INTRODUCTION TO THE THEORY OF LÉVY FLIGHTS

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Abstract

Lévy flights, also referred to as Lévy motion, stand for a class of non-Gaussian random processes whose stationary increments are distributed according to a Lévy stable distribution originally studied by French mathematician Paul Pierre Lévy. Lévy stable laws are important for three fundamental properties: (i) similar to the Gaussian law, Lévy stable laws form the basin of attraction for sums of random variables. This follows from the theory of stable laws, according to which a generalized central limit theorem exists for random variables with diverging variance. The Gaussian distribution is located at the boundary of the basin of attraction of stable laws; (ii) the probability density functions of Lévy stable laws decay in asymptotic power-law form with diverging variance and thus appear naturally in the description of many fluctuation processes with largely scattering statistics characterized by bursts or large outliers; (iii) Lévy flights are statistically self-affine, a property catering for the description of random fractal processes. Lévy stable laws appear as statistical description for a broad class of processes in physical, chemical, biological, geophysical, or financial contexts, among others. We here review the fundamental properties of Lévy flights and their underlying stable laws. Particular emphasis lies on recent developments such as the first passage time and leapover properties of Lévy flights, as well as the behavior of Lévy flights in external fields. These properties are discussed on the basis of analytical and numerical solutions of fractional kinetic equations as well as numerical solution of the Langevin equation with white Lévy noise.

Key words: Lévy stable probability density function, Lévy flights, fractional Fokker-Planck equation, Langevin equation.

Lévy flights are Markovian stochastic processes whose individual jumps have lengths that are distributed with the probability density function (PDF) $\lambda(x)$ decaying at large x as $\lambda(x) \simeq |x|^{-1-\alpha}$ with $0 < \alpha < 2$. Due to the divergence of their variance, $\left\langle x^2(t) \right\rangle \to \infty$, extremely long jumps may occur, and typical trajectories are self-similar, on all scales showing clusters of shorter jumps interspersed by long excursions. In fact, the trajectory of a Lévy flight has fractal dimension $d_f = \alpha$ [1]. Similar to the emergence of the Gaussian as limit distribution of independent identically distributed (iid) random variables with finite variance due to the central limit theorem, Lévy stable distributions represent the limit distributions of iid random variables with diverging variance [1-3]. In that sense, the Gaussian distribution represents the limiting case of the basin of attraction of the so-called generalized central limit theorem for $\alpha = 2$ [1-4].

Measurable quantities that follow Lévy statistics are widely known in nature. Thus, paleoclimatic time series exhibit a distinct behaviour, that bears clear signatures of Lévy noise [5]. The foraging behaviour of bacteria and higher animals relies on the advantages of Lévy distributed excursion lengths [6-13], which optimize the search compared to Brownian search. The latter tends to significant oversampling in one and two dimensions. In hopping processes along a polymer, where shortcuts across chemically remote segments, that due to looping come in close contact in the embedding space, Lévy flights occur in the annealed limit [14-16]. The spreading of diseases, due to the high degree of connectivity by airtraffic of remote geographic locations involves Lévy travel lengths and thus causes a fast worldwide spreading [17]. Similarly, extensive tracking of one dollar bills reveals Lévy statistics [18].

Signatures of Lévy statistics were also documented in the study of the position of a single ion in a one-dimensional optical lattice, in which diverging fluctuations could be observed in the kinetic energy [19]. The occurrence of Lévy flights in energy space for single molecules interacting with two level systems via long range interactions was discussed in Refs.[20], [21]. Lévy statistics has been identified in random single-molecule line shapes in glass-formers [22].

Finally, it has been revealed that Lévy statistics govern the distribution of trades in economical contexts [23-25]. Lévy flights are also connected to questions of ergodicity breaking due to the divergence of the variance of the jump length distribution [26].

In what follows we give a detailed introduction to the physical and mathematical foundation of Lévy processes, and explore their dynamic and stationary behaviour.

1. Lévy stable distributions

For sums of independent, iid random variables with proper normalization to the sample size, the generalized central limit theorem guarantees the convergence of the associated probability density function (PDF) to a Lévy stable (LS) PDF even though the variance of these random variables diverges [1-3], [27]. In general, an LS PDF is defined through its characteristic function of the probability density [28], [29], that is, its Fourier transform,

$$p_{\alpha,\beta}(k;\mu,\sigma) = \mathcal{F}\left\{p_{\alpha,\beta}(x;\mu,\sigma)\right\} \equiv \int_{-\infty}^{\infty} dx\, e^{ikx} p_{\alpha,\beta}(x;\mu,\sigma) =$$

$$= \exp\left[i\mu k - \sigma^{\alpha} |k|^{\alpha} \left(1 - i\beta \frac{k}{|k|} \varpi(k,\alpha)\right)\right] \quad , \tag{1}$$

where

$$\varpi(k,\alpha) = \begin{cases} \tan\dfrac{\pi\alpha}{2} \quad , \ \text{if} \quad \alpha \neq 1 \quad , \quad 0<\alpha<2 \, , \\ -\dfrac{2}{\pi}\ln|k| \ , \ \text{if} \quad \alpha = 1. \end{cases} \quad . \tag{2}$$

Thus, one can see that, in general, the characteristic function and, respectively, the LS PDF are determined by the four real parameters: α, β, μ, and σ. The exponent $\alpha \in [0,2]$ is the index of stability, or the Lévy index, $\beta \in [-1,1]$ is the skewness parameter, μ is the shift parameter, and $\sigma > 0$ is a scale parameter. The index α and the skewness parameter β play a major role in our considerations, since the former defines the asymptotic decay of the PDF, whereas the latter defines the asymmetry of the distribution. The shift and scale parameters play a lesser role in the sense that they can be eliminated by proper scale and shift transformations,

$$p_{\alpha,\beta}(x;\mu,\sigma) = \frac{1}{\sigma} p_{\alpha,\beta}\left(\frac{x-\mu}{\sigma};0,1\right). \tag{3}$$

Due to this property, in what follows we set μ = 0, σ = 1, and denote the Lévy stable PDF $p_{\alpha,\beta}(x;0,1)$ by $p_{\alpha,\beta}(x)$. We note the important symmetry property of the PDF, namely

$$p_{\alpha,-\beta}(x) = p_{\alpha,\beta}(-x). \tag{4}$$

For instance, the asymptotics of the PDF $p_{\alpha<1,1}(x),\ x \to 0\ (x>0)$ or $x \to \infty$ have the same behavior as $p_{\alpha<1,-1}(x),\ x \to 0\ (x<0)$ or $x \to -\infty$.

One easily recognizes that a stable distribution $p_{\alpha,\beta}(x)$ is *symmetric* if and only if $\beta = 0$. Stable distributions with skewness parameter $\beta = \pm 1$ are called *extremal*. One can prove that all extremal stable distributions with $0 < \alpha < 1$ are one-sided, the support being the positive semi-axis if $\beta = +1$, and the negative semi-axis if $\beta = -1$. For instance, $p_{0<\alpha<1,1}(x)$ is only defined for $x \geq 0$.

Only in three particular cases can the PDF $p_{\alpha,\beta}(x)$ be expressed in terms of elementary functions:

(i) Gaussian distribution, $\alpha = 2$,

$$p_2(x) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4}\right). \tag{5}$$

In the Gaussian case β is irrelevant since $\tan\pi = 0$, and the variance is equal to 2. In this case, the generalized central limit theorem coincides with the traditional central limit theorem.

(ii) Cauchy distribution, $\alpha = 1$, $\beta = 0$,

$$p_{1,0}(x) = \frac{1}{\pi\left(1+x^2\right)}\ ; \tag{6}$$

(iii) Lévy-Smirnov distribution, $\alpha = 1/2$, $\beta = 1$,

$$p_{1/2,1}(x) = \begin{cases} \dfrac{1}{\sqrt{2\pi}} x^{-3/2} \exp\left(-\dfrac{1}{2x}\right), & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{7}$$

This probability law appears, e.g., in the first passage problem for a Brownian particle [27], supporting that even in classical Brownian dynamics Lévy stable laws and therefore diverging moments appear.

Stable distributions for arbitrary α can be expressed via the Fox H – functions [30]. Such a representation for all stable densities was obtained in [31]. The properties of H – functions are presented in detail in [32], [33]. In our tutorial review we do not dwell on representations for general α.

Lévy stable laws are of interest for their asymptotic behaviour: The symmetric stable PDF scales asymptotically as [34], [35]

$$p_{\alpha,0}(x) \approx C_1(\alpha)/|x|^{1+\alpha} \quad , \quad x \to \pm\infty \quad , \tag{8}$$

where

$$C_1(\alpha) = \frac{1}{\pi}\sin\left(\pi\alpha/2\right)\Gamma(1+\alpha) \qquad . \tag{9}$$

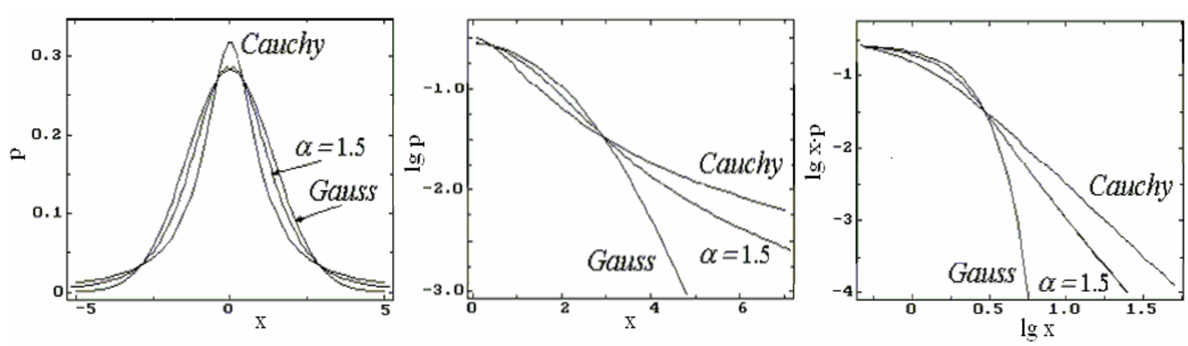

Fig.1. The shapes of symmetric stable laws are depicted for Gaussian, α = 1.5 and Cauchy PDFs, from the left to the right in linear, log-linear and log-log scales. One can see from the left figure showing the central parts of the PDFs that the Cauchy distribution has a more pronounced peak at the origin but basically all the three PDFs in the central part are similar to each other. The differences are becoming more pronounced in a log-linear scale which is well suited for showing the intermediate behavior between the central part of the PDF and the asymptotics. Note that in the log-linear scale the Cauchy distribution (in fact, as all the symmetric stable distributions) is concave far from its maximum at the origin, whereas the Gaussian distribution is convex. And, at last, in a log-log scale on the right figure one can see the large difference between the PDFs in the asymptotics, which are linear for α < 2 here, see Eq.(8). The slope in the log-log scale is determined by the Lévy index α according to Eq.(8). The Gaussian distribution is convex.

Respectively, for all LS PDFs with α < 2 the variance diverges, $\langle x^2 \rangle = \infty$. Conversely, all fractional moments $\langle |x|^q \rangle < \infty$ for all 0 < q < α < 2. The shapes of symmetric stable PDFs are shown in Fig.1 in a linear, log-linear and log-log scales.

