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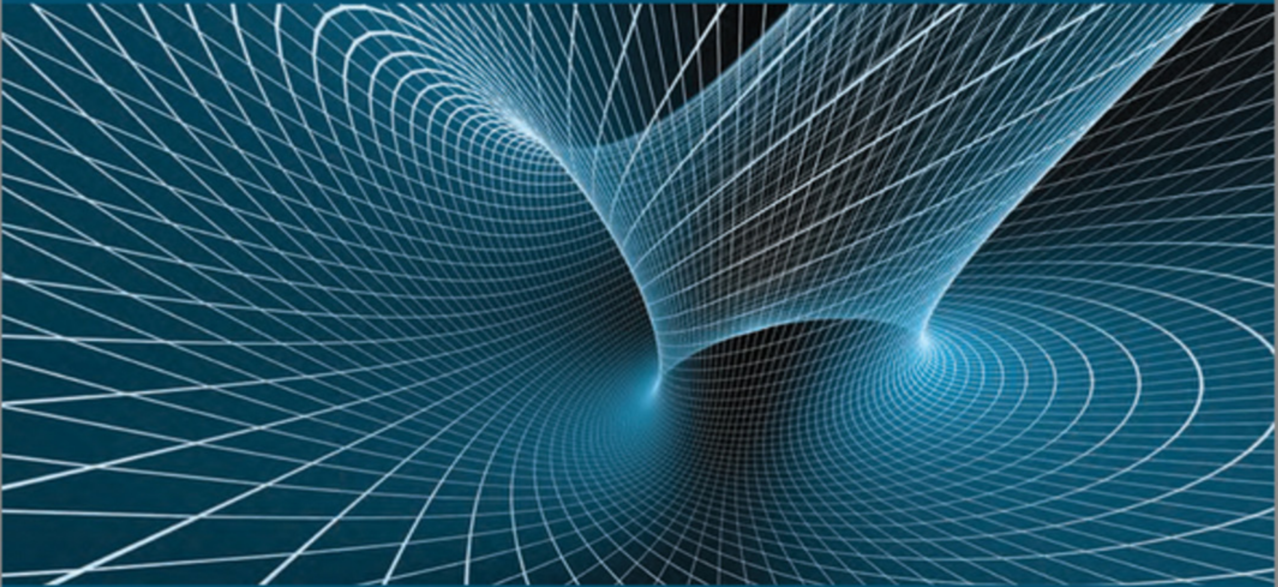

Fractional Calculus with Applications in Mechanics

Wave Propagation, Impact and Variational Principles

Teodor M. Atanacković, Stevan Pilipović
Bogoljub Stanković, Dušan Zorica

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Preface

The subject of this book is the application of the fractional calculus in mechanics. It is written so as to make fractional calculus acceptable to the engineering scientific community as well as to applied mathematicians who intend to use this calculus in their own research. The application of fractional calculus is mostly directed towards various areas of physics, engineering and biology. There are a number of monographs and a huge number of papers that cover various problems in fractional calculus. The list is large and is growing rapidly. The monograph by Oldham and Spanier [OLD 74], published in 1974, had a great influence on the subject. This was the first monograph devoted to fractional operators and their applications in problems of mass and heat transfer. Another important monograph was written by Miller and Ross in 1993 [MIL 93]. The encyclopedic treatise by Samko *et al.*, up to now, the most prominent book in the field. This book [SAM 87] was first published in Russian in 1987, and the English translation appeared in 1993 [SAM 93]. The monograph by Kilbas *et al.* [KIL 06] contains a detailed introduction to the theory and application of fractional differential equations, mostly given in references. It treats, in a mathematically sound way, the fractional differential equations. Kiryakova in [KIR 94] introduces a generalized fractional calculus. Diethelm in [DIE 10] gives a well-written introduction to fractional calculus before the main exposition on the Caputo-type fractional differential equations. This book has mathematically sound theory and relevant applications. In Russian, besides [SAM 87], we mention monographs by Nahushev [NAH 03], Pshu [PSH 05] and Uchaikin [UCH 08]. Variational calculus with fractional derivatives is analyzed by Klimek in [KLI 09] and by Malinowska and Torres in [MAL 12]. All of the above-mentioned monographs have had a great influence in the development of the fractional calculus. Also there are other influential books and articles in the field that are worth mentioning. The article by Gorenflo and Mainardi [GOR 97b] had a significant impact in the field of applications of fractional calculus in physics and mechanics. The book by Oustaloup [OUS 95] presents an application of fractional calculus in the control theory. The book edited by Hilfer [HIL 00] and the book by Hermann

[HER 11] contain applications of fractional calculus in physics. The article by Butzer and Westphal [BUT 00] contains a complete introduction to the fractional calculus. Podlubny's work [POD 99], which has become a standard reference in the field, contains applications of fractional calculus to various problems of mechanics, physics and engineering. Application of fractional calculus in bioengineering is presented by Magin in [MAG 06]. Baleanu *et al.* in [BAL 12a] present the models in which fractional calculus is used, together with the numerical procedures that are used for the solutions. It also contains an extensive review of the relevant literature. Tarasov in [TAR 11] presents, among other topics, an application of fractional calculus in statistical and condensed matter physics, as well as in quantum dynamics. Various applications, together with theoretical developments, are presented by Ortigueira in [ORT 11], Baleanu *et al.* in [BAL 12b], Petráš in [PET 11] and Sabatier *et al.* in [SAB 07]. The book by Mainardi [MAI 10] is the standard reference for the application of fractional calculus in viscoelasticity and for the study of wave motion. Finally, the book by Uchaikin [UCH 13] gives detailed motivation for fractional-order differential equations in various branches of physics. It also contains an introduction to the theory of fractional calculus.

Our book is devoted to the application of fractional calculus to (classical) mechanics. We have chosen to concentrate on more sophisticated constitutive equations that complement fundamental physical and geometrical principles. It is assumed that the reader has some basic knowledge of fractional calculus, i.e. calculus of integrals and derivatives of arbitrary real order. The main objective of this book is to complement, in a certain sense, the contents of the other books treating the theory of fractional calculus mentioned above. We will discuss non-local elasticity, viscoelasticity, heat conduction (diffusion) problems, elastic and viscoelastic rod theory, waves in viscoelastic rods and the impact of a viscoelastic rod against a rigid wall. The mathematical framework of the problems that we consider falls into different levels of abstraction. In our papers, on which the most part of the presentation is based, we use an approach to fractional calculus based on the functional analysis. In this way, we are able to use a well-developed method and techniques of the theory of generalized functions, especially of Schwartz space of tempered distributions and the space of exponential distributions, supported by $[0, \infty)$. The use of the generalized function setting only gives us flexibility in proving our results. If one deals with functions in $L^1_{loc}(\mathbb{R})$, equal zero on $(-\infty, 0)$, of polynomial or exponential growth, then their Laplace transform is the classical one. The same holds for the convolution of such functions. Namely, the convolution of such functions is again the locally integrable function, equal zero on $(-\infty, 0)$, that is of the polynomial or exponential growth. When we deal with the Fourier transform, our framework is S'. The deep connection with the real mechanical models is not lost. Even better explanations and correctness of the proofs can help the reader to understand mathematical models of the discussed problems. Our aim is not to make the book too complicated for the readers with less theoretical background in the quoted mathematical sense. The presentation, for the most part, is intended to avoid unnecessary details and state only major results that a detailed, and often abstract, analysis gives.

The related book Fractional Calculus Applications in Mechanics: Vibration and Diffusion Processes [ATA 14] is complementary to this book, and, indeed, they could have been presented together. However, for practical reasons, it has proved more convenient to present the book separately. There are 13 chapters in the two books combined.

This book, Fractional Calculus with Applications in Mechanics: Wave Propagation, Impact and Variational Principles, has a total of six chapters.

It begins with Part 1, entitled "Mathematical Preliminaries, Definitions and Properties of Fractional Derivatives", which presents an introduction to fractional calculus. It comprises two chapters. Chapter 1, entitled "Mathematical Preliminaries", is brief and gives definitions and notions that are used in later parts of the book. Chapter 2, entitled "Basic Definitions and Properties of Fractional Integral and Derivatives", presents definitions and some of the properties of fractional integral and derivatives. We give references where the presented results are proved. Some of the results, which are required in application, of our own research, such as expansion formulas for fractional derivatives and functional dependence of the fractional derivative on the order of derivative, are also presented in this chapter.

Part 2, entitled "Mechanical systems", is the central part of this book, containing four chapters. In Chapter 3, entitled "Waves in Viscoelastic Materials of Fractional-Order Type", we present an analysis of waves in fractional viscoelastic materials on infinite domain of space. A wave equation of the fractional Eringen-type is also studied. Stress relaxation, creep and forced oscillations of a viscoelastic of finite length are discussed in detail. In Chapter 4, entitled "Forced Oscillations of a System: Viscoelastic Rod and Body", the problem of oscillation of a rigid body, attached to viscoelastic rod and moving translatory, is analyzed in detail. The case of a light rod (mass of the rod is negligible with respect to the mass of the attached body) and the case of a heavy rod (mass of the rod is comparable with respect to the mass of the attached body) are discussed separately. Also, constitutive equations for solid-like and fluid-like bodies are distinguished. In Chapter 5, entitled "Impact of Viscoelastic Body against the Rigid Wall", we analyze a specific engineering problem of a viscoelastic rod impacting against the rigid wall. The case of a light viscoelastic rod sliding without friction is discussed first. Then, the more complicated case of a light viscoelastic rod attached to a rigid block that slides with dry friction is described. Finally, the case of a heavy viscoelastic rod attached to a rigid block that slides without friction is presented. In Chapter 6, entitled "Variational Problems with Fractional Derivatives", we present some results for the optimization of a functional containing fractional derivatives. We formulate the necessary conditions for optimality in standard and generalized formulation. We also present dual variational principles for Lagrangians having fractional derivatives. The necessary conditions for optimality are formulated in generalized problems where optimization is performed with respect to the order of the derivative and not only with respect to a given set of admissible functions. Invariance properties of variational principles and Nöther's theorem are discussed in this chapter, as well as the problem of approximation of Euler–Lagrange equations in two different ways. Also, in this chapter, we propose a constrained minimization problem in which the order of the derivative is considered as a constitutive quantity determined by the state and control variables.

The related book Fractional Calculus Applications in Mechanics: Vibration and Diffusion Processes has a total of seven chapters. It begins with Part 1, entitled "Mathematical Preliminaries, Definitions and Properties of Fractional Derivatives", which presents an introduction to fractional calculus. It contains two chapters. Chapter 1, entitled "Mathematical Preliminaries", is brief and gives definitions and notions that are used in later parts of the book. Chapter 2, entitled "Basic Definitions and Properties of Fractional Integral and Derivatives", presents definitions and some of the properties of fractional integral and derivatives. We give references where the presented results are proved. Some of the results, of our own research, which are required in applications, such as expansion formulas for fractional derivatives and functional dependence of the fractional derivative on the order of derivative, are also presented in this chapter.

Part 2, entitled "Mechanical Systems", is the central part of the book. It consists of five chapters. In Chapter 3, entitled "Restrictions Following from the Thermodynamics for Fractional Derivative Order Models of a Viscoelastic Body", the analysis of constitutive equations of fractional-order viscoelasticity is presented. The constitutive equations must satisfy two principles: the principle of material frame-indifference (objectivity), which asserts that the response of a material is the same for all observers, and the second law of thermodynamics, which in the case of isothermal processes reduces to dissipation inequality. Sometimes dissipation inequality is called the Volterra theorem for hereditary systems (see [UCH 13]). We analyze, in detail, various constitutive equations and determine the restrictions of the material functions or constants so that the dissipation inequality is satisfied. Material objectivity is treated for non-local elastic materials in section 3.3, Chapter 3 of the companion book Fractional Calculus with Applications in Mechanics: Wave Propagation, Impact and Variational Principles [ATA 14], for the simple one-dimensional spatial case. In Chapter 4, entitled "Vibrations with Fractional Dissipation", we analyze various vibration problems with single degree of freedom. Problems with single and multiple dissipation terms are discussed. Also, nonlinear vibrations with symmetrized fractional dissipation is analyzed in detail. Finally, the case of nonlinear vibrations with distributed-order fractional dissipation is analyzed. Existence uniqueness and regularity of solution are examined. Chapter 4 also contains an example from compartmental methods in pharmacokinetics, where the conservation of mass principle is observed. In Chapter 5, entitled “Lateral Vibrations and Stability of Viscoelastic Rods”, we present an analysis of lateral vibrations for several choices of constitutive functions for the rod. Thus, the cases of fractional Kelvin–Voigt, Zener and generalized Zener materials are discussed. Special attention is given to the stability conditions. The case of Beck’s column positioned on a fractional type of viscoelastic foundation is analyzed, as well as the case of a compressible rod on a fractional type of viscoelastic foundation. In Chapter 6, entitled “Fractional Diffusion-Wave Equations”, we present an analysis of the fractional partial differential equations with the order of time derivatives between 0 and 2. In the case of the generalized Burgers/Korteweg–deVries equation, we consider fractional derivatives with respect to space variable in the range between 2 and 3. By using similarity transformation, we even study some nonlinear cases of generalized heat equations. In Chapter 7, entitled “Fractional Heat Conduction Equations”, Cattaneo-type space-time fractional heat conduction equations and fractional Jeffreys-type heat conduction equations are presented.

A full bibliography of the two related titles is presented together at the end of each book. The bibliography does not pretend to be complete. It only contains references to the papers and books that we used. In every chapter, in the introductory section, we list the references used in that chapter. In this way, the presentation in every chapter is more readable.

We believe that a reader can find enough information for understanding the presented materials and apply the methods used in the book to his/her own investigations. We hope that this book may be useful for graduate students in mechanics and applied mathematics, as well as researchers in those fields.

We are grateful to our colleagues Nenad Grahovac, Alfio Grillo, Diana Dolićanin, Marko Janev, Sanja Konjik, Ljubica Oparnica, Dragan Spasić and Miodrag Žigi who worked with us on some problems presented in this book.

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Novi Sad, December 2013

PART 1

Mathematical Preliminaries, Definitions and Properties of Fractional Integrals and Derivatives

Chapter 1

Mathematical Preliminaries

1.1. Notation and definitions

Sets of natural, integer real and complex numbers are denoted, respectively, by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$.

Let Ω be an arbitrary subset of $\mathbb{R}$. We denote by $C_b(\Omega)$ the set of continuous functions on Ω such that

$$\|f\|_{C_b(\Omega)} = \sup_{x\in\Omega} |f(x)| < \infty.$$

It is well known that $C_b(\Omega)$ is a Banach space. If Ω is open, then we consider compact subsets of Ω, $K \subset\subset \Omega$, continuous functions f on Ω and the semi-norms

$$\|f\|_K = \sup_{x\in K} |f(x)| \,.$$

We can take a sequence of compact sets $K_1 \subset K_2 \subset, \ldots,$ so that $\cup_{i=1}^{\infty} K_i = \Omega$. Then, the sequence of semi-norms defines the Fréchet topology on $C(\Omega)$. This topology does not depend on a sequence $\{K_i\}_{i\in\mathbb{N}}$, with the given property. If K is compact, then $C(K)$ always denotes the set of continuous functions on K with the sup-norm over K.

Let Ω be open in $\mathbb{R}^n$. Then, we consider $C^k(\Omega) \subset C(\Omega)$: the space of functions having all the derivatives, up to order $k \in \mathbb{N}_0$, continuous. The Fréchet topology is defined by the semi-norms

$$\|f\|_{k,K} = \sup_{\substack{x \in K, \\ |i| \le k}} |f^{(i)}(x)|, \quad K \subset\subset \Omega, \ |i| = i_1 + \ldots + i_n. \tag{1.1}$$

The same topology is obtained if we again take, for compact sets in [1.1], a sequence of compact sets $K_1 \subset K_2 \subset \ldots$, so that $\cup_{i=1}^{\infty} K_i = \Omega$.

If K is compact, we use the notation $C^k(K)$ for the Banach space of functions with all derivatives continuous up to order k, with the norm [1.1]. In the case $k = 0$, we use the notation $C^0(\Omega) = C(\Omega)$ and $C^0(\mathbb{R}) = C(\mathbb{R})$. If $k = \infty$, then we call $C^\infty(\Omega)$ the space of smooth functions. It is the Fréchet space with the sequence of semi-norms $\|f\|_{p,K_p}$, $K_p \subset K_{p+1}$, $K_p \subset\subset \Omega$, $p \in \mathbb{N}$, $\cup_{p=1}^{\infty} K_p = \Omega$. Its subspace $C_0^\infty(\mathbb{R})$ consists of compactly supported smooth functions, that is of the smooth functions equal to zero outside the compact sets.

Analytic functions on $(a, b) \subset \mathbb{R}$ are smooth functions on (a, b), so that their Taylor series converges in any point a_0 of (a, b) on a suitable interval around a_0. A space $\mathcal{A}((a, b))$ of such functions is a Fréchet space under the convergence structure from $C^\infty((a, b))$.

$BV_{loc}(\mathbb{R}_+)$ denotes the space of functions f of locally bounded variations on $\mathbb{R}_+$. This means: for every interval $[a, b] \subset \mathbb{R}_+$, there exists a constant M such that $\sum_{i=0}^{n} |f(t_{i+1}) - f(t_i)| < M$ for every finite choice of points $t_0 = a, \ldots, t_n = b$.

$L^p((a, b)) = L^p([a, b])$, $p \ge 1$, is the space of measurable functions for which $\left(\int_a^b |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty$. We shorten the notation and use the symbol $L^p(a, b)$. In $L^p(a, b)$, $p \ge 1$, the norm is defined as

$$\|f\|_p = \left(\int_{(a,b)} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

In $L^\infty(a, b)$, we have $\|f\|_\infty = \operatorname{vrai}\sup_{x \in (a,b)} |f(x)|$.

More precisely, above we considered spaces of Lebesgue measurable functions on a Lebesgue measurable set $A \subset \mathbb{R}^n$ (above $A = (a, b)$) and identify them through the equivalence relation: $f \sim g$ over A if $f(x) = g(x)$, $x \in A \setminus N$ where N is of zero Lebesgue measure. This relation determines the classes $[f]$, ..., and in the sequel, we put f for $[f]$. In this sense, when $f \sim g$, we say that these functions are equal almost everywhere on A ($f = g$ almost everywhere (a.e.) on A) and we just identify f and g. So, the notation above $\operatorname{vrai\,sup}_{x\in(a,b)} f(x)$, for a measurable function bounded almost everywhere on (a, b), means: supremum up to a set of points in (a, b) with the zero measure. In the sequel, we will consider equality almost everywhere, as well as the integration in the sense of Lebesgue. $L^1_{loc}(a, b) = L^1_{loc}((a, b))$, $a, b \in \mathbb{R}^n$, $a < b$, is the space of measurable functions f on $(a, b) \in \mathbb{R}^n$ such that for every compact set $K \subset (a, b)$, there holds $\int_K |f(x)|\mathrm{d}x < \infty$. It is clear that $L^1_{loc}([a, b)) \neq L^1_{loc}(a, b)$.

If p and q are real numbers such that $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ (for $p = 1$, $q = \infty$), and if $f \in L^p(a, b),\ g \in L^q(a, b)$, then $fg \in L^1(a, b)$ and

$$\int_{(a,b)} |f(x)g(x)|\mathrm{d}x \leq \|f\|_p \|g\|_p. \qquad \text{[Hölder inequality]}$$

A real-valued function f defined on $[a, b] \subset \mathbb{R}$ is said to be absolutely continuous on $[a, b]$, if for given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \varepsilon$$

for every finite collection $\{(x_i', x_i)\}_{i\in\mathbb{N}}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

The space of absolutely continuous functions is denoted by $AC([a, b]) = AC^1([a, b])$. There holds $C([a, b]) \subset AC([a, b])$. Moreover, $f \in AC([a, b])$, if and only if there exists an integrable function g over $[a, b]$ such that

$$f(x) = c + \int_a^x g(t)\mathrm{d}t, \quad g = f' \text{ a.e. on } [a, b].$$

$AC^n([a, b])$, $n \in \mathbb{N}$, $n \geq 2$, is the space of functions f, which have continuous derivatives up to the order n 1 on $[a, b]$ and $f^{(n-1)} \in AC([a, b])$. Notation $AC^n_{loc}([0, \infty))$ means that the function $f \in AC^n([0, b])$, for every $b > 0$.

A function f on $[a,b]$ is Hölder continuous at $x_0 \in [a,b]$ if there exist $A > 0$ and $\lambda > 0$, such that

$$|f(x) - f(x_0)| \leq A\,|x - x_0|^{\lambda}$$

in a neighborhood of x_0. Hölder-type spaces on an interval $[a,b]$ are defined as subspaces of integrable functions on this interval with the following properties:

– $\mathcal{H}^{\lambda} \equiv \mathcal{H}^{\lambda}([a,b]) = \{f \mid |f(x_1) - f(x_2)| \leq A\,|x_1 - x_2|^{\lambda},\ x_1, x_2 \in [a,b]\},\ \lambda \in (0,1]$;

– $\mathcal{H} \equiv \mathcal{H}([a,b]) = \cup_{0<\lambda\leq 1}\mathcal{H}^{\lambda}([a,b])$;

– $\mathcal{H}^{*} \equiv \mathcal{H}^{*}([a,b]) = \Big\{f \mid f(x) = \frac{f^{*}(x)}{(x-a)^{1-\epsilon_1}(b-x)^{1-\epsilon_2}},\ x \in (a,b),\ \epsilon_1, \epsilon_2 > 0,\ f^{*} \in \mathcal{H}^{\lambda}([a,b]),\ \lambda \in (0,1]\Big\}$;

– $\mathcal{H}_0^{\lambda}(\epsilon_1, \epsilon_2) = \{f \in \mathcal{H}^{*} \mid f^{*}(0) = f^{*}(b) = 0\}$;

– $\mathcal{H}_{\alpha}^{*} = \cup_{\alpha<\lambda\leq 1,\ \epsilon_1,\epsilon_2>0}\mathcal{H}_0^{\lambda}(\epsilon_1, \epsilon_2)$;

– $h^{\lambda} \equiv h^{\lambda}([a,b]) = \Big\{f \mid \frac{f(x_1 - x_2)}{|x_1 - x_2|^{\lambda}} \to 0,\ x_2 \to x_1\Big\};\ h^{\lambda} \subset \mathcal{H}^{\lambda}$.

1.2. Laplace transform of a function

Let $f \in L_{loc}^{1}(\mathbb{R})$ and $f(t) = 0,\ t \in (-\infty, 0)$. The Laplace transform of f is defined by

$$\mathcal{L}\left[f(t)\right](s) = \tilde{f}(s) = \lim_{A\to\infty}\int_0^A f(t)\mathrm{e}^{-st}\mathrm{d}t, \qquad [1.2]$$

for those complex numbers s for which this limit exists. It is well known that the existence of the limit in [1.2] at $s = s_0$ implies the existence of this limit for any $s \in \mathbb{C}$ with the property $\operatorname{Re} s > \operatorname{Re} s_0$. We can consider the integral $\int_0^{\infty}|f(t)|\mathrm{e}^{-t\operatorname{Re} s}\mathrm{d}t$. If it is finite (we say an integral exists, or converges) for $s = s_1$, then f is called an absolutely convergent Laplace transformable function. In this case

$$\mathcal{L}\left[f(t)\right](s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\mathrm{d}t$$

is absolutely convergent for any $s \in \mathbb{C}$ such that $\operatorname{Re} s > \operatorname{Re} s_1$.

The number $a_e = \inf\{\operatorname{Re} s_0 \in \mathbb{R}\}$ representing the infimum of those $s_0 \in \mathbb{C}$ for which the Laplace transform is defined is called the abscissa of existence. The abscissa of absolute convergence a_a is defined in the same way. We have $a_a \geq a_e$.

It is clear that $\tilde{f}(s)$ exists (absolutely exists) for every $s \in \mathbb{C}$, $\operatorname{Re} s > a_e$ ($s \in \mathbb{C}$, $\operatorname{Re} s > a_a$). It is an analytic function in the half-plane $\operatorname{Re} s > a_e$, since, by partial integration, it can be represented as an absolutely convergent Laplace transform.

In the sequel, we consider the following class of Laplace transformable functions. Function $f \in L^1_{loc}([0,\infty))$ is called exponentially bounded if there exist constant $C = C_f > 0$, $r = r_f \in \mathbb{R}$ and $c = c_f \geq 0$ such that

$$|f(t)| \leq C\mathrm{e}^{rt}, \quad t > c. \qquad [1.3]$$

We denote by $L^{exp}([0,\infty))$ the space of such functions. The growth order r is greater or equal than the abscissa of absolute convergence, $r \geq a_a$.

The Laplace transform is a linear operation on the space of exponentially bounded functions. If a function and its derivatives on $[0,\infty)$ up to order k are of exponential growth, then

$$\mathcal{L}\left[f^{(k)}(t)\right](s) = s^k \tilde{f}(s) - s^{k-1} f(0) - \ldots - f^{k-1}(0), \quad \operatorname{Re} s > c,$$

for suitable $c > 0$. Let us mention several useful properties of the Laplace transform, based on appropriate assumptions and on corresponding domains

$$\begin{aligned} \mathcal{L}\left[f(t)\mathrm{e}^{at}\right](s) &= \tilde{f}(s-a), \quad \mathcal{L}\left[f(at)\right](s) = \frac{1}{a}\tilde{f}\left(\frac{s}{a}\right), \\ \mathcal{L}\left[t^n f(t)\right](s) &= (-1)^n \tilde{f}^{(n)}(s), \quad \mathcal{L}\left[f(t) * g(t)\right](s) = \tilde{f}(s)\tilde{g}(s), \end{aligned}$$

where the convolution of two locally integrable functions on $[0,\infty)$ is defined by

$$f(t) * g(t) = \int_0^t f(t-\tau) g(\tau) \mathrm{d}\tau, \quad t \geq 0.$$

The inverse Laplace transform is defined by

$$f(t) = \mathcal{L}^{-1}\left[\tilde{f}(s)\right](t) = \frac{1}{2\pi\mathrm{i}} \lim_{q\to\infty} \int_{p-\mathrm{i}q}^{p+\mathrm{i}q} \tilde{f}(s)\mathrm{e}^{st}\mathrm{d}s, \quad t \geq 0,$$

where $p > r$ (see [1.3]).

1.3. Spaces of distributions

The reader of this book has to have a knowledge of the theory of the generalized functions; here we call them distributions, as they are commonly known. This theory is a powerful tool used in mathematical theory and applications. Apart from books that discuss the basic theory, for example [SCH 51, VLA 73], there are a number of application-oriented textbooks such as [DUI 10].

We refer to [SCH 51, VLA 73] for the material of this section. By $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$, the well-known Schwartz spaces are denoted. Norms in the space $\mathcal{D}_K(\mathbb{R}^n)$ of smooth functions supported by K are

$$p_{K,m}(\varphi) = \sup_{x\in K, |\alpha|\leq m} |\varphi^{(\alpha)}(x)|, \quad m \in \mathbb{N}_0,$$

while in $\mathcal{S}(\mathbb{R}^n)$ are

$$q_m(\varphi) = \sup_{x\in \mathbb{R}, |\alpha|\leq m} (1+|x|)^m |\varphi^{(\alpha)}(x)|, \quad m \in \mathbb{N}_0.$$

Then, $\mathcal{D}(\mathbb{R})$ is the inductive limit

$$\mathcal{D}(\mathbb{R}) = \operatorname*{ind\,lim}_{K_n \subset\subset \mathbb{R}} \mathcal{D}_{K_n},$$

where K_n, $n \in \mathbb{N}$, is an increasing sequence of compact sets so that $\cup_n K_n = \mathbb{R}$.

The corresponding duals, spaces of continuous linear functionals, $\mathcal{D}'(\mathbb{R}^n)$ and its subspace $\mathcal{S}'(\mathbb{R}^n)$, with the strong topologies, are the space of distributions and the space of tempered distributions. The space of compactly supported distributions is denoted by $\mathcal{E}'(\mathbb{R}^n)$. It is the dual space for the Fréchet space see section 1.1.

Operations of multiplication and differentiation in $\mathcal{D}'(\mathbb{R})$ are defined in a usual way

$$\langle af, \varphi\rangle = \langle f, a\varphi\rangle, \quad \langle f^{(k)}, \varphi\rangle = (-1)^k \langle f, \varphi^{(k)}\rangle, \quad a \in C^\infty(\mathbb{R}), \; \varphi \in \mathcal{D}(\mathbb{R}), \; k \in \mathbb{N}.$$

Note $a(x)\,\delta(x - x_0) = a(x_0)\,\delta(x - x_0)$, where δ is the Dirac distribution,

$$\langle \delta(x - x_0), \varphi(x)\rangle = \varphi(x_0), \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

We note that $\mathcal{D}' = \mathcal{D}'(\mathbb{R})$ contains regular elements defined by $f \in L_{loc}(\mathbb{R})$; they are denoted by f_{reg} and defined by

$$f_{reg} : \varphi \mapsto \langle f_{reg}, \varphi \rangle = \int_{\mathbb{R}} f(x)\varphi(x)\mathrm{d}x, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

We can see that $\varphi_n \to 0$ in $\mathcal{D}$ implies $\langle f_{reg}, \varphi_n \rangle \to 0$, $n \to \infty$.

Polynomially bounded and locally integrable functions on $\mathbb{R}$ define, in the same way, regular tempered distributions. We will usually denote, by the same symbol f, a function and a corresponding distribution f_{reg}. Only if we want to explain in detail the relation between them do we use the symbol f_{reg}.

The Fourier transform of a function $\varphi \in \mathcal{S}$ ($\mathcal{S} = \mathcal{S}(\mathbb{R})$) is defined by

$$\mathcal{F}[\varphi(x)](\xi) = \hat{\varphi}(\xi) = \int_{\mathbb{R}} \varphi(x)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

The Fourier transform is an isomorphism on $\mathcal{S}$. If $f \in \mathcal{S}'$ ($\mathcal{S}' = \mathcal{S}'(\mathbb{R})$), then

$$\langle \mathcal{F}[f], \varphi \rangle = \langle f, \mathcal{F}[\varphi] \rangle, \quad \varphi \in \mathcal{S},$$

defines the Fourier transform of a tempered distribution. The Fourier transform is an isomorphism on $\mathcal{S}'$. The inverse Fourier transform of $\varphi \in \mathcal{S}$ is defined by

$$\varphi(x) = \mathcal{F}^{-1}[\hat{\varphi}(\xi)](x) = \frac{1}{2\pi}\int_{\mathbb{R}} \varphi(\xi)\mathrm{e}^{\mathrm{i}\xi x}\mathrm{d}\xi, \quad x \in \mathbb{R}.$$

If $\varphi, \psi \in \mathcal{S}$, their convolution is defined by

$$\varphi(x) * \psi(x) = \int_{\mathbb{R}} \varphi(x - \zeta)\,\psi(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}.$$

If $\operatorname{supp}\varphi, \operatorname{supp}\psi \in [0, \infty)$, which means that $\varphi = \psi = 0$ on $(-\infty, 0)$, then

$$\varphi(t) * \psi(t) = \int_0^t \varphi(t - \tau)\,\psi(\tau)\,\mathrm{d}\tau, \quad t \geq 0 \text{ and } \varphi(t) * \psi(t) = 0, \quad t < 0.$$

We know

$$\mathcal{F}\left[\varphi\left(x\right)*\psi\left(x\right)\right]\left(\xi\right)=\hat{\varphi}\left(\xi\right)*\hat{\psi}\left(\xi\right),\quad \varphi,\psi\in\mathcal{S}$$

and, as a consequence,

$$\mathcal{F}\left[f\left(x\right)*g\left(x\right)\right]\left(\xi\right)=\hat{f}\left(\xi\right)*\hat{g}\left(\xi\right),\quad f,g\in\mathcal{S}'.$$

Sobolev space $W^{k,p}(\mathbb{R})$, $p\in[1,\infty]$, $k\in\mathbb{N}_0$, is defined as the space of L^p-functions f with the property that all the distributional derivatives of f up to order k are elements of $L^p(\mathbb{R})$. It is a Banach space with the norm

$$\|f\|_{W^{k,p}}=\sum_{j=0}^{p}\left\|f^{(j)}\right\|_{L^p}.$$

Clearly, $W^{k,p}(\mathbb{R})\subset\mathcal{S}'(\mathbb{R})$.

$\mathcal{D}_{L^p},\ 1\leq p<\infty$ is a space of smooth functions with all derivatives belonging to L^p. Note $\mathcal{D}_{L^p}\subset\mathcal{D}_{L^q}$ if $p\leq q$. $\dot{\mathcal{B}}$ is a subspace of $\mathcal{D}_{L^\infty}=\mathcal{B}$, defined as follows: $\varphi\in\dot{\mathcal{B}}$ if and only if $|\varphi^{(\alpha)}(x)|\to 0$ as $|x|\to\infty$ for every $\alpha\in\mathbb{N}_0$.

$\mathcal{D}'_{L^p},\ 1<p\leq\infty$ is the dual space of $\mathcal{D}_{L^q},\ 1\leq q<\infty,\ \frac{1}{p}+\frac{1}{q}=1.$ $\mathcal{D}'_{L^1}$ is the dual of $\dot{\mathcal{B}}$ and $\mathcal{D}'_{L^\infty}$ is denoted by $\mathcal{B}'$ (see [SCH 51]).

$\mathcal{S}'_+$ denotes a subspace of tempered distributions consisting of distributions with supports in $[0,\infty)$. Note that $\mathcal{S}'_+$ is a convolution algebra.

The following structural theorem holds: $f\in\mathcal{S}'_+$ if and only if there exists a continuous function F on $\mathbb{R}$ such that $F(x)=0$, $x<0$, $|F(x)|\leq C(1+|x|)^k$ for some $C>0$, $k>0$ and there exists $p\in\mathbb{N}_0$ such that

$$f(x)=F^{(p)}(x),\quad x\in\mathbb{R},\tag{1.4}$$

where the derivative is taken in the sense of distributions.

Let $f\in\mathcal{S}'_+$. Its Laplace transform is defined by

$$\mathcal{L}\left[f\left(t\right)\right]\left(s\right)=\tilde{f}(s)=\langle f(t),\mathrm{e}^{-st}\rangle=s^p\mathcal{L}\left[F\left(t\right)\right]\left(s\right),\quad \operatorname{Re}s>0,\tag{1.5}$$

where we assume that f is of the form [1.4]. Clearly, $\tilde{f}(s)$ is a holomorphic function for $\mathrm{Re}\, s > 0$. We will often consider equations, with solutions u determining the tempered distributions, by the use of the Laplace transform. If we assume that u is of exponential growth, then we have $\tilde{u}(s)$, $\mathrm{Re}\, s > s_0$, for some $s_0 > 0$.

We consider the family $\{f_\alpha\}_{\alpha \in \mathbb{R}} \in \mathcal{S}'_+$ (see [VLA 84])

$$f_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)} H(t), & t \in \mathbb{R},\ \alpha > 0, \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m} f_{\alpha+m}(t), & \alpha \leq 0,\ \alpha + m > 0,\ m \in \mathbb{N}, \end{cases} \qquad [1.6]$$

where the mth derivative is understood in the distributional sense. Family $\{\check{f}_\alpha\}_{\alpha \in \mathbb{R}} \in \mathcal{S}'_-$ is defined by $\check{f}_\alpha(t) = f_\alpha(-t)$. The Heaviside function is defined as

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

Operators $f_\alpha *$ and $\check{f}_\alpha *$ are convolution operators

$$f_\alpha *, \check{f}_\alpha * : \mathcal{S}'_+ \to \mathcal{S}'_+ \ \text{ and } \ \check{f}_\alpha * : \mathcal{S}'_- \to \mathcal{S}'_-.$$

The semi-group property holds for f_α

$$f_\alpha * f_\beta = f_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R}.$$

The Laplace transform of f_α is

$$\mathcal{L}[f_\alpha(t)] = \frac{1}{s^\alpha} \tilde{f}(s), \quad \mathrm{Re}\, s > 0.$$

EXAMPLE 1.1.– Let f be an absolutely continuous function on $[0, \infty)$ so that $f(0^+) = p \neq 0$. Assume that f and f' have the (classical) Laplace transform (denoted by $\mathcal{L}_c$) in the domain $\mathrm{Re}\, \lambda > \lambda_0 > 0$. Put f_{reg}, and $(f')_{reg}$, for the corresponding distributions. Note that $f_{reg} = fH$, with symbol $\mathcal{L}_d$ for [1.5]. We have

$$\begin{aligned} \mathcal{L}_d \left[(f')_{reg}(t) \right](s) &= \mathcal{L}_d [f'(t) H(t)](s) = \mathcal{L}_d [(f(t) H(t))'](s) \\ &\quad - \mathcal{L}_d [f(0) \delta(t)](s) \\ &= s \mathcal{L}_d [f_{reg}(t)](s) - f(0), \end{aligned}$$

where

$$\mathcal{L}_d\left[\left(f'\right)_{reg}(t)\right](s) = \mathcal{L}_d\left[\left(f(t)\,H(t)\right)'\right](s) = s\mathcal{L}_d\left[f_{reg}(t)\right](s), \quad \mathrm{Re}\, s > \lambda_0,$$

while in the classical case,

$$\mathcal{L}_c\left[f'(t)\right](s) = \mathcal{L}_c\left[f(t)\right](s) - f(0), \quad \mathrm{Re}\, s > \lambda_0.$$

EXAMPLE 1.2.– Let $u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$, be a classical solution of the wave equation

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = f(x,t),\\
&u(x,0) = u_0(x), \qquad \frac{\partial}{\partial t}u(x,0) = v_0(x),
\end{aligned}$$

that is the second derivative above are locally integrable functions on $[0,\infty) \times \mathbb{R}^n$, equal to zero for $t < 0$, u_0, v_0 are locally integrable functions on $\mathbb{R}^n$ and $f \in L^1_{loc}([0,\infty) \times \mathbb{R}^n)$ so that it has the classical Laplace transform with respect to t in the domain $\mathrm{Re}\, s > 0$.

Writing

$$u_{reg}(x,t) = u(x,t)\,H(t), \ \text{ and } \ f_{reg}(x,t) = f(x,t)\,H(t)$$

for the corresponding distributions, we rewrite the wave equation in the space of distributions as

$$\begin{aligned}
&H(t)\frac{\partial^2}{\partial t^2}u(x,t) - H(t)\frac{\partial^2}{\partial x^2}u(x,t) = H(t)f(x,t),\\
&\frac{\partial^2}{\partial t^2}\left(H(t)u(x,t)\right) - \delta(t)\frac{\partial}{\partial t}u(x,t) - \delta'(t)u(x,t) - \frac{\partial^2}{\partial x^2}\left(H(t)u(x,t)\right)\\
&\qquad\qquad = H(t)f(x,t),\\
&\frac{\partial^2}{\partial t^2}u_{reg}(x,t) - \delta(t)v_0(x) - \delta'(t)u_0(x) - \frac{\partial^2}{\partial x^2}u_{reg}(x,t) = f_{reg}(x,t),\\
&\qquad\qquad t > 0, \ x \in \mathbb{R},
\end{aligned}$$

where the last equation is written in the space of distributions. So with the application of the distributional Laplace transform with respect to t, for $\mathrm{Re}\, s > 0$, we have

$$s^2 \tilde{u}_{reg}(x,s) - v_0(x) - su_0(x) - \frac{\partial^2}{\partial x^2}\tilde{u}_{reg}(x,s) = \tilde{f}_{reg}(x,s).$$

The space $\mathcal{K}(\mathbb{R})$ is the space of smooth functions φ with the property

$$\sup_{x\in\mathbb{R},\alpha\leq m} \left|\varphi^{(\alpha)}(x)\right| \mathrm{e}^{m|x|} < \infty, \quad m \in \mathbb{N}_0. \qquad [1.7]$$

The space $\mathcal{K}'(\mathbb{R})$ is the dual of $\mathcal{K}(\mathbb{R})$ and elements of $\mathcal{K}'(\mathbb{R})$ are of the form $f = \sum_{\alpha=0}^{r} \Phi_\alpha^{(\alpha)}$, where Φ_α are continuous functions with the property $|\Phi_\alpha(t)| \leq Ce^{k_0|t|}$, $\alpha \leq r,\ t \in \mathbb{R}$, for some $C > 0$, $r \in \mathbb{N}_0$ and some $k_0 \in \mathbb{N}_0$. $\mathcal{K}'_+(\mathbb{R}) = \mathcal{K}'_+$ is a subspace of $\mathcal{K}'(\mathbb{R})$ consisting of elements supported by $[0,\infty)$ (see [ABD 99, HAS 61]). Its elements are of the form

$$f(x) = (\Phi(x)\mathrm{e}^{kx})^{(p)}, \quad x \in \mathbb{R}, \qquad [1.8]$$

where Φ is a continuous bounded function such that $\Phi(t) = 0,\ t \leq 0$. Note that $\mathcal{S}$ and $\mathcal{S}'_+$ are subspaces of $\mathcal{K}'(\mathbb{R})$ and $\mathcal{K}'_+$, respectively. The construction implies that elements of $\mathcal{K}'_+$ have the Laplace transform, that is if f is of the form [1.8], then its Laplace transform $\tilde{f}$ is an analytic function in the domain $\mathrm{Re}\, s > k$.

The Lizorkin space of test functions Φ is introduced so that Riesz integro-differentiation (and therefore symmetrized fractional derivative) is well defined (see [SAM 93]). Let

$$\Psi = \{\psi \mid \psi \in \mathcal{S}(\mathbb{R}),\ \psi^{(j)}(0) = 0,\ j = 0,1,2...\},$$

and consider the space Φ consisting of the Fourier transforms of functions in Ψ, i.e. $\Phi = \mathcal{F}[\Psi]$. Then, Φ consists of those functions $\varphi \in \mathcal{S}(\mathbb{R})$ that are orthogonal to polynomials

$$\int_{\mathbb{R}} x^k \varphi(x)\mathrm{d}x = 0, \quad k \in \mathbb{N}_0.$$

The space Ψ' and the space of Lizorkin generalized functions Φ' are dual spaces of Ψ and Φ, respectively. Recall, for $f \in \Phi'$, we have

$$\langle \mathcal{F}[f], \psi\rangle = \langle f, \mathcal{F}[\psi]\rangle, \quad \psi \in \Psi.$$

Let $f \in C^\infty(\mathbb{R} \setminus \{0\})$ be such that it has all the derivatives bounded by the polynomials in $\mathbb{R} \setminus \{0\}$. Then, product $f \cdot u$ is defined by

$$\langle f \cdot u, \psi \rangle = \langle u, f \cdot \psi \rangle, \quad \psi \in \Psi.$$

1.4. Fundamental solution

Let P be a linear partial integro-differential operator with constant coefficients. A fundamental solution of P, denoted by E, is a distributional solution to the equation $Pu = \delta$. Once the fundamental solution is determined, we find a solution to $Pu = f$ as $u = E * f$, if this convolution exits.

The Cauchy problem for the second-order linear partial integro-differential operator with constant coefficients P is given by

$$Pu(x,t) = f(x,t), \quad x \in \mathbb{R}, \ t > 0, \qquad [1.9]$$

$$u(x,0) = u_0(x), \quad \frac{\partial}{\partial t} u(x,0) = v_0(x), \qquad [1.10]$$

where f is continuous for $t \geq 0$, $u_0 \in C^1(\mathbb{R})$ and $v_0 \in C(\mathbb{R})$. A classical solution $u(x,t)$ to the Cauchy problem [1.9], [1.10] is of class C^2 for $t > 0$ and of class C^1 for $t \geq 0$, satisfies equation [1.9] for $t > 0$, and initial conditions [1.10] when $t \to 0$. If functions u and f are continued by zero for $t < 0$, then the following equation is satisfied in $\mathcal{D}'(\mathbb{R}^2)$:

$$Pu(x,t) = f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t). \qquad [1.11]$$

The explanation is given in example 1.2 in the case of the wave equation. The problem of finding generalized solutions (in $\mathcal{D}'(\mathbb{R}^2)$) of equation [1.11] that vanish for $t < 0$ will be called the generalized Cauchy problem for the operator P. If there is a fundamental solution E of the operator P and if $f \in \mathcal{D}'(\mathbb{R}^2)$ vanishes for $t < 0$, then there exists a unique solution to the corresponding generalized Cauchy problem and is given by

$$u(x,t) = E(x,t) * (f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t)),$$

if the convolution $E * f$ exits. We refer to [DAU 00, TRE 75, VLA 84] for more details.

1.5. Some special functions

The Euler gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\mathrm{d}t, \quad \mathrm{Re}\, z > 0.$$

The gamma function can also be represented by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{\prod_{k=0}^{n}(z+k)}, \quad \mathrm{Re}\, z > 0.$$

It satisfies $\Gamma(z+1) = z\Gamma(z)\,, \mathrm{Re}\, z > 0$. By the analytic continuation, we have that $\Gamma(z)\,,\ z \neq -n,\ n \in \mathbb{N}_0$, is an analytic function. Gamma function has simple poles at $z = n,\ n \in \mathbb{N}_0$. Having $\Gamma(1) = 1$, we obtain $\Gamma(n+1) = n!, \quad n \in \mathbb{N}$. We refer to [POD 99] for the properties of the gamma function.

We refer to [GOR 97b, MAI 00] for the theory of Mittag-Leffler functions presented in this section. The one-parameter Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad z \in \mathbb{C},\ \alpha > 0. \qquad [1.12]$$

The one-parameter Mittag-Leffler function is an entire function of order $\rho = \frac{1}{\alpha}$ and type 1. In some special cases of α, the one-parameter Mittag-Leffler function becomes

$$E_2\left(z^2\right) = \cosh z, \quad E_2\left(-z^2\right) = \cos z, \quad z \in \mathbb{C},$$
$$E_{\frac{1}{2}}\left(\pm\sqrt{z}\right) = \mathrm{e}^z\left(1 + \mathrm{erf}\left(\pm\sqrt{z}\right)\right), \quad z \in \mathbb{C},$$

with $\mathrm{erf}\, z = \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-u^2}\mathrm{d}u$ being the error function.

The asymptotics of [1.12] are as follows:

$$E_\alpha(z) \approx \frac{1}{\alpha}\mathrm{e}^{\sqrt{z}} - \sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty,\ |\arg z| < \frac{\alpha\pi}{2},\ \alpha \in (0,2)\,,$$

$$E_\alpha(z) \approx -\sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad |z| \to \infty,\ \arg z \in \left(\frac{\alpha\pi}{2}, 2\pi - \frac{\alpha\pi}{2}\right),\ \alpha \in (0,2)\,,$$

$$E_\alpha(z) \approx \frac{1}{\alpha}\sum_m \mathrm{e}^{\sqrt[\alpha]{z}\,\mathrm{e}^{\frac{2\pi i m}{\alpha}}} - \sum_{k=1}^{\infty}\frac{z^{-k}}{\Gamma(1-\alpha k)}, \quad \begin{array}{c} |z| \to \infty,\ \arg z \in (-\pi,\pi)\,, \\ \alpha \geq 2, \\ m \in \mathbb{N},\ \arg(z + 2\pi m) \\ \in \left(-\frac{\alpha\pi}{2}, \frac{\alpha\pi}{2}\right). \end{array}$$

The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z \in \mathbb{C},\ \alpha > 0,\ \beta \in \mathbb{C}.$$

It is an entire function of order $\rho = \frac{1}{\alpha}$ and type 1. In some special cases of α and β, it becomes

$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z}, \qquad E_{2,2}(z) = \frac{\sinh\sqrt{z}}{\sqrt{z}}, \quad z \in \mathbb{C}.$$

We define one- and two-parameter Mittag-Leffler-type functions, respectively, by

$$e_\alpha(t,\lambda) = E_\alpha(-\lambda t^\alpha) \quad \text{and} \quad e_{\alpha,\beta}(t,\lambda) = t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha), \quad t \geq 0,\ \lambda \in \mathbb{C}.$$

In applications, we will often omit the parameter λ. According to [MAI 00], if $\alpha \in (0,1)$ and $\lambda > 0$, we have $e_\alpha \in C^\infty((0,\infty)) \cap C([0,\infty))$ and $\frac{\mathrm{d}}{\mathrm{d}t}e_\alpha \in C^\infty((0,\infty)) \cap L^1_{loc}([0,\infty))$. Also, e_α is a completely monotonic function, i.e. $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} e_\alpha(t) > 0$.

The Laplace transforms of e_α and $e_{\alpha,\beta}$ are

$$\mathcal{L}[e_\alpha(t,\lambda)](s) = \frac{s^{\alpha-1}}{s^\alpha+\lambda}, \qquad \mathcal{L}[e_{\alpha,\beta}(t,\lambda)](s) = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}, \quad \operatorname{Re} s > \sqrt[\alpha]{|\lambda|},$$

respectively.

Functions e_α and $e_{\alpha,\beta}$ admit integral representations given by

$$\begin{aligned} e_\alpha(t,\lambda) &= \frac{1}{\pi}\int_0^\infty \frac{\lambda q^{\alpha-1}\sin(\alpha\pi)}{q^{2\alpha}+2\lambda q^\alpha\cos(\alpha\pi)+\lambda^2}\mathrm{e}^{-qt}\mathrm{d}q, \quad t \geq 0,\ \alpha \in (0,1), \\ &\qquad \lambda > 0, \\ e_{\alpha,\beta}(t,\lambda) &= \frac{1}{\pi}\int_0^\infty \frac{\lambda\sin((\beta-\alpha)\pi)+q^\alpha\sin(\beta\pi)}{q^{2\alpha}+2\lambda q^\alpha\cos(\alpha\pi)+\lambda^2}q^{\alpha-\beta}\mathrm{e}^{-qt}\mathrm{d}q, \\ &\qquad t \geq 0,\ 0 < \alpha \leq \beta < 1,\ \lambda > 0. \end{aligned}$$

Chapter 2

Basic Definitions and Properties of Fractional Integrals and Derivatives

2.1. Definitions of fractional integrals and derivatives

In this section, we review some basic properties of fractional integrals and derivatives, which we will need later in the analysis of concrete problems. This section contains results from various books and papers [ALM 12, ATA 14a, ATA 13a, ATA 07a, ATA 09b, ATA 09d, ATA 08b, BUT 00, CAN 87, CAP 67, CAP 71b, DIE 10, HER 11, KIL 04, KIL 06, KIR 94, NAH 03, ODI 07, POO 12a, POO 12b, POO 13, ROS 93, SAM 95, SAM 93, TAR 06, TRU 99, UCH 08, WES 03].

2.1.1. *Riemann–Liouville fractional integrals and derivatives*

There are many possible generalizations of the notion of a derivative of a function that would lead to the answer of the question: what is $\frac{\mathrm{d}^n}{\mathrm{d}x^n}y\left(x\right)$ when n is any real number? We start from the Cauchy formula for an n-fold primitive of a function f given as

$$ {}_a\mathrm{I}_t^n f\left(t\right)=\frac{1}{\left(n-1\right)!}\int_a^t \left(t-\tau\right)^{n-1}f\left(\tau\right)\mathrm{d}\tau,\ \ t\in\left[a,b\right],\ n\in\mathbb{N}, \qquad [2.1]$$

where it is assumed that $f\left(t\right)=0$, for $t<a$. Note that $\left(n-1\right)!=\Gamma\left(n\right)$, where Γ is the Euler gamma function (see section 1.5).

DEFINITION 2.1.– *The left Riemann–Liouville fractional integral of order $\alpha \in \mathbb{C}$ is formally given by*

$$
{}_a\mathrm{I}_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad t \in [a,b], \ \operatorname{Re}\alpha > 0. \tag{2.2}
$$

In the special case of positive real α ($\alpha \in \mathbb{R}_+$) and $f \in L^1(a,b)$, the integral ${}_a\mathrm{I}_t^{\alpha} f$ exists for almost all $t \in [a,b]$. Also, ${}_a\mathrm{I}_t^{\alpha} f \in L^1(a,b)$ (see [DIE 10, p. 13]). For $\alpha = 0$, we define ${}_a\mathrm{I}_t^{0} f = f$. This definition is motivated by the following reasoning. Suppose that $f \in C^1([a,b])$. Then, after integration by parts, from [2.2], we have

$$
{}_a\mathrm{I}_t^{\alpha} f(t) = \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) + \frac{1}{\Gamma(\alpha+1)} \int_a^t (t-\tau)^{\alpha} f^{(1)}(\tau)\, \mathrm{d}\tau,
$$

so that

$$
\lim_{\alpha \to 0} {}_a\mathrm{I}_t^{\alpha} f(t) = f(a) + \int_a^t f^{(1)}(\tau)\, \mathrm{d}\tau = f(t).
$$

DEFINITION 2.2.– *The right Riemann–Liouville fractional integral of order $\alpha \in \mathbb{C}$ is formally given by*

$$
{}_t\mathrm{I}_b^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, \mathrm{d}\tau, \quad t \in [a,b], \ \operatorname{Re}\alpha > 0. \tag{2.3}
$$

The existence is the same as in the case of the left Riemann–Liouville fractional integral given above.

In the special case when $f(t) = (t-a)^{\beta-1}$ and $g(t) = (b-t)^{\beta-1}$, $t \in [a,b]$, $\alpha, \beta \in \mathbb{C}$, we have

$$
\begin{aligned}
{}_a\mathrm{I}_t^{\alpha} (t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (t-a)^{\beta+\alpha-1}, \quad \operatorname{Re}\alpha > 0, \ \operatorname{Re}\beta > 0, \\
{}_t\mathrm{I}_b^{\alpha} (b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (b-t)^{\beta+\alpha-1}, \quad \operatorname{Re}\alpha > 0, \ \operatorname{Re}\beta > 0.
\end{aligned}
$$

Operators ${}_a\mathrm{I}_t^{\alpha}$ and ${}_t\mathrm{I}_b^{\alpha}$ with $\operatorname{Re}\alpha > 0$ are bounded operators from $L^p(a,b)$ into $L^p(a,b)$, $p \geq 1$. The following estimates hold:

$$
\|\mathrm{I}_t^{\alpha} f\|_{L^p(a,b)} \leq \frac{(b-a)^{\operatorname{Re}\alpha}}{|\Gamma(\alpha)| \operatorname{Re}\alpha} \|f\|_{L^p(a,b)}, \quad \|{}_t\mathrm{I}_b^{\alpha} f\|_{L^p(0,b)} \leq \frac{(b-a)^{\operatorname{Re}\alpha}}{|\Gamma(\alpha)| \operatorname{Re}\alpha} \|f\|_{L^p(a,b)}, \tag{2.4}
$$

see [SAM 93, p. 48]. If $\alpha \in (0,1)$ and $1 < p < \frac{1}{\alpha}$, then the operators ${}_0\mathrm{I}_t^\alpha$ and ${}_t\mathrm{I}_b^\alpha$ are bounded from $L^p(a,b)$ into $L^q(a,b)$ for $q = \frac{p}{1-\alpha p}$ (see [SAM 93, p. 66]).

Introducing the function

$$f_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}, & t > a, \\ 0, & t < a, \end{cases} \quad \operatorname{Re}\alpha > 0, \tag{2.5}$$

we conclude that the integral [2.2] may be written in the form of convolution as

$$ {}_a\mathrm{I}_t^\alpha y(t) = f_\alpha(t) * y(t) = \int_a^t f_\alpha(t-\tau)\, y(\tau)\, \mathrm{d}\tau. \tag{2.6}$$

REMARK 2.1.– Expression [2.6] may be used to define the generalized fractional integral with the different choice of f_α. For example, in [KIL 04], various generalizations of the fractional integral were presented, including the generalization that uses the two-parameter Mittag-Leffler function

$$K^{(\alpha)} f(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} E_{\rho,\alpha}(\omega t^\rho) * f(t), \quad \omega \in \mathbb{R}.$$

The fractional integral of purely imaginary order is defined as

$$ {}_a\mathrm{I}_t^{\mathrm{i}\theta} y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left({}_a\mathrm{I}_t^{1+\mathrm{i}\theta} y(t)\right) = \frac{1}{\Gamma(1+\mathrm{i}\theta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\tau)^{\mathrm{i}\theta}\, y(\tau)\, \mathrm{d}\tau, \tag{2.7}$$

with $\theta \neq 0$.

The asymptotic behavior of the left Riemann–Liouville fractional integral may be characterized as follows.

PROPOSITION 2.1.– [UCH 08, p. 165] Suppose that $f \in L^1_{loc}([0,\infty))$ is an analytic function in $(0,\infty)$. Then

$$ {}_a\mathrm{I}_t^\alpha f(t) \sim {}_0\mathrm{I}_t^\alpha f(t) + \frac{a}{\pi}\Gamma(\alpha+1)\sin(\alpha\pi)\, f(0)\, t^{-\alpha-1} \sim {}_0\mathrm{I}_t^\alpha f(t), \quad \text{as } t \to \infty. \tag{2.8}$$

If ${}_a\mathrm{I}_t^\alpha f$ is used to model a hereditary process, then a physical meaning of [2.8] is that for large times, the importance of the initial state of the system is small.

DEFINITION 2.3.– *The left and right Riemann–Liouville fractional derivatives* ${}_a\mathrm{D}_t^\alpha f$ *and* ${}_t\mathrm{D}_b^\alpha f$ *of the order* $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \geq 0$, $n-1 \leq \operatorname{Re}\alpha < n$, $n \in \mathbb{N}$, *with the appropriate assumptions on* f *(see below), are defined as*

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_a\mathrm{I}_t^{n-\alpha} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad t \in (a,b), \\ {}_t\mathrm{D}_b^\alpha f(t) &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left({}_t\mathrm{I}_b^{n-\alpha} f(t)\right) = (-1)^n \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}}\mathrm{d}\tau, \\ & \qquad t \in (a,b). \end{aligned} \qquad [2.9]$$

If $f \in AC^n([a,b])$ and $n-1 \leq \operatorname{Re}\alpha < n$, $n \in \mathbb{N}$, then ${}_a\mathrm{D}_t^\alpha f$ and ${}_t\mathrm{D}_b^\alpha f$ exist almost everywhere on $[a,b]$ and

$${}_a\mathrm{D}_t^\alpha f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1+k-\alpha)}(t-a)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \qquad [2.10]$$

$${}_t\mathrm{D}_b^\alpha f(t) = \sum_{k=0}^{n-1}(-1)^k \frac{f^{(k)}(b)}{\Gamma(1+k-\alpha)}(b-t)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha-n+1}}\mathrm{d}\tau, \qquad [2.11]$$

see [KIL 06, p. 73]. From the definitions, it follows that in the special case when $f(t) = (t-a)^{\beta-1}$, $t > a$, and $f(t) = (b-t)^{\beta-1}$, $t < b$, $\beta \in \mathbb{C}$, we have

$$\begin{aligned} {}_a\mathrm{D}_t^\alpha (t-a)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1} \quad \text{and} \\ {}_t\mathrm{D}_b^\alpha (b-t)^{\beta-1} &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}. \end{aligned} \qquad [2.12]$$

Again, from [2.12], for constant function $f = C$, we have

$${}_a\mathrm{D}_t^\alpha C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \quad \text{and} \quad {}_t\mathrm{D}_b^\alpha C = \frac{C}{\Gamma(1-\alpha)}(b-t)^{-\alpha}.$$

Also, ${}_a\mathrm{D}_t^{\alpha} f(t) = 0$ and ${}_t\mathrm{D}_b^{\alpha} g(t) = 0$, $n-1 \leq \operatorname{Re}\alpha < n$, if and only if, respectively,

$$f(t) = \sum_{k=1}^{n} c_k (t-a)^{\alpha-k} \quad \text{and} \quad g(t) = \sum_{k=1}^{n} d_k (b-t)^{\alpha-k}, \tag{2.13}$$

where c_k and d_k, $k = 1, \ldots, n$, are arbitrary constants. Thus, functions f and g in [2.13] play the role of constants for the left and right Riemann–Liouville fractional derivatives, respectively.

Let $\alpha = k + \gamma,\ k \in \mathbb{N}_0,\ \gamma \in [0,1)$. Then, ${}_0\mathrm{D}_t^{\alpha}$ and ${}_t\mathrm{D}_b^{\alpha}$ may be written as

$$\begin{aligned}
{}_0\mathrm{D}_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_0^t \frac{f(\tau)}{(t-\tau)^{\gamma}} \mathrm{d}\tau, \quad t > 0, \\
{}_t\mathrm{D}_b^{\alpha} f(t) &= (-1)^{k+1} \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}} \int_t^b \frac{f(\tau)}{(\tau-t)^{\gamma}} \mathrm{d}\tau, \quad t < b.
\end{aligned}$$

Sometimes, in short, it is written ${}_a\mathrm{D}_t^{\alpha} f = f^{(\alpha)}$.

Let $\alpha \in [0,1)$. Then, for $t > a$ and $t < b$, we have

$$\begin{aligned}
{}_a\mathrm{D}_t^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau \quad \text{and} \\
{}_t\mathrm{D}_b^{\alpha} f(t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha}} \mathrm{d}\tau.
\end{aligned} \tag{2.14}$$

In the case when α is purely imaginary, i.e. $\alpha = \mathrm{i}\theta$, the left Riemann–Liouville fractional derivative is defined as

$${}_a\mathrm{D}_t^{\mathrm{i}\theta} f(t) = \frac{1}{\Gamma(1-\mathrm{i}\theta)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\tau)}{(t-\tau)^{\mathrm{i}\theta}} \mathrm{d}\tau, \quad t \geq a.$$

Consider the problem of determining $\lim_{\alpha \to 1^-} {}_a\mathrm{D}_t^{\alpha} f$. Then, we have the following proposition.

PROPOSITION 2.2.– [NAH 03, p. 174] Suppose that $f \in C^1([0,T])$. Then, $\lim_{\alpha \to 1^-} {}_a\mathrm{D}_t^{\alpha} f = f^{(1)}$.

We put $\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\cdot) = \mathrm{D}^n(\cdot)$. The index rule holds for the integer-order integrals and derivatives

$$\begin{aligned}
\left({}_a\mathrm{I}_t^n\, {}_a\mathrm{I}_t^m\right) f(t) &= \left({}_a\mathrm{I}_t^m\, {}_a\mathrm{I}_t^n\right) f(t) = {}_a\mathrm{I}_t^{n+m} f(t), \quad n,m \in \mathbb{N}_0, \\
\left(\mathrm{D}^n \mathrm{D}^m\right) f(t) &= \left(\mathrm{D}^m \mathrm{D}^n\right) f(t) = \mathrm{D}^{m+n} f(t), \quad n,m \in \mathbb{N}_0.
\end{aligned} \qquad [2.15]$$

The semi-group property $[2.15]_1$ holds for fractional integrals only.

PROPOSITION 2.3.– [DIE 10, p. 14] The fractional integral ${}_a\mathrm{I}_t^\alpha$ as a mapping from $L^1(a,b) \to L^1(a,b)$ forms a commutative semi-group with respect to orders of integrals. The identity operator ${}_a\mathrm{I}_t^0$ is the neutral element. Thus, if $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$

$$\begin{aligned}
\left({}_a\mathrm{I}_t^\alpha\, {}_a\mathrm{I}_t^\beta\right) f(t) &= \left({}_a\mathrm{I}_t^\beta\, {}_a\mathrm{I}_t^\alpha\right) f(t) = {}_a\mathrm{I}_t^{\alpha+\beta} f(t), \\
\left({}_t\mathrm{I}_b^\alpha\, {}_t\mathrm{I}_b^\beta\right) f(t) &= \left({}_t\mathrm{I}_b^\beta\, {}_t\mathrm{I}_b^\alpha\right) f(t) = {}_t\mathrm{I}_b^{\alpha+\beta} f(t),
\end{aligned}$$

holds for almost all $t \in [a,b]$ (almost everywhere (a.e.) in $[a,b]$) if $f \in L^p(a,b)$, $1 \leq p \leq \infty$.

Also, it can be shown that for $\operatorname{Re}\alpha > 0$, $f \in L^p(a,b)$, $1 \leq p \leq \infty$, the composition of fractional derivatives and fractional integrals holds, for almost all $t \in (a,b)$ (see [SAM 93, p. 44]),

$$\left({}_a\mathrm{D}_t^\alpha\, {}_a\mathrm{I}_t^\alpha\right) f(t) = f(t), \quad \text{and} \quad \left({}_t\mathrm{D}_b^\alpha\, {}_t\mathrm{I}_b^\alpha\right) f(t) = f(t),$$

showing that ${}_a\mathrm{D}_t^\alpha$, ${}_t\mathrm{D}_b^\alpha$ are the left inverses of ${}_a\mathrm{I}_t^\alpha$, ${}_t\mathrm{I}_b^\alpha$, respectively. However by applying ${}_a\mathrm{D}_t^\alpha$ and ${}_t\mathrm{D}_b^\alpha$ to the right of ${}_a\mathrm{I}_t^\alpha$ and ${}_t\mathrm{I}_b^\alpha$, we have different situation. To examine the resulting relations, we define the following spaces:

$$\begin{aligned}
{}_a\mathrm{I}_t^\alpha\left(L^p\right) &= \left\{f \mid f = {}_a\mathrm{I}_t^\alpha \varphi,\ \varphi \in L^p(a,b)\right\} \quad \text{and} \\
{}_t\mathrm{I}_b^\alpha\left(L^p\right) &= \left\{g \mid g = {}_t\mathrm{I}_b^\alpha \phi,\ \phi \in L^p(a,b)\right\}.
\end{aligned} \qquad [2.16]$$

PROPOSITION 2.4.– [KIL 06, p. 74] Let $\operatorname{Re}\alpha > 0$, $n-1 < \operatorname{Re}\alpha < n$. Then, the following holds:

i) If $f \in {}_a\mathrm{I}_t^\alpha\left(L^p\right)$, $1 \leq p \leq \infty$, then

$$\left({}_a\mathrm{I}_t^\alpha\, {}_a\mathrm{D}_t^\alpha\right) f(t) = f(t), \quad \text{a.e., in } [a,b]. \qquad [2.17]$$

ii) If $f \in L^1(a,b)$, ${}_a\mathrm{I}_t^{n-\alpha} f \in AC^n([a,b])$, then

$$\left({}_a\mathrm{I}_t^{\alpha}\, {}_a\mathrm{D}_t^{\alpha}\right) f(t) = f(t) - \sum_{j=1}^{n} \frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j+1)} \left[\frac{\mathrm{d}^{n-j}}{\mathrm{d}t^{n-j}} \left({}_a\mathrm{I}_t^{n-\alpha} f\right)\right]_{t=a} \qquad [2.18]$$

holds for almost all $t \in [a,b]$.

We state the results about the index rule for the fractional derivatives.

PROPOSITION 2.5.– [KIL 06, p. 75] Let $\alpha, \beta > 0$, $n-1 \le \alpha < n$, $m-1 \le \beta < m$ and $\alpha + \beta < n$. Let $f \in L^1(a,b)$ and ${}_a\mathrm{I}_t^{m-\alpha} f \in AC^m([a,b])$. Then, the following index rule holds:

$$\left({}_a\mathrm{D}_t^{\alpha}\, {}_a\mathrm{D}_t^{\beta}\right) f(t) = {}_a\mathrm{D}_t^{\alpha+\beta} f(t) - \sum_{j=1}^{m} \frac{(t-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)} \left[{}_a\mathrm{D}_t^{\beta-j} f(t)\right]_{t=a}, \quad t \in [a,b]. \qquad [2.19]$$

There are special cases when the index rule holds (see [KIL 06, p. 74]).

The composition rule for the left Riemann–Liouville derivative and the right Riemann–Liouville integral takes a rather complicated form (see [NAH 03, p. 22]). Suppose that f is Hölder continuous in $[a,b]$ and $f \in L^1(a,b)$. Then, for $\alpha \in (0,1)$, we have

$$\left({}_a\mathrm{D}_t^{\alpha}\, {}_t\mathrm{I}_b^{\alpha}\right) f(t) = f(t)\cos(\alpha\pi) + \mathrm{S}_{ab}^{\alpha} f(t)\cos(\alpha\pi), \quad \text{a.e. in } [a,b],$$

where

$$\mathrm{S}_{ab}^{\alpha} f(t) = \frac{1}{\pi} \int_a^b \left|\frac{u-a}{t-a}\right|^{\alpha} \frac{f(u)}{u-t} \mathrm{d}u. \qquad [2.20]$$

The integral in [2.20] should be taken as a Cauchy principal value.

In the variational problems, important result is integration by parts formula. We state it as follows.

PROPOSITION 2.6.– [SKM, pp. 46 and 67]

i) Suppose $0 < \alpha < 1$, $f \in L^p(a,b)$, $g \in L^q(a,b)$. Then

$$\int_a^b f(t)\left({}_a\mathrm{I}_t^{\alpha} g(t)\right) \mathrm{d}t = \int_a^b \left({}_t\mathrm{I}_b^{\alpha} f(t)\right) g(t)\, \mathrm{d}t, \qquad [2.21]$$

for $p \ge 1$, $q \ge 1$ and $\frac{1}{p} + \frac{1}{q} \le 1 + \alpha$.

ii) Suppose $0 < \operatorname{Re} \alpha < 1$, $f \in {}_t\mathrm{I}_b^\alpha\left(L^p\right)$ and $g \in {}_a\mathrm{I}_t^\alpha\left(L^q\right)$. Then

$$\int_a^b f\left(t\right)\left({}_a\mathrm{D}_t^\alpha g\left(t\right)\right)\mathrm{d}t = \int_a^b \left({}_t\mathrm{D}_b^\alpha f\left(t\right)\right) g\left(t\right)\mathrm{d}t, \tag{2.22}$$

for $p \geq 1$, $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$.

For a generalization of the integration by parts formula [2.22], see [11.81].

Fractional derivatives could be expressed in terms of integer-order derivatives through expansion formula.

PROPOSITION 2.7.– [SAM 93, p. 278] Suppose $\alpha \in \mathbb{R}_+$ and that f is an analytic function on (a,b). Then

$${}_a\mathrm{D}_t^\alpha f\left(t\right) = \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{\left(t-a\right)^{n-\alpha}}{\Gamma\left(n+1-\alpha\right)} f^{(n)}\left(t\right), \quad t \in \left(a,b\right), \tag{2.23}$$

where

$$\binom{\alpha}{n} = \left(-1\right)^{n-1}\frac{\alpha\Gamma\left(n-\alpha\right)}{\Gamma\left(1-\alpha\right)\Gamma\left(n+1\right)},$$

denotes the binomial coefficients.

The fractional derivatives can be expressed in terms of a function and its moments. The following expansion formula may be proved.

PROPOSITION 2.8.– [ATA 14a] Let $f \in C^1([0,T])$ and $0 < \alpha < 1$. Then

$${}_0\mathrm{D}_t^\alpha f(t) = \frac{f(t)}{t^\alpha}\mathcal{A}(N) - \sum_{p=1}^{N}\mathcal{C}_{p-1}\frac{V_{p-1}(f)(t)}{t^{p+\alpha}} + Q_{N+1}(f)(t), \quad t \in (0,T], \tag{2.24}$$

where

$$\mathcal{A}(N) = \frac{1}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\sum_{p=1}^{N}\frac{\Gamma(p+\alpha)}{p!} = \frac{\sin(\alpha\pi)}{\alpha\pi}\frac{\Gamma(N+1+\alpha)}{\Gamma\left(N+1\right)}, \tag{2.25}$$

$$\mathcal{C}_{p-1} = \frac{\Gamma(p+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(p)}, \tag{2.26}$$

$$V_{p-1}(f)(t) = \int_0^t \tau^{p-1} f(\tau)\mathrm{d}\tau, \quad t \in [0,T], \ p \in \mathbb{N}, \tag{2.27}$$

and the reminder term $Q_{N+1}(f)$ satisfies the estimate

$$|Q_{N+1}(f)(t)| \leq \frac{C \cdot M_t}{\Gamma(\alpha)\Gamma(1-\alpha)} \cdot \frac{t^{1-\alpha}}{N^{\alpha_1}}, \quad t \in [0,T], \ 0 < \alpha_1 < 1-\alpha, \quad [2.28]$$

with $M_t = \max_{0 \leq \tau \leq t} |y^{(1)}(\tau)|$ and certain constant $C > 0$.

