)$ are Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in I$ and $x \in B_r$. Thus, according to (F1), we get that $(t - s)^{\alpha - 1} F(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in I$ and $x \in B_r$.

Let $G(\theta, s) = (t - \theta)^{-\alpha} |\theta - s|^{\alpha - 1} m_r(s)$. Since $G(\theta, s)$ is a nonnegative, measurable function on $D = [t_0, t] \times [t_0, t]$ for $t \in I$, we have

$$\int_{t_0}^{t} \left[\int_{t_0}^{t} G(\theta, s) ds \right] d\theta = \int_{D} G(\theta, s) ds d\theta = \int_{t_0}^{t} \left[\int_{t_0}^{t} G(\theta, s) d\theta \right] ds$$

and

$$\begin{aligned}\int_D G(\theta,s)dsd\theta &= \int_{t_0}^{t}\left[\int_{t_0}^{t} G(\theta,s)ds\right]d\theta\\ &= \int_{t_0}^{t}(t-\theta)^{-\alpha}\left[\int_{t_0}^{t}|\theta-s|^{\alpha-1}m_r(s)ds\right]d\theta\\ &= \int_{t_0}^{t}(t-\theta)^{-\alpha}\left[\int_{t_0}^{\theta}(\theta-s)^{\alpha-1}m_r(s)ds\right]d\theta\\ &\quad +\int_{t_0}^{t}(t-\theta)^{-\alpha}\left[\int_{\theta}^{t}(s-\theta)^{\alpha-1}m_r(s)ds\right]d\theta\\ &\le \frac{2M_r}{(1+q_2)^{1-\beta}}\ \sigma^{(1+q_2)(1-\beta)}\int_{t_0}^{t}(t-\theta)^{-\alpha}d\theta\\ &\le \frac{2M_r}{(1-\alpha)(1+q_2)^{1-\beta}}\ \sigma^{(1+q_2)(1-\beta)+1-\alpha}.\end{aligned}$$

Therefore, $G_1(\theta,s)=(t-\theta)^{-\alpha}(\theta-s)^{\alpha-1}g(s,x_s)$ is a Lebesgue integrable function on $D=[t_0,t]\times[t_0,t]$, then we have

$$\int_{t_0}^{t}d\theta\int_{t_0}^{\theta}G_1(\theta,s)ds=\int_{t_0}^{t}ds\int_{s}^{t}G_1(\theta,s)d\theta.$$

We now prove that

$${}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}F(t,x_t)\Big)=F(t,x_t),\quad \text{for } t\in(t_0,t_0+\sigma].$$

Indeed, we have

$$\begin{aligned}{}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}g(t,x_t)\Big) &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_{t_0}^{t}(t-\theta)^{-\alpha}\left[\int_{t_0}^{\theta}(\theta-s)^{\alpha-1}g(s,x_s)ds\right]d\theta\\ &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_{t_0}^{t}d\theta\int_{t_0}^{\theta}G_1(\theta,s)ds\\ &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_{t_0}^{t}ds\int_{s}^{t}G_1(\theta,s)d\theta\\ &= \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\frac{d}{dt}\int_{t_0}^{t}g(s,x_s)ds\int_{s}^{t}(t-\theta)^{-\alpha}(\theta-s)^{\alpha-1}d\theta\\ &= \frac{d}{dt}\int_{t_0}^{t}g(s,x_s)ds\\ &= g(t,x_t)\ \ \text{for } t\in(t_0,t_0+\sigma].\end{aligned}$$

Similarly, we can get

$${}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}f(t,x_t)\Big)=f(t,x_t),\ {}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}h(t,x_t)\Big)=h(t,x_t),\quad \text{for } t\in(t_0,t_0+\sigma],$$

which implies

$${}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}F(t,x_t)\Big)=F(t,x_t),\quad \text{for } t\in(t_0,t_0+\sigma].$$

If x satisfies the relation (2.9), then we get that $x(t)-k(t,x_t)$ is absolutely continuous on I. In fact, for any disjoint family of open intervals $\{(a_i,b_i)\}_{1\le i\le n}$ on I with $\sum_{i=1}^n(b_i-a_i)\to 0$, we have

$$\begin{aligned}
&\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{b_i}(b_i-s)^{\alpha-1}g(s,x_s)ds-\int_{t_0}^{a_i}(a_i-s)^{\alpha-1}g(s,x_s)ds\Big|\\
\le&\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big|\int_{a_i}^{b_i}(b_i-s)^{\alpha-1}g(s,x_s)ds\Big|\\
&+\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{a_i}(b_i-s)^{\alpha-1}g(s,x_s)ds-\int_{t_0}^{a_i}(a_i-s)^{\alpha-1}g(s,x_s)ds\Big|\\
\le&\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\int_{a_i}^{b_i}(b_i-s)^{\alpha-1}m_r(s)ds\\
&+\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\int_{t_0}^{a_i}((a_i-s)^{\alpha-1}-(b_i-s)^{\alpha-1})m_r(s)ds\\
\le&\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big(\int_{a_i}^{b_i}(b_i-s)^{\frac{\alpha-1}{1-\beta}}ds\Big)^{1-\beta}\|m_r\|_{L^{\frac{1}{\beta}}I}\\
&+\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big(\int_{t_0}^{a_i}\Big((a_i-s)^{\frac{\alpha-1}{1-\beta}}-(b_i-s)^{\frac{\alpha-1}{1-\beta}}\Big)ds\Big)^{1-\beta}\|m_r\|_{L^{\frac{1}{\beta}}I}\\
\le&\sum_{i=1}^n \frac{(b_i-a_i)^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\|m_r\|_{L^{\frac{1}{\beta}}I}\\
&+\sum_{i=1}^n \frac{(a_i^{1+q_2}-b_i^{1+q_2}+(b_i-a_i)^{1+q_2})^{1-\beta}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\|m_r\|_{L^{\frac{1}{\beta}}I}\\
\le&\,2\sum_{i=1}^n \frac{(b_i-a_i)^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}M_r\to 0.
\end{aligned}$$

Using the similar method, as $\sum_{i=1}^n(b_i-a_i)\to 0$, we can get that

$$\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{b_i}(b_i-s)^{\alpha-1}f(s,x_s)ds-\int_{t_0}^{a_i}(a_i-s)^{\alpha-1}f(s,x_s)ds\Big|\to 0$$

and

$$\sum_{i=1}^n \frac{1}{\Gamma(\alpha)}\Big|\int_{t_0}^{b_i}(b_i-s)^{\alpha-1}h(s,x_s)ds-\int_{t_0}^{a_i}(a_i-s)^{\alpha-1}h(s,x_s)ds\Big|\to 0.$$

Hence, $\sum_{i=1}^n\|x(b_i)-k(b_i,x_{b_i})-x(a_i)+k(a_i,x_{a_i})\|\to 0$, as $\sum_{i=1}^n(b_i-a_i)\to 0$. Therefore, $x(t)-k(t,x_t)$ is absolutely continuous on I which implies that $x(t)-k(t,x_t)$ is differentiable for almost all $t\in I$. According to the argument above, for almost all $t\in(t_0,t_0+\sigma]$, we have

$${}_{t_0}^{C}D_t^{\alpha}(x(t)-k(t,x_t))={}_{t_0}^{C}D_t^{\alpha}\Big[\varphi(0)-k(t_0,\varphi)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F(s,x_s)ds\Big]$$

$$
\begin{aligned}
&= {}_{t_0}^{C}D_t^{\alpha}\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}F(s,x_s)ds\Big]\\
&= {}_{t_0}^{C}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}F(t,x_t)\Big)\\
&= {}_{t_0}D_t^{\alpha}\Big({}_{t_0}D_t^{-\alpha}F(t,x_t)\Big)-[{}_{t_0}D_t^{-\alpha}F(t,x_t)]_{t=t_0}\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\\
&= F(t,x_t)-[{}_{t_0}D_t^{-\alpha}F(t,x_t)]_{t=t_0}\frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Since $(t-s)^{\alpha-1}F(s,x_s)$ is Lebesgue integrable with respect to $s\in[t_0,t]$ for all $t\in I$, we know that ${}_{t_0}D_t^{-\alpha}F(t,x_t)]_{t=t_0}=0$, which means that ${}_{t_0}^{C}D_t^{q}x(t)=F(t,x_t)$, a.e. $t\in(t_0,t_0+\sigma]$. Hence, $x\in C(J,\mathbb{R}^n)$ is a solution of IVP (2.1). On the other hand, it is obvious that if $x\in C(J,\mathbb{R}^n)$ is a solution of IVP (2.1), then x satisfies the relation (2.9), and this completes the proof. □

Theorem 2.12. Assume that the hypotheses (F1), (F2), (k0)-(k2), (f1), (f2), (g1), (g2) and (h1) hold. Then IVP (2.1) has a minimal and a maximal solution in $[a,b]$ defined on J provided that

$$k_0+\frac{L\sigma^{(1+q_1)(1-\delta)}}{\Gamma(\alpha)(1+q_1)^{1-\delta}}<1. \tag{2.12}$$

Proof. Define three operators A, B and C on $C(J,\mathbb{R}^n)$ as follows

$$
\begin{cases}
(Ax)(t)=k(t,x_t)-k(t_0,\varphi)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_s)ds, & \text{for } t\in I,\\
(Ax)(t_0+\theta)=0, & \text{for } \theta\in I_0,
\end{cases}
$$

$$
\begin{cases}
(Bx)(t)=\varphi(0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}g(s,x_s)ds, & \text{for } t\in I,\\
(Bx)(t_0+\theta)=\varphi(\theta), & \text{for } \theta\in I_0,
\end{cases}
$$

and

$$
\begin{cases}
(Cx)(t)=\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1}h(s,x_s)ds, & \text{for } t\in I,\\
(Cx)(t_0+\theta)=0, & \text{for } \theta\in I_0,
\end{cases}
$$

where $x\in C(J,\mathbb{R}^n)$.

Obviously, $Ax+Bx+Cx\in C(J,\mathbb{R}^n)$ for every $x\in C(J,\mathbb{R}^n)$. From Lemma 2.11, we get that IVP (2.1) is equivalent to the operator equation $(Ax)(t)+(Bx)(t)+(Cx)(t)=x(t)$ for $t\in J$. Now we show that the operator equation $Ax+Bx+Cx=x$ has a least and a greatest solution in $[a,b]$. The proof is divided into three steps.

Step I. A is a contraction in $C(J,\mathbb{R}^n)$.

For any $x,y\in C(J,\mathbb{R}^n)$ and $t\in I$, according to (k1) and (f1), we have

$$
\begin{aligned}
|(Ax)(t)-(Ay)(t)| &\le |k(t,x_t)-k(t,y_t)|+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|f(s,x_s)-f(s,y_s)|ds\\
&\le k_0\|x_t-y_t\|_*+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}l(s)\|x_s-y_s\|_*\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\le k_0\|x-y\| + \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t}(t-s)^{\frac{\alpha-1}{1-\delta}}ds\right)^{1-\delta}\|l\|_{L^{\frac{1}{\delta}}[t_0,t]}\|x-y\| \\
&\le k_0\|x-y\| + \frac{L\sigma^{(1+q_1)(1-\delta)}}{\Gamma(\alpha)(1+q_1)^{1-\delta}}\|x-y\| \\
&= \left(k_0 + \frac{L\sigma^{(1+q_1)(1-\delta)}}{\Gamma(\alpha)(1+q_1)^{1-\delta}}\right)\|x-y\|,
\end{aligned}
$$

which implies that $\|Ax-Ay\| \le \left(k_0 + \frac{L\sigma^{(1+q_1)(1-\delta)}}{\Gamma(\alpha)(1+q_1)^{1-\delta}}\right)\|x-y\|$. Therefore, A is a contraction in $C(J,\mathbb{R}^n)$ according to (2.12).

Step II. B is a completely continuous operator and C is a totally bounded operator.

For any $x \in C(J,\mathbb{R}^n)$, we can choose a positive constant r such that $\|x\| \le r$. Firstly, we will prove that B is continuous on B_r. For $x^n, x \in B_r, n = 1,2,...$ with $\lim_{n\to\infty}\|x^n - x\|_C = 0$, we get

$$
\lim_{n\to\infty} x_s^n = x_s, \quad \text{for} \quad s \in I.
$$

Thus, by (g1) and Definition 2.6 (ii), and noting that x_s is continuous with respect to s on I, we have

$$
\lim_{n\to\infty} g(s,x_s^n) = g(s,x_s), \quad \text{a.e} \quad s \in I.
$$

On the other hand, noting that $|g(s,x_s^n) - g(s,x_s)| \le 2m_r(s)$, by Lebesgue's dominated convergence theorem, we have

$$
|(Bx^n)(t) - (Bx)(t)| \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|g(s,x_s^n) - g(s,x_s)|ds \to 0, \quad \text{as } n\to\infty,
$$

which implies

$$
\|Bx^n - Bx\| \to 0 \quad \text{as} \quad n\to\infty.
$$

This means that B is continuous.

Next, we will show that for any positive constant r, $\{Bx : \ x \in B_r\}$ is relatively compact. It suffices to show that the family of functions $\{Bx : \ x \in B_r\}$ is uniformly bounded and equicontinuous.

For any $x \in B_r$ and $t \in I$, we have

$$
\begin{aligned}
|(Bx)(t)| &\le |\varphi(0)| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}|g(s,x_s)|ds \\
&\le |\varphi(0)| + \frac{1}{\Gamma(\alpha)}\left(\int_{0}^{t}(t-s)^{\frac{\alpha-1}{1-\beta}}ds\right)^{1-\beta}\|m_r\|_{L^{\frac{1}{\beta}}[t_0,t]} \\
&\le |\varphi(0)| + \frac{M_r}{\Gamma(\alpha)}\left(\int_{0}^{t}(t-s)^{q_2}ds\right)^{1-\beta} \\
&\le |\varphi(0)| + \frac{M_r\sigma^{(1+q_2)(1-\beta)}}{\Gamma(\alpha)(1+q_2)^{1-\beta}}.
\end{aligned}
$$

For $\theta \in I_0$, we have $|(Bx)(t_0+\theta)| = |\varphi(\theta)|$. Thus $\{Bx : x \in B_r\}$ is uniformly bounded. In the following, we will show that $\{Bx : x \in B_r\}$ is a family of equicontinuous functions.

For any $x \in B_r$ and $t_0 \le t_1 < t_2 \le t_0 + \sigma$, we get

$$\begin{aligned}
&|(Bx)(t_2) - (Bx)(t_1)| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_{t_0}^{t_1}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})g(s,x_s)ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}g(s,x_s)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}|((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})g(s,x_s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}g(s,x_s)|ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})m_r(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}m_r(s)ds \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_{t_0}^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})^{\frac{1}{1-\beta}}ds\right)^{1-\beta}\|m_r\|_{L^{\frac{1}{\beta}}[t_0,t_1]} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}((t_2-s)^{\alpha-1})^{\frac{1}{1-\beta}}ds\right)^{1-\beta}\|m_r\|_{L^{\frac{1}{\beta}}[t_1,t_2]} \\
&\le \frac{M_r}{\Gamma(\alpha)}\left(\int_{t_0}^{t_1}((t_1-s)^{q_2}-(t_2-s)^{q_2})ds\right)^{1-\beta} + \frac{M_r}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{q_2}ds\right)^{1-\beta} \\
&\le \frac{M_r}{\Gamma(\alpha)(1+q_2)^{1-\beta}}\left((t_1-t_0)^{1+q_2}-(t_2-t_0)^{1+q_2}+(t_2-t_1)^{1+q_2}\right)^{1-\beta} \\
&\quad + \frac{M_r}{\Gamma(\alpha)(1+q_2)^{1-\beta}}(t_2-t_1)^{(1+q_2)(1-\beta)} \\
&\le \frac{2M_r}{\Gamma(\alpha)(1+q_2)^{1-\beta}}(t_2-t_1)^{(1+q_2)(1-\beta)}.
\end{aligned}$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. In view of the continuity of φ, we can get that $\{Bx : x \in B_r\}$ is a family of equicontinuous functions. Therefore, $\{Bx : x \in B_r\}$ is relatively compact by Ascoli-Arzela Theorem.

Using the similar argument, we can get that $\{Cx : x \in B_r\}$ is also relatively compact, which means that C is totally bounded.

Step III. A, B and C are three monotone increasing operators.

Since $x, y \in C(J, \mathbb{R}^n)$ with $x \le y$ implies that $x_t \le y_t$ for $t \in I$, according to (k2) and (f2), we have

$$\begin{aligned}
(Ax)(t) &= k(t,x_t) - k(t_0,\varphi) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,x_s)ds \\
&\le k(t,y_t) - k(t_0,\varphi) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}f(s,y_s)ds \\
&= (Ay)(t).
\end{aligned}$$

Hence A is a monotone increasing operator. Similarly, we can conclude that B and C are also monotone increasing operators according to (g2), (h1) and Definition 2.7 (ii).

Clearly, K is a normal cone. From (F2) and Definition 2.9, we have that $a \leq Aa + Ba + Ca$ and $b \geq Ab + Bb + Cb$ with $a \leq b$. Thus the operators A, B and C satisfy all the conditions of Theorem 1.47 and hence the operator equation $Ax + Bx + Cx = x$ has a least and a greatest solution in $[a, b]$. Therefore, IVP (2.1) has a minimal and a maximal solution on J. □

Example 2.13. Consider the following IVP of scalar discontinuous fractional functional differential equation

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}} x(t) = F(t, x_t) = f(t) + \zeta(t)x(t) \\ \qquad\qquad\qquad\qquad\qquad + \dfrac{1}{t^{1/3}} x(t-1) + \zeta(t)h(x(t)), \quad \text{a.e. } \ t \in (0, \sigma], \\ x(\theta) = 0, \quad \theta \in [-1, 0], \end{cases} \tag{2.13}$$

where $0 < \sigma \leq (\frac{1}{2\Gamma(3/2)})^2 = \frac{1}{\pi}$ and we take functions $f(t)$, $\zeta(t)$ and $h(x(t))$ as follows

$$f(t) = \begin{cases} t, & 0 \leq t \leq \frac{\sigma}{2}, \\ 0, & \frac{\sigma}{2} < t \leq \sigma, \end{cases} \qquad \zeta(t) = \begin{cases} 0, & 0 \leq t \leq \frac{\sigma}{2}, \\ 1, & \frac{\sigma}{2} < t \leq \sigma, \end{cases}$$

and

$$h(x(t)) = \begin{cases} x(t), & x(t) \geq 0, \\ x(t) - 1, & x(t) < 0. \end{cases}$$

Evidently, the function

$$F(t, \varphi) = f(t, \varphi) + g(t, \varphi) + h(t, \varphi), \quad \varphi \in C([-1, 0], \mathbb{R}),$$

where

$$f(t, \varphi) = f(t) + \zeta(t)\varphi(0), \quad g(t, \varphi) = \frac{1}{t^{1/3}} \varphi(-1) \quad \text{and} \quad h(t, \varphi) = \zeta(t)h(\varphi(0)).$$

One can easily check that $a(t) = 0$ is a lower solution of IVP (2.13). On the other hand, let

$$b(t) = \begin{cases} t, & t \in [0, \sigma], \\ 0, & t \in [-1, 0]. \end{cases}$$

Then, $b \in C([-1, \sigma], \mathbb{R})$ is a upper solution of IVP (2.13). In fact, direct calculation gives that

$${}_0^C D_t^{\frac{1}{2}} b(t) = \frac{t^{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \geq 2t \geq F(t, b_t) = \begin{cases} t, & 0 < t \leq \frac{\sigma}{2} \\ 2t, & \frac{\sigma}{2} < t \leq \sigma \end{cases} \quad \text{for } \ t \in (0, \sigma].$$

Moreover, noting that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, it is easy to verify that conditions (k0)-(k2), (f1)-(f2), (g1)-(g2), (h1) and (2.12) are satisfied. Therefore, Theorem 2.12 allows us to conclude that IVP (2.13) has a minimal and a maximal solution in $[0, b]$ defined on $[-1, \sigma]$.

2.3 p-Type Neutral Equations

2.3.1 *Introduction*

Let $\mathcal{C} = C([-1,0],\mathbb{R}^n)$ denote the space of continuous functions on $[-1,0]$. For any element $\varphi \in \mathcal{C}$, define the norm $\|\varphi\|_* = \sup_{\theta\in[-1,0]} |\varphi(\theta)|$.

Consider the IVP of p-type fractional neutral functional differential equations of the form

$$ {}_{t_0}^{C}D_t^q g(t,x_t) = f(t,x_t), \tag{2.14}$$

$$ x_{t_0} = \varphi, \quad (t_0,\varphi) \in \Omega, \tag{2.15}$$

where ${}_{t_0}^{C}D_t^q$ is Caputo fractional derivative of order $0 < q < 1$, Ω is an open subset of $[0,\infty)\times\mathcal{C}$ and $g, f : \Omega \to \mathbb{R}^n$ are given functionals satisfying some assumptions that will be specified later. $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(p(t,\theta))$, where $-1 \le \theta \le 0$, $p(t,\theta)$ is a p-function.

Definition 2.14. (Lakshmikantham, Wen and Zhang, 1994) A function $p \in C(J \times [-1,0],\mathbb{R})$ is called a p-function if it has the following properties:

(i) $p(t,0) = t$;
(ii) $p(t,-1)$ is a nondecreasing function of t;
(iii) there exists a $\sigma \ge -\infty$ such that $p(t,\theta)$ is an increasing function for θ for each $t \in (\sigma,\infty)$;
(iv) $p(t,0) - p(t,-1) > 0$ for $t \in (\sigma,\infty)$.

In the following, we suppose $t \in (\sigma,\infty)$.

Definition 2.15. (Lakshmikantham, Wen and Zhang, 1994) Let $t_0 \ge 0, A > 0$ and $x \in C([p(t_0,-1),t_0+A],\mathbb{R}^n)$. For any $t \in [t_0,t_0+A]$, we define x_t by

$$ x_t(\theta) = x(p(t,\theta)), \quad -1 \le \theta \le 0, $$

so that $x_t \in \mathcal{C} = C([-1,0],\mathbb{R}^n)$.

Note that the frequently used symbol "x_t" (in Hale, 1977; Lakshmikantham, 2008; Lakshmikantham, Wen and Zhang, 1994, $x_t(\theta) = x(t+\theta)$, where $-\tau \le \theta \le 0, r > 0, r = \text{const}$) in the theory of functional differential equations with bounded delay is a partial case of the above definition. Indeed, in this case we can put $p(t,\theta) = t + r\theta,\ \theta \in [-1,0]$.

Definition 2.16. A function x is said to be a solution of IVP (2.14)-(2.15) on $[p(t_0,-1),t_0+\alpha]$, if there are $t_0 \ge 0, \alpha > 0$, such that

(i) $x \in C([p(t_0,-1),t_0+\alpha],\mathbb{R}^n)$ and $(t,x_t) \in \Omega$, for $t \in [t_0,t_0+\alpha]$;
(ii) $x_{t_0} = \varphi$;
(iii) $g(t,x_t)$ is differentiable and (2.14) holds almost everywhere on $[t_0,t_0+\alpha]$.

We need the following lemma relative to p-function before we proceed further, which is taken from Lakshmikantham, Wen and Zhang, 1994.

Lemma 2.17. (Lakshmikantham, Wen and Zhang, 1994) Suppose that $p(t,\theta)$ is a p-function. For $A > 0$, $\tau \in (\sigma, \infty)$ (τ may be σ if $\sigma > -\infty$), let $x \in C([p(\tau,-1), \tau + A], \mathbb{R}^n)$ and $\varphi \in C([-1,0], \mathbb{R}^n)$. Then we have

(i) x_t is continuous in t on $[\tau, \tau + A]$ and $\tilde{p}(t,\theta) = p(\tau + t, \theta) - \tau$ is also a p-function;

(ii) if $p(\tau + t, -1) < \tau$ for $t > 0$, then there exists $-1 < s(\tau,t) < 0$ such that $p(\tau + t, s(\tau,t)) = \tau$ and

$$\begin{cases} p(\tau + t, -1) \leq p(\tau + t, \theta) \leq \tau, & \text{for } -1 \leq \theta \leq s(\tau,t), \\ \tau \leq p(\tau + t, \theta) \leq \tau + t, & \text{for } s(\tau,t) \leq \theta \leq 0. \end{cases}$$

Moreover, $s \to 0$ uniformly in τ as $t \to 0$;

(iii) there exists a function $\eta \in C([p(\tau,-1), \tau], \mathbb{R}^n)$ such that

$$\eta(p(\tau,\theta)) = \varphi(\theta) \quad \text{for } -1 \leq \theta \leq 0.$$

It is well known that a neutral functional differential equation (NFDE for short) is one in which the derivatives of the past history or derivatives of functionals of the past history are involved as well as the present state of the system. In other words, in order to guarantee that the equation (2.14) is NFDE, the coefficient of $x(t)$ that is contained in $g(t, x_t)$ can not be equal to zero. Then we need introduce the concept of atomic.

Let $g \in C(\mathbb{R}^+ \times \mathcal{C}, \mathbb{R}^n)$ and $g(t,\varphi)$ be linear in φ. Then Riesz representation theorem shows that there exists a $n \times n$ matrix function $\eta(t,\theta)$ of bounded variation such that

$$g(t,\varphi) = \int_{-\gamma}^{0} [d_\theta \eta(t,\theta)] \varphi(\theta).$$

For $t_0 \geq 0$ and $\theta_0 \in (-\gamma, 0)$, if

$$\det[\eta(t_0, \theta_0^+) - \eta(t_0, \theta_0^-)] \neq 0,$$

then we say that $g(t,\varphi)$ is atomic at θ_0 for t_0. Similarly, one can define $g(t,\varphi)$ to be atomic at the endpoints $-r$ and 0 for t_0. If for every $t \geq 0$, $g(t,\varphi)$ is atomic at θ_0 for t, then we say that $g(t,\varphi)$ is atomic at θ_0 for $\mathbb{R}^+$. If $g(t,\varphi)$ is not linear in φ, suppose that $g(t,\varphi)$ has a Frechet derivative with respect to φ, then $g'_\varphi(t,\varphi)\psi \in \mathbb{R}^n$ for $(t,\varphi) \in \mathbb{R}^+ \times \mathcal{C}$ and $\psi \in \mathcal{C}$, where g'_φ denote the Fréchet derivative of g with respect to φ. Then $g'_\varphi(t,\varphi)$ is a linear mapping from $\mathcal{C}$ into $\mathbb{R}^n$ and therefore

$$g'_\varphi(t,\varphi)\psi = \int_{-\gamma}^{0} [d_\theta \mu(t,\varphi,\theta)] \psi(\theta),$$

where $\mu(t,\varphi,\theta)$ is a matrix function of bounded variation. As before, if $\det[\mu(t_0,\varphi_0,\theta_0^+) - \mu(t_0,\varphi_0,\theta_0^-)] \neq 0$, for $t_0 \geq 0$, then we say that, the nonlinear $g(t,\varphi)$ is atomic at θ_0 for (t_0,φ_0). If $g(t,\varphi)$ is atomic at θ_0, for every (t,φ), then we say that $g(t,\varphi)$ is atomic at θ_0 for $\mathbb{R}^+ \times \mathcal{C}$.

For a detailed discussion on atomic concept we refer the reader to the books Hale, 1977; Lakshmikantham, Wen and Zhang, 1994.

Lemma 2.18. (Hale, 1977; Lakshmikantham, Wen and Zhang, 1994) Suppose that $g(t,\varphi)$ is atomic at zero on Ω. Then there are a continuous $n\times n$ matrix function $A(t,\varphi)$ with $detA(t,\varphi)\neq 0$ on Ω and a functional $L(t,\varphi,\psi)$ which is linear in ψ such that

$$g'_\varphi(t,\varphi)\psi = A(t,\varphi)\psi(0)+L(t,\varphi,\psi).$$

Moreover, there exists a continuous function $\gamma:\Omega\times[0,1]\to\mathbb{R}^+$ with $\gamma(t,\varphi,0)=0$ such that for every $s\in[0,1]$ and ψ with $(t,\psi)\in\Omega$, $\psi(\theta)=0$ for $-1\leq\theta\leq -s$,

$$|L(t,\varphi,\psi)|\leq\gamma(t,\varphi,s)\|\psi\|_*.$$

2.3.2 *Existence and Uniqueness*

Assume that the functional $f:\Omega\to\mathbb{R}^n$ satisfies the following conditions.
(H1) $f(t,\varphi)$ is Lebesgue measurable with respect to t for any $(t,\varphi)\in\Omega$;
(H2) $f(t,\varphi)$ is continuous with respect to φ for any $(t,\varphi)\in\Omega$;
(H3) there exist a constant $q_1\in(0,q)$ and a $L^{\frac{1}{q_1}}$-integrable function m such that $|f(t,\varphi)|\leq m(t)$ for any $(t,\varphi)\in\Omega$.

For each $(t_0,\varphi)\in\Omega$, let $\tilde{p}(t,\theta)=p(t_0+t,\theta)-t_0$. Define the function $\eta\in C([\tilde{p}(0,-1),\infty),\mathbb{R}^n)$ by

$$\begin{cases}\eta(\tilde{p}(0,\theta))=\varphi(\theta), & \text{for } \ \theta\in[-1,0],\\ \eta(t)=\varphi(0), & \text{for } \ t\in[0,\infty).\end{cases}$$

Let $x\in C([p(t_0,-1),t_0+\alpha],\mathbb{R}^n)$, $\alpha<A$ and let

$$x(t_0+t)=\eta(t)+z(t) \ \text{ for } \tilde{p}(0,-1)\leq t\leq\alpha. \tag{2.16}$$

Lemma 2.19. $x(t)$ is a solution of IVP (2.14)-(2.15) on $[p(t_0,-1),t_0+\alpha]$ if and only if $z(t)$ satisfies the relation

$$\begin{cases} g(t_0+t,\tilde{\eta}_t+\tilde{z}_t)-g(t_0,\varphi)=\dfrac{1}{\Gamma(q)}\displaystyle\int_0^t(t-s)^{q-1}f(t_0+s,\tilde{\eta}_s+\tilde{z}_s)ds, \text{ for } t\in[0,\alpha],\\ \tilde{z}_0=0,\end{cases} \tag{2.17}$$

where $\tilde{\eta}_t(\theta)=\eta(\tilde{p}(t,\theta))$, $\tilde{z}_t(\theta)=z(\tilde{p}(t,\theta))$, for $-1\leq\theta\leq 0$.

Proof. Since x_t is continuous in t, x_t is a measurable function, therefore according to conditions (H1) and (H2), $f(t,x_t)$ is Lebesgue measurable on $[t_0,t_0+\alpha]$. Direct calculation gives that $(t-s)^{q-1}\in L^{\frac{1}{1-q_1}}[t_0,t]$, for $t\in[t_0,t_0+\alpha]$ and $q_1\in(0,q)$. In light of Hölder inequality, we obtain that $(t-s)^{q-1}f(s,x_s)$ is Lebesgue integrable with respect to $s\in[t_0,t]$ for all $t\in[t_0,t_0+\alpha]$, and

$$\int_{t_0}^t|(t-s)^{q-1}f(s,x_s)|ds\leq\|(t-s)^{q-1}\|_{L^{\frac{1}{1-q_1}}[t_0,t]}\|m\|_{L^{\frac{1}{q_1}}[t_0,t_0+\alpha]}.$$

Hence $x(t)$ is the solution of IVP (2.14)-(2.15) if and only if it satisfies the relation

$$\begin{cases} g(t,x_t)-g(t_0,x_{t_0})=\dfrac{1}{\Gamma(q)}\displaystyle\int_{t_0}^t(t-u)^{q-1}f(u,x_u)du, \quad \text{for } \ t\in[t_0,t_0+\alpha],\\ x_{t_0}=\varphi,\end{cases}$$

or setting $u = t_0 + s$,

$$\begin{cases} g(t_0 + t, x_{t_0+t}) - g(t_0, x_{t_0}) = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(t_0 + s, x_{t_0+s}) ds, & \text{for } t \in [0, \alpha], \\ x_{t_0} = \varphi. \end{cases} \tag{2.18}$$

In view of (2.16), we have

$$\begin{aligned} x_{t_0+t}(\theta) &= x(p(t_0 + t, \theta)) = x(\tilde{p}(t, \theta) + t_0) = \eta(\tilde{p}(t, \theta)) + z(\tilde{p}(t, \theta)) \\ &= \tilde{\eta}_t(\theta) + \tilde{z}_t(\theta), \quad \text{for } t \in [0, \alpha]. \end{aligned}$$

In particular $x_{t_0}(\theta) = \tilde{\eta}_0(\theta) + \tilde{z}_0(\theta)$. Hence $x_{t_0} = \varphi$ if and only if $\tilde{z}_0 = 0$ according to $\tilde{\eta} = \varphi$. It is clear that $x(t)$ satisfies (2.18) if and only if $z(t)$ satisfies (2.17). $\square$

For any $\sigma, \xi > 0$, let

$$E(\sigma, \xi) = \{z \in C([\tilde{p}(0, -1), \sigma], \mathbb{R}^n) : \ \tilde{z}_0 = 0, \ \|\tilde{z}_t\|_* \leq \xi \ \text{ for } t \in [0, \sigma]\},$$

which is a bounded closed convex subset of the Banach space $C([\tilde{p}(0, -1), \sigma], \mathbb{R}^n)$ endowed with supremum norm $\| \cdot \|$.

Lemma 2.20. Suppose $\Omega \subseteq R \times \mathcal{C}$ is open, $W \subset \Omega$ is compact. For any a neighborhood $V' \subset \Omega$ of W, there is a neighborhood $V'' \subset V'$ of W and there exist positive numbers δ and ξ such that $(t_0 + t, \tilde{\eta}_t + \lambda \tilde{z}_t) \in V'$ with $0 \leq \lambda \leq 1$ for any $(t_0, \varphi) \in V'', t \in [0, \sigma]$ and $z \in E(\sigma, \xi)$.

The proof of Lemma 2.20 is similar to that of (iii) of Lemma 2.1.8 in Lakshmikantham, Wen and Zhang, 1994, thus it is omitted.

Suppose g is atomic at 0 on Ω. Define two operators S and T on $E(\alpha, \beta)$ as follows

$$\begin{cases} (Sz)(t) = 0, & \text{for } t \in [\tilde{p}(0, -1), 0], \\ A(t_0 + t, \tilde{\eta}_t)(Sz)(t) = g(t_0, \varphi) - g(t_0 + t, \tilde{\eta}_t + \tilde{z}_t) & \\ \qquad + g'_\varphi(t_0 + t, \tilde{\eta}_t)\tilde{z}_t - L(t_0 + t, \tilde{\eta}_t, \tilde{z}_t), & \text{for } t \in [0, \alpha] \end{cases} \tag{2.19}$$

and

$$\begin{cases} (Tz)(t) = 0, & \text{for } t \in [\tilde{p}(0, -1), 0], \\ A(t_0 + t, \tilde{\eta}_t)(Tz)(t) & \\ \qquad = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds, & \text{for } t \in [0, \alpha], \end{cases} \tag{2.20}$$

where $A(t_0 + t, \tilde{\eta}_t)$, $L(t_0 + t, \tilde{\eta}_t, \tilde{z}_t)$ are functions described in Lemma 2.18.

It is clear that the operator equation

$$z = Sz + Tz \tag{2.21}$$

has a solution $z \in E(\alpha, \beta)$ if and only if z is a solution of (2.17). Therefore the existence of a solution of IVP (2.14)-(2.15) is equivalent to determining $\alpha, \beta > 0$ such that $S + T$ has a fixed point on $E(\alpha, \beta)$.

We are now in a position to prove the following existence results, and the proof is based on Krasnoselskii's fixed point theorem.

Theorem 2.21. Suppose $g : \Omega \to \mathbb{R}^n$ is continuous together with its first Fréchet derivative with respect to the second argument, and g is atomic at 0 on Ω. $f : \Omega \to \mathbb{R}^n$ satisfies conditions (H1)-(H3). $W \subset \Omega$ is a compact set. Then there exist a neighborhood $V \subset \Omega$ of W and a constant $\alpha > 0$ such that for any $(t_0, \varphi) \in V$, IVP (2.14)-(2.15) has a solution which exists on $[p(t_0, -1), t_0 + \alpha]$.

Proof. As we have mentioned above, we only need to discuss operator equation (2.21). For any $(t, \varphi) \in \Omega$, the property of the matrix function $A(t, \varphi)$ which is nonsingular and continuous on Ω implies that its inverse matrix $A^{-1}(t, \varphi)$ exists and is continuous on Ω. Let $V_0 \subset \Omega$ be the neighborhood of W, suppose that there is an $M > 0$ such that

$$|A^{-1}(t^0, \varphi)| \leq M \quad \text{for every } (t^0, \varphi) \in V_0. \tag{2.22}$$

Note the complete continuity of the function $(m(t))^{\frac{1}{q_1}}$, hence, for a given positive number N, there must exist a number $\alpha_0 > 0$ satisfying

$$\left(\int_{t_0}^{t_0+\alpha_0} (m(s))^{\frac{1}{q_1}} ds \right)^{q_1} \leq N. \tag{2.23}$$

Due to the continuity of functions γ and g'_φ described in Lemma 2.18, there exist a neighborhood $V_1 \subset \Omega$ of W and constants $h_1 > 0$, $h_2 \in (0, 1]$ such that

$$|\gamma(t_0 + t, \tilde{\eta}_t, -s)| = |\gamma(t_0 + t, \tilde{\eta}_t, -s) - \gamma(t_0 + t, \tilde{\eta}_t, 0)| < \frac{1}{4M}, \tag{2.24}$$

$$|g'_\varphi(t_0 + t, \tilde{\eta}_t + \psi) - g'_\varphi(t_0 + t, \tilde{\eta}_t)| < \frac{1}{8M}, \tag{2.25}$$

whenever $(t_0 + t, \tilde{\eta}_t)$, $(t_0 + t, \tilde{\eta}_t + \psi) \in V_1$ and $\|\psi\|_* < h_1$, $-s \in [0, h_2]$.

Let $V_2 = V_0 \cap V_1$. According to Lemma 2.20, we can find a neighborhood $V \subset V_2$ of W and positive numbers α_1 and β with $\alpha_1 < \alpha_0$ and $\beta \leq h_1$ such that $(t_0+t, \tilde{\eta}_t+\lambda\tilde{z}_t) \in V_2$ with $0 \leq \lambda \leq 1$ for any $(t_0, \varphi) \in V$, $t \in [0, \alpha_1]$ and $z \in E(\alpha_1, \beta)$. Let

$$h(t_0 + t, \tilde{\eta}_t, \tilde{z}_t) = g(t_0 + t, \tilde{\eta}_t + \tilde{z}_t) - g(t_0 + t, \tilde{\eta}_t) - g'_\varphi(t_0 + t, \tilde{\eta}_t)\tilde{z}_t.$$

Then we have

$$\begin{aligned} |h(t_0 + t, \tilde{\eta}_t, \tilde{z}_t)| &= \left| \left[\int_0^1 g'_\varphi(t_0 + t, \tilde{\eta}_t + \lambda\tilde{z}_t) d\lambda - g'_\varphi(t_0 + t, \tilde{\eta}_t) \right] \tilde{z}_t \right| \\ &\leq \left| \int_0^1 [g'_\varphi(t_0 + t, \tilde{\eta}_t + \lambda\tilde{z}_t) - g'_\varphi(t_0 + t, \tilde{\eta}_t)] d\lambda \right| \parallel \tilde{z}_t \parallel_* . \end{aligned} \tag{2.26}$$

According to (2.22), (2.25) and (2.26), for any $(t_0, \varphi) \in V$, we have

$$|A^{-1}(t_0 + t, \tilde{\eta}_t) \; h(t_0 + t, \tilde{\eta}_t, \tilde{z}_t)| \leq \frac{\beta}{8}. \tag{2.27}$$

On the other hand, for any $z, w \in E(\alpha_1, \beta)$ and $t \in [0, \alpha_1]$

$$\|\lambda\tilde{z}_t + (1 - \lambda)\tilde{w}_t\|_* \leq \|\lambda\tilde{z}_t\|_* + \|(1 - \lambda)\tilde{w}_t\|_* \leq \lambda\beta + (1 - \lambda)\beta = \beta,$$

thus, $(t_0+t,\tilde{\eta}_t+\lambda\tilde{z}_t+(1-\lambda)\tilde{w}_t)\in V_2$, and

$$\begin{aligned}
&|h(t_0+t,\tilde{\eta}_t,\tilde{z}_t)-h(t_0+t,\tilde{\eta}_t,\tilde{w}_t)|\\
&=|g(t_0+t,\tilde{\eta}_t+\tilde{z}_t)-g(t_0+t,\tilde{\eta}_t+\tilde{w}_t)-g'_\varphi(t_0+t,\tilde{\eta}_t)(\tilde{z}_t-\tilde{w}_t)|\\
&=\left|\left[\int_0^1 g'_\varphi(t_0+t,\tilde{\eta}_t+\tilde{w}_t+\lambda(\tilde{z}_t-\tilde{w}_t))d\lambda-g'_\varphi(t_0+t,\tilde{\eta}_t)\right](\tilde{z}_t-\tilde{w}_t)\right| \qquad (2.28)\\
&\le\left|\int_0^1[g'_\varphi(t_0+t,\tilde{\eta}_t+\lambda\tilde{z}_t+(1-\lambda)\tilde{w}_t)-g'_\varphi(t_0+t,\tilde{\eta}_t)]d\lambda\right|\ \|\tilde{z}_t-\tilde{w}_t\|_*\ .
\end{aligned}$$

From (2.22), (2.25) and (2.28), we have

$$|A^{-1}(t_0+t,\tilde{\eta}_t)[h(t_0+t,\tilde{\eta}_t,\tilde{z}_t)-h(t_0+t,\tilde{\eta}_t,\tilde{w}_t)]|\le\frac{1}{8}\|\tilde{z}_t-\tilde{w}_t\|_*\ . \qquad (2.29)$$

By (ii) of Lemma 2.17, we can also choose $\alpha_2<\alpha_1$ such that for $t\in[0,\alpha_2]$, $-s(0,t)\in[0,h_2]$. From (2.22) and (2.24), we have

$$\begin{aligned}
&|A^{-1}(t_0+t,\tilde{\eta}_t)||L(t_0+t,\tilde{\eta}_t,\tilde{z}_t)|\\
&\le|A^{-1}(t_0+t,\tilde{\eta}_t)|\gamma(t_0+t,\tilde{\eta}_t,-s(0,t))\|\tilde{z}_t\|_* \qquad (2.30)\\
&\le\frac{1}{4}\|\tilde{z}_t\|_*,
\end{aligned}$$

whenever $t\in[0,\alpha_2]$ and $z\in E(\alpha_2,\beta)$.

Now consider the expression $g(t_0,\varphi)-g(t_0+t,\tilde{\eta}_t)$. Since g is continuous in Ω and noting the facts that $\tilde{\eta}_t$ is continuous in t and $\tilde{\eta}_0=\varphi$, there exists a constant $\alpha_3<\alpha_2$ such that

$$|g(t_0,\varphi)-g(t_0+t,\tilde{\eta}_t)|<\frac{\beta}{8M}, \qquad (2.31)$$

whenever $t\in[0,\alpha_3]$.

Set

$$\alpha=\min\left\{\alpha_3,(1+b)^{\frac{1}{1+b}}\left(\frac{\Gamma(q)\beta}{2MN}\right)^{\frac{1}{(1-q_1)(1+b)}}\right\}, \qquad (2.32)$$

where $b=\dfrac{q-1}{1-q_1}\in(-1,0)$.

Now we show that for any $(t_0,\varphi)\in V$, $S+T$ has a fixed point on $E(\alpha,\beta)$, where S and T are defined as in (2.19) and (2.20) respectively. The proof is divided into three steps.

Step I. $Sz+Tw\in E(\alpha,\beta)$ whenever $z,w\in E(\alpha,\beta)$.

Obviously, for every pair $z,w\in E(\alpha,\beta)$, $(Sz)(t)$ and $(Tw)(t)$ are continuous in $t\in[0,\alpha]$. From (2.27), (2.30) and (2.31), for $t\in[0,\alpha]$, we have

$$\begin{aligned}
|(Sz)(t)|\le{}&|A^{-1}(t_0+t,\tilde{\eta}_t)|\ \{|g(t_0,\varphi)-g(t_0+t,\tilde{\eta}_t)|+|L(t_0+t,\tilde{\eta}_t,\tilde{z}_t)|\\
&+|g(t_0+t,\tilde{\eta}_t)-g(t_0+t,\tilde{\eta}_t+\tilde{z}_t)+g'_\varphi(t_0+t,\tilde{\eta}_t)\tilde{z}_t|\}\\
\le{}&\frac{\beta}{2}\ .
\end{aligned}$$

For $t \in [0, \alpha]$, by using (2.22), (2.23), (2.32) and Hölder inequality, we have

$$\begin{aligned}|(Tw)(t)| &\le |A^{-1}(t_0+t,\tilde{\eta}_t)|\ \frac{1}{\Gamma(q)}\left|\int_0^t (t-s)^{q-1} f(t_0+s,\tilde{\eta}_s+\tilde{w}_s)ds\right| \\ &\le \frac{M}{\Gamma(q)}\left[\int_0^t ((t-s)^{q-1})^{\frac{1}{1-q_1}} ds\right]^{1-q_1}\left[\int_{t_0}^{t_0+\alpha} (m(s))^{\frac{1}{q_1}} ds\right]^{q_1} \\ &\le \frac{MN}{\Gamma(q)}\left[\frac{1}{1+b}\alpha^{1+b}\right]^{1-q_1} \\ &\le \frac{\beta}{2}.\end{aligned} \tag{2.33}$$

Thus $|(Sz)(t)+(Tw)(t)| \le \beta$ i.e. $Sz + Tw \in E(\alpha, \beta)$, whenever $z, w \in E(\alpha, \beta)$.

Step II. S is a contraction mapping from $E(\alpha, \beta)$ into itself whose contraction constant is independent of $(t_0, \varphi) \in V$.

For any $z, w \in E(\alpha, \beta)$, $\tilde{w}_0 - \tilde{z}_0 = 0$. Hence (ii) of Lemma 2.17 and Lemma 2.18 are applicable to $\tilde{w}_t - \tilde{z}_t$. For every pair $z, w \in E(\alpha, \beta)$, from (2.29), (2.30) and noting the fact that

$$\begin{aligned}\sup_{0\le t\le\alpha} \|\tilde{z}_t - \tilde{w}_t\|_* &= \sup_{0\le t\le \alpha}\ \sup_{-1\le\theta\le 0} |z(\tilde{p}(t,\theta) - w(\tilde{p}(t,\theta))| \\ &= \sup_{0\le t\le\alpha}\ \sup_{\tilde{p}(t,-1)\le s\le t} |z(s)-w(s)| \\ &= \sup_{\tilde{p}(0,-1)\le s\le\alpha} |z(s)-w(s)| \\ &= \|z-w\|,\end{aligned}$$

we have

$$\begin{aligned}\|Sz - Sw\| &= \sup_{\tilde{p}(0,-1)\le t\le\alpha} |(Sz)(t)-(Sw)(t)| \\ &= \sup_{0\le t\le\alpha} |(Sz)(t)-(Sw)(t)| \\ &\le \sup_{0\le t\le\alpha}\Big\{|A^{-1}(t_0+t,\tilde{\eta}_t)|\ [\ |L(t_0+t,\tilde{\eta}_t,\tilde{w}_t-\tilde{z}_t)| \\ &\quad +|h(t_0+t,\tilde{\eta}_t,\tilde{z}_t) - h(t_0+t,\tilde{\eta}_t,\tilde{w}_t)|\ ]\Big\} \\ &\le (\frac{1}{8}+\frac{1}{4})\sup_{0\le t\le\alpha} \|\tilde{z}_t-\tilde{w}_t\|_* \\ &\le \frac{3}{8}\|z-w\|.\end{aligned}$$

Therefore S is a contraction mapping from $E(\alpha, \beta)$ into itself whose contraction constant is independent of $(t_0, \varphi) \in V$.

Step III. Now we show that T is a completely continuous operator.

For any $z \in E(\alpha, \beta)$ and $0 \le t_1 < t_2 \le \alpha$, we get

$$
\begin{aligned}
&|(Tz)(t_2) - (Tz)(t_1)| \\
= &\bigg| A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \\
&- A^{-1}(t_0 + t_1, \tilde{\eta}_{t_1}) \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \bigg| \\
= &\bigg| A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \\
&+ A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) \frac{1}{\Gamma(q)} \int_0^{t_1} (t_2 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \\
&- A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \\
&+ A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \\
&- A^{-1}(t_0 + t_1, \tilde{\eta}_{t_1}) \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \bigg| \\
\le &\frac{|A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2})|}{\Gamma(q)} \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \right| \\
&+ \frac{|A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2})|}{\Gamma(q)} \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \right| \\
&+ \frac{|A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) - A^{-1}(t_0 + t_1, \tilde{\eta}_{t_1})|}{\Gamma(q)} \left| \int_0^{t_1} (t_1 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \right| \\
= &\frac{|A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2})|}{\Gamma(q)} (I_1 + I_2) + \frac{|A^{-1}(t_0 + t_2, \tilde{\eta}_{t_2}) - A^{-1}(t_0 + t_1, \tilde{\eta}_{t_1})|}{\Gamma(q)} \ I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \right|, \\
I_2 &= \left| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \right|, \\
I_3 &= \left| \int_0^{t_1} (t_1 - s)^{q-1} f(t_0 + s, \tilde{\eta}_s + \tilde{z}_s) ds \right|.
\end{aligned}
$$

By using analogous argument performed in (2.33), we can conclude that

$$
\begin{aligned}
I_1 &\le \frac{N}{(1+b)^{1-q_1}} \left[(t_2 - t_1)^{1+b} \right]^{1-q_1}, \\
I_3 &\le \frac{N}{(1+b)^{1-q_1}} \left({t_1}^{1+b} \right)^{1-q_1},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\leq \left[\int_0^{t_1} |(t_2-s)^{q-1}-(t_1-s)^{q-1}|^{\frac{1}{1-q_1}}ds\right]^{1-q_1}\left[\int_{t_0}^{t_0+t_1} |f(s,x_s)|^{\frac{1}{q_1}}ds\right]^{q_1}\\
&\leq N\left[\int_0^{t_1}((t_1-s)^b-(t_2-s)^b)ds\right]^{1-q_1}\\
&= \frac{N}{(1+b)^{1-q_1}}\left[{t_1}^{1+b}-{t_2}^{1+b}+(t_2-t_1)^{1+b}\right]^{1-q_1}\\
&\leq \frac{N}{(1+b)^{1-q_1}}\left[(t_2-t_1)^{1+b}\right]^{1-q_1},
\end{aligned}
$$

where $b=\frac{q-1}{1-q_1}\in(-1,0)$. Therefore

$$
\begin{aligned}
|(Tz)(t_2)-(Tz)(t_1)| &\leq \frac{|A^{-1}(t_0+t_2,\tilde{\eta}_{t_2})|}{\Gamma(q)}\frac{2N}{(1+b)^{1-q_1}}\left[(t_2-t_1)^{1+b}\right]^{1-q_1}\\
&+\frac{|A^{-1}(t_0+t_2,\tilde{\eta}_{t_2})-A^{-1}(t_0+t_1,\tilde{\eta}_{t_1})|}{\Gamma(q)}\frac{N}{(1+b)^{1-q_1}}\left({t_1}^{1+b}\right)^{1-q_1}.
\end{aligned}
$$

Since $A^{-1}(t_0+t,\tilde{\eta}_t)$ is continuous in $t\in[0,\alpha]$, then $\{Tz;z\in E(\alpha,\beta)\}$ is equicontinuous. In addition, T is continuous from the condition (H2) and $\{Tz;z\in E(\alpha,\beta)\}$ is uniformly bounded from (2.33), thus T is a completely continuous operator by Ascoli-Arzela Theorem.

Therefore, by Theorem 1.45, for every $(t_0,\varphi)\in V$, $S+T$ has a fixed point on $E(\alpha,\beta)$. Hence, IVP (2.14)-(2.15) has a solution defined on $[p(t_0,-1),t_0+\alpha]$. □

Corollary 2.22. Suppose that $(t_0,\varphi)\in\Omega$ is given, g,f are defined as in Theorem 2.21. Then there exists a solution of IVP (2.14)-(2.15).

Corollary 2.23. Suppose that Ω,f are defined as in Theorem 2.21. If $(t_0,\varphi)\in\Omega$ is given, then the IVP relative to fractional p-type retarded differential equations of the form

$$
\begin{cases}
{}_{t_0}^{C}D_t^q x(t)=f(t,x_t),\\
x_{t_0}=\varphi,
\end{cases}
$$

has a solution.

The following existence and uniqueness result for IVP (2.14)-(2.15) is based on Banach's contraction principle.

Theorem 2.24. Suppose $(t_0,\varphi)\in\Omega$ is given, g is defined as in Theorem 2.21. $f:\Omega\to\mathbb{R}^n$ satisfies the condition (H3) and

(H4) $f(t,x_t)$ is measurable for every $(t,x_t)\in\Omega$;

(H5) let $A>0$, there exists a nonnegative function $\ell:[0,A]\to[0,\infty)$ continuous at $t=0$ and $\ell(0)=0$ such that for any $(t,x_t),\ (t,y_t)\in\Omega$, $t\in[t_0,t_0+A]$, we have

$$
\left|\int_{t_0}^{t}(t-s)^{q-1}[f(s,x_s)-f(s,y_s)]ds\right|\leq \ell(t-t_0)\sup_{t_0\leq s\leq t}\|x_s-y_s\|_*.
$$

Then IVP (2.14)-(2.15) has a unique solution.

Proof. According to the argument of Theorem 2.21, it suffices to prove that $S+T$ has a unique fixed point on $E(\alpha,\beta)$, where $\alpha,\beta>0$ sufficiently small. Now, choose $\alpha\in(0,A]$ such that (2.32) holds and

$$c=\frac{3}{8}+\sup_{0\le s\le\alpha}\frac{|A^{-1}(t_0+s,\tilde{\eta}_s)|\ |\ell(s)|}{\Gamma(q)}<1. \tag{2.34}$$

Obviously, $S+T$ is a mapping from $E(\alpha,\beta)$ into itself. Using the same argument as that of Theorem 2.21, for any $z,w\in E(\alpha,\beta), t\in[0,\alpha]$, we get

$$|(Sz)(t)-(Sw)(t)|\le\frac{3}{8}\|z-w\|,$$

and

$$\begin{aligned}|(Tz)(t)-(Tw)(t)|&\le\frac{|A^{-1}(t_0+t,\tilde{\eta}_t)|}{\Gamma(q)}\left|\int_0^t(t-s)^{q-1}f(t_0+s,\tilde{\eta}_s+\tilde{z}_s)ds\right.\\&\quad\left.-\int_0^t(t-s)^{q-1}f(t_0+s,\tilde{\eta}_s+\tilde{w}_s)ds\right|\\&\le\frac{|A^{-1}(t_0+t,\tilde{\eta}_t)|}{\Gamma(q)}\ |\ell(t)|\ \sup_{0\le s\le t}\|\tilde{z}_s-\tilde{w}_s\|_*\\&\le\frac{\sup\limits_{0\le s\le\alpha}|A^{-1}(t_0+s,\tilde{\eta}_s)|\ |\ell(s)|}{\Gamma(q)}\ \|z-w\|.\end{aligned}$$

Therefore

$$\begin{aligned}|(S+T)z(t)-(S+T)w(t)|&\le\left[\frac{3}{8}+\sup_{0\le s\le\alpha}\frac{|A^{-1}(t_0+s,\tilde{\eta}_s)|\ |\ell(s)|}{\Gamma(q)}\right]\|z-w\|\\&=c\|z-w\|.\end{aligned}$$

Hence, we have

$$\|(S+T)z-(S+T)w\|\le c\|z-w\|,$$

where $c<1$. By applying Theorem 1.41, we know that $S+T$ has a unique fixed point on $E(\alpha,\beta)$. The proof is complete. □

Corollary 2.25. Suppose the condition (H5) of Theorem 2.24 is replaced by the following condition:

(H5)′ let $A>0$, there exist $q_2\in(0,q)$ and a real-valued function $\ell_1\in L^{\frac{1}{q_2}}[t_0,t_0+A]$ such that for any $(t,x_t),\ (t,y_t)\in\Omega,\ t\in[t_0,t_0+A]$, we have

$$|f(t,x_t)-f(t,y_t)|\le\ell_1(t)\sup_{t_0\le s\le t}\|x_s-y_s\|_*.$$

Then the result of Theorem 2.24 holds.

Proof. It suffices to prove that the condition (H5) of Theorem 2.24 holds. Note that $\ell_1\in L^{\frac{1}{q_2}}[t_0,t_0+A]$, let $K=\|\ell_1\|_{L^{\frac{1}{q_2}}[t_0,\ t_0+A]}$. Then for any $(t,x_t),\ (t,y_t)\in\Omega$

we have

$$
\begin{aligned}
&\left|\int_{t_0}^{t}(t-s)^{q-1}[f(s,x_s)-f(s,y_s)]ds\right| \\
&\le \int_{t_0}^{t}(t-s)^{q-1}|f(s,x_s)-f(s,y_s)|ds \\
&\le \int_{t_0}^{t}(t-s)^{q-1}\ell_1(s)\ ds \sup_{t_0\le s\le t}\|x_s-y_s\|_* \\
&\le \frac{K}{(1+b_1)^{1-q_2}}\ (t-t_0)^{(1+b_1)(1-q_2)} \sup_{t_0\le s\le t}\|x_s-y_s\|_*,
\end{aligned}
$$

where $b_1=\frac{q-1}{1-q_2}\in(-1,0)$. Let

$$
\ell(t-t_0)=\frac{K}{(1+b_1)^{1-q_2}}\ (t-t_0)^{(1+b_1)(1-q_2)}.
$$

Obviously, $\ell:[0,A]\to[0,\infty)$ continuous at $t=0$ and $\ell(0)=0$. Then the condition (H5) of Theorem 2.24 holds. □

The next result is concerned with the uniqueness of solutions.

Theorem 2.26. Suppose that g is defined as in Theorem 2.21 and the condition $(\mathrm{H5})'$ of Corollary 2.25 holds. If x is a solution of IVP (2.14)-(2.15), then x is unique.

Proof. Suppose (for contradiction) x and y are the solutions of IVP (2.14)-(2.15) on $[p(t_0,-1),t_0+A]$ with $x\ne y$, let

$$
t_1=\inf\{t\in[t_0,t_0+A]:x(t)\ne y(t)\}.
$$

Then $t_0\le t_1<t_0+A$ and

$$
x(t)=y(t)\ \ \text{for } p(t_0,-1)\le t<t_1,
$$

which implies that

$$
x_t(\theta)=x(p(t,\theta))=y(p(t,\theta))=y_t(\theta),\ t_0\le t<t_1,\ \ -1\le\theta\le 0. \tag{2.35}
$$

Choose $\alpha>0$ such that $t_1+\alpha<t_0+A$. According to (i) of Definition 2.16, we have

$$
\{(t,x_t),\ t_1\le t\le t_1+\alpha\}\cup\{(t,y_t),\ t_1\le t\le t_1+\alpha\}\subset\Omega.
$$

On the one hand, x and y satisfy (2.14)-(2.15) on $[t_0,t_0+A]$, thus from (2.35) and the condition $(\mathrm{H5})'$, for $t\in[t_0,t_1+\alpha]$, we have

$$
\begin{aligned}
|g(t,x_t)-g(t,y_t)| &\le \frac{1}{\Gamma(q)}\left|\int_{t_0}^{t}(t-s)^{q-1}[f(s,x_s)-f(s,y_s)]ds\right| \\
&= \frac{1}{\Gamma(q)}\left|\int_{t_1}^{t}(t-s)^{q-1}[f(s,x_s)-f(s,y_s)]ds\right| \\
&\le \frac{1}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}\ell_1(s)\ ds \sup_{t_0\le s\le t}\|x_s-y_s\|_* \\
&\le \frac{K}{\Gamma(q)(1+b_1)^{1-q_2}}\ \alpha^{(1+b_1)(1-q_2)} \sup_{t_1\le s\le t_1+\alpha}\|x_s-y_s\|_*,
\end{aligned} \tag{2.36}
$$

where $b_1 = \frac{q-1}{1-q_2} \in (-1,0), K = \|\ell_1\|_{L^{\frac{1}{q_2}}[t_0,\ t_0+A]}$.

On the other hand, since $g(t,\varphi)$ is continuously differentiable in φ, we have

$$g(t,x_t) - g(t,y_t) = g'_\varphi(t,y_t)(x_t - y_t) + k\|x_t - y_t\|_* \tag{2.37}$$

with $k \to 0$ as $\|x_t - y_t\|_* \to 0$.

By the hypothesis that $g(t,\varphi)$ is atomic at 0 on Ω, there exist a nonsingular continuous matrix function $A(t,y_t)$ and a function $L(t,y_t,\psi)$ which is linear in ψ such that

$$g'_\varphi(t,y_t)\psi = A(t,y_t)\psi(0) + L(t,y_t,\psi). \tag{2.38}$$

Moreover, there is a positive real-valued continuous function $\gamma(t,y_t,-s)$ such that for every $s \in [-1,0]$,

$$|L(t,y_t,\psi)| \le \gamma(t,y_t,-s)\|\psi\|_* \tag{2.39}$$

if $\psi(\theta) = 0$ for $-1 \le \theta \le s$.

Hence for every $t \in [t_1, t_1+\alpha]$, by (ii) of Lemma 2.17, there is $s(t_1, t-t_1) \in [-1,0]$ with $s(t_1, t-t_1) \to 0$ as $t \to t_1$ such that

$$|L(t,y_t,x_t - y_t)| \le \gamma(t,y_t,-s(t_1,t-t_1))\|x_t - y_t\|_*.$$

From (2.37)-(2.39), it follows that

$$g(t,x_t) - g(t,y_t) = A(t,y_t)(x(t) - y(t)) + L(t,y_t,x_t - y_t) + k\|x_t - y_t\|_*,$$

therefore

$$\begin{aligned}|x(t) - y(t)| \le |A^{-1}(t,y_t)|\{&|g(t,x_t) - g(t,y_t)| \\ &+ \gamma(t,y_t,-s(t_1,t-t_1))\|x_t - y_t\|_* + k\|x_t - y_t\|_*\}.\end{aligned}$$

Let $M_1 = \max\{|A^{-1}(t,y_t)| : t_1 \le t \le t_1 + \alpha\}$. Then by relation (2.36), for $t \in [t_1, t_1 + \alpha]$, we have

$$|x(t) - y(t)| \le c_1 \sup_{t_1 \le s \le t_1+\alpha} \|x_s - y_s\|_*,$$

where $c_1 = M_1 \left[\dfrac{K}{\Gamma(q)(1+b_1)^{1-q_2}}\ \alpha^{(1+b_1)(1-q_2)} + \gamma(t,y_t,-s(t_1,t-t_1)) + k \right]$.

Noting that

$$\begin{aligned}\sup_{t_1 \le s \le t_1+\alpha} \|x_s - y_s\|_* &= \sup_{t_1 \le s \le t_1+\alpha}\ \sup_{-1 \le \theta \le 0} |x(p(s,\theta)) - y(p(s,\theta))| \\ &= \sup_{t_1 \le s \le t_1+\alpha}\ \sup_{p(s,-1) \le \rho \le s} |x(\rho) - y(\rho)| \\ &= \sup_{p(t_1,-1) \le s \le t_1+\alpha} |x(s) - y(s)|,\end{aligned}$$

we have

$$\sup_{p(t_1,-1) \le s \le t_1+\alpha} |x(s) - y(s)| \le c_1 \sup_{p(t_1,-1) \le s \le t_1+\alpha} |x(s) - y(s)|.$$

Choose α so small that $c_1 < 1$. Thus

$$\sup_{p(t_1,-1) \le s \le t_1+\alpha} |x(s) - y(s)| = 0, \quad \text{i.e. } x(t) \equiv y(t), \quad \text{for } t_1 \le t \le t_1 + \alpha,$$

contradicting the definition of t_1. $\square$

2.3.3 *Continuous Dependence*

The following lemma is introduced in Lakshmikantham, Wen and Zhang, 1994. However, for the sake of completeness, we outline its proof here.

Lemma 2.27. Assume $x \in C([p(0,-1), A], \mathbb{R}^n)$. Then for every $t \in [0, A]$

$$\|x_t\| \le \sup_{0 \le s \le t} |x(s)| + \|x_0\|.$$

Proof. By definition, $\|x_0\| = \sup_{-1 \le \theta \le 0} |x(p(0,\theta))|$. If $p(t,-1) \ge 0$, then

$$0 \le p(t,\theta) \le t \quad \text{for} \quad -1 \le \theta \le 0.$$

Thus,

$$\sup_{-1 \le \theta \le 0} |x(p(t,\theta))| \le \sup_{0 \le s \le t} |x(s)| \le \sup_{0 \le s \le t} |x(s)| + \|x_0\|.$$

If $p(t,-1) < 0$, then by Lemma 2.17, there exists an $s \in [-1, 0]$ such that

$$p(t,-1) \le p(t,\theta) \le p(0,\theta) \quad \text{for} \quad -1 \le \theta \le s,$$

while

$$0 \le p(t,\theta) \le t \quad \text{for} \quad s \le \theta \le 0.$$

Hence

$$\begin{aligned}
\sup_{-1 \le \theta \le 0} |x(p(t,\theta))| &\le \sup_{-1 \le \theta \le s} |x(p(t,\theta))| + \sup_{s \le \theta \le 0} |x(p(t,\theta))| \\
&\le \sup_{-1 \le \theta \le 0} |x(p(0,\theta))| + \sup_{s \le \theta \le 0} |x(p(t,\theta))| \\
&= \|x_0\| + \sup_{0 \le s \le t} |x(s)|,
\end{aligned}$$

completing the proof. □

We can now prove the following result on continuous dependence.

Theorem 2.28. Let $(t_0, \varphi) \in \Omega$ be given. Suppose that the solution $x = x(t_0, \varphi)$ of (2.14) through (t_0, φ) defined on $[t_0, A]$ is unique. Then for every $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $(s, \psi) \in \Omega$, $|s - t_0| < \delta$ and $\|\psi - \varphi\| < \varphi$ imply

$$\|x_t(s,\psi) - x_t(t_0,\varphi)\| < \epsilon \quad \text{for all} \quad t \in [\sigma, A],$$

where $x(s,\psi)$ is the solution of (2.14) through (s, ψ) and $\sigma = \max\{s, t_0\}$.

Proof. In order to prove the theorem, it is enough to show that if $\{(t_k, \varphi^k)\} \subset \Omega$, with $t_k \to t_0$ and $\varphi^k \to \varphi$ as $k \to \infty$, then there is a natural number N such that each solution $x^k = x(t^k, \varphi^k)$ with $k \ge N$ of (2.14) through (t^k, φ^k) exists on $[p(t_k, -1), A]$ and $x^k(t) \to x(t)$ uniformly on $[p(\sigma, -1), A]$, where $\sigma = \sup\{t_0, t^k : k \ge N\}$.

Since $x_t(t_0, \varphi)$ is continuous in $t \in [t_0, A]$, the set $W = \{(t, x_t(t_0, \varphi)) : t \in [t_0, A]\}$ is compact in Ω. By Theorem 2.21, there exist a neighborhood V of W and number $\alpha > 0$ such that for any $(s, \psi) \in V$, there is a solution $x(s, \psi)$ of (2.14) through (s, ψ) which exists at least on $[s, s + \alpha]$. Without loss of generality, we let $V = V(W, r)$, choose N so large that $|t_k - t_0| < \frac{r}{2}$ and $\|\varphi^k - \varphi\| < \frac{r}{2}$, so that $(t_k, \varphi^k) \in V$ for $k \geq N$. Thus $x^k = x(t_k, \varphi^k)$ exists at least on $[t_k, t_k + \alpha]$. For convenience, we shall denote $\varphi = \varphi^0$, $x = x^0$ and $x^k = x(t_k, \varphi^k)$, $k = 0, 1, \ldots$.

Let $p_k(t, \theta) = p(t_k + t, \theta) - t_k$. Define η^k, y^k the same way as in Lemma 2.19. Recalling the proof of Lemma 2.19, we see that y^k satisfy:

$$g(t_k+t, \eta_t^k+y_t^k)-g(t_k, \varphi^k) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(t_k+s, \eta_s^k+y_s^k) ds, \quad t \in [0, \alpha] \tag{2.40}$$

if and only if x^k is the solution of (2.14) on $[p(t_k, -1), t_k+\alpha]$, where $\eta_t^k = \eta^k(p_k(t, \theta))$, $y_t^k = y^k(p_k(t, \theta))$.

Set $\bar{y}^k = y^k|_{[0,\alpha]}$, the restriction of y^k to $[0, \alpha]$. Let $\Lambda = \{\bar{y}^k : k = 0, 1, 2, \ldots\}$. For every $z_k = (t_k, \varphi^k)$, define operators $S(z_k) : \Lambda \to C([0, \alpha], \mathbb{R}^n)$ and $T(z_k) : \Lambda \to C([0, \alpha], \mathbb{R}^n)$ as follows:

$$\begin{aligned} S(z_k)z(t) = {} & A^{-1}(t_k + t, \eta_t^k)[g(t_k, \varphi^k) - g(t_k + t, \eta_t^k + z_t) \\ & + g'_\varphi(t_k + t, \eta_t^k) z_t - L(t_k + t, \eta_t^k + z_t)], \quad 0 \leq t \leq \alpha, \end{aligned}$$

and

$$T(z_k)z(t) = A^{-1}(t_k + t, \eta_t^k) \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(t_k + s, \eta_s^k + z_s), \quad 0 \leq t \leq \alpha,$$

where $z_t(\theta) = z(\bar{p}_k(t, \theta))$ with $z_0 = 0$.

It is easy to see that $\{T(z_k)\bar{y}^k\}$ is compact in $C([0, \alpha], \mathbb{R}^n)$. Recalling the Theorem 2.21, we see that there exists a constant $\gamma \in [0, 1)$ which is independent to z_k such that

$$\|Sz - Sy\| \leq \gamma \|z - y\| \quad \text{for any} \quad z, y \in \Lambda. \tag{2.41}$$

Let $\{z_k : k = 0, 1, 2, \ldots\} = \bar{\Lambda}$. Denote the Kuratowskii measure of $A \subset C([0, \alpha], \mathbb{R}^n)$ by $\alpha(A)$. Then (2.41) implies that

$$\alpha\Big(\bigcup_{z_k \in \bar{\Lambda}} S(z_k)(\Lambda)\Big) \leq \gamma \alpha(\Lambda).$$

Let $R = S + T$. Thus $\bar{y}^k = R(z_k)\bar{y}_k$. By the well-known properties of Kuratowskii measure α, we immediately obtain that

$$\begin{aligned} \alpha(\Lambda) &= \alpha(\{R(z_k)\bar{y}_k\}) \leq \alpha(\{S(z_k)\bar{y}^k\}) + \alpha(\{T(z_k)\bar{y}^k\}) \\ &= \alpha(\{S(z_k)\bar{y}^k\}) \leq \alpha\Big(\bigcup_{z_k \in \bar{\Lambda}} S(z_k)(\Lambda)\Big) \leq \gamma \alpha(\Lambda). \end{aligned}$$

This means that $\alpha(\Lambda) = 0$ which implies Λ is relatively compact in $C([0, \alpha], \mathbb{R}^n)$. Hence there exists a subsequence of Λ, say $\{\bar{y}^{k_i}\}$, which converges uniformly on $[0, \alpha]$. Assume that

$$\bar{y}^{k_i}(t) \to \bar{y}^*(t) \quad \text{uniformly on} \quad [0, \alpha].$$

Define a function $y^* : [p_0(0, -1), \alpha] \to \mathbb{R}^n$ by

$$\begin{cases} y^*(t) = \bar{y}^*(t), & \text{for } 0 \leq t \leq \alpha, \\ y_0^* = 0, \end{cases}$$

where p_0 is such a p-function that $p_0(t,\theta) = p(t+t_0,\theta) - t_0$. Let $\delta = \inf\{p_k(0,-1) : k = 0,1,2,...\}$ and $\hat{y}^k$ denote the extension of y^k to $[\delta,\alpha]$ which is defined by

$$\begin{cases} \hat{y}^{k_i}(t) = y^{k_i}(t) & \text{for } \ 0 \le t \le \alpha, \\ \hat{y}^{k_i}(t) = 0 & \text{for } \ \delta \le t \le 0. \end{cases}$$

Obviously, $\{\hat{y}^{k_i}(t)\}$ converges uniformly on $[\delta,\alpha]$ as $k_i \to \infty$. Consequently, $\{\hat{y}^{k_i}\}$ is a relatively compact set.

We claim that $y_t^{k_i} \to y_t^*$ uniformly in $t \in [0,\alpha]$. In fact,

$$\begin{aligned} |p_{k_i}(t,\theta) - p_0(t,\theta)| &= |[p(t_{k_i}+t,\theta) - t_{k_i}] - [p(t_0+t,\theta) - t_0]| \\ &\le |[p(t_{k_i}+t,\theta) - p(t_0+t,\theta)| + |t_0 - t_{k_i}|. \end{aligned}$$

Hence, for every $\mu > 0$ there exists a number L such that

$$|[p_{k_i}(t,\theta) - p_0(t,\theta)| < \mu \quad \text{whenever} \ \ k_i \ge L.$$

We have the inequality

$$\begin{aligned} \|y_t^{k_i} - y_t^*\| &= \sup_{-1\le\theta\le0} |\hat{y}^{k_i}(p_{k_i}(t,\theta)) - y^*(p_0(t,\theta))| \\ &= \sup_{-1\le\theta\le0} |\hat{y}^{k_i}(p_{k_i}(t,\theta)) - \hat{y}^{k_i}(p_0(t,\theta)) + \hat{y}^{k_i}(p_0(t,\theta)) - y^*(p_0(t,\theta))| \\ &\le \sup_{-1\le\theta\le0} |\hat{y}^{k_i}(p_{k_i}(t,\theta)) - \hat{y}^{k_i}(p_0(t,\theta))| \\ &\quad + \sup_{-1\le\theta\le0} |\hat{y}^{k_i}(p_0(t,\theta)) - y^*(p_0(t,\theta))|. \end{aligned}$$

By Lemma 2.27, we get

$$\begin{aligned} \sup_{-1\le\theta\le0} |\hat{y}^{k_i}(p_0(t,\theta)) - y^*(p_0(t,\theta))| &\le \sup_{0\le\theta\le\alpha} |y^{k_i}(t) - y^*(t)| + \|\hat{y}_0^{k_i} - y_0^*\| \\ &= \sup_{0\le\theta\le\alpha} |y^{k_i}(t) - y^*(t)|. \end{aligned}$$

For every $\epsilon > 0$ there exists a number L_1 such that

$$\sup_{0\le\theta\le\alpha} |y^{k_i}(t) - y^*(t)| < \frac{\epsilon}{2} \quad \text{for} \ \ k_i \ge L_1,$$

by the definition of y^*. On the other hand, since $\{\hat{y}^{k_i}\}$ is an equi-continuous set, for the given ϵ, there exists a $\mu > 0$ such that

$$|\hat{y}^{k_i}(t) - \hat{y}^{k_i}(\tau)| < \frac{\epsilon}{2} \quad \text{for} \ \ |t-\tau| < \mu. \tag{2.42}$$

We can choose $L \ge L_1$ so that $|p_{k_i}(t,\theta) - p_0(t,\theta)| < \mu$. Thus (2.42) holds as long as $k_i \ge L$. Furthermore,

$$\|y_t^{k_i} - y_t\| < \epsilon \quad \text{whenever} \ \ k_i \ge L,$$

which is just our claim. A similar argument shows that $\eta_t^{k_i} \to \eta_t$ uniformly in $t \in [0,\alpha]$. The limiting process upon (2.40) yields

$$\begin{cases} g(t_0+t,\eta_t+y_t^*) - g(t_0,\varphi) = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(t_0+s,\eta_s+y_s^*)ds, \ \ 0 \le t \le \alpha, \\ y_0^* = 0, \end{cases}$$

which demonstrates that y^* as well as y^0 is a solution of the initial problem

$$\begin{cases} g(t_0+t, \eta_t + y_t^*) - g(t_0, \varphi) = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(t_0+s, \eta_s + y_s) ds, \quad 0 \le t \le \alpha, \\ y_0 = 0. \end{cases}$$

The hypothesis that $x(t_0, \varphi)$ is unique, that is, y^0 is unique implies that $y^* = y^0$. Thus $y^{k_i}(t) \to y^0(t)$ uniformly on $[0, \alpha]$. The verified fact that every subsequence of sequence $\{y^k\}$ has a convergent subsequence with a same limit y^0 implies that the entire sequence $\{y^k\}$ converges to y^0. Translating these remarks back into x^k, we have indeed obtained the result stated in this theorem for the interval $[p(\sigma, -1), \sigma + \alpha]$.

Let $b = \sigma + \alpha$. If $b < A$, $(b, x_b) \in W$, we can choose $N_1 \ge N$ such that $(b, x_b^k) \in V$ as long as $k \ge N_1$. By Theorem 2.21, for every point (b, x_b^k) the solution $x^k(b, x_b^k)$ exists at least on $[p(b, -1), b + \alpha]$. The above argument can be adapted to this interval which yields the assertion that $x^k(b, x_b^k)(t) \to x^0(t_0, \varphi)(t)$ uniformly on the same interval. The conclusion stated in theorem can be verified by successive steps of finite intervals of length α. Hence the proof is completed. □

2.4 Neutral Equations with Infinite Delay

2.4.1 *Introduction*

In Section 2.4, we consider the initial value problem of fractional neutral functional differential equations with infinite delay of the form

$$ {}_{t_0}^{C}D_t^q g(t, x_t) = f(t, x_t), \qquad t \in [t_0, \infty), \tag{2.43}$$

$$ x_{t_0} = \varphi, \qquad (t_0, \varphi) \in [0, \infty) \times \Omega, \tag{2.44}$$

where ${}_{t_0}^{C}D_t^q$ is Caputo fractional derivative of order $0 < q < 1$, Ω is an open subset of B and $g, f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$ are given functionals satisfying some assumptions that will be specified later. B is called a phase space that will be defined later.

If $x : (-\infty, A) \to \mathbb{R}^n, A \in (0, \infty)$, then for any $t \in [0, A)$ define x_t by $x_t(\theta) = x(t + \theta)$, for $\theta \in (-\infty, 0]$.

Denote by $BC(J, \mathbb{R}^n)$ the Banach space of all continuous and bounded functions from J into $\mathbb{R}^n$ with the norm $\|\cdot\|$.

To describe fractional neutral functional differential equations with infinite delay, we need to discuss a phase space B in a convenient way. We shall provide a general description of phase spaces of neutral differential equations with infinite delay which is taken from Lakshmikantham, Wen and Zhang, 1994.

Let B be a real vector space either

(i) of continuous functions that map $(-\infty, 0]$ to $\mathbb{R}^n$ with $\varphi = \psi$ if $\varphi(s) = \psi(s)$ on $(-\infty, 0]$ or

(ii) of measurable functions that map $(-\infty, 0]$ to $\mathbb{R}^n$ with $\varphi = \psi$ (or φ is equivalent to ψ) in B if $\varphi(s) = \psi(s)$ almost everywhere on $(-\infty, 0]$, and $\varphi(0) = \psi(0)$.

Let B be endowed with a norm $\|\cdot\|_B$ such that B is complete with respect to $\|\cdot\|_B$. Thus B equipped with norm $\|\cdot\|_B$ is a Banach space. We denote this space by $(B, \|\cdot\|_B)$ or simply by B, whenever no confusion arises.

Let $0 \le a < A$. If $x : (-\infty, A) \to \mathbb{R}^n$ is given such that $x_a \in B$ and $x \in [a, A) \to \mathbb{R}^n$ is continuous, then $x_t \in B$ for all $t \in [a, A)$.

This is a very weak condition that the common admissible phase spaces and BC satisfy. For more details of the phase spaces, we refer the reader to Hino, Murakami and Naito, 1991; Lakshmikantham, Wen and Zhang, 1994.

Definition 2.29. A function $x : (-\infty, t_0 + \sigma) \to \mathbb{R}^n (t_0 \in [0, \infty), \sigma > 0)$ is said to be a solution of IVP (2.43)-(2.44) through (t_0, φ) on $[t_0, t_0 + \sigma)$, if

(i) $x_{t_0} = \varphi$;
(ii) x is continuous on $[t_0, t_0 + \sigma)$;
(iii) $g(t, x_t)$ is absolutely continuous on $[t_0, t_0 + \sigma)$;
(iv) (2.43) holds almost everywhere on $[t_0, t_0 + \sigma)$.

Let $\Omega \subseteq B$ be an open set such that for any $(t_0, \varphi) \in [0, \infty) \times \Omega$, there exist constants $\sigma_1, \gamma_1 > 0$ so that $x_t \in \Omega$ provided that $x \in A(t_0, \varphi, \sigma_1, \gamma_1)$ and $t \in [t_0, t_0 + \sigma_1]$, where $A(t_0, \varphi, \sigma_1, \gamma_1)$ is defined as

$$A(t_0, \varphi, \sigma_1, \gamma_1) = \left\{ x : (-\infty, t_0 + \sigma_1] \to \mathbb{R}^n,\ x_{t_0} = \varphi, \sup_{t_0 \le t \le t_0 + \sigma_1} |x(t) - \varphi(0)| \le \gamma_1 \right\}.$$

In order to guarantee that equation (2.43) is NFDE, the coefficient of $x(t)$ that is contained in $g(t, x_t)$ cannot be equal to zero. Then we need to introduce the generalized atomic concept.

Definition 2.30. (Lakshmikantham, Wen and Zhang, 1994) The functional $g : [0, \infty) \times \Omega \to \mathbb{R}^n$ is said to be generalized atomic on Ω, if

$$g(t, \varphi) - g(t, \psi) = K(t, \varphi, \psi)[\varphi(0) - \psi(0)] + L(t, \varphi, \psi)$$

where $(t, \varphi, \psi) \in [0, \infty) \times \Omega \times \Omega$, $K : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}^{n \times n}$ and $L : [0, \infty) \times \Omega \times \Omega \to \mathbb{R}^n$ satisfy

(i) $\det K(t, \varphi, \varphi) \neq 0$ for all $(t, \varphi) \in [0, \infty) \times \Omega$;
(ii) for any $(t_0, \varphi) \in [0, \infty) \times \Omega$, there exist constants $\delta_1, \gamma_1 > 0$, and $k_1, k_2 > 0$, with $2k_2 + k_1 < 1$ such that for all $x, y \in A(t_0, \varphi, \sigma_1, \gamma_1)$, $g(t, x_t), K(t, x_t, y_t)$ and $L(t, x_t, y_t)$ are continuous in $t \in [t_0, t_0 + \sigma_1]$, and

$$|K^{-1}(t_0, \varphi, \varphi) L(t, x_t, y_t)| \le k_1 \sup_{t_0 \le s \le t} |x(s) - y(s)|,$$
$$|K^{-1}(t_0, \varphi, \varphi) K(t, x_t, y_t) - I| \le k_2,$$

where I is the $n \times n$ unit matrix.

For a detailed discussion on the atomic concept we refer the reader to the books Hale, 1977; Lakshmikantham, Wen and Zhang, 1994.

In Subsection 2.4.2, we shall discuss existence and uniqueness of solutions for IVP (2.43)-(2.44) on a class of comparatively comprehensive phase spaces. We establish various criteria on existence and uniqueness of solutions for IVP (2.43)-(2.44).

2.4.2 *Existence and Uniqueness*

The following existence result for IVP (2.43)-(2.44) is based on Krasnoselskii's fixed point theorem.

Theorem 2.31. Assume that g is generalized atomic on Ω, and that for any $(t_0, \varphi) \in [0, \infty) \times \Omega$, there exist constants $\sigma_1, \gamma_1 \in (0, \infty), q_1 \in (0, q)$ and a real-valued function $m(t) \in L^{\frac{1}{q_1}}[t_0, t_0 + \sigma_1]$ such that
(H1) for any $x \in A(t_0, \varphi, \sigma_1, \gamma_1)$, $f(t, x_t)$ is measurable;
(H2) for any $x \in A(t_0, \varphi, \sigma_1, \gamma_1)$, $|f(t, x_t)| \le m(t)$, for $t \in [t_0, t_0 + \sigma_1]$;
(H3) $f(t, \phi)$ is continuous with respect to ϕ on Ω.
Then IVP (2.43)-(2.44) has a solution.

Proof. We know that $f(t, x_t)$ is Lebesgue measurable in $[t_0, t_0 + \sigma_1]$ according to conditions (H1). Direct calculation gives that $(t - s)^{q-1} \in L^{\frac{1}{1-q_1}}[t_0, t]$, for $t \in [t_0, t_0 + \sigma_1]$. In light of the Hölder inequality and the condition (H2), we obtain that $(t - s)^{q-1} f(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$ for all $t \in [t_0, t_0 + \sigma_1]$, and

$$\int_{t_0}^{t} |(t-s)^{q-1} f(s, x_s)| ds \le \|(t-s)^{q-1}\|_{L^{\frac{1}{1-q_1}}[t_0,t]} \|m\|_{L^{\frac{1}{q_1}}[t_0,t_0+\sigma_1]}. \tag{2.45}$$

According to Definition 2.29, IVP (2.43)-(2.44) is equivalent to the following equation

$$g(t, x_t) = g(t_0, \varphi) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x_s) ds \quad \text{for } t \in [t_0, t_0 + \sigma_1]. \tag{2.46}$$

Let $\hat{\varphi} \in A(t_0, \varphi, \sigma_1, \gamma_1)$ be defined as $\hat{\varphi}_{t_0} = \varphi$, $\hat{\varphi}(t_0 + t) = \varphi(0)$ for all $t \in [0, \sigma_1]$. If x is a solution of IVP (2.43)-(2.44), let $x(t_0 + t) = \hat{\varphi}(t_0 + t) + z(t)$, $t \in (-\infty, \sigma_1]$, then we have $x_{t_0+t} = \hat{\varphi}_{t_0+t} + z_t$, $t \in [0, \sigma_1]$. Thus (2.46) implies that z satisfies the equation

$$g(t_0 + t, \hat{\varphi}_{t_0+t} + z_t) = g(t_0, \varphi) + \frac{1}{\Gamma(q)} \int_{0}^{t} (t-s)^{q-1} f(t_0 + s, \hat{\varphi}_{t_0+s} + z_s) ds \tag{2.47}$$

for $0 \le t \le \sigma_1$.

Since g is generalized atomic on Ω, there exist positive constant $\alpha > 1$ and a positive function $\sigma_2(\gamma)$ defined in $(0, \gamma_1]$, such that for any $\gamma \in (0, \gamma_1]$, when $0 \le t \le \sigma_2(\gamma)$, we have

$$\alpha(2k_2 + k_1) < 1, \tag{2.48}$$

$$|K^{-1}(t_0, \varphi, \varphi)K(t_0 + t, x_{t_0+t}, y_{t_0+t}) - I| \leq k_2, \tag{2.49}$$

$$|I - K^{-1}(t_0 + t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})K(t_0, \varphi, \varphi)| \leq \min\{\alpha k_2, \alpha - 1\}, \tag{2.50}$$

$$|K^{-1}(t_0 + t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})||g(t_0 + t, \hat{\varphi}_{t_0+t}) - g(t_0, \varphi)| \leq \frac{1 - \alpha(2k_2 + k_1)}{2}\gamma. \tag{2.51}$$

Note the completely continuity of the function $(m(t))^{\frac{1}{q_1}}$. Hence, for a given positive number M, there must exist a number $h > 0$, satisfying

$$\int_{t_0}^{t_0+h} (m(s))^{\frac{1}{q_1}} ds \leq M.$$

For a given $\gamma \in (0, \gamma_1]$, choose

$$\sigma = \min\left\{\sigma_1, \sigma_2(\gamma), h, (1+\beta)^{\frac{1}{1+\beta}} \left[\frac{[1 - \alpha(2k_2 + k_1)]\Gamma(q)\gamma}{2\alpha|K^{-1}(t_0, \varphi, \varphi)|M^{q_1}}\right]^{\frac{1}{(1-q_1)(1+\beta)}}\right\}, \tag{2.52}$$

where $\beta = \frac{q-1}{1-q_1} \in (-1, 0)$.

For any $(t_0, \varphi) \in [0, \infty) \times \Omega$, define $E(\sigma, \gamma)$ as follows:

$E(\sigma, \gamma) = \{z : (-\infty, \sigma) \to \mathbb{R}^n$ is continuous; $z(s) = 0$ for $s \in (-\infty, 0]$ and $\|z\| \leq \gamma\}$

where $\|z\| = \sup_{0 \leq s \leq \sigma} |z(t)|$. Then $E(\sigma, \gamma)$ is a closed bounded and convex subset of Banach space $BC((-\infty, \sigma_1], \mathbb{R}^n)$.

Now, on $E(\sigma, \gamma)$ define two operators S and U as follows:

$$(Sz)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ K^{-1}(t_0 + t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})[-g(t_0 + t, \hat{\varphi}_{t_0+t} + z_t) + g(t_0, \varphi) \\ +K(t_0 + t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})z(t)], & t \in [0, \sigma], \end{cases}$$

and

$$(Uz)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ K^{-1}(t_0 + t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) \\ \times \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(t_0 + s, \hat{\varphi}_{t_0+s} + z_s) ds, & t \in [0, \sigma], \end{cases}$$

where $z \in E(\sigma, \gamma)$.

It is easy to see that the operator equation

$$z = Sz + Uz \tag{2.53}$$

has a solution $z \in E(\sigma, \gamma)$ if and only if z is a solution of the equation (2.47). Thus, $x_{t+t_0} = \hat{\varphi}_{t_0+t} + z_t$ is a solution of the equation (2.43) on $[0, \sigma]$. Therefore, the existence of a solution of IVP (2.43)-(2.44) is equivalent to determining $\sigma, \gamma > 0$ such that (2.53) has a fixed point in $E(\sigma, \gamma)$.

Now we show that $S + U$ has a fixed point in $E(\sigma, \gamma)$. The proof is divided into three steps.

Step I. $Sz + Uw \in E(\sigma, \gamma)$ for every pair $z, w \in E(\sigma, \gamma)$.

Obviously, for every pair $z, w \in E(\sigma, \gamma)$, $(Sz)(t)$ and $(Uw)(t)$ are continuous in $t \in [0, \sigma]$, and for $t \in [0, \sigma]$, by using the Hölder inequality and (2.50), we have

$$
\begin{aligned}
|(Uw)(t)| \le\, & |K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})\ K(t_0, \varphi, \varphi)\ K^{-1}(t_0, \varphi, \varphi)| \\
& \times \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} f(t_0+s, \hat{\varphi}_{t_0+s} + w_s) ds \right| \\
& \le \alpha |K^{-1}(t_0, \varphi, \varphi)|\ \frac{1}{\Gamma(q)}\ \left[\int_0^t [(t-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right]^{1-q_1} \\
& \times \left[\int_{t_0}^{t_0+\sigma} (m(s))^{\frac{1}{q_1}} ds \right]^{q_1} \\
& \le \alpha |K^{-1}(t_0, \varphi, \varphi)|\ \ \frac{M^{q_1}}{\Gamma(q)}\ \ \left[\frac{1}{1+\beta} \sigma^{1+\beta} \right]^{1-q_1} \\
& \le \frac{1-\alpha(2k_2+k_1)}{2} \gamma,
\end{aligned}
\tag{2.54}
$$

where $\beta = \dfrac{q-1}{1-q_1} \in (-1, 0)$, and

$$
\begin{aligned}
|(Sz)(t)| =\, & \big| K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})[-g(t_0+t, \hat{\varphi}_{t_0+t} + z_t) + g(t_0+t, \hat{\varphi}_{t_0+t}) \\
& -g(t_0+t, \hat{\varphi}_{t_0+t}) + g(t_0, \varphi) + K(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) z(t)] \big| \\
=\, & \big| K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})[-K(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t}) z(t) \\
& -L(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t}) - g(t_0+t, \hat{\varphi}_{t_0+t}) + g(t_0, \varphi) \\
& +K(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) z(t)] \big| \\
=\, & \big| K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})[K(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) \\
& -K(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t})] z(t) + K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) \\
& \times[-L(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t}) - g(t_0+t, \hat{\varphi}_{t_0+t}) + g(t_0, \varphi)] \big| \\
=\, & \big| K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) K(t_0, \varphi, \varphi) \\
& \times[K^{-1}(t_0, \varphi, \varphi) K(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) - I] z(t) \\
& -K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) K(t_0, \varphi, \varphi) \\
& \times[K^{-1}(t_0, \varphi, \varphi) K(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t}) - I] z(t) \\
& +K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})[-L(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t}) \\
& -g(t_0+t, \hat{\varphi}_{t_0+t}) + g(t_0, \varphi)] \big| \\
\le\, & |K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) K(t_0, \varphi, \varphi)| \\
& \times[(|K^{-1}(t_0, \varphi, \varphi) K(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t}) - I| \\
& +|K^{-1}(t_0, \varphi, \varphi) K(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t}) - I|) |z(t)| \\
& +|K^{-1}(t_0, \varphi, \varphi) L(t_0+t, \hat{\varphi}_{t_0+t} + z_t, \hat{\varphi}_{t_0+t})|\ ] \\
& +|K^{-1}(t_0+t, \hat{\varphi}_{t_0+t}, \hat{\varphi}_{t_0+t})| |g(t_0+t, \hat{\varphi}_{t_0+t}) - g(t_0, \varphi)|.
\end{aligned}
$$

According to (2.48)-(2.51), we have

$$
|(Sz)(t)| \le \alpha(2k_2+k_1)\gamma + \frac{1-\alpha(2k_2+k_1)}{2}\gamma = \frac{1+\alpha(2k_2+k_1)}{2}\gamma.
$$

Therefore, $|(Sz)(t)+(Uw)(t)| \leq \gamma$ for $t \in [0,\sigma]$. This means that $Sz+Uw \in E(\sigma,\gamma)$ whenever $z,w \in E(\sigma,\gamma)$.

Step II. S is a contraction mapping on $E(\sigma,\gamma)$.

For any $z,w \in E(\sigma,\gamma)$, we obtain

$$\begin{aligned}
&|(Sz)(t)-(Sw)(t)| \\
\leq\ & |K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})||K(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t}) \\
&-K(t_0+t,\hat{\varphi}_{t_0+t}+z_t,\hat{\varphi}_{t_0+t}+w_t)||z(t)-w(t)| \\
&+|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})L(t_0+t,\hat{\varphi}_{t_0+t}+z_t,\hat{\varphi}_{t_0+t}+w_t)| \\
\leq\ & \big|[I-K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})K(t_0,\varphi,\varphi)] \\
&-[K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})K(t_0,\varphi,\varphi)] \\
&\times[K^{-1}(t_0,\varphi,\varphi)K(t_0+t,\hat{\varphi}_{t_0+t}+z_t,\hat{\varphi}_{t_0+t}+w_t)-I]\big|\ |z(t)-w(t)| \\
&+|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})K(t_0,\varphi,\varphi)K^{-1}(t_0,\varphi,\varphi) \\
&\times L(t_0+t,\hat{\varphi}_{t_0+t}+z_t,\hat{\varphi}_{t_0+t}+w_t)| \\
\leq\ & (\alpha k_2+\alpha k_2)|z(t)-w(t)|+\alpha k_1 \sup_{0\leq s\leq t}|z(s)-w(s)| \\
\leq\ & \alpha(2k_2+k_1)\sup_{0\leq s\leq t}|z(s)-w(s)|,
\end{aligned}$$

where $\alpha(2k_2+k_1)<1$, and therefore S is a contraction mapping on $E(\sigma,\gamma)$.

Step III. Now we show that U is a completely continuous operator.

For any $z \in E(\sigma,\gamma), 0 \leq \tau < t \leq \sigma$, we get

$$\begin{aligned}
&|(Uz)(t)-(Uz)(\tau)| \\
=\ & \Big|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds \\
&-K^{-1}(t_0+\tau,\hat{\varphi}_{t_0+\tau},\hat{\varphi}_{t_0+\tau})\frac{1}{\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\Big| \\
=\ & \Big|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})\frac{1}{\Gamma(q)}\int_\tau^t (t-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds \\
&+K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})\frac{1}{\Gamma(q)}\int_0^\tau (t-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds \\
&-K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})\frac{1}{\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds \\
&+K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})\frac{1}{\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds \\
&-K^{-1}(t_0+\tau,\hat{\varphi}_{t_0+\tau},\hat{\varphi}_{t_0+\tau})\frac{1}{\Gamma(q)}\int_0^\tau (\tau-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\Big| \\
\leq\ & \frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})|}{\Gamma(q)}\left|\int_\tau^t (t-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right| \\
&+\frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})|}{\Gamma(q)}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times\left|\int_0^\tau [(t-s)^{q-1}-(\tau-s)^{q-1}]f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right| \\
&\quad +\frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})-K^{-1}(t_0+\tau,\hat{\varphi}_{t_0+\tau},\hat{\varphi}_{t_0+\tau})|}{\Gamma(q)} \\
&\quad \times\left|\int_0^\tau (\tau-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right| \\
&= \frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})|}{\Gamma(q)}(I_1+I_2) \\
&\quad +\frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})-K^{-1}(t_0+\tau,\hat{\varphi}_{t_0+\tau},\hat{\varphi}_{t_0+\tau})|}{\Gamma(q)}\ I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \left|\int_\tau^t (t-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right|, \\
I_2 &= \left|\int_0^\tau [(t-s)^{q-1}-(\tau-s)^{q-1}]f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right|, \\
I_3 &= \left|\int_0^\tau (\tau-s)^{q-1}f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right|.
\end{aligned}
$$

By using an analogous argument presented in (2.54), we can conclude that

$$
\begin{aligned}
I_1 &\le \frac{M^{q_1}}{(1+\beta)^{1-q_1}}\left[(t-\tau)^{1+\beta}\right]^{1-q_1}, \\
I_3 &\le \frac{M^{q_1}}{(1+\beta)^{1-q_1}}\left(\tau^{1+\beta}\right)^{1-q_1},
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &\le \left[\int_0^\tau |(t-s)^{q-1}-(\tau-s)^{q-1}|^{\frac{1}{1-q_1}}ds\right]^{1-q_1}\left[\int_{t_0}^{t_0+\tau}|f(s,x_s)|^{\frac{1}{q_1}}ds\right]^{q_1} \\
&\le M^{q_1}\left[\int_0^\tau ((\tau-s)^\beta-(t-s)^\beta)ds\right]^{1-q_1} \\
&\le \frac{M^{q_1}}{(1+\beta)^{1-q_1}}\left[\tau^{1+\beta}-t^{1+\beta}+(t-\tau)^{1+\beta}\right]^{1-q_1} \\
&\le \frac{M^{q_1}}{(1+\beta)^{1-q_1}}\left[(t-\tau)^{1+\beta}\right]^{1-q_1},
\end{aligned}
$$

where $\beta=\frac{q-1}{1-q_1}\in(-1,0)$. Therefore

$$
\begin{aligned}
|(Uz)(t)-(Uz)(\tau)| &\le \frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})|}{\Gamma(q)}\ \frac{2M^{q_1}}{(1+\beta)^{1-q_1}}\left[(t-\tau)^{1+\beta}\right]^{1-q_1} \\
&+\frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})-K^{-1}(t_0+\tau,\hat{\varphi}_{t_0+\tau},\hat{\varphi}_{t_0+\tau})|}{\Gamma(q)}\ \frac{M^{q_1}}{(1+\beta)^{1-q_1}}\left(\tau^{1+\beta}\right)^{1-q_1}.
\end{aligned}
$$

Since the property of the matrix function $K(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})$ which is nonsingular and continuous in $t\in[0,\sigma]$ implies that its inverse matrix $K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})$

exists and is continuous in $t \in [0, \sigma]$, then $\{Uz : \ z \in E(\sigma, \gamma)\}$ is equicontinuous. On the other hand, U is continuous from condition (H3) and $\{Uz : \ z \in E(\sigma, \gamma)\}$ is uniformly bounded from (2.54), thus U is a completely continuous operator by the Ascoli-Arzela Theorem.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\sigma, \gamma)$, and hence IVP (2.43)-(2.44) has a solution $x(t) = \varphi(0) + z(t - t_0)$ for all $t \in [t_0, t_0 + \sigma]$. □

Remark 2.32. If we replace condition (H1) by
(H1)′ $f(t, \phi)$ is measurable with respect to t on $[t_0, t_0 + \sigma_1]$.
Then we can also conclude that the result of Theorem 2.31 holds. In fact, for any $x \in A(t_0, \varphi, \sigma_1, \gamma_1)$, suppose $x_{t_0+t} = \hat{\varphi}_{t_0+t} + z_t$, $t \in [0, \sigma_1]$, then, according to the definition of $\hat{\varphi}_{t_0+t}$ and z_t, we know that x_{t_0+t} is a measurable function. It follows that from (H1)′ and (H3), $f(t, x_t)$ is measurable in t, where $x \in A(t_0, \varphi, \sigma_1, \gamma_1)$ and satisfies $x_{t_0+t} = \hat{\varphi}_{t_0+t} + z_t$, $t \in [0, \sigma_1]$.