Symmetric stable PDFs will be mostly used in our review. We also touch upon the behavior of extremal stable laws with β = 1, which find applications in different problems, for instance, as waiting time or first passage time distributions. At $x \to \infty$ their asymptotes have the same behavior as that of the symmetric PDFs, Eqs.(8), (9). At $x \to 0$ the extremal one-sided stable distribution with 0 < α < 1 behaves as [36], [37]

$$p_{\alpha,1}(x) \approx C_2(\alpha) x^{-1-\eta(\alpha)/2} \exp\left[-C_3(\alpha) x^{-\eta(\alpha)}\right] \quad , x \to 0 \, , \; 0 < \alpha < 1 \quad , \tag{10}$$

where $\eta(\alpha) = \alpha/(1-\alpha)$, and coefficients C_2, C_3 depend on α. Thus, at small x the PDF falls off exponentially. The extremal two-sided stable distribution with 1 < α < 2 exhibit a similar exponential decay at $x \to -\infty$,

$$p_{\alpha,1}(x) \approx \bar{C}_2(\alpha) x^{-1+\bar{\eta}(\alpha)/2} \exp\left[-\bar{C}_3(\alpha) x^{\bar{\eta}(\alpha)}\right] \quad , \; x \to -\infty \; , \mathbf{1} < \alpha < 2 \quad , \tag{11}$$

with $\bar{\eta}(\alpha) = \alpha/(\alpha-1)$, and another coefficients $\bar{C}_2$, $\bar{C}_3$. Both extremal stable PDFs are shown schematically in Fig.2.

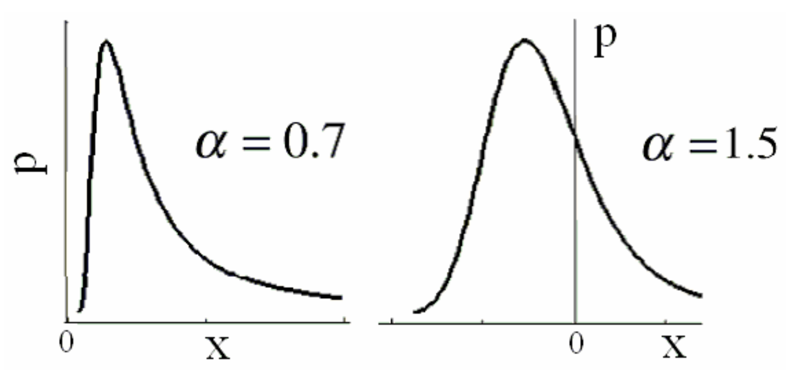

Fig.2. Sketch of one-sided (left) and two-sided (right) extremal LS PDFs with β = 1.

2. Underlying random walk processes

Consider a continuous time random walk process defined in terms of decoupled jump length and waiting time distributions $\lambda(\xi)$ and $\psi(\tau)$ [38], [39]. Each jump event of this random walk is characterized by a jump length ξ drawn from the distribution λ, and the time τ between two jump events is distributed according to ψ (note that an individual jump is supposed to occur spontaneously). As we will see, random processes whose jump lengths or waiting times are distributed according to an LS PDF exhibit anomalies, that is, no longer follow the laws of Brownian motion. In absence of an external bias, continuous time random walk theory connects $\lambda(\xi)$ and $\psi(\tau)$ with the probability distribution $f(x,t)$ to find the random walker at a position in the interval (x, $x+dx$) at time t. In Fourier–Laplace space, $f(k,u) \equiv \mathcal{F}\left\{\mathcal{L}\left\{f(x,t);t \to u;x \to k\right\}\right\}$, this relation reads [40]

$$f(k,u) = \frac{1-\psi(u)}{u}\frac{1}{1-\lambda(k)\psi(u)} \quad , \tag{12}$$

where $\mathcal{L}\left\{f(t)\right\} \equiv \int_0^\infty \exp(-ut) f(t)dt$.

The following cases can be distinguished:

(i) $\lambda(\xi)$ is Gaussian with variance σ^2 and $\psi(\tau) = \delta\left(\tau - \tau_0\right)$. Then, to a leading order in k^2 and u, respectively, one obtains $\lambda(k) \simeq 1-\sigma^2 k^2$ and $\psi(u) \simeq 1-u\tau_0$. From relation (12) one recovers the Gaussian probability density $P(x,t) = \sqrt{1/(4\pi Kt}\exp\left\{-x^2/(4Kt)\right\}$ with diffusion constant $K = \sigma^2/\tau_0$. The corresponding mean square displacement grows linearly with time:

$$\left\langle x^2(t)\right\rangle = 2Kt \quad . \tag{13}$$

This case corresponds to the continuum limit of regular Brownian motion. Note that here and in the following, we restrict the discussion to one dimension.

(ii) Assume $\lambda(\xi)$ is still a Gaussian, while for the waiting time distribution $\psi(\tau)$ we choose a one sided LS PDF with stable index $0 < \alpha < 1$. Consequently,

$\psi(u) \simeq 1-(u\tau_0)^{\alpha}$, and the characteristic waiting time $\int_0^{\infty} \psi(\tau)\tau d\tau$ diverges. Due to this lack of a time scale separating microscopic (single jump events) and macroscopic (on the level of $f(x,t)$) scales, $f(x,t)$ is no more Gaussian, but given by a more complex H – function [31], [41], [42]. In Fourier space, however, one finds quite simple analytical form of the characteristic function $f(k,t)$ in terms of the Mittag-Leffler function [41],

$$f(k,t) = E_{\alpha}\left(-K_{\alpha}k^2 t^{\alpha}\right) \quad , \tag{14}$$

which can be presented in the form of a series as [43]

$$E_{\alpha}\left(-K_{\alpha}k^2 t^{\alpha}\right) = \sum_{n=0}^{\infty} \frac{\left(K_{\alpha}k^2 t^{\alpha}\right)^n}{\Gamma(1+\alpha n)} \quad .$$

This function turns from an initial stretched exponential behavior $f(k,t) \approx 1 - K_{\alpha}k^2 t^{\alpha} / \Gamma(1+\alpha) \sim \exp\left\{-K_{\alpha}k^2 t^{\alpha} / \Gamma(1+\alpha)\right\}$ to a terminal power-law behavior $f(k,t) \approx \left(K_{\alpha}k^2 t^{\alpha}\Gamma(1-\alpha)\right)^{-1}$ [41]. In the limit $\alpha \to 1$, it reduces to the traditional exponential $f(k,t) = \exp\left(-Kk^2 t\right)$ with finite characteristic relaxation time. Also the mean squared displacement changes from its linear to the power-law dependence

$$\left\langle x^2(t)\right\rangle = \frac{2K_{\alpha}}{\Gamma(1+\alpha)} t^{\alpha} \quad , \tag{15}$$

with $K_{\alpha} = \sigma^2 / \tau_0^{\alpha}$. This is the case of subdiffusion. We note that in x,t space the dynamical equation is the diffusion-like equation with partial fractional derivative in time (time-fractional diffusion equation) [31]. In more detail the time-fractional diffusion equation is discussed in the Chapter written by **R. Gorenflo and F. Mainardi**. In the presence of an external potential, it generalizes to the time-fractional Fokker-Planck equation [44], [41], [42].

(iii) Finally, take $\psi(\tau)=\delta(\tau-\tau_0)$ sharply peaked, but $\lambda(\xi)$ of Lévy stable form with index $0 < \alpha < 2$. The resulting process is Markovian, but with diverging variance. It can be shown that the fractional moments scale like [45]

$$\left\langle |x(t)|^q \right\rangle \propto \left(K_\alpha t\right)^{q/\alpha} \quad , \tag{16}$$

where $K_\alpha = \sigma^\alpha / \tau_0$. We can therefore rescale this fractional moment according to $\left\langle |x(t)|^q \right\rangle^{2/q} \propto \left(K_\alpha t\right)^{2/\alpha}$ to see the "superdiffusive" character. Actual superdiffusion within the continuous time random walk framework requires the introduction of finite velocities and using the Lévy walk process [46]. From Eq.(12) one can immediately obtain the Fourier image of the associated PDF,

$$f\left(k,t\right) = \exp\left\{-K_\alpha |k|^\alpha t\right\} \quad . \tag{17}$$

From Eq.(1) this is but the characteristic function of a symmetric LS PDF with the index of stability α, and this type of random walk process is most aptly coined a Lévy flight. The characteristic function has regular exponential relaxation in time, and a Lévy flight process is in fact Markovian [47]. The PDF in position space is no more sharply localized like in the Gaussian or subdiffusive case, and it has the diverging variance. We will see below how the presence of steeper than harmonic external potentials cause a finite variance of the Lévy flight, although a power-law form of the probability density remains.

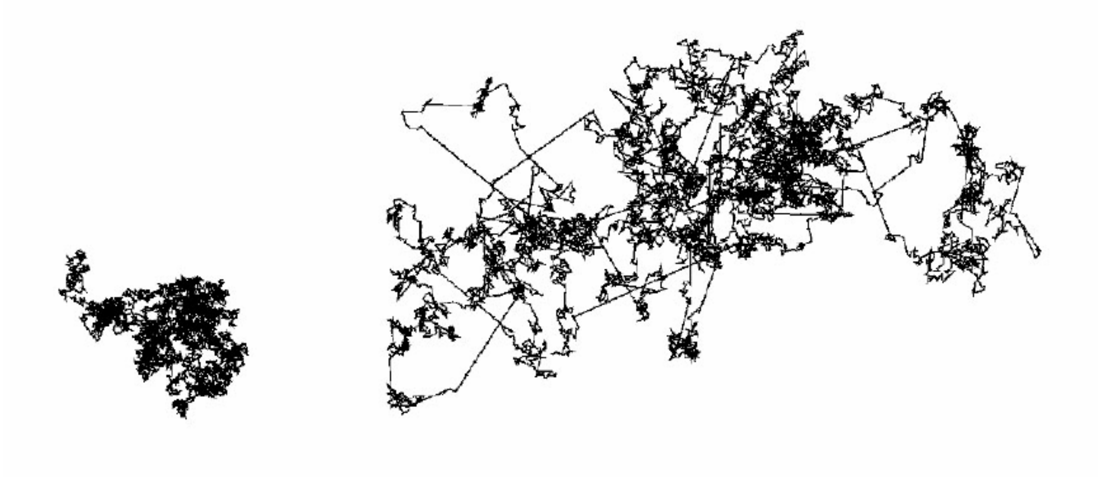

Fig.3. Comparison of the trajectories of a Gaussian (left) and a Lévy (right) process, the latter with index α = 1.5. Both trajectories are statistically self-similar. The Lévy process trajectory is characterized by the island structure of clusters of smaller steps, connected by a long step. Both walks are drawn for the same number of steps (≈7000).

For illustration, we show in Fig.3 the trajectory of a Lévy flight compared to a Gaussian walk. The clustered character of the Lévy flight separated by occasional long excursions is distinct. A Lévy flight trajectory has fractal dimension $d_f = \alpha$.