Thus

$$\lim_{N\to\infty} Q_{N+1}(f)(t) = 0 \ \text{ uniformly on } \ [0,T]$$

and the approximation formula for the left Riemann–Liouville fractional derivative becomes

$${}_0\mathrm{D}_t^\alpha f(t) \approx \frac{f(t)}{t^\alpha}\mathcal{A}(N) - \sum_{p=1}^{N} \mathcal{C}_{p-1} \frac{V_{p-1}(f)(t)}{t^{p+\alpha}}, \quad t \in (0,T]. \qquad [2.29]$$

From the expansion formula [2.24], approximation to the right fractional derivative could be derived.

PROPOSITION 2.9.– [ATA 14a] Let $g \in C^1([0,T])$ and $0 < \alpha < 1$. Then, the right Riemann–Liouville fractional derivative can be approximated by

$${}_t\mathrm{D}_T^\alpha g(t) \approx \frac{g(t)}{t^\alpha}\mathcal{A}(N) + \sum_{p=1}^{N} \mathcal{C}_{p-1} t^{p-1} W_{p-1}(g)(t), \quad t \in (0,T], \qquad [2.30]$$

where $A(N,\alpha)$ and $C_{p-1}(\alpha)$ are defined by [2.25] and [2.26], respectively, and $W_{p-2}(g)$ is

$$W_{p-1}(g)(t) = \int_t^T \frac{g(\tau)}{\tau^{p+\alpha}} \,\mathrm{d}\tau.$$

REMARK 2.2.– Expansion formula [2.24] may be expressed in a different form (see [ATA 08b]) in which the first derivative of a function appears. Thus, for $t \in (0,T]$,

$${}_0\mathrm{D}_t^\alpha f(t) = \mathcal{A}(N)\frac{f(t)}{t^\alpha} + \mathcal{B}(N) t^{1-\alpha} f^{(1)}(t) - \sum_{p=1}^{N} \mathcal{C}_{p-1} \frac{V_{p-1}(f)(t)}{t^{p+\alpha}} + R_{N+1}(t), [2.31]$$

where

$$\mathcal{A}(N) = \frac{1}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(\alpha-1)\Gamma(2-\alpha)} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha)}{(p-1)!} = \frac{\sin(\alpha\pi)}{\alpha\pi} \frac{\Gamma(N+\alpha)}{\Gamma(N)},$$

$$\mathcal{B}(N) = \frac{1}{\Gamma(2-\alpha)} \left(1 + \frac{1}{\Gamma(\alpha-1)} \sum_{p=1}^{N} \frac{\Gamma(p-1+\alpha)}{p!}\right) = \frac{\Gamma(N+\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)\Gamma(N+1)},$$

$$\mathcal{C}_{p-1} = \frac{\Gamma(p+\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(p)},$$

$$R_{N+1}(t) = \frac{t^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t y^{(2)}(\tau) \left(\sum_{p=N+1}^{\infty} \frac{\Gamma(p+\alpha)}{p!} \left(\frac{\tau}{t}\right)^p \mathrm{d}\tau\right), \quad t \in (0,T] \qquad [2.32]$$

and

$$V_p^{(1)}(f)(t) = t^p f(t), \quad V_p(f)(0) = 0, \;\; t \in [0,T], \; p \in \mathbb{N}.$$

There are several definitions of the fractional derivatives of variable order. In [ROS 93, SAM 95], the following definition was proposed. The left Riemann–Liouville fractional derivative of variable α for $0 \leq \alpha(t) < 1$ is

$${}_0\mathrm{D}_t^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha(t)}} \mathrm{d}\tau, \;\; t \in [0,T]. \qquad [2.33]$$

In [ATA 13a], the following expansion formula for [2.33] is proved.

PROPOSITION 2.10.– [ATA 13a] Suppose that $f \in C^2([0,T])$, $\alpha \in C^1([0,T])$. Then, the fractional derivative of the order α, defined by [2.33], may be written as

$$\begin{aligned} {}_0\mathrm{D}_t^{\alpha(t)} f(t) &= A_1\left(f(t), f^{(1)}(t), \alpha(t), t\right) \\ &\quad -\alpha^{(1)}(t) A_2\left(f(t), f^{(1)}(t), \alpha(t), t\right) + R_1^N(t) \\ &\quad + R_2^{N,M}(t), \;\; t \in (0,T], \end{aligned}$$

where

$$
\begin{aligned}
&A_1\left(f(t), f^{(1)}(t), \alpha(t), t\right) \\
&= \frac{f(t)}{t^{\alpha(t)}}\left(\frac{1}{\Gamma(1-\alpha(t))} - \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))}\sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha(t))}{(p-1)!}\right) \\
&+ \frac{f^{(1)}(t)t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))}\left(1 + \frac{1}{\Gamma(\alpha(t)-1)}\sum_{p=1}^{N}\frac{\Gamma(p-1+\alpha(t))}{p!}\right) \\
&- \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))}\sum_{p=0}^{N}\frac{\Gamma(p+1+\alpha(t))}{p!}\frac{V_p(f)(t)}{t^{p+1+\alpha(t)}}, \quad t \in (0,T],
\end{aligned}
$$

and ($t \in [0,T]$)

$$
\begin{aligned}
&A_2\left(f(t), f^{(1)}(t), \alpha(t), t\right) \\
&= \frac{1}{\Gamma(1-\alpha(t))}\left(\frac{f(t)\,t^{1-\alpha(t)}}{1-\alpha(t)}\left(\ln t - \frac{1}{1-\alpha(t)}\right)\right. \\
&- \frac{f^{(1)}(t)t^{2-\alpha(t)}}{2-\alpha(t)}\left(\ln t - \frac{1}{2-\alpha(t)}\right) \\
&+ \frac{t^{1-\alpha(t)}\ln t}{\Gamma(\alpha(t))}\sum_{k=0}^{N}\frac{\Gamma(k+\alpha(t))}{k!}\left(\frac{tf^{(1)}(t)}{(k+1)(k+2)} - \frac{f(t)}{k+1} + \frac{V_k(f)(t)}{t^{k+1}}\right) \\
&+ \frac{t^{1-\alpha(t)}}{\Gamma(\alpha(t))}\sum_{p=1}^{M}\frac{1}{p}\sum_{k=0}^{N}\frac{\Gamma(k+\alpha(t))}{k!}\left(\frac{tf^{(1)}(t)}{(k+p+1)(k+p+2)} - \frac{f(t)}{k+p+1}\right. \\
&\left.\left.+\frac{V_{k+p}(f)(t)}{t^{k+p+1}}\right)\right),
\end{aligned}
$$

with $V_p(f)(t) = \int_0^t \tau^p f(\tau)\,\mathrm{d}\tau$ being moments of the function f and satisfying

$$
V_p^{(1)}(f)(t) = t^p f(t), \quad V_p(0) = 0, \;\; t \in [0,T], \; p = 0, 1, \dots . \qquad [2.34]
$$

Also, there exists $N_\varepsilon \in \mathbb{N}$ such that for any $\varepsilon > 0$, and for $N, M > N_\varepsilon$, it holds that

$$
\left|R_1^N(t) + R_2^{N,M}(t)\right| < \varepsilon.
$$

Thus, the approximation formula for ${}_0\mathrm{D}_t^{\alpha(t)} f(t)$, $t \in (0,T]$, becomes

$$
\begin{aligned}
{}_0\mathrm{D}_t^{\alpha(t)} f(t) &\approx \widehat{{}_0\mathrm{D}_t^{\alpha(t)}} f(t) \\
&= \frac{f(t)}{t^{\alpha(t)}} \left(\frac{1}{\Gamma(1-\alpha(t))} - \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha(t))}{(p-1)!} \right) \\
&+ \frac{f^{(1)}(t) t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} \left(1 + \frac{1}{\Gamma(\alpha(t)-1)} \sum_{p=1}^{N} \frac{\Gamma(p-1+\alpha(t))}{p!} \right) \\
&- \frac{1}{\Gamma(\alpha(t)-1)\Gamma(2-\alpha(t))} \sum_{p=2}^{N} \frac{\Gamma(p-1+\alpha(t))}{(p-2)!} \frac{V_{p-2}(f)(t)}{t^{p-1+\alpha(t)}} \\
&- \frac{\alpha^{(1)}(t)}{\Gamma(1-\alpha(t))} \left(f(t) \left(\frac{1}{1-\alpha(t)} t^{1-\alpha(t)} \ln t - \frac{t^{1-\alpha(t)}}{(1-\alpha(t))^2} \right) \right. \\
&- \frac{f^{(1)}(t) t^{2-\alpha(t)}}{2-\alpha(t)} \left(\ln t - \frac{1}{2-\alpha(t)} \right) \\
&+ \frac{t^{1-\alpha(t)} \ln t}{\Gamma(\alpha(t))} \sum_{k=0}^{N} \frac{\Gamma(k+\alpha(t))}{k!} \left(\frac{t f^{(1)}(t)}{(k+1)(k+2)} - \frac{f(t)}{k+1} + \frac{V_k(f)(t)}{t^{k+1}} \right) \\
&+ \frac{t^{1-\alpha(t)}}{\Gamma(\alpha(t))} \sum_{p=1}^{M} \frac{1}{p} \sum_{k=0}^{N} \frac{\Gamma(k+\alpha(t))}{k!} \left(\frac{t f^{(1)}(t)}{(k+p+1)(k+p+2)} - \frac{f(t)}{k+p+1} \right. \\
&\left.\left. + \frac{V_{k+p}(f)(t)}{t^{k+p+1}} \right) \right).
\end{aligned}
\tag{2.35}
$$

Note that for the case $\alpha = const.$, expressions [2.29] and [2.35] coincide.

REMARK 2.3.– The procedure of expressing fractional derivatives in terms of function, its first derivative and moments of function is extended in different directions in a series of papers [POO 12a, POO 12b, POO 13].

2.1.1.1. *Laplace transform of Riemann–Liouville fractional integrals and derivatives*

Suppose that f is exponentially bounded (see section 1.2), that is $f \in L^1(0,\infty)$, $|f(t)| \leq A e^{s_0 t}$, $t > 0$, where $A > 0$, $s_0 > 0$. Then

$$
\mathcal{L}\left[{}_0\mathrm{I}_t^{\alpha} f(t)\right](s) = \frac{1}{s^{\alpha}} \tilde{f}(s), \quad \mathrm{Re}\, s > s_0, \tag{2.36}
$$

see [KIL 06, p. 84]. Expression [2.36] follows from the well-known property of the Laplace transform of convolution and $\mathcal{L}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right](s) = \frac{1}{s^{\alpha}}$ (see section 1.3).

For the fractional derivatives, we have the following result.

PROPOSITION 2.11.– [KIL 06, p. 84] Let $n-1 < \operatorname{Re}\alpha < n$, $f \in AC^n_{loc}([0,\infty))$ and f be of exponential growth. Suppose that there exist finite limits

$$\lim_{t\to 0}\left(\mathrm{D}^k\, {}_0\mathrm{I}_t^{n-\alpha} f(t)\right) \quad \text{and} \quad \lim_{t\to\infty}\left(\mathrm{D}^k\, {}_0\mathrm{I}_t^{n-\alpha} f(t)\right) = 0, \quad k = 0, 1, \ldots, n-1.$$

Then

$$\mathcal{L}\left[{}_0\mathrm{D}_t^{\alpha} f(t)\right](s) = s^{\alpha}\tilde{f}(s) - \sum_{k=0}^{n-1} s^{n-k-1}\left[\mathrm{D}^k\, {}_0\mathrm{I}_t^{n-\alpha} f(t)\right]_{t=0}, \quad \operatorname{Re}s > s_0. \qquad [2.37]$$

For $0 < \alpha < 1$

$$\mathcal{L}\left[{}_0\mathrm{D}_t^{\alpha} f(t)\right](s) = s^{\alpha}\tilde{f}(s) - \left[{}_0\mathrm{I}_t^{1-\alpha} f(t)\right]_{t=0} = s^{\alpha}\tilde{f}(s), \quad \operatorname{Re}s > s_0. \qquad [2.38]$$

Relation [2.38] could be used for the (heuristic) definition of the fractional derivative.

The Leibnitz rule for fractional derivatives does not hold in its usual form. It could be shown that for analytic functions, we have the following.

PROPOSITION 2.12.– [SAM 93, p. 280] Suppose that f and g are analytic for $t > 0$ and $\alpha > 0$. Then

$${}_a\mathrm{D}_t^{\alpha}\left(f(t)\, g(t)\right) = \sum_{k=0}^{\infty}\binom{\alpha}{k}\left(\mathrm{D}^k g(t)\right)\left({}_a\mathrm{D}_t^{\alpha-k} f(t)\right), \quad t > a. \qquad [2.39]$$

Note that in [2.39], on the right-hand side we have integer-order derivatives of g and fractional-order derivatives of f. There is an apparent lack of symmetry in the derivatives of the two functions. The left-hand side of [2.39] does not depend on the order of the functions f and g, while on the right-hand side there are only integer derivatives of g and non-integer derivatives (integrals) of f. It could be shown that the two functions f and g can be interchanged without changing the value of the fractional derivative of their product.

2.1.2. *Riemann–Liouville fractional integrals and derivatives on the real half-axis*

The Riemann–Liouville fractional integrals and derivatives defined on a finite interval $[a,b]$ can be naturally extended to a half-line $\mathbb{R}_+$ as

$$\begin{aligned}
\mathrm{I}_+^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ \operatorname{Re}\alpha > 0, \\
\mathrm{I}_-^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^{\infty} (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \quad t > 0,\ \operatorname{Re}\alpha > 0, \qquad [2.40]
\end{aligned}$$

and

$$\begin{aligned}\mathrm{D}_{+}^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\int_{0}^{t}\frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad t>0,\ n-1\leq \operatorname{Re}\alpha<n,\\ \mathrm{D}_{-}^{\alpha}f(t) &= (-1)^{n}\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}\int_{t}^{\infty}\frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}\mathrm{d}\tau, \quad t>0,\\ &\qquad n-1\leq \operatorname{Re}\alpha<n.\end{aligned}$$

Operator $[2.40]_2$ is sometimes called the Weyl integral.

Fourier transform of the Riemann–Liouville fractional integrals $\mathrm{I}_{+}^{\alpha}f$ and $\mathrm{I}_{-}^{\alpha}f$ are given as

$$\mathcal{F}\left[\mathrm{I}_{+}^{\alpha}f(t)\right](\omega)=\frac{1}{(\mathrm{i}\omega)^{\alpha}}\hat{f}(\omega) \text{ and } \mathcal{F}\left[\mathrm{I}_{-}^{\alpha}f(t)\right](\omega)=\frac{1}{(-\mathrm{i}\omega)^{\alpha}}\hat{f}(\omega), \quad \omega\in\mathbb{R}, \qquad [2.41]$$

for $0<\operatorname{Re}\alpha<1$ and for $f\in L^{1}(\mathbb{R})$. Equation [2.41] cannot be extended directly to the case $\operatorname{Re}\alpha\geq 1$. For the Riemann–Liouville derivatives, we have

$$\mathcal{F}\left[\mathrm{D}_{+}^{\alpha}f(t)\right](\omega)=(\mathrm{i}\omega)^{\alpha}\hat{f}(\omega) \text{ and } \mathcal{F}\left[\mathrm{D}_{-}^{\alpha}f(t)\right](\omega)=(-\mathrm{i}\omega)^{\alpha}\hat{f}(\omega), \quad \omega\in\mathbb{R}. \qquad [2.42]$$

In [2.41] and [2.42], we have $(\pm\mathrm{i}\omega)^{\alpha}=|\omega|^{\alpha}e^{\mp\frac{\alpha\pi}{2}\operatorname{sgn}\omega}$. In the case when $f\in\mathcal{S}'$, we have that [2.41] and [2.42] remain the same (see [VLA 73, p. 110]).

2.1.3. *Caputo fractional derivatives*

We present the definition of fractional derivative from Caputo [CAP 67] and Caputo and Mainardi [CAP 71b]. The left Caputo fractional derivative of a function of order α, denoted by ${}_{a}^{C}\mathrm{D}_{t}^{\alpha}f$, is

$${}_{a}^{C}\mathrm{D}_{t}^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}\mathrm{d}\tau, & n-1\leq\alpha<n,\\ \frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}f(t), & \alpha=n,\end{cases} \quad t\in[a,b]. \qquad [2.43]$$

Similarly, the right Caputo derivative is defined as

$${}_{t}^{C}\mathrm{D}_{b}^{\alpha}f(t)=\begin{cases}(-1)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{t}^{b}\frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha+1-n}}\mathrm{d}\tau, & n-1\leq\alpha<n,\\ (-1)^{n}\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}}f(t), & \alpha=n,\end{cases} \quad t\in[a,b]. \qquad [2.44]$$

It is easy to see that

$$ {}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t) = {}_{a}\mathrm{I}_{t}^{n-\alpha}\left(\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} f(t)\right) \quad \text{and} \quad {}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) = (-1)^{n}\, {}_{t}\mathrm{I}_{b}^{n-\alpha}\left(\frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} f(t)\right), $$

where ${}_{a}\mathrm{I}_{t}^{n-\alpha}$ and ${}_{t}\mathrm{I}_{b}^{n-\alpha}$ are the Riemann–Liouville fractional integrals [2.2] and [2.3], respectively.

Observe that [2.43] for $a = 0$ can be written as

$$ {}_{0}^{C}\mathrm{D}_{t}^{\alpha} f(t) = \frac{t^{n-1-\alpha}}{\Gamma(n-\alpha)} * \frac{\mathrm{d}^{n}}{\mathrm{d}t^{n}} f(t), \quad t > 0, \; n-1 \leq \operatorname{Re}\alpha < n. \qquad [2.45] $$

Note that the Caputo derivative of a constant function is zero

$$ {}_{a}^{C}\mathrm{D}_{t}^{\alpha} C = 0 \;\text{ and }\; {}_{t}^{C}\mathrm{D}_{b}^{\alpha} C = 0. \qquad [2.46] $$

For $n-1 \leq \alpha < n$, the Caputo derivatives ${}_{a}^{C}\mathrm{D}_{t}^{\alpha}$ and ${}_{t}^{C}\mathrm{D}_{b}^{\alpha}$ are operators mapping $C^{n}([a,b])$ into

$$ \begin{aligned} C_{a}([a,b]) &= \{f \mid f \in C([a,b]),\; f(a) = 0\}, \quad \|f\|_{C_{a}} = \|f\|_{C}, \\ C_{b}([a,b]) &= \{f \mid f \in C([a,b]),\; f(b) = 0\}, \quad \|f\|_{C_{b}} = \|f\|_{C}, \end{aligned} $$

respectively.

PROPOSITION 2.13.– [KIL 06, p. 94] Let $n-1 \leq \operatorname{Re}\alpha < n$, $\alpha \neq \mathbb{N}$. Then, the Caputo derivatives ${}_{a}^{C}\mathrm{D}_{t}^{\alpha}$ and ${}_{t}^{C}\mathrm{D}_{b}^{\alpha}$ are bounded operators from $C^{n}([a,b])$ into $C_{a}([a,b])$ and $C_{b}([a,b])$, respectively, and the following estimates hold

$$ \begin{aligned} \left\|{}_{a}^{C}\mathrm{D}_{t}^{\alpha} f\right\|_{C_{a}} &\leq \frac{(b-a)^{n-\operatorname{Re}\alpha}}{|\Gamma(n-\alpha)|\,(n-\operatorname{Re}\alpha+1)} \|f\|_{C^{n}}, \\ \left\|{}_{t}^{C}\mathrm{D}_{b}^{\alpha} f\right\|_{C_{b}} &\leq \frac{(b-a)^{n-\operatorname{Re}\alpha}}{|\Gamma(n-\alpha)|\,(n-\operatorname{Re}\alpha+1)} \|f\|_{C^{n}}. \end{aligned} $$

In general, the Caputo and the Riemann–Liouville fractional derivatives do not coincide. The connections between them are given as

$$
\begin{aligned}
{}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t) &= {}_{a}\mathrm{D}_{t}^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{(t-a)^{k}}{k!}f^{(k)}(a)\right), \quad t\in[a,b], \\
{}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) &= {}_{t}\mathrm{D}_{b}^{\alpha}\left(f(t)-\sum_{k=0}^{n-1}\frac{(b-t)^{k}}{k!}f^{(k)}(b)\right), \quad t\in[a,b].
\end{aligned}
$$

In particular, if $0<\operatorname{Re}\alpha<1$, for $t\in[a,b]$, we have

$$
{}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t) = {}_{a}\mathrm{D}_{t}^{\alpha}\left(f(t)-f(a)\right) \quad \text{and} \quad {}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) = {}_{t}\mathrm{D}_{b}^{\alpha}\left(f(t)-f(b)\right),
$$

or ($t\in[a,b]$)

$$
\begin{aligned}
{}_{a}^{C}\mathrm{D}_{t}^{\alpha} f(t) &= {}_{a}\mathrm{D}_{t}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)(t-a)^{\alpha}} \quad \text{and} \\
{}_{t}^{C}\mathrm{D}_{b}^{\alpha} f(t) &= {}_{t}\mathrm{D}_{b}^{\alpha} f(t) - \frac{f(b)}{\Gamma(1-\alpha)(b-t)^{\alpha}}. \qquad [2.47]
\end{aligned}
$$

Thus, the Caputo fractional derivatives are regularized Riemann–Liouville fractional derivatives.

The expansion formula [2.24] for the Caputo derivative ${}_{0}^{C}\mathrm{D}_{t}^{\alpha} f$, $0<\alpha<1$, with [2.47], becomes ($t\in(0,T]$)

$$
\begin{aligned}
{}_{0}\mathrm{D}_{t}^{\alpha} f(t) &= \frac{f(t)}{t^{\alpha}}\mathcal{A}(N,\alpha) - \frac{f(0)}{t^{\alpha}\Gamma(1-\alpha)} - \sum_{p=1}^{N}\mathcal{C}_{p-1}(\alpha)\frac{V_{p-1}(f)(t)}{t^{p+\alpha}} \\
&\quad + Q_{N+1}(f)(t), \qquad [2.48]
\end{aligned}
$$

where $\mathcal{A}$, $\mathcal{C}_{p-1}$ and V_p are given by [2.25], [2.26] and [2.27], respectively.

The case when α is close to 1 is called the case of low-level fractionality. For this case, we refer to [HER 11, TAR 06]. We have, for $t\in[a,b]$,

$$
\begin{aligned}
{}_{a}^{C}\mathrm{D}_{t}^{1-\varepsilon} f(t) &= f^{(1)}(t) + \varepsilon\left(f^{(1)}(0)\ln t + \int_{0}^{t} f^{(2)}(\xi)\ln(t-\xi)\,\mathrm{d}\xi\right) + o\left(\varepsilon^{2}\right), \\
&\qquad \varepsilon\to 0^{+}, \qquad [2.49]
\end{aligned}
$$

where o is the Landau symbol "small o". Recall that $a(x) = o(|x|^{\alpha})$ means $\frac{a(x)}{|x|^{\alpha}} \to 0$ as $|x| \to 0$, or $|x| \to \infty$.

We have the following integration by parts formula for the Caputo derivatives.

PROPOSITION 2.14.– [ALM 12, p. 112] Let $n-1 \leq \alpha < n$ and let $f, g \in C^n([a,b])$. Then

$$\begin{aligned}\int_a^b g(t)\left({}_t^C\mathrm{D}_b^\alpha f(t)\right)\mathrm{d}t &= \int_a^b \left({}_a^C\mathrm{D}_t^\alpha g(t)\right) f(t)\,\mathrm{d}t \\ &+ \sum_{j=0}^{n-1}(-1)^{n+j}\left[\left({}_a^C\mathrm{D}_t^{\alpha+j-n} g(t)\right)\left({}_a^C\mathrm{D}_t^{n-1-j} f(t)\right)\right]_{t=a}^{t=b}. \qquad [2.50]\end{aligned}$$

The Laplace transform of the left Caputo derivative is given as follows.

PROPOSITION 2.15.– [KIL 06, p. 98] Suppose that $n-1 < \alpha \leq n$ and let f be such that $f \in C^n(\mathbb{R}_+)$, $|f(t)|, \left|f^{(1)}(t)\right|, \ldots, \left|f^{(n)}(t)\right| \leq Be^{s_0 t}$, $B, s_0 > 0$, $t > 0$. Suppose that $\lim_{t\to\infty} f^{(k)}(t) = 0$, for $k = 0, 1, \ldots, n-1$. Then

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^\alpha f(t)\right](s) = s^\alpha \tilde{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad \operatorname{Re} s > s_0. \qquad [2.51]$$

For $0 < \alpha < 1$, expression [2.51] becomes

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^\alpha f(t)\right](s) = s^\alpha \tilde{f}(s) - s^{\alpha-1} f(0), \quad \operatorname{Re} s > s_0. \qquad [2.52]$$

2.1.4. *Riesz potentials and Riesz derivatives*

Let $0 < \alpha < 1$. Consider the following integrals:

$$\mathrm{R}^\alpha f(x) = \frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}\int_{-\infty}^{\infty}\frac{1}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}, \qquad [2.53]$$

$$\mathrm{H}^\alpha f(x) = \frac{1}{2\Gamma(\alpha)\sin\frac{\alpha\pi}{2}}\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(x-\zeta)}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}. \qquad [2.54]$$

Then, $\mathrm{R}^\alpha f$ is called the Riesz potential of f of the order α on $\mathbb{R}$, while $\mathrm{H}^\alpha f$ is the conjugate Riesz potential of the order α on $\mathbb{R}$. The following proposition holds for [2.53] and [2.54]. Note that equality is almost everywhere (see section 1.1).

PROPOSITION 2.16.– [UCH 08, p. 200] The Riemann–Liouville fractional integral and the Riesz potential, for $x \in \mathbb{R}$, are connected as follows:

$$\begin{aligned}
{}_{-\infty}\mathrm{I}_x^{\alpha} f(x) &= \cos\frac{\alpha\pi}{2}\,\mathrm{R}^{\alpha} f(x) + \sin\frac{\alpha\pi}{2}\,\mathrm{H}^{\alpha} f(x), \\
{}_{x}\mathrm{I}_{\infty}^{\alpha} f(x) &= \cos\frac{\alpha\pi}{2}\,\mathrm{R}^{\alpha} f(x) - \sin\frac{\alpha\pi}{2}\,\mathrm{H}^{\alpha} f(x), \\
\mathrm{R}^{\alpha} f(x) &= \frac{1}{2\cos\frac{\alpha\pi}{2}}\left({}_{-\infty}\mathrm{I}_x^{\alpha} f(x) + {}_{x}\mathrm{I}_{\infty}^{\alpha} f(x)\right), \\
\mathrm{H}^{\alpha} f(x) &= \frac{1}{2\sin\frac{\alpha\pi}{2}}\left({}_{-\infty}\mathrm{I}_x^{\alpha} f(x) - {}_{x}\mathrm{I}_{\infty}^{\alpha} f(x)\right).
\end{aligned} \qquad [2.55]$$

Also, if $\alpha, \beta > 0$, $\alpha + \beta < 1$, then

$$\mathrm{R}^{\alpha}\,\mathrm{R}^{\beta} f(x) = \mathrm{R}^{\alpha+\beta} f(x) \quad \text{and} \quad \mathrm{H}^{\alpha}\,\mathrm{H}^{\beta} f(x) = -\mathrm{R}^{\alpha+\beta} f(x), \quad x \in \mathbb{R}. \qquad [2.56]$$

Integrals [2.53] and [2.54] exist with the appropriate assumptions on f. For example, if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\mathrm{R}^{\alpha} f$ and $\mathrm{H}^{\alpha} f$ exist almost everywhere and belong to $L^1_{loc}(\mathbb{R})$, as stated in [BUT 00, p. 46]. More generally, the following result holds.

PROPOSITION 2.17.– [KIL 06, p. 129] Let $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. Then, R^{α} is a bounded operator from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ if and only if

$$0 < \alpha < 1, \quad 1 < p < \frac{1}{\alpha}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{\alpha}.$$

The Fourier transforms of R^{α} and H^{α} are

$$\begin{aligned}
\mathcal{F}\left[\mathrm{R}^{\alpha} f(x)\right](\omega) &= \frac{1}{|\omega|^{\alpha}}\hat{f}(\omega) \quad \text{and} \\
\mathcal{F}\left[\mathrm{H}^{\alpha} f(x)\right](\omega) &= -\mathrm{i}\frac{\operatorname{sgn}(\omega)}{|\omega|^{\alpha}}\hat{f}(\omega), \quad \omega \in \mathbb{R}\setminus\{0\},
\end{aligned} \qquad [2.57]$$

see [BUT 00].

There is another important property of R^{α}. Namely, it maps the Lizorkin space of test functions Φ into itself, i.e. $\mathrm{R}^{\alpha}(\Phi) = \Phi$ (see [SAM 93, p. 493]).

DEFINITION 2.4.– *[BUT 00, UCH 08] The Riesz fractional derivative of the order α is defined as*

$$
{}^{R}\mathrm{D}^{\alpha} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{H}^{1-\alpha} f(x) = \frac{1}{2\Gamma(1-\alpha)\cos\frac{\alpha\pi}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\frac{\operatorname{sgn}(x-\zeta)}{|x-\zeta|^{\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}, \qquad [2.58]
$$

while the conjugate Riesz derivative of the order α is defined as

$$
{}^{RC}\mathrm{D}^{\alpha} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{R}^{1-\alpha} f(x) = \frac{1}{2\Gamma(1-\alpha)\sin\frac{\alpha\pi}{2}}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\frac{1}{|x-\zeta|^{\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in \mathbb{R}. \qquad [2.59]
$$

It is easy to see that the Fourier transforms of ${}^{R}\mathrm{D}^{\alpha} f$ and ${}^{RC}\mathrm{D}^{\alpha} f$, for $0 < \alpha < 1$, are

$$
\begin{aligned}
\mathcal{F}\left[{}^{R}\mathrm{D}^{\alpha} f(x)\right](\omega) &= |\omega|^{\alpha}\hat{f}(\omega) \text{ and} \\
\mathcal{F}\left[{}^{RC}\mathrm{D}^{\alpha} f(x)\right](\omega) &= \mathrm{i}\,|\omega|^{\alpha}\operatorname{sgn}(\omega)\,\hat{f}(\omega), \quad \omega \in \mathbb{R},
\end{aligned} \qquad [2.60]
$$

see [BUT 00]. Therefore, ${}^{R}\mathrm{D}^{\alpha}$ inverts R^{α}, while ${}^{RC}\mathrm{D}^{\alpha}$ inverts H^{α}, if the functions involved satisfy certain additional conditions. Namely, ${}^{R}\mathrm{D}^{\alpha}\,\mathrm{R}^{\alpha} f = f$ holds for functions f belonging to the Lizorkin space Φ (see [SAM 93, p. 150]).

In various applications, there is a need to study the following potential:

$$
{}_{a}^{R}\mathrm{I}_{b}^{\alpha} f(x) = \int_{a}^{b}\frac{1}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta, \quad x \in (a,b), \qquad [2.61]
$$

where $0 < \alpha < 1$. The Carleman equation involves the potential of the type [2.61]. The inverse of ${}_{a}^{R}\mathrm{I}_{b}^{\alpha}$, i.e. the solution to the equation

$$
{}_{a}^{R}\mathrm{I}_{b}^{\alpha} f(x) = \int_{a}^{b}\frac{1}{|x-\zeta|^{1-\alpha}} f(\zeta)\,\mathrm{d}\zeta = g(x), \quad x \in (a,b), \qquad [2.62]
$$

for a specified class of functions f and g is given in [SAM 93, p. 627]. Let $g(x) = (x-a)^n$, $n = 0, 1, 2, \ldots$. Then, the solution to [2.62], for $x \in (a, b)$, becomes

$$\begin{aligned} f(x) &= \frac{n!}{\pi}(b-a)^n \sin\frac{\alpha\pi}{2} \frac{\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \frac{1}{((x-a)(b-x))^{\frac{\alpha}{2}}} \\ &\quad \times \sum_{k=0}^{n} \binom{n-\frac{\alpha}{2}}{n-k} \left(\frac{b-x}{x-a}\right)^k . \end{aligned} \quad [2.63]$$

If $g = 0$, then the solution to [2.62] is $f = 0$ (see [SAM 93, eq. (30.84)], or [NAH 03, p. 47]).

2.1.5. *Symmetrized Caputo derivative*

Let $0 \le \beta < 1$, $-\infty \le a < b \le \infty$. The symmetrized Caputo fractional derivative of an absolutely continuous function f is defined as

$$ {}_a^C\mathcal{E}_b^\beta f(x) = \frac{1}{2}\left({}_a^C\mathrm{D}_x^\beta - {}_x^C\mathrm{D}_b^\beta\right) f(x) = \frac{1}{2}\frac{1}{\Gamma(1-\beta)} \int_a^b \frac{f^{(1)}(\theta)}{|x-\theta|^\beta} \mathrm{d}\theta, \quad x \in [a,b] . \quad [2.64]$$

For $a = -\infty$ and $b = \infty$, we write $\mathcal{E}_x^\beta$ instead of ${}_a^C\mathcal{E}_b^\beta$ and then

$$\mathcal{E}_x^\beta f(x) = \frac{1}{2}\frac{1}{\Gamma(1-\beta)} |x|^{-\beta} * f^{(1)}(x), \quad x \in \mathbb{R}. \quad [2.65]$$

Note that $\mathcal{E}_x^0 f(x) = 0$ and $\mathcal{E}_x^\beta f(x) \to f^{(1)}(x)$, as $\beta \to 1$. This yields that the symmetrized fractional derivative generalizes the first derivative of a function, but it does not generalize the derivative of zero order: the zeroth-order symmetrized fractional derivative of a function is not a function itself, but zero.

For $f = const.$, we have that ${}_a^C\mathcal{E}_b^\beta f = 0$, and conversely, the fact that $f = const.$ is the unique solution to equation ${}_a^C\mathcal{E}_b^\beta f = 0$ is shown in [ATA 09f].

In studying [2.64], we need some properties of the function $|x|^{-\beta}$. It is an element of the Lizorkin space Ψ' for all $\beta \in \mathbb{R}$ and for $\beta \in [0, 1)$ it holds that

$$\begin{aligned} \mathcal{F}\left[|x|^{-\beta}\right](\xi) &= 2\Gamma(1-\beta)\sin\frac{\beta\pi}{2}\frac{1}{|\xi|^{1-\beta}}, \quad \xi \in \mathbb{R}, \\ \mathcal{F}\left[\mathcal{E}_x^\beta f(x)\right](\xi) &= \mathrm{i}\frac{\xi}{|\xi|^{1-\beta}} \sin\frac{\beta\pi}{2} \hat{f}(\xi), \quad \xi \in \mathbb{R}. \end{aligned}$$

2.1.6. *Other types of fractional derivatives*

In this section, we present other types of fractional integrals and derivatives. For extensive review of definitions, see [KIR 94].

2.1.6.1. *Canavati fractional derivative*

There is another definition of fractional derivatives that is useful in deriving inequalities. This is the Canavati fractional derivative. It is "between" the Riemann–Liouville derivative and the Caputo derivative. Let $n-1<\alpha<n$. Then, the Canavati derivative of order α is defined as

$$ {}_{a}^{Can}\mathrm{D}_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{a}^{t} \frac{f^{(n-1)}(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \qquad [2.66] $$

see [CAN 87]. The Canavati derivative [2.66] is used in [ANA 09] for $f \in C^{\alpha}([a,b])$ where

$$ C^{\alpha}([a,b]) = \left\{ f \in C^{n-1}([a,b]) \mid {}_{a}\mathrm{I}_{t}^{n-1} f^{(n-1)} \in C^{1}([a,b]) \right\}. $$

2.1.6.2. *Marchaud fractional derivatives*

The left Marchaud fractional derivative of the order $0<\alpha<1$ for $f \in \mathcal{H}^{\lambda}([a,b])$, $\lambda > \alpha$ (see section 1.1) is defined as

$$ {}_{a}^{M}\mathrm{D}_{t}^{\alpha} f(t) = \frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_{a}^{t} \frac{f(t)-f(\tau)}{(t-\tau)^{1+\alpha}} \mathrm{d}\tau, \ t \in [a,b]. \qquad [2.67] $$

The right Marchaud fractional derivative is defined as

$$ {}_{t}^{M}\mathrm{D}_{b}^{\alpha} f(t) = \frac{f(t)}{\Gamma(1-\alpha)(b-t)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_{t}^{b} \frac{f(t)-f(\tau)}{(\tau-t)^{1+\alpha}} \mathrm{d}\tau, \ t \in [a,b]. \qquad [2.68] $$

The integrals in [2.67] and [2.68] are assumed to be convergent. To make it precise, let

$$ \psi_{\varepsilon}(t) = \int_{a}^{t-\varepsilon} \frac{f(t)-f(\tau)}{(t-\tau)^{1+\alpha}} \mathrm{d}\tau, \quad \varepsilon > 0. \qquad [2.69] $$

Then

$$ {}_{t}^{M}\mathrm{D}_{t}^{\alpha} f(t) = \frac{f(t)}{\Gamma(1-\alpha)(t-a)^{\alpha}} + \lim_{\varepsilon \to 0} \psi_{\varepsilon}(t). \qquad [2.70] $$

If $f \in L^p(a,b)$, then the limit in [2.70] is considered in the norm of $L^p(a,b)$.

For functions belonging to $C^1([a,b])$, the Marchaud derivatives coincide with the corresponding Riemann–Liouville derivatives.

2.1.6.3. *Grünwald–Letnikov fractional derivatives*

The left Grünwald–Letnikov fractional derivative of the order α is, according to [KIL 06, p. 122], formally defined as

$$ {}^{G-L}_{a}\mathrm{D}_t^\alpha f(t) = \lim_{h\to 0}\left(\frac{1}{h^\alpha}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j}f(t-jh)\right), \quad t>a,\ \alpha>0. \qquad [2.71]$$

Similarly, the right Grünwald–Letnikov fractional derivative of the order α is defined as

$$ {}^{G-L}_{t}\mathrm{D}_b^\alpha f(t) = \lim_{h\to 0}\left(\frac{1}{h^\alpha}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j}f(t+jh)\right), \quad t>a,\ \alpha>0. \qquad [2.72]$$

There is a connection between the Marchaud and the Grünwald–Letnikov fractional derivatives.

PROPOSITION 2.18.– [SAM 93, p. 386] Let $f \in L^p(a,b)$, $1 \leq p < \infty$. Then, limit [2.71] exists in the sense of $L^p(a,b)$ convergence, if and only if there exists the Marchaud fractional derivative in sense [2.70]. Both limits, if they exist, are equal almost everywhere.

2.2. Some additional properties of fractional derivatives

2.2.1. *Fermat theorem for fractional derivative*

Let $0<\alpha<1$. As a motivation, following [SAM 93, p. 111], we start from

$$\begin{aligned}
{}_0\mathrm{D}_t^\alpha y(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{y(t-\tau)}{\tau^\alpha}\mathrm{d}\tau \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{y(0)}{t^\alpha} + \frac{1}{\Gamma(1-\alpha)}\int_0^t y^{(1)}(t-\tau)\left(\alpha\int_\tau^t \xi^{-1-\alpha}\mathrm{d}\xi + \frac{1}{t^\alpha}\right)\mathrm{d}\tau \\
&= \frac{y(t)}{\Gamma(1-\alpha)t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{y(t)-y(t-\tau)}{\tau^{1+\alpha}}\mathrm{d}\tau, \quad t>0. \qquad [2.73]
\end{aligned}$$

Let $0 < \alpha < 1$. Similarly, for the Caputo derivative of an integrable function, we have

$$\begin{aligned} {}_0^C\mathrm{D}_t^\alpha y(t) &= {}_0\mathrm{D}_t^\alpha y(t) - \frac{y(0)}{\Gamma(1-\alpha)\, t^\alpha} \\ &= \frac{y(t) - y(0)}{\Gamma(1-\alpha)\, t^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^t \frac{y(t) - y(t-\tau)}{\tau^{1+\alpha}} \mathrm{d}\tau, \quad t > 0. \end{aligned} \qquad [2.74]$$

Suppose that y is an increasing positive function with the maximum at $t^* \in (0, t)$. Then

$$y(t^*) - y(t) \geq 0, \quad t \in [0, t^*].$$

From [2.73] and [2.74], we conclude that

$$\left[{}_0\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \geq \frac{y(t^*)}{\Gamma(1-\alpha)(t^*)^\alpha} > 0 \text{ and } \left[{}_0^C\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \geq \frac{y(t^*) - y(0)}{\Gamma(1-\alpha)(t^*)^\alpha} > 0. \qquad [2.75]$$

The above results may be used to prove the following proposition.

PROPOSITION 2.19.– [NAH 03, p. 104] Suppose that y is an integrable function on $[A, B]$. Suppose further that there exists $\delta > 0$ such that $y \in \mathcal{H}^\lambda([t^* - \delta, t^*])$, $\lambda > \alpha$, and that y attains a maximum at a point $t^* \in [A, B]$. Then, for any $\alpha \in [0, 1]$ and $a \in [t^* - \delta, t^*]$, $a \neq t^*$, we have

$$\left[{}_a\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \leq \frac{y(t^*)}{\Gamma(1-\alpha)(t^* - a)^\alpha} \text{ and } \left[{}_a^C\mathrm{D}_t^\alpha y(t)\right]_{t=t^*} \leq \frac{y(x^*) - y(a)}{\Gamma(1-\alpha)(t^* - a)^\alpha}. \qquad [2.76]$$

Thus, at the point of maximum, fractional derivatives either satisfy [2.76] or do not exist.

It could be easily shown that for a minimum of a function, the inequalities in [2.76] become.

2.2.2. *Taylor theorem for fractional derivatives*

The mean value theorem for the Riemann–Liouville fractional derivative reads as follows.

PROPOSITION 2.20.– [TRU 99] Let $\alpha \in [0,1]$ and suppose that $f \in C([a,b])$, such that ${}_a\mathrm{D}_t^\alpha f \in C([a,b])$. Then

$$f(t) = (t-a)^{\alpha-1}\left[(t-a)^{1-\alpha} f(t)\right]_{t=a^+} + \left[{}_a\mathrm{D}_t^\alpha f(t)\right]_{t=\xi} \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}, t \in (a,b], \qquad [2.77]$$

with $a \leq \xi \leq b$.

The generalization of the Taylor formula for the Riemann–Liouville fractional derivative has several different forms. To state the formula, we need the following definition.

DEFINITION 2.5.– *A function* $f : [a,b] \to \mathbb{R}$ *is said to be* α*-continuous,* $0 \leq \alpha \leq 1$, *at* t_0 *if there exists* $\lambda \in [0, 1-\alpha)$ *such that* $g(t) = |t-t_0|^\lambda f(t)$ *is continuous at* t_0.

Function f is α-continuous in $[a,b]$ if it is α-continuous for every $t \in [a,b]$. Let $C_\alpha = \{f \mid [a,b] \to \mathbb{R},\ f \text{ is } \alpha\text{-continuous}\}$. Note that $C_1([a,b]) = C([a,b])$. Let ${}_a\mathrm{I}_b^\alpha(a,b) = \{f \mid [a,b] \to \mathbb{R},\ {}_a\mathrm{I}_t^n f(t) \text{ exists and it is finite for all } t \in [a,b]\}$. A function f is singular of order α at $t = t^*$ if $\lim_{t\to t^*} \frac{f(t)}{(t-t^*)^{\alpha-1}} = k < \infty$ and $k \neq 0$. Finally, we use ${}_a\mathrm{D}_t^{j\alpha}$ to denote the application of ${}_a\mathrm{D}_t^\alpha$ j-times, i.e. ${}_a\mathrm{D}_t^{j\alpha} = \underbrace{{}_a\mathrm{D}_t^\alpha \cdots {}_a\mathrm{D}_t^\alpha}_{j\text{-times}}$.

PROPOSITION 2.21.– [TRU 99] Let $\alpha \in [0,1]$, $n \in \mathbb{N}$. Let f be a continuous function in $(a,b]$ satisfying the following conditions:

i) ${}_a\mathrm{D}_t^{j\alpha} f \in C([a,b])$ and ${}_a\mathrm{D}_t^{j\alpha} f \in {}_a\mathrm{I}_b^\alpha(a,b)$, for all $j = 1, \ldots, n$;

ii) ${}_a\mathrm{D}_t^{(n+1)\alpha} f$ is continuous in $[a,b]$;

iii) if $\alpha < \frac{1}{2}$, then for each $j \in \mathbb{N}$, $1 \leq j \leq n$, such that $(j+1)\alpha < 1$, ${}_a\mathrm{D}_t^{(j+1)\alpha} f$ is γ-continuous at $t = a$ for some γ, $1-(j+1)\alpha \leq \gamma \leq 1$, or it is singular of order α at $t = a$.

Then, for $t \in (a,b]$,

$$f(t) = \sum_{j=0}^{n} \frac{c_j}{\Gamma((j+1)\alpha)}(t-a)^{(j+1)\alpha-1} + \frac{\left[{}_a\mathrm{D}_t^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}, \quad a \leq \xi \leq b, \qquad [2.78]$$

where

$$c_j = \Gamma(\alpha)\left[(t-a)^{1-\alpha}_a\, \mathrm{D}_t^{j\alpha} f(t)\right]_{t=a^+}, \quad j = 0, 1, \ldots, n. \qquad [2.79]$$

The Taylor formula for the Caputo derivative is given in the following proposition.

PROPOSITION 2.22.– [ODI 07, p. 289] Suppose that ${}_a^C\mathrm{D}_t^{j\alpha} f \in C\left((a,b]\right)$ for $j = 0, 1, \dots, n+1$, where $0 < \alpha \leq 1$. Then

$$f(t) = \sum_{j=0}^{n} \frac{(t-a)^{j\alpha}}{\Gamma(j\alpha+1)} \left[{}_a^C\mathrm{D}_t^{j\alpha} f(t)\right]_{t=a^+} + \frac{\left[{}_a^C\mathrm{D}_t^{(n+1)\alpha} f(t)\right]_{t=\xi}}{\Gamma((n+1)\alpha+1)} (t-a)^{(n+1)\alpha}, \quad t \in (a,b], \qquad [2.80]$$

with $a \leq \xi \leq b$.

REMARK 2.4.– [ODI 07, p. 288] In the special case for the Caputo derivative, the corresponding result is stated as follows. Suppose that $f \in C\left([a,b]\right)$ and ${}_a^C\mathrm{D}_t^{\alpha} f \in C\left((a,b]\right)$, for $0 < \alpha \leq 1$. Then

$$f(t) = f(a) + \frac{\left[{}_a^C\mathrm{D}_t^{\alpha} f(t)\right]_{t=\xi}}{\Gamma(\alpha+1)} (t-a)^{\alpha}, \quad t \in (a,b], \qquad [2.81]$$

where $a \leq \xi \leq b$.

2.3. Fractional derivatives in distributional setting

Throughout this section, we will assume that functions that appear determine the tempered distributions.

2.3.1. *Definition of the fractional integral and derivative*

We introduce the following definition.

DEFINITION 2.6.– *The convolution operator $f_\alpha *$ in $\mathcal{S}'_+$ ($\check{f}_\alpha *$ in $\mathcal{S}'_-$) is the operator of fractional integration for $\alpha > 0$ and the operator of the left (right) fractional differentiation for $\alpha < 0$*

$$\begin{aligned}
{}_D\mathrm{I}_t^{\alpha} u &= f_\alpha * u, \quad \alpha > 0, \\
{}_D\mathrm{D}_t^{\alpha} u &= f_{-\alpha} * u = \frac{\mathrm{d}^m}{\mathrm{d}t^m} f_{m-\alpha} * u = f_{m-\alpha} * \frac{\mathrm{d}^m}{\mathrm{d}t^m} u = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[f_{m-\alpha} * u\right], \quad \alpha > 0, \\
&\qquad m \in \mathbb{N}, \\
{}_D\check{\mathrm{D}}_t^{\alpha} u &= \check{f}_{-\alpha} * u = (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m} f_{m-\alpha} * u = (-1)^m f_{m-\alpha} * \frac{\mathrm{d}^m}{\mathrm{d}t^m} u, \quad \alpha > 0, \ m \in \mathbb{N},
\end{aligned} \qquad [2.82]$$

where f_α is given by [1.6] and $\check{f}_\alpha(t) = f_\alpha(-t)$.

Operator ${}_D\mathrm{I}_t^\alpha$ coincides with the operator of fractional derivation ${}_D\mathrm{D}_t^\alpha$ for $-\alpha \in \mathbb{N}$ and it is the operator of fractional integration for $\alpha \in \mathbb{N}$.

We have that the Laplace transforms of the distributional fractional integral and derivative are

$$\mathcal{L}\left[{}_D\mathrm{I}_t^\alpha u\left(t\right)\right]\left(s\right) = \frac{1}{s^\alpha}\tilde{u}(s) \text{ and } \mathcal{L}\left[{}_D\mathrm{D}_t^\alpha u\left(t\right)\right]\left(s\right) = s^\alpha \tilde{u}\left(s\right), \quad \operatorname{Re} s > 0.$$

We derive, in the following proposition, a connection between the Caputo fractional derivative of a function u belonging to $AC_{loc}^m\left([0,\infty)\right)$ and the distributional fractional derivative of a distribution u_{reg} belonging to $\mathcal{S}'_+$. Also, we derive the connection between the corresponding Laplace transforms. Recall that notation u_{reg} means that we consider u as a distribution, i.e. regular distribution u_{reg} is determined by u.

PROPOSITION 2.23.– [ATA 09d] Let $\alpha \in \left(m-1, m\right]$, $m \in \mathbb{N}$, $u \in AC_{loc}^m\left([0,\infty)\right)$ and put

$$u_{reg}\left(t\right) = u\left(t\right) H\left(t\right), \quad t \in \mathbb{R}. \qquad [2.83]$$

i) Then

$${}_D\mathrm{D}_t^\alpha u_{reg}\left(t\right) = {}_0^C\mathrm{D}_t^\alpha u\left(t\right) + \sum_{j=0}^{m-1} \frac{\mathrm{d}^{m-1-j}}{\mathrm{d}t^{m-1-j}} f_{m-\alpha}\left(t\right) \frac{\mathrm{d}^j}{\mathrm{d}t^j} u\left(0\right), \quad t > 0, \quad [2.84]$$

where ${}_0^C\mathrm{D}_t^\alpha$ is defined by [2.45].

ii) Also

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^\alpha u\left(t\right)\right]\left(s\right) = \mathcal{L}\left[{}_D\mathrm{D}_t^\alpha u_{reg}\left(t\right)\right]\left(s\right) - \sum_{j=0}^{m-1} s^{\alpha-1-j} \frac{\mathrm{d}^j}{\mathrm{d}t^j} u\left(0\right), \quad s \in \mathbb{C}_+. \quad [2.85]$$

Using the notation

$$\mathcal{L}\left[{}_D\mathrm{D}_t^\alpha u_{reg}\left(t\right)\right]\left(s\right) = s^\alpha \mathcal{L}\left[u_{reg}\left(t\right)\right]\left(s\right) = s^\alpha \tilde{u}\left(s\right), \quad s \in \mathbb{C}_+,$$

[2.85] can be written as

$$\mathcal{L}\left[{}_0^C\mathrm{D}_t^\alpha u\left(t\right)\right]\left(s\right) = s^\alpha \tilde{u}\left(s\right) - \sum_{j=0}^{m-1} s^{\alpha-1-j} \frac{\mathrm{d}^j}{\mathrm{d}t^j} u\left(0\right).$$

2.3.2. *Dependence of fractional derivative on order*

Recall that ${}_0\mathrm{D}_t^\alpha u = {}_D\mathrm{D}_t^\alpha u$ if $u \in AC^m\left([a,b]\right), m-1 \leq \alpha < m, m \in \mathbb{N}_0$.

We examine the mapping

$$\alpha \mapsto {}_D\mathrm{D}_t^\alpha u, \quad \alpha \in (-\infty, \infty),$$

for given $u \in L^1_{loc}\left(\mathbb{R}\right)$, such that $u\left(t\right) = 0$ for $t < 0$, i.e. $u \in L^1_{loc}\left([0,\infty)\right)$.

PROPOSITION 2.24.– [ATA 07a] Let $u \in L^1_{loc}\left(\mathbb{R}\right),\ u\left(t\right) = 0,\ t < 0$, so that it determines a tempered distribution. Then, $\alpha \mapsto {}_D\mathrm{D}_t^\alpha u$ is a smooth mapping from $(-\infty,\infty)$ to $\mathcal{S}'_+$. Also, for every $\alpha \in \mathbb{R}$ with $k > \alpha$,

$$\frac{\partial}{\partial \alpha}{}_D\mathrm{D}_t^\alpha u\left(t\right) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(f_k * u\right)\left(\alpha, t\right), \quad t \in \mathbb{R}, \qquad [2.86]$$

where the derivatives are understood in the sense of the tempered distributions,

$$f_k\left(\alpha, t\right) = \frac{t^{k-\alpha-1}}{\Gamma\left(k-\alpha\right)}\left[\psi\left(k-\alpha\right) - \ln\left(t\right)\right], \quad \alpha \in (-\infty, k),\ t > 0, \qquad [2.87]$$

$f_k(\alpha,t) = 0, t \leq 0$, and $\psi\left(x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma\left(x\right),\ x > 0$, is the Euler function.

Note that a locally integrable function f determines a tempered distribution in $\mathcal{S}'_+$ if it is polynomially bounded on $\mathbb{R}$ as $t \to \infty$. Thus, in sections 2.3.2 and 2.3.3 we will assume that f and its derivatives are polynomially bounded on $\mathbb{R}$ as $t \to \infty$.

PROPOSITION 2.25.– [ATA 07a] Let $u \in L^1_{loc}\left(\mathbb{R}\right),\ u(t) = 0,\ t < 0$, so that it determines a tempered distribution. Then

$$\left[\frac{\partial}{\partial \alpha}{}_D\mathrm{D}_t^\alpha u\left(t\right)\right]_{\alpha=0} = -\left(c + \ln t\right)u\left(t\right) + \int_0^t \frac{u\left(t\right) - u\left(t-\tau\right)}{\tau}\mathrm{d}\tau, \quad t > 0,$$

where $c = 0.5772$ is Euler's constant.

Note that for an analytic function, it holds that

$$\left[\frac{\partial}{\partial \alpha}{}_D\mathrm{D}_t^\alpha u\left(t\right)\right]_{\alpha=0} = -\left(c + \ln t\right)u\left(t\right) + \sum_{n=0}^{\infty}\frac{\left(-1\right)^k u^{(k)}\left(t\right)}{k\Gamma\left(k+1\right)}, \quad t > 0,$$

given in [WES 03, p. 112].

REMARK 2.5.– [ATA 07a] Proposition 2.25 allows the following representation of fractional derivative ${}_D\mathrm{D}_t^\alpha u$ for α small enough:

$$\begin{aligned} {}_D\mathrm{D}_t^\alpha u(t) &= u(t) + \alpha \left[\frac{\partial}{\partial \alpha} {}_D\mathrm{D}_t^\alpha u(t)\right]_{\alpha=0} + o(\alpha) \\ &= u(t) + \alpha \left[-(c+\ln t)\, u(t) + \int_0^t \frac{u(t) - u(t-\tau)}{\tau} \mathrm{d}\tau\right] + o(\alpha), t > 0. \end{aligned} \quad [2.88]$$

A relation similar to [2.88] was used in [DIE 02a] for the study of fractional differential equations through the change of the order of integration.

PROPOSITION 2.26.– [ATA 07a] Let $\alpha \in \mathbb{R}$, $u \in L^1_{loc}(\mathbb{R})$, $u(t) = 0$, $t < 0$, so that it determines a tempered distribution. Then

$$\mathcal{L}\left[\frac{\partial}{\partial \alpha} {}_D\mathrm{D}_t^\alpha u(t)\right](s) = s^\alpha \tilde{u}(s) \ln s, \quad \operatorname{Re} s > 0. \quad [2.89]$$

2.3.3. *Distributed-order fractional derivative*

Let u be an element of $\mathcal{S}'_+$. Then, it is proved in [ATA 09b] that the mappings

$$\alpha \mapsto {}_D\mathrm{D}_t^\alpha u : \mathbb{R} \mapsto \mathcal{S}'_+ \quad \text{and} \quad \alpha \mapsto \langle {}_D\mathrm{D}_t^\alpha u(t), \varphi(t)\rangle : \mathbb{R} \mapsto \mathbb{R} \quad [2.90]$$

are smooth (see proposition 2.24). We define the distributed-order fractional derivative by the use of the distributional fractional derivative.

DEFINITION 2.7.– *[ATA 09b] Let* $\phi \in \mathcal{E}'$, $\operatorname{supp}\phi \subset [0,2]$ *and* $u \in \mathcal{S}'_+$. *Then, the distributed-order fractional derivative of* u

$${}_D\mathrm{D}_\phi u(\cdot) = \int_{\operatorname{supp}\phi} \phi(\alpha)\, {}_D\mathrm{D}_t^\alpha u(\cdot)\, \mathrm{d}\alpha \quad [2.91]$$

is defined as element of $\mathcal{S}'_+$ *by*

$$\left\langle \int_{\operatorname{supp}\phi} \phi(\alpha)\, {}_D\mathrm{D}_t^\alpha u(t)\, \mathrm{d}\alpha, \varphi(t) \right\rangle = \langle \phi(\alpha), \langle {}_D\mathrm{D}_t^\alpha u(t), \varphi(t)\rangle\rangle, \quad \varphi \in \mathcal{S}. \quad [2.92]$$

Let $u \in AC^m_{loc}([0,\infty))$, $\alpha \in [0,m]$. Recall that the Caputo fractional derivative ${}_0^C\mathrm{D}_t^\alpha$ is defined on intervals $\alpha \in (j-1, j]$, $j \in \{1, \ldots, m\}$ and

$$\lim_{\alpha \to (j-1)+0} {}_0^C\mathrm{D}_t^\alpha u(t) = \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}} u(t) - \frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}} u(0), \quad t > 0.$$

Moreover, $\lim_{\alpha\to j-0}{}_0^C\mathrm{D}_t^\alpha u(t) = \frac{\mathrm{d}^j}{\mathrm{d}t^j}u(t)$, $t>0$. Thus, $[0,m]\ni\alpha\mapsto{}_0^C\mathrm{D}_t^\alpha u$ is continuous in intervals $\alpha\in(j-1,j)$, $j\in\{1,\ldots,m\}$, left continuous at j, $j\in\{1,\ldots,m\}$ and it has jumps that appear in the limit from the right at points $j-1$, $j\in\{1,\ldots,m\}$. For fixed $\alpha\in[0,m]$, the function $[0,\infty)\ni t\mapsto{}_0^C\mathrm{D}_t^\alpha u(t)$ is locally integrable on $[0,\infty)$.

For the sake of the next proposition, we introduce the following definition. Recall that we have assumed that function u, equal to zero on $(-\infty,0)$, has classical derivatives that are polynomially bounded as $t\to\infty$.

DEFINITION 2.8.– *[ATA 09d] Let $u\in AC_{loc}^2([0,\infty))$.*

i) Let $\alpha\mapsto\phi(\alpha)$ be continuous in $[0,2]$. Then, we define the distributed-order fractional derivative as

$$\mathrm{D}_\phi u(t) = \int_0^2 \phi(\alpha)\,{}_0^C\mathrm{D}_t^\alpha u(t)\,\mathrm{d}\alpha,\quad t>0.$$

ii) Let $\alpha=\{\alpha_j\}_{j\in\{0,\ldots,k\}}$, $\alpha_j\in[0,2]$, $j\in\{0,\ldots,k\}$. Then, we define the distributed-order fractional derivative as

$$\mathrm{D}_\phi u(t) = \sum_{j=0}^{k} a_j\,{}_0^C\mathrm{D}_t^{\alpha_j} u(t),\quad t>0.$$

iii) If ϕ is a continuous function in $[\mu,\eta]\subset[0,2]$ and $\phi(\alpha)=0$, $\alpha\in[0,2]\setminus[\mu,\eta]$, then we define the distributed-order fractional derivative as

$$\mathrm{D}_\phi u(\cdot,t) = \int_\mu^\eta \phi(\alpha)\,{}_0^C\mathrm{D}_t^\alpha u(\cdot,t)\,\mathrm{d}\alpha. \qquad [2.93]$$

Let us derive connections between the distributed-order fractional derivative of $u\in AC_{loc}^2([0,\infty))$ and the corresponding distribution $u_{reg}\in\mathcal{S}'_+$ (in the sense of [2.83]) in cases that are analyzed above.

PROPOSITION 2.27.– [ATA 09d]

i) If ϕ belongs to $C([0,2])$ and $u\in AC_{loc}^2([0,\infty))$, then

$$\begin{aligned}{}_D\mathrm{D}_\phi u_{reg}(t) &= \mathrm{D}_\phi u(t) + u(0)\int_0^2\phi(\alpha)\,f_{1-\alpha}(t)\,\mathrm{d}\alpha\\ &\quad+\frac{\mathrm{d}}{\mathrm{d}t}u(0)\int_1^2\phi(\alpha)\,f_{2-\alpha}(t)\,\mathrm{d}\alpha.\end{aligned} \qquad [2.94]$$

ii) Let $\phi = \sum\limits_{j=0}^{k} a_j \delta\left(\cdot - \alpha_j\right), \operatorname{supp}\phi \subset [0,2], a_j \in \mathbb{R}_+, 0 \leq \alpha_k \leq \alpha_j \leq \alpha_0 \leq 2,$ $\phi \in \mathcal{E}'$. Let $l \leq k$ be chosen so that $\alpha_l > 1$ and $\alpha_{l+1} \leq 1$. Then

$$\begin{aligned} {}_D\mathrm{D}_\phi u_{reg}\left(t\right) &= \mathrm{D}_\phi u\left(t\right) + u\left(0\right)\left(\sum_{j=l+1}^{k} a_j f_{1-\alpha_j}\left(t\right)\right. \\ &\left. + \sum_{j=0}^{l} a_j \frac{\mathrm{d}}{\mathrm{d}t} f_{2-\alpha_j}\left(t\right)\right) + \frac{\mathrm{d}}{\mathrm{d}t} u\left(0\right) \sum_{j=0}^{l} a_j f_{2-\alpha_j}\left(t\right). \end{aligned} \quad [2.95]$$

iii) Both cases will be summarized by the use of [2.92] as

$$\begin{aligned} {}_D\mathrm{D}_\phi u_{reg}\left(t\right) &= \mathrm{D}_\phi u\left(t\right) + u\left(0\right)\left(\int_{\alpha\in[0,1]} \phi\left(\alpha\right) f_{1-\alpha}\left(t\right) \mathrm{d}\alpha\right. \\ &\left. + \int_{\alpha\in(1,2]} \phi\left(\alpha\right) \frac{\mathrm{d}}{\mathrm{d}t} f_{2-\alpha}\left(t\right) \mathrm{d}\alpha\right) \\ &+ \frac{\mathrm{d}}{\mathrm{d}t} u\left(0\right) \int_{\alpha\in(1,2]} \phi\left(\alpha\right) f_{2-\alpha}\left(t\right) \mathrm{d}\alpha. \end{aligned} \quad [2.96]$$

REMARK 2.6.– We use the order of points $0 \leq \alpha_k \leq \alpha_j \leq \alpha_0 \leq 2$ because we will consider two cases separately. The first case is when $\alpha_j \leq 1, j \in \{0, \ldots, k\}$. The second case is when some of the α_j are in $(1, 2]$. So, this notation is helpful from this point of view.

REMARK 2.7.– If $\operatorname{supp}\phi \subset [0,1]$ and u belongs to $AC^1_{loc}\left([0,\infty)\right)$, then [2.96] reduces to

$${}_D\mathrm{D}_\phi u_{reg}\left(t\right) = \mathrm{D}_\phi u\left(t\right) + u\left(0\right) \int_{\alpha\in[0,1]} \phi\left(\alpha\right) f_{1-\alpha}\left(t\right) \mathrm{d}\alpha. \quad [2.97]$$

In the next proposition, we apply the Laplace transform to ${}_D\mathrm{D}_\phi u$, $u \in \mathcal{S}'_+$.

PROPOSITION 2.28.– [ATA 09b] Let $\phi \in \mathcal{E}'$, $\operatorname{supp}\phi \subset [0,2]$ and $u \in \mathcal{S}'_+$. Then:

i) $u \mapsto {}_D\mathrm{D}_\phi u$ is linear and continuous mapping from $\mathcal{S}'_+$ to $\mathcal{S}'_+$.

ii)

$$\mathcal{L}\left[{}_D\mathrm{D}_\phi u\right]\left(s\right) = \left\langle \phi\left(\alpha\right), s^\alpha \tilde{u}\left(s\right)\right\rangle, \quad s \in \mathbb{C}_+. \quad [2.98]$$

iii) Let $\phi \in C([\mu, \eta])$, $[\mu, \eta] \subset [0, 2]$ and $\phi(\alpha) = 0$, $\alpha \in [0, 2] \setminus [\mu, \eta]$. Then

$$\mathcal{L}\left[{}_{D}\mathrm{D}_{\phi}u\right](s) = \tilde{u}(s) \int_{\mu}^{\eta} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha, \quad s \in \mathbb{C}_{+}. \qquad [2.99]$$

Further, we use proposition 2.28 in order to derive the Laplace transform of a function $u \in AC_{loc}^{2}([0, \infty))$, using the connection between its distributed-order fractional derivative and distributed-order fractional derivative of the corresponding distribution $u_{reg} \in \mathcal{S}'_{+}$ (in the sense of [2.83]). Again, we have two cases.

PROPOSITION 2.29.– [ATA 09d]

i) Let $\phi \in C([0, 2])$ and $u \in AC_{loc}^{2}([0, \infty))$, then $(s \in \mathbb{C}_{+})$

$$\begin{aligned}\mathcal{L}\left[\mathrm{D}_{\phi}u(t)\right](s) &= \tilde{u}(s) \int_{0}^{2} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha - u(0)\, \frac{1}{s} \int_{0}^{2} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha - \frac{\mathrm{d}}{\mathrm{d}t} u(0)\, \frac{1}{s^{2}} \\ &\quad \times \int_{1}^{2} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha. \qquad [2.100]\end{aligned}$$

ii) Let $\phi = \sum_{j=0}^{k} a_{j} \delta(\cdot - \alpha_{j})$, $\operatorname{supp}\phi \subset [0, 2]$, $a_{j} \in \mathbb{R}_{+}$, $0 \leq \alpha_{k} \leq \alpha_{j} \leq \alpha_{0} \leq 2$, $\alpha_{l} > 1$ and $\alpha_{l+1} \leq 1$. Let $u \in AC_{loc}^{2}([0, \infty))$, then $(s \in \mathbb{C}_{+})$

$$\mathcal{L}\left[\mathrm{D}_{\phi}u(t)\right](s) = \tilde{u}(s) \sum_{j=0}^{k} a_{j} s^{\alpha_{j}} - u(0)\, \frac{1}{s} \sum_{j=0}^{k} a_{j} s^{\alpha_{j}} - \frac{\mathrm{d}}{\mathrm{d}t} u(0)\, \frac{1}{s^{2}} \sum_{j=0}^{l} a_{j} s^{\alpha_{j}}. \qquad [2.101]$$

iii) Both [2.100] and [2.101] are summarized by

$$\begin{aligned}\mathcal{L}\left[\mathrm{D}_{\phi}u(t)\right](s) &= \tilde{u}(s) \int_{\alpha \in [0,2]} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha - u(0)\, \frac{1}{s} \int_{\alpha \in [0,2]} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t} u(0)\, \frac{1}{s^{2}} \int_{\alpha \in (1,2]} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha, \quad s \in \mathbb{C}_{+}. \qquad [2.102]\end{aligned}$$

REMARK 2.8.– If $\operatorname{supp}\phi \subset [0, 1]$ and u belongs to $AC_{loc}^{1}([0, \infty))$, then equation [2.102] reduces to

$$\mathcal{L}\left[\mathrm{D}_{\phi}u(t)\right](s) = \tilde{u}(s) \int_{\alpha \in [0,1]} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha - u(0)\, \frac{1}{s} \int_{\alpha \in [0,1]} \phi(\alpha)\, s^{\alpha} \mathrm{d}\alpha, \quad s \in \mathbb{C}_{+}. \qquad [2.103]$$

PART 2

Mechanical Systems

Chapter 3

Waves in Viscoelastic Materials of Fractional-Order Type

The (one-dimensional) wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2 \frac{\partial^2}{\partial x^2}u(x,t), \quad c = \sqrt{\frac{E}{\rho}}, \qquad [3.1]$$

describes the motion, i.e. the change of displacement u during time t at point x, of the elastic medium. The elastic medium, which can be finite, semi-finite and infinite, is described by Young's modulus of elasticity E and mass density ρ. Parameter c is interpreted as the wave speed.

The wave equation [3.1] is obtained from a system of three equations (see [ATA 00]). The first one is the equation of motion of the one-dimensional deformable body whose axis coincides with the x-axis:

$$\frac{\partial}{\partial x}\sigma\left(x,t\right) + f\left(x,t\right) = \rho \frac{\partial^2}{\partial t^2}u\left(x,t\right), \qquad [3.2]$$

where σ is the stress, f is the body force, or the force coming from the surroundings, ρ is the (constant) mass density of the material and u is the displacement. Note that the body forces are assumed to be zero in obtaining the wave equation [3.1]. The second equation is Hooke's law as the constitutive (stress–strain) equation:

$$\sigma\left(x,t\right) = E\varepsilon\left(x,t\right), \qquad [3.3]$$

where E is Young's modulus of elasticity and ε is the strain measure. The third equation is the strain measure, connecting the strain and displacement, which in the case of local deformations reads:

$$\varepsilon(x,t) = \frac{\partial}{\partial x} u(x,t). \tag{3.4}$$

Contrary to the approach presented in Chapter 6 of [ATA 14b], where we generalized the wave equation [3.1] itself, in this chapter we write the wave equation [3.1] as a system and generalize the constitutive equation [3.3] and strain measure [3.4], while leaving the equation of motion [3.2] unchanged.

The constitutive equation reflects the response of the material to the applied stress. It is clear that Hooke's law [3.3], appropriate for the elastic materials, cannot be appropriate for the response description for all materials due to the differences in material properties. The constitutive equation of the time-fractional order may be used in order to model materials exhibiting the (fading) memory effect. The constitutive equation containing distributed-order fractional derivatives

$$\int_0^1 \phi_\sigma(\gamma)\, {}_0\mathrm{D}_t^\gamma \sigma(x,t)\,\mathrm{d}\gamma = E \int_0^1 \phi_\varepsilon(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(x,t)\,\mathrm{d}\gamma, \tag{3.5}$$

where E is the generalized Young's modulus and ϕ_σ, ϕ_ε are constitutive functions (or distributions) representing the material properties of the body, generalizes the constitutive equations of linear viscoelasticity of integer and fractional order and, with the appropriate choice of constitutive distributions, reduces to these equations. Note that [3.5] becomes Hooke's law for the following choice of constitutive distributions: $\phi_\sigma = \phi_\varepsilon = \delta$. We refer to Chapter 3 of [ATA 14b], for different constitutive equations that can be obtained from [3.5], as well as for the thermodynamical restrictions that parameters in these equations have to satisfy.