Remark 2.33. If we replace condition (H3) by a weaker condition
(H3)′ for any $x, y \in A(t_0, \varphi, \sigma, \gamma)$ with $\sup\limits_{t_0 \le s \le t_0+\sigma} |x(s) - y(s)| \to 0$,

$$\left| \int_{t_0}^{t} (t-s)^{q-1} [f(s, x_s) - f(s, y_s)] ds \right| \to 0, \quad t \in [t_0, t_0 + \sigma]$$

where σ satisfy (2.51), then we can also conclude that the result of Theorem 2.31 holds.

The following existence and uniqueness result for IVP (2.43)-(2.44) is based on the Banach's contraction principle.

Theorem 2.34. Assume that g is generalized atomic on Ω, and that for any $(t_0, \varphi) \in [0, \infty) \times \Omega$, there exist constants $\sigma_1, \gamma_1 \in (0, \infty), q_1 \in (0, q)$ and a real-valued function $m(t) \in L^{\frac{1}{q_1}}[t_0, t_0 + \sigma_1]$ such that conditions (H1)-(H2) of Theorem 2.31 hold. Further assume that:
(H4) there exists a nonnegative function $\ell : [0, \sigma_1] \to [0, \infty)$ continuous at $t = 0$ and $\ell(0) = 0$ such that for any $x, y \in A(t_0, \varphi, \sigma_1, \gamma_1)$ we have

$$\left| \int_{t_0}^{t} (t-s)^{q-1} [f(s, x_s) - f(s, y_s)] ds \right| \le \ell(t - t_0) \sup_{t_0 \le s \le t} |x(s) - y(s)|, \quad t \in [t_0, t_0 + \sigma_1],$$

then IVP (2.43)-(2.44) has a unique solution.

Proof. According to the argument of Theorem 2.31, it suffices to prove that $S + U$ has a unique fixed point on $E(\sigma, \gamma)$, where $\sigma, \gamma > 0$ are sufficiently small. Now, choose $\sigma \in (0, \sigma_1)$, $\gamma \in (0, \gamma_1]$, such that (2.52) holds and that

$$c = \alpha(2k_2 + k_1) + \sup_{0 \le s \le \sigma} \frac{|K^{-1}(t_0 + s, \hat{\varphi}_{t_0+s}, \hat{\varphi}_{t_0+s})||\ell(s)|}{\Gamma(q)} < 1. \tag{2.55}$$

Obviously, $S + U$ is a mapping from $E(\sigma, \gamma)$ into itself. Using the same argument as that of Theorem 2.31, for any $z, w \in E(\sigma, \gamma)$, we get

$$|(Sz)(t) - (Sw)(t)| \le \alpha(2k_2 + k_1) \sup_{0 \le s \le \sigma} |z(s) - w(s)|,$$

and

$$
\begin{aligned}
&|(Uz)(t)-(Uw)(t)| \\
&\le \frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})|}{\Gamma(q)}\left|\int_0^t (t-s)^{q-1} f(t_0+s,\hat{\varphi}_{t_0+s}+z_s)ds\right. \\
&\quad \left. -\int_0^t (t-s)^{q-1} f(t_0+s,\hat{\varphi}_{t_0+s}+w_s)ds\right| \\
&\le \frac{|K^{-1}(t_0+t,\hat{\varphi}_{t_0+t},\hat{\varphi}_{t_0+t})|}{\Gamma(q)}\,|\ell(t)|\sup_{0\le s\le t}|z(s)-w(s)| \\
&\le \frac{\sup\limits_{0\le s\le\sigma}|K^{-1}(t_0+s,\hat{\varphi}_{t_0+s},\hat{\varphi}_{t_0+s})||\ell(s)|}{\Gamma(q)}\sup_{0\le s\le\sigma}|z(s)-w(s)|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|(S+U)z(t)-(S+U)w(t)| \\
&\le \left[\alpha(2k_2+k_1)+\sup_{0\le s\le\sigma}\frac{|K^{-1}(t_0+s,\hat{\varphi}_{t_0+s},\hat{\varphi}_{t_0+s})|\;|\ell(s)|}{\Gamma(q)}\right]\sup_{0\le s\le\sigma}|z(s)-w(s)| \\
&= c\sup_{0\le s\le\sigma}|z(s)-w(s)|.
\end{aligned}
$$

Hence, we have

$$\|(S+U)z-(S+U)w\|\le c\|z-w\|,$$

where $c<1$. By applying the Banach contraction principle, we know that $S+U$ has a unique fixed point on $E(\sigma,\gamma)$. □

Corollary 2.35. If the condition (H4) of Theorem 2.34 is replaced by the following condition:
(H4)$'$ there exist $q_2\in(0,q)$ and a function $\ell_1\in L^{\frac{1}{q_2}}[t_0,t_0+\sigma_1]$, such that for any $x,y\in A(t_0,\varphi,\sigma_1,\gamma_1)$ we have

$$|f(t,x_t)-f(t,y_t)|\le \ell_1(t)\sup_{t_0\le s\le t}|x(s)-y(s)|,\quad t\in[t_0,t_0+\sigma_1].$$

Then the result of Theorem 2.34 holds.

Proof. It suffices to prove that the condition (H4) of Theorem 2.34 holds. Note that $\ell_1\in L^{\frac{1}{q_2}}[t_0,t_0+\sigma_1]$, hence, there must exist a positive number N, such that $N=\|\ell_1\|_{L^{\frac{1}{q_2}}[t_0,\ t_0+\sigma_1]}$. Then for any $x,y\in A(t_0,\varphi,\sigma_1,\gamma_1)$ we have

$$
\begin{aligned}
&\left|\int_{t_0}^t (t-s)^{q-1}[f(s,x_s)-f(s,y_s)]ds\right| \\
&\le \int_{t_0}^t (t-s)^{q-1}|f(s,x_s)-f(s,y_s)|ds \\
&\le \int_{t_0}^t (t-s)^{q-1}\ell_1(s)\,ds\sup_{t_0\le s\le t}|x(s)-y(s)| \\
&\le \frac{N}{(1+\beta')^{1-q_2}}\;(t-t_0)^{(1+\beta')(1-q_2)}\sup_{t_0\le s\le t}|x(s)-y(s)|,
\end{aligned}
$$

where $\beta' = \frac{q-1}{1-q_2} \in (-1, 0)$. Let

$$\ell(t-t_0) = \frac{N}{(1+\beta')^{1-q_2}} \, (t-t_0)^{(1+\beta')(1-q_2)}, \; t \in [t_0, t_0+\sigma_1].$$

Obviously, $\ell : [0, \sigma_1] \to [0, \infty)$ continuous at $t = 0$ and $\ell(0) = 0$. Then the condition (H4) of Theorem 2.34 holds. □

2.4.3 *Continuation of Solutions*

For any $t_0, \varphi \in [0, \infty) \times \Omega$, $\omega \subset \Omega$ and positive constants $\sigma, \gamma > 0$, define $B_\omega(t_0, \varphi, \sigma, \gamma)$ as the set of all maps $x : (-\infty, t_0 + \sigma) \to \mathbb{R}^n$ such that $x_{t_0} = \varphi$, $x : [t_0, t_0 + \sigma) \to \mathbb{R}^n$ is continuous with $|x(t) - \varphi(0)| \leq \gamma$ and $x_t \in \omega$ for all $t \in [t_0, t_0 + \sigma)$. In the following theorem, W is a set of all subsets of Ω such that for any $(t_0, \varphi) \in [0, \infty) \times \Omega$, constants $\sigma, \gamma > 0$ and a set $\omega \in W$, if $x \in B_\omega(t_0, \varphi, \sigma, \gamma)$ and if $x(t_0 + \sigma) = \lim\limits_{t \to (t_0+\sigma)^-} x(t)$ exists , then $x_{t_0+\sigma} \in \Omega$.

Theorem 2.36. Let all conditions of Theorem 2.31 hold. Besides, suppose that $\sigma \in (0, \sigma_1], \gamma \in (0, \gamma_1]$ and for any $x \in B_\omega(t_0, \varphi, \sigma, \gamma)$,
(H5) there exist constants $q_\omega \in (0, q)$ and a real-valued function $m_\omega(t) \in L^{\frac{1}{q_\omega}}[t_0, t_0 + \sigma]$ such that $f(t, x_t)$ is measurable and $|f(t, x_t)| \leq m_\omega(t)$ for $t \in [t_0, t_0 + \sigma)$;
(H6) $\lim\limits_{\tau \to 0^+} [g(t, x_{t-\tau}) - g(t - \tau, x_{t-\tau})] = 0$ uniformly for $t \in [t_0 + \tau, t_0 + \sigma)$;
(H7) $K(t, x_t, x_t) - K(t, x_t, x_{t-\tau}) \to 0$ uniformly for $t \in [t_0 + \tau, t_0 + \sigma)$ as $\tau \to 0^+$ and as $\sup\limits_{t_0+\tau \leq s \leq t} |x(s) - x(s - \tau)| \to 0$;
(H8) there exists a constant H such that $|K^{-1}(t, x_t, x_t)| \leq H$ for all $t \in [t_0, t_0 + \sigma)$;
(H9) there exists a continuous function $\ell_\omega : [0, \infty) \to [0, \infty)$ with $\ell_\omega(0) = 0$ such that

$$|L(t, x_t, x_{t-\tau}) - L_b^*(t, x_t, x_{t-\tau})| \leq \ell_\omega(b) \sup_{-b \leq \theta \leq 0} |x(t + \theta) - x(t - \tau + \theta)|$$

where for a given $b > 0$, $\lim\limits_{\tau \to 0^+} L_b^*(t, x_t, x_{t-\tau}) = 0$ uniformly for $t \in [t_0 + \tau, t_0 + \sigma)$.
Then for any $\omega \in W$ and any $\gamma > 0$, if $x(t)$ is a noncontinuable solution of IVP (2.43)-(2.44) defined on $[t_0, t_0 + \sigma)$, there exists a $t^* \in [t_0, t_0 + \sigma)$ such that $|x(t^*) - \varphi(0)| > \gamma$ or $x_{t^*} \notin \omega$.

Proof. By way of contradiction, if there exists a noncontinuable solution $x(t)$ of IVP (2.43)-(2.44) on $[t_0, t_0 + \sigma)$ such that $|x(t) - \varphi(0)| \leq \gamma$ and $x_t \in \omega$ for all $t \in [t_0, t_0 + \sigma)$, that is, $x \in B_\omega(t_0, \varphi, \sigma, \gamma)$, then first, $x(t)$ is not uniformly continuous on $[t_0, t_0 + \sigma)$. Otherwise, $x(t_0 + \sigma) = \lim\limits_{t \to (t_0+\sigma)^-} x(t)$ exists and thus $x_{t_0+\sigma} \in \Omega$. By Theorem 2.31, $x(t)$ can be continued beyond $t_0 + \sigma$.

Therefore, there exist a sufficiently small constant $\varepsilon > 0$, and sequences $\{t_k\} \subseteq [t_0, t_0 + \sigma)$, $\{\Delta_k\}$ with $\Delta_k \to 0^+$ as $k \to \infty$, such that

$$|x(t_k) - x(t_k - \Delta_k)| \geq \varepsilon, \quad \text{for all } k = 1, 2, \ldots .$$

Now choose a constant $H > 0$ so that

$$|K^{-1}(t, x_t, x_t)| \leq H, \quad \text{for all } t \in [t_0, t_0 + \sigma).$$

For given H and $\varepsilon > 0$, by (H5)-(H7) and (H9), we can find positive constants b and σ_0 so that

$$\frac{2HM_\omega}{\Gamma(q)(1+\beta_\omega)^{1-q_\omega}} \left(\sigma_0^{1+\beta_\omega}\right)^{1-q_\omega} < \frac{\varepsilon}{5},$$

where $\beta_\omega = \frac{q-1}{1-q_\omega} \in (-1, 0), M_\omega = \left(\int_{t_0}^{t_0+\sigma} (m_\omega(s))^{\frac{1}{q_\omega}} ds \right)^{q_\omega}$,

$$H|K(t, x_t, x_t) - K(t, x_t, x_{t-\tau})| < \frac{1}{5}, \quad \text{as} \sup_{t_0+\tau \leq s \leq t} |x(s) - x(s-\tau)| \leq \varepsilon,$$

and

$$\begin{aligned}
&H|g(t, x_{t-\tau}) - g(t-\tau, x_{t-\tau})| < \frac{\varepsilon}{5}, \\
&H\ell_\omega(b) < \frac{1}{5}, \ b < \frac{\delta - \sigma_0}{2}, \\
&H| \ L_b^*(t, x_t, x_{t-\tau}) \ | < \frac{\varepsilon}{5},
\end{aligned}$$

for all $t \in [t_0 + \tau, t_0 + \sigma)$ and $0 < \tau < \sigma_0$.

Since $x(t)$ is uniformly continuous on $[t_0, t_0 + \sigma - b]$, we can find a constant $H_1 > 0$ so that for all $k \geq H_1$, we have $\Delta_k < \sigma_0$ and $|x(t) - x(t - \Delta_k)| < \varepsilon$ for all $t \in [t_0 + \sigma_k, t_0 + \sigma - b]$. Now for all $k \geq H_1$, define a sequence $\{s_k\}$ in the following pattern

$$s_k = \inf\{t \in (t_0 + \sigma - b, t_0 + \sigma) : |x(t) - x(t - \Delta_k)| \geq \varepsilon\}.$$

Then

$$|x(s_k) - x(s_k - \Delta_k)| = \varepsilon.$$

Thus we get

$$\begin{aligned}
&\frac{2HM_\omega}{\Gamma(q)(1+\beta_\omega)^{1-q_\omega}} \left(\Delta_k^{1+\beta_\omega}\right)^{1-q_\omega} < \frac{\varepsilon}{5}, \\
&H|K(s_k, x_{s_k}, x_{s_k}) - K(s_k, x_{s_k}, x_{s_k - \Delta_k})| < \frac{1}{5}, \\
&H|g(s_k, x_{s_k-\Delta_k}) - g(s_k - \Delta_k, x_{s_k-\Delta_k})| < \frac{\varepsilon}{5}
\end{aligned}$$

and

$$\begin{aligned}
&H|L(s_k, x_{s_k}, x_{s_k-\Delta_k}) - L_b^*(s_k, x_{s_k}, x_{s_k-\Delta_k})| \\
&\leq \frac{1}{5} \sup_{-b \leq \theta \leq 0} |x(s_k + \theta) - x(s_k - \Delta_k + \theta)| \leq \frac{\varepsilon}{5}.
\end{aligned}$$

On the other hand, we see that

$$
\begin{aligned}
& g(s_k, x_{s_k}) - g(s_k - \Delta_k, x_{s_k - \Delta_k}) \\
= {} & g(s_k, x_{s_k}) - g(s_k, x_{s_k - \Delta_k}) + g(s_k, x_{s_k - \Delta_k}) - g(s_k - \Delta_k, x_{s_k - \Delta_k}) \\
= {} & [K(s_k, x_{s_k}, x_{s_k - \Delta_k}) - K(s_k, x_{s_k}, x_{s_k})]\ [x(s_k) - x(s_k - \Delta_k)] \\
& + K(s_k, x_{s_k}, x_{s_k})[x(s_k) - x(s_k - \Delta_k)] + L(s_k, x_{s_k}, x_{s_k - \Delta_k}) \\
& - L_b^*(s_k, x_{s_k}, x_{s_k - \Delta_k}) + L_b^*(s_k, x_{s_k}, x_{s_k - \Delta_k}) \\
& + g(s_k, x_{s_k - \Delta_k}) - g(s_k - \Delta_k, x_{s_k - \Delta_k}).
\end{aligned}
$$

By using the same argument as that of Step III in Theorem 2.31, we have

$$
\begin{aligned}
& |g(s_k, x_{s_k}) - g(s_k - \Delta_k, x_{s_k - \Delta_k})| \\
\leq {} & \frac{1}{\Gamma(q)} \left| \int_{s_k - \Delta_k}^{s_k} (s_k - s)^{q-1} f(s, x_s) ds \right| \\
& + \frac{1}{\Gamma(q)} \left| \int_{t_0}^{s_k - \Delta_k} [(s_k - s)^{q-1} - (s_k - \Delta_k - s)^{q-1}] f(s, x_s) ds \right| \\
\leq {} & \frac{2M_\omega}{\Gamma(q)(1+\beta_\omega)^{1-q_\omega}} \left(\Delta_k^{1+\beta_\omega} \right)^{1-q_\omega}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& |x(s_k) - x(s_k - \Delta_k)| \\
\leq {} & |K^{-1}(s_k, x_{s_k}, x_{s_k})| \{ |g(s_k, x_{s_k}) - g(s_k - \Delta_k, x_{s_k - \Delta_k})| \\
& + |K(s_k, x_{s_k}, x_{s_k - \Delta_k}) - K(s_k, x_{s_k}, x_{s_k})| |x(s_k) - x(s_k - \Delta_k)| \\
& + |L(s_k, x_{s_k}, x_{s_k - \Delta_k}) - L_b^*(s_k, x_{s_k}, x_{s_k - \Delta_k})| \\
& + |L_b^*(s_k, x_{s_k}, x_{s_k - \Delta_k})| + |g(s_k, x_{s_k - \Delta_k}) - g(s_k - \Delta_k, x_{s_k - \Delta_k})| \} \\
< {} & \varepsilon.
\end{aligned}
$$

This is contrary to $|x(s_k) - x(s_k - \Delta_k)| = \varepsilon$. The proof is completed. □

Remark 2.37. If we replace conditions of Theorem 2.31 by conditions of Remark 2.32, the result of Theorem 2.36 holds.

Remark 2.38. If we replace the condition (H3) of Theorem 2.31 by a weaker condition:

(H3)$''$ for any $x, y \in A(t_0, \varphi, \sigma, \gamma)$ with $\sup\limits_{t_0 \leq s \leq t_0 + \sigma} |x(s) - y(s)| \to 0$,

$$
\left| \int_{t_0}^{t} (t-s)^{q-1} [f(s, x_s) - f(s, y_s)] ds \right| \to 0, \quad t \in [t_0, t_0 + \sigma],
$$

where $\sigma \in (0, \sigma_1], \gamma \in (0, \gamma_1]$.

Then we can also conclude that the result of Theorem 2.36 holds.

In the following, for any $(t_0, \varphi) \in [0, \infty) \times \Omega$ and any constants $\varepsilon, \sigma, \gamma > 0$, $C_\varepsilon(t_0, \varphi, \delta, \gamma)$ denotes the set of all functions $x : (-\infty, t_0 + \sigma] \to \mathbb{R}^n$ so that $\|x_{t_0} - \varphi\|_B < \varepsilon,\ x : [t_0, t_0 + \sigma] \to \mathbb{R}^n$ is continuous and $|x(t) - \varphi(0)| \leq \gamma$.

Theorem 2.39. Suppose that for any $(t_0, \varphi) \in [0, \infty) \times \Omega$, the solution of IVP (2.43)-(2.44) is unique. Besides, suppose that $\sigma \in (0, \sigma_1], \gamma \in (0, \gamma_1]$ and for any $x \in C_\varepsilon(t_0, \varphi, \sigma, \gamma)$, (H5)-(H9) hold and

(H10) for any $x, y \in C_\varepsilon(t_0, \varphi, \sigma, \gamma)$, if $\|x_{t_0} - y_{t_0}\|_B \to 0$ and $\sup_{t_0 \le s \le t_0+\sigma} |x(s) - y(s)| \to 0$, then $g(t, x_t) \to g(t, y_t)$ and $|\int_{t_0}^t (t-s)^{q-1}[f(s, x_s) - f(s, y_s)]ds| \to 0$ for $t \in [t_0, t_0 + \sigma]$.

If x is a noncontinuable solution of IVP(2.43)-(2.44) defined on $[t_0, t_0+\sigma_1)$, then for any $\varepsilon > 0$ and $\sigma \in (0, \sigma_1)$, we can find a $\sigma > 0$ so that if $\|\varphi - \psi\|_B < \sigma$, then $|x(t) - y(t)| < \varepsilon$ for $t \in [t_0, t_0 + \sigma]$, where $y(t)$ is a solution of (2.43) through (t_0, ψ).

Proof. By way of contradiction, if the conclusion above is not true, then there exist $\varepsilon > 0$, sequences $\{t_k\} \subseteq [t_0, t_0 + \sigma]$ and $\{\varphi^k\} \subseteq \Omega$ such that

$$\|\varphi^k - \varphi\|_B < \frac{1}{k},$$
$$|y^k(t_k) - x(t_k)| = \varepsilon$$

and

$$|y^k(t) - x(t)| < \varepsilon \quad \text{for} \quad t \in [t_0, t_k),$$

where $y^k(t)$ is a solution of following IVP

$$ {}_{t_0}^{C}D_t^\alpha g(t, y_t) = f(t, y_t), \ y_{t_0} = \varphi^k. \tag{2.56}$$

Without loss of generality, we may assume $t_k \to \bar{t} \in [t_0, t_0 + \sigma]$. Now define a sequence of functions $\{z^k\}$ as follows:

$$z^k(t) = \begin{cases} y^k(t) & \text{for } t \in [t_0, t_k], \\ y^k(t_k) & \text{for } t \in [t_k, \bar{t}\,], \quad \text{if } t_k < \bar{t}. \end{cases}$$

Using the same argument as that of Theorem 2.31, we can assume that $\{z^k\}$ is equicontinuous in $t \in [t_0, \bar{t}\,]$. By Ascoli-Arzela theorem, without loss of generality, we can find a function $y : (-\infty, \bar{t}\,] \to \mathbb{R}^n$ such that $\lim_{k\to\infty} \sup_{t_0 \le s \le \bar{t}} |z^k(s) - y(s)| = 0$ and $y(s) = \varphi(s)$ for $s \le t_0$.

Now considering the equation (2.56), we get

$$g(t, y_t^k) - g(t_0, \varphi^k) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, y_s^k) ds, \quad \text{for } t \in [t_0, \bar{t}\,].$$

By (H10) and Lebesgue's dominated convergence theorem and let $k \to \infty$, we obtain

$$g(t, y_t) - g(t_0, \varphi) = \frac{1}{\Gamma(q)} \int_{t_0}^t (t-s)^{q-1} f(s, y_s) ds, \quad \text{for } t \in [t_0, \bar{t}\,].$$

This means that $y(t) = x(t)$ for $t \in [t_0, \bar{t}\,]$ by the uniqueness assumption of the solutions of IVP(2.43)-(2.44). This is contrary to

$$|y^k(t_k) - x(t_k)| = \varepsilon$$

and

$$\lim_{k\to\infty} \sup_{t_0 \le s \le \bar{t}} |z^k(s) - y(s)| = 0.$$

The proof is therefore complete. $\square$

2.5 Iterative Functional Differential Equations

2.5.1 *Introduction*

In Section 2.5, we consider the following fractional iterative functional differential equations with parameter

$$\begin{cases} {}_a^C D_t^q x(t) = f(t, x(t), x(x^v(t))) + \lambda, & t \in [a, b],\ v \in \mathbb{R} \setminus \{0\},\ q \in (0, 1),\ \lambda \in \mathbb{R}, \\ x(t) = \varphi(t), & t \in [a_1, a], \\ x(t) = \psi(t), & t \in [b, b_1], \end{cases} \tag{2.57}$$

where ${}_a^C D_t^q$ is Caputo fractional derivative of order q and
(C1) $a_1 \le a < b \le b_1$, $a_1 \le a_1^v$ and $b_1^v \le b_1$;
(C2) $f \in C([a, b] \times [a_1, b_1]^2, \mathbb{R})$;
(C3) $\varphi \in C([a_1, a], [a_1, b_1])$ and $\psi \in C([b, b_1], [a_1, b_1])$.
Definition 2.40. A function $x \in C([a_1, b_1], [a_1, b_1])$ is said to be a solution of the problem (2.57) if x satisfies the equation ${}_a^C D_t^q x(t) = f(t, x(t), x(x^v(t))) + \lambda$ on $[a, b]$, and the conditions $x(t) = \varphi(t), t \in [a_1, a]$, $x(t) = \psi(t), t \in [b, b_1]$.

The purpose of this section is to determine the pair (x, λ), $x \in C([a_1, b_1], [a_1, b_1])$ (or $\mathrm{C}_{\mathrm{L}}^{\mathrm{q}}([\mathrm{a}_1,b_1],\ [a_1, b_1])$), $\lambda \in \mathbb{R}$, which satisfies the problem (2.57). In Subsection 2.5.2, by using the Schauder's fixed point theorem, we establish existence theorems in $C([a_1, b_1], [a_1, b_1])$ and $C_L^q([a_1, b_1], [a_1, b_1])$ respectively. Unfortunately, uniqueness results can not be obtained since the solution operator is not Lipschitz continuous but only Hölder continuous. Meanwhile, data dependence results of solutions and parameters provide possible way to describe the error estimates between explicit and approximative solutions for such problems. In Subsection 2.5.4, We make some examples to illustrate our results and conclude some possible extensions to general parametrized iterative fractional functional differential equations.

2.5.2 *Existence*

We first give existence result in $C([a_1, b_1], [a_1, b_1])$. Let (x, λ) be a solution of the problem (2.57). Then this problem is equivalent to the following fixed point equation

$$x(t) = \begin{cases} \varphi(t), & \text{for } t \in [a_1, a], \\ \varphi(a) + \dfrac{1}{\Gamma(q)} \displaystyle\int_a^t (t - s)^{q-1} f(s, x(s), x(x^v(s))) ds & \\ \quad + \dfrac{\lambda}{\Gamma(q + 1)} (t - a)^q, & \text{for } t \in [a, b], \\ \psi(t), & \text{for } t \in [b, b_1]. \end{cases} \tag{2.58}$$

From the condition of continuity of x in $t = b$, we have that

$$\lambda = \frac{\Gamma(q + 1)(\psi(b) - \varphi(a))}{(b - a)^q} - \frac{q}{(b - a)^q} \int_a^b (b - s)^{q-1} f(s, x(s), x(x^v(s))) ds. \tag{2.59}$$

Now we consider the operator

$$A : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], \mathbb{R}), \tag{2.60}$$

where

$$(Ax)(t) := \begin{cases} \varphi(t), & \text{for } t \in [a_1, a], \\ \varphi(a) + \dfrac{(t-a)^q}{(b-a)^q}(\psi(b) - \varphi(a)) - \dfrac{(t-a)^q}{\Gamma(q)(b-a)^q} & \\ \quad \times \displaystyle\int_a^b (b-s)^{q-1} f(s, x(s), x(x^v(s)))ds & \\ \quad + \dfrac{1}{\Gamma(q)} \displaystyle\int_a^t (t-s)^{q-1} f(s, x(s), x(x^v(s)))ds, & \text{for } t \in [a, b], \\ \psi(t), & \text{for } t \in [b, b_1]. \end{cases} \tag{2.61}$$

It is clear that (x, λ) is a solution of the problem (2.57) if and only if x is a fixed point of the operator A and λ is given by (2.58). So, the problem is to study the fixed point equation

$$x = A(x).$$

Now, we are ready to state our first result in this section.

Theorem 2.41. We suppose that

(i) conditions (C1)-(C3) are satisfied;

(ii) there are m_f, $M_f \in \mathbb{R}$ such that

$$m_f \le f(t, u, w) \le M_f, \ \forall\ t \in [a, b], u, w \in [a_1, b_1],$$

along with

$$a_1 \le \min(\varphi(a), \psi(b)) - \max\left(0, \frac{M_f (b-a)^q}{\Gamma(q+1)}\right) + \min\left(0, \frac{m_f (b-a)^q}{\Gamma(q+1)}\right),$$

and

$$\max(\varphi(a), \psi(b)) - \min\left(0, \frac{m_f (b-a)^q}{\Gamma(q+1)}\right) + \max\left(0, \frac{M_f (b-a)^q}{\Gamma(q+1)}\right) \le b_1.$$

Then problem (2.57) has a solution in $C([a_1, b_1], [a_1, b_1])$.

Proof. In what follow we consider on $C([a_1, b_1], \mathbb{R})$ with the Chebyshev norm $\| \cdot \|_C$.

Condition (ii) assures that the set $C([a_1, b_1], [a_1, b_1])$ is an invariant subset for the operator A, that is, we have

$$A(C([a_1, b_1], [a_1, b_1])) \subset C([a_1, b_1], [a_1, b_1]).$$

Indeed, for $t \in [a_1, a] \cup [b, b_1]$, we have $A(x)(t) \in [a_1, b_1]$. Furthermore, we obtain

$$a_1 \le A(x)(t) \le b_1, \ \forall\ t \in [a, b],$$

if and only if

$$a_1 \le \min_{t \in [a,b]} A(x)(t) \tag{2.62}$$

and

$$\max_{t\in[a,b]} A(x)(t) \le b_1 \tag{2.63}$$

hold.

Since

$$\min_{t\in[a,b]} A(x)(t) \ge \min(\varphi(a),\psi(b)) - \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right) + \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right),$$

and

$$\max_{t\in[a,b]} A(x)(t) \le \max(\varphi(a),\psi(b)) - \min\left(0, \frac{m_f(b-a)^q}{\Gamma(q+1)}\right) + \max\left(0, \frac{M_f(b-a)^q}{\Gamma(q+1)}\right),$$

respectively, the requirements (2.62) and (2.63) are equivalent with the conditions appearing in (ii).

So, in the above conditions we have a selfmapping operator

$$A : C([a_1,b_1],[a_1,b_1]) \to C([a_1,b_1],[a_1,b_1]).$$

Further, we check A is a completely continuous operator.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([a_1,b_1],[a_1,b_1])$. Then for each $t \in [a_1,b_1]$, we have that

$$|(Ax_n)(t)-(Ax)(t)| \le \begin{cases} 0, & \text{for } t\in[a_1,a],\\ \dfrac{2(b-a)^q}{\Gamma(q+1)}\|f(\cdot,x_n(x_n^v(\cdot)))-f(\cdot,x(x^v(\cdot)))\|_C, & \text{for } t\in[a,b],\\ 0, & \text{for } t\in[b,b_1]. \end{cases}$$

Since $f \in C([a,b]\times[a_1,b_1]^2,\mathbb{R})$, we have that

$$\|Ax_n - Ax\|_C \to 0 \quad \text{as } n\to\infty.$$

Now, consider $a_1 \le t_1 < t_2 \le a$. Then,

$$|(Ax)(t_2)-(Ax)(t_1)| = |\varphi(t_2)-\varphi(t_1)|.$$

Similarly, for $b \le t_1 < t_2 \le b_1$,

$$|(Ax)(t_2)-(Ax)(t_1)| = |\psi(t_2)-\psi(t_1)|.$$

On the other hand, for $a \le t_1 < t_2 \le b$,

$$\begin{aligned}|(Ax)(t_2)-(Ax)(t_1)| \le{}& \frac{(t_2-t_1)^q}{(b-a)^q}|\psi(b)-\varphi(a)|\\ &+\frac{4(t_2-t_1)^q\max\{|m_f|,|M_f|\}}{\Gamma(q+1)}.\end{aligned} \tag{2.64}$$

Together with the Arzela-Ascoli theorem and A is a continuous operator, we can conclude that A is a completely continuous operator.

It is obvious that the set $C([a_1,b_1],[a_1,b_1]) \subseteq C([a_1,b_1],\mathbb{R})$ is a bounded convex closed subset of the Banach space $C([a_1,b_1],\mathbb{R})$. Thus, the operator A has a fixed point due to Schauder's fixed point theorem. This completes the proof. $\square$

In the following, we present the existence and estimate results in $C_L^q([a_1, b_1], [a_1, b_1])$. Let $L > 0$ and $I \subset \mathbb{R}$ be a compact interval, and introduce the following notation:

$$C_L^q(I, \mathbb{R}) = \{x \in C(I, \mathbb{R}) : |x(t_1) - x(t_2)| \leq L|t_1 - t_2|^q\}$$

for all $t_1, t_2 \in I$. Remark that $C_L^q(I, \mathbb{R}) \subseteq C(I, \mathbb{R})$ is a complete metric space. Then (2.64) implies that under assumptions of Theorem 2.41 any solution of problem (2.57) belongs to $C_{L_*}^q([a, b], \mathbb{R})$ for

$$L_* = \frac{|\psi(b) - \varphi(a)|}{(b-a)^q} + \frac{4\max\{|m_f|, |M_f|\}}{\Gamma(q+1)}. \tag{2.65}$$

Now we present our second result in this section.

Theorem 2.42. We suppose that

(i) conditions of Theorem 2.41 hold and $\varphi \in C_{L_\varphi}^q([a_1, a], [a_1, b_1])$, $\psi \in C_{L_\psi}^q([b, b_1], [a_1, b_1])$ for some $L_\varphi, L_\psi \geq 0$.

Then problem (2.57) has a solution in $X = C_L^q([a_1, b_1], [a_1, b_1])$ and all its solution belongs to X for

$$L = \left(\sqrt[1-q]{L_\varphi} + \sqrt[1-q]{L_\psi} + \sqrt[1-q]{L_*} \right)^{1-q},$$

where L_* is defined by (2.65). Assume in addition

(ii) there exist $L_u > 0$ and $L_w > 0$ such that

$$|f(t, u_1, w_1) - f(t, u_2, w_2)| \leq L_u|u_1 - u_2| + L_w|w_1 - w_2|,$$

for $\forall\ t \in [a, b],\ u_i, w_i \in [a_1, b_1],\ i = 1, 2$.

Then two solutions x_1 and x_2 of problem (2.57) satisfy

$$\|x_1 - x_2\|_C \leq L_A^{\frac{1}{1-q\min\{1,v\}}} \tag{2.66}$$

for

$$L_A := \frac{2(b-a)^q}{\Gamma(q+1)} \left((L_u + L_w) b_1^{1-q\min\{1,v\}} + \max\{1, v^q\} b_1^{q\max\{v-1,0\}} L_w L \right). \tag{2.67}$$

If in addition

$$\Gamma(q+1) > 2(b-a)^q L_u\ , \tag{2.68}$$

then

$$\|x_1 - x_2\|_C \leq \left(\frac{2(b-a)^q L_w \left(b_1^{1-q\min\{1,v\}} + \max\{1, v^q\} b_1^{q\max\{v-1,0\}} L \right)}{\Gamma(q+1) - 2(b-a)^q L_u} \right)^{\frac{1}{1-q\min\{1,v\}}}. \tag{2.69}$$

Proof. Consider the operator A given by (2.61). From Theorem 2.41, we have

$$A : C([a_1, b_1], [a_1, b_1]) \to C([a_1, b_1], [a_1, b_1])$$

and A has a fixed point in $C([a_1, b_1], [a_1, b_1])$.

Now, consider $a_1 \le t_1 < t_2 \le a$. Then,

$$|(Ax)(t_2) - (Ax)(t_1)| = |\varphi(t_2) - \varphi(t_1)| \le L_\varphi |t_1 - t_2|^q \le L_* |t_1 - t_2|^q$$

as $\varphi \in C^q_{L_\varphi}([a_1, a], [a_1, b_1])$, due to (i).

Similarly, for $b \le t_1 < t_2 \le b_1$,

$$|(Ax)(t_2) - (Ax)(t_1)| = |\psi(t_2) - \psi(t_1)| \le L_\psi |t_1 - t_2|^q \le L_* |t_1 - t_2|^q$$

that follows from (i), too.

On the other hand, for $a \le t_1 < t_2 \le b$, we already know (see (2.64))

$$|(Ax)(t_2) - (Ax)(t_1)| \le L_* |t_1 - t_2|^q.$$

Next, if $a_1 \le t_1 \le a \le t_2 \le b$, then by the Hölder inequality with $q' = \frac{1}{q}$ and $p' = \frac{1}{1-q}$ (note $q', p' > 1$)

$$\begin{aligned}
|(Ax)(t_2) - (Ax)(t_1)| &\le |(Ax)(a) - (Ax)(t_1)| + |(Ax)(t_2) - (Ax)(a)| \\
&\le L_\varphi (a - t_1)^q + L_* (t_2 - a)^q \\
&\le \sqrt[p']{L_\varphi^{p'} + L_*^{p'}} \sqrt[q']{(a - t_1)^{qq'} + (t_2 - a)^{qq'}} \\
&\le L |t_1 - t_2|^q.
\end{aligned}$$

Furthermore, if $a_1 \le t_1 \le a < b \le t_2 \le b_1$, then again by the Hölder inequality with $q' = \frac{1}{q}$ and $p' = \frac{1}{1-q}$

$$\begin{aligned}
&|(Ax)(t_2) - (Ax)(t_1)| \\
&\le |(Ax)(a) - (Ax)(t_1)| + |(Ax)(b) - (Ax)(a)| + |(Ax)(t_2) - (Ax)(b)| \\
&\le L_\varphi (a - t_1)^q + L_* (b - a)^q + L_\psi (t_2 - b)^q \\
&\le \sqrt[p']{L_\varphi^{p'} + L_*^{p'} + L_\psi^{p'}} \sqrt[q']{(a - t_1)^{qq'} + (b - a)^{qq'} + (t_2 - b)^{qq'}} \\
&= L |t_1 - t_2|^q.
\end{aligned}$$

Therefore, the function $A(x)(t)$ belongs to X. This proves the first statement.

Take $x_1, x_2 \in X$. Then for all $t \in [a_1, a] \cup [b, b_1]$, we have

$$|A(x_1)(t) - A(x_2)(t)| = 0.$$

Moreover, for $t \in [a, b]$, from our conditions, we get

$$\begin{aligned}
&|A(x_1)(t) - A(x_2)(t)| \\
&\le \frac{(t-a)^q}{\Gamma(q)(b-a)^q} \int_a^b (b-s)^{q-1} \left| f(s, x_1(s), x_1(x_1^v(s))) - f(s, x_2(s), x_2(x_2^v(s))) \right| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left| f(s, x_1(s), x_1(x_1^v(s))) - f(s, x_2(s), x_2(x_2^v(s))) \right| ds \\
&\le \frac{1}{\Gamma(q)} \int_a^b (b-s)^{q-1} \left[L_u |x_1(s) - x_2(s)| + L_w \left| x_1(x_1^v(s)) - x_2(x_2^v(s)) \right| \right] ds \\
&\quad + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} \left[L_u |x_1(s) - x_2(s)| + L_w \left| x_1(x_1^v(s)) - x_2(x_2^v(s)) \right| \right] ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(q)}\int_a^b (b-s)^{q-1}\Big[L_u|x_1(s)-x_2(s)| + L_w\,|x_1(x_1^v(s)) - x_1(x_2^v(s))| \\
&\quad + L_w\,|x_1(x_2^v(s)) - x_2(x_2^v(s))|\Big]ds \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\Big[L_u|x_1(s)-x_2(s)| + L_w\,|x_1(x_1^v(s)) - x_1(x_2^v(s))| \\
&\quad + L_w\,|x_1(x_2^v(s)) - x_2(x_2^v(s))|\Big]ds \\
&\le \frac{1}{\Gamma(q)}\int_a^b (b-s)^{q-1}\Big[(L_u+L_w)\|x_1-x_2\|_C + L_wL\,|x_1^v(s)-x_2^v(s)|^q\Big]ds \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\Big[(L_u+L_w)\|x_1-x_2\|_C + L_wL\,|x_1^v(s)-x_2^v(s)|^q\Big]ds \\
&\le \frac{1}{\Gamma(q)}\int_a^b (b-s)^{q-1}\Big[(L_u+L_w)\|x_1-x_2\|_C \\
&\quad + \max\{1,v^q\}b_1^{q\max\{v-1,0\}}L_wL\|x_1-x_2\|_C^{q\min\{1,v\}}\Big]ds \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\Big[(L_u+L_w)\|x_1-x_2\|_C \\
&\quad + \max\{1,v^q\}b_1^{q\max\{v-1,0\}}L_wL\|x_1-x_2\|_C^{q\min\{1,v\}}\Big]ds \\
&\le \frac{2(b-a)^q}{\Gamma(q+1)}((L_u+L_w)\|x_1-x_2\|_C \\
&\quad + \max\{1,v^q\}b_1^{q\max\{v-1,0\}}L_wL\|x_1-x_2\|_C^{q\min\{1,v\}}),
\end{aligned}
$$

where we use the inequality

$$r^v - s^v \le \max\{1,v\}r^{\max\{v-1,0\}}(r-s)^{\min\{1,v\}}$$

for any $r \ge s \ge 0$ and $v > 0$. From $\|x_1 - x_2\|_C \le b_1$ we get

$$
\begin{aligned}
\|A(x_1) - A(x_2)\|_C &\le \frac{2(b-a)^q}{\Gamma(q+1)}\Big((L_u+L_w)b_1^{1-q\min\{1,v\}} \\
&\quad + \max\{1,v^q\}b_1^{q\max\{v-1,0\}}L_wL\Big)\|x_1-x_2\|_C^{q\min\{1,v\}} \\
&= L_A\|x_1-x_2\|_C^{q\min\{1,v\}}.
\end{aligned}
\tag{2.70}
$$

So A is Hölder continuous but never Lipschitz continuous, since $q\min\{1,v\} \le q < 1$. If x_1 and x_2 are fixed points of A then

$$\|x_1 - x_2\|_C = \|A(x_1) - A(x_2)\|_C \le L_A\|x_1 - x_2\|_C^{q\min\{1,v\}}$$

which implies (2.66). In general, we have

$$
\begin{aligned}
\|x_1 - x_2\|_C &\le \frac{2(b-a)^q}{\Gamma(q+1)}L_u\|x_1-x_2\|_C \\
&\quad + \frac{2(b-a)^qL_w}{\Gamma(q+1)}\Big(b_1^{1-q\min\{1,v\}} \\
&\quad + \max\{1,v^q\}b_1^{q\max\{v-1,0\}}L\Big)\|x_1-x_2\|_C^{q\min\{1,v\}},
\end{aligned}
$$

which implies (2.69) under (2.68). The proof is completed. □

We do not know about uniqueness. But this is not so surprising, since A is not Lipschitzian in general. So we cannot apply metric fixed point theorems, only topological one. This can be simply illustrated on a simpler problem

$$ {}_0^C D_t^{\frac{1}{2}} x(t) = x(\sqrt{x(t)}), \quad x(0) = 0, \quad t \in [0,1]. \tag{2.71}$$

Rewriting (2.71) as

$$x(t) = B(x)(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t \frac{x(\sqrt{x(s)})}{\sqrt{t-s}} ds,$$

it follows that $B : C_{\frac{1}{2}}([0,1],[0,1]) \to C_{\frac{1}{2}}([0,1],\mathbb{R})$ satisfies

$$\|B(x_1) - B(x_2)\|_C \le \frac{2}{\sqrt{\pi}} \left(\|x_1 - x_2\|_C + \frac{1}{2}\sqrt{\|x_1 - x_2\|_C} \right),$$

so it is not Lipschitzian. Hence (2.71) should have a nonzero solution, and it does have $x(t) = \frac{4}{\pi} t$.

2.5.3 *Data Dependence*

Consider the following two problems

$$\begin{cases} {}_a^C D_t^q x(t) = f_i(t, x(t), x(x^v(t))) + \lambda_i, & t \in [a,b], \ v \in (0,1], \ q \in (0,1), \\ x(t) = \varphi_i(t), & t \in [a_1, a], \\ x(t) = \psi_i(t), & t \in [b, b_1], \end{cases} \tag{2.72}$$

where f_i, λ_i, φ_i and ψ_i, $i = 1,2$ be as in the Theorem 2.42.

Consider the operators

$$A_i : C([a_1,b_1],[a_1,b_1]) \to C([a_1,b_1],[a_1,b_1])$$

given by (2.61) when φ, ψ, f and λ are replaced by φ_i, ψ_i, f_i and λ_i, respectively.

We are ready to state the third result in this section.

Theorem 2.43. Suppose conditions of the Theorem 2.42, and, moreover

(i) there exists $\eta_1 > 0$ such that

$$|\varphi_1(t) - \varphi_2(t)| \le \eta_1, \ \forall \ t \in [a_1, a],$$

and

$$|\psi_1(t) - \psi_2(t)| \le \eta_1, \ \forall \ t \in [b, b_1];$$

(ii) there exists $\eta_2 > 0$ such that

$$|f_1(t,u,w) - f_2(t,u,w)| \le \eta_2, \ \forall \ t \in [a,b], \ u, w \in [a_1, b_1].$$

Let r_* be a positive root of equation

$$r_* = L^* r_*^{q \min\{1,v\}} + 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)} \eta_2, \tag{2.73}$$

where $L^* = \min\{L_{A_1}, L_{A_2}\}$ (see (2.67)). Then

$$\|x_1^* - x_2^*\|_C \le r_*, \tag{2.74}$$

and

$$|\lambda_1^* - \lambda_2^*| \le \frac{\Gamma(q+1)}{(b-a)^q}\left(2\eta_1 + \frac{L^*}{2} r_*^{q\min\{1,v\}}\right) + \eta_2, \tag{2.75}$$

where (x_i^*, λ_i^*), $i = 1, 2$ are solutions of the corresponding problems (2.72). Note r_* is uniquely defined.

Proof. Using the condition (i), it is easy to see that for $x \in C([a_1, b_1], [a_1, b_1])$ and $t \in [a_1, a] \cup [b, b_1]$, we have

$$\|A_1(x) - A_2(x)\|_C \le \eta_1.$$

On the other hand, for $t \in [a, b]$, using the condition (ii), we obtain

$$\begin{aligned}
&|A_1(x)(t) - A_2(x)(t)| \\
&\le |\varphi_1(a) - \varphi_2(a)| + \frac{(t-a)^q}{(b-a)^q}(|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)|) \\
&\quad + \frac{(t-a)^q}{\Gamma(q)(b-a)^q}\int_a^b (b-s)^{q-1}\left|f_1(s, x(s), x(x^v(s))) - f_2(s, x(s), x(x^v(s)))\right| ds \\
&\quad + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}\left|f_1(s, x(s), x(x^v(s))) - f_2(s, x(s), x(x^v(s)))\right| ds \\
&\le 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2.
\end{aligned}$$

So, we have

$$\|A_1(x) - A_2(x)\|_C \le 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2.$$

Next, (2.70) holds for both A_i with L_{f_i}. Without loss of generality, we may suppose that $L^* = L_{A_1} = \min\{L_{A_1}, L_{A_2}\}$. Consequently, we obtain

$$\begin{aligned}
\|x_1^* - x_2^*\|_C &= \|A_1(x_1^*) - A_2(x_2^*)\|_C \\
&\le \|A_1(x_1^*) - A_1(x_2^*)\|_C + \|A_1(x_2^*) - A_2(x_2^*)\|_C \\
&\le L^*\|x_1^* - x_2^*\|_C^{q\min\{1,v\}} + 3\eta_1 + \frac{2(b-a)^q}{\Gamma(q+1)}\eta_2,
\end{aligned}$$

which implies (2.74). Moreover, we get

$$\begin{aligned}
&|\lambda_1^* - \lambda_2^*| \\
&\le \frac{\Gamma(q+1)(|\psi_1(b) - \psi_2(b)| + |\varphi_1(a) - \varphi_2(a)|)}{(b-a)^q} \\
&\quad + \frac{q}{(b-a)^q}\int_a^b (b-s)^{q-1}\left|f_1(s, x_1^*(s), x_1^*(x_1^{*v}(s))) - f_1(s, x_2^*(s), x_2^*(x_2^{*v}(s)))\right| ds \\
&\quad + \frac{q}{(b-a)^q}\int_a^b (b-s)^{q-1}\left|f_1(s, x_2^*(s), x_2^*(x_2^{*v}(s))) - f_2(s, x_2^*(s), x_2^*(x_2^{*v}(s)))\right| ds \\
&\le \frac{\Gamma(q+1)}{(b-a)^q}\left(2\eta_1 + \frac{L^*}{2} r_*^{q\min\{1,v\}}\right) + \eta_2.
\end{aligned}$$

The proof is completed. □

2.5.4 Examples and General Cases

Example 2.44. Consider the following problem:

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}} x(t) = \mu x(x(t)) + \lambda, & t \in [0,1],\ \mu > 0,\ \lambda \in \mathbb{R}, \\ x(t) = 0, & t \in [-h, 0],\ h > 0, \\ x(t) = 1, & t \in [1, 1+h], \end{cases} \tag{2.76}$$

where $x \in C([-h, 1+h], [-h, 1+h])$.
Proposition 2.45. Suppose that

$$\mu \leq \frac{\Gamma(\frac{3}{2})h}{1+2h}.$$

Then the problem (2.76) has a solution in $C([-h, 1+h], [-h, 1+h])$.

Proof. First of all notice that accordingly to the Theorem 2.41 we have $v = 1$, $q = \frac{1}{2}$, $a = 0, b = 1$, $\psi(b) = 1, \varphi(a) = 0$ and $f(t, u_1, u_2) = \mu u_2$. Moreover, $a_1 = -h$ and $b_1 = 1 + h$ can be taken. Therefore, from the relation

$$m_f \leq f(t, u_1, u_2) \leq M_f,\ \forall\ t \in [0,1],\ u_1, u_2 \in [-h, 1+h],$$

we can choose $m_f = -h\mu$ and $M_f = (1+h)\mu$. For these data it can be easily verified that the conditions (ii) from the Theorem 2.41 are equivalent to the relation

$$\mu \leq \frac{\Gamma(\frac{3}{2})h}{1+2h},$$

consequently we complete the proof. $\square$

Example 2.46. Consider the following problem:

$$\begin{cases} {}_{2h}^{C} D_t^{\frac{1}{2}} x(t) = \mu x^2(x(t)) + \lambda, & t \in [2h, 3h],\ \mu > 0,\ \lambda \in \mathbb{R}, \\ x(t) = \frac{1}{2}, & t \in [h, 2h], \\ x(t) = \frac{1}{2}, & t \in [3h, 4h],\ h \in [\frac{1}{8}, \frac{1}{2}], \end{cases} \tag{2.77}$$

where $x \in C([h, 4h], [h, 4h])$. Note $\frac{1}{2} \in [h, 4h]$ for $h \in [\frac{1}{8}, \frac{1}{2}]$.
Proposition 2.47. We suppose that

$$0 < \mu \leq \frac{(-1+8h)\sqrt{\pi}}{64h^{5/2}}, \quad \text{for } h \in (\frac{1}{8}, \frac{1}{5}],$$
$$0 < \mu \leq \frac{(1-2h)\sqrt{\pi}}{64h^{5/2}}, \quad \text{for } h \in [\frac{1}{5}, \frac{1}{2}).$$

Then the problem (2.77) has a solution in $C_L^{\frac{1}{2}}([h, 4h], [h, 4h])$ with $L = \frac{128\mu h^2}{\sqrt{\pi}}$. Note $0 < \mu \leq \frac{15\sqrt{5\pi}}{64} \doteq 0.928905$. Furthermore, any two solutions $x_1, x_2 \in C_L^{\frac{1}{2}}([h, 4h], [h, 4h])$ of (2.77) satisfy

$$\|x_1 - x_2\|_C \leq \frac{4096h^4\mu^2\left(64h^{\frac{3}{2}}\mu + \sqrt{\pi}\right)^2}{\pi^2}. \tag{2.78}$$

Proof. First of all notice that accordingly to the Theorem 2.42 we have $v = 1$, $q = \frac{1}{2}$, $a = 2h, b = 3h$, $\psi(b) = \frac{1}{2}, \varphi(a) = \frac{1}{2}$, $a_1 = h$, $b_1 = 4h$. Observe that

$|f(t,u_1,u_2)-f(t,w_1,w_2)| = \mu|u_2+w_2||u_2-w_2| \le 8h\mu|u_2-w_2|,\ u_2,w_2 \in [h,4h]$. So $L_u = 0$ and $L_w = 8h\mu$. Next, we choose $m_f = \mu h^2$ and $M_f = 16\mu h^2$. By a common check in the conditions of Theorem 2.42 we can make sure that

$$a_1 \le \min(\varphi(a),\psi(b)) - \max\left(0, \tfrac{M_f(b-a)^q}{\Gamma(q+1)}\right) + \min\left(0, \tfrac{m_f(b-a)^q}{\Gamma(q+1)}\right)$$

$$\Longleftrightarrow h + \frac{16\mu h^{\frac{5}{2}}}{\Gamma(\frac{3}{2})} \le \frac{1}{2},$$

$$\max(\varphi(a),\psi(b)) - \min\left(0, \tfrac{m_f(b-a)^q}{\Gamma(q+1)}\right) + \max\left(0, \tfrac{M_f(b-a)^q}{\Gamma(q+1)}\right) \le b_1$$

$$\Longleftrightarrow \frac{1}{2} \le 4h - \frac{16\mu h^{\frac{5}{2}}}{\Gamma(\frac{3}{2})}.$$

These inequalities are equivalent to

$$0 < \mu \le \min\left\{\frac{(1-2h)\sqrt{\pi}}{64h^{\frac{5}{2}}}, \frac{(-1+8h)\sqrt{\pi}}{64h^{\frac{5}{2}}}\right\}.$$

The function $\kappa(h) = \min\left\{-\frac{(-1+2h)\sqrt{\pi}}{64h^{\frac{5}{2}}}, \frac{(-1+8h)\sqrt{\pi}}{64h^{\frac{5}{2}}}\right\}$ is increasing from 0 to $\frac{15\sqrt{5\pi}}{64} \doteq 0.928905$ on $[\frac{1}{8},\frac{1}{5}]$ and then it is decreasing to 0 on $[\frac{1}{5},\frac{1}{2}]$. Next, we derive $L_\varphi = L_\psi = 0$ and

$$L_* = \frac{|\psi(b)-\varphi(a)|}{(b-a)^q} + \frac{4\max\{|m_f|,|M_f|\}}{\Gamma(q+1)} = \frac{64\mu h^2}{\Gamma(\frac{3}{2})} = \frac{128\mu h^2}{\sqrt{\pi}},$$

so $L = L_*$. By (2.67) we derive

$$L_A = \frac{2\sqrt{h}}{\Gamma(\frac{3}{2})}\left(8h\mu\sqrt{4h} + 8h\mu\frac{128\mu h^2}{\sqrt{\pi}}\right) = \frac{64h^2\mu\left(64h^{\frac{3}{2}}\mu + \sqrt{\pi}\right)}{\pi}.$$

This gives (2.78) by (2.66). Therefore, by Theorem 2.42 the proof is completed. □

Example 2.48. Now take the following problems

$$\begin{cases} {}^{C}_{2h}D^{\frac{1}{2}}_{t}x(t) = \mu x^2(x(t)) + \lambda_i, & t \in [2h,3h],\ \mu_i = \mu,\ \lambda_i \in R, \\ x(t) = \varphi_i, & t \in [h,2h],\ h > 0, \\ x(t) = \psi_i, & t \in [3h,4h] \end{cases} \tag{2.79}$$

for $i = 1,2$. Suppose the following assumptions.
(H1) $\varphi_i \in C^{\frac{1}{2}}_{L_*}([h,2h],[h,4h])$, $\psi_i \in C^{\frac{1}{2}}_{L_*}([3h,4h],[h,4h])$ such that $\varphi_i(2h) = \frac{1}{2}$, $\psi_i(3h) = \frac{1}{2}$, $i = 1,2$ and $L_* = \frac{128\mu h^2}{\sqrt{\pi}}$;
(H2) we are in the conditions of Proposition 2.47 for both of the problems (2.79).

Let (x_i^*, λ_i^*) be solutions of the problems (2.79). We are looking for an estimation for $\|x_1^* - x_2^*\|_C$ and $|\lambda_1^* - \lambda_2^*|$.

Then, build upon Theorem 2.43, by a common substitution one can make sure that we have

Proposition 2.49. Consider the problems (2.79) and suppose the requirements (H1)-(H2) hold. Additionally, there exists $\eta_1 > 0$ such that

$$|\varphi_1(t) - \varphi_2(t)| \le \eta_1,\ \forall\, t \in [h,2h],$$

and

$$|\psi_1(t) - \psi_2(t)| \le \eta_1, \ \forall \ t \in [3h, 4h].$$

Then

$$\|x_1^* - x_2^*\|_C \le r_*,$$

and

$$|\lambda_1^* - \lambda_2^*| \le \frac{2}{\sqrt{\pi h}} \left(2\eta_1 + \frac{L^*}{2}\sqrt{r_*} \right),$$

where L^* and r_* are given by (2.80) and (2.81), respectively.

Proof. Results follow from Theorem 2.43 as follows. By Proposition 2.47, we have $L = \sqrt{3}L_* = \frac{128\mu h^2\sqrt{3}}{\sqrt{\pi}}$ and then (see (2.67))

$$L^* = L_{A_1} = L_{A_2} = \frac{64h^2\mu(64\sqrt{3}h^{3/2}\mu + \sqrt{\pi}))}{\pi}. \tag{2.80}$$

Realizing that now $\eta_2 = 0$, equation (2.73) has the form

$$r_* = L^*\sqrt{r_*} + 3\eta_1,$$

which has the positive solution

$$\begin{aligned} r_* = \frac{1}{\pi^2}\Bigg(& 25165824h^7\mu^4 + 64h^2\mu\pi^{\frac{3}{2}} + \eta_1\pi^2 \\ & +8\sqrt{2}\sqrt{4947802324992h^{14}\mu^8 + 25165824h^9\mu^5\pi^{\frac{3}{2}} + 393216h^7\mu^4\eta_1\pi^2}\Bigg). \end{aligned} \tag{2.81}$$

The estimate for $|\lambda_1^* - \lambda_2^*|$ follows directly from (2.75). The proof is finished. □

We conclude this section by considering a general fractional order iterative functional differential equations with parameter given by

$$\begin{cases} {}_a^C D_t^q x(t) = f(t, x(t), x(x^v(t)), \lambda), & t \in [a, b], \ v \in (0, 1], \ q \in (0, 1), \ \lambda \in J, \\ x(t) = \varphi(t), & t \in [a_1, a], \\ x(t) = \psi(t), & t \in [b, b_1], \end{cases} \tag{2.82}$$

when $J \subset R$ is an open interval, conditions (C1), (C3) are supposed and (C2) is extended to

(C4) $f \in C([a, b] \times [a_1, b_1]^2 \times J, \mathbb{R})$.

Then by (2.61) we have an operator $A(\lambda, x)$. It is easy to see that $A(\lambda, x) = A(x)$ for the problem (2.57). Supposing the assumptions of Theorem 2.41 for the problem (2.82) uniformly with respect to $\lambda \in J$, we can find its fixed point $x^*(\lambda, \cdot) \in C([a_1, b_1], [a_1, b_1])$. In order to get a solution of the problem (2.82), we need to solve

$$\Upsilon(\lambda) = \Gamma(q)(\psi(b) - \varphi(a)) - \int_a^b (b - s)^{q-1} f(s, x^*(\lambda, s), x^*(\lambda, x^*(\lambda, s)^v))ds = 0. \tag{2.83}$$

If there is an $\lambda_0 \in J$ solving (2.83) then $x^*(\lambda_0, t)$ is a solution of the problem (2.82). Since $x^*(\lambda, \cdot)$ is not unique in general, function $\Upsilon(\lambda)$ is multivalued. Consequently this way is not very useful. We propose another approach. The problem (2.82) is equivalent to the following fixed point equation

$$x(t) = \begin{cases} \varphi(t), & \text{for } t \in [a_1, a], \\ \varphi(a) + \dfrac{1}{\Gamma(q)} \displaystyle\int_a^t (t-s)^{q-1} f(s, x(s), x(x^v(s)), \lambda) ds, & \text{for } t \in [a, b], \\ \psi(t), & \text{for } t \in [b, b_1]. \end{cases} \tag{2.84}$$

From the condition of continuity of x in $t = b$, we have that

$$\psi(b) = \varphi(a) + \frac{1}{\Gamma(q)} \int_a^b (b-s)^{q-1} f(s, x(s), x(x^v(s)), \lambda) ds. \tag{2.85}$$

Now we consider the operator

$$A : C^b([a_1, b_1], [a_1, b_1]) \times J \to C^b([a_1, b_1], \mathbb{R}) \tag{2.86}$$

where

$$C^b([a_1, b_1], [a_1, b_1]) = \{x \in C([a_1, b], [a_1, b_1]) \cap C^b((b, b_1], [a_1, b_1]) : \exists \lim_{s \to b_+} x(s)\},$$
$$C^b([a_1, b_1], \mathbb{R}) = \{x \in C([a_1, b], \mathbb{R}) \cap C^b((b, b_1], \mathbb{R}) : \exists \lim_{s \to b_+} x(s)\}$$

and

$$A(x, \lambda)(t) := \begin{cases} \varphi(t), & \text{for } t \in [a_1, a], \\ \varphi(a) + \dfrac{1}{\Gamma(q)} \displaystyle\int_a^t (t-s)^{q-1} f(s, x(s), x(x^v(s)), \lambda) ds, & \text{for } t \in [a, b], \\ \psi(t), & \text{for } t \in (b, b_1]. \end{cases} \tag{2.87}$$

Now, we are ready to state the following result.
Theorem 2.50. Suppose that

(i) conditions (C1), (C3) and (C4) are satisfied;
(ii) there are m_f, $M_f \in R$ such that

$$m_f \le f(t, u, w, \lambda) \le M_f, \ \forall\ t \in [a, b], u, w \in [a_1, b_1], \lambda \in J$$

along with

$$a_1 \le \varphi(a) + \min\left(0, \frac{m_f (b-a)^q}{\Gamma(q+1)}\right),$$

and

$$\varphi(a) + \max\left(0, \frac{M_f (b-a)^q}{\Gamma(q+1)}\right) \le b_1.$$

Then operator $A(x,\lambda)$ has a fixed point in $C^b([a_1,b_1],[a_1,b_1])$ for any $\lambda \in J$.

Proof. Like in the proof of Theorem 2.41, condition (ii) assures that the set $C^b([a_1,b_1],[a_1,b_1])$ is an invariant subset for the operator A, that is, we have

$$A(C^b([a_1,b_1],[a_1,b_1]) \times J) \subset C^b([a_1,b_1],[a_1,b_1]). \tag{2.88}$$

Similarly, A is a completely continuous operator. It is obvious that the set $C^b([a_1,b_1],[a_1,b_1]) \subseteq C^b([a_1,b_1],\mathbb{R})$ is a bounded convex closed subset of the Banach space $C^b([a_1,b_1],\mathbb{R})$. Thus, the operator $A(x,\lambda)$ has a fixed point due to Schauder's fixed point theorem. This completes the proof. $\square$

We still do not have uniqueness result. For this purpose, we suppose

(C_5) f is nonnegative and nondecreasing, i.e. $m_f \geq 0$ and $0 \leq f(s_1,u_1,v_1,\lambda) \leq f(s_2,u_2,v_2,\lambda)$ for any $s_1 \leq s_2 \in [a,b]$, $u_1 \leq u_2, v_1 \leq v_2 \in [a_1,b_1]$ and $\lambda \in J$.

We introduce the Banach space

$$C^b_m([a_1,b_1],[a_1,b_1]) = \left\{x \in C^b([a_1,b_1],[a_1,b_1]) \mid x \text{ is nondecreasing on } [a_1,b_1]\right\}.$$

Now, we have the next result.

Theorem 2.51. We suppose conditions (i), (ii) of Theorem 2.50, (C_5) as well and $\varphi(t)$, $\psi(t)$ are nondecreasing with $\varphi(a) \leq \psi(b)$. Then operator $A(x,\lambda)$ is monotone nondecreasing in x on $C^b_m([a_1,b_1],[a_1,b_1])$ for any $\lambda \in J$. Consequently it has a unique smallest and largest fixed points $x_m(\lambda), x_M(\lambda)$ in $C^b_m([a_1,b_1],[a_1,b_1])$. Moreover, a nondecreasing sequence $\{A^k(a_1,\lambda)(t)\}_{k\geq 1}$ and a nonincreasing sequence $\{A^k(b_1,\lambda)(t)\}_{k\geq 1}$ satisfy

$$a_1 \leq A^k(a_1,\lambda)(t) \leq x_m(\lambda)(t) \leq x_M(\lambda)(t) \leq A^k(b_1,\lambda)(t) \leq b_1 \quad t \in J$$

for any $k \geq 1$ and $\lim_{k\to\infty} A^k(a_1,\lambda)(t) = x_m(\lambda)(t)$ and $\lim_{k\to\infty} A^k(b_1,\lambda)(t) = x_M(\lambda)(t)$ uniformly on $[a_1,b_1]$.

Proof. We already know (2.88). Let $x \in C_m([a_1,b_1],[a_1,b_1])$ then clearly $A(x,\lambda)(t_1) \leq A(x,\lambda)(t_2)$ for $t_1 \leq t_2 \in [a_1,a]$ and $t_1 \leq t_2 \in (b,b_1]$. Next for $s_1 \leq s_2 \in [a,b]$ we have $x(s_1) \leq x(s_2)$, $x^v(s_1) \leq x^v(s_2)$ and $x(x^v(s_1)) \leq x(x^v(s_2))$, which imply

$$f(s_1,x(s_1),x(x^v(s_1)),\lambda) \leq f(s_2,x(s_2),x(x^v(s_2)),\lambda). \tag{2.89}$$

Furthermore, for $t_1 \leq t_2 \in [a,b]$, following El-Sayed, 1995 and Darwish, 2008, and using (2.89), we derive

$$\begin{aligned}
&A(x,\lambda)(t_2) - A(x,\lambda)(t_1)\\
&= \frac{1}{\Gamma(q)}\int_a^{t_2}(t_2-s)^{q-1}f(s,x(s),x(x^v(s)),\lambda)ds\\
&\quad -\frac{1}{\Gamma(q)}\int_a^{t_1}(t_1-s)^{q-1}f(s,x(s),x(x^v(s)),\lambda)ds\\
&= \frac{1}{\Gamma(q)}\int_a^{t_1}\left((t_2-s)^{q-1}-(t_1-s)^{q-1}\right)f(s,x(s),x(x^v(s)),\lambda)ds\\
&\quad +\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s),x(x^v(s)),\lambda)ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\Gamma(q)} f(t_1, x(t_1), x(x^v(t_1)), \lambda) \\
&\quad \times \left(\int_a^{t_1} \left((t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) ds + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \right) \\
&= \frac{1}{\Gamma(q+1)} f(t_1, x(t_1), x(x^v(t_1)), \lambda) \left((t_2 - a)^q - (t_1 - a)^q \right) \\
&\geq 0.
\end{aligned}
$$

Consequently, we obtain

$$A(C_m^b([a_1, b_1], [a_1, b_1]), \lambda) \subset C_m^b([a_1, b_1], [a_1, b_1])$$

for any $\lambda \in J$.

Next, if $x_1, x_2 \in C_m^b([a_1, b_1], [a_1, b_1])$ with $x_1(t) \leq x_2(t)$, $t \in [a_1, b_1]$ then clearly we have $A(x_1, \lambda)(t) \leq A(x_2, \lambda)(t)$ for $t \in [a_1, a] \cup (b, b_1]$. For $s \in [a, b]$, we have $x_1(s) \leq x_2(s)$, $x_1^v(s) \leq x_2^v(s)$ and $x_1(x_1^v(s)) \leq x_2(x_2^v(s))$, which imply

$$f(s, x_1(s), x_1(x_1^v(s)), \lambda) \leq f(s, x_2(s), x_2(x_2^v(s)), \lambda). \tag{2.90}$$

Then for $t \in [a, b]$ we have

$$
\begin{aligned}
A(x_2, \lambda)(t) - A(x_1, \lambda)(t) = \frac{1}{\Gamma(q)} \int_a^t (t - s)^{q-1} \Big(& f(s, x_2(s), x_2(x_2^v(s)), \lambda) \\
& - f(s, x_1(s), x_1(x_1^v(s)), \lambda) \Big) ds \geq 0.
\end{aligned}
$$

This means that operator $A(x, \lambda)$ is monotone nondecreasing in x on $C_m^b([a_1, b_1], [a_1, b_1])$ for any $\lambda \in J$. We also know that A is a completely continuous operator. Then results follow from the general theory of nondecreasing compact operators in Banach spaces (see, e.g., Deimling, 1985). The proof is completed. $\square$

To get continuous solution, we need to solve either

$$\Upsilon_m(\lambda) = \psi(b) - \varphi(a) - \frac{1}{\Gamma(q)} \int_a^b (b - s)^{q-1} f(s, x_m(\lambda)(s), x_m(\lambda)(x_m^v(\lambda)(s)), \lambda) ds = 0$$

or

$$\Upsilon_M(\lambda) = \psi(b) - \varphi(a) - \frac{1}{\Gamma(q)} \int_a^b (b - s)^{q-1} f(s, x_M(\lambda)(s), x_M(\lambda)(x_M^v(\lambda)(s)), \lambda) ds = 0.$$

We can use to handle these equations also an analytical-numerical method like in Ronto, 2009. This means that first successive approximation is used

$$x_{n+1}(\lambda, t) = A(x_n(\lambda, t), \lambda)$$

for up to some order j with either $x_0(\lambda, t) = a_1$ or $x_0(\lambda, t) = b_1$. Then approximations

$$\Upsilon_j(\lambda) = \psi(b) - \varphi(a) - \frac{1}{\Gamma(q)} \int_a^b (b - s)^{q-1} f(s, x_j(\lambda, s), x_j(\lambda, x_j^v(\lambda, s)), \lambda) ds \tag{2.91}$$

of Υ_m and Υ_M are numerically drawn to check if they change the sign over J.

2.6 Notes and Remarks

The results in Section 2.2 are taken from Agarwal, Zhou and He, 2010. The main results in Section 2.3 are adopted from Zhou, Jiao and Li, 2009a. The material in Section 2.4 due to Zhou, Jiao and Li, 2009b. The results in Section 2.5 are taken from Wang, Fečkan and Zhou, 2013c.

Chapter 3

Fractional Ordinary Differential Equations in Banach Spaces

3.1 Introduction

In Chapter 3, we discuss the Cauchy problem of fractional ordinary differential equations in Banach spaces under hypotheses based on Carathéodory condition. The tools used include some classical and modern nonlinear analysis methods such as fixed point theory, measure of noncompactness method, topological degree method and Picard operators technique, etc.

Firstly, we give an example which show that the criteria on existence of solutions for the initial value problem of fractional differential equations in finite-dimensional spaces may not be true in infinite-dimensional cases. It is well known that Peano's theorem of integer order ordinary differential equations is not true in infinite-dimensional Banach spaces. The first result in this direction was obtained by Dieudonne, 1950. In Dieudonne, 1950, he produced an example which showed that Peano's theorem is not true in the space c_0 of sequences which converge to zero. In fact, Peano's theorem of fractional differential equations is also not true in infinite-dimensional Banach spaces. In the following, we shall show that the existence result of nonlocal Cauchy problem for fractional abstract differential equations which has been obtained in N'Guerekata, 2009 is not true in the space c_0.

Example 3.1. Let $E = c_0 = \{z = (z_1, z_2, z_3, ...) : z_n \to 0 \text{ as } n \to \infty\}$ with the norm $\|z\| = \sup_{n\geq 1} |z_n|$ and $f(z) = 2(\sqrt{|z_1|}, \sqrt{|z_2|}, \sqrt{|z_3|}, ...)$ with $z = (z_1, z_2, z_3, ...) \in c_0$. Consider the nonlocal Cauchy problem for fractional differential equations given by

$$ {}_0^C D_t^q x(t) = f(x(t)), \quad x(0) = \xi, \quad t \in (0, t_0] \tag{3.1}$$

where ${}_0^C D_t^q$ is Caputo fractional derivative of order $0 < q < 1$, $\xi = (1, 1/2^2, 1/3^2, ...) \in c_0$, $t_0 < (\frac{\Gamma(1+q)}{2})^{\frac{1}{q}}$.

It is obvious that $f : c_0 \to c_0$ is continuous. According to N'Guerekata, 2009, there exists a constant $k^* = \frac{\Gamma(1+q)}{\Gamma(1+q)-2t_0^q}$, such that IVP (3.1) possesses at least one continuous solution $x \in C([0, t_0], c_0)$ and $x(t) = (x_1(t), x_2(t), x_3(t), ...) \in c_0$ on $[0, t_0]$ with $\sup_{t\in[0,t_0]} \|x(t)\| \leq k^*$. According to the definition of the norm of c_0, we can conclude that

$$ {}_0^C D_t^q x_n(t) = 2\sqrt{|x_n(t)|}, \quad x_n(0) = \frac{1}{n^2}, \quad t \in (0, t_0], \quad n = 1, 2, 3, ..., \tag{3.2} $$

where x_n satisfies that $x_n \in C([0, t_0], \mathbb{R})$ with $\sup_{t\in[0,t_0]} |x_n(t)| \le k^*$.

Let us consider Eq. (3.2) which can be written as the following equivalent form

$$ x_n(t) = \frac{1}{n^2} + 2{}_0D_t^{-q}\sqrt{|x_n(t)|} = \frac{1}{n^2} + \frac{2}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sqrt{|x_n(s)|}ds, \quad t \in [0, t_0]. \tag{3.3} $$

Since $(t-s)^{q-1} > 1$ with $s \in [0, t)$ for $t \in (0, t_0]$, we have by (3.3)

$$ x_n(t) \ge \frac{1}{n^2} + \frac{2}{\Gamma(q)}\int_0^t \sqrt{|x_n(s)|}ds, \quad t \in [0, t_0], \quad n = 1, 2, 3, ... \tag{3.4} $$

Assume that $y_n \in C([0, t_0], \mathbb{R})$ is a solution of the following integral equation

$$ y_n(t) = \frac{1}{4n^2} + \frac{2}{\Gamma(q)}\int_0^t \sqrt{|y_n(s)|}ds, \quad t \in [0, t_0], \quad n = 1, 2, 3, ... \tag{3.5} $$

Then, we get

$$ x_n(t) \ge y_n(t), \quad t \in [0, t_0], \quad n = 1, 2, 3, ... \tag{3.6} $$

In fact, suppose (for contraction) that the conclusion (3.6) is not true. Then, because of the continuity of x and y, and that $x_n(0) > y_n(0)$, it follows that there exists a $t_1 \in (0, t_0]$ such that

$$ x_n(t_1) = y_n(t_1), \quad x_n(t) > y_n(t) \quad t \in [0, t_1), \quad n = 1, 2, 3, ... \tag{3.7} $$

Then using (3.4) and (3.7), we get

$$ \begin{aligned} y_n(t_1) &= \frac{1}{4n^2} + \frac{2}{\Gamma(q)}\int_0^{t_1} \sqrt{|y_n(s)|}ds \\ &< \frac{1}{n^2} + \frac{2}{\Gamma(q)}\int_0^{t_1} \sqrt{|x_n(s)|}ds \\ &\le x_n(t_1), \qquad n = 1, 2, 3, ..., \end{aligned} $$

which is a contraction in view of (3.7). Hence the conclusion (3.6) is valid.

Since the integral (3.5) is equivalent to the following IVP

$$ y_n'(t) = \frac{2}{\Gamma(q)}\sqrt{|y_n(t)|}, \quad y_n(0) = \frac{1}{4n^2}, \quad t \in [0, t_0], \quad n = 1, 2, 3, ..., \tag{3.8} $$

and noting $y_n(t) > 0, \ t \in [0, t_0]$, we can conclude that IVP (3.8) has a continuous solution

$$ y_n(t) = \left(\frac{t}{\Gamma(q)} + \frac{1}{2n}\right)^2, \quad t \in [0, t_0], \quad n = 1, 2, 3, ..., $$

which means that

$$ x_n(t) \ge y_n(t) = \left(\frac{t}{\Gamma(q)} + \frac{1}{2n}\right)^2, \quad t \in [0, t_0], \quad n = 1, 2, 3, ... \tag{3.9} $$

Therefore, for $t \in (0, t_0]$, $\lim_{n\to\infty} x_n(t) \ne 0$ by (3.9), contracting $x(t) \in c_0$. Hence IVP (3.1) has no nonlocal solution in c_0.

3.2 Cauchy Problems via Measure of Noncompactness Method

3.2.1 *Introduction*

In Section 3.2, we assume that X is a Banach space with the norm $|\cdot|$. Let $J \subset \mathbb{R}$. Denote $C(J, X)$ be the Banach space of continuous functions from J into X.

Let $r > 0$ and $\mathcal{C} = C([-r, 0], X)$ be the space of continuous functions from $[-r, 0]$ into X. For any element $z \in \mathcal{C}$, define the norm $\|z\|_* = \sup_{\theta\in[-r,0]} |z(\theta)|$.

Consider the initial value problem (IVP) for fractional functional differential equation given by

$$\begin{cases} {}_0^C D_t^q x(t) = f(t, x_t), \quad t \in (0, a), \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \tag{3.10}$$

where ${}_0^C D_t^q$ is Caputo fractional derivative of order $0 < q < 1$, f: $[0, a) \times \mathcal{C} \to X$ is a given function satisfying some assumptions and define x_t by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$.

In this section, we shall discuss the existence of the solutions for IVP (3.10) under assumptions that f satisfies Carathéodory condition and the condition on measure of noncompactness. Then, we give an example to illustrate the application of our abstract results.

Definition 3.2. A function $x \in C([-r, T], X)$ is a solution for IVP (3.10) on $[-r, T]$ for $T \in (0, a)$ if

(i) the function $x(t)$ is absolutely continuous on $[0, T]$;
(ii) $x_0 = \varphi$;
(iii) x satisfies the equation in (3.10).

3.2.2 *Existence*

We are now ready to prove the existence of the solutions for IVP (3.10) under the following hypotheses:

(H1) For almost all $t \in [0, a)$, the function $f(t, \cdot) : \mathcal{C} \to X$ is continuous and for each $z \in \mathcal{C}$, the function $f(\cdot, z) : [0, a) \to X$ is strongly measurable;

(H2) for each $\tau > 0$, there exist a constant $q_1 \in [0, q)$ and $m_1 \in L^{\frac{1}{q_1}}([0, a), \mathbb{R}^+)$ such that $|f(t, z)| \le m_1(t)$ for all $z \in \mathcal{C}$ with $\|z\|_* \le \tau$ and almost all $t \in [0, a)$;

(H3) there exist a constant $q_2 \in (0, q)$ and $m_2 \in L^{\frac{1}{q_2}}([0, a), \mathbb{R}^+)$ such that $\alpha(f(t, B)) \le m_2(t)\alpha(B)$ for almost all $t \in [0, a)$ and B a bounded set in $\mathcal{C}$.

In order to prove the existence theorem, we need the following lemma.

Lemma 3.3. Assume that the hypotheses (H1) and (H2) hold. $x \in C([-r, T], X)$ is a solution for IVP (3.10) on $[-r, T]$ for $T \in (0, a)$ if and only if x satisfies the following relation

$$\begin{cases} x(\theta) = \varphi(\theta), & \text{for } \theta \in [-r, 0], \\ x(t) = \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(s, x_s) ds, & \text{for } t \in [0, T]. \end{cases} \tag{3.11}$$

Proof. Since x_t is continuous in $t \in [0,a)$, according to (H1), $f(t,x_t)$ is a measurable function in $[0,a)$. Direct calculation gives that $(t-s)^{q-1} \in L^{\frac{1}{1-q_1}}[0,t]$ for $t \in (0,a)$ and $q_1 \in [0,q)$. Let

$$b_1 = \frac{q-1}{1-q_1} \in (-1,0), \qquad M = \|m_1\|_{L^{\frac{1}{q_1}}[0,a)}.$$

By using Hölder inequality and (H2), for $t \in (0,a)$, we obtain that

$$\begin{aligned} \int_0^t |(t-s)^{q-1} f(s,x_s)| ds &\le \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,t]} \\ &\le \frac{M}{(1+b_1)^{1-q_1}} \, a^{(1+b_1)(1-q_1)}. \end{aligned} \tag{3.12}$$

Thus, $|(t-s)^{q-1} f(s,x_s)|$ is Lebesgue integrable with respect to $s \in [0,t)$ for all $t \in (0,a)$. From Lemma 1.4 (Bochner's theorem), it follows that $(t-s)^{q-1} f(s,x_s)$ is Bochner integrable with respect to $s \in [0,t)$ for all $t \in (0,a)$.

Let $L(\tau,s) = (t-\tau)^{-q} |\tau - s|^{q-1} m_1(s)$. Since $L(\tau,s)$ is a nonnegative, measurable function on $D = [0,t] \times [0,t]$, then we have

$$\int_0^t \left[\int_0^t L(\tau,s) ds \right] d\tau = \int_D L(\tau,s) ds d\tau = \int_0^t \left[\int_0^t L(\tau,s) d\tau \right] ds$$

and

$$\begin{aligned} \int_D L(\tau,s) ds d\tau &= \int_0^t \left[\int_0^t L(\tau,s) ds \right] d\tau \\ &= \int_0^t (t-\tau)^{-q} \left[\int_0^t |\tau - s|^{q-1} m_1(s) ds \right] d\tau \\ &= \int_0^t (t-\tau)^{-q} \left[\int_0^\tau (\tau - s)^{q-1} m_1(s) ds \right] d\tau \\ &\quad + \int_0^t (t-\tau)^{-q} \left[\int_\tau^t (s-\tau)^{q-1} m_1(s) ds \right] d\tau \\ &\le \frac{2M}{(1+b_1)^{1-q_1}} \, a^{(1+b_1)(1-q_1)} \int_0^t (t-\tau)^{-q} d\tau \\ &\le \frac{2M}{(1-q)(1+b_1)^{1-q_1}} \, a^{(1+b_1)(1-q_1)+1-q}. \end{aligned}$$

Therefore, $L_1(\tau,s) = (t-\tau)^{-q}(\tau - s)^{q-1} f(s,x_s)$ is a Bochner integrable function on $D = [0,t] \times [0,t]$, then we have

$$\int_0^t d\tau \int_0^\tau L_1(\tau,s) ds = \int_0^t ds \int_s^t L_1(\tau,s) d\tau.$$

We now prove that

$${}_0D_t^q \Big({}_0D_t^{-q} f(t,x_t) \Big) = f(t,x_t), \quad \text{for } t \in (0,T],$$

where ${}_0D_t^q$ is Riemann-Liouville fractional derivative.

Indeed, we have

$$\begin{aligned}
{}_0D_t^q\Big({}_0D_t^{-q}f(t,x_t)\Big) &= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t (t-\tau)^{-q}\left[\int_0^\tau (\tau-s)^{q-1}f(s,x_s)ds\right]d\tau \\
&= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t d\tau\int_0^\tau L_1(\tau,s)ds \\
&= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t ds\int_s^t L_1(\tau,s)d\tau \\
&= \frac{1}{\Gamma(1-q)\Gamma(q)}\frac{d}{dt}\int_0^t f(s,x_s)ds\int_s^t (t-\tau)^{-q}(\tau-s)^{q-1}d\tau \\
&= \frac{d}{dt}\int_0^t f(s,x_s)ds \\
&= f(t,x_t) \qquad \text{for} \ \ t\in(0,T].
\end{aligned}$$

If x satisfies the relation (3.11), then we can get that $x(t)$ is absolutely continuous on $[0,T]$. In fact, for any disjoint family of open intervals $\{(c_i,d_i)\}_{1\le i\le n}$ in $[0,T]$ with $\sum_{i=1}^n(d_i-c_i)\to 0$, we have

$$\begin{aligned}
&\sum_{i=1}^n |x(d_i)-x(c_i)| \\
&= \sum_{i=1}^n \frac{1}{\Gamma(q)}\left|\int_0^{d_i}(d_i-s)^{q-1}f(s,x_s)ds - \int_0^{c_i}(c_i-s)^{q-1}f(s,x_s)ds\right| \\
&\le \sum_{i=1}^n \frac{1}{\Gamma(q)}\left|\int_{c_i}^{d_i}(d_i-s)^{q-1}f(s,x_s)ds\right| \\
&\quad + \sum_{i=1}^n \frac{1}{\Gamma(q)}\left|\int_0^{c_i}(d_i-s)^{q-1}f(s,x_s)ds - \int_0^{c_i}(c_i-s)^{q-1}f(s,x_s)ds\right| \\
&\le \sum_{i=1}^n \frac{1}{\Gamma(q)}\int_{c_i}^{d_i}(d_i-s)^{q-1}m_1(s)ds \\
&\quad + \sum_{i=1}^n \frac{1}{\Gamma(q)}\int_0^{c_i}((c_i-s)^{q-1}-(d_i-s)^{q-1})m_1(s)ds \\
&\le \sum_{i=1}^n \frac{1}{\Gamma(q)}\left(\int_{c_i}^{d_i}(d_i-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&\quad + \sum_{i=1}^n \frac{1}{\Gamma(q)}\left(\int_0^{c_i}(c_i-s)^{\frac{q-1}{1-q_1}}-(d_i-s)^{\frac{q-1}{1-q_1}}ds\right)^{1-q_1}\|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&= \sum_{i=1}^n \frac{(d_i-c_i)^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}}\ \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&\quad + \sum_{i=1}^n \frac{(c_i^{1+b_1}-d_i^{1+b_1}+(d_i-c_i)^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}}\ \|m_1\|_{L^{\frac{1}{q_1}}[0,T]}
\end{aligned}$$

$$\leq 2\sum_{i=1}^{n} \frac{(d_i - c_i)^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \, \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \to 0.$$

Therefore, $x(t)$ is absolutely continuous on $[0,T]$, which implies that $x(t)$ is differentiable a.e. on $[0,T]$. According to the argument above and Definition 1.8, for $t \in (0,T]$, we have

$$\begin{aligned}
{}_0^C D_t^q x(t) &= {}_0^C D_t^q \left[\varphi(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x_s)ds\right] \\
&= {}_0^C D_t^q \left[\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x_s)ds\right] \\
&= {}_0^C D_t^q \Big({}_0D_t^{-q} f(t,x_t)\Big) \\
&= {}_0D_t^q \Big({}_0D_t^{-q} f(t,x_t)\Big) - [{}_0D_t^{-q} f(t,x_t)]_{t=0} \frac{t^{-q}}{\Gamma(1-q)} \\
&= f(t,x_t) - [{}_0D_t^{-q} f(t,x_t)]_{t=0} \frac{t^{-q}}{\Gamma(1-q)} \, .
\end{aligned}$$

Since $(t-s)^{q-1} f(s,x_s)$ is Lebesgue integrable with respect to $s \in [0,t)$ for all $t \in (0,T]$, we know that $[{}_0D_t^{-q} f(t,x_t)]_{t=0} = 0$, which means that ${}_0^C D_t^q x(t) = f(t,x_t)$, for $t \in (0,T]$. Hence, $x \in C([-r,T],X)$ is a solution of IVP (3.10). On the other hand, it is obvious that if $x \in C([-r,T],X)$ is a solution of IVP (3.10), then x satisfies the relation (3.11), and this completes the proof. □

Theorem 3.4. Assume that hypotheses (H1)-(H3) hold. Then, for every $\varphi \in \mathcal{C}$, there exists a solution $x \in C([-r,T],X)$ for IVP (3.10) with some $T \in (0,a)$.

Proof. Let $k > 0$ be any number and we can choose $T \in (0,a)$ such that

$$\frac{T^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \, \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \leq k \tag{3.13}$$

and

$$\frac{T^{(1+b_2)(1-q_2)}}{\Gamma(q)(1+b_2)^{1-q_2}} \, \|m_2\|_{L^{\frac{1}{q_2}}[0,T]} < 1, \tag{3.14}$$

where $b_i = \dfrac{q-1}{1-q_i} \in (-1,0), \ i = 1,2.$

Consider the set B_k defined as follows

$$B_k = \left\{x \in C([-r,T],X) : x_0 = \varphi, \sup_{s\in[0,T]} |x(s) - \varphi(0)| \leq k\right\}.$$

Define the operator F on B_k as follows

$$\begin{cases} Fx(\theta) = \varphi(\theta), & \text{for } \theta \in [-r,0], \\ Fx(t) = \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(t,x_s)ds, & \text{for } t \in [0,T], \end{cases}$$

where $x \in B_k$. We prove that the operator equation $x = Fx$ has a solution $x \in B_k$, which means that x is a solution of IVP (3.10).

Firstly, we observe that for every $y \in B_k$, $(Fy)(t)$ is continuous on $t \in [-r, T]$ and for $t \in [0, T]$, by (3.13) and Hölder inequality, we have

$$\begin{aligned} |(Fy)(t) - \varphi(0)| &\leq \frac{1}{\Gamma(q)} \int_0^t |(t-s)^{q-1} f(s, y_s)| ds \\ &\leq \frac{1}{\Gamma(q)} \left(\int_0^t (t-s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\ &\leq \frac{T^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \ \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\ &\leq k, \end{aligned} \tag{3.15}$$

where $b_1 = \frac{q-1}{1-q_1} \in (-1, 0)$. Thus, $\sup_{t \in [0,T]} |(Fy)(t) - \varphi(0)| \leq k$, which implies that $F : B_k \to B_k$.

Further, we prove that F is a continuous operator on B_k. Let $\{y^n\} \subseteq B_k$ with $y^n \to y$ on B_k. Then by (H1) and the fact that $y_t^n \to y_t$, $t \in [0, T]$, we have

$$f(s, y_s^n) \to f(s, y_s), \quad \text{a.e. } s \in [0, T], \quad \text{as } n \to \infty.$$

Noting that $(t-s)^{q-1}|f(s, y_s^n) - f(s, y_s)| \leq (t-s)^{q-1} 2m_1(s)$, by Lebesgue's dominated convergence theorem, as $n \to \infty$, we have

$$|(Fy^n)(t) - (Fy)(t)| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, y_s^n) - f(s, y_s)| ds \to 0.$$

Therefore $Fy^n \to Fy$ as $n \to \infty$ which implies that F is continuous.

For each $n \geq 1$, we define a sequence $\{x^n : n \geq 1\}$ in the following way

$$x^n(t) = \begin{cases} \varphi^0(t), & \text{for } \ t \in [-r, \frac{T}{n}], \\ \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^{t-\frac{T}{n}} (t-s)^{q-1} f(t, x_s^n) ds, & \text{for } \ t \in [\frac{T}{n}, T], \end{cases}$$

where $\varphi^0 \in C([-r, a), X)$ denotes the function defined by

$$\varphi^0(t) = \begin{cases} \varphi(t), & \text{for } t \in [-r, 0], \\ \varphi(0), & \text{for } t \in [0, a). \end{cases}$$

Using the similar method as we did in (3.15), we get that $x^n \in B_k$ for all $n \geq 1$.

Let $A = \{x^n : n \geq 1\}$. It follows that the set A is uniformly bounded. Further, we show that the set A is equicontinuous on $[-r, T]$.

If $-r \leq t_1 < t_2 \leq \frac{T}{n}$, then for each $x^n \in A$, we have $\lim_{t_1 \to t_2} |x^n(t_2) - x^n(t_1)| = \lim_{t_1 \to t_2} |\varphi^0(t_2) - \varphi^0(t_1)| = 0$ independently of $x^n \in A$. Next, if $-r \leq t_1 \leq \frac{T}{n} <$

$t_2 \leq T$, then for each $x^n \in A$, by using Hölder inequality, we have

$$
\begin{aligned}
&|x^n(t_2) - x^n(t_1)| \\
&\leq |\varphi(0) - \varphi^0(t_1)| + \left| \frac{1}{\Gamma(q)} \int_0^{t_2 - \frac{T}{n}} (t_2 - s)^{q-1} f(s, x_s^n) ds \right| \\
&\leq |\varphi(0) - \varphi^0(t_1)| + \frac{1}{\Gamma(q)} \int_0^{t_2 - \frac{T}{n}} (t_2 - s)^{q-1} m_1(s) ds \\
&\leq |\varphi(0) - \varphi^0(t_1)| + \frac{1}{\Gamma(q)} \left(\int_0^{t_2 - \frac{T}{n}} (t_2 - s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&= |\varphi(0) - \varphi^0(t_1)| + \frac{(t_2^{1+b_1} - (\frac{T}{n})^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]}.
\end{aligned}
$$

According to the definition of φ^0, and using the last inequality, we obtain that $|x^n(t_2) - x^n(t_1)| \to 0$ independently of $x^n \in A$, as $t_1 \to t_2$.

Finally, if $\frac{T}{n} \leq t_1 < t_2 \leq T$, then for each $x^n \in A$, by using Hölder inequality, we have

$$
\begin{aligned}
&|x^n(t_2) - x^n(t_1)| \\
&= \left| \frac{1}{\Gamma(q)} \int_0^{t_2 - \frac{T}{n}} (t_2 - s)^{q-1} f(s, x_s^n) ds - \frac{1}{\Gamma(q)} \int_0^{t_1 - \frac{T}{n}} (t_1 - s)^{q-1} f(s, x_s^n) ds \right| \\
&\leq \left| \frac{1}{\Gamma(q)} \int_{t_1 - \frac{T}{n}}^{t_2 - \frac{T}{n}} (t_2 - s)^{q-1} f(s, x_s^n) ds \right| + \left| \frac{1}{\Gamma(q)} \int_0^{t_1 - \frac{T}{n}} (t_2 - s)^{q-1} f(s, x_s^n) ds \right. \\
&\quad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1 - \frac{T}{n}} (t_1 - s)^{q-1} f(s, x_s^n) ds \right| \\
&\leq \frac{1}{\Gamma(q)} \int_{t_1 - \frac{T}{n}}^{t_2 - \frac{T}{n}} (t_2 - s)^{q-1} m_1(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^{t_1 - \frac{T}{n}} ((t_1 - s)^{q-1} - (t_2 - s)^{q-1}) m_1(s) ds \\
&\leq \frac{1}{\Gamma(q)} \left(\int_{t_1 - \frac{T}{n}}^{t_2 - \frac{T}{n}} (t_2 - s)^{\frac{q-1}{1-q_1}} ds \right)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&\quad + \frac{1}{\Gamma(q)} \left(\int_0^{t_1 - \frac{T}{n}} \left((t_1 - s)^{\frac{q-1}{1-q_1}} - (t_2 - s)^{\frac{q-1}{1-q_1}} \right) ds \right)^{1-q_1} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&= \frac{((t_2 - t_1 + \frac{T}{n})^{1+b_1} - (\frac{T}{n})^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&\quad + \frac{(t_1^{1+b_1} - (\frac{T}{n})^{1+b_1} - t_2^{1+b_1} + (t_2 - t_1 + \frac{T}{n})^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} \\
&\leq 2 \frac{((t_2 - t_1 + \frac{T}{n})^{1+b_1} - (\frac{T}{n})^{1+b_1})^{1-q_1}}{\Gamma(q)(1+b_1)^{1-q_1}} \|m_1\|_{L^{\frac{1}{q_1}}[0,T]}.
\end{aligned}
$$

It is easy to see that the last inequality tends to zero independently of $x^n \in A$, as $t_1 \to t_2$, which means that the set A is equicontinuous.

Set $A(t) = \{x^n(t) : n \geq 1\}$ and $A_t = \{x_t^n : n \geq 1\}$ for any $t \in [0,T]$. By the properties (iv) and (vi) of the measure of noncompactness, for any fixed $t \in (0,T]$ and $\delta \in (0,t)$, we have

$$\begin{aligned}\alpha(A(t)) \leq \alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_s^n)ds : n \geq 1\bigg\}\bigg)\\ + \alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-\frac{T}{n}}^t (t-s)^{q-1} f(s,x_s^n)ds : n \geq 1\bigg\}\bigg),\end{aligned}$$

$\forall\epsilon > 0$, we can find δ sufficiently small such that

$$\frac{\delta^{(1+b_1)(1-q_1)}}{\Gamma(q)(1+b_1)^{1-q_1}} \ \|m_1\|_{L^{\frac{1}{q_1}}[0,T]} < \frac{\epsilon}{2}.$$

Therefore, for each $t \in (0,T]$, we can choose $N_\delta \geq 1$ such that $\frac{T}{n} \leq \delta$ for $n \geq N_\delta$. Then we obtain that

$$\begin{aligned}&\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-\frac{T}{n}}^t (t-s)^{q-1} f(s,x_s^n)ds : n \geq N_\delta\bigg\}\bigg)\\ &\leq \frac{2}{\Gamma(q)}\sup_{n\geq N_\delta}\int_{t-\frac{T}{n}}^t (t-s)^{q-1} m_1(s)ds\\ &< \epsilon,\end{aligned}$$

for each $t \in (0,T]$. Hence, by the properties (iii) and (v) of the measure of noncompactness, it follows that

$$\alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_{t-\frac{T}{n}}^t (t-s)^{q-1} f(s,x_s^n)ds : n \geq 1\bigg\}\bigg) < \epsilon.$$

Then, we obtain that

$$\alpha(A(t)) \leq \alpha\bigg(\bigg\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_s^n)ds : n \geq 1\bigg\}\bigg) + \epsilon,$$

for $t \in (0,T]$. By Property 1.28 and (H3), we have that

$$\begin{aligned}\alpha(A(t)) &\leq \frac{2}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(f(s,A_s))ds + \epsilon\\ &\leq \frac{2}{\Gamma(q)}\int_0^t (t-s)^{q-1} m_2(s)\alpha(A_s)ds + \epsilon,\end{aligned}$$

where $t \in (0,T]$. Since $x^n(\theta) = \varphi(\theta),\ \theta \in [-r,0]$, we have $\alpha(\{x^n(\theta) : n \geq 1\}) = 0$ for $\theta \in [-r,0]$. Moreover, by Property 1.27, for $s \in [0,t]$ with $t \in (0,T]$, we deduce that

$$\alpha(A_s) = \max_{\theta\in[-r,0]}\alpha(\{x_s^n(\theta) : n \geq 1\}) \leq \sup_{s\in[0,t]}\alpha(\{x^n(s) : n \geq 1\}) = \sup_{s\in[0,t]}\alpha(A(s)).$$

Since ϵ is arbitrary, we have that

$$\alpha(A(t)) \le \frac{2T^{(1+b_2)(1-q_2)}}{\Gamma(q)(1+b_2)^{1-q_2}} \, \|m_2\|_{L^{\frac{1}{q_2}}[0,T]} \sup_{s\in[0,t]} \alpha(A(s)),$$

where $t \in (0,T]$ and $b_2 = \frac{q-1}{1-q_2} \in (-1,0)$.

Since (3.14) and $x_0^n = \varphi$, we must have that $\alpha(A(t)) = 0$ for every $t \in [-r,T]$. Then, by Property 1.27, we have that $\alpha(A) = \sup_{t\in[-r,T]} \alpha(A(t)) = 0$. Therefore, A is a relatively compact subset of B_k. Then, there exists a subsequence if necessary, we may assume that the sequence $\{x^n\}_{n\ge 1}$ converges uniformly on $[-r,T]$ to a continuous function $x \in B_k$ with $x(\theta) = \varphi(\theta),\ \theta \in [-r,0]$.