3. Space Fractional Fokker-Planck Equation

To derive the dynamic equation of a Lévy flight in the presence of an external force field $F(x) = -dU(x)/dx$, we pursue two different routes.
One starts from the Langevin equation

$$\frac{dx}{dt} = -\frac{1}{m\gamma}\frac{dU(x)}{dx} + \xi_\alpha(t) \quad , \tag{18}$$

driven by white Lévy stable noise $\xi_\alpha(t)$, defined through $L(\Delta t) = \int_t^{t+\Delta t} \xi_\alpha(t')dt'$ being a symmetric LS PDF of index α with characteristic function $p_{\alpha,0}(k,\Delta t) = \exp\left(-K_\alpha |k|^\alpha \Delta t\right)$ for $0 < \alpha \le 2$. As with standard Langevin equations, K_α denotes the *noise intensity* of physical dimension $[K_\alpha] = cm^\alpha/\sec$, m is the mass of the diffusing (test) particle, and γ is the friction constant characterizing the dissipative interaction with the bath of surrounding particles. It was shown [48], [49], [50], [51] that the kinetic equation corresponding to the Langevin description Eq.(18) has the form

$$\frac{\partial}{\partial t} f(x,t) = \left(-\frac{\partial}{\partial x} \frac{F(x)}{m\gamma} + K_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha} \right) f(x,t) \quad , \tag{19}$$

which is called *space-fractional Fokker-Planck equation* (space - FFPE). Remarkably, the presence of the Lévy stable $\lambda(\xi)$ only affects the diffusion term, while the drift term remains unchanged [48], [49]. The symbol $\partial^\alpha / \partial |x|^\alpha$ introduced in Ref. [52] denotes the *symmetric Riesz space fractional derivative*, which represents an integro-differential operator defined through

$$\frac{\partial^\alpha}{\partial |x|^\alpha} f(x,t) = \frac{-1}{2\cos(\pi\alpha/2)\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{f(x',t)}{|x-x'|^{\alpha-1}} dx' \tag{20}$$

for $1 < \alpha < 2$, $-\infty < x, x' < \infty$, and a similar form for $0 < \alpha < 1$ [53], [54]. In Fourier space, for all $0 < \alpha \leq 2$ the simple relation

$$F\left\{ \frac{\partial^\alpha}{\partial |x|^\alpha} f(x,t) \right\} = -|k|^\alpha f(k,t) \tag{21}$$

holds. For more details about fractional derivatives see the Chapter written by **R. Hilfer**. The space fractional diffusion equation with asymmetric derivatives, its numerical solution and application to plasma physics are discussed in the Chapter by **D. del-Castillo-Negrete**. In the Gaussian limit $\alpha = 2$, all relations above reduce to the familiar second-order derivatives in x and thus the corresponding $f(x,t)$ is governed by the standard Fokker-Planck equation.

Another approach starts with a generalized version of the continuous time random walk, compare Ref. [49] for details. To include the local asymmetry of the jump length distribution due to the force field $F(x)$, we introduce [49], [55] the generalized transfer kernel $\Lambda(x,x') = \lambda(x-x')\left[A(x')\Theta(x-x') + B(x')\Theta(x'-x)\right]$ (and, therefore $\Lambda(x,x') = \Lambda(x'; x-x')$). As in standard random walk theory (compare [56]), the coefficients $A(x)$ and $B(x)$ define the local asymmetry for jumping left and right, depending on the value of $F(x)$. Here, $\Theta(x)$ is the Heaviside jump function. With the normalization $\int \Lambda(x',\Delta)d\Delta = 1$, the space FFPE, Eq.(19), is obtained.

A subtle point about the FFPE (19) is that it does not uniquely define the underlying trajectory [57]; however, *starting from* our definition of the process in terms of the stable jump length distribution $\lambda(\xi) \sim |\xi|^{-1-\alpha}$, or its generalized pendant $\Lambda(x,x')$, the FFPE (19) truly represents a Levy flight in the presence of the force $F(x)$.

We also note that the other forms of space FFPE exist [58], [59], which describe different physical situations observed in experiment [60]; in particular, they describe the systems relaxing to the Boltzmann distribution, which is not the case for the systems obeying Eqs.(18) and (19), as we will see below.

4. Free Lévy Flights in the semi-infinite domain

First passage time and leapover properties. We consider two coupled properties of Lévy flights: The first passage time (FPT) and the first passage leapover (FPL). Considering a particle that starts at the origin and performs random jumps with independent increments chosen from a Lévy stable probability law with the PDF $p_{\alpha,\beta}(x)$, the FPT measures how long it takes the particle to arrive at or cross a target. For processes with broad jump length distributions, another quantity is of interest, namely, the statistics of the first passage leapovers, that is the distance the random walker overshoots the threshold value $x = d$ in a single jump. Figure 4 illustrates schematically the leapover event.

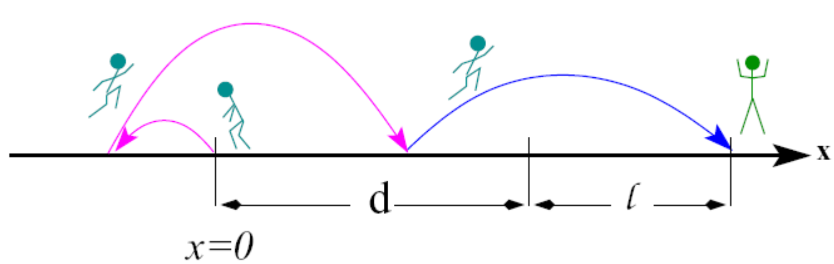

Fig.4. Schematic representation of the leapover problem: the random walker starts at $x = 0$ and after a number of jumps crosses the point $x = d$, overshooting it by a distance l. For narrow jump length distributions, each jump is so small that crossing the point d is equal to arriving at this point [61].

Information on the leapover behavior is important to the understanding of how far search processes of animals for food [8], [10] or of proteins for their specific binding site along DNA overshoot their target [16], or to define better stock market strategies determining when to buy or sell a certain share instead of a given threshold price [25]. Examples of the leapovers for the trajectories of symmetric and extremal one-sided Lévy motion are demonstrated in Fig.5. It is quite obvious that the smaller α is the larger are the jumps and, as one might expect, the larger is also the size of leapover.

One might naively assume that for Lévy flights the FPT PDF should decay quicker, than for a narrow jump length distribution. However, as we have a symmetric jump length distribution PDF, the long outliers characteristic for these Lévy flights can occur both toward and away from the absorbing barrier. From this point of view it is not totally surprising that for all Markovian processes with a symmetric jump length distribution the celebrated *Sparre Andersen theorem* [62], [63] proves, without knowledge of any details of $\lambda(\xi)$, that the asymptotic behaviour of the first passage time density universally follows $p(t) \propto t^{-3/2}$, and thus the mean FPT is infinite. The details of the specific form of $\lambda(\xi)$ only enter the prefactor, and the pre-asymptotic behaviour. A special case of the Sparre Andersen theorem was proved in Ref.[64] when the particle is released at $x_0 = 0$ at time $t = 0$, and after the first jump an

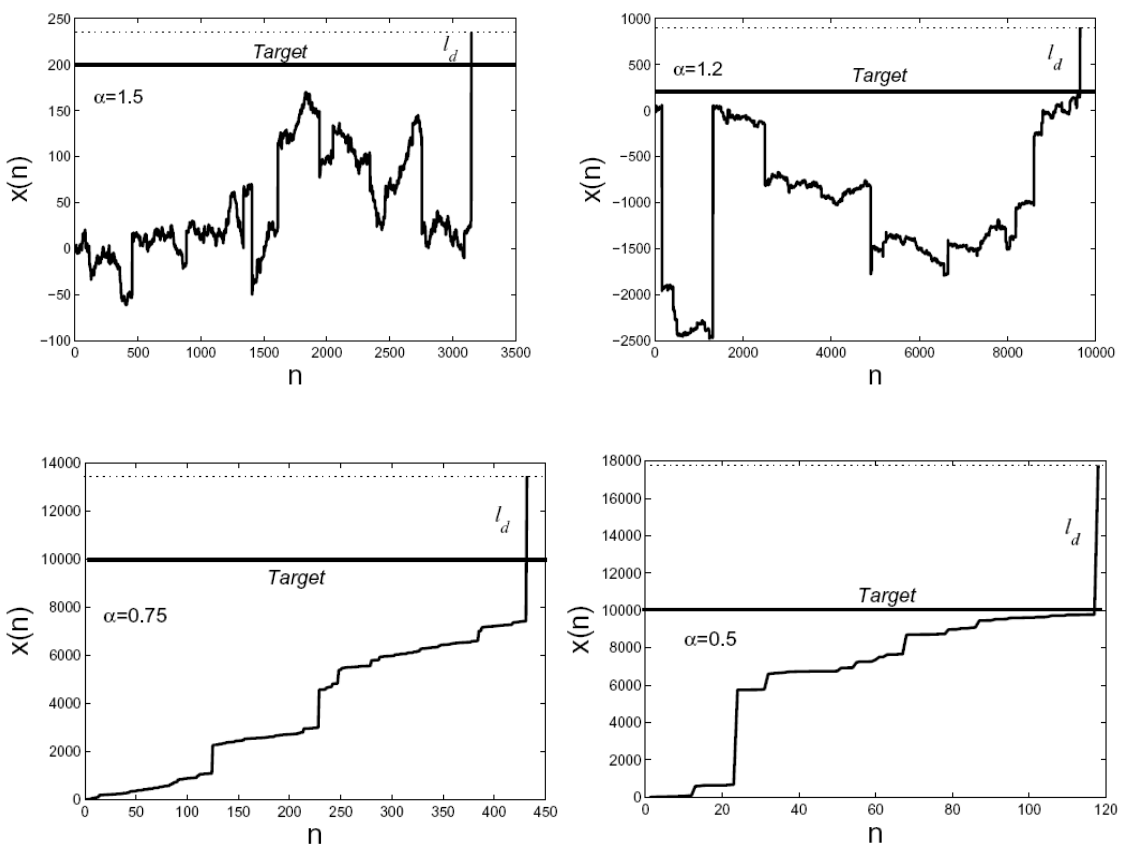

Fig.5. Trajectories obtained from numerical simulations with symmetric Lévy motion (top) and extremal one-sided Lévy motion with positive jumps (bottom); n is the number of time steps. As *α* becomes smaller larger jumps are more probable. The target is located at *d* = 200 in case of symmetric motion and at *d* = 10^4 in case of one-sided motion. Note the different scales for the different *α* values. The target location is shown by the full line and the leapover distance by the broken line.

absorbing boundary is installed at $x = 0$. This latter case was simulated extensively in Ref. [65].