Apart from the memory effects, constitutive equations may reflect the non-local effects also. The constitutive equations of non-local type were first used for modeling the elastic wave dispersion in crystals, or in heterogeneous materials (see [MAU 99]). Also, non-locality has found applications in modeling scale effects in small-scale structures, such as micro- and nanostructures (see [ELI 12]). We refer to [ERI 83] and point out the Eringen non-local model

$$\sigma(x,t) - l_c \frac{\partial^2}{\partial x^2} \sigma(x,t) = E\varepsilon(x,t). \tag{3.6}$$

This model depends on one length scale l_c which has to be calibrated. The application of non-local model [3.6] has been able to quite accurately reproduce the dispersive wave properties of the Born–Kármán model of lattice dynamics (see [BOR 12]), a specific property that cannot be captured by a local stress–strain model. We refer to [ERI 83, ERI 87, ERI 02] for the calibration of this non-local model with respect to the lattice dynamics model.

Within the framework of the fractional calculus, the Eringen constitutive equation can be generalized by

$$\sigma(x,t) - l_c^\alpha \, \mathcal{E}_x^\alpha \sigma(x,t) = E\varepsilon(x,t), \tag{3.7}$$

where $\alpha \in [1,3)$. Thus, the symmetrized Caputo derivative $\mathcal{E}_x^\alpha$ is used in the form

$$\mathcal{E}_x^\alpha f(x) = \frac{1}{2\Gamma(2-\alpha)} \frac{1}{|x|^{\alpha-1}} * \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x), \tag{3.8}$$

in the case when the order is $\alpha \in [1,2)$ and in the form

$$\mathcal{E}_x^\alpha f(x) = \frac{1}{2\Gamma(3-\alpha)} \frac{\operatorname{sgn}(x)}{|x|^{\alpha-2}} * \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x), \tag{3.9}$$

in the case $\alpha \in [2,3)$. Both [3.8] and [3.9] generalize the second derivative so that the fractional Eringen constitutive equation [3.7] generalizes the Eringen model [3.6], obtained for $\alpha = 2$. We also refer to [CAR 11, CHA 13, DIP 09, DIP 08] for following the approach of introducing non-locality in the constitutive equation within the framework of the fractional calculus.

Description of the non-locality in materials can also be introduced in the strain measure, i.e. the strain-displacement relation, as done in [ATA 09f]. Well-defined strain measure needs to satisfy the condition arising from physics. Namely, the quantity ε is the strain measure if it is invariant under the change of the inertial reference frame. In the one-dimensional case, it means that if we suppose displacement to be

$$u(x,t) = c(t), \quad \text{then} \quad \varepsilon(x,t) = 0, \quad x \in \mathbb{R}, \; t > 0,$$

and vice versa, with c being an arbitrary function of time. Let us define, for $x \in \mathbb{R}$, $t > 0$, a quantity ε by

$$\begin{aligned} \varepsilon(x,t) &= \mathcal{E}_x^\beta u(x,t) = \frac{1}{2}\left({}_{-\infty}^{\;\;C}\mathrm{D}_x^\beta - {}_x^C\mathrm{D}_\infty^\beta\right) u(x,t) \\ &= \frac{1}{2\Gamma(1-\beta)} \frac{1}{|x|^\beta} * \frac{\partial}{\partial x} u(x,t), \end{aligned} \tag{3.10}$$

where $\beta \in (0,1)$. Quantity ε is the strain measure, since by plugging $u\left(x,t\right) = c\left(t\right)$ into [3.10], we have $\varepsilon\left(x,t\right) = 0$ and also, from lemma 3.1, the only solution to $\varepsilon\left(x,t\right) = 0$ is $u\left(x,t\right) = c\left(t\right)$.

We make an analysis in the framework of $\mathcal{S}'$, $\mathcal{S}'_{+}$, or $\mathcal{K}'$, $\mathcal{K}'_{+}$, if it is necessary. If one works with locally integrable functions of exponential growth, then the classical Laplace transform is already enough for the computation. Concerning the convolution, we use the fact that it is well defined if the distributions (functions) are supported by $[0,\infty)$. This is always the case for the time variable t. If one considers the convolution with respect to the spatial variable x, then the analysis should be carefully performed.

Let $u\left(x,t\right)$ be a locally integrable function equal to zero for x out of a compact set K and for $t<0$, i.e. $\operatorname{supp} u \subset K \times [0,\infty)$. We assume this condition in order to avoid problems related to the convolution of two distributions. Actually, some of the assertions to be followed hold with less restrictive conditions, but we will not discuss such cases.

LEMMA 3.1.– Let $\beta \in (0,1)$. Assume that u is such that $\operatorname{supp} u \subset K \times [0,\infty)$. Then, the only solution to

$$\frac{1}{|x|^{\beta}} * \frac{\partial}{\partial x} u(x,t) = 0, \quad x \in \mathbb{R},\ t > 0,$$

is $u\left(x,t\right) = c\left(t\right)$.

PROOF.– We have

$$\frac{1}{|x|^{\beta}} * \frac{\partial}{\partial x} u(x,t) = \frac{\partial}{\partial x}\left(\frac{1}{|x|^{\beta}} * u(x,t)\right) = 0, \quad x \in \mathbb{R},\ t > 0.$$

This means that $\frac{1}{|x|^{\beta}} * u(x,t)$ does not depend on x, i.e. u does not depend on x, thus $u\left(x,t\right) = c\left(t\right)$. ■

The strain measure [3.10] is non-local, since it involves the weighted classical strain $\frac{\partial}{\partial x} u$ integrated on the whole space. Note that the strain measure [3.10], when $\beta = 1$, becomes classical, while when $\beta = 0$, then $\varepsilon = 0$.

REMARK 3.1.–

(i) Fractional strain measure

$$\mathcal{K}^{\alpha}(x,t) = \frac{1}{2}\left({}_0\mathrm{D}_x^{\alpha} - {}_x\mathrm{D}_L^{\alpha}\right) u(x,t),$$

called the symmetric fractional derivative, was used in [KLI 01].

(ii) Fractional strain measure

$$\mathcal{L}^{\alpha}(x,t) = \frac{\partial u(x,t)}{\partial x}\left({}_0\mathrm{D}_x^{\alpha} + {}_x\mathrm{D}_L^{\alpha}\right)\frac{\partial}{\partial x}u(x,t). \quad [3.11]$$

was introduced in the strain energy function in [LAZ 06]. Strain measure [3.11] leads to the possibility of modeling coexistence of phases in solids.

The results presented in this chapter are taken from [ATA 11b, ATA 11d, ATA 11e, ATA 09f, CHA 13, KON 10, KON 11].

3.1. Time-fractional wave equation on unbounded domain

Following [KON 10, KON 11], we study the wave motion in the infinite viscoelastic medium without body forces being present. We assume that the deformations are local. Then, the wave equation is written as the system consisting of the equation of motion [3.2] with $f = 0$, time-fractional constitutive equation [3.5] and local strain measure [3.4]:

$$\frac{\partial}{\partial x}\sigma(x,t) = \rho\frac{\partial^2}{\partial t^2}u(x,t), \quad x \in \mathbb{R},\ t > 0, \quad [3.12]$$

$$\int_0^1 \phi_\sigma(\gamma)\ {}_0\mathrm{D}_t^{\gamma}\sigma(x,t)\,\mathrm{d}\gamma = E\int_0^1 \phi_\varepsilon(\gamma)\ {}_0\mathrm{D}_t^{\gamma}\varepsilon(x,t)\,\mathrm{d}\gamma, \quad x \in \mathbb{R},\ t > 0, \quad [3.13]$$

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x \in \mathbb{R},\ t > 0, \quad [3.14]$$

In sections 3.1.1 and 3.1.2, we will consider time-fractional wave equations in two special cases of the constitutive equation [3.13].

1) The fractional Zener model of the viscoelastic body is obtained from [3.13] by choosing the constitutive distributions in the form

$$\phi_\sigma(\gamma) = \delta(\gamma) + \tau_\sigma\,\delta(\gamma - \alpha), \quad \phi_\varepsilon(\gamma) = \delta(\gamma) + \tau_\varepsilon\,\delta(\gamma - \alpha).$$

Thus, it reads

$$\sigma(x,t) + \tau_{\sigma 0}\mathrm{D}_t^{\alpha}\sigma(x,t) = E\left(\varepsilon(x,t) + \tau_{\varepsilon 0}\mathrm{D}_t^{\alpha}\varepsilon(x,t)\right), \qquad [3.15]$$

where τ_σ and τ_ε are relaxation times satisfying $\tau_\varepsilon > \tau_\sigma > 0$, which follows from the second law of thermodynamics (see [ATA 02b], or [3.34] of [ATA 14b]. The first authors who introduced the fractional Zener model were Caputo and Mainardi [CAP 71a, CAP 71b]. Such a model was also considered in [RAB 48] but without using fractional calculus. A similar problem of stress waves in a viscoelastic medium was investigated in [GON 73] by the inversion formula of the Laplace transform. For a review of the fractional Zener-type wave equation, see [NÄS 13].

2) The linear fractional model of the solid-like viscoelastic body is obtained from [3.13] by choosing the constitutive distributions in the form

$$\phi_\sigma(\gamma) = \sum_{k=0}^{n} a_k\,\delta(\gamma - \alpha_k), \quad \phi_\varepsilon(\gamma) = \sum_{k=0}^{n} b_k\,\delta(\gamma - \alpha_k).$$

Thus, it reads

$$\sum_{k=0}^{n} a_k\,{}_0\mathrm{D}_t^{\alpha_k}\sigma(x,t) = E\sum_{k=0}^{n} b_k\,{}_0\mathrm{D}_t^{\alpha_k}\varepsilon(x,t), \qquad [3.16]$$

where $0 \le \alpha_0 < \alpha_1 < \ldots < \alpha_n \le 1$, and $a_k, b_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$, are positive constants which satisfy

$$\frac{a_0}{b_0} \ge \frac{a_1}{b_1} \ge \ldots \ge \frac{a_n}{b_n} \ge 0,$$

as shown in [ATA 11b]. Similar generalization of the constitutive equation was done in [ROS 01b] in the case of bounded domain.

We subject system [3.12]–[3.14] to initial conditions

$$u(x,0) = u_0(x), \ \ \frac{\partial}{\partial t}u(x,0) = v_0(x), \ \ \sigma(x,0) = 0, \ \ \varepsilon(x,0) = 0, \ \ x \in \mathbb{R}, \qquad [3.17]$$

where $x \in \mathbb{R}$, and u_0 and v_0 are the initial displacement and velocity, respectively. Note that the initial stress and strain do not exist. We also supply boundary conditions

$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad \lim_{x\to\pm\infty} \sigma(x,t) = 0, \ \ t \ge 0. \qquad [3.18]$$

We introduce dimensionless quantities in [3.12]–[3.14], [3.17] and [3.18] as

$$\begin{aligned}\bar{x} &= \frac{x}{X^*}, \ \bar{t} = \frac{t}{T^*}, \ \bar{u} = \frac{u}{X^*}, \ \bar{\sigma} = \frac{\sigma}{E}, \\ \bar{\phi}_\sigma(\gamma) &= \frac{\phi_\sigma(\gamma)}{(T^*)^\gamma}, \ \bar{\phi}_\varepsilon(\gamma) = \frac{\phi_\varepsilon(\gamma)}{(T^*)^\gamma}, \ \bar{u}_0 = \frac{u_0}{X^*}, \ \bar{v}_0 = v_0 \frac{T^*}{X^*}.\end{aligned}$$

Note that ε is already a dimensionless quantity. Quantities X^* and T^* are constants (measured in meters and seconds, respectively) which satisfy $\left(\frac{X^*}{T^*}\right)^2 \frac{\rho}{E} = 1$.

1) In the case of the fractional Zener model [3.15], we chose $T^* = (\tau_\varepsilon)^{\frac{1}{\alpha}}$ and $X^* = (\tau_\varepsilon)^{\frac{1}{\alpha}} \sqrt{\frac{E}{\rho}}$. This implies $\bar{\tau} = \frac{\tau_\sigma}{\tau_\varepsilon}$. Obviously, condition $0 < \tau_\sigma < \tau_\varepsilon$ implies $0 < \tau < 1$.

2) In the case of the general linear model [3.16], we chose $T^* = (a_0)^{\frac{1}{\alpha_0}}$ and $X^* = (a_0)^{\frac{1}{\alpha_0}} \sqrt{\frac{E}{\rho}}$. This implies $\bar{a}_0 = 1$, $\bar{a}_k = \frac{a_k}{a_0}$, $\bar{b}_k = \frac{b_k}{a_0}$, $k = 0, 1, \ldots, n$.

In order to simplify the notation in sections 3.1.1 and 3.1.2, the bar over the dimensionless quantities will be dropped.

3.1.1. *Time-fractional Zener wave equation*

In this section, the aim is to study the wave equation for viscoelastic infinite media described by the fractional Zener model. Following [KON 10], we write systems [3.12], [3.14] and [3.15], with initial [3.17] and boundary [3.18] data in the dimensionless form:

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t), \quad x \in \mathbb{R}, \ t > 0, \tag{3.19}$$

$$\sigma(x,t) + \tau {}_0\mathrm{D}_t^\alpha \sigma(x,t) = \varepsilon(x,t) + {}_0\mathrm{D}_t^\alpha \varepsilon(x,t), \quad x \in \mathbb{R}, \ t > 0, \tag{3.20}$$

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x \in \mathbb{R}, \ t > 0, \tag{3.21}$$

and study the existence and uniqueness of solutions. Systems [3.19]– [3.21] are subject to initial and boundary conditions

$$u(x,0) = u_0(x), \ \frac{\partial}{\partial t}u(x,0) = v_0(x), \ \sigma(x,0) = 0, \ \varepsilon(x,0) = 0, \tag{3.22}$$

$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad \lim_{x\to\pm\infty} \sigma(x,t) = 0. \tag{3.23}$$

As we will show, systems [3.19]–[3.21] can be reduced to

$$\frac{\partial^2}{\partial t^2}u(x,t) = L(t)\frac{\partial^2}{\partial x^2}u(x,t), \quad x \in \mathbb{R}, \ t > 0, \tag{3.24}$$

which will be called the wave equation for the fractional Zener-type viscoelastic media or, for short, the fractional Zener wave equation (FZWE). Here, L denotes a linear operator (of convolution type) acting on $\mathcal{S}'$, whose explicit form is given by

$$L(t) = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t, \quad t > 0.$$

In fact, equation [3.24] will be the subject of our consideration. Systems [3.19]–[3.21], and in particular [3.24], generalize the classical wave equation.

The appropriate setting for studying systems [3.19]–[3.21] will be a distributional one. In fact, we will look for a fundamental solution to the generalized Cauchy problem for FZWE [3.24] in $\mathcal{S}'(\mathbb{R}^2)$, with support in $\mathbb{R} \times [0,\infty)$. This suffices to obtain a solution, since [3.19]–[3.21] and [3.24] are equivalent. Indeed, by applying the Laplace transform, with respect to the time variable t, to the second equation in [3.19]–[3.21], we obtain

$$(1+\tau s^\alpha)\tilde{\sigma}(x,s) = (1+s^\alpha)\tilde{\varepsilon}(x,s), \quad x \in \mathbb{R}, \ \operatorname{Re} s > 0.$$

According to [OPA 02], $\mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right]$ is a well-defined element in $\mathcal{S}'_+$; hence

$$\sigma(t) = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t \varepsilon(t), \quad t > 0. \tag{3.25}$$

Inserting ε from [3.21] into [3.25] and then σ into [3.19], we obtain

$$\frac{\partial^2}{\partial t^2}u(x,t) = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t \frac{\partial^2}{\partial x^2}u(x,t), \quad x \in \mathbb{R}, \ t > 0. \tag{3.26}$$

Setting $L(t) = \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t$ we come to [3.24]. Therefore, we have proved that [3.19]–[3.21] and [3.24] are equivalent. Note that equation [3.26] is of the form $Pu = 0$, with

$$P = \frac{\partial^2}{\partial t^2} - \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t \frac{\partial^2}{\partial x^2}. \tag{3.27}$$

We find a solution to the generalized Cauchy problem (see section 1.4, to [3.24]), i.e.

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t \frac{\partial^2}{\partial x^2}u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t),\\ &\qquad x\in\mathbb{R},\ t>0,\end{aligned} \qquad [3.28]$$

or equivalently

$$Pu(x,t) = u_0(x)\delta'(t) + v_0(x)\delta(t),\ \ x\in\mathbb{R},\ t>0,$$

where P is given by [3.27].

We state the main theorem.

THEOREM 3.1.– Let $u_0, v_0 \in \mathcal{E}'$. Then there exists the unique solution $u\in\mathcal{S}'(\mathbb{R}^2)$ to [3.28] given by

$$\begin{aligned}u(x,t) &= S(x,t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)),\ \ |x| < \frac{t}{\sqrt{\tau}},\ t>0, && [3.29]\\ u(x,t) &= 0,\ \ |x| > \frac{t}{\sqrt{\tau}},\ t>0,\end{aligned}$$

where

$$\begin{aligned}S(x,t) &= \frac{1}{2} + \frac{1}{4\pi\mathrm{i}}\int_0^\infty \left(\sqrt{\frac{1+\tau q^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^\alpha\mathrm{e}^{\mathrm{i}\alpha\pi}}}\mathrm{e}^{|x|q\sqrt{\frac{1+\tau q^\alpha\mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^\alpha\mathrm{e}^{\mathrm{i}\alpha\pi}}}}\right.\\ &\quad \left. -\sqrt{\frac{1+\tau q^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^\alpha\mathrm{e}^{-\mathrm{i}\alpha\pi}}}\mathrm{e}^{|x|q\sqrt{\frac{1+\tau q^\alpha\mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^\alpha\mathrm{e}^{-\mathrm{i}\alpha\pi}}}}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q, \end{aligned} \qquad [3.30]$$

is the fundamental solution of operator P, $S\in\mathcal{S}'(\mathbb{R}^2)$, with support in the cone $|x| < \frac{t}{\sqrt{\tau}}$, $t>0$.

We will need the following lemma.

LEMMA 3.2.– Let $f\in\mathcal{E}'$. Then the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}v\left(x\right) - \omega v\left(x\right) = -f\left(x\right),\ \ x\in\mathbb{R}, \qquad [3.31]$$

has a solution $v \in \mathcal{S}'$ for all $\omega \in \mathbb{C} \setminus (-\infty, 0]$, which is of the form

$$v(x) = \frac{e^{-\sqrt{\omega}|x|}}{2\sqrt{\omega}} * f(x), \quad x \in \mathbb{R},$$

where $\sqrt{\omega}$ represents the main branch of the complex square root.

PROOF OF THEOREM 3.1.– Applying the Laplace transform to [3.28] with respect to t, we obtain:

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\tilde{u}(x,s) - s^2\frac{1+\tau s^\alpha}{1+s^\alpha}\tilde{u}(x,s) &= -\frac{1+\tau s^\alpha}{1+s^\alpha}(su_0(x)+v_0(x)), \\ & \qquad x \in \mathbb{R},\ \operatorname{Re} s > 0. \end{aligned} \qquad [3.32]$$

Set $\omega(s) = s^2\frac{1+\tau s^\alpha}{1+s^\alpha}$ and $f(x,s) = \frac{1+\tau s^\alpha}{1+s^\alpha}(su_0(x)+v_0(x))$, $s \in \mathbb{C}_+$. In order to apply lemma 3.2 to [3.32], we have to show first that $f(\cdot, s) \in \mathcal{E}'$, which follows from assumptions that $u_0, v_0 \in \mathcal{E}'$ and second, that $\omega(s) \in \mathbb{C} \setminus (-\infty, 0]$, for all s, $\operatorname{Re} s > 0$. The latter can be verified in the following way. Arbitrarily take $s = \rho e^{i\varphi}$, $\rho > 0$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$. Then $\omega(s) = \rho^2 e^{2i\varphi}\frac{1+\tau\rho^\alpha e^{i\alpha\varphi}}{1+\rho^\alpha e^{i\alpha\varphi}}$, and after a straightforward calculation we obtain

$$\begin{aligned} \operatorname{Re}\omega(s) &= \frac{\rho^2}{A}\left[\left(1+\rho^\alpha(1+\tau)\cos(\alpha\varphi)+\tau\rho^{2\alpha}\right)\right. \\ &\quad \left.\times\cos(2\varphi)+\rho^\alpha(1-\tau)\sin(\alpha\varphi)\sin(2\varphi)\right], \end{aligned} \qquad [3.33]$$

$$\begin{aligned} \operatorname{Im}\omega(s) &= \frac{\rho^2}{A}\left[\left(1+\rho^\alpha(1+\tau)\cos(\alpha\varphi)+\tau\rho^{2\alpha}\right)\right. \\ &\quad \left.\times\sin(2\varphi)-\rho^\alpha(1-\tau)\sin(\alpha\varphi)\cos(2\varphi)\right], \end{aligned} \qquad [3.34]$$

where $A = (1+\rho^\alpha\cos(\alpha\varphi))^2 + \rho^{2\alpha}\sin^2(\alpha\varphi) > 0$. Suppose that $\omega(s) \in (-\infty, 0]$, for some s, $\operatorname{Re} s > 0$. Then, $\operatorname{Im}\omega(s) = 0$ and [3.34] yields

$$\left(1+\rho^\alpha(1+\tau)\cos(\alpha\varphi)+\tau\rho^{2\alpha}\right) = \rho^\alpha(1-\tau)\sin(\alpha\varphi)\frac{\cos(2\varphi)}{\sin(2\varphi)}. \qquad [3.35]$$

From [3.33], we obtain

$$\operatorname{Re}\omega(s) = \frac{\rho^2}{A}\rho^\alpha(1-\tau)\sin(\alpha\varphi)\sin(2\varphi)\left(\frac{\cos^2(2\varphi)}{\sin^2(2\varphi)}+1\right) \le 0.$$

where we have used [3.35] and the assumption $\omega(s) \in (-\infty, 0]$, for all s, $\operatorname{Re} s > 0$.

Since $\frac{\rho^2}{A}\rho^\alpha(1-\tau)\sin(\alpha\varphi)\left(\frac{\cos^2(2\varphi)}{\sin^2(2\varphi)}+1\right) > 0$, for $\rho > 0$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, it follows that $\sin(2\varphi) < 0$, and therefore $\varphi \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right] \cup \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$. However, [3.35] cannot be satisfied for those φ. (Indeed, for $\varphi \in \left(\frac{\pi}{2}, -\frac{\pi}{4}\right]$, $\cos(\alpha\varphi), \cos(2\varphi), \sin(\alpha\varphi) > 0$, but $\sin(2\varphi) < 0$ and similarly for $\varphi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$). Hence, $\omega(s) \in \mathbb{C} \setminus (-\infty, 0]$, for all s, $\operatorname{Re} s > 0$.

Now we can apply lemma 3.2 to obtain a solution of [3.32]:

$$
\begin{aligned}
\tilde{u}(x,s) &= \frac{\mathrm{e}^{-\sqrt{\omega(s)}|x|}}{2\sqrt{\omega(s)}} *_x \frac{\omega(s)}{s^2}(su_0(x)+v_0(x)) \\
&= \frac{\sqrt{\omega(s)}\mathrm{e}^{-\sqrt{\omega(s)}|x|}}{2s^2} *_x (su_0(x)+v_0(x)), \quad x \in \mathbb{R}, \ \operatorname{Re} s > 0. \qquad [3.36]
\end{aligned}
$$

We need to calculate the inverse Laplace transform of [3.36] of [ATA 14b] in order to obtain solution u to [3.24]. For that purpose, set

$$
\begin{aligned}
\tilde{S}(x,s) &= \frac{\sqrt{\omega(s)}\mathrm{e}^{-\sqrt{\omega(s)}|x|}}{2s^2} = \frac{1}{2s}\sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}\mathrm{e}^{-|x|s\sqrt{\frac{1+\tau s^\alpha}{1+s^\alpha}}}, \\
&\qquad x \in \mathbb{R}, \ \operatorname{Re} s > 0, \qquad [3.37] \\
S(x,t) &= \mathcal{L}^{-1}\left[\tilde{S}(x,s)\right](x,t), \quad x \in \mathbb{R}, \ t > 0.
\end{aligned}
$$

Note that S is a fundamental solution of P. Then the solution u is given by [3.29]. If S is rewritten as $S = \left(\frac{\mathrm{d}}{\mathrm{d}t}+1\right)\mathcal{L}^{-1}\left[\frac{\tilde{S}(x,s)}{s+1}\right]$, one can prove that $\mathcal{L}^{-1}\left[\frac{\tilde{S}(x,s)}{s+1}\right](x,t)$ is a continuous function for $t > 0$ and zero for $t < 0$.

We will determine S by considering two domains: $|x| < \frac{t}{\sqrt{\tau}}$ and $|x| > \frac{t}{\sqrt{\tau}}$, $t > 0$.

Function $\tilde{S}$, given by [3.37], has branch points at $s = 0$ and $s = \infty$ and has no singularities. In the domain $|x| < \frac{t}{\sqrt{\tau}}$, $t > 0$, $S = \mathcal{L}^{-1}\left[\tilde{S}\right]$ can be evaluated by using the Cauchy integral formula

$$
\oint_\Gamma \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s = 0, \quad x \in \mathbb{R}, \ t > 0, \qquad [3.38]
$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\varepsilon \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$ is a contour given in Figure 3.1. More precisely, for arbitrarily chosen $R > 0$, $0 < \varepsilon < R$ and $a > 0$, Γ is defined by

$$\begin{array}{ll} \Gamma_1 : \ s = R\mathrm{e}^{\mathrm{i}\varphi}, \varphi_0 < \varphi < \pi; & \Gamma_3 : \ s = q\mathrm{e}^{-\mathrm{i}\pi}, \varepsilon < q < R; \\ \Gamma_2 : \ s = q\mathrm{e}^{\mathrm{i}\pi}, -R < -q < -\varepsilon; & \Gamma_4 : \ s = R\mathrm{e}^{\mathrm{i}\varphi}, -\pi < \varphi < -\varphi_0; \\ \Gamma_\varepsilon : \ s = \varepsilon\mathrm{e}^{\mathrm{i}\varphi}, -\pi < \varphi < \pi; & \gamma_0 : \ s = a(1 + \mathrm{i}\tan\varphi), -\varphi_0 < \varphi < \varphi_0, \end{array} \quad [3.39]$$

where $\varphi_0 = \arccos \frac{a}{R}$. Note that $\lim_{R\to\infty} \varphi_0 = \frac{\pi}{2}$. In the limit when $R \to \infty$, the integral along contour Γ_1 reads ($x \in \mathbb{R},\ t > 0$)

$$\begin{aligned} &\lim_{R\to\infty} \int_{\Gamma_1} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \\ &\quad = \frac{1}{2} \lim_{R\to\infty} \int_{\varphi_0}^{\pi} \sqrt{\frac{1+\tau R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}} \mathrm{e}^{-|x|R\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{1+\tau R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}}+Rt\mathrm{e}^{\mathrm{i}\varphi}} \mathrm{i}\,\mathrm{d}\varphi. \end{aligned} \quad [3.40]$$

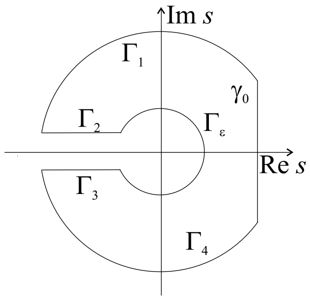

Figure 3.1. *Integration contour* Γ

Evaluating the absolute value of $\int_{\Gamma_1} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s$, we obtain ($x \in \mathbb{R},\ t > 0$)

$$\begin{aligned} &\lim_{R\to\infty} \left| \int_{\Gamma_1} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \right| \\ &\quad \leq \frac{1}{2} \lim_{R\to\infty} \int_{\varphi_0}^{\pi} \left| \sqrt{\frac{1+\tau R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}} \right| \left| \mathrm{e}^{-|x|R\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{1+\tau R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}}} \right| \mathrm{e}^{Rt\cos\varphi}\mathrm{d}\varphi. \end{aligned}$$

In the limit when $R \to \infty$, the expression $\sqrt{\frac{1+\tau R^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}}}$ tends to $\sqrt{\tau}$ and therefore

$$\begin{aligned}&\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s\right|\\&\quad\leq \frac{\sqrt{\tau}}{2}\lim_{R\to\infty}\int_{\varphi_0}^{\pi}\mathrm{e}^{R\cos\varphi(t-|x|\sqrt{\tau})}\mathrm{d}\varphi = 0,\ \text{if}\ |x| < \frac{t}{\sqrt{\tau}},\ t>0,\end{aligned}$$

since $\cos\varphi < 0$ for $\varphi \in \left(\frac{\pi}{2},\pi\right)$. A similar argument is valid for the integral along Γ_4.

In the limit when $\varepsilon \to 0$, the integral along Γ_ε is given by the formula similar to [3.40] and it is calculated as ($x\in\mathbb{R},\ t>0$)

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s\\&\quad=\frac{1}{2}\lim_{\varepsilon\to 0}\int_{\pi}^{-\pi}\sqrt{\frac{1+\tau\varepsilon^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+\varepsilon^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}}}\mathrm{e}^{-|x|\varepsilon\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{1+\tau\varepsilon^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+\varepsilon^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}}}+\varepsilon t\mathrm{e}^{\mathrm{i}\varphi}}\mathrm{i}\,\mathrm{d}\varphi=-\mathrm{i}\pi.\end{aligned}$$

Integrals along contours Γ_2, Γ_3 and γ_0, in the limit when $R\to\infty$, $\varepsilon\to 0$, give ($|x| < \frac{t}{\sqrt{\tau}}$, $t>0$)

$$\begin{aligned}\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_2}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s &= -\frac{1}{2}\int_0^{\infty}\sqrt{\frac{1+\tau q^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi}}}\mathrm{e}^{-q\left(t-|x|\sqrt{\frac{1+\tau q^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi}}{1+q^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi}}}\right)}\frac{\mathrm{d}q}{q},\\ \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_3}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s &= \frac{1}{2}\int_0^{\infty}\sqrt{\frac{1+\tau q^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi}}}\mathrm{e}^{-q\left(t-|x|\sqrt{\frac{1+\tau q^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+q^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi}}}\right)}\frac{\mathrm{d}q}{q},\\ \lim_{R\to\infty}\int_{\gamma_0}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i}S(x,t).\end{aligned}$$

Now, by the Cauchy integral formula [3.38], we obtain S as in [3.30].

In the domain $|x| > \frac{t}{\sqrt{\tau}}$, $t>0$, $S=\mathcal{L}^{-1}\left[\tilde{S}\right]$ can be evaluated by using the Cauchy integral formula

$$\oint_{\bar{\Gamma}}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s = 0,\quad x\in\mathbb{R},\ t>0, \qquad [3.41]$$

where $\bar{\Gamma} = \bar{\Gamma}_1 \cup \gamma_0$ is the contour parameterized by

$$\begin{aligned} \bar{\Gamma}_1 : \quad & s = a + R\mathrm{e}^{\mathrm{i}\varphi}, -\frac{\pi}{2} < \varphi < \frac{\pi}{2}; \\ \gamma_0 : \quad & s = a(1 + \mathrm{i}\tan\varphi), -\varphi_0 < \varphi < \varphi_0, \end{aligned} \tag{3.42}$$

where $\varphi_0 = \arccos \frac{a}{R}$. In the limit when $R \to \infty$, the integral along contour $\bar{\Gamma}_1$ reads

$$\begin{aligned} & \lim_{R\to\infty} \left| \int_{\Gamma_1} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \right| \\ & = \frac{1}{2} \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sqrt{\frac{1+\tau R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}} \right| \left| \mathrm{e}^{-|x|R\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{1+\tau R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}{1+R^\alpha \mathrm{e}^{\mathrm{i}\alpha\varphi}}} + Rt\mathrm{e}^{\mathrm{i}\varphi}} \right| \mathrm{d}\varphi \\ & \leq \frac{\sqrt{\tau}}{2} \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{e}^{-R\cos\varphi(|x|\sqrt{\tau}-t)} \mathrm{d}\varphi = 0, \text{ if } |x| > \frac{t}{\sqrt{\tau}}, \ t > 0, \end{aligned}$$

since $\cos\varphi > 0$ for $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, by [3.41], we have $S(x,t) = 0$ and therefore $u(x,t) = 0$, for $|x| > \frac{t}{\sqrt{\tau}}$, $t > 0$. ■

As a consequence of the results that we proved, we have the following corollary.

COROLLARY 3.1.– Let u be given by [3.29]. Then

$$(u,\varepsilon,\sigma)(x,t) = \left(u(x,t), \frac{\partial}{\partial x}u(x,t), \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t \frac{\partial}{\partial x}u(x,t) \right),$$

where u, ε, σ belong to $\mathcal{S}'(\mathbb{R}^2)$, with support in $\mathbb{R} \times [0,\infty)$, and (u,ε,σ) is the unique solution to systems [3.19]–[3.21].

In a similar way, we can also consider the non-homogeneous case.

REMARK 3.2.– Let the equation of motion [3.19] be replaced by

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t) + f(x,t), \quad x \in \mathbb{R}, \ t > 0,$$

where $f \in \mathcal{S}'(\mathbb{R}^2)$, with support in $\mathbb{R} \times [0,\infty)$. This is a case of a rod under the influence of body forces. Then, the solution of the generalized Cauchy problem

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u(x,t) &= \mathcal{L}^{-1}\left[\frac{1+s^\alpha}{1+\tau s^\alpha}\right](t) *_t \frac{\partial^2}{\partial x^2}u(x,t) + f(x,t)\\ &\quad +u_0(x)\delta'(t) + v_0(x)\delta(t),\ \ x\in\mathbb{R},\ t>0,\end{aligned}$$

is

$$u(x,t) = S(x,t) *_{x,t} (f(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t)),\ \ x\in\mathbb{R},\ t>0,$$

where S is as it was in [3.30].

REMARK 3.3.– The dimensionless condition $|x| < \frac{t}{\sqrt{\tau}}$, or $|x| < \sqrt{\frac{\rho}{E}}\sqrt{\frac{\tau_\sigma}{\tau_\varepsilon}}t$ in the dimensional form, can be physically interpreted as the finite wave speed property. Namely, at moment t, the wave caused by the initial disturbance $u_0(x) = \delta(x)$, $x \in \mathbb{R}$, has reached the point which is at distance $|x|$ from the origin, where the initial disturbance was applied. Thus, constant $c_Z = \sqrt{\frac{\rho}{E}}\sqrt{\frac{\tau_\sigma}{\tau_\varepsilon}}$ is the wave speed in the fractional viscoelastic media of Zener type.

3.1.2. *Time-fractional general linear wave equation*

This section is based on the results presented in [KON 11]. The aim is to study the wave equation for viscoelastic infinite media described by the linear fractional model. More precisely, we will study the dimensionless form of systems [3.12], [3.14] and [3.16], with initial [3.17] and boundary [3.18] data:

$$\frac{\partial}{\partial x}\sigma(x,t) = \frac{\partial^2}{\partial t^2}u(x,t),\ \ x\in\mathbb{R},\ t>0, \qquad [3.43]$$

$$\sum_{k=0}^{n} a_k\, {}_0\mathrm{D}_t^{\alpha_k}\sigma(x,t) = \sum_{k=0}^{n} b_k\, {}_0\mathrm{D}_t^{\alpha_k}\varepsilon(x,t),\ \ x\in\mathbb{R},\ t>0, \qquad [3.44]$$

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t),\ \ x\in\mathbb{R},\ t>0, \qquad [3.45]$$

where $0 \le \alpha_0 < \alpha_1 < \ldots < \alpha_n \le 1$, and $a_k, b_k \in \mathbb{R}$, $k = 0,1,\ldots,n$, are positive constants which satisfy

$$\sum_{k,j=\{0,..,n\},k\ge j} \xi^{\alpha_k+\alpha_j}(a_k b_j - a_j b_k)\sin\left(\frac{\pi(\alpha_j-\alpha_k)}{2}\right) > 0,\ \ \xi\in\mathbb{R}. \qquad [3.46]$$

The fact that derivatives on both sides of the constitutive equation [3.44] are of the same order reflects the well-known mechanical principle of equipresence. Condition [3.46] on coefficients a_k and b_k comes as a result of the second law of thermodynamics.

We will look for solutions to systems [3.43]–[3.45] which satisfy the initial and boundary conditions:

$$u(x,0)=u_0(x),\ \ \frac{\partial}{\partial t}u(x,0)=v_0(x),\ \ \sigma(x,0)=0,\ \ \varepsilon(x,0)=0,\ \ x\in\mathbb{R}, \quad [3.47]$$

$$\lim_{x\to\pm\infty}u(x,t)=0,\quad \lim_{x\to\pm\infty}\sigma(x,t)=0,\ \ t>0. \quad [3.48]$$

It will be shown that, under certain assumptions on coefficients in the constitutive equation, systems [3.43]–[3.45] can be reduced to the time-fractional general linear wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t)=L(t)\frac{\partial^2}{\partial x^2}u(x,t),\ \ x\in\mathbb{R},\ t>0. \quad [3.49]$$

Here, L denotes the linear operator (of convolution type) acting on $\mathcal{S}'$, whose explicit form will be derived in the following.

Systems [3.43]–[3.45], and in particular [3.49], generalize the classical wave equation, as well as the fractional Zener wave equation, studied in section 3.1.1, which is obtained by setting $\alpha_0=0$, $\alpha_1=\alpha$, $a_0=b_0=b_1=1$ and $a_1=\tau$.

For later use, we state the following simple lemma.

LEMMA 3.3.– Let $a_k>0$, $k=0,1,\ldots,n$, and $0\leq\alpha_0<\alpha_1<\ldots<\alpha_n\leq 1$. Then

$$\sum_{k=0}^{n}a_k s^{\alpha_k}\neq 0,\ \ \operatorname{Re}s>0. \quad [3.50]$$

PROOF.– If $\operatorname{Re}s>0$, $a_k>0$ and $\alpha_k\in[0,1]$, $k=0,1,\ldots,n$, then $\operatorname{Re}(a_k s^{\alpha_k})>0$. Since the real part of a finite sum of complex numbers with strictly positive real parts is again strictly positive, the claim follows. ■

REMARK 3.4.– It has been proved in [ATA 10c, theorem 3.1] that [3.50] implies the existence of

$$\mathcal{L}^{-1}\left[\left(\sum_{k=0}^{n}a_k s^{\alpha_k}\right)^{-1}\right]\in\mathcal{S}'_+,$$

a fundamental solution to the equation of the form

$$\sum_{k=0}^{n} a_k \, {}_0\mathrm{D}_t^{\alpha_k} u = f \in \mathcal{S}'_+,$$

which is a locally integrable function.

Now, equivalence of systems [3.43]–[3.45] and equation [3.49] can be shown as follows. We first apply the Laplace transform with respect to t to [3.44] and obtain

$$\tilde{\sigma}(x,s) \sum_{k=0}^{n} a_k s^{\alpha_k} = \tilde{\varepsilon}(x,s) \sum_{k=0}^{n} b_k s^{\alpha_k}, \quad x \in \mathbb{R}, \ \mathrm{Re}\, s > 0.$$

Then, since $a_k > 0$, $k = 0, 1, \ldots, n$, $0 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n \leq 1$, and

$$\mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right] = \mathcal{L}^{-1}\left[\frac{1}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right] *_t \mathcal{L}^{-1}\left[\sum_{k=0}^{n} b_k s^{\alpha_k}\right],$$

it follows from lemma 3.3 and remark 3.4 that there exists the operator

$$L(t) = \mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right](t) *_t, \quad t > 0,$$

where $\mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right]$ is an element in $\mathcal{S}'_+$. Therefore, we have

$$\sigma(t) = \mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right](t) *_t \varepsilon(t), \quad t > 0. \qquad [3.51]$$

Now we replace ε from [3.45] into [3.51] and insert such σ into [3.43]. As a result, we obtain exactly [3.49], and hence equivalence of systems [3.43]–[3.45] with equation [3.49] follows. Defining the operator

$$P = \frac{\partial^2}{\partial t^2} - L(t)\frac{\partial^2}{\partial x^2}, \qquad [3.52]$$

we can write [3.49] in the form $Pu = 0$.

In the following, we will study equation [3.49] in the distributional setting. In fact, we will seek a fundamental solution to the generalized Cauchy problem for [3.52] in $\mathcal{S}'(\mathbb{R}^2)$, with support in $\mathbb{R} \times [0, \infty)$. This will also lead to solutions of problems [3.43]–[3.48], due to the equivalence of [3.43]– [3.45] and [3.49]. For the setup of the generalized Cauchy problem, see section 1.4.

The generalized Cauchy problem for the operator P given in [3.52] takes the form $Pu(x,t) = u_0(x)\delta'(t) + v_0(x)\delta(t)$, $x \in \mathbb{R}$, $t > 0$, or equivalently

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x,t) &= \mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right](t) *_t \frac{\partial^2}{\partial x^2} u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \\ & \quad x \in \mathbb{R},\ t > 0. \end{aligned} \tag{3.53}$$

Clearly, initial conditions [3.47] are included into the generalized Cauchy problem.

The following result is crucial for the proof of the main theorem.

LEMMA 3.4.– Let $a_k, b_k > 0$ and $0 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n \leq 1$ satisfy condition [3.46]. Then for all $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$

$$\omega(s) = s^2 \frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}} \in \mathbb{C} \setminus (-\infty, 0].$$

PROOF.– Writing $s = r\mathrm{e}^{\mathrm{i}\varphi}$ with $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain

$$\omega(s) = r^2(\cos(2\varphi) + \mathrm{i}\sin(2\varphi)) \frac{\sum_{k=0}^{n} a_k r^{\alpha_k}\cos(\alpha_k\varphi) + \mathrm{i}\sum_{k=0}^{n} a_k r^{\alpha_k}\sin(\alpha_k\varphi)}{\sum_{k=0}^{n} b_k r^{\alpha_k}\cos(\alpha_k\varphi) + \mathrm{i}\sum_{k=0}^{n} b_k r^{\alpha_k}\sin(\alpha_k\varphi)}.$$

Set $A = \sum_{k=0}^{n} a_k r^{\alpha_k}\cos(\alpha_k\varphi)$, $B = \sum_{k=0}^{n} a_k r^{\alpha_k}\sin(\alpha_k\varphi)$, $C = \sum_{k=0}^{n} b_k r^{\alpha_k}\cos(\alpha_k\varphi)$, $D = \sum_{k=0}^{n} b_k r^{\alpha_k}\sin(\alpha_k\varphi)$. Then we have

$$\begin{aligned} \omega(s) &= \frac{r^2}{C^2 + D^2}\left((AC + BD)\cos(2\varphi) + (AD - BC)\sin(2\varphi)\right. \\ &\quad \left. + \mathrm{i}((AC + BD)\sin(2\varphi) - (AD - BC)\cos(2\varphi))\right). \end{aligned}$$

Consider first the case $\varphi \in [0, \frac{\pi}{2})$. Then $\sin(\varphi(\alpha_j - \alpha_k))$ equals $\mu_{k,j} \sin \frac{\pi(\alpha_j - \alpha_k)}{2}$, for some $0 \leq \mu_{k,j} < 1$ and setting ξ to be $r \cdot \sqrt[\alpha_k+\alpha_j]{\min_{k\geq j} \mu_{k,j}}$, we obtain

$$\begin{aligned} AD - BC = & \sum_{k,j=\{0,..,n\},k\geq j} r^{\alpha_k+\alpha_j}(a_k b_j - a_j b_k)\sin(\varphi(\alpha_j - \alpha_k)) \\ = & \sum_{k,j=\{0,..,n\},k\geq j} r^{\alpha_k+\alpha_j}(a_k b_j - a_j b_k)\mu_{k,j}\sin\left(\frac{\pi(\alpha_j - \alpha_k)}{2}\right) \\ \geq & \sum_{k,j=\{0,..,n\},k\geq j} \xi^{\alpha_k+\alpha_j}(a_k b_j - a_j b_k)\sin\left(\frac{\pi(\alpha_j - \alpha_k)}{2}\right) > 0. \end{aligned}$$

Now, if $\varphi \in [0, \frac{\pi}{4}]$, then $\sin(2\varphi), \cos(2\varphi) \geq 0$, $A, B, C, D > 0$ and $AD - BC > 0$; hence, $\mathrm{Re}\,\omega(s) > 0$. In the case $\varphi \in (\frac{\pi}{4}, \frac{\pi}{2})$, we again have $A, B, C, D > 0$, $AD - BC > 0$ and $\sin(2\varphi) > 0$, but $\cos(2\varphi) < 0$; thus, $\mathrm{Im}\,\omega(s)$ cannot be zero.

In a similar way, we obtain the result for $\varphi \in (-\frac{\pi}{2}, 0)$. In this case, $AD - BC < 0$, which in fact implies that $\omega(s) \notin (-\infty, 0]$. ■

We show the existence and uniqueness of a solution to the generalized Cauchy problem [3.53].

THEOREM 3.2.– Let $u_0, v_0 \in \mathcal{E}'$. Then there exists the unique solution $u \in \mathcal{S}'(\mathbb{R}^2)$, to [3.53] given by

$$\begin{aligned} u(x,t) &= S(x,t) *_{x,t} (u_0(x)\delta'(t) + v_0(x)\delta(t)), \;\; |x| < t\sqrt{\frac{b_n}{a_n}}, \; t > 0, \quad [3.54] \\ u(x,t) &= 0, \;\; |x| > t\sqrt{\frac{b_n}{a_n}}, \; t > 0, \end{aligned}$$

where $S \in \mathcal{S}'(\mathbb{R}^2)$, with support in the cone $|x| < t\sqrt{\frac{b_n}{a_n}}$, $t > 0$, is the fundamental solution of the operator P.

Moreover, the fundamental solution can be calculated as

$$\begin{aligned} S(x,t) = & \frac{1}{2}\sqrt{\frac{a_0}{b_0}} + \frac{1}{4\pi\mathrm{i}}\int_0^\infty \left(\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha}}{\sum_{k=0}^n b_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha}}}\,\mathrm{e}^{|x|q\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha}}{\sum_{k=0}^n b_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha}}}}\right. \\ & \left. - \sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{\sum_{k=0}^n b_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha}}}\,\mathrm{e}^{|x|q\sqrt{\frac{\sum_{k=0}^n a_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha}}{\sum_{k=0}^n b_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha}}}}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q \quad [3.55] \end{aligned}$$

and it has support in the cone $|x| < t\sqrt{\frac{b_n}{a_n}}$, $t > 0$.

PROOF.– We apply the Laplace transform, with respect to t, to [3.53] and obtain

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x,s) - s^2 \frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}} \tilde{u}(x,s) = -\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}} (s u_0(x) + v_0(x)),$$
$$x \in \mathbb{R},\ \mathrm{Re}\, s > 0. \qquad [3.56]$$

If we set $\omega(s) = s^2 \frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}$ and $f(x,s) = \frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}} (s u_0(x) + v_0(x))$, $x \in \mathbb{R}$, $\mathrm{Re}\, s > 0$, we obtain equation of the form [3.31], i.e. $\frac{\mathrm{d}^2}{\mathrm{d}x^2} v(x) - \omega v(x) = -f(x)$, $x \in \mathbb{R}$. According to lemma 3.2, it has a solution $v(x) = \frac{\mathrm{e}^{-\sqrt{\omega}|x|}}{2\sqrt{\omega}} * f(x)$, $x \in \mathbb{R}$, provided that $f(\cdot, s) \in \mathcal{E}'$ and $\omega(s) \in \mathbb{C} \setminus (-\infty, 0]$ for $\mathrm{Re}\, s > 0$. The first condition is clearly satisfied, since by assumption $u_0, v_0 \in \mathcal{E}'$, while the second condition is proved in lemma 3.4. Therefore, the corresponding solution to [3.56] is

$$\begin{aligned}\tilde{u}(x,s) &= \frac{\mathrm{e}^{-\sqrt{\omega(s)}|x|}}{2\sqrt{\omega(s)}} *_x \frac{\omega(s)}{s^2} (s u_0(x) + v_0(x)) \\ &= \frac{\sqrt{\omega(s)}\mathrm{e}^{-\sqrt{\omega(s)}|x|}}{2s^2} *_x (s u_0(x) + v_0(x)),\quad x \in \mathbb{R},\ \mathrm{Re}\, s > 0. \qquad [3.57]\end{aligned}$$

Solution u to [3.49] is obtained by applying the inverse Laplace transform to [3.57]. Set

$$\begin{aligned}\tilde{S}(x,s) &= \frac{\sqrt{\omega(s)}\mathrm{e}^{-\sqrt{\omega(s)}|x|}}{2s^2} = \frac{1}{2s}\sqrt{\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}}\, \mathrm{e}^{-|x| s \sqrt{\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}}}, \\ &\qquad x \in \mathbb{R},\ \mathrm{Re}\, s > 0, \qquad [3.58] \\ S(x,t) &= \mathcal{L}^{-1}\left[\tilde{S}(x,s)\right](t),\quad x \in \mathbb{R},\ t > 0.\end{aligned}$$

We prove the existence of the inverse Laplace transform of $\tilde{S}$ as follows. The existence of the inverse Laplace transform of $\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}$ follows from lemma 3.3 and remark 3.4. Hence, $\left|\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}\right| \leq C\frac{(1+|s|^a)}{(\mathrm{Re}\, s)^b}$, for some constants $a, b \in \mathbb{R}$,

$C > 0$, and then

$$\begin{aligned}\left|\frac{1}{2s}\sqrt{\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}}\mathrm{e}^{-|x|s\sqrt{\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}}}\right| &\leq C_1\frac{(1+|s|^p)}{(\mathrm{Re}\, s)^q}\mathrm{e}^{-|x|\,\mathrm{Re}\left(s\sqrt{\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}}\right)} \\ &\leq C_1\frac{(1+|s|^p)}{(\mathrm{Re}\, s)^q},\end{aligned}$$

for some $p, q \in \mathbb{R}$ and $C_1 > 0$. The last inequality follows from the fact that $\mathrm{Re}\left(s\sqrt{\frac{\sum_{k=0}^{n} a_k s^{\alpha_k}}{\sum_{k=0}^{n} b_k s^{\alpha_k}}}\right) > 0$, which is a result of lemma 3.4. Therefore, S is a fundamental solution of the operator P, and it is a well-defined element in $\mathcal{S}'_+$. Hence, the solution u to [3.53] is given by [3.54], and the first part of the theorem is proved.

In order to calculate the fundamental solution, we first note that the multiform function $\tilde{S}$ given by [3.58] has branch points at $s = 0$ and $s = \infty$ and has no singularities. For $|x| < t\sqrt{\frac{b_n}{a_n}},\ t > 0$, the inverse Laplace transform of $\tilde{S}$ can be evaluated by using the Cauchy integral theorem:

$$\oint_{\Gamma} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s = 0, \quad x \in \mathbb{R},\ t > 0.$$

Here, $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\varepsilon \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$ is a contour given in Figure 3.1, parameterized by [3.39]. In the limit when $R \to \infty$, the integral along contour Γ_1 reads ($x \in \mathbb{R},\ t > 0$)

$$\begin{aligned}&\lim_{R\to\infty}\int_{\Gamma_1}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \\ &\quad = \frac{1}{2}\lim_{R\to\infty}\int_{\varphi_0}^{\pi}\sqrt{\frac{\sum_{k=0}^{n} a_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}{\sum_{k=0}^{n} b_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}}\mathrm{e}^{-|x|R\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{\sum_{k=0}^{n} a_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}{\sum_{k=0}^{n} b_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}}+Rt\mathrm{e}^{\mathrm{i}\varphi}}\,\mathrm{i}\,\mathrm{d}\varphi.\end{aligned}$$

Thus

$$\begin{aligned}
&\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s\right| \\
&\leq \frac{1}{2}\lim_{R\to\infty}\int_{\varphi_0}^{\pi}\left|\sqrt{\frac{\sum_{k=0}^{n}a_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}{\sum_{k=0}^{n}b_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}}\right|\left|\mathrm{e}^{-|x|R\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{\sum_{k=0}^{n}a_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}{\sum_{k=0}^{n}b_k R^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}}}\right|\mathrm{e}^{Rt\cos\varphi}\mathrm{d}\varphi \\
&\leq \frac{1}{2}\sqrt{\frac{a_n}{b_n}}\lim_{R\to\infty}\int_{\varphi_0}^{\pi}\mathrm{e}^{R\cos\varphi\left(t-|x|\sqrt{\frac{a_n}{b_n}}\right)}\mathrm{d}\varphi = 0,\ \text{if}\ |x|<t\sqrt{\frac{b_n}{a_n}},\ t>0,
\end{aligned}$$

where the equality follows from the fact that $\cos\varphi < 0$ for $\varphi \in (\frac{\pi}{2},\pi)$. A similar argument is valid for the integral along Γ_4. Integral along Γ_ε is calculated as

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \\
&\quad = \frac{1}{2}\lim_{\varepsilon\to 0}\int_{\pi}^{-\pi}\sqrt{\frac{\sum_{k=0}^{n}a_k\varepsilon^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}{\sum_{k=0}^{n}b_k\varepsilon^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}}\mathrm{e}^{-|x|\varepsilon\mathrm{e}^{\mathrm{i}\varphi}\sqrt{\frac{\sum_{k=0}^{n}a_k\varepsilon^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}{\sum_{k=0}^{n}b_k\varepsilon^{\alpha_k}\mathrm{e}^{\mathrm{i}\alpha_k\varphi}}}+\varepsilon t\mathrm{e}^{\mathrm{i}\varphi}}\mathrm{i}\,\mathrm{d}\varphi \\
&\quad = -\mathrm{i}\sqrt{\frac{a_0}{b_0}}\pi,\quad x\in\mathbb{R},\ t>0.
\end{aligned}$$

In the limit when $R\to\infty$ and $\varepsilon\to 0$, integrals along contours Γ_2, Γ_3 and γ_0 read ($|x|<t\sqrt{\frac{b_n}{a_n}}$, $t>0$)

$$\begin{aligned}
&\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_2}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \\
&\quad = -\frac{1}{2}\int_0^\infty\sqrt{\frac{\sum_{k=0}^{n}a_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^{n}b_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha_k}}}\mathrm{e}^{-q\left(t-|x|\sqrt{\frac{\sum_{k=0}^{n}a_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^{n}b_k q^{\alpha_k}\mathrm{e}^{\mathrm{i}\pi\alpha_k}}}\right)}\frac{\mathrm{d}q}{q}, \\
&\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_3}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s \\
&\quad = \frac{1}{2}\int_0^\infty\sqrt{\frac{\sum_{k=0}^{n}a_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^{n}b_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha_k}}}\mathrm{e}^{-q\left(t-|x|\sqrt{\frac{\sum_{k=0}^{n}a_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha_k}}{\sum_{k=0}^{n}b_k q^{\alpha_k}\mathrm{e}^{-\mathrm{i}\pi\alpha_k}}}\right)}\frac{\mathrm{d}q}{q}, \\
&\lim_{R\to\infty}\int_{\gamma_0}\tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}S(x,t).
\end{aligned}$$

Thus, it follows from the Cauchy integral theorem that S takes the form as it did in [3.55].

For $|x| > t\sqrt{\frac{b_n}{a_n}},\ t > 0$, the inverse Laplace transform of $\tilde{S}$ can be evaluated by using the Cauchy integral theorem:

$$\oint_{\bar{\Gamma}} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s = 0, \quad x \in \mathbb{R},\ t > 0,$$

where $\bar{\Gamma} = \bar{\Gamma}_1 \cup \gamma_0$ is the contour parameterized by [3.42]. In the limit when $R \to \infty$, the integral along contour $\bar{\Gamma}_1$ reads

$$\lim_{R\to\infty}\left|\int_{\Gamma_1} \tilde{S}(x,s)\mathrm{e}^{st}\mathrm{d}s\right| \quad \le \quad \frac{1}{2}\sqrt{\frac{a_n}{b_n}}\lim_{R\to\infty}\int_{\varphi_0}^{\pi} \mathrm{e}^{-R\cos\varphi\left(|x|\sqrt{\frac{a_n}{b_n}}-t\right)}\mathrm{d}\varphi = 0,$$
$$\text{if } |x| > t\sqrt{\frac{b_n}{a_n}},\ t > 0,$$

since $\cos\varphi > 0$ for $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus, by the Cauchy integral theorem, we have $S\,(x,t) = 0$ and therefore $u\,(x,t) = 0$, for $|x| > t\sqrt{\frac{b_n}{a_n}}$, $t > 0$. ■

Once the existence and the uniqueness of a solution to the generalized Cauchy problem are proved, one immediately has the following corollary.

COROLLARY 3.2.– Let u be given by [3.54]. Then

$$(u,\varepsilon,\sigma)(x,t) = \left(u(x,t), \frac{\partial u}{\partial x}(x,t), \mathcal{L}^{-1}\left[\frac{\sum_{k=0}^{n} b_k s^{\alpha_k}}{\sum_{k=0}^{n} a_k s^{\alpha_k}}\right](t) *_t \frac{\partial u}{\partial x}(x,t)\right)$$

is the unique solution to systems [3.43]–[3.48] and u,ε,σ belong to $\mathcal{S}'(\mathbb{R}^2)$, with support in $\mathbb{R}\times[0,\infty)$.

REMARK 3.5.–

(i) The non-homogeneous case of a rod under the influence of body forces can be analogously treated (see [KON 10]).

(ii) The constant c in the cone $|x| < c \cdot t$, which is the domain of the fundamental solution S, can be interpreted as the wave speed in viscoelastic media described by linear fractional model.

(iii) For the case of the fractional Zener model, i.e. when we take $\alpha_0 = 0$, $\alpha_1 = \alpha$, $a_0 = b_0 = b_1 = 1$ and $a_1 = \tau$, [3.46] becomes $\tau < 1$. That was also the crucial condition in proving the existence of the fundamental solution in [KON 10].

3.1.3. *Numerical examples*

Let $u_0 = \delta$ and $v_0 = 0$ in solution [3.29] to the Cauchy problem of wave equation for fractional Zener-type viscoelastic media. Note that this type of initial condition describes the sudden, but short-lasting, initial disturbance. Then, the solution reads

$$u(x,t) = \frac{\partial}{\partial t} S(x,t), \quad x \in \mathbb{R}, \; t > 0, \qquad [3.59]$$

i.e. the fundamental solution represents the solution itself.

3.1.3.1. *Case of time-fractional Zener wave equation*

Figure 3.2 presents the plots of $u(x,t)$, given by [3.59], for $x \in (0,3)$, $t \in \{0.5, 1, 1.5\}$, $\alpha = 0.23$ and $\tau = 0.004$.

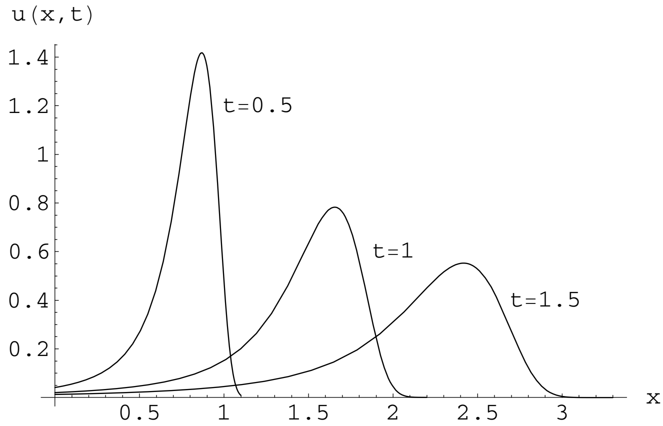

Figure 3.2. *Solution* $u(x,t)$, $x \in (0,3)$, $t \in \{0.5, 1, 1.5\}$

In order to see the effect of a change in the order of a fractional derivative, we present the following figures. Figure 3.3 presents plots of $u(x,t)$, $x \in (0,3)$, $t \in \{0.5, 1, 1.5\}$, given by [3.59], for different values of the parameter α. The plot of u, denoted by the dashed line, corresponds to $\alpha = 0.25$, The dot-dashed line is used for $\alpha = 0.5$, while the solid line denotes the plot for $\alpha = 0.75$. The parameter $\tau = 0.004$ is the same in all three figures.

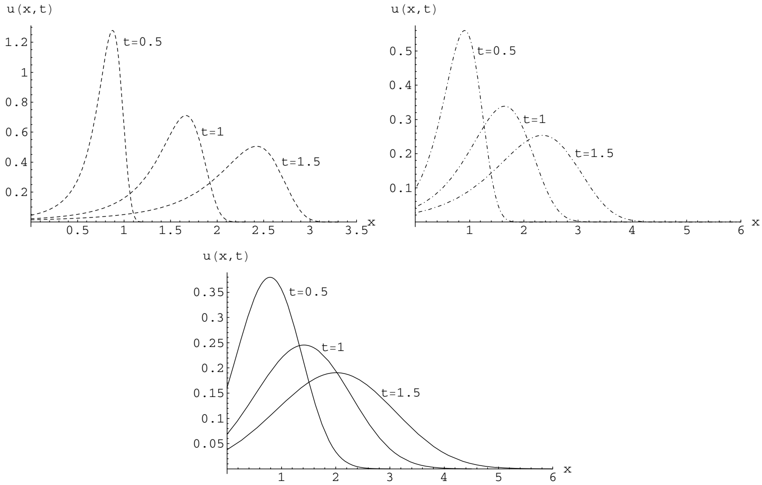

Figure 3.3. *Solution* $u(x,t)$*,* $x \in (0,6)$*,* $t \in \{0.5, 1, 1.5\}$*,* $\alpha \in \{0.25, 0.5, 0.75\}$

We can see from Figure 3.3 that for each value of the order of fractional derivative α, as time increases, the maximum value of u decreases, which is the consequence of the dissipative model of a medium.

Figure 3.4 presents plots of $u(x,t)$, $x \in (0,3)$, $\alpha \in \{0.25, 0.5, 0.75\}$, given by [3.59], for different time moments, i.e. for $t \in \{0.5, 1, 1.5\}$. Parameter τ and line styles are the same as in Figure 3.3. From Figure 3.4, we can see that at the same time instant, as the value of α increases, the maximum value of u decreases, while the width of the maximum increases. This is the consequence of the fact that the constitutive equation of the fractional type in [3.19]–[3.21] describes the medium that tends to be elastic as α tends to 0. Therefore, the fundamental solution [3.59] should tend to the Dirac distribution as $\alpha \to 0$. Also, the medium tends to be more viscous as α in [3.19]–[3.21] tends to 1. Thus, in the case $\alpha \to 1$, the fundamental solution [3.59] should tend to the solution of the wave equation.

Figure 3.5 presents plots of u for both different time moments and values of α. Parameter τ and line styles are the same as in Figure 3.3.

3.1.3.2. *Case of time-fractional general linear wave equation*

Figure 3.6 shows the plot of displacement u given by [3.59]. The numerical example presents u as a function of the coordinate for $t \in \{1, 2, 3, 4\}$ and the following set of parameters: $\alpha_0 = 0.25$, $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $a_0 = 1.25$, $a_1 = 1.1$, $a_2 = 1.2$, $b_0 = 1.4$, $b_1 = 1.3$, $b_2 = 1.5$.

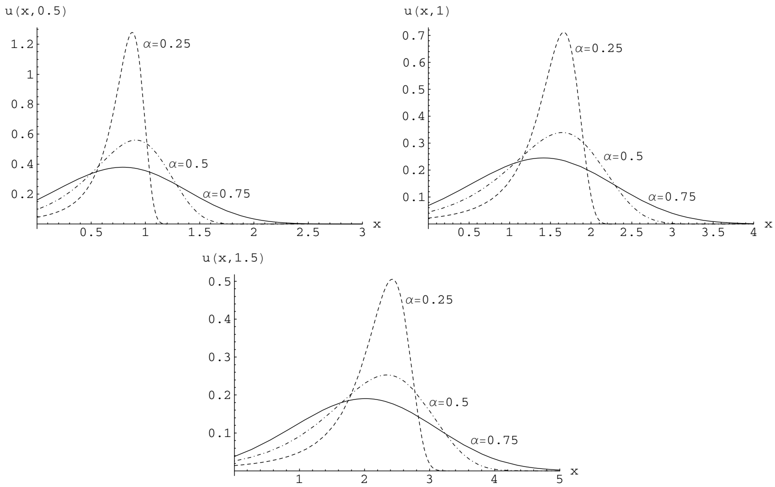

Figure 3.4. *Solution* $u(x,t)$, $x \in (0,5)$, $t \in \{0.5, 1, 1.5\}$, $\alpha \in \{0.25, 0.5, 0.75\}$

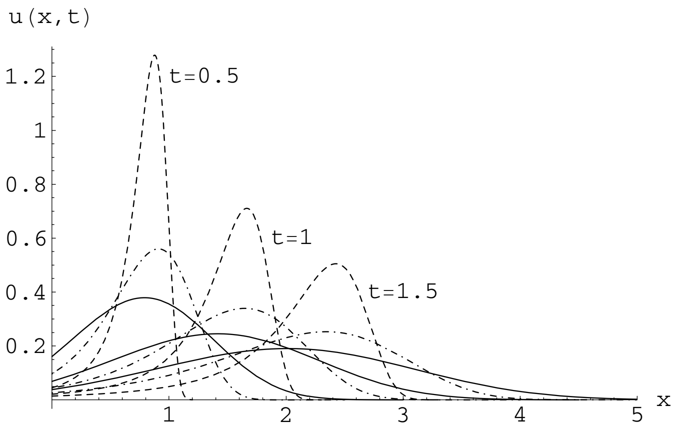

Figure 3.5. *Solution* $u(x,t)$, $x \in (0,5)$, $t \in \{0.5, 1, 1.5\}$, $\alpha \in \{0.25, 0.5, 0.75\}$

The numerical example in Figure 3.7 presents the displacement as a function of coordinates for $t \in \{0.5, 1, 1.5, 2, 2.5\}$ and for the following set of parameters: $\alpha_0 = 0.25$, $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, $a_0 = 0.008$, $a_1 = 0.006$, $a_2 = 0.004$, $b_0 = 1.6$, $b_1 = 1.4$, $b_2 = 1.2$. It follows from [3.44] that if $a_k = b_k$, $k = 0, 1, \ldots, n$, we obtain Hooke's law, i.e. one has elastic, rather than viscoelastic, material. Since values of parameters a_k and b_k, in the case of the example in the Figure 3.6, are chosen to be very close, one obtains a fundamental solution that is closer to the Dirac distribution $\delta(x-t)$, the fundamental solution of the classical wave equation. We remark that, as time increases, the effect of the dissipation increases, since the peak at $t=1$ is narrow and it is positioned close to $x=1$ (for the classical wave equation, the wave speed is equal to 1), while at later time instances the peak becomes smaller, wider and further from the position $x=t$. The same effect of the decrease of the peak's height and increase of its width, as time increases, is also noticeable in Figure 3.7. Since values of parameters a_k and b_k in this case differ significantly, the effect of the dissipation is more noticeable.

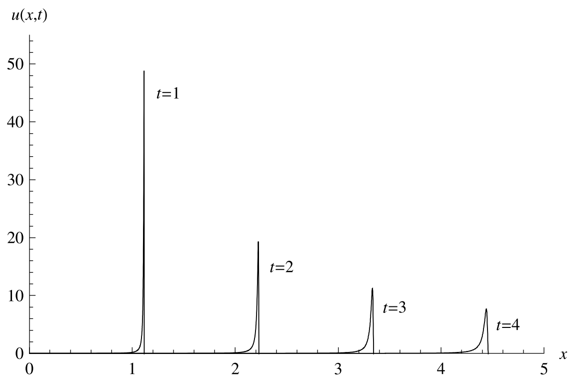

Figure 3.6. *Solution* $u(x,t)$, $x \in (0,5)$, $t \in \{1,2,3,4\}$

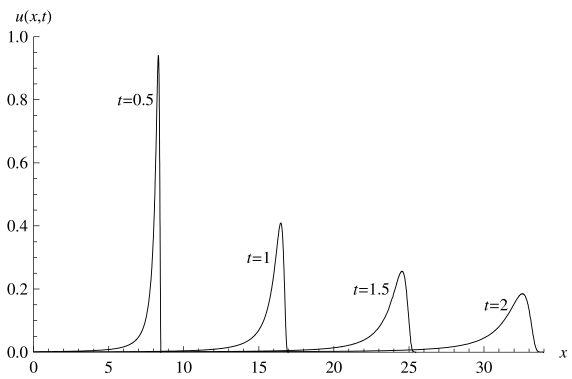

Figure 3.7. *Solution* $u(x,t)$, $x \in (0,35)$, $t \in \{0.5,1,1.5,2,2.5\}$

3.2. Wave equation of the fractional Eringen-type

Following [CHA 13], we consider the wave equation for the non-local media of the fractional Eringen-type and write it as a system of the equation of motion with zero body forces [3.2], fractional Eringen model of non-local media [3.7] and strain measure [3.4] as

$$\frac{\partial}{\partial x}\sigma(x,t) = \rho\frac{\partial^2}{\partial t^2}u(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{3.60}$$

$$\sigma(x,t) - l_c^\alpha\, \mathcal{E}_x^\alpha \sigma(x,t) = E\varepsilon(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{3.61}$$

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x \in \mathbb{R},\ t > 0. \tag{3.62}$$

The wave equation obtained from the system consisting of the equation of motion, fractional Eringen-type constitutive equation and classical strain measure is also studied in [COT 09, SAP 13], while in [MIC 11, MIC 12] the generalization of the classical theory is performed on the level of the potential energy and the wave equation was analyzed using a non-local Laplacian operator in the wave equation.