Moreover, for $t \in [0, \frac{T}{n}]$, we have

$$|(Fx^n)(t) - x^n(t)| \le \frac{1}{\Gamma(q)} \int_0^{\frac{T}{n}} (t-s)^{q-1} |f(t,x_s^n)| ds \le \frac{1}{\Gamma(q)} \int_0^{\frac{T}{n}} (t-s)^{q-1} m_1(s) ds$$

and for $t \in [\frac{T}{n}, T]$, we have

$$\begin{aligned} |(Fx^n)(t) - x^n(t)| &= \frac{1}{\Gamma(q)} \left| \int_0^t (t-s)^{q-1} f(t,x_s^n) ds - \int_0^{t-\frac{T}{n}} (t-s)^{q-1} f(t,x_s^n) ds \right| \\ &= \frac{1}{\Gamma(q)} \left| \int_{t-\frac{T}{n}}^t (t-s)^{q-1} f(t,x_s^n) ds \right| \\ &\le \frac{1}{\Gamma(q)} \int_{t-\frac{T}{n}}^t (t-s)^{q-1} m_1(s) ds. \end{aligned}$$

Therefore, it follows that

$$\sup_{t\in[0,T]} |(Fx^n)(t) - x^n(t)| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.16}$$

Since

$$\begin{aligned} \sup_{t\in[0,T]} |(Fx)(t) - x(t)| \le & \sup_{t\in[0,T]} |(Fx)(t) - (Fx^n)(t)| \\ & + \sup_{t\in[0,T]} |(Fx^n)(t) - x^n(t)| + \sup_{t\in[0,T]} |x^n(t) - x(t)|, \end{aligned}$$

then, by (3.16) and the fact that F is a continuous operator, we obtain that $\sup_{t\in[0,T]} |(Fx)(t) - x(t)| = 0$. It follows that $x(t) = (Fx)(t)$ for every $t \in [0,T]$. Hence

$$x(t) = \begin{cases} \varphi(t), & \text{for } \ t \in [-r,0], \\ \varphi(0) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} f(t,x_s) ds, & \text{for } \ t \in [0,T] \end{cases}$$

solve IVP (3.10), and this completes the proof. □

Corollary 3.5. Assume that hypotheses (H1)-(H3) hold. Then, for every $\varphi \in \mathcal{C}$, there exist $T \in (0,a)$ and a sequence of continuous function $x^n : [-r,T] \to X$, such that

(i) $x^n(t)$ are absolutely continuous on $[0,T]$;
(ii) $x_0^n = \varphi$, for every $n \geq 1$, and
(iii) extracting a subsequence which is labeled in the same way such that $x^n(t) \to x(t)$ uniformly on $[-r,T]$ and $x : [-r,T] \to X$ is a solution for IVP (3.10).

We now give an example to illustrate the application of our abstract results.

Example 3.6. Consider the infinite system of fractional functional differential equations

$$\begin{cases} {}_0^C D_t^{\frac{1}{2}} x_n(t) = \dfrac{1}{nt^{1/3}} x_n^2(t-r), & \text{for } t \in (0,a), \\ x_n(\theta) = \varphi(\theta) = \dfrac{\theta}{n}, & \text{for } \theta \in [-r,0],\ n = 1,2,3,\dots . \end{cases} \tag{3.17}$$

Let $E = c_0 = \{x = (x_1, x_2, x_3, \dots) : x_n \to 0\}$ with norm $|x| = \sup_{n\geq 1} |x_n|$. Then the infinite system (3.17) can be regarded as a IVP of form (3.10) in E. In this situation, $q = \frac{1}{2}$, $x = (x_1, \dots, x_n, \dots)$, $x_t = x(t-r) = (x_1(t-r), \dots, x_n(t-r), \dots)$, $\varphi(\theta) = (\theta, \frac{\theta}{2}, \dots, \frac{\theta}{n}, \dots)$ for $\theta \in [-r,0]$ and $f = (f_1, \dots, f_n, \dots)$, in which

$$f_n(t, x_t) = \frac{1}{nt^{1/3}} x_n^2(t-r). \tag{3.18}$$

It is obvious that conditions (H1) and (H2) are satisfied. Now, we check the condition (H3) and the argument is similar to Section 2.4. Let $t \in (0,a)$, $R > 0$ be given and $\{w^{(m)}\}$ be any sequence in $f(t,B)$, where $w^{(m)} = (w_1^{(m)}, \dots, w_n^{(m)}, \dots)$ and $B = \{z \in \mathcal{C}\colon \|z\|_* \leq R\}$ is a bounded set in $\mathcal{C}$. By (3.18), we have

$$0 \leq w_n^{(m)} \leq \frac{R^2}{nt^{1/3}}, \quad n, m = 1,2,3,\dots . \tag{3.19}$$

So, $\{w_n^{(m)}\}$ is bounded and, by the diagonal method, we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$w_n^{(m_i)} \to w_n \ \text{ as } \ i \to \infty, \quad n = 1,2,3,\dots, \tag{3.20}$$

which implies by virtue of (3.19) that

$$0 \leq w_n \leq \frac{R^2}{nt^{1/3}}, \quad n = 1,2,3,\dots . \tag{3.21}$$

Hence $w = (w_1, \dots, w_n, \dots) \in c_0$. It is easy to see from (3.19)-(3.21) that

$$|w^{(m_i)} - w| = \sup_n |w_n^{(m_i)} - w_n| \to 0 \quad \text{as } i \to \infty. \tag{3.22}$$

Thus, we have proved that $f(t,B)$ is relatively compact in c_0 for $t \in (0,a)$, which means that $f(t,B) = 0$ for almost all $t \in [0,a)$ and B a bounded set in $\mathcal{C}$. Hence, the condition (H3) is satisfied. Finally, from Theorem 3.4, we can conclude that the infinite system (3.17) has a continuous solution.

3.3 Cauchy Problems via Topological Degree Method

3.3.1 *Introduction*

It is well known that the topological methods proved to be a powerful tool in the study of various problems which appear in nonlinear analysis. Particularly, a priori estimate method has been often used together with the Brouwer degree, the Leray-Schauder degree or the coincidence degree in order to prove the existence of solutions for some boundary value problems and bifurcation problems for nonlinear differential equations or nonlinear partial differential equations. See, for example, Fečkan, 2008; Mawhin, 1979.

In Section 3.3, we will consider the following nonlocal problem via a coincidence degree for condensing mapping in a Banach space X

$$\begin{cases} {}_0^C D_t^q u(t) = f(t, u(t)),\ t \in J := [0, T], \\ u(0) + g(u) = u_0, \end{cases} \tag{3.23}$$

where ${}_0^C D_t^q$ is Caputo fractional derivative of order $q \in (0, 1)$, u_0 is an element of X, $f : J \times X \to X$ is continuous. The nonlocal term $g : C(J, X) \to X$ is a given function, here $C(J, X)$ is the Banach space of all continuous functions from J into X with the norm $\|u\| := \sup_{t\in J} |u(t)|$ for $u \in C(J, X)$.

3.3.2 *Qualitative Analysis*

This subsection deals with existence of solutions for the nonlocal problem (3.23).

Definition 3.7. A function $u \in C^1(J, X)$ is said to be a solution of the nonlocal problem (3.23.) if u satisfies the equation ${}_0^C D_t^q u(t) = f(t, u(t))$ a.e. on J, and the condition $u(0) + g(u) = u_0$.

Lemma 3.8. A function $u \in C(J, X)$ is a solution of the fractional integral equation

$$u(t) = u_0 - g(u) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s, u(s))ds, \tag{3.24}$$

if and only if u is a solution of the nonlocal problem (3.23).

We make some following assumptions:

(H1) for arbitrary $u, v \in C(J, X)$, there exists a constant $K_g \in [0, 1)$ such that

$$|g(u) - g(v)| \le K_g \|u - v\|;$$

(H2) for arbitrary $u \in C(J, X)$, there exist $C_g, M_g > 0$, $q_1 \in [0, 1)$ such that

$$|g(u)| \le C_g \|u\|^{q_1} + M_g;$$

(H3) for arbitrary $(t, u) \in J \times X$, there exist $C_f, M_f > 0$, $q_2 \in [0, 1)$ such that

$$|f(t, u)| \le C_f |u|^{q_2} + M_f;$$

(H4) for any $r > 0$, there exists a constant $\beta_r > 0$ such that

$$\alpha(f(s, \mathcal{M})) \le \beta_r \alpha(\mathcal{M}),$$

for all $t \in J$, $\mathcal{M} \subset \mathfrak{B}_r := \{\|u\| \leq r : u \in C(J, X)\}$ and

$$\frac{2T^q \beta_r}{\Gamma(q+1)} < 1.$$

Under the assumptions (H1)-(H4), we will show that fractional integral equation (3.24) has at least one solution $u \in C(J, X)$.

Define operators

$$\begin{aligned}
&F : C(J,X) \to C(J,X), \quad (Fu)(t) = u_0 - g(u), && t \in J,\\
&G : C(J,X) \to C(J,X), \quad (Gu)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, && t \in J,\\
&\mathbb{T} : C(J,X) \to C(J,X), \quad \mathbb{T}u = Fu + Gu.
\end{aligned}$$

It is obvious that $\mathbb{T}$ is well defined. Then, fractional integral equation (3.24) can be written as the following operator equation

$$u = \mathbb{T}u = Fu + Gu. \tag{3.25}$$

Thus, the existence of a solution for the nonlocal problem (3.23) is equivalent to the existence of a fixed point for operator $\mathbb{T}$.

Lemma 3.9. The operator $F : C(J,X) \to C(J,X)$ is Lipschitz with constant K_g. Consequently F is α-Lipschitz with the same constant K_g. Moreover, F satisfies the following growth condition:

$$\|Fu\| \leq |u_0| + C_g \|u\|^{q_1} + M_g, \tag{3.26}$$

for every $u \in C(J, X)$.

Proof. Using (H1), we have $\|Fu - Fv\| \leq |g(u) - g(v)| \leq K_g \|u - v\|$, for every $u, v \in C(J, X)$. By Property 1.33, F is α-Lipschitz with constant K_g. Relation (3.26) is a simple consequence of (H2). □

Lemma 3.10. The operator $G : C(J,X) \to C(J,X)$ is continuous. Moreover, G satisfies the following growth condition:

$$\|Gu\| \leq \frac{T^q (C_f \|u\|^{q_2} + M_f)}{\Gamma(q+1)}, \tag{3.27}$$

for every $u \in C(J, X)$.

Proof. For that, let $\{u_n\}$ be a sequence of a bounded set $\mathfrak{B}_K \subseteq C(J,X)$ such that $u_n \to u$ in $\mathfrak{B}_K (K > 0)$. We have to show that $\|Gu_n - Gu\| \to 0$.

It is easy to see that $f(s, u_n(s)) \to f(s, u(s))$ as $n \to \infty$ due to the continuity of f. On the one hand, using (H3), we get for each $t \in J$, $(t-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))| \leq (t-s)^{q-1} 2(C_f K^{q_2} + M_f)$. On the other hand, using the fact that the function $s \to (t-s)^{q-1} 2(C_f K^{q_2} + M_f)$ is integrable for $s \in [0,t]$, $t \in J$, the Lebesgue's dominated convergence theorem yields $\int_0^t (t-s)^{q-1} |f(s,u_n(s)) - f(s,u(s))| ds \to 0$ as $n \to \infty$. Then, for all $t \in J$,

$$|(Gu_n)(t) - (Gu)(t)| \leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,u_n(s)) - f(s,u(s))| ds \to 0.$$

Therefore, $Gu_n \to Gu$ as $n \to \infty$ which implies that G is continuous. Relation (3.27) is a simple consequence of (H3). □

Lemma 3.11. The operator $G : C(J,X) \to C(J,X)$ is compact. Consequently G is α-Lipschitz with zero constant.

Proof. In order to prove the compactness of G, we consider a bounded set $\mathcal{M} \subseteq C(J,X)$ and the key step is to show that $G(\mathcal{M})$ is relatively compact in $C(J,X)$.

Let $\{u_n\}$ be a sequence on $\mathcal{M} \subset \mathfrak{B}_K$, for every $u_n \in \mathcal{M}$. By relation (3.27), we have

$$\|Gu_n\| \leq \frac{T^q(C_f K^{q_2} + M_f)}{\Gamma(q+1)} =: r,$$

for every $u_n \in \mathcal{M}$, so $G(\mathcal{M})$ is bounded in $\mathfrak{B}_r$.

Now we prove that $\{Gu_n\}$ is equicontinuous. For $0 \leq t_1 < t_2 \leq T$, we get

$$\begin{aligned}
&|(Gu_n)(t_1) - (Gu_n)(t_2)| \\
\leq\ & \frac{1}{\Gamma(q)} \int_0^{t_1} ((t_1 - s)^{q-1} - (t_2 - s)^{q-1})|f(s, u_n(s))|ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1}|f(s, u_n(s))|ds \\
\leq\ & \frac{1}{\Gamma(q)} \int_0^{t_1} ((t_1 - s)^{q-1} - (t_2 - s)^{q-1})(C_f|u_n(s)|^{q_2} + M_f)ds \\
&+ \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1}(C_f|u_n(s)|^{q_2} + M_f)ds \\
\leq\ & \frac{(C_f K^{q_2} + M_f)}{\Gamma(q)} \left[\frac{t_1^q}{q} - \frac{t_2^q}{q} + \frac{(t_2 - t_1)^q}{q} + \frac{(t_2 - t_1)^q}{q}\right] \\
\leq\ & \frac{2(C_f K^{q_2} + M_f)(t_2 - t_1)^q}{\Gamma(q+1)}.
\end{aligned}$$

As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero. Therefore $\{Gu_n\}$ is equicontinuous.

Consider a bounded set

$$\mathcal{M}(t) := \left\{ v_n(t) : v_n(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, v_n(s))ds \right\} \subset \mathfrak{B}_r.$$

Applying Property 1.27, we know that the function $t \to \alpha(\mathcal{M}(t))$ is continuous on J. Moreover,

$$(t-s)^{q-1}|f(s, v_n(s))| \leq (t-s)^{q-1}(C_f r^{q_2} + M_f) \in L^1(J, \mathbb{R}_+),\ \ for\ s \in [0,t],\ t \in J.$$

Using (H4) and Property 1.28, we have

$$\begin{aligned}
\alpha(\mathcal{M}(t)) &\leq \alpha\left(\left\{\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, \mathcal{M}(s))ds\right\}\right) \\
&\leq \frac{2}{\Gamma(q)} \int_0^t (t-s)^{q-1} \alpha\left(f(s, \mathcal{M}(s))\right) ds \\
&\leq \frac{2\beta_r}{\Gamma(q)} \int_0^t (t-s)^{q-1} \alpha(\mathcal{M}(s))ds,
\end{aligned}$$

which implies that

$$\begin{aligned}\alpha(\mathcal{M}) &\leq \left[\frac{2\beta_r}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds\right]\alpha(\mathcal{M})\\ &\leq \frac{2T^q\beta_r}{\Gamma(q+1)}\alpha(\mathcal{M})\\ &< \alpha(\mathcal{M}),\end{aligned}$$

due to the condition

$$\frac{2T^q\beta_r}{\Gamma(q+1)} < 1.$$

Then we can deduce that $\alpha(\mathcal{M}) = 0$. Therefore, $G(\mathcal{M})$ is a relatively compact subset of $C(J,X)$, and so, there exists a subsequence v_n which converge uniformly on J to some $v_* \in C(J,X)$ together with the Arzela-Ascoli theorem. By Property 1.32, G is α-Lipschitz with zero constant. □

Theorem 3.12. Assume that (H1)-(H4) hold, then the nonlocal problem (3.23) has at least one solution $u \in C(J,X)$ and the set of the solutions of system (3.23) is bounded in $C(J,X)$.

Proof. Let $F, G, \mathbb{T} : C(J,X) \to C(J,X)$ be the operators defined in the beginning of this section. They are continuous and bounded. Moreover, F is α-Lipschitz with constant $K_g \in [0,1)$ and G is α-Lipschitz with zero constant (see Lemmas 3.9-3.11). Property 1.31 shows that $\mathbb{T}$ is a strict α-contraction with constant K_g.

Set

$$S_0 = \{u \in C(J,X) : \ (\exists)\ \lambda \in [0,1] \ \text{ such that } \ u = \lambda\mathbb{T}u\}.$$

Next, we prove that S_0 is bounded in $C(J,X)$. Consider $u \in S_0$ and $\lambda \in [0,1]$ such that $u = \lambda\mathbb{T}u$. It follows from (3.26) and (3.27) that

$$\begin{aligned}\|u\| = \lambda\|\mathbb{T}u\| &\leq \lambda(\|Fu\| + \|Gu\|)\\ &\leq |u_0| + C_g\|u\|^{q_1} + M_g + \tfrac{T^q(C_f\|u\|^{q_2}+M_f)}{\Gamma(q+1)}.\end{aligned} \tag{3.28}$$

This inequality (3.28), together with $q_1 < 1$ and $q_2 < 1$, shows that S_0 is bounded in $C(J,X)$. If not, we suppose by contradiction, $\rho := \|u\| \to \infty$. Dividing both sides of (3.28) by ρ, and taking $\rho \to \infty$, we have

$$1 \leq \lim_{\rho\to\infty} \frac{|u_0| + C_g\rho^{q_1} + M_g + \frac{T^q(C_f\rho^{q_2}+M_f)}{\Gamma(q+1)}}{\rho} = 0. \tag{3.29}$$

This is a contradiction. Consequently, by Theorem 1.35 we deduce that $\mathbb{T}$ has at least one fixed point and the set of the fixed points of $\mathbb{T}$ is bounded in $C(J,X)$. □

Remark 3.13.

(i) If the growth condition (H2) is formulated for $q_1 = 1$, then the conclusions of Theorem 3.12 remain valid provided that $C_g < 1$;

(ii) If the growth condition (H3) is formulated for $q_2 = 1$, then the conclusions of Theorem 3.12 remain valid provided that $\frac{T^qC_f}{\Gamma(q+1)} < 1$;

(iii) If the growth conditions (H2) and (H3) are formulated for $q_1 = 1$ and $q_2 = 1$, then the conclusions of Theorem 3.12 remain valid provided that $C_g + \frac{T^qC_f}{\Gamma(q+1)} < 1$.

3.4 Cauchy Problems via Picard Operators Technique

3.4.1 *Introduction*

Assume that $(X, |\cdot|)$ is a Banach space, and $J := [0, T]$, $T > 0$. Let $C(J, X)$ be the Banach space of all continuous functions from J into X with the norm $\|x\| := \sup\{|x(t)| : t \in J\}$ for $x \in C(J, X)$.

Consider the following Cauchy problem of fractional differential equation

$$\begin{cases} {}_0^C D_t^q x(t) = f(t, x(t)), \quad \text{a.e. } t \in J, \\ x(0) = x_0 \in X, \end{cases} \tag{3.30}$$

where ${}_0^C D_t^q$ is Caputo fractional derivative of order $q \in (0, 1)$, the function $f : J \times X \to X$ satisfies some assumptions that will be specified later.

To our knowledge, Picard and weakly Picard operators technique due to Rus 1979, 1987, 1993, 2003; Rus and Muresan, 2000 have been used to study the existence for the solutions of some integer differential equations (see, Mureşan, 2004; Şerban, Rus and Petruşel, 2010). In the present section we consider suitable Bielecki norms in some convenient spaces and obtain existence, uniqueness and data dependence results for the solutions of the fractional Cauchy problem (3.30) via Picard and weakly Picard operators technique.

Definition 3.14. A function $x \in C^1(J, X)$ is said to be a solution of the fractional Cauchy problem (3.30) if x satisfies the equation ${}_0^C D_t^q x(t) = f(t, x(t))$ a.e. on J, and the condition $x(0) = x_0$.

3.4.2 *Results via Picard Operators*

Consider a Banach space $(X, |\cdot|)$, let $\|\cdot\|_B$ and $\|\cdot\|_C$ be the Bielecki and Chebyshev norms on $C(J, X)$ defined by

$$\|x\|_B = \max_{t \in J} |x(t)| e^{-\tau t} (\tau > 0) \quad \text{and} \quad \|x\|_C = \max_{t \in J} |x(t)|$$

and denote by d_B and d_C their corresponding metrics. We consider the set

$$C_L^{q-q^*}(J, X) = \left\{ x \in C(J, X) : |x(t_1) - x(t_2| \le L |t_1 - t_2|^{q-q^*} \text{ for all } t_1, t_2 \in J \right\}$$

where $L > 0$, $q^* \in (0, q)$, and

$$C_{\bar{L}}^q(J, X) = \left\{ x \in C(J, X) : |x(t_1) - x(t_2)| \le \bar{L} |t_1 - t_2|^q \text{ for all } t_1, t_2 \in J \right\}$$

where $\bar{L} > 0$, and

$$C_{\bar{L}}^q(J, B_R) = \left\{ x \in C(J, B_R) : |x(t_1) - x(t_2)| \le \bar{L} |t_1 - t_2|^q \text{ for all } t_1, t_2 \in J \right\}$$

where $B_R = \{x \in X : |x| \le R\}$ with $R > 0$.

If $d \in \{d_C, d_B\}$, then $(C(J, X), d)$, $(C_L^{q-q^*}(J, X), d)$, $(C_{\bar{L}}^q(J, X), d)$ and $(C_{\bar{L}}^q(J, B_R), d)$ are complete metric spaces.

Let $q_i \in (0, q)$, $i = 1, 2, 3$ and the functions $m(t) \in L^{\frac{1}{q_1}}(J, \mathbb{R}_+)$, $\eta(t) \in L^{\frac{1}{q_2}}(J, \mathbb{R}_+)$, $\mu(t) \in L^{\frac{1}{q_3}}(J, \mathbb{R}_+)$ and $l(t) \in C(J, \mathbb{R}_+)$.

For brevity, let

$$M = \|m\|_{L^{\frac{1}{q_1}}J}, \ N = \|\eta\|_{L^{\frac{1}{q_2}}J}, \ V = \|\mu\|_{L^{\frac{1}{q_2}}J}, \ L_0 = \max_{t \in J}\{l(t)\},$$
$$\beta = \frac{q-1}{1-q_1} \in (-1,0), \ \gamma = \frac{q-1}{1-q_2} \in (-1,0), \ \nu = \frac{q-1}{1-q_3} \in (-1,0).$$

Lemma 3.15. A function $x \in C(J,X)$ is a solution of the fractional integral equation

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))ds, \tag{3.31}$$

if and only if x is a solution of the fractional Cauchy problem (3.30).
Theorem 3.16. Suppose the following conditions hold:
(C1) $f \in C(J \times X, X)$;
(C2) there exist a constant $q_1 \in (0,q)$ and function $m(\cdot) \in L^{\frac{1}{q_1}}(J,\mathbb{R}_+)$ such that

$$|f(t,x)| \le m(t)$$

for all $x \in X$ and all $t \in J$;
(C3) There exists a constant $L > 0$ such that

$$L \ge \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}};$$

(C4) there exists a function $l(\cdot) \in C(J,\mathbb{R}_+)$ such that

$$|f(t,u_1) - f(t,u_2)| \le l(t)|u_1 - u_2|$$

for all $u_i \in X$ $(i = 1,2)$ and all $t \in J$;
(C5) there exist constants q_1 and τ such that

$$\frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1.$$

Then the fractional Cauchy problem (3.30) has a unique solution x^* in $C_L^{q-q_1}(J,X)$, and this solution can be obtained by the successive approximation method, starting from any element of $C_L^{q-q_1}(J,X)$.

Proof. Consider the operator

$$A : (C_L^{q-q_1}(J,X), \|\cdot\|_B) \to (C_L^{q-q_1}(J,X), \|\cdot\|_B)$$

defined by

$$Ax(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s,x(s))\, ds.$$

It is easy to see the operator A is well defined due to (C1).

Firstly, we check that $Ax \in C(J,X)$ for every $x \in C_L^{q-q_1}(J,X)$.

For any $\delta > 0$, every $x \in C_L^{q-q_1}(J, X)$, by (C2) and Hölder inequality,

$$
\begin{aligned}
& |(Ax)(t+\delta) - (Ax)(t)| \\
\leq & \frac{1}{\Gamma(q)} \int_0^t ((t-s)^{q-1} - (t+\delta-s)^{q-1})|f(s,x(s))|ds \\
& + \frac{1}{\Gamma(q)} \int_t^{t+\delta} (t+\delta-s)^{q-1}|f(s,x(s))|ds \\
\leq & \frac{1}{\Gamma(q)} \int_0^t ((t-s)^{q-1} - (t+\delta-s)^{q-1})m(s)ds + \frac{1}{\Gamma(q)} \int_t^{t+\delta} (t+\delta-s)^{q-1}m(s)ds \\
\leq & \frac{1}{\Gamma(q)} \left(\int_0^t [(t-s)^{q-1} - (t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^t (m(s))^{\frac{1}{q_1}} ds \right)^{q_1} \\
& + \frac{1}{\Gamma(q)} \left(\int_t^{t+\delta} [(t+\delta-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_t^{t+\delta} (m(s))^{\frac{1}{q_1}} ds \right)^{q_1} \\
\leq & \frac{M}{\Gamma(q)} \left(\int_0^t \left((t-s)^\beta - (t+\delta-s)^\beta \right) ds \right)^{1-q_1} + \frac{M}{\Gamma(q)} \left(\int_t^{t+\delta} (t+\delta-s)^\beta ds \right)^{1-q_1} \\
\leq & \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}} \left(|t^{1+\beta} - (t+\delta)^{1+\beta}| + \delta^{1+\beta} \right)^{1-q_1} + \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}} \delta^{(1+\beta)(1-q_1)} \\
\leq & \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}} \delta^{(1+\beta)(1-q_1)} + \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}} \delta^{(1+\beta)(1-q_1)} \\
\leq & \frac{3M}{\Gamma(q)(1+\beta)^{1-q_1}} \delta^{(1+\beta)(1-q_1)}.
\end{aligned}
$$

It is easy to see that the right-hand side of the above inequality tends to zero as $\delta \to 0$. Therefore $Ax \in C(J, X)$.

Secondly, we show that $Ax \in C_L^{q-q_1}(J, X)$.

Without loss of generality, for any $t_1 < t_2$, $t_1, t_2 \in J$, applying (C2) and Hölder inequality, we have

$$
\begin{aligned}
& |(Ax)(t_2) - (Ax)(t_1)| \\
\leq & \frac{1}{\Gamma(q)} \left| \int_0^{t_1} [(t_2-s)^{q-1} - (t_1-s)^{q-1}] f(s,x(s))ds + \int_{t_1}^{t_2} (t_2-s)^{q-1} f(s,x(s))ds \right| \\
\leq & \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}]|f(s,x(s))|ds \\
& + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1}|f(s,x(s))|ds \\
\leq & \frac{1}{\Gamma(q)} \int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}]m(s)ds + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2-s)^{q-1}m(s)ds \\
\leq & \frac{1}{\Gamma(q)} \left(\int_0^{t_1} [(t_1-s)^{q-1} - (t_2-s)^{q-1}]^{\frac{1}{1-q_1}} ds \right)^{1-q_1} \left(\int_0^{t_1} (m(s))^{\frac{1}{q_1}} ds \right)^{q_1}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q)}\left(\int_{t_1}^{t_2}[(t_2-s)^{q-1}]^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_{t_1}^{t_2}(m(s))^{\frac{1}{q_1}}ds\right)^{q_1}\\
&\le \frac{M}{\Gamma(q)}\left(\int_0^{t_1}\left((t_1-s)^{\beta}-(t_2-s)^{\beta}\right)ds\right)^{1-q_1}+\frac{M}{\Gamma(q)}\left(\int_{t_1}^{t_2}(t_2-s)^{\beta}ds\right)^{1-q_1}\\
&\le \frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}\left(t_1^{1+\beta}-t_2^{1+\beta}+(t_2-t_1)^{1+\beta}\right)^{1-q_1}\\
&\quad+\frac{M}{\Gamma(q)(1+\beta)^{1-q_1}}(t_2-t_1)^{(1+\beta)(1-q_1)}\\
&\le \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}|t_1-t_2|^{(1+\beta)(1-q_1)}\\
&\le \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}|t_1-t_2|^{q-q_1}.
\end{aligned}
$$

Similarly, for any $t_1 > t_2$, $t_1, t_2 \in J$, we also have the above inequality. This implies that Ax is belong to $C_L^{q-q_1}(J,X)$ due to (C3).

Thirdly, A is continuous.

For that, let $\{x_n\}$ be a sequence of B_R such that $x_n \to x$ in B_R. Then, $f(s,x_n(s)) \to f(s,x(s))$ as $n \to \infty$ due to (C1). On the one hand, by using (C2), we get for each $s \in [0,t]$, $|f(s,x_n(s)) - f(s,x(s))| \le 2m(s)$. On the other hand, using the fact that the function $s \to 2(t-s)^{q-1}m(s)$ is integrable on $[0,t]$, the Lebesgue's dominated convergence theorem yields

$$\int_0^t (t-s)^{q-1}|f(s,x_n(s)) - f(s,x(s))|ds \to 0.$$

For all $t \in J$, we have

$$|(Ax_n)(t) - (Ax)(t)| \le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x_n(s)) - f(s,x(s))|ds.$$

Thus, $Ax_n \to Ax$ as $n \to \infty$ which implies that A is continuous.

Moreover, for all $x, z \in C_L^{q-q_1}(J,X)$, using (C4) and Hölder inequality we have

$$
\begin{aligned}
|(Ax)(t)-(Az)(t)| &\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))-f(s,z(s))|ds\\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}l(s)|x(s)-z(s)|ds\\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\max_{s\in[0,t]}\{l(s)\}\left[|x(s)-z(s)|e^{-\tau s}\right]e^{\tau s}ds\\
&\le \frac{L_0}{\Gamma(q)}\|x-z\|_B\int_0^t (t-s)^{q-1}e^{\tau s}ds\\
&\le \frac{L_0}{\Gamma(q)}\|x-z\|_B\left(\int_0^t (t-s)^{\beta}ds\right)^{1-q_1}\left(\int_0^t e^{\frac{\tau s}{q_1}}ds\right)^{q_1}\\
&\le \frac{L_0}{\Gamma(q)}\frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}}\left(\frac{q_1}{\tau}\right)^{q_1}e^{\tau t}\|x-z\|_B.
\end{aligned}
$$

It follows that

$$|(Ax)(t) - (Az)(t)|e^{-\tau t} \leq \frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \|x - z\|_B$$

for all $t \in J$. So we have

$$\|Ax - Az\|_B \leq \frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \|x - z\|_B$$

for all x, $z \in C_L^{q-q_1}(J, X)$. The operator A is of Lipschitz type with constant

$$L_A = \frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \tag{3.32}$$

and $0 < L_A < 1$ due to (C5). By applying the Contraction Principle to this operator we obtain that A is a Picard operator. This completes the proof. □

Example 3.17. Consider the fractional Cauchy problem

$$\begin{cases} {}_0^C D_t^q x(t) = x(t), \ q = \frac{1}{2}, \\ x(0) = 0 \in X, \end{cases} \tag{3.33}$$

on $[0, 1]$. Set $L_0 = 1$, $T = 1$, $q_1 = \frac{1}{3}$, then $\beta = -\frac{3}{4}$. Indeed

$$\frac{L_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1 \iff \frac{qL_0}{\Gamma(q+1)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1,$$

which implies that we must choose a suitable $\tau_0 > 0$ such that $\frac{\frac{1}{2}}{\Gamma(\frac{3}{2})} \frac{1}{(\frac{1}{4})^{\frac{2}{3}}} \left(\frac{\frac{1}{3}}{\tau_0}\right)^{\frac{1}{3}} < 1$.
Noting that $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, for $\tau_0 = \frac{16}{9} > \frac{16}{3\sqrt{\pi^3}}$ we have the condition (C5) in Theorem 3.16.

Theorem 3.18. Suppose the following conditions hold:
(C1) $f \in C(J \times X, X)$;
(C2)$'$ there exists a constant $\bar{M} > 0$ such that $|f(t, x)| \leq \bar{M}$ for all $x \in X$ and all $t \in J$;
(C3)$'$ there exists a constant $\bar{L} > 0$ such that $\bar{L} \geq \frac{2\bar{M}}{\Gamma(q+1)}$;
(C4)$'$ there exists a constant $\bar{L}_0 > 0$ such that $|f(t, u_1) - f(t, u_2)| \leq \bar{L}_0|u_1 - u_2|$ for all $u_i \in X$ $(i = 1, 2)$ and all $t \in J$;
(C5)$'$ there exist constants q_1 and τ such that $\bar{L}_{\bar{A}} = \frac{\bar{L}_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1$.
Then the fractional Cauchy problem (3.30) has a unique solution x^* in $C_{\bar{L}}^q(J, X)$, and this solution can be obtained by the successive approximation method, starting from any element of $C_{\bar{L}}^q(J, X)$.

Proof. Consider the following continuous operator

$$\bar{A} : (C_{\bar{L}}^q(J, X), \|\cdot\|_B) \to (C_{\bar{L}}^q(J, X), \|\cdot\|_B)$$

defined by

$$\bar{A}x(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\left(s, x(s)\right) ds.$$

As the proof in Theorem 3.16, applying the given conditions one can verify that

$$\|\bar{A}(x) - \bar{A}(z)\|_B \leq \frac{\bar{L}_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} \|x - z\|_B$$

for all x, $z \in C_{\bar{L}}^q(J, X)$. So, the operator $\bar{A}$ is a Picard operator. □

Similarly, we can prove

Theorem 3.19. Suppose the following conditions hold:
(C1)$'$ $f \in C(J \times B_R, X)$;
(C2)$''$ there exists a constant $\bar{M}(R) > 0$ such that $|f(t,x)| \leq \bar{M}(R)$ for all $x \in B_R$ and all $t \in J$ with $R \geq |x_0| + \frac{\bar{M}(R)T^q}{\Gamma(q+1)}$;
(C3)$''$ there exists a constant $\bar{L} > 0$ such that $\bar{L} \geq \frac{2\bar{M}(R)}{\Gamma(q+1)}$;
(C4)$''$ there exists a constant $\bar{L}_0 > 0$ such that $|f(t,u_1) - f(t,u_2)| \leq \bar{L}_0 |u_1 - u_2|$ for all $u_i \in B_R$ $(i = 1,2)$ and all $t \in J$;
(C5)$'$ there exist constants q_1 and τ such that $\bar{L}_{\bar{A}} = \frac{\bar{L}_0}{\Gamma(q)} \frac{T^{(1+\beta)(1-q_1)}}{(1+\beta)^{1-q_1}} \left(\frac{q_1}{\tau}\right)^{q_1} < 1$.
Then the fractional Cauchy problem (3.30) has a unique solution x^* in $C_{\bar{L}}^q(J, B_R)$, and this solution can be obtained by the successive approximation method, starting from any element of $C_{\bar{L}}^q(J, B_R)$.

Consider the following fractional Cauchy problem

$$\begin{cases} {}_0^C D_t^q x(t) = g(t, x(t)),\ t \in J, \\ x(0) = y_0 \in X, \end{cases} \tag{3.34}$$

where $g \in C(J \times X, X)$. By Lemma 3.15, a function $x \in C(J, X)$ is a solution of the fractional integral equation

$$x(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(s, x(s)\right) ds, \tag{3.35}$$

if and only if x is a solution of the fractional Cauchy problem (3.34).

Now, we consider both fractional integral equation (3.31) and (3.35).

Theorem 3.20. Suppose the following:
(D1) all conditions in Theorem 3.16 are satisfied and $x^* \in C_L^{q-q_1}(J, X)$ is the unique solution of the fractional integral equation (3.31);
(D2) with the same function $m(\cdot)$ as in Theorem 3.16, $|g(t,x)| \leq m(t)$ for all $x \in X$ and all $t \in J$;
(D3) with the same function $l(\cdot)$ as in Theorem 3.16, $|g(t,u_1) - g(t,u_2)| \leq l(t)|u_1 - u_2|$ for all $u_i \in X$ $(i = 1,2)$ and all $t \in J$;
(D4) $L \geq \frac{2M}{\Gamma(q)(1+\beta)^{1-q_1}}$;
(D5) there exists a constant $q_2 \in (0,q)$ and function $\eta(\cdot) \in L^{\frac{1}{q_2}}(J, \mathbb{R}_+)$ such that $|f(t,u) - g(t,u)| \leq \eta(t)$ for all $u \in X$ and all $t \in J$.
If y^* is the solution of the fractional integral equation (3.35), then

$$\|x^* - y^*\|_B \leq \frac{|x_0 - y_0| + \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}}{1 - L_A}, \tag{3.36}$$

where L_A is given by (3.32) with $\tau = \tau_0 > 0$ such that $0 < L_A < 1$.

Proof. Consider the following two operators

$$A,\ B : (C_L^{q-q_1}(J, X), \|\cdot\|_B) \to (C_L^{q-q_1}(J, X), \|\cdot\|_B)$$

defined by

$$Ax(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds,$$
$$Bx(t) = y_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\, ds,$$

on J. We have

$$\begin{aligned} |Ax(t) - Bx(t)| &\le |x_0 - y_0| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s)) - g(s, x(s))| ds \\ &\le |x_0 - y_0| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \eta(s) ds \\ &\le |x_0 - y_0| + \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}, \end{aligned}$$

for $t \in J$. It follows that

$$\|Ax - Bx\|_B \le |x_0 - y_0| + \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}}.$$

So we can apply Theorem 1.38 to obtain (3.36) which completes the proof. □

Remark 3.21. All the results obtained in Theorem 3.16 hold even if the condition (C2) is replaced by the following:
(C2-E) there exist a constant $q_1 \in [0, q)$ and function $m(\cdot) \in L^{\frac{1}{q_1}}(J, \mathbb{R}_+)$ such that $|f(t, x)| \le m(t)$ for all $x \in X$ and all $t \in J$.

In fact, we only need extend the space $L^p(J, \mathbb{R}_+)$ $(1 < p < \infty)$ to $L^p(J, \mathbb{R}_+)$ $(1 \le p \le \infty)$ where $L^p(J, \mathbb{R}_+)$ $(1 \le p \le \infty)$ be the Banach space of all Lebesgue measurable functions $\phi : J \to \mathbb{R}_+$ with $\|\phi\|_{L^p J} < \infty$.

3.4.3 *Results via Weakly Picard Operators*

Now, we consider another fractional integral equation

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds \tag{3.37}$$

on J, where $f \in C(J \times X, X)$ is as in the fractional Cauchy problem (3.30).

Theorem 3.22. Suppose that for the fractional integral equation (3.37) the same conditions as in Theorem 3.16 are satisfied. Then this equation has solutions in $C_L^{q-q_1}(J, X)$. If $\mathcal{S} \subset C_L^{q-q_1}(J, X)$ is its solutions set, then *card* $\mathcal{S} =$ *card* X.

Proof. Consider the operator

$$A_* : (C_L^{q-q_1}(J, X), \|\cdot\|_B) \to (C_L^{q-q_1}(J, X), \|\cdot\|_B)$$

defined by

$$A_* x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s))\, ds.$$

This is a continuous operator, but not a Lipschitz one. We can write

$$C_L^{q-q_1}(J,X) = \bigcup_{\alpha \in X} X_\alpha, \qquad X_\alpha = \left\{x \in C_L^{q-q_1}(J,X) : x(0) = \alpha\right\}.$$

We have that X_α is an invariant set of A_* and we apply Theorem 3.16 to $A_*|_{X_\alpha}$. By using Theorem 1.40 we obtain that A_* is a weakly Picard operator.

Consider the operator

$$A_*^\infty : C_L^{q-q_1}(J,X) \to C_L^{q-q_1}(J,X), \quad A_*^\infty x = \lim_{n\to\infty} A_*^n x.$$

From $A_*^{n+1}(x) = A_*(A_*^n(x))$ and the continuity of A_*, $A_*^\infty(x) \in F_{A_*}$. Then

$$A_*^\infty(C_L^{q-q_1}(J,X)) = F_{A_*} = \mathcal{S} \text{ and } \mathcal{S} \neq \varnothing.$$

So, *card* $\mathcal{S} =$ *card* X. □

Theorem 3.23. Suppose that for the fractional integral equation (3.37) the same conditions as in Theorem 3.18 are satisfied. Then this equation has solutions in $C_{\bar{L}}^q(J,X)$. If $\mathcal{S} \subset C_{\bar{L}}^q(J,X)$ is its solutions set, then *card* $\mathcal{S} =$ *card* X.

Proof. As the proof in Theorem 3.22, we need to consider the continuous operator (but not a Lipschitz one)

$$\bar{A}_* : (C_{\bar{L}}^q(J,X), \|\cdot\|_B) \to (C_{\bar{L}}^q(J,X), \|\cdot\|_B)$$

defined by

$$\bar{A}_* x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f\left(s, x(s)\right) ds.$$

We can write $C_{\bar{L}}^q(J,X) = \bigcup_{\alpha\in X} \bar{X}_\alpha$, $\bar{X}_\alpha = \left\{x \in C_{\bar{L}}^q(J,X) : x(0) = \alpha\right\}$. We have that $\bar{X}_\alpha$ is an invariant set of $\bar{A}_*$ and we apply Theorem 3.18 to $\bar{A}_*|_{X_\alpha}$. By using Theorem 1.40 we obtain that $\bar{A}_*$ is a weakly Picard operator. Consider the operator $\bar{A}_*^\infty : C_{\bar{L}}^q(J,X) \to C_{\bar{L}}^q(J,X)$, $\bar{A}_*^\infty(x) = \lim_{n\to\infty} \bar{A}_*^n(x)$. From $\bar{A}_*^{n+1}(x) = \bar{A}_*(\bar{A}_*^n(x))$ and the continuity of $\bar{A}_*$, $\bar{A}_*^\infty(x) \in F_{\bar{A}_*}$. Then $\bar{A}_*^\infty(C_{\bar{L}}^q(J,X)) = F_{\bar{A}_*} = \mathcal{S}$ and $\mathcal{S} \neq \varnothing$. So, *card* $\mathcal{S} =$ *card* X. □

Similarly as above, we can prove

Theorem 3.24. Suppose that for the fractional integral equation (3.37) the same conditions as in Theorem 3.19 are satisfied. Then this equation has solutions in $C_{\bar{L}}^q(J,B_R)$. If $\mathcal{S} \subset C_{\bar{L}}^q(J,B_R)$ is its solutions set, then *card* $\mathcal{S} =$ *card* B_R.

In order to study data dependence for the solutions set of the fractional integral equation (3.37), we consider both (3.37) and the following fractional integral equation

$$x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(s, x(s)\right) ds$$

on J where $g \in C(J \times X, X)$. Let $\mathcal{S}_1$ be the solutions set of this equation.

Theorem 3.25. Suppose the following conditions:

(E1) there exists a function $l(t) \in C(J, \mathbb{R}_+)$ such that

$$|f(t,u_1) - f(t,u_2)| \leq l(t)|u_1 - u_2| \;\text{ and }\; |g(t,u_1) - g(t,u_2)| \leq l(t)|u_1 - u_2|$$

for all $u_i \in X$ $(i = 1, 2)$ and all $t \in J$;
(E2) there exist $q_1, q_3 \in (0, q)$ and functions $m(t) \in L^{\frac{1}{q_1}}(J, \mathbb{R}_+)$, $\mu(t) \in L^{\frac{1}{q_3}}(J, \mathbb{R}_+)$ such that

$$|f(t, x)| \leq m(t) \quad \text{and} \quad |g(t, x)| \leq \mu(t)$$

for all $x \in X$ and all $t \in J$;
(E3) there exists a constant $L > 0$ such that

$$L \geq \frac{2 \max\{M, V\}}{\Gamma(q) \min\left\{(1+\beta)^{1-q_1}, (1+\nu)^{1-q_3}\right\}};$$

(E4) there exist a constant $q_2 \in (0, q)$ and function $\eta \in L^{\frac{1}{q_2}}(J, \mathbb{R}_+)$

$$|f(t, u) - g(t, u)| \leq \eta(t)$$

for all $u \in X$ and all $t \in J$;
(E5) $\frac{L_0 T^q}{\Gamma(q+1)} < 1$.
Then

$$H_{\|\cdot\|_C}(\mathcal{S}, \mathcal{S}_1) \leq \frac{q N T^{(1+\gamma)(1-q_2)}}{(\Gamma(q+1) - L_0 T^q)(1+\gamma)^{1-q_2}}$$

where by $H_{\|\cdot\|_C}$ we denote the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C_L^{q-q_1}(J, X)$.

Proof. Consider the operator

$$B_* : (C_L^{q-q_1}(J, X), \|\cdot\|_B) \to (C_L^{q-q_1}(J, X), \|\cdot\|_B)$$

defined by

$$B_* x(t) = x(0) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g\left(s, x(s)\right) ds, \text{ for } t \in J.$$

Because of (E1)-(E3), $A_*, B_* : (C_L^{q-q_1}(J, X), \|\cdot\|_B) \to (C_L^{q-q_1}(J, X), \|\cdot\|_B)$ are two orbitally continuous operators. Moreover, we have

$$\begin{aligned} |A_*^2 x(t) - A_* x(t)| &\leq \frac{L_0}{\Gamma(q)} \int_0^t (t-s)^{q-1} |A_* x(s) - x(s)| ds \\ &\leq \frac{L_0 T^q}{\Gamma(q+1)} \|A_* x - x\|_C, \end{aligned}$$

for all $x \in C_L^{q-q_1}(J, X)$. Similarly,

$$|B_*^2 x(t) - B_* x(t)| \;\leq\; \frac{L_0 T^q}{\Gamma(q+1)} \|B_* x - x\|_C$$

for all $x \in C_L^{q-q_1}(J, X)$. It follows that

$$\|A_*^2 x - A_* x\|_C \leq \frac{L_0 T^q}{\Gamma(q+1)} \|A_* x - x\|_C,$$

$$\|B_*^2 x - B_* x\|_C \leq \frac{L_0 T^q}{\Gamma(q+1)} \|B_* x - x\|_C.$$

Because of (E4),

$$\begin{aligned}\|A_*x - B_*x\|_C &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\eta(s)ds \\ &\leq \frac{NT^{(1+\gamma)(1-q_2)}}{\Gamma(q)(1+\gamma)^{1-q_2}},\end{aligned}$$

for all $x \in C_L^{q-q_1}(J, X)$.

By (E5) and applying Theorem 1.39, we obtain

$$H_{\|\cdot\|_C}(F_{A_*}, F_{B_*}) \leq \frac{qNT^{(1+\gamma)(1-q_2)}}{(\Gamma(q+1) - L_0T^q)(1+\gamma)^{1-q_2}}$$

and the theorem is proved. □

Theorem 3.26. Suppose the following conditions:
(E1)′ there exists a constant $L_* > 0$ such that

$$|f(t,u_1) - f(t,u_2)| \leq L_*|u_1 - u_2| \text{ and } |g(t,u_1) - g(t,u_2)| \leq L_*|u_1 - u_2|$$

for all $u_i \in X$ $(i = 1, 2)$ and all $t \in J$;
(E2)′ there exists a constant $M_* > 0$ such that

$$|f(t,x)| \leq M_* \text{ and } |g(t,x)| \leq M_*$$

for all $x \in X$ and all $t \in J$;
(E3)′ there exists a constant $\bar{L} > 0$ such that

$$\bar{L} \geq \frac{2M_*}{\Gamma(q+1)};$$

(E4)′ there exists a constant $\eta_* > 0$ such that

$$|f(t,u) - g(t,u)| \leq \eta_*$$

for all $u \in X$ and all $t \in J$;
(E5)′ $\frac{L_*T^q}{\Gamma(q+1)} < 1$.
Then we have

$$\bar{H}_{\|\cdot\|_C}(\mathcal{S}, \mathcal{S}_1) \leq \frac{\eta_*T^q}{\Gamma(q+1) - L_*T^q}$$

where by $\bar{H}_{\|\cdot\|_C}$ we denote the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C_{\bar{L}}^q(J, X)$.

Proof. Consider the operator

$$\bar{B}_* : (C_{\bar{L}}^q(J, X), \|\cdot\|_B) \to (C_{\bar{L}}^q(J, X), \|\cdot\|_B)$$

defined by

$$\bar{B}_*x(t) = x(0) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g\left(s, x(s)\right) ds, \text{ for } t \in J.$$

Applying (E1)$'$-(E3)$'$, $\bar{A}_*, \bar{B}_* : (C_{\bar{L}}^{q-q_1}(J,X), \|\cdot\|_B) \to (C_{\bar{L}}^{q-q_1}(J,X), \|\cdot\|_B)$ are two orbitally continuous operators. Moreover, we have

$$|\bar{A}_*^2 x(t) - \bar{A}_* x(t)| \le \frac{L_* T^q}{\Gamma(q+1)} \|\bar{A}_*(x) - x\|_C,$$
$$|\bar{B}_*^2 x(t) - \bar{B}_* x(t)| \le \frac{L_* T^q}{\Gamma(q+1)} \|\bar{B}_*(x) - x\|_C,$$

for all $x \in C_{\bar{L}}^q(J,X)$. It follows that

$$\|\bar{A}_*^2(x) - \bar{A}_*(x)\|_C \le \frac{L_* T^q}{\Gamma(q+1)} \|\bar{A}_*(x) - x\|_C$$
$$\|\bar{B}_*^2(x) - \bar{B}_*(x)\|_C \le \frac{L_* T^q}{\Gamma(q+1)} \|\bar{B}_*(x) - x\|_C.$$

Because of (E4)$'$, we obtain

$$\|\bar{A}_*(x) - \bar{B}_*(x)\|_C \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \eta_* ds \ \le \ \frac{\eta_* T^q}{\Gamma(q+1)},$$

for all $x \in C_{\bar{L}}^q(J,X)$.

By (E5)$'$ and applying Theorem 1.39, we obtain the result and the theorem is proved. □

Similarly, we can prove

Theorem 3.27. Suppose the following:

(E1)$''$ there exists a constant $L_* > 0$ such that

$$|f(t,u_1) - f(t,u_2)| \le L_*|u_1 - u_2| \ \text{ and } |g(t,u_1) - g(t,u_2)| \le L_*|u_1 - u_2|$$

for all $u_i \in B_R$ $(i = 1,2)$ and all $t \in J$;

(E2)$''$ there exists a constant $M_*(R) > 0$ such that

$$|f(t,x)| \le M_*(R) \ \text{ and } |g(t,x)| \le M_*(R)$$

for all $x \in B_R$ and all $t \in J$ with

$$R \ge |x(0)| + \frac{M_*(R) T^q}{\Gamma(q+1)};$$

(E3)$''$ there exists a constant $\bar{L} > 0$ such that

$$\bar{L} \ge \frac{2M_*(R)}{\Gamma(q+1)};$$

(E4)$''$ there exists a constant $\eta_* > 0$ such that

$$|f(t,u) - g(t,u)| \le \eta_*$$

for all $u \in B_R$ and all $t \in J$;

(E5)$''$ $\frac{L_* T^q}{\Gamma(q+1)} < 1$.

Then

$$\bar{H}_{\|\cdot\|_C}(\mathcal{S}, \mathcal{S}_1) \le \frac{\eta_* T^q}{\Gamma(q+1) - L_* T^q}$$

where by $\bar{H}_{\|\cdot\|_C}$ we denote the Pompeiu-Hausdorff functional with respect to $\|\cdot\|_C$ on $C_{\bar{L}}^q(J, B_R)$.

3.5 Notes and Remarks

The results in Sections 3.1 and 3.2 are taken from Zhou, Jiao and Pecaric, 2013. The material in Section 3.3 due to Wang, Zhou and Medved, 2012. The main results in Section 3.4 are adopted from Wang, Zhou and Wei, 2013.

Chapter 4

Fractional Abstract Evolution Equations

4.1 Introduction

The existence of mild solutions for the Cauchy problem of fractional evolution equations has been considered in several recent papers (see, e.g., Agarwal and Shmad, 2011; Belmekki and Benchohra, 2010; Chang, Kavitha and Mallika, 2009; Darwish, Henderson and Ntouyas, 2009; Hernandez, O'Regan and Balachandran, 2010; Hu, Ren and Sakthivel, 2009; Kumar and Sukavanam, 2012; Li, Peng and Jia, 2012; Shu, Lai and Chen, 2011; Wang, Chen and Xiao, 2012; Wang and Zhou, 2011; Wang, Fečkan and Zhou, 2011; Zhou and Jiao, 2010), much less is known about the fractional evolution equations with Riemann-Liouville derivative. In most of the existing articles, Schauder's fixed point theorem, Krasnoselskii's fixed point theorem or Darbo's fixed point theorem, Kuratowski measure of noncompactness are employed to obtain the fixed points of the solution operator of the Cauchy problems under some restrictive conditions. In order to show that the solution operator is compact, a very common approach is to use Arzela-Ascoli's theorem. However, it is difficult to check the relatively compactness of the solution operator and the equicontinuity of certain family of functions which is given by the solution operator.

In this chapter, we discuss the existence of mild solutions of fractional abstract evolution equations. The suitable mild solutions of fractional evolution equations with Riemann-Liouville derivative and Caputo derivative are introduced respectively. In Sections 4.2 and 4.3, by using the theory of Hausdorff measure of noncompactness, we investigate the existence of mild solutions for the Cauchy problems in the cases C_0-semigroup is compact or noncompact. Section 4.4 devoted to study the evolution equations with almost sectorial operators. In Section 4.5, the existence results of mild solutions of nonlocal problem of fractional evolution equations are presented.

4.2 Evolution Equations with Riemann-Liouville Derivative

4.2.1 *Introduction*

Assume that X is a Banach space with the norm $|\cdot|$. Let $a \in \mathbb{R}^+, J = [0, a]$ and $J' = (0, a]$. Denote $C(J, X)$ be the Banach space of continuous functions from J into X with the norm

$$\|x\| = \sup_{t\in[0,a]} |x(t)|,$$

where $x \in C(J, X)$, and $B(X)$ be the space of all bounded linear operators from X to X with the norm $\|Q\|_{B(X)} = \sup\{|Q(x)| : |x| = 1\}$, where $Q \in B(X)$ and $x \in X$.

Consider the following nonlocal Cauchy problem of fractional evolution equation with Riemann-Liouville derivative

$$\begin{cases} {}_0D_t^q x(t) = Ax(t) + f(t, x(t)), \quad \text{a.e. } t \in [0, a], \\ {}_0D_t^{q-1} x(0) + g(x) = x_0, \end{cases} \tag{4.1}$$

where ${}_0D_t^q$ is Riemann-Liouville derivative of order q, ${}_0D_t^{q-1}$ is Riemann-Liouville integral of order $1 - q$, $0 < q < 1$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. C_0-semigroup) $\{Q(t)\}_{t\geq 0}$ in Banach space X, $f : J \times X \to X$ is a given function, $g : C(J, X) \to L(J, X)$ is a given operator satisfying some assumptions and x_0 is an element of the Banach space X.

A strong motivation for investigating the nonlocal Cauchy problem (4.1) comes from physics. For example, fractional diffusion equations are abstract partial differential equations that involve fractional derivatives in space and time. They are useful to model anomalous diffusion, where a plume of particles spreads in a different manner than the classical diffusion equation predicts. The time fractional diffusion equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $q \in (0, 1)$, namely

$$\partial_t^q u(z, t) = Au(z, t), \quad t \geq 0, \quad z \in \mathbb{R}.$$

We can take $A = \partial_z^{\beta_1}$, for $\beta_1 \in (0, 1]$, or $A = \partial_z + \partial_z^{\beta_2}$ for $\beta_2 \in (1, 2]$, where $\partial_t^q, \partial_z^{\beta_1}, \partial_z^{\beta_2}$ are the fractional derivatives of order q, β_1, β_2 respectively. We refer the interested reader to Eidelman and Kochubei, 2004; Hanyga, 2002; Hayashi, Kaikina and Naumkin, 2005; Meerschaert, Benson, Scheffler *et al.*, 2002; Schneider and Wayes, 1989; Zaslavsky, 1994 and the references therein for more details.

The nonlocal conditions ${}_0D_t^{q-1} x(0) + g(x) = x_0$ and $x(0) + g(x) = x_0$ can be applied in physics with better effect than the classical initial conditions ${}_0D_t^{q-1} x(0) = x_0$ and $x(0) = x_0$ respectively. For example, $g(x)$ may be given by

$$g(x) = \sum_{i=1}^{m} c_i x(t_i),$$

where c_i $(i = 1, 2, \ldots, n)$ are given constants and $0 < t_1 < t_2 < \cdots < t_n \leq a$. Nonlocal conditions were initiated by Byszewski, 1991, which he proved the existence and uniqueness of mild and classical solutions for nonlocal Cauchy problems. As remarked by Byszewski and Lakshmikantham, 1991, the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

In this section, we study the nonlocal Cauchy problems of fractional evolution equations with Riemann-Liouville derivative by considering an integral equation which is given in terms of probability density. By using the theory of Hausdorff measure of noncompactness, we establish various existence theorems of mild solutions for the Cauchy problem (4.1) in the cases C_0-semigroup is compact or noncompact. Subsection 4.2.2 is devoted to obtain the appropriate definition on the mild solutions of the problem (4.1) by considering an integral equation which is given in terms of probability density. In Subsection 4.2.3, we give some preliminary lemmas. Subsection 4.2.4 provides various existence theorems of mild solutions for the Cauchy problem (4.1) in the case C_0-semigroup is compact. In Subsection 4.2.5, we establish various existence theorems of mild solutions for the Cauchy problem (4.1) in the case C_0-semigroup is noncompact.

4.2.2 *Definition of Mild Solutions*

Definition 4.1. (Mainardi, Paraddisi and Forenflo, 2000) The Wright function $M_q(\varrho)$ is defined by

$$M_q(\varrho) = \sum_{n=1}^{\infty} \frac{(-\varrho)^{n-1}}{(n-1)!\Gamma(1-qn)}, \ 0 < q < 1, \ \varrho \in \mathbb{C}.$$

It is known that $M_q(\varrho)$ satisfies the following equality (see Mönch, 1980)

$$\int_0^{\infty} \theta^{\delta} M_q(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+q\delta)}, \quad \text{for } \delta \geq 0.$$

Lemma 4.2. (Tazali, 1982)

(i) Let $\xi, \eta \in \mathbb{R}$ such that $\eta > -1$. If $t > 0$, then

$$_0D_t^{-\xi} \frac{t^{\eta}}{\Gamma(\eta+1)} = \begin{cases} \frac{t^{\xi+\eta}}{\Gamma(\xi+\eta+1)}, & \text{if } \xi+\eta \neq -n \\ 0, & \text{if } \xi+\eta = -n \end{cases} \ (n \in \mathbb{N}).$$

(ii) Let $\xi > 0$ and $\varphi \in L((0,a), X)$. Define

$$G_{\xi}(t) = {_0D_t^{-\xi}} \varphi, \quad \text{for } t \in (0,a),$$

then

$$_0D_t^{-\eta} G_{\xi}(t) = {_0D_t^{-(\xi+\eta)}} \varphi(t), \quad \eta > 0, \quad \text{almost all } t \in [0,a].$$

Lemma 4.3. The nonlocal Cauchy problem (4.1) is equivalent to the integral equation

$$x(t) = \frac{t^{q-1}}{\Gamma(q)}(x_0 - g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} [Ax(s) + f(s, x(s))] ds, \quad \text{for } t \in (0,a], \tag{4.2}$$

provided that the integral in (4.2) exists.

Proof. Suppose (4.2) is true, then

$$_0D_t^{q-1}x(t) = {_0D_t^{q-1}}\left(\frac{t^{q-1}}{\Gamma(q)}(x_0 - g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-\tau)^{q-1}[Ax(\tau) + f(\tau, x(\tau))]d\tau\right),$$

applying Lemma 4.2 we obtain that

$$_0D_t^{q-1}x(t) = x_0 - g(x) + \int_0^t [Ax(s) + f(s, x(s))]ds, \text{ almost all } t \in [0, a],$$

and this proves that $_0D_t^{q-1}x(t)$ is absolutely continuous on $[0, a]$. Then we have

$$_0D_t^q x(t) = \frac{d}{dt}\, {_0D_t^{q-1}}x(t) = Ax(t) + f(t, x(t)), \text{ almost all } t \in [0, a]$$

and

$$_0D_t^{q-1}x(0) + g(x) = x_0.$$

The proof of the converse is given as follows.

Suppose (4.1) is true, then

$$_0D_t^{-q}({_0D_t^q}x(t)) = {_0D_t^{-q}}(Ax(t) + f(t, x(t))).$$

Since

$$\begin{aligned} _0D_t^{-q}({_0D_t^q}x(t)) &= x(t) - \frac{t^{q-1}}{\Gamma(q)}\, {_0D_t^{q-1}}x(0) \\ &= x(t) - \frac{t^{q-1}}{\Gamma(q)}(x_0 - g(x)), \quad \text{for } t \in (0, a], \end{aligned}$$

then we have

$$\begin{aligned} x(t) &= \frac{t^{q-1}}{\Gamma(q)}(x_0 - g(x)) + {_0D_t^{-q}}(Ax(t) + f(t, x(t))) \\ &= \frac{t^{q-1}}{\Gamma(q)}(x_0 - g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[Ax(s) + f(s, x(s))]ds, \quad \text{for } t \in (0, a]. \end{aligned}$$

The proof is completed. □

Before giving the definition of mild solution of (4.1), we firstly prove the following lemma.

Lemma 4.4. If

$$x(t) = \frac{t^{q-1}}{\Gamma(q)}(x_0 - g(x)) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[Ax(s) + f(s, x(s))]ds, \quad \text{for } t > 0 \quad (4.3)$$

holds, then we have

$$x(t) = t^{q-1}P_q(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s, x(s))ds, \quad \text{for } t > 0, \quad (4.4)$$

where

$$P_q(t) = \int_0^\infty q\theta M_q(\theta)Q(t^q\theta)d\theta.$$

Proof. Let $\lambda > 0$. Applying the Laplace transform

$$\nu(\lambda) = \int_0^{\infty} e^{-\lambda s} x(s) ds \quad \text{and} \quad \omega(\lambda) = \int_0^{\infty} e^{-\lambda s} f(s, x(s)) ds, \quad \text{for } \lambda > 0$$

to (4.3), we have

$$\begin{aligned} \nu(\lambda) &= \frac{1}{\lambda^q}(x_0 - g(x)) + \frac{1}{\lambda^q} A\nu(\lambda) + \frac{1}{\lambda^q}\,\omega(\lambda) \\ &= (\lambda^q I - A)^{-1}(x_0 - g(x)) + (\lambda^q I - A)^{-1}\omega(\lambda) \\ &= \int_0^{\infty} e^{-\lambda^q s} Q(s)(x_0 - g(x)) ds + \int_0^{\infty} e^{-\lambda^q s} Q(s)\omega(\lambda) ds, \end{aligned} \tag{4.5}$$

provided that the integrals in (4.5) exist, where I is the identity operator defined on X.

Set

$$\psi_q(\theta) = \frac{q}{\theta^{q+1}} M_q(\theta^{-q}),$$

whose Laplace transform is given by

$$\int_0^{\infty} e^{-\lambda\theta} \psi_q(\theta) d\theta = e^{-\lambda^q}, \quad \text{where } q \in (0, 1). \tag{4.6}$$

Using (4.6), we get

$$\begin{aligned} &\int_0^{\infty} e^{-\lambda^q s} Q(s)(x_0 - g(x)) ds = \int_0^{\infty} q t^{q-1} e^{-(\lambda t)^q} Q(t^q)(x_0 - g(x)) dt \\ &= \int_0^{\infty}\int_0^{\infty} q\psi_q(\theta) e^{-(\lambda t\theta)} Q(t^q) t^{q-1}(x_0 - g(x)) d\theta dt \\ &= \int_0^{\infty}\int_0^{\infty} q\psi_q(\theta) e^{-\lambda t} Q\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q}(x_0 - g(x)) d\theta dt \\ &= \int_0^{\infty} e^{-\lambda t}\left[q\int_0^{\infty} \psi_q(\theta) Q\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q}(x_0 - g(x)) d\theta\right] dt, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\int_0^{\infty} e^{-\lambda^q s} Q(s)\omega(\lambda) ds \\ &= \int_0^{\infty}\int_0^{\infty} q t^{q-1} e^{-(\lambda t)^q} Q(t^q) e^{-\lambda s} f(s, x(s)) ds dt \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} q\psi_q(\theta) e^{-(\lambda t\theta)} Q(t^q) e^{-\lambda s} t^{q-1} f(s, x(s)) d\theta ds dt \\ &= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} q\psi_q(\theta) e^{-\lambda(t+s)} Q\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q} f(s, x(s)) d\theta ds dt \\ &= \int_0^{\infty} e^{-\lambda t}\left[q\int_0^{t}\int_0^{\infty} \psi_q(\theta) Q\left(\frac{(t-s)^q}{\theta^q}\right)\frac{(t-s)^{q-1}}{\theta^q} f(s, x(s)) d\theta ds\right] dt. \end{aligned} \tag{4.8}$$

According to (4.7) and (4.8), we have

$$\begin{aligned} \nu(\lambda) = \int_0^{\infty} e^{-\lambda t}\Bigg[& q\int_0^{\infty} \psi_q(\theta) Q\left(\frac{t^q}{\theta^q}\right)\frac{t^{q-1}}{\theta^q}(x_0 - g(x)) d\theta \\ &+ q\int_0^{t}\int_0^{\infty} \psi_q(\theta) Q\left(\frac{(t-s)^q}{\theta^q}\right)\frac{(t-s)^{q-1}}{\theta^q} f(s, x(s)) d\theta ds\Bigg] dt. \end{aligned}$$

Now we can invert the last Laplace transform to get

$$\begin{aligned} x(t) &= q\int_0^\infty \theta t^{q-1} M_q(\theta) Q(t^q\theta)(x_0 - g(x))d\theta \\ &\quad + q\int_0^t \int_0^\infty \theta(t-s)^{q-1} M_q(\theta) Q((t-s)^q\theta) f(s, x(s)) d\theta ds \\ &= t^{q-1} P_q(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s)) ds. \end{aligned}$$

The proof is completed. □

Due to Lemma 4.4, we give the following definition of the mild solution of (4.1).

Definition 4.5. By the mild solution of the nonlocal Cauchy problem (4.1), we mean that the function $x \in C(J', X)$ which satisfies

$$x(t) = t^{q-1} P_q(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s)) ds, \quad \text{for } t \in (0, a].$$

Suppose that A is the infinitesimal generator of a C_0 semigroup $\{Q(t)\}_{t \geq 0}$ of uniformly bounded linear operators on Banach space X. This means that there exists $M > 1$ such that $M = \sup_{t \in [0,\infty)} \|Q(t)\|_{B(X)} < \infty$.

Proposition 4.6. (Zhou and Jiao, 2010a) For any fixed $t > 0$, $P_q(t)$ is linear and bounded operator, i.e., for any $x \in X$

$$|P_q(t)x| \leq \frac{M}{\Gamma(q)} |x|.$$

Proposition 4.7. (Zhou and Jiao, 2010a) Operator $\{P_q(t)\}_{t>0}$ is strongly continuous, which means that, for $\forall x \in X$ and $0 < t' < t'' \leq a$, we have

$$|P_q(t'')x - P_q(t')x| \to 0 \text{ as } t'' \to t'.$$

Proposition 4.8. (Zhou and Jiao, 2010a) Assume that $\{Q(t)\}_{t>0}$ is compact operator. Then $\{P_q(t)\}_{t>0}$ is also compact operator.

Proposition 4.9. (Pazy, 1983) Assume that $\{Q(t)\}_{t>0}$ is compact operator. Then $\{Q(t)\}_{t>0}$ is equicontinuous.

4.2.3 *Preliminary Lemmas*

Define

$$X^{(q)}(J') = \left\{ x \in C(J', X) : \lim_{t \to 0+} t^{1-q} x(t) \text{ exists and is finite} \right\}.$$

For any $x \in X^{(q)}(J')$, let the norm $\|\cdot\|_q$ defined by

$$\|x\|_q = \sup_{t \in (0,a]} \{t^{1-q}|x(t)|\}.$$

Then $(X^{(q)}(J'), \|\cdot\|_q)$ is a Banach space.

For $r > 0$, define a closed subset $B_r^{(q)}(J') \subset X^{(q)}(J')$ as follows

$$B_r^{(q)}(J') = \{x \in X^{(q)}(J') : \|x\|_q \leq r\}.$$

Thus, $B_r^{(q)}(J')$ is a bounded closed and convex subset of $X^{(q)}(J')$.

Let $B(J)$ be the closed ball of the space $C(J,X)$ with radius r and center at 0, that is

$$B(J) = \{y \in C(J,X) : \|y\| \leq r\}.$$

Thus $B(J)$ is a bounded closed and convex subset of $C(J,X)$.

We introduce the following hypotheses:

(H0) $Q(t)(t>0)$ is equicontinuous, i.e., $Q(t)$ is continuous in the uniform operator topology for $t>0$;

(H1) for each $t \in J'$, the function $f(t,\cdot) : X \to X$ is continuous and for each $x \in X$, the function $f(\cdot,x) : J' \to X$ is strongly measurable;

(H2) there exists a function $m \in L(J',\mathbb{R}^+)$ such that

$${}_0D_t^{-q}m \in C(J',\mathbb{R}^+), \quad \lim_{t\to 0+} t^{1-q}{}_0D_t^{-q}m(t) = 0,$$

and

$$|f(t,x)| \leq m(t) \;\text{ for all } x \in B_r^{(q)}(J') \;\text{ and almost all } t \in [0,a];$$

(H3) there exists a constant $L \in (0, \frac{\Gamma(q)}{M})$ such that the operator $g : C(J',X) \to L(J',X)$ satisfies

$$|g(x_1) - g(x_2)| \leq L\|x_1 - x_2\|_q, \;\text{ for } x_1, x_2 \in B_r^{(q)}(J');$$

(H4) there exists a constant $r>0$ such that

$$\frac{M}{\Gamma(q) - ML}\left(|x_0| + |g(0)| + \sup_{t\in(0,a]}\left\{t^{1-q}\int_0^t (t-s)^{q-1}m(s)ds\right\}\right) \leq r;$$

(H3)$'$ the operator $g : C(J',X) \to L(J',X)$ is a continuous and compact map, and there exist positive constants L_1, L_2 such that $L_1 \in (0, \frac{\Gamma(q)}{M})$ and $|g(x)| \leq L_1\|x\|_q + L_2$ for all $x \in B_r^{(q)}(J')$;

(H4)$'$ there exists a constant $r>0$ such that

$$\frac{M}{\Gamma(q) - ML_1}\left(|x_0| + L_2 + \sup_{t\in(0,a]}\left\{t^{1-q}\int_0^t (t-s)^{q-1}m(s)ds\right\}\right) \leq r.$$

For any $x \in B_r^{(q)}(J')$, define an operator T as follows

$$(Tx)(t) = (T_1x)(t) + (T_2x)(t),$$

where

$$\begin{aligned}(T_1x)(t) &= t^{q-1}P_q(t)(x_0 - g(x)), && \text{for } t \in (0,a],\\ (T_2x)(t) &= \int_0^t (t-s)^{q-1}P_q(t-s)f(s,x(s))ds, && \text{for } t \in (0,a].\end{aligned}$$

It is easy to see that $\lim_{t\to 0+} t^{1-q}(Tx)(t) = \frac{x_0 - g(x)}{\Gamma(q)}$. For any $y \in B(J)$, set

$$x(t) = t^{q-1}y(t), \;\text{ for } t \in (0,a].$$

Then, $x \in B_r^{(q)}(J')$. Define $\mathscr{T}$ as follows

$$(\mathscr{T}y)(t) = (\mathscr{T}_1 y)(t) + (\mathscr{T}_2 y)(t),$$

where

$$(\mathscr{T}_1 y)(t) = \begin{cases} t^{1-q}(T_1 x)(t), & \text{for } t \in (0,a], \\ \dfrac{x_0 - g(x)}{\Gamma(q)}, & \text{for } t = 0, \end{cases}$$

$$(\mathscr{T}_2 y)(t) = \begin{cases} t^{1-q}(T_2 x)(t), & \text{for } t \in (0,a], \\ 0, & \text{for } t = 0. \end{cases}$$

Obviously, x is a mild solution of (4.1) in $B_r^{(q)}(J')$ if and only if the operator equation $x = Tx$ has a solution $x \in B_r^{(q)}(J')$. Before giving the main results, we firstly prove the following lemmas.

Lemma 4.10. Assume that (H0)-(H4) hold, then $\{\mathscr{T}y : \ y \in B(J)\}$ is equicontinuous.

Proof. **Step I.** $\{\mathscr{T}_1 y : \ y \in B(J)\}$ is equicontinuous. For any $y \in B(J)$, let $x(t) = t^{q-1}y(t),\ t \in (0,a]$. Then $x \in B_r^{(q)}(J')$. For $t_1 = 0$, $0 < t_2 \le a$, we get

$$\begin{aligned} |(\mathscr{T}_1 y)(t_2) - (\mathscr{T}_1 y)(0)| &\le \left| P_q(t_2)(x_0 - g(x)) - \frac{x_0 - g(x)}{\Gamma(q)} \right| \\ &\le \left| (P_q(t_2) - \frac{1}{\Gamma(q)})(x_0 - g(x)) \right| \\ &\le \left| (P_q(t_2) - \frac{1}{\Gamma(q)}) \right| (|x_0| + L\|x\|_q + |g(0)|) \\ &\le \left| (P_q(t_2) - \frac{1}{\Gamma(q)}) \right| (|x_0| + Lr + |g(0)|) \\ &\to 0, \quad \text{as } t_2 \to 0. \end{aligned}$$

For $0 < t_1 < t_2 \le a$, we get

$$\begin{aligned} |(\mathscr{T}_1 y)(t_2) - (\mathscr{T}_1 y)(t_1)| &\le |P_q(t_2)(x_0 - g(x)) - P_q(t_1)(x_0 - g(x))| \\ &\le |(P_q(t_2) - P_q(t_1))(x_0 - g(x))| \\ &\le |(P_q(t_2) - P_q(t_1))|(|x_0| + L\|x\|_q + |g(0)|) \\ &\le |(P_q(t_2) - P_q(t_1))|(|x_0| + Lr + |g(0)|) \\ &\to 0, \quad \text{as } t_2 \to t_1. \end{aligned}$$

Hence, $\{\mathscr{T}_1 y : \ y \in B(J)\}$ is equicontinuous.

Step II. $\{\mathscr{T}_2 y : \ y \in B(J)\}$ is equicontinuous. For any $y \in B(J)$, let $x(t) = t^{q-1}y(t),\ t \in (0,a]$. Then $x \in B_r^{(q)}(J')$. For $t_1 = 0$, $0 < t_2 \le a$, we get

$$\begin{aligned} |(\mathscr{T}_2 y)(t_2) - (\mathscr{T}_2 y)(0)| &= \left| t_2^{1-q} \int_0^{t_2} (t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right| \\ &\le \frac{M}{\Gamma(q)} t_2^{1-q} \int_0^{t_2} (t_2 - s)^{q-1} m(s) ds \to 0, \quad \text{as } t_2 \to 0. \end{aligned}$$

For $0 < t_1 < t_2 \le a$, we have

$$\begin{aligned}
|(\mathscr{T}_2 y)(t_2) - (\mathscr{T}_2 y)(t_1)| &\le \left| \int_{t_1}^{t_2} t_2^{1-q}(t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right| \\
&\quad + \left| \int_0^{t_1} t_2^{1-q}(t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right. \\
&\quad \left. - \int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right| \\
&\quad + \left| \int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right. \\
&\quad \left. - \int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} P_q(t_1 - s) f(s, x(s)) ds \right| \\
&\le \frac{M}{\Gamma(q)} \left| \int_{t_1}^{t_2} t_2^{1-q}(t_2 - s)^{q-1} m(s) ds \right| \\
&\quad + \frac{M}{\Gamma(q)} \int_0^{t_1} \left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] m(s) ds \\
&\quad + \left| \int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} [P_q(t_2 - s) f(s, x(s)) - P_q(t_1 - s) f(s, x(s))] ds \right| \\
&\le I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \frac{M}{\Gamma(q)} \left| \int_0^{t_2} t_2^{1-q}(t_2 - s)^{q-1} m(s) ds - \int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} m(s) ds \right|, \\
I_2 &= \frac{2M}{\Gamma(q)} \int_0^{t_1} \left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] m(s) ds, \\
I_3 &= \left| \int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} [P_q(t_2 - s) - P_q(t_1 - s)] f(s, x(s)) ds \right|.
\end{aligned}$$

One can reduce that $\lim_{t_2 \to t_1} I_1 = 0$, since ${}_0D_t^{-q} m \in C(J', \mathbb{R}^+)$. Noting that

$$\left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] m(s) \le t_1^{1-q}(t_1 - s)^{q-1} m(s),$$

and $\int_0^{t_1} t_1^{1-q}(t_1 - s)^{q-1} m(s) ds$ exists ($s \in [0, t_1]$), then by Lebesgue's dominated convergence theorem, we have

$$\int_0^{t_1} \left[t_1^{1-q}(t_1 - s)^{q-1} - t_2^{1-q}(t_2 - s)^{q-1} \right] m(s) ds \to 0, \quad \text{as } t_2 \to t_1,$$

then one can deduce that $\lim_{t_2 \to t_1} I_2 = 0$.

For $\varepsilon > 0$ be enough small, we have

$$\begin{aligned}
I_3 &\le \int_0^{t_1-\varepsilon} t_1^{1-q}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}\ |f(s,x(s))|ds \\
&\quad + \int_{t_1-\varepsilon}^{t_1} t_1^{1-q}(t_1-s)^{q-1}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}\ |f(s,x(s))|ds \\
&\le t_1^{1-q}\int_0^{t_1}(t_1-s)^{q-1}m(s)ds \sup_{s\in[0,t_1-\varepsilon]}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)} \\
&\quad + \frac{2M}{\Gamma(q)}\int_{t_1-\varepsilon}^{t_1} t_1^{1-q}(t_1-s)^{q-1}m(s)ds \\
&\le I_{31}+I_{32}+I_{33},
\end{aligned}$$

where

$$\begin{aligned}
I_{31} &= \frac{r\Gamma(q)}{M}\sup_{s\in[0,t_1-\varepsilon]}\|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)}, \\
I_{32} &= \frac{2M}{\Gamma(q)}\left|\int_0^{t_1} t_1^{1-q}(t_1-s)^{q-1}m(s)ds - \int_0^{t_1-\varepsilon}(t_1-\varepsilon)^{1-q}(t_1-\varepsilon-s)^{q-1}m(s)ds\right|, \\
I_{33} &= \frac{2M}{\Gamma(q)}\int_0^{t_1-\varepsilon}[(t_1-\varepsilon)^{1-q}(t_1-\varepsilon-s)^{q-1}-t_1^{1-q}(t_1-s)^{q-1}]m(s)ds.
\end{aligned}$$

By (H0), it is easy to see that $I_{31}\to 0$ as $t_2\to t_1$. Similar to the proof that I_1, I_2 tend to zero, we get $I_{32}\to 0$ and $I_{33}\to 0$ as $\varepsilon\to 0$. Thus, I_3 tends to zero independently of $y\in B(J)$ as $t_2\to t_1,\ \varepsilon\to 0$. Therefore, $|(\mathscr{T}_2y)(t_2)-(\mathscr{T}_2y)(t_1)|$ tends to zero independently of $y\in B(J)$ as $t_2\to t_1$, which means that $\{\mathscr{T}_2y:\ y\in B(J)\}$ is equicontinuous.

Therefore, $\{\mathscr{T}y:\ y\in B(J)\}$ is equicontinuous. □

Lemma 4.11. Assume that (H1)-(H4) hold. Then $\mathscr{T}$ maps $B(J)$ into $B(J)$, and $\mathscr{T}$ is continuous in $B(J)$.

Proof. **Step I.** $\mathscr{T}$ maps $B(J)$ into $B(J)$. For any $y\in B(J)$, let $x(t)=t^{q-1}y(t)$. Then $x\in B_r^{(q)}(J')$.

For $t\in[0,a]$, by (H1)-(H4), we have

$$\begin{aligned}
|(\mathscr{T}y)(t)| &\le |P_q(t)(x_0-g(x))| + t^{1-q}\left|\int_0^t (t-s)^{q-1}P_q(t-s)f(s,x(s))ds\right| \\
&\le \frac{M}{\Gamma(q)}(|x_0|+L\|x\|_q+|g(0)|) + \frac{Mt^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s))|ds \\
&\le \frac{M}{\Gamma(q)}\left(|x_0|+Lr+|g(0)|+\sup_{t\in[0,a]}\left\{t^{1-q}\int_0^t(t-s)^{q-1}m(s)ds\right\}\right) \\
&\le r.
\end{aligned}$$

Hence, $\|\mathscr{T}y\|\le r$, for any $y\in B(J)$.

Step II. $\mathscr{T}$ is continuous in $B(J)$. For any $y_m, y \in B(J)$, $m = 1, 2, ...$, with $\lim_{m\to\infty} y_m = y$, we have

$$\lim_{m\to\infty} y_m(t) = y(t) \;\text{ and }\; \lim_{m\to\infty} t^{q-1} y_m(t) = t^{q-1} y(t), \quad \text{for } t \in (0, a].$$

Then by (H1), we have

$$f(t, x_m(t)) = f(t, t^{q-1} y_m(t)) \to f(t, t^{q-1} y(t)) = f(t, x(t)), \quad \text{as } m \to \infty,$$

where $x_m(t) = t^{q-1} y_m(t)$ and $x(t) = t^{q-1} y(t)$.

On the one hand, using (H2), we get for each $t \in J'$,

$$(t-s)^{q-1} |f(s, x_m(s)) - f(s, x(s))| \le (t-s)^{q-1} 2m(s), \quad \text{a.e. in } [0, t].$$

On the other hand, the function $s \to (t-s)^{q-1} 2m(s)$ is integrable for $s \in [0, t]$ and $t \in J$. By Lebesgue's dominated convergence theorem, we get

$$\int_0^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x(s))| ds \to 0, \quad \text{as } m \to 0.$$

For $t \in [0, a]$

$$\begin{aligned}
&|(\mathscr{T} y_m)(t) - (\mathscr{T} y)(t)| = |t^{1-q}(T x_m(t) - T x(t))| \\
&\le |P_q(t)(g(x_m) - g(x))| + t^{1-q} \left| \int_0^t (t-s)^{q-1} P_q(t-s)(f(s, x_m(s)) - f(s, x(s))) ds \right| \\
&\le \frac{ML}{\Gamma(q)} \|x_m - x\|_q + \frac{M t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x(s))| ds \\
&\le \frac{ML}{\Gamma(q)} \|y_m - y\| + \frac{M t^{1-q}}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x(s))| ds.
\end{aligned}$$

Therefore, $\mathscr{T} y_m \to \mathscr{T} y$ pointwise on J as $m \to \infty$, by which Lemma 4.10 implies that $\mathscr{T} y_m \to \mathscr{T} y$ uniformly on J as $m \to \infty$ and so $\mathscr{T}$ is continuous. □

Lemma 4.12. Assume that (H0)-(H2), (H3)$'$ and (H4)$'$ hold. Then $\{\mathscr{T} y : \; y \in B(J)\}$ is equicontinuous.

Proof. For any $y \in B(J)$, for $t_1 = 0$, $0 < t_2 \le a$, then, we get

$$\begin{aligned}
&|(\mathscr{T} y)(t_2) - (\mathscr{T} y)(0)| \\
&\le \left| P_q(t_2)(x_0 - g(x)) - \frac{x_0 - g(x)}{\Gamma(q)} \right| + \left| t_2^{1-q} \int_0^{t_2} (t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right| \\
&\le \left| P_q(t_2)(x_0 - g(x)) - \frac{x_0 - g(x)}{\Gamma(q)} \right| + \frac{M}{\Gamma(q)} t_2^{1-q} \int_0^{t_2} (t_2 - s)^{q-1} m(s) ds \\
&\to 0, \quad \text{as } t_2 \to 0.
\end{aligned}$$

For any $y \in B(J)$ and $0 < t_1 < t_2 \le a$, we get

$$\begin{aligned}
|(\mathscr{T} y)(t_2) - (\mathscr{T} y)(t_1)| &\le |(\mathscr{T}_1 y)(t_2) - (\mathscr{T}_1 y)(t_1)| + |(\mathscr{T}_2 y)(t_2) - (\mathscr{T}_2 y)(t_1)| \\
&\le |(P_q(t_2) - P_q(t_1))(x_0 - g(x))| + I_1 + I_2 + I_3,
\end{aligned}$$

where I_1, I_2 and I_3 are defined as in the proof of Lemma 4.10. According to Proposition 4.7, we know that $|(\mathscr{T} y)(t_2) - (\mathscr{T} y)(t_1)|$ tends to zero independently of $y \in B(J)$ as $t_2 \to t_1$, which means that $\{\mathscr{T} y : \; y \in B(J)\}$ is equicontinuous. □

Lemma 4.13. Assume that (H1), (H2), (H3)′ and (H4)′ hold. Then $\mathscr{T}$ maps $B(J)$ into $B(J)$, and $\mathscr{T}$ is continuous in $B(J)$.

Proof. For any $y \in B(J)$, we have

$$|(\mathscr{T}y)(t)| \le \frac{|x_0| + L_1 r + L_2}{\Gamma(q)} \le r, \quad \text{for } t = 0$$

and

$$|(\mathscr{T}y)(t)| = t^{1-q}|(Tx)(t)| \le r, \quad \text{for } t \in (0, a].$$

Hence, $\|\mathscr{T}y\|_B \le r$, for any $y \in B(J)$. Using the similar argument as that we did in the proof of Lemma 4.11, we know that $\mathscr{T}$ is continuous in $B(J)$. □

4.2.4 *Compact Semigroup Case*

In the following, we suppose that the operator A generates a compact C_0-semigroup $\{Q(t)\}_{t\ge 0}$ on X, that is, for any $t > 0$, the operator $Q(t)$ is compact.

Theorem 4.14. Assume that $Q(t)(t > 0)$ is compact. Furthermore assume that (H1)-(H4) hold. Then the nonlocal Cauchy problem (4.1) has at least one mild solution in $B_r^{(q)}(J')$.

Proof. Obviously, x is a mild solution of (4.1) in $B_r^{(q)}(J')$ if and only if y is a fixed point of $y = \mathscr{T}y$ in $B(J)$, where $x(t) = t^{q-1}y(t)$. So, it is enough to prove that $y = \mathscr{T}y$ has a fixed point in $B(J)$.

For any $y_1, y_2 \in B(J)$, according to (H3), we have

$$\begin{aligned}
|\mathscr{T}_1 y_1(t) - \mathscr{T}_1 y_2(t)| &= t^{1-q}|(T_1 x_1)(t) - (T_1 x_2)(t)| \\
&\le \frac{M}{\Gamma(q)}|g(x_1) - g(x_2)| \\
&\le \frac{ML}{\Gamma(q)}\|x_1 - x_2\|_q \\
&= \frac{ML}{\Gamma(q)}\|y_1 - y_2\|
\end{aligned}$$

which implies that $\|\mathscr{T}_1 y_1 - \mathscr{T}_1 y_2\| \le \frac{ML}{\Gamma(q)}\|y_1 - y_2\|$. Thus, we obtain that

$$\alpha(\mathscr{T}_1(B(J))) \le \frac{ML}{\Gamma(q)}\alpha(B(J)). \tag{4.9}$$

Next, we will show that for any $t \in [0, a]$, $V(t) = \{(\mathscr{T}_2 y)(t), y \in B(J)\}$ is relatively compact in X. Obviously, $V(0)$ is relatively compact in X. Let $t \in (0, a]$ be fixed. For $\forall\, \varepsilon \in (0, t)$ and $\forall\, \delta > 0$, define an operator $\mathscr{T}_{\varepsilon,\delta}$ on $B(J)$ by the formula

$$\begin{aligned}
(\mathscr{T}_{\varepsilon,\delta} y)(t) &= qt^{1-q}\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s, x(s))d\theta ds \\
&= qt^{1-q}Q(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta - \varepsilon^q\delta)f(s, x(s))d\theta ds,
\end{aligned}$$

where $x \in B_r^{(q)}(J')$. Then from the compactness of $Q(\varepsilon^q\delta)(\varepsilon^q\delta > 0)$, we obtain that the set $V_{\varepsilon,\delta}(t) = \{(\mathscr{T}_{\varepsilon,\delta}y)(t),\ y \in B(J)\}$ is relatively compact in X for $\forall\ \varepsilon \in (0,t)$ and $\forall\ \delta > 0$. Moreover, for every $y \in B(J)$, we have

$$
\begin{aligned}
&|(\mathscr{T}_2 y)(t) - (\mathscr{T}_{\varepsilon,\delta}y)(t)| \\
\le{}& \left|qt^{1-q}\int_0^t\int_0^\delta \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s,x(s))d\theta ds\right| \\
&+\left|qt^{1-q}\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s,x(s))d\theta ds\right| \\
\le{}& qMt^{1-q}\int_0^t (t-s)^{q-1}m(s)ds\int_0^\delta \theta M_q(\theta)d\theta \\
&+qMt^{1-q}\int_{t-\varepsilon}^t (t-s)^{q-1}m(s)ds\int_0^\infty \theta M_q(\theta)d\theta \\
\le{}& qMt^{1-q}\int_0^t (t-s)^{q-1}m(s)ds\int_0^\delta \theta M_q(\theta)d\theta + \frac{M}{\Gamma(q)}t^{1-q}\int_{t-\varepsilon}^t (t-s)^{q-1}m(s)ds \\
\to{}& 0, \quad \text{as } \varepsilon \to 0,\ \delta \to 0.
\end{aligned}
$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t),\ t > 0$. Hence the set $V(t),\ t > 0$ is also relatively compact in X. Therefore, $\{(\mathscr{T}_2 y)(t),\ y \in B(J)\}$ is relatively compact by Ascoli-Arzela Theorem. Thus, we have $\alpha(T_2(B_r^{(q)}(J'))) = 0$. By (4.9), we have

$$
\begin{aligned}
\alpha(\mathscr{T}(B(J))) &\le \alpha(\mathscr{T}_1(B(J))) + \alpha(\mathscr{T}_2(B(J))) \\
&\le \frac{ML}{\Gamma(q)}\alpha(B(J)).
\end{aligned}
$$

Thus, the operator $\mathscr{T}$ is an α-contraction in $B(J)$. By Lemma 4.11, we know that $\mathscr{T}$ is continuous. Hence, Theorem 1.44 shows that $\mathscr{T}$ has a fixed point $y^* \in B(J)$. Let $x^*(t) = t^{q-1}y^*(t)$. Then x^* is a mild solution of (4.1). $\square$

Theorem 4.15. Assume that $Q(t)(t > 0)$ is compact. Furthermore assume that (H1), (H2), (H3)$'$ and (H4)$'$ hold. Then the nonlocal Cauchy problem (4.1) has at least one mild solution in $B_r^{(q)}(J')$.

Proof. Since Proposition 4.9, $Q(t)(t > 0)$ is equicontinuous, which implies (H0) is satisfied. Then, by Lemmas 4.10-4.11, we know that $\mathscr{T} : B(J) \to B(J)$ is bounded, continuous and $\{\mathscr{T}y :\ y \in B(J)\}$ is equicontinuous.

According to the argument of Theorem 4.14, we only need prove that for any $t \in J$, the set $V_1(t) = \{(\mathscr{T}_1 y)(t),\ y \in B(J)\}$ is relatively compact in X. Obviously, $V_1(0)$ is relatively compact in X. Let $0 < t \le a$ be fixed. For $\forall\ \delta > 0$, define an operator $\mathscr{T}_1^\delta$ on $B(J)$ by the formula

$$
\begin{aligned}
(\mathscr{T}_1^\delta y)(t) &= q\int_\delta^\infty \theta M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta \\
&= qQ(t^q\delta)\int_\delta^\infty \theta M_q(\theta)Q(t^q\theta - t^q\delta)(x_0 - g(x))d\theta,
\end{aligned}
$$

where $x(t) = t^{q-1}y(t),\ t \in (0,a]$. From the compactness of $Q(t^q\delta)$ $(t^q\delta > 0)$, we obtain that the set $V_1^\delta(t) = \{(\mathscr{T}_1^\delta y)(t),\ y \in B(J)\}$ is relatively compact in X for $\forall\ \delta > 0$. Moreover, for any $y \in B(J)$, we have

$$
\begin{aligned}
|(\mathscr{T}_1 y)(t) - (\mathscr{T}_1^\delta y)(t)| &= \left| q\int_0^\delta \theta M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta \right| \\
&\leq qM(|x_0| + L_1 r + L_2)\int_0^\delta \theta M_q(\theta)d\theta.
\end{aligned}
$$

Therefore, there are relatively compact sets arbitrarily close to the set $V_1(t),\ t > 0$. Hence the set $V_1(t),\ t > 0$ is also relatively compact in X. Moreover, $\{\mathscr{T}y :\ y \in B(J)\}$ is uniformly bounded by Lemmas 4.13. Therefore, $\{(\mathscr{T}y)(t),\ y \in B(J)\}$ is relatively compact by Ascoli-Arzela Theorem. Hence, Theorem 1.44 shows that $\mathscr{T}$ has a fixed point $y^* \in B(J)$. Let $x^*(t) = t^{q-1}y^*(t)$. Then x^* is a mild solution of (4.1). $\square$

Remark 4.16. If g is not a compact map, we use another method given in Zhu and Li, 2008 to consider the following integral equations

$$
x(t) = t^{q-1}P_q\left(t + \frac{1}{n}\right)(x_0 - g(x)) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s,x(s))ds, \quad t \in (0,a]. \tag{4.10}
$$

For any $n \in \mathbb{N}$, noticing that the operator $Q(\frac{1}{n})$ is compact, one can easily derive the relative compactness of $V(0)$ and $V(t)(t > 0)$. Then, (4.10) has a solution in $B_r^{(q)}(J')$. By passing the limit, as $n \to \infty$, one obtains a mild solution of the nonlocal Cauchy problem (4.1). However, because $Q(t)$ is replaced by $Q(\frac{1}{n})$, one needs a more restrictive condition than (H4)$'$, such as

(H4)$''$ there exists a constant $r > 0$ such that

$$
\frac{M_\varepsilon}{\Gamma(q)}\left(|x_0| + L_1 r + L_2 + \sup_{t\in(0,a]}\left\{t^{1-q}\int_0^t (t-s)^{q-1}m(s)ds\right\}\right) \leq r,
$$

where $M_\varepsilon = \sup_{t\in[0,a+\varepsilon]}\|Q(t)\|_{B(X)}$, ε is a small constant.

Remark 4.17. The condition (H2) of Theorems 4.14-4.15 can be replaced by the following condition.

(H2)$'$ There exist a constant $q_1 \in (0,q)$ and $m \in L^{\frac{1}{q_1}}(J,\mathbb{R}^+)$ such that

$$
|f(t,x)| \leq m(t) \quad \text{for all } x \in B_r^{(q)}(J') \ \text{ and almost all } t \in [0,a].
$$

In fact, if (H2)$'$ holds, by using the Hölder inequality, for any $t_1, t_2 \in J'$ and $t_1 < t_2$, we obtain

$$
\begin{aligned}
&|{}_0D_t^{-q}m(t_2) - {}_0D_t^{-q}m(t_1)| \\
&= \frac{1}{\Gamma(q)}\left|\int_0^{t_1}((t_2-s)^{q-1} - (t_1-s)^{q-1})m(s)ds + \int_{t_1}^{t_2}(t_2-s)^{q-1}m(s)ds\right| \\
&\leq \frac{1}{\Gamma(q)}\left(\int_0^{t_1}((t_1-s)^{q-1} - (t_2-s)^{q-1})^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_0^{t_1}(m(s))^{\frac{1}{q_1}}ds\right)^{q_1} \\
&\quad + \frac{1}{\Gamma(q)}\left(\int_{t_1}^{t_2}((t_2-s)^{q-1})^{\frac{1}{1-q_1}}ds\right)^{1-q_1}\left(\int_{t_1}^{t_2}(m(s))^{\frac{1}{q_1}}ds\right)^{q_1}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(q)}\bigg(\int_0^{t_1}((t_1-s)^{\frac{q-1}{1-q_1}}-(t_2-s)^{\frac{q-1}{1-q_1}})ds\bigg)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}}\\
&\quad+\frac{1}{\Gamma(q)}\bigg(\int_{t_1}^{t_2}(t_2-s)^{\frac{q-1}{1-q_1}}ds\bigg)^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}}\\
&\leq \frac{\|m\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left(t_1^{\frac{q-q_1}{1-q_1}}+(t_2-t_1)^{\frac{q-q_1}{1-q_1}}-t_2^{\frac{q-q_1}{1-q_1}}\right)^{1-q_1} \qquad (4.11)\\
&\quad+\frac{\|m\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}\left((t_2-t_1)^{\frac{q-q_1}{1-q_1}}\right)^{1-q_1}\\
&\leq \frac{2\|m\|_{L^{\frac{1}{q_1}}}}{\Gamma(q)}\left(\frac{1-q_1}{q-q_1}\right)^{1-q_1}(t_2-t_1)^{q-q_1}\to 0,\quad \text{as } t_2\to t_1,
\end{aligned}
$$

where

$$
\|m\|_{L^{\frac{1}{q_1}}}=\left(\int_0^a (m(t))^{\frac{1}{q_1}}dt\right)^{q_1}.
$$

Furthermore,

$$
\begin{aligned}
&t^{1-q}\int_0^t (t-s)^{q-1}m(s)ds\\
&\leq t^{1-q}\bigg(\int_0^t (t-s)^{\frac{q-1}{1-q_1}}ds\bigg)^{1-q_1}\bigg(\int_0^t (m(s))^{\frac{1}{q_1}}ds\bigg)^{q_1} \qquad (4.12)\\
&\leq \left(\frac{1-q_1}{q-q_1}\right)^{1-q_1} t^{1-q_1}\|m\|_{L^{\frac{1}{q_1}}}\to 0,\quad \text{as } t\to 0.
\end{aligned}
$$

Thus, (4.11) and (4.12) mean that ${}_0D_t^{-q}m\in C(J',\mathbb{R}^+)$, and $\lim_{t\to 0^+}t^{1-q}{}_0D_t^{-q}m(t) = 0$. Hence, (H2) holds.

Example 4.18. Let $X=L^2([0,\pi],\mathbb{R})$. Consider the following fractional partial differential equations.

$$
\begin{cases}
\partial_t^q u(t,z)=\partial_z^2 u(t,z)+\partial_z G(t,u(t,z)), & z\in[0,\pi],\ t\in(0,a],\\
u(t,0)=u(t,\pi)=0, & t\in(0,a],\\
u(0,z)+\displaystyle\sum_{i=0}^{n}\int_0^{\pi}k(z,y)u(t_i,y)dy=u_0(z), & z\in[0,\pi],
\end{cases} \qquad (4.13)
$$

where ∂_t^q is Riemann-Liouville fractional partial derivative of order $0<q<1$, $a>0$, G is a given function, n is a positive integer, $0<t_0<t_1<\cdots<t_n\leq a$, $u_0(z)\in X=L^2([0,\pi],\mathbb{R})$, $k(z,y)\in L^2([0,\pi]\times[0,\pi],\mathbb{R}^+)$.

We define an operator A by $Av=v''$ with the domain

$$
D(A)=\{v(\cdot)\in X:\ v,v' \text{ absolutly continuous},\ v''\in X,\ v(0)=v(\pi)=0\}.
$$

Then A generates a strongly continuous semigroup $\{Q(t)\}_{t\geq 0}$ which is compact, analytic and self-adjoint. Clearly the nonlocal Cauchy problem (4.2) and (H1) are satisfied.