Using the general theorem of FPT and FPL properties of the homogeneous processes with independent increments [47], the corresponding properties for symmetric and extremal one-sided LFs were derived. The basic results are as follows. For the symmetric LFs with index α and intensity K_α, starting at distance d from the boundary the asymptotics of the FPT PDF reads as [61]

$$p(\tau_d) \approx \frac{d^{\alpha/2}}{\alpha\sqrt{\pi K_\alpha}\,\Gamma(\alpha/2)}\tau_d^{-3/2} \tag{22}$$

Thus, not only the $t^{-3/2}$ is reproduced, but the prefactor also. The distribution of FPL l_d reads [61]

$$f(l_d) = \frac{\sin(\pi\alpha/2)}{\pi}\frac{d^{\alpha/2}}{l_d^{\alpha/2}(d+l_d)} \quad . \tag{23}$$

Surprisingly, for symmetric LFs with jump length distribution $p_{\alpha,0}(\xi) \sim |\xi|^{-1-\alpha}$ $(0 < \alpha < 2)$ the distribution of leapover lengths across d is distributed like $f(l_d) \sim l_d^{-1-\alpha/2}$ at large l_d, i.e., it is much broader than the original jump length distribution. For a phenomenological explanation of the leapover asymptotic behavior we use the "superdiffusive" nature of the motion of the Lévy particle during time interval τ_d, $x \propto \tau_d^{1/\alpha}$, where τ_d has a PDF with the Sparre Andersen "-3/2" asymptotics. The leapover is therefore $l_d \sim Q\tau_d^{1/\alpha} - d$, where Q is a (dimensional) constant. Applying $f(l_d) = f(\tau_d)d\tau_d / dl_d$, indeed leads to $f(l_d) \sim l_d^{-1-\alpha/2}$ at $l_d >> d$. This is a remarkable finding: while $p_{\alpha,0}(x)$ for 1 < α < 2 has a finite characteristic length $\langle |x| \rangle$, the mean leapover value is infinite for all α's. Another interesting (however, obvious from dimensional arguments) feature is that the leapover PDF does not depend on the intensity of the Lévy flights K_α. Both features are confirmed by extensive simulations, for more details refer to [61]. Note that $f(l_d)$ is normalized. In the limit $\alpha \to 2$, $f(l_d)$ tends to zero if $l_d \neq 0$ and to infinity at $l_d = 0$ corresponding to the absence of leapovers in the Gaussian continuum limit.

In contrast, the FPL PDF for extremal LFs with $0 < \alpha < 1$ obtained from the general theorem has the form [61]

$$f(l_d) = \frac{\sin(\pi\alpha)}{\pi} \frac{d^\alpha}{l_d^\alpha (d + l_d)} \quad , \tag{24}$$

which corresponds to the results obtained in Ref.[66] from a different method. Thus, for the extremal one-sided LF, the scaling of the leapover is exactly the same as for the jump length distribution, namely, with exponent α. The FPT PDF is expressed via the so-called *M*-function [61] (special case of the Wright function [67], [54]), and the mean value of the FPT is finite and reads

$$\langle \tau_d \rangle = \frac{d^\alpha \cos(\pi\alpha/2)}{K_\alpha \Gamma(1+\alpha)} \quad . \tag{25}$$

Again, the results (24), (25) compare favorably with simulations [68], [61].

Lévy flights and the method of images. As it is well-known in the theory of Brownian motion, the method of images allows one to get the solution of the diffusion equation in semi-infinite and finite domains as combination of the solutions on the infinite axis [27], [63]. That is, given the initial condition $\delta(x - x_0)$, $x_0 > 0$, the solution $f_{\mathrm{im}}(x,t)$ for the absorbing boundary problem, $f_{\mathrm{im}}(0,t) = 0$, according to the method of images in the semi-infinite domain corresponds to the difference

$$f_{im}(x,t) = W(x - x_0, t) - W(x + x_0, t) \tag{26}$$

in terms of free propagator W, i.e., a negative image solution originating at $-x_0$ balances the probability flux across the absorbing boundary. However, it was demonstrated in Ref.[69] that being applied to the LFs, the method of images produces a result, which is inconsistent with the universal behavior of the FPT PDF. Indeed, the corresponding (pseudo-) FPT PDF $p_{im}(t)$ is calculated as a negative time derivative of the survival probability [27],

$$p_{im}(t) = -\frac{d}{dt}\int_0^\infty f_{im}(x,t)dx \quad , \tag{27}$$

which leads to the long – t form [69]

$$p_{im}(t) \sim 2\Gamma\left(1/\alpha\right)x_0 / \left(\pi\alpha K_\alpha^{1/\alpha} t^{1+1/\alpha}\right) \tag{28}$$

for the image method. In the Gaussian limit $\alpha = 2$, Eq.(28) produces $p_{im}(t) \sim x_0 / \sqrt{4\pi K t^3}$, in accordance with Eq.(22). Conversely, for general $1 < \alpha < 2$, $p_{im}(t)$ according to Eq.(28) would decay faster than $t^{-3/2}$. Therefore, the method of images breaks down for LFs due to their non-local nature, displayed by the integrals in Eqs.(19), (21), see the detailed discussion in [69], [57].

5. Lévy Flights in External Fields

An important point is to understand the behavior of LFs in the presence of external potentials, that is "*confined*" LFs or the barrier crossing of LFs. Although the stage has been set for the study of such properties of LFs, rather limited information is available. In what follows, we briefly review the description of LFs in external fields in terms of the Langevin equation with white Lévy noise and the space-fractional Fokker–Planck equation, and demonstrate interesting and *a priori* unexpected statistical properties of confined LFs.

***Reminder: stationary solution of the Fokker-Planck equation*, $\boldsymbol{\alpha}$ = 2.** We remind that the stationary solution of the (standard) Fokker-Planck equation (in dimensionless variables) on the infinite axis,

$$\frac{d}{dx}\left(\frac{dU}{dx}f\right) + \frac{d^2 f}{dx^2} = 0 \qquad , \qquad -\infty < x < \infty \quad , \tag{29}$$

is given by the well-known Boltzmann formula,

$$f(x) = C\exp\left(-U(x)\right) \quad , \quad \int_{-\infty}^{\infty} dx f(x) = 1 \qquad . \tag{30}$$

In particular, for one-well potentials the stationary solution is unimodal, that is, it has one hump whose location coincides with the location of the well, and exhibits fast exponential decay away from the origin.

Lévy flights in harmonic potential. Let us find the stationary solution of space-FFPE with harmonic potential, $U(x) = x^2/2$. The direct way to solve Eq.(18) is applying the Fourier transform (we remind that in Fourier space the Riesz derivative $d^\alpha f(x)/d|x|^\alpha$ turns into $\left(-|k|^\alpha\right) f(k)$), which for the stationary state gives

$$\frac{df}{dk} = -\mathrm{sgn}(k)|k|^{\alpha-1} f(k) \quad , \quad f(k=0) = 1 \quad . \tag{31}$$

The solution is

$$f(k) = \exp\left(-|k|^\alpha / \alpha\right) \quad , \tag{32}$$

that is the characteristic function of symmetric stable law, see Eq.(1). The two properties of the stationary PDF in an harmonic potential follow immediately from Eq.(32): (i) unimodality (one hump at the origin), and (ii) slowly decaying power-law asymptotics, $f(|x| \to \infty) \approx C_\alpha |x|^{-1-\alpha}$, where $C_\alpha = \pi^{-1}\Gamma(\alpha)\sin(\pi\alpha/2)$ (compare with Eqs. (8), (9)). It implies that the second moment is infinite, and that the harmonic force is not "strong" enough to "confine" LFs. The LFs does not leave their basin of attraction defined by the external Lévy noise of index α.

***Lévy flights in quartic potential*, 1≤ α < 2.** With quartic potential, $U(x) = x^4/4$, the equation for the characteristic function in the stationary state follows from Eq.(19),

$$\frac{d^3 f}{dk^3} = \mathrm{sgn}(k)|k|^{\alpha-1} f(k) \quad . \tag{33}$$

This equation is solved with the use of the following boundary conditions and properties of the characteristic function: (i) $f(|k| = \pm\infty) = 0$; (ii) $f(k=0) = 1$ (normalization); (iii) $f(k) = f^*(k) = f(-k)$, where * stands for complex conjugation; here the first equality is a consequence of the Khintchin theorem (reality of characteristic function for symmetric PDF), whereas the second equality is the consequence of the Bochner-Khintchin theorem (positive definiteness of characteristic function); and (iv) $df^{(p)}(0)/dk^p = 0, \quad p = 1,3,5,...$, because odd moments (if exist) equal zero due to symmetry of the PDF, and we restrict ourselves to the class of characteristic functions which are "smooth enough" in the origin.

Let us first consider the particular Cauchy case, $\alpha = 1$, for which the solution of Eq.(33) is straightforward [70],

$$\hat{f}(k) = \frac{2}{\sqrt{3}} \exp\left(-\frac{|k|}{2}\right) \cos\left(\frac{\sqrt{3}|k|}{2} - \frac{\pi}{6}\right) \quad . \tag{34}$$

Inverse Fourier transform yields the PDF,

$$f(x) = \frac{1}{\pi(1 - x^2 + x^4)} \quad . \tag{35}$$

We observe surprisingly that the variance of the solution (35) is finite,

$$\left\langle x^2 \right\rangle = 1 \quad , \tag{36}$$

due to long-tailed asymptotics $f(x) \propto x^{-4}$. In addition, as shown in Fig.6, this solution has two global maxima at $x_{\max} = \pm 1/\sqrt{2}$ along with the local minimum at the origin (that is the position of the initial condition).

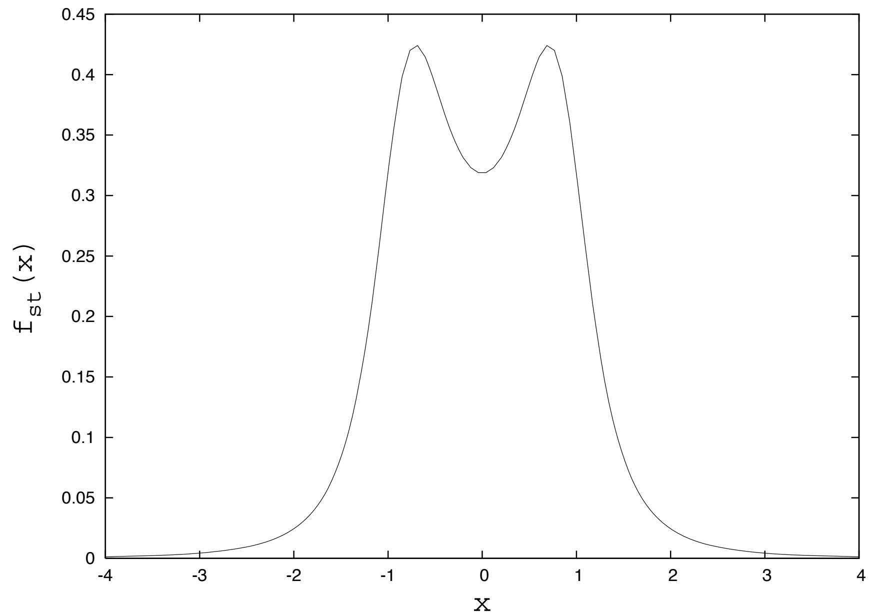

Fig.6. Stationary PDF (35) of the Cauchy- Lévy flight in a quartic potential.

Equation (33) was analyzed in [71] in detail, and the two properties, bimodality and steep asymptotics, $f(x) \approx C_\alpha |x|^{-\alpha-3}, \quad |x| \to \infty$, that is the finite variance, were discovered for the Lévy flights in quartic potential for the whole range of the Lévy indices between 1 and 2. In Fig.7 the profiles of stationary PDFs (obtained by an inverse Fourier transformation) are shown for the different Lévy indices from $\alpha = 1$ at the top of the figure up to $\alpha = 2$ at the bottom. It is seen that the bimodality is most strongly expressed for $\alpha = 1$. With the Lévy index increasing, the bimodal profile smoothes out and, finally, it turns to a unimodal one at $\alpha = 2$, that is, for the Boltzmann distribution.