The model of the non-local wave equation, i.e. systems [3.60]–[3.62], is expected to match the Born–Kármán model better than the classical theory, since parameter α represents an additional degree-of-freedom. In order to obtain the dispersive equation, we assume the displacement to be in a form of the harmonic wave, having a complex form, as

$$u(x,t) = u_0 \mathrm{e}^{\mathrm{i}(\omega t - kx)}, \quad x \in \mathbb{R},\ t > 0,$$

where $\omega \in (0,\infty)$ denotes the angular frequency and $k \in \mathbb{R}$ is the wave number. Inserting this expression into [3.60] and [3.62], we obtain

$$\frac{\partial}{\partial x}\sigma(x,t) = -\rho\omega^2 u(x,t) \quad \text{and} \quad \varepsilon(x,t) = -\mathrm{i}ku(x,t). \tag{3.63}$$

Further, we apply the Fourier transform to [3.63] and obtain

$$\frac{\hat{\sigma}(k,t)}{\hat{\varepsilon}(k,t)} = \frac{\rho\omega^2}{k}, \quad k \in \mathbb{R},\ t > 0. \tag{3.64}$$

Applying the Fourier transform to the fractional Eringen constitutive equation [3.61], we obtain

$$\frac{\hat{\sigma}(k,t)}{\hat{\varepsilon}(k,t)} = \frac{E}{1 - l_c^\alpha\ |k|^\alpha \cos\frac{\alpha\pi}{2}}, \quad k \in \mathbb{R},\ t > 0, \tag{3.65}$$

where we used the fact that the Fourier transform of the symmetrized Caputo derivative in both cases, [3.8] and [3.9], i.e. for $\alpha \in [1,3)$, yields

$$\mathcal{F}\left[\mathcal{E}_x^\alpha f(x)\right](k) = \hat{f}(k)\,|k|^\alpha \cos\frac{\alpha\pi}{2}, \quad k \in \mathbb{R}.$$

By combining [3.64] and [3.65], we finally obtain the dispersive equation

$$\omega(k) = \pm k c_0 \frac{1}{\sqrt{1 - l_c^\alpha\ |k|^\alpha \cos\frac{\alpha\pi}{2}}}, \qquad [3.66]$$

where $c_0 = \sqrt{\frac{E}{\rho}}$ is the speed of the longitudinal wave. Dispersive equation [3.66] holds true for all ranges of parameters $\alpha \in [1,3)$. This dispersive wave equation can be rewritten for positive wave numbers k as

$$\frac{a}{c_0}\omega(k) = ka \frac{1}{\sqrt{1 - \left(\frac{l_c}{a}\right)^\alpha (ka)^\alpha \cos\frac{\alpha\pi}{2}}}, \qquad [3.67]$$

where a is the distance between atoms, with reference to the lattice dynamics behavior. The present non-local model can be compared to the Born–Kármán model of lattice dynamics, where the nearest neighbor interactions are accounted for (see [ERI 83, ERI 02]), which gives the following dispersion relation

$$\frac{a}{c_0}\omega_{bk}(k) = 2\sin\frac{ka}{2}. \qquad [3.68]$$

The foregoing non-local model is able to predict the Born–Kármán model for specified values of the length scale parameter l_c. This parameter could be identified from the characteristic prediction of the dispersive curve of lattice dynamics and equations [3.67] and [3.68] as

$$\begin{aligned} \frac{a}{c_0}\left.\omega(k)\right|_{k=\frac{\pi}{a}} &= \frac{a}{c_0}\left.\omega_{bk}(k)\right|_{k=\frac{\pi}{a}} = 2 \ \text{ implying } \left(\frac{l_c\pi}{a}\right)^\alpha \cos\frac{\alpha\pi}{2} \\ &= 1 - \frac{\pi^2}{4} \leq 0, \end{aligned} \qquad [3.69]$$

valid for $\alpha \in [1,3)$. Inserting the solution of [3.69] into [3.67], we obtain the dispersive wave relation

$$\frac{a}{c_0}\omega(k) = ka\sqrt{\frac{1}{1 + \left(\frac{\pi^2}{4} - 1\right)\left(\frac{ka}{\pi}\right)^\alpha}}. \qquad [3.70]$$

Figure 3.8 shows the dispersive curve parameterized by the fractional order α. It is shown that the nonlinearity of the dispersive curve in Figure 3.8 is more pronounced for higher values of the fractional order α.

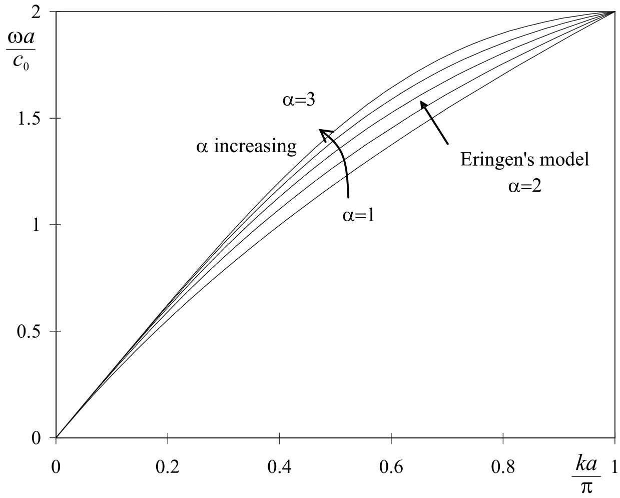

Figure 3.8. *Dispersion curve for the fractional non-local elasticity model – parameterized study for* $\alpha \in \{1, 1.5, 2, 2.5, 3\}$

It is possible to adequately match the Born–Kármán model by adding an optimality condition based on the mean square error. Introducing the dimensionless parameters

$$\bar{\omega} = \frac{a}{c_0}\omega, \quad \text{and} \quad \bar{k} = \frac{ka}{\pi} \qquad [3.71]$$

in [3.68] and [3.70], we obtain the dimensionless dispersion functions

$$\bar{\omega}\left(\bar{k}\right) = \bar{k}\pi\sqrt{\frac{1}{1+\left(\frac{\pi^2}{4}-1\right)\bar{k}^{\alpha}}}, \quad \text{and} \quad \bar{\omega}_{bk}\left(\bar{k}\right) = 2\sin\frac{\bar{k}\pi}{2}. \qquad [3.72]$$

The optimal fitting is obtained from the integral equation

$$\frac{\partial}{\partial\alpha}\left(\int_0^1 \left(\bar{\omega}_{bk}\left(\bar{k}\right) - \bar{\omega}\left(\bar{k}\right)\right)^2 \mathrm{d}\bar{k}\right) = 0$$

$$\frac{\partial}{\partial\alpha}\left(\int_0^1 \left(2\sin\frac{\pi\bar{k}}{2} - \frac{\pi\bar{k}}{\sqrt{1+\left(\frac{\pi^2}{4}-1\right)\bar{k}^{\alpha}}}\right)^2 \mathrm{d}\bar{k}\right) = 0. \qquad [3.73]$$

The criterion expressing the optimality is found from [3.73] as

$$\int_0^1 \left(2\frac{\bar{k}^{\alpha+1}\sin\frac{\pi\bar{k}}{2}\ln\bar{k}}{\left(\sqrt{1+\left(\frac{\pi^2}{4}-1\right)\bar{k}^{\alpha}}\right)^3} - \frac{\pi\bar{k}^{\alpha+2}\ln\bar{k}}{\left(1+\left(\frac{\pi^2}{4}-1\right)\bar{k}^{\alpha}\right)^2} \right)\mathrm{d}\bar{k} = 0$$

which leads to $\alpha_0 \approx 2.833$.

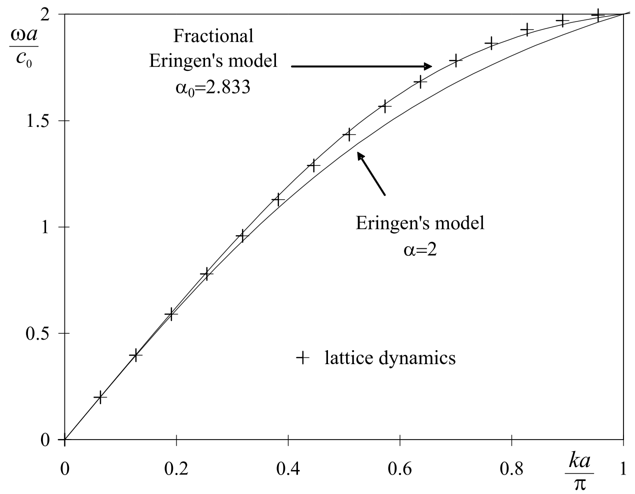

Figure 3.9. *Dispersion curve for the fractional non-local elasticity model – comparison with the Born–Kármán model of lattice dynamics*

As shown in Figure 3.9, an excellent matching of the dispersive curve of the Born–Kármán model of the lattice dynamics is obtained with such generalized integral-based non-local model, using the numerical optimum $\alpha_0 = 2.833$ for the order of the fractional derivative. Fractional derivative theory seems to be an efficient engineering tool for calibrating the dispersive wave properties of the Born–Kármán model.

Furthermore, the length scale parameter can be easily identified due to [3.69] as

$$\left(\frac{l_c}{a}\right)_0 = \frac{1}{\pi}\sqrt[\alpha_0]{\frac{4-\pi^2}{4\cos\frac{\alpha_0\pi}{2}}} \approx 0.587,$$

for $\alpha_0 = 2.833$. This value $l_c \cong 0.587a$ is not so far from the well-known value $l_c \cong 0.386a$ found in the case of Eringen's non-local model based on $\alpha = 2$ (see

[CHA 09, ERI 83, ERI 87]. More generally, the length scale parameter is shown to be dependent on the order of the fractional derivative, as highlighted by Figure 3.10.

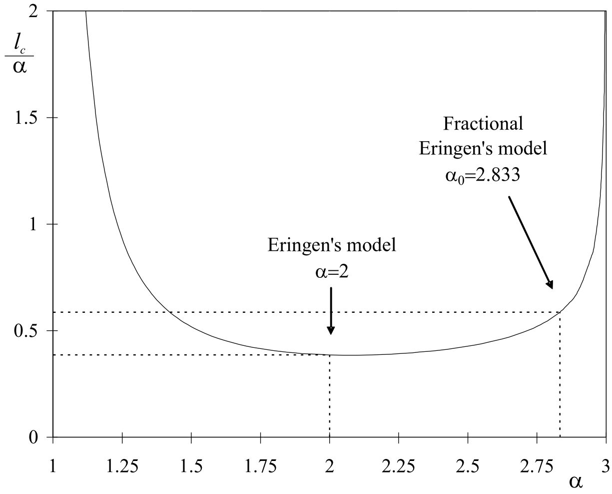

Figure 3.10. *Evolution of the length scale parameters with respect to the fractional derivative order*

The length scale $l_c \approx 0.386a$ obtained for $\alpha = 2$ is almost the minimum value, even if the exact minimum is obtained for a slightly larger value $\alpha \approx 2.076$ associated with $l_c \approx 0.384a$.

We note that the present fractional generalization of Eringen's model has the advantage and the simplicity to be based on only one length scale, whereas the recent model of [CHA 09], which also fits well the Born–Kármán model, is based on two length scales.

3.3. Space-fractional wave equation on unbounded domain

Following [ATA 09f], we study the wave motion in the infinite elastic rod of non-local type, with the dissipative body forces present due to the surroundings of the rod and assumed to be of viscous type:

$$f(x,t) = -\eta\, {}_0^C\mathrm{D}_t^{\alpha} u(x,t), \quad x \in \mathbb{R},\ t > 0, \qquad [3.74]$$

where η is a generalized viscosity coefficient and $\alpha \in (0,1)$. Then, the wave equation is written as the system consisting of the equation of motion [3.2], with f given by

[3.74], Hooke's law [3.3] as the constitutive equation and the non-local strain measure [3.10]:

$$\frac{\partial}{\partial x}\sigma(x,t) - \eta\, {}_0^C\mathrm{D}_t^{\alpha} u(x,t) = \rho \frac{\partial^2}{\partial t^2} u(x,t), \quad x \in \mathbb{R},\ t > 0, \qquad [3.75]$$

$$\sigma(x,t) = E\varepsilon(x,t), \quad x \in \mathbb{R},\ t > 0, \qquad [3.76]$$

$$\varepsilon(x,t) = \mathcal{E}_x^{\beta} u(x,t), \quad x \in \mathbb{R},\ t > 0. \qquad [3.77]$$

Note that E is the generalized Young's modulus, measured in $\frac{\mathrm{Pa}}{\mathrm{m}^{1-\beta}}$.

The fractionally damped space-fractional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) + \tau\, {}_0^C\mathrm{D}_t^{\alpha} u(x,t) = c^2 \frac{\partial}{\partial x}\mathcal{E}_x^{\beta} u(x,t), \quad x \in \mathbb{R},\ t > 0, \qquad [3.78]$$

where $\tau = \frac{\eta}{\rho}$ and $c = \sqrt{\frac{E}{\rho}}$, is obtained from systems [3.75]–[3.77].

REMARK 3.6.–

1) In the case when there is no damping present ($\tau = 0$), [3.78] reduces to the space-fractional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial}{\partial x}\mathcal{E}_x^{\beta} u(x,t), \quad x \in \mathbb{R},\ t > 0. \qquad [3.79]$$

a) If, additionally, there are no non-local effects ($\beta = 1$), then [3.78] reduces to the classical wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c^2 \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R},\ t > 0. \qquad [3.80]$$

b) If, additionally, we have the case $\beta = 0$, then [3.78] becomes

$$\frac{\partial^2}{\partial t^2} u(x,t) = 0, \quad x \in \mathbb{R},\ t > 0. \qquad [3.81]$$

The solution to [3.81] is

$$u(x,t) = u_0(x) + v_0(x)\, t, \quad x \in \mathbb{R},\ t > 0.$$

This is the case of the non-propagating wave.

2) Consider the case when there is damping present ($\tau \neq 0$) in [3.78].

a) If $\alpha = 1$ and $\beta \in (0,1)$, then [3.78] reduces to the space-fractional telegraph equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) = c^2\frac{\partial}{\partial x}\mathcal{E}_x^\beta u(x,t), \quad x \in \mathbb{R},\ t > 0.$$

b) If $\alpha \in (0,1)$ and $\beta = 1$, then [3.78] reduces to the time-fractional telegraph equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + \tau\, {}_0^C\mathrm{D}_t^\alpha u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \quad x \in \mathbb{R},\ t > 0.$$

c) If $\alpha = 1$ and $\beta = 1$, then [3.78] reduces to the telegraph equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + \tau\frac{\partial}{\partial t}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \quad x \in \mathbb{R},\ t > 0.$$

We subject [3.78] to the initial conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial}{\partial t}u(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{3.82}$$

as well as the boundary conditions

$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad t > 0. \tag{3.83}$$

3.3.1. *Solution to Cauchy problem for space-fractional wave equation*

We investigate the Cauchy problem given by [3.79], [3.82] and [3.83] in the space of tempered distributions. In this way, we obtain not only the classical solutions, if they exist, but also the generalized solutions. The equation in $\mathcal{S}'(\mathbb{R}^2)$ which corresponds to [3.79], [3.82] and [3.83] is

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial}{\partial x}\mathcal{E}_x^\beta u(x,t) + u_0(x)\delta'(t) + v_0(x)\delta(t), \tag{3.84}$$

where $\beta \in (0,1)$, $u \in \mathcal{S}'(\mathbb{R}^2)$, $\operatorname{supp} u \subset \mathbb{R} \times [0,\infty)$ and $u_0, v_0 \in \mathcal{E}'$. We look for solutions that are regular tempered distributions defined by the locally integrable function u.

We state the theorem on the existence of the solution to [3.84].

THEOREM 3.3.– Suppose $u_0, v_0 \in \mathcal{E}'$. Then the distribution

$$u(x,t) = \mathcal{F}^{-1}\left[\hat{u}_0(\xi)\cos\left(a|\xi|^{\gamma}t\right)\right](x,t) + \mathcal{F}^{-1}\left[\hat{v}_0(\xi)\frac{\sin\left(a|\xi|^{\gamma}t\right)}{a|\xi|^{\gamma}}\right](x,t), \quad x \in \mathbb{R},\ t > 0. \qquad [3.85]$$

is a solution to [3.84].

PROOF.– Applying the Fourier and Laplace transforms to [3.84], we obtain

$$s^2\hat{\tilde{u}}(\xi,s) = -c^2\sin\frac{\beta\pi}{2}|\xi|^{1+\beta}\hat{\tilde{u}}(\xi,s) + \hat{u}_0(\xi)s + \hat{v}_0(\xi), \quad \xi \in \mathbb{R},\ \mathrm{Re}\, s > 0. \qquad [3.86]$$

Consequently, we have

$$\hat{\tilde{u}}(\xi,s) = \frac{\hat{u}_0(\xi)s + \hat{v}_0(\xi)}{s^2 + c^2\sin\frac{\beta\pi}{2}|\xi|^{1+\beta}}, \quad \xi \in \mathbb{R},\ \mathrm{Re}\, s > 0. \qquad [3.87]$$

The inverse Laplace transform gives

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)\cos\left(a|\xi|^{\gamma}t\right) + \hat{v}_0(\xi)\frac{\sin\left(a|\xi|^{\gamma}t\right)}{a|\xi|^{\gamma}}, \quad \xi \in \mathbb{R},\ t > 0, \qquad [3.88]$$

see [MAG 48, p. 171], where $a = c\sqrt{\sin\frac{\beta\pi}{2}}$ and $\gamma = \frac{1+\beta}{2}$. Since $\cos\left(a|\xi|^{\gamma}t\right)$ and $\frac{\sin(a|\xi|^{\gamma}t)}{a|\xi|^{\gamma}}$ are continuous with respect to ξ in zero (and smooth out), we know that the products with $\hat{u}_0$ and $\hat{v}_0$ are well defined if $\hat{u}_0$ and $\hat{v}_0$ are measures (continuous linear operators on the space of continuous functions with suitable norm). Finally, the distribution u, given by [3.85], is a solution to [3.84]. ■

3.3.1.1. *Limiting case* $\beta \to 1$

From the solution given in the form [3.88], we have

$$\begin{aligned}\lim_{\beta\to 1}\mathcal{F}\left[u(x,t)\right](\xi,t) &= \hat{u}_0(\xi)\cos(\xi ct) + \hat{v}_0(\xi)\frac{\sin(\xi ct)}{ct}\\ &= \frac{1}{2}\left(\hat{u}_0(\xi) + \frac{\hat{v}_0(\xi)}{\mathrm{i}\xi}\right)\mathrm{e}^{\mathrm{i}\xi ct} + \frac{1}{2}\left(\hat{u}_0(\xi) - \frac{\hat{v}_0(\xi)}{\mathrm{i}\xi}\right)\mathrm{e}^{-\mathrm{i}\xi ct}.\end{aligned}$$

If $v_0(x) = \frac{\mathrm{d}}{\mathrm{d}x} v_1(x)$, then $\hat{v}_0(\xi) = -\mathrm{i}\xi\hat{v}_1(\xi)$, so that the previous expression becomes

$$\lim_{\beta\to 1} u(x,t) = \frac{1}{2}\left(u_0(x-ct) - v_1(x-ct)\right) + \frac{1}{2}\left(u_0(x+ct) + v_1(x+ct)\right).$$

This is the well-known form of the solution to the wave equation [3.80], with the initial condition [3.82] (see [SCH 61, p. 280-281]).

3.3.1.2. *Case* $u_0(x) = \frac{1}{x^2+d^2},\ v_0(x) = 0,\ x \in \mathbb{R},\ d > 0$

Then we have $\hat{u}_0(\xi) = \frac{\pi}{d}\mathrm{e}^{-d|\xi|}$ (see [MAG 48, p. 162]) so that [3.88] becomes

$$\hat{u}(\xi,t) = \frac{\pi}{d}\cos\left(a\left|\xi\right|^{\gamma} t\right)\mathrm{e}^{-d|\xi|} \in L^1 \ \text{ in }\ \xi,\ t \geq 0.$$

Since $\hat{u}$ is an even function, from the inverse Fourier transform, we have

$$u(x,t) = \frac{1}{d}\int_0^{\infty}\cos\left(a\xi^{\gamma}t\right)\cos\left(\xi x\right)\mathrm{e}^{-d\xi}\mathrm{d}\xi. \qquad [3.89]$$

There holds:

(i) The initial conditions are satisfied since

$$u(x,0) = \frac{1}{d}\int_0^{\infty}\cos\left(\xi x\right)\mathrm{e}^{-d\xi}\mathrm{d}\xi = \frac{1}{x^2+d^2},$$

see [MAG 48, p. 171] and

$$\frac{\partial}{\partial t}u(x,t) = \frac{a}{d}\int_0^{\infty}\xi^{\gamma}\sin\left(a\xi^{\gamma}t\right)\cos\left(\xi x\right)\mathrm{e}^{-d\xi}\mathrm{d}\xi$$

implies $\frac{\partial}{\partial t}u(x,0) = 0$.

(ii) $\hat{u}_0 \in L^1$.

(iii) Since $\hat{u} \in L^1$, by the Riemann–Lebesgue lemma, from [3.89] it follows that $\lim_{x\to\pm\infty} u(x,t) = 0,\ t \geq 0$.

Hence, u given by [3.89] is a generalized solution to [3.79], [3.82] and [3.83]. In Figure 3.11, we show the solution given by [3.89] for several values of parameters. As could be seen, the solution has the form of a decaying wave traveling in both $x > 0$ and $x < 0$ directions.

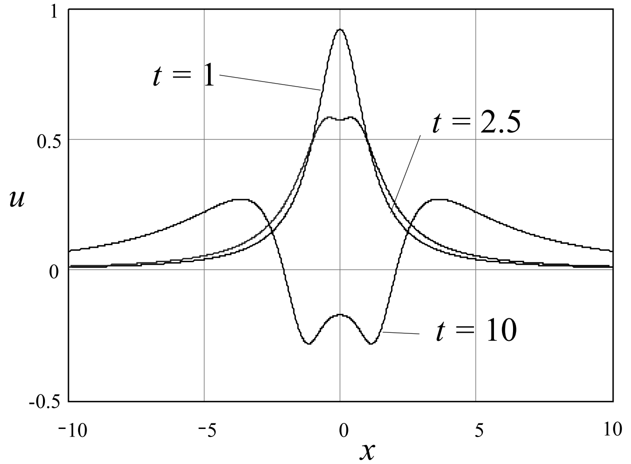

Figure 3.11. *Displacement u, given by* [3.89], *for $\beta = 0.1$, $d = 1$, $c = 1$*

3.3.1.3. *Case* $u_0(x) = 0,\, v_0(x) = \begin{cases} A\pi, & |x| \le r, \\ 0, & |x| > r, \end{cases} \quad x \in \mathbb{R}$

Then we have $\hat{v}_0(\xi) = A\frac{\sin(\xi r)}{\xi}$ (see [MAG 48, p. 161]) so that [3.88] becomes

$$\hat{u}(\xi, t) = A\frac{\sin(\xi r)}{\xi}\frac{\sin\left(a\,|\xi|^{\gamma}\,t\right)}{a\,|\xi|^{\gamma}} \in L^1 \text{ in } \xi,\ t \ge 0.$$

Therefore, inverting the Fourier transform in the previous expression, we obtain

$$u(x,t) = \frac{A}{\pi}\int_0^{\infty}\frac{\sin(\xi r)}{\xi}\frac{\sin\left(a\,|\xi|^{\gamma}\,t\right)}{a\,|\xi|^{\gamma}}\cos(\xi x)\,\mathrm{d}\xi. \qquad [3.90]$$

It is easily seen that u given by [3.90] is also a generalized solution to [3.79], [3.82] and [3.83].

3.3.1.4. *Case* $u_0(x) = A\delta(x), v_0(x) = 0,\ x \in \mathbb{R}, \nu \in (1-\gamma, 1+\gamma)$

Then we have $\hat{u}_0(\xi) = A$, so that [3.88] becomes

$$\begin{aligned}
\hat{u}(\xi,t) &= A\cos\left(a\left|\xi\right|^{\gamma} t\right) \\
&= A\left(\frac{at}{2}\right)^2 \left|\xi\right|^{\nu+\gamma} \frac{\cos\left(a\left|\xi\right|^{\gamma} t\right) - 1}{\left(\frac{at}{2}\right)^2 \left|\xi\right|^{\nu+\gamma}} + A \\
&= A - 2A\left(\frac{at}{2}\right)^2 \left|\xi\right|^{\nu+\gamma} \frac{\sin^2 \frac{a|\xi|^{\gamma} t}{2}}{\left|\xi\right|^{\nu-\gamma}\left(\frac{a|\xi|^{\gamma} t}{2}\right)^2} \\
&= A - 2A\left(\frac{at}{2}\right)^2 \frac{\sin^2 \frac{a|\xi|^{\gamma} t}{2}}{\left|\xi\right|^{\nu-\gamma}\left(\frac{a|\xi|^{\gamma} t}{2}\right)^2} \left(\mathrm{i}\xi\right)^{\nu+\gamma} \mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi(\nu+\gamma)\,\mathrm{sgn}(\xi)}.
\end{aligned}$$

By the inverse Fourier transform, we obtain

$$\begin{aligned}
u(x,t) &= A\delta(x) - 2A\left(\frac{at}{2}\right)^2 \\
&\quad \times {}_{-\infty}^{\;\;C}\mathrm{D}_x^{\nu+\gamma}\left(\mathcal{F}^{-1}\left[\frac{\sin^2 \frac{a|\xi|^{\gamma} t}{2}}{\left|\xi\right|^{\nu-\gamma}\left(\frac{a|\xi|^{\gamma} t}{2}\right)^2} \mathrm{e}^{-\frac{1}{2}\mathrm{i}\pi(\nu+\gamma)\,\mathrm{sgn}(\xi)}\right](x,t)\right).
\end{aligned}$$

This is a solution to [3.84], but not a generalized solution to [3.79], [3.82] and [3.83].

3.3.2. *Solution to Cauchy problem for fractionally damped space-fractional wave equation*

We analyze solution u to [3.78], [3.82] and [3.83] in the space of tempered distributions, more precisely $u \in \mathcal{S}'(\mathbb{R}^2)$, $\operatorname{supp} u \subset \mathbb{R} \times [0,\infty)$. The equation which corresponds to [3.78], [3.82] and [3.83] is

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) + \tau\, {}_0^C\mathrm{D}_t^{\alpha} u(x,t) &= c^2 \frac{\partial}{\partial x}\mathcal{E}_x^{\beta} u(x,t) + u_0(x)\left(\delta^{(1)}(t) + \tau\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right) \\
&\quad + v_0(x)\delta(t). \qquad [3.91]
\end{aligned}$$

We state the theorem on the existence and uniqueness of the solution to [3.91] and therefore to [3.78], [3.82] and [3.83].

THEOREM 3.4.– Let $u_0, v_0 \in \mathcal{E}'$. Then equation [3.91] has one and only one solution $u \in \mathcal{S}'(\mathbb{R}^2)$, $\operatorname{supp} u \subset \mathbb{R} \times [0, \infty)$, given by

$$\begin{aligned} u\,(x,t) \quad &= \quad u_0(x) - a\mathcal{F}^{-1}\left[\hat{u}_0(\xi)\,|\xi|^{1+\beta}\int_0^t \phi\,(\xi,\tau)\,\mathrm{d}\tau\right](x,t) \\ &\quad\; +\mathcal{F}^{-1}\left[\hat{v}_0\,(\xi)\,\phi\,(\xi,t)\right](x,t)\,, \end{aligned} \qquad [3.92]$$

where $\theta = c^2 \sin\frac{\beta\pi}{2}$ and

$$\phi\,(\xi,t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + \tau s^\alpha + \theta\,|\xi|^{1+\beta}}\right](\xi,t)\,.$$

PROOF.– We apply the Fourier transform in x and the Laplace transform in t to [3.91] and obtain

$$(s^2 + \tau s^\alpha + \theta\,|\xi|^{1+\beta})\widehat{\tilde{u}}\,(\xi,s) = \hat{u}_0\,(\xi)\,(s + \tau s^{\alpha-1}) + \hat{v}_0\,(\xi)\,.$$

From the previous equation, it follows

$$\widehat{\tilde{u}}\,(\xi,s) = \hat{u}_0\,(\xi)\,\frac{s + \tau s^{\alpha-1}}{s^2 + \tau s^\alpha + \theta\,|\xi|^{1+\beta}} + \hat{v}_0\,(\xi)\,\frac{1}{s^2 + \tau s^\alpha + \theta\,|\xi|^{1+\beta}}. \qquad [3.93]$$

It is easily seen that the function $\mathcal{L}^{-1}\left[\frac{1}{s^2+\tau s^\alpha+\theta|\xi|^{1+\beta}}\right]$ is a key factor to find u, defined by [3.93]. This function has been examined in many papers [EID 04, GOR 97b, KIL 06]. But we have a specific case, namely the term $\theta\,|\xi|^{1+\beta}$ changes with $\xi \in \mathbb{R}$. Therefore, we give two analytic forms for this function.

Let F and ϕ denote the functions

$$F\,(\xi,s) = \frac{1}{s^2 + \tau s^\alpha + \theta\,|\xi|^{1+\beta}} \qquad [3.94]$$

and

$$\phi\,(\xi,t) = \mathcal{L}^{-1}\left[F\,(\xi,s)\right](\xi,t)\,. \qquad [3.95]$$

We examine the properties of the functions F and ϕ.

1) There exists $\sigma_0(\xi) \in \mathbb{R}_+$ such that $F(\xi, s)$ is analytic for $\mathrm{Re}\, s > \sigma_0(\xi)$ and

$$|F(\xi, s)| \leq \frac{2}{|s|^2}, \quad \xi \in \mathbb{R}, \ \mathrm{Re}\, s > \sigma_0(\xi). \qquad [3.96]$$

We prove this as follows. Since,

$$|F(\xi, s)| = \frac{1}{|s|^2} \frac{1}{\left|1 + \frac{\tau s^\alpha + \theta |\xi|^{1+\beta}}{s^2}\right|} < \frac{1}{|s|^2} \frac{1}{\left|1 - \left|\frac{\tau s^\alpha + \theta |\xi|^{1+\beta}}{s^2}\right|\right|},$$

for fixed ξ, there exists $\sigma_0(\xi) \in \mathbb{R}_+$ such that $\left|\frac{\tau s^\alpha + \theta|\xi|^{1+\beta}}{s^2}\right| < \frac{1}{2}, \ \mathrm{Re}\, s > \sigma_0(\xi)$. Consequently, $|F(\xi, s)| \leq \frac{2}{|s|^2}, \ \mathrm{Re}\, s > \sigma_0(\xi)$. Also it follows that $F(\xi, s)$ is analytic in the half-plane $\mathrm{Re}\, s > \sigma_0(\xi), \ \xi \in \mathbb{R}$.

2) Function F is the Laplace transform of the function:

$$\phi(\xi, t) = \frac{1}{2\pi \mathrm{i}} \int_{s_0 - \mathrm{i}\infty}^{s_0 + \mathrm{i}\infty} F(\xi, s) \, \mathrm{e}^{st} \mathrm{d}s, \quad s_0 \geq \sigma_0(\xi), \ \xi \in \mathbb{R}, \ t \geq 0. \qquad [3.97]$$

This follows from [3.96].

3) Function ϕ, given by [3.97], is continuous in t for $t > 0, \ \xi \in \mathbb{R}$, see theorem 6 in [DOE 55, p. 267], 1. and 3.

4) $\phi(\xi, t) = O(t)$, as $t \to 0$, $\xi \in \mathbb{R}$, since $F(s) \sim \frac{1}{s^2}$, as $s \to \infty$ in the half-plane $\mathrm{Re}\, s > \sigma_0(\xi)$ (see Satz 3 in [DOE 55, p. 503]).

Thus, we proved the following properties of function ϕ. It is defined on $[0, \infty)$ continuous in t on $(0, \infty)$ and $\phi(\xi, t) = O(t)$, as $t \to 0$.

For the second addend in [3.93], we have:

$$\frac{s + \tau s^{\alpha - 1}}{s^2 + \tau s^\alpha + \theta |\xi|^{1+\beta}} = \frac{1}{s}\left(1 - \frac{\theta |\xi|^{1+\beta}}{s^2 + \tau s^\alpha + \theta |\xi|^{1+\beta}}\right) = \frac{1}{s} - \frac{\theta |\xi|^{1+\beta}}{s} F(\xi, s),$$

hence,

$$\mathcal{L}^{-1}\left[\frac{s + \tau s^{\alpha - 1}}{s^2 + \tau s^\alpha + \theta |\xi|^{1+\beta}}\right](t) = 1 - \theta |\xi|^{1+\beta} \int_0^t \phi(\xi, \tau) \mathrm{d}\tau. \qquad [3.98]$$

We can find another form of the function ϕ. We have

$$\frac{1}{s^2+\tau s^\alpha+\theta\left|\xi\right|^{1+\beta}}=\frac{1}{s^2+\theta\left|\xi\right|^{1+\beta}}\frac{1}{1+\frac{\tau s^\alpha}{s^2+\theta\left|\xi\right|^{1+\beta}}}$$
$$=\frac{1}{s^2+\theta\left|\xi\right|^{1+\beta}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\tau s^\alpha}{s^2+\theta\left|\xi\right|^{1+\beta}}\right)^k, \qquad [3.99]$$

since for $\sigma_0(\xi)$ large enough, we have $\left|\frac{s^\alpha}{s^2+\theta|\xi|^{1+\beta}}\right|<1$, if $\operatorname{Re}s>\sigma_0(\xi)$. It is well known

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\theta\left|\xi\right|^{1+\beta}}\right](\xi,t)=\frac{\sin\left(\sqrt{\theta\left|\xi\right|^{1+\beta}}t\right)}{\sqrt{\theta\left|\xi\right|^{1+\beta}}},\ \ \xi\in\mathbb{R},\ t>0,$$

(see [MAG 48, p. 171]) as well as ($\xi\in\mathbb{R},\ t>0$)

$$\left|\mathcal{L}^{-1}\left[\frac{s^\alpha}{s^2+\theta\left|\xi\right|^{1+\beta}}\right](\xi,t)\right|=\left|\mathcal{L}^{-1}\left[s^{\alpha-1}\frac{s}{s^2+\theta\left|\xi\right|^{1+\beta}}\right](\xi,t)\right|$$
$$=\frac{1}{\Gamma(1-\alpha)}\left|\int_0^t\frac{\cos\left(\sqrt{\theta\left|\xi\right|^{1+\beta}}\tau\right)}{(t-\tau)^\alpha}\mathrm{d}\tau\right|\leq\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{1}{\tau^\alpha}\mathrm{d}\tau=\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}. \qquad [3.100]$$

Let

$$\phi_k=\mathcal{L}^{-1}\left[\left(\frac{\tau s^\alpha}{s^2+\theta\left|\xi\right|^{1+\beta}}\right)^k\right].$$

Then, from [3.100], we have

$$\left|\phi_k(\xi,t)\right|\leq\left|\left(\frac{\tau t^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{*k}\right|=\frac{\tau^k t^{k(1-\alpha)}}{\Gamma(k(1-\alpha)+1)},\ \ \xi\in\mathbb{R},\ t\geq 0, \qquad [3.101]$$

where f^{*k} means $f*f*...*f$, k-times. From

$$\int_0^\infty\phi_k(\xi,t)\,\mathrm{e}^{-st}\mathrm{d}t=\left(\frac{\tau s^\alpha}{s^2+\theta\left|\xi\right|^{1+\beta}}\right)^k,\ \ \xi\in\mathbb{R},\ \operatorname{Re}s>\sigma_0>0.$$

and taking sufficiently large σ_0 so that $\left|\frac{\tau}{\sigma_0^{2-\alpha}}\right| < 1$ and we have

$$\int_0^\infty |\phi_k(\xi,t)| \, \mathrm{e}^{-\sigma_0 t} \mathrm{d}t \leq C \left(\frac{\tau}{\sigma_0^{2-\alpha}}\right)^k .$$

Hence, the series $\sum_{k=0}^\infty \int_0^\infty |\phi_k(\xi,t)| \, \mathrm{e}^{-\sigma_0 t} \mathrm{d}t$ converges for $\sigma_0 > 1$. From Satz 2 in [DOE 55, p. 305], the series

$$g(\xi,t) = \sum_{k=0}^\infty \phi_k(\xi,t) \qquad [3.102]$$

converges absolutely and uniformly in $\xi \in \mathbb{R}$. Consequently,

$$\begin{aligned}
\phi(\xi,t) &= \mathcal{L}^{-1}\left[\frac{1}{s^2 + \tau s^\alpha + \theta |\xi|^{1+\beta}}\right](\xi,t) \\
&= \frac{\sin\left(\sqrt{\theta |\xi|^{1+\beta}}\, t\right)}{\sqrt{\theta |\xi|^{1+\beta}}} *_t g(\xi,t) .
\end{aligned} \qquad [3.103]$$

Let us summarize the properties of the function ϕ.

1) It is defined on $\mathbb{R} \times [0,\infty)$, continuous in t on $(0,\infty)$ for every $\xi \in \mathbb{R}$ and $\phi(\xi,t) = O(t)$, as $t \to 0$, for every $\xi \in \mathbb{R}$.

2) The analytical forms of ϕ are:

a) $$\phi(\xi,t) = \frac{1}{2\pi \mathrm{i}} \int_{x-\mathrm{i}\infty}^{x+\mathrm{i}\infty} F(\xi,s)\, \mathrm{e}^{st} \mathrm{d}s, \quad x > \sigma_0(\xi), \ \xi \in \mathbb{R}, \ t \geq 0, \qquad [3.104]$$

where $\sigma_0(\xi)$ is such that

$$\left|\frac{\tau s^\alpha + \theta |\xi|^{1+\beta}}{s^2}\right| < \frac{1}{2}, \quad \operatorname{Re} s > \sigma_0(\xi),$$

and F is given by [3.94];

b) $$\phi(\xi,t) = \frac{\sin\left(\sqrt{\theta |\xi|^{1+\beta}}\, t\right)}{\sqrt{\theta |\xi|^{1+\beta}}} *_t g(\xi,t), \qquad [3.105]$$

where g is given by [3.102]. It follows from [3.101] that

$$|g(\xi,t)| \leq E_{1-\alpha}(\tau t^{1-\alpha}), \quad \xi \in \mathbb{R}, \; t > 0, \tag{3.106}$$

where $E_{1-\alpha}$ is the one-parameter Mittag-Leffler function. As far as function $\frac{\sin\left(\sqrt{\theta|\xi|^{1+\beta}}t\right)}{\sqrt{\theta|\xi|^{1+\beta}}}$ is concerned, we have from the well-known inequality:

$$\cos x < \frac{\sin x}{x} < 1, \quad 0 < x < \frac{\pi}{2},$$

and from the inequality

$$\left|\frac{\sin x}{x}\right| < \frac{1}{x} \leq \frac{2}{\pi} < 1, \quad x \geq \frac{\pi}{2},$$

that

$$\left|\frac{\sin\left(\sqrt{\theta\,|\xi|^{1+\beta}}t\right)}{\sqrt{\theta\,|\xi|^{1+\beta}}}\right| \leq t, \quad t > 0. \tag{3.107}$$

c) From [3.105], [3.106] and [3.107], we have:

$$|\phi(\xi,t)| \leq \int_0^t (t-\tau)E_{1-\alpha}(\tau\tau^{1-\alpha})\mathrm{d}\tau, \quad \xi \in \mathbb{R}, \; t > 0. \tag{3.108}$$

Finally, we can go back to [3.93]. Applying the inverse Laplace transform and using [3.95] and [3.98], we have

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)\left(1 - \theta\,|\xi|^{1+\beta}\int_0^t \phi(\xi,\tau)\,\mathrm{d}\tau\right) + \hat{v}_0(\xi)\,\phi(\xi,\tau)\,. \tag{3.109}$$

■

The function g is given by the series [3.102]. In the case when only a finite number of terms are taken into account, it may be written as

$$g(\xi,t) = \sum_{k=0}^{\infty}\phi_n(\xi,t) = \sum_{k=0}^{j}\phi_k(\xi,t) + R_j(\xi,t)\,.$$

We estimate R_j as follows:

$$
\begin{aligned}
|R_j(\xi,t)| &\leq \sum_{k=j+1}^{\infty} |\phi_k(\xi,t)| \leq \sum_{k=j+1}^{\infty} \frac{(\tau t^{1-\alpha})^k}{\Gamma(k(1-\alpha)+1)} \\
&\leq \frac{(\tau t^{1-\alpha})^{j+1}}{\Gamma(j+1)(1-\alpha))} \sum_{k=j+1}^{\infty} \frac{(\tau t^{1-\alpha})^{k-j-1}}{\Gamma(k(1-\alpha)+1)} \Gamma((j+1)(1-\alpha)) \\
&\leq \frac{(\tau t^{1-\alpha})^{j+1}}{\Gamma((j+1)(1-\alpha))} \sum_{i=0}^{\infty} \frac{(\tau t^{1-\alpha})^{i} \Gamma((j+1)(1-\alpha))}{\Gamma((i+j+1)(1-\alpha))} \\
&\leq \frac{\pi(\tau t^{1-\alpha})^{j+1}}{\Gamma((j+1)(1-\alpha))} \sum_{i=0}^{\infty} \frac{(\tau t^{1-\alpha})^{i}}{\Gamma(i(1-\alpha)+1)} \\
&\leq \frac{\pi(\tau t^{1-\alpha})^{j+1}}{\Gamma((j+1)(1-\alpha))} E_{1-\alpha}(\tau t^{1-\alpha}), \quad \xi \in \mathbb{R}, \ t > 0,
\end{aligned}
$$

where $E_{1-\alpha}$ is the one-parameter Mittag-Leffler function. In the above, we used

$$
\frac{\Gamma(q)\Gamma(p)}{\Gamma(p+q)} = 2\int_0^{\frac{\pi}{2}} \cos^{2p-1}\theta \sin^{2q-1}\theta \mathrm{d}\theta, \quad p,q>0,
$$

see [SCH 61, p. 347].

EXAMPLE 3.1.– Let $u_0(x) = 0$ and $v_0(x) = \mathrm{e}^{-\lambda x^2}$, $\lambda > 0$. Then $\hat{v}_0(\xi) = \sqrt{\frac{\pi}{\lambda}}\,\mathrm{e}^{-\frac{\xi^2}{4\lambda}}$, $\sqrt{\lambda} > 0$. Since in this case v_0 satisfy conditions of theorem 3.4, the solution to equation [3.91] reads

$$
u(x,t) = \frac{1}{2\sqrt{\pi\lambda}} \int_{-\infty}^{\infty} \phi(\xi,t)\, \mathrm{e}^{-\left(\mathrm{i}x\xi + \frac{\xi^2}{4\lambda}\right)} \mathrm{d}\xi.
$$

3.4. Stress relaxation, creep and forced oscillations of a viscoelastic rod

Following [ATA 11b, ATA 11d, ATA 11e], we investigate the wave motion in the rod of finite length L, whose one end is fixed and the other end is subject to the prescribed displacement, or stress. In particular, we are interested in the stress relaxation, which corresponds to the case of the prescribed displacement and in the creep, which corresponds to the case of the prescribed stress. These two effects, usually studied in the case of light rod, i.e. on the level of the constitutive equation (see section 3.1.4 of [ATA 14b], will be investigated in the case of a rod of non-negligible mass. Also, the analysis will be presented for solid and fluid-like viscoelastic bodies modeled by the constitutive equations of fractional derivative type.

We write the wave equation of the viscoelastic body as a system of equations of motion [3.2] with $f = 0$, time-fractional constitutive equation [3.5] and the local strain measure [3.4] as:

$$\frac{\partial}{\partial x}\sigma(x,t) = \rho\frac{\partial^2}{\partial t^2}u(x,t), \quad x \in [0,L], \; t > 0, \qquad [3.110]$$

$$\int_0^1 \phi_\sigma(\eta)\, {}_0\mathrm{D}_t^\eta \sigma(x,t)\,\mathrm{d}\eta = E\int_0^1 \phi_\varepsilon(\eta)\, {}_0\mathrm{D}_t^\eta \varepsilon(x,t)\,\mathrm{d}\eta, \quad x \in [0,L], \; t > 0, \qquad [3.111]$$

$$\varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x \in [0,L], \; t > 0. \qquad [3.112]$$

We prescribe initial conditions for systems [3.110]–[3.112]

$$u(x,0) = 0, \quad \frac{\partial}{\partial t}u(x,0) = 0, \quad \sigma(x,0) = 0, \quad \varepsilon(x,0) = 0, \quad x \in [0,L], \qquad [3.113]$$

as well as with two types of boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = \Upsilon(t), \quad t \in \mathbb{R}, \qquad [3.114]$$

$$u(0,t) = 0, \quad \sigma(L,t) = \Sigma(t), \quad t \in \mathbb{R}. \qquad [3.115]$$

Functions Υ and Σ are locally integrable functions supported in $[0,\infty)$. Note that $[3.114]_1$, or $[3.115]_1$, means that the one rod's end is fixed at $x = 0$, while $[3.114]_2$ and ($[3.115]_2$) describe the prescribed displacement (stress) of the other rod's end. If $\Upsilon = \Upsilon_0 H$, we have the case of stress relaxation, while if $\Sigma = \Sigma_0 H$ we have the case of creep, where H is the Heaviside function.

In the following, we will treat systems [3.110]–[3.112], subject to [3.113] and either [3.114] or [3.115], for two types of viscoelastic bodies.

1) Solid-like viscoelastic body can be modeled by the distributed-order constitutive equation [3.111] with the choice of constitutive (weight) functions

$$\phi_\sigma(\eta) = a^\eta, \quad \phi_\varepsilon(\eta) = b^\eta, \quad \eta \in (0,1), \; a \le b. \qquad [3.116]$$

The restriction $a \le b$ follows from the second law of thermodynamics (see [ATA 02b, ATA 03]). If $a = b$, then [3.111], with [3.116], reduces to Hooke's law.

The choice of ϕ_σ and ϕ_ε in the form [3.116] is the simplest choice guaranteeing the dimensional homogeneity.

2) Fluid-like viscoelastic body can be modeled by the constitutive equation [3.111] with the choice of constitutive (weight) distributions

$$\begin{aligned}\phi_\sigma(\eta) &= \delta(\eta) + \frac{a}{b}\delta(\eta - (\alpha - \beta)),\\ \phi_\varepsilon(\eta) &= a\delta(\eta - \alpha) + c\delta(\eta - \gamma) + \frac{ac}{b}\delta(\eta - (\alpha + \gamma - \beta)),\end{aligned} \quad [3.117]$$

as proposed in [SCH 00], where a, b and c are given positive constants, while $0 < \beta < \alpha < \gamma < \frac{1}{2}$. It should be stressed that in [SCH 00] there is no discussion concerning restrictions on the parameters a, b, c, α, β and γ. Instead, only $a, b, c \geq 0$, $0 \leq \beta < \alpha \leq 1$ and $0 \leq \gamma \leq 1$ were assumed. The constitutive equation [3.111], with [3.117], is obtained in [SCH 00] via the rheological model that generalizes the classical Zener rheological model by substituting spring and dashpot elements by fractional elements. It is assumed that the stress–strain relation for the fractional element is given by $\sigma(t) = {}_0\mathrm{D}_t^\eta \varepsilon(t)$, $t > 0$, where $\eta \in [0,1]$. We refer to [SCH 00] for more details of the derivation, creep compliance and relaxation modulus.

REMARK 3.7.– Physically, the difference between solid and fluid-like materials is observed in the creep test, i.e. when the material is subject to a sudden, but later constant force on its free end. Namely, the solid-like materials creep to a finite value of displacement, while the fluid-like materials creep to an infinite value of displacement.

We refer to [MAI 10, ROS 10a, ROS 10b] for the detailed account of applications of the fractional calculus in viscoelasticity. Problems similar to [3.110]–[3.112] were also discussed in [ROS 01b, ROS 01b] with the constitutive equations related to the distributed-order model [3.111] in the special cases.

3.4.1. *Formal solution to systems [3.110]–[3.112], [3.113] and either [3.114] or [3.115]*

In order to solve systems [3.110]–[3.112] subject to initial conditions [3.113] and boundary conditions assumed either as [3.114] or as [3.115], we introduce the dimensionless quantities

$$\begin{aligned}\bar{x} &= \frac{x}{L},\ \bar{t} = \frac{t}{L\sqrt{\frac{\rho}{E}}},\ \bar{u} = \frac{u}{L},\ \bar{\sigma} = \frac{\sigma}{E},\ \bar{\Upsilon} = \frac{\Upsilon}{L},\ \bar{\Sigma} = \frac{\Sigma}{E},\ \bar{\phi}_\sigma\\ &= \frac{\phi_\sigma}{\left(L\sqrt{\frac{\rho}{E}}\right)^\alpha},\ \bar{\phi}_\varepsilon = \frac{\phi_\varepsilon}{\left(L\sqrt{\frac{\rho}{E}}\right)^\alpha},\end{aligned}$$

so that systems [3.110]–[3.112], after omitting the bar over dimensionless quantities, become

$$\frac{\partial}{\partial x}\sigma\left(x,t\right)=\frac{\partial^2}{\partial t^2}u\left(x,t\right),\quad x\in\left[0,1\right],\ t>0,$$
$$\int_0^1\phi_\sigma\left(\alpha\right){}_0\mathrm{D}_t^\alpha\sigma\left(x,t\right)\mathrm{d}\alpha=\int_0^1\phi_\varepsilon\left(\alpha\right){}_0\mathrm{D}_t^\alpha\varepsilon\left(x,t\right)\mathrm{d}\alpha,\quad x\in\left[0,1\right],\ t>0,\qquad [3.118]$$
$$\varepsilon\left(x,t\right)=\frac{\partial}{\partial x}u\left(x,t\right),\quad x\in\left[0,1\right],\ t>0.$$

System [3.118] is subject to initial condition

$$u\left(x,0\right)=0,\quad \frac{\partial}{\partial t}u\left(x,0\right)=0,\quad \sigma\left(x,0\right)=0,\quad \varepsilon\left(x,0\right)=0,\quad x\in\left[0,1\right],\qquad [3.119]$$

and two types of boundary conditions

$$u(0,t)=0,\quad u(1,t)=\Upsilon(t),\quad t\in\mathbb{R},\qquad [3.120]$$
$$u(0,t)=0,\quad \sigma(1,t)=\Sigma(t),\quad t\in\mathbb{R}.\qquad [3.121]$$

Initial and boundary data are obtained as the dimensionless forms of [3.113], [3.114] and [3.115].

Formally applying the Laplace transform to [3.118] and [3.119], we obtain

$$\frac{\partial}{\partial x}\tilde{\sigma}\left(x,s\right)=s^2\tilde{u}\left(x,s\right),\quad x\in\left[0,1\right],\ s\in D,$$
$$\tilde{\sigma}\left(x,s\right)\int_0^1\phi_\sigma\left(\alpha\right)s^\alpha\mathrm{d}\alpha=\tilde{\varepsilon}\left(x,s\right)\int_0^1\phi_\varepsilon\left(\alpha\right)s^\alpha\mathrm{d}\alpha,\quad x\in\left[0,1\right],\ s\in D,\qquad [3.122]$$
$$\tilde{\varepsilon}\left(x,s\right)=\frac{\partial}{\partial x}\tilde{u}\left(x,s\right),\quad x\in\left[0,1\right],\ s\in D.$$

Domain D for [3.122] is determined after [3.126], below.

System [3.122] reduces to

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x,s)-(sM(s))^2\tilde{u}(x,s)=0,\quad x\in[0,1],\ s\in D,\qquad [3.123]$$

where we introduced

$$M\left(s\right)=\sqrt{\frac{\int_{0}^{1}\phi_{\sigma}\left(\alpha\right)s^{\alpha}\mathrm{d}\alpha}{\int_{0}^{1}\phi_{\varepsilon}\left(\alpha\right)s^{\alpha}\mathrm{d}\alpha}},\quad s\in D.$$

Since our aim is to consider the prescribed displacement (stress) response of two types of viscoelastic bodies, for the solid-like viscoelastic body described by the constitutive functions, [3.116] function M becomes

$$M_{s}\left(s\right)=\sqrt{\frac{\ln\left(bs\right)}{\ln\left(as\right)}\frac{as-1}{bs-1}},\quad s\in\mathbb{C}\backslash\left(-\infty,0\right], \qquad [3.124]$$

while for the fluid-like viscoelastic body described by the constitutive functions [3.117], function M takes the form

$$M_{f}(s) = \sqrt{\frac{1+\frac{a}{b}s^{\alpha-\beta}}{as^{\alpha}+cs^{\gamma}+\frac{ac}{b}s^{\alpha+\gamma-\beta}}}=\frac{1}{\sqrt{as^{\alpha}}}\sqrt{\frac{1+\frac{a}{b}s^{\alpha-\beta}}{1+\frac{c}{a}s^{\gamma-\alpha}+\frac{c}{b}s^{\gamma-\beta}}},$$
$$s\in\mathbb{C}\backslash\left(-\infty,0\right]. \qquad [3.125]$$

Solution of [3.123] is of the form

$$\tilde{u}(x,s)=C_{1}(s)\mathrm{e}^{xsM(s)}+C_{2}(s)\mathrm{e}^{-xsM(s)},\quad x\in[0,1],\ s\in D, \qquad [3.126]$$

where C_1 and C_2 are functions of s, which will be determined from the boundary conditions. Since the power function s^{η} is analytic on the complex plane except the branch cut along the negative axis (including the origin), we take $D=\mathbb{C}\backslash(-\infty,0]$ to be the domain for variable s in [3.122]. Applying either $[3.120]_1$ or $[3.121]_1$, we obtain $C_1=-C_2=C$, and thus

$$\tilde{u}(x,s)=C(s)(\mathrm{e}^{xsM(s)}-\mathrm{e}^{-xsM(s)}),\quad x\in[0,1],\ s\in\mathbb{C}\backslash(-\infty,0]. \qquad [3.127]$$

From [3.122] and [3.125], it follows that

$$\tilde{\sigma}(x,s)=\frac{1}{M^{2}(s)}\frac{\partial}{\partial x}\tilde{u}(x,s),\quad x\in[0,1],\ s\in\mathbb{C}\backslash(-\infty,0]. \qquad [3.128]$$

We will separately seek solutions in different cases: displacement u and stress σ in the case of prescribed displacement (e.g. stress relaxation), and displacement u in the case of prescribed stress (e.g. creep). For the former, we supply to the system boundary conditions [3.120], while for the latter we assume [3.121]. In the following, we derive convolution forms of solutions in all these cases.

3.4.1.1. *Displacement of rod's end Υ is prescribed by [3.120]*

In the case of prescribed displacement Υ, substituting [3.127] into [3.126] and using [3.120], we obtain

$$\tilde{u}(x,s) = \tilde{\Upsilon}(s)\tilde{P}(x,s), \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0], \tag{3.129}$$

where

$$\tilde{P}(x,s) = \frac{\sinh(xsM(s))}{\sinh(sM(s))}, \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0]. \tag{3.130}$$

Clearly, $P(1,t) = \delta(t)$, $t \in \mathbb{R}$.

Since Υ and P are supported in $[0,\infty)$, the displacement u is given by

$$\begin{aligned} u(x,t) &= \Upsilon(t) * P(x,t), \quad x \in [0,1], \; t \in \mathbb{R}, \\ u(x,t) &= 0, \quad x \in [0,1], \; t < 0, \end{aligned} \tag{3.131}$$

where $*$ denotes the convolution with respect to t. Explicit calculation of [3.131] will be done by using the Laplace inversion formula applied to [3.129].

Further, from [3.128], [3.129] and [3.130], it follows that

$$\tilde{\sigma}(x,s) = s\tilde{\Upsilon}(s)\tilde{T}(x,s), \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0], \tag{3.132}$$

where

$$\tilde{T}(x,s) = \frac{\cosh(xsM(s))}{M(s)\sinh(sM(s))}, \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0]. \tag{3.133}$$

Applying the Laplace inversion formula to [3.132], we obtain

$$\sigma(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}(\Upsilon(t) * T(x,t)), \quad x \in [0,1], \; t \in \mathbb{R}, \tag{3.134}$$

where the derivative is understood in the sense of distributions. Again, $\sigma(x,t) = 0$ for $x \in [0,1]$, $t < 0$.

3.4.1.2. *Stress at rod's end Σ is prescribed by [3.121]*

In the case of prescribed stress Σ, using [3.128] at $x = 1$ and [3.121], we obtain

$$\frac{\partial}{\partial x}\tilde{u}(1,s) = \tilde{\Sigma}(s)M^2(s), \quad s \in \mathbb{C}\backslash(-\infty,0].$$

This combined with [3.127] gives

$$\tilde{u}(x,s) = \tilde{\Sigma}(s)\tilde{Q}(x,s), \quad x \in [0,1],\ s \in \mathbb{C}\backslash(-\infty,0], \tag{3.135}$$

where

$$\tilde{Q}(x,s) = \frac{1}{s}M(s)\frac{\sinh(xsM(s))}{\cosh(sM(s))}, \quad x \in [0,1],\ s \in \mathbb{C}\backslash(-\infty,0]. \tag{3.136}$$

Again, applying the Laplace inversion formula to [3.135], the displacement reads

$$\begin{aligned} u(x,t) &= \Sigma(t) * Q(x,t), \quad x \in [0,1],\ t > 0, \\ u(x,t) &= 0, \quad x \in [0,1],\ t < 0. \end{aligned} \tag{3.137}$$

3.4.2. *Case of solid-like viscoelastic body*

We consider system [3.118] with either [3.120] or [3.121] in the case of the solid-like viscoelastic body, described by the constitutive functions [3.116]. The fact that the body is solid-like is reflected in function M_s, given by [3.124]. Our aim is to find the displacement u and stress σ as functions of coordinate and time, when Υ is prescribed, in the form given by [3.131] and [3.134], respectively. In particular, we are interested in the case $\Upsilon = \Upsilon_0 H$, which corresponds to the stress relaxation. We also aim to find the displacement u as a function of coordinate and time in the case when Σ is prescribed by [3.121] in the form given by [3.137].

In this section, we will calculate the inverse Laplace transforms of distributions and functions on $\mathbb{R}$ supported by $[0,\infty)$. In the following, we will write $A(x) \sim B(x)$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$. Propositions 3.1, 3.2 and 3.3 are crucial in proving the existence of the solution to systems [3.118], [3.119], either [3.120], or [3.121]. They will be used in the proofs of theorems 3.5 and 3.6.

We examine the properties of function M_s, given by [3.124].

PROPOSITION 3.1.–

1) M_s is an analytic function in $s \in \mathbb{C}\backslash(-\infty, 0]$;

2) $\lim\limits_{\substack{s \to 0 \\ s\in\mathbb{C}\backslash(-\infty,0]}} M_s(s) = 1$ and $\lim\limits_{\substack{|s| \to \infty \\ s\in\mathbb{C}\backslash(-\infty,0]}} M_s(s) = \sqrt{\frac{a}{b}}$.

3) Let $p \in (0, s_0)$, $s_0 > 0$. Then

$$M_s(p \pm \mathrm{i}R)$$
$$\sim \sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \sqrt[4]{(\ln(aR)\ln(bR))^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2} \mathrm{e}^{\mp \mathrm{i} \arctan \frac{\frac{\pi}{2}\ln\frac{b}{a}}{\ln(aR)\ln(bR)}}$$

as $R \to \infty$.

PROOF.– We first prove point 1. The only points where M_s could be singular are $s = \frac{1}{a}$ and $s = \frac{1}{b}$. Since

$$\frac{\ln(bs)}{bs-1} = \frac{\ln(1+(bs-1))}{bs-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (bs-1)^n}{n+1}, \quad |bs-1| \in (-1, 1],$$

it is obvious that $s = \frac{1}{b}$ is a regular point of M_s. Similar arguments hold for $s = \frac{1}{a}$. Limits in point 2 can easily be calculated. In order to prove point 3, let us introduce $\mu = \sqrt{p^2 + R^2}$ and $\nu = \arctan\frac{R}{p}$. It is obvious that $\mu \sim R$, $\nu \sim \frac{\pi}{2}$, as $R \to \infty$. Then $M_s(p \pm \mathrm{i}R)$ becomes

$$M_s(p \pm \mathrm{i}R)$$
$$= \sqrt{\frac{\ln(a\mu)\ln(b\mu) + \nu^2 \mp \mathrm{i}\nu\ln\frac{b}{a}}{\ln^2(a\mu) + \nu^2}} \sqrt{\frac{(ap-1)(bp-1) + abR^2 \pm \mathrm{i}R(b-a)}{(bp-1)^2 + (bR)^2}}$$
$$= \left(\frac{\left[abR^2 + (ap-1)(bp-1)\right]\left[\ln(a\mu)\ln(b\mu) + \nu^2\right] + R\nu(b-a)\ln\frac{b}{a}}{\left((bR)^2 + (bp-1)^2\right)\left(\ln^2(a\mu) + \nu^2\right)} \right.$$
$$\left. \pm \mathrm{i} \frac{R(b-a)\left[\ln(a\mu)\ln(b\mu) + \nu^2\right] - \nu\ln\frac{b}{a}\left[abR^2 + (ap-1)(bp-1)\right]}{\left((bR)^2 + (bp-1)^2\right)\left(\ln^2(a\mu) + \nu^2\right)} \right)^{\frac{1}{2}}$$
$$\sim \sqrt{\frac{a}{b}} \frac{1}{\ln(aR)} \sqrt{\ln(aR)\ln(bR) \mp \mathrm{i}\frac{\pi}{2}\ln\frac{b}{a}}, \quad \text{as } R \to \infty. \quad \blacksquare$$

Let us examine the properties of $\tilde{P}$, given by [3.130], with M_s given by [3.124]. Clearly, it has complex conjugated poles at ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, which are solutions of

$$\sinh\left(sM_s\left(s\right)\right) = 0, \quad \text{i.e.} \quad sM_s\left(s\right) = \pm n\mathrm{i}\pi. \qquad [3.138]$$

The position and the multiplicity of the solutions to [3.138] are as follows.

PROPOSITION 3.2.– There are an infinite number of solutions ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, of [3.138], such that

$$\operatorname{Re}\left({}_P s_n^{(\pm)}\right) \approx -\frac{\frac{\pi}{4}\ln\frac{b}{a}\sqrt{\frac{b}{a}}n\pi}{\ln\left(\sqrt{ab}n\pi\right)\ln\left(b\sqrt{\frac{b}{a}}n\pi\right)}, \qquad [3.139]$$

$$\operatorname{Im}\left({}_P s_n^{(\pm)}\right) \approx \pm R \approx \pm\sqrt{\frac{b}{a}}n\pi, \qquad [3.140]$$

as $n \to \infty$. Moreover, there exists $n_0 \in \mathbb{N}$, such that poles ${}_P s_n^{(\pm)}$, for $n > n_0$, are simple.

PROOF.– Let us square [3.138] and put ${}_P s_n^{(\pm)} = R\mathrm{e}^{\mathrm{i}\phi}$, $\phi \in (-\pi, \pi)$. Then, after separation of real and imaginary parts, we obtain

$$R^2\cos(2\phi)\operatorname{Re}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) - R^2\sin(2\phi)\operatorname{Im}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) = -n^2\pi^2, \qquad [3.141]$$

$$R^2\sin(2\phi)\operatorname{Re}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) + R^2\cos(2\phi)\operatorname{Im}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) = 0. \qquad [3.142]$$

By using [3.124], the real and imaginary parts of $M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)$ are

$$\begin{aligned}
\operatorname{Re}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) &= \frac{\left(\ln(aR)\ln(bR)+\phi^2\right)\left(abR^2-(a+b)R\cos\phi+1\right)}{\left(\ln^2(aR)+\phi^2\right)\left(b^2R^2-2bR\cos\phi+1\right)} \\
&\quad + \frac{\ln\frac{b}{a}(b-a)R\phi\sin\phi}{\left(\ln^2(aR)+\phi^2\right)\left(b^2R^2-2bR\cos\phi+1\right)}, \\
\operatorname{Im}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) &= -\frac{\phi\ln\frac{b}{a}\left(abR^2-(a+b)R\cos\phi+1\right)}{\left(\ln^2(aR)+\phi^2\right)\left(b^2R^2-2bR\cos\phi+1\right)} \\
&\quad + \frac{+(b-a)R\sin\phi\left(\ln(aR)\ln(bR)+\phi^2\right)}{\left(\ln^2(aR)+\phi^2\right)\left(b^2R^2-2bR\cos\phi+1\right)}.
\end{aligned}$$

Letting $R \to \infty$, previous expressions are written as

$$\operatorname{Re}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) \approx \frac{abR^2 \ln(aR)\ln(bR)}{b^2R^2\ln^2(aR)} = \frac{a}{b}\frac{\ln(bR)}{\ln(aR)}, \quad [3.143]$$

$$\operatorname{Im}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right) \approx -\frac{ab\ln\frac{b}{a}R^2\phi}{b^2R^2\ln^2(aR)} = -\frac{a}{b}\ln\frac{b}{a}\phi\frac{1}{\ln^2(aR)}. \quad [3.144]$$

Using [3.142], [3.143] and [3.144], we obtain

$$\tan(2\phi) = -\frac{\operatorname{Im}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right)}{\operatorname{Re}\left(M_s^2\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right)} \approx \phi\frac{\ln\frac{b}{a}}{\ln(aR)\ln(bR)}. \quad [3.145]$$

Let $\phi \in (0,\pi)$. Then $\frac{\tan(2\phi)}{\phi} > 0$ and $\frac{\tan(2\phi)}{\phi} \to 0$ as $R \to \infty$. Hence, $\phi \in \left(0, \frac{\pi}{4}\right)$ or $\phi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Since $\phi \neq 0$ and $\tan(2\phi) \to 0$, it follows that $\phi \to \frac{\pi}{2}$ from the interval $\phi \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$. Therefore, by [3.145], we have

$$\sin\phi \approx 1 - \left(\frac{\frac{\pi}{2}\ln\frac{b}{a}}{2\ln(aR)\ln(bR)}\right)^2 \approx 1, \quad \cos\phi \approx -\frac{\frac{\pi}{2}\ln\frac{b}{a}}{2\ln(aR)\ln(bR)}. \quad [3.146]$$

Inserting [3.143], [3.144] and [3.146] into [3.141], we obtain

$$\frac{1}{\sqrt{\ln^2(aR)\ln^2(bR) + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}}\left(\ln^2(bR) + \frac{\left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}{\ln^2(aR)}\right) \approx \frac{b}{a}\frac{n^2\pi^2}{R^2} \approx 1. \quad [3.147]$$

Thus, the real and imaginary parts of ${}_Ps_n^{(\pm)}$, as $R \to \infty$, obtained by [3.146] and [3.147], are as stated in the proposition.

In order to prove that solutions to [3.138] are simple for $n > n_0$, we define

$$f(s) = \sinh(sM_s(s)), \quad s \in \mathbb{C}\backslash(-\infty, 0].$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}f(s) = M_s(s)\left(1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)}\right)\cosh(sM_s(s)).$$

Solutions to $f(s)=0$ are given by [3.138], and so, as $\left|{}_P s_n^{(\pm)}\right| \to \infty$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} f(s)\right|_{s={}_P s_n^{(\pm)}} \sim (-1)^n \left[M_s(s) \left(1 - \frac{\ln \frac{b}{a}}{2 \ln(as) \ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)} \right) \right]_{s={}_P s_n^{(\pm)}}.$$

From proposition 3.1, $M_s \sim \sqrt{\frac{a}{b}}$ and this implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s} f(s)\right|_{s={}_P s_n^{(\pm)}} \sim (-1)^n \sqrt{\frac{a}{b}} \quad \text{as} \quad \left|{}_P s_n^{(\pm)}\right| \to \infty.$$

Thus, for large $\left|{}_P s_n^{(\pm)}\right|$, we have $\left.\frac{\mathrm{d}}{\mathrm{d}s} f(s)\right|_{s={}_P s_n^{(\pm)}} \neq 0$ and solutions are simple for $n > n_0$. ■

Function $\tilde{Q}$ given by [3.136], with M_s given by [3.124], contains the natural logarithm and consequently has a branch point at $s=0$. It also has poles that are solutions of

$$\cosh(sM_s(s)) = 0 \quad \text{i.e.} \quad sM_s(s) = \pm \frac{2n+1}{2} \mathrm{i}\pi, \quad n \in \mathbb{N}_0. \qquad [3.148]$$

We state the proposition analogous to proposition 3.1.

PROPOSITION 3.3.– There are an infinite number of solutions ${}_Q s_n^{(\pm)}$, $n \in \mathbb{N}_0$, of [3.148], such that

$$\operatorname{Re}\left({}_Q s_n^{(\pm)}\right) \approx -\frac{\frac{\pi}{4} \ln \frac{b}{a} \sqrt{\frac{b}{a}} \frac{2n+1}{2} \pi}{\ln\left(\sqrt{ab} \frac{2n+1}{2} \pi\right) \ln\left(b \sqrt{\frac{b}{a}} \frac{2n+1}{2} \pi\right)},$$

$$\operatorname{Im}\left({}_Q s_n^{(\pm)}\right) \approx \pm \sqrt{\frac{b}{a}} \frac{2n+1}{2} \pi,$$

as $n \to \infty$. Moreover, there exists $n_0 \in \mathbb{N}_0$ such that the poles ${}_Q s_n^{(\pm)}$ are simple for $n > n_0$.

3.4.2.1. *Determination of the displacement u in a stress relaxation test*

We investigate properties of $\tilde{P}$, given by [3.130] with M_s taken in the form [3.124], and find a solution to [3.118], [3.119] and [3.120], with weight functions given by [3.116] in two cases.

1) The case when the boundary condition [3.120] is given by

$$\Upsilon(t) = \Upsilon_0 H(t), \quad \Upsilon_0 > 0, \ t \in \mathbb{R}, \tag{3.149}$$

which is discussed in section 3.4.2.1.1.

2) The case when the boundary condition [3.120] is given by

$$\Upsilon(t) = \Upsilon_0 H(t) + F(t), \quad t \in \mathbb{R}, \tag{3.150}$$

where F is a locally integrable function equal to zero on $(-\infty, 0]$, which is discussed in section 3.4.2.2.

3.4.2.1.1. Case $\boldsymbol{\Upsilon = \Upsilon_0 H}$

This is the case that has physical importance, since we obtain displacement in the case of a stress relaxation test. Formally, we write [3.131] with [3.149] as

$$u_H(x,t) = \Upsilon_0 H(t) * P(x,t), \quad x \in [0,1], \ t \in \mathbb{R}. \tag{3.151}$$

The following theorem is on existence and properties of u_H.

THEOREM 3.5.– Let $\Upsilon = \Upsilon_0 H$ and let ϕ_σ and ϕ_ε be given by [3.116]. Then the solution to [3.118], [3.119] and [3.120] is given by [3.151], where, for $x \in [0,1]$, $t > 0$,

$$\begin{aligned} P(x,t) &= \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left(\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)} - \frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)} \right) \mathrm{e}^{-qt} \mathrm{d}q \\ &\quad + \sum_{n=1}^{\infty} \left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(-)}\right) \right), \end{aligned} \tag{3.152}$$

$$P(x,t) = 0, \quad x \in [0,1], \ t < 0. \tag{3.153}$$

The residues are given by

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(\pm)}\right) = \left[\frac{\sinh\left(xsM_s(s)\right)}{\frac{\mathrm{d}}{\mathrm{d}s}\left[\sinh\left(sM_s(s)\right)\right]} \mathrm{e}^{st} \right]_{s = {}_P s_n^{(\pm)}} \tag{3.154}$$

and simple poles ${}_{P}s_n^{(\pm)}$, for $n > n_0$, are solutions of [3.138]. Function P is real-valued, locally integrable on $\mathbb{R}$ and smooth for $t > 0$.

The explicit form of solution, for $x \in [0,1]$, $t > 0$, is

$$\begin{aligned} u_H(x,t) &= \frac{\Upsilon_0}{2\pi\mathrm{i}} \int_0^\infty \left(\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)} - \frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)} \right) \frac{1-\mathrm{e}^{-qt}}{q} \mathrm{d}q \\ &+ \int_0^t \sum_{n=1}^\infty \left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{s\tau}, {}_{P}s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{s\tau}, {}_{P}s_n^{(-)}\right)\right) \mathrm{d}\tau, \end{aligned} \quad [3.155]$$

$$u_H(x,t) = 0, \quad x \in [0,1], \ t < 0. \quad [3.156]$$

Function u_H is continuous at $t = 0$.

PROOF.– We calculate $P(x,t)$, $x \in [0,1]$, $t \in \mathbb{R}$, by the integration over a suitable contour.

Let $t > 0$. The Cauchy residues theorem yields

$$\begin{aligned} \oint_\Gamma \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i} \sum_{n=1}^\infty \left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(+)}\right) \right. \\ &\left. + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(-)}\right)\right), \end{aligned} \quad [3.157]$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$ so that all poles lie inside the contour Γ (see Figure 3.12).