The system (4.13) can be reformulated as the following nonlocal Cauchy problem in X

$$\begin{cases} {}_0D_t^q x(t) = Ax(t) + f(t, x(t)), & \text{almost all } t \in [0, a], \\ {}_0D_t^{q-1} x(0) + g(x) = x_0, \end{cases}$$

where $x(t) = u(t, \cdot)$, that is $x(t)(z) = u(t, z),\ t \in (0, a],\ z \in [0, \pi]$. The function $f : J' \times X \to X$ is given by

$$f(t, x(t))(z) = \partial_z G(t, u(t, z)),$$

and the operator $g : C(J', X) \to L(J', X)$ is given by

$$g(x)(z) = \sum_{i=0}^{n} K_g x(t_i)(z),$$

where $K_g v(z) = \displaystyle\int_0^\pi k(z, y)v(y)dy$, for $v \in X = L^2([0, \pi], \mathbb{R}),\ z \in [0, \pi]$.

We can take $q = 1/3$ and $f(t, x(t)) = t^{-1/4} \sin x(t)$, and choose

$$m(t) = t^{-1/4}, \quad L = (n + 1)\left(\int_0^\pi \int_0^\pi k^2(z, y)dydz \right)^{\frac{1}{2}}$$

and

$$r = \frac{M}{\Gamma(\frac{1}{3}) - ML}\left(|x_0| + g(0) + \frac{\Gamma(\frac{1}{3})\Gamma(\frac{3}{4})}{\Gamma(\frac{12}{13})} a^{\frac{3}{4}} \right).$$

Then, (H1)-(H4) are satisfied (noting that $K_g : X \to X$ is completely continuous). According to Theorem 4.14, system (4.13) has a mild solution in $B_r^{(1/3)}((0, a])$ provided that $\frac{ML}{\Gamma(1/3)} < 1$.

4.2.5 *Noncompact Semigroup Case*

If $Q(t)$ is noncompact, we give an assumption as follows.
(H5) There exists a constant $\ell > 0$ such that for any bounded $D \subset X$,

$$\alpha(f(t, D)) \le \ell\alpha(D).$$

Theorem 4.19. Assume that (H0)-(H5) hold. Then the nonlocal Cauchy problem (4.1) has at least one mild solution in $B_r^{(q)}(J')$.

Proof. By Lemmas 4.11-4.12, we know that $\mathscr{T}_2 : B(J) \to B(J)$ is bounded, continuous and $\{\mathscr{T}_2 y :\ y \in B(J)\}$ is equicontinuous. Next, we will show that $\mathscr{T}_2$ is compact in a subset of $B(J)$.

For each bounded subset $B_0 \subset B(J)$, set

$$\mathscr{T}^1(B_0) = \mathscr{T}_2(B_0),\ \ \mathscr{T}^n(B_0) = \mathscr{T}_2\left(\overline{co}(\mathscr{T}^{n-1}(B_0))\right),\ n = 2, 3, \dots .$$

Then, from Propositions 1.28-1.29, for any $\varepsilon > 0$, there is a sequence $\{y_n^{(1)}\}_{n=1}^{\infty} \subset B_0$ such that

$$\begin{aligned}
\alpha(\mathscr{T}^1(B_0(t))) &= \alpha(\mathscr{T}_2(B_0(t))) \\
&\le 2\alpha\left(t^{1-q}\int_0^t (t-s)^{q-1}P_q(t-s)f(s,\{s^{q-1}y_n^{(1)}(s)\}_{n=1}^{\infty})ds\right)+\varepsilon \\
&\le \frac{4M}{\Gamma(q)}t^{1-q}\int_0^t (t-s)^{q-1}\alpha\left(f(s,\{s^{q-1}y_n^{(1)}(s)\}_{n=1}^{\infty})\right)ds+\varepsilon \\
&\le \frac{4M\ell}{\Gamma(q)}t^{1-q}\int_0^t (t-s)^{q-1}\alpha(\{s^{q-1}y_n^{(1)}(s)\}_{n=1}^{\infty})ds+\varepsilon \\
&\le \frac{4M\ell\alpha(B_0)}{\Gamma(q)}t^{1-q}\int_0^t (t-s)^{q-1}s^{q-1}ds+\varepsilon \\
&\le \frac{4M\ell\Gamma(q)t^q\alpha(B_0)}{\Gamma(2q)}+\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha(\mathscr{T}^1(B_0(t))) \le \frac{4M\ell\Gamma(q)t^q}{\Gamma(2q)}\alpha(B_0).$$

From Propositions 1.28-1.29, for any $\varepsilon > 0$, there is a sequence $\{y_n^{(2)}\}_{n=1}^{\infty} \subset \overline{co}(\mathscr{T}^1(B_0))$ such that

$$\begin{aligned}
\alpha(\mathscr{T}^2(B_0(t))) &= \alpha(\mathscr{T}_2(\overline{co}(\mathscr{T}^1(B_0(t))))) \\
&\le 2\alpha\left(t^{1-q}\int_0^t (t-s)^{q-1}P_q(t-s)f(s,\{s^{q-1}y_n^{(2)}(s)\}_{n=1}^{\infty})ds\right)+\varepsilon \\
&\le \frac{4Mt^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left((f(s,\{s^{q-1}y_n^{(2)}(s)\}_{n=1}^{\infty})\right)ds+\varepsilon \\
&\le \frac{4M\ell t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(\{s^{q-1}y_n^{(2)}(s)\}_{n=1}^{\infty})ds+\varepsilon \\
&\le \frac{4M\ell t^{1-q}}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^{q-1}\alpha(\{y_n^{(2)}(s)\}_{n=1}^{\infty})ds+\varepsilon \\
&\le \frac{(4M\ell)^2t^{1-q}}{\Gamma(2q)}\int_0^t (t-s)^{q-1}s^{2q-1}ds+\varepsilon \\
&= \frac{(4M\ell)^2\Gamma(q)}{\Gamma(3q)}t^{2q}\alpha(B_0)+\varepsilon.
\end{aligned}$$

It can be shown, by mathematical induction, that for every $\bar{n} \in \mathbb{N}$,

$$\alpha(\mathscr{T}^{\bar{n}}(B_0(t))) \le \frac{(4M\ell)^{\bar{n}}\Gamma(q)}{\Gamma((\bar{n}+1)q)}t^{\bar{n}q}\alpha(B_0).$$

Since

$$\lim_{\bar{n}\to\infty}\frac{(4M\ell a^q)^{\bar{n}}\Gamma(q)}{\Gamma((\bar{n}+1)q)} = 0,$$

there exists a positive integer $\hat{n}$ such that

$$\frac{(4M\ell)^{\hat{n}}\Gamma(q)}{\Gamma((\hat{n}+1)q)}t^{\hat{n}q} \leq \frac{(4M\ell a^q)^{\hat{n}}\Gamma(q)}{\Gamma((\hat{n}+1)q)} = k < 1.$$

Then

$$\alpha(\mathscr{T}^{\hat{n}}(B_0(t))) \leq k\alpha(B_0).$$

We know from Proposition 1.26, $\mathscr{T}^{\hat{n}}(B_0(t))$ is bounded and equicontinuous. Then, from Proposition 1.27, we have

$$\alpha(\mathscr{T}^{\hat{n}}(B_0)) = \max_{t\in[0,a]} \alpha(\mathscr{T}^{\hat{n}}(B_0(t))).$$

Hence

$$\alpha(\mathscr{T}^{\hat{n}}(B_0)) \leq k\alpha(B_0).$$

Let

$$D_0 = B(J),\ D_1 = \overline{co}(\mathscr{T}^{\hat{n}}(D)), ...,\ D_n = \overline{co}(\mathscr{T}^{\hat{n}}(D_{n-1})),\ n = 2, 3, ... \, .$$

Then, we can get

(i) $D_0 \supset D_1 \supset D_2 \supset \cdots \supset D_{n-1} \supset D_n \supset \cdots$;
(ii) $\lim_{n\to\infty} \alpha(D_n) = 0$.

Then $\hat{D} = \bigcap_{n=0}^{\infty} D_n$ is a nonempty, compact and convex subset in $B(J)$.

We will prove $\mathscr{T}_2(\hat{D}) \subset \hat{D}$. Firstly, we show

$$\mathscr{T}_2(D_n) \subset D_n, \quad n = 0, 1, 2, ... \, . \tag{4.14}$$

From $\mathscr{T}^1(D_0) = \mathscr{T}_2(D_0) \subset D_0$, we know $\overline{co}(\mathscr{T}^1(D_0)) \subset D_0$. Therefore

$$\mathscr{T}^2(D_0) = \mathscr{T}_2(\overline{co}(\mathscr{T}^1(D_0))) \subset \mathscr{T}_2(D_0) = \mathscr{T}^1(D_0),$$

$$\mathscr{T}^3(D_0) = \mathscr{T}_2(\overline{co}(\mathscr{T}^2(D_0))) \subset \mathscr{T}_2(\overline{co}(\mathscr{T}^1(D_0))) = \mathscr{T}^2(D_0),$$

$$...$$

$$\mathscr{T}^{\hat{n}}(D_0) = \mathscr{T}_2(\overline{co}(\mathscr{T}^{\hat{n}-1}(D_0))) \subset \mathscr{T}_2(\overline{co}(\mathscr{T}^{\hat{n}-2}(D_0))) = \mathscr{T}^{\hat{n}-1}(D_0).$$

Hence, $D_1 = \overline{co}(\mathscr{T}^{\hat{n}}(D_0)) \subset \overline{co}(\mathscr{T}^{\hat{n}-1}(D_0))$, so $\mathscr{T}(D_1) \subset \mathscr{T}(\overline{co}(\mathscr{T}^{\hat{n}-1}(D_0))) = \mathscr{T}^{\hat{n}}(D_0) \subset \overline{co}(\mathscr{T}^{\hat{n}}(D_0)) = D_1$. Employing the same method, we can prove $\mathscr{T}_2(D_n) \subset D_n (n = 0, 1, 2, ...)$. By (4.14), we get $\mathscr{T}_2(\hat{D}) \subset \bigcap_{n=0}^{\infty} \mathscr{T}_2(D_n) \subset \bigcap_{n=0}^{\infty} D_n = \hat{D}$. Then $\mathscr{T}_2(\hat{D})$ is compact. Hence, $\alpha(\mathscr{T}_2(\hat{D})) = 0$.

On the other hand, for any $y_1, y_2 \in \hat{D}$ and $t \in (0, a]$, according to (H3), we have

$$\begin{aligned} |\mathscr{T}_1 y_1(t) - \mathscr{T}_1 y_2(t)| &= t^{1-q}|(T_1x_1)(t) - (T_1x_2)(t)| \\ &\leq \frac{M}{\Gamma(q)}|g(x_1) - g(x_2)| \\ &\leq \frac{ML}{\Gamma(q)}\|x_1 - x_2\|_q \\ &= \frac{ML}{\Gamma(q)}\|y_1 - y_2\|, \end{aligned} \tag{4.15}$$

which implies that $\|\mathscr{T}_1 y_1 - \mathscr{T}_1 y_2\| \leq \frac{ML}{\Gamma(q)}\|y_1 - y_2\|$. Thus, we obtain that

$$\alpha(\mathscr{T}_1(\hat{D})) \leq \frac{ML}{\Gamma(q)}\alpha(\hat{D}). \tag{4.16}$$

By (4.16), we have

$$\begin{aligned}\alpha(\mathscr{T}(\hat{D})) &\leq \alpha(\mathscr{T}_1(\hat{D})) + \alpha(\mathscr{T}_2(\hat{D}))\\ &\leq \frac{ML}{\Gamma(q)}\alpha(\hat{D}).\end{aligned}$$

Thus, the operator $\mathscr{T}$ is an α-contraction in $\hat{D}$. By Lemma 4.11, we know that $\mathscr{T}$ is continuous. Hence, Theorem 1.44 shows that $\mathscr{T}$ has a fixed point $y^* \in B(J)$. Let $x^*(t) = t^{q-1}y^*(t)$. Then x^* is a mild solution of (4.1). □

Theorem 4.20. Assume that (H0)-(H2), (H3)$'$, (H4)$'$ and (H5) hold, then the nonlocal Cauchy problem (4.1) has at least one mild solution in $B_r^{(q)}(J')$.

Proof. Since $g(x)$ is compact and $P_q(t)$ is bounded, for every $t > 0$, $\{(\mathscr{T}_1 y)(t),\ y \in B(J)\}$ is relatively compact. Thus, we have $\alpha(\mathscr{T}_1(B(J))) = 0$.

By the proof of Theorem 4.19, we know that there exists a $\hat{D} \subset B(J)$ such that $\mathscr{T}_2(\hat{D})$ is relatively compact, i.e., $\alpha(\mathscr{T}_2(\hat{D})) = 0$. Hence, we have

$$\alpha(\mathscr{T}(\hat{D})) \leq \alpha(\mathscr{T}_1(\hat{D})) + \alpha(\mathscr{T}_2(\hat{D})) = 0.$$

Hence, Theorem 1.44 shows that $\mathscr{T}$ has a fixed point $y^* \in B(J)$. Let $x^*(t) = t^{q-1}y^*(t)$. Then x^* is a mild solution of (4.1). □

4.3 Evolution Equations with Caputo Derivative

4.3.1 *Introduction*

Consider the following nonlocal Cauchy problems of fractional evolution equation with Caputo derivative

$$\begin{cases} {}_0^C D_t^q x(t) = Ax(t) + f(t, x(t)), \quad \text{a.e. } t \in [0, a], \\ x(0) + g(x) = x_0, \end{cases} \tag{4.17}$$

where ${}_0^C D_t^q$ is Caputo derivative of order q, $0 < q < 1$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. C_0 semigroup) $\{Q(t)\}_{t\geq 0}$ in Banach space X, $f : J \times X \to X$, $g : C(J, X) \to L(J, X)$ are given operators satisfying some assumptions and x_0 is an element of the Banach space X.

In this section, by using the theory of Hausdorff measure of noncompactness and fixed point theorems, we study the nonlocal Cauchy problem (4.17) in the cases $Q(t)$ is compact or noncompact. Subsection 4.3.2 is devoted to obtain the appropriate definition on the mild solutions of the problem (4.17) by considering a integral equation which is given in terms of probability density. In Subsection 4.3.3, we give some preliminary lemmas. Subsection 4.3.4 provides various existence theorems of mild solutions for the Cauchy problem (4.17) in the case $Q(t)$ is compact. In Subsection 4.3.5, we establish various existence theorems of mild solutions for the Cauchy problem (4.17) in the case $Q(t)$ is noncompact.

4.3.2 *Definition of Mild Solutions*

Lemmas 4.21. If

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}[Ax(s) + f(s, x(s))]ds, \quad \text{for } t \geq 0 \tag{4.18}$$

holds, then we have

$$x(t) = S_q(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s))ds, \quad \text{for } t \geq 0, \tag{4.19}$$

where

$$S_q(t) = \int_0^\infty M_q(\theta) Q(t^q\theta)d\theta, \quad P_q(t) = \int_0^\infty q\theta M_q(\theta) Q(t^q\theta)d\theta.$$

Proof. Let $\lambda > 0$. Applying the Laplace transform

$$\nu(\lambda) = \int_0^\infty e^{-\lambda s} x(s)ds \ \text{ and } \omega(\lambda) = \int_0^\infty e^{-\lambda s} f(s, x(s))ds, \quad \lambda > 0$$

to (4.18), we have

$$\begin{aligned} \nu(\lambda) &= \frac{1}{\lambda}(x_0 - g(x)) + \frac{1}{\lambda^q} A\nu(\lambda) + \frac{1}{\lambda^q}\, \omega(\lambda) \\ &= \lambda^{q-1}(\lambda^q I - A)^{-1}(x_0 - g(x)) + (\lambda^q I - A)^{-1}\omega(\lambda) \\ &= \lambda^{q-1}\int_0^\infty e^{-\lambda^q s} Q(s)(x_0 - g(x))ds + \int_0^\infty e^{-\lambda^q s} Q(s)\omega(\lambda)ds, \end{aligned} \tag{4.20}$$

provided that the integrals in (4.20) exist, where I is the identity operator defined on X.

Using (4.6) and (4.20), we get

$$\begin{aligned} &\lambda^{q-1}\int_0^\infty e^{-\lambda^q s} Q(s)(x_0 - g(x))ds \\ &= \int_0^\infty q(\lambda t)^{q-1} e^{-(\lambda t)^q} Q(t^q)(x_0 - g(x))dt \\ &= \int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}[e^{-(\lambda t)^q}] Q(t^q)(x_0 - g(x))dt \\ &= \int_0^\infty \int_0^\infty \theta\psi_q(\theta) e^{-\lambda t\theta} Q(t^q)(x_0 - g(x))d\theta dt \\ &= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_q(\theta) Q\left(\frac{t^q}{\theta^q}\right)(x_0 - g(x))d\theta\right]dt. \end{aligned} \tag{4.21}$$

According to (4.8), (4.20) and (4.21), we have

$$\nu(\lambda) = \int_0^\infty e^{-\lambda t}\Big[\int_0^\infty \psi_q(\theta)Q\Big(\frac{t^q}{\theta^q}\Big)(x_0 - g(x))d\theta \\ + q\int_0^t\int_0^\infty \psi_q(\theta)Q\Big(\frac{(t-s)^q}{\theta^q}\Big)f(s,x(s))\frac{(t-s)^{q-1}}{\theta^q}d\theta ds\Big]dt.$$

Now we can invert the last Laplace transform to get

$$\begin{aligned} x(t) &= \int_0^\infty \psi_q(\theta)Q\Big(\frac{t^q}{\theta^q}\Big)(x_0 - g(x))d\theta \\ &\quad + q\int_0^t\int_0^\infty \psi_q(\theta)Q\Big(\frac{(t-s)^q}{\theta^q}\Big)f(s,x(s))\frac{(t-s)^{q-1}}{\theta^q}d\theta ds \\ &= \int_0^\infty M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta \\ &\quad + q\int_0^t\int_0^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s,x(s))d\theta ds \\ &= S_q(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s,x(s))ds. \end{aligned}$$

The proof is completed. $\square$

Due to Lemma 4.21, we give the following definition of the mild solution of (4.17).

Definition 4.22. By the mild solution of the nonlocal Cauchy problem (4.17), we mean that the function $x \in C(J, X)$ which satisfies

$$x(t) = S_q(t)(x_0 - g(x)) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s,x(s))ds, \quad \text{for } t \in [0,a].$$

Suppose that A is the infinitesimal generator of a C_0-semigroup $\{Q(t)\}_{t\geq 0}$ of uniformly bounded linear operators on Banach space X. This means that there exists $M > 1$ such that $M = \sup_{t\in[0,\infty)} \|Q(t)\|_{B(X)} < \infty$.

Proposition 4.23. (Zhou and Jiao, 2010a) For any fixed $t > 0$, $\{S_q(t)\}_{t>0}$ and $\{P_q(t)\}_{t>0}$ are linear and bounded operators, i.e., for any $x \in X$

$$|S_q(t)x| \leq M|x|, \quad |P_q(t)x| \leq \frac{M}{\Gamma(q)}|x|.$$

Proposition 4.24. (Zhou and Jiao, 2010a) Operators $\{S_q(t)\}_{t>0}$ and $\{P_q(t)\}_{t>0}$ are strongly continuous, which means that, for $\forall x \in X$ and $0 < t' < t'' \leq a$, we have

$$|S_q(t'')x - S_q(t')x| \to 0, \quad |P_q(t'')x - P_q(t')x| \to 0, \quad \text{as } t'' \to t'.$$

Proposition 4.25. (Zhou and Jiao, 2010a) Assume that $\{Q(t)\}_{t>0}$ is compact operator. Then $\{S_q(t)\}_{t>0}$ and $\{P_q(t)\}_{t>0}$ are also compact operators.

4.3.3 *Preliminary Lemmas*

For $r > 0$, let $B_r(J)$ be the closed ball of the space $C(J, X)$ with radius r and center at 0, that is,

$$B_r(J) = \{x \in C(J, X) : \ \|x\| \leq r\},$$

where $\|x\| = \sup\limits_{t\in[0,a]} |x(t)|$.

We introduce the following hypotheses:

(H0) $Q(t)(t > 0)$ is equicontinuous, i.e., $Q(t)$ is continuous in the uniform operator topology for $t > 0$;

(H1) for each $t \in J$, the function $f(t, \cdot) : X \to X$ is continuous and for each $x \in X$, the function $f(\cdot, x) : J \to X$ is strongly measurable;

(H2) there exists a function $m \in L(J, \mathbb{R}^+)$ such that

$${}_0D_t^{-q} m \in C(J, \mathbb{R}^+), \quad \lim_{t\to 0+} {}_0D_t^{-q} m(t) = 0$$

and

$$|f(t, x)| \leq m(t) \ \text{ for all } x \in B_r(J) \text{ and almost all } t \in [0, a];$$

(H3) there exists a constant $L \in (0, \frac{1}{M})$, the operator $g : C(J, X) \to L(J, X)$ satisfies

$$|g(x_1) - g(x_2)| \leq L\|x_1 - x_2\|, \text{ for } x_1, x_2 \in B_r(J);$$

(H4) there exists a constant $r > 0$ such that

$$\frac{M}{1 - ML}\left(|x_0| + |g(0)| + \sup_{t\in[0,a]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} m(s)ds\right\}\right) \leq r;$$

(H3)$'$ the operator $g : C(J, X) \to L(J, X)$ is a continuous and compact map, and there exist positive constants L_1, L_2 such that $|g(x)| \leq L_1\|x\| + L_2$ for all $x \in B_r(J)$;

(H4)$'$ there exists a constant $r > 0$ such that

$$\frac{M}{1 - ML_1}\left(|x_0| + L_2 + \sup_{t\in[0,a]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} m(s)ds\right\}\right) \leq r.$$

For any $x \in B_r(J)$, we define an operator T as follows

$$(Tx)(t) = (T_1x)(t) + (T_2x)(t),$$

where

$$\begin{aligned} (T_1x)(t) &= S_q(t)(x_0 - g(x)), && \text{for } t \in [0, a], \\ (T_2x)(t) &= \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s))ds, && \text{for } t \in [0, a]. \end{aligned}$$

Obviously, x is a mild solution of (4.17) in $B_r(J)$ if and only if the operator equation $x = Tx$ has a solution $x \in B_r(J)$.

Lemma 4.26. Assume that (H0)-(H3) hold. Then $\{T_2x : \ x \in B_r(J)\}$ is equicontinuous.

Proof. For any $x \in B_r(J)$, for $t_1 = 0$, $0 < t_2 \le a$, by (H2), we get

$$|(T_2x)(t_2) - (T_2x)(0)| = \left| \int_0^{t_2} (t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right|$$
$$\le \frac{M}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} m(s) ds \to 0 \ \text{ as } t_2 \to 0.$$

For $0 < t_1 < t_2 \le a$, we have

$$\begin{aligned}
&|(T_2x)(t_2) - (T_2x)(t_1)| \\
= &\left| \int_0^{t_2} (t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} P_q(t_1 - s) f(s, x(s)) ds \right| \\
= &\left| \int_{t_1}^{t_2} (t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right| \\
&+ \left| \int_0^{t_1} (t_2 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds \right| \\
&+ \left| \int_0^{t_1} (t_1 - s)^{q-1} P_q(t_2 - s) f(s, x(s)) ds - \int_0^{t_1} (t_1 - s)^{q-1} P_q(t_1 - s) f(s, x(s)) ds \right| \\
\le &\frac{M}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} m(s) ds + \frac{M}{\Gamma(q)} \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] m(s) ds \\
&+ \int_0^{t_1} (t_1 - s)^{q-1} |P_q(t_2 - s) f(s, x(s)) - P_q(t_1 - s) f(s, x(s))| ds \\
\le &\frac{M}{\Gamma(q)} \left| \int_0^{t_2} (t_2 - s)^{q-1} m(s) ds - \int_0^{t_1} (t_1 - s)^{q-1} m(s) ds \right| \\
&+ \frac{2M}{\Gamma(q)} \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] m(s) ds \\
&+ \int_0^{t_1} (t_1 - s)^{q-1} |P_q(t_2 - s) - P_q(t_1 - s)| m(s) ds \\
=: &I_1 + I_2 + I_3.
\end{aligned}$$

Since ${}_0D_t^{-q} m \in C(J, \mathbb{R}^+)$, thus $I_1 \to 0$ as $t_2 \to t_1$.

For $t_1 < t_2$,

$$I_2 \le \frac{2M}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} m(s) ds,$$

then by Lebesgue's dominated convergence theorem, we have that $I_2 \to 0$ as $t_2 \to t_1$.

For $\varepsilon > 0$ be small enough, we have

$$\begin{aligned}
I_3 \le &\int_0^{t_1 - \varepsilon} (t_1 - s)^{q-1} \|P_q(t_2 - s) - P_q(t_1 - s)\|_{B(X)} \ m(s) ds \\
&+ \int_{t_1 - \varepsilon}^{t_1} (t_1 - s)^{q-1} \|P_q(t_2 - s) - P_q(t_1 - s)\|_{B(X)} \ m(s) ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^{t_1} (t_1-s)^{q-1} m(s)ds \sup_{s\in[0,t_1-\varepsilon]} \|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)} \\
&\quad + \frac{2M}{\Gamma(q)} \int_{t_1-\varepsilon}^{t_1} (t_1-s)^{q-1} m(s)ds \\
&\le \int_0^{t_1} (t_1-s)^{q-1} m(s)ds \sup_{s\in[0,t_1-\varepsilon]} \|P_q(t_2-s)-P_q(t_1-s)\|_{B(X)} \\
&\quad + \frac{2M}{\Gamma(q)} \left| \int_0^{t_1} (t_1-s)^{q-1} m(s)ds - \int_0^{t_1-\varepsilon} (t_1-\varepsilon-s)^{q-1} m(s)ds \right| \\
&\quad + \frac{2M}{\Gamma(q)} \int_0^{t_1-\varepsilon} [(t_1-\varepsilon-s)^{q-1}-(t_1-s)^{q-1}] m(s)ds. \\
&=: I_{31}+I_{32}+I_{33}.
\end{aligned}
$$

By (H0), it is easy to see that $I_{31} \to 0$ as $t_2 \to t_1$. Similar to the proof that I_1, I_2 tend to zero, we get $I_{32} \to 0$ and $I_{33} \to 0$ as $\varepsilon \to 0$. Thus, I_3 tends to zero independently of $x \in B_r(J)$ as $t_2 \to t_1,\ \varepsilon \to 0$. Therefore, $|(T_2x)(t_1)-(T_2x)(t_2)|$ tends to zero independently of $x \in B_r(J)$ as $t_2 \to t_1$, which means that $\{T_2x : x \in B_r(J)\}$ is equicontinuous. □

Lemma 4.27. Assume that (H1)-(H4) hold. Then T maps $B_r(J)$ into $B_r(J)$, and T is continuous in $B_r(J)$.

Proof. **Step I.** T maps $B_r(J)$ into $B_r(J)$.

For any $x \in B_r(J)$ and $t \in J$, by using (H1)-(H4), we have

$$
\begin{aligned}
|(Tx)(t)| &= |(T_1x)(t)+(T_2x)(t)| \\
&\le |S_q(t)(x_0-g(x))| + \left| \int_0^t (t-s)^{q-1} P_q(t-s) f(s,x(s))ds \right| \\
&\le M(|x_0| + L\|x-0\| + |g(0)|) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s,x(s))|ds \\
&\le M\left(|x_0| + Lr + |g(0)| + \sup_{t\in[0,a]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} m(s)ds \right\} \right) \\
&\le r.
\end{aligned}
$$

Hence, $\|Tx\| \le r$ for any $x \in B_r(J)$.

Step II. T is continuous in $B_r(J)$.

For any $\{x_m\}_{m=1}^{\infty} \subseteq B_r(J), x \in B_r(J)$ with $\lim_{n\to\infty} \|x_m - x\| = 0$, by the condition (H1), we have

$$
\lim_{m\to\infty} f(s,x_m(s)) = f(s,x(s)).
$$

On the one hand, using (H2), we get for each $t \in J$, $(t-s)^{q-1}|f(s,x_m(s)) - f(s,x(s))| \le (t-s)^{q-1}2m(s)$. On the other hand, the function $s \to (t-s)^{q-1}2m(s)$ is integrable for $s \in [0,t]$ and $t \in J$. By Lebesgue's dominated convergence theorem, we get

$$
\int_0^t (t-s)^{q-1} |f(s,x_m(s)) - f(s,x(s))|ds \to 0, \quad \text{as } m \to \infty.
$$

Then, for $t \in J$,

$$\begin{aligned}
&|(Tx_m)(t) - (Tx)(t)| \\
&\leq |S_q(t)(g(x_m) - g(x))| + \left| \int_0^t (t-s)^{q-1} P_q(t-s)[f(s, x_m(s)) - f(s, x(s))] ds \right| \\
&\leq ML\|x_m - x\| + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_m(s)) - f(s, x(s))| ds, \quad \text{as } m \to \infty.
\end{aligned}$$

Therefore, $Tx_m \to Tx$ pointwise on J as $m \to \infty$, by which Lemma 4.26 implies that $Tx_m \to Tx$ uniformly on J as $m \to \infty$ and so T is continuous. $\square$

Lemma 4.28. Assume that there exists a constant $r > 0$ such that the conditions (H0)-(H2) and (H3)$'$ are satisfied. Then $\{Tx : \ x \in B_r(J)\}$ is equicontinuous.

Proof. For any $x \in B_r(J)$ and $0 \leq t_1 < t_2 \leq a$, we get

$$|(Tx)(t_2) - (Tx)(t_1)| \leq |(S_q(t_2) - S_q(t_1))(x_0 - g(x))| + I_1 + I_2 + I_3,$$

where I_1, I_2 and I_3 are defined as in the proof of Lemma 4.26. According to Proposition 4.24, we know that $|(Tx)(t_2) - (Tx)(t_1)|$ tends to zero independently of $x \in B_r(J)$ as $t_2 \to t_1$, which means that $\{Tx, x \in B_r(J)\}$ is equicontinuous. $\square$

Lemma 4.29. Assume that there exists a constant $r > 0$ such that the conditions (H1), (H2), (H3)$'$ and (H4)$'$ are satisfied. Then T maps $B_r(J)$ into $B_r(J)$, and T is continuous in $B_r(J)$.

Proof. For any $x \in B_r(J)$ and $t \in J$, by using (H1), (H2), (H3)$'$ and (H4)$'$, we have

$$\begin{aligned}
|(Tx)(t)| &= |(T_1x)(t) + (T_2x)(t)| \\
&\leq |S_q(t)(x_0 - g(x))| + \left| \int_0^t (t-s)^{q-1} P_q(t-s) f(s, x(s)) ds \right| \\
&\leq M(|x_0| + L_1 r + L_2) + \frac{M}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s))| ds \qquad (4.22) \\
&\leq M\left(|x_0| + L_1 r + L_2 + \sup_{t \in [0,a]} \left\{ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} m(s) ds \right\} \right) \\
&\leq r.
\end{aligned}$$

Hence, $\|Tx\| \leq r$ for any $x \in B_r(J)$. Using the similar argument as that we did in the proof of Lemma 4.27, we know that T is continuous in $B_r(J)$. $\square$

4.3.4 *Compact Semigroup Case*

In the following, we suppose that the operator A generates a compact C_0-semigroup $\{Q(t)\}_{t \geq 0}$ on X, that is, for any $t > 0$, the operator $Q(t)$ is compact.

Theorem 4.30. Assume that $Q(t)(t > 0)$ is compact. Furthermore assume that there exists a constant $r > 0$ such that the conditions (H1)-(H4) are satisfied. Then the nonlocal Cauchy problem (4.17) has at least one mild solution in $B_r(J)$.

Proof. Since Proposition 4.9, $Q(t)(t > 0)$ is equicontinuous, which implies (H0) is satisfied.

For any $x_1, x_2 \in B_r(J)$ and $t \in J$, according to (H3), we have

$$\begin{aligned}|(T_1x_1)(t) - (T_1x_2)(t)| &\leq M|g(x_1) - g(x_2)| \\ &\leq ML\|x_1 - x_2\|,\end{aligned}$$

which implies that $\|T_1x_1 - T_1x_2\| \leq ML\|x_1 - x_2\|$. Thus, we obtain that

$$\alpha(T_1B_r(J)) \leq ML\alpha(B_r(J)). \tag{4.23}$$

Next, we will show that $\{T_2x,\ x \in B_r(J)\}$ is relatively compact, i.e. $\alpha(T_2(B_r(J))) = 0$. It suffices to show that the family of functions $\{T_2x :\ x \in B_r(J)\}$ is uniformly bounded and equicontinuous, and for any $t \in J$, $\{(T_2x)(t),\ x \in B_r(J)\}$ is relatively compact in X.

By Lemma 4.27, we have $\|T_2x\| \leq r$, for any $x \in B_r(J)$, which means that $\{T_2x :\ x \in B_r(J)\}$ is uniformly bounded. By Lemma 4.26, $\{T_2x :\ x \in B_r(J)\}$ is equicontinuous.

It remains to prove that for any $t \in J$, $V(t) = \{(T_2x)(t),\ x \in B_r(J)\}$ is relatively compact in X.

Obviously, $V(0)$ is relatively compact in X. Let $0 < t \leq a$ be fixed. For $\forall\ \varepsilon \in (0, t)$ and $\forall\ \delta > 0$, define an operator T_ε^δ on $B_r(J)$ by the formula

$$\begin{aligned}(T_\varepsilon^\delta x)(t) &= q\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s,x(s))d\theta ds \\ &= q\ Q(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta - \varepsilon^q\delta)f(s,x(s))d\theta ds,\end{aligned}$$

where $x \in B_r(J)$. Then from the compactness of $Q(\varepsilon^q\delta)(\varepsilon^q\delta > 0)$, we obtain that the set $V_\varepsilon^\delta(t) = \{(T_\varepsilon^\delta x)(t),\ x \in B_r(J)\}$ is relatively compact in X for $\forall\ \varepsilon \in (0, t)$ and $\forall\ \delta > 0$. Moreover, for any $x \in B_r(J)$, we have

$$\begin{aligned}|(T_2x)(t) - (T_\varepsilon^\delta x)(t)| &\leq q\left|\int_0^t\int_0^\delta \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s,x(s))d\theta ds\right| \\ &\quad + q\left|\int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}M_q(\theta)Q((t-s)^q\theta)f(s,x(s))d\theta ds\right| \\ &\leq qM\int_0^t (t-s)^{q-1}m(s)ds\int_0^\delta \theta M_q(\theta)d\theta \\ &\quad + \frac{M}{\Gamma(q)}\int_{t-\varepsilon}^t (t-s)^{q-1}m(s)ds \to 0, \quad \text{as } \varepsilon \to 0, \quad \delta \to 0.\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V(t),\ t > 0$. Hence the set $V(t),\ t > 0$ is also relatively compact in X.

Therefore, $\{(T_2x)(t),\ x \in B_r(J)\}$ is relatively compact by Ascoli-Arzela Theorem. Thus, we have $\alpha(T_2(B_r(J))) = 0$. By (4.23), we have

$$\alpha(T(B_r(J))) \leq \alpha(T_1(B_r(J))) + \alpha(T_2(B_r(J))) \leq ML\alpha(B_r(J)).$$

Thus, the operator T is an α-contraction in $B_r(J)$. By Lemma 4.27, we know that T is continuous. Hence, Theorem 1.44 shows that T has a fixed point in $B_r(J)$. Therefore, the nonlocal Cauchy problem (4.17) has a mild solution in $B_r(J)$. $\square$

Theorem 4.31. Assume that $Q(t)(t > 0)$ is compact. Furthermore assume that (H1), (H2), (H3)$'$ and (H4)$'$ hold. Then the nonlocal Cauchy problem (4.17) has at least a mild solution in $B_r(J)$.

Proof. Since Proposition 4.9, $Q(t)(t > 0)$ is equicontinuous, which implies (H0) is satisfied. By Lemma 4.29, we have $\|Tx\| \leq r$, for any $x \in B_r(J)$, which means that $\{Tx: \ x \in B_r(J)\}$ is uniformly bounded. By Lemmas 4.28-4.29, we know that T is continuous, $\{Tx, x \in B_r(J)\}$ is equicontinuous. It remains to prove that for $t \in J$, the set $\{(Tx)(t), x \in B_r(J)\}$ is relatively compact in X.

According to the argument of Theorem 4.30, we only need to prove that for any $t \in J$, the set $V_1(t) = \{(T_1x)(t), \ x \in B_r(J)\}$ is relatively compact in X.

Obviously, $V_1(0)$ is relatively compact in X. Let $0 < t \leq a$ be fixed. For $\forall\ \delta > 0$, define an operator T_1^δ on $B_r(J)$ by the formula

$$\begin{aligned}(T_1^\delta x)(t) &= \int_\delta^\infty M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta \\ &= Q(t^q\delta)\int_\delta^\infty M_q(\theta)Q(t^q\theta - t^q\delta)(x_0 - g(x))d\theta\end{aligned}$$

where $x \in B_r(J)$. From the compactness of $Q(t^q\delta)(t^q\delta > 0)$, we obtain that the set $V_1^\delta(t) = \{(T_1^\delta x)(t), \ x \in B_r(J)\}$ is relatively compact in X for $\forall\ \delta > 0$. Moreover, for every $x \in B_r(J)$, we have

$$\begin{aligned}&|(T_1x)(t) - (T_1^\delta x)(t)| \\ &= \left|\int_0^\infty M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta - \int_\delta^\infty M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta\right| \\ &= \left|\int_0^\delta M_q(\theta)Q(t^q\theta)(x_0 - g(x))d\theta\right| \\ &\leq M(|x_0| + L_1r + L_2)\int_0^\delta M_q(\theta)d\theta.\end{aligned}$$

Therefore, there are relatively compact sets arbitrarily close to the set $V_1(t), \ t > 0$. Hence the set $V_1(t), \ t > 0$ is also relatively compact in X. Moreover, $\{Tx: \ x \in B_r(J)\}$ is uniformly bounded by Lemma 4.27. Therefore, $\{Tx, \ x \in B_r(J)\}$ is relatively compact by Ascoli-Arzela Theorem. Therefore, $\alpha(T(B_r(J))) = 0$. Hence, Theorem 1.44 shows that T has a fixed point in $B_r(J)$, which means that the nonlocal Cauchy problem (4.17) has a mild solution. $\square$

Remark 4.32. If g is not a compact mapping, we consider the following integral equations

$$x(t) = t^{q-1}P_q\left(t + \frac{1}{n}\right)(x_0 - g(x)) + \int_0^t (t-s)^{q-1}P_q(t-s)f(s, x(s))ds, \quad t \in (0, a]. \tag{4.24}$$

For any $n \in \mathbb{N}$, noticing that the operator $Q(\frac{1}{n})$ is compact, one can easily derive the relative compactness of $V(0)$ and $V(t)(t > 0)$. Then, (4.24) has a solution in $B_r^{(q)}(J')$. By passing the limit, as $n \to \infty$, one obtains a mild solution of the nonlocal Cauchy problem (4.17). However, because $Q(t)$ is replaced by $Q(\frac{1}{n})$, one needs a more restrictive condition than (H4)$'$, such as
(H4)$''$ there exists a constant $r > 0$ such that

$$M_\varepsilon\left(|x_0| + L_1 r + L_2 + \sup_{t\in[0,a]}\left\{\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}m(s)ds\right\}\right) \le r,$$

where $M_\varepsilon = \sup_{t\in[0,a+\varepsilon]} \|Q(t)\|_{B(X)}$, ε is a small constant.
Remark 4.33. The condition (H2) of Theorems 4.30-4.31 can be replaced by the following condition.
(H2)$'$ There exist a constant $q_1 \in (0, q)$ and $m \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that

$$|f(t,x)| \le m(t) \quad \text{for all } x \in B_r(J) \quad \text{and almost all } t \in [0,a].$$

We emphasize that (H2) is more weak than the condition (H2)$'$.

4.3.5 *Noncompact Semigroup Case*

If $Q(t)$ is noncompact, we give an assumption as follows.
(H5) There exists $\ell > 0$ such that for any bounded $D \subset X$,

$$\alpha(f(t,D)) \le \ell\alpha(D).$$

Theorem 4.34. Assume that (H0)-(H5) hold. Then the nonlocal Cauchy problem (4.17) has at least one mild solution in $B_r(J)$.

Proof. By Lemmas 4.26-4.27, we know that $T : B_r(J) \to B_r(J)$ is bounded, continuous and $\{T_2 x : \ x \in B_r(J)\}$ is equicontinuous. For each bounded subset $B_0 \subset B_r(J)$, set

$$T^1(B_0) = T_2(B_0),\ T^n(B_0) = T_2\left(\overline{co}(T^{n-1}(B_0))\right),\ n = 2, 3, \ldots .$$

Then for any $\varepsilon > 0$, there is a sequence $\{x_n^{(1)}\}_{n=1}^\infty$ such that

$$\begin{aligned}
\alpha(T^1(B_0(t))) &= \alpha(T_2(B_0(t))) \\
&\le 2\alpha\left(\int_0^t (t-s)^{q-1}P_q(t-s)f(s,\{x_n^{(1)}(s)\}_{n=1}^\infty)ds\right) + \varepsilon \\
&\le \frac{4M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left(f(s,\{x_n^{(1)}(s)\}_{n=1}^\infty)\right)ds + \varepsilon \\
&\le \frac{4M\ell\alpha(B_0)}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds + \varepsilon \\
&= \frac{4M\ell t^q\alpha(B_0)}{\Gamma(q+1)} + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\alpha(T^1(B_0(t))) \leq \frac{4M\ell t^q}{\Gamma(q+1)}\alpha(B_0).$$

From Propositions 1.28-1.29, for any $\varepsilon > 0$, there is a sequence $\{x_n^{(2)}\}_{n=1}^{\infty} \subset \overline{co}(T^1(B_0))$ such that

$$\begin{aligned}
\alpha(T^2(B_0(t))) &= \alpha(T_2(\overline{co}(T^1(B_0(t))))) \\
&\leq 2\alpha\left(\int_0^t (t-s)^{q-1}P_q(t-s)f(s,\{x_n^{(2)}(s)\}_{n=1}^{\infty})ds\right) + \varepsilon \\
&\leq \frac{4M}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha\left(f(s,\{x_n^{(2)}(s)\}_{n=1}^{\infty}\right)ds + \varepsilon \\
&\leq \frac{4M\ell}{\Gamma(q)}\int_0^t (t-s)^{q-1}\alpha(\{x_n^{(2)}(s)\}_{n=1}^{\infty})ds + \varepsilon \\
&\leq \frac{(4M\ell)^2\alpha(B_0)}{\Gamma(q)\Gamma(q+1)}\int_0^t (t-s)^{q-1}s^q ds + \varepsilon \\
&= \frac{(4M\ell)^2 t^{2q}\alpha(B_0)}{\Gamma(2q+1)} + \varepsilon.
\end{aligned}$$

It can be shown, by mathematical induction, that for every $\bar{n} \in \mathbb{N}$,

$$\alpha(T^{\bar{n}}(B_0(t))) \leq \frac{(4M\ell)^{\bar{n}}t^{\bar{n}q}\alpha(B_0)}{\Gamma(\bar{n}q+1)}.$$

Since

$$\lim_{\bar{n}\to\infty}\frac{(4M\ell a^q)^{\bar{n}}}{\Gamma(\bar{n}q+1)} = 0,$$

there exists a positive integer $\hat{n}$ such that

$$\frac{(4M\ell)^{\hat{n}}t^{\hat{n}q}}{\Gamma(\hat{n}q+1)} \leq \frac{(4M\ell a^q)^{\hat{n}}}{\Gamma(\hat{n}q+1)} = k < 1.$$

Then

$$\alpha(T^{\hat{n}}(B_0(t))) \leq k\alpha(B_0).$$

We know from Proposition 1.26, $T^{\hat{n}}(B_0(t))$ is bounded and equicontinuous, Then, from Proposition 1.27, we have

$$\alpha(T^{\hat{n}}(B_0)) = \max_{t\in[0,a]}\alpha(T^{\hat{n}}(B_0(t))).$$

Hence

$$\alpha(T^{\hat{n}}(B_0)) \leq k\alpha(B_0).$$

By using the similar method as in the proof of Theorem 4.19, we can prove that there exists a $D \subset B_r(J)$ such that

$$\alpha(T_2(D)) = 0. \tag{4.25}$$

On the other hand, for any $x_1, x_2 \in D$ and $t \in J$, according to (H3), we have

$$\begin{aligned}|(T_1x_1)(t) - (T_1x_2)(t)| &\le M|g(x_1) - g(x_2)| \\ &\le ML\|x_1 - x_2\|,\end{aligned}$$

which implies that $\|T_1x_1 - T_1x_2\| \le ML\|x_1 - x_2\|$. Thus, we obtain that

$$\alpha(T_1D) \le ML\alpha(D). \tag{4.26}$$

By (4.25) and (4.26), we have

$$\alpha(T(D)) \le \alpha(T_1(D)) + \alpha(T_2(D)) \le ML\alpha(T(D)).$$

Thus, the operator T is an α-contraction in D. By Lemma 4.27, we know that T is continuous. Hence, Theorem 1.44 shows that T has a fixed point in $D \subset B_r(J)$. Therefore, the nonlocal Cauchy problem (4.17) has a mild solution in $B_r(J)$. □

Theorem 4.35. Assume that (H0)-(H2), (H3)$'$, (H4)$'$ and (H5) hold, then the nonlocal Cauchy problem (4.17) has at least a mild solution in $B_r(J)$.

Proof. By the proof of Theorem 4.34, we know that there exists a $D \subset B_r(J)$ such that $T_2(D)$ is relatively compact, i.e., $\alpha(T_2(D)) = 0$. Clearly, $\alpha(T_1(D)) = 0$, since $g(x)$ is compact and $S_q(t)$ is bounded. Hence, we have

$$\alpha(T(D)) \le \alpha(T_1(D)) + \alpha(T_2(D)) = 0.$$

Therefore, Theorem 1.44 shows that T has a fixed point in $D \subset B_r(J)$. Therefore, the nonlocal Cauchy problem (4.17) has a mild solution in $B_r(J)$. □

4.4 Nonlocal Cauchy Problems for Evolution Equations

4.4.1 *Introduction*

The nonlocal condition has a better effect on the solution and is more precise for physical measurements than the classical condition alone. In this section, we discuss the existence of mild solutions of Cauchy problem for fractional order evolution equations with nonlocal conditions

$$\begin{cases} {}_0^C D_t^\alpha x(t) = Ax(t) + f\left(t, x(t)\right), \ \alpha \in (0,1), \ t \in J = [0,1], \\ x(0) = \sum_{k=1}^m a_k x(t_k), \ k = 1, 2, ..., m, \end{cases} \tag{4.27}$$

where $A: \ D(A) \subset X \to X$ is the generator of a C_0-semigroup $\{Q(t), \ t \ge 0\}$ on a Banach space X, $f: \ J \times X \to X$ is a given function and a_k $(k = 1, 2, ..., m)$ are real numbers with $\Sigma_{k=1}^m a_k \ne 1$ and $t_k, \ k = 1, 2, ...m$ are given points satisfying $0 \le t_1 \le t_2 \le \cdots \le t_m < 1$.

A suitable definition of mild solution of the equation (4.27) will be introduced by defining a bounded operator B. Meanwhile, two sufficient conditions are given to guarantee such B exists. We mention that our first existence result relies on a growth condition on J and second existence result relies on a growth condition involving two parts, one for $[0, t_m]$, and the other for $[t_m, 1]$.

4.4.2 *Definition of Mild Solutions*

Assume that P_α, S_α are defined as in Subsection 4.3.2. Suppose that there exists a bounded operator $B : X \to X$ given by

$$B = \left[I - \sum_{k=1}^{m} a_k S_\alpha(t_k) \right]^{-1}. \tag{4.28}$$

We can give two sufficient conditions to guarantee such B exists and is bounded.
Lemma 4.36. The operator B defined in (4.28) exists and is bounded, if one of the following two conditions holds:
(C1) there exist some real numbers a_k such that

$$M \sum_{k=1}^{m} |a_k| < 1, \tag{4.29}$$

where $M = \sup_{t\in(0,\infty)} ||Q(t)||_{B(X)} < \infty$.
(C2) $Q(t)$ is compact for each $t > 0$ and homogeneous linear nonlocal problems

$$\begin{cases} {}_0^C D_t^\alpha x(t) = Ax(t), \ \alpha \in (0,1), \ t \in J, \\ x(0) = \sum_{k=1}^{m} a_k x(t_k), \end{cases} \tag{4.30}$$

has no non-trivial mild solutions.

Proof. For (C1), it is easy to see

$$\left\| \sum_{k=1}^{m} a_k S_\alpha(t_k) \right\|_{B(X)} \leq M \sum_{k=1}^{m} |a_k| < 1,$$

where Proposition 4.23 and (4.29) are used. Thus by Neumann theorem, B defined by (4.28) exists and it is bounded.

For (C2), it is obvious that the mild solutions of the problem (4.30) is given by

$$x(t) = S_\alpha(t)x(0),$$

which implies that

$$x(0) = \sum_{k=1}^{m} a_k x(t_k) = \sum_{k=1}^{m} a_k S_\alpha(t_k)x(0).$$

By Proposition 4.25, $S_\alpha(t_k)$ is compact for each $t_k > 0$, $k = 1, 2, ..., m$. Then $\sum_{k=1}^{m} a_k S_\alpha(t_k)$ is also compact. Since the problem (4.30) has no non-trivial mild solutions, one can obtain the desired result via Fredholm alternative theorem. $\square$

Now we introduce the following definition of mild solutions of the equation (4.27).
Definition 4.37. By a mild solution of the equation (4.27), we mean a function $x \in C(J, X)$ satisfying

$$x(t) = S_\alpha(t) \sum_{k=1}^{m} a_k B(g(t_k)) + g(t), \ t \in J, \tag{4.31}$$

where

$$g(t_k) = \int_0^{t_k} (t_k - s)^{\alpha-1} P_\alpha(t_k - s) f(s, x(s)) ds, \tag{4.32}$$

and

$$g(t) = \int_0^{t} (t - s)^{\alpha-1} P_\alpha(t - s) f(s, x(s)) ds, \ t \in J. \tag{4.33}$$

Remark 4.38. To explain the formula (4.31), we note that a mild solution of the fractional evolution equation (4.27) with the initial condition is just $x(t) = S_\alpha(t)x(0) + g(t)$, so taking into account also the nonlocal condition, we get

$$x(0) = \sum_{k=1}^{m} a_k S_\alpha(t_k) x(0) + \sum_{k=1}^{m} a_k g(t_k).$$

So $x(0) = \sum_{k=1}^m a_k B(g(t_k))$ and hence $x(t) = S_\alpha(t) \sum_{k=1}^m a_k B(g(t_k)) + g(t)$ which is just (4.31).

4.4.3 *Existence*

Our first existence result is based on the well-known Schaefer fixed point theorem.

In this subsection, we will study our problem under the following assumptions:
(H1) $f : J \times X \to X$ satisfies the Carathéodory type conditions;
(H2) there exists a function h such that ${}_0D_t^{-\alpha} h(t)$ exists for all $t \in J$ and ${}_0D_\cdot^{-\alpha} h(\cdot) \in C((0,1], \mathbb{R}^+)$ with $\lim_{t\to 0^+} {}_0D_t^{-\alpha} h(t) = 0$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,x)| \le h(t)\Omega(|x|)$$

for all $x \in X$ and all $t \in J$;
(H3) the inequality

$$\limsup_{\rho\to\infty} \frac{\rho}{M^2 B \Omega(\rho) \sum_{k=1}^m |a_k| {}_0D_t^{-\alpha} h(t_k) + M\Omega(\rho) \sup_{t\in J} {}_0D_t^{-\alpha} h(t)} > 1$$

hold;
(H4) $Q(t)$ is compact for each $t > 0$.

We begin to consider the following problem

$$\begin{cases} {}_0^C D_t^\alpha x(t) = Ax(t) + \lambda f(t, x(t)), \ \alpha \in (0,1], \ \lambda, t \in J, \\ x(0) = \sum_{k=1}^m a_k x(t_k). \end{cases} \tag{4.34}$$

Define an operator $F : C(J,X) \to C(J,X)$ as follows

$$(Fx)(t) = (F_1 x)(t) + (F_2 x)(t), \ t \in J,$$

where $F_i : C(J,X) \to C(J,X)$, $i = 1, 2$ are given by the formulas

$$(F_1 x)(t) = S_\alpha(t) \sum_{k=1}^m a_k B(g(t_k)), \ (F_2 x)(t) = g(t),$$

where B is the operator defined in (4.28), $g(t_k)$ is defined in (4.32) and $g(t)$ is defined in (4.33).

Obviously, a mild solution of the equation (4.34) is a solution of the operator equation

$$x = \lambda F x \tag{4.35}$$

and conversely. Thus, we can apply Schaefer's fixed point theorem to derive the existence results of solutions of the equation (4.27).

Lemma 4.39. Let x be any solution of the equation (4.35). Then, there exists $R^* > 0$ such that $\|x\|_C \leq R^*$ which is independent of the parameter $\lambda \in J$.

Proof. Denote $R_0 := \|x\|$. Taking into accounts our conditions and Lemma Proposition 4.23 and Proposition 4.24, it follows from (4.31) that

$$\begin{aligned} |x(t)| &\leq |(F_1 x)(t)| + |(F_2 x)(t)| \\ &\leq M \textstyle\sum_{k=1}^{m} |a_k| \|B\|_{B(X)} |g(t_k)| + |g(t)|, \ t \in J. \end{aligned} \tag{4.36}$$

Note that

$$\begin{aligned} |g(t_k)| &\leq \int_0^{t_k} (t_k - s)^{\alpha-1} \|P_\alpha(t_k - s)\|_{B(X)} |f(s, x(s))| ds \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} h(s) \Omega(\|x\|) ds \\ &\leq \frac{M\Omega(R_0)}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} h(s) ds \\ &= M\Omega(R_0) {}_0D_{t_k}^{-\alpha} h(t_k), \ k = 1, 2, ..., m, \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} |g(t)| &\leq \frac{M\Omega(R_0)}{\Gamma(\alpha)} \int_0^{t} (t - s)^{\alpha-1} h(s) ds \\ &= M\Omega(R_0) \sup_{t \in J} {}_0D_t^{-\alpha} h(t), \ t \in J. \end{aligned} \tag{4.38}$$

In view of (4.36)-(4.38), one can obtain

$$R_0 := \|x\| \leq M^2 \|B\|_{B(X)} \Omega(R_0) \sum_{k=1}^{m} |a_k| {}_0D_{t_k}^{-\alpha} h(t_k) + M\Omega(R_0) \sup_{t \in J} {}_0D_t^{-\alpha} h(t), \ t \in J,$$

which implies that

$$\frac{R_0}{M^2 \|B\|_{B(X)} \Omega(R_0) \sum_{k=1}^{m} |a_k| {}_0D_{t_k}^{-\alpha} h(t_k) + M\Omega(R_0) \sup_{t \in J} {}_0D_t^{-\alpha} h(t)} \leq 1. \tag{4.39}$$

However, it follows (H3) that there exists a $R^* > 0$ such that for all $R > R^*$ we can derive that

$$\frac{R}{M^2 \|B\|_{B(X)} \Omega(R) \sum_{k=1}^{m} |a_k| {}_0D_{t_k}^{-\alpha} h(t_k) + M\Omega(R) \sup_{t \in J} {}_0D_t^{-\alpha} h(t)} > 1. \tag{4.40}$$

Now, comparing the equalities (4.39) and (4.40), we claim that $R_0 \leq R^*$. As a result, we find that $\|x\| \leq R^*$ which independents the parameter λ. This completes the proof. □

Let

$$\mathfrak{B}_{R^*} = \{x \in C(J, X) : \|x\| \leq R^*\}.$$

Then $\mathfrak{B}_{R^*}$ is a bounded closed and convex subset in $C(J, X)$.

By Lemma 4.39, we can derive the following result.

Lemma 4.40. The operator F maps $\mathfrak{B}_{R^*}$ into itself.

Lemma 4.41. The operator $F : \mathfrak{B}_{R^*} \to \mathfrak{B}_{R^*}$ is completely continuous.

Proof. For our purpose, we only need to check that $F_i : \mathfrak{B}_{R^*} \to \mathfrak{B}_{R^*}$, $i = 1, 2$ is completely continuous. Firstly, by repeating the same producers of Lemma 4.27 and Theorem 4.30, one can obtain $F_2 : \mathfrak{B}_{R^*} \to \mathfrak{B}_{R^*}$ is completely continuous.

Secondly, one can check that $F_1 : \mathfrak{B}_{R^*} \to \mathfrak{B}_{R^*}$ is continuous (by (H1), (H2) and Proposition 4.23) and $F_1 : \mathfrak{B}_{R^*} \to \mathfrak{B}_{R^*}$ is compact in viewing of $S_\alpha(t)$ is compact for each $t > 0$ (by (H4) and Proposition 4.25). The proof is completed. □

Now, we can state the main result of this section.

Theorem 4.42. Assume that (H1)-(H4) hold and the condition (C1) (or (C2)) is satisfied. Then the equation (4.27) has at least one solution $u \in C(J, X)$ and the set of the solutions of the equation (4.27) is bounded in $C(J, X)$.

Proof. Obviously, the set $\{x \in C(J, X) : x = \lambda F x, 0 < \lambda < 1\}$ is bounded due to Lemma 4.40. Now we can apply Theorem 1.43 to derive that F has a fixed point in $\mathfrak{B}_{R^*}$ which is just the mild solution of the equation (4.27). This completes the proof. □

Our second existence result is based on O'Regan fixed point theorem.

In addition to (H1), (H4) and (C1) (or (C2)), motivated by Boucherif and Precup, 2003, 2007, we introduce the following two assumptions:

(H5) There exists a function h such that ${}_0D_t^{-\alpha}h(t)$ exists for all $t \in [0, t_m]$ and ${}_0D_\cdot^{-\alpha}h(\cdot) \in C((0, t_m], \mathbb{R}^+)$ with $\lim_{t\to 0^+} {}_0D_t^{-\alpha}h(t) = 0$ and a nondecreasing continuous function $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t, x)| \leq h(t)\Omega(|x|)$$

for all $x \in X$, and for all $t \in [t_m, 1]$ there exists a function l such that ${}_{t_m}D_t^{-\alpha}l(t)$ exists and ${}_{t_m}D_\cdot^{-\alpha}l(\cdot) \in C([t_m, 1], \mathbb{R}^+)$ such that

$$|f(t, x)| \leq l(t), \tag{4.41}$$

for all $x \in X$. Moreover, Ω has the property

$$r > M\Omega(r)\left(\sum_{k=1}^{m} |a_k|\|B\|_{B(X)} + 1\right) \sup_{t\in[0,t_m]} {}_0D_t^{-\alpha}h(t) \tag{4.42}$$

for all $r > R_1^* > 0$;

(H6) there exists a function q such that ${}_{t_m}D_t^{-\alpha}q(t)$ exists for all $t \in [t_m, 1]$ and ${}_{t_m}D_\cdot^{-\alpha}q(\cdot) \in C([t_m, 1], \mathbb{R}^+)$ with $M \sup_{t\in[t_m,1]} {}_0D_t^{-\alpha}q(t) \leq 1$ and a nondecreasing continuous function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(r) < r$ for $r > 0$ such that

$$|f(t, x) - f(t, y)| \leq q(t)\Psi(|x - y|)$$

for all $t \in [t_m, 1]$ and $x, y \in X$.

Consider the equation (4.34) again and the equivalent equation

$$x = \lambda T x \tag{4.43}$$

where $T : C(J, X) \to C(J, X)$ is defined by

$$(Tx)(t) = (T_1 x)(t) + (T_2 x)(t), \ t \in J,$$

where $T_i : C(J, X) \to C(J, X)$, $i = 1, 2$ given by

$$(T_1 x)(t) = \begin{cases} S_\alpha(t) \displaystyle\sum_{k=1}^{m} a_k B(g(t_k)) + g(t), & t \in [0, t_m), \\ S_\alpha(t) \displaystyle\sum_{k=1}^{m} a_k B(g(t_k)) & \\ \qquad + \displaystyle\int_0^{t_m} (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x(s)) ds, & t \in [t_m, 1], \end{cases}$$

and

$$(T_2 x)(t) = \begin{cases} 0, & t \in [0, t_m), \\ \displaystyle\int_{t_m}^{t} (t-s)^{\alpha-1} P_\alpha(t-s) f(s, x(s)) ds, & t \in [t_m, 1]. \end{cases}$$

We first prove that any solutions of the equation (4.43) are a priori bounded.

Lemma 4.43. Let x be any solution of the equation (4.43). Then, there exist $R_i^* > 0$ $i = 1, 2$ such that $\|x\|_{C([0,t_m],X)} \leq R_1^*$ and $\|x\|_{C([t_m,1],X)} \leq R_2^*$. In other words, $\|x\| \leq R^* = \max\{R_1^*, R_2^*\}$ which is independent of the parameter λ.

Proof. **Case I.** We prove that there exists $R_1^* > 0$ such that $\|x\|_{C([0,t_m],X)} \leq R_1^*$.

For $t \in [0, t_m]$ and $\lambda \in J$, denote $R_{[0,t_m]} := \|x\|_{C([0,t_m],X)}$, we have

$$\begin{aligned} |x(t)| &\leq \lambda |(T_1 x)(t)| + |(T_2 x)(t)| \\ &\leq M \sum_{k=1}^{m} |a_k| \|B\|_{B(X)} |g(t_k)| + |g(t)| \\ &\leq M \sum_{k=1}^{m} |a_k| \|B\|_{B(X)} \frac{M}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} h(s) \Omega(R_{[0,t_m]}) ds \\ &\quad + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) \Omega(R_{[0,t_m]}) ds \\ &\leq M \Omega(R_{[0,t_m]}) \left(\sum_{k=1}^{m} |a_k| \|B\|_{B(X)} + 1 \right) \sup_{t \in [0,t_m]} {}_0D_t^{-\alpha} h(t), \end{aligned}$$

which implies that

$$R_{[0,t_m]} \leq M \Omega(R_{[0,t_m]}) \left(\sum_{k=1}^{m} |a_k| \|B\|_{B(X)} + 1 \right) \sup_{t \in [0,t_m]} {}_0D_t^{-\alpha} h(t).$$

From (4.42) we find that there exists $R_1^* \geq R_{[0,t_m]} > 0$ such that
$$\|x\|_{C([0,t_m],X)} \leq R_1^*.$$

Case II. We prove that there exists $R_2^* > 0$ such that $\|x\|_{C([t_m,1],X)} \leq R_2^*$.

For $t \in [t_m, 1]$ and $\lambda \in J$, keeping in mind our assumptions, we find that
$$\begin{aligned}
|x(t)| &\leq M \sum_{k=1}^{m} |a_k| \|B\|_{B(X)} \frac{M}{\Gamma(\alpha)} \int_0^{t_k} (t_k - s)^{\alpha-1} h(s) \Omega(R_1^*) ds \\
&\quad + \frac{M}{\Gamma(\alpha)} \int_0^{t_m} (t-s)^{\alpha-1} h(s) \Omega(R_1^*) ds + \frac{M}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} h(s) ds \\
&\leq M\Omega(R_1^*) \left(\sum_{k=1}^{m} |a_k| \|B\|_{B(X)} + 1 \right) \sup_{t\in[0,t_m]} {}_0D_t^{-\alpha} h(t) \\
&\quad + M \sup_{t\in[t_m,1]} {}_{t_m}D_t^{-\alpha} l(t),
\end{aligned}$$
which implies that
$$\|x\|_{C([t_m,1],X)} \leq R_2^*,$$
where
$$R_2^* = M \left[\Omega(R_1^*) \left(\sum_{k=1}^{m} |a_k| \|B\|_{B(X)} + 1 \right) \sup_{t\in[0,t_m]} {}_0D_t^{-\alpha} h(t) + \sup_{t\in[t_m,1]} {}_{t_m}D_t^{-\alpha} l(t) \right].$$

Let $R^* = \max\{R_1^*, R_2^*\}$. Then we will find that any possible solutions of the equation (4.43) satisfy $\|x\| \leq R^*$ which are independent of the parameter λ. This completes the proof. □

Denote
$$\mathcal{D} = \{x \in C(J,X) : \ \|x\| < R^* + 1\}.$$

We can proceed as in the proof of Lemma 4.43 to derive the following result.

Lemma 4.44. $T(\overline{\mathcal{D}})$ is bounded.

One can proceed as in the proof of Lemma 4.41 to obtain the following result.

Lemma 4.45. The operator $T_1 : \overline{\mathcal{D}} \to C(J,X)$ is completely continuous.

Next, we show the following result.

Lemma 4.46. The operator $T_2 : \overline{\mathcal{D}} \to C(J,X)$ is nonlinear contraction.

Proof. It follows from the definition of T_2 we only need to show $T_2 : \overline{\mathcal{D}} \to C([t_m,1],X)$ is a nonlinear contraction.

In fact, for any $x, y \in \overline{\mathcal{D}}$ and $t \in [t_m, 1]$, we have
$$\begin{aligned}
|(T_2x)(t) - (T_2y)(t)| &\leq \int_{t_m}^{t} (t-s)^{\alpha-1} \|P_\alpha(t-s)[f(s,x(s)) - f(s,y(s))]\| ds \\
&\leq \frac{M}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} q(s) \Psi(|x(s) - y(s)|) ds \\
&\leq \frac{M\Psi(\|x-y\|)}{\Gamma(\alpha)} \int_{t_m}^{t} (t-s)^{\alpha-1} q(s) ds \\
&\leq \left(M \sup_{t\in[t_m,1]} {}_{t_m}D_t^{-\alpha} q(t) \right) \Psi(\|x-y\|),
\end{aligned}$$

which implies that

$$\|T_2x - T_2y\| \le \Psi(\|x-y\|).$$

This completes the proof. □

Now, we are ready to present the main result of this section.

Theorem 4.47. Assume that (H1), (H4), (H5) and (H6) hold and the condition (C1) (or (C2)) is satisfied. Then the equation (4.27) has at least a solution $u \in C(J,X)$.

Proof. By Lemma 4.43 we see that (ii) of Theorem 1.46 does not hold. Thus, there is no solution of the equation (4.43) with $x \in \partial\mathcal{D}$. Therefore, one can apply Theorem 1.46 to derive that T has a fixed point in $\mathcal{D}$ which is just the mild solution of the equation (4.27). This completes the proof. □

Remark 4.48. Theorem 4.47 also holds even if (H5) and (H6) are replaced by the following conditions:

(H5)$'$ Condition (H5) is assumed without (4.41);

(H6)$'$ denoting $\delta := \lim_{r\to+\infty}\inf\frac{\Psi(r)}{r} \le 1$, condition (H6) is assumed in addition with

$$M\delta \sup_{t\in[t_m,1]} {}_{t_m}D_t^{-\alpha}q(t) < 1.$$

Indeed, we can modify Case II in the proof of Lemma 4.43 as follows

$$\begin{aligned}
|x(t)| &\le M\sum_{k=1}^{m}|a_k|\|B\|_{B(X)}\frac{M}{\Gamma(\alpha)}\int_0^{t_k}(t_k-s)^{\alpha-1}h(s)\Omega(R_1^*)ds\\
&\quad+\frac{M}{\Gamma(\alpha)}\int_0^{t_m}(t-s)^{\alpha-1}h(s)\Omega(R_1^*)ds\\
&\quad+\frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}|f(s,0)|ds+\frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}q(s)\Psi(|x(s)|)ds\\
&\le M\Omega(R_1^*)\left(\sum_{k=1}^{m}|a_k|\|B\|_{B(X)}+1\right)\sup_{t\in[0,t_m]} {}_{t_m}D_t^{-\alpha}h(t)\\
&\quad+\frac{M\sup_{t\in[t_m,t]}|f(t,0)|(1-t_m)^\alpha}{\Gamma(\alpha+1)}\\
&\quad+\frac{M}{\Gamma(\alpha)}\int_{t_m}^{t}(t-s)^{\alpha-1}q(s)(\delta|x(s)|+\delta_1)ds,
\end{aligned}$$

for some $\delta_1 \ge 0$.

Then we have

$$\begin{aligned}
R_2^* &= \frac{1}{1-M\delta\sup_{t\in[t_m,1]} {}_{t_m}D_t^{-\alpha}q(t)}\\
&\quad\times\left(M\Omega(R_1^*)\left(\sum_{k=1}^{m}|a_k|\|B\|_{B(X)}+1\right)\sup_{t\in[0,t_m]} {}_{t_m}D_t^{-\alpha}h(t)\right.\\
&\quad\left.+\frac{M\sup_{t\in[t_m,t]}|f(t,0)|(1-t_m)^\alpha}{\Gamma(\alpha+1)}+M\delta_1\sup_{t\in[t_m,1]} {}_{t_m}D_t^{-\alpha}q(t)\right).
\end{aligned}$$

4.5 Abstract Cauchy Problems with Almost Sectorial Operators

4.5.1 *Introduction*

Let X be a complex Banach space with norm $|\cdot|$. As usual, for a linear operator A, we denote by $D(A)$ the domain of A, by $\sigma(A)$ its spectrum, while $\rho(A) := \mathbb{C} - \sigma(A)$ is the resolvent set of A, and denote by the family $R(z; A) = (zI - A)^{-1}$, $z \in \rho(A)$ of bounded linear operators the resolvent of A. Moreover, we denote by $\mathscr{L}(X, Y)$ the space of all bounded linear operators from Banach space X to Banach space Y with the usual operator norm $\|\cdot\|_{\mathscr{L}(X,Y)}$, and we abbreviate this notation to $\mathscr{L}(X)$ when $Y = X$, and write $\|T\|_{\mathscr{L}(X)}$ as $\|T\|$ for every $T \in \mathscr{L}(X)$.

When dealing with parabolic evolution equations, it is usually assumed that the partial differential operator in the linear part is a sectorial operator, stimulated by the fact that this class of operators appears very often in the applications. For example, one can find from Henry, 1981; Lunardi, 1995 and Pazy, 1983 that many elliptic differential operators equipped homogeneous boundary conditions are sectorial when they are considered in the Lebesgue spaces (e.g. L^p-spaces) or in the space of continuous functions. We here mention that the operator A_ε in Example 4.49, which acts on a domain of "dumb-bell with a thin handle", is sectorial on V_ϵ^p. However, as presented in Example 4.49 and Example 4.51, though the resolvent set of some partial differential operators considered in some special domains such as the limit "domain" of dumb-bell with a thin handle or in some spaces of more regular functions such as the space of Hölder continuous functions, contains a sector, but for which the resolvent operators do not satisfy the required estimate to be a sectorial operator.

Example 4.49. In this notation the "dumb-bell with a thin handle" has the form

$$\Omega_\varepsilon = D_1 \cup Q_\varepsilon \cup D_2 \quad (\varepsilon \in (0,1];\ \text{small}),$$

where D_1 and D_2 are mutually disjoint bounded domains in $\mathbb{R}^N (N \geq 2)$ with smooth boundaries, joined by a thin channel, Q_ε (which is not required to be cylindrical), which degenerates to a 1-dim line segment Q_0 as ε approaches zero. This implies that passing to the limit as $\varepsilon \to 0$, the limit "domain" of Ω_ε consists of the fixed part D_1, D_2 and the line segment Q_0. Without loss of generality, we may assume that $Q_0 = \{(x, 0, \ldots, 0); 0 < x < 1\}$. Let $P_0 = (0, 0, \ldots, 0), P_1 = (1, 0, \ldots, 0)$ be the points where the line segment touches the boundary of D_1 and D_2. Put $\Omega = D_1 \cup D_2$.

Firstly, consider the evolution equation of parabolic type equipped with Neumann boundary condition in the form

$$\begin{cases} u_t - \Delta u + u = f(u), \quad x \in \Omega_\epsilon,\ t > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega_\epsilon, \end{cases} \tag{4.44}$$

where Δ stands for the Laplacian operator with respect to the spatial variable $x \in \Omega_\epsilon$, $\partial\Omega_\epsilon$ is the boundary of Ω_ϵ, $\frac{\partial}{\partial n}$ denotes the outward normal derivative on

$\partial\Omega_\epsilon$ and $f:\mathbb{R}\to\mathbb{R}$ is a nonlinearity. Let V_ϵ^p $(1\le p<\infty)$ denote the family of spaces based on $L^p(\Omega_\epsilon)$, equipped with the norm

$$\|u\|_{V_\epsilon^p}=\Big(\int_\Omega |u|^p+\frac{1}{\varepsilon^{N-1}}\int_{Q_\epsilon}|u|^p\Big)^{1/p}.$$

Define the linear operator $A_\varepsilon : D(A_\varepsilon)\subset V_\epsilon^p\mapsto V_\epsilon^p$ by

$$D(A_\varepsilon)=\Big\{u\in W^{2,p}(\Omega_\epsilon);\Delta u\in V_\epsilon^p,\frac{\partial u}{\partial n}\Big|_{\partial\Omega_\epsilon}=0\Big\},$$

$$A_\varepsilon u=-\Delta u+u,\quad u\in D(A_\varepsilon).$$

It follows from a standard argument that the operator A_ε generates an analytic semigroup on V_ϵ^p. Moreover, the following estimate holds

$$\|R(\lambda;-A_\varepsilon)\|_{\mathscr{L}(L^p(\Omega_\epsilon))}\le\frac{C}{|\lambda|},\quad\text{for }\lambda\in\Sigma'_\theta,$$

where $\Sigma'_\theta=\{\lambda\in\mathbb{C};|\arg(\lambda-1)|\le\theta\}$ with $\theta>\frac{\pi}{2}$, and C is a constant that does not depend on ε (see, e.g., Henry, 1981 and Pazy, 1983).

The limit problem of (4.44) as $\varepsilon\to0$ is the following problem studied in Carvalho, Dlotko and Neseimento, 2008

$$\begin{cases} w_t-\Delta w+w=f(w), \quad x\in\Omega,\ t>0,\\ \dfrac{\partial w}{\partial n}=0,\quad x\in\partial\Omega,\\ v_t-\dfrac{1}{g}(gv_x)_x+v=f(v),\quad x\in Q_0=(0,1),\\ v(0)=w(P_0)\,,v(1)=w(P_1),\end{cases}$$

where w is a function that lives in Ω and v lives in the line segment Q_0, the function $g:[0,1]\to(0,\infty)$ is a smooth function related to the geometry of the channel Q_ε, more exactly, on the way the channel Q_ε collapses to the segment line Q_0. Observe that the vector (w,v) is continuous in the junction between Ω and Q_0 and the variable w does not depend on the variable v, but v depends on w.

We identify V_0^p with $L^p(\Omega)\oplus L_g^p(0,1)$ $(1\le p<\infty)$ endowed with the norm

$$\|(w,v)\|_{V_0^p}=\Big(\int_\Omega|w|^p+\int_0^1 g|v|^p\Big)^{1/p}.$$

Consider the operator $A_0 : D(A_0)\subset V_0^p\mapsto V_0^p$ defined by

$$D(A_0)=\{(w,v)\in V_0^P;w\in D(\Delta_\Omega),v\in L_g^p(0,1),w(P_0)=v(0),w(P_1)=v(1)\},$$

$$A_0(w,v)=\Big(-\Delta w+w,-\frac{1}{g}(gv')'+v\Big),\quad (w,v)\in V_0^p, \tag{4.45}$$

where Δ_Ω is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ and

$$D(\Delta_\Omega)=\Big\{u\in W^{2,p}(\Omega);\frac{\partial u}{\partial n}\Big|_{\partial\Omega}=0\Big\}.$$

As pointed out by Arrieta, Carvalho and Lozada-Cruz, 2009a, the operator A_0 defined by (4.45) is not a sectorial operator. Its spectrum is all real and, therefore, it is contained in a sector but the resolvent estimate is different from the case of sectorial operator. More precisely, the operator A_0 has the following properties (see also Arrieta, Carvalho and Lozada-Cruz, 2006, 2009a):
(a) the domain $D(A_0)$ is dense in V_0^P,
(b) if $p > \frac{N}{2}$, then A_0 is a closed operator,
(c) A_0 has compact resolvent,
(d) for some $\mu \in (0, \frac{\pi}{2})$, $\Sigma_\mu := \{\lambda \in \mathbb{C} \setminus \{0\}; |\arg \lambda| \leq \pi - \mu\} \cup \{0\} \subset \rho(-A_0)$, and for $\frac{N}{2} < q \leq p$, the following estimate holds:

$$\|R(\lambda; -A_0)\|_{\mathscr{L}(V_0^q, V_0^p)} \leq \frac{C}{1 + |\lambda|^{\gamma'}}, \quad \lambda \in \Sigma_\mu, \tag{4.46}$$

for each $0 < \gamma' < 1 - \frac{N}{2q} - \frac{1}{2}(\frac{1}{q} - \frac{1}{p}) < 1$, where C is a positive constant.

Remark 4.50. In fact, it is easy to prove that the estimate (4.46) with $p = q > \frac{N}{2}$ is equivalent to

$$\|R(\lambda; -A_0)\|_{\mathscr{L}(V_0^p)} \leq \frac{\widetilde{C}}{|\lambda|^{\gamma'}}, \quad \lambda \in \Sigma_\mu \setminus \{0\},$$

for $0 < \gamma' < 1 - \frac{N}{2p}$, where $\widetilde{C}$ is a positive constant.

We refer to Section 2 of Arrieta, Carvalho and Lozada-Cruz, 2006 for a complete and rigorous definition of the dumb-bell domain, and to Arrieta, 1995; Arrieta, Carvalho and Lozada-Cruz, 2006, 2009a,b; Dancer and Daners, 1997; Gadyl'shin, 2005; Jimbo, 1989 for related studies of partial differential equations involving dumb-bell domain.

Example 4.51. Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with boundary $\partial\Omega$ of class C^{4m} $(m \in \mathbb{N})$. Let $C^l(\overline{\Omega})$, $l \in (0, 1)$, denote the usual Banach space with norm $|\cdot|_l$. Consider the elliptic differential operator $A' : D(A') \subset C^l(\Omega) \mapsto C^l(\Omega)$ in the form

$$D(A') = \{u \in C^{2m+l}(\overline{\Omega}); D^\beta u|_{\partial\Omega} = 0, |\beta| \leq m - 1\},$$

$$A'u = \sum_{|\beta| \leq 2m} a_{_\beta}(x) D^\beta u(x), \quad u \in D(A'),$$

where β is a multiindex in $(\mathbb{N} \bigcup \{0\})^n$,

$$|\beta| = \sum_{j=1}^{n} \beta_j, \quad D^\beta = \prod_{j=1}^{n} (-i \frac{\partial}{\partial x_j})^{\beta_j}.$$

The coefficients $a_{_\beta} : \overline{\Omega} \mapsto \mathbb{C}$ of A' are assumed to satisfy

(i) $a_{_\beta} \in C^l(\overline{\Omega})$ for all $|\beta| \leq 2m$,
(ii) $a_{_\beta}(x) \in \mathbb{R}$ for all $x \in \overline{\Omega}$ and $|\beta| = 2m$,

(iii) there exists a constant $M > 0$ such that

$$M^{-1}|\xi|^2 \leq \sum_{|\beta|=2m} a_{\beta}(x)\xi^{\beta} \leq M|\beta|^2, \quad \text{for all } \ \xi \in \mathbb{R}^N \quad \text{and} \quad x \in \overline{\Omega}.$$

Then, the following statements hold.
(a) A' is not densely defined in $C^l(\overline{\Omega})$,
(b) there exist $\nu,\ \varepsilon > 0$ such that

$$\sigma(A'+\nu) \subset S_{\frac{\pi}{2}-\varepsilon} = \{\lambda \in \mathbb{C} \setminus \{0\}; |\arg\lambda| \leq \frac{\pi}{2} - \varepsilon\} \cup \{0\},$$

$$\|R(\lambda; A'+\nu)\|_{\mathscr{L}(C^l(\overline{\Omega}))} \leq \frac{C}{|\lambda|^{1-\frac{l}{2m}}}, \quad \lambda \in \mathbb{C} \setminus S_{\frac{\pi}{2}-\varepsilon},$$

(c) the exponent $\frac{l}{2m} - 1 \in (-1, 0)$ is sharp. In particular, the operator $A' + \nu$ is not sectorial.

Notice in particular that the Laplace operator satisfies the conditions (a)-(c) in Example 4.51. For more details we refer to Wahl, 1972.

Observe that from Example 4.49 and Remark 4.50, if $p > \frac{N}{2}$, then $A_0 \in \Theta_{\mu}^{-\gamma'}(V_0^p)$ for some $\gamma' \in (0, 1 - \frac{N}{2p})$ and $\mu \in (0, \frac{\pi}{2})$, that is, A_0 is an almost sectorial operator on V_0^p. Also, from Example 4.51 one can find that $(A' + \nu) \in \Theta_{\frac{\pi}{2}-\varepsilon}^{\frac{l}{2m}-1}(C^l(\overline{\Omega}))$, which implies that $A' + \nu$ is an almost sectorial operator on $C^l(\overline{\Omega})$.

In this section, motivated by the above consideration, we are interested in studying the Cauchy problem for the linear evolution equation

$$\begin{cases} {}_0^C D_t^{\alpha} u(t) + Au(t) = f(t), & t > 0, \\ u(0) = u_0, \end{cases} \tag{4.47}$$

as well as the Cauchy problem for the corresponding semilinear fractional evolution equation

$$\begin{cases} {}_0^C D_t^{\alpha} u(t) + Au(t) = f(t, u(t)), & t > 0, \\ u(0) = u_0 \end{cases} \tag{4.48}$$

in X, where ${}_0^C D_t^{\alpha}$, $0 < \alpha < 1$, is Caputo fractional derivative of order α and A is an almost sectorial operator, that is, $A \in \Theta_{\omega}^{\gamma}(X)$ $(-1 < \gamma < 0,\ 0 < \omega < \pi/2)$. The main purpose is to study the existence and uniqueness of mild solutions and classical solutions of Cauchy problems (4.47) and (4.48). To do this, we construct two operator families based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A, present deep anatomy on basic properties for these families consisting on the study of the compactness, and prove that, under natural assumptions, reasonable concept of solutions can be given to problems (4.47) and (4.48), which in turn is used to find solutions to the Cauchy problems.

Remark 4.52. We make no assumption on the density of the domain of A.

Remark 4.53.

(i) M. M. Dzhrbashyan and A. B. Nersessyan in Dzhrbashyan and Nersessyan, 1968 (see also Miller and Ross, 1993) showed that the solution of the Cauchy problem

$$\begin{cases} {}_0^C D_t^\alpha u(t) + \lambda u(t) = 0, \quad t > 0, \\ u(0) = 1, \quad 0 < \alpha < 1, \end{cases}$$

has the form $u(t) = E_\alpha(-\lambda t^\alpha)$, where E_α is the known Mittag-Leffler function. This result issues a warning to us that no matter how smooth the data u_0 is, it is inappropriate to define the mild solution of problem (4.47) as follows

$$u(t) = T(t)u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) ds,$$

where $T(t)$ is the semigroup generated by A (see Remark 1.64 (i)), though this fashion was used in some situations of previous research (see, e.g., Jaradat, Ao-Omari and Momani, 2008).

(ii) Let us point out that in the treatment of problems (4.47) and (4.48), one of the difficult points is to give reasonable concept of solutions (see also the works of Zhou and Jiao, 2010a; Hernandez, O'Regan and Balachandran, 2010). Another is that even though the operator A generates a semigroup $T(t)$ in X, it will not be continuous at $t = 0$ for nonsmooth initial data u_0.

(iii) It is worth mentioning that if it is the case when A is a matrix (or even bounded linear operators) then Kilbas, Srivastava and Trujill, 2006, obtained an explicit representation of mild solution to problem (4.47).

Let us now give a short summary of this section, which is organized in a way close to that given by Carvalho, Dlotko and Nescimento, 2008. In Subsection 4.5.2 we give brief overview of the construction of functional calculus about almost sectorial operators, state some results about the analytic semigroups of growth order $1+\gamma$, and summarize some properties on two special functions. In Subsection 4.5.3, we construct a pair of families of operators and present deep anatomy on the properties for these families. Based on the families of operators defined in Subsection 4.5.3, a reasonable concept of solution will be given in Subsection 4.5.4 to problems (4.47), which in turn is used to analyze the existence of mild solutions and classical solutions to the Cauchy problem. The corresponding semilinear problem (4.48) is studied in Subsection 4.5.5. We investigate the existence of mild solutions, and then the existence of classical solutions. Finally, based mainly in Carvalho, Dlotko and Nescimento, 2008; Periago and Straub, 2002, we present three examples in Subsection 4.5.6 to illustrate our results.

Remark 4.54. Let us note that results in this section can be easily extended to the case of (general) sectorial operators.

4.5.2 *Preliminaries*

We first introduce some special functions and classes of functions which will be used in the following, for more details, we refer to Markus, 2006; Periago and

Straub, 2002. Let $-1 < \gamma < 0$, and let S^0_μ with $0 < \mu < \pi$ be the open sector $\{z \in \mathbb{C} \setminus \{0\}; |\arg z| < \mu\}$ and S_μ be its closure, that is $S_\mu = \{z \in \mathbb{C} \setminus \{0\}; |\arg z| \leq \mu\} \cup \{0\}$. Set

$$\mathcal{F}^\gamma_0(S^0_\mu) = \bigcup_{s<0} \Psi^\gamma_s(S^0_\mu) \bigcup \Psi_0(S^0_\mu),$$
$$\mathcal{F}(S^0_\mu) = \{f \in \mathcal{H}(S^0_\mu); \text{ there } k, n \in \mathbb{N} \text{ such that } f\psi^k_n \in \mathcal{F}_0(S^0_\mu)\},$$

where

$$\mathcal{H}(S^0_\mu) = \{f : S^0_\mu \mapsto \mathbb{C}; \ f \text{ is holomorphic}\},$$
$$\mathcal{H}^\infty(S^0_\mu) = \{f \in \mathcal{H}(S^0_\mu); \ f \text{ is bounded}\},$$
$$\varphi_0(z) = \frac{1}{1+z}, \quad \psi_n(z) := \frac{z}{(1+z)^n}, \quad z \in \mathbb{C}\backslash\{-1\}, n \in \mathbb{N} \cup \{0\},$$
$$\Psi_0(S^0_\mu) = \{f \in \mathcal{H}(S^0_\mu); \sup_{z \in S^0_\mu} \left|\frac{f(z)}{\varphi_0(z)}\right| < \infty\},$$

and for each $s < 0$,

$$\Psi^\gamma_s(S^0_\mu) := \{f \in \mathcal{H}(S^0_\mu); \sup_{z \in S^0_\mu} |\psi^s_n(z) f(z)| < \infty\},$$

where n is the smallest integer such that $n \geq 2$ and $\gamma + 1 < -(n-1)s$.

Observe that the classes of functions introduced above satisfy the inclusions

$$\mathcal{F}^\gamma_0(S^0_\mu) \subset \mathcal{H}^\infty(S^0_\mu) \subset \mathcal{F}(S^0_\mu) \subset \mathcal{H}(S^0_\mu).$$

Moreover, taking $k, n \in \mathbb{N} \cup \{0\}$ with $n > k$, one easily sees that $\psi^k_n \in \mathcal{F}^\gamma_0(S^0_\mu)$.

Assume that $A \in \Theta^\gamma_\omega(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Following Periago and Straub, 2002 (see also McIntosh, 1986; Cowling, Doust, McIntosh et al., 1996), a closed linear operator $f \to f(A)$ can be constructed for every $f \in \mathcal{F}(S^0_\mu)$ via a extended functional calculus. In the following we give a short overview to this construction.

For $f \in \mathcal{F}^\gamma_0(S^0_\mu)$, via the Dunford-Riesz integral, the operator $f(A)$ is defined by

$$f(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} f(z) R(z; A) dz, \tag{4.49}$$

where the integral contour $\Gamma_\theta := \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$, is oriented counter-clockwise and $\omega < \theta < \mu < \pi$. It follows that the integral is absolutely convergent and defines a bounded linear operator on X, and its value does not depend on the choice of θ.

Notice in particular that for $k, n \in \mathbb{N} \cup \{0\}$ with $n > k$,

$$\psi^k_n(A) = A^k (A+1)^{-n}$$

and the operator $\psi^k_n(A)$ is injective. Notice also that if $f \in \mathcal{F}(S^0_\mu)$, then there exist $k, n \in \mathbb{N}$ such that $f\psi^k_n \in \mathcal{F}^\gamma_0(S^0_\mu)$. Hence, for $f \in \mathcal{F}(S^0_\mu)$, one can define a closed linear operator, still denoted by $f(A)$,

$$D(f(A)) = \{x \in X; (f\psi^k_n)(A)x \in D(A^{(n-1)k})\},$$

$$f(A) = (\psi^k_n(A))^{-1} (f\psi^k_n)(A),$$

and the definition of $f(A)$ does not depend on the choice of k and n. We emphasize that $f(A)$ is indeed an extension of the original and the triple $(\mathcal{F}_0^\gamma(S_\mu^0), \mathcal{F}(S_\mu^0), f(A))$ is called an *abstract functional calculus* on X (see Markus, 2006).

With respect to this construction we collect some basic properties. For more details, we refer to Periago and Straub, 2002.

Proposition 4.55. The following assertions hold.

(i) $\alpha f(A) + \beta g(A) = (\alpha f + \beta g)(A)$, $(fg)(A) = f(A)g(A)$ for $\forall\ f, g \in \mathcal{F}_0^\gamma(S_\mu^0)$, $\alpha, \beta \in \mathbb{C}$;

(ii) $f(A)g(A) \subset (fg)(A)$ for $\forall\ f, g \in \mathcal{F}(S_\mu^0)$, and

(iii) $f(A)g(A) = (fg)(A)$, provided that $g(A)$ is bounded or $D((fg)(A)) \subset D(g(A))$.

Since for each $\beta \in \mathbb{C}$, $z^\beta \in \mathcal{F}(S_\mu^0)$ $(z \in \mathbb{C} \setminus (-\infty, 0],\ 0 < \mu < \pi)$, one can define, via the triple $(\mathcal{F}_0^\gamma(S_\mu^0), \mathcal{F}(S_\mu^0), f(A))$, the complex powers of A which are closed by

$$A^\beta = z^\beta(A), \quad \beta \in \mathbb{C},$$

However, in difference to the case of sectorial operators, having $0 \in \rho(A)$ does not imply that the complex powers $A^{-\beta}$ with $Re(\beta) > 0$, are bounded. The operator $A^{-\beta}$ belongs to $\mathscr{L}(X)$ whenever $Re(\beta) > 1+\gamma$. So, in this situation, the linear space $X^\beta := D(A^\beta)$, $\beta > 1 + \gamma$, endowed with the graph norm $|x|_\beta = |A^\beta x|$, $\quad x \in X^\beta$, is a Banach space.

Next, we turn our attention to the semigroup associated with A. Since given $t \in S^0_{\frac{\pi}{2}-\omega}$, $e^{-tz} \in \mathcal{H}^\infty(S_\mu^0)$ satisfies the conditions (a) and (b) of Lemma 2.13 of Periago and Straub, 2002, the family

$$T(t) = e^{-tz}(A) = \frac{1}{2\pi i}\int_{\Gamma_\theta} e^{-tz}R(z;A)dz, \quad t \in S^0_{\frac{\pi}{2}-\omega} \tag{4.50}$$

here $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$, forms a analytic semigroup of growth order $1 + \gamma$. For more properties on $T(t)$, please see the following proposition.

Proposition 4.56. (Periago and Straub, 2002) Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then the following properties remain true.

(i) $T(t)$ is analytic in $S^0_{\frac{\pi}{2}-\omega}$ and
$$\frac{d^n}{dt^n}T(t) = (-A)^n T(t), \quad \text{for all } t \in S^0_{\frac{\pi}{2}-\omega};$$

(ii) The functional equation $T(s+t) = T(s)T(t)$ for all $s,\ t \in S^0_{\frac{\pi}{2}-\omega}$ holds;

(iii) There exists a constant $C_0 = C_0(\gamma) > 0$ such that
$$\|T(t)\|_{\mathscr{L}(X)} \le C_0 t^{-\gamma-1}, \quad \text{for all } t > 0;$$

(iv) The range $R(T(t))$ of $T(t)$, $t \in S^0_{\frac{\pi}{2}-\omega}$ is contained in $D(A^\infty)$. Particularly, $R(T(t)) \subset D(A^\beta)$ for all $\beta \in \mathbb{C}$ with $Re(\beta) > 0$,
$$A^\beta T(t)x = \frac{1}{2\pi i}\int_{\Gamma_\theta} z^\beta e^{-tz}R(z;A)x dz, \quad \text{for all } x \in X,$$
and hence there exists a constant $C' = C'(\gamma, \beta) > 0$ such that
$$\|A^\beta T(t)\|_{\mathscr{L}(X)} \le C' t^{-\gamma-Re(\beta)-1}, \quad \text{for all } t > 0;$$

(v) If $\beta > 1+\gamma$, then $D(A^\beta) \subset \Sigma_T$, where Σ_T is the continuity set of the semigroup $\{T(t)\}_{t\geq 0}$, that is,

$$\Sigma_T = \left\{ x \in X; \lim_{t\to 0;\ t>0} T(t)x = x \right\}.$$

Remark 4.57. We note that the condition (ii) of the proposition does not satisfy for $t = 0$ or $s = 0$.

Recall that semigroups of growth $1+\gamma$ were investigated earlier in deLaubenfels, 1994 and Toropova, 2003.

The relation between the resolvent operators of A and the semigroup $T(t)$ is characterized by

Proposition 4.58. (Periago and Straub, 2002) Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. Then for every $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, one has $R(\lambda, -A) = \int_0^\infty e^{-\lambda t} T(t)dt$.

At the end of this section, we present some properties of two special functions.

Definition 4.59. (Miller and Ross, 1993; Podlubny, 1999) The generalized Mittag-Leffler special function $E_{\alpha,\beta}$ is defined by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} = \frac{1}{2\pi i}\int_{\Upsilon} \frac{\lambda^{\alpha-\beta}e^\lambda}{\lambda^\alpha - z} d\lambda, \quad \alpha,\ \beta > 0, z \in \mathbb{C},$$

where Υ is a contour which starts and ends as $-\infty$ and encircles the disc $|\lambda| \leq |z|^{1/\alpha}$ counter-clockwise.

If $0 < \alpha < 1,\ \beta > 0$, then the asymptotic expansion of $E_{\alpha,\beta}$ as $z \to \infty$ is given by

$$E_{\alpha,\beta}(z) = \begin{cases} \dfrac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), & |\arg z| \leq \frac{1}{2}\alpha\pi, \\ \varepsilon_{\alpha,\beta}(z), & |\arg(-z)| < (1-\frac{1}{2}\alpha)\pi, \end{cases} \tag{4.51}$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + O(|z|^{-N}), \quad \text{as} \quad z \to \infty.$$

For short, set

$$E_\alpha(z) := E_{\alpha,1}(z), \quad e_\alpha(z) := E_{\alpha,\alpha}(z).$$

Then one has

$${}_0^C D_t^\alpha E(\omega t^\alpha) = \omega E(\omega t^\alpha), \quad {}_0D_t^{\alpha-1}(t^{\alpha-1}e_\alpha(\omega t^\alpha)) = E_\alpha(\omega t^\alpha).$$

Consider the function of Wright-type

$$\begin{aligned}\Psi_\alpha(z) :&= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\alpha n + 1 - \alpha)} \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-z)^n}{(n-1)!}\Gamma(n\alpha)\sin(n\pi\alpha), \quad z \in \mathbb{C}\end{aligned}$$

with $0 < \alpha < 1$. For $-1 < r < \infty, \lambda > 0$, the following results hold.

Property 4.60.

(W1) $\Psi_\alpha(t) \geq 0, \quad t > 0;$

(W2) $\displaystyle\int_0^\infty \frac{\alpha}{t^{\alpha+1}} \Psi_\alpha(\frac{1}{t^\alpha}) e^{-\lambda t} dt = e^{-\lambda^\alpha};$

(W3) $\displaystyle\int_0^\infty \Psi_\alpha(t) t^r dt = \frac{\Gamma(1+r)}{\Gamma(1+\alpha r)};$

(W4) $\displaystyle\int_0^\infty \Psi_\alpha(t) e^{-zt} dt = E_\alpha(-z), \quad z \in \mathbb{C};$

(W5) $\displaystyle\int_0^\infty \alpha t \Psi_\alpha(t) e^{-zt} dt = e_\alpha(-z), \quad z \in \mathbb{C}.$

4.5.3 *Properties of Operators*

Throughout this subsection we let A be an operator in the class $\Theta_\omega^\gamma(X)$ and $-1 < \gamma < 0$, $0 < \omega < \pi/2$. In the sequel, we will succeed in defining two families of operators based on the generalized Mittag-Leffler-type functions and the resolvent operators associated with A. They will be two families of linear and bounded operators. In order to check the properties of the families, we will need a third object, namely the semigroup associated with A. We stress that these families will be used very frequently throughout the rest of this section. Below the letter C will denote various positive constants.

Define operator families $\{\mathcal{S}_\alpha(t)\}|_{t \in S^0_{\frac{\pi}{2}-\omega}}$, $\{\mathcal{P}_\alpha(t)\}|_{t \in S^0_{\frac{\pi}{2}-\omega}}$ by

$$\mathcal{S}_\alpha(t) := E_\alpha(-zt^\alpha)(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} E_\alpha(-zt^\alpha) R(z;A) dz,$$

$$\mathcal{P}_\alpha(t) := e_\alpha(-zt^\alpha)(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} e_\alpha(-zt^\alpha) R(z;A) dz,$$

where the integral contour $\Gamma_\theta := \{\mathbb{R}_+ e^{i\theta}\} \cup \{\mathbb{R}_+ e^{-i\theta}\}$, is oriented counter-clockwise and $\omega < \theta < \mu < \frac{\pi}{2} - |\arg t|$.

We need some basic properties of these families which are used further in this section.

Theorem 4.61. For each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are linear and bounded operators on X. Moreover, there exists constants $C_s = C(\alpha,\gamma) > 0$, $C_p = C(\alpha,\gamma) > 0$ such that for all $t > 0$,

$$\|\mathcal{S}_\alpha(t)\| \leq C_s t^{-\alpha(1+\gamma)}, \quad \|\mathcal{P}_\alpha(t)\| \leq C_p t^{-\alpha(1+\gamma)}. \tag{4.52}$$

Proof. Note, from the asymptotic expansion of $E_{\alpha,\beta}$ that for each fixed $t \in S^0_{\frac{\pi}{2}-\omega}$,

$$E_\alpha(-zt^\alpha),\ e_\alpha(-zt^\alpha) \in \mathcal{F}_0^\gamma(S_\mu^0).$$

Therefore, by (4.49), the operators families $\{\mathcal{S}_\alpha(t)\}|_{t \in S^0_{\frac{\pi}{2}-\omega}}$, $\{\mathcal{P}_\alpha(t)\}|_{t \in S^0_{\frac{\pi}{2}-\omega}}$ are well-defined, and for each $t \in S^0_{\frac{\pi}{2}-\omega}$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are linear bounded operators on X. So, to prove the theorem, it is sufficient to prove that the estimates in (4.52) hold.