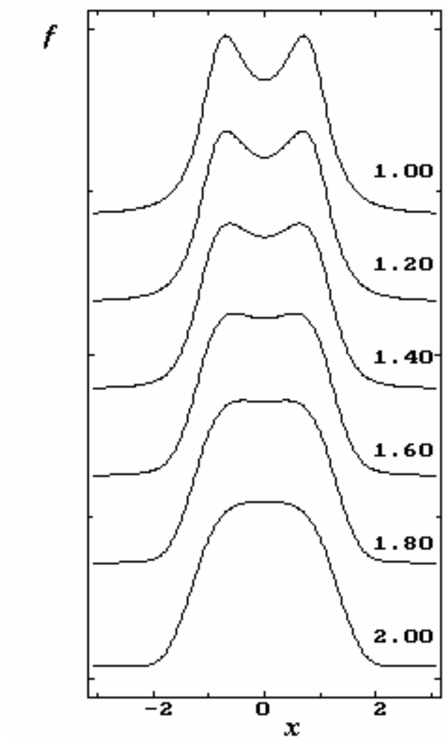

Fig.7. Profiles of stationary PDFs of the quartic oscillator for different Lévy indices, from α = 1 (at the top) till α = 2 (at the bottom).

Lévy flights in a more general potential well. In more general one well potentials of the type

$$U(x) = \frac{|x|^c}{c} \quad , \tag{37}$$

the turnover from unimodal to bimodal form of *f*(*x*) in stationary state occurs exactly when *c* becomes larger than 2. Moreover, the asymptotics of the PDF in the potential (37) are given by $f(x) \approx C_\alpha / |x|^{\alpha+c-1}, \quad |x| \to \infty$. It implies that the variance is finite only if $c > 4 - \alpha$, that is the potential wall is "steep enough". Remarkably, C_α appears to be a "universal" constant, i.e., it does not depend on *c*, analogously to harmonic and quartic potentials. Both properties of stationary states can be obtained by using different representation of the Riesz derivative in Eq.(18) [72]. An interesting effect is observed during relaxation to the bimodal stationary state, starting at t = 0 from narrow a Gaussian-like distribution in the origin. At $2 < c \leq 4$ the bifurcation occurs from a unimodal to a bimodal state, and the corresponding bifurcation time as function of α, $1 \leq \alpha < 2$, has a minimum at some intermediate value. In the potential well with *c*

> 4 two bifurcations occurs: first, from unimodal to trimodal transient state and then from trimodal to bimodal one. In the stationary state a bimodal distribution is always observed. The details of the Lévy flights behavior in a general potential well are given in [72]. From a reverse engineering point of view, Lévy flights in confining potentials are studied in [73].

Kramers problem for Lévy flights. Many physical and chemical problems are related to the cross of an energetic barrier, driven by external noise, such as dissociation of molecules, nucleation processes, or the escape from an external, confining potential of finite height [74]. A particular example of LFs barrier crossing in a double well potential was proposed for a long-time series of paleoclimatic data [5]. Detailed numerical investigations of the LFs barrier crossing were performed in [75], [76], [77]. In [77] an analytical theory of the escape over a barrier in a symmetric bistable potential was developed based on the space-FFPE for particular Cauchy case α = 1. Another analytical approach which is different from space-FFPE and is purely probabilistic was developed in [78]. The results of analytical and numerical approaches are the following. Similar to Brownian motion, an exponential decay of the survival probability in the initial well was found, $p(t) = T_c^{-1} \exp\left(-t / T_c\right)$, as demonstrated in Fig.8.

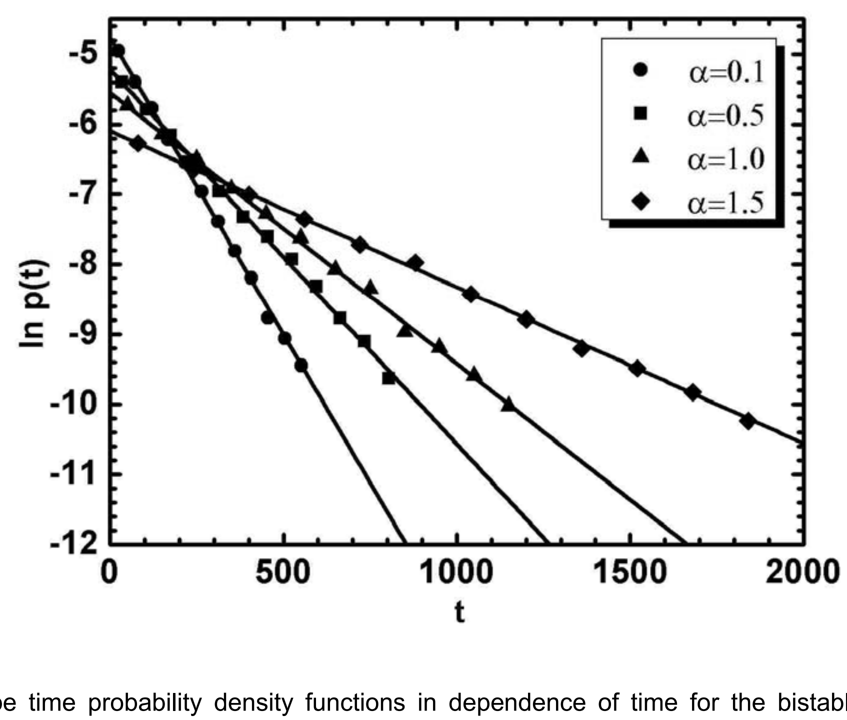

Fig.8. Escape time probability density functions in dependence of time for the bistable potentials. In logarithmic versus linear plot, the exponential dependence is obvious.

More interesting is the question how the mean escape time T_c behaves as function of the noise parameters K_α and α. While in the regular Kramers problem with Gaussian driving noise the Arrhenius-type activation $T_c = A\exp\left(h/D\right)$ is followed, where h is the barrier height and the prefactor A includes details of the potential, in the case of Lévy noise driven escape, a power-law form

$$T_c\left(\alpha, D\right) = \frac{C(\alpha)}{D^{\mu(\alpha)}} \tag{38}$$

was assumed [79], [75]. This behavior is distinctly visible in Fig.9 in form of the parallel lines in the log–log scale. Indeed, detailed investigations [77] show that the scaling exponent $\mu(\alpha)$ is approximately 1 for all α strictly smaller than 2, see Fig.10.

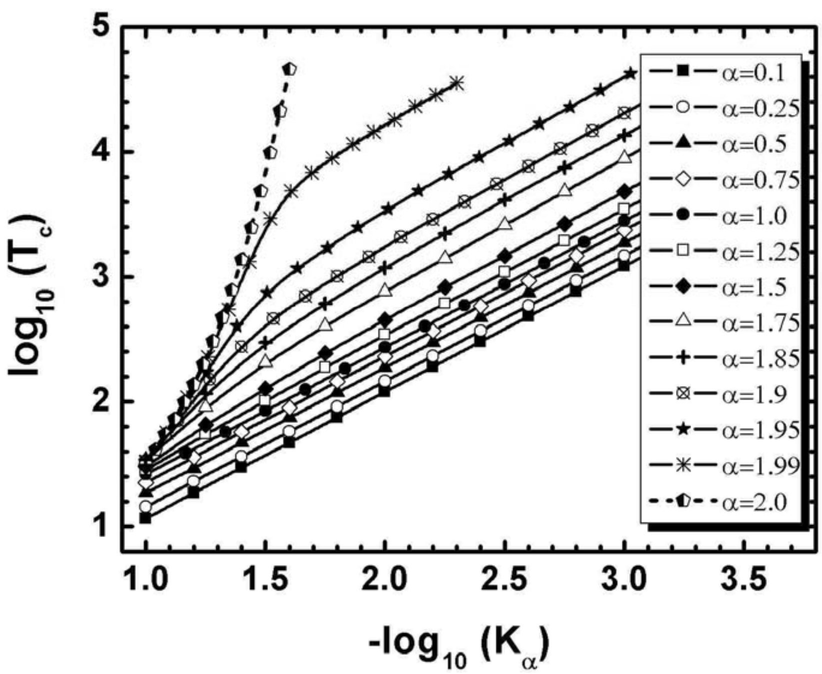

Fig.9. Characteristic escape time as function of the diffusivity D for the symmetric bistable potential.

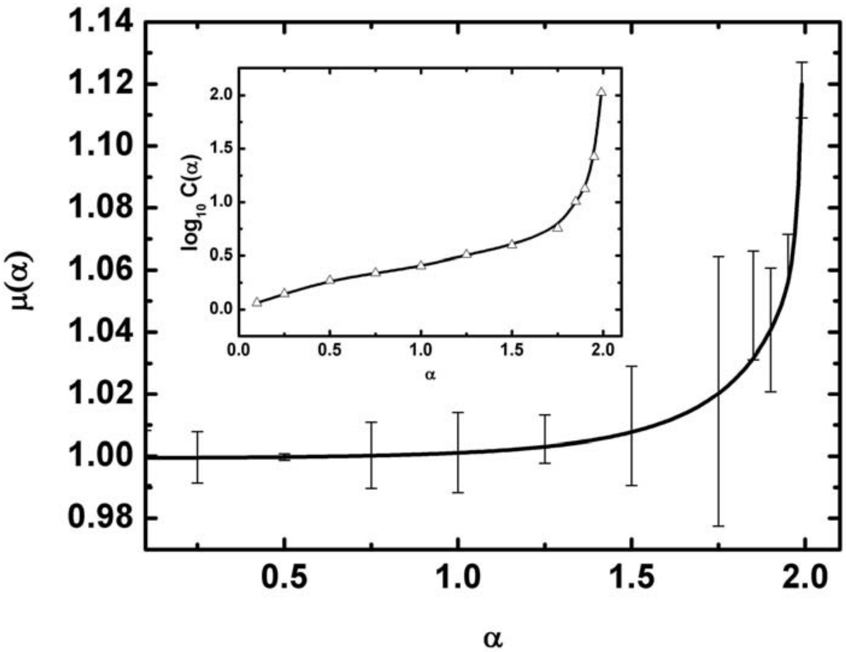

Fig.10. Dependencies of $\mu(\alpha)$ and $C(\alpha)$ for the symmetric bistable potential.

As already proposed in Ref.[79] and derived in [78] in a different model, this means that, apart from a prefactor, the Lévy flight is insensitive to the external potential for the barrier crossing. For large values of D deviations from the scaling are observed: eventually it will only take a single jump to cross the barrier when $D \to \infty$.We also note that in Ref.[76] the escape problem was investigated numerically for asymmetric LFs, $\beta \neq 0$.

6. Lévy Flights in Phase Space

In the phase space spanned by position and velocity coordinates x and v the basic equation of the Brownian motion is the Klein-Kramers equation for the PDF $f(x,\mathrm{v},t)$ [80], [81]. The equivalent Langevin description is based on the two first order differential equations for space coordinate and velocity, with white Gaussian noise source. The Langevin description can be generalized by including white Lévy noise source instead of the Gaussian. Correspondingly, we arrive at the generalized velocity-fractional Klein-Kramers equation.