First, we show that the series of residues in [3.152] is convergent. From proposition 3.2, poles ${}_{P}s_n^{(\pm)}$ of $\tilde{P}$, given by [3.130] with M_s in the form [3.124], are simple for $n > n_0$. Then the residues in [3.157] can be calculated using [3.138] and [3.154] as

$$\begin{aligned} &\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(\pm)}\right) \\ &= (-1)^n \frac{\sin(n\pi x)}{n\pi} \left[\frac{s\mathrm{e}^{st}}{1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}} \right]_{s={}_{P}s_n^{(\pm)}}, \quad n > n_0. \end{aligned} \quad [3.158]$$

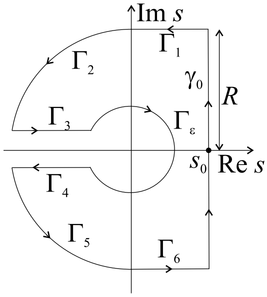

Figure 3.12. *Integration contour* Γ

If ${}_P s_n^{(\pm)} = R\mathrm{e}^{\pm \mathrm{i}\phi}$, then [3.158] transforms into

$$\begin{aligned}
&\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(\pm)}\right) \\
&\quad = (-1)^n \frac{\sin(n\pi x)}{n\pi} \frac{R\mathrm{e}^{Rt\cos\phi}\mathrm{e}^{\pm\mathrm{i}(\phi + Rt\sin\phi)}}{\left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}\right]_{s=R\mathrm{e}^{\pm\mathrm{i}\phi}}} \\
&\quad = (-1)^n \frac{\sin(n\pi x)}{n\pi} \frac{R\mathrm{e}^{Rt\cos\phi}\left[\cos(\phi + Rt\sin\phi) \pm \mathrm{i}\sin(\phi + Rt\sin\phi)\right]}{\left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}\right]_{s=R\mathrm{e}^{\pm\mathrm{i}\phi}}},
\end{aligned}$$

and therefore, for $n > n_0$, we have

$$\begin{aligned}
&\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(-)}\right) = (-1)^n \frac{\sin(n\pi x)}{n\pi} R\mathrm{e}^{Rt\cos\phi} \\
&\quad \times \left(\frac{\cos(\phi + Rt\sin\phi) + \mathrm{i}\sin(\phi + Rt\sin\phi)}{\left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}\right]_{s=R\mathrm{e}^{\mathrm{i}\phi}}} \right. \\
&\quad \left. + \frac{\cos(\phi + Rt\sin\phi) - \mathrm{i}\sin(\phi + Rt\sin\phi)}{\left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}\right]_{s=R\mathrm{e}^{-\mathrm{i}\phi}}} \right). \qquad [3.159]
\end{aligned}$$

When $n \to \infty$, $\left|{}_P s_n^{(\pm)}\right| \to \infty$, i.e. $R \to \infty$, then

$$\left| \left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{s(b-a)}{2(as-1)(bs-1)}\right]_{s=R\mathrm{e}^{\pm\mathrm{i}\phi}} \right| \to 1.$$

Also, as $n \to \infty$, [3.159] becomes

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(-)}\right)\right|$$
$$\approx 2\left|\frac{\sin(n\pi x)}{n\pi} R\mathrm{e}^{Rt\cos\phi}\cos(\phi + Rt\sin\phi)\right|.$$

Formula [3.139] implies that

$$\operatorname{Re}\left({}_P s_n^{(\pm)}\right) \approx -\frac{\pi}{4}\ln\frac{b}{a}\sqrt{\frac{b}{a}}\pi\frac{n}{\ln\left(\sqrt{ab}n\pi\right)\ln\left(b\sqrt{\frac{b}{a}}n\pi\right)} \leq -C\sqrt{n}, \quad n > n_0.$$

Also, from [3.140], we have $\frac{R}{n} \approx \sqrt{\frac{b}{a}}\pi$. This implies that summands in [3.157] can be estimated by $K\mathrm{e}^{-Ct\sqrt{n}}$, which implies the convergence of the sum of residues in [3.157].

We calculate now the integral over Γ in [3.157]. Consider the integral along contour Γ_1. Then

$$\left|\int_{\Gamma_1}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq \int_0^{s_0}\left|\tilde{P}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p.$$

Let $R \to \infty$. In order to investigate the asymptotic behavior of $\left|\tilde{P}(x,p\pm\mathrm{i}R)\right|$ as $R \to \infty$, we use proposition 3.1, point 3, and write

$$M_s(p \pm \mathrm{i}R) \sim v \pm \mathrm{i}w,$$
$$v = \sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\frac{\ln(aR)\ln(bR)}{\sqrt[4]{(\ln(aR)\ln(bR))^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}},$$
$$w = -\sqrt{\frac{a}{b}}\frac{1}{\ln(aR)}\frac{\frac{\pi}{2}\ln\frac{b}{a}}{\sqrt[4]{(\ln(aR)\ln(bR))^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}}.$$

Then, as $R \to \infty$,

$$\begin{aligned} \left|\tilde{P}\left(x, p \pm \mathrm{i}R\right)\right| &\sim \left|\frac{\sinh\left[x\left(pv - Rw\right) \pm \mathrm{i}x\left(pw + Rv\right)\right]}{\sinh\left[\left(pv - Rw\right) \pm \mathrm{i}\left(pw + Rv\right)\right]}\right| \\ &\leq \frac{\mathrm{e}^{x(pv-Rw)} + \mathrm{e}^{-x(pv-Rw)}}{\left|\mathrm{e}^{pv-Rw} - \mathrm{e}^{-(pv-Rw)}\right|} = \mathrm{e}^{-(1-x)(pv-Rw)} \\ &\quad \times \frac{1 + \mathrm{e}^{-2x(pv-Rw)}}{\left|1 - \mathrm{e}^{-2(pv-Rw)}\right|} \to 0. \end{aligned} \quad [3.160]$$

The previous statement is valid since, as $R \to \infty$,

$$\begin{aligned} pv - Rw &= \sqrt{\frac{a}{b}} \frac{1}{\ln\left(aR\right)} \frac{1}{\sqrt[4]{\left(\ln\left(aR\right)\ln\left(bR\right)\right)^2 + \left(\frac{\pi}{2}\ln\frac{b}{a}\right)^2}} \\ &\quad \times \left(p\ln\left(aR\right)\ln\left(bR\right) + R\frac{\pi}{2}\ln\frac{b}{a}\right) \\ &\sim \sqrt{\frac{a}{b}}\left(p\sqrt{\frac{\ln\left(bR\right)}{\ln\left(aR\right)}} + \frac{\pi}{2}\ln\frac{b}{a}\frac{R}{\ln\left(aR\right)\sqrt{\ln\left(aR\right)\ln\left(bR\right)}}\right) \to \infty. \end{aligned}$$

Therefore, according to [3.160], we have $\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{P}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right| = 0$. By using [3.160], we conclude that similar arguments are valid for the integral along the contour Γ_6. Thus, we have $\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{P}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right| = 0$. Next, we consider the integral along the contour Γ_2

$$\left|\int_{\Gamma_2}\tilde{P}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right| \leq \int_{\frac{\pi}{2}}^{\pi} R\left|\mathrm{e}^{R(1-x)\mathrm{e}^{\mathrm{i}\phi}M_s\left(R\mathrm{e}^{\mathrm{i}\phi}\right)}\right|\left|\frac{\mathrm{e}^{2xR\mathrm{e}^{\mathrm{i}\phi}M_s\left(R\mathrm{e}^{\mathrm{i}\phi}\right)} - 1}{\mathrm{e}^{2R\mathrm{e}^{\mathrm{i}\phi}M_s\left(R\mathrm{e}^{\mathrm{i}\phi}\right)} - 1}\right|\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi.$$

Since $M_s \sim \sqrt{\frac{a}{b}}$ as $|s| \to \infty$ and $\cos\phi \leq 0$ for $\phi \in \left[\frac{\pi}{2}, \pi\right]$, by the Lebesgue theorem, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{P}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right| \leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi} R\,\mathrm{e}^{R\cos\phi\left(t+(1-x)\sqrt{\frac{a}{b}}\right)}\mathrm{d}\phi = 0.$$

Similar arguments are valid for the integral along the contour Γ_5; so $\lim_{R\to\infty}\left|\int_{\Gamma_5}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0$. The integration along the contour Γ_ε gives

$$\begin{aligned}\lim_{\varepsilon\to 0}\left|\int_{\Gamma_\varepsilon}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &= \lim_{\varepsilon\to 0}\int_{\pi}^{-\pi}\varepsilon\left|\mathrm{e}^{-\varepsilon(1-x)\mathrm{e}^{\mathrm{i}\phi}M_s\left(\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)}\right| \\ &\quad \left|\frac{1-\mathrm{e}^{-2x\varepsilon\mathrm{e}^{\mathrm{i}\phi}M_s\left(\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)}}{1-\mathrm{e}^{-2\varepsilon\mathrm{e}^{\mathrm{i}\phi}M_s\left(\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)}}\right|\mathrm{e}^{\varepsilon t\cos\phi}\mathrm{d}\phi = 0.\end{aligned}$$

Integrals along Γ_3, Γ_4 and γ_0 give

$$\begin{aligned}\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_3}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= \int_0^\infty\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\mathrm{e}^{-qt}\mathrm{d}q, \\ \lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_4}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= -\int_0^\infty\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\mathrm{e}^{-qt}\mathrm{d}q, \\ \lim_{R\to\infty}\int_{\gamma_0}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i}P(x,t)\,.\end{aligned}$$

Now, by the Cauchy residues theorem [3.157], function P is determined by [3.152].

In order to see that P is a real-valued function, we use the fact that for $q \in [0,\infty)$, $M_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right) = \overline{M_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)}$, where the bar denotes the complex conjugation. Due to the exponential in the hyperbolic sine, we have $\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right) = \overline{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}$ and, therefore, the integrand in [3.152] is of the form

$$\begin{aligned}&\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}-\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)} \\ &\qquad = \overline{\left(\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right)}-\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)} \\ &= -2\mathrm{i}\,\mathrm{Im}\left(\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right),\end{aligned}$$

which implies that the first term in [3.152] is real.

Next, we examine $\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(\pm)}\right)$ in order to prove that the sum of residues is also real. By [3.158] and the fact that

$$\left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{(b-a)\,s}{2\,(as-1)\,(bs-1)}\right]_{s={}_{P}s_n^{(-)}}$$
$$= \overline{\left(\left[1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{(b-a)\,s}{2\,(as-1)\,(bs-1)}\right]_{s={}_{P}s_n^{(+)}}\right)}$$

we obtain that $\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(-)}\right) = \overline{\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(+)}\right)}$. It is clear that

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(-)}\right)$$
$$= 2\operatorname{Re}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, {}_{P}s_n^{(+)}\right)\right).$$

This implies that the second term in [3.152] is also real for $n \in \mathbb{N}$. Hence, [3.152] is a real-valued function.

Let $t < 0$. We prove that the integral over γ_0 does not depend on the choice of s_0 (see Figure 3.12). Let $\bar{\Gamma} = \gamma_0 \cup \gamma_1 \cup \gamma_0' \cup \gamma_2$ (see Figure 3.13), where s_0 and s_0' are chosen so that all poles, i.e., solutions of [3.138], lie on the left of γ_0. The Cauchy residues theorem yields for $x \in [0,1]$

$$\oint_{\bar{\Gamma}} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 0.$$

This and [3.160] imply

$$\lim_{R\to\infty}\left|\int_{\gamma_1} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq \lim_{R\to\infty}\int_{s_0}^{s_0'}\left|\tilde{P}(x,v+\mathrm{i}R)\right|\left|\mathrm{e}^{(v+\mathrm{i}R)t}\right|\mathrm{d}v = 0.$$

Similar arguments hold for the integral along γ_2. Therefore, from the Cauchy residues theorem, integrals along γ_0 and γ_0' are the same and the inversion of the Laplace transformation does not depend either on the choice of s_0 or on the choice of s_0'.

Again, the Cauchy residues theorem yields for $x \in [0,1]$

$$\oint_{\tilde{\Gamma}} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 0,$$

where $\tilde{\Gamma} = \gamma_0 \cup \Gamma_r$ (see Figure 3.14), with the assumption that all poles, i.e. solutions of [3.138], lie on the left of γ_0.

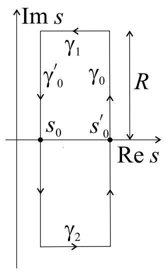

Figure 3.13. *Integration contour* $\bar{\Gamma}$

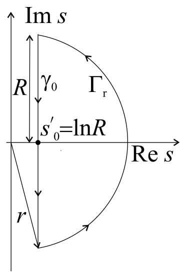

Figure 3.14. *Integration contour* $\tilde{\Gamma}$

Let $r(R) = \sqrt{R^2 + s_0^2}$. Then

$$\left|\int_{\Gamma_r} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq \int_{-\phi_0(r(R))}^{\phi_0(r(R))} \sqrt{R^2+s_0^2} \times \left|\tilde{P}\left(x, \sqrt{R^2+s_0^2}\mathrm{e}^{\mathrm{i}\phi}\right)\right| \mathrm{e}^{t\sqrt{R^2+s_0^2}\cos\phi}\mathrm{d}\phi,$$

where $\lim_{R\to\infty} \phi_0(r(R)) = \frac{\pi}{2}$. Since $M_s \sim \sqrt{\frac{a}{b}}$ as $|s| \to \infty$, from [3.130] and [3.124], we have, as $|s| \to \infty$,

$$\left|\tilde{P}(x,s)\right| = \left|\mathrm{e}^{-(1-x)sM_s(s)}\frac{1-\mathrm{e}^{-2xsM_s(s)}}{1-\mathrm{e}^{-2sM_s(s)}}\right| \leq C, \quad x \in [0,1],\ s \in \mathbb{C}\backslash(-\infty,0]. \qquad [3.161]$$

From [3.161], we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_r} \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq C \lim_{R\to\infty}\int_{-\phi_0(r(R))}^{\phi_0(r(R))} \sqrt{R^2+s_0^2}\mathrm{e}^{t\sqrt{R^2+s_0^2}\cos\phi}\mathrm{d}\phi = 0,$$

since $t < 0$ and $\cos\phi > 0$. Therefore, we proved [3.153].

By using [3.152] and [3.153] in [3.151] and by calculating the convolution, we obtain [3.155] and [3.156].

In order to prove that u_H is a continuous function at $t = 0$, we will use the Lebesgue dominated convergence theorem. Let

$$f(q,x) = \frac{\Upsilon_0}{2\pi\mathrm{i}}\left(\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)} - \frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right),$$
$$q \in (0,\infty),\ x \in [0,1].$$

Then

$$\left|\int_0^\infty f(q,x)\,\frac{1-\mathrm{e}^{-qt}}{q}\mathrm{d}q\right| \to 0 \ \text{ as } \ t \to 0. \qquad [3.162]$$

From simple calculations, we have $\frac{1-\mathrm{e}^{-qt}}{q} \leq Ct$ if $0 < q < 1$ and $\frac{1-\mathrm{e}^{-qt}}{q} \leq 1-\mathrm{e}^{-qt}$ if $q \geq 1$. Thus

$$f(q,x)\,\frac{1-\mathrm{e}^{-qt}}{q} \leq Cf(q,x)\,,\quad q > 0$$

and since $\frac{1-\mathrm{e}^{-qt}}{q} \to 0$ as $t \to 0$, [3.162] follows.

In proving the continuity of u_H at $t = 0$, by [3.151], we estimated

$$\int_0^t P(x,\tau)\,\mathrm{d}\tau,\quad x \in [0,1]\,,\ t > 0,$$

and actually proved that P is an integrable function on any interval $[0,T]\,, T > 0$. Thus, P is locally integrable on $\mathbb{R}$. ■

3.4.2.2. *Case* $\Upsilon = \Upsilon_0 H + F$

CONDITION 3.1.– Let F be a locally integrable function, equal to zero for $t \leq 0$, such that its Laplace transform exists in $\mathbb{C}\backslash(-\infty,0]$. Assume the following:

1) $\tilde{F}$ is analytic and $\tilde{F} \neq 0$ in $\mathbb{C}\backslash(-\infty,0]$;

2) for some $\alpha > 1$, $\tilde{F}(s) \sim \frac{1}{|s|^\alpha}$, $s \in \mathbb{C}\backslash(-\infty,0]$, as $|s| \to \infty$;

3) $s\tilde{F}(s) \sim o(1)$, $s \in \mathbb{C} \backslash (-\infty, 0]$, as $|s| \to 0$.

If the boundary condition [3.120] is given by [3.150], then the solution to [3.118], [3.119] and [3.120], given by [3.129] in the Laplace domain, reads formally

$$\tilde{u}(x,s) = \tilde{u}_H(x,s) + \tilde{F}(s)\tilde{P}(x,s), \quad x \in [0,1], \ s \in \mathbb{C} \backslash (-\infty, 0],$$

which in the time domain is

$$u(x,t) = u_H(x,t) + F(t) * P(x,t), \quad x \in [0,1], \ t \in \mathbb{R}.$$

The existence of u_H is shown in section 3.4.2.1.1; therefore, what remains is to show the existence of

$$u_F(x,t) = F(t) * P(x,t), \quad x \in [0,1], \ t \in \mathbb{R}.$$

For $t > 0$, $x \in [0,1]$, we apply the Cauchy residues theorem and obtain

$$\begin{aligned} \oint_\Gamma \tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{u}_F(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(+)}\right)\right. \\ &\quad \left. + \operatorname{Res}\left(\tilde{u}_F(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(-)}\right)\right), \end{aligned} \qquad [3.163]$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$ (see Figure 3.12. Since poles ${}_P s_n^{(\pm)}$ of $\tilde{u}_F$ are actually the poles of $\tilde{P}$, which are obtained from [3.138] and are simple for $n > n_0$, the residues in [3.163] can be calculated as

$$\operatorname{Res}\left(\tilde{u}_F(x,s)\,\mathrm{e}^{st}, {}_P s_n^{(\pm)}\right) = \left[\tilde{F}(s)\,\frac{\sinh(xsM_s(s))}{\frac{\mathrm{d}}{\mathrm{d}s}\left[\sinh(sM_s(s))\right]}\mathrm{e}^{st}\right]_{s={}_P s_n^{(\pm)}}. \qquad [3.164]$$

The proof that the sum in [3.163] converges is analogous to the one presented in section 3.4.2.1.1.

Consider the integral along the contour Γ_1. It reads

$$\left|\int_{\Gamma_1} \tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq \int_0^{s_0} \left|\tilde{F}(p+\mathrm{i}R)\right| \left|\tilde{P}(x,p+\mathrm{i}R)\right| \left|\mathrm{e}^{(p+\mathrm{i}R)t}\right| \mathrm{d}p.$$

According to [3.161] and condition 3.1, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right|\leq C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{\left(\sqrt{p^2+R^2}\right)^{\alpha}}\mathrm{e}^{pt}\mathrm{d}p=0.$$

The integral along the contour Γ_2 reads

$$\begin{aligned}\left|\int_{\Gamma_2}\tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \int_{\frac{\pi}{2}}^{\pi}\left|\tilde{F}\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{R(1-x)\mathrm{e}^{\mathrm{i}\phi}M_s\left(R\mathrm{e}^{\mathrm{i}\phi}\right)}\right| \\ &\quad\times\left|\frac{\mathrm{e}^{2xR\mathrm{e}^{\mathrm{i}\phi}M_s\left(R\mathrm{e}^{\mathrm{i}\phi}\right)}-1}{\mathrm{e}^{2R\mathrm{e}^{\mathrm{i}\phi}M_s(R\mathrm{e}^{\mathrm{i}\phi})}-1}\right|\mathrm{e}^{Rt\cos\phi}R\,\mathrm{d}\phi.\end{aligned}$$

In order to apply the Lebesgue theorem, we need $\alpha>1$ in condition 3.1 (actually it is enough to have $\alpha\geq 1$, but the case $\alpha=1$ is already considered). Since $M_s\sim\sqrt{\frac{a}{b}}$ as $|s|\to\infty$ and $\cos\phi\leq 0$ for $\phi\in\left[\frac{\pi}{2},\pi\right]$, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right|\leq C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}R^{1-\alpha}\mathrm{e}^{R\cos\phi\left(t+(1-x)\sqrt{\frac{a}{b}}\right)}\mathrm{d}\phi=0.$$

Similar arguments are valid for the integral along the contour Γ_5. Thus

$$\lim_{R\to\infty}\left|\int_{\Gamma_5}\tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right|=0.$$

The integration along the contour Γ_ε gives

$$\begin{aligned}\left|\int_{\Gamma_\varepsilon}\tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \int_{-\pi}^{\pi}\left|\tilde{F}\left(\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{-\varepsilon(1-x)\mathrm{e}^{\mathrm{i}\phi}M_s\left(\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)}\right| \\ &\quad\left|\frac{1-\mathrm{e}^{-2x\varepsilon\mathrm{e}^{\mathrm{i}\phi}M_s\left(\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)}}{1-\mathrm{e}^{-2\varepsilon\mathrm{e}^{\mathrm{i}\phi}M_s(\varepsilon\mathrm{e}^{\mathrm{i}\phi})}}\right|\mathrm{e}^{\varepsilon t\cos\phi}\varepsilon\mathrm{d}\phi,\end{aligned}$$

and this tends to 0 as $\varepsilon \to 0$, according to condition 3.1. Integrals along the contours Γ_3, Γ_4 and γ_0 give

$$\begin{aligned}
\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\Gamma_3} \tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= \int_0^\infty \tilde{F}\left(q\mathrm{e}^{\mathrm{i}\pi}\right) \frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)} \mathrm{e}^{-qt}\mathrm{d}q, \\
\lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_{\Gamma_4} \tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= -\int_0^\infty \tilde{F}\left(q\mathrm{e}^{-\mathrm{i}\pi}\right) \frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)} \mathrm{e}^{-qt}\mathrm{d}q, \\
\lim_{R\to\infty} \int_{\gamma_0} \tilde{u}_F(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i}\,u_F(x,t).
\end{aligned}$$

Now, from the Cauchy residues theorem [3.163], u_F is determined as

$$\begin{aligned}
u_F(x,t) &= \frac{1}{2\pi\mathrm{i}} \int_0^\infty \left(\tilde{F}\left(q\mathrm{e}^{-\mathrm{i}\pi}\right) \frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)} \right. \\
&\quad \left. -\tilde{F}\left(q\mathrm{e}^{\mathrm{i}\pi}\right) \frac{\sinh\left(xM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)} \right) \mathrm{e}^{-qt}\mathrm{d}q \\
&\quad + \sum_{n=1}^{\infty} \left(\operatorname{Res}\left(\tilde{u}_F \mathrm{e}^{st}, {}_P s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{u}_F \mathrm{e}^{st}, {}_P s_n^{(-)}\right) \right), \\
&\qquad x \in [0,1],\ t > 0, \qquad [3.165] \\
u_F(x,t) &= 0, \quad x \in [0,1],\ t < 0,
\end{aligned}$$

where the residues are given by [3.164]. Note that u_F is a locally integrable, real-valued function, which can be shown similarly in section 3.4.2.1.1.

Therefore, in the case when the boundary condition takes the form [3.150], the solution to systems [3.118], [3.119] and [3.120] reads

$$u(x,t) = u_H(x,t) + u_F(x,t), \quad x \in [0,1],\ t > 0,$$

where u_H and u_F are given by [3.155] and [3.165], respectively. Note that u_H and u_F are equal to 0 for $t < 0$. Again, u is a smooth function for $x \in [0,1]$, $t > 0$.

3.4.2.3. *Determination of the stress σ in a stress relaxation test*

We see that $\tilde{T}$, given by [3.133] with M_s in the form [3.124], has the branch point at $s=0$ and poles at the same points as $\tilde{P}$. Therefore, the poles of $\tilde{T}$ are solutions to [3.138]. Using the Cauchy residues theorem

$$\begin{aligned}\oint_\Gamma \tilde{T}(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{T}(x,s)\,\mathrm{e}^{st},{}_Ps_n^{(+)}\right)\right.\\ &\quad\left.+\operatorname{Res}\left(\tilde{T}(x,s)\,\mathrm{e}^{st},{}_Ps_n^{(-)}\right)\right), \end{aligned} \qquad [3.166]$$

where the contour Γ is given in Figure 3.12, we obtain T in the following way. The residues in [3.166] are given by

$$\operatorname{Res}\left(\tilde{T}(x,s)\,\mathrm{e}^{st},{}_Ps_n^{(\pm)}\right)=\left[\frac{\cosh(xsM_s(s))}{M_s(s)\,\frac{\mathrm{d}}{\mathrm{d}s}\left[\sinh(sM_s(s))\right]}\mathrm{e}^{st}\right]_{s={}_Ps_n^{(\pm)}}, \qquad [3.167]$$

where ${}_Ps_n^{(\pm)}$, $n\in\mathbb{N}$, are solutions of [3.138].

Evaluating the integral at the left-hand side of [3.166] in the same way as in section 3.4.2.1.1, we obtain

$$\begin{aligned}T(x,t) &= 1+\frac{1}{2\pi\mathrm{i}}\int_0^\infty\left(\frac{\cosh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{M_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right.\\ &\quad\left.-\frac{\cosh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{M_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\right)\mathrm{e}^{-qt}\mathrm{d}q\\ &\quad+\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{T}(x,s)\,\mathrm{e}^{st},{}_Ps_n^{(+)}\right)+\operatorname{Res}\left(\tilde{T}(x,s)\,\mathrm{e}^{st},{}_Ps_n^{(-)}\right)\right),\\ &\qquad x\in[0,1],\ t>0,\\ T(x,t) &= 0,\quad x\in[0,1],\ t<0,\end{aligned}$$

where the residues are given by [3.167]. The proof is analogous to the one presented in section 3.4.2.1.1.

Thus, from [3.134], we have

$$\sigma_H(x,t)=\Upsilon_0T(x,t),\quad x\in[0,1],\ t>0, \qquad [3.168]$$

if the boundary conditions [3.120] are given by [3.149]. Note that σ_H is a locally integrable function with the jump at $t = 0$ and smooth for $t > 0$. Also in the case when the boundary conditions are given by [3.120] and [3.150], we have

$$\begin{aligned} \sigma_F(x,t) &= \sigma_H(x,t) + \frac{\mathrm{d}}{\mathrm{d}t}\big(F(t) * T(x,t)\big), \;\; x \in [0,1], \; t > 0 \\ \sigma_F(x,t) &= 0, \;\; x \in [0,1], \; t < 0. \end{aligned}$$

This is a smooth function for $t > 0$. Note that σ_H and σ_F are real-valued functions, which can be shown similarly in section 3.4.2.1.1.

3.4.2.4. *Determination of displacement* u *in the case of prescribed stress*

We are now in a position to state our result concerning the existence of a solution to systems [3.118], [3.119] and [3.121]. Recall that this is the case of the prescribed stress.

THEOREM 3.6.– Let ϕ_σ and ϕ_ε be given by [3.116]. Then the solution to [3.118], [3.119] and [3.121] is given by [3.137], i.e. by

$$u(x,t) = \Sigma(t) * Q(x,t), \;\; x \in [0,1], \; t > 0,$$

where, for $x \in [0,1]$, $t > 0$,

$$\begin{aligned} Q(x,t) &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \Bigg(M_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{\cosh\left(qM_s\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)} \\ &\quad - M_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\frac{\sinh\left(xqM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{\cosh\left(qM_s\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\Bigg)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q \\ &\quad + \sum_{n=0}^{\infty}\left(\operatorname{Res}\left(\tilde{Q}(x,s)\,\mathrm{e}^{st}, {}_Q s_n^{(+)}\right) + \operatorname{Res}\left(\tilde{Q}(x,s)\,\mathrm{e}^{st}, {}_Q s_n^{(-)}\right)\right), \end{aligned} \tag{3.169}$$

$$Q(x,t) = 0, \;\; x \in [0,1], \; t < 0.$$

The residues are

$$\operatorname{Res}\left(\tilde{Q}(x,s)\,\mathrm{e}^{st}, {}_Q s_n^{(\pm)}\right) = \left[\frac{1}{s}M_s(s)\frac{\sinh(xsM_s(s))}{\frac{\mathrm{d}}{\mathrm{d}s}\left[\cosh(sM_s(s))\right]}\mathrm{e}^{st}\right]_{s={}_Q s_n^{(\pm)}}, \;\; n > n_0,$$

where ${}_{Q}s_n^{(\pm)}$ are simple poles for $n > n_0$, and they are solutions of [3.148]. For every fixed $x \in [0, 1]$, the function Q is real-valued, locally integrable on $\mathbb{R}$ and smooth for $t > 0$.

The proof of theorem 3.6 is analogous to the proof of theorem 3.5. For details, we refer the reader to [ATA 11d].

3.4.2.5. *Numerical examples*

3.4.2.5.1. Stress relaxation experiments

We give several examples in which the displacement u_H and the stress σ_H are determined numerically from [3.155] and [3.168], respectively. In Figure 3.15, we show displacements, determined according to [3.155], for three different positions. Parameters in [3.155] are chosen as follows: $\Upsilon_0 = 1$, $a = 0.045$, $b = 0.5$. From Figure 3.15, one can see that the further the point is from the free end of the rod, the greater the delay in the start of its oscillation. This indicates that the speed of the disturbance propagation is finite. The displacements of the points close to the free end are shown in Figure 3.16. The figure shows that the wave profiles of the waves approaching the free end deform in shape, so that the amplitudes of displacement do not exceed the displacement of the free end, i.e. $u(1, t) = 1$. Also, from Figures 3.15 and 3.16, for large times, one sees that $\lim_{t\to\infty} u_H(x, t) = x$.

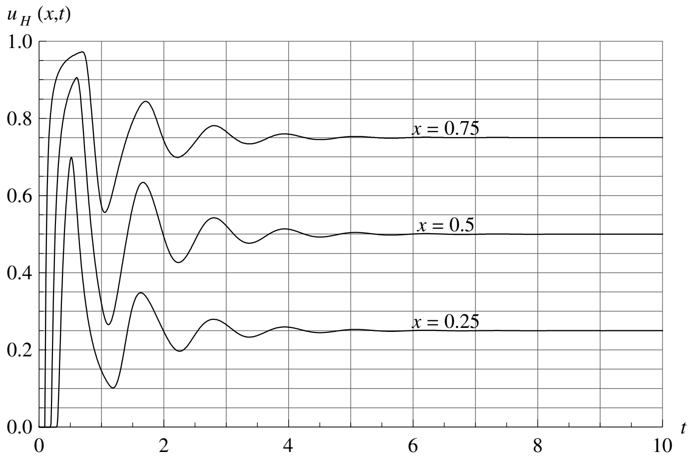

Figure 3.15. *Displacements $u_H(x, t)$ in a stress relaxation test as functions of time at $x \in \{0.25, 0.5, 0.75\}$ for $t \in (0, 10)$*

In Figures 3.17, 3.18, 3.19 and 3.20, we show the stresses determined according to [3.168] for the following values of parameters: $\Upsilon_0 = 1$, $a = 0.045$ and $b = 0.5$. In order to emphasize the stress relaxation process, we also show the curve corresponding to quasistatic deformation, usually used in the classical stress relaxation tests (see [ATA 02b, ATA 11b]). The quasistatic case is described by $[3.118]_2$ and $[3.118]_3$. In this case, we have $u(x,t) = x \cdot u(1,t) = x \cdot u_{QS}(t)$, $t > 0$, $x \in [0,1]$, i.e. $[3.118]_3$ implies $\varepsilon = u_{QS}$. Next, we use the Laplace transform of the boundary condition [3.120], given in the form [3.149], in $[3.122]_2$ and obtain

$$\tilde{\sigma}(x,s) = \tilde{\varepsilon}(x,s) \frac{\int_0^1 (bs)^\alpha \, \mathrm{d}\alpha}{\int_0^1 (as)^\alpha \, \mathrm{d}\alpha},$$

$$\tilde{\sigma}_{QS}(s) = \Upsilon_0 \frac{1}{s M_s^2(s)}, \quad s \in \mathbb{C} \backslash (-\infty, 0].$$

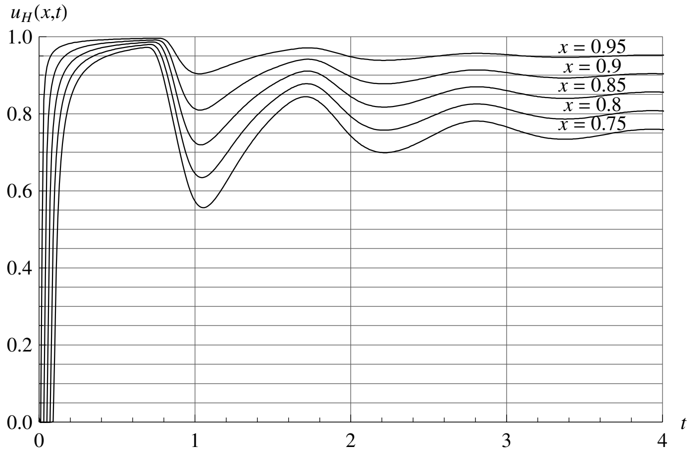

Figure 3.16. *Displacements $u_H(x,t)$ in a stress relaxation test as functions of time at $x \in \{0.75, 0.8, 0.85, 0.9, 0.95\}$ for $t \in (0,4)$*

Inverting the Laplace transform $\tilde{\sigma}_{QS}$, as in theorem 3.5, we obtain

$$\sigma_{QS}(t) = \Upsilon_0 + \frac{\Upsilon_0}{2\pi \mathrm{i}} \int_0^\infty \left(\frac{1}{M_s^2(q\mathrm{e}^{\mathrm{i}\pi})} - \frac{1}{M_s^2(q\mathrm{e}^{-\mathrm{i}\pi})} \right) \frac{\mathrm{e}^{-qt}}{q} \mathrm{d}q, \quad t > 0.$$

Figure 3.17 presents the time evolution of the stress at three points close to the fixed end of the rod. From Figure 3.17, one sees that there is a delay in the occurrence of the stress and that the delay is greater as the point is closer to the fixed end of the rod. This is a consequence of the finite wave speed. Also, it is evident that the stresses display the damped oscillatory character and that the curves tend to the curve of σ_{QS}, which is monotonically decreasing. Figure 3.18 shows stresses for the points that are in the middle part of the rod. One sees that the amplitude of the stress increases as the point is closer to the free end of the rod. Finally, in Figures 3.19 and 3.20, we present the time evolution of stresses at the points close to the free end. Figure 3.19 displays stresses at three different points. As can be seen, for the points closer to the free end, the time lag is smaller and the amplitude of stress is higher. For all three points shown, the stresses are positive (extension). Figure 3.20 displays stresses of the same points as Figure 3.19, but at the later time instant. For these points, in the beginning, we have positive stresses, while after some time the stresses become negative, indicating the compressive phase in the axial vibrations of the rod.

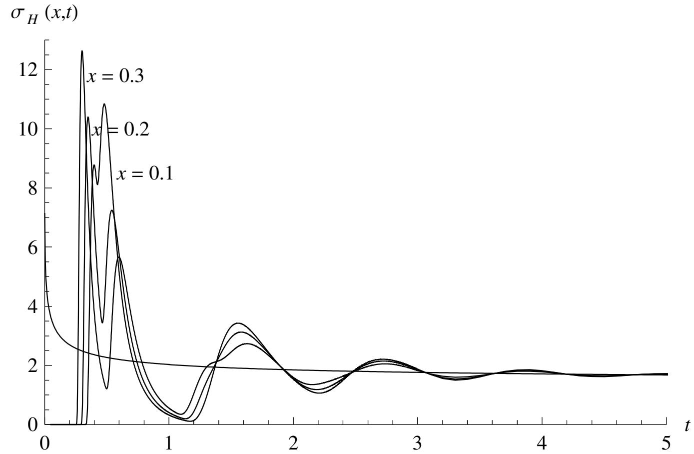

Figure 3.17. *Stresses $\sigma_H\,(x,t)$ in stress relaxation test as functions of time at $x \in \{0.1, 0.2, 0.3\}$ for $t \in (0,5)$*

All figures show the oscillatory character of both displacements and stresses. Oscillations are damped, and for large time displacements they show linear dependence on the distance of a particle from a fixed end, while stresses approach the limiting value independently of the position of the particle. In Figure 3.15, the first minimum of $u_H(x,t)$ at $x = 0.75$ is closer to the origin than the first minimum corresponding to $x = 0.25$. This shows that the position of the minima is not determined by the waves reflected for the ends of the rod only. The accumulation of material near the ends of the rod (ends are rigid) also influences the position of the minima, in the sense that the curves are deformed in shape, and the maxima and minima are not on the positions where one would expect them to be. This is seen from the fact that $u_H\,(1,t) = H\,(t)$. Therefore, $\frac{\partial}{\partial t} u_H\,(1,t) = \delta\,(t)$ and, in the limit when $x \to 1$, both minima and maxima of $u_H(x,t)$ are positioned arbitrarily close to the origin, since $\delta(t) = 0$ for $t \in (0,\infty)$. For large times, stress relaxation curves show a behavior similar to that of the curves obtained in the quasistatic case (see [ATA 02b, DRO 98]).

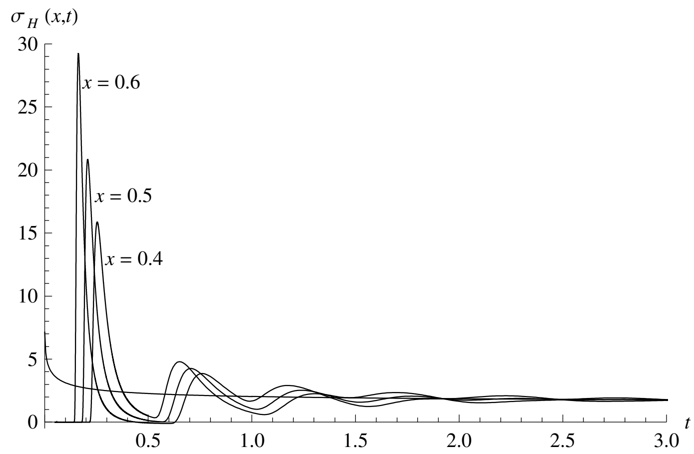

Figure 3.18. *Stresses $\sigma_H\ (x,t)$ in stress relaxation test as functions of time at $x \in \{0.4, 0.5, 0.6\}$ for $t \in (0, 2.5)$*

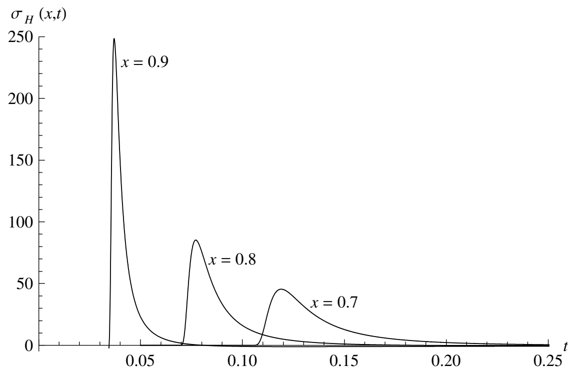

Figure 3.19. *Stresses $\sigma_H\ (x,t)$ in stress relaxation test as functions of time at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 0.25)$*

3.4.2.5.2. Creep experiments

In this section, we examine the solutions given by [3.137], which correspond to displacement u in the cases of creep and forced oscillations. Choosing $\Sigma(t) = \Sigma_0 H(t)$, $t > 0$, the displacement u, given by [3.137], corresponds to creep.

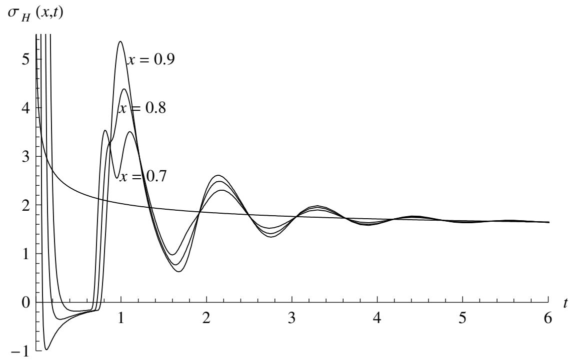

Figure 3.20. *Stresses $\sigma_H(x,t)$ in stress relaxation test as functions of time at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0,6)$*

If $\Sigma(t) = \Sigma_0 \sin(\omega t)$, $t > 0$, $\omega > 0$, then [3.137] describes the displacement u in the case of the forced oscillations of a rod, with angular frequency ω.

Let us consider the case when sudden but constant stress is applied at the rod's free end. It is mathematically described through boundary condition [3.121] given by

$$\Sigma(t) = \Sigma_0 H(t), \quad t \in \mathbb{R}, \tag{3.170}$$

where Σ_0 is a positive constant. When Σ is given by [3.170], the displacement is determined from [3.137] as

$$u(x,t) = \Sigma_0 H(t) * Q(x,t), \quad x \in [0,1], \ t \in \mathbb{R}, \tag{3.171}$$

where Q is given by [3.169]. In Figure 3.21, we present u given by [3.171] with parameters $\Sigma_0 = 1$, $a = 0.045$, $b = 0.5$; the upper bound in the integral is $1,000$ and the number of residues in the sum is 400. The time interval is $(0,5)$. From Figure 3.21, we see that the displacement depends on the position of a point. Also, we conclude that displacements are zero for a certain time interval at points far from the free end. Note this delay in displacement of a point $x = 0.5$. This is a consequence of the fact that waves in a rod have a finite speed of propagation.

The values of a and b are chosen arbitrarily with the only restriction $a \leq b$ (see [3.116]). Since $a = b$ corresponds to the linearly elastic (Hookean) body, we selected $a \ll b$. Although there are experimental data for the fractional order Kelvin–Voigt, Maxwell and Zener viscoelastic bodies (see [GRA 10, HEY 03, LIU 06, WEL 99]), we are not aware of any experimental data corresponding to constitutive equation [3.111] with constitutive functions given by [3.116].

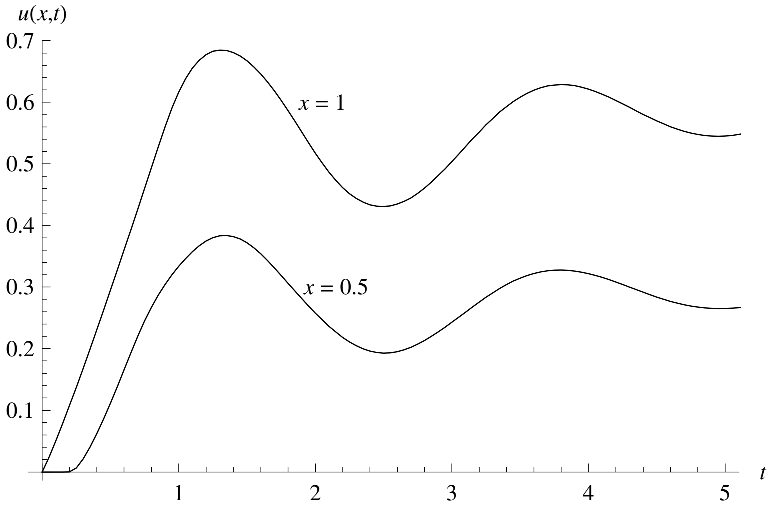

Figure 3.21. *Displacements* $u(x,t)$ *in a creep experiment as functions of time at* $x = 0.5$, $x = 1$ *for* $t \in (0,5)$

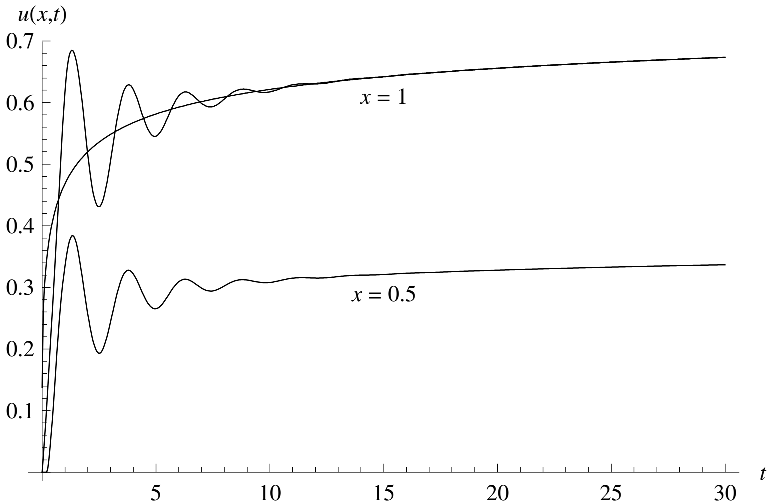

Figure 3.22. *Displacements* $u(x,t)$ *in a creep experiment as functions of time at* $x = 0.5$, $x = 1$ *for* $t \in (0,30)$

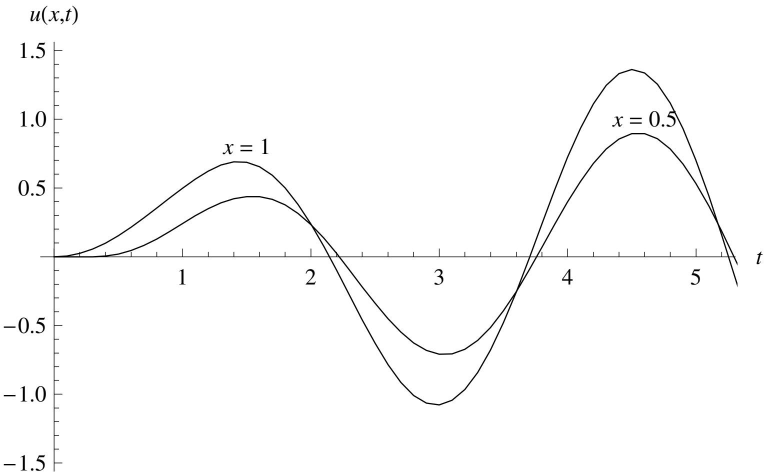

Figure 3.23. *Displacements* $u\,(x,t)$ *in a forced oscillations experiment as functions of time at* $x = 0.5$, $x = 1$ *for* $t \in (0,5)$

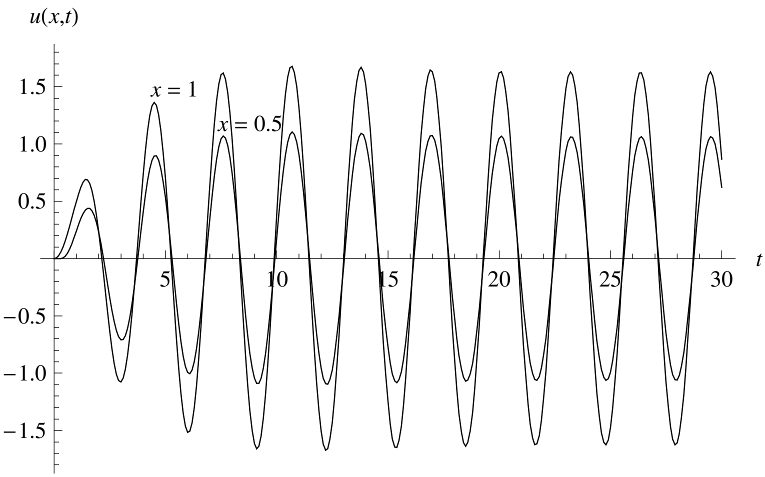

Figure 3.24. *Displacements* $u\,(x,t)$ *in a forced oscillations experiment as functions of time at* $x = 0.5$, $x = 1$ *for* $t \in (0,30)$

Next, we compare the behavior of displacement of a point at $x = 1$ for large times with the quasistatic case, which is described by [3.118]$_2$ and [3.118]$_3$. In this case, we have $u(x,t) = x \cdot u(1,t) = x \cdot u_{QS}(t)$, $t > 0$, $x \in [0,1]$, i.e. [3.118]$_3$ implies $\varepsilon = u_{QS}$. Using the Laplace transform of [3.170] in [3.122]$_2$, we obtain

$$\begin{aligned}
&\tilde{\varepsilon}(x,s) = \tilde{\sigma}(x,s)\, M_s^2(s), \\
&\tilde{u}_{QS}(s) = \Sigma_0 \frac{1}{s} M_s^2(s), \quad s \in \mathbb{C} \backslash (-\infty, 0].
\end{aligned}$$

This implies, as in theorem 3.6

$$u_{QS}(t) = \Sigma_0 + \frac{\Sigma_0}{2\pi \mathrm{i}} \int_0^\infty \left(M_s^2\left(q\mathrm{e}^{\mathrm{i}\pi}\right) - M_s^2\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right) \frac{\mathrm{e}^{-qt}}{q} \mathrm{d}q, \quad t > 0. \qquad [3.172]$$

In Figure 3.22, we show the displacement $u(\cdot, t)$, $t \in (0, 30)$, given by [3.171], as well as the quasistatic displacement u_{QS}, given by [3.172], in the same time interval. We see that the curve that represents u displays damped oscillatory character and that the curve tends to the curve of u_{QS}, which monotonically increases. It is evident that the displacement is exhibiting creep in the rod and tends to a finite value for a large time. Also, the quasistatic approximation [3.172] agrees well with the dynamic solution [3.171] for large times. Thus, creep curves tend to curves corresponding to quasistatic analysis (see [ATA 02b, DRO 98]).

Let us consider the case when the periodic stress is applied at the rod's free end. It is mathematically described through boundary condition [3.121] as

$$\Sigma(t) = \begin{cases} \Sigma_0 \sin(\omega t), & t \geq 0, \\ 0, & t < 0, \end{cases} \qquad [3.173]$$

where Σ_0 is the amplitude and ω is the angular frequency of the applied force. When Σ is given by [3.173], the displacement is determined from [3.137] as

$$u(x,t) = \Sigma_0 \sin(\omega t) * Q(x,t), \quad x \in [0,1], \ t \in \mathbb{R}, \qquad [3.174]$$

where again Q is given by [3.169]. In Figures 3.23 and 3.24, we present u given by [3.174] with parameters $\Sigma_0 = 1$, $a = 0.1$, $b = 0.5$, $\omega = 2$; the upper bound in the integral is $1,000$ and the number of residues in the sum is 400. Figure 3.23 shows the delay in the displacement function for $x = 0.5$. In Figure 3.24, we present oscillations for large times. Again, the delay in displacement depends on the position of a point. After the delay, the motion is oscillatory, with the increasing amplitude and angular frequency equal to the angular frequency of the force applied at the free end. For large t, the amplitude tends to a constant finite value.

3.4.3. *Case of fluid-like viscoelastic body*

We consider systems [3.118], [3.119], either [3.120], or [3.121] in the case of the fluid-like viscoelastic body, described by the constitutive distributions [3.117]. The fact that the body is fluid-like is reflected in function M_f, given by [3.125]. Our aim is to find the displacement u and stress σ as functions of coordinate and time in the case when Υ is prescribed by [3.120] in the form given by [3.131] and [3.134], respectively. In particular, we are interested in the case $\Upsilon = \Upsilon_0 H$, which corresponds to the stress relaxation. We also aim to find the displacement u as a function of coordinate and time in the case when Σ is prescribed by [3.121] in the form given by [3.137].

We begin with examining some basic properties of M_f given in [3.125], which will be important for further investigations. In the following, we will again write $A(x) \sim B(x)$ if $\lim_{x\to\infty} \frac{A(x)}{B(x)} = 1$.

PROPOSITION 3.4.– Let M_f be the function defined by [3.125].

1) Then M_f is an analytic function on $\mathbb{C}\backslash(-\infty, 0]$ if $0 < \beta < \alpha < \gamma < 1$.

2) For $s \in \mathbb{C}\backslash(-\infty, 0]$, $\lim_{|s|\to 0} M_f(s) = \infty$, $\lim_{|s|\to 0} sM_f(s) = 0$, $\lim_{|s|\to\infty} M_f(s) = 0$, and $\lim_{|s|\to\infty} sM_f(s) = \infty$.

PROOF.– 1) Since $1 + \frac{a}{b}s^{\alpha-\beta} \neq 0$ and $1 + \frac{c}{a}s^{\gamma-\alpha} + \frac{c}{b}s^{\gamma-\beta} \neq 0$ if $\arg s \in (-\pi, \pi)$ and $0 < \beta < \alpha < \gamma < 1$, it follows that the function M_f given in [3.125] is analytic on the complex plane except the branch cut along the negative axis, i.e. on $\mathbb{C}\backslash(-\infty, 0]$.

2) Limits in point 2 can easily be calculated. ■

To begin with, we examine properties of $\tilde{P}$ given by [3.130]. $\tilde{P}$ has isolated singularities at ${}_Ps_n^{(\pm)}$, $n \in \mathbb{N}$, where ${}_Ps_n^{(\pm)}$ denotes the solutions of the equation

$$\sinh(sM_f(s)) = 0, \quad \text{i.e.,} \quad sM_f(s) = \pm n\mathrm{i}\pi. \qquad [3.175]$$

We examine their position and multiplicity.

PROPOSITION 3.5.– There are an infinite number of complex conjugated solutions ${}_Ps_n^{(\pm)}$, $n \in \mathbb{N}$, of [3.175], which all lie in the left complex half-plane. Moreover, each ${}_Ps_n^{(\pm)}$, $n \in \mathbb{N}$, is a simple zero of [3.175].

PROOF.– Let us square [3.175] and define

$$\Phi(s,n) = (sM_f(s))^2 + (n\pi)^2 = s^2 \frac{1 + \frac{a}{b}s^{\alpha-\beta}}{as^\alpha + cs^\gamma + \frac{ac}{b}s^{\alpha+\gamma-\beta}} + (n\pi)^2.$$

Writing $s = R\mathrm{e}^{\mathrm{i}\varphi}$, it follows that

$$
\begin{aligned}
\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) &= R^2\mathrm{e}^{2\mathrm{i}\varphi}\frac{1+\frac{a}{b}R^{\alpha-\beta}\mathrm{e}^{\mathrm{i}(\alpha-\beta)\varphi}}{aR^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\varphi}+cR^{\gamma}\mathrm{e}^{\mathrm{i}\gamma\varphi}+\frac{ac}{b}R^{\alpha+\gamma-\beta}\mathrm{e}^{\mathrm{i}(\alpha+\gamma-\beta)\varphi}}+(n\pi)^2 \\
&= R^2(\cos(2\varphi)+\mathrm{i}\sin(2\varphi))\frac{A+\mathrm{i}B}{C+\mathrm{i}D}+(n\pi)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= 1+\frac{a}{b}R^{\alpha-\beta}\cos((\alpha-\beta)\varphi), \quad B=\frac{a}{b}R^{\alpha-\beta}\sin((\alpha-\beta)\varphi), \\
C &= aR^{\alpha}\cos(\alpha\varphi)+cR^{\gamma}\cos(\gamma\varphi)+\frac{ac}{b}R^{\alpha+\gamma-\beta}\cos((\alpha+\gamma-\beta)\varphi), \\
D &= aR^{\alpha}\sin(\alpha\varphi)+cR^{\gamma}\sin(\gamma\varphi)+\frac{ac}{b}R^{\alpha+\gamma-\beta}\sin((\alpha+\gamma-\beta)\varphi).
\end{aligned}
$$

Real and imaginary parts of Φ are then given by

$$
\begin{aligned}
\operatorname{Re}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) &= \frac{R^2}{C^2+D^2}\left[(AC+BD)\cos(2\varphi)\right. \\
&\quad \left.+(AD-BC)\sin(2\varphi)\right]+(n\pi)^2, && [3.176] \\
\operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) &= \frac{R^2}{C^2+D^2}\left[(AC+BD)\sin(2\varphi)\right. \\
&\quad \left.-(AD-BC)\cos(2\varphi)\right], && [3.177]
\end{aligned}
$$

and

$$
\begin{aligned}
AC+BD &= aR^{\alpha}\cos(\alpha\varphi)+cR^{\gamma}\cos(\gamma\varphi)+\frac{a^2}{b}R^{2\alpha-\beta}\cos(\beta\varphi) \\
&\quad +\frac{a^2c}{b^2}R^{2(\alpha-\beta)+\gamma}\cos(\gamma\varphi) \\
&\quad +2\frac{ac}{b}R^{\alpha+\gamma-\beta}\cos((\alpha-\beta)\varphi)\cos(\gamma\varphi), \\
AD-BC &= aR^{\alpha}\sin(\alpha\varphi)+cR^{\gamma}\sin(\gamma\varphi)+\frac{a^2}{b}R^{2\alpha-\beta}\sin(\beta\varphi) \\
&\quad +\frac{a^2c}{b^2}R^{2(\alpha-\beta)+\gamma}\sin(\gamma\varphi) \\
&\quad +2\frac{ac}{b}R^{\alpha+\gamma-\beta}\cos((\alpha-\beta)\varphi)\sin(\gamma\varphi).
\end{aligned}
$$

Let (R, φ) be a solution to $\Phi(s, n) = 0$ (or equivalently, $\operatorname{Re}\Phi = \operatorname{Im}\Phi = 0$). Then changing $\varphi \to -\varphi$, we again obtain that $\operatorname{Re}\Phi = \operatorname{Im}\Phi = 0$, which implies that solutions of [3.175] are complex conjugated.

Further, we will prove that the function Φ has no zeros in the half-plane $\arg s \in [0, \frac{\pi}{2}]$. For that purpose, we will use the argument principle. Recall that if Φ is an analytic function inside and on a regular closed curve c and non-zero on c, then the number of zeros of Φ is given by $N_Z = \frac{1}{2\pi}\Delta \arg \Phi(s)$. Let $\gamma = \gamma_a \cup \gamma_b \cup \gamma_c$ be parameterized as

$$\gamma_a : s = x,\ x \in [0, R],\ \ \gamma_b : s = R\mathrm{e}^{\mathrm{i}\varphi},\ \varphi \in \left[0, \frac{\pi}{2}\right],\ \ \gamma_c : s = x\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\ x \in [0, R],$$

and let $R \to \infty$. Along γ_a, function Φ becomes a real-valued function; hence, $\Delta \arg \Phi(s, n) = 0$. Along γ_b, we have $AC + BD, AD - BC \geq 0$ for $\varphi \in [0, \pi]$, since $\sin(\eta\varphi), \cos(\eta\varphi) > 0$, for $\eta \in \{\alpha, \beta, \gamma, \alpha - \beta\} \subset (0, \frac{1}{2})$, and $\varphi \in [0, \pi]$. Therefore, [3.176] and [3.177] imply that

$$\operatorname{Re}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) > 0,\ \ \varphi \in \left[0, \frac{\pi}{4}\right], \qquad [3.178]$$

$$\operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) > 0,\ \ \varphi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]. \qquad [3.179]$$

Inequalities at boundary points are easily checked by inserting them into [3.176] and [3.177]. Along γ_c, we obtain

$$\operatorname{Re}\Phi(x\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, n) = -\left.\frac{R^2}{C^2 + D^2}\right|_{R=x,\varphi=\frac{\pi}{2}} (AC + BD)|_{R=x,\varphi=\frac{\pi}{2}} + (n\pi)^2, \qquad [3.180]$$

$$\operatorname{Im}\Phi(x\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, n) = \left.\frac{R^2}{C^2 + D^2}\right|_{R=x,\varphi=\frac{\pi}{2}} (AD - BC)|_{R=x,\varphi=\frac{\pi}{2}} > 0. \qquad [3.181]$$

From [3.178], [3.179] and [3.181], we may now conclude that along γ_b and γ_c,

$$\Delta \arg \Phi(s, n) = 0.$$

Indeed, this follows from the following: along γ_b, in the case $\varphi \in [0, \frac{\pi}{4}]$, $\operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n)$ can change its sign, but $\operatorname{Re}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) > 0$, while for $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$, $\operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) > 0$. Along γ_c, $\operatorname{Im}\Phi(x\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, n) > 0$. Therefore, there is no change in the argument of Φ along the whole γ, which implies that Φ has no zeros for

$\varphi \in [0, \frac{\pi}{2}]$. This further implies that [3.175] has no solutions in the right complex half-plane, since its solutions are complex conjugated.

In order to prove that for fixed $n \in \mathbb{N}$ equation [3.175] has one solution (and its complex conjugate), we again use function Φ and the argument principle. Now consider the contour $\Gamma = \Gamma_A \cup \Gamma_B \cup \Gamma_C$ parameterized by

$$\begin{aligned}\Gamma_A : s &= x\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},\ x \in [0, R],\ \ \Gamma_B : s = R\mathrm{e}^{\mathrm{i}\varphi},\ \varphi \in \left[\frac{\pi}{2}, \pi\right],\ \ \Gamma_C : s = x\mathrm{e}^{\mathrm{i}\pi},\\ &\quad x \in [0, R],\end{aligned}$$

and let $R \to \infty$. Along Γ_A, real and imaginary parts of Φ are given by [3.180] and [3.181]. Along Γ_B, using [3.176] and [3.177], we conclude that

$$\operatorname{Re}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) < 0,\ \ \varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{4}\right] \ \text{ and }\ \operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) < 0,\ \ \varphi \in \left[\frac{3\pi}{4}, \pi\right],$$

for $R \to \infty$, and n fixed. Along Γ_C we have

$$\begin{aligned}\operatorname{Re}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) &= \left.\frac{R^2}{C^2 + D^2}\right|_{R=x,\varphi=\pi} (AC + BD)|_{R=x,\varphi=\pi} + (n\pi)^2 > 0,\\ \operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\varphi}, n) &= -\left.\frac{R^2}{C^2 + D^2}\right|_{R=x,\varphi=\pi} (AD - BC)|_{R=x,\varphi=\pi} < 0.\end{aligned}$$

Along Γ_A, we have $\operatorname{Im}\Phi > 0$, while $\operatorname{Re}\Phi$ changes its sign (since $\operatorname{Re}\Phi(0, n) = (n\pi)^2$ and $\lim_{x\to\infty} \operatorname{Re}\Phi(x\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}, n) = -\infty$, for fixed $n \in \mathbb{N}$). Along the part of Γ_B where $\varphi \in [\frac{3\pi}{4}, \pi]$, and along Γ_C, $\operatorname{Im}\Phi < 0$. Also, $\lim_{R\to\infty} \operatorname{Re}\Phi(R\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}, n) = -\infty$ and $\operatorname{Im}\Phi(R\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}, n) < 0$. This implies that the argument of Φ changes from 0 to 2π.

As a conclusion, we obtain that along Γ

$$\Delta \arg \Phi(s, n) = 2\pi,$$

which implies, by the argument principle, that function Φ has exactly one zero in the upper left complex plane, for each fixed $n \in \mathbb{N}$. Since the zeros of Φ are complex conjugated, it follows that Φ also has one zero in the lower left complex plane, for each fixed $n \in \mathbb{N}$. ■

In proposition 3.6, we examine the behavior of simple poles ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$.

PROPOSITION 3.6.– Solutions ${}_P s_n^{(\pm)}$, $n \in \mathbb{N}$, of [3.175] are such that

$$\operatorname{Re}\left({}_P s_n^{(\pm)}\right) = R\cos\varphi \sim \sqrt[2-\gamma]{c(n\pi)^2}\cos\left(\frac{\pi}{2-\gamma}\right) < 0,$$
$$\operatorname{Im}\left({}_P s_n^{(\pm)}\right) = R\sin\varphi \sim \pm\sqrt[2-\gamma]{c(n\pi)^2}\sin\left(\frac{\pi}{2-\gamma}\right),$$

as $n \to \infty$.

PROOF.– Let us square [3.175] and insert ${}_P s_n^{(\pm)} = R\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in (-\pi, \pi)$. Then, after separation of real and imaginary parts, we obtain

$$R^2\cos(2\varphi)\operatorname{Re}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) - R^2\sin(2\varphi)\operatorname{Im}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) = -(n\pi)^2, \qquad [3.182]$$
$$R^2\sin(2\varphi)\operatorname{Re}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) + R^2\cos(2\varphi)\operatorname{Im}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) = 0. \qquad [3.183]$$

Using notation from the proof of proposition 3.5, we can write

$$\operatorname{Re}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) = \frac{AC+BD}{C^2+D^2} \quad \text{and} \quad \operatorname{Im}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) = \frac{BC-AD}{C^2+D^2}.$$

Letting $R \to \infty$, we have

$$\operatorname{Re}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) \sim \frac{\frac{a^2c}{b^2}R^{2(\alpha-\beta)+\gamma}\cos(\gamma\varphi)}{\frac{a^2c^2}{b^2}R^{2(\alpha-\beta)+2\gamma}} = \frac{1}{cR^\gamma}\cos(\gamma\varphi), \qquad [3.184]$$
$$\operatorname{Im}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi})) \sim -\frac{\frac{a^2c}{b^2}R^{2(\alpha-\beta)+\gamma}\sin(\gamma\varphi)}{\frac{a^2c^2}{b^2}R^{2(\alpha-\beta)+2\gamma}} = -\frac{1}{cR^\gamma}\sin(\gamma\varphi). \qquad [3.185]$$

It now follows from [3.183], [3.184] and [3.185] that

$$\tan(2\varphi) = -\frac{\operatorname{Im}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi}))}{\operatorname{Re}(M_f^2(R\mathrm{e}^{\mathrm{i}\varphi}))} \sim \tan(\gamma\varphi) \quad \Rightarrow \quad \frac{\sin((2-\gamma)\varphi)}{\cos(2\varphi)\cos(\gamma\varphi)} \sim 0$$
$$\Rightarrow \quad \varphi \sim \pm\frac{\pi}{2-\gamma} \qquad [3.186]$$

Inserting [3.186] into [3.184] and [3.185], and subsequently into [3.182], we obtain

$$\frac{R^{2-\gamma}}{c}\cos\left(\frac{2\pi}{2-\gamma}\right)\cos\left(\frac{\gamma\pi}{2-\gamma}\right)+\frac{R^{2-\gamma}}{c}\sin\left(\frac{2\pi}{2-\gamma}\right)\sin\left(\frac{\gamma\pi}{2-\gamma}\right)\sim-(n\pi)^2$$
$$R\sim\sqrt[2-\gamma]{c(n\pi)^2}. \qquad [3.187]$$

Thus, the real and imaginary parts of ${}_{P}s_n^{(\pm)}$, as $R\to\infty$, are as claimed. ■

PROPOSITION 3.7.– Let $p\in(0,s_0)$, $s_0>0$. Then

$$M_f(p\pm\mathrm{i}R)\sim\frac{1}{\sqrt{cR^\gamma}}\mathrm{e}^{\mp\mathrm{i}\frac{\gamma\pi}{4}}$$

as $R\to\infty$.

PROOF.– Set $\mu=\sqrt{p^2+R^2}$ and $\nu=\arctan\frac{\pm R}{p}$. Then $\mu\sim R$ and $\nu\sim\pm\frac{\pi}{2}$, as $R\to\infty$. By [3.184] and [3.185], we have

$$M_f(\mu\mathrm{e}^{\mathrm{i}\nu})\sim\frac{1}{\sqrt{c\mu^\gamma}}\mathrm{e}^{\mp\mathrm{i}\frac{\gamma\nu}{4}},\quad \mu\to\infty,$$

as claimed. ■

Function $\tilde{Q}$ given by [3.136] is analytic on the complex plane except the branch cut $(-\infty,0]$, and has isolated singularities at solutions ${}_{Q}s_n^{(\pm)}$ of the equation

$$\cosh(sM_f(s))=0,\quad\text{i.e.,}\quad sM_f(s)=\pm\frac{2n+1}{2}\mathrm{i}\pi,\quad n\in\mathbb{N}_0. \qquad [3.188]$$

We state a proposition that is analogous to propositions 3.5 and 3.6. The proof is omitted since it follows the same lines as those of propositions 3.5 and 3.6.

PROPOSITION 3.8.–

1) There are an infinite number of complex conjugated solutions ${}_{Q}s_n^{(\pm)}$, $n\in\mathbb{N}_0$, of [3.188], which all lie in the left complex half-plane. Moreover, each ${}_{Q}s_n^{(\pm)}$, $n\in\mathbb{N}_0$, is a simple pole.

2) Solutions ${}_Qs_n^{(\pm)}$, $n \in \mathbb{N}_0$, of [3.188] are such that

$$\mathrm{Re}({}_Qs_n^{(\pm)}) = R\cos\varphi \sim \sqrt[2-\gamma]{c\left(\frac{2n+1}{2}\pi\right)^2}\cos\left(\frac{\pi}{2-\gamma}\right) < 0,$$

$$\mathrm{Im}({}_Qs_n^{(\pm)}) = R\sin\varphi \sim \pm\sqrt[2-\gamma]{c\left(\frac{2n+1}{2}\pi\right)^2}\sin\left(\frac{\pi}{2-\gamma}\right),$$

as $n \to \infty$.

3.4.3.1. *Determination of the displacement u in a stress relaxation test*

In order to obtain the explicit form of the solution u to initial-boundary value problems [3.118]–[3.120], we require the calculation of function P.

THEOREM 3.7.– The solution u to initial-boundary value problem [3.118], [3.119] and [3.120] is given by [3.131], i.e. $u(x,t) = \Upsilon(t) * P(x,t)$, where P, for $x \in [0,1]$, $t > 0$, takes the form

$$\begin{aligned} P(x,t) &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \left(\frac{\sinh(xqM_f(q\mathrm{e}^{-\mathrm{i}\pi}))}{\sinh(qM_f(q\mathrm{e}^{-\mathrm{i}\pi}))} - \frac{\sinh(xqM_f(q\mathrm{e}^{\mathrm{i}\pi}))}{\sinh(qM_f(q\mathrm{e}^{\mathrm{i}\pi}))}\right)\mathrm{e}^{-qt}\mathrm{d}q \\ &\quad + \sum_{n=1}^{\infty}\left(\mathrm{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(+)}\right) + \mathrm{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(-)}\right)\right). \end{aligned} \quad [3.189]$$

The residues at simple poles ${}_Ps_n^{(\pm)}$, $n \in \mathbb{N}$, are given by

$$\mathrm{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(\pm)}\right) = \left[\frac{\sinh(xsM_f(s))}{\frac{\mathrm{d}}{\mathrm{d}s}[\sinh(sM_f(s))]}\mathrm{e}^{st}\right]_{s={}_Ps_n^{(\pm)}}. \quad [3.190]$$

PROOF.– Function $P(x,t)$, $x \in [0,1]$, $t > 0$, will be calculated by the integration over a suitable contour. The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\mathrm{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(+)}\right) + \mathrm{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(-)}\right)\right), \quad [3.191]$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$ is such a contour that all poles lie inside the contour Γ (see Figure 3.12).