Let $T(t),\ t \in S^0_{\frac{\pi}{2}-\omega}$, be the semigroup defined by (4.50). Then by (W4) and the Fubini Theorem, we get

$$\begin{aligned}\mathcal{S}_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_\theta} E_\alpha(-zt^\alpha)R(z;A)xdz \\ &= \frac{1}{2\pi i}\int_0^\infty \Psi_\alpha(\lambda)\int_{\Gamma_\theta} e^{-\lambda z t^\alpha} R(z;A)xdzd\lambda \\ &= \int_0^\infty \Psi_\alpha(s)T(st^\alpha)xds, \quad t \in S^0_{\frac{\pi}{2}-\omega},\ x \in X.\end{aligned} \tag{4.53}$$

A similar argument shows that

$$\mathcal{P}_\alpha(t)x = \int_0^\infty \alpha s\Psi_\alpha(s)T(st^\alpha)xds, \quad t \in S^0_{\frac{\pi}{2}-\omega},\ x \in X. \tag{4.54}$$

Hence, by (4.53), (4.54), Proposition 4.56 (iii), (W1) and (W3), we have

$$\begin{aligned}|\mathcal{S}_\alpha(t)x| &\le C_0\int_0^\infty \Psi_\alpha(s)s^{-(1+\gamma)}t^{-\alpha(1+\gamma)}|x|ds \\ &\le C_0\frac{\Gamma(-\gamma)}{\Gamma(1-\alpha(1+\gamma))}t^{-\alpha(1+\gamma)}|x|, \quad t>0,\ x\in X,\end{aligned}$$

$$\begin{aligned}|\mathcal{P}_\alpha(t)x| &\le \alpha C_0\int_0^\infty \Psi_\alpha(s)s^{-\gamma}t^{-\alpha(1+\gamma)}|x|ds \\ &\le \alpha C_0\frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha\gamma)}t^{-\alpha(1+\gamma)}|x| \quad t>0,\ x\in X.\end{aligned}$$

Therefore, the estimates in (4.52) hold. This completes the proof. □

From now on, we will frequently use the representations (4.53) and (4.54) for operators $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$, respectively.

Theorem 4.62. For $t>0$, $\mathcal{S}_\alpha(t)$ and $\mathcal{P}_\alpha(t)$ are continuous in the uniform operator topology. Moreover, for every $r>0$, the continuity is uniform on $[r,\infty)$.

Proof. Let $\epsilon>0$ be given. For every $r>0$, it follows from (W3) that we may choose $\delta_1,\delta_2>0$ such that

$$\frac{2C_0}{r^{\alpha(1+\gamma)}}\int_0^{\delta_1}\Psi_\alpha(s)s^{-(1+\gamma)}ds \le \frac{\epsilon}{3}, \quad \frac{2C_0}{r^{\alpha(1+\gamma)}}\int_{\delta_2}^\infty \Psi_\alpha(s)s^{-(1+\gamma)}ds \le \frac{\epsilon}{3}. \tag{4.55}$$

Then we deduce, by Proposition 4.56 (i), that there exists a positive constant δ such that

$$\int_{\delta_1}^{\delta_2}\Psi_\alpha(s)\|T(t_1^\alpha s)-T(t_2^\alpha s)\|ds \le \frac{\epsilon}{3}, \tag{4.56}$$

for $t_1,t_2\ge r$ and $|t_1-t_2|<\delta$.

On the other hand, using (4.55), (4.56) and Theorem 4.61, we get

$$\begin{aligned}
|\mathcal{S}_\alpha(t_1)x - \mathcal{S}_\alpha(t_2)x| &\leq \int_0^{\delta_1} \Psi_\alpha(s)(\|T(t_1^\alpha s)\| + \|T(t_2^\alpha s)\|)|x|ds \\
&\quad + \int_{\delta_1}^{\delta_2} \Psi_\alpha(s)\|T(t_1^\alpha s) - T(t_2^\alpha s)\||x|ds \\
&\quad + \int_{\delta_2}^{\infty} \Psi_\alpha(s)(\|T(t_1^\alpha s)\| + \|T(t_2^\alpha s)\|)|x|ds \\
&\leq \frac{2C_0}{r^{\alpha(1+\gamma)}} \int_0^{\delta_1} \Psi_\alpha(s)s^{-(1+\gamma)}|x|ds \\
&\quad + \int_{\delta_1}^{\delta_2} \Psi_\alpha(s)\|T(t_1^\alpha s) - T(t_2^\alpha s)\||x|ds \\
&\quad + \frac{2C_0}{r^{\alpha(1+\gamma)}} \int_{\delta_2}^{\infty} \Psi_\alpha(s)s^{-(1+\gamma)}|x|ds \\
&\leq \epsilon|x|, \quad \text{for any } x \in X,
\end{aligned}$$

that is,

$$\|\mathcal{S}_\alpha(t_1) - \mathcal{S}_\alpha(t_2)\| \leq \epsilon,$$

which implies that $\mathcal{S}_\alpha(t)$ is uniformly continuous on $[r, \infty)$ in the uniform operator topology and hence, by the arbitrariness of $r > 0$, $\mathcal{S}_\alpha(t)$ is continuous in the uniform operator topology for $t > 0$. A similar argument enables us to give the characterization of continuity on $\mathcal{P}_\alpha(t)$. This completes the proof. □

Theorem 4.63. Let $0 < \beta < 1 - \gamma$. Then

(i) The range $R(\mathcal{P}_\alpha(t))$ of $\mathcal{P}_\alpha(t)$ for $t > 0$, is contained in $D(A^\beta)$;
(ii) $\mathcal{S}'_\alpha(t)x = -t^{\alpha-1}A\mathcal{P}_\alpha(t)x$ $(x \in X)$, and $\mathcal{S}'_\alpha(t)x$ for $x \in D(A)$ is locally integrable on $(0, \infty)$;
(iii) for all $x \in D(A)$ and $t > 0$, $\|A\mathcal{S}_\alpha(t)x\| \leq Ct^{-\alpha(1+\gamma)}\|Ax\|$, here C is a constant depending on γ, α.

Proof. It follows from Proposition 4.56 (iv) that for all $x \in X$, $t > 0$, $T(t)x \in D(A^\beta)$ with $\beta > 0$. Therefore, in view of (4.54), Proposition 4.56 (iv) and (W3) we have

$$\begin{aligned}
|A^\beta \mathcal{P}_\alpha(t)x| &\leq \int_0^\infty \alpha s \Psi_\alpha(s)\|A^\beta T(t^\alpha s)\||x|ds \\
&\leq \alpha C' t^{-\alpha(\gamma+\beta+1)} \int_0^\infty \Psi_\alpha(s)s^{-(\beta+\gamma)}ds|x| \\
&\leq \alpha C' \frac{\Gamma(1-\beta-\gamma)}{\Gamma(1-\alpha(\beta+\gamma+1))} t^{-\alpha(1+\beta+\gamma)}|x|,
\end{aligned}$$

which implies that the assertion (i) holds.

From (i), it is easy to see that for all $x \in X$,

$$\mathcal{S}'_\alpha(t)x = -t^{\alpha-1}A\mathcal{P}_\alpha(t)x.$$

Moreover, for every $x \in D(A)$, one has by Proposition 4.56 (iv),

$$|t^{\alpha-1} A\mathcal{P}_\alpha(t)x| \leq t^{\alpha-1} \int_0^\infty \alpha s \Psi_\alpha(s) \|T(t^\alpha s)\| |Ax| ds$$
$$\leq \alpha C_0 \frac{\Gamma(1-\gamma)}{\Gamma(1-\alpha\gamma)} t^{-\alpha\gamma-1} |Ax|.$$

Since $-\alpha\gamma - 1 > -1$, this shows that $\mathcal{S}'_\alpha(t)x$ for each $x \in D(A)$, is locally integrable on $(0, \infty)$, that is , (ii) is true.

Moreover, Proposition 4.56 (iv) and (4.53) imply that

$$|A\mathcal{S}_\alpha(t)x| \leq C_0 t^{-\alpha(1+\gamma)} \int_0^\infty \Psi_\alpha(s) s^{-1-\gamma} ds |Ax|$$
$$\leq C_0 \frac{\Gamma(-\gamma)}{\Gamma(1-\alpha(1+\gamma))} t^{-\alpha(1+\gamma)} |Ax|, \quad x \in D(A).$$

This means that (iii) holds, and completes the proof. □

Remark 4.64. Particularly, from the proof of Theorem 4.63 (i), we can conclude that

$$\|A\mathcal{P}_\alpha(t)\| \leq C t^{-\alpha(2+\gamma)},$$

where C is a constant depending on γ, α. Moreover, using a similar argument with that in Theorem 4.62, we have that $A\mathcal{P}_\alpha(t)$ for $t > 0$ is continuous in the uniform operator topology.

Theorem 4.65. The following properties hold.

(i) Let $\beta > 1 + \gamma$. For all $x \in D(A^\beta)$, $\lim_{t \to 0; t > 0} \mathcal{S}_\alpha(t)x = x$,
(ii) for all $x \in D(A)$,$(\mathcal{S}_\alpha(t) - I)x = \int_0^t -s^{\alpha-1} A\mathcal{P}_\alpha(s)x ds$,
(iii) for all $x \in D(A)$, $t > 0$, ${}_0D_t^\alpha \mathcal{S}_\alpha(t)x = -A\mathcal{S}_\alpha(t)x$,
(iv) for all $t > 0$, $\mathcal{S}_\alpha(t) = {}_0D_t^{\alpha-1}(t^{\alpha-1}\mathcal{P}_\alpha(t))$.

Proof. For any $x \in X$, note by (4.53) and (W_3) that

$$\mathcal{S}_\alpha(t)x - x = \int_0^\infty \Psi_\alpha(s)(T(t^\alpha s)x - x) ds.$$

On the other hand, by Proposition 4.56 (v) it follows that $D(A^\beta) \subset \Sigma_T$ in view of $\beta > 1 + \gamma$. Therefore, we deduce, using Proposition 4.56 (iii), that for any $x \in D(A^\beta)$, there exists a function $\eta(s) \in L^1(0, +\infty)$ depending on $\Psi_\alpha(s)$ such that

$$\| \Psi_\alpha(s)(T(t^\alpha s)x - x) \| \leq \eta(s).$$

Hence, by means of the Lebesgue's dominated convergence theorem we obtain that

$$\mathcal{S}_\alpha(t)x - x \to 0, \quad \text{as } t \to 0,$$

that is, the assertion (i) remains true.

From (i) and Theorem 4.63 (ii) we get for all $x \in D(A)$,

$$(\mathcal{S}_\alpha(t) - I)x = \lim_{s \to 0} (\mathcal{S}_\alpha(t)x - \mathcal{S}_\alpha(s)x) = \int_0^t -\lambda^{\alpha-1} A\mathcal{P}_\alpha(\lambda)x d\lambda,$$

which implies that the assertion (ii) holds.

To prove (iii), first it is easy to see that $\frac{1}{\varphi_0} \in \mathcal{F}(S_\mu^0)$ and the operator $\varphi_0(A)$ is injective. Taking $x \in D(A)$, by Proposition 4.55 (iii) one has

$$\mathcal{S}_\alpha(t)x = E_\alpha(-zt^\alpha)(A)x = (E_\alpha(-zt^\alpha)\varphi_0)(A)\left(\frac{1}{\varphi_0}\right)(A)x.$$

Moreover, by (4.51) we have $\sup_{z\to\infty} |zt^\alpha E_\alpha(-zt^\alpha)| < \infty$, which implies that

$$|zE_\alpha(-zt^\alpha)(1+ z)^{-1}| \le C|z|^{-1}t^{-\alpha}, \quad \text{as } z \to \infty,$$

where C is a constant which is independent of t. Consequently,

$$-zE_\alpha(-zt^\alpha)(1 + z)^{-1} \in \mathcal{F}_0^\gamma(S_\mu^0). \tag{4.57}$$

Notice also that

$${}_0^C D_t^\alpha E_\alpha(-zt^\alpha)(1+z)^{-1}R(z; A) = (-z)E_\alpha(-zt^\alpha)(1+z)^{-1}R(z; A).$$

Combining Proposition 4.55 (ii) and (4.57), we get

$$\begin{aligned}{}_0^C D_t^\alpha((E_\alpha(-zt^\alpha)(1 + z^\beta)^{-1})(A)) &= \frac{1}{2\pi i}\int_{\Gamma_\theta} (-z)E_\alpha(-zt^\alpha)(1+ z)^{-1}R(z; A)dz \\ &= (-z)(A)\big(E_\alpha(-zt^\alpha)(1 + z)^{-1}\big)(A) \\ &= -A\big(E_\alpha(-zt^\alpha)(1+ z)^{-1}\big)(A).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}{}_0D_t^\alpha \mathcal{S}_\alpha(t)x &= -A\big(E_\alpha(-zt^\alpha)(1 + z)^{-1}\big)(A)(1 + z)(A)x \\ &= -A(E_\alpha(-zt^\alpha))(A)x \\ &= -A\mathcal{S}_\alpha(t)x.\end{aligned}$$

This proves (iii).

For (iv), by a similar argument with (iii), one can prove that $t^{\alpha-1}e_\alpha(-zt^\alpha)$ belongs to $\mathcal{F}_0^\gamma(S_\mu^0)$ for $t > 0$ and hence

$${}_0D_t^{\alpha-1}(t^{\alpha-1}\mathcal{P}_\alpha(t)) = {}_0D_t^{\alpha-1}((t^{\alpha-1}e_\alpha(-zt^\alpha)(A)) = (E_\alpha(-zt^\alpha))(A) = \mathcal{S}_\alpha(t),$$

in view of

$${}_0D_t^{\alpha-1}\big(t^{\alpha-1}e_\alpha(-zt^\alpha)\big) = E_\alpha(-zt^\alpha).$$

This completes the proof. □

Before proceeding with our theory further, we present the following result.

Lemma 4.66. If $R(\lambda, -A)$ is compact for every $\lambda > 0$, then $T(t)$ is compact for every $t > 0$.

Proof. Note first that as a consequence of Theorem 3.13 in Periago and Straub, 2002., for every $\lambda \in \mathbb{C}$ with $Re(\lambda) > 0$, $R(\lambda; -A) = \int_0^\infty e^{-\lambda s}T(s)ds$ defines a bounded linear operator on X. Therefore, we obtain

$$\lambda R(\lambda; -A)T(t) - T(t) = \lambda \int_0^\infty e^{-\lambda s}(T(t + s) - T(t))ds. \tag{4.58}$$

Let $\epsilon > 0$ be given. For every $\lambda > 0$ and $t > 0$, it follows from Theorem 4.62 that there exists a $\nu > 0$ such that

$$\sup_{s\in[0,\nu]} \|T(s+t) - T(t)\| \le \frac{\epsilon}{2}.$$

So

$$\lambda \int_0^{\nu} e^{-s\lambda}\|T(t+s) - T(t)\|ds \le \frac{\epsilon}{2}. \tag{4.59}$$

On the other hand, by Proposition 4.56 (iii) we get

$$\begin{aligned}\lambda\left\| \int_{\nu}^{\infty} e^{-s\lambda}(T(s+t) - T(t))ds\right\| &\le \lambda C \int_{\nu}^{\infty} e^{-s\lambda}((t+s)^{-1-\gamma} + t^{-\gamma-1})ds \\ &\le 2Ct^{-\gamma-1}e^{-\lambda\nu},\end{aligned}$$

which implies that there exists a $\lambda_0 > 0$ such that

$$\lambda\left\| \int_{\nu}^{\infty} e^{-s\lambda}(T(s+t) - T(t))ds\right\| \le \frac{\epsilon}{2}, \quad \lambda \ge \lambda_0. \tag{4.60}$$

Thus, for all $\lambda \ge \lambda_0$, using (4.58), (4.59) and (4.60) we deduce that

$$\begin{aligned}\|\lambda R(\lambda; -A)T(t) - T(t)\| &\le \lambda \int_0^{\nu} e^{-s\lambda}\|T(t+s) - T(t)\|ds \\ &\quad + \lambda \int_{\nu}^{\infty} e^{-s\lambda}\|T(s+t) - T(t)\|ds \\ &\le \epsilon.\end{aligned}$$

It follows from the arbitrariness of $\nu > 0$ that

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda; -A)T(t) - T(t)\| = 0.$$

Since $\lambda R(\lambda; -A)T(t)$ is compact for every $\lambda > 0$ and $t > 0$, $T(t)$ is compact for every $t > 0$. □

With the help of this lemma we now show the following result.

Theorem 4.67. If $R(\lambda, -A)$ is compact for every $\lambda > 0$, then $\mathcal{S}_\alpha(t)$, $\mathcal{P}_\alpha(t)$ are compact for every $t > 0$.

Proof. Let $\epsilon > 0$ be arbitrary. Put

$$\varsigma_\epsilon(t) = \int_{\epsilon}^{\infty} \Psi_\alpha(s)T(st^\alpha - \epsilon t^\alpha)ds, \quad \zeta_\epsilon(t) = \int_{\epsilon}^{\infty} \Psi_\alpha(s)T(st^\alpha)ds.$$

Then, one has $\zeta_\epsilon(t) = T(\epsilon t^\alpha)\varsigma_\epsilon(t)$, and it is easy to prove that for every $t > 0$, $\varsigma_\epsilon(t)$ is bounded linear operators on X. Therefore, from the compactness of $T(t), t > 0$, we see that $\zeta_\epsilon(t)$ is compact for every $t > 0$.

On the other hand, note that

$$\|\zeta_\epsilon(t) - \mathcal{S}_\alpha(t)\| \le \left\| \int_0^{\epsilon} \Psi_\alpha(s)T(st^\alpha)ds\right\| \le C_0 t^{-\alpha(1+\gamma)} \int_0^{\epsilon} \Psi_\alpha(s)s^{-1-\gamma}ds.$$

Hence, it follows from the compactness of $\zeta_\epsilon(t), t > 0$ that $\mathcal{S}_\alpha(t)$ is compact for every $t > 0$. By a similar technique we can conclude that $\mathcal{P}_\alpha(t)$ is compact for every $t > 0$. The proof is completed. □

4.5.4 *Linear Problems*

Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \pi/2$. We discuss the existence and uniqueness of mild solution and classical solutions for inhomogeneous linear abstract Cauchy problem (4.47). We assume the following condition.
(H*) $u \in C([0,T];X)$, $g_{1-\alpha} * u \in C^1((0,T];X)$, $u(t) \in D(A)$ for $t \in (0,T]$, $Au \in L^1((0,T);X)$, and u satisfies (4.47).

Then, by Definitions 1.5 and 1.8, one can rewrite (4.47) as

$$u(t) = u_0 - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Au(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds, \qquad (4.61)$$

for $t \in [0,T]$.

Before presenting the definition of mild solution of problem (4.47), we first prove the following lemma.

Lemma 4.68. If $u : [0,T] \to X$ is a function satisfying the assumption (H*), then $u(t)$ satisfies the following integral equation

$$u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds, \quad t \in (0,T].$$

Proof. Note that the Laplace transform of a abstract function $f \in L^1(\mathbb{R}^+, X)$ is defined by $\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t}f(t)dt, \lambda > 0$. Applying Laplace transfer to (4.61) we get $\widehat{u}(\lambda) = \frac{u_0}{\lambda} - \frac{1}{\lambda^\alpha}A\widehat{u}(\lambda) + \frac{\widehat{f}(\lambda)}{\lambda^\alpha}$, that is,

$$\widehat{u}(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}u_0 + (\lambda^\alpha + A)^{-1}\widehat{f}(\lambda).$$

On the other hand, using Proposition 4.58 and (W2) we deduce that

$$\begin{aligned}
&\lambda^{\alpha-1}(\lambda^\alpha + A)^{-1}u_0 + (\lambda^\alpha + A)^{-1}\widehat{f}(\lambda)\\
&= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha t}T(t)u_0dt + \int_0^\infty e^{-\lambda^\alpha t}T(t)\widehat{f}(\lambda)dt\\
&= \int_0^\infty \frac{d}{d\lambda}e^{-(\lambda t)^\alpha}T(t^\alpha)u_0dt + \int_0^\infty\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha t}T(t^\alpha)f(s)e^{-s\lambda}dsdt\\
&= \int_0^\infty\int_0^\infty \frac{\alpha t}{\tau^\alpha}\Psi_\alpha\left(\frac{1}{\tau^\alpha}\right)e^{-\lambda t\tau}T(t^\alpha)d\tau dt\\
&\quad + \int_0^\infty\int_0^\infty\int_0^\infty \frac{\alpha}{\tau^{2\alpha}}t^{\alpha-1}\Psi\left(\frac{1}{\tau^\alpha}\right)e^{-\lambda t}T(\frac{t^\alpha}{\tau^\alpha})f(s)e^{-s\lambda}d\tau dsdt\\
&= \int_0^\infty\int_0^\infty \frac{\alpha}{\tau^{\alpha+1}}\Psi_\alpha\left(\frac{1}{\tau^\alpha}\right)e^{-\lambda t}T(\frac{t^\alpha}{\tau^\alpha})d\tau dt\\
&\quad + \int_0^\infty\int_0^\infty\int_0^\infty \alpha\tau t^{\alpha-1}\Psi(\tau)T(t^\alpha\tau)f(s)e^{-(s+t)\lambda}d\tau dsdt\\
&= \int_0^\infty e^{-\lambda t}\int_0^\infty \Psi_\alpha(\tau)T(t^\alpha\tau)d\tau\\
&\quad + \int_0^\infty e^{-t\lambda}\int_0^t (t-s)^{\alpha-1}f(s)\left(\int_0^\infty \alpha\tau\Psi(\tau)T((t-s)^\alpha\tau)d\tau\right)dsdt
\end{aligned}$$

$$= \int_0^\infty e^{-\lambda t} \mathcal{S}_\alpha(t) dt + \int_0^\infty e^{-\lambda t} \int_0^t (t-s)^\alpha \mathcal{P}_\alpha(t-s) f(s) ds dt$$
$$= \int_0^\infty e^{-\lambda t} \Big(\mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s) ds \Big) dt.$$

This implies that

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} \Big(\mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s) ds \Big) dt.$$

Now using the uniqueness of the Laplace transform (cf. Theorem 1.1.6 of Xiao and Liang, 1998), we deduce that the assertion of lemma holds. This completes this proof. □

Motivated by Lemma 4.68, we adopt the following concept of mild solution to problem (4.47).

Definition 4.69. By a mild solution of problem (4.47), we mean a function $u \in C((0,T];X)$ satisfying

$$u(t) = \mathcal{S}_\alpha(t) u_0 + \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s) f(s) ds, \quad t \in (0,T].$$

Remark 4.70. It is to be noted that

(a) unlike the case of strongly continuous operator semigroups, we do not require the mild solution of problem (4.47) to be continuous at $t = 0$. Moreover, in general, since the operator $\mathcal{S}_\alpha(t)$ is singular at $t = 0$, solutions to problem (4.47) are assumed to have the same kind of singularity at $t = 0$ as the operator $\mathcal{S}_\alpha(t)$. This is the case, for instance, if f $\equiv 0$ so that we have that $u(t) = \mathcal{S}_\alpha(t) u_0$, which presents a discontinuity at the initial time;

(b) When $u_0 \in D(A^\beta)$, $\beta > 1 + \gamma$, it follows from Theorem 4.65 (i) that the mild solution is continuous at $t = 0$.

For $f \in L^1((0,T);X)$, the initial problem (4.47) has an unique mild solution for every $u_0 \in X$. We will now be interested in imposing further condition on f and u_0 so that the mild solution will become a classical solution. To this end we first introduce the following definition.

Definition 4.71. By a classical solution to problem (4.47), we mean a function $u(t) \in C([0,T];X)$ with ${}_0^C D_t^\alpha u(t) \in C((0,T];X)$, which, for all $t \in (0,T]$, takes values in $D(A)$ and satisfies (4.47).

We are now ready to state our main result in this subsection.

Theorem 4.72. Let $A \in \Theta_\omega^\gamma(X)$ with $0 < \omega < \frac{\pi}{2}$. Suppose that $f(t) \in D(A)$ for all $0 < t \le T$, $Af(t) \in L^\infty((0,T);X)$, and $f(t)$ is Hölder continuous with an exponent $\theta' > \alpha(1+\gamma)$, that is,

$$|f(t) - f(s)| \le K|t-s|^{\theta'}, \text{ for all } 0 < t, s \le T.$$

Then, for every $u_0 \in D(A)$, there exists a classical solution to problem (4.47) and this solution is unique.

Proof. For $u_0 \in D(A)$, let $u(t) = \mathcal{S}_\alpha(t)u_0$ $(t > 0)$. Then it follows from Theorem 4.65 (i, iii) that $u(t)$ is a classical solution of the following problem

$$\begin{cases} {}_0^C D_t^\alpha u(t) + Au(t) = 0, & 0 < t \le T, \\ u(0) = u_0. \end{cases} \tag{4.62}$$

Moreover, from Lemma 4.68, it is easy to see that $u(t)$ is the only solution to problem (4.62). Put

$$w(t) = \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s)ds, \quad 0 < t \le T.$$

Then from the assumptions on f and Theorem 4.61 we obtain

$$\begin{aligned} \|Aw(t)\| &\le \int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\|\|Af(t)\|_{L^\infty((0,T);X)}ds \\ &\le C_p\|Af(t)\|_{L^\infty((0,T);X)}\frac{1}{-\alpha\gamma}t^{-\gamma\alpha}, \end{aligned}$$

which implies that $w(t) \in D(A)$ for all $0 < t \le T$.

Next, we show ${}_0^C D_t^\alpha w(t) \in C((0,T;X)$. Since $w(0) = 0$ and hence

$${}_0^C D_t^\alpha w(t) = {}_0D_t^1\, {}_0D_t^{\alpha-1}w(t) = \frac{d}{dt}\int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, \tag{4.63}$$

in view of Properties 1.17, 1.21, 1.22 and Theorem 4.65 (iv). Let $v(t) = \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds$, it remains to prove $v(t) \in C^1((0,T];X)$. Let $h > 0$ and $h \le T - t$. Then it is easy to verify the identity

$$\begin{aligned} \frac{v(t+h)-v(t)}{h} = &\int_0^t \frac{\mathcal{S}_\alpha(t+h-s)-\mathcal{S}_\alpha(t-s)}{h}f(s)ds \\ &+ \frac{1}{h}\int_t^{t+h} \mathcal{S}_\alpha(t+h-s)f(s)ds. \end{aligned}$$

Again by the assumptions on f and Theorem 4.61, we have, for $t > 0$ fixed,

$$\|(t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)f(s)\| \le C_p(t-s)^{-\alpha\gamma-1}\|Af(s)\| \in L^1((0,T);X),$$

for all $s \in [0,t)$. Therefore, using Theorem 4.63 (ii) and the Dominated Convergence Theorem we get

$$\begin{aligned} &\lim_{h\to 0}\int_0^t \frac{\mathcal{S}_\alpha(t+h-s)-\mathcal{S}_\alpha(t-s)}{h}f(s)ds \\ &= \int_0^t (t-s)^{\alpha-1}(-A)\mathcal{P}_\alpha(t-s)f(s)ds \\ &= -Aw(t). \end{aligned} \tag{4.64}$$

Furthermore, note that

$$\begin{aligned} &\frac{1}{h}\int_t^{t+h}\mathcal{S}_\alpha(t+h-s)f(s)ds \\ &= \frac{1}{h}\int_0^h \mathcal{S}_\alpha(s)f(t+h-s)ds \\ &= \frac{1}{h}\int_0^h \mathcal{S}_\alpha(s)(f(t+h-s)-f(t-s))ds \\ &\quad + \frac{1}{h}\int_0^h \mathcal{S}_\alpha(s)(f(t-s)-f(t))ds + \frac{1}{h}\int_0^h \mathcal{S}_\alpha(s)f(t)ds. \end{aligned}$$

From Theorem 4.61 and the Hölder continuity on f we have

$$\frac{1}{h}\left|\int_0^h \mathcal{S}_\alpha(s)(f(t+h-s)-f(t-s))ds\right| \le \frac{C_s K h^{\theta'-\alpha(1+\gamma)}}{1-\alpha(1+\gamma)},$$

and

$$\frac{1}{h}\left|\int_0^h \mathcal{S}_\alpha(s)(f(t-s)-f(t))ds\right| \le \frac{C_s K h^{\theta'-\alpha(1+\gamma)}}{1+\theta-\alpha(1+\gamma)}.$$

And since $f(t)\in D(A)$ $(0<t\le T)$, $\lim_{h\to 0}\frac{1}{h}\int_0^h \mathcal{S}_\alpha(s)f(t)ds = f(t)$ in view of Theorem 4.65 (i). Hence,

$$\frac{1}{h}\int_t^{t+h} \mathcal{S}_\alpha(t+h-s)f(s)ds \to f(t), \quad \text{as } h\to 0^+. \tag{4.65}$$

Combining (4.64) and (4.65) we deduce that v is differentiable from the right at t and $v'_+(t)=f(t)-Aw(t),\ t\in(0,T]$. By a similar argument with the above one has that v is differentiable from the left at t and $v'_-(t)=f(t)-Aw(t),\ t\in(0,T]$. Next, we prove $Aw(t)\in C((0,T];X)$. To the end, let $Aw(t)=I_1(t)+I_2(t)$, where

$$I_1(t) = \int_0^t (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)\big(f(s)-f(t)\big)ds,$$
$$I_2(t) = \int_0^t A(t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(t)ds.$$

By Theorem 4.65 (ii), we obtain

$$I_2(t) = -(\mathcal{S}_\alpha(t)-I)f(t).$$

So, by the assumption of f and Theorem 4.62 note that $I_2(t)$ is continuous for $0<t\le T$. To prove the same conclusion for $I_1(t)$, we let $0<h\le T-t$ and write

$$\begin{aligned}
&I_1(t+h)-I_1(t)\\
&= \int_0^t \Big((t+h-s)^{\alpha-1}A\mathcal{P}_\alpha(t+h-s)-(t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)\Big)\big(f(s)-f(t)\big)ds\\
&\quad + \int_0^t (t+h-s)^{\alpha-1}A\mathcal{P}_\alpha(t+h-s)(f(t)-f(t+h))ds\\
&\quad + \int_t^{t+h} (t+h-s)^{\alpha-1}A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t+h)\big)ds\\
&=: h_1(t)+h_2(t)+h_3(t).
\end{aligned}$$

For $h_1(t)$, on the one hand, it follows from Theorem 4.62 that

$$\begin{aligned}
&\lim_{h\to 0}(t+h-s)^{\alpha-1}A\mathcal{P}_\alpha(t+h-s)(f(s)-f(t))\\
&= (t-s)^{\alpha-1}A\mathcal{P}_\alpha(t-s)(f(s)-f(t)).
\end{aligned}$$

On the other hand, for $t \in (0,T]$ fixed, by Remark 4.64 and the assumption on f, we get

$$\begin{aligned}
&|(t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t)\big)| \\
&\le C_p' K(t+h-s)^{-\alpha(1+\gamma)-1}(t-s)^{\theta'} \\
&\le C_p' K(t-s)^{(\theta'-\alpha-\alpha\gamma)-1} \in L^1((0,t);X).
\end{aligned}$$

Thus, by mean of the Dominated Convergence Theorem one has

$$\begin{aligned}
&\lim_{h\to 0}\int_0^t (t+h-s)^{\alpha-1} A\mathcal{P}_\alpha(t+h-s)\big(f(s)-f(t)\big)ds \\
&= \int_0^t (t-s)^{\alpha-1} A\mathcal{P}_\alpha(t-s)\big(f(s)-f(t)\big)ds,
\end{aligned}$$

which implies that $h_1(t) \to 0$ as $h \to 0^+$.

For $h_2(t)$, using Theorem 4.63 (i), Remark 4.64,

$$\begin{aligned}
&\int_0^t (t+h-s)^{\alpha-1}\|A\mathcal{P}_\alpha(t+h-s)\||f(t)-f(t+h)|ds \\
&\le \int_0^t C_p' K(t+h-s)^{-\alpha(1+\gamma)-1} h^{\theta'} ds \\
&= \frac{C_p' K h^{\theta'}}{\alpha(1+\gamma)}\big(h^{-\alpha(1+\gamma)} - (h+t)^{-\alpha(1+\gamma)}\big).
\end{aligned}$$

This yields $h_2(t) \to 0$ as $h \to 0^+$.

Moreover,$h_3(t) \to 0$ as $h \to 0^+$ by the following estimate

$$\begin{aligned}
&\left|\int_t^{t+h} (t+h-s)^{\alpha-1}\mathcal{P}_\alpha(t+h-s)\big(Af(s) - Af(t+h)\big)ds\right| \\
&\le \frac{2C_p}{-\alpha\gamma}\|Af(s)\|_{L^\infty(0,T;X)} h^{-\alpha\gamma}
\end{aligned}$$

in view of $Af(s) \in L^\infty((0,T);X)$ and Theorem 4.62.

The same reasoning establishes $I_1(t-h) - I_1(h) \to 0$ as $h \to 0^+$. Consequently, $Aw \in C((0,T];X)$, which implies that $v' \in C((0,T];X)$, provided that f is continuous on $(0,T]$. Then, by (4.63) we have ${}_0^C D_t^\alpha w \in C((0,T];X)$. Hence, we prove that $u+w$ is a classical solution to problem (4.47), and Lemma 4.68 implies that it is unique. This completes the proof. □

4.5.5 *Nonlinear Problems*

In this subsection we apply the theory developed in the previous sections to nonlinear abstract Cauchy problem (4.48).

Definition 4.73. By a mild solution to problem (4.48), we mean a function $u \in C((0,T];X)$ satisfying

$$u(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s,u(s))ds, \quad t \in (0,T].$$

Theorem 4.74. Let $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \frac{\pi}{2}$. Suppose that the nonlinear mapping $f : (0, T] \times X \to X$ is continuous with respect to t and there exist constants $M, N > 0$ such that

$$|f(t,x) - f(t,y)| \le M(1 + |x|^{\nu-1} + |y|^{\nu-1})|x - y|,$$

$$|f(t,x)| \le N(1 + |x|^\nu),$$

for all $t \in (0, T]$ and for each $x, y \in X$, where ν is a constant in $[1, -\frac{\gamma}{1+\gamma})$. Then, for every $u_0 \in X$, there exists a $T_0 > 0$ such that the problem (4.48) has a unique mild solution defined on $(0, T_0]$.

Proof. For fixed $r > 0$, we introduce the metric space

$$F_r(T, u_0) = \{u \in C((0, T]; X); \rho_T(u, \mathcal{S}_\alpha(t)u_0) \le r\},$$

$$\rho_T(u_1, u_2) = \sup_{t \in (0,T]} |u_1(t) - u_2(t)|.$$

It is not difficult to see that, with this metric, $F_r(T, u_0)$ is a complete metric space. Take $L = T^{\alpha(1+\gamma)}r + C_s\|u_0\|$, then for any $u \in F_r(T, u_0)$, we have

$$|s^{\alpha(1+\gamma)}u(s)| \le s^{\alpha(1+\gamma)}|u - \mathcal{S}_\alpha(t)u_0| + s^{\alpha(1+\gamma)}|\mathcal{S}_\alpha(t)u_0| \le L.$$

Choose $0 < T_0 \le T$ such that

$$C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_pNL^\nu T_0^{-\alpha(\nu(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1 - \nu\alpha(1+\gamma)) \le r, \tag{4.66}$$

$$MC_p\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + 2L^{\rho-1}T_0^{-\alpha(\gamma+(1+\gamma)(\nu-1))}\beta(-\alpha\gamma, 1 - \alpha(1+\gamma)(\nu-1)) \le \frac{1}{2}, \tag{4.67}$$

where $\beta(\eta_1, \eta_2)$ with $\eta_i > 0, i = 1, 2$, denotes the Beta function. Assume that $u_0 \in X$. Consider the mapping Γ^α given by

$$(\Gamma^\alpha u)(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\mathcal{P}_\alpha(t-s)f(s, u(s))ds, \quad u \in F_r(T_0, u_0).$$

By the assumptions on f, Theorem 4.61 and Theorem 4.62, we see that $(\Gamma^\alpha u)(t) \in C((0, T]; X)$ and

$$\begin{aligned}
|(\Gamma^\alpha u)(t) - \mathcal{S}_\alpha(t)u_0| &\le C_pN\int_0^t (t-s)^{-\alpha\gamma-1}(1 + \|u(s)\|^\nu)ds \\
&\le C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + \int_0^t C_pNL^\nu(t-s)^{-\alpha\gamma-1}s^{-\nu\alpha(1+\gamma)}ds \\
&\le C_pN\frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} + C_pNL^\nu T_0^{-\alpha(\nu(1+\gamma)+\gamma)}\beta(-\gamma\alpha, 1 - \nu\alpha(1+\gamma)) \\
&\le r,
\end{aligned}$$

in view of (4.66). So, Γ^α maps $F_r(T_0, u_0)$ into itself. Next, for any $u, v \in F_r(T_0, u_0)$, by the assumptions on f and Theorem 4.61, we have

$$\begin{aligned}
&|(\Gamma^\alpha u)(t) - (\Gamma^\alpha v)(t)| \\
&= \Big| \int_0^t (t-s)^{\alpha-1} \mathcal{P}_\alpha(t-s)\big(f(s,u(s)) - f(s,v(s))\big) ds \Big| \\
&\le C_p M \int_0^t (t-s)^{-\alpha\gamma-1}(1 + |u(s)|^{\rho-1} + |v(s)|^{\rho-1})|u(s) - v(s)| ds \\
&\le C_p M \rho_t(u,v) \int_0^t (t-s)^{-\alpha\gamma-1}(1 + 2L^{\nu-1} s^{-\alpha(\nu-1)(1+\gamma)}) ds \\
&\le 2L^{\rho-1} T_0^{-\alpha(\gamma+(1+\gamma)(\nu-1))} \beta(-\alpha\gamma, 1-\alpha(1+\gamma)(\nu-1)) \rho_{T_0}(u,v) \\
&\quad + MC_p \frac{T_0^{-\alpha\gamma}}{-\alpha\gamma} \rho_{T_0}(u,v).
\end{aligned}$$

This yields that Γ^α is a contraction on $F_r(T_0, u_0)$ due to (4.67). So, Γ_α has a unique fixed point $u \in F_r(T_0, u_0)$ in view of the Banach Fixed Point Theorem, this means that u is a mild solution to problem (4.48) defined on $(0, T_0]$. The proof is completed. □

By a similar argument with the proof of Theorem 4.74 we have:

Corollary 4.75. Assume that $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < -\frac{2}{3}$ and $0 < \omega < \frac{\pi}{2}$. Suppose in addition that the nonlinear mapping $f : (0,T] \times X^\beta \to X$, $\beta \in (1+\gamma, -1-2\gamma)$, is continuous with respect to t and there exist constants $M, N > 0$ such that

$$|f(t,x) - f(t,y)| \le M(1 + |x|_\beta^{\nu-1} + |y|_\beta^{\nu-1})|x-y|_\beta,$$
$$|f(t,x)| \le N(1 + |x|_\beta^\nu),$$

for all $t \in (0,T]$ and for each $x, y \in X^\beta$, where ν is a constant in $[1, -\frac{\gamma+\beta}{1+\gamma})$. Then, for every $u_0 \in X^\beta$, there exists a $T_0 > 0$ such that the problem (4.48) has a unique mild solution $u \in C((0,T_0]; X^\beta)$.

Remark 4.76. If $A \in \Theta_\omega^\gamma(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$, then we can derive the local existence and uniqueness of mild solutions to problem (4.48), under the conditions:

(i) $u_0 \in X^\beta$ with $\beta > 1 + \gamma$;
(ii) the nonlinear mapping $f : [0,T] \times X \to X$ is continuous with respect to t and there exists a continuous function $L_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t,x) - f(t,y)| \le L_f(r)|x-y|,$$

for all $0 \le t \le T$ and for each $x, y \in X$ satisfying $|x|, |y| \le r$.

Indeed, for $r > \frac{C_p T_0^{-\alpha\gamma}}{-\alpha\gamma} \sup_{t\in[0,T]} |f(t,u_0)|$ fixed, we may choose $0 < T_0 \le T$ such that

$$\sup_{t\in[0,T_0]} \|(\mathcal{S}_\alpha(t) - I)u_0\| + \frac{C_p T_0^{-\alpha\gamma}}{-\alpha\gamma}\Big(L_f(r)r + \sup_{t\in[0,T_0]} |f(t,u_0)|\Big) < r \qquad (4.68)$$

in view of Theorem 4.65 (i). Assume that the map Γ^α is defined the same as in Theorem 4.74 and the space $F_r(T_0, u_0)$ is replaced by the following Banach space:

$$F'_r(T_0, u_0) = \{u \in C([0, T_0]; X); u(0) = u_0 \text{ and } \sup_{t\in[0,T_0]} |u - u_0| \le r\}.$$

Then, it is easy to verify, thanks to the assumptions on f and (4.68), that Γ^α maps $F'_r(T_0, u_0)$ into itself and is a contraction on $F'_r(T_0, u_0)$, which implies that the problem (4.48) has a unique mild solution defined on $[0, T_0]$.

Since $1 > 1 + \gamma$ $(-1 < \gamma < -\frac{1}{2})$, $X^1 = D(A)$ is a Banach space endowed with the graph norm $|x|_{X^1} = |Ax|$, for $x \in X^1$.

The following is the existence of X^1-smooth solutions.

Theorem 4.77. Let $A \in \Theta^\gamma_\omega(X)$ with $-1 < \gamma < -\frac{1}{2}$ and $0 < \omega < \frac{\pi}{2}$ and $u_0 \in X^1$. Assume that there exists a continuous function $M_f(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $N_f > 0$ such that the nonlinear mapping $f : (0, T] \times X^1 \to X^1$ satisfies

$$|f(t, x) - f(t, y)|_{X^1} \le M_f(r)|x - y|_{X^1},$$

$$|f(t, \mathcal{S}_\alpha(t)u_0)|_{X^1} \le N_f(1 + t^{-\alpha(1+\gamma)}|u_0|_{X^1}),$$

for all $0 < t \le T$ and for each $x, y \in X^1$ satisfying $\sup_{t\in(0,T]} |x(t) - \mathcal{S}_\alpha(t)u_0|_{X^1} \le r$, $\sup_{t\in(0,T]} |y(t) - \mathcal{S}_\alpha(t)u_0|_{X^1} \le r$. Then there exists a $T_0 > 0$ such that the problem (4.48) has a unique mild solution defined on $(0, T_0]$.

Proof. For $u_0 \in X^1$ and $r > 0$, set

$$F''_r(T, u_0) = \{u \in C((0, T]; X^1); \sup_{t\in(0,T]} |u - \mathcal{S}_\alpha(t)u_0|_{X^1} \le r\}.$$

For any $u \in F''_r(T, u_0)$, by the assumptions on f and Theorem 4.61 we have

$$\begin{aligned}
&|(\Gamma^\alpha u)(t) - \mathcal{S}_\alpha(t)u_0|_{X^1} \\
&\le \int_0^t (t - s)^{\alpha-1}\|\mathcal{P}_\alpha(t - s)\||f(s, u(s)) - f(s, \mathcal{S}_\alpha(t)u_0)|_{X^1} ds \\
&\quad + \int_0^t (t - s)^{\alpha-1}\|\mathcal{P}_\alpha(t - s)\||f(s, \mathcal{S}_\alpha(t)u_0))|_{X^1} ds \\
&\le C_p \int_0^t (t - s)^{-\alpha\gamma-1}(M_f(r)r + N_f + N_f s^{-\alpha(1+\gamma)}|u_0|)ds \\
&\le C_p(M_f(r)r + N_f)\frac{T^{-\alpha\gamma}}{-\alpha\gamma} + C_p N_f T^{-\alpha(1+2\gamma)}\beta(-\gamma\alpha, 1 - \alpha(1 + \gamma))|u_0|.
\end{aligned}$$

Using this result, it follows from an analogous idea with Theorem 4.74 that the claim of theorem follows. Here we omit the details. $\square$

Next, we will derive mild solutions under the condition of compactness on the resolvent of A.

Theorem 4.78. Let $A \in \Theta^\gamma_\omega(X)$ with $-1 < \gamma < 0$ and $0 < \omega < \frac{\pi}{2}$. Let

(H1) $R(\lambda, -A)$ is compact for every $\lambda > 0$;

(H2) $f : [0,T] \times X \to X$ is a Carathéodory function and for any $r > 0$, there exists a function $m_r(t) \in L^p((0,T);\mathbb{R}^+)$ with $p > -\frac{1}{\alpha\gamma}$ such that

$$|f(t,x)| \leq m_r(t), \quad \text{and } \liminf_{r\to+\infty} \frac{|m_r(t)|_{L^p(0,T)}}{r} = \sigma < \infty$$

for a.e. $t \in [0,T]$ and all $x \in X$ satisfying $|x| \leq r$.

Then for every $u_0 \in D(A^\beta)$ with $\beta > 1+\gamma$, the problem (4.48) has at least a mild solution, provided that

$$C_p\sigma\Big(\frac{T^{1-(1+\alpha\gamma)\,q}}{1-(1+\alpha\gamma)q}\Big)^{1/q} < 1, \tag{4.69}$$

where $q = p/(p-1)$.

Proof. Assume that $u_0 \in D(A^\beta)$. On $C([0,T];X)$ define the map

$$(\Gamma^\alpha u)(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}\,\mathcal{P}_\alpha(t-s)f(s,u(s))ds.$$

From our assumptions it is easy to see that Γ_μ is well defined and maps $C([0,T];X)$ into itself. Put

$$\Omega_r = \{u \in C([0,T];X); \|u\| \leq r, \quad \text{for all } 0 \leq t \leq T\},$$

for $r > 0$ as selected below. We seek for solutions in Ω_r. We claim that there exists an integer $r > 0$ such that Γ^α maps Ω_r into Ω_r. In fact, if this is not the case, then for each $r > 0$, there would exist $u^r \in \Omega_r$ and $t^r \in [0,T]$ such that $\|(\Gamma^\alpha u^r)(t^r)\| > r$. On the other hand, by (H2) and Theorem 4.61 we get

$$\begin{aligned}
r &< |(\Gamma^\alpha u^r)(t^r)| \\
&\leq |\mathcal{S}_\alpha(t^r)u_0| + \int_0^{t^r} |(t^r-s)^{\alpha-1}\mathcal{P}_\alpha(t^r-s)f(s,u(s))|ds \\
&\leq \sup_{t\in[0,T]} |\mathcal{S}_\alpha(t)u_0| + \int_0^{t^r} C_p(t^r-s)^{-1-\alpha\gamma}m_r(s)ds \\
&\leq \sup_{t\in[0,T]} |\mathcal{S}_\alpha(t)u_0| + C_p\Big(\int_0^{t^r} s^{-(1+\alpha\gamma)q}ds\Big)^{\frac{1}{q}}\Big(\int_0^{t^r} m_r^p(s)ds\Big)^{\frac{1}{p}} \\
&\leq \sup_{t\in[0,T]} |\mathcal{S}_\alpha(t)u_0| + C_p\|m_r\|_{L^p(0,T)}\Big(\frac{T^{1-(1+\alpha\gamma)\,q}}{1-(1+\alpha\gamma)q}\Big)^{\frac{1}{q}},
\end{aligned}$$

where $q = p/(p-1)$. Dividing on both sides by r and taking the lower limit as $r \to \infty$, one has

$$1 \leq C_p\sigma\Big(\frac{T^{1-(1+\alpha\gamma)\,q}}{1-(1+\alpha\gamma)q}\Big)^{1/q},$$

which contradicts (4.69). Hence for some positive integer r, $\Gamma^\alpha(\Omega_r) \subset \Omega_r$.

The rest of the proof is divided into three steps.

Step I. Γ^α is continuous on Ω_r.

Take $\{u_n\}_{n=1}^{\infty} \subset \Omega_r$ with $u_n \to u$ in $C([0,T];X)$. Then by the continuity of f with respect to the second argument we deduce that

$$f(s, u_n(s)) \to f(s, u(s)) \text{ a.e. } s \in [0,T].$$

Moreover, observe from (H2) and Theorem 4.61, that for fixed $0 < t \leq T$,

$$(t-s)^{\alpha-1}|\mathcal{P}_\alpha(t-s)f(s,u_n(s))| \leq C_p(t-s)^{-1-\alpha\gamma}m_r(s).$$

Thus, by means of the Lebesgue's dominated convergence theorem we obtain that

$$\int_0^t (t-s)^{\alpha-1}\|\mathcal{P}_\alpha(t-s)\||f(s,u_n(s)) - f(s,u(s))|ds \to 0,$$

which means that $\lim_{n\to\infty}\|\Gamma^\alpha u_n - \Gamma^\alpha u\|_\infty = 0$, that is, Γ^α is continuous on Ω_r.

Step II. $P = \{(\Gamma^\alpha u)(\cdot); \cdot \in [0,T], u \in \Omega_r\}$ is equicontinuous.

For $0 < t_1 < t_2 \leq T$ and $\delta > 0$ small enough, we have

$$|(\Gamma^\alpha u)(t_1) - (\Gamma^\alpha u)(t_2)| \leq I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned}
I_1 &= |\mathcal{S}_\alpha(t_1)u_0 - \mathcal{S}_\alpha(t_2)u_0|,\\
I_2 &= \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|\mathcal{P}_\alpha(t_2-s)f(s,u(s))\|ds,\\
I_3 &= \int_0^{t_1-\delta}(t_1-s)^{\alpha-1}\|\mathcal{P}_\alpha(t_2-s) - \mathcal{P}_\alpha(t_1-s)\||f(s,u(s))|ds,\\
I_4 &= \int_{t_1-\delta}^{t_1}(t_1-s)^{\alpha-1}\|\mathcal{P}_\alpha(t_2-s) - \mathcal{P}_\alpha(t_1-s)\||f(s,u(s))|ds,\\
I_5 &= \int_0^{t_1}|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|\|\mathcal{P}_\alpha(t_2-s)\||f(s,u(s))|ds.
\end{aligned}$$

From Theorem 4.62 and Theorem 4.65 (i) it is easy to see that $I_1 \to 0$ when $t_1 \to t_2$. Moreover, using (H2) and Theorem 4.61 we get

$$\begin{aligned}
I_2 &\leq C_p\left(\frac{(t_2-t_1)^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\right)^{1/q}\|m_r\|_{L^p(0,T)},\\
I_3 &\leq \sup_{s\in[0,t_1-\delta]}\|\mathcal{P}_\alpha(t_2-s) - \mathcal{P}_\alpha(t_1-s)\|\Big(\int_0^{t_1-\delta}(t_1-s)^{q\alpha-q}qds\Big)^{1/q}\|m_r\|_{L^p(0,T)}\\
&\leq \sup_{s\in[0,t_1-\delta]}\|\mathcal{P}_\alpha(t_2-s) - \mathcal{P}_\alpha(t_1-s)\|\Big(\frac{t_1^{1+q(\alpha-1)} - \delta^{1+q(\alpha-1)}}{1+q(\alpha-1)}\Big)\|m_r\|_{L^p(0,T)},
\end{aligned}$$

$$
\begin{aligned}
I_4 &\leq C_p \int_{t_1-\delta}^{t_1} (t_1-s)^{\alpha-1} \cdot 2(t_1-s)^{-\alpha(\gamma+1)} m_r(s)ds \\
&\leq 2C_p \frac{\delta^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q} \|m_r\|_{L^p(0,T)}, \\
I_5 &\leq \int_0^{t_1} C_p((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})(t_2-s)^{-\alpha(1+\gamma)} m_r(s)ds \\
&\leq \int_0^{t_1} C_p((t_1-s)^{-\gamma\alpha-1} - (t_2-s)^{-\alpha\gamma-1}) m_r(s)ds \\
&\leq C_p\Big(\int_0^{t_1} ((t_1-s)^{-q(\gamma\alpha+1)} - (t_2-s)^{-q(\alpha\gamma+1)})ds\Big)^{1/q} \|m_r\|_{L^p(0,T)} \\
&= C_p\Big(\frac{(t_2-t_1)^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q} + \frac{t_1^{1-(1+\alpha\gamma)q} - t_2^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\Big)^{1/q} \|m_r\|_{L^p(0,T)}.
\end{aligned}
$$

It follows from Theorem 4.62 that I_i $(i = 2, 3, 4, 5)$ tends to zero independent of $u \in \Omega_r$ as $t_2 - t_1 \to 0$, $\delta \to 0$. Hence, we can conclude that

$$
\|(\Gamma^\alpha u)(t_1) - (\Gamma^\alpha u)(t_2)\| \to 0, \quad \text{as} \quad t_2 - t_1 \to 0,
$$

and the limit is independent of $u \in \Omega_r$. For the case when $0 = t_1 < t_2 \leq T$, since

$$
\int_0^{t_2} (t_2-s)^{\alpha-1} \|P(t_2-s)f(s,u(s))\|ds \leq C_p\left(\frac{t_2^{1-q(\alpha\gamma+1)}}{1-q(\alpha\gamma+1)}\right)^{1/q} \|m_r\|_{L^p(0,T)},
$$

in view of (H2) and Theorem 4.61, $\|(\Gamma^\alpha u)(t_2)\|$ can be made small when t_2 is small independently of $u \in \Omega_r$. Thus, we prove that the assertion in Step II holds.

Step III. For each $t \in [0, T]$, $\{(\Gamma^\alpha u)(t); u \in \Omega_r\}$ is precompact in X.

For the case when $t = 0$, it is not difficult to see that $\{(\Gamma^\alpha u)(0); u \in \Omega_r\} = \{u_0 : u \in \Omega_r\}$ is compact. Let $t \in (0, T]$ be fixed and $\epsilon, \delta > 0$. For $u \in \Omega_r$, define the map $\Gamma^\alpha_{\epsilon,\delta}$ by

$$
(\Gamma^\alpha_{\epsilon,\delta} u)(t) = \mathcal{S}_\alpha(t)u_0 + \int_0^{t-\epsilon}\int_\delta^\infty \alpha\tau(t-s)^{\alpha-1}\Psi_\alpha(\tau)T((t-s)^\alpha\tau)f(s,u(s))d\tau ds.
$$

Since A has compact resolvent, $\{T(t)\}_{t>0}$ is compact in view of Theorem 4.67. Thus, for each $t \in (0, T]$, $\{(\mathcal{F}_{\epsilon,\delta}u)(t); u \in \Omega_r,\ \delta > 0,\ 0 < \epsilon < t\}$ is precompact in X. On the other hand, using (H2) and Theorem 4.61, a direct calculation yields

$$
\begin{aligned}
&|(\Gamma^\alpha u)(t) - (\Gamma^\alpha_{\epsilon,\delta} u)(t)| \\
&\leq \left|\int_0^t \int_0^\delta \alpha\tau(t-s)^{\alpha-1}\Psi_\alpha(\tau)T((t-s)^\alpha\tau)f(s,u(s))d\tau ds\right| \\
&\quad + \left|\int_{t-\epsilon}^t \int_\delta^\infty \alpha\tau(t-s)^{\alpha-1}\Psi_\alpha(\tau)T((t-s)^\alpha\tau)f(s,u(s))d\tau ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^t C_p(t-s)^{-1-\alpha\gamma} m_r(s)ds \int_0^\delta \tau^{-\gamma}\Psi_\alpha(\tau)d\tau \\
&\quad + \int_{t-\epsilon}^t C_p(t-s)^{-1-\alpha\gamma} m_r(s)ds \int_\delta^\infty \tau^{-\gamma}\Psi_\alpha(\tau)d\tau \\
&\le C_p\Big(\frac{T^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\Big)^{1/q} \|m_r\|_{L^p(0,T)} \int_0^\delta \tau^{-\gamma}\Psi_\alpha(\tau)d\tau \\
&\quad + C_p\Big(\frac{\epsilon^{1-(1+\alpha\gamma)q}}{1-(1+\alpha\gamma)q}\Big)^{1/q} \|m_r\|_{L^p(0,T)} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma\alpha)}.
\end{aligned}
$$

Using the total boundedness we have that for each $t \in (0,T]$ $\{(\Gamma^\alpha u)(t); u \in \Omega_r\}$ is precompact in X. Therefore, for each $t \in [0,T]$, $\{(\Gamma^\alpha u)(t); u \in \Omega_r\}$ is precompact in X.

Finally, by Steps I-III and the Arzela-Ascoli theorem, we conclude that Γ^α is a compact operator. Hence, from Schauder's second fixed point theorem it follows that Γ^α has a fixed point, which gives rise to a mild solution. This completes the proof. □

Theorem 4.79. Let $A \in \Theta_\omega^\gamma(X)$ with $0 < \omega < \frac{\pi}{2}$ and $-1 < \gamma < -\frac{1}{2}$. Suppose that there exists a continuous function $M_f'(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $\kappa > \alpha(1+\gamma)$ such that the nonlinear mapping $f : [0,T] \times X \to X$ satisfies

$$|f(t,x) - f(s,y)| \le M_f'(r)(|t-s|^\kappa + |x-y|),$$

for all $0 \le t \le T$ and $x, y \in X$ satisfying $|x|, |y| \le r$. In addition, let the assumptions of Theorem 4.77 be satisfies and u be a mild solution corresponding to u_0, defined on $[0, T_0]$. Then u is in fact the unique classical solution to problem (4.48), existing on $[0, T_0]$, provided that $u_0 \in D(A)$ with $Au_0 \in D(A^\beta)$, $\beta > (1+\gamma)$.

Proof. In order to prove that u is a classical solution, by Theorem 4.72 and the condition on f, we only have to verify that u is Hölder continuous with an exponent $\varsigma > \alpha(1+\gamma)$ on $(0, T_0]$. For fixed $t \in (0, T_0]$, take $0 < h < 1$ such that $h + t \le T_0$. We estimate the difference

$$
\begin{aligned}
|u(t+h) - u(t)| &\le |\mathcal{S}_\alpha(t+h)u_0 - \mathcal{S}_\alpha(t)u_0| \\
&\quad + \left| \int_0^h (t+h-s)^{\alpha-1}\mathcal{P}(t+h-s)f(s,u(s))ds \right| \\
&\quad + \left| \int_0^t (t-s)^{\alpha-1}\mathcal{P}(t-s)[f(s+h,u(s+h)) - f(s,u(s))]ds \right| \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

According to Theorem 4.61, Theorem 4.63 (ii) and the assumptions on f we obtain

$$I_1 = \left| \int_0^t -s^{\alpha-1} A\mathcal{P}_\alpha(s)u_0 ds \right| \le \frac{C_p}{-\alpha\gamma}\big((t+h)^{-\alpha\gamma} - t^{-\alpha\gamma}\big),$$

$$
\begin{aligned}
I_3 &\le M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}(|h|^\kappa + |u(s+h)-u(s)|)ds \\
&\le \frac{M'C_p}{-\alpha\gamma} T_0^{-\alpha\gamma} h^\kappa + M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}|u(s+h)-u(s)|ds.
\end{aligned}
$$

Put $N_2 = \sup_{t\in(0,T_0)} |f(t,u(t))|$. Then, it follows from Theorem 4.61 that

$$
\begin{aligned}
I_2 &\le C_p \int_0^h (t+h-s)^{-\alpha\gamma-1}|f(s,u(s))|ds \\
&\le \frac{C_p N_2}{-\alpha\gamma}\big((t+h)^{-\alpha\gamma} - t^{-\alpha\gamma}\big).
\end{aligned}
$$

Collecting these estimates and using the inequality $(t+h)^{-\alpha\gamma} - t^{-\alpha\gamma} \le h^{-\alpha\gamma}$ $(0 < -\alpha\gamma < 1)$ we have

$$
\begin{aligned}
&|u(t+h)-u(t)| \\
&\le \frac{C_p N_2 + C_p}{-\alpha\gamma}\big((t+h)^{-\alpha\gamma} - t^{-\alpha\gamma}\big) + \frac{M'_p}{-\alpha\gamma} T_0^{-\alpha\gamma} h^\kappa \\
&\quad + M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}|u(s+h)-u(s)|ds \\
&\le \frac{C_p N_2 + C_p + M'C_p}{-\alpha\gamma} h^\varsigma + M'C_p \int_0^t (t-s)^{-\alpha\gamma-1}|u(s+h)-u(s)|ds,
\end{aligned}
$$

where $\varsigma = \min\{\kappa, -\alpha\gamma\} > \alpha(\gamma+1)$. Now, it follows from the usual Gronwall's inequality that u has Hölder continuity on $(0, T_0]$. This completes the proof of theorem. □

4.5.6 *Applications*

In this subsection, we present three examples (Examples 4.80-4.82) motivated from physics, which do not aim at generality but indicate how our theorems can be applied to concrete problems. Examples 4.80 and 4.81 are inspired directly from the work of Carvalho, Dlotko and Nescimento, 2008, and they describe anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see Anh and Leonenko, 2001; Metzler and Klafter, 2000 and references therein). Example 4.80 is the limit problem of certain fractional diffusion equations in complex systems on domains of "dumb-bell with a thin handle"(see, e.g., Anh and Leonenko, 2001; Metzler and Klafter, 2000). Example 4.81 displays anomalous dynamical behavior of anomalous transport processes (see, e.g., Anh and Leonenko, 2001; Metzler and Klafter, 2000). Example 4.82 is a modified fractional Schrödinger equation with fractional Laplacians whose physical background is statistical physics and fractional quantum mechanics (see, e.g., Hu and Kallianpur, 2000; Podlubny, 1999). We refer the reader to Kirane, Laskri and Tatar, 2005 and references therein for more research results related to fractional Laplacians.

Example 4.80. Consider the system of fractional partial differential equations in the form

$$\begin{cases} {}_0^C D_t^\alpha w - \Delta w + w = f(w), \quad x \in \Omega,\ t > 0, \\ \dfrac{\partial w}{\partial n} = 0, \quad x \in \partial\Omega, \\ {}_0^C D_t^\alpha v - \dfrac{1}{g}(g v_x)_x + v = f(v), \quad x \in (0,1), \\ v(0) = w(P_0)\,, v(1) = w(P_1), \\ w(x,0) = w_0(x) \quad x \in \Omega, \quad v(x,0) = v_0(x), \quad x \in (0,1), \end{cases} \tag{4.70}$$

where $\Omega = D_1 \cup D_2$ and D_1 and D_2 are mutually disjoint bounded domains in $\mathbb{R}^N (N \geq 2)$ with smooth boundaries, joined by the line segment Q_0, and ${}_0^C D_t^\alpha$, $0 < \alpha < 1$, is the regularized Caputo fractional derivative of order α, that is,

$$({}_0^C D_t^\alpha u)(t,x) = \frac{1}{\Gamma(1-\alpha)} \Big(\frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(s,x) ds - t^{-\alpha} u(0,x) \Big). \tag{4.71}$$

When $\alpha = 1$, we regard (4.70) as the limit problem of (4.44) as $\varepsilon \to 0$, which is described in more detail in Example 4.49. Here, our objective is to show that system (4.70) is well posed in $V_0^p = L^p(\Omega) \oplus L_g^p(0,1)$ $(1 \leq p < \infty)$.

Let the operators $A_0 : D(A_0) \subset V_0^p \mapsto V_0^p$ be defined by

$$D(A_0) = \{(w,v) \in V_0^P; w \in D(\Delta_\Omega), v \in L_g^p(0,1), w(P_0) = v(0), w(P_1) = v(1)\},$$

$$A_0(w,v) = (-\Delta w + w, -\frac{1}{g}(g v')' + v), \quad (w,v) \in V_0^p,$$

where Δ_Ω is the Laplace operator with homogeneous Neumann boundary conditions in $L^p(\Omega)$ and

$$D(\Delta_\Omega) = \Big\{ u \in W^{2,p}(\Omega); \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0 \Big\}.$$

From Example 4.49, if $p > \frac{N}{2}$, then $A_0 \in \Theta_\mu^{-\gamma'}(V_0^p)$ for some $\gamma' \in (0, 1 - \frac{N}{2p})$ and $\mu \in (0, \frac{\pi}{2})$. Therefore, system (4.70) can be seen as an abstract evolution equation in the form

$$\begin{cases} {}_0^C D_t^\alpha u + A_0 u = f(u), \quad t > 0, \\ u(0) = u_0 = (w_0, v_0) \in V_0^p. \end{cases} \tag{4.72}$$

We assume that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz continuous. It can define a Nemitskiĭ operator from V_0^p into itself by $f(w,v) = (f_\Omega(w), f_I(v))$ with $f_\Omega(w)(x) = f(w(x))$, $x \in \Omega$ and $f_I(v)(x) = f(v(x))$, $x \in (0,1)$ such that

$$|f(u) - f(u')|_{V_0^p} \leq L''(r)|u - u'|_{V_0^p}$$

for all $u, u' \in V_0^p$ satisfying $|u|_{V_0^p}, |u'|_{V_0^p} \leq r$. Hence, from Remark 4.76, (4.72) (that is, (4.70)) has a unique mild solution provided that $u_0 \in D(A_0^\beta)$ with $\beta > 1 - \gamma'$ (in particular, $u_0 \in D(A_0)$).

Example 4.81. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \geq 1$) with boundary $\partial\Omega$ of class C^4. Consider the fractional initial-boundary value problem of form

$$\begin{cases} ({}_0^C D_t^\alpha u)(t,x) - \Delta u(t,x) = f(u(t,x)), \quad t > 0,\ x \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{4.73}$$

in the space $C^l(\overline{\Omega})$ ($0 < l < 1$), where Δ stands for the Laplacian operator with respect to the spatial variable and ${}_0^C D_t^\alpha$, representing the regularized Caputo fractional derivative of order α ($0 < \alpha < 1$), is given by (4.71). Set

$$\widetilde{A} = -\Delta, \quad D(\widetilde{A}) = \{u \in C^{2+l}(\overline{\Omega}); u = 0 \ \text{ on } \partial\Omega\}.$$

It follows from Example 4.51 that there exist $\nu,\ \varepsilon > 0$ such that $\widetilde{A} + \nu \in \Theta^{\frac{l}{2}-1}_{\frac{\pi}{2}-\varepsilon}(C^l(\overline{\Omega}))$. Then, problem (4.73) can be written abstractly as

$${}_0^C D_t^\alpha u(t) + \widetilde{A}u(t) = f(u), \quad t > 0.$$

With respect to the nonlinearity f, we assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies the condition

$$|f(x) - f(y)| \leq \frac{k(r)}{r}|x - y|, \quad |x|, |y| \leq r \tag{4.74}$$

for any $r > 0$. It defines a Nemitskiĭ operator from $C^l(\overline{\Omega})$ into itself by $f(u)(x) = f(u(x))$ with

$$|f(u) - f(v)|_{C^l(\overline{\Omega})} \leq k(r)|u - v|_{C^l(\overline{\Omega})}, \quad |v|_{C^l(\overline{\Omega})}, |u|_{C^l(\overline{\Omega})} \leq r.$$

Noting $\frac{l}{2} - 1 \in (-1, -\frac{1}{2})$, we then obtain the following conclusion: (i) according to Remark 4.76, (4.73) has a unique mild solution for each $u_0 \in D(\widetilde{A}^\beta)$ with $\beta > \frac{l}{2}$. Moreover, (ii) if f', f'' are continuously differentiable functions satisfying the condition (4.74), then one finds that the Nemitskiĭ operator satisfies the assumptions of Theorem 4.77 and Theorem 4.79, which implies that for each $u_0 \in D(\widetilde{A})$ with $\widetilde{A}u_0 \in D(\widetilde{A}^\beta)$ ($\beta > \frac{l}{2}$), the corresponding mild solution to (4.73) is also a unique classical solution.

Example 4.82. Consider the following fractional Cauchy problem

$$\begin{cases} ({}_0^C D_t^\alpha y)(t,x) + (-i\Delta + \sigma)^{1/2} u(t,x) = f(u(t,x)), \quad t > 0,\ x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^2, \end{cases} \tag{4.75}$$

in $L^3(\mathbb{R}^2)$, where $\sigma > 0$ is a suitable constant, $i\Delta$ is the Schrödinger operator and ${}_0^C D_t^\alpha (0 < \alpha < 1)$ is given by (4.71). Let

$$\widehat{A} = (-i\Delta + \sigma)^{1/2}, \quad D(\widehat{A}) = W^{1,3}(\mathbb{R}^2) \quad \text{(a Sobolev space)}.$$

Then $i\Delta$ generates a β-times integrated semigroup $S^\beta(t)$ with $\beta = \frac{5}{12}$ on $L^3(\mathbb{R}^2)$ such that $\|S^\beta(t)\|_{\mathscr{L}(L^3(\mathbb{R}^2))} \leq \widehat{M}t^\beta$ for all $t \geq 0$ and some constants $\widehat{M} > 0$ (see Neerven and Straub, 1998). Therefore, by virtue of Theorem 1.3.5 and Definition 1.3.1 for $C = I$ of Xiao and Liang, 1998, we deduce that the operator $-i\Delta + \sigma$

belongs to $\Theta_{\frac{\pi}{2}}^{\beta-1}(L^3(\mathbb{R}^2))$, which denotes the family of all linear closed operators $A : D(A) \subset L^3(\mathbb{R}^2) \to L^3(\mathbb{R}^2)$ satisfying $\sigma(A) \subset S_{\frac{\pi}{2}} = \{z \in \mathbb{C} \setminus \{0\};\ |\arg z| \leq \frac{\pi}{2}\} \cup \{0\}$, and for every $\frac{\pi}{2} < \mu < \tau$ there exists a constant C_μ such that $\|R(z;A)\| \leq C_\mu |z|^{\beta-1}$. Thus, it follows from Proposition 3.6 of Periago and Straub, 2002 that $\widehat{A} \in \Theta_\omega^{-1+2\beta}(L^3(\mathbb{R}^2))$ for some $0 < \omega < \frac{2}{\pi}$. Moreover, the system (4.75) can be rewritten as follows:

$$\begin{cases} ({}_0^C D_t^\alpha y)(t,x) + \widehat{A}u = f(u), \quad t > 0, \\ u(0,x) = u_0 \in L^3(\mathbb{R}^2). \end{cases}$$

Assume that $f : \mathbb{C} \to \mathbb{C}$ is globally Lipschitz continuous. Then we have a Nemitskiĭ operator from $L^3(\mathbb{R}^2)$ to itself given by $f(u)(x) = f(u(x))$, and for a constant $\widehat{L}(r)$ and all $u, v \in L^3(\mathbb{R}^2)$ such that $|u|_{L^3(\mathbb{R}^2)} \leq r$ and $|v|_{L^3(\mathbb{R}^2)} \leq r$. Consequently, it follows from Remark 4.76 that (4.75) has a unique mild solution provided $u_0 \in D(\widehat{A})^\tau$ with $\tau > \frac{5}{6}$.

4.6 Notes and Remarks

The results in Section 4.2 are taken from Zhou, Zhang and Shen, 2013. The material in Section 4.3 due to Zhou, Shen and Zhang, 2013. The main results in Section 4.4 is taken from Wang, Zhou and Fečkan, 2014. The contents of Section 4.5 are adopted from Wang, Chen and Xiao, 2012.

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Chapter 5

Fractional Boundary Value Problems via Critical Point Theory

5.1 Introduction

The main purpose of this chapter is to present a new approach via critical point theory to study the existence of solutions for the boundary value problem of fractional differential equations. This approach is, to the best of our knowledge, novel and it may open a new approach to deal with some types of nonlinear fractional differential equations with certain boundary conditions.

5.2 Existence of Solution for BVP with Left and Right Fractional Integrals

5.2.1 *Introduction*

In Section 5.2, we consider the fractional boundary value problem (BVP for short) of the following form

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u(t)) = 0, \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{5.1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$ respectively, $F\colon\ [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function satisfying some assumptions and $\nabla F(t,x)$ is the gradient of F at x. In particular, if $\beta = 0$, BVP (5.1) reduces to the standard second order BVP.

Physical models containing fractional differential operators have recently renewed attention from scientists which is mainly due to applications as models for physical phenomena exhibiting anomalous diffusion. A strong motivation for investigating the fractional BVP (5.1) comes from fractional advection-dispersion equation (ADE for short). A fractional ADE is a generalization of the classical ADE in which the second-order derivative is replaced with a fractional-order derivative. In contrast to the classical ADE, the fractional ADE has solutions that resemble the highly skewed and heavy-tailed breakthrough curves observed in field and laboratory studies (see, Benson, Schumer, Meerschaert *et al.*, 2001; Benson, Wheatcraft and Meerschaert, 2000a), in particular in contaminant transport of ground-water flow (see, Benson, Wheatcraft and Meerschaert, 2000b). Benson et al. stated that solutes moving through a highly heterogeneous aquifer violations violates the basic assumptions of local second-order theories because of large deviations from the stochastic process of Brownian motion.

Let $\phi(t,x)$ represents the concentration of a solute at a point x at time t in an arbitrary bounded connected set $\Omega \subset \mathbb{R}^N$. According to Benson, Wheatcraft and Meerschaert, 2000a; Fix and Roop, 2004, the N-dimensional form of the fractional ADE can be written as

$$\frac{\partial \phi}{\partial t} = -\nabla(v\phi) - \nabla(\nabla^{-\beta}(-k\nabla\phi)) + f, \quad \text{in } \Omega, \tag{5.2}$$

where v is a constant mean velocity, k is a constant dispersion coefficient, $v\phi$ and $-k\nabla\phi$ denote the mass flux from advection and dispersion respectively. The components of $\nabla^{-\beta}$ in (5.2) are linear combination of the left and right Riemann-Liouville fractional integral operators

$$(\nabla^{-\beta}(-k\nabla\phi))_i = (q\ {}_{-\infty}D_{x_i}^{-\beta} + (1-q)\ {}_{x_i}D_{+\infty}^{-\beta})\left(-k\frac{\partial \phi}{\partial x_i}\right), \quad i = 1, ..., N, \tag{5.3}$$

where $q \in [0,1]$ describes the skewness of the transport process, and $\beta \in [0,1)$ is the order of the Liouville-Weyl left and right fractional integral operators on the real line (see Definition 1.12). This equation may be interpreted as stating that the mass flux of a particle is related to the negative gradient via a combination of the left and right fractional integrals. Eq. (5.3) is physically interpreted as a Fick's law for concentrations of particles with a strong nonlocal interaction.

For discussions of Eq. (5.2), see Benson, Wheatcraft and Meerschaert, 2000b; Fix and Roop, 2004. When $\beta = 0$, the dispersion operators in (5.2) are identical and the classical ADE is recovered. In a more general version of (5.2), k is replaced by a symmetric positive definite matrix.

A special case of the fractional ADE (Eq. (5.2)) describes symmetric transitions. In this case, $\nabla^{-\beta}$ is equivalent to the symmetric operator

$$(\nabla^{-\beta})_i = \frac{1}{2}\ {}_{-\infty}D_{x_i}^{-\beta} + \frac{1}{2}\ {}_{x_i}D_{+\infty}^{-\beta}, \quad i = 1, ..., N. \tag{5.4}$$

Combining (5.2) and (5.4) gives the mass balance equation for advection and symmetric fractional dispersion.

The fractional ADE has been studied in one dimension (see, e.g., Benson, Wheatcraft and Meerschaert, 2000b), and in three dimension (see Lu, Molz and Fix, 2002), over infinite domains by using the Fourier transform of fractional differential operators to determine a classical solution. Variational methods, especially the Galerkin approximation has been investigated to find the solutions of fractional BVP (see, e.g., Fix and Roop, 2004) and fractional ADE (see, e.g., Ervin and Roop, 2006) on a finite domain by establishing some suitable fractional derivative spaces. A Lagrangian structure for some partial differential equations is obtained by using the fractional embedding theory of continuous Lagrangian systems (see, Cresson, 2010).

We note that for nonlinear fractional BVP, some fixed point theorems were already applied successfully to investigate the existence of solutions (see, e.g., Agarwal, Benchohra and Hamani, 2010; Ahmad and Nieto, 2009; Benchohra, Hamani and Ntouyas, 2009; Zhang, 2010). However, it seems that fixed point theorem is not appropriate for discussing BVP (5.1) since the equivalent integral equation is not easy to be obtained. On the other hand, there is another effective approach, calculus of variation, which proved to be very useful in determining the existence of solutions for integer order differential equation provided that equation with certain boundary conditions possesses a variational structure on some suitable Sobolev spaces, for example, one can refer to Corvellec, Motreanu and Saccon, 2010; Li, Liang and Zhang, 2005; Mawhin and Willem, 1989; Rabinowitz, 1986; Tang and Wu, 2010 and the references therein for detailed discussions.

However, to the best of author's knowledge, there are few results on the solutions to fractional BVP which were established by the critical point theory, since it is often very difficult to establish a suitable space and variational functional for fractional differential equations with some boundary-conditions. These difficulties are mainly caused by the following properties of fractional integral and fractional derivative operators. These are:

(i) the composition rule in general fails to be satisfied by fractional integral and fractional derivative operators (e.g., Lemma 2.21 in Kilbas, Srivastava and Trujillo, 2006);
(ii) the fractional integral is a singular integral operator and fractional derivative operator is non-local (see Definitions 1.5, 1.6 and 1.8), and
(iii) the adjoint of a fractional differential operator is not the negative of itself (e.g., Lemma 2.7 in Kilbas, Srivastava and Trujillo, 2006).

It should be mentioned here that the fractional variational principles were started to be investigated deeply. The fractional calculus of variations was introduced by Riewe, 1996, where he presented a new approach to mechanics that allows one to obtain the equations for a nonconservative system using certain functionals. Klimek, 2002, gave another approach by considering fractional derivatives, and corresponding Euler-Lagrange equations were obtained, using both Lagrangian and Hamiltonian formalisms. Agrawal, 2002, presented Euler-Lagrange equations for unconstrained and constrained fractional variational problems, and as a continuation of Agrawal's work, the generalized mechanics are considered to obtain the Hamiltonian formulation for the Lagrangian depending on fractional derivative of coordinates (see, Rabei, Nawafleh and Hijjawi *et al.*, 2007). The recent book by Malinowska and Torres, 2012, provides a broad introduction to the important subject of fractional calculus of variations.

In Section 5.2, we investigate the existence of solutions for BVP (5.1). The technical tool is the critical point theory. In Subsection 5.2.2, we develop a fractional derivative space and some propositions are proven which will aid in our analysis, and in Subsection 5.2.3, we shall exhibit a variational structure for BVP (5.1). The results presented in Subsections 5.2.2 and 5.2.3 are basic, but crucial to limpidly reveal that under some suitable assumptions, the critical points of the variational functional defined on a suitable Hilbert space are the solutions of BVP (5.1). In Subsection 5.2.4, we will introduce some critical point theorems. Also, various criteria on the existence of solutions for BVP (5.1) will be established.

As it was already mentioned, if $\beta = 0$, then BVP (5.1) reduces to the standard second order BVP of the following form

$$\begin{cases} u''(t) + \nabla F(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

where F: $[0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t, x)$ is the gradient of F at x. Although many excellent results have been worked out on the existence of solutions for second order BVP (e.g., Li, Liang and Zhang, 2005; Rabinowitz, 1986), it seems that no similar results were obtained in the literature for fractional BVP. The present results in Section 5.2 are to show that the critical point theory is an effective approach to tackle the existence of solutions for fractional BVP.

5.2.2 *Fractional Derivative Space*

Let us recall that for any fixed $t \in [0, T]$ and $1 \le p < \infty$,

$$\|u\|_{L^p[0,t]} = \left(\int_0^t |u(\xi)|^p d\xi \right)^{\frac{1}{p}}, \quad \|u\|_{L^p} = \left(\int_0^T |u(t)|^p dt \right)^{\frac{1}{p}} \quad \text{and } \|u\| = \max_{t \in [0,T]} |u(t)|.$$

The following result yields the boundedness of the Riemann-Liouville fractional integral operators from the space $L^p([0, T], \mathbb{R}^N)$ to the space $L^p([0, T], \mathbb{R}^N)$, where $1 \le p < \infty$. It should be mentioned here that the similar results have been presented in Fix and Roop, 2004; Kilbas, Srivastava and Trujillo, 2006; Samko, Kilbas and Marichev, 1993.

Lemma 5.1. Let $0 < \alpha \le 1$ and $1 \le p < \infty$. For any $f \in L^p([0, T], \mathbb{R}^N)$, we have

$$\|{}_0D_\xi^{-\alpha} f\|_{L^p[0,t]} \le \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p[0,t]}, \quad \text{for } \xi \in [0, t], \quad t \in [0, T]. \tag{5.5}$$

Proof. Inspired by the proof of the Young's theorem in Adams, 1975, we can prove (5.5).

In fact, if $p = 1$, we have

$$\begin{aligned} \|{}_0D_\xi^{-\alpha} f\|_{L^1[0,t]} &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t \int_0^\xi (\xi - \tau)^{\alpha-1} f(\tau) d\tau d\xi \right| \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t \int_0^\xi (\xi - \tau)^{\alpha-1} |f(\tau)| d\tau d\xi \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t |f(\tau)| d\tau \int_\tau^t (\xi - \tau)^{\alpha-1} d\xi \end{aligned} \tag{5.6}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha+1)} \int_0^t |f(\tau)|(t-\tau)^\alpha d\tau \\
&\le \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^1[0,t]}, \quad \text{for } \ t \in [0,T].
\end{aligned}
$$

Now, suppose that $1 < p < \infty$ and $g \in L^q([0,T], \mathbb{R}^N)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$
\begin{aligned}
\left| \int_0^t g(\xi) \int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau) d\tau d\xi \right| &= \left| \int_0^t g(\xi) \int_0^\xi \tau^{\alpha-1} f(\xi-\tau) d\tau d\xi \right| \\
&\le \int_0^t |g(\xi)| \int_0^\xi \tau^{\alpha-1} |f(\xi-\tau)| d\tau d\xi \\
&= \int_0^t \tau^{\alpha-1} d\tau \int_\tau^t |g(\xi)| |f(\xi-\tau)| d\xi \\
&\le \int_0^t \tau^{\alpha-1} d\tau \Big(\int_\tau^t |g(\xi)|^q d\xi \Big)^{\frac{1}{q}} \Big(\int_\tau^t |f(\xi-\tau)|^p d\xi \Big)^{\frac{1}{p}} \\
&\le \frac{t^\alpha}{\alpha} \|f\|_{L^p[0,t]} \|g\|_{L^q[0,t]}, \quad \text{for } \ t \in [0,T].
\end{aligned} \tag{5.7}
$$

For any fixed $t \in [0,T]$, consider the functional $H_{\xi * f} : L^q([0,T], \mathbb{R}^N) \to \mathbb{R}$

$$
H_{\xi * f}(g) = \int_0^t \left[\int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau) d\tau \right] g(\xi) d\xi. \tag{5.8}
$$

According to (5.7), it is obvious that $H_{\xi * f} \in (L^q([0,T], \mathbb{R}^N))^*$, where $(L^q([0,T], \mathbb{R}^N))^*$ denotes the dual space of $L^q([0,T], \mathbb{R}^N)$. Therefore, by (5.7), (5.8) and Riesz representation theorem, there exists $h \in L^p([0,T], \mathbb{R}^N)$ such that

$$
\int_0^t h(\xi) g(\xi) d\xi = \int_0^t \left[\int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau) d\tau \right] g(\xi) d\xi \tag{5.9}
$$

and

$$
\|h\|_{L^p[0,t]} \le \frac{t^\alpha}{\alpha} \|f\|_{L^p[0,t]} \tag{5.10}
$$

for all $g \in L^q([0,T], \mathbb{R}^N)$. Hence, we have by (5.9)

$$
\frac{1}{\Gamma(\alpha)} h(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^\xi (\xi-\tau)^{\alpha-1} f(\tau) d\tau = {}_0D_\xi^{-\alpha} f(\xi), \quad \text{for } \xi \in [0,t],
$$

which means that

$$
\|{}_0D_\xi^{-\alpha} f\|_{L^p[0,t]} = \frac{1}{\Gamma(\alpha)} \|h\|_{L^p[0,t]} \le \frac{t^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^p[0,t]} \tag{5.11}
$$

according to (5.10). Combining (5.6) and (5.11), we obtain the inequality (5.5). □

In order to establish a variational structure for BVP (5.1), it is necessary to construct appropriate function spaces. Denote by $C_0^\infty([0,T], \mathbb{R}^N)$ the set of all functions $h \in C^\infty([0,T], \mathbb{R}^N)$ with $h(0) = h(T) = 0$. According to Lemma 5.1, for any $h \in C_0^\infty([0,T], \mathbb{R}^N)$ and $1 < p < \infty$, we have $h \in L^p([0,T], \mathbb{R}^N)$ and

${}_0^CD_t^\alpha h \in L^p([0,T],\mathbb{R}^N)$. Therefore, one can construct a set of space $E_0^{\alpha,p}$, which depend on L^p-integrability of the Caputo fractional derivative of a function.

Definition 5.2. Let $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\alpha,p}=\left(\int_0^T |u(t)|^p dt+\int_0^T |{}_0^CD_t^\alpha u(t)|^p dt\right)^{\frac{1}{p}},\quad \forall u\in E_0^{\alpha,p}. \tag{5.12}$$

Remark 5.3.

(i) It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u\in L^p([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^CD_t^\alpha u\in L^p([0,T],\mathbb{R}^N)$ and $u(0)=u(T)=0$.

(ii) For any $u\in E_0^{\alpha,p}$, noting the fact that $u(0)=0$, we have ${}_0^CD_t^\alpha u(t)={}_0D_t^\alpha u(t),\ t\in[0,T]$ according to Property 1.9.

(iii) It is easy to verify that $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Proposition 5.4. Let $0<\alpha\le 1$ and $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.

Proof. In fact, owing to $L^p([0,T],\mathbb{R}^N)$ be reflexive and separable, the Cartesian product space

$$L_2^p([0,T],\mathbb{R}^N)=L^p([0,T],\mathbb{R}^N)\times L^p([0,T],\mathbb{R}^N)$$

is also a reflexive and separable Banach space with respect to the norm

$$\|v\|_{L_2^p}=\left(\sum_{i=1}^2\|v_i\|_{L^p}^p\right)^{\frac{1}{p}}, \tag{5.13}$$

where $v=(v_1,v_2)\in L_2^p([0,T],\mathbb{R}^N)$.

Consider the space $\Omega=\{(u,{}_0^CD_t^\alpha u):u\in E_0^{\alpha,p}\}$, which is a closed subset of $L_2^p([0,T],\mathbb{R}^N)$ as $E_0^{\alpha,p}$ is closed. Therefore, Ω is also a reflexive and separable Banach space with respect to the norm (5.13) for $v=(v_1,v_2)\in\Omega$.

We form the operator $A:E_0^{\alpha,p}\to\Omega$ as follows

$$A:u\to(u,\ {}_0^CD_t^\alpha u),\quad \forall u\in E_0^{\alpha,p}.$$

It is obvious that

$$\|u\|_{\alpha,p}=\|Au\|_{L_2^p},$$

which means that the operator $A:u\to(u,{}_0^CD_t^\alpha u)$ is an isometric isomorphic mapping and the space $E_0^{\alpha,p}$ is isometric isomorphic to the space Ω. Thus $E_0^{\alpha,p}$ is a reflexive and separable Banach space, and this completes the proof. □

Applying Property 1.22 and Lemma 5.1, we now can give the following useful estimates.

Proposition 5.5. Let $0<\alpha\le 1$ and $1<p<\infty$. For all $u\in E_0^{\alpha,p}$, we have

$$\|u\|_{L^p}\le\frac{T^\alpha}{\Gamma(\alpha+1)}\|{}_0^CD_t^\alpha u\|_{L^p}. \tag{5.14}$$

Moreover, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\| \le \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}_0^C D_t^\alpha u\|_{L^p}. \tag{5.15}$$

Proof. For any $u \in E_0^{\alpha,p}$, according to (1.38) and noting the fact that $u(0) = 0$, we have that

$${}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t)) = u(t), \qquad t \in [0, T].$$

Therefore, in order to prove inequalities (5.14) and (5.15), we only need to prove that

$$\|{}_0D_t^{-\alpha}({}_0^C D_t^\alpha u)\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0^C D_t^\alpha u\|_{L^p}, \tag{5.16}$$

where $0 < \alpha \le 1$ and $1 < p < \infty$, and

$$\|{}_0D_t^{-\alpha}({}_0^C D_t^\alpha u)\| \le \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}_0^C D_t^\alpha u\|_{L^p}, \tag{5.17}$$

where $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Firstly, we note that ${}_0^C D_t^\alpha u \in L^p([0,T], \mathbb{R}^N)$, the inequality (5.16) follows from (5.5) directly.

We are now in a position to prove (5.17). For $\alpha > \frac{1}{p}$, choose q such that $\frac{1}{p} + \frac{1}{q} = 1$. $\forall u \in E_0^{\alpha,p}$, we have

$$\begin{aligned}
|{}_0D_t^{-\alpha}({}_0^C D_t^\alpha u(t))| &= \frac{1}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1}\ {}_0^C D_s^\alpha u(s) ds \right| \\
&\le \frac{1}{\Gamma(\alpha)} \left(\int_0^t (t-s)^{(\alpha-1)q} ds \right)^{\frac{1}{q}} \|{}_0^C D_t^\alpha u\|_{L^p} \\
&\le \frac{T^{\frac{1}{q}+\alpha-1}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}_0^C D_t^\alpha u\|_{L^p} \\
&= \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}_0^C D_t^\alpha u\|_{L^p},
\end{aligned}$$

and this completes the proof. $\square$

According to (5.14), we can consider $E_0^{\alpha,p}$ with respect to the norm

$$\|u\|_{\alpha,p} = \|{}_0^C D_t^\alpha u\|_{L^p} = \left(\int_0^T |{}_0^C D_t^\alpha u(t)|^p dt \right)^{\frac{1}{p}} \tag{5.18}$$

in the following analysis.

Proposition 5.6. Let $0 < \alpha \le 1$ and $1 < p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T], \mathbb{R}^N)$, i.e., $\|u - u_k\| \to 0$, as $k \to \infty$.

Proof. If $\alpha > \frac{1}{p}$, then by (5.15) and (5.18), the injection of $E_0^{\alpha,p}$ into $C([0,T],\mathbb{R}^N)$, with its natural norm $\|\cdot\|$, is continuous, i.e., if $u_k \to u$ in $E_0^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$.

Since $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, it follows that $u_k \rightharpoonup u$ in $C([0,T],\mathbb{R}^N)$. In fact, For any $h \in (C([0,T],\mathbb{R}^N))^*$, if $u_k \to u$ in $E_0^{\alpha,p}$, then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, and thus $h(u_k) \to h(u)$. Therefore, $h \in (E_0^{\alpha,p})^*$, which means that $(C([0,T],\mathbb{R}^N))^* \subseteq (E_0^{\alpha,p})^*$.

Hence, if $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, then for any $h \in (C([0,T],\mathbb{R}^N))^*$, we have $h \in (E_0^{\alpha,p})^*$, and thus $h(u_k) \to h(u)$, i.e., $u_k \rightharpoonup u$ in $C([0,T],\mathbb{R}^N)$.

By the Banach-Steinhaus theorem, $\{u_k\}$ is bounded in $E_0^{\alpha,p}$ and, hence, in $C([0,T],\mathbb{R}^N)$. We are now in a position to prove that the sequence $\{u_k\}$ is equi-uniformly continuous.

Let $\frac{1}{p} + \frac{1}{q} = 1$ and $0 \le t_1 < t_2 \le T$. $\forall f \in L^p([0,T],\mathbb{R}^N)$, by using the Hölder inequality and noting that $\alpha > \frac{1}{p}$, we have

$$\begin{aligned}
&|{}_0D_{t_1}^{-\alpha} f(t_1) - {}_0D_{t_2}^{-\alpha} f(t_2)| \\
&= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} (t_1-s)^{\alpha-1} f(s)ds - \int_0^{t_2} (t_2-s)^{\alpha-1} f(s)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_1} (t_1-s)^{\alpha-1} f(s)ds - \int_0^{t_1} (t_2-s)^{\alpha-1} f(s)ds\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} f(s)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})|f(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}|f(s)|ds \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})^q ds\right)^{\frac{1}{q}} \|f\|_{L^p} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2} (t_2-s)^{(\alpha-1)q} ds\right)^{\frac{1}{q}} \|f\|_{L^p} \\
&\le \frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1} ((t_1-s)^{(\alpha-1)q} - (t_2-s)^{(\alpha-1)q})ds\right)^{\frac{1}{q}} \|f\|_{L^p} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2} (t_2-s)^{(\alpha-1)q} ds\right)^{\frac{1}{q}} \|f\|_{L^p} \\
&= \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_1^{(\alpha-1)q+1} - t_2^{(\alpha-1)q+1} + (t_2-t_1)^{(\alpha-1)q+1})^{\frac{1}{q}} \\
&\quad + \frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}((t_2-t_1)^{(\alpha-1)q+1})^{\frac{1}{q}} \\
&\le \frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-1+\frac{1}{q}} \\
&= \frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-\frac{1}{p}}.
\end{aligned} \tag{5.19}$$

Therefore, the sequence $\{u_k\}$ is equi-uniformly continuous since, for $0 \leq t_1 < t_2 \leq T$, by applying (5.19) and in view of (5.18), we have

$$\begin{aligned} |u_k(t_1) - u_k(t_2)| &= |{}_0D_{t_1}^{-\alpha}({}_0^C D_{t_1}^\alpha u_k(t_1)) - {}_0D_{t_2}^{-\alpha}({}_0^C D_{t_2}^\alpha u_k(t_2))| \\ &\leq \frac{2(t_2 - t_1)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(1 + (\alpha - 1)q)^{\frac{1}{q}}} \|{}_0^C D_t^\alpha u_k\|_{L^p} \\ &= \frac{2(t_2 - t_1)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)(1 + (\alpha - 1)q)^{\frac{1}{q}}} \|u_k\|_{\alpha,p} \\ &\leq c(t_2 - t_1)^{\alpha - \frac{1}{p}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $c \in \mathbb{R}^+$ is a constant. By the Ascoli-Arzela theorem, $\{u_k\}$ is relatively compact in $C([0,T], \mathbb{R}^N)$. By the uniqueness of the weak limit in $C([0,T], \mathbb{R}^N)$, every uniformly convergent subsequence of $\{u_k\}$ converges uniformly on $[0,T]$ to u. The proof is completed. □

5.2.3 *Variational Structure*

In this section, we will establish a variational structure which enables us to reduce the existence of solutions of BVP (5.1) to the one of critical points of corresponding functional defined on the space $E_0^{\alpha,p}$ with $p = 2$ and $\frac{1}{2} < \alpha \leq 1$.

First of all, making use of Property 1.17, for any $u \in AC([0,T], \mathbb{R}^N)$, BVP (5.1) transforms to

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\ {}_0D_t^{-\frac{\beta}{2}}({}_0D_t^{-\frac{\beta}{2}} u'(t)) + \dfrac{1}{2}\ {}_tD_T^{-\frac{\beta}{2}}({}_tD_T^{-\frac{\beta}{2}} u'(t))\right) + \nabla F(t, u(t)) = 0, \\ u(0) = u(T) = 0, \end{cases} \tag{5.20}$$

for almost every $t \in [0,T]$, where $\beta \in [0,1)$.

Furthermore, in view of Definition 1.8 and Property 1.10, it is obvious that $u \in AC([0,T], \mathbb{R}^N)$ is a solution of BVP (5.20) if and only if u is a solution of the following problem

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^C D_t^\alpha u(t)) - \dfrac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^C D_T^\alpha u(t))\right) + \nabla F(t, u(t)) = 0, \\ u(0) = u(T) = 0, \end{cases} \tag{5.21}$$

for almost every $t \in [0,T]$, where $\alpha = 1 - \frac{\beta}{2} \in (\frac{1}{2}, 1]$. Therefore, we seek a solution u of BVP (5.21) which, of course, corresponds to the solutions u of BVP (5.1) provided that $u \in AC([0,T], \mathbb{R}^N)$.

Let us denote by

$$D^\alpha(u(t)) = \frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^C D_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^C D_T^\alpha u(t)). \tag{5.22}$$

We are now in a position to give a definition of the solution of BVP (5.21).

Definition 5.7. A function $u \in AC([0,T], \mathbb{R}^N)$ is called a solution of BVP (5.21) if

(i) $D^\alpha(u(t))$ is derivable for almost every $t \in [0,T]$, and
(ii) u satisfies (5.21).

In the sequel, we will treat BVP (5.21) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$ which we defined in (5.18).

Consider the functional $u \to -\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt$ on E^α. The following estimate is useful for our further discussion.

Proposition 5.8. If $\frac{1}{2} < \alpha \leq 1$, then for any $u \in E^\alpha$, we have

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leq -\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt \leq \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{5.23}$$

Proof. Let $u \in E^\alpha$ and $\tilde{u}$ be the extension of u by zero on $\mathbb{R} \setminus [0,T]$. Then $\mathrm{supp}(\tilde{u}) \subseteq [0,T]$. However, as the left and right fractional derivatives are nonlocal,

$$\mathrm{supp}({}_{-\infty}D_t^\alpha \tilde{u}) \subseteq [0,\infty) \ \text{ and } \mathrm{supp}({}_t D_{+\infty}^\alpha \tilde{u}) \subseteq (-\infty, T].$$

Nonetheless, the product $({}_{-\infty}D_t^\alpha \tilde{u}, {}_t D_{+\infty}^\alpha \tilde{u})$ has support in $[0,T]$.

On the other hand, according to Theorem 2.3 and Lemma 2.4 in Ervin and Roop, 2006, we have

$$\begin{aligned}\int_{-\infty}^{\infty} ({}_{-\infty}D_t^\alpha \tilde{u}(t), {}_t D_{+\infty}^\alpha \tilde{u}(t))dt &= \cos(\pi\alpha)\int_{-\infty}^{\infty} |{}_{-\infty}D_t^\alpha \tilde{u}(t)|^2 dt \\ &= \cos(\pi\alpha)\int_{-\infty}^{\infty} |{}_t D_{+\infty}^\alpha \tilde{u}(t)|^2 dt,\end{aligned} \tag{5.24}$$

where ${}_{-\infty}D_t^\alpha$ and ${}_t D_{+\infty}^\alpha$ are the Liouville-Weyl fractional derivatives on the real line (see Definition 1.12). Helpful in establishing (5.24) is the Fourier transform of the Liouville-Weyl fractional derivative on the real line (see, Podlubny, 1999). Hence, according to Remark 5.3, (5.24) and noting that $\cos(\pi\alpha) \in [-1,0)$ as $\alpha \in (\frac{1}{2}, 1]$, we have

$$\begin{aligned}-\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt &= -\int_0^T ({}_0 D_t^\alpha u(t), {}_t D_T^\alpha u(t))dt \\ &= -\int_0^T ({}_{-\infty}D_t^\alpha \tilde{u}(t), {}_t D_{+\infty}^\alpha \tilde{u}(t))dt \\ &= -\int_{-\infty}^{\infty} ({}_{-\infty}D_t^\alpha \tilde{u}(t), {}_t D_{+\infty}^\alpha \tilde{u}(t))dt \\ &= -\cos(\pi\alpha)\int_{-\infty}^{\infty} |{}_{-\infty}D_t^\alpha \tilde{u}(t)|^2 dt \\ &= -\cos(\pi\alpha)\int_0^{\infty} |{}_0 D_t^\alpha \tilde{u}(t)|^2 dt \\ &\geq -\cos(\pi\alpha)\int_0^T |{}_0 D_t^\alpha u(t)|^2 dt \\ &= |\cos(\pi\alpha)|\int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt \\ &= |\cos(\pi\alpha)|\|u\|_\alpha^2.\end{aligned} \tag{5.25}$$

On the other hand, by using the Young's inequality, we obtain

$$\begin{aligned}
&\left|\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt\right| \\
&= \left|\int_0^T ({}_0 D_t^\alpha u(t), {}_t D_T^\alpha u(t))dt\right| \\
&\leq \int_0^T \frac{1}{\sqrt{2\varepsilon}}\ |{}_0 D_t^\alpha u(t)| \sqrt{2\varepsilon}\ |{}_t D_T^\alpha u(t)| dt \\
&\leq \frac{1}{4\varepsilon}\int_0^T |{}_0 D_t^\alpha u(t)|^2 dt + \varepsilon \int_0^T |{}_t D_T^\alpha u(t)|^2 dt \\
&= \frac{1}{4\varepsilon}\int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt + \varepsilon \int_0^\infty |{}_t D_{+\infty}^\alpha \tilde{u}(t)|^2 dt \\
&\leq \frac{1}{4\varepsilon}\|u\|_\alpha^2 + \varepsilon \int_{-\infty}^\infty |{}_t D_{+\infty}^\alpha \tilde{u}(t)|^2 dt \\
&= \frac{1}{4\varepsilon}\|u\|_\alpha^2 + \frac{\varepsilon}{|\cos(\pi\alpha)|}\left|\int_{-\infty}^\infty ({}_{-\infty} D_t^\alpha \tilde{u}(t), {}_t D_{+\infty}^\alpha \tilde{u}(t))dt\right| \\
&= \frac{1}{4\varepsilon}\|u\|_\alpha^2 + \frac{\varepsilon}{|\cos(\pi\alpha)|}\left|\int_0^T ({}_0 D_t^\alpha u(t), {}_t D_T^\alpha u(t))dt\right| \\
&= \frac{1}{4\varepsilon}\|u\|_\alpha^2 + \frac{\varepsilon}{|\cos(\pi\alpha)|}\left|\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt\right|.
\end{aligned}$$

Therefore, by taking $\varepsilon = |\cos(\pi\alpha)|/2$, we have

$$\left|\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt\right| \leq \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{5.26}$$

The inequality (5.23) follows then from (5.25) and (5.26), and the proof is complete. □

Remark 5.9. According to (5.23) and (5.24), for any $u \in E^\alpha$, it is obvious that

$$\begin{aligned}
\int_0^T |{}_t^C D_T^\alpha u(t)|^2 dt &\leq \int_{-\infty}^\infty |{}_t D_{+\infty}^\alpha \tilde{u}(t)|^2 dt \\
&= -\int_0^T \frac{({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))}{|\cos(\pi\alpha)|} dt \\
&\leq \frac{1}{|\cos(\pi\alpha)|^2}\|u\|_\alpha^2,
\end{aligned}$$

which means that ${}_t^C D_T^\alpha u \in L^2([0,T], \mathbb{R}^N)$.