Langevin description. Our starting point is the coupled Langevin equations in the phase space (x,v), which are written as

$$\frac{dx}{dt} = \mathrm{v} \quad , \quad \frac{d\mathrm{v}}{dt} = -\gamma \mathrm{v} + \frac{F}{m} + \zeta_\alpha(t) \quad , \quad , \tag{39}$$

where γ is the friction constant, which in general may be v – dependent, $F = -dU/dx$ is a deterministic force, U is a potential energy, and $\zeta_\alpha(t)$ is stationary white Lévy noise with the Lévy index α. The white Lévy noise is defined in way analogous to Sect.3, that is $L(\Delta t) = \int_t^{t+\Delta t} \zeta_\alpha(t')dt'$ being a symmetric LS PDF of index α with characteristic function $p_{\alpha,0}(\kappa,\Delta t) = \exp\left(-D_\alpha |\kappa|^\alpha \Delta t\right)$ for $0 < \alpha \leq 2$, D_α is the noise intensity, $[D_\alpha] = cm^\alpha / \sec^{1+\alpha}$. If we neglect inertia effects in Eq.(39), $d\mathrm{v}/dt = 0$, we arrive at the overdamped Langevin equation and the kinetic description presented in Sect.3, where $\xi_\alpha(t) = \zeta_\alpha(t)/\gamma$.

Velocity-fractional Klein-Kramers equation. It can be shown that the kinetic description equivalent to the Langevin description, Eq.(39), is given by the *velocity - fractional Klein-Kramers equation* (velocity-FKKE) [82], [83], [84],

$$\frac{\partial f}{\partial t} + \mathrm{v}\frac{\partial f}{\partial x} + \frac{F}{m}\frac{\partial f}{\partial \mathrm{v}} = \frac{\partial}{\partial \mathrm{v}}(\gamma \mathrm{v} f) + D_\alpha \frac{\partial^\alpha f}{\partial |\mathrm{v}|^\alpha} \quad , \tag{40}$$

where the last term in the right hand side is the Riesz fractional velocity derivative, defined, in analogy with definition Eq.(21), via the velocity Fourier transform of the PDF, whereas the other terms are the usual terms of the Klein-Kramers equation. At $\alpha = 2$ Eq.(40) is the (standard) Klein-Kramers equation for the systems driven by white Gaussian noise.

Space-homogeneous relaxation in absence of external field. We are looking for the solution $f(\mathrm{v},t)$ of the *velocity-fractional Rayleigh equation*

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v}(\gamma \mathrm{v} f) + D_\alpha \frac{\partial^\alpha f}{\partial |\mathrm{v}|^\alpha} \quad , \tag{41}$$

with the initial condition $f(\mathrm{v}, t=0) = \delta(\mathrm{v} - \mathrm{v}_0)$. Passing to the velocity - Fourier transform of the PDF,

$$f(\mathrm{v},t) = \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi} \exp(-i\kappa \mathrm{v}) f(\kappa,t) \quad , \tag{42}$$

we get the solution for the characteristic function [82],

$$f(\kappa,t) = \exp\left\{ i\kappa v_0 e^{-\gamma t} - |\kappa|^{\alpha} \chi(t) \right\} \quad , \tag{43}$$

where $\chi(t) = \left(D_\alpha / \alpha\gamma\right)(1 - e^{-\alpha\gamma t})$. We see that the PDF tends to a stationary solution which is a Lévy stable distribution in velocity, with relaxation time $\tau_v = 1/\alpha\nu$. The stationary Maxwell solution appears only in the Gaussian case, $\alpha = 2$, $D_2 \equiv D$,

$$f_{st}(v;\alpha = 2) = \left(\frac{\gamma}{2\pi D}\right)^{1/2} \exp\left(-\frac{\gamma v^2}{2D}\right) \quad . \tag{44}$$

This is the equilibrium PDF of Brownian motion. It is characterized by the temperature of the surrounding medium T. For Brownian motion, there is a well-known relation between noise intensity D in the Langevin equation, friction coefficient γ and temperature T: $D = \gamma k_B T / m$, where m is the mass of Brownian particle, k_B is Boltzmann constant. Temperature is a measure of a mean kinetic energy of a Brownian particle, $\langle E_{kin} \rangle = m\langle v^2 \rangle / 2 = k_B T / 2$. The last two relations are examples of fluctuation-dissipation relations. In this case the random source in the Langevin equation is called the source of internal fluctuations. These relations may not take place, as it happens, for example in autooscillatory systems. In this case it is said that $\zeta_2(t)$ is a source of external (relative to the system considered) fluctuations. However, the stationary Maxwell-Boltzmann distribution still exists [81]. For Lévy motion, there are no fluctuation-dissipation relations, that is why one can talk about $\zeta_\alpha(t)$ as a source of external fluctuations, only. Moreover, the stationary PDF is not a Maxwellian one but instead has the more general form of a stable distribution. At present no theory of the equilibrium state does exist, which would be based on stable distributions.

Space-inhomogeneous relaxation in absence of external field. Similar to Brownian motion, the space-inhomogeneous relaxation in a force - free case can be divided into two stages which are described in the framework of the velocity-FKKE: a "fast" stage, at which a stationary stable PDF over velocity is formed, and a "slow" diffusion stage, at which relaxation in the position coordinate occurs [84]. The latter process can be described asymptotically as the Lévy stable process with independent increments. The characteristic time of the velocity relaxation is τ_v, whereas the relaxation time in position coordinate is $\tau_x \approx (\gamma L)^{\alpha} / D_\alpha$, where L is an external size of the system considered. For systems large enough one has $\tau_x >> \tau_{\mathrm{v}}$, and the two-stage representation of relaxation process is valid. The diffusion stage of relaxation can be described in the framework of the space-FFPE, with the fractional moments of the PDF scaling according to Eq.(16) with $K_\alpha = D_\alpha / \gamma^\alpha$.

Relaxation of linear oscillator [84]. When studying the relaxation behaviour of a linear oscillator, $F(x) = -m\omega^2 x$, it is worthwhile to distinguish between two cases: an overdamped oscillator, $\omega/\gamma << 1$; and a weakly damped oscillator, $\omega/\gamma >> 1$. Both cases are of special importance in the kinetic theory. The relaxation process is very different in the two cases.

The relaxation of an overdamped oscillator, similar to space-inhomogeneous relaxation, has two stages, which are described in the framework of Kramers equation: a "fast" stage, at which during the time interval $\tau_{\mathrm{v}} = 1/\alpha\gamma$ a stationary stable PDF in velocity is formed; and a "slow" diffusion stage, at which during the time interval $\tau_x = \gamma/\alpha\omega^2$ a stationary stable PDF in position coordinate emerges. At the diffusion stage the relaxation of an overdamped oscillator can be described in the framework of the space-FFPE.

For a weakly damped oscillator in the theory of Brownian motion a method was developed, which allows one to simplify the kinetic description by using slowly varying (within period of oscillations) random variables [81]. The generalization of this approach to the case of a weakly damped Lévy oscillator leads to the fractional kinetic equation in the slow variables $\tilde{x}, \tilde{\mathrm{v}}$,

$$\frac{\partial}{\partial t} f(\tilde{x}, \tilde{\mathrm{v}}, t) = \frac{\gamma}{2} \frac{\partial}{\partial \tilde{x}} (\tilde{x} f) + \frac{\gamma}{2} \frac{\partial}{\partial \tilde{\mathrm{v}}} (\tilde{\mathrm{v}} f) + D_{\tilde{x}} \frac{\partial^\alpha f}{\partial |\tilde{x}|^\alpha} + D_{\tilde{\mathrm{v}}} \frac{\partial^\alpha f}{\partial |\tilde{\mathrm{v}}|^\alpha} \quad , \tag{45}$$

where $D_{\tilde{x}} = D_\alpha / 2\omega^\alpha$, and $D_{\tilde{v}} = D_\alpha / 2$ (for derivation see the Appendix in Ref.[84]). It follows from Eq.(45) that for a weakly damped oscillator both velocity and coordinate relax with the same relaxation time $\tau_{\mathrm{v}} = \tau_x = \tau = 2/\alpha\gamma$, and thus there is no separation between the kinetic and diffusion stages. In a stationary state the PDF is a Lévy stable in space and velocity, with the characteristic function

$$f_{st}(k, \kappa) = \exp\left(-\frac{2D_{\tilde{x}}}{\alpha\gamma} |k|^\alpha - \frac{2D_{\tilde{\mathrm{v}}}}{\alpha\gamma} |\kappa|^\alpha \right) \tag{46}$$

Eq.(46) reduces to Maxwell-Boltzmann distribution for $\alpha = 2$.

Relaxation in magnetized plasma. We here present a model example of magnetized plasma relaxation in the presence of a random electric field obeying Lévy statistics [85]. We consider a test charged particle with mass *m* and charge *e*, embedded in a constant external magnetic field $\vec{B} \parallel \vec{e}_z$ and subjected to a stochastic electric field $\vec{\varepsilon}(t)$. We also assume, as in the classical problem for a charged Brownian particle [86], [87], that the particle is influenced by the linear friction force $-\gamma m \vec{\mathrm{v}}$. The Langevin equations of motion read

$$\frac{d\vec{r}}{dt} = \vec{\mathrm{v}}, \qquad \frac{d\vec{\mathrm{v}}}{dt} = \frac{e}{mc}\left(\vec{\mathrm{v}} \times \vec{B}\right) - \gamma \vec{\mathrm{v}} + \frac{e}{m}\vec{\varepsilon} \quad , \tag{47}$$

where the field $\vec{\varepsilon}(t)$ is assumed to be (i) homogenous and isotropic; (ii) stationary white Lévy noise with intensity D_ε. The assumption about non-Gaussian statistics allows us to consider anomalous diffusion and non-Maxwellian heavy-tailed PDFs, both properties shown to be inherent to a strongly non-equilibrium cosmic and laboratory magnetized plasmas, see, e.g., [88] and references therein. Fig.11 shows typical trajectories of the Brownian and Lévy particles in the plane perpendicular to the magnetic field.

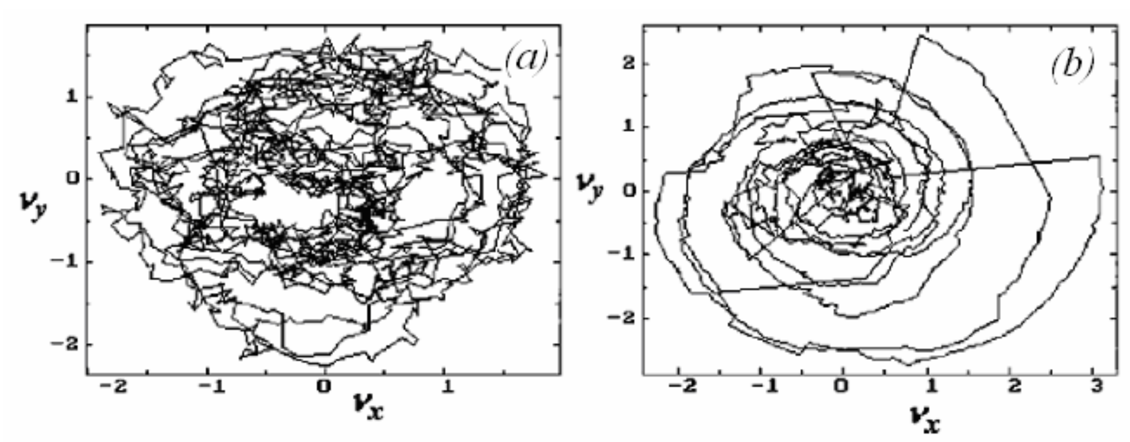

Fig.11. Numerical solution to the Langevin equations: trajectory on (v_x-v_y) plane perpendicular to the ambient magnetic field for (a) α = 2, and (b) α = 1.2. For the latter, Lévy flights for a charged particle are clearly seen [85].