First, we show that the series of residues in [3.189] is convergent. From proposition 3.6, ${}_Ps_n^{(\pm)}$ are simple poles of $\tilde{P}$, and therefore also simple poles of $\mathrm{e}^{st}\tilde{P}$. The residues in [3.191] can be calculated as is given in [3.190], so

$$\operatorname{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(\pm)}\right) = \left[\frac{1}{\frac{\mathrm{d}}{\mathrm{d}s}(sM_f(s))} \frac{\sinh(xsM_f(s))}{\cosh(sM_f(s))}\mathrm{e}^{st}\right]_{s={}_Ps_n^{(\pm)}}.$$

Using [3.175], we have $\sinh(xsM_f(s)) = \pm\mathrm{i}\sin(xn\pi)$, while $\cosh(sM_f(s)) = (-1)^n$. Also, the calculation gives $\frac{\mathrm{d}}{\mathrm{d}s}(sM_f(s)) = M_f(s) + M_f(s)\cdot A(s)$, where

$$A(s) = 1 + \frac{\frac{a}{b}(\alpha-\beta)s^{\alpha-\beta}}{2(1+\frac{a}{b}s^{\alpha-\beta})} - \frac{a\alpha s^{\alpha} + c\gamma s^{\gamma} + \frac{ac}{b}(\alpha+\gamma-\beta)s^{\alpha+\gamma-\beta}}{2(as^{\alpha}+cs^{\gamma}+\frac{ac}{b}s^{\alpha+\gamma-\beta})}.$$

Now take ${}_Ps_n^{(\pm)} = R\mathrm{e}^{\pm\mathrm{i}\varphi}$. Then

$$\operatorname{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(\pm)}\right) = (-1)^n \frac{\sin(n\pi x)}{n\pi} \frac{R\mathrm{e}^{Rt\cos\varphi}\mathrm{e}^{\pm\mathrm{i}(\varphi+Rt\sin\varphi)}}{A(R\mathrm{e}^{\pm\mathrm{i}\varphi})}.$$

Let $n\to\infty$. Then also $|{}_Ps_n^{(\pm)}|\to\infty$, i.e. $R\to\infty$, and $|A(R\mathrm{e}^{\pm\mathrm{i}\varphi})|\to 1-\frac{\gamma}{2}$. Further, it follows from proposition 3.6 that

$$\operatorname{Re}({}_Ps_n^{(\pm)}) = R\cos\varphi \sim \sqrt[2-\gamma]{c(n\pi)^2}\cos\left(\frac{\pi}{2-\gamma}\right) \leq -Cn, \quad \text{for some } C>0,$$

and by [3.187], $\frac{R}{n} \sim \sqrt[2-\gamma]{c\pi^2}\cdot n^{\frac{\gamma}{2-\gamma}}$. Therefore, as $n\to\infty$,

$$\begin{aligned}
&\left|\operatorname{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(+)}\right) + \operatorname{Res}\left(\tilde{P}(x,s)\mathrm{e}^{st}, {}_Ps_n^{(-)}\right)\right| \\
&\qquad\leq \left|\frac{\sin(n\pi x)}{n\pi}\frac{R\mathrm{e}^{Rt\cos\varphi}}{A(R\mathrm{e}^{+\mathrm{i}\varphi})}\right| + \left|\frac{\sin(n\pi x)}{n\pi}\frac{R\mathrm{e}^{Rt\cos\varphi}}{A(R\mathrm{e}^{-\mathrm{i}\varphi})}\right| \\
&\qquad\leq \frac{1}{\pi}\frac{R}{n}\mathrm{e}^{-Cnt}\left(\frac{1}{|A(R\mathrm{e}^{+\mathrm{i}\varphi})|} + \frac{1}{|A(R\mathrm{e}^{-\mathrm{i}\varphi})|}\right) \\
&\qquad\sim \frac{4}{\pi(2-\gamma)}\sqrt[2-\gamma]{c\pi^2}\cdot n^{\frac{\gamma}{2-\gamma}}\mathrm{e}^{-Cnt},
\end{aligned}$$

which implies the convergence of the sum of residues in [3.191].

The integral over Γ in [3.191] remain to be calculated. Consider the integral along the contour $\Gamma_1 : s = p + \mathrm{i}R$, $s_0 > p > 0$. Then

$$\left| \int_{\Gamma_1} \tilde{P}(x,s) \mathrm{e}^{st} \mathrm{d}s \right| \leq \int_0^{s_0} \left| \tilde{P}(x, p + \mathrm{i}R) \right| \left| \mathrm{e}^{(p+\mathrm{i}R)t} \right| \mathrm{d}p.$$

Let $R \to \infty$. In order to estimate $|\tilde{P}(x, p \pm \mathrm{i}R)|$, using proposition 3.7, we write

$$M_f(p \pm \mathrm{i}R) \sim v \pm \mathrm{i}w, \quad v = \frac{1}{\sqrt{cR^\gamma}} \cos\left(\frac{\gamma\pi}{4}\right), \quad w = -\frac{1}{\sqrt{cR^\gamma}} \sin\left(\frac{\gamma\pi}{4}\right).$$

Then

$$\begin{aligned} \left| \tilde{P}(x, p \pm \mathrm{i}R) \right| &\sim \left| \frac{\sinh[x(pv - Rw) \pm \mathrm{i}x(pw + Rv)]}{\sinh[(pv - Rw) \pm \mathrm{i}(pw + Rv)]} \right| \\ &\leq \frac{\mathrm{e}^{x(pv-Rw)} + \mathrm{e}^{-x(pv-Rw)}}{|\mathrm{e}^{pv-Rw} - \mathrm{e}^{-(pv-Rw)}|} \\ &= \mathrm{e}^{-(1-x)(pv-Rw)} \frac{1 + \mathrm{e}^{-2x(pv-Rw)}}{|1 - \mathrm{e}^{-2(pv-Rw)}|} \to 0, \quad \text{as} \quad R \to \infty. \end{aligned}$$

[3.192]

The above convergence is valid since

$$pv - Rw = p\frac{1}{\sqrt{cR^\gamma}} \cos\left(\frac{\gamma\pi}{4}\right) + R\frac{1}{\sqrt{cR^\gamma}} \sin\left(\frac{\gamma\pi}{4}\right) \to \infty, \quad \text{as} \quad R \to \infty.$$

Therefore, according to [3.192], we have $\lim_{R\to\infty} \left| \int_{\Gamma_1} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s \right| = 0$. A similar argument is valid for the integral along Γ_6, thus $\lim_{R\to\infty} \left| \int_{\Gamma_6} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s \right| = 0$. Next, we consider the integral along the contour $\Gamma_2 : s = R\mathrm{e}^{\mathrm{i}\varphi}$, $\frac{\pi}{2} < \varphi < \pi$:

$$\begin{aligned} \left| \int_{\Gamma_2} \tilde{P}(x,s) \mathrm{e}^{st} \mathrm{d}s \right| &\leq \int_{\frac{\pi}{2}}^{\pi} R |\mathrm{e}^{-R(1-x)\mathrm{e}^{\mathrm{i}\varphi} M_f(R\mathrm{e}^{\mathrm{i}\varphi})}| \\ &\quad \times \left| \frac{1 - \mathrm{e}^{-2xR\mathrm{e}^{\mathrm{i}\varphi} M_f(R\mathrm{e}^{\mathrm{i}\varphi})}}{1 - \mathrm{e}^{-2R\mathrm{e}^{\mathrm{i}\varphi} M_f(R\mathrm{e}^{\mathrm{i}\varphi})}} \right| \mathrm{e}^{Rt\cos\varphi} \mathrm{d}\varphi. \end{aligned}$$

Since $sM_f(s) \to \infty$ as $|s| \to \infty$ (see Proposition 3.4 (ii)) and $\cos\varphi \leq 0$ for $\varphi \in [\frac{\pi}{2}, \pi]$, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_2} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s\right| \leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi} R|\mathrm{e}^{-R(1-x)\mathrm{e}^{\mathrm{i}\varphi}M_f(R\mathrm{e}^{\mathrm{i}\varphi})}|\mathrm{e}^{Rt\cos\varphi}\mathrm{d}\varphi = 0.$$

A similar argument is valid for the integral along Γ_5, thus $\lim_{R\to\infty}\left|\int_{\Gamma_5}\tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s\right| = 0$. Since $sM_f(s) \to 0$ as $|s| \to 0$ (see proposition 3.4, point 2), the integration along the contour $\Gamma_\varepsilon : s = \varepsilon\mathrm{e}^{\mathrm{i}\varphi}$, $\pi > \varphi > -\pi$, gives

$$\lim_{\varepsilon\to 0}\left|\int_{\Gamma_\varepsilon} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s\right| \leq \lim_{\varepsilon\to 0}\int_{\pi}^{-\pi} \varepsilon\left|\frac{\sinh(x\varepsilon\mathrm{e}^{\mathrm{i}\varphi}M_f(\varepsilon\mathrm{e}^{\mathrm{i}\varphi}))}{\sinh(\varepsilon\mathrm{e}^{\mathrm{i}\varphi}M_f(\varepsilon\mathrm{e}^{\mathrm{i}\varphi}))}\right|\mathrm{e}^{\varepsilon t\cos\varphi}\mathrm{d}\varphi = 0.$$

Integrals along the contours $\Gamma_3 : s = q\mathrm{e}^{\mathrm{i}\pi}$, $R > q > \varepsilon$, $\Gamma_4 : s = q\mathrm{e}^{-\mathrm{i}\pi}$, $\varepsilon < q < R$, and $\gamma_0 : s = s_0 + \mathrm{i}r$, $-R < r < R$, give

$$\begin{aligned}
\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_3} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s &= \int_0^\infty \frac{\sinh(xqM_f(q\mathrm{e}^{\mathrm{i}\pi}))}{\sinh(qM_f(q\mathrm{e}^{\mathrm{i}\pi}))}\mathrm{e}^{-qt}\mathrm{d}q,\\
\lim_{\substack{R\to\infty\\ \varepsilon\to 0}}\int_{\Gamma_4} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s &= -\int_0^\infty \frac{\sinh(xqM_f(q\mathrm{e}^{-\mathrm{i}\pi}))}{\sinh(qM_f(q\mathrm{e}^{-\mathrm{i}\pi}))}\mathrm{e}^{-qt}\mathrm{d}q,\\
\lim_{R\to\infty}\int_{\gamma_0} \tilde{P}(x,s)\mathrm{e}^{st}\mathrm{d}s &= 2\pi\mathrm{i}P(x,t).
\end{aligned}$$

[3.189] now follows from [3.191]. ∎

COROLLARY 3.3.– In the case of the stress relaxation, i.e. when $\Upsilon(t) = \Upsilon_0 H(t)$, $\Upsilon_0 > 0$, $t \in \mathbb{R}$, the solution takes the form

$$u_H(x,t) = \Upsilon_0 H(t) * P(x,t), \quad x \in [0,1],\ t \in \mathbb{R}. \qquad [3.193]$$

We will numerically examine it in the following.

3.4.3.2. *Determination of the stress σ in a stress relaxation test*

We determined the stress σ (see [3.134]) that is a solution to [3.118], [3.119] and [3.120]. In order to obtain an explicit form of σ, we need to calculate function T. As in the previous section 3.4.3.1, it will be done by inversion of the Laplace transform of $\tilde{T}$.

Function $\tilde{T}$, which is given by [3.133], is analytic on the complex plane except the branch cut $(-\infty, 0]$, and has simple poles at the same points as $\tilde{P}$, i.e. ${}_{P}s_n^{(\pm)}$, $n \in \mathbb{N}$.

Using similar arguments as those in the proof of theorem 3.7, we can prove the following theorem:

THEOREM 3.8.– The solution σ to initial-boundary value problems [3.118]–[3.120] is given by [3.134], i.e. $\sigma(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}(\Upsilon(t) * T(x,t))$, where T, for $x \in [0,1]$, $t > 0$, takes the form

$$\begin{aligned} T(x,t) \quad = \quad & \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left(\frac{\cosh(xqM_f(q\mathrm{e}^{\mathrm{i}\pi}))}{M_f(q\mathrm{e}^{\mathrm{i}\pi}) \sinh(qM_f(q\mathrm{e}^{\mathrm{i}\pi}))} \right. \\ & \left. - \frac{\cosh(xqM_f(q\mathrm{e}^{-\mathrm{i}\pi}))}{M_f(q\mathrm{e}^{-\mathrm{i}\pi}) \sinh(qM_f(q\mathrm{e}^{-\mathrm{i}\pi}))} \right) \mathrm{e}^{-qt} \mathrm{d}q \\ & + \sum_{n=1}^{\infty} \left(\operatorname{Res}\left(\tilde{T}(x,s)\mathrm{e}^{st}, {}_{P}s_n^{(+)} \right) + \operatorname{Res}\left(\tilde{T}(x,s)\mathrm{e}^{st}, {}_{P}s_n^{(-)} \right) \right). \end{aligned}$$

The residues at simple poles ${}_{P}s_n^{(\pm)}$, $n \in \mathbb{N}$, are given by

$$\operatorname{Res}\left(\tilde{T}(x,s)\mathrm{e}^{st}, {}_{P}s_n^{(\pm)} \right) = \left[\frac{\cosh(xsM_f(s))}{M_f(s) \frac{\mathrm{d}}{\mathrm{d}s}[\sinh(sM_f(s))]} \mathrm{e}^{st} \right]_{s={}_{P}s_n^{(\pm)}}.$$

COROLLARY 3.4.– In the case of stress relaxation $\Upsilon = \Upsilon_0 H$, we obtain the solution

$$\sigma_H(x,t) = \Upsilon_0 T(x,t), \quad x \in [0,1], \; t > 0, \qquad [3.194]$$

In order to check our results for large times, we compare them with the quasistatic case. In the quasistatic case, one uses only the constitutive equation $[3.118]_3$ with the constitutive distributions taken in as in [3.117], i.e. the dynamics of the process is neglected. Taking the Laplace transform of the constitutive equation, we obtain $[3.122]_2$, and define the relaxation modulus G via its Laplace transform as follows:

$$\begin{aligned} G(t) &= \mathcal{L}^{-1}[\tilde{G}(s)](t), \quad t > 0, \\ \tilde{G}(s) &= \frac{\tilde{\sigma}^{(QS)}(s)}{\tilde{\varepsilon}^{(QS)}(s)} = \frac{1}{M_f^2(s)}, \quad s \in \mathbb{C} \backslash (-\infty, 0], \end{aligned}$$

where M_f is given by [3.125]. Then the stress in the quasistatic case is

$$\sigma^{(QS)} = G * \varepsilon^{(QS)}. \qquad [3.195]$$

Following the proof of theorem 3.7, we obtain that

$$G(t) = \frac{1}{2\pi \mathrm{i}} \int_0^{\infty} \left(\tilde{G}(q\mathrm{e}^{-\mathrm{i}\pi}) - \tilde{G}(q\mathrm{e}^{\mathrm{i}\pi}) \right) \mathrm{e}^{-qt} \mathrm{d}t, \;\; t > 0.$$

In the quasistatic case, it holds that

$$u(x,t) = x \cdot u(1,t), \;\; x \in [0,1], \; t > 0,$$

and consequently by [3.118],

$$\varepsilon(x,t) = u(1,t) = \varepsilon^{(QS)}(t), \;\; x \in [0,1], \; t > 0.$$

Since according to boundary condition [3.120], $u(1,t) = \Upsilon(t)$, it follows from [3.195] that

$$\sigma^{(QS)} = \Upsilon * G,$$

which, in the case of stress relaxation, i.e. when $\Upsilon = \Upsilon_0 H$, $\Upsilon_0 > 0$, becomes

$$\sigma_H^{(QS)} = \Upsilon_0 H * G. \qquad [3.196]$$

3.4.3.3. *Determination of the displacement u in a creep test*

We determined the displacement u (see [3.137]) that is a solution to [3.118], [3.119] and [3.121]. As above, we now want to find u explicitly by calculating the inverse Laplace transform of $\tilde{Q}$.

In the following theorem, we present the displacement u in the explicit form.

THEOREM 3.9.– The solution u to initial-boundary value problems [3.118]–[3.121] is given by [3.137], i.e. $u(x,t) = \Sigma(t) * Q(x,t)$, where Q, for $x \in [0,1]$, $t > 0$, takes the form

$$\begin{aligned} Q(x,t) \;\; = \;\; & \frac{1}{2\pi \mathrm{i}} \int_0^{\infty} \left(M_f(q\mathrm{e}^{-\mathrm{i}\pi}) \frac{\sinh(xqM_f(q\mathrm{e}^{-\mathrm{i}\pi}))}{\cosh(qM_f(q\mathrm{e}^{-\mathrm{i}\pi}))} \right. \\ & \left. - M_f(q\mathrm{e}^{\mathrm{i}\pi}) \frac{\sinh(xqM_f(q\mathrm{e}^{\mathrm{i}\pi}))}{\cosh(qM_f(q\mathrm{e}^{\mathrm{i}\pi}))} \right) \frac{\mathrm{e}^{-qt}}{q} \mathrm{d}q \\ & + \sum_{n=0}^{\infty} \left(\mathrm{Res}\left(\tilde{Q}(x,s)\mathrm{e}^{st}, {}_{Q}s_n^{(+)} \right) + \mathrm{Res}\left(\tilde{Q}(x,s)\mathrm{e}^{st}, {}_{Q}s_n^{(-)} \right) \right). \end{aligned}$$

The residues at simple poles ${}_Qs_n^{(\pm)}$, $n \in \mathbb{N}_0$, are given by

$$\operatorname{Res}\left(\tilde{Q}(x,s)\mathrm{e}^{st}, {}_Qs_n^{(\pm)}\right) = \left[\frac{1}{s} M_f(s) \frac{\sinh(xsM_f(s))}{\frac{\mathrm{d}}{\mathrm{d}s}[\cosh(sM_f(s))]} \mathrm{e}^{st}\right]_{s={}_Qs_n^{(\pm)}}.$$

COROLLARY 3.5.– The case of creep is described by the boundary condition $\Sigma(t) = \Sigma_0 H(t)$, $\Sigma_0 > 0$, $t \in \mathbb{R}$, in which the displacement u, given by [3.137], reads

$$u(t) = \Sigma_0 H(t) * Q(x,t), \quad x \in [0,1], \ t \in \mathbb{R}. \qquad [3.197]$$

Similarly to section 3.4.3.2, we examine the quasistatic case that corresponds to displacement u. Again, using only the constitutive equation $[3.118]_3$ and its Laplace transform $[3.122]_2$, we define the creep compliance J via its Laplace transform as

$$J(t) = \mathcal{L}^{-1}[\tilde{J}(s)](t), \quad t > 0,$$
$$\tilde{J}(s) = \frac{\tilde{\varepsilon}^{(QS)}(s)}{\tilde{\sigma}^{(QS)}(s)} = M_f^2(s), \quad s \in \mathbb{C}\backslash(-\infty, 0],$$

where M_f is given by [3.125]. Strain measure in the quasistatic case now equals

$$\varepsilon^{(QS)} = J * \sigma^{(QS)}. \qquad [3.198]$$

Following the proof of theorem 3.7, we obtain

$$J(t) = \frac{1}{2\pi \mathrm{i}} \int_0^\infty \left(\tilde{J}(q\mathrm{e}^{-\mathrm{i}\pi}) - \tilde{J}(q\mathrm{e}^{\mathrm{i}\pi})\right) \mathrm{e}^{-qt} \mathrm{d}t, \quad t > 0.$$

In the quasistatic case, it holds that

$$u(x,t) = x \cdot u(1,t) = x \cdot u^{(QS)}(t), \quad x \in [0,1], \ t > 0,$$

and consequently from [3.118],

$$\varepsilon^{(QS)} = u^{(QS)}. \qquad [3.199]$$

Also, $\sigma^{(QS)}(t) = \sigma(1,t) = \Sigma(t)$, $t > 0$, which is the boundary condition [3.121]; hence, from [3.198] and [3.199],

$$u^{(QS)} = \Sigma * J.$$

In the case of creep (see corollary 3.5), we have

$$u_H^{(QS)} = \Sigma_0 H * J. \qquad [3.200]$$

3.4.3.4. *Numerical examples*

In this section, we give several numerical examples of displacement u_H and stress σ_H, given by [3.193] and [3.194], respectively, which correspond to the case of stress relaxation, and examine the solution [3.197], which correspond to displacement u in the case of creep. In addition, we investigate solutions u_H, σ_H and u for different orders of fractional derivatives.

Figure 3.25 presents displacements in a stress relaxation experiment, determined according to [3.193], for three different positions. Parameters in [3.193] are chosen as follows: $\Upsilon_0 = 1$, $a = 0.2$, $b = 0.6$, $c = 0.45$, $\alpha = 0.3$, $\beta = 0.1$ and $\gamma = 0.4$. From Figure 3.25, it is seen that the displacements in the case of stress relaxation show the damped oscillatory character and that they tend to a constant value for large times, namely $\lim_{t\to\infty} u_H(x,t) = x$, $x \in [0,1]$. Figure 3.26 presents the same displacements as Figure 3.25, but close to the initial time instant. It is evident that there is a delay in the displacement that increases as the point is further from the end where the prescribed displacement is applied. This is a consequence of the finite wave propagation speed.

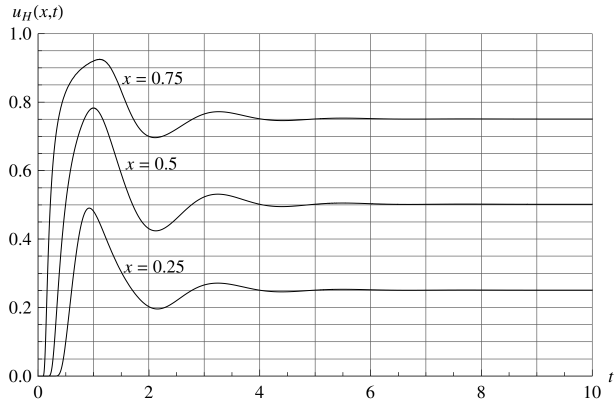

Figure 3.25. *Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time at $x \in \{0.25, 0.5, 0.75\}$ for $t \in (0,10)$*

Figure 3.27 presents the displacements u_H of the points close to the end of the rod, where the sudden but afterward constant displacement is applied, i.e. $u(1,t) = H(t)$, $t > 0$. It is seen that the amplitudes of these points deform in shape so that displacements do not exceed the prescribed value of the displacement of the rod's free end.

In order to examine the influence of the orders of fractional derivatives in the constitutive equation $[3.118]_2$, with [3.117], on the displacement u_H and stress σ_H

in a stress relaxation experiment, we plot the displacement u_H and stress σ_H obtained by [3.193] and [3.194] for the following sets of parameters $(\alpha, \beta, \gamma) \in \{(0.1, 0.05, 0.15), (0.3, 0.1, 0.4), (0.45, 0.4, 0.49)\}$, while we fix $x = 0.5$ and leave other parameters as before. In the case of the first set, the constitutive equation $[3.118]_2$, with [3.117], describes a body in which the elastic properties are dominant, since the orders of the fractional derivatives of stress and strain are close to zero, i.e. the fractional derivatives of stress and strain almost coincide with the stress and strain.

This is also evident from Figure 3.28, since oscillations of the point $x = 0.5$ for the first set of parameters vanish quite slowly compared to the second set and, in particular, compared to the third set of parameters. Note that the third set of parameters describes a body in which the fluid properties of the fractional type dominate, since in the constitutive equation $[3.118]_2$, with [3.117], we have the low-order derivative of the stress (almost the stress itself), while almost all the derivatives of strain are of order 0.5. Figure 3.28 also shows that the dissipative properties of a material grow as the orders of the fractional derivatives increase. Figure 3.29 shows that the delay in displacement depends on the order of the fractional derivative so that a material that has dominant elastic properties (the first set) has the longest delay, compared to a material with the dominant fluid properties (the third set).

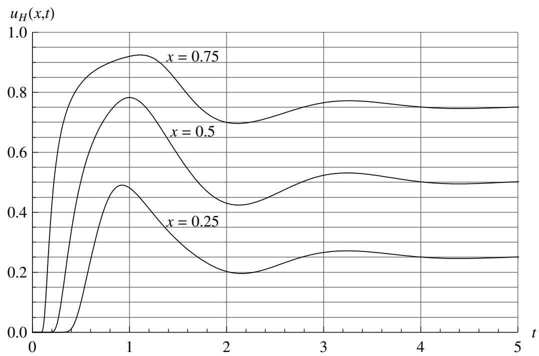

Figure 3.26. *Displacements $u_H(x, t)$ in a stress relaxation experiment as functions of time at $x \in \{0.25, 0.5, 0.75\}$ for $t \in (0, 5)$*

Figure 3.30 presents stresses σ_H in the case of stress relaxation, determined according to [3.194], for different points of the rod. Parameters are the same as the previous case. Also, Figure 3.30 presents the quasistatic curve $\sigma_H^{(QS)}$ obtained by [3.196].

Stresses, as can be seen in Figure 3.30, show the damped oscillatory character and for large times, in each point $x \in [0, 1]$, tend to the quasistatic curve, i.e. to the same value. Eventually, the stresses in all points of the rod tend to zero, namely $\lim_{t\to\infty} \sigma_H(x,t) = 0$, $x \in [0,1]$. From Figure 3.30, it is evident that the further the point is from the rod's free end, the greater the delay. This is again the consequence of the finite wave speed.

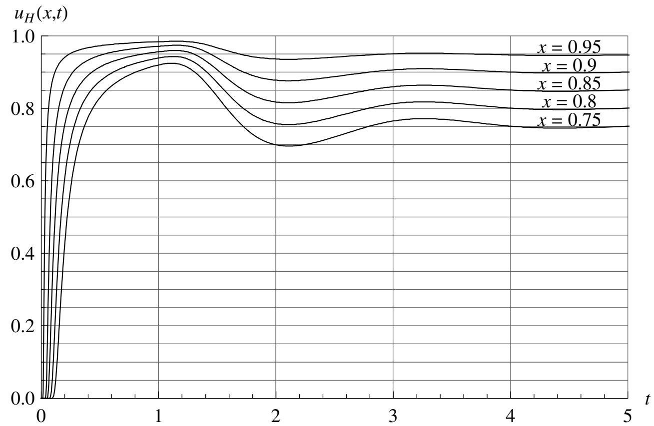

Figure 3.27. *Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time at $x \in \{0.75, 0.8, 0.85, 0.9, 0.95\}$ for $t \in (0,5)$*

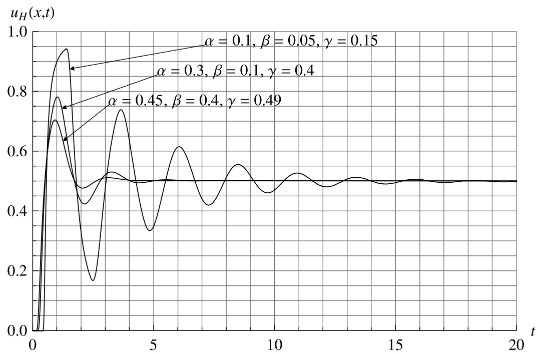

Figure 3.28. *Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time at $x = 0.5$ for $t \in (0,20)$*

Figures 3.31 and 3.32 present the stresses of the points close to the free end. We notice from Figures 3.30 and 3.31 that as the point is closer to the end whose displacement is prescribed, the peak of the stress is higher and its width is smaller. Figure 3.32 presents the compressive phase in the stress relaxation process as well as the quasistatic curve.

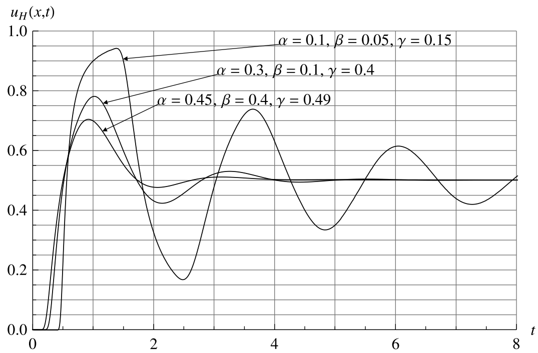

Figure 3.29. *Displacements $u_H(x,t)$ in a stress relaxation experiment as functions of time at $x = 0.5$ for $t \in (0,8)$*

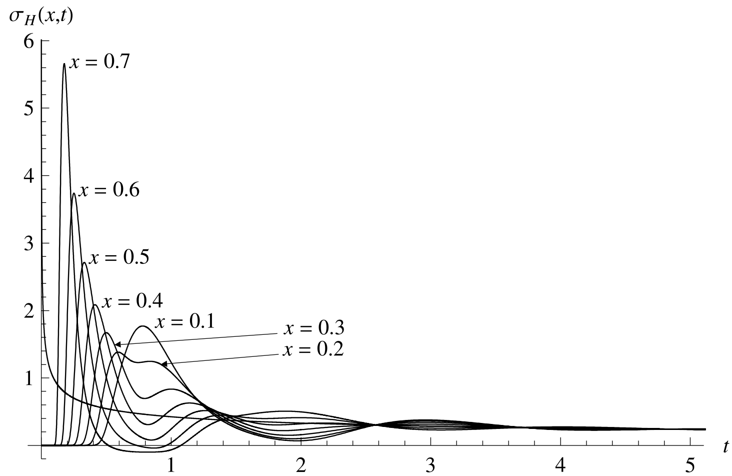

Figure 3.30. *Stresses $\sigma_H(x,t)$ and $\sigma_H^{(QS)}(t)$ in a stress relaxation experiment as functions of time at $x \in \{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7\}$ for $t \in (0,5)$*

Again, we fix the midpoint of the rod and investigate the influence of the change of orders of the fractional derivatives (for the same three sets as before) on the stress σ_H in a stress relaxation experiment. From Figure 3.33, we notice that there is a stress relaxation in a material regardless of the order of fractional derivatives. However, the relaxed stress depends on the order of the derivatives, as expected from the analysis of the stress $\sigma_H^{(QS)}$ in the quasistatic case, given by [3.196]. The material with the dominant elastic properties (the first set) relaxes to the highest stress, and as the fluid properties of the material become more and more dominant, the relaxed stress decreases. Again, oscillations of the value of the stress for the first set (material with dominant elastic properties) are the least damped compared to the second and third sets. Figure 3.34 shows that the conclusion about the dependence of the delay on the orders of fractional derivatives drawn earlier holds.

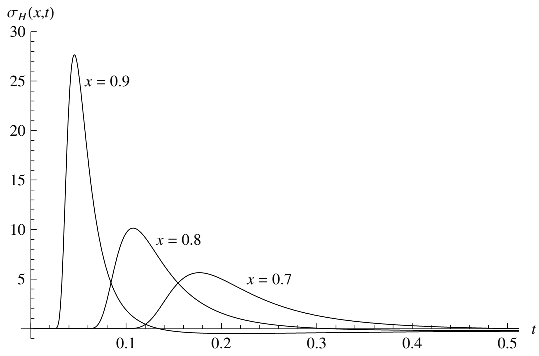

Figure 3.31. *Stresses $\sigma_H(x,t)$ in stress relaxation experiment as functions of time at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 0.5)$*

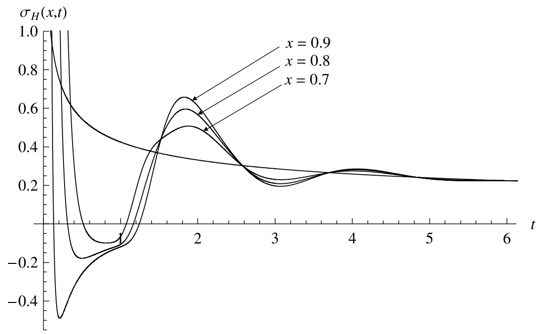

Figure 3.32. *Stresses $\sigma_H(x,t)$ and $\sigma_H^{(QS)}(t)$ in stress relaxation experiment as functions of time at $x \in \{0.7, 0.8, 0.9\}$ for $t \in (0, 6)$*

Figure 3.35 presents displacements u in the creep experiment, determined according to [3.197], for four different points. Parameters are the same as the previous cases (except that in this case instead of $\Upsilon_0 = 1$ we have $\Sigma_0 = 1$). For large times, as can be seen in Figure 3.35, the displacement curves monotonically increase. This indicates that we are dealing with the viscoelastic fluid. Figure 3.35 also shows good agreement between the displacements obtained by the dynamic model (displacement is given by [3.197]) and the quasistatic model (displacement is given by [3.200]). Figure 3.36 presents the same displacements as Figure 3.35, but close to the initial time instant and it is evident that, again, there is a delay due to the finite speed of the wave propagation.

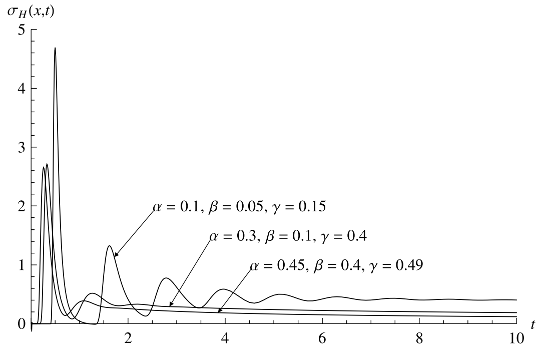

Figure 3.33. *Stresses $\sigma_H(x,t)$ in a stress relaxation experiment as functions of time at $x = 0.5$ for $t \in (0, 10)$*

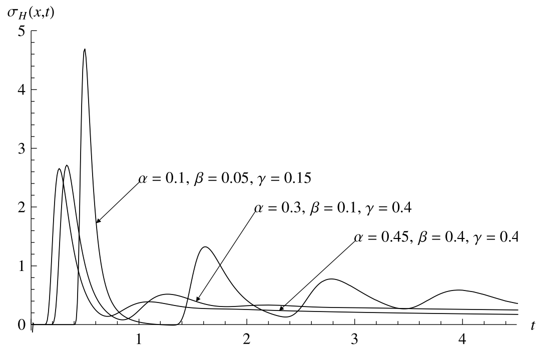

Figure 3.34. *Stresses $\sigma_H(x,t)$ in a stress relaxation experiment as functions of time at $x = 0.5$ for $t \in (0, 4.5)$*

In order to examine the dependence of the displacement u in the case of a creep experiment, given by [3.197], on the orders of fractional derivatives in the constitutive equation [3.117], we again fix the point $x = 0.5$ of the rod and plot the displacement u for the same set of values of α, β and γ as before. Figure 3.37 clearly shows that all of the materials exhibit creep, but the displacements do not tend to a constant value. However, the material which has the dominant elastic properties (the first set) creeps more slowly compared to the material with the dominant fluid properties (the third set). From Figure 3.38, we conclude that the higher the elasticity of the material, the greater the delay.

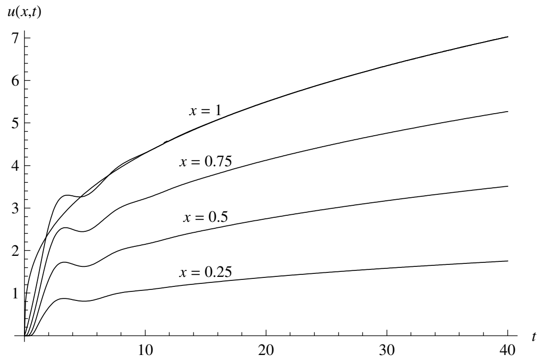

Figure 3.35. *Displacements* $u(x,t)$ *and* $u_H^{(QS)}(t)$ *in a creep experiment as functions of time at* $x \in \{0.25, 0.5, 0.75, 1\}$ *for* $t \in (0, 40)$

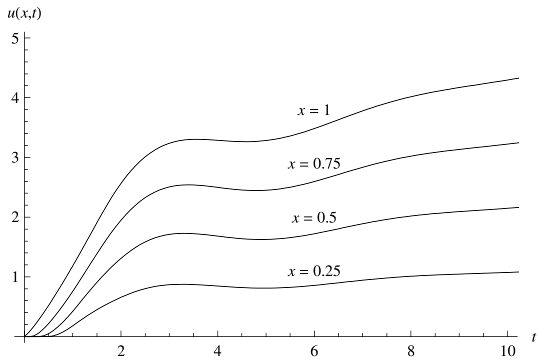

Figure 3.36. *Displacements* $u(x,t)$ *in a creep experiment as functions of time at* $x \in \{0.25, 0.5, 0.75, 1\}$ *for* $t \in (0, 10)$

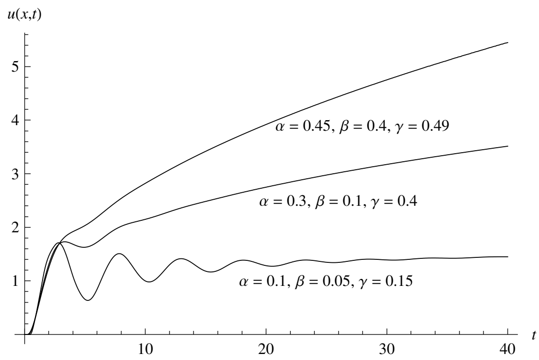

Figure 3.37. *Displacements* $u(x,t)$ *in a creep experiment as functions of time at* $x = 0.5$ *for* $t \in (0, 40)$

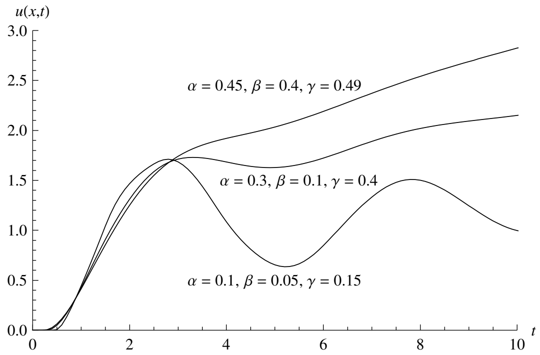

Figure 3.38. *Displacements $u(x,t)$ in a creep experiment as functions of time at $x = 0.5$ for $t \in (0, 10)$*

Chapter 4

Forced Oscillations of a System: Viscoelastic Rod and Body

This chapter is devoted to the analysis of a problem of forced oscillations of a body attached to a viscoelastic rod, i.e. a rod-body system shown in Figure 4.1. We refer to [NOW 63] for the analysis of oscillations of an elastic rod with the mass attached to its end. Analysis will be conducted in the cases when the rod is considered to be heavy (i.e. the mass of a rod is comparable with the mass of a body) as well as when the rod is considered to be light (i.e. the mass of a rod is negligible in comparison to the mass of a body).

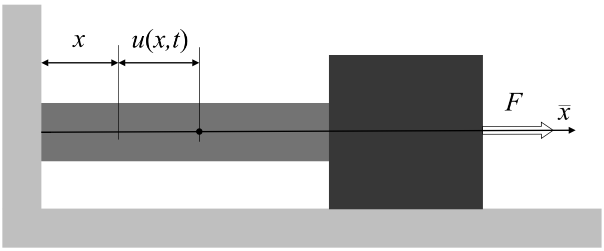

Figure 4.1. *System rod-body*

Let m be the mass of a body attached to a heavy rod. The length of the rod in undeformed state is L and its axis, at the initial time moment as well as during the motion, coincides with the $\bar{x}$ axis (see Figure 4.1). Let x denote a position of a material point of the rod at the initial time $t_0 = 0$. The position of this point at time $t > 0$ is

$x+u(x,t)$. The equations of motion of the rod-body system are:

$$\frac{\partial}{\partial x}\sigma(x,t)=\rho\frac{\partial^2}{\partial t^2}u(x,t),\quad \varepsilon(x,t)=\frac{\partial}{\partial x}u(x,t),\ \ x\in[0,L],\ t>0, \tag{4.1}$$

$$\int_0^1\phi_\sigma(\gamma)\,{}_0\mathrm{D}_t^\gamma\sigma(x,t)\,\mathrm{d}\gamma=E\int_0^1\phi_\varepsilon(\gamma)\,{}_0\mathrm{D}_t^\gamma\varepsilon(x,t)\,\mathrm{d}\gamma,\ \ x\in[0,L],\ t>0, \tag{4.2}$$

$$u(0,t)=0,\quad -A\sigma(L,t)+F(t)=m\frac{\partial^2}{\partial t^2}u(L,t),\ \ t>0, \tag{4.3}$$

$$u(x,0)=0,\quad \frac{\partial}{\partial t}u(x,0)=0,\quad \sigma(x,0)=0,\quad \varepsilon(x,0)=0,\ \ x\in[0,L]. \tag{4.4}$$

We use symbols σ, u and ε in the equation of motion $[4.1]_1$ and in the strain $[4.1]_2$, to denote stress, displacement and strain, respectively, depending on the initial position x at time t, while ρ denotes the density of a material. Constitutive equation [4.2] corresponds to the distributed-order fractional derivative model of a viscoelastic body, E is a generalized Young modulus, ϕ_σ and ϕ_ε are given constitutive functions or distributions. Boundary condition $[4.3]_1$ means that one end of the rod is fixed. The other boundary condition $[4.3]_2$ is the equation of translatory motion along the $\bar{x}$ axis of the body attached to the free end of the rod. In [4.3], A stands for the cross-sectional area of the rod and F stands for the known external force acting on the body. Initial conditions in [4.4] specify that the rod-body system is unstressed and in a state of rest at the initial time instant.

In the case of the light rod, the density of the rod is zero, i.e. $\rho=0$. Then, from $[4.1]_1$, the stress does not depend on the spatial coordinate, i.e.

$$\sigma=\sigma(t),\ \ t>0.$$

Then, [4.2] implies that the strain also does not depend on the spatial coordinate since the left-hand side of [4.2] is not a function of x. Therefore

$$\varepsilon=\varepsilon(t),\ \ t>0,$$

and from $[4.1]_2$, we conclude that

$$u(x,t)=x\varepsilon(t),\ \ x\in[0,L],\ t>0 \tag{4.5}$$

since u satisfies $[4.3]_1$. Taking into account these considerations, system [4.1]–[4.3] become

$$\int_0^1 \phi_\sigma(\gamma)\, {}_0\mathrm{D}_t^\gamma \sigma(t)\,\mathrm{d}\gamma = E\int_0^1 \phi_\varepsilon(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(t)\,\mathrm{d}\gamma, \quad t>0, \tag{4.6}$$

$$-A\sigma(t) + F(t) = mL\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t), \quad t>0. \tag{4.7}$$

Following [ATA 05a, ATA 05b, ATA 12e, ATA 13c, ATA 13d], we will consider forced oscillations of the rod-body system for different types of viscoelastic materials modeled by the fractional derivatives.

Our approach in the investigation of the dynamics of linear viscoelastic rods of fractional type is based on the properties of a specially defined function of complex variable M (see [4.26]), which reflects the inherent properties of a material of a rod. Function M is associated with the Laplace transform of the constitutive equation for the material of a rod. It is defined as a square root of the reciprocal complex modulus (a quantity obtained after application of the Fourier transform to the constitutive equation) calculated in $\mathrm{i}\omega = s$.

4.1. Heavy viscoelastic rod – body system

Following [ATA 12e, ATA 13c], we investigate forced oscillations of a heavy rod-body system in the case when the material of a rod is modeled by the distributed-order model. The initial-boundary value problem [4.1]–[4.4] describing such a system reads

$$\frac{\partial}{\partial x}\sigma(x,t) = \rho\frac{\partial^2}{\partial t^2}u(x,t), \quad \varepsilon(x,t) = \frac{\partial}{\partial x}u(x,t), \quad x\in[0,L],\ t>0, \tag{4.8}$$

$$\int_0^1 \phi_\sigma(\gamma)\ {}_0\mathrm{D}_t^\gamma \sigma(x,t)\,\mathrm{d}\gamma = E\int_0^1 \phi_\varepsilon(\gamma)\ {}_0\mathrm{D}_t^\gamma \varepsilon(x,t)\,\mathrm{d}\gamma, \quad x\in[0,L],\ t>0, \tag{4.9}$$

$$u(0,t)=0, \quad -A\sigma(L,t) + F(t) = m\frac{\partial^2}{\partial t^2}u(L,t), \quad t>0, \tag{4.10}$$

$$u(x,0)=0, \quad \frac{\partial}{\partial t}u(x,0)=0, \quad \sigma(x,0)=0, \quad \varepsilon(x,0)=0, \quad x\in[0,L]. \tag{4.11}$$

Systems [4.1]–[4.4] will be formally solved for the general case of the constitutive equation [4.9]. The existence and uniqueness analysis of the formally obtained solution will be given in section 4.1.2 for the case of the solid-like viscoelastic body. We will also consider the following three cases of the constitutive equation [4.9].

1) The fractional Zener model of a viscoelastic body

$$\left(1 + a\,{}_0\mathrm{D}_t^{\alpha}\right)\sigma\left(x,t\right) = E\left(1 + b\,{}_0\mathrm{D}_t^{\alpha}\right)\varepsilon\left(x,t\right). \qquad [4.12]$$

It is obtained from [4.9] by choosing

$$\begin{aligned}\phi_\sigma\left(\gamma\right) &= \delta\left(\gamma\right) + a\,\delta\left(\gamma-\alpha\right), \quad \phi_\varepsilon\left(\gamma\right) = \delta\left(\gamma\right) + b\,\delta\left(\gamma-\alpha\right),\\ &\qquad\qquad \alpha \in \left(0,1\right), \ a \leq b,\end{aligned} \qquad [4.13]$$

where δ denotes the Dirac delta distribution.

2) The distributed-order model of a viscoelastic body

$$\int_0^1 a^{\gamma}\,{}_0\mathrm{D}_t^{\gamma}\sigma\left(x,t\right)\mathrm{d}\gamma = E\int_0^1 b^{\gamma}\,{}_0\mathrm{D}_t^{\gamma}\varepsilon\left(x,t\right)\mathrm{d}\gamma. \qquad [4.14]$$

It is obtained from [4.9] by choosing

$$\phi_\sigma\left(\gamma\right) = a^{\gamma}, \quad \phi_\varepsilon\left(\gamma\right) = b^{\gamma}, \quad \gamma \in \left(0,1\right), \ a \leq b. \qquad [4.15]$$

We note that [4.15] is the most simple form of ϕ_σ and ϕ_ε providing the dimensional homogeneity.

3) The fractional linear model of a fluid-like viscoelastic body

$$\left(1 + a\,{}_0\mathrm{D}_t^{\alpha}\right)\sigma\left(x,t\right) = E\left(b_0\,{}_0\mathrm{D}_t^{\beta_0} + b_1\,{}_0\mathrm{D}_t^{\beta_1} + b_2\,{}_0\mathrm{D}_t^{\beta_2}\right)\varepsilon\left(x,t\right). \qquad [4.16]$$

It is obtained from [4.9] by choosing

$$\begin{aligned}&\phi_\sigma\left(\gamma\right) = \delta\left(\gamma\right) + a\,\delta\left(\gamma-\alpha\right),\\ &\phi_\varepsilon\left(\gamma\right) = b_0\,\delta\left(\gamma-\beta_0\right) + b_1\,\delta\left(\gamma-\beta_1\right) + b_2\,\delta\left(\gamma-\beta_2\right),\end{aligned} \qquad [4.17]$$

where a, b_0, b_1, b_2 are positive constants and $0 < \alpha < \beta_0 < \beta_1 < \beta_2 \leq 1$.

The results of the existence and uniqueness analysis, given in section 4.1.2, will be applied to the case of the fractional Zener [4.12] and distributed-order model [4.14] in section 4.1.3. Parallel analysis of the forced oscillations problem of a heavy rod-body system in the case of solid-like (Zener and distributed-order models) and fluid-like [4.16] models of the viscoelastic rod will be considered in section 4.1.4.

4.1.1. *Formal solutions*

We start from systems [4.8]–[4.11] and write them in the dimensionless form. Then, by using the Laplace transform method, we obtain the displacement u and the stress σ as the convolution of the external force F and solution kernels P and Q, respectively. Determination of P and Q will be given in section 4.1.2.

Systems [4.8]–[4.11] transform into

$$\frac{\partial}{\partial x}\sigma(x,t)=\kappa^2\frac{\partial^2}{\partial t^2}u(x,t),\quad \varepsilon(x,t)=\frac{\partial}{\partial x}u(x,t),\ \ x\in[0,1],\ t>0, \qquad [4.18]$$

$$\int_0^1\phi_\sigma(\gamma)\,{}_0\mathrm{D}_t^\gamma\sigma(x,t)\,\mathrm{d}\gamma=\int_0^1\phi_\varepsilon(\gamma)\,{}_0\mathrm{D}_t^\gamma\varepsilon(x,t)\,\mathrm{d}\gamma,\ \ x\in[0,1],\ t>0, \qquad [4.19]$$

$$u(0,t)=0,\quad -\sigma(1,t)+F(t)=\frac{\partial^2}{\partial t^2}u(1,t),\ \ t>0, \qquad [4.20]$$

$$u(x,0)=0,\quad \frac{\partial}{\partial t}u(x,0)=0,\quad \sigma(x,0)=0,\quad \varepsilon(x,0)=0,\ \ x\in[0,1]. \qquad [4.21]$$

This is done by introducing the square root of the ratio between the masses of a rod and a body

$$\kappa=\sqrt{\frac{\rho AL}{m}}$$

and dimensionless quantities

$$\bar{x}=\frac{x}{L},\qquad \bar{t}=\frac{t}{\sqrt{\frac{mL}{AE}}},\ \ \bar{u}=\frac{u}{L},\ \ \bar{\sigma}=\frac{\sigma}{E},\ \ \bar{\phi}_\sigma=\frac{\phi_\sigma}{\left(\sqrt{\frac{mL}{AE}}\right)^\gamma},$$
$$\bar{\phi}_\varepsilon=\frac{\phi_\varepsilon}{\left(\sqrt{\frac{mL}{AE}}\right)^\gamma},\ \ \bar{F}=\frac{F}{AE}.$$

In writing [4.18]–[4.21], we omitted the bar over dimensionless quantities. Note that the choice of the dimensionless quantities implies that the case of a rod without the attached mass ($m=0$) cannot be studied as a special case of equations [4.18]–[4.21].

In order to solve systems [4.18], [4.19] with the boundary [4.20] and initial data [4.21], we use the Laplace transform method. Formally applying the Laplace transform to [4.18]–[4.21] we obtain

$$\frac{\partial}{\partial x}\tilde{\sigma}(x,s)=\kappa^2 s^2\tilde{u}(x,s),\quad \tilde{\varepsilon}(x,s)=\frac{\partial}{\partial x}\tilde{u}(x,s),\ \ x\in[0,1],\ s\in D, \qquad [4.22]$$

$$\tilde{\sigma}(x,s)\int_0^1\phi_\sigma(\gamma)\,s^\gamma\mathrm{d}\gamma=\tilde{\varepsilon}(x,s)\int_0^1\phi_\varepsilon(\gamma)\,s^\gamma\mathrm{d}\gamma,\ \ x\in[0,1],\ s\in D, \qquad [4.23]$$

$$\tilde{u}(0,s)=0,\ \ \tilde{\sigma}(1,s)+s^2\tilde{u}(1,s)=\tilde{F}(s),\ \ s\in D. \qquad [4.24]$$

From [4.23] we have

$$\tilde{\sigma}(x,s) = \tilde{E}(s)\,\tilde{\varepsilon}(x,s) = \frac{1}{M^2(s)}\tilde{\varepsilon}(x,s), \quad x \in [0,1], \; s \in D, \qquad [4.25]$$

where we introduced the complex modulus $\tilde{E}$ and associated function M with

$$\tilde{E}(s) = \frac{\int_0^1 \phi_\varepsilon(\gamma)\, s^\gamma \mathrm{d}\gamma}{\int_0^1 \phi_\sigma(\gamma)\, s^\gamma \mathrm{d}\gamma}, \quad M(s) = \frac{1}{\sqrt{\tilde{E}(s)}} = \sqrt{\frac{\int_0^1 \phi_\sigma(\gamma)\, s^\gamma \mathrm{d}\gamma}{\int_0^1 \phi_\varepsilon(\gamma)\, s^\gamma \mathrm{d}\gamma}},$$
$$s \in D. \qquad [4.26]$$

We have that [4.26] in the cases of constitutive equations [4.12], [4.14] and [4.16] yields ($s \in \mathbb{C}\backslash(-\infty,0]$)

$$\tilde{E}(s) = \frac{1+bs^\alpha}{1+as^\alpha}, \quad M(s) = \sqrt{\frac{1+as^\alpha}{1+bs^\alpha}}, \qquad [4.27]$$

$$\tilde{E}(s) = \frac{\ln(as)}{\ln(bs)}\frac{bs-1}{as-1}, \quad M(s) = \sqrt{\frac{\ln(bs)}{\ln(as)}\frac{as-1}{bs-1}}, \qquad [4.28]$$

$$\tilde{E}(s) = \frac{b_0 s^{\beta_0} + b_1 s^{\beta_1} + b_2 s^{\beta_2}}{1+as^\alpha}, \quad M(s) = \sqrt{\frac{1+as^\alpha}{b_0 s^{\beta_0} + b_1 s^{\beta_1} + b_2 s^{\beta_2}}}. \qquad [4.29]$$

In order to obtain the displacement u, we use [4.22], [4.25] and obtain

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x,s) - \kappa^2 s^2 M^2(s)\,\tilde{u}(x,s) = 0, \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0]. \qquad [4.30]$$

The solution of [4.30] is

$$\tilde{u}(x,s) = C_1(s)\,\mathrm{e}^{\kappa x s M(s)} + C_2(s)\,\mathrm{e}^{-\kappa x s M(s)}, \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0];$$

C_1 and C_2 are arbitrary functions which are determined from $[4.24]_1$ as $2C = C_1 = -C_2$. Therefore,

$$\tilde{u}(x,s) = C(s)\sinh(\kappa x s M(s)), \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0]. \qquad [4.31]$$

From $[4.22]_2$, [4.25] and [4.31], we have

$$\tilde{\sigma}(x,s) = C(s)\frac{\kappa s}{M(s)}\cosh(\kappa x s M(s)), \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty,0]. \qquad [4.32]$$

Using [4.31] and [4.32] at $x = 1$ as well as $[4.24]_2$ we obtain

$$C(s) = \frac{M(s)\tilde{F}(s)}{s(sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s)))}, \quad s \in \mathbb{C}\backslash(-\infty, 0].$$

Therefore, the Laplace transforms of the displacement and stress from [4.31] and [4.32] are

$$\begin{aligned} \tilde{u}(x,s) &= \tilde{F}(s)\tilde{P}(x,s), \quad \tilde{\sigma}(x,s) = \tilde{F}(s)\tilde{Q}(x,s), \quad x \in [0,1], \\ & s \in \mathbb{C}\backslash(-\infty, 0]. \end{aligned} \qquad [4.33]$$

where

$$\tilde{P}(x,s) = \frac{1}{s}\frac{M(s)\sinh(\kappa x s M(s))}{sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s))}, \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty, 0], \qquad [4.34]$$

$$\tilde{Q}(x,s) = \frac{\kappa\cosh(\kappa x s M(s))}{sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s))}, \quad x \in [0,1], \; s \in \mathbb{C}\backslash(-\infty, 0]. \qquad [4.35]$$

Applying the inverse Laplace transform to [4.33] we obtain the displacement and stress as

$$u(x,t) = F(t) * P(x,t), \quad \sigma(x,t) = F(t) * Q(x,t), \quad x \in [0,1], \; t > 0. \qquad [4.36]$$

The validity of these formal expressions will be proved in the following.

4.1.2. *Existence and uniqueness of the solution to [4.18]–[4.21]*

In this section, we prove the existence and uniqueness of the displacement u and stress σ as solutions to [4.8]–[4.11]. The results are given for the general case of constitutive equation [4.19] in the case of the solid-like viscoelastic body. We will also present the case of the elastic body as the limiting case of the analysis.

4.1.2.1. *Notation and assumptions*

In the sequel, we consider analytic functions in

$$V = \mathbb{C} \backslash (-\infty, 0] = \left\{ z = r\mathrm{e}^{\mathrm{i}\varphi} \mid r > 0,\ \varphi \in (-\pi, \pi) \right\}.$$

We often use notation $|s| \to \infty$ and $|s| \to 0$, where we assume that $s \in V$.

In the analysis that follows, we will need the properties of an analytic function

$$M(s) = \sqrt{\frac{\int_0^1 \phi_\sigma(\gamma)\, s^\gamma \mathrm{d}\gamma}{\int_0^1 \phi_\varepsilon(\gamma)\, s^\gamma \mathrm{d}\gamma}}, \quad s \in \mathcal{V}.$$

defined in appropriate domain $\mathcal{V} \subset \mathbb{C}$. We will have $\mathcal{V} = V$.

The following function has a special role

$$f(s) = sM(s)\sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s)), \quad s \in V. \tag{4.37}$$

As will be seen from [4.34] and [4.35], f is a denominator of functions $\tilde{P}$ and $\tilde{Q}$, which, after the inversion of the Laplace transform, represent solution kernels of u and σ, respectively.

We summarize all the assumptions that we use. Let M be of the form

$$M(s) = r(s) + \mathrm{i}h(s), \quad \text{as} \quad |s| \to \infty.$$

We assume the following:

(A1)

$$\lim_{|s|\to\infty} r(s) = c_\infty > 0, \quad \lim_{|s|\to\infty} h(s) = 0, \quad \lim_{|s|\to 0} M(s) = c_0,$$

for some constants $c_\infty, c_0 > 0$.

Let $s_n = \xi_n + \mathrm{i}\zeta_n$, $n \in \mathbb{N}$, satisfy the equation

$$f(s) = 0, \quad s \in V, \tag{4.38}$$

where f is given by [4.37].

(A2) There exists $n_0 > 0$, such that for $n > n_0$

$$\operatorname{Im} s_n \in \mathbb{R}_+ \Rightarrow h(s_n) \leq 0, \quad \operatorname{Im} s_n \in \mathbb{R}_- \Rightarrow h(s_n) \geq 0,$$

where $h = \operatorname{Im} M$.

(A3) There exist $s_0 > 0$ and $c > 0$ such that

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right| \geq c, \quad |s| > s_0.$$

(A4) For every $\gamma > 0$, there exist $\theta > 0$ and s_0 such that

$$|(s + \Delta s) M(s + \Delta s) - sM(s)| \leq \gamma, \text{ if } |\Delta s| < \theta \text{ and } |s| > s_0.$$

Alternatively to $(\mathrm{A}2)$, in proposition 4.3, we will consider the following assumption.

(B) $|h(s)| \leq \frac{C}{|s|}$, $|s| > s_0$, for some constants $C > 0$ and $s_0 > 0$.

It is shown in section 4.1.3 that (A1)–(A4) hold for M given by [4.28] and [4.27].

4.1.2.2. *Theorems on the existence and uniqueness*

Our central results on the existence, uniqueness and properties of u and σ are stated in theorems 4.1 and 4.2. Recall that f is given by [4.37] and s_n, $n \in \mathbb{N}$, are solutions of [4.38].

THEOREM 4.1.– Let $F \in \mathcal{S}'_+$ and suppose that M satisfies assumptions $(\mathrm{A}1)$–$(\mathrm{A}4)$. Then, the unique solution u to [4.18]–[4.21] is given by

$$u(x,t) = F(t) * P(x,t), \quad x \in [0,1], \ t > 0, \qquad [4.39]$$

where

$$P(x,t) = \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa x q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right) + \kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q$$

$$+2\sum_{n=1}^\infty \operatorname{Re}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right), \quad x \in [0,1], \ t > 0, \qquad [4.40]$$

$$P(x,t) = 0, \quad x \in [0,1], \ t < 0.$$

The residues are given by

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) = \left[\frac{1}{s}\frac{M(s)\sinh(\kappa x s M(s))}{\frac{\mathrm{d}}{\mathrm{d}s}f(s)}\mathrm{e}^{st}\right]_{s=s_n}, \quad x\in[0,1],\ t>0, \qquad [4.41]$$

Then, $P \in C([0,1]\times[0,\infty))$ and $u \in C\left([0,1], \mathcal{S}'_+\right)$. In particular, if $F \in L^1_{loc}([0,\infty))$, then u is continuous on $[0,1]\times[0,\infty)$.

The following theorem is related to the stress σ. We formulate this theorem with $F = H$, where H denotes the Heaviside function, while the more general cases of F are discussed in remark 4.1, below.

THEOREM 4.2.– Let $F = H$ and suppose that M satisfies assumptions (A1)–(A4). Then, the unique solution σ_H to [4.18]–[4.21], is given by

$$\begin{aligned}&\sigma_H(x,t)\\ &= H(t) + \frac{\kappa}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{\cosh\left(\kappa x q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right)\\ &\quad\times\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q + 2\sum_{n=1}^{\infty}\operatorname{Re}\left(\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st},s_n\right)\right), \quad x\in[0,1],\ t>0, \qquad [4.42]\\ &\sigma_H(x,t) = 0, \quad x\in[0,1],\ t<0. \qquad [4.43]\end{aligned}$$

The residues are given by

$$\operatorname{Res}\left(\sigma_H(x,s)\,\mathrm{e}^{st}, s_n\right) = \left[\frac{\kappa\cosh(\kappa x s M(s))}{s\frac{\mathrm{d}}{\mathrm{d}s}f(s)}\mathrm{e}^{st}\right]_{s=s_n}, \quad x\in[0,1],\ t>0. \qquad [4.44]$$

In particular, σ_H is continuous on $[0,1]\times[0,\infty)$.

REMARK 4.1.–

1) The assumption $F = H$ in theorem 4.2 can be relaxed by requiring that F is locally integrable and

$$\tilde{F}(s) \approx \frac{1}{s^\alpha}, \quad \text{as} \quad |s|\to\infty,$$

for some $\alpha\in(0,1)$. This condition ensures the convergence of the series in [4.42].

2) If $F = \delta$, or even $F(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta(t)$, one uses σ_H, given by [4.42], in order to obtain σ as the $(k+1)$ th distributional derivative:

$$\sigma = \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}\sigma_H \in C\left([0,1], \mathcal{S}'_+\right).$$

4.1.2.3. *Auxiliary results*

4.1.2.3.1. Zeros of the function $\boldsymbol{f}$

The following propositions establish the location and the multiplicity of poles of functions $\tilde{P}$ and $\tilde{Q}$, given by [4.34] and [4.35], respectively.

PROPOSITION 4.1.– Assume (A1). Equation [4.38] has a countable number of solutions s_n, $n \in \mathbb{N}$, with the properties

$$s_n M(s_n) = \mathrm{i}w_n, \quad \tan(\kappa w_n) = \frac{\kappa}{w_n}, \quad w_n \in \mathbb{R}, \; w_n \neq 0. \qquad [4.45]$$

Complex conjugate $\bar{s}_n$ also satisfies [4.38], $n \in \mathbb{N}$.

Note $\mathrm{Im}\, s_n \neq 0$ in [4.45]; so, all the solutions belong to V.

PROOF.– We seek the solutions of [4.38], or equivalently of

$$\mathrm{e}^{2\kappa s M(s)} = \frac{sM(s) - \kappa}{sM(s) + \kappa}, \quad s \in V. \qquad [4.46]$$

Put $sM(s) = v(s) + \mathrm{i}w(s)$, $s \in V$, where v, w are real-valued functions. Taking the modulus of [4.46], we obtain

$$\mathrm{e}^{2\kappa v} = \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2}. \qquad [4.47]$$

Fix w and let $v < 0$. Then,

$$\mathrm{e}^{2\kappa v} < 1 \;\text{ and }\; \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2} > 1.$$

Now let $v > 0$. Then,

$$\mathrm{e}^{2\kappa v} > 1 \;\text{ and }\; \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2} < 1.$$

Thus, in both cases, we have a contradiction and we conclude that the solutions of [4.47] satisfy $v(s) = 0$. Therefore, the solutions of [4.46] satisfy

$$sM(s) = \mathrm{i}w(s), \quad s \in V.$$

Inserting this into [4.38] yields

$$\tan(\kappa w) = \frac{\kappa}{w}, \quad w \in \mathbb{R}. \qquad [4.48]$$

Since the tangent function is periodic, we conclude that there are a countable number of values of w, denoted by w_n, $n \in \mathbb{N}$, satisfying [4.48]. Hence, we have [4.45].

In order to prove that the solutions $s_n \in V$ of [4.38] are complex conjugated, we note that $M(\bar{s}) = \overline{M(s)}$, $s \in V$. By [4.45], $\bar{s}_n M(\bar{s}_n) = \overline{s_n M(s_n)} = -\mathrm{i}w_n$. Thus, $\bar{s}_n$ also solves [4.38]. ■

PROPOSITION 4.2.– Assume (A1). Positive (negative) solutions of $\tan(\kappa w_n) = \frac{\kappa}{w_n}$ satisfy $w_n \approx \frac{n\pi}{\kappa}$ ($w_n \approx -\frac{n\pi}{\kappa}$) as $n \to \infty$.

PROOF.– As we noted, if w_n satisfies [4.45], then $-w_n$ also satisfies [4.45]. Since $\frac{\kappa}{w_n}$ monotonically decreases to zero for all $w_n > 0$, from [4.45], κw_n behave as zeros of the tangent function, i.e. that

$$w_n \approx \frac{n\pi}{\kappa} \ \left(w_n \approx -\frac{n\pi}{\kappa}\right) \text{ as } n \to \infty. \quad ■$$

PROPOSITION 4.3.– Assume (A1), (A2) or (A1), (B). Then, there exist $\xi_0 > 0$ and $n_0 \in \mathbb{N}$ so that the real part of s_n, $n \in \mathbb{N}$, denoted by ξ_n satisfies $\xi_n < \xi_0$, $n > n_0$. Moreover, if we additionally assume (A3), then the solutions s_n of [4.38] are of multiplicity one for $n > n_0$.

PROOF.– Assume (A1) and (A2). From [4.45], we have

$$(\xi_n + \mathrm{i}\zeta_n)(r(s_n) + \mathrm{i}h(s_n)) \approx \mathrm{i}w_n, \quad n > n_0.$$

This implies

$$\xi_n r(s_n) = \zeta_n h(s_n), \quad n \in \mathbb{N}, \qquad [4.49]$$

$$\xi_n h(s_n) + \zeta_n r(s_n) \approx w_n, \quad n > n_0. \qquad [4.50]$$

Inserting [4.49] into [4.50], we obtain

$$\zeta_n \approx \frac{n\pi}{c_\infty \kappa} \text{ because of } w_n \approx \frac{n\pi}{\kappa}, \quad n > n_0, \tag{4.51}$$

or

$$\zeta_n \approx -\frac{n\pi}{c_\infty \kappa} \text{ because of } w_n \approx -\frac{n\pi}{\kappa}, \quad n > n_0. \tag{4.52}$$

In the case of [4.51], we have

$$\xi_n \approx \frac{\zeta_n}{c_\infty} h(s_n) \leq 0, \quad n > n_0,$$

since s_n belongs to the upper complex half-plane. In the case of [4.52], we have

$$\xi_n \approx \frac{\zeta_n}{c_\infty} h(s_n) \leq 0, \quad n > n_0,$$

since then s_n belongs to the lower complex half-plane. Thus, in both cases, $\xi_n \leq 0$ for the sufficiently large n. This proves the first assertion.

Assume $(\mathrm{A1})$ and (B). This and [4.45] imply that $s_n \approx \mathrm{i}\frac{w_n}{c_\infty}$, for $n > n_0$. Thus, from [4.49] and [4.50], we obtain

$$|\xi_n| \leq \frac{|w_n|}{c_\infty^2} \frac{C}{|s_n|} \leq \frac{C}{c_\infty}, \quad n > n_0.$$

So, the real parts ξ_n of solutions s_n of [4.38] satisfy $\xi_n \in \left[-\frac{C}{c_\infty}, \frac{C}{c_\infty}\right]$ for $n > n_0$ This is an even stronger condition for the zeros, but it will not be used in the following.

In order to prove that the solutions s_n, $n > n_0$, of f are of the multiplicity one, we use $(\mathrm{A3})$, differentiate [4.38] and obtain

$$\begin{aligned} \frac{\mathrm{d}f(s)}{\mathrm{d}s} &= \left(\left(1+\kappa^2\right)\sinh\left(\kappa s M(s)\right)\right. \\ &\qquad \left. +\kappa s M(s)\cosh\left(\kappa s M(s)\right)\right) \frac{\mathrm{d}}{\mathrm{d}s}\left(s M(s)\right), \quad s \in V. \end{aligned}$$

Calculating the previous expression at s_n, we have that

$$\left| \left. \frac{\mathrm{d}f(s)}{\mathrm{d}s} \right|_{s=s_n} \right| = \left| \left(1+\kappa^2\right) \sin(\kappa w_n) + \kappa w_n \cos(\kappa w_n) \right| \left| \left[\frac{\mathrm{d}}{\mathrm{d}s} (sM(s)) \right]_{s=s_n} \right|$$

that is different from zero by $(\mathrm{A3})$. ■

4.1.2.3.2. Estimates

In the following, we assume $(\mathrm{A1})$–$(\mathrm{A4})$.

Let $R > 0$. A quarter of a disc, denoted by D, and its boundary Γ are defined by

$$\begin{aligned} D &= D_R = \left\{ s = \rho \mathrm{e}^{\mathrm{i}\varphi} \mid \rho \leq R,\ \varphi \in \left(\frac{\pi}{2}, \pi \right) \right\}, \\ \Gamma &= \Gamma_R = \left\{ s = R\mathrm{e}^{\mathrm{i}\varphi} \mid \varphi \in \left(\frac{\pi}{2}, \pi \right) \right\}. \end{aligned}$$

Let S be the set of all solutions of [4.38] in D.

In the calculation of P and Q in the next sections, we will need the estimates given in the lemmas 4.1 and 4.2. Recall that S is the set of zeros.

LEMMA 4.1.– Let $\eta > 0$ and

$$D_\eta = \{ s \in D \mid |s - s_j| > \eta,\ s_j \in S \}.$$

Then, there exist $s_0 > 0$ and $p_\eta > 0$ such that

$$|f(s)| > p_\eta, \quad \text{if} \quad s \in D_\eta,\ |s| > s_0. \qquad [4.53]$$

REMARK 4.2.– We will have in the following certain assertions that hold for $n > n_0$. This is related to the subindices of the solutions to [4.38], but it also implies that we consider domains in D where $|s| > s_0$, where s_0 depends on n_0.

PROOF.– If [4.53] does not hold, then there exists a sequence $\{\tilde{s}_n\}_{n\in\mathbb{N}} \in D_\eta$ such that

$$|f(\tilde{s}_n)| = \eta_n \to 0, \quad n \to \infty. \qquad [4.54]$$

This implies

$$\left|\mathrm{Re}\left(\tilde{s}_n M\left(\tilde{s}_n\right)\right)\right| \to 0, \quad \left|\mathrm{Im}\left(\tilde{s}_n M\left(\tilde{s}_n\right)\right)\right| \to \infty, \text{ as } n \to \infty.$$

Our aim is to show that there exist $N \in \mathbb{N}$ and $s_N \in S$ so that

$$\left|\tilde{s}_N - s_N\right| \leq \eta \qquad [4.55]$$

and this will be the contradiction. Let $\delta < \frac{\pi}{c_\infty \kappa}$ and $\delta \ll \eta$. Recall that there exists n_0 such that for $s_n \in S$, $n > n_0$, there holds

$$\mathrm{Re}\left(s_n M\left(s_n\right)\right) = 0, \quad \left|\mathrm{Im}\left(s_n M\left(s_n\right)\right) - \frac{n\pi}{c_\infty \kappa}\right| < \frac{\delta}{2}.$$

Now consider the intervals

$$\begin{aligned} I_n &= \left(\frac{n\pi}{c_\infty \kappa} + \frac{\delta}{2}, \frac{(n+1)\pi}{c_\infty \kappa} - \frac{\delta}{2}\right), \\ I_{n+1} &= \left(\frac{(n+1)\pi}{c_\infty \kappa} + \frac{\delta}{2}, \frac{(n+2)\pi}{c_\infty \kappa} - \frac{\delta}{2}\right), \ldots. \end{aligned}$$

Because $\left|\mathrm{Re}\left(\tilde{s}_n M\left(\tilde{s}_n\right)\right)\right| \to 0$, we see that

$$\left|\mathrm{Im}\left(\tilde{s}_n M\left(\tilde{s}_n\right)\right)\right| \in I_n \cup I_{n+1} \cup \ldots, \quad n > n_0.$$

Put $\kappa \tilde{s}_n M\left(\tilde{s}_n\right) = t_n + \mathrm{i}\tau_n$, $n \in \mathbb{N}$. We have

$$\begin{aligned} 2\kappa f\left(\tilde{s}_n\right) &= t_n\left(\mathrm{e}^{t_n} - \mathrm{e}^{-t_n}\right)\cos\tau_n - \tau_n\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin\tau_n \\ &\quad + \kappa^2\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos\tau_n \\ &\quad + \mathrm{i}\left(t_n\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin\tau_n + \tau_n\left(\mathrm{e}^{t_n} - \mathrm{e}^{-t_n}\right)\cos\tau_n\right. \\ &\quad \left. + \kappa^2\left(\mathrm{e}^{t_n} - \mathrm{e}^{-t_n}\right)\sin\tau_n\right). \end{aligned} \qquad [4.56]$$

Put $\tau_n = k_n\pi + r_n$, $r_n < \pi$, where $k_n, n \in \mathbb{N}$ is an increasing sequence of natural numbers. We will show that r_n must have a subsequence tending to 0 and this will lead to [4.55], i.e. to the contradiction with $\tilde{s}_n \in D_\eta$.