In the following, we establish a variational structure on E^α with $\alpha \in (\frac{1}{2}, 1]$. Also, we will show that the critical points of that functional are indeed solutions of BVP(5.21), and therefore, are solutions of BVP (5.1).

Theorem 5.10. Let $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$L(t,x,y,z) = -\frac{1}{2}(y,z) - F(t,x),$$

where $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:
(A) $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^N$, continuously differentiable in x for almost every $t \in [0,T]$ and there exist $m_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $m_2 \in L^1([0,T], \mathbb{R}^+)$ such that

$$|F(t,x)| \le m_1(|x|)m_2(t), \quad |\nabla F(t,x)| \le m_1(|x|)m_2(t)$$

for all $x \in \mathbb{R}^N$ and a.e. in $t \in [0,T]$.

If $\frac{1}{2} < \alpha \le 1$, then the functional defined by

$$\begin{aligned}\varphi(u) &= \int_0^T L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt \\ &= \int_0^T \Big[-\frac{1}{2}({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)) - F(t,u(t))\Big] dt\end{aligned} \tag{5.27}$$

is continuously differentiable on E^α, and $\forall u, v \in E^\alpha$, we have

$$\begin{aligned}\langle \varphi'(u), v\rangle &= \int_0^T (D_x L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), v(t))dt \\ &\quad + \int_0^T (D_y L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), {}_0^C D_t^\alpha v(t))dt \\ &\quad + \int_0^T (D_z L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), {}_t^C D_T^\alpha v(t))dt \\ &= -\int_0^T \frac{1}{2}[({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha v(t)) + ({}_t^C D_T^\alpha u(t), {}_0^C D_t^\alpha v(t))]dt \\ &\quad - \int_0^T (\nabla F(t,u(t)), v(t))dt.\end{aligned} \tag{5.28}$$

Proof. First, we note that for a.e. $t \in [0,T]$ and every $[x,y,z] \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, one has

$$|L(t,x,y,z)| \le m_1(|x|)m_2(t) + \frac{1}{4}(|y|^2 + |z|^2), \tag{5.29}$$

$$|D_x L(t,x,y,z)| \le m_1(|x|)m_2(t), \tag{5.30}$$

$$|D_y L(t,x,y,z)| \le \frac{1}{2}|z| \quad \text{and} \quad |D_z L(t,x,y,z)| \le \frac{1}{2}|y|. \tag{5.31}$$

Then, inspired by the proof of Theorem 1.4 in Mawhin and Willem, 1989, it suffices to prove that at every point u,φ has a directional derivative $\varphi'(u) \in (E^\alpha)^*$ given by (5.28) and that the mapping

$$\varphi' : E^\alpha \to (E^\alpha)^*, \quad u \to \varphi'(u)$$

is continuous.

1) It follows easily from Remark 5.9 and (5.29) that φ is everywhere finite on E^α. Let us define, for u and v fixed in E^α, $t \in [0,T]$, $\lambda \in [-1,1]$,

$$G(\lambda, t) = L(t, u(t) + \lambda v(t), {}_0^C D_t^\alpha u(t) + \lambda\, {}_0^C D_t^\alpha v(t), {}_t^C D_T^\alpha u(t) + \lambda\, {}_t^C D_T^\alpha v(t))$$

and

$$\psi(\lambda) = \int_0^T G(\lambda, t)dt = \varphi(u + \lambda v).$$

We shall apply Leibniz formula of differentiation under integral sign to ψ. By (5.30) and (5.31), we have

$$\begin{aligned}
&|D_\lambda G(\lambda, t)| \\
= &|(D_x L(t, u(t) + \lambda v(t), {}_0^C D_t^\alpha u(t) + \lambda\, {}_0^C D_t^\alpha v(t), {}_t^C D_T^\alpha u(t) + \lambda\, {}_t^C D_T^\alpha v(t)), v(t))| \\
&+|(D_y L(t, u(t) + \lambda v(t), {}_0^C D_t^\alpha u(t) + \lambda\, {}_0^C D_t^\alpha v(t), \\
&\quad {}_t^C D_T^\alpha u(t) + \lambda\, {}_t^C D_T^\alpha v(t)), {}_0^C D_t^\alpha v(t))| \\
&+|(D_z L(t, u(t) + \lambda v(t), {}_0^C D_t^\alpha u(t) + \lambda\, {}_0^C D_t^\alpha v(t), \\
&\quad {}_t^C D_T^\alpha u(t) + \lambda\, {}_t^C D_T^\alpha v(t)), {}_t^C D_T^\alpha v(t))| \\
\leq &\, m_1(|u(t) + \lambda v(t)|)m_2(t)|v(t)| + \frac{1}{2}|{}_t^C D_T^\alpha u(t) + \lambda\, {}_t^C D_T^\alpha v(t)||{}_0^C D_t^\alpha v(t)| \\
&+\frac{1}{2}|{}_0^C D_t^\alpha u(t) + \lambda\, {}_0^C D_t^\alpha v(t)||{}_t^C D_T^\alpha v(t)| \\
\leq &\, m_0 m_2(t)|v(t)| + \frac{1}{2}|{}_t^C D_T^\alpha u(t)||{}_0^C D_t^\alpha v(t)| + \frac{1}{2}|{}_0^C D_t^\alpha u(t)||{}_t^C D_T^\alpha v(t)| \\
&+|{}_0^C D_t^\alpha v(t)||{}_t^C D_T^\alpha v(t)|,
\end{aligned}$$

where

$$m_0 = \max_{(\lambda,t)\in[-1,1]\times[0,T]} m_1(|u(t) + \lambda v(t)|).$$

Since $m_2 \in L^1([0, T], \mathbb{R}^+)$, v is continuous on $[0, T]$, and in view of Remark 5.9, we have

$$|D_\lambda G(\lambda, t)| \leq d(t),$$

where $d \in L^1([0, T], \mathbb{R}^+)$. Thus Leibniz formula is applicable and

$$\begin{aligned}
\frac{d}{d\lambda}\psi(0) &= \int_0^T D_\lambda G(0, t)dt \\
&= \int_0^T (D_x L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), v(t))dt \\
&\quad + \int_0^T (D_y L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), {}_0^C D_t^\alpha v(t))dt \\
&\quad + \int_0^T (D_z L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), {}_t^C D_T^\alpha v(t))dt.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|D_x L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))| &\leq m_1(|u(t)|)m_2(t), \\
|D_y L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))| &\leq \frac{1}{2}|{}_t^C D_T^\alpha u(t)|
\end{aligned}$$

and

$$|D_z L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))| \leq \frac{1}{2} |{}_0^C D_t^\alpha u(t)|.$$

Thus, by Remark 5.9 and (5.15),

$$\begin{aligned}
\langle \varphi'(u), v \rangle &= \int_0^T (D_x L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), v(t)) dt \\
&\quad + \int_0^T (D_y L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), {}_0^C D_t^\alpha v(t)) dt \\
&\quad + \int_0^T (D_z L(t, u(t), {}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)), {}_t^C D_T^\alpha v(t)) dt \\
&\leq c_1 \|v\| + c_2 \|{}_0^C D_t^\alpha v(t)\|_{L^2} + c_3 \|{}_t^C D_T^\alpha v(t)\|_{L^2} \\
&\leq c_1 \|v\| + c_2 \|v\|_\alpha + \frac{c_3}{|\cos(\pi\alpha)|} \|v\|_\alpha \\
&\leq c_4 \|v\|_\alpha,
\end{aligned}$$

where c_1, c_2, c_3 and c_4 are some positive constants. Therefore, φ has, at u, a directional derivative $\varphi'(u) \in (E^\alpha)^*$ given by (5.28).

2) By a theorem of Krasnoselskii, (5.30) and (5.31) imply that the mapping from E^α into $L^1([0,T], \mathbb{R}^N) \times L^2([0,T], \mathbb{R}^N) \times L^2([0,T], \mathbb{R}^N)$ defined by

$$u \to (D_x L(\cdot, u, {}_0^C D_t^\alpha u, {}_t^C D_T^\alpha u), D_y L(\cdot, u, {}_0^C D_t^\alpha u, {}_t^C D_T^\alpha u), D_z L(\cdot, u, {}_0^C D_t^\alpha u, {}_t^C D_T^\alpha u))$$

is continuous, so that φ' is continuous from E^α into $(E^\alpha)^*$, and the proof is completed. □

Theorem 5.11. Let $\frac{1}{2} < \alpha \leq 1$ and φ be defined by (5.27). If assumption (A) is satisfied and $u \in E^\alpha$ is a solution of corresponding Euler equation $\varphi'(u) = 0$, then u is a solution of BVP (5.21) which, of course, corresponding to the solution of BVP (5.1).

Proof. By Theorem 5.10 and Property 1.23, we have

$$\begin{aligned}
0 = \langle \varphi'(u), v \rangle &= -\int_0^T \frac{1}{2} [({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha v(t)) + ({}_t^C D_T^\alpha u(t), {}_0^C D_t^\alpha v(t))] dt \\
&\quad - \int_0^T (\nabla F(t, u(t)), v(t)) dt \\
&= \int_0^T \left[\frac{1}{2} ({}_0 D_t^{\alpha-1} ({}_0^C D_t^\alpha u(t)), v'(t)) - \frac{1}{2} ({}_t D_T^{\alpha-1} ({}_t^C D_T^\alpha u(t)), v'(t)) \right] dt \\
&\quad - \int_0^T (\nabla F(t, u(t)), v(t)) dt
\end{aligned} \tag{5.32}$$

for all $v \in E^\alpha$.

Let us define $w \in C([0,T], \mathbb{R}^N)$ by

$$w(t) = \int_0^t \nabla F(s, u(s)) ds, \quad t \in [0, T],$$

so that

$$\int_0^T (w(t), v'(t))dt = \int_0^T \left[\int_0^t (\nabla F(s, u(s)), v'(t))ds\right] dt.$$

By the Fubini theorem and noting that $v(T) = 0$, we obtain

$$\begin{aligned}\int_0^T (w(t), v'(t))dt &= \int_0^T \left[\int_s^T (\nabla F(s, u(s)), v'(t))dt\right] ds \\ &= \int_0^T (\nabla F(s, u(s)), v(T) - v(s))ds \\ &= -\int_0^T (\nabla F(s, u(s)), v(s))ds.\end{aligned}$$

Hence, by (5.32) we have, for every $v \in E^\alpha$,

$$\int_0^T \left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^CD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^CD_T^\alpha u(t)) + w(t), v'(t)\right) dt = 0. \tag{5.33}$$

If (e_j) denotes the canonical basis of $\mathbb{R}^N$, we can choose $v \in E^\alpha$ such that

$$v(t) = \sin\frac{2k\pi t}{T} e_j \quad \text{or} \quad v(t) = e_j - \cos\frac{2k\pi t}{T} e_j, \qquad k = 1, 2, ... \text{ and } j = 1, ..., N.$$

The theory of Fourier series and (5.33) imply that

$$\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^CD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^CD_T^\alpha u(t)) + w(t) = C,$$

a.e. $t \in [0, T]$, for some $C \in \mathbb{R}^N$. According to the definition of $w \in C([0, T], \mathbb{R}^N)$, we have

$$\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^CD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^CD_T^\alpha u(t)) = -\int_0^t \nabla F(s, u(s))ds + C, \tag{5.34}$$

a.e. $t \in [0, T]$, for some $C \in \mathbb{R}^N$.

In view of $\nabla F(\cdot, u(\cdot)) \in L^1([0, T], \mathbb{R}^N)$, we shall identify the equivalence class $D^\alpha(u(t))$ given by (5.22) and its continuous representation

$$\begin{aligned}D^\alpha(u(t)) &= \frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^CD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^CD_T^\alpha u(t)) \\ &= -\int_0^t \nabla F(s, u(s))ds + C\end{aligned} \tag{5.35}$$

for $t \in [0, T]$.

Therefore, it follows from (5.35) and a classical result of Lebesgue theory that $-\nabla F(\cdot, u(\cdot))$ is the classical derivative of $D^\alpha(u(t))$ a.e. on $[0, T]$ which means that (i) in Definition 5.7 is verified.

Since $u \in E^\alpha$ implies that $u \in AC([0, T], \mathbb{R}^N)$, it remains to show that u satisfies (5.21). In fact, according to (5.35), we can get that

$$\frac{d}{dt} D^\alpha(u(t)) = \frac{d}{dt}\left(\frac{1}{2}\ {}_0D_t^{\alpha-1}({}_0^CD_t^\alpha u(t)) - \frac{1}{2}\ {}_tD_T^{\alpha-1}({}_t^CD_T^\alpha u(t))\right) = -\nabla F(t, u(t)).$$

Moreover, $u \in E^\alpha$ implies that $u(0) = u(T) = 0$, and therefore (5.1) is verified. The proof is completed. □

From now on, φ given by (5.27) will be considered as a functional on E^α with $\frac{1}{2} < \alpha \leq 1$.

5.2.4 *Existence under Ambrosetti-Rabinowitz Condition*

According to Theorem 5.11, we know that in order to find solutions of BVP (5.1), it suffices to obtain the critical points of functional φ given by (5.27). We need to use some critical point theorems.

First, we use Theorem 1.50 to consider the existence of solutions for BVP (5.1). Assume that condition (A) is satisfied. Recall that, in our setting in (5.27), the corresponding functional φ on E^α given by

$$\varphi(u) = \int_0^T \left[-\frac{1}{2}({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)) - F(t, u(t))\right] dt$$

is continuously differentiable according to Theorem 5.10 and is also weakly lower semi-continuous functional on E^α as the sum of a convex continuous function (see Theorem 1.2 in Mawhin and Willem, 1989) and of a weakly continuous one (see Proposition 1.2 in Mawhin and Willem, 1989).

In fact, according to Proposition 5.6, if $u_k \rightharpoonup u$ in E^α, then $u_k \to u$ in $C([0,T], \mathbb{R}^N)$. Therefore, $F(t, u_k(t)) \to F(t, u(t))$ a.e. $t \in [0,T]$. By Lebesgue's dominated convergence theorem, we have $\int_0^T F(t, u_k(t))dt \to \int_0^T F(t, u(t))dt$, which means that the functional $u \to \int_0^T F(t, u(t))dt$ is weakly continuous on E^α. Moreover, the following lemma implies that the functional $u \to -\int_0^T [({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))/2]dt$ is convex and continuous on E^α.

Lemma 5.12. Let $\frac{1}{2} < \alpha \le 1$ and assumption (A) be satisfied. If $u \in E^\alpha$, then the functional $H : E^\alpha \to \mathbb{R}^N$ denoted by

$$H(u) = -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt$$

is convex and continuous on E^α.

Proof. The continuity follows from (5.23) and (5.18) directly. We are now in a position to prove the convexity of H.

Let $\lambda \in (0,1)$, $u, v \in E^\alpha$ and $\tilde{u}, \tilde{v}$ be the extension of u and v by zero on $\mathbb{R}/[0,T]$ respectively. Since the Caputo fractional derivative operator is linear operator, we have by Remark 5.9 and (5.24) that

$$\begin{aligned}
&H((1-\lambda)u + \lambda v) \\
&= -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha((1-\lambda)u(t) + \lambda v(t)), {}_t^C D_T^\alpha((1-\lambda)u(t) + \lambda v(t)))dt \\
&= -\frac{1}{2}\int_0^T ({}_0 D_t^\alpha((1-\lambda)u(t) + \lambda v(t)), {}_t D_T^\alpha((1-\lambda)u(t) + \lambda v(t)))dt \\
&= -\frac{1}{2}\int_{-\infty}^\infty ({}_{-\infty} D_t^\alpha((1-\lambda)\tilde{u}(t) + \lambda \tilde{v}(t)), {}_t D_{+\infty}^\alpha((1-\lambda)\tilde{u}(t) + \lambda \tilde{v}(t)))dt \\
&= \frac{|\cos(\pi\alpha)|}{2}\int_{-\infty}^\infty |{}_{-\infty} D_t^\alpha((1-\lambda)\tilde{u}(t) + \lambda \tilde{v}(t))|^2 dt \\
&\le \frac{|\cos(\pi\alpha)|}{2}\int_{-\infty}^\infty [(1-\lambda)|{}_{-\infty} D_t^\alpha \tilde{u}(t)|^2 + \lambda|{}_{-\infty} D_t^\alpha \tilde{v}(t)|^2]dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\left[-\frac{1-\lambda}{2}({}_{-\infty}D_t^\alpha \tilde{u}(t), {}_tD_{+\infty}^\alpha \tilde{u}(t)) - \frac{\lambda}{2}({}_{-\infty}D_t^\alpha \tilde{v}(t), {}_tD_{+\infty}^\alpha \tilde{v}(t))\right]dt\\
&= \int_0^T\left[-\frac{1-\lambda}{2}({}_0^CD_t^\alpha u(t), {}_t^CD_T^\alpha u(t)) - \frac{\lambda}{2}({}_0^CD_t^\alpha v(t), {}_t^CD_T^\alpha v(t))\right]dt\\
&= (1-\lambda)H(u) + \lambda H(v),
\end{aligned}
$$

which implies that H is a convex functional defined on E^α. This completes the proof. □

According to the arguments above, if φ is coercive, by Theorem 1.50, φ has a minimum so that BVP (5.1) is solvable. It remains to find conditions under which φ is coercive on E^α, i.e., $\lim_{\|u\|_\alpha\to\infty}\varphi(u) = +\infty$, for $u \in E^\alpha$. We shall see that it suffices to require that $F(t,x)$ is bounded by a function for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

Theorem 5.13. Let $\alpha \in (\frac{1}{2}, 1]$ and assume that F satisfies condition (A). If

$$|F(t,x)| \le \bar{a}|x|^2 + \bar{b}(t)|x|^{2-\gamma} + \bar{c}(t), \quad t \in [0,T],\ x \in \mathbb{R}^N, \tag{5.36}$$

where $\bar{a} \in [0, |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/2T^{2\alpha})$, $\gamma \in (0,2)$, $\bar{b} \in L^{2/\gamma}([0,T],\mathbb{R})$, and $\bar{c} \in L^1([0,T],\mathbb{R})$, then BVP (5.1) has at least one solution which minimizes φ on E^α.

Proof. According to arguments above, our problem reduces to prove that φ is coercive on E^α. For $u \in E^\alpha$, it follows from (5.23), (5.36) and (5.14) that

$$
\begin{aligned}
\varphi(u) &= -\frac{1}{2}\int_0^T ({}_0^CD_t^\alpha u(t), {}_t^CD_T^\alpha u(t))dt - \int_0^T F(t,u(t))dt\\
&\ge \frac{|\cos(\pi\alpha)|}{2}\int_0^T |{}_0^CD_t^\alpha u(t)|^2dt - \bar{a}\int_0^T |u(t)|^2dt\\
&\quad - \int_0^T \bar{b}(t)|u(t)|^{2-\gamma}dt - \int_0^T \bar{c}(t)dt\\
&= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \int_0^T \bar{b}(t)|u(t)|^{2-\gamma}dt - \bar{c}_1\\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \left(\int_0^T |\bar{b}(t)|^{2/\gamma}dt\right)^{\gamma/2}\left(\int_0^T |u(t)|^2dt\right)^{1-\gamma/2} - \bar{c}_1\\
&= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \bar{b}_1\|u\|_{L^2}^{2-\gamma} - \bar{c}_1\\
&\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{\bar{a}T^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|_\alpha^2 - \bar{b}_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|_\alpha^{2-\gamma} - \bar{c}_1\\
&= \left(\frac{|\cos(\pi\alpha)|}{2} - \frac{\bar{a}T^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|u\|_\alpha^2 - \bar{b}_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|_\alpha^{2-\gamma} - \bar{c}_1,
\end{aligned}
$$

where $\bar{b}_1 = (\int_0^T |\bar{b}(t)|^{2/\gamma}dt)^{\gamma/2}$ and $\bar{c}_1 = \int_0^T \bar{c}(t)dt$.

Noting that $\bar{a} \in [0, |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/2T^{2\alpha})$ and $\gamma \in (0,2)$, we have

$$\varphi(u) = +\infty \ \text{ as } \|u\|_\alpha \to \infty,$$

and hence φ is coercive, which completes the proof. □

Our task is now to use Theorem 1.51 (Mountain pass theorem) to find a nonzero critical point of functional φ on E^α.

Theorem 5.14. Let $\alpha \in (\frac{1}{2}, 1]$ and suppose that F satisfies condition (A). If
(A1) $F \in C([0,T] \times \mathbb{R}^N, \mathbb{R})$ and there exists $\mu \in [0, \frac{1}{2})$ and $M > 0$ such that $0 < F(t,x) \le \mu(\nabla F(t,x), x)$ for all $x \in \mathbb{R}^N$ with $|x| \ge M$ and $t \in [0,T]$;
(A2) $\limsup_{|x| \to 0} F(t,x)/|x|^2 < |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/2T^{2\alpha}$ uniformly for $t \in [0,T]$ and $x \in \mathbb{R}^N$;
are satisfied, then BVP (5.1) has at least one nonzero solution on E^α.

Proof. We will verify that φ satisfies all conditions of Theorem 1.51.

First, we will prove that φ satisfies (PS) condition. Since $F(t,x) - \mu(\nabla F(t,x), x)$ is continuous for $t \in [0,T]$ and $|x| \le M$, there exists $c \in \mathbb{R}^+$, such that

$$F(t,x) \le \mu(\nabla F(t,x), x) + c, \quad t \in [0,T], \quad |x| \le M.$$

By assumption (A1), we obtain

$$F(t,x) \le \mu(\nabla F(t,x), x) + c, \quad t \in [0,T], \quad x \in \mathbb{R}^N. \tag{5.37}$$

Let $\{u_k\} \subset E^\alpha$, $|\varphi(u_k)| \le K$, $k = 1, 2, ...,$ $\varphi'(u_k) \to 0$. Notice that

$$\langle \varphi'(u_k), u_k \rangle = -\int_0^T [({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t)) + (\nabla F(t, u_k(t)), u_k(t))]dt. \tag{5.38}$$

It follows from (5.37), (5.38) and (5.23) that

$$\begin{aligned} K \ge \varphi(u_k) &= -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t))dt - \int_0^T F(t, u_k(t))dt \\ &\ge -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t))dt - \mu\int_0^T (\nabla F(t, u_k(t)), u_k(t))dt - cT \\ &= \left(\mu - \frac{1}{2}\right)\int_0^T ({}_0^C D_t^\alpha u_k(t), {}_t^C D_T^\alpha u_k(t))dt + \mu\langle \varphi'(u_k), u_k \rangle - cT \\ &\ge \left(\frac{1}{2} - \mu\right)|\cos(\pi\alpha)|\|u_k\|_\alpha^2 - \mu\|\varphi'(u_k)\|_\alpha\|u_k\|_\alpha - cT, \quad k = 1, 2, ... \end{aligned}$$

Since $\varphi'(u_k) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$K \ge \left(\frac{1}{2} - \mu\right)|\cos(\pi\alpha)|\|u_k\|_\alpha^2 - \|u_k\|_\alpha - cT, \quad k > N_0,$$

and this implies that $\{u_k\} \subset E^\alpha$ is bounded. Since E^α is a reflexive space, going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ weakly in E^α, thus we have

$$\begin{aligned} \langle \varphi'(u_k) - \varphi'(u), u_k - u \rangle &= \langle \varphi'(u_k), u_k - u \rangle - \langle \varphi'(u), u_k - u \rangle \\ &\le \|\varphi'(u_k)\|_\alpha \|u_k - u\|_\alpha - \langle \varphi'(u), u_k - u \rangle \to 0, \end{aligned} \tag{5.39}$$

as $k \to \infty$. Moreover, according to (5.15) and Proposition 5.6, we have u_k is bounded in $C([0,T], \mathbb{R}^N)$ and $\|u_k - u\| \to 0$ as $k \to \infty$. Hence, we have

$$\int_0^T \nabla F(t, u_k(t))dt \to \int_0^T \nabla F(t, u(t))dt \quad \text{as} \quad k \to \infty. \tag{5.40}$$

Noting that

$$\begin{aligned}\langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle &= -\int_0^T ({}_0^C D_t^\alpha (u_k(t) - u(t)), {}_t^C D_T^\alpha (u_k(t) - u(t)))dt \\ &\quad - \int_0^T \Big((\nabla F(t, u_k(t)) - \nabla F(t, u(t))), (u_k(t) - u(t))\Big)dt \\ &\geq |\cos(\pi\alpha)| \|u_k - u\|_\alpha^2 - \left| \int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)))dt \right| \|u_k - u\|.\end{aligned}$$

Combining (5.39) and (5.40), it is easy to verify that $\|u_k - u\|_\alpha^2 \to 0$ as $k \to \infty$, and hence that $u_k \to u$ in E^α. Thus, we obtain the desired convergence property.

From $\limsup_{|x|\to 0} F(t,x)/|x|^2 < |\cos(\pi\alpha)|\Gamma^2(\alpha+1)/2T^{2\alpha}$ uniformly for $t \in [0,T]$, there exists $\epsilon \in (0, |\cos(\pi\alpha)|)$ and $\delta > 0$ such that $F(t,x) \leq (|\cos(\pi\alpha)| - \epsilon)(\Gamma^2(\alpha+1)/2T^{2\alpha})|x|^2$ for all $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| \leq \delta$.

Let $\rho = \frac{\Gamma(\alpha)((\alpha-1)/2+1)^{\frac{1}{2}}}{T^{\alpha-\frac{1}{2}}}\delta$ and $\sigma = \epsilon\rho^2/2 > 0$. Then it follows from (5.15) that

$$\|u\| \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)((\alpha-1)/2+1)^{\frac{1}{2}}}\|u\|_\alpha = \delta$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. Therefore, we have

$$\begin{aligned}\varphi(u) &= -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt - \int_0^T F(t, u(t))dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - (|\cos(\pi\alpha)| - \epsilon)\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}}\int_0^T |u(t)|^2 dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{1}{2}(|\cos(\pi\alpha)| - \epsilon)\|u\|_\alpha^2 \\ &= \frac{1}{2}\epsilon\|u\|_\alpha^2 \\ &= \sigma\end{aligned}$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. This implies (ii) in Theorem 1.51 is satisfied.

It is obvious from the definition of φ and (A2) that $\varphi(0) = 0$, and therefore, it suffices to show that φ satisfies (iii) in Theorem 1.51.

Since $0 < F(t,x) \leq \mu(\nabla F(t,x), x)$ for all $x \in \mathbb{R}^N$ and $|x| \geq M$, a simple regularity argument then shows that there exists $r_1, r_2 > 0$ such that

$$F(t,x) \geq r_1|x|^{1/\mu} - r_2, \quad x \in \mathbb{R}^N, \ \ t \in [0,T].$$

For any $u \in E^\alpha$ with $u \neq 0$, $\kappa > 0$ and noting that $\mu \in [0, \frac{1}{2})$ and (5.23), we have

$$\begin{aligned}\varphi(\kappa u) &= -\frac{1}{2}\int_0^T ({}_0^C D_t^\alpha \kappa u(t), {}_t^C D_T^\alpha \kappa u(t))dt - \int_0^T F(t, \kappa u(t))dt \\ &\leq \frac{\kappa^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - r_1\int_0^T |\kappa u(t)|^{1/\mu}dt + r_2 T \\ &= \frac{\kappa^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - r_1\kappa^{1/\mu}\|u\|_{L^{1/\mu}}^{1/\mu} + r_2 T \to -\infty\end{aligned}$$

as $\kappa \to \infty$. Then there exists a sufficiently large κ_0 such that $\varphi(\kappa_0 u) \le 0$. Hence (iii) in Theorem 1.51 holds.

Lastly noting that $\varphi(0) = 0$ while for our critical point u, $\varphi(u) \ge \sigma > 0$. Hence u is a nontrivial weak solution of BVP (5.1), and this completes the proof. $\square$

Corollary 5.15. $\forall \alpha \in (\frac{1}{2}, 1]$, suppose that F satisfies conditions (A) and (A1). If (A2)$'$ $F(t,x) = o(|x|^2)$, as $|x| \to 0$ uniformly for $t \in [0,T]$ and $x \in \mathbb{R}^N$
is satisfied, then BVP (5.1) has at least one nonzero solution on E^α.

5.2.5 *Superquadratic Case*

Under the usual Ambrosetti-Rabinowitz condition, it is easy to show that the energy functional associated with the system has the Mountain Pass geometry and satisfies the (PS) condition. However, the A.R. condition is so strong that many potential functions can not satisfy it, then the problem becomes more delicate and complicated.

Assume that $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the condition (A) which is assumed as in Subsection 5.2.3.

In the following, we introduce the function space E^α, where $\alpha \in (\frac{1}{2}, 1]$. For $u \in E^\alpha$, where

$$E^\alpha := \big\{u \in L^2(0,T;\mathbb{R}^N) : \ {}_0^C D_t^\alpha u \in L^2(0,T;\mathbb{R}^N)\big\}$$

is a reflexive Banach space with the norm defined by

$$\|u\|_\alpha = \|{}_0^C D_t^\alpha u\|_{L_2}$$

and

$$\|u\| = \max_{t\in[0,T]} |u(t)|.$$

It follows from Theorem 5.10 that the functional φ on E^α given by

$$\varphi(u) = \int_0^T \Big[-\frac{1}{2}\big({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t)\big) - F(t,u(t))\Big]dt \tag{5.41}$$

is continuously differentiable on E^α. Moreover, we have

$$\begin{aligned}\langle \varphi'(u), v\rangle = &-\int_0^T \frac{1}{2}\big[\big({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha v(t)\big) + \big({}_t^C D_T^\alpha u(t), {}_0^C D_t^\alpha v(t)\big)\big]dt \\ &-\int_0^T \big(\nabla F\big(t,u(t)\big), v(t)\big)dt.\end{aligned} \tag{5.42}$$

Recall that a sequence $\{u_n\} \subset E^\alpha$ is said to be a (C) sequence of φ if $\varphi(u_n)$ is bounded and $(1 + \|u_n\|_\alpha)\|\varphi(u_n)\|_\alpha \to 0$ as $n \to \infty$. The functional φ satisfies condition (C) if every (C) sequence of φ has a convergent subsequence. This condition is due to Cerami, 1978.

For the superquadratic case, we make the following assumptions:

(A3) $\lim\limits_{|x|\to 0} \dfrac{F(t,x)}{|x|^2} = 0$, $\liminf\limits_{|x|\to\infty} \dfrac{F(t,x)}{|x|^2} \geq L > \dfrac{\pi^2}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)T^{2\alpha}(3-2\alpha)}$ uniformly for some $L > 0$ and a.e. $t \in [0,T]$;

(A4) $\limsup\limits_{|x|\to+\infty} \dfrac{F(t,x)}{|x|^r} \leq M < +\infty$ uniformly for some $M > 0$ and a.e. $t \in [0,T]$;

(A5) $\liminf\limits_{|x|\to+\infty} \dfrac{(\nabla F(t,x),x) - 2F(t,x)}{|x|^\mu} \geq Q > 0$ uniformly for some $Q > 0$ and a.e. $t \in [0,T]$, where $r > 2$ and $\mu > r - 2$.

We will first establish the following lemma.

Lemma 5.16. Assume (A), (A4), (A5) hold, then the functional φ satisfies condition (C).

Proof. Let $\{u_n\} \subset E^\alpha$ is a (C) sequence of φ, that is $\varphi(u_n)$ is bounded and $(1 + \|u_n\|_\alpha)\|\varphi'(u_n)\|_\alpha \to 0$ as $n \to \infty$. Then there exists M_0 such that

$$|\varphi(u_n)| \leq M_0 \quad \text{and} \quad (1 + \|u_n\|_\alpha)\|\varphi'(u_n)\|_\alpha \leq M_0 \tag{5.43}$$

for all $n \in \mathbb{N}$.

By (A4), there exist positive constants B_1 and M_1 such that

$$F(t,x) \leq B_1|x|^r \tag{5.44}$$

for all $|x| \geq M_1$ and a.e. $t \in [0,T]$.

It follows from (A) that

$$|F(t,x)| \leq \max_{s\in[0,\,M_1]} a(s)b(t)$$

for all $|x| \leq M_1$ and a.e. $t \in [0,T]$. Therefore, we obtain

$$F(t,x) \leq B_1|x|^r + \max_{s\in[0,\,M_1]} a(s)b(t) \tag{5.45}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Combining (5.23) and (5.45), we get

$$\begin{aligned} \frac{|\cos(\pi\alpha)|}{2}\|u_n\|_\alpha^2 &\leq \varphi(u_n) + \int_0^T F(t,u_n(t))dt \\ &\leq M_0 + \max_{s\in[0,M_1]} a(s)\int_0^T b(t)dt + B_1\int_0^T |u_n(t)|^r dt. \end{aligned} \tag{5.46}$$

On the other hand, by (A5), there exist $\eta > 0$ and $M_2 > 0$ such that

$$\big(\nabla F(t,x),x\big) - 2F(t,x) \geq \eta|x|^\mu \tag{5.47}$$

for a.e. $t \in [0,T]$ and $|x| \geq M_2$.

By (A), we have

$$|\left(\nabla F(t,x),x\right) - 2F(t,x)| \leq (2 + M_2)\max_{s\in[0,\,M_2]} a(s)b(t)$$

for all $|x| \leq M_2$ and a.e. $t \in [0,T]$.

Therefore, we obtain

$$(\nabla F(t,x),x)-2F(t,x)\geq \eta|x|^{\mu}-(2+M_2)\max_{s\in[0,\,M_2]}a(s)b(t) \tag{5.48}$$

for all $x\in\mathbb{R}^N$ and a.e. $t\in[0,T]$.

It follows from (5.43) and (5.48) that

$$\begin{aligned}
3M_0 &\geq 2\varphi(u_n)-\langle\varphi'(u_n),u_n\rangle\\
&=2\int_0^T\big[-\frac{1}{2}({}_0^CD_t^{\alpha}u_n(t),\,{}_t^CD_T^{\alpha}u_n(t))-F(t,u_n(t))\big]dt\\
&\quad-\int_0^T\big[-({}_0^CD_t^{\alpha}u_n(t),{}_t^CD_T^{\alpha}u_n(t))-(\nabla F(t,u_n(t)),u_n(t))\big]dt\\
&=\int_0^T\big[\big(\nabla F(t,u_n(t)),u_n(t)\big)-2F(t,u_n(t))\big]dt\\
&\geq \eta\int_0^T|u_n(t)|^{\mu}dt-(2+M_2)\max_{s\in[0,\,M_2]}a(s)\int_0^T b(t)dt,
\end{aligned}$$

thus, $\int_0^T|u_n(t)|^{\mu}dt$ is bounded.

If $\mu>r$, then

$$\int_0^T|u_n(t)|^r dt\leq T^{\frac{\mu-r}{\mu}}\Big(\int_0^T|u_n(t)|^{\mu}dt\Big)^{r/\mu},$$

which combining (5.46) implies that $\|u_n\|_{\alpha}$ is bounded.

If $\mu\leq r$, then

$$\int_0^T|u_n(t)|^r dt\leq\|u_n\|_{\infty}^{r-\mu}\int_0^T|u_n(t)|^{\mu}dt\leq C_1^{r-\mu}\|u_n\|_{\alpha}^{r-\mu}\int_0^T|u_n(t)|^{\mu}dt,$$

where

$$C_1:=\frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$$

by (5.15).

Since $\mu>r-2$, it follows from (5.46) that $\|u_n\|_{\alpha}$ is bounded too. Thus $\|u_n\|_{\alpha}$ is bounded in E^{α}.

By Proposition 5.6, the sequence $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, such that

$$u_n\rightharpoonup u\ \text{ in }\ E^{\alpha}\quad\text{and}\ \ u_n\to u\ \text{ in }\ C([0,T],\mathbb{R}^N). \tag{5.49}$$

Then we obtain $u_n\to u$ in E^{α} by use of the same argument of Theorem 5.14. The proof of Lemma 5.16 is completed. □

We state our first existence result as follows.

Theorem 5.17. Assume that (A3)-(A5) hold and that $F(t,x)$ satisfies the condition (A). Then BVP (5.1) has at least one solution on E^{α}.

Proof. By (A3), there exist $\epsilon_1 \in (0, |\cos(\pi\alpha)|)$ and $\delta > 0$ such that

$$F(t,x) \leq (|\cos(\pi\alpha)| - \epsilon_1)\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}}|x|^2$$

for a.e. $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| \leq \delta$.

Let

$$\rho = \frac{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}{T^{\alpha-\frac{1}{2}}}\,\delta \quad \text{and} \quad \sigma = \frac{\epsilon_1 \rho^2}{2} > 0.$$

Then it follows from (5.15) that

$$\|u\| \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2(\alpha-1)+1)^{\frac{1}{2}}}\|u\|_\alpha = \delta$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$.

Therefore, we have

$$\begin{aligned}
\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\left({}_0^CD_t^\alpha u(t), {}_t^CD_T^\alpha u(t)\right) - F(t,u(t))\Big] dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - (|\cos(\pi\alpha)| - \epsilon_1)\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}}\int_0^T |u(t)|^2 dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{(|\cos(\pi\alpha)| - \epsilon_1}{2}\|u\|_\alpha^2 \\
&= \frac{\epsilon_1}{2}\|u\|_\alpha^2 \\
&= \sigma
\end{aligned} \tag{5.50}$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. This implies that (ii) in Theorem 1.51 is satisfied.

It is obvious from the definition of φ and (A3) that $\varphi(0) = 0$, and therefore, it suffices to show that φ satisfies (iii) in Theorem 1.51.

By (A3), there exist $\epsilon_2 > 0$ and $M_3 > 0$ such that

$$F(t,x) > \left(\frac{\pi^2}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)T^{2\alpha}(3-2\alpha)} + \epsilon_2\right)|x|^2 \tag{5.51}$$

for all $|x| \geq M_3$ and a.e. $t \in [0,T]$.

It follows from (A) that

$$|F(t,x)| \leq \max_{s\in[0,\, M_3]} a(s)b(t),$$

for all $|x| \leq M_3$ and a.e. $t \in [0,T]$.

Therefore, we obtain

$$\begin{aligned}
F(t,x) \geq &\left(\frac{\pi^2}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)T^{2\alpha}(3-2\alpha)} + \epsilon_2\right)(|x|^2 - M_3^2) \\
&- \max\nolimits_{s\in[0,\, M_3]} a(s)b(t)
\end{aligned} \tag{5.52}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$.

Choosing $u_0 = \left(\frac{T}{\pi}\sin\frac{\pi t}{T}, 0, ..., 0\right) \in E^\alpha$, then

$$\|u_0\|_{L^2}^2 = \frac{T^3}{2\pi^2} \quad \text{and} \quad \|u_0\|_\alpha^2 \le \frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)}. \tag{5.53}$$

For $\varsigma > 0$ and noting that (5.52) and (5.53), we have

$$\begin{aligned}
\varphi(\varsigma u_0) &= \int_0^T \Big[-\frac{1}{2}\left({}_0^C D_t^\alpha \varsigma u_0(t), {}_t^C D_T^\alpha \varsigma u_0(t)\right) - F(t, \varsigma u_0(t))\Big]dt \\
&\le \frac{\varsigma^2}{2|\cos(\pi\alpha)|}\|u_0\|_\alpha^2 \\
&\quad -\left(\frac{\varsigma^2\pi^2}{|\cos(\pi\alpha)|T^{2\alpha}\Gamma^2(2-\alpha)(3-2\alpha)} + \varsigma^2\epsilon_2\right)\int_0^T |u_0(t)|^2 dt + C_2 \qquad (5.54)\\
&\le \frac{\varsigma^2}{2|\cos(\pi\alpha)|}\cdot\frac{T^{3-2\alpha}}{\Gamma^2(2-\alpha)(3-2\alpha)} \\
&\quad -\frac{\varsigma^2\pi^2}{|\cos(\pi\alpha)|T^{2\alpha}\Gamma^2(2-\alpha)(3-2\alpha)}\cdot\frac{T^3}{2\pi^2} - \frac{\varsigma^2\epsilon_2 T^3}{2\pi^2} + C_2 \\
&\to -\infty
\end{aligned}$$

as $\varsigma \to \infty$, where C_2 is a positive constant. Then there exists a sufficiently large ς_0 such that $\varphi(\varsigma_0 u_0) \le 0$. Hence (iii) in Theorem 1.51 holds.

Finally, noting that $\varphi(0) = 0$ while for critical point u, $\varphi(u) \ge \sigma > 0$. Hence u is a nontrivial solution of BVP (5.1), and this completes the proof. □

We give an example to illustrate our results.

Example 5.18. In BVP (5.1), let

$$F(t, x) = \ln(1 + 2|x|^2)|x|^2.$$

These show that all conditions of Theorem 5.17 are satisfied, where

$$r = 2.5, \quad \mu = 2.$$

By Theorem 5.17, BVP (5.1) has at least one solution $u \in E^\alpha$.

5.2.6 *Asymptotically Quadratic Case*

For the asymptotically quadratic case, we assume:

(A4)′ $\limsup_{|x|\to+\infty} \frac{F(t,x)}{|x|^2} \le M < +\infty$ uniformly for some $M > 0$ and a.e. $t \in [0, T]$;

(A6) there exists $\tau(t) \in L^1([0, T], \mathbb{R}^+)$ such that $(\nabla F(t, x), x) - 2F(t, x) \ge \tau(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$;

(A7) $\lim_{|x|\to+\infty} [(\nabla F(t, x), x) - 2F(t, x)] = +\infty$ for a.e. $t \in [0, T]$.

Theorem 5.19. Assume that $F(t, x)$ satisfies (A), (A3), (A4)′, (A6) and (A7). Then BVP (5.1) has at least one solution on E^α.

The following lemmas are needed in the proof of Theorem 5.19.

Lemma 5.20. Assume that (A7) holds. Then for any $\varepsilon > 0$, there exists a subset $E_\varepsilon \subset [0,T]$ with $\alpha([0,T]\backslash E_\varepsilon) < \varepsilon$ such that

$$\lim_{|x|\to\infty}[(\nabla F(t,x),x) - 2F(t,x)] = +\infty$$

uniformly for $t \in E_\varepsilon$.

The proof is similar to that of Lemma 2 in Tang and Wu, 2001, and is omitted.

Lemma 5.21. Assume that (A), (A4)$'$, (A6) and (A7) hold. Then the functional φ satisfies condition (C).

Proof. Suppose that $\{u_n\} \subset E^\alpha$ is a (C) sequence of φ, that is $\varphi(u_n)$ is bounded and $(1+\|u_n\|_\alpha)\|\varphi'(u_n)\|_\alpha \to 0$ as $n \to \infty$. Then we have

$$\liminf_{n\to\infty}[\langle\varphi'(u_n),u_n\rangle - 2\varphi(u_n)] > -\infty,$$

which implies that

$$\limsup_{n\to\infty}\int_0^T [(\nabla F(t,u_n),\ u_n) - 2F(t,u_n)]dt < +\infty. \tag{5.55}$$

We only need to show that $\{u_n\}$ is bounded in E^α. If $\{u_n\}$ is unbounded, we may assume, without loss of generality, that $\|u_n\|_\alpha \to \infty$ as $n \to \infty$. Put $z_n = \dfrac{u_n}{\|u_n\|_\alpha}$, we then have $\|z_n\|_\alpha = 1$. Going to a sequence if necessary, we assume that $z_n \rightharpoonup z$ in E^α, $z_n \to z$ in $C([0,T],\mathbb{R}^N)$ and $L^2([0,T],\mathbb{R}^N)$.

By (A2)$'$, it follows that there exist constants $B_2 > 0$ and $M_4 > 0$ such that

$$F(t,x) \le B_2|x|^2$$

for all $|x| \ge M_4$ and a.e. $t \in [0,T]$.

By assumption (A), it follows that

$$|F(t,x)| \le \max_{s\in[0,\,M_4]} a(s)b(t)$$

for all $|x| \le M_4$ and a.e. $t \in [0,T]$. Therefore, we obtain

$$F(t,x) \le B_2|x|^2 + \max_{s\in[0,\,M_4]} a(s)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Therefore, we have

$$\begin{aligned}\varphi(u) &= \int_0^T \Big[-\frac{1}{2}\left({}_0^cD_t^\alpha u(t),\ {}_t^cD_T^\alpha u(t)\right) - F(t,u(t))\Big]dt \\ &\ge \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - B_2\int_0^T |u|^2dt - \max_{s\in[0,\,M_4]} a(s)\int_0^T b(t)dt,\end{aligned} \tag{5.56}$$

from which, it follows that

$$\frac{\varphi(u_n)}{\|u_n\|_\alpha^2} \ge \frac{|\cos(\pi\alpha)|}{2} - B_2\|z_n\|_{L_2}^2 - \frac{1}{\|u_n\|_\alpha^2}\max_{s\in[0,\,M_4]} a(s)\int_0^T b(t)dt.$$

Passing to the limit in the last inequality, we get

$$\frac{|\cos(\pi\alpha)|}{2} - B_2\|z\|_{L_2}^2 \le 0,$$

which yields $z \ne 0$. Therefore, there exists a subset $E \subset [0,T]$ with $\alpha(E) > 0$ such that $z(t) \ne 0$ on E.

By virtue of Lemma 5.20, for $\varepsilon = \dfrac{1}{2}\alpha(E) > 0$, we can choose a subset $E_\varepsilon \subset [0,T]$ with $\alpha([0,T]\backslash E_\varepsilon) < \varepsilon$ such that

$$\lim_{|x|\to\infty}[(\nabla F(t,x),x) - 2F(t,x)] = +\infty \tag{5.57}$$

uniformly for $t \in E_\varepsilon$.

We assert that $\alpha(E\bigcap E_\varepsilon) > 0$. If not, $\alpha(E\bigcap E_\varepsilon) = 0$.

Since $E = (E\bigcap E_\varepsilon)\bigcup(E\backslash E_\varepsilon)$, it follows that

$$\begin{aligned} 0 < \alpha(E) &= \alpha(E\textstyle\bigcap E_\varepsilon) + \alpha(E\backslash E_\varepsilon) \\ &\le \alpha([0,T]\backslash E_\varepsilon) \\ &< \varepsilon = \tfrac{1}{2}\alpha(E), \end{aligned}$$

which leads to a contradiction and establishes the assertion.

By (A6), we obtain

$$\begin{aligned} &\int_0^T [(\nabla F(t,u_n),u_n) - 2F(t,u_n)]dt \\ &= \int_{E\bigcap E_\varepsilon} [(\nabla F(t,u_n),u_n) - 2F(t,u_n)]dt \\ &\quad + \int_{[0,T]\backslash(E\bigcap E_\varepsilon)} [(\nabla F(t,u_n),u_n) - 2F(t,u_n)]dt \\ &\ge \int_{E\bigcap E_\varepsilon} [(\nabla F(t,u_n),u_n) - 2F(t,u_n)]dt - \int_0^T |\tau(t)|dt. \end{aligned} \tag{5.58}$$

By (5.57), (5.58) and Fatou's lemma, it follows that

$$\lim_{n\to\infty}\int_0^T [(\nabla F(t,u_n),u_n) - 2F(t,u_n)]dt = +\infty,$$

which contradicts (5.55). This contradiction shows that $\|u_n\|_\alpha$ is bounded in E^α and this completes the proof. □

Theorem 5.22. Assume that $F(t,x)$ satisfies (A), (A3), (A4)$'$ and the following conditions:

(A6)$'$ there exists $\tau(t) \in L^1(0,T;\mathbb{R}^+)$ such that $(\nabla F(t,x),x) - 2F(t,x) \le \tau(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(A7)$'$ $\lim\limits_{|x|\to+\infty}[(\nabla F(t,x),x) - 2F(t,x)] = -\infty$ for a.e. $t \in [0,T]$.

Then BVP (5.1) has at least one solution on E^α.

By virtue of Lemma 5.20 and Lemma 5.21, similar to Theorem 5.17, we can complete the proof of Theorem 5.19 by using the similar proof of Theorem 5.17. Theorem 5.22 can be proved similarly.

We give an example to illustrate our results.

Example 5.23. In BVP (5.1), let $T = 2\pi$ and $F(t,x) = \kappa f(x)(2+\sin t)\arctan|x|^2$, where $\kappa > 0$ and $f(x)$ will be specified below.

Let $f(x) = |x|^2 + \ln(1+|x|^2)$. Noting that $0 \le \ln(1+|x|^2) \le |x|^2$, we see that (A) and (A4)$'$ hold. It is also easy to see that (A3) hold for

$$\kappa > \frac{(2\pi)^{1-2\alpha}}{|\cos(\pi\alpha)|\Gamma^2(2-\alpha)(3-2\alpha)}.$$

Furthermore, we have

$$(\nabla f(x), x) - 2f(x) = \frac{2|x|^2}{1+|x|^2} - 2\ln(1+|x|^2) \to -\infty$$

as $|x| \to +\infty$. Therefore, we have

$$\begin{aligned}
&(\nabla F(t,x), x) - 2F(t,x)\\
={}& \kappa\frac{2|x|^2}{1+|x|^4} f(x)(2+\sin t) + \kappa[(\nabla f(x), x) - 2f(x)](2+\sin t)\arctan|x|^2\\
\to{}& -\infty
\end{aligned}$$

uniformly for all $t \in [0, 2\pi]$ as $|x| \to +\infty$. Thus (A6)$'$ and (A7)$'$ hold. By virtue of Theorem 5.22, we conclude that BVP (5.1) has at least one solution on E^α.

If $f(x) = |x|^2 - \ln(1+|x|^2)$, then exact the same conclusions as above hold true by Theorem 5.19.

5.3 Multiple Solutions for BVP with Parameters

5.3.1 *Introduction*

In this section, we study the existence of three solutions to fractional BVP of the form

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \lambda\nabla F(t, u(t)) = 0, \quad t \in [0,T],\\ u(0) = u(T) = 0, \end{cases} \tag{5.59}$$

where $T > 0$, $\lambda > 0$ is a parameter, $0 \le \beta < 1$, ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order β, respectively, $N \ge 1$ is an integer, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function such that $F(t, \mathbf{x})$ is measurable in t for each $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and continuously differentiable in $\mathbf{x}$ for a.e. $t \in [0,T]$, $F(t, 0, \ldots, 0) \equiv 0$ on $[0,T]$, and $\nabla F(t, \mathbf{x}) = (\partial F/\partial x_1, \ldots, \partial F/\partial x_N)$ is the gradient of F at $\mathbf{x}$. By a solution of (5.59), we mean an absolutely continuous function $u : [0,T] \to \mathbb{R}^N$ such that $u(t)$ satisfies both equation for a.e. $t \in [0,T]$ and the boundary conditions in (5.59). We notice that when $\beta = 0$, problem (5.59) has the form

$$\begin{cases} u''(t) + \lambda\nabla F(t, u(t)) = 0, \quad t \in [0,T],\\ u(0) = u(T) = 0, \end{cases} \tag{5.60}$$

which has been extensively studied.

The equation in (5.59) is motivated by the steady fractional advection dispersion equation studied in Ervin and Roop, 2006,

$$-D\,a\,(p\,{}_0D_t^{-\beta} + q\,{}_tD_T^{-\beta})Du + b(t)Du + c(t)u = f, \tag{5.61}$$

where D represents a single spatial derivative, $0 \le p, q \le 1$ satisfying $p + q = 1$, $a > 0$ is a constant, and b, c, f are functions satisfying some suitable conditions. The interest in (5.61) arises from its application as a model for physical phenomena exhibiting anomalous diffusion; i.e., diffusion not accurately modeled by the usual advection dispersion equation. Anomalous diffusion has been used in modeling turbulent flow (see, Carreras, Lynch and Zaslavsky, 2001; Shlesinger, West and Klafter, 1987), and chaotic dynamics of classical conservative systems (see, Zaslavsky, Stevens and Weitzner, 1993). The reader may find more background information and applications on (5.61) in Benson, Wheatcraft and Meerschaert, 2000a; Ervin and Roop, 2006.

Remark 5.24. When $N = 1$, problem (5.59) reduces to the scalar BVP

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}{}_tD_T^{-\beta}(u'(t))\Big) + \lambda f(t, u(t)) = 0, \quad t \in [0, T], \\ u(0) = u(T) = 0, \end{cases} \tag{5.62}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that $f(t, x)$ is measurable in t for each $x \in \mathbb{R}$ and continuous in x for a.e. $t \in [0, T]$.

It is clear that the equation in (5.62) is of the special form of (5.61) with $D = d/dt$, $a = 1$, $p = q = \frac{1}{2}$, $b(t) = c(t) = 0$, and $f = \lambda f(t, u)$.

We also notice that since (5.61) is the steady fractional advection dispersion equation, it has no dependence on the time variable and it just depends on the space variable t (here, the notation t stands for the space variable in (5.61)). Since the space we studied is one dimensional and has the form of an interval, say $[0, T]$, the boundary conditions in the space reduce to the conditions at the two endpoints $t = 0$ and $t = T$ of the interval. In our system, we study the Dirichlet type boundary conditions.

5.3.2 *Existence*

For $0 \le \beta < 1$ given in (5.59), let $\alpha = 1 - \frac{\beta}{2} \in (\frac{1}{2}, 1]$ and define

$$\rho_\alpha = \frac{16N}{T^2\Gamma^2(2-\alpha)}\left(\frac{1}{3-2\alpha}\Big(\frac{T}{4}\Big)^{3-2\alpha} + \int_{T/4}^{3T/4} g^2(t)dt + \int_{3T/4}^{T} h^2(t)dt\right), \tag{5.63}$$

where

$$g(t) = t^{1-\alpha} - (t - T/4)^{1-\alpha}, \tag{5.64}$$

$$h(t) = t^{1-\alpha} - (t - T/4)^{1-\alpha} - (t - 3T/4)^{1-\alpha}. \tag{5.65}$$

In the remainder of this section, for some $c, d, l, m, p \in \mathbb{R}$, let the bold letters $\mathbf{c}$, $\mathbf{d}$, $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{p}$ be the constant vectors in $\mathbb{R}^N$ defined by

$$\mathbf{c} = (c, ..., c), \quad \mathbf{d} = (d, ..., d), \quad \mathbf{l} = (l, ..., l), \quad \mathbf{m} = (m, ..., m), \quad \mathbf{p} = (p, ..., p),$$

and any other bold letter, such as $\mathbf{x}$, is used to denote an arbitrary vector in $\mathbb{R}^N$.

Let E^α be the space of functions $u \in L^2([0,T], \mathbb{R}^N)$ having an α-order Caputo fractional derivatives ${}_0^C D_t^\alpha u \in L^2([0,T], \mathbb{R}^N)$ and $u(0) = u(T) = 0$. Then, by Remark 5.3 (i) and Proposition 5.4, E^α is a reflexive and separable Banach space with the norm

$$\|u\|_\alpha = \Big(\int_0^T |u(t)|^2 dt + \int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}} \quad \text{for any } u \in E^\alpha.$$

We see that the norm $\|u\|_\alpha$ is equivalent to the norm defined as the follow

$$\|u\|_\alpha = \Big(\int_0^T |{}_0^C D_t^\alpha u(t)|^2 dt \Big)^{\frac{1}{2}} \quad \text{for any } u \in E^\alpha.$$

We recall the norms

$$\|u\|_{L^2} = \Big(\int_0^T |u(t)|^2 dt \Big)^{\frac{1}{2}} \quad \text{and} \quad \|u\| = \max_{t \in [0,T]} |u(t)|.$$

For $u \in E^\alpha$, let the functionals Φ and Ψ be defined as follows

$$\Phi(u) = -\frac{1}{2} \int_0^T \big({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t) \big) dt, \tag{5.66}$$

$$\Psi(u) = \int_0^T F(t, u(t)) dt. \tag{5.67}$$

Then, by Theorem 5.10, we see that Φ and Ψ are continuously differentiable, and for any $u, v \in E^\alpha$, we have

$$\langle \Phi'(u), v \rangle = -\frac{1}{2} \int_0^T \big[\big({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha v(t) \big) + \big({}_t^C D_T^\alpha u(t), {}_0^C D_t^\alpha v(t) \big) \big] dt, \tag{5.68}$$

$$\langle \Psi'(u), v \rangle = \int_0^T \big(\nabla F(t, u(t)), v(t) \big) dt. \tag{5.69}$$

Parts (i) and (ii) of Lemma 5.25 below are taken from Lemma 5.12 and Theorem 5.11, respectively.

Lemma 5.25. We have that

(i) The functional Φ is convex and continuous on E^α.
(ii) If $u \in E^\alpha$ is a critical point of the functional $\Phi - \lambda\Psi$, then u is a solution of BVP (5.59).

We now state the results of this section.

Theorem 5.26. Assume that there exist four positive constants c, d, l and m, with

$$d < m \quad \text{and } c < \frac{T^{\alpha-\frac{1}{2}}\rho_\alpha^{\frac{1}{2}} d}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}} < |\cos(\pi\alpha)|l < |\cos(\pi\alpha)|m, \tag{5.70}$$

such that

$$F(t,\mathbf{x}) \geq 0, \quad \text{for } (t,\mathbf{x}) \in [0,T] \times [-m,m]^N, \tag{5.71}$$

$$\max_{|\mathbf{x}|\leq c} F(t,\mathbf{x}) \leq F(t,\mathbf{c}), \quad \max_{|\mathbf{x}|\leq l} F(t,\mathbf{x}) \leq F(t,\mathbf{l}), \quad \max_{|\mathbf{x}|\leq m} F(t,\mathbf{x}) \leq F(t,\mathbf{m}), \tag{5.72}$$

$$\frac{\int_0^T F(t,\mathbf{c})dt}{c^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big), \tag{5.73}$$

$$\frac{\int_0^T F(t,\mathbf{l})dt}{l^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big), \tag{5.74}$$

$$\frac{\int_0^T F(t,\mathbf{m})dt}{m^2-l^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big). \tag{5.75}$$

Then, for each $\lambda \in (\underline{\lambda}, \bar{\lambda})$, the system (5.59) has at least three solutions u_1, u_2 and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < c$, $\max_{t\in[0,T]} |u_2(t)| < l$, and $\max_{t\in[0,T]} |u_3(t)| < m$, where

$$\underline{\lambda} = \frac{\rho_\alpha d^2}{2|\cos(\pi\alpha)|\big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\big)} \tag{5.76}$$

and

$$\begin{aligned}\bar{\lambda} = \min\Big\{&\frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|c^2}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{c})dt}, \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|l^2}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{l})dt}, \\ &\frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|(m^2-l^2)}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{m})dt}\Big\}.\end{aligned} \tag{5.77}$$

Proof. For any $x \in \mathbb{R}$, let $p(x) = \max\{-m, \min\{x,m\}\}$. For any $\mathbf{x} = (x_1,\ldots,x_N) \in E^\alpha$, let $\tilde{F}(t,\mathbf{x}) = F(t,\tilde{\mathbf{x}})$, where $\tilde{\mathbf{x}} = (p(x_1),\ldots,p(x_N))$. Then, $\tilde{F}(t,\mathbf{x})$ is measurable in t for each $\mathbf{x} \in \mathbb{R}^N$ and continuously differentiable in $\mathbf{x}$ for a.e. $t \in [0,T]$, and $\tilde{F}(t,0,\ldots,0) = 0$ on $[0,T]$. Note that $-m \leq p(u_i) \leq m$ for any $u = (u_1,\ldots,u_N) \in E^\alpha$ and $i = 1,\ldots,N$. Then, (5.71) implies that

$$\tilde{F}(t,u) \geq 0, \quad \text{for } (t,u) \in [0,T] \times E^\alpha. \tag{5.78}$$

Note that $d < m$ and $c < l < m$ by (5.70). Then, we have

$$\begin{aligned}&\tilde{F}(t,\mathbf{x}) = F(t,\mathbf{x}), \quad \text{for } (t,\mathbf{x}) \in [0,T] \times \mathbb{R}^N \quad \text{with } |\mathbf{x}| < m,\\ &\tilde{F}(t,\mathbf{c}) = F(t,\mathbf{c}), \quad \tilde{F}(t,\mathbf{d}) = F(t,\mathbf{d}), \quad \tilde{F}(t,\mathbf{l}) = F(t,\mathbf{l}), \quad \tilde{F}(t,\mathbf{m}) = F(t,\mathbf{m}).\end{aligned} \tag{5.79}$$

Let the continuously differentiable functional Φ be given by (5.66) and the functional $\tilde{\Psi}$ be defined by

$$\tilde{\Psi}(u) = \int_0^T \tilde{F}(t, u(t))dt, \quad \text{for} \quad u \in E^\alpha. \tag{5.80}$$

Then, by Proposition 5.8 and (5.66), we have

$$\frac{1}{2}|\cos(\pi\alpha)|\,\|u\|_\alpha^2 \le \Phi(u) \le \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2, \quad \text{for } u \in E^\alpha. \tag{5.81}$$

Moreover, $\tilde{\Psi}$ is continuously differentiable, and for any $u, v \in E^\alpha$, in view of (5.78), we have

$$\tilde{\Psi}(u) \ge 0 \quad \text{and} \quad \langle \tilde{\Psi}'(u), v\rangle = \int_0^T \big(\nabla \tilde{F}(t, u(t)), v(t)\big)dt. \tag{5.82}$$

In the following, we will apply Lemma 1.57 with $X = E^\alpha$ to the functionals Φ and $\tilde{\Psi}$.

We first show that some basic assumptions of Lemma 1.57 are satisfied. The convexity and coercivity of Φ follow from Lemma 5.25 (i) and (5.81), respectively. For any $u, v \in E^\alpha$, from Proposition 5.8 and (5.68),

$$\begin{aligned}
&\langle \Phi'(u) - \Phi'(v), u - v\rangle \\
&= -\frac{1}{2}\int_0^T \Big[\Big({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha (u(t) - v(t))\Big) + \Big({}_t^C D_T^\alpha u(t), {}_0^C D_t^\alpha (u(t) - v(t))\Big)\Big]dt \\
&\quad + \frac{1}{2}\int_0^T \Big[\Big({}_0^C D_t^\alpha v(t), {}_t^C D_T^\alpha (u(t) - v(t))\Big) + \Big({}_t^C D_T^\alpha v(t), {}_0^C D_t^\alpha (u(t) - v(t))\Big)\Big]dt \\
&= -\int_0^T \Big({}_0^C D_t^\alpha (u(t) - v(t)), {}_t^C D_T^\alpha (u(t) - v(t))\Big)dt \\
&\ge |\cos(\pi\alpha)|\,\|u - v\|_\alpha^2.
\end{aligned}$$

Thus, Φ' is uniformly monotone. Hence, by Theorem 26.A (d) in Zeidler, 1990, $(\Phi')^{-1} : (E^\alpha)^* \to E^\alpha$ exists and is continuous. Suppose that $u_n \rightharpoonup u \in E^\alpha$. Then, by Proposition 5.6 $u_n \to u$ in $C([0,T], \mathbb{R}^N)$. Since $\tilde{F}(t, \mathbf{x})$ is continuously differentiable in $\mathbf{x}$ for a.e. $t \in [0, 1]$, from the derivative formula in (5.82), we have $\tilde{\Psi}'(u_n) \to \tilde{\Psi}'(u)$, i.e., $\tilde{\Psi}'$ is strongly continuous. Therefore, $\tilde{\Psi}'$ is a compact operator by Proposition 26.2 in Zeidler, 1990.

Next, note that the facts that $\tilde{F}(t, 0, ..., 0) = 0$ on $[0, T]$ and the inequality in (5.82), from Proposition 5.8, (5.66) and (5.80), we see that conditions (i) and (ii) of Lemma 1.57 are satisfied.

Now, we show that condition (iii) of Lemma 1.57 holds. For $i = 1, ..., N$, let

$$w_i(t) = \begin{cases} \dfrac{4d}{T}t, & t \in [0, T/4), \\ d, & t \in [T/4, 3T/4], \\ \dfrac{4d}{T}(T - t), & t \in (3T/4, T], \end{cases}$$

and $w(t) = (w_1(t), ..., w_N(t))$. Then, $w \in E^\alpha$ and

$$ {}_0^C D_t^\alpha w_i(t) = \frac{4d}{T\Gamma(2-\alpha)} \begin{cases} t^{1-\alpha}, & t \in [0, T/4), \\ g(t), & t \in [T/4, 3T/4], \\ h(t), & t \in (3T/4, T], \end{cases} \tag{5.83} $$

where $g(t)$ and $h(t)$ are defined by (5.64) and (5.65). From (5.63) and (5.83),

$$ \begin{aligned} &\int_0^T |{}_0^C D_t^\alpha w(t)|^2 dt \\ &= N\left(\int_0^T |{}_0^C D_t^\alpha w_1(t)|^2 dt + \int_{T/4}^{3T/4} |{}_0^C D_t^\alpha w_1(t)|^2 dt + \int_{3T/4}^{T} |{}_0^C D_t^\alpha w_1(t)|^2 dt \right) \\ &= \frac{16Nd^2}{T^2\Gamma^2(2-\alpha)} \left(\int_0^{T/4} t^{2-2\alpha} dt + \int_{T/4}^{3T/4} g^2(t) dt + \int_{3T/4}^{T} |h(t)|^2 dt \right) \\ &= \frac{16Nd^2}{T^2\Gamma^2(2-\alpha)} \left(\frac{1}{3-2\alpha} (\frac{T}{4})^{3-2\alpha} + \int_{T/4}^{3T/4} g^2(t) dt + \int_{3T/4}^{T} h^2(t) dt \right) \\ &= \rho_\alpha d^2. \end{aligned} $$

Then, $\|w\|_\alpha^2 = \rho_\alpha d^2$. Thus, from (5.81) with $u = w$,

$$ \frac{1}{2} |\cos(\pi\alpha)| \rho_\alpha d^2 \le \Phi(w) \le \frac{1}{2|\cos(\pi\alpha)|} \rho_\alpha d^2. \tag{5.84} $$

Let

$$ r_1 = \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}{2T^{2\alpha-1}} c^2, \quad r_2 = \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}{2T^{2\alpha-1}} l^2, \tag{5.85} $$

$$ r_3 = \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}{2T^{2\alpha-1}} (m^2 - l^2). \tag{5.86} $$

Then, from (5.70) and (5.84), we have $r_1 < \Phi(w) < r_2$ and $r_3 > 0$. For any $u \in E^\alpha$, from the first inequality in (5.81), we see that $\|u\|_\alpha^2 \le 2\Phi(u)/|\cos(\pi\alpha)|$. Then, by (5.15) and (5.18), we have

$$ \|u\|^2 \le \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)} \|u\|_\alpha^2 \le \frac{2T^{2\alpha-1}\Phi(u)}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|}. $$

Thus, by (5.85) and (5.86), we have the following implications

$$ \begin{aligned} &\Phi(u) < r_1 \Rightarrow \|u\| < c, \\ &\Phi(u) < r_2 \Rightarrow \|u\| < l, \\ &\Phi(u) < r_2 + r_3 \Rightarrow \|u\| < m. \end{aligned} \tag{5.87} $$

This, together with (5.72) and (5.79), implies

$$ \sup_{u \in \Phi^{-1}(-\infty, r_1)} \int_0^T \tilde{F}(t, u(t)) dt \le \int_0^T \max_{|\mathbf{x}| \le c} F(t, \mathbf{x}) dt \le \int_0^T F(t, \mathbf{c}) dt, \tag{5.88} $$

$$ \sup_{u \in \Phi^{-1}(-\infty, r_2)} \int_0^T \tilde{F}(t, u(t)) dt \le \int_0^T \max_{|\mathbf{x}| \le l} F(t, \mathbf{x}) dt \le \int_0^T F(t, \mathbf{l}) dt, $$

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_0^T \tilde{F}(t,u(t))dt \le \int_0^T \max_{|\mathbf{x}|\le m} F(t,\mathbf{x})dt \le \int_0^T F(t,\mathbf{m})dt.$$

Let φ, β, γ and α be defined by (1.46)-(1.49). Then, taking into account the fact that $0\in\Phi^{-1}(-\infty,r_i)$, $i=1,2$, from (5.80), (5.85) and (5.86), it follows that

$$\varphi(r_1) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\tilde{\Psi}(u)}{r_1} \le \frac{2T^{2\alpha-1}\int_0^T F(t,\mathbf{c})dt}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|c^2}, \tag{5.89}$$

$$\varphi(r_2) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\tilde{\Psi}(u)}{r_2} \le \frac{2T^{2\alpha-1}\int_0^T F(t,\mathbf{l})dt}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|l^2}, \tag{5.90}$$

$$\gamma(r_2,r_3) = \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\tilde{\Psi}(u)}{r_3} \le \frac{2T^{2\alpha-1}\int_0^T F(t,\mathbf{m})dt}{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|(m^2-l^2)}. \tag{5.91}$$

On the other hand, in view of the fact that $w(t)=\mathbf{d}<\mathbf{m}$ on $[T/4,3T/4]$ and from (5.78) and (5.79),

$$\int_0^T \tilde{F}(t,w(t))dt \ge \int_{T/4}^{3T/4} \tilde{F}(t,w(t))dt = \int_{T/4}^{3T/4} \tilde{F}(t,\mathbf{d})dt.$$

Note that $w\in\Phi^{-1}[r_1,r_2)$, from (1.47) and (5.88), we obtain

$$\begin{aligned}\beta(r_1,r_2) &\ge \inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{\tilde{\Psi}(w)-\tilde{\Psi}(u)}{\Phi(w)-\Phi(u)} \ge \inf_{u\in\Phi^{-1}(-\infty,r_1)}\frac{\tilde{\Psi}(w)-\tilde{\Psi}(u)}{\Phi(w)}\\ &\ge \frac{\int_{T/4}^{3T/4}\tilde{F}(t,\mathbf{d})dt-\int_0^T\tilde{F}(t,\mathbf{c})dt}{\Phi(w)}.\end{aligned}$$

By (5.84), $1/\Phi(w)\ge 2|\cos(\pi\alpha)|/(\rho_\alpha d^2)$. Then

$$\beta(r_1,r_2) \ge \frac{2|\cos(\pi\alpha)|}{\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4}\tilde{F}(t,\mathbf{d})dt - \int_0^T \tilde{F}(t,\mathbf{c})dt\Big). \tag{5.92}$$

For $\underline{\lambda}$ and $\bar{\lambda}$ defined by (5.76) and (5.77), from (5.73)-(5.75) and (5.89)-(5.92), we have

$$\begin{aligned}\varphi(r_1) &< \frac{1}{\bar{\lambda}} < \frac{1}{\underline{\lambda}} < \beta(r_1,r_2),\\ \varphi(r_2) &< \frac{1}{\bar{\lambda}} < \frac{1}{\underline{\lambda}} < \beta(r_1,r_2),\\ \gamma(r_2,r_3) &< \frac{1}{\bar{\lambda}} < \frac{1}{\underline{\lambda}} < \beta(r_1,r_2).\end{aligned}$$

In view of (1.49), $\alpha(r_1,r_2,r_3)<1/\bar{\lambda}<1/\underline{\lambda}<\beta(r_1,r_2)$; i.e., condition (iii) of Lemma 1.57 holds. Hence, all the assumptions of Lemma 1.57 are satisfied. Then, by Lemma 1.57, for each $\lambda\in\big(\underline{\lambda},\bar{\lambda}\big)$, the functional $\Phi-\lambda\tilde{\Psi}$ has three distinct critical points u_1, u_2 and u_3 such that $u_1\in\Phi^{-1}(-\infty,r_1)$, $u_2\in\Phi^{-1}[r_1,r_2)$, and $u_3\in\Phi^{-1}(-\infty,r_2+r_3)$. From (5.87), we have

$$\|u_1\|<c, \quad \|u_2\|<l, \quad \|u_3\|<m.$$

Then, in view of (5.67), (5.79) and (5.80), we have $\tilde{\Psi}(u)=\Psi(u)$. Therefore, u_1, u_2, and u_3 are three distinct critical points of the functional $\Phi-\lambda\Psi$. Thus, by Proposition 5.25 (ii), u_1, u_2 and u_3 are three distinct solutions of (5.59). This completes the proof of the theorem. □

The following results are consequences of Theorem 5.26. In particular, Corollaries 5.27 and 5.29 give some conditions for the system (5.60) to have at least three solutions, and Corollary 5.28 provide some relatively simpler existence criteria for the system (5.59).

Corollary 5.27. Assume that there exist four positive constants c, d, l and m, with

$$c < (8N)^{\frac{1}{2}} d < l < m,$$

such that (5.71) and (5.72) hold, and

$$\frac{\int_0^T F(t,\mathbf{c})dt}{c^2} < \frac{1}{8Nd^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big),$$
$$\frac{\int_0^T F(t,\mathbf{l})dt}{l^2} < \frac{1}{8Nd^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big),$$
$$\frac{\int_0^T F(t,\mathbf{m})dt}{m^2 - l^2} < \frac{1}{8Nd^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big).$$

Then, for each $\lambda \in (\underline{\lambda}_1, \bar{\lambda}_1)$, system (5.60) has at least three solutions u_1, u_2 and u_3 such that $\max_{t\in[0,T]}|u_1(t)| < c$, $\max_{t\in[0,T]}|u_2(t)| < l$, and $\max_{t\in[0,T]}|u_3(t)| < m$, where

$$\underline{\lambda}_1 = \frac{4Nd^2}{T\big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\big)},$$
$$\bar{\lambda}_1 = \min\Big\{\frac{c^2}{2T\int_0^T F(t,\mathbf{c})dt}, \frac{l^2}{2T\int_0^T F(t,\mathbf{l})dt}, \frac{m^2 - l^2}{2T\int_0^T F(t,\mathbf{m})dt}\Big\}.$$

Proof. When $\alpha = 1$, from (5.63), we have $\rho_\alpha = 8N/T$. Then, under the assumptions of Corollary 5.27, it is easy to see that all the conditions of Theorem 5.26 hold for $\alpha = 1$. Note that the system (5.60) is a special case of the system (5.59) with $\alpha = 1$. The conclusion then follows directly from Theorem 5.26. The proof is completed. $\square$

Corollary 5.28. Assume that there exist three positive constants c, d and p, with

$$d < p \text{ and } c < \frac{T^{\alpha-\frac{1}{2}}\rho_\alpha^{\frac{1}{2}} d}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}} < \frac{|\cos(\pi\alpha)|p}{\sqrt{2}}, \tag{5.93}$$

such that

$$F(t,\mathbf{x}) \geq 0 \quad \text{for}(t,\mathbf{x}) \in [0,T]\times[-p,p]^N, \tag{5.94}$$

$$\max_{|\mathbf{x}|\leq c} F(t,\mathbf{x}) \leq F(t,\mathbf{c}), \quad \max_{|\mathbf{x}|\leq p/\sqrt{2}} F(t,\mathbf{x}) \leq F(t,\frac{\mathbf{p}}{\sqrt{2}}), \ \max_{|\mathbf{x}|\leq p} F(t,\mathbf{x}) \leq F(t,\mathbf{p}), \tag{5.95}$$

$$\frac{\int_0^T F(t,\mathbf{c})dt}{c^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt, \tag{5.96}$$

$$\frac{\int_0^T F(t,\mathbf{p})dt}{p^2} < \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{2T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt. \tag{5.97}$$

Then, for each $\lambda \in (\underline{\lambda}_2, \bar{\lambda}_2)$, system (5.59) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]}|u_1(t)| < c$, $\max_{t\in[0,T]}|u_2(t)| < p/\sqrt{2}$, and $\max_{t\in[0,T]}|u_3(t)| < p$, where

$$\underline{\lambda}_2 = \frac{\rho_\alpha d^2(1+\cos^2(\pi\alpha))}{2|\cos(\pi\alpha)|\int_{T/4}^{3T/4} F(t,\mathbf{d})dt}, \tag{5.98}$$

$$\bar{\lambda}_2 = \min\Big\{\frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|c^2}{2T^{2\alpha-1}\int_0^T F(t,\mathbf{c})dt},\ \frac{\Gamma^2(\alpha)(2\alpha-1)|\cos(\pi\alpha)|p^2}{4T^{2\alpha-1}\int_0^T F(t,\mathbf{p})dt}\Big\}. \tag{5.99}$$

Proof. Let $l = p/\sqrt{2}$ and $m = p$. Then, from (5.93)-(5.95), we see that (5.70)-(5.72) hold. By (5.95) and (5.97), we have

$$\begin{aligned}\frac{\int_0^T F(t,\mathbf{l})dt}{l^2} &= \frac{2\int_0^T F(t,\mathbf{p}/\sqrt{2})dt}{p^2} \le \frac{2\int_0^T F(t,\mathbf{p})dt}{p^2}\\ &< \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt,\end{aligned} \tag{5.100}$$

and

$$\begin{aligned}\frac{\int_0^T F(t,\mathbf{m})dt}{m^2-l^2} &= \frac{2\int_0^T F(t,\mathbf{p})dt}{p^2}\\ &< \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt.\end{aligned} \tag{5.101}$$

Note from (5.93) it follows that

$$\frac{\Gamma^2(\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2} < \frac{1}{c^2}.$$

Combining this inequality with (5.96), we obtain

$$\begin{aligned}&\frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\Big(\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \int_0^T F(t,\mathbf{c})dt\Big)\\ &> \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt - \frac{\cos^2(\pi\alpha)}{c^2}\int_0^T F(t,\mathbf{c})dt\\ &> \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt\\ &\quad - \frac{\Gamma^2(\alpha)\cos^4(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{c})dt\\ &= \frac{\Gamma^2(\alpha)\cos^2(\pi\alpha)(2\alpha-1)}{T^{2\alpha-1}\rho_\alpha d^2(1+\cos^2(\pi\alpha))}\int_{T/4}^{3T/4} F(t,\mathbf{d})dt.\end{aligned} \tag{5.102}$$

By (5.96) and (5.100)-(5.102), we see that (5.73)-(5.74) hold. From (5.76), (5.77), (5.98), (5.99) and (5.102), we have $\underline{\lambda} < \underline{\lambda}_2$ and $\bar{\lambda} = \bar{\lambda}_2$. Therefore, the conclusion now follows from Theorem 5.26. The proof is completed. □

Corollary 5.29. Assume that there exist three positive constants c, d and p, with

$$c < (8N)^{\frac{1}{2}} d < \frac{p}{\sqrt{2}}, \tag{5.103}$$

such that (5.94) and (5.95) hold, and

$$\frac{\int_0^T F(t, \mathbf{c})dt}{c^2} < \frac{1}{16Nd^2} \int_{T/4}^{3T/4} F(t, \mathbf{d})dt, \tag{5.104}$$

and

$$\frac{\int_0^T F(t, \mathbf{p})dt}{p^2} < \frac{1}{32Nd^2} \int_{T/4}^{3T/4} F(t, \mathbf{d})dt. \tag{5.105}$$

Then, for each $\lambda \in (\underline{\lambda}_3, \bar{\lambda}_3)$, system (5.60) has at least three solutions u_1, u_2, and u_3 such that $\max_{t\in[0,T]} |u_1(t)| < c$, $\max_{t\in[0,T]} |u_2(t)| < p/\sqrt{2}$, and $\max_{t\in[0,T]} |u_3(t)| < p$, where

$$\underline{\lambda}_3 = \frac{8Nd^2}{T \int_{T/4}^{3T/4} F(t, \mathbf{d})dt},$$

$$\bar{\lambda}_3 = \min\left\{ \frac{c^2}{2T \int_0^T F(t, \mathbf{c})dt}, \frac{p^2}{4T \int_0^T F(t, \mathbf{p})dt} \right\}.$$

Proof. When $\alpha = 1$, from (5.63), we have $\rho_\alpha = 8N/T$. Under the assumptions of Corollary 5.29, it is easy to see that all the conditions of Corollary 5.28 hold for $\alpha = 1$. Note that system (5.60) is a special case of system (5.59) with $\alpha = 1$. The conclusion then follows directly from Corollary 5.28. The proof is completed. $\square$

Remark 5.30. We want to point out that when F does not depend on t, (5.104) and (5.105) reduce to

$$\frac{F(\mathbf{c})}{c^2} < \frac{F(\mathbf{d})}{32Nd^2} \quad \text{and} \quad \frac{F(\mathbf{p})}{p^2} < \frac{F(\mathbf{d})}{64Nd^2}, \tag{5.106}$$

and $\underline{\lambda}_3$ and $\bar{\lambda}_3$ become

$$\underline{\lambda}_3 = \frac{16Nd^2}{T^2 F(\mathbf{d})} \quad \text{and} \quad \bar{\lambda}_3 = \min\left\{ \frac{c^2}{2T^2 F(\mathbf{c})}, \frac{p^2}{4T^2 F(\mathbf{p})} \right\}. \tag{5.107}$$

Remark 5.31. We observe that, in our results, no asymptotic condition on F is needed and only local conditions on F are imposed to guarantee the existence of solutions. Moreover, in the conclusions of the above results, one of the three solutions may be trivial since $\nabla F(t, 0, ..., 0)$ may be zero.

In the remainder of this subsection, we give two examples to illustrate the applicability of our results.

Example 5.32. Let $T > 0$. For $(t, x, y) \in [0, T] \times \mathbb{R}^2$, let $F(t, x, y) = tG(x, y)$, where $G : \mathbb{R}^2 \to \mathbb{R}$ satisfies that $G(-x, -y) = G(x, y)$, and that for $x \in [0, \infty)$ and $y \in \mathbb{R}$,

$$G(x, y) = \begin{cases} x^3 + |y|^3, & 0 \le x \le 1,\ 0 \le |y| \le 1, \\ x^3 + 2|y|^{3/2} - 1, & 0 \le x \le 1,\ |y| > 1, \\ 2x^{3/2} + |y|^3 - 1, & x > 1,\ 0 \le |y| \le 1, \\ 2x^{3/2} + 2|y|^{3/2} - 2, & x > 1,\ |y| > 1. \end{cases} \tag{5.108}$$

It is easy to verify that $F : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in t for $(x,y) \in \mathbb{R}^2$ and continuously differentiable in x and y for $t \in [0,T]$, and $F(t,0,0) \equiv 0$ on $[0,T]$.

Let $0 \leq \beta < 1$, $\alpha = 1 - \frac{\beta}{2} \in (\frac{1}{2}, 1]$, ρ_α be defined by (5.63), and $u(t) = (u_1(t), u_2(t))$. We claim that for each

$$\lambda \in \Big(\frac{\rho_\alpha(1+\cos^2(\pi\alpha))}{T^2|\cos(\pi\alpha)|}, \infty\Big),$$

the system

$$\begin{cases} \dfrac{d}{dt}\Big(\dfrac{1}{2}\, {}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\, {}_tD_T^{-\beta}(u'(t))\Big) + \lambda \nabla F(t,u(t)) = 0, \quad t \in [0,T], \\ u(0) = u(T) = 0 \end{cases} \tag{5.109}$$

has at least three solutions.

In fact, system (5.109) is a special case of system (5.59) with $N = 2$. For $0 < c < 1$ and $p > 1$, in view of (5.108), we have

$$\frac{\int_0^T F(t,c,c)dt}{c^2} = \frac{2c^3 \int_0^T tdt}{c^2} = T^2 c, \tag{5.110}$$

$$\frac{\int_0^T F(t,p,p)dt}{p^2} = \frac{(4p^{3/2}-2)\int_0^T tdt}{p^2} = \frac{T^2(2p^{3/2}-1)}{p^2}. \tag{5.111}$$

Choose $d = 1$. Then,

$$\int_{T/4}^{3T/4} F(t,d,d)dt = 2\int_{T/4}^{3T/4} tdt = \frac{1}{2}T^2. \tag{5.112}$$

By (5.110)-(5.112), we see that there exist $0 < c^* < 1$ and $p^* > 1$ such that (5.93), (5.96) and (5.97) hold for any $0 < c < c^*$ and $p > p^*$. Moreover, (5.94) and (5.95) hold for any $c, p > 0$. Finally, note from (5.98) and (5.99) that

$$\underline{\lambda}_2 = \frac{\rho_\alpha(1+\cos^2(\pi\alpha))}{T^2|\cos(\pi\alpha)|},$$
$$\bar{\lambda}_2 \to \infty \quad \text{as} \quad c \to 0^+ \quad \text{and} \quad p \to \infty.$$

Then, the claim follows from Corollary 5.28.

Example 5.33. Let $F : \mathbb{R}^2 \to \mathbb{R}$ satisfy that $F(-x,-y) = F(x,y)$, and that for $x \in [0,\infty)$ and $y \in \mathbb{R}$,

$$F(x,y) = \begin{cases} x^3, & 0 \leq x \leq 1,\ 0 \leq |y| \leq 1, \\ x^3 + 2|y|^{3/2} - 3|y| + 1, & 0 \leq x \leq 1,\ |y| > 1, \\ 2x^{3/2} - 1, & x > 1,\ 0 \leq |y| \leq 1, \\ 2x^{3/2} + 2|y|^{3/2} - 3|y|, & x > 1,\ |y| > 1. \end{cases} \tag{5.113}$$

It is easy to verify that $F : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable in x and y and $F(0,0) = 0$.

Let $T > 0$ and $u(t) = (u_1(t), u_2(t))$. We claim that for each $\lambda \in (32/T^2, \infty)$, the system

$$\begin{aligned} &u''(t) + \lambda \nabla F(u(t)) = 0, \quad t \in [0, T], \\ &u(0) = u(T) = 0 \end{aligned} \tag{5.114}$$

has at least three solutions. In fact, the system (5.114) is a special case of the system (5.60) with $N = 2$. For $0 < c < 1$ and $p > 1$, from (5.113), we have

$$\frac{F(c,c)}{c^2} = \frac{c^3}{c^2} = c, \tag{5.115}$$

$$\frac{F(p,p)}{p^2} = \frac{4p^{3/2} - 3p}{p^2} = \frac{4p^{\frac{1}{2}} - 3}{p}. \tag{5.116}$$

Choose $d = 1$. Then

$$\frac{F(d,d)}{32Nd^2} = \frac{1}{64} \quad \text{and} \quad \frac{F(d,d)}{64Nd^2} = \frac{1}{128}. \tag{5.117}$$

By (5.115)-(5.117), we see that there exist $0 < c^* < 1$ and $p^* > 1$ such that (5.103) and (5.106) hold for any $0 < c < c^*$ and $p > p^*$. Moreover, (5.94) and (5.95) hold for any $c, p > 0$. Finally, note from (5.107) that

$$\underline{\lambda}_3 = \frac{32}{T^2} \quad \text{and} \quad \bar{\lambda}_3 \to \infty \quad \text{as} \;\; c \to 0^+ \;\; \text{and} \;\; p \to \infty.$$

Then, the claim follows from Corollary 5.29 and Remark 5.30.

Remark 5.34. As noted in Remark 5.31, one of the three solutions in the conclusions of the above examples may be trivial.

5.4 Infinite Solutions for BVP with Left and Right Fractional Integrals

5.4.1 *Introduction*

In this section, we consider BVP (5.1), i.e.,

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{1}{2}\,{}_0D_t^{-\beta}(u'(t)) + \dfrac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \le \beta < 1$ respectively. Assume that $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the condition (A) which is assumed as in Subsection 5.2.3.

In particular, if $\beta = 0$, BVP (5.1) reduces to the standard second-order BVP.

In the Subsection 5.4.2, using variational methods we prove the multiplicity results for the solutions of problem (5.1).

5.4.2 *Existence*

Making use of the Property 1.17 and Definition 1.8, for any $u \in AC([0,T],\mathbb{R}^N)$, BVP (5.1) is equivalent to (5.21).

In the following, we will treat BVP (5.21) in the Hilbert space $E^\alpha = E_0^{\alpha,2}$ with the corresponding norm $\|u\|_\alpha = \|u\|_{\alpha,2}$.

As E^α is a reflexive and separable Banach space, then there are $e_j \in E^\alpha$ and $e_j^* \in (E^\alpha)^*$ such that

$$E^\alpha = \overline{\mathrm{span}\{e_j : j = 1, 2, ...\}} \quad \text{and} \quad (E^\alpha)^* = \overline{\mathrm{span}\{e_j{}^* : j = 1, 2, ...\}}.$$

For $k = 1, 2, ...$, denote

$$X_j := \mathrm{span}\{e_j\}, \quad Y_k := \bigoplus_{j=1}^{k} X_j, \quad Z_k := \overline{\bigoplus_{j=k}^{\infty} X_j}.$$

Theorem 5.35. Assume that $F(t,x)$ satisfies the condition (A), and suppose the following conditions hold:
(A1) there exist $\kappa > 2$ and $r > 0$ such that

$$\kappa F(t,x) \le (\nabla F(t,x), x)$$

for a.e. $t \in [0,T]$ and all $|x| \ge r$ in $\mathbb{R}^N$;
(A2) there exist positive constants $\mu > 2$ and $Q > 0$ such that

$$\limsup_{|x| \to +\infty} \frac{F(t,x)}{|x|^\mu} \le Q$$

uniformly for a.e. $t \in [0,T]$;
(A3) there exist $\mu' > 2$ and $Q' > 0$ such that

$$\limsup_{|x| \to +\infty} \frac{F(t,x)}{|x|^{\mu'}} \ge Q'$$

uniformly for a.e. $t \in [0,T]$;
(A4) $F(t,x) = F(t,-x)$ for $t \in [0,T]$ and all x in $\mathbb{R}^N$.
Then BVP (5.1) has infinite solutions $\{u_n\}$ on E^α for every positive integer n such that $\|u_n\|_\infty \to \infty$, as $n \to \infty$.

Proof. Let $\{u_n\} \subset E^\alpha$ such that $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$, where $\varphi(u)$ and $\varphi'(u)$ are defined by (5.27) and (5.28) respectively. First we prove $\{u_n\}$ is a bounded sequence, otherwise, $\{u_n\}$ would be unbounded sequence, passing to a subsequence, still denoted by $\{u_n\}$, such that $\|u_n\|_\alpha \ge 1$ and $\|u_n\|_\alpha \to \infty$, as $n \to \infty$.

Noting that

$$\langle \varphi'(u_n), u_n \rangle = -\int_0^T [({}_0^C D_t^\alpha u_n(t), {}_t^C D_T^\alpha u_n(t)) + (\nabla F(t, u_n(t)), u_n(t))]dt.$$

In view of the condition (A1) and (5.23) that

$$\begin{aligned}\varphi(u_n)-\frac{1}{\kappa}\langle\varphi'(u_n),u_n\rangle &= \left(\frac{1}{\kappa}-\frac{1}{2}\right)\int_0^T ({}_0^CD_t^\alpha u_n(t), {}_t^CD_T^\alpha u_n(t))dt\\ &\quad+\int_0^T\left[\frac{1}{\kappa}(\nabla F(t,u_n(t)),u_n(t))-F(t,u_n(t))\right]dt\\ &\ge\left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2\\ &\quad+\left(\int_{\Omega_1}+\int_{\Omega_2}\right)\left[\frac{1}{\kappa}(\nabla F(t,u_n(t)),u_n(t))-F(t,u_n(t))\right]dt\\ &\ge\left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2-C_1,\end{aligned}\tag{5.118}$$

where $\Omega_1:=\{t\in[0,T];|u_n(t)|\le r\},\Omega_2:=[0,T]\backslash\Omega_1$ and C_1 is a positive constant.

Since $\varphi(u_n)$ is bounded, there exists a positive constant C_2, such that $|\varphi(u_n)|\le C_2$. Hence, we have

$$\begin{aligned}C_2\ge\varphi(u_n)&\ge\left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2+\frac{1}{\kappa}\langle\varphi'(u_n),u_n\rangle-C_1\\ &\ge\left(\frac{1}{2}-\frac{1}{\kappa}\right)|\cos(\pi\alpha)|\|u_n\|_\alpha^2-\frac{1}{\kappa}\|\varphi'(u_n)\|_\alpha\|u_n\|_\alpha-C_1,\end{aligned}\tag{5.119}$$

so $\{u_n\}$ is a bounded sequence in E^α by (5.119).

Since E^α is a reflexive space, going to a subsequence if necessary, we may assume that $u_n\rightharpoonup u$ weakly in E^α, thus we have

$$\begin{aligned}\langle\varphi'(u_n)-\varphi'(u),u_n-u\rangle&=\langle\varphi'(u_n),u_n-u\rangle-\langle\varphi'(u),u_n-u\rangle\\ &\le\|\varphi'(u_n)\|_\alpha\|u_n-u\|_\alpha-\langle\varphi'(u),u_n-u\rangle\to 0\end{aligned}\tag{5.120}$$

as $n\to\infty$. Moreover, according to (5.15) and Proposition 5.6, we have u_n is bounded in $C([0,T],\mathbb{R}^N)$ and $\|u_n-u\|\to 0$ as $n\to\infty$.

Observing that

$$\begin{aligned}&\langle\varphi'(u_n)-\varphi'(u),u_n-u\rangle\\ &=-\int_0^T({}_0^CD_t^\alpha(u_n(t)-u(t)),{}_t^CD_T^\alpha(u_n(t)-u(t)))dt\\ &\quad-\int_0^T(\nabla F(t,u_n(t))-\nabla F(t,u(t)),u_n(t)-u(t))dt\\ &\ge|\cos(\pi\alpha)|\|u_n(t)-u(t)\|_\alpha^2\\ &\quad-\left|\int_0^T(\nabla F(t,u_n(t))-\nabla F(t,u(t)))dt\right|\|u_n(t)-u(t)\|.\end{aligned}\tag{5.121}$$

Combining this with (5.120), it is easy to verify that $\|u_n(t)-u(t)\|_\alpha\to 0$ as $n\to\infty$, and hence that $u_n\to u$ in E^α. Thus, $\{u_n\}$ admits a convergent subsequence.

For any $u\in Y_k$, let

$$\|u\|_*:=\left(\int_0^T|u(t)|^{\mu'}dt\right)^{1/\mu'},\tag{5.122}$$

and it is easy to verify that $\|\cdot\|_*$ define by (5.122) is a norm of Y_k. Since all the norms of a finite dimensional normed space are equivalent, so there exists positive constant C_3 such that

$$C_3\|u\|_\alpha \leq \|u\|_* \quad \text{for} \quad u \in Y_k. \tag{5.123}$$

In view of (A3), there exist two positive constants M_1 and C_4 such that

$$F(t,x) \geq M_1|x|^{\mu'} \tag{5.124}$$

for a.e. $t \in [0,T]$ and $|x| \geq C_4$.

It follows from (5.23), (5.123) and (5.124) that

$$\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2}({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt - \int_0^T F(t,u(t))dt \\
&\leq \frac{1}{|2\cos(\pi\alpha)|}\|u\|_\alpha^2 - \int_{\Omega_3} F(t,u(t))dt - \int_{\Omega_4} F(t,u(t))dt \\
&\leq \frac{1}{|2\cos(\pi\alpha)|}\|u\|_\alpha^2 - M_1\int_{\Omega_3} |u(t)|^{\mu'}dt - \int_{\Omega_4} F(t,u(t))dt \\
&= \frac{1}{|2\cos(\pi\alpha)|}\|u\|_\alpha^2 - M_1\int_0^T |u(t)|^{\mu'}dt + M_1\int_{\Omega_4} |u(t)|^{\mu'}dt - \int_{\Omega_4} F(t,u(t))dt \\
&\leq \frac{1}{|2\cos(\pi\alpha)|}\|u\|_\alpha^2 - C_3^{\mu'}M_1\|u\|_\alpha^{\mu'} + C_5,
\end{aligned}$$

where $\Omega_3 := \{t \in [0,T] : \ |u(t)| \geq C_4\}$, $\Omega_4 := [0,T]\backslash\Omega_3$ and C_5 is a positive constant.

Since $\mu' > 2$, then there exist positive constants d_k such that

$$\varphi(u) \leq 0, \quad \text{for } u \in Y_k, \quad \text{and } \|u\|_\alpha \geq d_k. \tag{5.125}$$

For any $u \in Z_k$, let

$$\|u\|_\mu := \left(\int_0^T |u(t)|^\mu dt\right)^{1/\mu} \quad \text{and} \quad \beta_k := \sup_{\substack{u\in Z_k \\ \|u\|_\alpha = 1}} \|u\|_\mu, \tag{5.126}$$

then we conclude $\beta_k \to 0$ as $k \to \infty$.

In fact, it is obvious that $\beta_k \geq \beta_{k+1} > 0$, so $\beta_k \to \beta$. For every $k \in \mathbb{N}$, there exists $u_k \in Z_k$ such that

$$\|u_k\|_\alpha = 1 \quad \text{and} \quad \|u_k\|_\mu > \beta_k/2. \tag{5.127}$$

As E^α is reflexive, $\{u_k\}$ has a weakly convergent subsequence, still denoted by $\{u_k\}$, such that $u_k \rightharpoonup u$. We claim $u = 0$.

In fact, for any $f_m \in \{f_n : n = 1, 2, ...\}$, we have $f_m(u_k) = 0$, when $k > m$, so

$$f_m(u_k) \to 0, \quad \text{as} \quad k \to \infty$$

for any $f_m \in \{f_n : n = 1, 2, ...\}$, therefore $u = 0$.

By Proposition 5.6, when $u_k \rightharpoonup 0$ in E^α, then $u_k \to 0$ strongly in $C([0,T], \mathbb{R}^N)$. So we conclude $\beta = 0$ by (5.127).

In view of (A2), there exist two positive constants M_2 and C_6 such that

$$F(t,x) \leq M_2|x|^\mu \tag{5.128}$$

uniformly for a.e. $t \in [0,T]$ and $|x| \geq C_6$. We have

$$\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2}({}_0^C D_t^\alpha u(t), {}_t^C D_T^\alpha u(t))dt - \int_0^T F(t,u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \int_{\Omega_5} F(t,u(t))dt - \int_{\Omega_6} F(t,u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - M_2\int_{\Omega_5} |u(t)|^\mu dt - \int_{\Omega_6} F(t,u(t))dt \\
&= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - M_2\int_0^T |u(t)|^\mu dt + M_2\int_{\Omega_6} |u(t)|^\mu dt - \int_{\Omega_6} F(t,u(t))dt \\
&\geq \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - M_2{\beta_k}^\mu\|u\|_\alpha^\mu - C_7,
\end{aligned}$$

where $\Omega_5 := \{t \in [0,T] : |u(t)| \geq C_6\}$, $\Omega_6 := [0,T]\backslash\Omega_5$ and C_7 is a positive constant.

Choosing $r_k = 1/\beta_k$, it is obvious that $r_k \to \infty$ as $k \to \infty$, then

$$b_k := \inf_{\substack{u\in Z_k \\ \|u\|_\alpha=\rho_k}} \varphi(u) \to \infty \quad \text{as} \quad k \to \infty, \tag{5.129}$$

that is, the condition (H3) in Theorem 1.54 is satisfied.