The kinetic equation, equivalent to the Langevin description has the form

$$\frac{\partial f}{\partial t} + \vec{\mathrm{v}}\frac{\partial f}{\partial \vec{r}} + \Omega(\vec{\mathrm{v}} \times \vec{e}_z)\frac{\partial f}{\partial \vec{\mathrm{v}}} = \gamma \frac{\partial}{\partial \vec{\mathrm{v}}}(\vec{\mathrm{v}} f) - D_\alpha (-\Delta_{\vec{\mathrm{v}}})^{\alpha/2} f \quad , \tag{48}$$

where $\Omega = eB/mc$, $B \equiv |\vec{B}|$, $D_\alpha = e^\alpha D_\varepsilon / m^\alpha$, and $(-\Delta_{\vec{\mathrm{v}}})^{\alpha/2}$ is the fractional Riesz potential, which is defined via its Fourier transform in velocity as (compare with Eq.(21))

$$\mathcal{F}_{\mathrm{V}}\left\{(-\Delta_{\vec{\mathrm{v}}})^{\alpha/2} f(\vec{r},\vec{\mathrm{v}},t)\right\} = |\vec{\kappa}|^\alpha f(\vec{r},\vec{\kappa},t) \ . \tag{49}$$

At α = 2 Eq.(48) is reduced to Klein-Kramers equation for a charged Brownian particle in a magnetic field [87].

The general solution of Eq.(48) is obtained by the method of characteristics [85]. In the absence of a magnetic field there are two stages of relaxation in this problem: in the first, "fast" stage, the velocity relaxation leads to non-Maxwellian stationary states, for which the velocity PDF is the Lévy stable distribution. The PDF of energy has power law tails decaying as $f_{st}(E) \propto E^{-(1+\alpha/2)}$. At the second, "diffusion" stage, the charged particle exhibit superdiffusion with the typical displacement behaving as

$$\Delta r \sim \left\langle r^q \right\rangle^{1/q} \propto \frac{t^{1/\alpha}}{B} \ . \tag{50}$$

Recalling that the classical diffusion law across a magnetic field gives

$$\left\langle r^2 \right\rangle \propto \frac{t}{B^2} \ , \tag{51}$$

we conclude that the diffusion described by the velocity-fractional Klein-Kramers equation demonstrates an anomalous behavior with time and remains classical with respect to the magnetic field dependence.

Damped Lévy flights [89]**.** At higher velocities the friction experienced by a moving body starts to depend on the velocity itself [90]. Such nonlinear friction is known from the classical Riccati equation $md\mathrm{v}/dt = mg - A\mathrm{v}^2$ for the fall of a particle of mass m in a gravitational field with acceleration g [91] (A is a positive constant), or autonomous oscillatory systems with a friction that is non-linear in the velocity [90], [92]. The occurrence of a non-constant friction coefficient $\gamma(\mathrm{v})$ leading to a non-linear dissipative force was highlighted in Klimontovich's theory of non-linear Brownian motion [93]. It is therefore natural that higher order, non-linear friction terms also occur in the case of Lévy processes.

We consider the Lévy flights in velocity space as governed by the Langevin equation (compare with Eq.(39))

$$\frac{d\mathrm{v}}{dt} = -\gamma(\mathrm{v})\mathrm{v} + \zeta_{\alpha}(t) \ , \tag{52}$$

where the non-linear friction coefficient is represented as a series in even powers of v,

$$\gamma(\mathrm{v}) = \gamma(-\mathrm{v}) = \gamma_0 + \gamma_2 \mathrm{v}^2 + ... + \gamma_{2n} \mathrm{v}^{2n} + , \quad \gamma_{2n} > 0 \quad . \tag{53}$$

This Langevin picture is equivalent to the velocity-fractional Rayleigh equation, Eq.(41), but with the friction coefficient being a function of v. The analysis of this equation is similar to that of space-FFPE for LFs in steep potentials, see Sect.4. The presence of the first higher order correction, $\gamma_2 \mathrm{v}^2$, rectifies the Lévy motion such that the asymptotic power-law becomes steeper and the variance finite. When even higher order corrections are taken into consideration, also higher order moments become finite. We show an example in Fig.12 for the second moment: compare the "smooth" behavior of the finite moments with the scattering behavior of the infinite moment. Here, the strong fluctuations observed in numerical simulations are manifestation of the theoretical infinity.

The effect on the stationary velocity distribution $f(\mathrm{v}) = \lim_{t \to \infty} f(\mathrm{v}, t)$ for higher order corrections in Eq.(53) is demonstrated in Fig.13: while for smaller v the character of the original Lévy stable behaviour is preserved (the original power-law behaviour, that is, persists to intermediately large v), for even larger v the corrections due to the dissipative non-linearity are visible in the transition(s) to steeper slope(s).

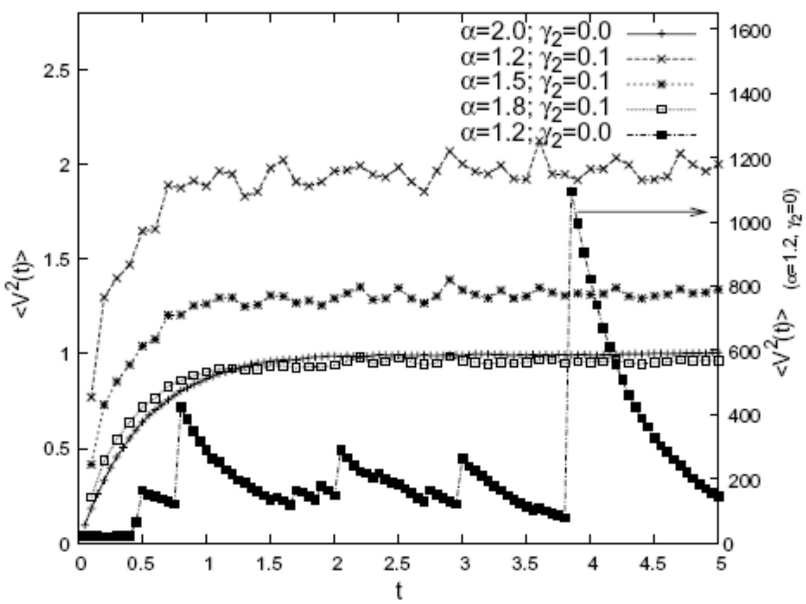

Fig.12. Variance $\left\langle \mathrm{v}^2(t) \right\rangle$ as function of time *t*, with the quartic and higher velocity terms set to zero in Eq.(6.16), and γ_0 = 1.0 for all cases. The variance is finite for the cases α = 2, γ_2 = 0.0; α = 1.2, γ_2 = 0.1; α = 1.5, γ_2 = 0.1; α = 1.8, γ_2 = 0.1. These correspond to the left ordinate. For the case α = 1.2, γ_2 = 0.0, the variance diverges, strong fluctuations are visible; note the large values of this curve corresponding to the right ordinate.

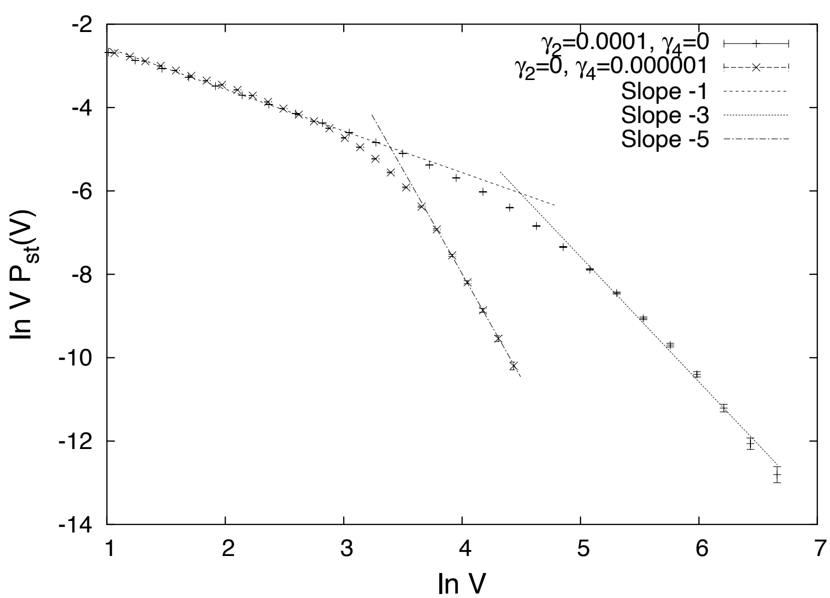

Fig.13. Stationary PDF *f*(*v*) for γ_0 = 1.0 and (i) γ_2 = 10^{-4} and γ_4 = 0; and (ii) γ_2 = 0 and γ_4 = 10^{-6}; with α = 1.0. The lines indicate the slopes -1, -3, and -5.

7. Power - Law Truncated Lévy Flights

Since Lévy flights have been introduced into statistical physics, it has become clear that special attention must be given to the fact that due to their heavy tails they are characterized by diverging moments. While the hopping along a coiled polymer allows for shortcuts at polymer loops, leading to an LF in the chemical coordinate along the chain but local jumps in the physical embedding space [14], [15], [16]; or while diffusion in energy space with diverging variance does not violate physical principles [20], particles of finite mass always must have a finite variance. A few approaches have been suggested to overcome this divergence. These include the introduction of the concept of Lévy walks including a spatiotemporal coupling such that at finite times only finite windows in space may be explored [94]; confining Lévy flights by external potentials (Section 5); damped Lévy flights (Section 6); and introduction of truncation procedures [95], [96]. Each of the approaches represents a different physical situation, but they all made it possible for Lévy processes to be applicable in a variety of areas. In many cases, the Lévy flight behavior corresponds to intermediate asymptotics. At very large values of the variable some cutoffs enter, so that the moments exist. Truncated Lévy flights, a process showing a slow convergence to a Gaussian, were introduced by Mantegna and Stanley [95] and have been used in econophysics ever since, see Refs. [97], [98], in turbulence [99] and in plasma physics [100]. The truncated Lévy flight is a Markovian jump process, with the length of jumps showing a power-law behavior up to some large scale. At larger scales the power-law behavior crosses over to a faster decay, so that the second moment of the jump lengths exists. In this case the central limit theorem applies, so that at very long times the distribution of displacements converges to a Gaussian; this convergence however may be extremely slow and cannot be observed in many experimental realizations. In such cases, a statistical description in terms of LFs is perfectly in order. In those cases where cutoffs become relevant, the transition of the dynamics needs to be studied. The original work concentrated on numerical simulations of the process which assumed a Θ-function cutoff. Koponen [96] slightly changed the model by replacing the Θ-function cutoff by an exponential one and obtained a useful analytical representation for the model. However, further investigations have shown that the models with sharp (Θ-function or exponential) cutoffs predicting a Gaussian or an exponential tail of the PDF are not always appropriate [98].

The equation proposed for truncated Lévy flights with power-law cutoff has the following form [101]:

$$\left(1-K_\alpha \frac{\partial^{2-\alpha}}{\partial |x|^{2-\alpha}}\right)\frac{\partial f(x,t)}{\partial t} = K\frac{\partial^2 f(x,t)}{\partial x^2} \quad . \tag{54}$$

Equation (54) is in fact a special case of distributed-order diffusion equations [102], which were introduced for the description of anomalous non-scaling behavior. The positivity of the solution is proved in [101].