So let us assume that r_n does not have a subsequence converging to zero; so, $r_n > \frac{\delta}{2}$, $n > n_0$. Bearing in mind that $t_n \to 0$ and dropping summands tending to 0 in [4.56], we obtain $(n \to \infty)$

$$\begin{aligned}
|2\kappa f(\tilde{s}_n)| &\sim |-\tau_n \left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin\tau_n + \kappa^2 \left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos\tau_n| \\
&= |-(k_n\pi + r_n)\left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin r_n + \kappa^2 \left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos r_n| \\
&\sim |-k_n\pi \left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin r_n - r_n \left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\sin r_n \\
&\quad +\kappa^2 \left(\mathrm{e}^{t_n} + \mathrm{e}^{-t_n}\right)\cos r_n|.
\end{aligned}$$

Now, we see that the first summand on the last right-hand side tends to infinity, while the second and the third summands are bounded. This is in contradiction with [4.54]; so, lemma is proved. ■

In the next proposition, we will find estimates on f needed for the later calculation of integrals.

PROPOSITION 4.4.–

1)] Let D_0 be a subdomain of D. If $|\mathrm{Re}\,(sM(s))| > d$, $s \in D_0 \subset D$, then there exist $c > 0$ and $s_0 > 0$ such that

$$|f(s)| \geq c|sM(s)|\,|\sinh(\kappa sM(s))|, \quad s \in D_0, \ |s| > s_0.$$

2) If $s \in D_\eta$, $|s| > s_0$ (see lemma 4.1), then $|f(s)| \geq p_\eta$.

Note that in case (2) the condition $|\mathrm{Re}\,(sM(s))| \leq d$ is not assumed, although we consider this case in (2), since this part is already a consequence of lemma 4.1.

PROOF.– (1) follows from the fact that $sM(s)\sinh(\kappa sM(s))$ tends faster to infinity than $\cosh(\kappa sM(s))$ when $|s| \to \infty$, $s \in D_0$. So, for some $c > 0$ and $|s| > s_0$,

$$\begin{aligned}
|f(s)| &\geq |sM(s)|\,|\sinh(\kappa sM(s))| - \kappa^2 |\cosh(\kappa sM(s))| \\
&\geq c\,|sM(s)|\,|\sinh(\kappa sM(s))|.
\end{aligned}$$

This implies the assertion. ■

We need one more estimate of f in the case when we have to control how s is close to the zero set S of f. This is needed for the small deformation of the circle arc Γ_R

near the point of Γ_R which is close to some zero of f. We need assumptions (A3) and (A4).

For later use, we choose $\varepsilon > 0$ such that $\varepsilon < \frac{\theta}{2}$ (θ is from (A4)) and that the difference $|s_1 M(s_1) - s_2 M(s_2)| \leq \gamma$ implies small differences

$$|\cosh(s_1 M(s_1)) - \cosh(s_2 M(s_2))| < \delta_1, \ |\sinh(s_1 M(s_1)) - \sinh(s_2 M(s_2))| < \delta_1$$

as we will need in the proof of the lemma (in [4.63]).

LEMMA 4.2.– Let $0 < \varepsilon < \frac{\theta}{2}$. Then there exist $n_0 \in \mathbb{N}$ and $d > 0$ such that, for $s_j \in S$,

$$j > n_0, \ \varepsilon < |s - s_j| \leq 2\varepsilon \Rightarrow \mathrm{Re}\,(sM\,(s)) > d. \qquad [4.57]$$

PROOF.– Since for suitable n_0

$$|s_{n+1} - s_n| \geq |\mathrm{Im}\,(s_{n+1} - s_n)| \approx \frac{\pi}{c_\infty \kappa} > \frac{\pi}{2c_\infty \kappa}, \quad n > n_0,$$

the balls $L\,(s_j, 2\varepsilon)$ are disjoint for $j > n_0$. Let $j > n_0$ and $\varepsilon < |s - s_j| \leq 2\varepsilon$. From the Taylor formula, we have

$$|f\,(s) - f\,(s_j)| = \left|\frac{\mathrm{d}f\,(\bar{s})}{\mathrm{d}s}\right| |s - s_j| > \varepsilon \left|\frac{\mathrm{d}f\,(\bar{s} - s_j)}{\mathrm{d}s}\right|, \quad \varepsilon < |\bar{s}| \leq 2\varepsilon. \quad [4.58]$$

So, with n_0 large enough, we have $|s| > s_0$ so that (A3) implies

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\,(sM\,(s))\right| \geq c, \quad \text{for} \quad |s| > s_0.$$

In the following, we will refer to the following set of conditions

$$j > n_0, \ |s| > s_0, \ \varepsilon < |s - s_j| \leq 2\varepsilon, \ \ \varepsilon < |\bar{s}| \leq 2\varepsilon. \qquad [4.59]$$

Assuming [4.59], it follows that

$$\begin{aligned}\left|\frac{\mathrm{d}f\left(\bar{s}\right)}{\mathrm{d}s}\right| &= \left|\left(1+\kappa^{2}\right)\sinh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right.\\ &\quad \left.+\kappa\bar{s}M\left(\bar{s}\right)\cosh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|\left|\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(sM\left(s\right)\right)\right]_{s=\bar{s}}\right|\\ &\geq c\left(\kappa\left|\bar{s}M\left(\bar{s}\right)\right|\left|\cosh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|-\left(1+\kappa^{2}\right)\left|\sinh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|\right). \end{aligned} \tag{4.60}$$

Now, we estimate $|f|$, assuming [4.59], and obtain

$$\left|f\left(s\right)\right|\leq\left|sM\left(s\right)\right|\left|\sinh\left(\kappa sM\left(s\right)\right)\right|+\kappa\left|\cosh\left(\kappa sM\left(s\right)\right)\right|. \tag{4.61}$$

Using [4.60], [4.61] with [4.59] in [4.58], we have

$$\begin{aligned}&\left|sM\left(s\right)\right|\left|\sinh\left(\kappa sM\left(s\right)\right)\right|+\kappa\left|\cosh\left(\kappa sM\left(s\right)\right)\right|\\ &\quad>\varepsilon c\left(\kappa\left|\bar{s}M\left(\bar{s}\right)\right|\left|\cosh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|-\left(1+\kappa^{2}\right)\left|\sinh\left(\kappa\bar{s}M\left(\bar{s}\right)\right)\right|\right).\end{aligned} \tag{4.62}$$

The final part of the proof of the lemma is to show that [4.62] implies that there exists d such that [4.57] holds if [4.59] is satisfied. Contrary to [4.57], assume that there exist sequences $\tilde{s}_n$, s_{j_n} and $d_n\to 0$ such that

$$\left|\operatorname{Re}\left(\tilde{s}_n\right)M\left(\tilde{s}_n\right)\right|\leq d_n,\quad\text{if }\ \varepsilon<\left|\tilde{s}_n-s_{j_n}\right|\leq 2\varepsilon, n>n_0.$$

Since $\tilde{s}_n$, $n>n_0$ satisfies [4.59], from [4.62], we would have

$$\begin{aligned}&\left|\tilde{s}_nM\left(\tilde{s}_n\right)\right|\left|\sinh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right|+\kappa\left|\cosh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right|\\ &\quad>\varepsilon c\left(\kappa\left|\bar{s}_nM\left(\bar{s}_n\right)\right|\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|-\left(1+\kappa^{2}\right)\left|\sinh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|\right).\end{aligned} \tag{4.63}$$

The second addend on the right-hand side tends to 0; thus, we neglect it. Moreover, we note that $\kappa\left|\cosh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right|$ cannot estimate $\varepsilon c\kappa\left|\bar{s}_nM\left(\bar{s}_n\right)\right|\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|$ because the second one tends to infinity while the first one is finite. Thus, in [4.63], leading terms on both sides are the first ones (with $\tilde{s}_nM\left(\tilde{s}_n\right)$ and $\bar{s}_nM\left(\bar{s}_n\right)$) and we skip the second terms on both sides of [4.63]. It follows with another $c_0>0$,

$$\left|\tilde{s}_nM\left(\tilde{s}_n\right)\right|\left|\sinh\left(\kappa\tilde{s}_nM\left(\tilde{s}_n\right)\right)\right|\geq c_0\left|\bar{s}_nM\left(\bar{s}_n\right)\right|\left|\cosh\left(\kappa\bar{s}_nM\left(\bar{s}_n\right)\right)\right|. \tag{4.64}$$

Now according to [4.64] we can choose γ (which then determines θ) and ε so that with suitable δ_1 and c_1, [4.63] implies

$$|\sinh(\kappa\tilde{s}_n M(\tilde{s}_n))| > c_1 |\cosh(\kappa\bar{s}_n M(\bar{s}_n))|,$$

on the domain [4.59]. However, this leads to the contradiction because $|\sinh(\kappa\tilde{s}_n M(\tilde{s}_n))| \to 0$, while $|\cosh(\kappa\bar{s}_n M(\bar{s}_n))|$ is close to 1, under the assumptions. This proves the lemma. ■

We note that we can choose η in lemma 4.1 to be equal to 2ε of lemma 4.2. In this way, we obtain that lemma 4.2 and proposition 4.4 imply:

PROPOSITION 4.5.–

1) There exist $\varepsilon > 0$ and $d > 0$ such that for $\varepsilon < |s - s_n| \leq 2\varepsilon$, $s_n \in S$ and $|s| > s_0$,

$$|\mathrm{Re}(sM(s))| > d$$

and

$$|f(s)| \geq c\,|sM(s)|\,|\sinh(\kappa sM(s))|, \quad \varepsilon < |s - s_n| \leq 2\varepsilon, \ s_n \in S, \ |s| > s_0.$$

2) Let $|s - s_n| > 2\varepsilon$, $s_n \in S$, $|s| > s_0$ and d, ε, c be as in (1).

$a)$ If $|\mathrm{Re}(sM(s))| > d$, then

$$|f(s)| \geq c\,|sM(s)|\,|\sinh(\kappa sM(s))|.$$

$b)$ If $|\mathrm{Re}(sM(s))| \leq d$, then

$$|f(s)| > p_{2\varepsilon}$$

(see the comment before the proof of proposition 4.4).

With the notation of the previous proposition, we finally come to corollaries which will be used in the subsequent sections. We keep the notation from proposition 4.5.

COROLLARY 4.1.–

1) There exist $\varepsilon > 0$ and $C > 0$ such that for $\varepsilon < |s - s_n| \leq 2\varepsilon$, $s_n \in S$ and $|s| > s_0$,

$$\frac{|sM(s)\sinh(\kappa x s M(s))|}{|f(s)|} \leq \frac{1}{c}\mathrm{e}^{-\kappa(1-x)\operatorname{Re}(sM(s))} \leq C;$$

2) for $|s - s_n| > 2\varepsilon$, $s_n \in S$, $|s| > s_0$

$$\frac{|sM(s)\sinh(\kappa x s M(s))|}{|f(s)|} \leq \frac{1}{c}\mathrm{e}^{-\kappa(1-x)\operatorname{Re}(sM(s))} \leq C,$$

or

$$\frac{|sM(s)\sinh(\kappa x s M(s))|}{|f(s)|} \leq \frac{d\mathrm{e}^{\kappa x d}\left|1-\mathrm{e}^{-2\kappa x s M(s)}\right|}{p_{2\varepsilon}} \leq C.$$

Because we also need to estimate $\frac{|\cosh(\kappa x s M(s))|}{|f(s)|}$, we again use proposition 4.5. Moreover, we use the fact that in the case $|\operatorname{Re}(sM(s))| > d$, there exists $c > 0$ such that

$$|\cosh(\kappa x s M(s))| \leq c\,|\sinh(\kappa x s M(s))|\,, \quad |s| \to \infty.$$

COROLLARY 4.2.–

1) There exist $\varepsilon > 0$ and $C > 0$ such that for $\varepsilon < |s - s_n| \leq 2\varepsilon$, $s_n \in S$ $|s| > s_0$,

$$\frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \leq \frac{1}{c \cdot c_\infty}\frac{1}{|s|}\mathrm{e}^{-\kappa(1-x)\operatorname{Re}(sM(s))} \leq \frac{C}{|s|};$$

2) for $|s - s_n| > 2\varepsilon$, $s_n \in S$, $|s| > s_0$

$$\frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \leq \frac{1}{c \cdot c_\infty}\frac{1}{|s|}\mathrm{e}^{-\kappa(1-x)\operatorname{Re}(sM(s))} \leq \frac{C}{|s|},$$

or

$$\frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \leq \frac{\mathrm{e}^{\kappa x d}\left|1+\mathrm{e}^{-2\kappa x s M(s)}\right|}{p_{2\varepsilon}} \leq C.$$

4.1.2.4. *Proofs of theorems 4.1 and 4.2*

In this section, we prove theorems 4.1 and 4.2. First, we prove theorem 4.1.

PROOF OF THEOREM 4.1.– We calculate $P(x,t)$, $x \in [0,1]$, $t \in \mathbb{R}$, by the integration over the contour given in Figure 4.2. Small, inside or outside half-circles, depending on the zeros of f near Γ_2 and Γ_6, have radius ε determined in corollaries 4.1 and 4.2. This will be explained in the proof. The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, \bar{s}_n\right)\right), \qquad [4.65]$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \Gamma_7 \cup \gamma_0$ so that poles of $\tilde{P}$ lie inside the contour Γ.

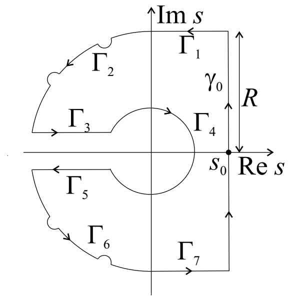

Figure 4.2. *Integration contour* Γ

First we show that the series of residues in [4.40] is real-valued and convergent. Proposition 4.3 implies that the poles s_n, $n \in \mathbb{N}$, of $\tilde{P}$, given by [4.34], are simple for sufficiently large n. Then the residues in [4.65] can be calculated using [4.41] as

$$\begin{aligned}
&\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) \\
&= \left[\frac{M(s)\sinh(\kappa x s M(s))}{(1+\kappa^2)\sinh(\kappa s M(s)) + \kappa s M(s)\cosh(\kappa s M(s))}\right]_{s=s_n} \\
&\times \left[\frac{\mathrm{e}^{st}}{s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))}\right]_{s=s_n} \quad x \in [0,1],\ t > 0.
\end{aligned} \qquad [4.66]$$

Substituting [4.45] in [4.66], we obtain ($x \in [0,1]$, $t > 0$)

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) = \frac{w_n \sin(\kappa w_n x)}{(1+\kappa^2)\sin(\kappa w_n) + \kappa w_n \cos(\kappa w_n)} \frac{\mathrm{e}^{s_n t}}{\left[s^2 \frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}}.$$

Proposition 4.3 implies

$$\left|\mathrm{e}^{s_n t}\right| < \bar{c}\mathrm{e}^{at}, \quad t > 0,$$

for some $a \in \mathbb{R}$. Because $|s_n| \approx \frac{n\pi}{c_\infty \kappa}$ and $|w_n| \approx \frac{n\pi}{\kappa}$, for $n > n_0$, it follows that

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le \frac{\bar{c}}{\kappa} \frac{\mathrm{e}^{at}}{\left|\left[s^2 \frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}\right|}, \quad x \in [0,1], \ t > 0.$$

Now we use assumption (A3). This implies that there exists $K > 0$ such that

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le K \frac{\mathrm{e}^{at}}{n^2}, \quad x \in [0,1], \ t > 0.$$

This implies that the series of residues in [4.65], i.e. in [4.40], is convergent.

Now, we calculate the integral over Γ in [4.65]. First, we consider the integral along contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0], \ R > 0\}$, where R is defined as follows. Take n_0 so that $\left|\operatorname{Im} s_n - \frac{n\pi}{c_\infty \kappa}\right| < \eta$, where $0 < \eta \ll \frac{1}{2}\frac{\pi}{c_\infty \kappa}$, for $n > n_0$, and put

$$R = \frac{n\pi}{c_\infty \kappa} + \frac{1}{2}\frac{\pi}{c_\infty \kappa}, \quad n > n_0. \qquad [4.67]$$

From [4.34] and corollary 4.1, we have

$$\left|\tilde{P}(x,s)\right| \le \frac{C}{|s|^2}, \quad |s| \to \infty. \qquad [4.68]$$

Using [4.68], we calculate the integral over Γ_1 as

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\le \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{P}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\
&\le C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0,\quad x\in[0,1],\ t>0.
\end{aligned}$$

Similar arguments are valid for the integral along contour Γ_7. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_7}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0,\quad x\in[0,1],\ t>0.$$

Next, we consider the integral along contour Γ_2. As has been said for Figure 4.2, the contour Γ_2 consists of parts of the contour $\Gamma_R = \left\{s = R\,\mathrm{e}^{\mathrm{i}\phi} \mid \phi\in\left[\frac{\pi}{2},\pi\right]\right\}$ and of finite number of contours $\Gamma_\varepsilon = \{|s-s_k| = \varepsilon \mid f(s_k) = 0\}$ encircling the poles s_k either from inside or from outside of Γ_R. (Note that the distances between poles are greater than ε, $n>n_0$). More precisely, if a pole is inside D_R, then Γ_ε is outside D_R, and if a pole is outside D_R, then Γ_ε is inside D_R. From [4.68], the integral over the contour Γ_2 becomes

$$\begin{aligned}
\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\le \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{P}\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\
&\le C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0,\quad x\in[0,1],\ t>0
\end{aligned}$$

because $\cos\phi\le 0$ for $\phi\in[\frac{\pi}{2},\pi]$. Similar arguments are valid for the integral along Γ_6. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0,\quad x\in[0,1],\ t>0.$$

Consider the integral along Γ_4. Let $|s|\to 0$. Then, by $(\mathrm{A}1)$, $sM(s)\to 0$, $\cosh(\kappa sM(s))\to 1$, $\sinh(\kappa sM(s))\to 0$ and $\sinh(\kappa xsM(s))\approx\kappa xsM(s)$. Hence, from [4.34], we have

$$\left|\tilde{P}(x,s)\right|\approx x\left|M(s)\right|^2\approx c_0^2 x,\quad x\in[0,1],\ s\in V, |s|\to 0. \tag{4.69}$$

The integration along the contour Γ_4 gives

$$\lim_{r\to 0}\left|\int_{\Gamma_4}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq \lim_{r\to 0}\int_{-\pi}^{\pi}\left|\tilde{P}\left(x,r\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi$$
$$\leq c_0^2 x\lim_{r\to 0}\int_{-\pi}^{\pi} r\mathrm{e}^{rt\cos\phi}\mathrm{d}\phi = 0,\quad x\in[0,1],\ t>0,$$

where we used [4.69].

Integrals along Γ_3, Γ_5 and γ_0 give ($x\in[0,1]$, $t>0$)

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_3}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = \int_0^{\infty}\frac{M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa xqM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q, \tag{4.70}$$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_5}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = -\int_0^{\infty}\frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa xqM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q, \tag{4.71}$$

$$\lim_{R\to\infty}\int_{\gamma_0}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}P(x,t). \tag{4.72}$$

We note that [4.72] is valid if the inversion of the Laplace transform exists, which is true because all the singularities of $\tilde{P}$ are left from the line γ_0 and the estimates on $\tilde{P}$ over γ_0 imply the convergence of the integral. Summing up [4.70], [4.71] and [4.72] we obtain the left-hand side of [4.65] and finally P in the form given by [4.40]. Separately analyzing

$$\frac{1}{\pi}\int_0^{\infty}\mathrm{Im}\left(\frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa xqM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q,$$

$$2\sum_{n=1}^{\infty}\mathrm{Re}\left(\mathrm{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)\right),$$

we conclude that both terms appearing in [4.40] are continuous functions on $t \in [0, \infty)$ for every $x \in [0, 1]$. The continuity also holds with respect to $x \in [0, 1]$ if we fix $t \in [0, \infty)$. This implies that u is a continuous function on $[0, 1] \times [0, \infty)$. From the uniqueness of the Laplace transform, it follows that u is unique. Since F belongs to $\mathcal{S}'_+$, it follows that

$$u(x, \cdot) = F(\cdot) * P(x, \cdot) \in \mathcal{S}'_+$$

for every $x \in [0, 1]$ and $u \in C\left([0, 1], \mathcal{S}'_+\right)$. Moreover, if $F \in L^1_{loc}([0, \infty))$, then $u \in C([0, 1] \times [0, \infty))$ because P is continuous. ■

Next, we prove theorem 4.2.

PROOF OF THEOREM 4.2.– Because $\tilde{F}(s) = \tilde{H}(s) = \frac{1}{s}$, $s \neq 0$, by [4.33] and [4.35] we obtain

$$\begin{aligned} \tilde{\sigma}_H(x, s) &= \frac{1}{s} \frac{\kappa \cosh(\kappa x s M(s))}{s M(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s))}, \\ &\quad x \in [0, 1], \ s \in V. \end{aligned} \qquad [4.73]$$

We calculate $\sigma_H(x, t)$, $x \in [0, 1]$, $t \in \mathbb{R}$ by the integration over the same contour from Figure 4.2. The Cauchy residues theorem yields

$$\begin{aligned} \oint_\Gamma \tilde{\sigma}_H(x, s) \mathrm{e}^{st} \mathrm{d}s &= 2\pi \mathrm{i} \sum_{n=1}^{\infty} \left(\operatorname{Res}\left(\tilde{\sigma}_H(x, s) \mathrm{e}^{st}, s_n\right) \right. \\ &\quad \left. + \operatorname{Res}\left(\tilde{\sigma}_H(x, s) \mathrm{e}^{st}, \bar{s}_n\right) \right) \end{aligned} \qquad [4.74]$$

so that poles of $\tilde{Q}$ lie inside the contour Γ.

First we show that the series of residues in [4.42] is convergent and real-valued. The poles s_n, $n \in \mathbb{N}$, of $\tilde{\sigma}_H$ given by [4.73] are the same as for the function $\tilde{P}$, [4.34]. From proposition 4.3, the poles s_n, $n \in \mathbb{N}$, are simple for sufficiently large n. Then, for $n > n_0$, the residues in [4.74] can be calculated using [4.44] as

$$\begin{aligned}
&\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right) \\
&\quad = \left[\frac{\kappa\cosh(\kappa x s M(s))}{(1+\kappa^2)\sinh(\kappa s M(s)) + \kappa s M(s)\cosh(\kappa s M(s))}\right]_{s=s_n} \\
&\qquad \times \left[\frac{\mathrm{e}^{st}}{s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))}\right]_{s=s_n}, \quad x\in[0,1],\ t>0.
\end{aligned}$$

By using [4.45], we obtain ($x \in [0,1]$, $t > 0$)

$$\begin{aligned}
&\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right) \\
&\quad = \frac{\kappa\cos(\kappa w_n x)}{(1+\kappa^2)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)}\frac{\mathrm{e}^{s_n t}}{\left[s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}}.
\end{aligned}$$

Proposition 4.3 implies

$$\left|\mathrm{e}^{s_n t}\right| < \bar{c}\mathrm{e}^{at}, \quad t > 0,$$

for some $a \in \mathbb{R}$. Because $|s_n| \approx \frac{n\pi}{c_\infty \kappa}$ and $|w_n| \approx \frac{n\pi}{\kappa}$, for $n > n_0$, it follows

$$\begin{aligned}
\left|\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \quad &\le \quad \frac{\kappa}{n\pi}\frac{\bar{c}\mathrm{e}^{at}}{\left|\left[s\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right]_{s=s_n}\right|}, \\
& \qquad x\in[0,1],\ t>0,\ n>n_0.
\end{aligned}$$

Now we use assumption (A3) and conclude that there exists $K > 0$ such that

$$\left|\operatorname{Res}\left(\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \le K\frac{\mathrm{e}^{at}}{n^2}, \quad x\in[0,1],\ t>0,\ n>n_0.$$

This implies that the series of residues in [4.74] (i.e. in [4.42]) is convergent.

Let us calculate the integral over Γ in [4.74]. Consider the integral along contour

$$\Gamma_1 = \{s = p + \mathrm{i}R \mid p\in[0,s_0],\ R>0\},$$

where R is defined by [4.67]. We use estimates

$$\frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \leq \frac{C}{|s|}, \quad \text{or} \quad \frac{|\cosh(\kappa x s M(s))|}{|f(s)|} \leq C, \qquad [4.75]$$

from corollary 4.2 in [4.73]. With the first estimate in [4.75] we calculate the integral over Γ_1 as

$$\begin{aligned} \lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}|\tilde{\sigma}_H(x,p+\mathrm{i}R)|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\kappa\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p=0, \quad x\in[0,1],\ t>0, \end{aligned}$$

while with the second estimate in [4.75], we have

$$\begin{aligned} \lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}|\tilde{\sigma}_H(x,p+\mathrm{i}R)|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\kappa\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R}\mathrm{e}^{pt}\mathrm{d}p=0, \\ &\qquad x\in[0,1],\ t>0. \end{aligned}$$

Similar arguments are valid for the integral along Γ_7. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_7}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right|=0, \quad x\in[0,1],\ t>0.$$

Next, we consider the integral along the contour Γ_2, defined as in the proof of theorem 4.1. With the first estimate in [4.75], the integral over Γ_2 becomes

$$\begin{aligned} \lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{\sigma}_H\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\kappa\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi=0, \\ &\qquad x\in[0,1],\ t>0, \end{aligned}$$

since $\cos\phi \le 0$ for $\phi \in \left[\frac{\pi}{2}, \pi\right]$. The integral over Γ_2 in the case of the second estimate in [4.75] becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\le \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{\sigma}_H\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi\\ &\le C\kappa\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0,\quad x\in[0,1],\ t>0,\end{aligned}$$

since $\cos\phi \le 0$ for $\phi \in \left[\frac{\pi}{2}, \pi\right]$. Similar arguments are valid for the integral along Γ_6. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0,\quad x\in[0,1],\ t>0.$$

Consider the integral along the contour Γ_4. Let $|s|\to 0$. Then, by $(\mathrm{A1})$, we have $sM(s)\to 0$, $\cosh(\kappa sM(s))\to 1$, $\sinh(\kappa sM(s))\to 0$ and $\cosh(\kappa xsM(s))\to 1$. Hence, from [4.73], we have

$$s\sigma_H(x,s)\approx 1.$$

The integration along the contour Γ_4 gives

$$\begin{aligned}\lim_{r\to 0}\int_{\Gamma_4}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= \lim_{r\to 0}\int_{\pi}^{-\pi}\tilde{\sigma}_H\left(x,r\mathrm{e}^{\mathrm{i}\phi}\right)\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\phi}}\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\mathrm{d}\phi\\ &= \mathrm{i}\int_{\pi}^{-\pi}\mathrm{d}\phi = -2\pi\mathrm{i},\quad x\in[0,1],\ t>0.\end{aligned}\qquad [4.76]$$

Integrals along Γ_3, Γ_5 and γ_0 give ($x\in[0,1]$, $t>0$)

$$\begin{aligned}&\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_3}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\\ &= -\int_0^{\infty}\frac{\kappa\cosh\left(\kappa xqM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q,\end{aligned}\qquad [4.77]$$

$$\begin{aligned}&\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_5}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\\ &= \int_0^{\infty}\frac{\kappa\cosh\left(\kappa xqM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\mathrm{e}^{-qt}}{q\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)\right)}\mathrm{d}q,\end{aligned}\qquad [4.78]$$

$$\lim_{R\to\infty}\int_{\gamma_0}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sigma_H(x,t).\qquad [4.79]$$

Through the same arguments as in the proof of theorem 4.1, [4.79] is valid if the inversion of the Laplace transform exists. This is true since all the singularities of $\tilde{\sigma}_H$ are left from the line γ_0 and appropriate estimates on $\tilde{\sigma}_H$ are satisfied. Adding [4.76], [4.77], [4.78] and [4.79] we obtain the left-hand side of [4.74] and finally σ_H in the form given by [4.42].

Function σ_H is a sum of three addends: H and

$$\frac{\kappa}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{\cosh\left(\kappa x q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa q M\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q,$$

$$2\sum_{n=1}^{\infty}\operatorname{Re}\left(\operatorname{Res}\left(\tilde{\sigma}_H\left(x,s\right)\mathrm{e}^{st},s_n\right)\right).$$

As in the case of theorem 4.1, the explicit form of solution implies that σ_H is continuous on $[0,1]\times[0,\infty)$. ■

4.1.2.5. *The case of the elastic rod*

We treat the case of the elastic rod separately. Then, for $s\in V$, $M(s)=1$ ($r(s)=1$ and $h(s)\equiv 0$) and clearly, all the conditions (A1)–(A4) hold. From [4.34] and [4.35], we have

$$\tilde{P}_{el}\left(x,s\right) = \frac{1}{s}\frac{\sinh\left(\kappa s x\right)}{s\sinh\left(\kappa s\right)+\kappa\cosh\left(\kappa s\right)}, \quad x\in[0,1],\; s\in V, \qquad [4.80]$$

$$\tilde{Q}_{el}\left(x,s\right) = \frac{\kappa\cosh\left(\kappa s x\right)}{s\sinh\left(\kappa s\right)+\kappa\cosh\left(\kappa s\right)}, \quad x\in[0,1],\; s\in V. \qquad [4.81]$$

We apply the results of the previous section. Propositions 4.1 and 4.2 imply that the zeros of

$$f_{el}\left(s\right)=s\sinh\left(\kappa s\right)+\kappa\cosh\left(\kappa s\right)=0,\quad s\in V,$$

are of the form

$$s_n=\mathrm{i}w_n,\quad \tan\left(\kappa w_n\right)=\frac{\kappa}{w_n},\quad w_n\approx\pm\frac{n\pi}{\kappa},\quad \text{as}\;\; n\to\infty.$$

Each of these zeros is of multiplicity one for $n > n_0$. Moreover, all the zeros s_n, $n > n_0$, of f_{el} lie on the imaginary axis; so, we do not have the branch point at $s = 0$. This implies that the integrals over Γ_3 and Γ_5 (see Figure 4.2) are equal to zero. So, we have the following modifications.

THEOREM 4.3.– Let $F \in \mathcal{S}'_+$. Then the unique solution u to [4.18]–[4.21] is given by

$$u(x,t) = F(t) * P_{el}(x,t), \quad x \in [0,1], \ t > 0,$$

where

$$\begin{aligned} P_{el}(x,t) &= 2\sum_{n=1}^{\infty} \frac{\sin(\kappa w_n x)\sin(w_n t)}{w_n\left(\left(1+\kappa^2\right)\sin(\kappa w_n) + \kappa w_n \cos(\kappa w_n)\right)}, \\ &\qquad x \in [0,1], \ t > 0. \end{aligned} \qquad [4.82]$$

In particular, $u \in C\left([0,1], \mathcal{S}'_+\right)$. Moreover, if $F \in L^1_{loc}([0,\infty))$, then $u \in C([0,1] \times [0,\infty))$.

PROOF.– The explicit form of P_{el} is obtained from [4.80] by the use of the Cauchy residues theorem ($x \in [0,1]$, $t > 0$)

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{el}} \tilde{P}_{el}(x,s)\,\mathrm{e}^{st}\mathrm{d}s &= \sum_{n=1}^{\infty}\Big(\mathrm{Res}\left(\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}, s_n\right) \\ &\quad + \mathrm{Res}\left(\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}, \bar{s}_n\right)\Big), \end{aligned} \qquad [4.83]$$

where the integration contour $\Gamma_{el} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \gamma_0$ is presented in Figure 4.3.

First, we show that the series of residues in [4.83] is real-valued and convergent. Since the poles $s_n = \mathrm{i}w_n$ (and $\bar{s}_n = -\mathrm{i}w_n$) of $\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}$ are simple for $n > n_0$, the residues in [4.83] are calculated by ($x \in [0,1]$, $t > 0$)

$$\begin{aligned} \mathrm{Res}\left(\tilde{P}_{el}(x,s)\,\mathrm{e}^{st}, s_n\right) &= \left[\frac{1}{s}\frac{\sinh(\kappa s x)\,\mathrm{e}^{st}}{\frac{\mathrm{d}}{\mathrm{d}s}\left(s\sinh(\kappa s) + \kappa\cosh(\kappa s)\right)}\right]_{s=\mathrm{i}w_n} \\ &= \frac{\sin(\kappa w_n x)\left(\sin(w_n t) - \mathrm{i}\cos(w_n t)\right)}{w_n\left(\left(1+\kappa^2\right)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)\right)}. \end{aligned}$$

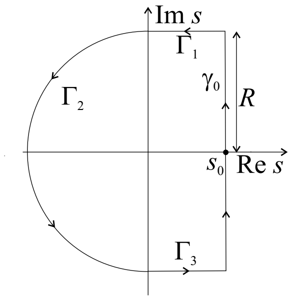

Figure 4.3. *Integration contour* Γ_{el}

Since $w_n \approx \pm\frac{n\pi}{\kappa}$, as $n \to \infty$, the previous expression becomes

$$\left|\operatorname{Res}\left(\tilde{P}_{el}\left(x,s\right)\mathrm{e}^{st}, s_n\right)\right| \le \frac{k}{n^2}, \quad x \in [0,1]\,,\ t>0, n>n_0.$$

We conclude that the series of residues is convergent.

Now, we calculate the integral over Γ in [4.83]. First, we consider the integral along the contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0]\,,\ R > 0\}$, where R is defined as

$$R = \frac{n\pi}{\kappa} + \frac{1}{2}\frac{\pi}{\kappa}, \quad n > n_0. \tag{4.84}$$

Let $x \in [0,1]\,, t>0$. From [4.80] and corollary 4.1, we have

$$\left|\tilde{P}_{el}\left(x,s\right)\right| \le \frac{C}{|s|^2}, \quad |s| \to \infty. \tag{4.85}$$

Using [4.85], we calculate the integral over Γ_1 as

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{P}_{el}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right| &\le \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{P}_{el}\left(x,p+\mathrm{i}R\right)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\le C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0.\end{aligned}$$

Similar arguments are valid for the integral along Γ_3. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_3}\tilde{P}_{el}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right| = 0, \quad x \in [0,1]\,,\ t>0.$$

Next, we consider the integral along the contour $\Gamma_2 = \left\{ s = R\,\mathrm{e}^{\mathrm{i}\phi} \mid \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \right\}$. By using [4.85], the integral over the contour Γ_2 becomes

$$\begin{aligned}
\lim_{R\to\infty} \left| \int_{\Gamma_2} \tilde{P}_{el}\left(x,s\right) \mathrm{e}^{st} \mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \tilde{P}_{el}\left(x, R\mathrm{e}^{\mathrm{i}\phi}\right) \right| \left| \mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}} \right| \left| \mathrm{i}R\mathrm{e}^{\mathrm{i}\phi} \right| \mathrm{d}\phi \\
&\leq C \lim_{R\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{R} \mathrm{e}^{Rt\cos\phi} \mathrm{d}\phi = 0, \\
& \qquad x \in [0,1]\,,\ t > 0
\end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Integrating along the Bromwich contour, we have

$$\lim_{R\to\infty} \int_{\gamma_0} \tilde{P}_{el}\left(x,s\right) \mathrm{e}^{st} \mathrm{d}s = 2\pi \mathrm{i} P_{el}\left(x,t\right), \quad x \in [0,1]\,,\ t > 0.$$

Therefore, [4.83] yields P_{el} in the form [4.82]. The last assertion of the theorem follows from the proof of theorem 4.1. ■

THEOREM 4.4.– Let $F = H$. Then the unique solution $\sigma_H^{(el)}$ to [4.18]–[4.21], is given by

$$\begin{aligned}
\sigma_H^{(el)}\left(x,t\right) &= -2\kappa \sum_{n=1}^{\infty} \frac{\cos\left(\kappa w_n x\right)\cos\left(w_n t\right)}{w_n\left(\left(1+\kappa^2\right)\sin\left(\kappa w_n\right) + \kappa w_n \cos\left(\kappa w_n\right)\right)}, \\
& \qquad x \in [0,1]\,,\ t > 0.
\end{aligned} \qquad [4.86]$$

In particular, $\sigma_H^{(el)}$ is continuous on $[0,1] \times [0,\infty)$.

PROOF.– The explicit forms of $\sigma_H^{(el)}$ are obtained from [4.81] and [4.33]$_2$, with $\tilde{F} = \frac{1}{s}$, i.e.

$$\tilde{\sigma}_H^{(el)}\left(x,s\right) = \frac{1}{s} \frac{\kappa\cosh\left(\kappa s x\right)}{s\sinh\left(\kappa s\right) + \kappa\cosh\left(\kappa s\right)}, \quad x \in [0,1]\,,\ s \in V,$$

by the use of the Cauchy residues theorem ($x \in [0,1]\,, t > 0$)

$$\frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_{el}} \tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = \sum_{n=1}^{\infty} \Big(\mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, s_n\right) + \mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, \bar{s}_n\right)\Big), \quad [4.87]$$

where the integration contour Γ_{el} is the contour from Figure 4.3.

First we show that the series of residues in [4.87] is convergent and real-valued. Since the poles $s_n = \mathrm{i}w_n$ (and $\bar{s}_n = -\mathrm{i}w_n$) of $\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}$ are simple for $n > n_0$, the residues in [4.87] are calculated by

$$\begin{aligned}\mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, s_n\right) &= \left[\frac{1}{s}\frac{\kappa\cosh(\kappa s x)\,\mathrm{e}^{st}}{\frac{\mathrm{d}}{\mathrm{d}s}\left(s\sinh(\kappa s) + \kappa\cosh(\kappa s)\right)}\right]_{s=\mathrm{i}w_n} \\ &= -\frac{\kappa\cos(\kappa w_n x)\left(\cos(w_n t) + \mathrm{i}\sin(w_n t)\right)}{w_n\left((1+\kappa^2)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)\right)}, \\ &\quad n > n_0.\end{aligned}$$

As $n \to \infty$ $w_n \approx \pm\frac{n\pi}{\kappa}$, the previous expression becomes

$$\left|\mathrm{Res}\left(\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}, s_n\right)\right| \leq \frac{k}{n^2}, \quad x \in [0,1]\,, \ t > 0, \ n > n_0.$$

We conclude that the series of residues is convergent.

Let us calculate the integral over Γ_{el} in [4.87]. Consider the integral along the contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0]\,, \ R > 0\}$, where R is defined by [4.84]. We use estimates

$$\frac{|\cosh(\kappa x s)|}{|f_{el}(s)|} \leq \frac{C}{|s|}, \quad \text{or} \quad \frac{|\cosh(\kappa x s)|}{|f_{el}(s)|} \leq C, \quad [4.88]$$

from corollary 4.2 in [4.86]. With the first estimate in [4.88], we calculate the integral over Γ_1 as

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_1} \tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{\sigma}_H^{(el)}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\kappa\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0, \quad x \in [0,1]\,, \ t > 0,\end{aligned}$$

while with the second estimate in [4.88], we have

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{\sigma}_H^{(el)}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\kappa\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R}\mathrm{e}^{pt}\mathrm{d}p=0,\quad x\in[0,1],\ t>0.\end{aligned}$$

Similar arguments are valid for the integral along Γ_3. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma 3}\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right|=0,\quad x\in[0,1],\ t>0.$$

Next, we consider the integral along the contour $\Gamma_2=\left\{s=R\,\mathrm{e}^{\mathrm{i}\phi}\mid\phi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}$. With the first estimate in [4.88] the integral over Γ_2 becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\tilde{\sigma}_H^{(el)}\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\kappa\lim_{R\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi=0, \\ &\qquad \in[0,1],\ t>0,\end{aligned}$$

since $\cos\phi\leq 0$ for $\phi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. The integral over Γ_2, in the case of the second estimate in [4.88] becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{\sigma}_H(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left|\tilde{\sigma}_H\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\kappa\lim_{R\to\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi=0, \\ &\qquad x\in[0,1],\ t>0,\end{aligned}$$

because $\cos\phi\leq 0$ for $\phi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Integrating along the Bromwich contour, we have

$$\lim_{R\to\infty}\int_{\gamma_0}\tilde{\sigma}_H^{(el)}(x,s)\,\mathrm{e}^{st}\mathrm{d}s=2\pi\mathrm{i}\sigma_H^{(el)}(x,t),\quad x\in[0,1],\ t>0.$$

Therefore, [4.87] yields $\sigma_H^{(el)}$ in the form [4.86]. The last assertion of the theorem follows from the proof of theorem 4.2. ■

4.1.3. *Distributed-order and fractional Zener model as special cases*

It is well-known that the distributed-order and fractional Zener model describe the solid-like viscoelastic body. Thus, we apply the analysis of the previous section to these two models. First, we examine the behavior of the function M, given by [4.26], in the limiting cases as $|s| \to \infty$ and $|s| \to 0$ (by this we mean only those s that belong to $\mathbb{C} \backslash (-\infty, 0]$) in the special cases when M takes any of the forms given by [4.27] and [4.28].

If M is given by [4.27] or [4.28] we have

$$|M(s)| \approx \sqrt{\frac{a}{b}}, \quad \text{as} \quad |s| \to \infty, \quad \text{and} \quad |M(s)| \approx 1, \quad \text{as} \quad |s| \to 0. \qquad [4.89]$$

We will analyze a function of the complex variable

$$f(s) = sM(s) \sinh(\kappa s M(s)) + \kappa \cosh(\kappa s M(s)), \quad s \in \mathbb{C} \qquad [4.90]$$

and in particular the equation

$$f(s) = 0, \quad s \in V, \qquad [4.91]$$

where f is given by [4.90].

Let M be of the form $M(s) = r(s) + \mathrm{i}h(s)$, as $|s| \to \infty$. In order to write M as required, we start from $M^2 = u + \mathrm{i}v$ and obtain the system

$$r^2 - h^2 = u, \quad 2rh = v.$$

Solutions of the previous system belonging to the set of real numbers are

$$r = \pm\frac{\sqrt{2}}{2}\sqrt{\sqrt{u^2+v^2}+u}, \qquad [4.92]$$

$$h = \pm\frac{\sqrt{2}}{2}\sqrt{\sqrt{u^2+v^2}-u}. \qquad [4.93]$$

Assume $\frac{v}{u} \to 0$, $u > 0$. Then by using the approximation $(1+x)^a \approx 1 + ax$, as $x \to 0^+$, from [4.92] and [4.93], we have

$$r \approx \pm\sqrt{u}, \quad h \approx \pm\frac{1}{2}\frac{|v|}{\sqrt{u}}. \qquad [4.94]$$

PROPOSITION 4.6.– Functions M given by [4.27] and [4.28] satisfy $(\mathrm{A1})$–$(\mathrm{A4})$.

PROOF.– Consider M given by [4.27]. If we write $s = r\mathrm{e}^{\mathrm{i}\varphi}$, by [4.27], we have

$$\begin{aligned} M^2(s) &= \frac{1 + as^\alpha}{1 + bs^\alpha}, \quad a \leq b \\ &= \frac{1 + (a+b)\, r^\alpha \cos(\alpha\varphi) + abr^{2\alpha}}{1 + 2br^\alpha \cos(\alpha\varphi) + (br^\alpha)^2} - \mathrm{i}\frac{(b-a)\, r^\alpha \sin(\alpha\varphi)}{1 + 2br^\alpha \cos(\alpha\varphi) + (br^\alpha)^2} \\ &\approx \frac{a}{b} - \mathrm{i}\frac{b-a}{b^2}\frac{\sin(\alpha\varphi)}{r^\alpha}, \quad \text{as} \quad |s| \to \infty. \end{aligned}$$

Using [4.94] we obtain

$$r(s) \approx \pm\sqrt{\frac{a}{b}}, \quad h(s) \approx \pm\frac{1}{2}\sqrt{\frac{b}{a}}\frac{b-a}{b^2}\frac{|\sin(\alpha\varphi)|}{r^\alpha}, \quad \text{as} \quad |s| \to \infty.$$

Let $\varphi \in (0, \pi)$, then $\sin(\alpha\varphi) > 0$. This implies $\operatorname{Re}\left(M^2(s)\right) > 0$ and $\operatorname{Im}\left(M^2(s)\right) < 0$. Therefore, we also have $\operatorname{Re}(M(s)) > 0$ and $\operatorname{Im}(M(s)) < 0$. Similar arguments are valid if $\varphi \in (-\pi, 0)$. Hence, we finally have

$$r(s) \approx \sqrt{\frac{a}{b}}, \quad h(s) \approx -\frac{1}{2}\sqrt{\frac{b}{a}}\frac{b-a}{b^2}\frac{\sin(\alpha\varphi)}{r^\alpha}, \quad \text{as} \quad |s| \to \infty. \qquad [4.95]$$

Next we prove that M in [4.27] satisfies $(\mathrm{A1})$. By [4.95] and [4.27], we have

$$\lim_{|s|\to\infty} r(s) = \sqrt{\frac{a}{b}}, \quad \lim_{|s|\to\infty} h(s) = 0, \quad \lim_{|s|\to 0} |M(s)| = 1.$$

Validity of assumption $(\mathrm{A2})$ follows from [4.95].

In order to show that M in [4.27] satisfies $(\mathrm{A3})$, we use [4.27] and obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}(sM(s)) = M(s)\left(1 - \frac{\alpha(b-a)\, s^\alpha}{2(1 + as^\alpha)(1 + bs^\alpha)}\right).$$

Thus, by [4.95]

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\left(sM\left(s\right)\right)\right| \approx \sqrt{\frac{a}{b}}, \quad \text{as} \quad |s| \to \infty. \tag{4.96}$$

We have that there exists ξ such that

$$\left|\left(s+\Delta s\right)M\left(s+\Delta s\right) - sM\left(s\right)\right| \leq \left|\Delta s\right| \left|\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(sM\left(s\right)\right)\right]_{s=\xi}\right|.$$

Becasue $\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))$, by [4.96], is bounded as $|s| \to \infty$ and if $|\Delta s| < \theta$ for some $\theta > 0$, we have that $(\mathrm{A}4)$ is satisfied.

Now consider M given by [4.28]. With $s = r\mathrm{e}^{\mathrm{i}\varphi}$, we have

$$\begin{aligned}
M^2\left(s\right) &= \frac{\ln\left(bs\right)}{\ln\left(as\right)}\frac{as-1}{bs-1}, \quad a \leq b \\
&= \frac{\ln\left(ar\right)\ln\left(br\right)+\varphi^2}{\ln^2\left(ar\right)+\varphi^2}\frac{abr^2-\left(a+b\right)r\cos\varphi+1}{\left(br\right)^2-2br\cos\varphi+1} \\
&\quad -\frac{\varphi\ln\frac{b}{a}}{\ln^2\left(ar\right)+\varphi^2}\frac{\left(b-a\right)r\sin\varphi}{\left(br\right)^2-2br\cos\varphi+1} \\
&\quad -\mathrm{i}\left(\frac{\ln\left(ar\right)\ln\left(br\right)+\varphi^2}{\ln^2\left(ar\right)+\varphi^2}\frac{\left(b-a\right)r\sin\varphi}{\left(br\right)^2-2br\cos\varphi+1}\right. \\
&\quad \left.+\frac{\varphi\ln\frac{b}{a}}{\ln^2\left(ar\right)+\varphi^2}\frac{abr^2-\left(a+b\right)r\cos\varphi+1}{\left(br\right)^2-2br\cos\varphi+1}\right) \\
&\approx \frac{a}{b}-\mathrm{i}\frac{a}{b}\ln\frac{b}{a}\frac{\varphi}{\ln^2\left(ar\right)}, \quad \text{as} \quad |s| \to \infty.
\end{aligned}$$

Using [4.94], we obtain

$$r\left(s\right) = \pm\sqrt{\frac{a}{b}}, \qquad h\left(s\right) = \pm\frac{1}{2}\sqrt{\frac{a}{b}}\ln\frac{b}{a}\frac{|\varphi|}{\ln^2\left(ar\right)}.$$

Let $\varphi \in (0,\pi)$, then $\sin(\alpha\varphi) > 0$. This implies $\operatorname{Re}\left(M^2(s)\right) > 0$ and $\operatorname{Im}\left(M^2(s)\right) < 0$. Therefore, we also have $\operatorname{Re}(M(s)) > 0$ and $\operatorname{Im}(M(s)) < 0$. Similar arguments are valid if $\varphi \in (-\pi, 0)$. Hence, we finally have

$$r(s) = \sqrt{\frac{a}{b}}, \quad h(s) = -\frac{1}{2}\sqrt{\frac{a}{b}}\ln\frac{b}{a}\frac{\varphi}{\ln^2(ar)}. \tag{4.97}$$

Next we prove that [4.28] satisfies $(\mathrm{A1})$. From [4.97] and [4.28], we have

$$\lim_{|s|\to\infty} r(s) = \sqrt{\frac{a}{b}}, \quad \lim_{|s|\to\infty} h(s) = 0, \quad \lim_{|s|\to 0} |M(s)| = 1.$$

Validity of assumption $(\mathrm{A2})$ follows from [4.97].

In order to show M in [4.28] satisfies $(\mathrm{A3})$, we use [4.28] and obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}(sM(s)) = M(s)\left(1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)}\right).$$

Thus, by [4.97]

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}(sM(s))\right| \approx \sqrt{\frac{a}{b}}, \quad \text{as} \quad |s| \to \infty. \tag{4.98}$$

Using the same arguments as above, $(\mathrm{A4})$ is satisfied because $sM(s)$, $s \in V$, from [4.98], has the bounded first derivative. ■

The existence and the uniqueness of u and σ, as solutions to systems [4.18]–[4.21], are guaranteed by the fact that M in both cases satisfies $(\mathrm{A1})$–$(\mathrm{A4})$. Recall that f is given by [4.90] and s_n, $n \in \mathbb{N}$, are solutions of [4.91]. The multiplicity of zeros s_n is one for n large enough.

4.1.4. *Solid and fluid-like model of a rod. A parallel analysis of systems [4.18]–[4.21]*

In section 4.1.2, we presented the existence and uniqueness theorems valid in the general case of the linear solid-like fractional-order viscoelastic material, while in section 4.1.3, we applied these results to two specific modes: distributed-order and the fractional Zener model. In this section, we parallelly treat the same problem for the mentioned two models of solid-like viscoelastic bodies and for a linear model of fluid-like body.

4.1.4.1. *Auxiliary results*

We examine the behavior of the function M, given by [4.26], in the limiting cases as $|s| \to \infty$ and $|s| \to 0$ in the special cases when M takes any of the forms given by [4.27], [4.28] and [4.29]. Note that we will use the "terms" "solid" and "fluid-like" viscoelastic material as stated in remark 3.7.

PROPOSITION 4.7.–

1) If M is given by [4.27] or [4.28] we have

$$|M(s)| \approx \sqrt{\frac{a}{b}}, \quad \text{as} \quad |s| \to \infty, \quad \text{and} \quad |M(s)| \approx 1, \quad \text{as} \quad |s| \to 0. \qquad [4.99]$$

2) If M is given by [4.29] we have

$$|M(s)| \approx \frac{1}{|s|^{\frac{\beta_2-\alpha}{2}}}\sqrt{\frac{a}{b_2}}, \quad \text{as} \quad |s| \to \infty, \quad \text{and} \quad |M(s)| \approx \frac{1}{|s|^{\frac{\beta_0}{2}}\sqrt{b_0}}, \quad \text{as} \quad |s| \to 0. \qquad [4.100]$$

Now, we establish the location and the multiplicity of poles of $\tilde{P}$, given by [4.34] and of $\tilde{Q}$, given by [4.35] in the three special cases of ϕ_σ and ϕ_ε, given by [4.13], [4.15] and [4.17].

In the following, we say that some assertion holds for sufficiently large $n \in \mathbb{N}$, if there exists $n_0 \in \mathbb{N}$ such that the assertion holds for $n > n_0$.

PROPOSITION 4.8.– Let M of the form [4.27], [4.28] and [4.29]. Let

$$f(s) = sM(s)\sinh(\kappa s M(s)) + \kappa\cosh(\kappa s M(s)) = 0, \quad s \in \mathbb{C}. \qquad [4.101]$$

Then the following cases hold:

1) There are a countable number of complex conjugated zeros of f, denoted by s_n and $\bar{s}_n$, $n \in \mathbb{N}$, respectively.

2) The zeros s_n (and $\bar{s}_n$) of f are located in the left complex half-plane for sufficiently large $n \in \mathbb{N}$. Moreover, for sufficiently large $n \in \mathbb{N}$, the zeros s_n take the following forms.

a) If M is given by [4.27], then

$$|s_n| \approx \frac{n\pi}{\kappa}\sqrt{\frac{b}{a}}, \quad \operatorname{Re} s_n \approx -\frac{b-a}{2ab}|s_n|^{1-\alpha}\sin\frac{\alpha\pi}{2}, \quad \operatorname{Im} s_n \approx |s_n|. \qquad [4.102]$$

b) If M is given by [4.28], then

$$|s_n| \approx \frac{n\pi}{\kappa}\sqrt{\frac{b}{a}}, \quad \operatorname{Re} s_n \approx -\frac{\pi}{4}\frac{|s_n|}{\ln^2(b|s_n|)}\ln\frac{b}{a}, \quad \operatorname{Im} s_n \approx |s_n|.$$

c) If M is given by [4.29], then

$$|s_n| \approx \left(\frac{n\pi}{\kappa}\sqrt{\frac{b_2}{a}}\right)^{\frac{2}{2-\beta_2+\alpha}},$$

$$\operatorname{Re} s_n \approx |s_n|\cos\frac{\pi}{2-\beta_2+\alpha} < 0, \quad \operatorname{Im} s_n \approx |s_n|\sin\frac{\pi}{2-\beta_2+\alpha}.$$

3) The zero s_n (and $\bar{s}_n$) of f is of multiplicity one for sufficiently large $n \in \mathbb{N}$.

REMARK 4.3.– Note that $\operatorname{Re} s_n = 0$ in [4.102] for the case of the elastic body $b = a$.

PROOF.– Put

$$sM(s) = v(s) + \mathrm{i}w(s), \quad s \in \mathbb{C} \qquad [4.103]$$

in [4.101], where v, w are real-valued functions depending on the real and imaginary parts of s. By taking the modulus of [4.101] we obtain

$$\mathrm{e}^{2\kappa v} = \frac{(v-\kappa)^2 + w^2}{(v+\kappa)^2 + w^2}. \qquad [4.104]$$

Let $v < 0$ for fixed w. Then, $\mathrm{e}^{2\kappa v} < 1$ and $\frac{(v-\kappa)^2+w^2}{(v+\kappa)^2+w^2} > 1$. Let $v > 0$ for fixed w. Then $\mathrm{e}^{2\kappa v} > 1$ and $\frac{(v-\kappa)^2+w^2}{(v+\kappa)^2+w^2} < 1$. Thus, we conclude that the only solution to [4.104] is $v = 0$. Therefore, solutions of [4.101] are of the form $sM(s) = \mathrm{i}w(s)$, $s \in \mathbb{C}$. Inserting this into [4.101] yields $\tan(\kappa w) = \frac{\kappa}{w}$, $w \in \mathbb{R}$. We see that if w solves the previous equation, then $-w$ does it also, which allows us to consider the positive w only. Since $\frac{\kappa}{w}$ monotonically decreases for all $w \neq 0$ and the since the tangent is a periodic function, we conclude that there are a countable number of values of w (thus of s also), which will be denoted by w_n, $n \in \mathbb{N}$. This proves the first part of (1). Moreover, we see that

$$w_n \approx \frac{n\pi}{\kappa} \quad \text{as} \quad n \to \infty. \qquad [4.105]$$

Therefore, [4.103] becomes

$$s_n M(s_n) = \pm \mathrm{i} w_n \text{ i.e. } s_n = \pm \mathrm{i} w_n \sqrt{\tilde{E}(s_n)}, \quad s_n \in \mathbb{C}, \ n \in \mathbb{N}, \qquad [4.106]$$

where $\tilde{E}$ is given by [4.25] and

$$\tan(\kappa w_n) = \frac{\kappa}{w_n}, \quad w_n > 0.$$

Writing $s = r\mathrm{e}^{\mathrm{i}\varphi}$, $r \in (0, \infty)$, $\varphi \in (-\pi, \pi)$, $\tilde{E} = E_R + \mathrm{i}E_I$ and by separating the real and imaginary parts in [4.106] we obtain

$$r_n^2 \cos(2\varphi_n) = -w_n^2 E_R(r_n, \varphi_n), \quad r_n^2 \sin(2\varphi_n) = -w_n^2 E_I(r_n, \varphi_n). [4.107]$$

a) In the case of the fractional Zener model of a viscoelastic body, by [4.27], we have

$$\begin{aligned} E_R(r,\varphi) &= \frac{1 + (a+b) r^\alpha \cos(\alpha\varphi) + abr^{2\alpha}}{1 + 2ar^\alpha \cos(\alpha\varphi) + (ar^\alpha)^2}, \\ E_I(r,\varphi) &= \frac{(b-a) r^\alpha \sin(\alpha\varphi)}{1 + 2ar^\alpha \cos(\alpha\varphi) + (ar^\alpha)^2}, \end{aligned}$$

where $\alpha \in (0,1)$, $a \leq b$. Let $r \to \infty$, then from the previous expressions we have

$$E_R(r,\varphi) \approx \frac{b}{a}, \quad E_I(r,\varphi) \approx \frac{b-a}{a^2} \sin(\alpha\varphi) \frac{1}{r^\alpha}. \qquad [4.108]$$

b) In the case of the distributed-order model of a solid-like viscoelastic body, from [4.28], we have

$$\begin{aligned} E_R(r,\varphi) &= \frac{abr^2 - (a+b) r\cos\varphi + 1}{(ar)^2 - 2ar\cos\varphi + 1} \frac{\ln(ar)\ln(br) + \varphi^2}{\ln^2(br) + \varphi^2} \\ &\quad + \frac{(b-a) r\sin\varphi}{(ar)^2 - 2ar\cos\varphi + 1} \frac{\varphi \ln\frac{b}{a}}{\ln^2(br) + \varphi^2}, \\ E_I(r,\varphi) &= \frac{abr^2 - (a+b) r\cos\varphi + 1}{(ar)^2 - 2ar\cos\varphi + 1} \frac{\varphi \ln\frac{b}{a}}{\ln^2(br) + \varphi^2} \\ &\quad - \frac{(b-a) r\sin\varphi}{(ar)^2 - 2ar\cos\varphi + 1} \frac{\ln(ar)\ln(br) + \varphi^2}{\ln^2(br) + \varphi^2}, \end{aligned}$$

where $a \leq b$. Let $r \to \infty$, then from the previous expressions we have

$$E_R(r,\varphi) \approx \frac{b}{a}, \quad E_I(r,\varphi) \approx \varphi \frac{b}{a} \ln \frac{b}{a} \frac{1}{\ln^2(br)}. \tag{4.109}$$

c) In the case of the fractional linear model of a fluid-like viscoelastic body, from [4.29], we have

$$\begin{aligned}
&E_R(r,\varphi) \\
&= \frac{b_0 r^{\beta_0} \cos(\beta_0\varphi) + b_1 r^{\beta_1} \cos(\beta_1\varphi) + b_2 r^{\beta_2} \cos(\beta_2\varphi) + ab_0 r^{\alpha+\beta_0} \cos((\beta_0-\alpha)\varphi)}{1 + 2ar^\alpha \cos(\alpha\varphi) + (ar^\alpha)^2} \\
&\quad + \frac{ab_1 r^{\alpha+\beta_1} \cos((\beta_1-\alpha)\varphi) + ab_2 r^{\alpha+\beta_2} \cos((\beta_2-\alpha)\varphi)}{1 + 2ar^\alpha \cos(\alpha\varphi) + (ar^\alpha)^2} \\
&E_I(r,\varphi) \\
&= \frac{b_0 r^{\beta_0} \sin(\beta_0\varphi) + b_1 r^{\beta_1} \sin(\beta_1\varphi) + b_2 r^{\beta_2} \sin(\beta_2\varphi) + ab_0 r^{\alpha+\beta_0} \sin((\beta_0-\alpha)\varphi)}{1 + 2ar^\alpha \cos(\alpha\varphi) + (ar^\alpha)^2} \\
&\quad + \frac{ab_1 r^{\alpha+\beta_1} \sin((\beta_1-\alpha)\varphi) + ab_2 r^{\alpha+\beta_2} \sin((\beta_2-\alpha)\varphi)}{1 + 2ar^\alpha \cos(\alpha\varphi) + (ar^\alpha)^2},
\end{aligned}$$

where $0 < \alpha < \beta_0 < \beta_1 < \beta_2 \leq 1$. Let $r \to \infty$, then from the previous expressions, we have

$$\begin{aligned}
E_R(r,\varphi) &\approx \frac{b_2}{a} \cos((\beta_2-\alpha)\varphi)\, r^{\beta_2-\alpha}, \\
E_I(r,\varphi) &\approx \frac{b_2}{a} \sin((\beta_2-\alpha)\varphi)\, r^{\beta_2-\alpha}.
\end{aligned} \tag{4.110}$$

In the following, we assume that w_n is given by [4.105]. Let $\varphi \to -\varphi$. Then, by [4.108], [4.109] and [4.110], we have

$$E_R(r,-\varphi) = E_R(r,\varphi) \quad \text{and} \quad E_I(r,-\varphi) = -E_I(r,\varphi).$$

Using this in [4.107] we conclude that the zeros of f are complex conjugated for $n \in \mathbb{N}$ large enough. This proves the second statement of (1). This also allows us to work in the upper complex half-plane ($\varphi \in (0,\pi)$) only.

Let $\varphi \in (0, \frac{\pi}{2})$. Then, from [4.108], [4.109] and [4.110], we have

$$E_R(r,\varphi) > 0 \quad \text{and} \quad E_I(r,\varphi) > 0.$$

In this case, $\sin(2\varphi) > 0$; so we conclude that [4.107]$_2$ cannot be satisfied. Thus, the zeros of [4.101], if they exist, lie in the left complex half-plane, for $n \in \mathbb{N}$ large enough, in all three cases considered.

Let us obtain the zeros of f for sufficiently large $n \in \mathbb{N}$. In order to do so, we write [4.107] in the form

$$r_n = w_n \sqrt[4]{E_R^2(r_n, \varphi_n) + E_I^2(r_n, \varphi_n)}, \quad \tan(2\varphi_n) = \frac{E_I(r_n, \varphi_n)}{E_R(r_n, \varphi_n)}. \quad [4.111]$$

By [4.107] we also have

$$\tan\varphi_n = \frac{\left(\frac{w_n}{r_n}\right)^2 (E_I(r_n, \varphi_n) - E_R(r_n, \varphi_n)) - 1}{\left(\frac{w_n}{r_n}\right)^2 (E_I(r_n, \varphi_n) + E_R(r_n, \varphi_n)) - 1}. \quad [4.112]$$

Suppose that $\varphi_n \in \left(\frac{\pi}{2}, \pi\right)$ and that $\tan\varphi_n \to \infty$, as $r_n \to \infty$. Then, by $\cos\varphi_n = -\frac{1}{\sqrt{1+\tan^2\varphi_n}}$ and $\sin\varphi_n = \frac{|\tan\varphi_n|}{\sqrt{1+\tan^2\varphi_n}}$, we have $\cos\varphi_n \approx -\frac{1}{|\tan\varphi_n|}$, $\sin\varphi_n \approx 1$, and therefore,

$$\operatorname{Re} s_n = r_n \cos\varphi_n \approx -r_n \frac{1}{|\tan\varphi_n|}, \quad \operatorname{Im} s_n = r_n \sin\varphi_n \approx r_n. \quad [4.113]$$

Specifically, the cases of (2) are as follows:

a) In the case of the fractional Zener model, from [4.105], [4.108] and [4.111]$_1$, as $r_n \to \infty$, we have

$$|s_n| = r_n \approx \frac{n\pi}{\kappa}\sqrt{\frac{b}{a}}. \quad [4.114]$$

On the other hand, from [4.108] and [4.112], we have

$$\tan\varphi_n \approx 1 - \frac{2ab}{b-a}\frac{r_n^\alpha}{\sin(\alpha\varphi_n)}.$$

Since $\varphi_n \in \left(\frac{\pi}{2}, \pi\right)$ and $\alpha \in (0,1)$, then $\alpha\varphi_n \in (0,\pi)$ and $\sin(\alpha\varphi_n) > 0$ so that as $r_n \to \infty$ we have $\tan\varphi_n \to -\infty$. Thus, $\varphi_n \approx \frac{\pi}{2}$. The previous relation therefore becomes

$$|\tan\varphi_n| \approx \frac{2ab}{b-a}\frac{r_n^\alpha}{\sin\frac{\alpha\pi}{2}}.$$

This, along with [4.113] and [4.114], yields $\operatorname{Re} s_n$ and $\operatorname{Im} s_n$ as stated in (2).

b) In the case of the distributed-order model, from [4.105], [4.109] and $[4.111]_1$, as $r_n \to \infty$, we have

$$|s_n| = r_n \approx \frac{n\pi}{\kappa}\sqrt{\frac{b}{a}}. \qquad [4.115]$$

On the other hand, from [4.109] and [4.112], we have

$$\tan\varphi_n \approx 1 - \frac{2}{\ln\frac{b}{a}}\frac{\ln^2(br_n)}{\varphi_n}.$$

Because $\varphi_n \in \left(\frac{\pi}{2}, \pi\right)$, as $r_n \to \infty$ we have $\tan\varphi_n \to -\infty$. Thus, $\varphi_n \approx \frac{\pi}{2}$. The previous relation therefore becomes

$$|\tan\varphi_n| \approx \frac{4}{\pi\ln\frac{b}{a}}\ln^2(br_n).$$

This, along with [4.113] and [4.115], yields $\operatorname{Re} s_n$ and $\operatorname{Im} s_n$ are as stated in (2).

c) In the case of the fractional linear model of a fluid-like viscoelastic body, by [4.105], [4.110] and [4.111], as $r_n \to \infty$, we have

$$|s_n| = r_n \approx \left(\frac{n\pi}{\kappa}\sqrt{\frac{b_2}{a}}\right)^{\frac{2}{2-\beta_2+\alpha}}, \quad \tan(2\varphi_n) \approx \tan((\beta_2-\alpha)\varphi_n). \qquad [4.116]$$

Solving $[4.116]_2$, we obtain $\varphi_n \approx \frac{\pi}{2-\beta_2+\alpha}$, where $\varphi_n \in \left(\frac{\pi}{2}, \pi\right)$, so that by [4.113] and $[4.116]_1$, $\operatorname{Re} s_n$ and $\operatorname{Im} s_n$ are as stated in (2).

In order to prove 3), i.e. that the zeros of f are of multiplicity one, we differentiate [4.101] and obtain

$$\begin{aligned}\frac{\mathrm{d}f(s)}{\mathrm{d}s} &= \left(\left(1+\kappa^2\right)\sinh(\kappa s M(s))\right. \\ &\qquad \left.+\kappa s M(s)\cosh(\kappa s M(s))\right)\frac{\mathrm{d}}{\mathrm{d}s}(sM(s)). \qquad [4.117]\end{aligned}$$

By the use of [4.26], we have

$$\frac{\mathrm{d}}{\mathrm{d}s}(sM(s)) = M(s)F(s), \qquad [4.118]$$

where

$$F(s) = \left(1 - \frac{\alpha(b-a)s^{\alpha}}{2(1+as^{\alpha})(1+bs^{\alpha})}\right),$$

$$F(s) = \left(1 - \frac{\ln\frac{b}{a}}{2\ln(as)\ln(bs)} + \frac{(b-a)s}{2(as-1)(bs-1)}\right),$$

$$F(s) = \left(1 - \frac{b_0\beta_0 s^{\beta_0} + b_1\beta_1 s^{\beta_1} + b_2\beta_2 s^{\beta_2}}{2\left(b_0 s^{\beta_0} + b_1 s^{\beta_1} + b_2 s^{\beta_2}\right)} + \frac{a\alpha s^{\alpha}}{2(1+as^{\alpha})}\right),$$

in the cases when M is given by [4.27], [4.28] and [4.29], respectively. From the previous expressions, we have

$$|F(s)| \approx 1 \quad \text{as} \quad |s| \to \infty, \qquad [4.119]$$

in the cases of [4.27] and [4.28], while in the case of [4.29] we have

$$|F(s)| \approx 1 - \frac{\beta_2 - \alpha}{2} \quad \text{as} \quad |s| \to \infty. \qquad [4.120]$$

Calculating [4.117] with [4.118] at the point s_n we obtain

$$\left.\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right|_{s=s_n} = \pm\,\mathrm{i}\left(\left(1+\kappa^2\right)\sin(\kappa w_n) + \kappa w_n\cos(\kappa w_n)\right) M(s_n)\, F(s_n)\,, \qquad [4.121]$$

where we used [4.106]. Taking the absolute value of the previous expression, in the limit $|n| \to \infty$, with w_n given by [4.105] we obtain

$$\left|\left.\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right|_{s=s_n}\right| \approx n\pi\,|M(s_n)|\,|F(s_n)| \quad \text{as} \quad n \to \infty. \qquad [4.122]$$

In the cases when M is given either by [4.27], or by [4.28], using $[4.99]_1$ and [4.119], [4.122] becomes

$$\left|\left.\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right|_{s=s_n}\right| \approx n\pi\sqrt{\frac{a}{b}} \to \infty \quad \text{as} \quad n \to \infty.$$

In the case when M is given by [4.29], using $[4.100]_1$ and [4.120], then [4.122] becomes

$$\left|\left.\frac{\mathrm{d}f(s)}{\mathrm{d}s}\right|_{s=s_n}\right| \approx (n\pi)^{2\frac{1-\beta_2+\alpha}{2-\beta_2+\alpha}}\kappa^{\frac{\beta_2-\alpha}{2-\beta_2+\alpha}}\left(\sqrt{\frac{a}{b_2}}\right)^{\frac{2}{2-\beta_2+\alpha}} \times\left(1-\frac{\beta_2-\alpha}{2}\right)\rightarrow\infty \text{ as } n\rightarrow\infty.$$

Thus, we conclude that the multiplicity of the zeros s_n of f is 1 for $n\in\mathbb{N}$ sufficiently large. ■

4.1.4.2. *Existence theorems*

The existence and the uniqueness of u and σ, as solutions to systems [4.18]–[4.21], are given in this section. The next two theorems are in general framework given in section 4.1.2.