In view of (5.125), let $\rho_k := \max\{d_k, r_k + 1\}$, then

$$a_k := \max_{\substack{u\in Y_k \\ \|u\|_\alpha=\rho_k}} \varphi(u) \leq 0,$$

and this shows the condition of (H2) in Theorem 1.54 is satisfied.

We have proved the functional φ satisfies all the conditions of Theorem 1.54, then φ has an unbounded sequence of critical values $c_n = \varphi(u_n)$ by Theorem 1.54. We only need to show $\|u_n\| \to \infty$ as $n \to \infty$.

In fact, since u_n is a critical point of the functional φ, that is

$$\langle\varphi^{'}(u_n), u_n\rangle = -\int_0^T [({}_0^C D_t^\alpha u_n(t), {}_t^C D_T^\alpha u_n(t)) + (\nabla F(t,u_n(t)), u_n(t))]dt = 0. \tag{5.130}$$

Hence, we have

$$\begin{aligned}
c_n = \varphi(u_n) &= -\int_0^T \frac{1}{2}({}_0^C D_t^\alpha u_n(t), {}_t^C D_T^\alpha u_n(t))dt - \int_0^T F(t,u_n(t))dt, \\
&= \frac{1}{2}\int_0^T (\nabla F(t,u_n(t)), u_n(t))dt - \int_0^T F(t,u_n(t))dt, \\
&\leq \frac{1}{2}\int_0^T |\nabla F(t,u_n(t))||u_n(t)|dt + \int_0^T |F(t,u_n(t))|dt,
\end{aligned} \tag{5.131}$$

since $c_n \to \infty$, we conclude

$$\|u_n\| \to \infty, \quad \text{as } n \to \infty$$

by (5.131). In fact, if not, going to a subsequence if necessary, we may assume that

$$\|u_n\| \leq M_3,$$

for all $n \in \mathbb{N}$ and some positive constant M_3.

Combining assumption (A) and (5.131), we have

$$\begin{aligned} c_n &\leq \frac{1}{2}\int_0^T |\nabla F(t, u_n(t))||u_n(t)|dt + \int_0^T |F(t, u_n(t))|dt \\ &\leq \frac{1}{2}(M_3+1) \max_{0 \leq s \leq M_3} m_1(s) \int_0^T m_2(t)dt, \end{aligned}$$

which contradicts the unboundness of c_n. This completes the proof of Theorem 5.35. $\square$

Example 5.36. In BVP (5.1), let $F(t,x) = |x|^4$, and choose

$$\kappa = 4, \quad r = 2, \quad \mu = \mu' = 4 \ \text{ and } \ Q = Q' = 1,$$

so it is easy to verify that all the conditions (A1)-(A4) are satisfied. Then by Theorem 5.35, BVP (5.1) has infinite solutions $\{u_k\}$ on E^α for every positive integer k such that $\|u_k\| \to \infty$, as $k \to \infty$.

Theorem 5.37. Assume that $F(t,x)$ satisfies the following assumption:
(A5) $F(t,x) := a(t)|x|^\gamma$, where $a(t) \in L^\infty([0,T], \mathbb{R}^+)$ and $1 < \gamma < 2$ is a constant.
Then BVP (5.1) has infinite solutions $\{u_n\}$ on E^α for every positive integer n with $\|u_n\|_\alpha$ bounded.

Proof. Let us show that φ satisfies conditions in the Theorem 1.55 item by item. First, we show that φ satisfies the $(\mathrm{PS})_c^*$ condition for every $c \in \mathbb{R}$.

Suppose $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $\varphi(u_n) \to c$ and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ is a bounded sequence, otherwise, $\{u_{n_j}\}$ would be unbounded sequence, passing to a subsequence, still denoted by $\{u_{n_j}\}$ such that $\|u_{n_j}\|_\alpha \geq 1$ and $\|u_{n_j}\|_\alpha \to \infty$. Note that

$$\langle \varphi'(u_{n_j}), u_{n_j}\rangle - \gamma\varphi(u_{n_j}) = (-1 + \frac{\gamma}{2}) \int_0^T ({}_0^C D_t^\alpha u_{n_j}(t), {}_t^C D_T^\alpha u_{n_j}(t))dt. \tag{5.132}$$

However, from (5.132), we have

$$-\gamma\varphi(u_{n_j}) \geq (1 - \frac{\gamma}{2})|\cos(\pi\alpha)|\|u_{n_j}\|_\alpha^2 - \|(\varphi|_{Y_{n_j}})'(u_{n_j})\|\|u_{n_j}\|_\alpha, \tag{5.133}$$

thus $\|u_{n_j}\|_\alpha$ is a bounded sequence in E^α. Going, if necessary, to a subsequence, we can assume that $u_{n_j} \rightharpoonup u$ in E^α. As $E^\alpha = \overline{\bigcup_{n_j} Y_{n_j}}$, we can choose $v_{n_j} \in Y_{n_j}$ such that $v_{n_j} \to u$.

Hence

$$\begin{aligned} &\lim_{n_j \to \infty} \langle \varphi'(u_{n_j}), u_{n_j} - u\rangle \\ &= \lim_{n_j \to \infty} \langle \varphi'(u_{n_j}), u_{n_j} - v_{n_j}\rangle + \lim_{n_j \to \infty} \langle \varphi'(u_{n_j}), v_{n_j} - u\rangle \\ &= \lim_{n_j \to \infty} \langle (\varphi|_{Y_{n_j}})'(u_{n_j}), u_{n_j} - v_{n_j}\rangle \\ &= 0. \end{aligned}$$

So we have

$$\begin{aligned}
&\lim_{n_j\to\infty}\langle\varphi'(u_{n_j})-\varphi'(u),u_{n_j}-u\rangle\\
&=\lim_{n_j\to\infty}\langle\varphi'(u_{n_j}),u_{n_j}-u\rangle-\lim_{n_j\to\infty}\langle\varphi'(u),u_{n_j}-u\rangle\\
&=0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle\varphi'(u_{n_j})-\varphi'(u),u_{n_j}-u\rangle\\
&=-\int_0^T({}_0^CD_t^\alpha(u_{n_j}(t)-u(t)),{}_t^CD_T^\alpha(u_{n_j}(t)-u(t)))dt\\
&\quad-\int_0^T(\nabla F(t,u_{n_j}(t))-\nabla F(t,u(t)),u_{n_j}(t)-u(t))dt\\
&\geq|\cos(\pi\alpha)|\|u_{n_j}(t)-u(t)\|_\alpha^2\\
&\quad-\left|\int_0^T(\nabla F(t,u_{n_j}(t))-\nabla F(t,u(t)))dt\right|\|u_{n_j}(t)-u(t)\|,
\end{aligned}$$

we can conclude $u_{n_j}\to u$ in E^α, furthermore, we have $\varphi'(u_{n_j})\to\varphi'(u)$.

Let us prove $\varphi'(u)=0$ below. Taking arbitrarily $w_k\in Y_k$, notice when $n_j\geq k$, we have

$$\begin{aligned}
\langle\varphi'(u),w_k\rangle&=\langle\varphi'(u)-\varphi'(u_{n_j}),w_k\rangle+\langle\varphi'(u_{n_j}),w_k\rangle\\
&=\langle\varphi'(u)-\varphi'(u_{n_j}),w_k\rangle+\langle(\varphi|_{Y_{n_j}})'(u_{n_j}),w_k\rangle.
\end{aligned}$$

Let $n_j\to\infty$ in the right side of above equation. Then

$$\langle\varphi'(u),w_k\rangle=0,\quad\forall w_k\in Y_k,$$

so $\varphi'(u)=0$, this shows that φ satisfies the $(\mathrm{PS})_c^*$ for every $c\in\mathbb{R}$.

For any finite dimensional subspace $E\subset E^\alpha$, there exists $\varepsilon>0$ such that

$$\alpha\{t\in[0,T]:a(t)|u(t)|^\gamma\geq\varepsilon\|u\|_\alpha^\gamma\}\geq\varepsilon,\quad\forall u\in E\backslash\{0\}.\tag{5.134}$$

Otherwise, for any positive integer n, there exists $u_n\in E\backslash\{0\}$ such that

$$\alpha\{t\in[0,T]:a(t)|u_n(t)|^\gamma\geq\frac{1}{n}\|u_n\|_\alpha^\gamma\}<\frac{1}{n}.\tag{5.135}$$

Set $v_n:=\frac{u_n(t)}{\|u_n\|_\alpha}\in E\backslash\{0\}$, then $\|v_n\|_\alpha=1$ for all $n\in\mathbb{N}$ and

$$\alpha\{t\in[0,T]:a(t)|v_n(t)|^\gamma\geq\frac{1}{n}\}<\frac{1}{n}.\tag{5.136}$$

Since $\dim E<\infty$, it follows from the compactness of the unit sphere of E that there exists a subsequence, denoted also by $\{v_n\}$, such that $\{v_n\}$ converges to some v_0 in E. It is obvious that $\|v_0\|_\alpha=1$.

By the equivalence of the norms on the finite dimensional space, we have $v_n\to v_0$ in $L^2([0,T],\mathbb{R}^N)$, i.e.,

$$\int_0^T|v_n-v_0|^2dt\to0,\quad\text{as}\ \ n\to\infty.\tag{5.137}$$

By (5.137) and Hölder inequality, we have

$$\begin{aligned}\int_0^T a(t)|v_n-v_0|^\gamma dt &\le \left(\int_0^T a(t)^{\frac{2}{2-\gamma}}dt\right)^{\frac{2-\gamma}{2}}\left(\int_0^T |v_n-v_0|^2dt\right)^{\frac{\gamma}{2}}\\ &= \|a\|_{\frac{2}{2-\gamma}}\left(\int_0^T |v_n-v_0|^2dt\right)^{\frac{\gamma}{2}} \to 0, \quad \text{as } n\to\infty.\end{aligned} \tag{5.138}$$

Thus, there exist $\xi_1, \xi_2 > 0$ such that

$$\alpha\{t\in[0,T]: a(t)|v_0(t)|^\gamma \ge \xi_1\} \ge \xi_2. \tag{5.139}$$

In fact, if not, we have

$$\alpha\{t\in[0,T]: a(t)|v_0(t)|^\gamma \ge \frac{1}{n}\} = 0$$

for all positive integer n.

It implies that

$$0 \le \int_0^T a(t)|v_0|^{\gamma+2}dt < \frac{T}{n}\|v_0\|^2 \le \frac{C_8^2T}{n}\|v_0\|_\alpha^2 \to 0$$

as $n\to\infty$, where

$$C_8 := \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(2\alpha-1)^{\frac{1}{2}}}$$

by (5.15). Hence $v_0 = 0$ which contradicts that $\|v_0\|_\alpha = 1$. Therefore, (5.139) holds.

Now let

$$\Omega_0 = \{t\in[0,T]: a(t)|v_0(t)|^\gamma \ge \xi_1\}, \quad \Omega_n = \{t\in[0,T]: a(t)|v_n(t)|^\gamma < \frac{1}{n}\},$$

and $\Omega_n^c = [0,T]\setminus\Omega_n = \{t\in[0,T]: a(t)|v_n(t)|^\gamma \ge \frac{1}{n}\}$.

By (5.136) and (5.139), we have

$$\begin{aligned}\alpha(\Omega_n\cap\Omega_0) &= \alpha(\Omega_0\setminus(\Omega_n^c\cap\Omega_0))\\ &\ge \alpha(\Omega_0) - \alpha(\Omega_n^c\cap\Omega_0)\\ &\ge \xi_2 - \frac{1}{n}\end{aligned}$$

for all positive integer n. Let n be large enough such that

$$\xi_2 - \frac{1}{n} \ge \frac{1}{2}\xi_2 \quad \text{and} \quad \frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n} \ge \frac{1}{2^\gamma}\xi_1,$$

then we have

$$\begin{aligned}\int_0^T a(t)|v_n-v_0|^\gamma dt &\ge \int_{\Omega_n\cap\Omega_0} a(t)|v_n-v_0|^\gamma dt\\ &\ge \frac{1}{2^{\gamma-1}}\int_{\Omega_n\cap\Omega_0} a(t)|v_0|^\gamma dt - \int_{\Omega_n\cap\Omega_0} a(t)|v_n|^\gamma dt\\ &\ge \left(\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n}\right)\alpha(\Omega_n\cap\Omega_0)\\ &\ge \frac{\xi_1}{2^\gamma}\frac{\xi_2}{2} = \frac{\xi_1\xi_2}{2^{\gamma+1}} > 0\end{aligned}$$

for all large n, which is a contradiction to (5.138). Therefore, (5.134) holds.

For any $u \in Z_k$, let

$$\|u\|_2 := \left(\int_0^T |u(t)|^2 dt \right)^{\frac{1}{2}} \quad \text{and} \quad \gamma_k := \sup_{\substack{u \in Z_k \\ \|u\|_\alpha = 1}} \|u\|_2,$$

then we conclude $\gamma_k \to 0$ as $k \to \infty$ in the same way as in the proof of Theorem 5.35.

$$\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2} ({}_0^C D_t^\alpha u(t),\ {}_t^C D_T^\alpha u(t)) dt - \int_0^T F(t, u(t)) dt \\
&\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \int_0^T a(t) |u(t)|^\gamma dt \\
&\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \left(\int_0^T a(t)^{\frac{2}{2-\gamma}} dt \right)^{\frac{2-\gamma}{2}} \|u\|_2^\gamma \\
&\geq \frac{1}{2} |\cos(\pi\alpha)| \|u\|_\alpha^2 - \left(\int_0^T a(t)^{\frac{2}{2-\gamma}} dt \right)^{\frac{2-\gamma}{2}} \gamma_k^\gamma \|u\|_\alpha^\gamma.
\end{aligned} \tag{5.140}$$

Let $\rho_k := \left(\frac{4c\gamma_k^\gamma}{|\cos(\pi\alpha)|} \right)^{\frac{1}{2-\gamma}}$, where $c := \left(\int_0^T a(t)^{\frac{2}{2-\gamma}} dt \right)^{\frac{1}{2-\gamma}}$, it is obvious that $\rho_k \to 0$, as $k \to \infty$.

In view of (5.140), we conclude

$$\inf_{\substack{u \in Z_k \\ \|u\|_\alpha = \rho_k}} \varphi(u) \geq \frac{|\cos(\pi\alpha)|}{4} \rho_k^2 > 0,$$

so the condition (H7) in Theorem 1.55 is satisfied.

Furthermore, by (5.140), for any $u \in Z_k$ with $\|u\|_\alpha \leq \rho_k$, we have

$$\varphi(u) \geq -c\gamma_k^\alpha \|u\|_\alpha^\gamma.$$

Therefore,

$$-c\gamma_k^\gamma \rho_k^\gamma \leq \inf_{\substack{u \in Z_k \\ \|u\|_\alpha \leq \rho_k}} \varphi(u) \leq 0.$$

So we have

$$\inf_{\substack{u \in Z_k \\ \|u\|_\alpha \leq \rho_k}} \varphi(u) \to 0,$$

for $\rho_k, \gamma_k \to 0$, as $k \to \infty$. Hence (H5) in Theorem 1.55 is satisfied.

For any $u \in Y_k \setminus \{0\}$,

$$\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2} ({}_0^C D_t^\alpha u(t),\ {}_t^C D_T^\alpha u(t)) dt - \int_0^T F(t, u(t)) dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 - \int_0^T a(t) |u(t)|^\gamma dt \\
&\leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 - \varepsilon \|u\|_\alpha^\gamma \alpha(\Omega_u) \\
&\leq \frac{1}{2|\cos(\pi\alpha)|} \|u\|_\alpha^2 - \varepsilon^2 \|u\|_\alpha^\gamma,
\end{aligned}$$

where ε is given in (5.134), and $\Omega_u := \{t \in [0,T] : a(t)|u(t)|^\gamma \geq \varepsilon \|u\|_\alpha^\gamma\}$.

Choosing $0 < r_k < \min\{\rho_k, (|\cos(\pi\alpha)|\varepsilon^2)^{\frac{1}{2-\gamma}}\}$, we conclude

$$i_k := \max_{\substack{u \in Y_k \\ \|u\|_\alpha = r_k}} \varphi(u) < -\frac{1}{2|\cos(\pi\alpha)|} r_k^2 < 0, \quad \forall\, k \in \mathbb{N},$$

that is, the condition (H6) in Theorem 1.55 is satisfied.

We have proved the functional φ satisfies all the conditions of Theorem 1.55, then φ has a bounded sequence of negative critical values $c_n = \varphi(u_n)$ converging to 0 by Theorem 1.55, we only need to show $\|u_n\|_\alpha$ is bounded as for every positive integer n. Since

$$\begin{aligned} c_n = \varphi(u_n) &= -\int_0^T \frac{1}{2}({}_0^C D_t^\alpha u_n(t),\ {}_t^C D_T^\alpha u_n(t))dt - \int_0^T F(t, u_n(t))dt \\ &= -\int_0^T \frac{1}{2}({}_0^C D_t^\alpha u_n(t),\ {}_t^C D_T^\alpha u_n(t))dt - \int_0^T a(t)|u_n(t)|^\gamma dt \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u_n\|_\alpha^2 - a_0\|u_n\|^\gamma T \\ &\geq \frac{|\cos(\pi\alpha)|}{2}\|u_n\|_\alpha^2 - a_0 T C_8^\gamma \|u_n\|_\alpha^\gamma, \end{aligned} \tag{5.141}$$

where $a_0 = \operatorname{ess\,sup}\{a(t) : t \in [0,T]\}$, by Theorem 1.55, $c_n \to 0$ as $n \to \infty$. If $\|u_n\|_\alpha$ has an unbounded sequence, then c_n is unbounded by (5.141), which is a contradiction. The proof is completed. $\square$

Example 5.38. In BVP (5.1), let $F(t,x) = a(t)|x|^{\frac{3}{2}}$, where

$$a(t) = \begin{cases} T, & t = 0, \\ t, & 0 < t \leq T. \end{cases}$$

By Theorem 5.37, BVP (5.1) has infinite solutions $\{u_k\}$ on E^α for every positive integer k with $\|u_k\|_\alpha$ bounded.

5.5 Existence of Solutions for BVP with Left and Right Fractional Derivatives

5.5.1 *Introduction*

In Section 5.5, we consider the fractional BVP of the following form

$$\begin{cases} {}_t D_T^\alpha({}_0 D_t^\alpha u(t)) = \nabla F(t, u(t)), & \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{5.142}$$

where ${}_t D_T^\alpha$ and ${}_0 D_t^\alpha$ are the right and left Riemann-Liouville fractional derivatives of order $0 < \alpha \leq 1$ respectively, F: $[0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function satisfying some assumptions and $\nabla F(t,x)$ is the gradient of F at x.

In particular, if $\alpha = 1$, BVP (5.142) reduces to the standard second order boundary value problem of the following form

$$\begin{cases} u''(t) + \nabla F(t, u(t)) = 0, & \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0, \end{cases}$$

where $F\colon [0, T] \times \mathbb{R}^N \to \mathbb{R}$ is a given function and $\nabla F(t, x)$ is the gradient of F at x. Although many excellent results have been worked out on the existence of solutions for second order BVP (e.g., Li, Liang and Zhang, 2005; Nieto and O'Regan, 2009; Rabinowitz, 1986), it seems that no similar results were obtained in the literature for fractional BVP.

According to Benson, Wheatcraft and Meerschaert, 2000a, the one-dimensional form of the fractional ADE can be written as

$$\frac{\partial \mathcal{C}}{\partial t} = -v\frac{\partial \mathcal{C}}{\partial x} + \mathcal{D}j\frac{\partial^\gamma \mathcal{C}}{\partial x^\gamma} + \mathcal{D}(1-j)\frac{\partial^\gamma \mathcal{C}}{\partial(-x)^\gamma}, \tag{5.143}$$

where $\mathcal{C}$ is the expected concentration, t is time, v is a constant mean velocity, x is distance in the direction of mean velocity, $\mathcal{D}$ is a constant dispersion coefficient, $0 \le j \le 1$ describes the skewness of the transport process, and γ is the order of left and right fractional differential operators. For discussions of this equation, see Benson, Wheatcraft and Meerschaert, 2000b; Fix and Roop, 2004, when $\gamma = 2$, the dispersion operators are identical and the classical ADE is recovered. Fundamental (Green function) solutions are Lévy's γ-stable densities.

A special case of the fractional ADE (Eq. (5.143)) describes symmetric transitions, where $j = \frac{1}{2}$. Defining the symmetric operator equivalent to the Riesz potential in Samko, Kilbas and Marichev, 1993,

$$2\nabla^\gamma \equiv D_+^\gamma + D_-^\gamma \tag{5.144}$$

gives the mass balance equation for advection and symmetric fractional dispersion

$$\frac{\partial \mathcal{C}}{\partial t} = -v\nabla\mathcal{C} + \mathcal{D}\nabla^\gamma\mathcal{C}. \tag{5.145}$$

In Subsection 5.5.2, we shall establish a variational structure for BVP (5.142). We show that under some suitable assumptions, the critical points of the variational functional defined on a suitable Hilbert space are the solutions of BVP (5.142). In Subsection 5.5.3, the existence of weak solutions for BVP (5.142) with $\frac{1}{2} < \alpha \le 1$ will be established, where α is the order of fractional derivative in BVP (5.142). In Subsection 5.5.4, we will give some existence results of solutions for BVP (5.142).

5.5.2 *Variational Structure*

Proposition 5.39. Let $0 < \alpha \le 1$ and $1 < p < \infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > \frac{1}{p}$, we have ${}_0D_t^{-\alpha}({}_0D_t^\alpha u(t)) = u(t)$. Moreover, we can get that $E_0^{\alpha,p} \in C_0([0,T], \mathbb{R}^N)$.

Proof. Let $\frac{1}{p}+\frac{1}{q}=1$ and $0\le t_1<t_2\le T$. $\forall f\in L^p([0,T],\mathbb{R}^N)$, by using Hölder inequality and noting that $\alpha>\frac{1}{p}$, we have

$$\begin{aligned}
&|{}_0D_{t_1}^{-\alpha}f(t_1)-{}_0D_{t_2}^{-\alpha}f(t_2)|\\
&=\frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds-\int_0^{t_2}(t_2-s)^{\alpha-1}f(s)ds\Big|\\
&=\frac{1}{\Gamma(\alpha)}\Big|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s)ds-\int_0^{t_1}(t_2-s)^{\alpha-1}f(s)ds\Big|\\
&\quad+\frac{1}{\Gamma(\alpha)}\Big|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s)ds\Big|\\
&\le\frac{1}{\Gamma(\alpha)}\int_0^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})|f(s)|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|f(s)|ds\\
&\le\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}((t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1})^q ds\right)^{\frac{1}{q}}\|f\|_{L^p}\\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}ds\right)^{\frac{1}{q}}\|f\|_{L^p}\\
&\le\frac{1}{\Gamma(\alpha)}\left(\int_0^{t_1}((t_1-s)^{(\alpha-1)q}-(t_2-s)^{(\alpha-1)q})ds\right)^{\frac{1}{q}}\|f\|_{L^p}\\
&\quad+\frac{1}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{(\alpha-1)q}ds\right)^{\frac{1}{q}}\|f\|_{L^p}\\
&=\frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left(t_1^{(\alpha-1)q+1}-t_2^{(\alpha-1)q+1}+(t_2-t_1)^{(\alpha-1)q+1}\right)^{\frac{1}{q}}\\
&\quad+\frac{\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}\left((t_2-t_1)^{(\alpha-1)q+1}\right)^{\frac{1}{q}}\\
&\le\frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-1+\frac{1}{q}}\\
&=\frac{2\|f\|_{L^p}}{\Gamma(\alpha)(1+(\alpha-1)q)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-\frac{1}{p}}.
\end{aligned}\tag{5.146}$$

For any $u\in E_0^{\alpha,p}$, as ${}_0D_t^{\alpha}u(t)\in L^p([0,T],\mathbb{R}^N)$, we apply (5.146) to obtain the continuity of the function ${}_0D_t^{-\alpha}({}_0D_t^{\alpha}u(t))$ on $[0,T]$. We complete the argument by using Properties 1.19-1.20, and we have

$$ {}_0D_t^{-\alpha}({}_0D_t^{\alpha}u(t))=u(t)+Ct^{\alpha-1},\quad t\in[0,T],$$

where $C\in\mathbb{R}^N$.

Since $u(0)=0$ and ${}_0D_t^{-\alpha}({}_0D_t^{\alpha}u(t))$ is continuous in $[0,T]$, we can get that $C=0$, which means that ${}_0D_t^{-\alpha}({}_0D_t^{\alpha}u(t))=u(t)$ and u is continuous in $[0,T]$. $\square$

Remark 5.40. In the case that $1-\alpha\ge\frac{1}{p}$, for any $u\in E_0^{\alpha,p}$, we also have ${}_0D_t^{-\alpha}({}_0D_t^{\alpha}u(t))=u(t)$. In fact, set $f(t)={}_0D_t^{\alpha-1}u(t)$. According to Properties

1.19-1.20, we only need to prove that $f(0) = [{}_0D_t^{\alpha-1}u(t)]_{t=0} = 0$. Noting that $1-\alpha \geq \frac{1}{p}$, by using Hölder inequality, Lemma 5.1 and the similar method in the proof of Lemma 7 in Fix and Roop, 2004, we can obtain the desired result, i.e. $f(0) = 0$. We skip the proof since it is similar to Lemma 7 in Fix and Roop, 2004.

If $\alpha > \frac{1}{p}$, the following theorem is useful for us to establish the variational structure on the space $E_0^{\alpha,p}$ for BVP (5.142).

Theorem 5.41. Let $1 < p < \infty$, $\frac{1}{p} < \alpha \leq 1$ and $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $(t,x,y) \to L(t,x,y)$ be measurable in t for each $[x,y] \in \mathbb{R}^N \times \mathbb{R}^N$ and continuously differentiable in $[x,y]$ for almost every $t \in [0,T]$. If there exist $m_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$, $m_2 \in L^1([0,T], \mathbb{R}^+)$ and $m_3 \in L^q([0,T], \mathbb{R}^+)$, $1 < q < \infty$, such that, for a.e. $t \in [0,T]$ and every $[x,y] \in \mathbb{R}^N \times \mathbb{R}^N$, one has

$$\begin{aligned} |L(t,x,y)| &\leq m_1(|x|)(m_2(t) + |y|^p), \\ |D_xL(t,x,y)| &\leq m_1(|x|)(m_2(t) + |y|^p), \\ |D_yL(t,x,y)| &\leq m_1(|x|)(m_3(t) + |y|^{p-1}), \end{aligned} \tag{5.147}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, then the functional φ defined by

$$\varphi(u) = \int_0^T L(t, u(t), {}_0D_t^\alpha u(t))dt$$

is continuously differentiable on $E_0^{\alpha,p}$, and $\forall u, v \in E_0^{\alpha,p}$, we have

$$\begin{aligned} \langle \varphi'(u), v\rangle = \int_0^T &[(D_xL(t,u(t),{}_0D_t^\alpha u(t)), v(t)) \\ &+ (D_yL(t,u(t),{}_0D_t^\alpha u(t)), {}_0D_t^\alpha v(t))]dt. \end{aligned} \tag{5.148}$$

Proof. It suffices to prove that φ has at every point u a directional derivative $\varphi'(u) \in (E_0^{\alpha,p})^*$ given by (5.148) and that the mapping

$$\varphi' : E_0^{\alpha,p} \to (E_0^{\alpha,p})^*, \quad u \to \varphi'(u)$$

is continuous.

We omit the rather technical proof which is similar to the proof of Theorem 1.4 in Mawhin and Willem, 1989. In fact, the only change we need is to replace the weak derivatives for u and v of Theorem 1.4 in Mawhin and Willem, 1989, by ${}_0D_t^\alpha u$ and ${}_0D_t^\alpha v$ respectively. The proof is completed. □

We are now in a position to give the definition for the solution of BVP (5.142).

Definition 5.42. A function $u : [0,T] \to \mathbb{R}^N$ is called a solution of BVP (5.142) if
(i) ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t))$ and ${}_0D_t^{\alpha-1}u(t)$ are differentiable for almost every $t \in [0,T]$, and
(ii) u satisfies (5.142).

For a solution $u \in E^\alpha$ of BVP (5.142) such that $\nabla F(\cdot, u(\cdot)) \in L^1([0,T], \mathbb{R}^N)$, multiplying (5.142) by $v \in C_0^\infty([0,T], \mathbb{R}^N)$ yields

$$\begin{aligned} &\int_0^T [({}_tD_T^\alpha({}_0D_t^\alpha u(t)), v(t)) - (\nabla F(t,u(t)), v(t))]dt \\ &= \int_0^T [({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t)) - (\nabla F(t,u(t)), v(t))]dt \\ &= 0 \end{aligned} \tag{5.149}$$

after applying (1.40) and Definition 5.42. Therefore, we can give the definition of weak solution for BVP (5.142) as follows.

Definition 5.43. By the weak solution of BVP (5.142), we mean that the function $u \in E^\alpha$ such that $\nabla F(\cdot, u(\cdot)) \in L^1([0,T], \mathbb{R}^N)$ and satisfies (5.149) for all $v \in C_0^\infty([0,T], \mathbb{R}^N)$.

Any solution $u \in E^\alpha$ of BVP (5.142) is a weak solution provided that $\nabla F(\cdot, u(\cdot)) \in L^1([0,T], \mathbb{R}^N)$. Our task is now to establish a variational structure on E^α with $\alpha \in (\frac{1}{2}, 1]$, which enables us to reduce the existence of weak solutions of BVP (5.142) to the one of finding critical points of corresponding functional.

Corollary 5.44. Let $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by

$$L(t, x, y) = \frac{1}{2}|y|^2 - F(t, x),$$

where $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following condition (A) which is assumed as in Subsection 5.2.3.

If $\frac{1}{2} < \alpha \leq 1$ and $u \in E^\alpha$ is a solution of corresponding Euler equation $\varphi'(u) = 0$, where φ is defined as

$$\varphi(u) = \int_0^T \left[\frac{1}{2}|{}_0D_t^\alpha u(t)|^2 - F(t, u(t))\right] dt, \quad \text{for } u \in E^\alpha, \tag{5.150}$$

then u is a weak solution of BVP (5.142) with $\frac{1}{2} < \alpha \leq 1$.

Proof. By Theorem 5.41, we have

$$0 = \langle \varphi'(u), v \rangle = \int_0^T [({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t)) - (\nabla F(t, u(t)), v(t))] dt$$

for all $v \in E^\alpha$ and hence for all $v \in C_0^\infty([0,T], \mathbb{R}^N)$. Thus, according to Definition 5.43, u is a weak solution of BVP (5.142). The proof is completed. □

Remark 5.45. Generally speaking, a critical point u of φ on E^α will be a weak solution of BVP (5.142). However, we shall show that every weak solution is also a solution of BVP (5.142).

5.5.3 *Existence of Weak Solutions*

According to Corollary 5.44, we know that in order to find weak solutions of BVP (5.142), it suffices to obtain the critical points of functional φ given by (5.150). We need to use some critical point theorems.

First, we use Theorem 1.50 to consider the existence of weak solutions for BVP (5.142). Assume that the condition (A) is satisfied. Recall that, in our setting in (5.150), the corresponding functional φ on E^α is continuously differentiable according to Corollary 5.44 and is also weakly lower semi-continuous functional on E^α as the sum of a convex continuous function (see Theorem 1.2 in Mawhin and Willem, 1989) and of a weakly continuous one (see Proposition 1.2 in Mawhin and Willem, 1989).

In fact, according to Proposition 5.6, if $u_k \rightharpoonup u$ in E^α, then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$. Therefore, $F(t,u_k(t)) \to F(t,u(t))$ a.e. $t \in [0,T]$. By Lebesgue's dominated convergence theorem, we have $\int_0^T F(t,u_k(t))dt \to \int_0^T F(t,u(t))dt$, which means that the functional $u \to \int_0^T F(t,u(t))dt$ is weakly continuous on E^α. Moreover, since fractional derivative operator is linear operator, the functional $u \to \int_0^T (|{}_0D_t^\alpha u(t)|^2/2)dt$ is convex and continuous on E^α.

If φ is coercive, by Theorem 1.50, φ has a minimum so that BVP (5.142) is solvable. It remains to find conditions under which φ is coercive on E^α, i.e. $\lim_{\|u\|_\alpha\to\infty}\varphi(u) = +\infty$, for $u \in E^\alpha$. We shall see that it suffices to require that $F(t,x)$ is bounded by a function for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^N$.

Theorem 5.46. Let $\alpha \in (\frac{1}{2},1]$ and assume that F satisfies (A). If

$$|F(t,x)| \le \bar{a}|x|^2 + \bar{b}(t)|x|^{2-\gamma} + \bar{c}(t), \quad \text{a.e. } t \in [0,T],\ x \in \mathbb{R}^N, \tag{5.151}$$

where $\bar{a} \in [0,\Gamma^2(\alpha+1)/2T^{2\alpha})$, $\gamma \in (0,2)$, $\bar{b} \in L^{2/\gamma}([0,T],\mathbb{R})$, and $\bar{c} \in L^1([0,T],\mathbb{R})$, then BVP (5.142) has at least one weak solution which minimizes φ on E^α.

Proof. According to the arguments above, our problem reduces to prove that φ is coercive on E^α. For $u \in E^\alpha$, it follows from (5.151) and (5.14) that

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2dt - \int_0^T F(t,u(t))dt \\
&\ge \frac{1}{2}\int_0^T |{}_0D_t^\alpha u(t)|^2dt - \bar{a}\int_0^T |u(t)|^2dt - \int_0^T \bar{b}(t)|u(t)|^{2-\gamma}dt - \int_0^T \bar{c}(t)dt \\
&= \frac{1}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \int_0^T \bar{b}(t)|u(t)|^{2-\gamma}dt - \bar{c}_1 \\
&\ge \frac{1}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \left(\int_0^T |\bar{b}(t)|^{2/\gamma}dt\right)^{\gamma/2}\left(\int_0^T |u(t)|^2dt\right)^{1-\gamma/2} - \bar{c}_1 \\
&= \frac{1}{2}\|u\|_\alpha^2 - \bar{a}\|u\|_{L^2}^2 - \bar{b}_1\|u\|_{L^2}^{2-\gamma} - \bar{c}_1 \\
&\ge \frac{1}{2}\|u\|_\alpha^2 - \frac{\bar{a}T^{2\alpha}}{\Gamma^2(\alpha+1)}\|u\|_\alpha^2 - \bar{b}_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|_\alpha^{2-\gamma} - \bar{c}_1 \\
&= \left(\frac{1}{2} - \frac{\bar{a}T^{2\alpha}}{\Gamma^2(\alpha+1)}\right)\|u\|_\alpha^2 - \bar{b}_1\left(\frac{T^\alpha}{\Gamma(\alpha+1)}\right)^{2-\gamma}\|u\|_\alpha^{2-\gamma} - \bar{c}_1,
\end{aligned}$$

where $\bar{b}_1 = (\int_0^T |\bar{b}(t)|^{2/\gamma}dt)^{\gamma/2}$ and $\bar{c}_1 = \int_0^T \bar{c}(t)dt$. Noting that $\bar{a} \in [0,\Gamma^2(\alpha+1)/2T^{2\alpha})$ and $\gamma \in (0,2)$, we have $\varphi(u) = +\infty$ as $\|u\|_\alpha \to \infty$, and hence φ is coercive, which completes the proof. □

Let $a_0 = \min_{\lambda\in[\frac{1}{2},1]}\{\Gamma^2(\lambda+1)/2T^{2\lambda}\}$. The following result follows immediately from Theorem 5.46.

Corollary 5.47. $\forall\alpha \in (\frac{1}{2},1]$ and if F satisfies the condition (A) and (5.151) with $a \in [0,a_0)$, then BVP (5.142) has at least one weak solution which minimizes φ on E^α.

Our task is now to use Theorem 1.51(Mountain pass theorem) to find a nonzero critical point of functional φ on E^α.

Theorem 5.48. Let $\alpha \in (\frac{1}{2}, 1]$ and suppose that F satisfies the condition (A). If
(A1) $F \in C([0,T] \times \mathbb{R}^N, \mathbb{R})$ and there exists $\mu \in [0, \frac{1}{2})$ and $M > 0$ such that $0 < F(t,x) \leq \mu(\nabla F(t,x), x)$ for all $x \in \mathbb{R}^N$ with $|x| \geq M$ and $t \in [0,T]$;
(A2) $\limsup_{|x| \to 0} F(t,x)/|x|^2 < \Gamma^2(\alpha+1)/2T^{2\alpha}$ uniformly for $t \in [0,T]$ and $x \in \mathbb{R}^N$
are satisfied, then BVP (5.142) has at least one nonzero weak solution on E^α.

Proof. We will verify that φ satisfies all the conditions of Theorem 1.51.

First, we will prove that φ satisfies (PS) condition. Since $F(t,x) - \mu(\nabla F(t,x), x)$ is continuous for $t \in [0,T]$ and $|x| \leq M$, there exists $c \in \mathbb{R}^+$, such that

$$F(t,x) \leq \mu(\nabla F(t,x), x) + c, \quad t \in [0,T], \quad |x| \leq M.$$

By assumption (A1), we obtain

$$F(t,x) \leq \mu(\nabla F(t,x), x) + c, \quad t \in [0,T], \quad x \in \mathbb{R}^N. \tag{5.152}$$

Let $\{u_k\} \subset E^\alpha$, $|\varphi(u_k)| \leq K,\ k = 1, 2, ...,\ \varphi'(u_k) \to 0.$ Notice that

$$\begin{aligned} \langle \varphi'(u_k), u_k \rangle &= \int_0^T [({}_0D_t^\alpha u_k(t), {}_0D_t^\alpha u_k(t)) - (\nabla F(t, u_k(t)), u_k(t))]dt \\ &= \|u_k\|_\alpha^2 - \int_0^T (\nabla F(t, u_k(t)), u_k(t))dt. \end{aligned} \tag{5.153}$$

It follows from (5.152) and (5.153) that

$$\begin{aligned} K \geq \varphi(u_k) &= \frac{1}{2}\|u_k\|_\alpha^2 - \int_0^T F(t, u_k(t))dt \\ &\geq \frac{1}{2}\|u_k\|_\alpha^2 - \mu\int_0^T (\nabla F(t, u_k(t)), u_k(t))dt - cT \\ &= \left(\frac{1}{2} - \mu\right)\|u_k\|_\alpha^2 + \mu\langle \varphi'(u_k), u_k\rangle - cT \\ &\geq \left(\frac{1}{2} - \mu\right)\|u_k\|_\alpha^2 - \mu\|\varphi'(u_k)\|_\alpha \|u_k\|_\alpha - cT, \quad k = 1, 2, ... \end{aligned}$$

Since $\varphi'(u_k) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$K \geq \left(\frac{1}{2} - \mu\right)\|u_k\|_\alpha^2 - \|u_k\|_\alpha - cT, \quad k > N_0,$$

and this implies that $\{u_k\} \subset E^\alpha$ is bounded. Since E^α is a reflexive space, going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ weakly in E^α, thus we have

$$\begin{aligned} \langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle &= \langle \varphi'(u_k), u_k - u\rangle - \langle \varphi'(u), u_k - u\rangle \\ &\leq \|\varphi'(u_k)\|_\alpha \|u_k - u\|_\alpha - \langle \varphi'(u), u_k - u\rangle \to 0, \end{aligned} \tag{5.154}$$

as $k \to \infty$. Moreover, according to (5.15) and Proposition 5.6, we get that u_k is bounded in $C([0,T],\mathbb{R}^N)$ and $\|u_k - u\| = 0$ as $k \to \infty$. Hence, we have

$$\int_0^T \nabla F(t, u_k(t))dt \to \int_0^T \nabla F(t, u(t))dt, \quad \text{as } k \to \infty. \tag{5.155}$$

Noting that

$$\begin{aligned}
&\langle \varphi'(u_k) - \varphi'(u), u_k - u\rangle \\
&= \int_0^T ({}_0D_t^\alpha u_k(t) - {}_0D_t^\alpha u(t))^2 dt - \int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)))(u_k(t) - u(t))dt \\
&\geq \|u_k - u\|_\alpha^2 - \left| \int_0^T (\nabla F(t, u_k(t)) - \nabla F(t, u(t)))dt \right| \|u_k - u\|.
\end{aligned}$$

Combining (5.154) and (5.155), it is easy to verify that $\|u_k - u\|_\alpha^2 \to 0$ as $k \to \infty$, and hence $u_k \to u$ in E_α. Thus, we obtain the desired convergence property.

From $\limsup_{|x|\to 0} F(t,x)/|x|^2 < \Gamma^2(\alpha+1)/2T^{2\alpha}$ uniformly for $t \in [0,T]$, there exists $\epsilon \in (0,1)$ and $\delta > 0$ such that $F(t,x) \leq (1-\epsilon)(\Gamma^2(\alpha+1)/2T^{2\alpha})|x|^2$ for all $t \in [0,T]$ and $x \in \mathbb{R}^N$ with $|x| \leq \delta$.

Let $\rho = \frac{\Gamma(\alpha)((\alpha-1)/2+1)^{\frac{1}{2}}}{T^{\alpha-\frac{1}{2}}}\delta$ and $\sigma = \epsilon\rho^2/2 > 0$. Then it follows from (5.15) that

$$\|u\| \leq \frac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)((\alpha-1)/2+1)^{\frac{1}{2}}}\|u\|_\alpha = \delta$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. Therefore, we have

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\|u\|_\alpha^2 - \int_0^T F(t, u(t))dt \\
&\geq \frac{1}{2}\|u\|_\alpha^2 - (1-\epsilon)\frac{\Gamma^2(\alpha+1)}{2T^{2\alpha}}\int_0^T |u(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|_\alpha^2 - \frac{1}{2}(1-\epsilon)\|u\|_\alpha^2 \\
&= \frac{1}{2}\epsilon\|u\|_\alpha^2 \\
&= \sigma
\end{aligned}$$

for all $u \in E^\alpha$ with $\|u\|_\alpha = \rho$. This implies (ii) in Theorem 1.51 is satisfied.

It is obvious from the definition of φ and (A2) that $\varphi(0) = 0$, and therefore, it suffices to show that φ satisfies (iii) in Theorem 1.51.

Since $0 < F(t,x) \leq \mu(\nabla F(t,x), x)$ for all $x \in \mathbb{R}^N$ and $|x| \geq M$, a simple regularity argument then shows that there exists $r_1, r_2 > 0$ such that

$$F(t,x) \geq r_1|x|^{1/\mu} - r_2, \quad x \in \mathbb{R}^N, \quad t \in [0,T].$$

For any $u \in E^\alpha$ with $u \neq 0$, $\kappa > 0$ and noting that $\mu \in [0, \frac{1}{2})$, we have

$$\begin{aligned}
\varphi(\kappa u) &= \frac{1}{2}\|\kappa u\|_\alpha^2 - \int_0^T F(t, \kappa u(t))dt \\
&\leq \frac{1}{2}\kappa^2\|u\|_\alpha^2 - r_1\int_0^T |\kappa u(t)|^{1/\mu}dt + r_2 T \\
&= \frac{1}{2}\kappa^2\|u\|_\alpha^2 - r_1\kappa^{1/\mu}\|u\|_{L^{1/\mu}}^{1/\mu} + r_2 T \to -\infty
\end{aligned}$$

as $\kappa \to \infty$. Then there exists a sufficiently large κ_0 such that $\varphi(\kappa_0 u) \leq 0$. Hence (iii) holds.

Lastly noting that $\varphi(0) = 0$ while for our critical point u, $\varphi(u) \geq \sigma > 0$. Hence u is a nontrivial weak solution of BVP (5.142), and this completes the proof. $\square$

Theorem 5.49. $\forall \alpha \in (\frac{1}{2}, 1]$, suppose that F satisfies conditions (A) and (A1). If (A2)′ $F(t,x) = o(|x|^2)$, as $|x| \to 0$ uniformly for $t \in [0,T]$ and $x \in \mathbb{R}^N$ is satisfied, then BVP (5.142) has at least one nonzero weak solution on E^α.

Remark 5.50. The assumptions in Theorem 5.46 and Theorem 5.48 are classical and the examples can be found in many papers which use critical point theory to discuss differential equations, see, e.g., Li, Liang and Zhang, 2005; Mawhin and Willem, 1989; Rabinowitz, 1989 and references therein.

5.5.4 *Existence of Solutions*

We firstly give the following lemma which is useful for our further discussion.

Lemma 5.51. Let $0 < \alpha \leq 1$. If $u \in E^\alpha$ is a weak solution of BVP (5.142), then there exists a constant $C \in \mathbb{R}^N$ such that

$$ {}_0D_t^\alpha u(t) = {}_tD_T^{-\alpha} \nabla F(t, u(t)) + C(T-t)^{\alpha-1}, \quad \text{a.e. } t \in [0,T]. \tag{5.156}$$

Proof. Since $u \in E^\alpha$ is a weak solution of BVP (5.142), i.e. $\forall h \in C_0^\infty([0,T], \mathbb{R}^N)$, we have

$$ \int_0^T [({}_0D_t^\alpha u(t), {}_0D_t^\alpha h(t)) - (\nabla F(t, u(t)), h(t))]dt = 0. \tag{5.157}$$

Noting that $\nabla F(\cdot, u(\cdot)) \in L^1([0,T], \mathbb{R}^N)$, and applying a similar argument as that for (5.5) in the proof of Lemma 5.1, we get that ${}_tD_T^{-\alpha} \nabla F(\cdot, u(\cdot)) \in L^1([0,T], \mathbb{R}^N)$. Let us define $w \in L^1([0,T], \mathbb{R}^N)$ by

$$ w(t) = {}_tD_T^{-\alpha} \nabla F(t, u(t)), \quad t \in [0,T], $$

so that

$$ \begin{aligned} \int_0^T (w(t), {}_0D_t^\alpha h(t))dt &= \int_0^T ({}_tD_T^\alpha w(t), h(t))dt \\ &= \int_0^T ({}_tD_T^\alpha({}_tD_T^{-\alpha} \nabla F(t, u(t))), h(t))dt \\ &= \int_0^T (\nabla F(t, u(t)), h(t))dt \end{aligned} $$

by applying (1.40) and Property 1.18.

Hence, by (5.157) we have, for every $h \in C_0^\infty([0,T], \mathbb{R}^N)$,

$$ \int_0^T ({}_0D_t^\alpha u(t) - w(t), {}_0D_t^\alpha h(t))dt = 0. \tag{5.158}$$

According to Properties 1.9 and in view of $h \in C_0^\infty([0,T],\mathbb{R}^N)$, we have ${}_0D_t^\alpha h(t) = {}_0D_t^{\alpha-1}h'(t)$. Since ${}_0D_t^\alpha u \in L^2([0,T],\mathbb{R}^N)$ and $w \in L^1([0,T],\mathbb{R}^N)$, using (1.39) and (5.158), we get that

$$\int_0^T ({}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t) - w(t)), h'(t))dt = 0.$$

If (e_j) denotes the Canonical basis of $\mathbb{R}^N$, we can choose

$$h(t) = \sin\frac{2k\pi t}{T}e_j \quad \text{or} \quad h(t) = e_j - \cos\frac{2k\pi t}{T}e_j, \qquad k = 1,... \text{ and } j = 1,...,N.$$

In view of ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u - w) \in L^1([0,T],\mathbb{R}^N)$, and the theory of Fourier series implies that

$$ {}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t) - w(t)) = \tilde{C} \tag{5.159}$$

a.e. on $[0,T]$ for some $\tilde{C} \in \mathbb{R}^N$. Using Property 1.18 and Property 1.16, we can get that

$$ {}_0D_t^\alpha u(t) = w(t) + C(T-t)^{\alpha-1}, \quad \text{a.e. } t \in [0,T],$$

for some $C \in \mathbb{R}^N$ and this completes the proof. □

Remark 5.52.

(i) According to (5.159) and Property 1.17, we have

$$ {}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t)) = {}_tD_T^{\alpha-1}({}_tD_T^{-\alpha}\nabla F(t,u(t))) + \tilde{C} = {}_tD_T^{-1}\nabla F(t,u(t)) + \tilde{C}$$

a.e. on $[0,T]$ for some $\tilde{C} \in \mathbb{R}^N$. In view of Definition 1.5 and $\nabla F(\cdot,u(\cdot)) \in L^1([0,T],\mathbb{R}^N)$, we shall identify the equivalence class ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u)$ and its continuous representant

$$ {}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t)) = \int_t^T \nabla F(s,u(s))ds + \tilde{C} \tag{5.160}$$

for $t \in [0,T]$.

(ii) It follows from (5.160) and a classical result of Lebesgue theory that $-\nabla F(\cdot,u(\cdot))$ is the classical derivative of ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u)$ a.e. on $[0,T]$.

We are now in a position to show that every weak solution of BVP (5.142) is also a solution of BVP (5.142).

Theorem 5.53. Let $0 < \alpha \leq 1$. If $u \in E^\alpha$ is a weak solution of BVP (5.142), then u is also a solution of BVP (5.142).

Proof. Firstly, We notice that ${}_0D_t^{\alpha-1}u(t)$ is derivative for almost every $t \in [0,T]$ and $({}_0D_t^{\alpha-1}u(t))' = {}_0D_t^\alpha u(t) \in L^2([0,T],\mathbb{R}^N)$ as $u \in E^\alpha$. On the other hand, Remark 5.52 implies that ${}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t))$ is derivative a.e. on $[0,T]$ and $({}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t)))' \in L^1([0,T],\mathbb{R}^N)$. Therefore, (i) in Definition 5.42 is verified.

It remains to show that u satisfies (5.142). In fact, according to Definition 1.6 and (5.160), we can get that

$$ {}_tD_T^\alpha({}_0D_t^\alpha u(t)) = -({}_tD_T^{\alpha-1}({}_0D_t^\alpha u(t)))' = \nabla F(t,u(t)), \quad \text{a.e. } t \in [0,T].$$

Moreover, $u \in E^\alpha$ implies that $u(0) = u(T) = 0$, and therefore (5.142) is verified. The proof is completed. □

The conclusions in Subsection 5.5.3 and Theorem 5.53 imply that BVP (5.142) with $\alpha \in (\frac{1}{2}, 1]$ possesses at least one solution if F satisfies some hypotheses. However, we would like to consider the existence of solutions for BVP (5.142) with $\alpha = \frac{1}{2}$ under the same hypotheses.

For any given $\epsilon_0 \in (0, \frac{1}{2})$, let $\epsilon \in (0, \epsilon_0)$ and $\delta = \delta(\epsilon) = \frac{1}{2} + \epsilon$. According to Corollary 5.47 and Theorem 5.49, if (A) and (5.151) with $a \in [0, a_0)$, or (A), (A1) and (A2)$'$ are satisfied, then $\forall \epsilon \in (0, \epsilon_0)$, the following BVP

$$\begin{cases} {}_tD_T^{\delta}\Big({}_0D_t^{\delta}u(t)\Big) = \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\ u(0) = u(T) = 0 \end{cases} \tag{5.161}$$

has at least a weak solution $u_\epsilon \in E^{\delta}$. Moreover, according to Theorem 5.53, u_ϵ is also the solution of BVP (5.161). Now, our idea is to obtain the solution of BVP (5.142) with $\delta = \frac{1}{2}$ by considering the approximation of u_ϵ as $\epsilon \to 0$.

Theorem 5.54. Assume that there exists $\epsilon_0 \in (0, \frac{1}{2})$ such that $\forall \epsilon \in (0, \epsilon_0)$ and $\delta = \delta(\epsilon) = \frac{1}{2} + \epsilon$, BVP (5.161) possesses a weak solution $u_\epsilon \in E^{\delta}$. Moreover, if the following conditions are satisfied

(A3) there exist $\beta > 2$ and $m \in L^{\beta}([0, T], \mathbb{R}^+)$ such that $|\nabla F(t, u_\epsilon(t))| \le m(t)$;

(A4) there exists $\beta_1 > 1/(\frac{1}{2} - \epsilon_0)$ such that ${}_0D_t^{\delta}u_\epsilon \in L^{\beta_1}([0, T], \mathbb{R}^N)$.

Then there exists a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n \to 0$ as $n \to \infty$, $u(t) = \lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly on $[0, T]$ and u is a solution of BVP (5.142) with $\alpha = \frac{1}{2}$.

Proof. According to Theorem 5.53, u_ϵ is also a solution of BVP (5.161). Thus, we have

$${}_tD_T^{\delta}\Big({}_0D_t^{\delta}u_\epsilon(t)\Big) = \nabla F(t, u_\epsilon(t)), \quad \text{a.e. } t \in [0, T]. \tag{5.162}$$

Properties 1.19-1.20 implies that the equation (5.162) is equivalent to the integral equation

$${}_0D_t^{\delta}u_\epsilon(t) = {}_tD_T^{-\delta}(\nabla F(t, u_\epsilon(t))) + C(T-t)^{\delta-1}, \quad \text{a.e. } t \in [0, T], \tag{5.163}$$

where $C = (1/\Gamma(\delta))[{}_tD_T^{\delta-1}({}_0D_t^{\delta}u_\epsilon(t))]_{t=T}$. Noting that ${}_0D_t^{\delta}u_\epsilon \in L^{\beta_1}([0, T], \mathbb{R}^N)$ according to (A4), direct calculation gives that

$$\begin{aligned} \Big|{}_tD_T^{\delta-1}\Big({}_0D_t^{\delta}u_\epsilon(t)\Big)\Big| &\le \frac{1}{\Gamma(1-\delta)} \int_t^T (s-t)^{-\delta} |{}_0D_s^{\delta}u_\epsilon(s)| ds \\ &\le \frac{1}{\Gamma(1-\delta)} \left(\int_t^T (s-t)^{\frac{-\delta\beta_1}{\beta_1-1}} ds \right)^{1-1/\beta_1} \|{}_0D_t^{\delta}u_\epsilon\|_{L^{\beta_1}} \\ &\le c(T-t)^{1-\delta-1/\beta_1} \|{}_0D_t^{\delta}u_\epsilon\|_{L^{\beta_1}}, \end{aligned}$$

where $c \in \mathbb{R}^+$ is a constant. It is obvious that $1-\delta-1/\beta_1 > 0$ since $\beta_1 > 1/(\frac{1}{2}-\epsilon_0) > 1/(1-\delta)$, then we have $C = (1/\Gamma(\delta))[{}_tD_T^{\delta-1}({}_0D_t^{\delta}u_\epsilon(t))]_{t=T} = 0$. Therefore, (5.163) can be written as

$${}_0D_t^{\delta}u_\epsilon(t) = {}_tD_T^{-\delta}(\nabla F(t, u_\epsilon(t))), \qquad \text{a.e. } t \in [0, T]. \tag{5.164}$$

According to Proposition 5.39 and in view of the continuity of $u_\epsilon \in E^\delta$, (5.164) is equivalent to the integral equation

$$u_\epsilon(t) = {}_0D_t^{-\delta}({}_tD_T^{-\delta}\nabla F(t, u_\epsilon(t))), \qquad t \in [0, T]. \tag{5.165}$$

On the other hand, we observe that $m \in L^\beta([0, T], \mathbb{R}^+)$ and $\beta > 2$ in (A3) imply that

$$\begin{aligned} |{}_tD_T^{-\delta}m(t)| &\leq \frac{1}{\Gamma(\delta)}\int_t^T (s-t)^{\delta-1}|m(s)|ds \\ &\leq \frac{1}{\Gamma(\delta)}\left(\int_t^T (s-t)^{\frac{(\delta-1)\beta}{\beta-1}}ds\right)^{1-1/\beta}\|m\|_{L^\beta} \\ &\leq c_1 T^{\delta-1/\beta}\|m\|_{L^\beta} \\ &\leq c_1\|m\|_{L^\beta}\max_{\lambda\in[\frac{1}{2},1]}\{T^{\lambda-1/\beta}\}, \quad t \in [0, T], \end{aligned}$$

where $c_1 \in \mathbb{R}^+$ is a constant. Therefore, there exists a constant $M \in \mathbb{R}^+$ such that $\|{}_tD_T^{-\delta}m\| \leq M$, which means that $|{}_tD_T^{-\delta}\nabla F(t, u_\epsilon(t))| \leq M$ on $[0, T]$ since $|\nabla F(t, u_\epsilon(t))| \leq m(t)$.

Set $G(t, u_\epsilon(t)) = {}_tD_T^{-\delta}\nabla F(t, u_\epsilon(t))$, and we have by (5.165)

$$\begin{aligned} |u_\epsilon(t)| &\leq \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}|G(s, u_\epsilon(s))|ds \\ &\leq \frac{M}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}ds \\ &\leq \frac{M}{\Gamma(\delta+1)}T^\delta \\ &\leq M\max_{\lambda\in[\frac{1}{2},1]}\left\{\frac{T^\lambda}{\Gamma(\lambda+1)}\right\}, \quad t \in [0, T]. \end{aligned} \tag{5.166}$$

The last inequality follows from the continuity of $T^\lambda/\Gamma(\lambda+1)$ with respect to $\lambda > 0$ and the fact that $\Gamma(\lambda) > 0$ for $\lambda > 0$. Furthermore, letting $0 \leq t_1 < t_2 \leq T$, we see that

$$\begin{aligned} &|u_\epsilon(t_1) - u_\epsilon(t_2)| \\ &= \frac{1}{\Gamma(\delta)}\left|\int_0^{t_1}(t_1-s)^{\delta-1}G(s, u_\epsilon(s))ds - \int_0^{t_2}(t_2-s)^{\delta-1}G(s, u_\epsilon(s))ds\right| \\ &= \frac{1}{\Gamma(\delta)}\left|\int_0^{t_1}[(t_1-s)^{\delta-1} - (t_2-s)^{\delta-1}]G(s, u_\epsilon(s))ds\right. \\ &\quad \left. + \int_{t_1}^{t_2}(t_2-s)^{\delta-1}G(s, u_\epsilon(s))ds\right| \\ &\leq \frac{M}{\Gamma(\delta)}\left|\int_0^{t_1}[(t_1-s)^{\delta-1} - (t_2-s)^{\delta-1}]ds + \int_{t_1}^{t_2}(t_2-s)^{\delta-1}ds\right| \\ &= \frac{M}{\Gamma(\delta+1)}[2(t_2-t_1)^\delta + t_1^\delta - t_2^\delta] \end{aligned} \tag{5.167}$$

$$\leq \frac{2M}{\Gamma(\delta+1)}(t_2 - t_1)^\delta$$
$$\leq 2M \max_{\lambda\in[\frac{1}{2},1]} \left\{ \frac{(t_2 - t_1)^\lambda}{\Gamma(\lambda+1)} \right\}.$$

It then follows from (5.166) and (5.167) that the family $\{u_\epsilon\}$ forms an equicontinuous and uniformly bounded functions. Application of the Ascoli-Arzela theorem shows the existence of a sequence $\{\epsilon_n\}$ such that $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_n \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} u_{\epsilon_n}(t)$ exists uniformly on $[0, T]$. Since the continuity and boundness of $\nabla F(t, \cdot)$ imply the continuity of ${}_tD_T^{-\delta}\nabla F(t, \cdot)$, we obtain that

$${}_tD_T^{-\delta}\nabla F(t, u_{\epsilon_n}(t)) \to {}_tD_T^{-\delta}\nabla F(t, u(t)), \quad \text{as } n \to \infty,$$

and combining (5.164) yields

$$u(t) = {}_0D_t^{-\delta}({}_tD_T^{-\delta}\nabla F(t, u(t))), \qquad t \in [0, T].$$

This proves that u is a solution of BVP (5.142) by using the Property 1.18 and Lemma 5.1. The proof is completed. □

Example 5.55. Set $F(t, x) = m(t)\sin(|x|)$, where $m \in L^\beta([0, T], \mathbb{R}^+)$ and $x \in \mathbb{R}^N$. Then (A3) is verified since $|F(t, x)| \leq m(t)$ for $x \in \mathbb{R}^N$. If for any $\epsilon \in (0, \epsilon_0)$, we have $u_\epsilon \in AC([0, T], \mathbb{R}^N)$ and $u'_\epsilon \in L^{\beta_1}([0, T], \mathbb{R}^N)$, then ${}_0D_t^\delta u_\epsilon \in L^{\beta_1}([0, T], \mathbb{R}^N)$ by using Property 1.9 and (5.5). Thus, (A4) is satisfied.

5.6 Notes and Remarks

The results in Subsections 5.2.1-5.2.4 are taken from Jiao and Zhou, 2011. The material in Subsections 5.2.5-5.2.6 due to Chen and Tang, 2012. The results in Section 5.3 are adopted from Kong, 2013. The main results of Section 5.4 are from Chen and Tang, 2013. The material in Section 5.5 due to Jiao and Zhou, 2012.

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Chapter 6

Fractional Partial Differential Equations

6.1 Introduction

The main objective of this chapter is to investigate the existence theory for a variety of fractional partial differential equations with applications. Section 6.2 is devoted to study the existence of a weak solution for Euler-Lagrange equations. In Section 6.3, we investigate the regularity and unique existence of the solution for initial-boundary value problems of diffusion equation with multiple time-fractional derivatives. In Section 6.4, we discuss existence and multiplicity of solutions of fractional Hamiltonian systems. And in the last Subsection, we present some results on existence of mild solution for fractional Schrödinger equations.

6.2 Fractional Euler-Lagrange Equations

6.2.1 *Introduction*

In this section, we consider $a < b$ two reals, $d \in \mathbb{N}$ and the following Lagrangian functional

$$\mathfrak{L}(u) = \int_a^b L(u, {}_aD_t^\alpha u, t)dt,$$

where L is a Lagrangian, i.e., a map of the form:

$$\begin{aligned} L : \mathbb{R}^d \times \mathbb{R}^d \times [a,b] &\to \mathbb{R} \\ (x, y, t) &\to L(x, y, t), \end{aligned}$$

where ${}_aD_t^\alpha$ is the left fractional derivative of Riemann-Liouville of order $0 < \alpha < 1$ and where the variable u is a function defined almost everywhere on (a, b) with values in $\mathbb{R}^d$. It is well-known that critical points of the functional L are characterized by the solutions of the fractional Euler-Lagrange equation:

$$\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t) + {}_tD_b^\alpha \left(\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t) \right) = 0, \tag{6.1}$$

where ${}_tD_b^\alpha$ is the right fractional derivative of Riemann-Liouville, see detailed proofs in Agrawal, 2002; Baleanu and Muslih, 2005 for example.

For any $p \geq 1, L^p := L^p((a,b),\mathbb{R}^d)$ denotes the classical Lebesgue space of p-integrable functions endowed with its usual norm $\|\cdot\|_{L^p}$. We denote by $|\cdot|$ the Euclidean norm of $\mathbb{R}^d$ and $C := C([a,b],\mathbb{R}^d)$ the space of continuous functions endowed with its usual norm $\|\cdot\|$. We remind that a function f is an element of AC if and only if $f' \in L^1$ and the following equality holds

$$\forall t \in [a,b], \quad f(t) = f(a) + \int_a^t f'(\xi)d\xi,$$

where f' denotes the derivative of f. We refer to Kolmogorov, Fomine and Tihomirov, 1974 for more details concerning the absolutely continuous functions. In addition, we denote by C_{a} (resp. AC_{a} or C_{a}^∞) the space of functions $f \in C$ (resp. AC or C^∞) such that $f(a) = 0$. In particular, $C_c^\infty \subset C_a^\infty \subset AC_a$.

Remark 6.1. In the whole section, an equality between functions must be understood as an equality holding for almost all $t \in (a,b)$. When it is not the case, the interval on which the equality is valid will be specified.

Definition 6.2. A function u is said to be a weak solution of (6.1) if $u \in C$ and if u satisfies (6.1) a.e. on [a,b].

In the following, we will provide some properties concerning the left fractional operators of Riemann-Liouville. One can easily derive the analogous versions for the right ones. Property 6.3 is well-known and one can find their proofs in the classical literature on the subject (see Lemma 2.1 in Kilbas, Srivastava and Trujillo, 2006).

Property 6.3. For any $\alpha > 0$ and any $p \geq 1$, ${}_aD_t^{-\alpha}$ is linear and continuous from L^p to L^p. Precisely, the following inequality holds

$$\|{}_aD_t^{-\alpha} f\|_{L^p} \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|f\|_{L^p}, \quad \text{for } f \in L^p.$$

The following classical property concerns the integration of fractional integrals. It is occasionally called fractional integration by parts:

Property 6.4. Let $0 < \frac{1}{p} < \alpha < 1$ and $q = \frac{p}{p-1}$. Then, for any $f \in L^p$, we have

(i) ${}_aD_t^{-\alpha}$ is Hölder continuous on $[a,b]$ with exponent $\alpha - \frac{1}{p} > 0$;

(ii) $\lim\limits_{t\to a} {}_aD_t^{-\alpha} f(t) = 0$.

Consequently, ${}_aD_t^{-\alpha} f(t)$ can be continuously extended by 0 in $t = a$. Finally, for any $f \in L^p$, we have ${}_aD_t^{-\alpha} f \in C_a$. Moreover, the following inequality holds

$$\|{}_aD_t^{-\alpha} f\| \leq \frac{(b-a)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|f\|_{L^p}, \quad \text{for } f \in L^p.$$

Proof. Let us note that this result is mainly proved in Section 5.2. Let $f \in L^p$. We first remind the following inequality

$$(\xi_1 - \xi_2)^q \leq \xi_1^q - \xi_2^q, \quad \text{for } \xi_1 \geq \xi_2 \geq 0.$$

Let us prove that ${}_aD_t^{-\alpha}f(t)$ is Hölder continuous on $[a,b]$. For any $a < t_1 < t_2 \leq b$, using Hölder's inequality, we have

$$\begin{aligned}
|{}_aD_t^{-\alpha}f(t_2) - {}_aD_t^{-\alpha}f(t_1)| &= \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_2}(t_2-\xi)^{\alpha-1}f(\xi)d\xi - \int_a^{t_1}(t_1-\xi)^{\alpha-1}f(\xi)d\xi\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-\xi)^{\alpha-1}f(\xi)d\xi\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}((t_2-\xi)^{\alpha-1}-(t_1-\xi)^{\alpha-1})f(\xi)d\xi\right| \\
&\leq \frac{\|f\|_{L^p}}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-\xi)^{(\alpha-1)q}d\xi\right)^{\frac{1}{q}} \\
&\quad + \frac{\|f\|_{L^p}}{\Gamma(\alpha)}\left(\int_a^{t_1}((t_1-\xi)^{\alpha-1}-(t_2-\xi)^{\alpha-1})^q d\xi\right)^{\frac{1}{q}} \\
&\leq \frac{\|f\|_{L^p}}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-\xi)^{(\alpha-1)q}d\xi\right)^{\frac{1}{q}} \\
&\quad + \frac{\|f\|_{L^p}}{\Gamma(\alpha)}\left(\int_a^{t_1}(t_1-\xi)^{(\alpha-1)q}-(t_2-\xi)^{(\alpha-1)q}d\xi\right)^{\frac{1}{q}} \\
&\leq \frac{2\|f\|_{L^p}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}(t_2-t_1)^{\alpha-\frac{1}{p}}.
\end{aligned}$$

The proof of the first point is complete. Let us consider the second point. For any $t \in [a,b]$, we can prove in the same manner that

$$|{}_aD_t^{-\alpha}f(t)| \leq \frac{\|f\|_{L^p}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}(t-a)^{\alpha-\frac{1}{p}}, \quad \text{as} \quad t \to 0.$$

The proof is now complete. □

6.2.2 *Functional Spaces*

In order to prove the existence of a weak solution of (6.1) using a variational method, we need the introduction of an appropriate space of functions. This space has to present some properties like reflexivity, see Dacorogna, 2008.

For any $0 < \alpha < 1$ and any $p \geq 1$, we define the following space of functions

$$E_{\alpha,p} := \{u \in L^p \quad \text{satisfying} \quad {}_aD_t^{\alpha}u \in L^p \quad \text{and} \quad {}_aD_t^{-\alpha}({}_aD_t^{\alpha}u) = u \quad \text{a.e.}\}.$$

We endow $E_{\alpha,p}$ with the following norm

$$\begin{aligned}
&\|\cdot\|_{\alpha,p} : E_{\alpha,p} \to \mathbb{R}^+, \\
&u \mapsto (\|u\|_{L^p}^p + \|{}_aD_t^{\alpha}u\|_{L^p}^p)^{\frac{1}{p}}.
\end{aligned}$$

Let us note that

$$\begin{aligned}
&|\cdot|_{\alpha,p} : E_{\alpha,p} \to \mathbb{R}^+, \\
&u \mapsto \|{}_aD_t^{\alpha}u\|_{L^p}
\end{aligned}$$

is an equivalent norm to $\|\cdot\|_{\alpha,p}$ for $E_{\alpha,p}$. Indeed, Property 6.3 leads to

$$\|u\|_{L^p} = \|{}_aD_t^{-\alpha}({}_aD_t^\alpha u)\|_{L^p} \leq \frac{(b-a)^\alpha}{\Gamma(1+\alpha)}\|{}_aD_t^\alpha u\|_{L^p}, \quad \text{for } u \in E_{\alpha,p}. \tag{6.2}$$

The goal of this section is to prove the following proposition:

Proposition 6.5. Assuming $0 < \frac{1}{p} < \alpha < 1$, $E_{\alpha,p}$ is a reflexive separable Banach space and the compact embedding $E_{\alpha,p} \hookrightarrow C_a$ holds.

Proof. Consider that

$$0 < \frac{1}{p} < \alpha < 1 \quad \text{and} \quad q = \frac{p}{p-1}.$$

Now, we divide the proof into several steps.

Step I. $E_{\alpha,p}$ is a reflexive separable Banach space.

Let us consider $(L^p)^2$ the set $L^p \times L^p$ endowed with the norm $\|(u,v)\|_{(L_p)^2} = (\|u\|_{L^p}^p + \|v\|_{L^p}^p)^{\frac{1}{p}}$. Since $p > 1, (L^p, \|\cdot\|_{L^p})$ is a reflexive separable Banach space and therefore, $((L^p)^2, \|\cdot\|_{(L^p)^2})$ is also a reflexive separable Banach space. We define $\Omega := \{(u, {}_aD_t^\alpha u) : \ u \in E_{\alpha,p}\}$. Let us prove that Ω is a closed subspace of $((L^p)^2, \|\cdot\|_{(L^p)^2})$. Let $(u_n, v_n)_{n\in\mathbb{N}} \subset \Omega$ such that

$$(u_n, v_n) \xrightarrow{(L^p)^2} (u,v).$$

Then, we prove that $(u,v) \in \Omega$. For any $n \in \mathbb{N}, (u_n, v_n) \in \Omega$. Thus, $u_n \in E_{\alpha,p}$ and $v_n = {}_aD_t^\alpha u_n$. Consequently, we have

$$u_n \xrightarrow{L^p} u \quad \text{and} \quad {}_aD_t^\alpha u_n \xrightarrow{L^p} v.$$

For any $n \in \mathbb{N}$, since $u_n \in E_{\alpha,p}$ and ${}_aD_t^{-\alpha}$ is continuous from L^p to L^p, we have

$$u_n = {}_aD_t^{-\alpha}({}_aD_t^\alpha u_n) \xrightarrow{L^p} {}_aD_t^{-\alpha}v.$$

Thus, $u = {}_aD_t^{-\alpha}v,\ {}_aD_t^\alpha u = {}_aD_t^\alpha({}_aD_t^{-\alpha}v) = v \in L^p$ and ${}_aD_t^{-\alpha}({}_aD_t^\alpha u) = {}_aD_t^{-\alpha}v = u$. Hence, $u \in E_{\alpha,p}$ and $(u,v) = (u, {}_aD_t^\alpha u) \in \Omega$. In conclusion, Ω is a closed subspace of $((L^p)^2,\ \|\cdot\|_{(L^p)^2})$ and then Ω is a reflexive separable Banach space. Finally, defining the following operator

$$\begin{aligned} &A : E_{\alpha,p} \to \Omega, \\ &u \mapsto (u, {}_aD_t^\alpha u), \end{aligned}$$

we prove that $E_{\alpha,p}$ is isometric isomorphic to Ω. This completes the proof of Step I.

Step II. The continuous embedding $E_{\alpha,p} \hookrightarrow C_a$.

Let $u \in E_{\alpha,p}$ and then ${}_aD_t^\alpha u \in L^p$. Since $0 < \frac{1}{p} < \alpha < 1$, Property 6.4 leads to ${}_aD_t^{-\alpha}({}_aD_t^\alpha u) \in C_a$. Furthermore, $u = {}_aD_t^{-\alpha}({}_aD_t^\alpha u)$ and consequently, u can be identified to its continuous representative. Finally, Property 6.4 also gives

$$\|u\| = \|{}_aD_t^{-\alpha}({}_aD_t^\alpha u)\| \leq \frac{(b-a)^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}|u|_{\alpha,p}, \quad \text{for } u \in E_{\alpha,p}.$$

Since $\|\cdot\|_{\alpha,p}$ and $|\cdot|_{\alpha,p}$ are equivalent norms, the proof of Step II is complete.

Step III. The compact embedding $E_{\alpha,p} \hookrightarrow C_a$.

Since $E_{\alpha,p}$ is a reflexive Banach space, we only have to prove that

$$\forall (u_n)_{n\in\mathbb{N}} \subset E_{\alpha,p} \quad \text{such that} \quad u_n \overset{E_{\alpha,p}}{\rightharpoonup} u, \quad \text{then} \quad u_n \overset{\mathcal{C}}{\to} u.$$

Let $(u_n)_{n\in\mathbb{N}} \subset E_{\alpha,p}$ such that

$$u_n \overset{E_{\alpha,p}}{\rightharpoonup} u.$$

Since $E_{\alpha,p} \hookrightarrow C_a$, we have

$$u_n \overset{\mathcal{C}}{\rightharpoonup} u.$$

Since $(u_n)_{n\in\mathbb{N}}$ converges weakly in $E_{\alpha,p}$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $E_{\alpha,p}$. Consequently, $({}_aD_t^{\alpha}u_n)_{n\in\mathbb{N}}$ is bounded in L_p by a constant $M \geq 0$. Let us prove that $(u_n)_{n\in\mathbb{N}} \subset C_a$ is uniformly Lipschitzian on $[a,b]$. According to the proof of Property 6.4, for $\forall n \in \mathbb{N}, \forall a \leq t_1 < t_2 \leq b$, we have,

$$\begin{aligned}
|u_n(t_2) - u_n(t_1)| &\leq |{}_aD_t^{-\alpha}({}_aD_t^{\alpha}u_n(t_2)) - {}_aD_t^{-\alpha}({}_aD_t^{\alpha}u_n(t_1))| \\
&\leq \frac{2\|{}_aD_t^{\alpha}u_n\|_{L^p}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{p}}}(t_2-t_1)^{\alpha-\frac{1}{p}} \\
&\leq \frac{2M}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{p}}}(t_2-t_1)^{\alpha-\frac{1}{p}}.
\end{aligned}$$

Hence, from Ascoli's theorem, $(u_n)_{n\in\mathbb{N}}$ is relatively compact in C. Consequently, there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging strongly in C and the limit is u by uniqueness of the weak limit.

Now, let us prove by contradiction that the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly to u in C. If not, there exist $\varepsilon > 0$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$\|u_{n_k} - u\| > \varepsilon > 0, \quad \text{for } k \in \mathbb{N}. \tag{6.3}$$

Nevertheless, since $(u_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(u_n)_{n\in\mathbb{N}}$, then it satisfies

$$u_{n_k} \overset{E_{\alpha,p}}{\rightharpoonup} u.$$

In the same way (using Ascoli's theorem), we can construct a subsequence of $(u_{n_k})_{k\in\mathbb{N}}$ converging strongly to u in C which is a contradiction to (6.3). The proof of Step III is now complete. □

Let us remind the following property

$${}_aD_t^{-\alpha}\varphi \in C_a^{\infty}, \quad \text{for } \varphi \in C_c^{\infty}.$$

From this result, we get the following results.

Proposition 6.6. C_a^{∞} is dense in $E_{\alpha,p}$.

Proof. Indeed, let us first prove that $C_a^{\infty} \subset E_{\alpha,p}$. Let $u \in C_a^{\infty} \subset L^p$. Since $u \in AC_a$ and $u' \in L^p$, we have ${}_aD_t^{\alpha}u = {}_aD_t^{\alpha-1}u' \in L^p$. Since $u \in AC$, we also have

${}_aD_t^{-\alpha}({}_aD_t^\alpha u) = u$. Finally, $u \in E_{\alpha,p}$. Now, let us prove that C_a^∞ is dense in $E_{\alpha,p}$. Let $u \in E_{\alpha,p}$, then ${}_aD_t^\alpha u \in L^p$. Consequently, there exists $(v_n)_{n\in\mathbb{N}} \subset C_c^\infty$ such that

$$v_n \xrightarrow{L^p} {}_aD_t^\alpha u \quad \text{and then} \quad {}_aD_t^{-\alpha} v_n \xrightarrow{L^p} {}_aD_t^{-\alpha}({}_aD_t^\alpha u) = u,$$

since ${}_aD_t^{-\alpha}$ is continuous from L^p to L^p. Defining $u_n := {}_aD_t^{-\alpha} v_n \in C_a^\infty$ for any $n \in \mathbb{N}$, we obtain

$$u_n \xrightarrow{L^p} u \quad \text{and} \quad {}_aD_t^\alpha u_n = {}_aD_t^\alpha({}_aD_t^{-\alpha} v_n) = v_n \xrightarrow{L^p} {}_aD_t^\alpha u.$$

Finally, $(u_n)_{n\in\mathbb{N}} \subset C_a^\infty$ and converges to u in $E_{\alpha,p}$. The proof is completed. □

Proposition 6.7. If $\frac{1}{p} < \min(\alpha, 1-\alpha)$, then $E_{\alpha,p} = \{u \in L^p \text{ satisfying } {}_aD_t^\alpha u \in L^p\}$.

Proof. Indeed, let $u \in L^p$ satisfying ${}_aD_t^\alpha u \in L^p$ and let us prove that ${}_aD_t^{-\alpha}({}_aD_t^\alpha u) = u$. Let $\varphi \in C_c^\infty \subset L^1$. Since ${}_aD_t^\alpha u \in L^p$, Property 1.23 leads to

$$\int_a^b {}_aD_t^{-\alpha}({}_aD_t^\alpha u) \cdot \varphi dt = \int_a^b {}_aD_t^\alpha u \cdot {}_tD_b^{-\alpha} \varphi dt = \int_a^b \frac{d}{dt}({}_aD_t^{\alpha-1} u) \cdot {}_tD_b^{-\alpha} \varphi dt.$$

Then, an integration by parts gives

$$\int_a^b {}_aD_t^{-\alpha}({}_aD_t^\alpha u) \cdot \varphi dt = \int_a^b {}_aD_t^{\alpha-1} u \cdot {}_tD_b^{1-\alpha} u dt.$$

Indeed, ${}_tD_b^{-\alpha}\varphi(b) = 0$ since $\varphi \in C_c^\infty$ and ${}_aD_t^{\alpha-1}u(a) = 0$ since $u \in L^p$ and $\frac{1}{p} < 1-\alpha$. Finally, using Property 1.23 again, we obtain

$$\int_a^b {}_aD_t^{-\alpha}({}_aD_t^\alpha u) \cdot \varphi dt = \int_a^b u \cdot {}_tD_b^{\alpha-1}({}_tD_b^{1-\alpha}\varphi) dt = \int_a^b u \cdot \varphi dt,$$

this completes the proof. □

Remark 6.8. In the Proposition 6.7, let us note that such a definition of $E_{\alpha,p}$ could lead us to name it fractional Sobolev space and to denote it by $W^{\alpha,p}$. Nevertheless, these notions and notations are already used, see Brezis, 2011.

6.2.3 *Variational Structure*

In this subsection, we assume that Lagrangian L is of class C^1 and we define the Lagrangian functional $\mathfrak{L}$ on $E_{\alpha,p}$ (with $0 < \frac{1}{p} < \alpha < 1$). Precisely, we define

$$\mathfrak{L} : E_{\alpha,p} \to \mathbb{R},$$

$$u \mapsto \int_a^b L(u, {}_aD_t^\alpha u, t) dt.$$

$\mathfrak{L}$ is said to be Gâteaux-differentiable in $u \in E_{\alpha,p}$ if the map

$$D\mathfrak{L}(u) : E_{\alpha,p} \to \mathbb{R},$$

$$v \mapsto D\mathfrak{L}(u)(v) := \lim_{h\to 0} \frac{\mathfrak{L}(u+hv) - \mathfrak{L}(u)}{h}$$

is well-defined for any $v \in E_{\alpha,p}$ and if it is linear and continuous. A critical point $u \in E_{\alpha,p}$ of $\mathfrak{L}$ is defined by $D\mathfrak{L}(u) = 0$.

We introduce the following hypotheses:

(H1) there exist $0 \le d_1 \le p$ and $r_1, s_1 \in C(\mathbb{R}^d \times [a,b], \mathbb{R}^+)$ such that

$$|L(x,y,t) - L(x,0,t)| \le r_1(x,t)\|y\|^{d_1} + s_1(x,t), \quad \text{for } (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a,b];$$

(H2) there exist $0 \le d_2 \le p$ and $r_2, s_2 \in C(\mathbb{R}^d \times [a,b], \mathbb{R}^+)$ such that

$$\left\|\frac{\partial L}{\partial x}(x,y,t)\right\| \le r_2(x,t)\|y\|^{d_2} + s_2(x,t), \quad \text{for } (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a,b];$$

(H3) there exist $0 \le d_3 \le p-1$ and $r_3, s_3 \in C(\mathbb{R}^d \times [a,b], \mathbb{R}^+)$ such that

$$\left\|\frac{\partial L}{\partial y}(x,y,t)\right\| \le r_3(x,t)\|y\|^{d_3} + s_3(x,t), \quad \text{for } (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a,b];$$

(H4) *coercivity condition:* there exist $\gamma > 0, 1 \le d_4 < p, c_1 \in C(\mathbb{R}^d \times [a,b], [\gamma,\infty))$, $c_2, c_3 \in C([a,b],\mathbb{R})$ such that

$$\forall (x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times [a,b], \quad L(x,y,t) \ge c_1(x,t)\|y\|^p + c_2(t)\|x\|^{d_4} + c_3(t);$$

(H5) *convexity condition:*

$$\forall t \in [a,b], L(\cdot,\cdot,t) \ \text{ is convex.}$$

Hypotheses denoted by (H1)-(H3) are usually called regularity hypotheses (see Cesari, 1983; Dacorogna, 2008).

Let us prove the following results.

Lemma 6.9. The following implications hold

(i) L satisfies $(H1) \Rightarrow$ for any $u \in E_{\alpha,p}, L(u, {}_aD_t^\alpha u, t) \in L^1$ and then $\mathfrak{L}(u)$ exists in $\mathbb{R}$;

(ii) L satisfies $(H2) \Rightarrow$ for any $u \in E_{\alpha,p}, \partial L/\partial x(u, {}_aD_t^\alpha u, t) \in L^1$;

(iii) L satisfies $(H3) \Rightarrow$ for any $u \in E_{\alpha,p}, \partial L/\partial y(u, {}_aD_t^\alpha u, t) \in L^q$, where $q = \frac{p}{p-1}$.

Proof. Let us assume that $\mathfrak{L}$ satisfies (H1) and let $u \in E_{\alpha,p} \subset C_a$. Then, $\|{}_aD_t^\alpha u\|^{d_1} \in L^{p/d_1} \subset L^1$ and the three maps $t \to r_1(u(t),t)$, $s_1(u(t),t)$, $|L(u(t),0,t)| \in C([a,b],\mathbb{R}^+) \subset L^\infty \subset L^1$. Hypothesis (H1) implies for almost all $t \in [a,b]$

$$|L(u(t), {}_aD_t^\alpha u(t), t)| \le r_1(u(t),t)\|{}_aD_t^\alpha u(t)\|^{d_1} + s_1(u(t),t) + |L(u(t),0,t)|.$$

Hence, $L(u, {}_aD_t^\alpha u(t), t) \in L^1$ and then $L(u)$ exists in $\mathbb{R}$. We proceed in the same manner in order to prove the second point of Lemma 6.9. Now, assuming that L satisfies (H3), we have $\|{}_aD_t^\alpha u\|^{d_3} \in L^{p/d_3} \subset L^q$ for any $u \in E_{\alpha,p}$. An analogous argument gives the third point of Lemma 6.9. This completes the proof. □

Lemma 6.10. Assuming that L satisfies Hypotheses (H1)-(H3), $\mathfrak{L}$ is Gâteaux-differentiable in any $u \in E_{\alpha,p}$ and

$$D\mathfrak{L}(u)(v) = \int_a^b \left[\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t) \cdot v + \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t) \cdot {}_aD_t^\alpha v\right] dt, \quad \text{for } u, v \in E_{\alpha,p}.$$

Proof. Let $u,v \in E_{\alpha,p} \subset C_a$. Let $\psi_{u,v}$ defined for any $h \in [-1,1]$ and for almost all $t \in [a,b]$ by

$$\psi_{u,v}(t,h) := L\Big(u(t) + hv(t), {}_aD_t^\alpha u(t) + h{}_aD_t^\alpha v(t), t\Big).$$

Then, we define the following mapping

$$\begin{aligned}&\phi_{u,v} : [-1,1] \to \mathbb{R},\\ &h \mapsto \int_a^b L\Big(u + hv(t), {}_aD_t^\alpha u + h{}_aD_t^\alpha v, t\Big)dt = \int_a^b \psi_{u,v}(t,h)dt.\end{aligned}$$

Our aim is to prove that the following term

$$D\mathfrak{L}(u)(v) = \lim_{h\to 0} \frac{\mathfrak{L}(u+hv) - \mathfrak{L}(u)}{h} = \lim_{h\to 0} \frac{\phi_{u,v}(h) - \phi_{u,v}(0)}{h} = \phi'_{u,v}(0)$$

exists in $\mathbb{R}$. In order to differentiate $\phi_{u,v}$, we use the theorem of differentiation under the integral sign. Indeed, we have for almost all $t \in [a,b]$, $\psi_{u,v}(t,\cdot)$ is differentiable on $[-1,1]$ with

$$\begin{aligned}\frac{\partial \psi_{u,v}}{\partial h}(t,h) = {}&\frac{\partial L}{\partial x}\Big(u(t) + hv(t), {}_aD_t^\alpha u(t) + h{}_aD_t^\alpha v(t), t\Big)\cdot v(t)\\ &+\frac{\partial L}{\partial y}\Big(u(t) + hv(t), {}_aD_t^\alpha u(t) + h{}_aD_t^\alpha v(t), t\Big)\cdot {}_aD_t^\alpha v(t).\end{aligned}$$

Then, from Hypotheses (H2) and (H3), we have for any $h \in [-1,1]$ and for almost all $t \in [a,b]$

$$\begin{aligned}&\left|\frac{\partial \psi_{u,v}}{\partial h}(t,h)\right|\\ &\le \big[r_2(u(t)+hv(t),t)\|{}_aD_t^\alpha u(t) + h{}_aD_t^\alpha v(t)\|^{d_2} + s_2(u(t)+hv(t),t)\big]\|v(t)\|\\ &\quad+\big[r_3(u(t)+hv(t),t)\|{}_aD_t^\alpha u(t) + h{}_aD_t^\alpha v(t)\|^{d_3} + s_3(u(t)+hv(t),t)\big]\|{}_aD_t^\alpha v(t)\|.\end{aligned}$$

We define

$$r_{2,0} := \max_{(t,h)\in[a,b]\times[-1,1]} r_2(u(t)+hv(t),t)$$

and we define similarly $s_{2,0}, r_{3,0}, s_{3,0}$. Finally, it holds

$$\begin{aligned}\left|\frac{\partial \psi_{u,v}}{\partial h}(t,h)\right| \le {}&2^{d_2} r_{2,0}\underbrace{\left(\|{}_aD_t^\alpha u(t)\|^{d_2} + \|{}_aD_t^\alpha v(t)\|^{d_2}\right)}_{\in L^{p/d_2}\subset L^1}\underbrace{\|v(t)\|}_{\in C_a\subset L^\infty} + s_{2,0}\underbrace{\|v(t)\|}_{\in C_a\subset L^1}\\ &+2^{d_3} r_{3,0}\underbrace{\left(\|{}_aD_t^\alpha u(t)\|^{d_3} + \|{}_aD_t^\alpha v(t)\|^{d_3}\right)}_{\in L^{p/d_3}\subset L^q}\underbrace{\|{}_aD_t^\alpha v(t)\|}_{\in L^p} + s_{3,0}\underbrace{\|{}_aD_t^\alpha v(t)\|}_{\in L^p\subset L^1}.\end{aligned}$$

The right term is then a L^1 function independent of h. Consequently, applying the theorem of differentiation under the integral sign, $\phi_{u,v}$ is differentiable with

$$\phi'_{u,v}(h) = \int_a^b \frac{\partial \psi_{u,v}}{h}(t,h)dt, \quad \text{for } h \in [-1,1].$$

Hence

$$\begin{aligned} D\mathfrak{L}(u)(v) = \phi'_{u,v}(0) &= \int_a^b \frac{\partial \psi_{u,v}}{h}(t,0)dt \\ &= \int_a^b \left[\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha, t)\ v + \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\ {}_aD_t^\alpha v\right] dt. \end{aligned}$$

From Lemma 6.9, it holds

$$\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t) \in L^1 \quad \text{and} \quad \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t) \in L^q.$$

Since $v \in C_a \subset L^\infty$ and ${}_aD_t^\alpha \in L^p, D\mathfrak{L}(u)(v)$ exists in $\mathbb{R}$. Moreover, we have

$$\begin{aligned} |D\mathfrak{L}(u)(v)| &\le \left\| \frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t)\right\|_{L^1} \|v\| + \left\|\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\right\|_{L^q} \|{}_aD_t^\alpha v\|_{L^p} \\ &\le \left(\frac{(b-a)^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\left\|\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t)\right\|_{L^1} + \left\|\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\right\|_{L^q}\right)|v|_{\alpha,p}. \end{aligned}$$

Consequently, $D\mathfrak{L}(u)$ is linear and continuous from $E_{\alpha,p}$ to $\mathbb{R}$. The proof is completed. □

6.2.4 *Existence of Weak Solution*

In this subsection, we will present the existence theorem of weak solution for (6.1). We firstly give two preliminary theorems.

Theorem 6.11. Assume that L satisfies Hypotheses (H1)-(H3). If u is a critical point of $\mathfrak{L}$, u is a weak solution of (6.1).

Proof. Let u be a critical point of $\mathfrak{L}$. Then, we have in particular

$$D\mathfrak{L}(u)(v) = \int_a^b \left[\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t)\cdot v + \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\cdot {}_aD_t^\alpha v\right] dt = 0, \quad \text{for } v \in C_c^\infty.$$

For any $v \in C_c^\infty \subset AC_a,\ {}_aD_t^\alpha v = {}_aD_t^{\alpha-1} v' \in C_a^\infty$. Since $\partial L/\partial y(u, {}_aD_t^\alpha u, t) \in L^q$, Property 1.23 gives

$$\int_a^b \left[\frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t)\cdot v + {}_tD_b^{\alpha-1}\left(\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\right)\cdot v'\right] dt = 0, \quad \text{for } v \in C_c^\infty.$$

Finally, we define

$$w_u(t) = \int_a^t \frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t)dt, \quad \text{for } t \in [a,b].$$

Since $\partial L/\partial x(u, {}_aD_t^\alpha, t) \in L^1, w_u \in AC_a$ and $w_u' = \partial L/\partial x(u, {}_aD_t^\alpha u, t)$. Then, an integration by parts leads to

$$\int_a^b \left({}_tD_b^{\alpha-1}\Big(\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\Big) - w_u\right) v' dt = 0, \quad \text{for } v \in C_c^\infty.$$

Consequently, there exists a constant $C \in \mathbb{R}^d$ such that

$$ {}_tD_b^{\alpha-1}\left(\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\right) = C + w_u \in AC. $$

By differentiation, we obtain

$$ -{}_tD_b^{\alpha}\left(\frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\right) = \frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t), $$

and then $u \in E_{\alpha,p} \subset C$ satisfies (6.1) a.e. on $[a, b]$. The proof is completed. □

As usual in a variational method, in order to prove the existence of a global minimizer of a functional, coercivity and convexity hypotheses need to be added on the Lagrangian. We have already define Hypotheses (H4) (coercivity) and (H5) (convexity). Next, we introduce two different convexity hypotheses (H5)$'$ and (H5)$''$:

(H5)$'$ $\forall (x,t) \in \mathbb{R}^d \times [a,b]$, $L(x, \cdot, t)$ is convex and $(L(\cdot, y, t))_{(y,t)\in\mathbb{R}^d\times[a,b]}$ is uniformly equicontinuous on $\mathbb{R}^d$, i.e.,

$$ \begin{aligned} &\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall (x_1, x_2) \in (\mathbb{R}^d)^2,\ \|x_2 - x_1\| < \delta \\ &\Rightarrow \forall (y,t) \in \mathbb{R}^d \times [a,b],\ |L(x_2, y, t) - L(x_1, y, t)| < \varepsilon. \end{aligned} $$

(H5)$''$ $\forall (x,t) \in \mathbb{R}^d \times [a,b]$, $L(x, \cdot, t)$ is convex.

Let us note that Hypotheses (H5) and (H5)$'$ are independent. Hypothesis (H5)$''$ is the weakest. Nevertheless, in this case, the detailed proof of Theorem 6.13 is more complicated. Consequently, in the case of Hypothesis (H5)$''$, we do not develop the proof and we use a strong result proved in Dacorogna, 2008. Let us prove the following preliminary result.

Lemma 6.12. Assume that L satisfies Hypothesis (H4). Then, $\mathfrak{L}$ is coercive in the sense that

$$ \lim_{\|u\|_{\alpha,p}\to+\infty} \mathfrak{L}(u) = +\infty. $$

Proof. Let $u \in E_{\alpha,p}$, we have

$$ \mathfrak{L}(u) = \int_a^b L(u, {}_aD_t^\alpha u, t)dt \ge \int_a^b c_1(u,t)\|{}_aD_t^\alpha u\|^p + c_2(t)\|u\|^{d_4} + c_3(t)dt. $$

Eq.(6.2) implies that

$$ \|u\|_{L^{d_4}}^{d_4} \le (b-a)^{1-\frac{d_4}{p}}\|u\|_{L^p}^{d_4} \le \frac{(b-a)^{\alpha+1-\frac{d_4}{p}}}{\Gamma(\alpha+1)}\|{}_aD_t^\alpha u\|_{L^p}^{d_4} = \frac{(b-a)^{\alpha+1-\frac{d_4}{p}}}{\Gamma(\alpha+1)}|u|_{\alpha,p}^{d_4}. $$

Finally, we conclude that

$$ \begin{aligned} \mathfrak{L}(u) &\ge \gamma\|{}_aD_t^\alpha u\|_{L^p}^p - \|c_2\|\|u\|_{L^{d_4}}^{d_4} - (b-a)\|c_3\| \\ &\ge \gamma|u|_{\alpha,p}^p - \frac{\|c_2\|(b-a)^{\alpha+1-\frac{d_4}{p}}}{\Gamma(\alpha+1)}|u|_{\alpha,p}^{d_4} - (b-a)\|c_3\|, \quad \text{for } u \in E_{\alpha,p}. \end{aligned} $$

Since $d_4 < p$ and the norms $|\cdot|_{\alpha,p}$ and $\|\cdot\|_{\alpha,p}$ are equivalent, the proof is completed. □

Theorem 6.13. Assume that L satisfies Hypotheses (H1)-(H4) and one of Hypotheses (H5), (H5)$'$ or (H5)$''$. Then, $\mathfrak{L}$ admits a global minimizer.

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $E_{\alpha,p}$ satisfying

$$\mathfrak{L}(u_n) \to \inf_{v\in E_{\alpha,p}} \mathfrak{L}(v) =: K.$$

Since L satisfies Hypothesis (H1), $\mathfrak{L}(u) \in \mathbb{R}$ for any $u \in E_{\alpha,p}$. Hence, $K < +\infty$. Let us prove by contradiction that $(u_n)_{n\in\mathbb{N}}$ is bounded in $E_{\alpha,p}$. In the negative case, we can construct a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ satisfying $\|u_{n_k}\|_{\alpha,p} \to \infty$. Since L satisfies Hypothesis (H4), Lemma 6.12 gives:

$$K = \lim_{k\in\mathbb{N}} \mathfrak{L}(u_{n_k}) = +\infty,$$

which is a contradiction. Hence, $(u_n)_{n\in\mathbb{N}}$ is bounded in $E_{\alpha,p}$. Since $E_{\alpha,p}$ is reflexive, there exists a subsequence still denoted by $(u_n)_{n\in\mathbb{N}}$ converging weakly in $E_{\alpha,p}$ to an element denoted by $u \in E_{\alpha,p}$. Let us prove that u is a global minimizer of $\mathfrak{L}$. Since

$$u_n \overset{E_{\alpha,p}}{\rightharpoonup} u \quad \text{and} \quad E_{\alpha,p} \hookrightarrow\!\!\!\to C_a,$$

we have

$$u_n \xrightarrow{C} u \quad \text{and} \quad {}_aD_t^\alpha u_n \overset{L^p}{\rightharpoonup} {}_aD_t^\alpha u. \tag{6.4}$$

Case L satisfies (H5): by convexity, it holds for any $n \in \mathbb{N}$

$$\begin{aligned}\mathfrak{L}(u_n) = \int_a^b L(u_n, {}_aD_t^\alpha u_n, t)dt &\geq \int_a^b L(u, {}_aD_t^\alpha u, t)dt \\ &+ \int_a^b \frac{\partial L}{\partial x}(u, {}_aD_t^\alpha u, t)\ (u_n - u)dt + \int_a^b \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\ ({}_aD_t^\alpha u_n - {}_aD_t^\alpha u)dt.\end{aligned}$$

Since L satisfies Hypotheses (H2) and (H3), $\partial L/\partial x(u, {}_aD_t^\alpha u, t) \in L^1$ and $\partial L/\partial y(u, {}_aD_t^\alpha u, t) \in L^q$. Consequently, using (6.4) and making n tend to $+\infty$, we obtain

$$K = \inf_{v\in E_{\alpha,p}} \mathfrak{L}(v) \geq \int_a^b L(u, {}_aD_t^\alpha u, t)dt = \mathfrak{L}(u).$$

Consequently, u is a global minimizer of $\mathfrak{L}$.

Case L satisfies (H5)$'$: let $\varepsilon > 0$. Since $(u_n)_{n\in\mathbb{N}}$ converges strongly in C to u, we have

$$\exists N \in \mathbb{N}, \forall n \geq N, \quad \|u_n - u\| < \delta,$$

where δ is given in the definition of (H5)$'$. In consequence, it holds a.e. on [a,b]

$$|L(u_n(t), {}_aD_t^\alpha u_n(t), t) - L(u(t), {}_aD_t^\alpha u_n(t), t)| < \varepsilon, \quad \text{for } n \geq N. \tag{6.5}$$

Moreover, for any $n \geq N$, we have

$$\begin{aligned}\mathfrak{L}(u_n) = \int_a^b L(u, {}_aD_t^\alpha u, t)dt &+ \int_a^b \left[L(u_n, {}_aD_t^\alpha u_n, t) - L(u, {}_aD_t^\alpha u_n, t)\right] dt \\ &+ \int_a^b \left[L(u, {}_aD_t^\alpha u_n, t) - L(u, {}_aD_t^\alpha u, t)\right] dt.\end{aligned}$$

Then, for any $n \geq N$, it holds by convexity

$$\mathfrak{L}(u_n) \geq \int_a^b L(u, {}_aD_t^\alpha u, t)dt - \int_a^b |L(u_n, {}_aD_t^\alpha u_n, t) - L(u, {}_aD_t^\alpha u_n, t)|dt$$
$$+ \int_a^b \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\ ({}_aD_t^\alpha u_n - {}_aD_t^\alpha u)dt.$$

And, using Eq. (6.5), we obtain for any $n \geq N$

$$\mathfrak{L}(u_n) \geq \int_a^b L(u, {}_aD_t^\alpha u, t)dt - \varepsilon(b-a) + \int_a^b \frac{\partial L}{\partial y}(u, {}_aD_t^\alpha u, t)\ ({}_aD_t^\alpha u_n - {}_aD_t^\alpha u)dt.$$

We remind that $\partial L/\partial y(u, {}_aD_t^\alpha u, t) \in L^q$ since L satisfies (H3). Since $({}_aD_t^\alpha u_n)_{n\in\mathbb{N}}$ converges weakly in L^p to ${}_aD_t^\alpha u$ we obtain by making n tend to $+\infty$ and then by making ε tend to 0

$$K = \inf_{v\in E_{\alpha,p}} \mathfrak{L}(v) \geq \int_a^b L(u, {}_aD_t^\alpha u, t)dt = \mathfrak{L}(u).$$

Consequently, u is a global minimizer of $\mathfrak{L}$.