Eq.(54) can be easily solved in Fourier space. For the characteristic function we have the solution corresponding to the initial condition $f(x,t=0)=\delta(x)$

$$f(k,t) = \exp\left(-\frac{Kk^2}{1+K_\alpha |k|^{2-\alpha}}\right) \quad . \tag{55}$$

The second moment of the solution evolves as in normal diffusion: $\left\langle x^2(t)\right\rangle = -\left(\partial^2 f(k,t)/\partial k^2\right)_{k=0} = 2Kt$. However, in the intermediate domain of x the distribution shows the behavior typical for Lévy flights; namely for k large enough, i.e., for $K_\alpha |k|^{2-\alpha} >> 1$, the characteristic function has the form

$$f(k,t) = \exp\left(-\frac{K}{K_\alpha}|k|^\alpha t\right) \quad , \tag{56}$$

i.e., it corresponds to the characteristic function of the Lévy distribution. Assuming $\alpha < 2$, we get the following expansion for $f(k,t)$ near $k = 0$:

$$f(k,t) \simeq 1 - Kk^2 t + KK_\alpha t|k|^{4-\alpha} + \ldots \quad . \tag{57}$$

From this expression it is evident that $f(k,t)$ always lacks the fourth derivative at $k = 0$ (for $1 < \alpha < 2$ it even lacks the third derivative), which means that the fourth moment of the corresponding distribution diverges. The absence of higher moments of the distribution explains the particular nature of the truncation implied by our model: the Lévy distribution is truncated by a power-law with a power between 3 and 5.

Thus, for all $0 < \alpha < 2$ the corresponding distributions have a finite second moment and, according to the central limit theorem (slowly!) converge to a Gaussian. For the case $0 < \alpha < 1$

(when $\left\langle |x|^3 \right\rangle < \infty$) the speed of this convergence is given by the Berry–Esseen theorem, as noted in Ref. [103]. The convergence criteria for $1 < \alpha < 2$ can be obtained using theorems of Chapter XVI of Ref. [27].

This transition from the initial Lévy-like distribution to a Gaussian is illustrated in Fig.14, obtained by a numerical inverse Fourier-transform of the characteristic function, Eq.(55). Here the case $\alpha = 1$, $K = K_\alpha = 1$ is shown. To combine the functions for $t = 0.001$ and $t = 1000$ in the same plot we rescale them in such a way that the characteristic width of the distribution (defined by $\int_0^{W(t)} f(x,t)dx = 1/4$) is the same. The behavior of the PDF to be at the origin $f(0,t)$ as a function of t is shown on the double logarithmic scales in Fig.15. Note the crossover from the initially fast decay $f(0,t) \propto t^{-1/\alpha}$ (Lévy superdiffusion) to the final form $f(0,t) \propto t^{-1/2}$ typical for normal diffusion. This crossover was actually observed when analyzing experimental data in tokamak edge turbulence [100]. Thus, Eq. (54) with the exponent α obtained from fluctuations data may be applied to describe the evolution of the PDF of the potential and electric field fluctuations measured in tokamak edge plasmas.

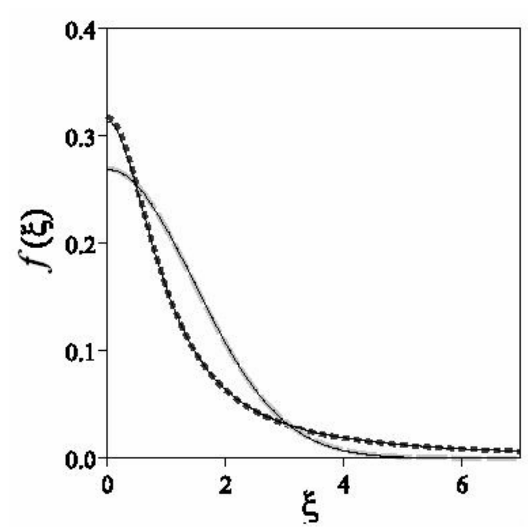

Fig.14. On the left: the rescaled PDF $f(\xi) = W(t) f(x,t)$ is shown for t = 0.001 (dotted line) and for t = 1000 (full line) as a function of a rescaled displacement $\xi = x / W(t)$. The corresponding thin lines denote the limiting Cauchy and Gaussian distributions under the same rescaling.

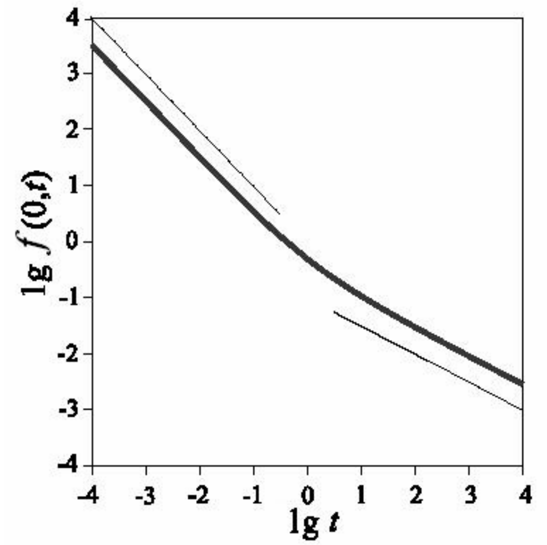

Fig.15. The PDF to be at the origin $f(0,t)$ as a function of time. Note the double logarithmic scales. The thick solid line has the slope -1 (with α = 1), and corresponds to the superdiffusive decay at small times, and the slope -1/2, as in the case of normal diffusion, at large times.

The asymptotics of the PDF at large x is determined by the first non-analytical term in the expansion, Eq.(57), i.e., by $KK_\alpha t|k|^{4-\alpha}$. By making the inverse Fourier transformation of this term and using the Abel method of summation of an improper integral, we get

$$f(x,t) \simeq \frac{\Gamma(5-\alpha)\sin(\pi\alpha/2)}{\pi}\frac{KK_\alpha t}{x^{5-\alpha}}, \quad x \to \infty \quad , \tag{58}$$

Thus, in this case the Lévy distribution is truncated not by a Θ- or an exponential function, but by a steeper power-law, with a power $\beta = 5 - \alpha$.

This truncation is clearly shown in Fig.16 in a log-log scale. At small times the Cauchy asymptote x^{-2} is replaced by the faster decaying asymptote x^{-4}. At large times the Gaussian-like central part of the PDF has a power law asymptotic behaviour, which, again decays as x^{-4}. With increasing time, the central Gaussian part enlarges, and the region of power law asymptotics shifts towards larger values of x.

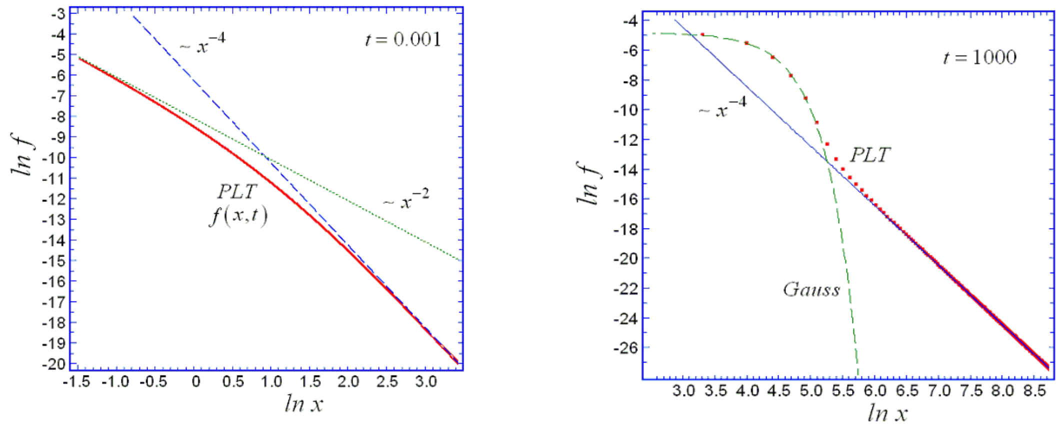

Fig.16. Time evolution of power-law truncated (PLT) Lévy flights. On the left: the solution of Eq.(54) at small time is shown by the thick solid line. Thin solid and dashed lines shows the asymptotics x^{-2} and x^{-4}, respectively. On the right: The solution of Eq.(54) at large time is shown by dots. The Gaussian distribution is depicted by dashed line. The thin solid line demonstrates the asymptote x^{-4}.

Another example for a possible application, a Lévy flight truncated by another, with faster decaying power-law, is a much better model for the behavior of commodity prices. Thus, the discussion in Ref. [98] shows that the cumulative distribution function of cotton prices may correspond to a power-law behavior of $1-F(x)=\int_x^{\infty} f(y)dy \propto x^{-\alpha}$ with the power α = 1.7 in its middle part and the heavy tail decaying as a power-law $1-F(x) \propto x^{-\beta}$ with $\beta \approx 3$. Thus, our equation (which is the simplest form for an equation for truncated Lévy flights) adequately describes this very interesting case giving $\beta = 3.3$. It is likely that fractional equations of the type considered here might be a valuable tool in economic research.

It is worthwhile to note that the power-law truncated Lévy flights possess an interesting property: they can be considered as a process subordinated to a Wiener process under the operational time given by the truncated one-sided (extreme) Lévy law of index $\alpha/2$ [101]. The subordination property also sheds light on the possible nature of truncated Lévy distributions in economic processes. The truncated Lévy process can be interpreted as a simple random walk with a finite variance. However, the number of steps of the random walk (the number of transactions) per unit time is not fixed, but fluctuates strongly. The implications of such models to economics were considered in [104]. In our case the distribution function of the number of steps itself has a form of a truncated one-sided Lévy law.

9. Summary

Lévy flights represent a widely used tool in the description of anomalous stochastic processes. By their mathematical definition, Lévy flights are Markovian and their statistical limit distribution emerges from independent identically distributed random variables, by virtue of the generalized central limit theorem. Despite their popularity, comparatively long history, and their Markovian nature, Lévy flights are not fully understood. The proper formulation in the presence of non-trivial boundary conditions, their behaviour in external potentials both infinitely high and finite, as well as their thermodynamical meaning are under ongoing investigation. This is discussed in the reviews [41], [42], [105], with numerous examples listed. We also recommend the introductory articles [106] and [107] devoted to these and closely related topics.

In this review, we have addressed some of the fundamental properties of random processes, these being the behaviour in external force fields, the first passage behaviour, as well as the Kramers-like escape over a potential barrier. We have shown that dissipative non-linear mechanisms cause a natural cutoff in the PDF, so that within a finite experimental range the untruncated Lévy flight provides a reasonable physical description. We also considered power-law truncated Lévy flight showing transition from the Lévy to Gaussian law behaviour in the course of time.

While the continuous time random walk model for Lévy flights in the absence of non-trivial boundary conditions or external potentials is a convenient description, in all other cases the fractional Fokker–Planck equation or, equivalently, the Langevin equation with white Lévy stable noise are the description of choice. These equations in most cases cannot be solved exactly, however, it is usually straightforward to obtain the asymptotic behaviour, or to solve them numerically.

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