Case L satisfies (H5)$''$: we refer to Theorem 3.23 in Bacorogna, 2008. □

Finally, we give the existence theorem of weak solution for (6.1).

Theorem 6.14. Let L be a Lagrangian of class C^1 and $0 < \frac{1}{p} < \alpha < 1$. If L satisfies the hypotheses denoted by (H1)-(H5). Then (6.1) admits a weak solution.

Combining Theorems 6.11 and 6.13, the proof of Theorem 6.14 is obvious.

Let us consider some examples of Lagrangian L satisfying Hypotheses of Theorem 6.14. Consequently, the fractional Euler-Lagrange equation (6.1) associated admits a weak solution $u \in E_{\alpha,p}$.

Examples 6.15. The most classical example is the Dirichlet integral, i.e. the Lagrangian functional associated to the Lagrangian L given by

$$L(x, y, t) = \frac{1}{2}\|y\|^2.$$

In this case, L satisfies Hypotheses (H1)-(H5) for $p = 2$. Hence, the fractional Euler-Lagrange equation (6.1) associated admits a weak solution in $E_{\alpha,p}$ for $\frac{1}{2} < \alpha < 1$.

In a more general case, the following Lagrangian L

$$L(x, y, t) = \frac{1}{p}\|y\|^p + a(x, t),$$

where $p > 1$ and $a \in C^1(\mathbb{R}^d \times [a, b], \mathbb{R}^+)$, satisfies Hypotheses (H1)-(H4) and (H5)$''$. Consequently, the fractional Euler-Lagrange equation (6.1) associated to L admits a weak solution in $E_{\alpha,p}$ for any $\frac{1}{p} < \alpha < 1$. Let us note that if for any $t \in [a, b]$, $a(\cdot, t)$ is convex, then L satisfies Hypothesis (H5).

In the unidimensional case $d = 1$, let us take a Lagrangian with a second term linear in its first variable, i.e.

$$L(x, y, t) = \frac{1}{p}|y|^p + f(t)x,$$

where $p > 1$ and $f \in C^1([a,b],\mathbb{R})$. Then, L satisfies Hypotheses (H1)-(H5). Then, the fractional Euler-Lagrange equation (6.1) associated admits a weak solution in $E_{\alpha,p}$ for any $\frac{1}{p} < \alpha < 1$.

Theorem 6.14 is a result based on strong conditions on Lagrangian L. Consequently, some Lagrangian do not satisfy all hypotheses of Theorem 6.14. We can cite Bolza's example in dimension $d = 1$ given by

$$L(x,y,t) = (y^2 - 1)^2 + x^4.$$

L does not satisfy Hypothesis (H4) neither Hypothesis (H5)$''$. Nevertheless, as usual with variational methods, the conditions of regularity, coercivity and/or convexity can often be replaced by weaker assumptions specific to the studied problem. As an example, we can cite Ammi and Torres, 2008 and references therein about higher-order integrals of the calculus of variations. Indeed, in this Subsection, it is proved that calculus of variations is still valid with weaker regularity assumptions.

6.3 Time-Fractional Diffusion Equations

6.3.1 *Introduction*

We assume Ω to be a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$. We consider an initial-boundary value problem for a diffusion equation with two fractional time derivatives

$$\begin{cases} \partial_t^{\alpha_1} u(x,t) + q(x)\partial_t^{\alpha_2} u(x,t) = (-\mathcal{A}u)(x,t), & x \in \Omega, \quad t \in (0,T), \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \\ u(x,0) = a(x), \quad x \in \Omega. \end{cases} \tag{6.6}$$

Here $0 < \alpha_2 < \alpha_1 < 1$. For $\alpha \in (0,1)$, by ∂_t^α we denote the Caputo fractional derivative with respect to t

$$\partial_t^\alpha g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{dt} g(\tau) d\tau$$

and Γ is the Gamma function and $q \in W^{2,\infty}(\Omega)$. The space $W^{2,\infty}(\Omega)$ is the usual Sobolev space (see, e.g., Adams, 1999).

The operator $\mathcal{A}$ denotes a second-order partial differential operator in the following form

$$(-\mathcal{A}u)(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + b(x)u(x), \quad x \in \Omega,$$

for $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$, and we assume that $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$, $1 \le i,j \le d, b \in C(\bar{\Omega})$, $b(x) \le 0$ for $x \in \bar{\Omega}$ and that there exists a constant $\nu > 0$ such that

$$\nu \sum_{j=1}^{d} \xi_j^2 \le \sum_{j,\ k=1}^{d} a_{jk}(x)\xi_j\xi_k, \quad x \in \bar{\Omega}, \quad \xi \in \mathbb{R}^d.$$

The classic diffusion models (diffusion equation with integer-order derivative) have played important roles in modelling contaminants diffusion processes. However, in recent two decades, more and more experimental data (e.g., the diffusion process in the highly heterogeneous media) showed that the classical model are inadequate to explain the phenomenon described by the experimental data, e.g., Adams and Gelhar pointed out that the field data in the saturated zone of a highly heterogeneous aquifer indicated the long-tailed profile in the spatial distribution of densities as the time passes, which is very different from the classical one (see, Adams and Gelhar, 1992). The above phenomenon of long-tailed profile has been investigated by many researchers, see Berkowitz, Scher and Silliman, 2000; Giona, Gerbelli and Roman, 1992; Hatano and Hatano, 1980, and the references therein. In the many researches, there is an effective one that being used to explain the long-tailed profile phenomenon, that is to replace the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$ since the fractional derivative possesses the memory effect which leads to the not too fast diffusion. This modified model is presented as a useful approach for the description of transport dynamics in complex system that are governed by anomalous diffusion and non-exponential relaxation patterns, and attracted great attention from different areas. For numerical calculation, see Benson, Schumer, Meerschaert *et al.*, 2001; Meerschaert and Tadjeran *et al.*, 2004; Diethelm and Luchko, 2004 and the references therein. For the theoretics, see Gorenflo, Luchko and Zabrejko, 1999; Hanyaga, 2002; Luchko, 2009a,b, 2010; Luchko and Gorenflo, 1999; Sakamoto and Yamamoto, 2011; Xu, Cheng and Yamamoto, 2011, etc. For the stochastic analysis, one can regard the time-fractional diffusion equation as a macroscopic model derived from the continuous-time random walk. Metzler and Klafter, 2000b demonstrated that a fractional diffusion equation describes a non-Markovian diffusion process with a memory. Roman and Alemany, 1994 investigated continuous-time random walks on fractals and showed that the average probability density of random walks on fractals obeys a diffusion equation with a fractional time derivative asymptotically. As for diffusion equations with multiple fractional time derivatives, see Jiang, Liu, Turner *et al.*, 2012; Daftardar-Gejji and Bhalekar, 2008; Luchko, 2011 and the references therein.

In this Section, we consider the case of multiple fractional time derivatives. Such equations can be considered as more feasible model equations than equations with a single fractional time derivative in modeling diffusion in porous media. We apply the perturbation method and the theory of evolution equations to prove regularity as well as unique existence of solution to (6.6).

6.3.2 *Regularity and Unique Existence*

Let $L^2(\Omega)$ be a usual L^2-space with the scalar product $(\cdot,\cdot)$, and $H^l(\Omega)$, $H_0^m(\Omega)$ denote the usual Sobolev spaces (e.g., Adams, 1999). We set $\|a\|_{L^2(\Omega)} = (a,a)^{\frac{1}{2}}$.

We define the operator A in $L^2(\Omega)$ by

$$(Au)(x) = (\mathcal{A}u)(x), \quad x \in \Omega, \quad D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Then the fractional power A^γ is defined for $\gamma \in \mathbb{R}$ (see, e.g., Pazy, 1983), and $D(A^\gamma) \subset H^{2\gamma}(\Omega)$, $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$ for example. We note that $\|u\|_{D(A^\gamma)} := \|A^\gamma u\|_{L^2(\Omega)}$ is stronger than $\|u\|_{L^2(\Omega)}$ for $\gamma > 0$.

Since $-A$ is a symmetric uniformly elliptic operator, the spectrum of A is entirely composed of eigenvalues and counting according to the multiplicities, we can set $0 < \lambda_1 \le \lambda_2 \le \cdots$. By $\phi_n \in D(A)$, we denote the orthonormal eigenfunction corresponding to $\lambda_n : A\phi_n = \lambda_n \phi_n$. Then the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is orthonormal basis in $L^2(\Omega)$. Then we see that

$$D(A^\gamma) = \left\{ \psi \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \phi_n)|^2 < \infty \right\}$$

and that $D(A^\gamma)$ is a Hilbert space with the norm

$$\|\psi\|_{D(A^\gamma)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} |(\psi, \phi_n)|^2 \right)^{\frac{1}{2}}.$$

Henceforth we associate with $u(x,t)$, provided that it is well-defined, a map $u(\cdot) : (0,T) \to L^2(\Omega)$ by $u(t)(x) = u(x,t)$, $0 < t < T$, $x \in \Omega$. Then we can write (6.6) as

$$\begin{cases} \partial_t^{\alpha_1} u(t) + q\partial_t^{\alpha_2} u(t) = -Au(t), \quad t > 0 \text{ in } L^2(\Omega), \\ u(0) = a \in L^2(\Omega). \end{cases} \tag{6.7}$$

Remark 6.16. The interpretation of the initial condition should be made in a suitable function space. In our case, as Theorem 6.17 asserts, we have $\lim_{t\to 0} \|u(t) - a\|_{L^2(\Omega)} = 0$.

Moreover we define the Mittag-Leffler function $E_{\alpha,\beta}(z)$ by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$ are arbitrary constants. By the power series, we can directly verify that $E_{\alpha,\beta}(z)$ is an entire function of $z \in \mathbb{C}$.

Now we define the operator $S(t) : L^2(\Omega) \to L^2(\Omega)$, $t \ge 0$, by

$$S(t)a := \sum_{n=1}^{\infty} (a, \phi_n) E_{\alpha_1,1}(-\lambda_n t^{\alpha_1}) \phi_n \quad \text{in} \quad L^2(\Omega) \tag{6.8}$$

for $a \in L^2(\Omega)$. Then we can prove that $S(t) : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator for $t \ge 0$ (see, e.g., Sakamoto and Yamamoto, 2011). Moreover the term-wise differentiations are possible and give

$$S'(t)a := -\sum_{n=1}^{\infty} \lambda_n (a, \phi_n) t^{\alpha_1 - 1} E_{\alpha_1,\alpha_1}(-\lambda_n t^{\alpha_1}) \phi_n \quad \text{in} \quad L^2(\Omega) \tag{6.9}$$

and

$$S''(t)a := -\sum_{n=1}^{\infty} \lambda_n (a, \phi_n) t^{\alpha_1 - 2} E_{\alpha_1, \alpha_1 - 1}(-\lambda_n t^{\alpha_1}) \phi_n \quad \text{in} \quad L^2(\Omega) \tag{6.10}$$

for $a \in L^2(\Omega)$.

For $F \in L^2(\Omega \times (0,T))$ and $a \in L^2(\Omega)$, there exists a unique solution in a suitable class (see, e.g., Sakamoto and Yamamoto, 2011) to the problem

$$\begin{cases} \partial_t^{\alpha_1} u(t) = -Au(t) + F, \quad 0 < t < T, \\ u(0) = a. \end{cases} \tag{6.11}$$

This solution is given by

$$u(t) = \int_0^t A^{-1} S'(t - \tau) F(\tau) d\tau + S(t)a, \quad t > 0. \tag{6.12}$$

In view of (6.12), we mainly discuss the equation

$$u(t) = S(t)a - \int_0^t A^{-1} S'(t - \tau) q \partial_t^{\alpha_2} u(\tau) d\tau, \quad 0 < t < T, \tag{6.13}$$

in order to establish unique existence of solutions to (6.7). Henceforth, C denotes generic positive constants which are independent of a in (6.6), but may depend on T, α_1, α_2 and the coefficients of the operator A and q.

Now we are ready to state first main result in this section.

Theorem 6.17. We assume that $u \in C((0,T], L^2(\Omega))$ satisfy (6.13) and

$$\alpha_1 + \alpha_2 > 1.$$

Then

$$\|u(t)\|_{H^{2\gamma}(\Omega)} \le C t^{-\alpha_1 \gamma} \|a\|_{L^2(\Omega)}, \quad 0 < t \le T$$

for any $\gamma \in (0,1)$.

Proof. First we have

$$A^{\gamma - 1} S'(t)a = -t^{\alpha_1 - 1} \sum_{n=1}^{\infty} \lambda_n^{\gamma} (a, \phi_n) E_{\alpha_1, \alpha_1}(-\lambda_n t^{\alpha_1}) \phi_n \quad \text{in} \quad L^2(\Omega) \tag{6.14}$$

for $a \in L^2(\Omega)$ and $\gamma \ge 0$. Moreover, since

$$|E_{\alpha_1, \alpha_1}(-\eta)| \le \frac{C}{1 + \eta}, \quad \eta > 0$$

(see, e.g., Theorem 1.6 in Podlubny, 1999), we can prove

$$\|A^{\gamma - 1} S'(t)\| \le C t^{\alpha_1 - 1 - \alpha_1 \gamma}, \quad t > 0 \tag{6.15}$$

and

$$\|A^{-1} S''(t)\| \le C t^{\alpha_1 - 2}, \quad t > 0. \tag{6.16}$$

Now we proceed to the proof of Theorem 6.17. We set

$$v(t) := \int_0^t A^{\gamma-1} S'(t-\eta) q \partial_t^{\alpha_2} u(\eta) d\eta, \quad 0 < t < T.$$

By (6.13), we have

$$A^\gamma u(t) = A^\gamma S(t) a - v(t), \quad 0 < t < T.$$

Therefore, using

$$\|u(t)\|_{H^{2\gamma}(\Omega)} \le C \|A^\gamma u(t)\|_{L^2(\Omega)},$$

it is sufficient to estimate $\|A^\gamma S(t) a\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}$. First we will estimate $\|v(t)\|_{L^2(\Omega)}$. Substituting the definition of $\partial_t^{\alpha_2} u$ and changing the order of integration, we have

$$\begin{aligned} v(t) &= \int_0^t A^{\gamma-1} S'(t-\eta) \frac{1}{\Gamma(1-\alpha_2)} \left(\int_0^\eta (\eta-\tau)^{-\alpha_2} q u'(\tau) d\tau \right) d\eta \\ &= \frac{1}{\Gamma(1-\alpha_2)} \int_0^t H(t,\tau) q u'(\tau) d\tau, \quad 0 < t < T. \end{aligned} \tag{6.17}$$

Here we have set

$$H(t,\tau) = \int_\tau^t A^{\gamma-1} S'(t-\eta)(\eta-\tau)^{-\alpha_2} d\eta.$$

Decomposing the integrand and introducing the change of variables $\eta - \tau \to \eta$ we obtain

$$\begin{aligned} H(t,\tau) &= \int_\tau^t A^{\gamma-1} S'(t-\eta)(\eta-\tau)^{-\alpha_2} d\eta, \\ &= \int_\tau^t A^{\gamma-1} S'(t-\eta)[(\eta-\tau)^{-\alpha_2} - (t-\tau)^{-\alpha_2}] d\eta \\ &\quad + \int_\tau^t A^{\gamma-1} S'(t-\eta) d\eta (t-\tau)^{-\alpha_2} \\ &= \int_0^{t-\tau} A^{\gamma-1} S'(t-\eta-\tau)[\eta^{-\alpha_2} - (t-\tau)^{-\alpha_2}] d\eta \\ &\quad + \int_\tau^t A^{\gamma-1} S'(t-\eta) d\eta (t-\tau)^{-\alpha_2} \\ &= \int_0^{t-\tau} A^{\gamma-1} S'(t-\eta-\tau)[\eta^{-\alpha_2} - (t-\tau)^{-\alpha_2}] d\eta \\ &\quad + A^{\gamma-1} S(0)(t-\tau)^{-\alpha_2} - A^{\gamma-1} S(t-\tau)(t-\tau)^{-\alpha_2} \\ &=: I_1(t,\tau) + I_2(t,\tau). \end{aligned} \tag{6.18}$$

On the other hand, we have

$$\begin{aligned} \partial_\tau I_1(t,\tau) = &- \int_0^{t-\tau} A^{\gamma-1} S''(t-\eta-\tau)(\eta^{-\alpha_2} - (t-\tau)^{-\alpha_2}) d\eta \\ &- \alpha_2 \int_0^{t-\tau} A^{\gamma-1} S'(t-\eta-\tau)(t-\tau)^{-\alpha_2-1} d\eta \\ &- \lim_{\eta \to t-\tau} A^{\gamma-1} S'(t-\tau-\eta)[\eta^{-\alpha_2} - (t-\tau)^{-\alpha_2}]. \end{aligned}$$

By the estimate (6.15) we obtain

$$\|A^{\gamma-1}S'(t-\tau-\eta)(\eta^{-\alpha_2} - (t-\tau)^{-\alpha_2})\|_{L^2(\Omega)} \\ \leq C(t-\tau-\eta)^{\alpha_1-1-\alpha_1\gamma}\frac{|(t-\tau)^{\alpha_2}-\eta^{\alpha_2}|}{\eta^{\alpha_2}(t-\tau)^{\alpha_2}}.$$

According to the mean value theorem, we can choose $\theta \in (\eta, t-\tau)$ such that

$$|(t-\tau)^{\alpha_2}-\eta^{\alpha_2}| = |\alpha_2\theta^{\alpha_2-1}(t-\tau-\eta)| \leq \alpha_2\eta^{\alpha_2-1}(t-\tau-\eta).$$

Hence we obtain

$$\|A^{\gamma-1}S'(t-\tau-\eta)(\eta^{-\alpha_2}-(t-\tau)^{-\alpha_2})\|_{L^2(\Omega)}$$

$$\leq C\alpha_2\eta^{-1}(t-\tau)^{-\alpha_2}(t-\tau-\eta)^{\alpha_1-\alpha_1\gamma} \to 0 \quad \text{as } \eta \to t-\tau$$

by $\alpha_1 - \alpha_1\gamma > 0$. This implies

$$\begin{aligned} \partial_\tau I_1(t,\tau) = &-\int_0^{t-\tau} A^{\gamma-1}S''(t-\eta-\tau)(\eta^{-\alpha_2}-(t-\tau)^{-\alpha_2})d\eta \\ &-\alpha_2\int_0^{t-\tau} A^{\gamma-1}S'(t-\eta-\tau)(t-\tau)^{-\alpha_2-1}d\eta, \quad 0<t<T. \end{aligned} \tag{6.19}$$

On the other hand, we have

$$\begin{aligned} \partial_\tau I_2(t,\tau) &= -\alpha_2A^{\gamma-1}S(t-\tau)(t-\tau)^{-\alpha_2-1} + \alpha_2A^{\gamma-1}S(0)(t-\tau)^{-\alpha_2-1} \\ &\quad +A^{\gamma-1}S'(t-\tau)(t-\tau)^{-\alpha_2} \\ &= \alpha_2\int_0^{t-\tau} A^{\gamma-1}S'(t-\eta-\tau)(t-\tau)^{-\alpha_2-1}d\eta \\ &\quad +A^{\gamma-1}S'(t-\tau)(t-\tau)^{-\alpha_2}. \end{aligned}$$

Adding this and (6.19) we obtain

$$\begin{aligned} \partial_\tau H(t,\tau) = &-\int_0^{t-\tau} A^{\gamma-1}S''(t-\eta-\tau)(\eta^{-\alpha_2}-(t-\tau)^{-\alpha_2})d\eta \\ &+A^{\gamma-1}S'(t-\tau)(t-\tau)^{-\alpha_2}. \end{aligned} \tag{6.20}$$

Using (6.20) in (6.17), integrating by parts and using $H(t,t)=0$ we obtain

$$\begin{aligned} (\Gamma(1-\alpha_2))\ v(t) &= \int_0^t H(t,\tau)qu'(\tau)d\tau \\ &= -H(t,0)qa + \int_0^t \Bigg[\int_0^{t-\tau} A^{\gamma-1}S''(t-\eta-\tau)(\eta^{-\alpha_2}-(t-\tau)^{-\alpha_2})d\eta \\ &\quad -A^{\gamma-1}S'(t-\tau)(t-\tau)^{-\alpha_2}\Bigg]qu(\tau)d\tau \\ &=: I_3(t)+I_4(t). \end{aligned}$$

We set

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \alpha,\beta>0.$$

First, by (6.15) and $q \in W^{2,\infty}(\Omega)$ we have

$$\begin{aligned}\|I_3(t)\|_{L^2(\Omega)} &= \| - H(t,0)qa\|_{L^2(\Omega)} \\ &= \left\| - \int_0^t A^{\gamma-1}S'(t-\eta)\eta^{-\alpha_2}d\eta qa\right\|_{L^2(\Omega)} \\ &\le C\|a\|_{L^2(\Omega)}\int_0^t (t-\eta)^{\alpha_1-\alpha_1\gamma-1}\eta^{-\alpha_2}d\eta \\ &= C\|a\|_{L^2(\Omega)}B(1-\alpha_2,\alpha_1-\alpha_1\gamma)t^{\alpha_1-\alpha_1\gamma-\alpha_2},\end{aligned} \tag{6.21}$$

since $1-\alpha_2 > 0$ and $\alpha_1 - \alpha_1\gamma > 0$.

On the other hand, by $q \in W^{2,\infty}(\Omega)$ and $u|_{\partial\Omega} = 0$, we have

$$\|A(qu(\tau))\|_{L^2(\Omega)} \le C\|qu(\tau)\|_{H^2(\Omega)} \le C\|u(\tau)\|_{H^2(\Omega)} \le C\|Au(\tau)\|_{L^2(\Omega)}$$

and $\|qu(\tau)\|_{L^2(\Omega)} \le C\|u(\tau)\|_{L^2(\Omega)}$, that is,

$$\|A^0(qu(\tau))\|_{L^2(\Omega)} \le C\|A^0u(\tau)\|_{L^2(\Omega)}.$$

Hence the interpolation theorem (see, e.g., Theorem 5.1 in Lions and Magenes, 1972) we obtain

$$\|A^\gamma(qu(\tau))\|_{L^2(\Omega)} \le C\|A^\gamma u(\tau)\|_{L^2(\Omega)}.$$

Therefore by (6.15) and (6.16), the second term of $I_4(t)$ can be estimated as follows:

$$\begin{aligned}\|I_4(t)\|_{L^2(\Omega)} &\le C\int_0^t \left[\int_0^{t-\tau}(t-\eta-\tau)^{\alpha_1-2}(\eta^{-\alpha_2}-(t-\tau)^{-\alpha_2})d\eta \right.\\ &\quad \left. +(t-\tau)^{\alpha_1-1-\alpha_2}\right]\|A^\gamma(qu(\tau))\|_{L^2(\Omega)}d\tau \\ &\le C\int_0^t \left[\int_0^{t-\tau}(t-\eta-\tau)^{\alpha_1-2}\frac{(t-\tau-\eta)^{\alpha_2}}{\eta^{\alpha_2}(t-\tau)^{\alpha_2}}d\eta \right.\\ &\quad \left. +(t-\tau)^{\alpha_1-1-\alpha_2}\right]\|A^\gamma u(\tau)\|_{L^2(\Omega)}d\tau \\ &\le C\int_0^t \left[\int_0^{t-\tau}(t-\eta-\tau)^{\alpha_1+\alpha_2-2}\eta^{-\alpha_2}d\eta \right.\\ &\quad \left. +(t-\tau)^{\alpha_1-1-\alpha_2}\right]\|A^\gamma u(\tau)\|_{L^2(\Omega)}d\tau \\ &= C\int_0^t \big(B(1-\alpha_2,\alpha_1+\alpha_2-1)(t-\tau)^{\alpha_1-1} \\ &\quad +(t-\tau)^{\alpha_1-1-\alpha_2}\big)\|A^\gamma u(\tau)\|_{L^2(\Omega)}d\tau.\end{aligned}$$

For the last equality, we used $\alpha_1+\alpha_2 > 1$. Therefore we have

$$\begin{aligned}\|\Gamma(1-\alpha_2)v(t)\|_{L^2(\Omega)} &\le C\|a\|^2_{L^2(\Omega)}B(1-\alpha_2,\alpha_1-\alpha_1\gamma)t^{\alpha_1-\alpha_1\gamma-\alpha_2} \\ &\quad + C\int_0^t (t-\tau)^{\alpha_1-1-\alpha_2}\|A^\gamma u(\tau)\|_{L^2(\Omega)}d\tau.\end{aligned}$$

Thus the estimate of $\|v(t)\|_{L^2(\Omega)}$ is completed.

Next we estimate $\|A^\gamma S(t)a\|_{L^2(\Omega)}$. By Theorem 1.6 in Podlubny, 1999, we obtain

$$\begin{aligned}\|A^\gamma S(t)a\|^2_{L^2(\Omega)} &= \left\| \sum_{n=1}^{\infty}(a,\phi_n)\lambda_n^\gamma E_{\alpha_1,1}(-\lambda_n t^{\alpha_1})\phi_n \right\|^2_{L^2(\Omega)} \\ &\le C\sum_{n=1}^{\infty}(a,\phi_n)^2 t^{-2\alpha_1\gamma}\left(\frac{(\lambda_n t^{\alpha_1})^\gamma}{1+\lambda_n t^{\alpha_1}}\right)^2 \\ &\le Ct^{-2\alpha_1\gamma}\|a\|^2_{L^2(\Omega)},\end{aligned}$$

and hence

$$\begin{aligned}\|A^\gamma u(t)\ \|_{L^2(\Omega)} &\le C\|a\|_{L^2(\Omega)}(t^{-\alpha_1\gamma}+t^{\alpha_1-\alpha_1\gamma-\alpha_2}) \\ &\quad +C\int_0^t (t-\tau)^{\alpha_1-1-\alpha_2}\|A^\gamma u(\tau)\|_{L^2(\Omega)}d\tau \\ &\le C\|a\|_{L^2(\Omega)}t^{-\alpha_1\gamma}+C\int_0^t (t-\tau)^{\alpha_1-1-\alpha_2}\|A^\gamma u(\tau)\|_{L^2(\Omega)}d\tau, \quad 0<t<T.\end{aligned}$$

Therefore by an inequality of Gronwall type (see, Exercise 3 (p.190) in Henry, 1981), we obtain

$$\|A^\gamma u(t)\|_{L^2(\Omega)} \le C\|a\|_{L^2(\Omega)}t^{-\alpha_1\gamma}, \quad 0<t\le T.$$

Thus the proof is completed. □

Remark 6.18. We may be able to remove the condition $\alpha_1+\alpha_2>1$. On the other hand, Prüss established regularity in case $\gamma=1$ for general $\alpha_1,\alpha_2\in(0,1)$ under a strong condition on $a\in\mathcal{D}(A)$ (see, Prüss, 1993).

On the basis of Theorem 6.17, a standard argument (see, Henry, 1981) yields:

Theorem 6.19. For any $\gamma\in(0,1)$ there exists a mild solution to (6.13) in the space $u\in C((0,T],\mathcal{D}(A^\gamma))\bigcap C((0,T],L^2(\Omega))$.

The above results shall now be extended to the solution of linear diffusion equation with multiple fractional time derivatives

$$\partial_t^{\alpha_1}u(t)+\sum_{j=2}^{l}q_j\partial_t^{\alpha_j}u(t)=-Au(t), \quad t>0$$

and

$$u(0)=a\in L^2(\Omega),$$

where $0<\alpha_l<\cdots<\alpha_2<\alpha_1<1$ and $q_j\in W^{2,\infty}(\Omega)$, $2\le j\le l$.

As before the lower-order derivatives are regarded as source terms and we consider

$$u(t)=S(t)a-\int_0^t A^{-1}S'(t-\tau)\sum_{j=2}^{l}q_j\partial_t^{\alpha_j}u(\tau)d\tau, \quad 0<t<T. \tag{6.22}$$

Similarly to Theorems 6.17 and 6.19, we can prove

Theorem 6.20. Assume that $u\in C((0,T],L^2(\Omega))$ satisfies (6.22) and

$$0<\alpha_l<\cdots<\alpha_1, \quad \alpha_1+\alpha_l>1.$$

Then

$$\|u(t)\|_{H^{2\gamma}(\Omega)}\le Ct^{-\alpha_1\gamma}\|a\|_{L^2(\Omega)}, \quad 0<t\le T$$

for any $\gamma\in(0,1)$. Moreover there exists a mild solution to (6.22) in the space $C((0,T],\mathcal{D}(A^\gamma))\cap C([0,T],L^2(\Omega))$ with $\gamma\in(0,1)$.

6.4 Fractional Hamiltonian Systems

6.4.1 *Introduction*

Consider the following fractional Hamiltonian system

$$\begin{cases} {}_tD_\infty^\alpha({}_{-\infty}D_t^\alpha u(t)) + L(t)u(t) = \nabla W(t, u(t)), \quad t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}), \end{cases} \tag{6.23}$$

where $\alpha \in (\frac{1}{2}, 1]$, ${}_{-\infty}D_t^\alpha$ and ${}_tD_\infty^\alpha$ are left and right Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ respectively, $u \in \mathbb{R}^n$, $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and $W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a function that satisfy conditions which will be stated later and $\nabla W(t, u)$ is the gradient of W at u.

In particular, if $\alpha = 1$, (6.23) reduces to the standard second order Hamiltonian system of the following form

$$u''(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad t \in \mathbb{R}. \tag{6.24}$$

The existence of homoclinic orbits is one of the most important problems in the history of Hamiltonian systems. It has been intensively studied by many mathematicians (see Ambrosetti and Zelati, 1993; Ding and Jeanjean, 2007; Izydorek and Janczewska, 2005; Makita, 2012; Omana and Willem, 1992; Paturel, 2001; Rabinowitz, 1990; Séré, 1992; Szulkin and Zou, 2001; Zelati, Ekeland and Séré, 1990; Zelati and Rabinowitz, 1991; Zou and Li, 2002). Variational methods to prove the existence of homoclinic orbits for second order Hamiltonian systems were first used by Rabinowitz, 1990, while the first multiplicity result, later improved by Paturel, 2001, is due to Ambrosetti and Zelati, 1993. In the recent years, the existence and multiplicity of homoclinic orbits for the second order Hamiltonian systems have been extensively studied via variational methods in many papers (e.g. Ambrosetti and Zelati, 1993; Ding and Jeanjean, 2007; Izydorek and Janczewska, 2005; Makita, 2012; Paturel, 2001; Rabinowitz, 1990; Zelati and Rabinowitz, 1991; Zou and Li, 2002).

It is worth to mention that the fractional Hamiltonian is not uniquely defined and many researchers have explored this area giving new insight into this problem (e.g. Baleanu, Golmankaneh and Golmankaneh, 2009, Tarasov 2010, Toress, 2013). In Subsection 6.4.2, we introduce the fractional space that we use in our work and some proposition are proven which will aid in our analysis. In Subsection 6.4.3, we discuss existence and multiplicity of solutions of fractional Hamiltonian systems.

6.4.2 *Fractional Derivative Space*

In this section we introduce some fractional spaces, for more detail, see Ervin and Roop, 2006. Let $\alpha > 0$. Define the semi-norm

$$|u|_{I_{-\infty}^\alpha} = \|{}_{-\infty}D_x^\alpha u\|_{L^2}$$

and norm

$$\|u\|_{I_{-\infty}^{\alpha}} = (\|u\|_{L^2}^2 + |u|_{I_{-\infty}^{\alpha}}^2)^{\frac{1}{2}}, \tag{6.25}$$

and let

$$I_{-\infty}^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_{I_{-\infty}^{\alpha}}}.$$

Now we define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ in terms of the Fourier transform. Let $0 < \alpha < 1$, let the semi-norm

$$|u|_{\alpha} = \||w|^{\alpha}\widehat{u}\|_{L^2} \tag{6.26}$$

and norm

$$\|u\|_{\alpha} = (\|u\|_{L^2}^2 + |u|_{\alpha}^2)^{\frac{1}{2}},$$

and let

$$H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_{\alpha}}.$$

We note a function $u \in L^2(\mathbb{R})$ belong to $I_{-\infty}^{\alpha}(R)$ if and only if

$$|w|^{\alpha}\widehat{u} \in L^2(\mathbb{R}). \tag{6.27}$$

Especially

$$|u|_{I_{-\infty}^{\alpha}} = \||w|^{\alpha}\hat{u}\|_{L^2}. \tag{6.28}$$

Therefore $I_{-\infty}^{\alpha}(\mathbb{R})$ and $H^{\alpha}(\mathbb{R})$ are equivalent with equivalent semi-norm and norm. Analogous to $I_{-\infty}^{\alpha}(\mathbb{R})$ we introduce $I_{\infty}^{\alpha}(\mathbb{R})$. Let the semi-norm

$$|u|_{I_{\infty}^{\alpha}} = \|{}_xD_{\infty}^{\alpha}u\|_{L^2}$$

and norm

$$\|u\|_{I_{\infty}^{\alpha}} = (\|u\|_{L^2}^2 + |u|_{I_{\infty}^{\alpha}}^2)^{\frac{1}{2}}, \tag{6.29}$$

and let

$$I_{\infty}^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R}, \mathbb{R}^n)}^{\|\cdot\|_{I_{\infty}^{\alpha}}}.$$

Moreover $I_{-\infty}^{\alpha}(\mathbb{R})$ and $I_{\infty}^{\alpha}(\mathbb{R})$ are equivalent , with equivalent semi-norm and norm (see, Ervin and Roop, 2006). Now we give the proof of the Sobolev lemma.

Theorem 6.21. If $\alpha > \frac{1}{2}$, then $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$ and there is a constant $C = C_{\alpha}$ such that

$$\|u\| = \sup_{x \in \mathbb{R}} |u(x)| \leq C\|u\|_{\alpha}. \tag{6.30}$$

Proof. By the Fourier inversion theorem, if $\hat{u} \in L^1(R)$ then u is continuous and

$$\sup_{x \in \mathbb{R}} |u(x)| \leq \|\hat{u}\|_{L^1}.$$

Hence, to prove the theorem it is enough to prove that

$$\|\hat{u}\|_{L^1} \leq \|u\|_{\alpha},$$

so by Schwarz inequality, we have

$$\int_{\mathbb{R}} |\hat{u}(w)|dw = \int_{\mathbb{R}} (1+|w|^2)^{\alpha/2}|\hat{u}(w)|\frac{1}{(1+|w|^2)^{\alpha/2}}dw$$
$$\leq \left(\int_{\mathbb{R}} (1+|w|^2)^{\alpha}|\hat{u}(w)|^2 dw\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} (1+|w|^2)^{-\alpha} dw\right)^{\frac{1}{2}}.$$

The first integral on the right is $\|u\|_{\alpha}^2$, so the theorem boils down to the fact

$$\int_{\mathbb{R}} (1+|w|^2)^{-\alpha} dw = \int_0^{\infty} (1+r^2)^{-\alpha} r^{n-1} dr < \infty$$

precisely when $\alpha > \frac{1}{2}$. $\square$

Remark 6.22. If $u \in H^{\alpha}(\mathbb{R})$, then $u \in L^q(\mathbb{R})$ for all $q \in [2, \infty]$, since

$$\int_{\mathbb{R}} |u(x)|^q dx \leq \|u\|_{\infty}^{q-2}\|u\|_{L^2}^2.$$

Now we introduce a new fractional space. Let

$$X^{\alpha} = \{u \in H^{\alpha}(\mathbb{R}, \mathbb{R}^n) : \int_{\mathbb{R}} \left[|_{-\infty}D_t^{\alpha}u(t)|^2 + (L(t)u(t), u(t))\right] dt < \infty\}.$$

The space X^{α} is a Hilbert space with the inner product

$$\langle u, v\rangle_{X^{\alpha}} = \int_{\mathbb{R}} [({}_{-\infty}D_t^{\alpha}u(t), {}_{-\infty}D_t^{\alpha}v(t)) + (L(t)u(t), v(t))]\, dt$$

and the corresponding norm

$$\|u\|_{X^{\alpha}}^2 = \langle u, u\rangle_{X^{\alpha}}\ .$$

Before give the main existence theorem, we firstly state the conditions:
(L) $L(t)$ is positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists an $l \in C(\mathbb{R}, (0, \infty))$ such that $l(t) \to +\infty$ as $t \to +\infty$ and

$$(L(t)x, x) \geq l(t)|x|^2, \quad \text{for all } \ t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n;$$

(W1) $W \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and there is a constant $\mu > 2$ such that

$$0 < \mu W(t, x) \leq (x, \nabla W(t, x)), \quad \text{for all } \ t \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^n \backslash \{0\};$$

(W2) $|\nabla W(t, x)| = o(|x|)$ as $x \to 0$ uniformly with respect to $t \in \mathbb{R}$;
(W3) there exists $\overline{W} \in C(\mathbb{R}^n, \mathbb{R})$ such that

$$|W(t, x)| + |\nabla W(t, x)| \leq \overline{W}(x), \quad \text{for every } \ x \in \mathbb{R}^n \quad \text{and} \quad t \in \mathbb{R}.$$

Lemma 6.23. Suppose L satisfies (L). Then X^{α} is continuously embedded in $H^{\alpha}(\mathbb{R}, \mathbb{R}^n)$.

Proof. Since $l \in C(\mathbb{R}, (0, \infty))$ and l is coercive, then $l_{\min} = \min_{t \in \mathbb{R}} l(t)$ exists, so we have

$$(L(t)u(t), u(t)) \geq l(t)|u(t)|^2 \geq l_{\min}|u(t)|^2, \quad \forall t \in \mathbb{R}.$$

Then

$$\begin{aligned} l_{\min}\|u(t)\|_\alpha^2 &= l_{\min}\left(\int_{\mathbb{R}}\left[|_{-\infty}D_t^\alpha u(t)|^2+|u(t)|^2\right]dt\right) \\ &\leq l_{\min}\int_{\mathbb{R}}|_{-\infty}D_t^\alpha u(t)|^2dt+\int_{\mathbb{R}}(L(t)u(t),u(t))dt\ . \end{aligned}$$

So

$$\|u\|_\alpha^2 \leq K\|u\|_{X^\alpha}^2, \tag{6.31}$$

where $K=\frac{\max\{l_{\min},1\}}{l_{\min}}$. □

Lemma 6.24. Suppose L satisfies (L). Then the imbedding of X^α in $L^2(\mathbb{R})$ is compact.

Proof. We note first that by Lemma 6.23 and Remark 6.22 we have

$$X^\alpha \hookrightarrow L^2(\mathbb{R}) \quad \text{is continuous.}$$

Now, let $(u_k)\in X^\alpha$ be a sequence such that $u_k \rightharpoonup u$ in X^α. We will show that $u_k \to u$ in $L^2(\mathbb{R})$. Suppose, without loss of generality, that $u_k \rightharpoonup 0$ in X^α. The Banach-Steinhaus theorem implies that

$$A=\sup_k \|u\|_{X^\alpha}<+\infty\ .$$

Let $\epsilon>0$, there is $T_0<0$ such that $\frac{1}{l(t)}\leq\epsilon$ for all t such that $t\leq T_0$. Similarly, there is $T_1>0$, such that $\frac{1}{l(t)}\leq\epsilon$ for all $t\geq T_1$. Sobolev's theorem (see Stuart, 1995) implies that $u_k\to 0$ uniformly on $\bar{\Omega}=[T_0,T_1]$, so there is a k_0 such that

$$\int_\Omega |u_k(t)|^2dt\leq\epsilon, \quad \text{for all} \ \ k\geq k_0. \tag{6.32}$$

Since $\frac{1}{l(t)}\leq\epsilon$ on $(-\infty,T_0]$ we have

$$\int_{-\infty}^{T_0}|u_k(t)|^2dt\leq\epsilon\int_{-\infty}^{T_0}l(t)|u_k(t)|^2dt\leq\epsilon A^2. \tag{6.33}$$

Similarly, since $\frac{1}{l(t)}\leq\epsilon$ on $(T_1,+\infty]$, we have

$$\int_{T_1}^{+\infty}|u_k(t)|^2dt\leq\epsilon A^2. \tag{6.34}$$

Combining (6.32), (6.33) and (6.34), we get $u_k\to 0$ in $L^2(\mathbb{R},\mathbb{R}^n)$. □

Lemma 6.25. There are constants $c_1>0$ and $c_2>0$ such that

$$W(t,u)\geq c_1|u|^\mu, \quad |u|\geq 1 \tag{6.35}$$

and

$$W(t,u)\leq c_2|u|^\mu, \quad |u|\leq 1. \tag{6.36}$$

Proof. By (W1) we note that

$$\mu W(t,\sigma u)\leq(\sigma u,\nabla W(t,\sigma u)).$$

Let $f(\sigma) = W(t, \sigma u)$, then

$$\frac{d}{d\sigma}(f(\sigma)\sigma^{-\mu}) \geq 0. \tag{6.37}$$

Now we consider two cases:

Case 1. $|u| \leq 1$. In this case we integrate (6.37), from 1 until $\frac{1}{|u|}$ and we get

$$W(t, u) \leq W(t, \frac{u}{|u|})|u|^{\mu}. \tag{6.38}$$

Case 2. $|u| \geq 1$. In this case we integrate (6.37), from $\dfrac{1}{|u|}$ until 1 and we get

$$W(t, u) \geq |u|^{\mu} W(t, \frac{u}{|u|}). \tag{6.39}$$

Now, since $u \in \mathbb{R}^n$, $\dfrac{u}{|u|} \in B(0, 1)$. So, since W is continuous and $B(0, 1)$ is compact, there are $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \leq W(t, u) \leq c_2, \quad \text{for every} \;\; u \in B(0, 1).$$

Therefore we get the affirmation of the lemma. □

Remark 6.26. By Lemma 6.25, we have

$$W(t, u) = o(|u|^2), \quad \text{as} \;\; u \to 0 \quad \text{uniformly in} \;\; t \in \mathbb{R}. \tag{6.40}$$

In addition, by (W2), we have, for any $u \in \mathbb{R}^n$ such that $|u| \leq M_1$, there exists some constant $d > 0$ (dependent on (W1)) such that

$$|\nabla W(t, u(t))| \leq d|u(t)|. \tag{6.41}$$

Similar to Lemma 2 of Zaslavsky, 2002, we can get the following result.

Lemma 6.27. Suppose that (L), (W1)-(W2) are satisfied. If $u_k \rightharpoonup u$ in X^{α}, then $\nabla W(t, u_k) \to \nabla W(t, u)$ in $L^2(\mathbb{R}, \mathbb{R}^n)$.

Proof. Assume that $u_k \rightharpoonup u$ in X^{α}. Then there exists a constant $d_1 > 0$ such that, by Banach-Steinhaus theorem and (6.30),

$$\sup_{k \in \mathbb{R}} \|u_k\|_{\infty} \leq d_1, \quad \|u\|_{\infty} \leq d_1.$$

By (W2), for any $\epsilon > 0$ there is $\delta > 0$ such that

$$|u_k| < \delta \quad \text{implies} \;\; |\nabla W(t, u_k)| \leq \epsilon |u_k|$$

and by (W3) there is $M > 0$ such that

$$|\nabla W(t, u_k)| \leq M, \quad \text{for all } \delta < |u_k| \leq d_1.$$

Therefore, there exists a constant $d_2 > 0$ (dependent on d_1) such that

$$|\nabla W(t, u_k(t))| \leq d_2 |u_k(t)|, \quad |\nabla W(t, u(t))| \leq d_2 |u(t)|$$

for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$. Hence,

$$|\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq d_2(|u_k(t)| + |u(t)|) \leq d_2(|u_k(t) - u(t)| + 2|u(t)|).$$

Since, by Lemma 6.24, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$, passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_{L^2} < \infty.$$

But this implies $u_k \to u$ almost everywhere $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} |u_k(t) - u(t)| = v(t) \in L^2(\mathbb{R}, \mathbb{R}^n).$$

Therefore

$$|\nabla W(t, u_k(t)) - \nabla W(t, u(t))| \leq d_2(v(t) + 2|u(t)|).$$

Then, using Lebesgue's dominated convergence theorem, the lemma is proved. □

In the following, we consider the case L is uniformly bounded from below. Here we do not need that L satisfies the corecive condition (L). Explicitly, we assume that

($\overline{\mathrm{L}}$) $L \in C(\mathbb{R}, \mathbb{R}^{n\times n})$ is a symmetric and positive definite symmetric matrix for all $t \in \mathbb{R}$ and there exists a $M > 0$ such that

$$(L(t)x, x) \geq M|x|^2, \quad \text{for all } t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

In order to prove our main results, we assume that $W(t,x) = b(t)\pi(x)$ satisfies the following conditions:

($\overline{\mathrm{W}}$1) $b : \mathbb{R} \to (0, +\infty)$ is a continuous function such that $b(t) \to 0$ as $|t| \to \infty$;

($\overline{\mathrm{W}}$2) $\pi \in C^1(\mathbb{R}^n, \mathbb{R})$ and there is a constant $\mu > 2$ such that

$$0 < \mu\pi(x) \leq (\nabla\pi(x), x), \qquad \forall\, x \in \mathbb{R}^n \setminus \{0\};$$

($\overline{\mathrm{W}}$3) $\nabla\pi(x) = o(|x|)$ as $|x| \to 0$;

($\overline{\mathrm{W}}$4) $\pi(-x) = \pi(x)$, for all $x \in \mathbb{R}^n$;

($\overline{\mathrm{W}}$5) For any $r > 0$, there exists $\alpha_0, \beta_0 > 0$ and $\varrho < 2$ such that

$$0 \leq \left(2 + \frac{1}{\alpha_0 + \beta_0|x|^{\varrho}}\right) W(t,x) \leq (\nabla W(t,x), x), \quad \forall (t,x) \in \mathbb{R}\times\{x \in \mathbb{R}^n : |x| \geq r\}.$$

Let $L_b^p(\mathbb{R}, \mathbb{R}^n)$ denote the weighted space of measurable functions $u : \mathbb{R} \to \mathbb{R}^n$ with the norm

$$\|u\|_{p,b} = \left(\int_{\mathbb{R}} b(t)|u(t)|^p dt\right)^{\frac{1}{p}}.$$

Lemma 6.28. Suppose that ($\overline{\mathrm{L}}$) and ($\overline{\mathrm{W}}$1) hold. Then the imbedding of X^α in $L_b^2(\mathbb{R}, \mathbb{R}^n)$ is compact.

Proof. The proof is standard (cf., Ervin and Roop, 2006). For readers convenience, we give the rough proof. It is easy to check that the embedding of $X^\alpha \hookrightarrow L_b^2(\mathbb{R}, \mathbb{R}^n)$ is continuous. Next, we prove that the embedding is compact. Let $\{u_n\}_{n\in\mathbb{N}} \subset X^\alpha$ be a sequence such that $u_n \rightharpoonup u$ in X^α, we show that $u_n \to u$ in

$L^2_b(\mathbb{R}, \mathbb{R}^n)$. Suppose, without loss of generality, that $u_n \rightharpoonup 0$ in X^α. The Banach-Steinhaus Theorem implies that

$$A = \sup_{n\in\mathbb{N}} \|u_n\| < \infty.$$

For any $\varepsilon > 0$, there is $T_0 < 0$ such that $b(t) \le \varepsilon$ for all t such that $t \le T_0$. Similarly, there is $T_1 > 0$ such that $b(t) \le \varepsilon$ for all $t \ge T_1$. Sobolev's theorem (see Yuan and Zhang, 2012) implies that $u_n \to u$ uniformly on $\bar{\Omega} = [T_0, T_1]$, so there is a k_0 such that

$$\int_\Omega b(t)|u_n(t)|^2 dt < \varepsilon, \quad \forall\, k \ge k_0. \tag{6.42}$$

Since $b(t) \le \varepsilon$ on $(-\infty, T_0]$ we have

$$\int_{-\infty}^{T_0} b(t)|u_n(t)|^2 dt \le \frac{\varepsilon}{M} \int_{-\infty}^{T_0} M|u_n(t)|^2 dt < \frac{\varepsilon}{M} A^2. \tag{6.43}$$

Similarly, since $b(t) \le \varepsilon$ on $(T_1, +\infty)$, one can get

$$\int_{T_1}^{+\infty} b(t)|u_n(t)|^2 dt \le \frac{\varepsilon}{M} \int_{T_1}^{+\infty} M|u_n(t)|^2 dt < \frac{\varepsilon}{M} A^2. \tag{6.44}$$

Combining (6.42)-(6.44) we get $u_n \to 0$ in $L^2_b(\mathbb{R}, \mathbb{R}^n)$. □

Remark 6.29. From Lemma 6.28, there is a constant C_b such that

$$\|u\|_{2,b} \le C_b \|u\|_{X^\alpha}, \quad \forall\, u \in X^\alpha. \tag{6.45}$$

Lemma 6.30. Suppose that ($\overline{\mathrm{L}}$), ($\overline{\mathrm{W}}$1) and ($\overline{\mathrm{W}}$3) are satisfied. If $u_n \rightharpoonup u$ in X^α, then $\nabla\pi(u_n) \to \nabla\pi(u)$ in $L^2_b(\mathbb{R}, \mathbb{R}^n)$.

The proof is similar to Lemma 6.27 and is omitted.

Now, from Theorem 6.21, it is well known that for $\alpha > \frac{1}{2}$, $X^\alpha \subset H^\alpha(\mathbb{R}, \mathbb{R}^n) \subset C(\mathbb{R}, \mathbb{R}^n)$, the space of continuous functions u on $\mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$.

Lemma 6.31. We have

$$\pi(u) \le \pi\left(\frac{u}{|u|}\right)|u|^\mu, \quad |u| \le 1, \tag{6.46}$$

$$\pi(u) \ge \pi\left(\frac{u}{|u|}\right)|u|^\mu, \quad |u| \ge 1. \tag{6.47}$$

The proof is similar to Lemma 6.25 and is omitted.

6.4.3 *Existence and Multiplicity*

Now we are going to establish the corresponding variational framework to obtain the existence and multiplicity of solutions for (6.23).

Definition 6.32. We say that $u \in I^\alpha_{-\infty}(\mathbb{R})$ is a weak solution of (6.23) if

$$\int_{-\infty}^{\infty} [({}_{-\infty}D^\alpha_t u(t), {}_{-\infty}D^\alpha_t v(t)) + (L(t)u(t), u(t))]dt = \int_{-\infty}^{\infty} (\nabla W(t, u(t)), v(t))dt,$$

for all $v \in I^{\alpha}_{-\infty}(\mathbb{R})$.

For $u \in I^{\alpha}_{-\infty}(\mathbb{R})$, we may define the functional $I : X^{\alpha} \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_{\mathbb{R}} \left[\frac{1}{2}|{}_{-\infty}D^{\alpha}_{t}u(t)|^2 + \frac{1}{2}(L(t)u(t), u(t)) - W(t, u(t))\right] dt \\ &= \frac{1}{2}\|u\|^2_{X^{\alpha}} - \int_{\mathbb{R}} W(t, u(t))dt, \end{aligned} \tag{6.48}$$

which is of class C^1. We say that $u \in X^{\alpha}$ is a weak solution of (6.23) if u is a critical point of I.

Lemma 6.33. Suppose that (L), (W1)-(W2) hold, we have

$$I'(u)v = \int_{\mathbb{R}} [({}_{-\infty}D^{\alpha}_{t}u(t), {}_{-\infty}D^{\alpha}_{t}v(t)) + (L(t)u(t), v(t)) - (\nabla W(t, u(t)), v(t))]dt, \tag{6.49}$$

for all $u, v \in X^{\alpha}$, which yields that

$$I'(u)u = \|u\|^2_{X^{\alpha}} - \int_{\mathbb{R}} (\nabla W(t, u(t)), u(t))dt. \tag{6.50}$$

Moreover, I is a continuously Fréchet-differentiable functional defined on X^{α}, i.e., $I \in C^1(X^{\alpha}, \mathbb{R})$.

Proof. We firstly show that $I : X^{\alpha} \to \mathbb{R}$. By (6.40), there is a $\delta > 0$ such that $|u| \le \delta$ implies that

$$W(t, u) \le \epsilon|u|^2 \quad \text{for } t \in \mathbb{R}. \tag{6.51}$$

Let $u \in X^{\alpha}$, $\alpha > \frac{1}{2}$. Then $u \in C(\mathbb{R}, \mathbb{R}^n)$, the space of continuous function $u \in \mathbb{R}$ such that $u(t) \to 0$ as $|t| \to +\infty$. Therefore there is a constant $R > 0$ such that $|t| \ge R$ implies $|u(t)| \le \delta$. Hence, by (6.51), we have

$$\int_{\mathbb{R}} W(t, u(t))dt \le \int_{-R}^{R} W(t, u(t))dt + \epsilon \int_{|t| \ge R} |u(t)|^2 dt < +\infty. \tag{6.52}$$

Combining (6.48) and (6.52), we show that $I : X^{\alpha} \to \mathbb{R}$.

Now we prove that $I \in C^1(X^{\alpha}, \mathbb{R})$. Rewrite I as follows $I = I_1 - I_2$, where

$$I_1 = \frac{1}{2}\int_{\mathbb{R}} [|{}_{-\infty}D^{\alpha}_{t}u(t)|^2 + (L(t)u(t), u(t))]dt, \quad I_2 = \int_{\mathbb{R}} W(t, u(t))dt.$$

It is easy to check that $I_1 \in C^1(X^{\alpha}, \mathbb{R})$ and

$$I'_1(u)v = \int_{\mathbb{R}} [({}_{-\infty}D^{\alpha}_{t}u(t), {}_{-\infty}D^{\alpha}_{t}v(t)) + (L(t)u(t), v(t))]dt. \tag{6.53}$$

Thus it is sufficient to show this is the case for I_2. In the process we will see that

$$I'_2(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt, \tag{6.54}$$

which is defined for all $u, v \in X^{\alpha}$. For any given $u \in X^{\alpha}$, let us define $J(u) : X^{\alpha} \to \mathbb{R}$ as follows

$$J(u)v = \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t))dt, \quad \forall v \in X^{\alpha}.$$

It is obvious that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in X^\alpha$, by (6.41), there is a constant $d_3 > 0$ such that

$$|\nabla W(t, u(t))| \leq d_3 |u(t)|,$$

which yields that, by the Hölder inequality and Lemma 6.23

$$\begin{aligned} |J(u)v| &= \left| \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t)) dt \right| \leq d_3 \int_{\mathbb{R}} |u(t)||v(t)| dt \\ &\leq \frac{d_3}{l_{\min}} \|u\|_{X^\alpha} \|v\|_{X^\alpha}. \end{aligned} \tag{6.55}$$

Moreover, for u and $v \in X^\alpha$, by Mean Value theorem, we have

$$\int_{\mathbb{R}} W(t, u(t) + v(t)) dt - \int_{\mathbb{R}} W(t, u(t)) dt = \int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t))) dt,$$

where $h(t) \in (0,1)$. Therefore, by Lemma 6.24 and the Hölder inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t)), v(t)) dt - \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t)) dt \\ &= \int_{\mathbb{R}} (\nabla W(t, u(t) + h(t)v(t) - \nabla W(t, u(t)), v(t)) dt \to 0 \end{aligned} \tag{6.56}$$

as $v \to 0$ in X^α. Combining (6.55) and (6.56), we see that (6.54) holds. It remains to prove that I_2' is continuous. Suppose that $u \to u_0$ in X^α and note that

$$\begin{aligned} \sup_{\|v\|_{X^\alpha}=1} |I_2'(u)v - I_2'(u_0)v| &= \sup_{\|v\|_{X^\alpha}=1} \left| \int_{\mathbb{R}} (\nabla W(t, u(t)) - \nabla W(t, u_0(t)), v(t)) dt \right| \\ &\leq \sup_{\|v\|_{X^\alpha}=1} \|\nabla W(., u(.)) - \nabla W(., u_0(.))\|_{L^2} \|v\|_{L^2} \\ &\leq \frac{1}{\sqrt{l_{\min}}} \|\nabla W(., u(.)) - \nabla W(., u_0(.))\|_{L^2}. \end{aligned}$$

By Lemma 6.24, we obtain that $I_2'(u)v - I_2'(u_0)v \to 0$ as $u \to u_0$ uniformly with respect to v, which implies the continuity of I_2' and $I \in C^1(X^\alpha, \mathbb{R})$. The proof is completed. □

Lemma 6.34. Under the conditions of (L) and (W1)-(W2), I satisfies the (PS) condition.

Proof. Assume that $\{u_k\}_{k\in\mathbb{N}} \in X^\alpha$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $C_1 > 0$ such that

$$|I(u_k)| \leq C_1, \quad \|I'(u_k)\|_{(X^\alpha)^*} \leq C_1 \tag{6.57}$$

for every $k \in \mathbb{N}$.

We firstly prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in X^α. By (6.48), (6.50) and (W1), we have

$$\begin{aligned} C_1 + \|u_k\|_{X^\alpha} &\geq I(u_k) - \frac{1}{\mu} I'(u_k) u_k \\ &= (\frac{\mu}{2} - 1) \|u_k\|_{X^\alpha}^2 - \int_{\mathbb{R}} [(W(t, u_k(t)) - \frac{1}{\mu} (\nabla W(t, u_k(t)), u_k(t))] dt \\ &\geq (\frac{\mu}{2} - 1) \|u_k\|_{X^\alpha}^2. \end{aligned} \tag{6.58}$$

Since $\mu > 2$, the inequality (6.58) shows that $(u_k)_{k\in N}$ is bounded in X^α. So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in X^α and hence, by Lemma 6.24, $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. It follows from the definition of I that

$$(I'(u_k)-I'(u))(u_k-u) = \|u_k-u\|^2_{X^\alpha} - \int_{\mathbb{R}} [\nabla W(t,u_k)-\nabla W(t,u)](u_k-u)dt. \quad (6.59)$$

Since $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^n)$, we have (see Lemma 6.27) $\nabla W(t, u_k(t)) \to \nabla W(t, u(t))$ in $L^2(\mathbb{R}, \mathbb{R}^n)$. Hence

$$\int_{\mathbb{R}} (\nabla W(t, u_k) - \nabla W(t, u), u_k - u)dt \to 0$$

as $k \to \infty$. So (6.59) implies

$$\|u_k - u\|_{X^\alpha} \to 0, \quad \text{as } k \to \infty.$$

The proof is completed. □

Theorem 6.35. Suppose that (L), (W1)-(W2) hold, then (6.23) possesses at least one nontrivial solution.

Proof. We divide the proof into several steps:

Step I. It is clear that $I(0) = 0$ and $I \in C^1(X^\alpha, \mathbb{R})$ satisfies the (PS) condition by Lemma 6.33 and Lemma 6.34.

Step II. Now We show that there exist constant $\rho > 0$ and $\beta > 0$ such that I satisfies the condition (i) of Theorem 1.51. By Lemma 6.24, there is a $C_0 > 0$ such that

$$\|u\|_{L^2} \le C_0\|u\|_{X^\alpha}.$$

On the other hand, by Theorem 6.21 there is $C_\alpha > 0$ such that

$$\|u\| \le C_\alpha\|u\|_{X^\alpha}.$$

By (6.40), for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$W(t, u(t)) \le \epsilon|u(t)|^2 \quad \text{wherever} \quad |u(t)| < \delta.$$

Let $\rho = \frac{\delta}{C_\alpha}$ and $\|u\|_{X^\alpha} \le \rho$; we have $\|u\|_\infty \le \frac{\delta}{C_\alpha} \cdot C_\alpha = \delta$. Hence

$$|W(t, u(t))| \le \epsilon|u(t)|^2 \quad \text{for all} \quad t \in \mathbb{R}.$$

Integrating on $\mathbb{R}$, we get

$$\int_{\mathbb{R}} W(t, u(t))dt \le \varepsilon\|u\|^2_{L^2} \le \epsilon C_0^2\|u\|^2_{X^\alpha}.$$

So, if $\|u\|_{X^\alpha} = \rho$, then

$$I(u) = \frac{1}{2}\|u\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, u(t))dt \ge (\frac{1}{2} - \epsilon C_0^2)\|u\|^2_{X^\alpha} = (\frac{1}{2} - \epsilon C_0^2)\rho^2.$$

And it suffices to choose $\epsilon = \frac{1}{4C_0^2}$ to get

$$I(u) \ge \frac{\rho^2}{4C_0^2} = \beta > 0. \quad (6.60)$$

Step III. It remains to prove that there exists an $e \in X^\alpha$ such that $\|e\|_{X^\alpha} > \rho$ and $I(e) \le 0$, where ρ is defined in Step II. Consider

$$I(\sigma u) = \frac{\sigma^2}{2}\|u\|^2_{X^\alpha} - \int_{\mathbb{R}} W(t, \sigma u(t))dt$$

for all $\sigma \in \mathbb{R}$. By (6.35), there is $c_1 > 0$ such that

$$W(t, u(t)) \ge c_1|u(t)|^\mu \quad \text{for all} \quad |u(t)| \ge 1. \tag{6.61}$$

Take some $u \in X^\alpha$ such that $\|u\|_{X^\alpha} = 1$. Then there exists a subset Ω of positive measure of $\mathbb{R}$ such that $u(t) \ne 0$ for $t \in \Omega$. Take $\sigma > 0$ such that $\sigma|u(t)| \ge 1$ for $t \in \Omega$. Then by (6.61), we obtain

$$I(\sigma u) \le \frac{\sigma^2}{2} - c_1\sigma^\mu \int_\Omega |u(t)|^\mu dt. \tag{6.62}$$

Since $c_1 > 0$ and $\mu > 2$, (6.62) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma|u(t)| \ge 1$ for $t \in \Omega$ and $\|\sigma u\|_{X^\alpha} > \rho$, where ρ is defined in Step II. By Theorem 1.51, I possesses a critical value $c \ge \beta > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} I(\gamma(s)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X^\alpha) : \quad \gamma(0) = 0, \quad \gamma(1) = e\}.$$

Hence there is $u \in X^\alpha$ such that

$$I(u) = c, \quad I'(u) = 0.$$

□

Similar to Lemma 6.33, we give the following lemma.

Lemma 6.36. Suppose that $(\overline{\mathrm{L}})$, $(\overline{\mathrm{W}}1)$ and $(\overline{\mathrm{W}}4)$ are satisfied. Then the conclusions of Lemma 6.33 hold.

In the following, we give the result on multiplicity of solutions of fractional Hamiltonian systems.

Theorem 6.37. Assume that L and W satisfy $(\overline{\mathrm{L}})$ and $(\overline{\mathrm{W}}1)$-$(\overline{\mathrm{W}}3)$. Then system (6.23) possesses a nontrivial homoclinic orbit.

Proof. It is clear that $I(0) = 0$. We show that I satisfies the hypotheses of the Theorem 1.51.

Step I. We show that I satisfies the (PS) condition. Assume that $\{u_n\}_{n\in\mathbb{N}} \subset X^\alpha$ is a sequence such that $\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded and $I'(u_n) \to 0$ as $n \to +\infty$. Then there exists a constant $c > 0$ such that

$$|I(u_n)| \le c, \qquad \|I'(u_n)\|_{X^\alpha} \le c, \quad \text{for every } n \in \mathbb{N}. \tag{6.63}$$

We firstly prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X^α. By (6.48), (6.49), (6.63) and $(\overline{\mathrm{W}}2)$, one can get

$$\begin{aligned}
c+\frac{c}{\mu}\|u_n\|_{X^\alpha} &\geq I(u_n)-\frac{1}{\mu}I'(u_n)u_n \\
&=\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}}\Big(|_{-\infty}D_t^\alpha u_n(t)|^2+(L(t)u_n(t),u_n(t))\Big)dt \\
&\quad -\int_{\mathbb{R}}W(t,u_n(t))dt+\frac{1}{\mu}\int_{\mathbb{R}}(\nabla W(t,u_n(t)),u_n(t))dt \\
&=\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_{X^\alpha}^2+\frac{1}{\mu}\int_{\mathbb{R}}\left[(\nabla W(t,u_n(t)),u_n(t))-\mu W(t,u_n(t))\right]dt \\
&\geq\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_{X^\alpha}^2, \quad n\in\mathbb{N}.
\end{aligned}$$

Since $\mu>2$, the above inequality shows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X^α, i.e., that there exists a constant $\theta>0$ such that

$$\|u_n\|_{X^\alpha}\leq\theta, \quad \text{for every } n\in\mathbb{N}. \tag{6.64}$$

Since X^α is a reflexive space (X^α is a Hilbert space), thus passing to a subsequence if necessary, by Lemma 6.28, we may assume that

$$\begin{cases} u_n\rightharpoonup u, \text{ weakly in } X^\alpha, \\ u_n\to u, \text{ a.e. in } L_b^2(\mathbb{R},\mathbb{R}^n). \end{cases} \tag{6.65}$$

Thus,

$$(I'(u_n)-I'(u))(u_n-u)\to 0,$$

and by Lemma 6.30 and the Hölder inequality, one can get

$$\int_{\mathbb{R}}(\nabla W(t,u_n(t))-\nabla W(t,u(t)),u_n(t)-u(t))dt\to 0,$$

as $n\to+\infty$. On the other hand we have

$$\begin{aligned}
(I'(u_n)-I'(u))(u_n-u)&=\|u_n-u\|_{X^\alpha}^2 \\
&\quad -\int_{\mathbb{R}}(\nabla W(t,u_n(t))-\nabla W(t,u(t)),u_n(t)-u(t))dt.
\end{aligned}$$

Hence, $\|u_n-u\|_{X^\alpha}\to 0$ as $n\to+\infty$. Therefore, I satisfies (PS) condition.

Step II. We now show that there exist constants $\rho>0$ and $\alpha>0$ such that I satisfies assumption (ii) of Theorem 1.51. For any give number $\varepsilon>0$, from $(\overline{\mathrm{W}}3)$, we can choose $\delta>0$ such that

$$\pi(x)\leq\varepsilon|x|^2, \quad \text{for every } |x|\leq\delta. \tag{6.66}$$

By (6.30), if $\|u\|_{X^\alpha}=\frac{\delta}{C_\alpha}=:\rho$, then $|u(t)|\leq C_\alpha\cdot\rho=\delta$, so $\pi(u(t))\leq\varepsilon|u(t)|^2$ for all $t\in\mathbb{R}$. Integrating on $\mathbb{R}$ and by Remark 3 and Lemma 6.31, we have

$$\int_{\mathbb{R}}W(t,u(t))dt\leq\varepsilon\|u\|_{2,b}^2\leq\varepsilon C_b^2\|u\|_{X^\alpha}^2. \tag{6.67}$$

Let

$$\beta = \frac{1}{4}\left(\frac{\delta}{C_\alpha}\right)^2.$$

For $\|u\|_{X^\alpha} = \rho \leq 1$, from (6.48) and (6.67), we obtain

$$\begin{aligned} I(u) &= \frac{1}{2}\int_{\mathbb{R}}\Big(|_{-\infty}D_t^\alpha u(t)|^2 + (b(t)u(t), u(t))\Big)dt - \int_{\mathbb{R}} W(t, u(t))dt \\ &\geq \frac{1}{2}\|u\|^2_{X^\alpha} - \varepsilon C_b^2\|u\|^2_{X^\alpha} \\ &= \left(\frac{1}{2} - \varepsilon C_b^2\right)\|u\|^2_{X^\alpha}. \end{aligned} \tag{6.68}$$

Setting $\varepsilon = \frac{1}{4C_b^2}$, the inequality (6.68) implies that

$$I|_{\partial B_\rho} \geq \frac{1}{4}\left(\frac{\delta}{C_\alpha}\right)^2 = \beta. \tag{6.69}$$

Clearly, (6.69) shows that $\|u\|_{X^\alpha} = \rho$ implies that $I(u) \geq \beta$, i.e., I satisfies assumption (ii) of Theorem 1.51.

Step III. We shall prove (iii) of Theorem 1.51, i.e., there exists an $e \in X^\alpha$ such that $\|e\|_{X^\alpha} > \rho$ and $I(e) \leq 0$, where ρ is defined in Step II. By (6.47), there is $c_1 > 0$ such that

$$\pi(u(t)) \geq c_1|u(t)|^\mu, \quad \text{for all } |u(t)| \geq 1. \tag{6.70}$$

Take some $u \in X^\alpha$ such that $\|u\|_{X^\alpha} = 1$. Then there exists a subset Ω of positive measure $|\Omega| < \infty$ of $\mathbb{R}$ such that $u(t) \neq 0$ for $t \in \Omega$. Take $\sigma > 0$ such that $\sigma|u(t)| \geq 1$ for $t \in \Omega$. Then by (6.48) and (6.70), one can get

$$I(\sigma u) \leq \frac{\sigma^2}{2} - c_1\sigma^\mu \int_\Omega b(t)|u(t)|^\mu dt. \tag{6.71}$$

Since $\mu > 2$, $b(t) > 0$ and $\int_\Omega b(t)|u(t)|^\mu dt > 0$, (6.71) implies that $I(\sigma u) < 0$ for some $\sigma > 0$ with $\sigma|u(t)| \geq 1$ for $t \in \Omega$ and $\|\sigma u\|_{X^\alpha} > \rho$. Therefore, I possesses a critical value $c \geq \beta$ given by

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0, and

$$\Gamma = \{g \in C([0,1], X^\alpha) : \quad g(0) = 0, \quad g(1) = e\}.$$

Here there is $u^* \in X^\alpha$ such that

$$I(u^*) = c, \quad I'(u^*) = 0.$$

Since $c > 0$, u^* is a nontrivial homoclinic solution. The proof is complete. □

Theorem 6.38. Assume that L and W satisfy $(\overline{\text{L}})$ and $(\overline{\text{W}}1)$-$(\overline{\text{W}}4)$. Then there exists an unbounded sequence of homoclinic orbits for system (6.23).

Proof. The conditions $(\overline{\text{W}}1)$ and $(\overline{\text{W}}4)$ imply that I is even. In view of the proof of Theorem 6.37, we see that $I \in C^1(X^\alpha, \mathbb{R})$, and I satisfies the (PS) condition and assumptions (i) and (ii) of Theorem 1.52. To apply Theorem 1.52, it suffices to prove that I satisfies the condition (iii′) of Theorem 1.52. Let $H' \subset X^\alpha$ be a finite dimensional subspace. From Step III of Theorem 6.37, we know that, for any $u_0 \in H' \subset X^\alpha$ such that $\|u_0\|_{X^\alpha} = 1$, there is $m_{u_0} > 0$ such that

$$I(m_{u_0}) < 0, \quad \text{for all } |m| \geq m_{u_0} > 0.$$

Since $H' \subset X^\alpha$ is a finite dimensional subspace, we can choose an $R = r(H') > 0$ such that

$$I(\omega) < 0, \quad \text{for } \omega \in H' \setminus B_R(0).$$

Therefore, by Theorem 1.52, I possesses an unbounded sequence of critical values $\{c_j\}_{j\in\mathbb{N}}$ with $c_j \to +\infty$. Let u_j be the critical point of I corresponding to c_j, then (6.23) has infinitely many distinct homoclinic solutions. □

Theorem 6.39. Let $\alpha > \frac{1}{2}$. Assume that L and W satisfy $(\overline{\text{L}})$ and $(\overline{\text{W}}1)$, $(\overline{\text{W}}3)$-$(\overline{\text{W}}5)$. Then there exists an unbounded sequence of homoclinic orbits for system (6.23).

Proof. In view of the proof of Theorem 6.38, we see that $I \in C^1(X^\alpha, \mathbb{R})$, and I satisfies assumptions (i), (ii) and (iii′) of Theorem 1.52. To apply Theorem 1.52, it suffices to prove that I satisfies the condition (C). Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset X^\alpha$ is a (C) sequence of I, that is, $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|_{X^\alpha})\|I'(u_n)\|_{(X^\alpha)^*} \to 0$ as $n \to \infty$. Then, in view of (6.48) and (6.49), For a constant $C_0 > 0$, we have

$$\begin{aligned} C_0 &\geq 2I(u_n) - I'(u_n)u_n \\ &= \int_{\mathbb{R}} [(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))]\, dt. \end{aligned} \tag{6.72}$$

Since $\pi(0) = 0$, then from $(\overline{\text{W}}3)$ that there exists $\eta \in (0, 1)$ such that

$$|W(t, x)| \leq \frac{1}{4} b(t)|x|^2, \quad \text{for every } t \in \mathbb{R}, \quad |x| \leq \eta. \tag{6.73}$$

By $(\overline{\text{W}}5)$, we have

$$(\nabla W(t, x), x) \geq 2W(t, x) \geq 0, \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{6.74}$$

$$W(t,x) \leq (\alpha_0+\beta_0|x|^\varrho)\,[(\nabla W(t,x),x) - 2W(t,x)]\,, \;\; \forall (t,x) \in \mathbb{R}\times\{x \in \mathbb{R}^n : \; |x| > \eta\}. \tag{6.75}$$

Now from (6.30), (6.48), (6.72)-(6.75) and Remark 6.29, we get

$$\begin{aligned} \frac{1}{2}\|u_n\|^2_{X^\alpha} &= I(u_n) + \int_{\mathbb{R}} W(t, u_n(t))dt \\ &= I(u_n) + \int_{\{t\in\mathbb{R},\ |u_n(t)|\leq\eta\}} W(t, u_n(t))dt + \int_{\{t\in\mathbb{R},\ |u_n(t)|>\eta\}} W(t, u_n(t))dt \end{aligned}$$

$$
\begin{aligned}
&= I(u_n) + \frac{1}{4}\int_{\{t\in\mathbb{R},\ |u_n(t)|\le\eta\}} b(t)|u_n(t)|^2 dt \\
&\quad + \int_{\{t\in\mathbb{R},\ |u_n(t)|>\eta\}} (\alpha_0 + \beta_0|u_n(t)|^{\varrho})\left[(\nabla W(t,u_n(t)),u_n(t)) - 2W(t,u_n(t))\right] dt \\
&\le C_1 + \frac{1}{4}\|u_n\|_{2,b}^2 + \int_{\mathbb{R}} (\alpha_0 + \beta_0|u_n(t)|^{\varrho})\left[(\nabla W(t,u_n(t)),u_n(t)) - 2W(t,u_n(t))\right] dt \\
&\le C_1 + \frac{1}{4}C_b^2\|u_n\|_{X^\alpha}^2 + C_0(\alpha_0 + \beta_0\|u_n\|_\infty^{\varrho}) \\
&\le C_1 + \frac{1}{4}C_b^2\|u_n\|_{X^\alpha}^2 + C_0(\alpha_0 + \beta_0 C_\alpha^{\varrho}\|u_n\|_{X^\alpha}^{\varrho}).
\end{aligned}
\tag{6.76}
$$

Since $\varrho < 2$, it follows that $\{\|u_n\|\}$ is bounded. Next, similar to the proof of Theorem 6.37, we can also prove that $\{u_n\}$ has a convergent subsequence in X^α. Thus, I satisfies condition (C). Therefore, the proof is complete. □

6.5 Fractional Schrödinger Equations

6.5.1 *Introduction*

Schrödinger equations have received a great deal of interest from the mathematicians in the past twenty years or so, due in particular to their applications to optics. Indeed, simplified versions or limits of Zakharov's system lead to certain Schrödinger equations. Schrödinger equations also arise in quantum field theory, and in particular in the Hartree-Fock theory. From the mathematical point of view, Schrödinger equations appears a delicate problem, since it possesses a mixture of the properties of parabolic and hyperbolic equations. Indeed, it is almost reversible, it has conservation laws and also some dispersive properties like the Klein-Gordon equation, but it has an infinite speed of propagation. On the other hand, Schrödinger equations has a kind of smoothing effect shared by parabolic problems but the time-reversibility it from generating an analytic semigroup. For more details, one can see the monographs Cazenave, 2003; Sulem and Sulem, 1999 and the papers Buslaev and Sulem, 2013; Cazenave, 1983; Cazenave and Lions, 1982; Cuccagna, 2001; Eid, Muslih and Baleanu *et al.*, 2009; Fibich, 2011; Floer and Weinstein, 1986; Guo and Wu, 1995; Tsai, 1995; Wang, 2008 and etc.

In this section we study the initial value problem of the following fractional Schrödinger equations with potential

$$
\begin{cases}
\frac{1}{i}\,{}_0^C D_t^\alpha x(t,y) - \Delta x(t,y) + kV(y)x(t,y) = 0,\ \alpha\in(0,1),\ y\in\Omega,\ t\in(0,T], \\
x(0,y) = x_0(y),\ y\in\Omega,
\end{cases}
\tag{6.77}
$$

where ${}_0^C D_t^\alpha$ is Caputo fractional derivative of order α in time t, $\Omega\subseteq\mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$, Δ denotes the Laplace operator in $\mathbb{R}^2$, x is a complex valued function in $[0,T]\times\mathbb{R}^2$, $k := \max_{t\in[0,T]}|\chi(t)|$ with $\chi\in C(J,\mathbb{R})$, is a positive constant, and the function V is called potential.

In Subsection 6.5.2, a suitable concept on a mild solution for our problems is introduced and existence and uniqueness of mild solutions are presented.

We recall the following initial value problem for linear Schrödinger equations

$$\begin{cases} \dfrac{1}{i}\dfrac{\partial}{\partial t}x(t,y) - \Delta x(t,y) = 0, \ y \in \Omega, \ t \in (0,T], \\ x(0,y) = x_0(y), \ y \in \Omega, \end{cases} \tag{6.78}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with a smooth boundary $\partial\Omega$, Δ denotes the Laplace operator in $\mathbb{R}^2$, and x is a complex valued function which defined in $[0,T] \times \mathbb{R}^2$.

Take $X = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $x \in D(A)$, define $Ax = i\Delta x$. By virtue of the well known Hille-Yosida theorem, it is obvious that A is the infinitesimal generator of a strongly continuous group $\{S(t), -\infty < t < \infty\}$ in X. Moreover, $\{S(t), -\infty < t < \infty\}$ can be given by

$$(S(t)x)(y) = \frac{1}{4\pi it}\int_\Omega e^{\frac{i|y-z|^2}{4t}}x(z)dz. \tag{6.79}$$

In order to derive the expression (6.79), we define

$$\bar{x}(t,y) = \begin{cases} x(t,y), & y \in \Omega, \\ 0, & y \in \mathbb{R}^2 \setminus \Omega, \end{cases} \tag{6.80}$$

and

$$\bar{x}_0(y) = \begin{cases} x_0(y), & y \in \Omega, \\ 0, & y \in \mathbb{R}^2 \setminus \Omega. \end{cases} \tag{6.81}$$

Then the equation (6.78) can be rewritten as

$$\begin{cases} \dfrac{\partial}{\partial t}\bar{x}(t,y) = i\Delta\bar{x}(t,y), \ y \in \mathbb{R}^2, \ t \in (0,T], \\ \bar{x}(0,y) = \bar{x}_0(y), \ y \in \mathbb{R}^2, \ \bar{x}_0 \in H^2(\mathbb{R}^2). \end{cases} \tag{6.82}$$

Applying Fourier transformation to the equation (6.82), we obtain

$$\begin{cases} \dfrac{d}{dt}\tilde{x}(t,z) = -iz^2\tilde{x}(t,z), \ z \in \mathbb{R}^2, \ t \in (0,T], \\ \tilde{x}(0,z) = \tilde{x}_0(z), \ z \in \mathbb{R}^2. \end{cases} \tag{6.83}$$

Thus, the classical solution of the equation (6.83) can be given by

$$\tilde{x}(t,z) \ = \ e^{-i|z|^2 t}\tilde{x}_0(z).$$

Thus,

$$\begin{aligned} \bar{x}(t,y) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{(i(y-\tau)z - i|z|^2 t)}\bar{x}_0(\tau)d\tau dz \\ &= \frac{1}{4\pi it}\int_{\mathbb{R}^2} e^{\frac{i|y-\xi|^2}{4t}}\bar{x}_0(\xi)d\xi. \end{aligned}$$

It comes from the inverse of Fourier transformation that

$$x(t,y) = \frac{1}{4\pi it}\int_\Omega e^{\frac{i|y-z|^2}{4t}}\bar{x}_0(z)dz. \tag{6.84}$$

On the other hand, the equation (6.82) has a unique classical solution

$$\bar{x}(t,y)=S(t)\bar{x}_0(y),\ y\in\mathbb{R}^2.$$

Keeping in mind of (6.80) and (6.81), we have

$$\bar{x}(t,y)=S(t)\bar{x}_0(y),\ y\in\Omega. \tag{6.85}$$

Combined (6.84) and (6.85), we obtain

$$(S(t)x_0)(y)=\frac{1}{4\pi it}\int_\Omega e^{\frac{i|y-z|^2}{4t}}x_0(z)dz. \tag{6.86}$$

By Lemma 1.1 in Pazy, 1983, we have the estimation immediately.

Lemma 6.40. Let $\{S(t),t\geq 0\}$ be the strongly continuous semigroup given by (6.79). Then $S(\cdot)$ can be extended in a unique way to a bounded operator from $L^2(\Omega)$ into $L^2(\Omega)$ and

$$\|S(t)x\|_{L^2(\Omega)}\leq\|x\|_{L^2(\Omega)}.$$

6.5.2 *Existence and Uniqueness*

In this section, we study the existence and uniqueness of mild solutions for system (6.77). To achieve our aim, we adopt the idea of Zhou and Jiao, 2010a,b and introduce the following two characteristic solution operators:

$$\begin{aligned}(\mathscr{T}(t)x)(y)&:=\int_0^\infty\xi_\alpha(\theta)(S(t^\alpha\theta)x)(y)d\theta\\&=\int_0^\infty\xi_\alpha(\theta)\left(\frac{1}{4\pi it^\alpha\theta}\int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}}x(z)dz\right)d\theta,\end{aligned} \tag{6.87}$$

and

$$\begin{aligned}(\mathscr{S}(t)x)(y)&:=\alpha\int_0^\infty\theta\xi_\alpha(\theta)(S(t^\alpha\theta)x)(y)d\theta\\&=\alpha\int_0^\infty\theta\xi_\alpha(\theta)\left(\frac{1}{4\pi it^\alpha\theta}\int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}}x(z)dz\right)d\theta,\end{aligned} \tag{6.88}$$

where

$$\xi_\alpha(\theta)=\frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi_\alpha(\theta^{-\frac{1}{\alpha}})\geq 0,$$

and

$$\varpi_\alpha(\theta)=\frac{1}{\pi}\sum_{n=1}^\infty(-1)^{n-1}\theta^{-\alpha n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\quad \theta\in(0,\infty).$$

Here, ξ_α satisfies $\lim_{\theta\to\infty}\xi_\alpha(\theta)=0$ and it is a probability density function defined on $(0,\infty)$, that is

$$\xi_\alpha(\theta)\geq 0,\quad \theta\in(0,\infty)\ \text{ and }\quad \int_0^\infty\xi_\alpha(\theta)d\theta=1.$$

Then, we can introduce the following definition for our problem.

Definition 6.41. By a mild solution of the system (6.77), we mean that a continuous function $x : J \to L^2(\Omega)$ which satisfies

$$\begin{aligned} x(t,y) &= \int_0^\infty \xi_\alpha(\theta)(S(t^\alpha\theta)x_0)(y)d\theta \\ &\quad + \int_0^t (t-s)^{\alpha-1}\left(\alpha\int_0^\infty \theta\xi_\alpha(\theta)S((t-s)^\alpha\theta)(kVx)(s)(y)d\theta\right)ds \\ &= \int_0^\infty \xi_\alpha(\theta)\left(\frac{1}{4\pi i t^\alpha\theta}\int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}}x_0(z)dz\right)d\theta \\ &\quad + \int_0^t (t-s)^{\alpha-1}\Big(\alpha\int_0^\infty \theta\xi_\alpha(\theta)\Big(\frac{1}{4\pi i(t-s)^\alpha\theta} \\ &\quad \times \int_\Omega e^{\frac{i|y-z|^2}{4(t-s)^\alpha\theta}}kV(z)x(s,z)dz\Big)d\theta\Big)ds, \end{aligned} \tag{6.89}$$

for $t \in J$.

Remark 6.42. Keeping in mind of that $\int_0^\infty \xi_\alpha(\theta)d\theta$ and $\int_0^\infty \theta\xi_\alpha(\theta)d\theta$ are absolutely convergence respectively, and $\lim_{\theta\to\infty}\xi_\alpha(\theta) = 0$ and $\lim_{\theta\to\infty}\theta\xi_\alpha(\theta) = 0$, there exist two constants $M_1 > 0$ and $M_2 > 0$ respectively such that

$$\int_0^\infty \xi_\alpha^2(\theta)d\theta \le M_1, \quad \int_0^\infty \theta^2\xi_\alpha^2(\theta)d\theta \le M_2.$$

The following properties of $\{\mathscr{T}(t), t \ge 0\}$ and $\{\mathscr{S}(t), t \ge 0\}$ are widely used in the sequel.

Lemma 6.43. Let $\{\mathscr{T}(t), t \ge 0\}$ and $\{\mathscr{S}(t), t \ge 0\}$ be two solution operators defined by (6.87) and (6.88) respectively. Then $\mathscr{T}(t)$ and $\mathscr{S}(t)$ can be extended in a unique way to the bounded operators from $L^2(\Omega)$ into $L^2(\Omega)$ and

$$\|\mathscr{T}(t)x\|_{L^2(\Omega)} \le \sqrt{M_1}\|x\|_{L^2(\Omega)}, \quad \|\mathscr{S}(t)x\|_{L^2(\Omega)} \le \alpha\sqrt{M_2}\|x\|_{L^2(\Omega)}.$$

Proof. For any $x \in L^2(\Omega)$, keeping in mind of Lemma 6.40, we obtain

$$\begin{aligned} \|\mathscr{T}(t)x\|^2_{L^2(\Omega)} &= \int_\Omega \left|\int_0^\infty \xi_\alpha(\theta)\left(\frac{1}{4\pi i t^\alpha\theta}\int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}}x(z)dz\right)d\theta\right|^2 dy \\ &\le \int_0^\infty \xi_\alpha^2(\theta)d\theta \cdot \int_\Omega\int_0^\infty \left|\left(\frac{1}{4\pi i t^\alpha\theta}\int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}}x(z)dz\right)\right|^2 d\theta dy \\ &\le M_1\|S(t^\alpha)x\|^2_{L^2(\Omega)} \\ &\le M_1\|x\|^2_{L^2(\Omega)}. \end{aligned}$$

Thus, one can obtain the first estimation immediately. Using the similar method, one can derive the second estimation. □

Lemma 6.44. Operators $\{\mathscr{T}(t), t \ge 0\}$ and $\{\mathscr{S}(t), t \ge 0\}$ are strongly continuous, which means that for all $x \in L^2(\Omega)$ and $0 \le t' < t'' \le T$, we have

$$\|\mathscr{T}(t'')x - \mathscr{T}(t')x\|_{L^2(\Omega)} \to 0 \ \text{ and } \|\mathscr{S}(t'')x - \mathscr{S}(t')x\|_{L^2(\Omega)} \to 0, \ \text{ as } t' \to t''.$$

Proof. For any $x \in L^2(\Omega)$ and $0 \le t' < t'' \le T$, we get that

$$\begin{aligned}\|\mathscr{T}(t'')x - \mathscr{T}(t')x\|_{L^2(\Omega)}^2 &= \int_\Omega \left| \int_0^\infty \xi_\alpha(\theta)[S((t'')^\alpha\theta) - S((t')^\alpha\theta)]x(y)d\theta \right|^2 dy \\ &\le M_1 \left\| [S((t'')^q\theta - (t')^q\theta) - I]x \right\|_{L^2(\Omega)}^2.\end{aligned}$$

According to the strongly continuity of $\{S(t), t \ge 0\}$, we know that $\|\mathscr{T}(t'')x - \mathscr{T}(t')x\|_{L^2(\Omega)}$ tends to zero as $t''-t' \to 0$, which means that $\{\mathscr{T}(t), t \ge 0\}$ is strongly continuous. Using the similar method, we can also obtain that $\{\mathscr{S}(t), t \ge 0\}$ is also strongly continuous. □

Lemma 6.45. Let Ω be a measurable subset of $\mathbb{R}^2$, $k = \max_{t\in[0,T]} |\chi(t)|$ and $V \in H^2(\Omega)$. Then, we have

$$\|kVx\|_{L^2(\Omega)} \le k\|V\|_{L^\infty(\Omega)}\|x\|_{L^2(\Omega)}. \tag{6.90}$$

Proof. Since $H^2(\Omega)$ can be continuous embedded in the space $C^0(\Omega) = \{V \in C(\Omega) : V \in L^\infty(\Omega)\}$ and $V \in L^\infty(\Omega)$, for arbitrary $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset \Omega$, such that $\alpha(\Omega_\varepsilon) = 0$ and

$$\sup_{\Omega\backslash\Omega_\varepsilon} |V(y)| < \|V\|_{L^\infty(\Omega)} + \varepsilon.$$

Thus,

$$\begin{aligned}\|kVx\|_{L^2(\Omega)}^2 &= \int_{\Omega_\varepsilon} |kV(y)x(y)|^2 dy + \int_{\Omega\backslash\Omega_\varepsilon} |kV(y)x(y)|^2 dy \\ &= k\int_{\Omega\backslash\Omega_\varepsilon} |V(y)x(y)|^2 dy \\ &\le k\left(\|V\|_{L^\infty(\Omega)} + \varepsilon\right)^2 \int_\Omega |x(y)|^2 dy \\ &\le k\left(\|V\|_{L^\infty(\Omega)} + \varepsilon\right)^2 \|x\|_{L^2(\Omega)}^2.\end{aligned}$$

Let $\varepsilon \to 0$ and taking the limit in the above inequality, one can obtain the inequality (6.90) immediately. □

In what follows, we collect the Henry-Gronwall inequality (see Lemma 7.1.1 in Henry, 1981), which can be used in fractional differential equations and integral equations with singular kernel.

Lemma 6.46. Let z, $\omega : [0,T) \to [0,+\infty)$ be continuous functions where $T \le \infty$. If ω is nondecreasing and there are constants $\kappa \ge 0$ and $q > 0$ such that

$$z(t) \le \omega(t) + \kappa\int_0^t (t-s)^{q-1}z(s)ds, \ t \in [0,T),$$

then

$$z(t) \le \omega(t) + \int_0^t \left[\sum_{n=1}^\infty \frac{(\kappa\Gamma(q))^n}{\Gamma(nq)}(t-s)^{nq-1}\omega(s)\right] ds, \ t \in [0,T).$$

If $\omega(t) = \bar{a}$, constant on $0 \le t < T$, then the above inequality is reduce to

$$z(t) \le \bar{a} E_q(\kappa \Gamma(q) t^q), \ 0 \le t < T,$$

where E_q is the Mittag-Leffler function defined by

$$E_\beta(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(k\beta + 1)}, \ y \in \mathbb{C}, \ Re(\beta) > 0.$$

Remark 6.47.

(i) There exists a constant $M_\kappa^* > 0$ independent of $\bar{a}$ such that

$$z(t) \le M_\kappa^* \bar{a} \text{ for all } 0 \le t < T.$$

(ii) For more generalized Henry-Gronwall inequalities, see Ye, Gao and Ding, 2007.

In order to discuss the existence of mild solutions for system (6.77), we need the following important priori estimate.

Lemma 6.48. Let $V \in H^2(\Omega)$. Suppose system (6.77) has a mild solution on $[0, T]$, then there exists a constant $\rho > 0$ such that

$$\|x(t)\|_{L^2(\Omega)} \le \rho \text{ for all } t \in [0, T].$$

Proof. If x is a mild solution x of system (6.77) on $[0, T]$, then x satisfies (6.89). Keeping in mind of Lemma 6.43 and Lemma 6.45, we have

$$\begin{aligned}
\|x(t)\|^2_{L^2(\Omega)} \le & \int_\Omega \left| \int_0^\infty \xi_\alpha(\theta) \left(\frac{1}{4\pi i t^\alpha \theta} \int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha \theta}} x_0(z) dz \right) d\theta \right|^2 dy \\
& + \int_\Omega \left| \int_0^t (t-s)^{\alpha-1} \left(\alpha \int_0^\infty \theta \xi_\alpha(\theta) \left(\frac{1}{4\pi i (t-s)^\alpha \theta} \right.\right.\right. \\
& \times \left.\left.\left. \int_\Omega e^{\frac{i|y-z|^2}{4(t-s)^\alpha \theta}} kV(z) x(s, z) dz \right) d\theta \right) ds \right|^2 dy \\
& \le M_1 \|x_0\|^2_{L^2(\Omega)} + \alpha^2 M_2 k^2 \|V\|^2_{L^\infty(\Omega)} \int_0^t (t-s)^{\alpha-1} \|x(s)\|^2_{L^2(\Omega)} ds.
\end{aligned} \tag{6.91}$$

By Lemma 6.46 and Remark 6.47, there exists a constant $M_{\alpha\sqrt{M_2} k \|V\|_{L^\infty(\Omega)}} > 0$ such that

$$\|x(t)\|^2_{L^2(\Omega)} \le M_1 \|x_0\|^2_{L^2(\Omega)} M_{\alpha\sqrt{M_2} k \|V\|_{L^\infty(\Omega)}} := \rho^2, \quad \text{for all } t \in [0, T].$$

The proof is completed. □

Theorem 6.49. Let $V \in H^2(\Omega)$. System (6.77) has a unique mild solution $x \in C([0, T], L^2(\Omega))$.

Proof. Fixed $x_0 \in L^2(\Omega)$, define

$$\mathcal{B}(x_0, 1) = \left\{ x \in C([0, T_1], L^2(\Omega)) : \|x(t) - x_0\|_{L^2_\Omega} \le 1, \ t \in [0, T_1] \right\},$$

where T_1 will be chosen latter. It is obvious that $\mathcal{B}(x_0, 1)$ is a closed and convex subset of $C([0, T_1], L^2(\Omega))$.

Define an operator $P : \mathcal{B}(x_0, 1) \to \mathcal{B}(x_0, 1)$ as follows:

$$(Px)(t,y) = \int_0^\infty \xi_\alpha(\theta) \left(\frac{1}{4\pi i t^\alpha \theta} \int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}} x_0(z) dz \right) d\theta$$
$$+ \int_0^t (t-s)^{\alpha-1} \left(\alpha \int_0^\infty \theta \xi_\alpha(\theta) \left(\frac{1}{4\pi i (t-s)^\alpha \theta} \int_\Omega e^{\frac{i|y-z|^2}{4(t-s)^\alpha\theta}} kV(z) x(s,z) dz \right) d\theta \right) ds.$$

It is easy to see that P is well defined.

By Lemma 6.43, for all $x \in C([0,T_1], L^2(\Omega))$,

$$\begin{aligned}
&\|(Px)(t) - x_0\|_{L^2(\Omega)} \\
&\le \int_\Omega \left| \int_0^\infty \xi_q(\theta) \left(\frac{1}{4\pi i t^\alpha \theta} \int_\Omega e^{\frac{i|y-z|^2}{4t^\alpha\theta}} x_0(z) dz \right) d\theta - x_0(y) \right|^2 dy \\
&\quad + \int_\Omega \left| \int_0^t (t-s)^{\alpha-1} \left(\alpha \int_0^\infty \theta \xi_\alpha(\theta) \left(\frac{1}{4\pi i (t-s)^\alpha \theta} \right.\right.\right. \\
&\quad \times \left.\left.\left. \int_\Omega e^{\frac{i|y-z|^2}{4(t-s)^\alpha\theta}} kV(z) x(s,z) dz \right) d\theta \right) ds \right|^2 dy \\
&\le \|\mathscr{T}(t)x_0 - x_0\|_{L^2(\Omega)} + \alpha\sqrt{M_2} \int_0^t (t-s)^{\alpha-1} \|(kVx)(s)\|_{L^2(\Omega)} ds \\
&\le \|\mathscr{T}(t)x_0 - x_0\|_{L^2(\Omega)} \\
&\quad + \alpha\sqrt{M_2} k \|V\|_{L^\infty(\Omega)} \int_0^t (t-s)^{\alpha-1} (1 + \|x_0\|_{L^2(\Omega)}) ds \\
&= \|\mathscr{T}(t)x_0 - x_0\|_{L^2(\Omega)} + t^\alpha \sqrt{M_2} k \|V\|_{L^\infty(\Omega)} (1 + \|x_0\|_{L^2(\Omega)}).
\end{aligned} \tag{6.92}$$

By Lemma 6.44, $\{\mathscr{T}(t), t \ge 0\}$ is a strongly continuous operator in $L^2(\Omega)$. Thus, there exists a $\bar{t} > 0$ such that

$$\|\mathscr{T}(t)x_0 - x_0\|_{L^2(\Omega)} \le \frac{1}{2}, \quad t \le \bar{t}.$$

Let

$$T_{11} = \min \left\{ \bar{t}, \left[\frac{1}{2\sqrt{M_2} k \|V\|_{L^\infty(\Omega)} (1 + \|x_0\|_{L^2(\Omega)})} \right]^{\frac{1}{\alpha}} \right\},$$

then for all $t \le T_{11}$, it comes from (6.92) that

$$\|(Px)(t) - x_0\|_{L^2(\Omega)} \le 1.$$

Let x_1, $x_2 \in \mathcal{B}(x_0, 1)$, we have

$$\begin{aligned}
&\|(Px_1)(t) - (Px_2)(t)\|_{L^2(\Omega)} \\
&\le \int_\Omega \left| \int_0^t (t-s)^{\alpha-1} \left(\alpha \int_0^\infty \theta \xi_\alpha(\theta) \left(\frac{1}{4\pi i (t-s)^\alpha \theta} \right.\right.\right. \\
&\quad \times \left.\left.\left. \int_\Omega e^{\frac{i|y-z|^2}{4(t-s)^\alpha\theta}} kV(z) [x_1(s,z) - x_2(s,z)] dz \right) d\theta \right) ds \right|^2 dy \\
&\le \alpha\sqrt{M_2} k \|V\|_{L^\infty(\Omega)} \int_0^t (t-s)^{\alpha-1} \|x_1(s) - x_2(s)\|_{L^2(\Omega)} ds \\
&\le t^\alpha \sqrt{M_2} k \|V\|_{L^\infty(\Omega)} \|x_1 - x_2\|_{C([0,T_1], L^2(\Omega))}.
\end{aligned} \tag{6.93}$$

Let

$$T_{12} = \frac{1}{2}\left(\frac{1}{2\sqrt{M_2}k\|V\|_{L^\infty(\Omega)}}\right)^{\frac{1}{\alpha}}, \quad T_1 = \min\{T_{11}, T_{12}\},$$

then P is a contraction map on $\mathcal{B}(x_0, 1)$. It follows from the contraction mapping principle that P has a unique fixed point $x \in \mathcal{B}(x_0, 1)$, and x is the unique mild solution of system (6.77) on $[0, T_1]$. □

6.6 Notes and Remarks

The results in Section 6.2 are adopted from Bourdin, 2013. The results in Section 6.3 are taken from Beckers and Yamamoto, 2013. The material in Subsection 6.4.2 and Theorem 6.35 due to Torres, 2013. Theorems 6.37, 6.38 and 6.39 are adopted Nyamoradi and Zhou, 2014. The results of Section 6.5 are from Wang, Zhou and Wei, 2012d.

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