THEOREM 4.5.– Let F be a tempered distribution supported by $[0,\infty)$. Then the displacement u, as a part of the solution to [4.18]–[4.21], is given by

$$u(x,t) = F(t) * P(x,t), \quad x\in[0,1],\ t>0,$$

where

$$\begin{aligned} P(x,t) &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{M\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa xqM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)}\right)\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q \\ &+2\sum_{n=1}^{\infty}\operatorname{Re}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)\right), \quad x\in[0,1],\ t>0, \qquad [4.123] \\ P(x,t) &= 0, \quad x\in[0,1],\ t<0. \end{aligned}$$

The residues are given by

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right) = \left[\frac{1}{s}\frac{M(s)\sinh(\kappa xsM(s))}{\frac{\mathrm{d}}{\mathrm{d}s}f(s)}\mathrm{e}^{st}\right]_{s=s_n}, \quad x\in[0,1],\ t>0, \qquad [4.124]$$

where f is given by [4.101] and s_n, for $n \in \mathbb{N}$ large enough, are zeros of f having multiplicity one. Function P is real-valued and both $P(x,\cdot)$ and $u(x,\cdot)$ are continuous on $[0,\infty)$.

PROOF.– First we show using theorem 2 of [DOE 55, p. 293] that P is a real-valued function. Namely, because $\tilde{P}$, given by [4.34], is a real-valued function for all real s (which will be denoted by ξ) in the complex half-plane right from $\operatorname{Re} s = 0$, then the function P is also real-valued almost everywhere. Using [4.34], we have

$$\tilde{P}(x,\xi) = \frac{1}{\xi} \frac{M(\xi)\sinh(\kappa x \xi M(\xi))}{\xi M(\xi)\sinh(\kappa \xi M(\xi)) + \kappa \cosh(\kappa \xi M(\xi))}, \quad \xi > 0,$$

which is real for M, given by [4.27], [4.28] and [4.29].

Next, we calculate P by the integration over a suitable contour shown in Figure 4.4. The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sum_{n=1}^{\infty}\left(\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, s_n\right) + \operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st}, \bar{s}_n\right)\right), \tag{4.125}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_\varepsilon \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \gamma_0$, so that all poles, i.e., the zeros of f, [4.101], lie inside the contour Γ shown in Figure 4.4.

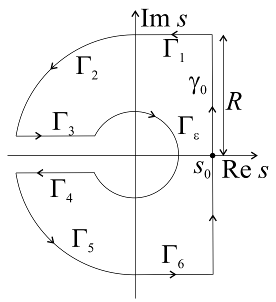

Figure 4.4. *Integration contour* Γ

We show that the series of residues in [4.123] is convergent. By proposition 4.8 the poles s_n, $n \in \mathbb{N}$, of $\tilde{P}$, given by [4.34], are simple for n large enough. Then, the residues in [4.125] can be calculated using [4.124]. By substituting [4.121] into [4.124] and using [4.106], we obtain

$$\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)=\frac{\sin(\kappa w_n x)}{(1+\kappa^2)\sin(\kappa w_n)+\kappa w_n\cos(\kappa w_n)}\frac{\mathrm{e}^{s_n t}}{s_n F(s_n)}.$$

Taking the modulus of the previous expression, we obtain

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)\right|=\left|\frac{\sin(\kappa w_n x)}{(1+\kappa^2)\sin(\kappa w_n)+\kappa w_n\cos(\kappa w_n)}\right|\frac{\mathrm{e}^{t\operatorname{Re}s_n}}{|s_n|\,|F(s_n)|}. \quad [4.126]$$

Let $n\to\infty$. From proposition 4.8 (2) we see that in all three cases considered, $|s_n|\approx Cn^a$, $a\in(0,1]$, $\operatorname{Re}s_n\approx -C'\,|s_n|^b$, $b\in(0,1]$; for some constants $C,C'>0$. From [4.119] and [4.120] it follows that $|F(s_n)|\approx C''>0$. Thus, from the previously mentioned observations and [4.105], [4.126] becomes

$$\left|\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)\right|\approx K\frac{\mathrm{e}^{-K'tn^{ab}}}{n^{a+1}},\quad \text{as } n\to\infty,$$

where $K=\frac{1}{\pi CC''}$, $K'=C'C^b$. This implies that the series of residues in [4.125] is convergent. The case of residues with the complex conjugated poles $\bar{s}_n$ is analogous because $\overline{M(s)}=M(\bar{s})$, $\overline{f(s)}=f(\bar{s})$, and therefore, $\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},\bar{s}_n\right)=\overline{\operatorname{Res}\left(\tilde{P}(x,s)\,\mathrm{e}^{st},s_n\right)}$.

Now, we calculate the integral over Γ in [4.125]. Consider the integral along the contour Γ_1:

$$\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right|\leq\lim_{R\to\infty}\int_0^{s_0}\left|\tilde{P}(x,p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p.$$

Let us estimate $\tilde{P}$ for $|s|\to\infty$. By [4.34], we have

$$\left|\tilde{P}(x,s)\right|\leq\frac{1}{|s|^2}\frac{|sM(s)|\,|\sinh(\kappa xsM(s))|}{||sM(s)|\,|\sinh(\kappa sM(s))|-\kappa\,|\cosh(\kappa sM(s))||}\leq\frac{C}{|s|^2} \quad [4.127]$$

because, by $[4.99]_1$ and $[4.100]_1$, $|sM(s)| \to \infty$, $\left|\frac{\sinh(\kappa s M(s))}{\cos(\kappa s M(s))}\right| \approx 1$ as $|s| \to \infty$ for $x \in [0,1]$. Then, the previous estimate becomes

$$\lim_{R\to\infty}\left|\int_{\Gamma_1}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0,$$

where we used [4.127]. Similar arguments are valid for the integral along the contour Γ_6. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0.$$

Next, we consider the integral along the contour Γ_2

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{P}\left(x,R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0\end{aligned}$$

since [4.127] holds and $\cos\phi \leq 0$ for $\phi \in \left[\frac{\pi}{2},\pi\right]$. Similar arguments are valid for the integral along the contour Γ_5. Thus, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_5}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0.$$

The integration along the contour Γ_ε gives

$$\lim_{\varepsilon\to 0}\left|\int_{\Gamma_\varepsilon}\tilde{P}(x,s)\,\mathrm{e}^{st}\mathrm{d}s\right| \leq \lim_{\varepsilon\to 0}\int_{-\pi}^{\pi}\left|\tilde{P}\left(x,\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{\varepsilon t\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}\varepsilon\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi.$$

Let $|s| \to 0$. Then, by $[4.99]_2$ and $[4.100]_2$, $M(s) \approx \frac{C}{|s|^a}$, $a \in \left[0,\frac{1}{2}\right)$, $sM(s) \approx C|s|^{1-a} \approx 0$ so that $\cosh(\kappa s M(s)) \approx 1$, $\sinh(\kappa s M(s)) \approx \kappa s M(s)$ and $\sinh(\kappa x s M(s)) \approx \kappa x s M(s)$. Hence, from [4.34] we have

$$\left|\tilde{P}(x,s)\right| \approx x\,|M(s)|^2 \approx \frac{C^2 x}{|s|^{2a}} \quad \text{as} \quad |s| \to 0.$$

Thus, the previous integral becomes

$$\lim_{\varepsilon\to 0}\left|\int_{\Gamma_\varepsilon}\tilde{P}\left(x,s\right)\mathrm{e}^{st}\mathrm{d}s\right|\leq C^2 x\lim_{\varepsilon\to 0}\int_{-\pi}^{\pi}\varepsilon^{1-2a}\mathrm{e}^{\varepsilon t\cos\phi}\mathrm{d}\phi=0.$$

From the Cauchy residues theorem [4.125], integrals along Γ_3 and Γ_4 give the summand in [4.123] represented by an integral, while the integral along γ_0 gives the function P itself. ■

The following theorem is on the existence of the stress σ as a solution of systems [4.18]–[4.21]. Note that in this case the existence result is given for $F = H$, while the more general case of F is discussed in remark 4.1. Here, we present the theorem without the proof, which is analogous to the proof of theorem 4.5.

THEOREM 4.6.– Let $F = H$. Then the stress σ_H as a part of the solution to [4.18]–[4.21] is given by

$$\begin{aligned}&\sigma_H\left(x,t\right)\\&\quad=H\left(t\right)+\frac{\kappa}{\pi}\int_0^\infty\mathrm{Im}\left(\frac{\cosh\left(\kappa xqM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}{qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\sinh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)+\kappa\cosh\left(\kappa qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)}\right)\\&\quad\times\frac{\mathrm{e}^{-qt}}{q}\mathrm{d}q+2\sum_{n=1}^{\infty}\mathrm{Re}\left(\mathrm{Res}\left(\tilde{\sigma}_H\left(x,s\right)\mathrm{e}^{st},s_n\right)\right),\quad x\in[0,1],\ t>0,\qquad [4.128]\\&\sigma_H\left(x,t\right)=0,\quad x\in[0,1],\ t<0.\end{aligned}$$

The residues are given by

$$\mathrm{Res}\left(\sigma_H\left(x,s\right)\mathrm{e}^{st},s_n\right)=\left[\frac{\kappa\cosh\left(\kappa xsM\left(s\right)\right)}{s\frac{\mathrm{d}}{\mathrm{d}s}f\left(s\right)}\mathrm{e}^{st}\right]_{s=s_n},\quad x\in[0,1],t>0,\quad [4.129]$$

where f is given by [4.101] and s_n, for $n\in\mathbb{N}$ large enough, are zeros of f having multiplicity one. The function $\sigma_H\left(x,\cdot\right)$ is real-valued and continuous on $[0,\infty)$.

4.1.5. *Numerical examples*

The displacement u as a solution to [4.18]–[4.21] is given in theorem 4.5. We present various numerical examples for constitutive models: fractional Zener and distributed-order models of a solid-like viscoelastic body that are distinguished by the form of M: [4.27] and [4.28], respectively.

4.1.5.1. *The case $F = \delta$*

In order to plot the time dependence of the displacement u for the several points of the rod as well as for the body attached to its free end, we chose the fractional Zener model and the force acting on the body to be the Dirac delta distribution, i.e. $F = \delta$. We fix the parameters describing the rod: $a = 0.2$, $b = 0.6$, $\alpha = 0.45$ and also fix the ratio between the masses of rod and body $\varkappa = \kappa^2 = 1$. The plot of u as a function of time t for various points of a rod is shown in Figure 4.5. It is evident that the oscillations of the rod and a body are damped because the material is viscoelastic. One notices that initially there is a transitional regime of the oscillations. Afterwards, the curves resemble the curves of the damped linear oscillator.

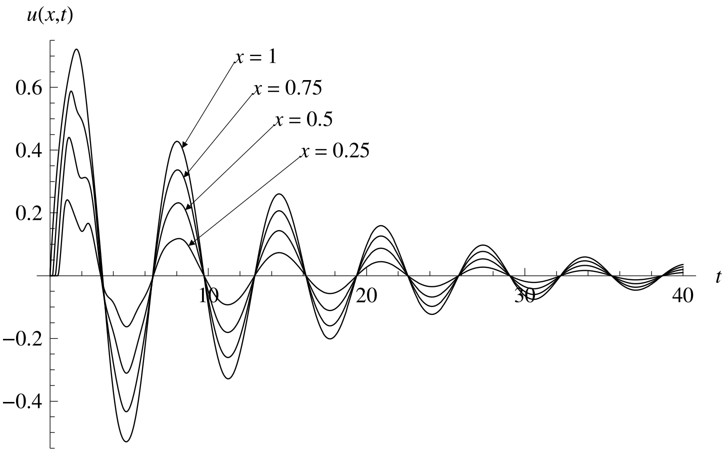

Figure 4.5. *Displacement $u(x, t)$ in the case $F = \delta$ for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 40)$*

In order to examine the transitional regime more closely, in Figures 4.6–4.8 we present the plots of u for smaller values of time, but for different values of $\varkappa \in \{0.5, 1, 2\}$. We notice that the shape the curves depends on the ratio between the masses $\varkappa$, while later, the shape resembles the shape of curves for damped oscillations. It could be noticed that regardless of the value of $\varkappa$, there is a delay in the starting of oscillation of points that are further away from the free end of a rod. This is due to the finite speed of wave propagation. Namely, the body ($x = 1$), which is subject to the action of the force, starts oscillating at $t = 0$, while the delay in the starting time-instant of the oscillation is greater as the point is further from the point where the force acts. Moreover, we see that different points of the rod do not come to their initial position, for the first time, at the same time-instant (see Figure 4.8). This as well as the initial delay depend on the mass ratio $\varkappa$. For the influence of $\varkappa$ on the initial delay, compare Figures 4.6–4.8. Later, again depending on $\varkappa$, the motion of the points becomes synchronized.

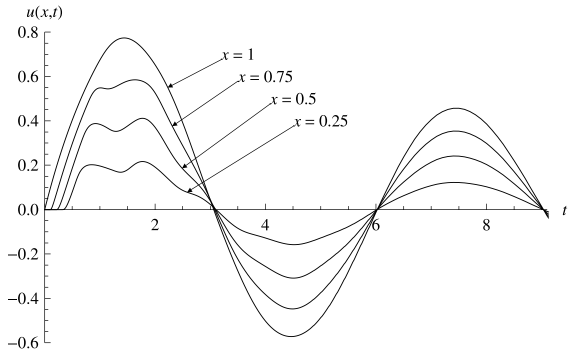

Figure 4.6. *Displacement $u(x,t)$ in the case $F = \delta$ for $\varkappa = 0.5$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 9)$*

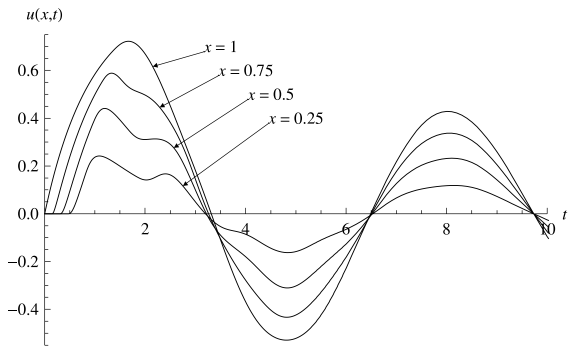

Figure 4.7. *Displacement $u(x,t)$ in the case $F = \delta$ for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 10)$*

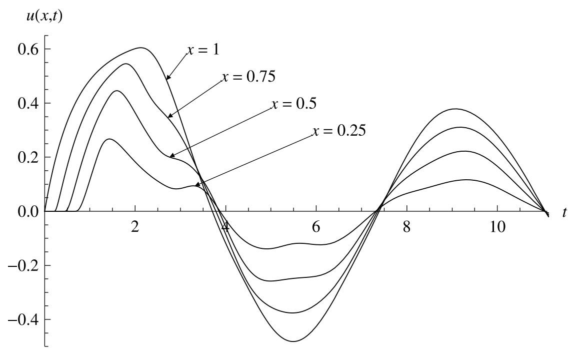

Figure 4.8. *Displacement $u(x,t)$ in the case $F = \delta$ for $\varkappa = 2$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 11)$*

Figures 4.9 and 4.10 present the plots of the displacement u for the fixed point of the rod $x = 0.5$ if the ratio between masses $\varkappa$ varies. We see from Figure 4.9 that the larger the mass of a rod (then the value of $\varkappa$ is greater), the greater the quasi-period (time between two consecutive passages of a fixed point through its initial position) of the oscillations. This is due to the increased inertia of the rod. Also, for larger times, there is no significant influence of $\varkappa$ on the heights and widths of the peaks, which indicates that the damping effects are only due to the parameters a, b and α figuring in the constitutive equation. Figure 4.10 shows that the shape of the curve in transitional regime strongly depends on $\varkappa$. Moreover, the delay in oscillations increases as the $\varkappa$ increases, which indicates that the speed of the wave propagation depends on the mass ratio.

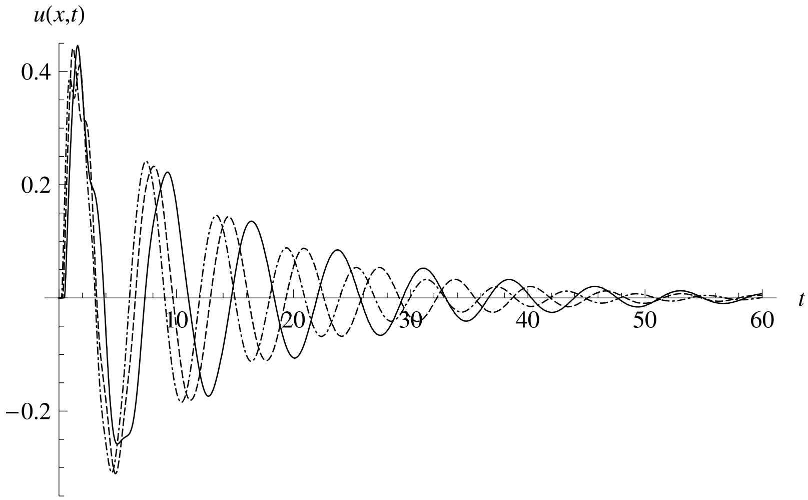

Figure 4.9. *Displacement* $u(x,t)$ *in the case* $F = \delta$ *at* $x = 0.5$ *as a function of time* $t \in (0, 40)$ *for* $\varkappa = 0.5$ *- dot-dashed line,* $\varkappa = 1$ *- dashed line and* $\varkappa = 2$ *- solid line*

4.1.5.2. *The case* $F = H$

The aim of this section is the qualitative analysis of the behavior of a displacement u when there is a force, given in the form of the Heaviside function, i.e. $F = H$, acting on the attached body. Thus, our results correspond to a creep experiment. The rod is modeled by the fractional Zener model, i.e., the function M is given by [4.27]. The parameters describing the rod are: $a = 0.1$, $b = 0.9$, $\alpha = 0.8$. We present plots of u for the ratio between the masses of the rod and body $\varkappa = 1$.

Figure 4.11 presents the long-time behavior of the displacement u. One notices that the rod creeps to a finite value of the displacement so that $\lim_{t\to\infty} u(x,t) = x$,

$x \in [0, 1]$. Figure 4.12 presents the short-time behavior of u. We see that there is a delay in starting time-instant of a point of a rod.

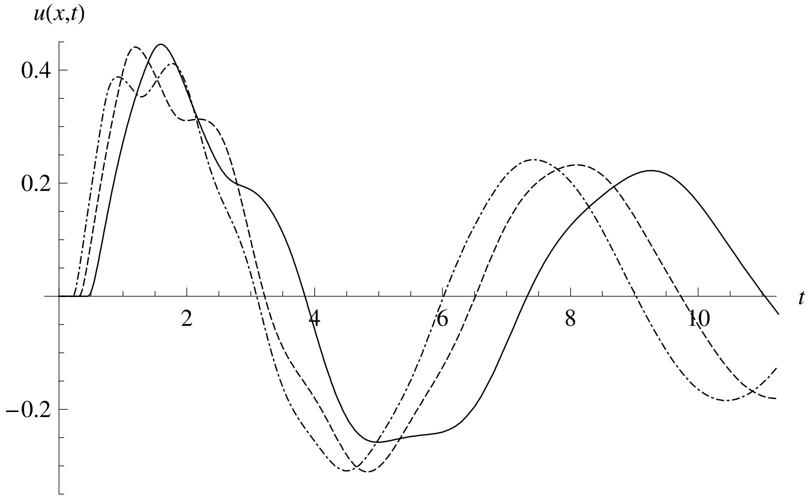

Figure 4.10. *Displacement* $u(x, t)$ *in the case* $F = \delta$ *at* $x = 0.5$ *as a function of time* $t \in (0, 11)$ *for* $\varkappa = 0.5$ *- dot-dashed line,* $\varkappa = 1$ *- dashed line and* $\varkappa = 2$ *- solid line*

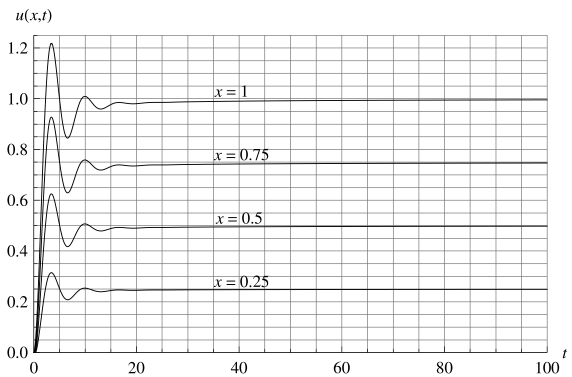

Figure 4.11. *Displacement* $u(x, t)$ *in the creep experiment for* $\varkappa = 1$ *as a function of time* t *at* $x \in \{0.25, 0.5, 0.75, 1\}$ *for* $t \in (0, 100)$

We compare the behavior of u in the cases of two different constitutive models of a rod. Namely, we compare the displacement if the rod is modeled by the fractional Zener model of a (solid-like) viscoelastic body and the fractional linear model of a fluid-like viscoelastic body. These cases are distinguished by the function M that is given by [4.27] and [4.29], respectively. In the case of the Zener model, the parameters describing the rod are: $a = 0.1$, $b = 0.9$, $\alpha = 0.8$, while in the case of the fractional linear model of the fluid-like viscoelastic body, the parameters describing the rod are: $a = 0.2$, $b_0 = 0.8$, $b_1 = 0.4$, $b_2 = 0.6$, $\alpha = 0.1$, $\beta_0 = 0.2$, $\beta_1 = 0.3$, $\beta_2 = 0.4$. We present plots of u for the ratio between the masses of the rod and body $\varkappa = 1$.

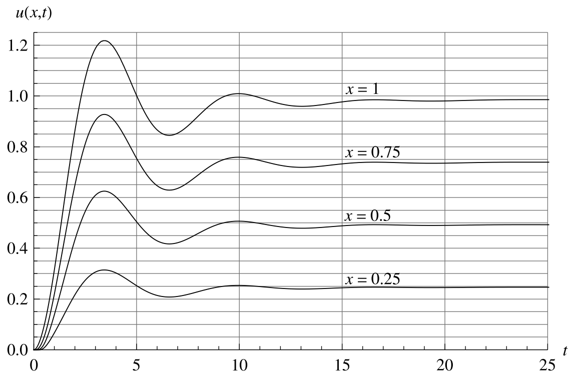

Figure 4.12. *Displacement $u(x,t)$ in the creep experiment for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 25)$*

Figures 4.13 and 4.14 present the long-time behavior of the displacement u for the two mentioned models. In the case of the Zener model, we notice that the rod creeps to a finite value of displacement so that $\lim_{t\to\infty} u(x,t) = x$, $x \in [0,1]$ (see Figure 4.13). In the case of the fluid-like model of viscoelastic body the rod creeps to an infinite value of displacement (see Figure 4.14).

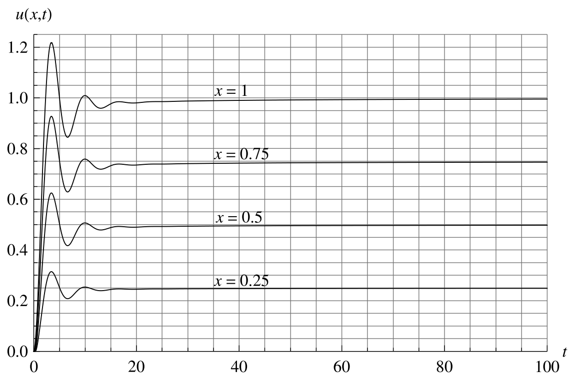

Figure 4.13. *Displacement $u(x,t)$ in the creep experiment in the case of solid-like viscoelastic material for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 100)$*

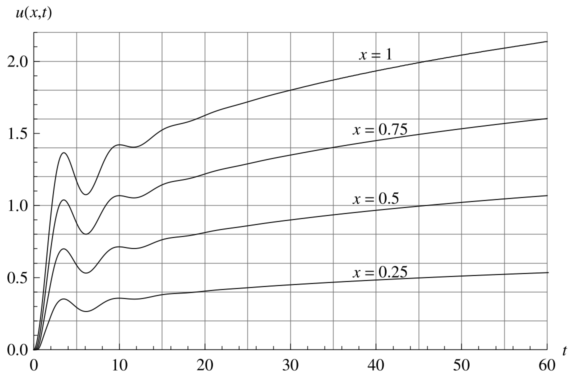

Figure 4.14. *Displacement $u(x,t)$ in the creep experiment in the case of fluid-like viscoelastic material for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 60)$*

Figures 4.15 and 4.16 present the short-time behavior of u for the two mentioned models. We see that in both cases there is a delay in the starting time-instant of a point of a rod.

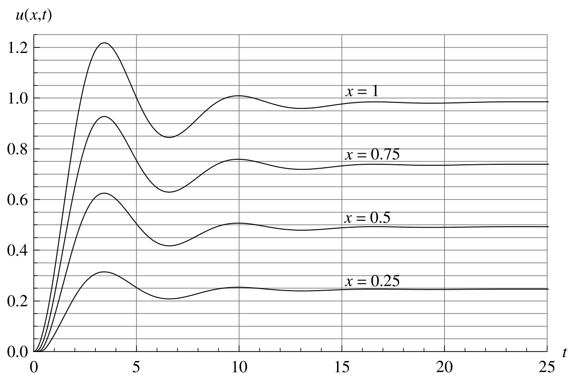

Figure 4.15. *Displacement $u(x,t)$ in the creep experiment in the case of solid-like viscoelastic material for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0, 25)$*

Figures 4.17 and 4.18 present the plots of time evolution of displacement u of a rod described by the Zener model for a fixed point of the rod $x = 0.5$ if the ratio between masses $\varkappa$ varies. Here, the parameters are $a = 0.2$, $b = 0.6$, $\alpha = 0.45$. We see, in Figure 4.17 that the finite value of displacement in creep does not depend on the value of the mass ratio $\varkappa$. We see in Figure 4.18 that for small times there is an influence of $\varkappa$ on the height of the peaks such that its height increases as $\varkappa$ increases. This is due to the inertia, while for larger times, the viscoelastic properties of the rod prevail. Similarly, as in the previous section, the delay in the oscillations' starting time-instant increases as $\varkappa$ increases.

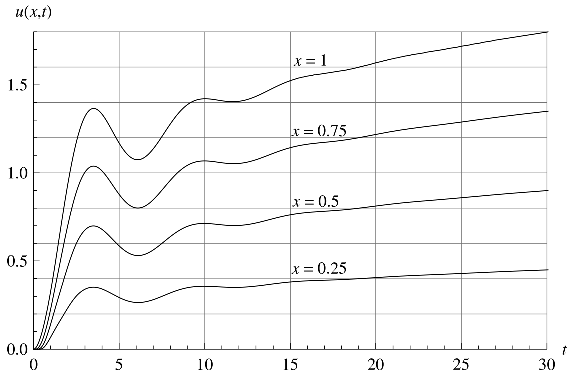

Figure 4.16. *Displacement $u(x,t)$ in the creep experiment in the case of fluid-like viscoelastic material for $\varkappa = 1$ as a function of time t at $x \in \{0.25, 0.5, 0.75, 1\}$ for $t \in (0,30)$*

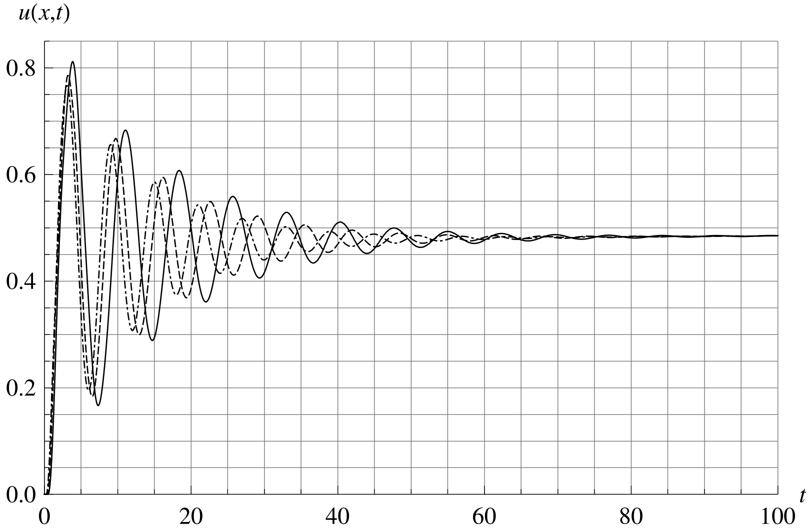

Figure 4.17. *Displacement $u(x,t)$ in the creep experiment at $x = 0.5$ as a function of time $t \in (0,100)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line*

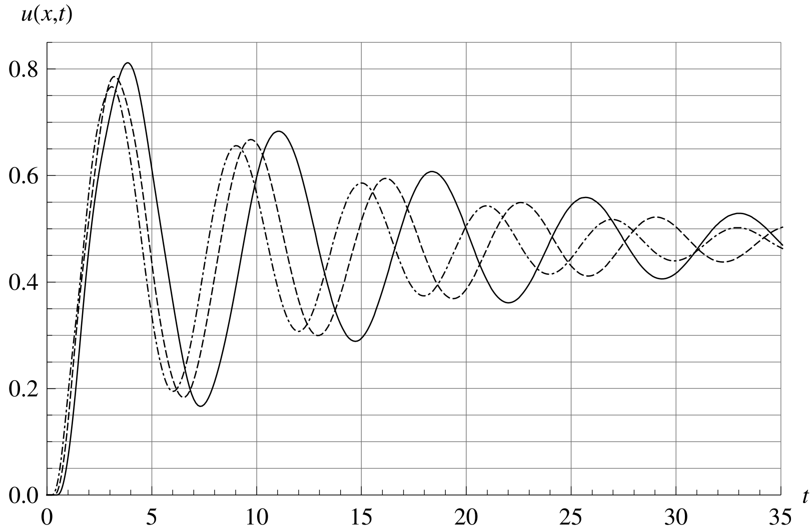

Figure 4.18. *Displacement $u(x,t)$ in the creep experiment at $x = 0.5$ as a function of time $t \in (0,35)$ for $\varkappa = 0.5$ - dot-dashed line, $\varkappa = 1$ - dashed line and $\varkappa = 2$ - solid line*

4.2. Light viscoelastic rod – body system

Following [ATA 05a, ATA 05b, ATA 13d], we investigate forced oscillations of the body attached to the light viscoelastic rod. Recall that systems [4.5]–[4.7] describing the forced oscillations of the light rod-body system can be written as

$$u(x,t) = x\varepsilon(t), \quad x \in [0,L], \ t > 0, \tag{4.130}$$

$$\int_0^1 \phi_\sigma(\gamma)\, {}_0\mathrm{D}_t^\gamma \sigma(t)\, \mathrm{d}\gamma = E\int_0^1 \phi_\varepsilon(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(t)\, \mathrm{d}\gamma, \quad t > 0, \tag{4.131}$$

$$-A\sigma(t) + F(t) = mL\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t), \quad t > 0. \tag{4.132}$$

Introducing the dimensionless quantities

$$\begin{aligned} \bar{x} &= \frac{x}{L}, \ \bar{t} = \frac{t}{\sqrt{\frac{mL}{AE}}}, \ \bar{u} = \frac{u}{L}, \ \bar{\sigma} = \frac{\sigma}{E}, \ \bar{\phi}_\sigma = \frac{\phi_\sigma}{\left(\sqrt{\frac{mL}{AE}}\right)^\gamma}, \\ \bar{\phi}_\varepsilon &= \frac{\phi_\varepsilon}{\left(\sqrt{\frac{mL}{AE}}\right)^\gamma}, \ \bar{F} = \frac{F}{AE}, \end{aligned}$$

substituting them into [4.130], [4.131], [4.132] and by omitting the bar, we obtain

$$u(x,t) = x\varepsilon(t), \quad x \in [0,1], \; t > 0, \tag{4.133}$$

$$\int_0^1 \phi_\sigma(\gamma)\, {}_0\mathrm{D}_t^\gamma \sigma(t)\, \mathrm{d}\gamma = \int_0^1 \phi_\varepsilon(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(t)\, \mathrm{d}\gamma, \quad t > 0, \tag{4.134}$$

$$-\sigma(t) + F(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t), \quad t > 0. \tag{4.135}$$

Systems [4.134], [4.135] will be the subject of further analysis. Namely, in section 4.2.1 we analyze the existence and uniqueness of a solution to systems [4.134], [4.135] for the general case of the constitutive equation [4.134] corresponding to the solid-like viscoelastic rod, while the case of the distributed-order Kelvin–Voigt model is considered in section 4.2.3. Forced oscillations of a body attached to a parallel connection of spring and light viscoelastic rod, described by the distributed-order model, are considered in section 4.2.2.

4.2.1. ***The case of the solid-like viscoelastic rod***

The analysis of systems [4.134] and [4.135], subject to zero initial conditions, is presented along the lines of [ATA 13d]. Thus, we consider

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t) + \sigma(t) = F(t), \quad t > 0, \tag{4.136}$$

$$\int_0^1 \phi_\sigma(\gamma)\, {}_0\mathrm{D}_t^\gamma \sigma(t)\, \mathrm{d}\gamma = \int_0^1 \phi_\varepsilon(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(t)\, \mathrm{d}\gamma, \quad t > 0, \tag{4.137}$$

$$\varepsilon(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(0) = 0. \tag{4.138}$$

4.2.1.1. *Analysis of the problem*

Formally, by applying the Laplace transform to [4.136]–[4.138] we obtain

$$s^2\tilde{\varepsilon}(s) + \tilde{\sigma}(s) = \tilde{F}(s), \quad s \in D, \tag{4.139}$$

$$\tilde{\sigma}(s)\int_0^1 \phi_\sigma(\gamma)\, s^\gamma \mathrm{d}\gamma = \tilde{\varepsilon}(s)\int_0^1 \phi_\varepsilon(\gamma)\, s^\gamma \mathrm{d}\gamma, \quad s \in D. \tag{4.140}$$

By [4.140], we have

$$\tilde{\sigma}(x,s) = \frac{1}{M^2(s)}\tilde{\varepsilon}(x,s), \quad s \in D, \tag{4.141}$$

where we introduce M by

$$M(s) = \sqrt{\frac{\int_0^1 \phi_\sigma(\gamma) s^\gamma \mathrm{d}\gamma}{\int_0^1 \phi_\varepsilon(\gamma) s^\gamma \mathrm{d}\gamma}}, \quad s \in D \subset \mathbb{C}. \qquad [4.142]$$

Note that for $s = \mathrm{i}\omega$

$$E(\omega) = E'(\omega) + \mathrm{i}E''(\omega) = \frac{1}{M^2(\mathrm{i}\omega)} = \frac{\int_0^1 \phi_\varepsilon(\gamma)(\mathrm{i}\omega)^\gamma \mathrm{d}\gamma}{\int_0^1 \phi_\sigma(\gamma)(\mathrm{i}\omega)^\gamma \mathrm{d}\gamma}, \quad \omega \in (0, \infty), \qquad [4.143]$$

is a complex modulus (see [BAG 86]). Functions E' and E'' are real-valued and represent the storage and loss moduli, respectively.

From [4.139] and [4.141], we have

$$\tilde{\varepsilon}(s) = \tilde{F}(s)\tilde{P}(s), \quad \tilde{\sigma}(s) = \tilde{F}(s)\tilde{Q}(s), \quad s \in D, \qquad [4.144]$$

where

$$\tilde{P}(s) = \frac{M^2(s)}{1 + (sM(s))^2}, \quad \tilde{Q}(s) = \frac{1}{1 + (sM(s))^2}, \quad s \in D, \qquad [4.145]$$

and M is given by [4.142].

Inverting the Laplace transform in [4.144], we obtain

$$\varepsilon(t) = F(t) * P(t), \quad \sigma(t) = F(t) * Q(t), \quad t > 0. \qquad [4.146]$$

We now discuss restrictions that ϕ_σ and ϕ_ε must satisfy. There are two conditions that those functions must satisfy. First, ϕ_σ and ϕ_ε must be such that, P and Q are real-valued functions so that the strain ε and stress σ, given by [4.146], are real. The second condition on ϕ_σ and ϕ_ε imposes the second law of thermodynamics, which requires that (in the isothermal case) the dissipation work must be positive. Mathematically, these conditions read as follows:

CONDITION 4.1.–

1) There exists $x_0 \in \mathbb{R}$ such that

$$M(x) = \sqrt{\frac{\int_0^1 \phi_\sigma(\gamma) x^\gamma \mathrm{d}\gamma}{\int_0^1 \phi_\varepsilon(\gamma) x^\gamma \mathrm{d}\gamma}} \in \mathbb{R}, \quad \text{for all } x > x_0.$$

2) For all $\omega \in (0, \infty)$, we have

$$E'(\omega) \geq 0, \quad E''(\omega) \geq 0,$$

where E' and E'' are the storage and loss moduli, respectively, given by [4.143] (see [BAG 86]).

The motivation for (1) of condition 4.1 follows from the following theorem of Doetsch.

THEOREM 4.7.– [DOE 55, p. 293, theorem 2]– Let $f(s) = \mathcal{L}[F](s)$, $\operatorname{Re} s > x_0 \in \mathbb{R}$, be real-valued on the real half-line $s \in (x_0, \infty)$. Then, the function F is real-valued almost everywhere.

Alternatively, if f is real-valued at a sequence of equidistant points on the real axis, then the function F is real-valued almost everywhere.

If ϕ_σ and ϕ_ε are such that (1) of condition 4.1 is satisfied, theorem 4.7 ensures that inversions of [4.144] with [4.145], given by [4.146], are real. As is well known, (2) of condition 4.1 guarantees that the second law of thermodynamics for the isothermal process is satisfied (see [BAG 86]). We will see from proposition 4.9 below that condition 4.1 along with an additional assumption on the asymptotics of M (assumption 4.1 below) guarantees that the poles of the solution kernel in the Laplace domain [4.145] belong to the left complex half-plane. In this case, the amplitude of the solution decreases with the time; this is a characteristic behavior for a dissipative process.

REMARK 4.4.–

1) Condition 4.1 is satisfied for the fractional Zener model

$$\left(1 + a\, {}_0\mathrm{D}_t^{\alpha}\right) \sigma(t) = \left(1 + b\, {}_0\mathrm{D}_t^{\alpha}\right) \varepsilon(t), \qquad [4.147]$$

which is obtained from [4.137] by choosing

$$\begin{aligned} \phi_\sigma(\gamma) &= \delta(\gamma) + a\,\delta(\gamma - \alpha), \quad \phi_\varepsilon(\gamma) = \delta(\gamma) + b\,\delta(\gamma - \alpha), \\ &\alpha \in (0,1), \; 0 < a \leq b \end{aligned} \qquad [4.148]$$

and for the distributed-order model

$$\int_0^1 a^\gamma\, {}_0\mathrm{D}_t^{\gamma} \sigma(x,t)\, \mathrm{d}\gamma = E \int_0^1 b^\gamma\, {}_0\mathrm{D}_t^{\gamma} \varepsilon(x,t)\, \mathrm{d}\gamma, \qquad [4.149]$$

which is obtained from [4.137] by choosing

$$\phi_\sigma(\gamma) = a^\gamma, \;\; \phi_\varepsilon(\gamma) = b^\gamma, \;\; \gamma \in (0,1), \; 0 < a \le b. \qquad [4.150]$$

In the cases of [4.147] and [4.149], function M has a form

$$M(s) = \sqrt{\frac{1+as^\alpha}{1+bs^\alpha}}, \;\; s \in \mathbb{C} \backslash (-\infty, 0], \; 0 < a \le b, \; \alpha \in (0,1), \qquad [4.151]$$

$$M(s) = \sqrt{\frac{\ln(bs)}{\ln(as)} \frac{as-1}{bs-1}}, \;\; s \in \mathbb{C} \backslash (-\infty, 0], \; 0 < a \le b, \qquad [4.152]$$

respectively. Putting $s = x \in \mathbb{R}$ in the previous expressions we see that (1) of condition 4.1 is satisfied with $x_0 = 0$, while $0 < a \le b$ ensures that (2) of condition 4.1 is satisfied (see [ATA 02b, ATA 03, ATA 11b, BAG 86]).

2) In general, the stress σ as a function of a real-valued strain ε may not be real-valued. In [ATA 12c], we have such a situation, because (1) of condition 4.1 is not satisfied. This shows the role of condition 4.1, (1).

In order to obtain ε and σ from [4.144], we have to obtain functions P and Q, i.e. to invert the Laplace transform in [4.145]. Let us make an additional assumption on the function M, [4.142].

ASSUMPTION 4.1.– Let M be of the form

$$M(s) = r(s) + \mathrm{i}h(s), \;\; \text{as} \;\; |s| \to \infty.$$

Suppose that

$$\lim_{|s|\to\infty} r(s) = c_\infty > 0, \;\; \lim_{|s|\to\infty} h(s) = 0, \;\; \lim_{|s|\to 0} M(s) = c_0,$$

for some constants $c_\infty, c_0 > 0$.

Assumption 4.1 is motivated by the fractional Zener [4.147] and distributed-order model [4.149]. Note that both of these models describe the viscoelastic solid-like body. For both models mentioned above, we have $c_\infty = \sqrt{\frac{a}{b}}$, and $c_0 = 1$.

PROPOSITION 4.9.– Let M satisfy condition 4.1 with $x_0 = 0$ and assumption 4.1. Let

$$f(s) = 1 + (sM(s))^2, \;\; s \in \mathbb{C}, \qquad [4.153]$$

then f has two different zeros: s_0 and its complex conjugate $\bar{s}_0$, located in the left complex half-plane ($\operatorname{Re} s < 0$). The multiplicity of each zero is one.

PROOF.– Since $\bar{s}M(\bar{s}) = \overline{sM(s)}$ (the bar denotes the complex conjugation), it is clear from [4.153] that if s_0 is zero of [4.153], then its complex conjugate $\bar{s}_0$ is also zero.

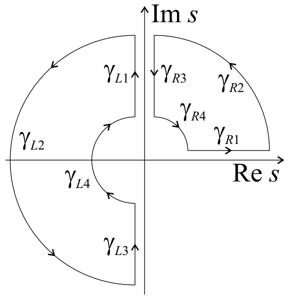

Figure 4.19. *Integration contours γ_L and γ_R*

We use the argument principle in order to show that there are no zeros of [4.153] in the upper right complex half-plane. Let us consider the contour $\gamma_R = \gamma_{R1} \cup \gamma_{R2} \cup \gamma_{R3} \cup \gamma_{R4}$, presented in Figure 4.19. Let $\gamma_{R1} : s = x,\ x \in [r, R]$, where r is the radius of the inner quarter of the circle and R is the radius of the outer quarter of the circle. Then, by (1) of condition 4.1 (with $x_0 = 0$), we have $\operatorname{Re} f(x) > 0$ and $\operatorname{Im} f(x) \equiv 0$. assumption 4.1 implies $\lim_{r\to 0} f(x) = 1$ and $\lim_{R\to\infty} f(x) = \infty$. Hence, $\Delta \arg f(s) = 0$ for $s \in \gamma_{R1}$, as $r \to 0$, $R \to \infty$. Let $\gamma_{R2} : s = R\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in \left[0, \frac{\pi}{2}\right]$. Then, by assumption 4.1, for $\varphi = 0$ we have

$$\operatorname{Re} f(R) \approx c_\infty^2 R^2 \to \infty \quad \text{and} \quad \operatorname{Im} f(R) \to 0 \quad \text{as} \quad R \to \infty.$$

For $\varphi = \frac{\pi}{2}$, Assumption 4.1 implies

$$\operatorname{Re} f\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) \approx -c_\infty^2 R^2 \to -\infty \quad \text{and} \quad \operatorname{Im} f\left(R\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) \to 0 \quad \text{as} \quad R \to \infty. \qquad [4.154]$$

By using assumption 4.1 in [4.153] we obtain

$$\operatorname{Im} f\left(R\mathrm{e}^{\mathrm{i}\varphi}\right) \approx c_\infty^2 R^2 \sin(2\varphi) \geq 0, \quad \varphi \in \left[0, \frac{\pi}{2}\right], \quad \text{as} \quad R \to \infty. \qquad [4.155]$$

Therefore, $\Delta \arg f(s) = \pi$ for $s \in \gamma_{R2}$, as $R \to \infty$. Let $\gamma_{R3} : s = \omega\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}\omega$, $\omega \in [r, R]$. Then we have

$$f(\mathrm{i}\omega) = 1 - \omega^2 M^2(\mathrm{i}\omega), \quad \omega \in [r, R]. \qquad [4.156]$$

Using [4.143] in [4.156] we obtain

$$f(\mathrm{i}\omega) = 1 - \omega^2 \frac{E'(\omega) - \mathrm{i}E''(\omega)}{(E'(\omega))^2 + (E''(\omega))^2}, \quad \omega \in [r, R].$$

Thus,

$$\operatorname{Im} f(\mathrm{i}\omega) = \omega^2 \frac{E''(\omega)}{(E'(\omega))^2 + (E''(\omega))^2} > 0, \quad \omega \in [r, R], \tag{4.157}$$

due to (2) of condition 4.1. Moreover, assumption 4.1 applied to [4.156] implies

$$\operatorname{Re} f(\mathrm{i}\omega) \approx -c_\infty^2 \omega^2 \to -\infty \text{ and } \operatorname{Im} f(\mathrm{i}\omega) \to 0, \text{ as } R \to \infty, \tag{4.158}$$

$$\operatorname{Re} f(\mathrm{i}\omega) \approx 1 - c_0^2 \omega^2 \to 1 \text{ and } \operatorname{Im} f(\mathrm{i}\omega) \to 0, \text{ as } r \to 0. \tag{4.159}$$

We conclude that $\Delta \arg f(s) = -\pi$ for $s \in \gamma_{R3}$, as $r \to 0$, $R \to \infty$. Let $\gamma_{R4} : s = r\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in \left[0, \frac{\pi}{2}\right]$. Assumption 4.1, for $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ implies

$$\begin{aligned} &\operatorname{Re} f(r) \approx 1 + c_0^2 r^2 \to 1 \text{ and } \operatorname{Im} f(r) \to 0 \text{ as } r \to 0, \\ &\operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) \approx 1 - c_0^2 r^2 \to 1 \text{ and } \operatorname{Im} f\left(r\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\right) \to 0 \text{ as } r \to 0, \end{aligned} \tag{4.160}$$

as well as

$$\begin{aligned} \operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\varphi}\right) &\approx 1 + c_0^2 r^2 \cos(2\varphi) \to 1 \text{ and} \\ \operatorname{Im} f\left(r\mathrm{e}^{\mathrm{i}\varphi}\right) &\approx c_0^2 r^2 \sin(2\varphi) \to 0, \end{aligned} \tag{4.161}$$

for $\varphi \in \left[0, \frac{\pi}{2}\right]$, as $r \to 0$. We see that $\Delta \arg f(s) = 0$ for $s \in \gamma_{R4}$, as $r \to 0$. Thus, we conclude that

$$\Delta \arg f(s) = 0 \text{ for } s \in \gamma_R, \text{ as } r \to 0, \quad R \to \infty.$$

By the argument principle and that fact that the zeros are complex conjugated, we conclude that f has no zeros in the right complex half-plane. We will again use the argument principle and show that there are two zeros of [4.153] in the left complex half-plane. Let us consider the contour $\gamma_L = \gamma_{L1} \cup \gamma_{L2} \cup \gamma_{L3} \cup \gamma_{L4}$, presented in Figure 4.19. Let $\gamma_{L1} : s = \omega\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = \mathrm{i}\omega$, $\omega \in [r, R]$. Then [4.157], [4.158] and [4.159] hold. Thus, $\Delta \arg f(s) = \pi$ for $s \in \gamma_{L1}$, as $r \to 0$, $R \to \infty$. Let $\gamma_{L2} : s = R\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Then, for $\varphi = \frac{\pi}{2}$, [4.154] holds. For $\varphi = \pi$, assumption 4.1 implies

$$\operatorname{Re} f\left(R\mathrm{e}^{\mathrm{i}\pi}\right) \approx c_\infty^2 R^2 \to \infty \text{ and } \operatorname{Im} f\left(R\mathrm{e}^{\mathrm{i}\pi}\right) \to 0 \text{ as } R \to \infty,$$

while for $\varphi = \frac{3\pi}{2}$, it implies

$$\operatorname{Re} f\left(R\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}\right) \approx -c_\infty^2 R^2 \to -\infty \text{ and } \operatorname{Im} f\left(R\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}\right) \to 0 \text{ as } R \to \infty.$$

From [4.155], we have

$$\begin{aligned}
\operatorname{Im} f\left(R\mathrm{e}^{\mathrm{i}\varphi}\right) &\approx c_\infty^2 R^2 \sin(2\varphi) \leq 0, \quad \varphi \in \left[\frac{\pi}{2}, \pi\right) \text{ as } R \to \infty, \\
\operatorname{Im} f\left(R\mathrm{e}^{\mathrm{i}\varphi}\right) &\approx c_\infty^2 R^2 \sin(2\varphi) \geq 0, \quad \varphi \in \left(\pi, \frac{3\pi}{2}\right] \text{ as } R \to \infty.
\end{aligned}$$

We conclude that $\Delta \arg f(s) = 2\pi$ for $s \in \gamma_{L2}$, as $R \to \infty$. Let $\gamma_{L3} : s = \omega\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}} = -\mathrm{i}\omega$, $\omega \in [r, R]$. By [4.153] we have

$$f(-\mathrm{i}\omega) = 1 - \omega^2 \overline{M^2(\mathrm{i}\omega)}, \quad \omega \in [r, R], \qquad [4.162]$$

since $M(\bar{s}) = \overline{M(s)}$. Using [4.143] in [4.162] we obtain

$$f(-\mathrm{i}\omega) = 1 - \omega^2 \frac{E'(\omega) + \mathrm{i}E''(\omega)}{(E'(\omega))^2 + (E''(\omega))^2}, \quad \omega \in [r, R].$$

Thus,

$$\operatorname{Im} f(-\mathrm{i}\omega) = -\omega^2 \frac{E''(\omega)}{(E'(\omega))^2 + (E''(\omega))^2} < 0, \quad \omega \in [r, R],$$

due to (2) of condition 4.1. Note that assumption 4.1 applied to [4.162] implies

$$\begin{aligned}
\operatorname{Re} f(-\mathrm{i}\omega) &\approx -c_\infty^2 \omega^2 \to -\infty \text{ and } \operatorname{Im} f(\mathrm{i}\omega) \to 0, \text{ as } R \to \infty, \\
\operatorname{Re} f(-\mathrm{i}\omega) &\approx 1 - c_0^2 \omega^2 \to 1 \text{ and } \operatorname{Im} f(\mathrm{i}\omega) \to 0, \text{ as } r \to 0.
\end{aligned}$$

We conclude that $\Delta \arg f(s) = \pi$ for $s \in \gamma_{L3}$, as $r \to 0$, $R \to \infty$. Let γ_{L4} : $s = r\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Then, for $\varphi = \frac{\pi}{2}$, [4.160] holds. For $\varphi = \frac{3\pi}{2}$, assumption 4.1 implies

$$\operatorname{Re} f\left(r\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}\right) \approx 1 - c_0^2 r^2 \to 1 \text{ and } \operatorname{Im} f\left(r\mathrm{e}^{\mathrm{i}\frac{3\pi}{2}}\right) \to 0 \text{ as } r \to 0.$$

[4.161] also holds for $\varphi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, as $r \to 0$, so that $\Delta \arg f(s) = 0$ for $s \in \gamma_{L4}$, as $r \to 0$. Thus, the conclusion is that

$$\Delta \arg f(s) = 4\pi \text{ for } s \in \gamma_L, \text{ as } r \to 0, \quad R \to \infty.$$

This implies that f has two zeros in the left complex half-plane. ■

The following theorem is related to the existence of solutions to systems [4.137]–[4.138].

THEOREM 4.8.– Let M satisfy condition 4.1 with $x_0 = 0$ and assumption 4.1. Let $F \in \mathcal{S}'_+$.

1) The displacement u as a solution to [4.136]–[4.138], is given by

$$u(x,t) = x\varepsilon(t), \quad \text{where } \varepsilon(t) = F(t) * P(t), \quad x \in [0,1], \ t > 0, \qquad [4.163]$$

and

$$\begin{aligned} P(t) &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{M^2\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)}{1 + \left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)^2}\right)\mathrm{e}^{-qt}\mathrm{d}q \\ &\quad +2\operatorname{Re}\left(\operatorname{Res}\left(\tilde{P}(s)\,\mathrm{e}^{st}, s_0\right)\right), \quad t > 0, \qquad [4.164] \\ P(t) &= 0, \quad t < 0. \end{aligned}$$

The residue is given by

$$\operatorname{Res}\left(\tilde{P}(s)\,\mathrm{e}^{st}, s_0\right) = \left[\frac{M^2(s)}{\frac{\mathrm{d}}{\mathrm{d}s} f(s)}\mathrm{e}^{st}\right]_{s=s_0}, \quad t > 0, \qquad [4.165]$$

where f is given by [4.153] and s_0 is the zero of f.

Function P is real-valued, continuous on $[0,\infty)$ and $u \in C\left([0,1], \mathcal{S}'_+\right)$. Moreover, if F is locally integrable on $\mathbb{R}$ (and equals zero on $(-\infty,0]$), then $u \in C([0,1] \times [0,\infty))$.

2) The stress σ as a solution to [4.136]–[4.138], is given by

$$\sigma(t) = F(t) * Q(t), \quad x \in [0,1], \; t > 0,$$

where

$$\begin{aligned} Q(t) &= \frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{1}{1+\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)^2}\right)\mathrm{e}^{-qt}\mathrm{d}q \\ &\quad +2\operatorname{Re}\left(\operatorname{Res}\left(\tilde{Q}(s)\,\mathrm{e}^{st}, s_0\right)\right), \quad t>0, \\ Q(t) &= 0, \quad t<0. \end{aligned} \qquad [4.166]$$

The residue is given by

$$\operatorname{Res}\left(\tilde{Q}(s)\,\mathrm{e}^{st}, s_0\right) = \left[\frac{1}{\frac{\mathrm{d}}{\mathrm{d}s}f(s)}\mathrm{e}^{st}\right]_{s=s_0}, \quad t>0, \qquad [4.167]$$

where f is given by [4.153] and s_0 is the zero of f.

Function Q is real-valued, continuous on $[0,\infty)$. Moreover, if F is locally integrable on $\mathbb{R}$ (and equals zero on $(-\infty,0]$), then $\sigma \in C([0,\infty))$.

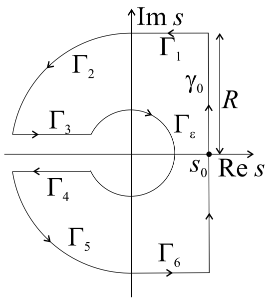

Figure 4.20. *Integration contour* Γ

PROOF.– First, we prove (1). Condition 4.1 ensures that P is a real-valued function. We calculate $P(t)$, $t \in \mathbb{R}$, by the integration over the contour given in Figure 4.20. The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\left(\operatorname{Res}\left(\tilde{P}(s)\,\mathrm{e}^{st}, s_0\right) + \operatorname{Res}\left(\tilde{P}(s)\,\mathrm{e}^{st}, \bar{s}_0\right)\right), \qquad [4.168]$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \cup \Gamma_7 \cup \gamma_0$, so that poles of $\tilde{P}$, given by [4.145], lie inside the contour Γ. Proposition 4.9 implies that the pole s_0, and its complex conjugate $\bar{s}_0$, of $\tilde{P}$ are simple. Then the residues in [4.168] can be calculated using [4.165].

Now, we calculate the integral over Γ in [4.168]. First, we consider the integral along the contour $\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0],\ R > 0\}$, where R is such that the poles s_0 and $\bar{s}_0$ lie inside the contour Γ. By [4.145] and assumption 4.1 we have

$$\left|\tilde{P}(s)\right| \leq \frac{C}{|s|^2}, \quad |s| \to \infty. \qquad [4.169]$$

Using [4.169], we calculate the integral over Γ_1 as

$$\begin{aligned} \lim_{R\to\infty}\left|\int_{\Gamma_1} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_0^{s_0}\left|\tilde{P}(p+\mathrm{i}R)\right|\left|\mathrm{e}^{(p+\mathrm{i}R)t}\right|\mathrm{d}p \\ &\leq C\lim_{R\to\infty}\int_0^{s_0}\frac{1}{R^2}\mathrm{e}^{pt}\mathrm{d}p = 0, \quad t > 0. \end{aligned}$$

Similar arguments are valid for the integral along the contour Γ_7:

$$\lim_{R\to\infty}\left|\int_{\Gamma_7} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0, \quad t > 0.$$

Next, we consider the integral along the contour Γ_2. From [4.169], we obtain

$$\begin{aligned} \lim_{R\to\infty}\left|\int_{\Gamma_2} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{P}\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0, \quad t > 0 \end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[\frac{\pi}{2}, \pi\right]$. Similarly, we have

$$\lim_{R\to\infty}\left|\int_{\Gamma_6} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0, \quad t > 0.$$

Consider the integral along Γ_4. Let $|s| \to 0$. Then, by assumption 4.1, $M(s) \to c_0$ and $sM(s) \to 0$. Hence, from [4.145], we have

$$\left|\tilde{P}(s)\right| \approx |M(s)|^2 \approx c_0^2, \text{ as } |s| \to 0. \qquad [4.170]$$

The integration along the contour Γ_4 gives

$$\begin{aligned}\lim_{r\to 0}\left|\int_{\Gamma_4} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{r\to 0}\int_{-\pi}^{\pi}\left|\tilde{P}\left(r\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq c_0^2\lim_{r\to 0}\int_{-\pi}^{\pi} r\mathrm{e}^{rt\cos\phi}\mathrm{d}\phi = 0, \quad t>0,\end{aligned}$$

where we used [4.170].

Integrals along Γ_3, Γ_5 and γ_0 give $(t>0)$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_3} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s = \int_0^{\infty}\frac{M^2\left(q\mathrm{e}^{\mathrm{i}\pi}\right)}{1+\left(qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)^2}\mathrm{e}^{-qt}\mathrm{d}q, \qquad [4.171]$$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_5} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s = -\int_0^{\infty}\frac{M^2\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)}{1+\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)^2}\mathrm{e}^{-qt}\mathrm{d}q, \qquad [4.172]$$

$$\lim_{R\to\infty}\int_{\gamma_0} \tilde{P}(s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}P(t)\,. \qquad [4.173]$$

We note that [4.173] is valid if the inversion of the Laplace transform exists, which is true since all the singularities of $\tilde{P}$ are left from the line γ_0 and the estimates on $\tilde{P}$ over γ_0 imply the convergence of the integral. Summing up [4.171], [4.172] and [4.173], we obtain the left-hand side of [4.168] and finally P in the form given by [4.164]. Analyzing separately

$$\frac{1}{\pi}\int_0^{\infty}\operatorname{Im}\left(\frac{M^2\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)}{1+\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)^2}\right)\mathrm{e}^{-qt}\mathrm{d}q, \quad 2\sum_{n=1}^{\infty}\operatorname{Re}\left(\operatorname{Res}\left(\tilde{P}(s)\,\mathrm{e}^{st}, s_0\right)\right),$$

we conclude that both terms appearing in [4.164] are continuous functions on $t \in [0,\infty)$. This implies that u is a continuous function on $[0,1]\times[0,\infty)$. From the uniqueness of the Laplace transform, it follows that u is unique. Since F belongs to $\mathcal{S}'_+$, it follows that

$$u(x,\cdot) = x\left(F(\cdot) * P(\cdot)\right) \in \mathcal{S}'_+$$

for every $x \in [0,1]$ and $u \in C\left([0,1], \mathcal{S}'_+\right)$. Moreover, if $F \in L^1_{loc}([0,\infty))$, then $u \in C([0,1] \times [0,\infty))$ since P is continuous.

Second, we prove (2). Again, condition 4.1 ensures that Q is a real-valued function. We calculate $Q(t)$, $t \in \mathbb{R}$, by the integration over the same contour from Figure 4.20. The Cauchy residues theorem yields

$$\oint_\Gamma \tilde{Q}(s)\, \mathrm{e}^{st} \mathrm{d}s = 2\pi\mathrm{i}\left(\operatorname{Res}\left(\tilde{Q}(s)\, \mathrm{e}^{st}, s_0\right) + \operatorname{Res}\left(\tilde{Q}(s)\, \mathrm{e}^{st}, \bar{s}_0\right)\right), \qquad [4.174]$$

so that the poles of $\tilde{Q}$ lie inside the contour Γ. The poles s_0 and $\bar{s}_0$ of $\tilde{Q}$ given by [4.145] are the same as for the function $\tilde{P}$. Since the poles s_0 and $\bar{s}_0$ are simple, the residues in [4.174] can be calculated using [4.167].

Let us calculate the integral over Γ in [4.174]. Consider the integral along the contour

$$\Gamma_1 = \{s = p + \mathrm{i}R \mid p \in [0, s_0],\ R > 0\}.$$

From [4.145] and assumption 4.1, we have

$$\tilde{Q}(s) \leq \frac{C}{|s|^2}, \quad |s| \to \infty. \qquad [4.175]$$

Using [4.175] we calculate the integral over Γ_1, as

$$\begin{aligned} \lim_{R\to\infty} \left| \int_{\Gamma_1} \tilde{Q}(s)\, \mathrm{e}^{st} \mathrm{d}s \right| &\leq \lim_{R\to\infty} \int_0^{s_0} \left|\tilde{Q}(p + \mathrm{i}R)\right| \left|\mathrm{e}^{(p+\mathrm{i}R)t}\right| \mathrm{d}p \\ &\leq C \lim_{R\to\infty} \int_0^{s_0} \frac{1}{R^2} \mathrm{e}^{pt} \mathrm{d}p = 0, \quad t > 0. \end{aligned}$$

Similar arguments are valid for the integral along Γ_7. Thus, we have

$$\lim_{R\to\infty} \left| \int_{\Gamma_7} \tilde{Q}(s)\, \mathrm{e}^{st} \mathrm{d}s \right| = 0, \quad t > 0.$$

With [4.175], the integral over Γ_2 becomes

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_2}\tilde{Q}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\left|\tilde{Q}\left(R\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{Rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}R\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq C\lim_{R\to\infty}\int_{\frac{\pi}{2}}^{\pi}\frac{1}{R}\mathrm{e}^{Rt\cos\phi}\mathrm{d}\phi = 0,\quad t>0,\end{aligned}$$

since $\cos\phi \leq 0$ for $\phi \in \left[\frac{\pi}{2},\pi\right]$. Similar arguments are valid for the integral along Γ_6:

$$\lim_{R\to\infty}\left|\int_{\Gamma_6}\tilde{Q}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| = 0,\quad t>0.$$

Since $M(s) \to c_0$ and $sM(s) \to 0$ as $|s| \to 0$, [4.145] implies $\tilde{Q}(s) \approx 1$, as $|s| \to 0$, the integration along the contour Γ_4 gives

$$\begin{aligned}\lim_{r\to 0}\left|\int_{\Gamma_4}\tilde{Q}(s)\,\mathrm{e}^{st}\mathrm{d}s\right| &\leq \lim_{r\to 0}\int_{-\pi}^{\pi}\left|\tilde{Q}\left(r\mathrm{e}^{\mathrm{i}\phi}\right)\right|\left|\mathrm{e}^{rt\mathrm{e}^{\mathrm{i}\phi}}\right|\left|\mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\right|\mathrm{d}\phi \\ &\leq \lim_{r\to 0}\int_{-\pi}^{\pi} r\mathrm{e}^{rt\cos\phi}\mathrm{d}\phi = 0,\quad t>0.\end{aligned}$$

Integrals along Γ_3, Γ_5 and γ_0 give $(t>0)$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_3}\tilde{Q}(s)\,\mathrm{e}^{st}\mathrm{d}s = \int_0^{\infty}\frac{1}{1+\left(qM\left(q\mathrm{e}^{\mathrm{i}\pi}\right)\right)^2}\mathrm{e}^{-qt}\mathrm{d}q, \qquad [4.176]$$

$$\lim_{\substack{R\to\infty\\ r\to 0}}\int_{\Gamma_5}\tilde{Q}(s)\,\mathrm{e}^{st}\mathrm{d}s = -\int_0^{\infty}\frac{1}{1+\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)^2}\mathrm{e}^{-qt}\mathrm{d}q, \qquad [4.177]$$

$$\lim_{R\to\infty}\int_{\gamma_0}\tilde{Q}(s)\,\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}Q(t). \qquad [4.178]$$

Through the same arguments as in the proof of (i), [4.178] is valid if the inversion of the Laplace transform exists. This is true because all the singularities of $\tilde{Q}$ are left from the line γ_0 and appropriate estimates on $\tilde{Q}$ are satisfied. Adding [4.176], [4.177] and [4.178] we obtain the left-hand side of [4.174] and finally Q in the form given by [4.166].

Because

$$\frac{1}{\pi}\int_0^\infty \operatorname{Im}\left(\frac{1}{1+\left(qM\left(q\mathrm{e}^{-\mathrm{i}\pi}\right)\right)^2}\right)\mathrm{e}^{-qt}\mathrm{d}q, \quad 2\operatorname{Res}\left(\tilde{Q}\left(s\right)\mathrm{e}^{st}, s_0\right), \quad t>0$$

are continuous, it follows that Q is continuous on $[0,\infty)$. ■

4.2.1.2. *Numerical examples*

Suppose that F is harmonic, i.e. $F\left(t\right)=F_0\cos\left(\omega t\right)$ and that ϕ_σ and ϕ_ε are given by [4.148]. Then $\tilde{\varepsilon}$ and $\tilde{\sigma}$, given by [4.144], become

$$\tilde{\varepsilon}\left(s\right)=F_0\frac{s}{s^2+\omega^2}\frac{\frac{1+as^\alpha}{1+bs^\alpha}}{1+s^2\frac{1+as^\alpha}{1+bs^\alpha}}, \quad \tilde{\sigma}\left(s\right)=F_0\frac{s}{s^2+\omega^2}\frac{1}{1+s^2\frac{1+as^\alpha}{1+bs^\alpha}}, \quad s\in D.$$

In the special case $a=b$, which corresponds to an elastic body, we obtain

$$\tilde{\varepsilon}\left(s\right)=F_0\frac{s}{s^2+\omega^2}\frac{1}{1+s^2}, \quad \tilde{\sigma}\left(s\right)=F_0\frac{s}{s^2+\omega^2}\frac{1}{1+s^2}.$$

After inverting the Laplace transform, we have

$$\begin{aligned}
\varepsilon\left(t\right) &= \frac{F_0}{\omega^2-1}\cos\left(\omega t\right)*\sin t, \quad \sigma\left(t\right)=\frac{F_0}{\omega^2-1}\cos\left(\omega t\right)*\sin t,\\
\varepsilon\left(t\right) &= \frac{2F_0}{\omega^2-1}\sin\frac{\left(\omega+1\right)t}{2}\sin\frac{\left(\omega-1\right)t}{2},\\
\sigma\left(t\right) &= \frac{2F_0}{\omega^2-1}\sin\frac{\left(\omega+1\right)t}{2}\sin\frac{\left(\omega-1\right)t}{2}.
\end{aligned} \qquad [4.179]$$

For $\omega\to 1$, we obtain

$$\varepsilon\left(t\right)=\frac{1}{2}t\sin t, \quad \sigma\left(t\right)=\frac{1}{2}t\sin t,$$

a resonance. Also, in the case when $\omega\approx 1$, we observe pulsation.

We will present several plots of ε, obtained from theorem 4.8, in the case of the fractional Zener model of viscoelastic body [4.148]. We fix the parameters of the model: $a = 0.2$ and $b = 0.6$, and in Figures 4.21–4.24 show the plots of ε in the cases when the forcing term is given as $F = \delta$ (δ is the Dirac delta distribution), $F = H$ (H is the Heaviside function) and $F(t) = \cos(\omega t)$, respectively.

We see from Figure 4.21 that the oscillations of the body are damped because the rod is viscoelastic. The curve resembles the curve of the damped oscillations of the linear harmonic oscillator. We also see that the change of $\alpha \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$ makes the amplitudes of the curves decrease more slowly with time as α becomes smaller. This is due the fact that $\alpha = 1$ corresponds to the standard linear viscoelastic body and $\alpha = 0$ corresponds to the elastic body.

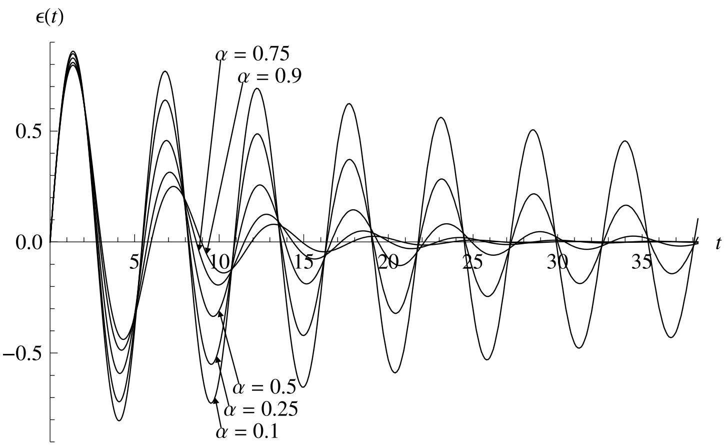

Figure 4.21. *Strain $\varepsilon(t)$ in the case $F = \delta$ as a function of time $t \in (0, 38)$*

In the case of the forcing term given as the Heaviside function, from Figures 4.22 and 4.23 we observe that the body creeps to the finite value of the displacement regardless of the value of $\alpha \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$. Creeping to the finite value of the displacement is due to the fact that the fractional Zener model describes the solid-like viscoelastic material. If viscoelastic properties of the material are dominant (the value of α is closer to one), then the time required for body to reach the limiting value of the strain is smaller (see Figure 4.22), compared to time in the case when elastic properties of the material are dominant (see Figure 4.23).

Figure 4.24 shows the expected behavior of the body in the case of the harmonic forcing term. Namely, the oscillations of the body die out and the body oscillates in the phase with the harmonic function. For this plot, we took: $\alpha = 0.45$ and $\omega = 1.1$.

Now, we examine the case when values of the coefficients a and b are close to each other. We fix them to be $a = 0.58$ and $b = 0.6$. In this case, the elastic properties of the material prevail because in the limiting case $a = b$ the fractional Zener model [4.147], becomes Hooke's law for the arbitrary value of $\alpha \in (0,1)$. We present in Figure 4.25 the plot of ε in the case when $\alpha = 0.45$ and $\omega = 1.1$. In the elastic case, as can be seen from [4.179], the frequency of the free oscillations of the body is $\omega_f = 1$. Since the frequency of the forcing function ($\omega = 1.1$) is close to the frequency of the free oscillations, we will have the pulsation, as can be observed from Figure 4.25. Since there is still some damping left ($a \neq b$), the amplitude of the wave-package decreases in time, as shown in Figure 4.26.

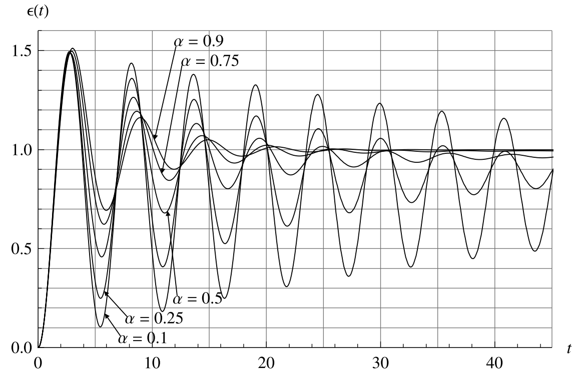

Figure 4.22. *Strain $\varepsilon(t)$ in the case $F = H$ as a function of time $t \in (0, 45)$*

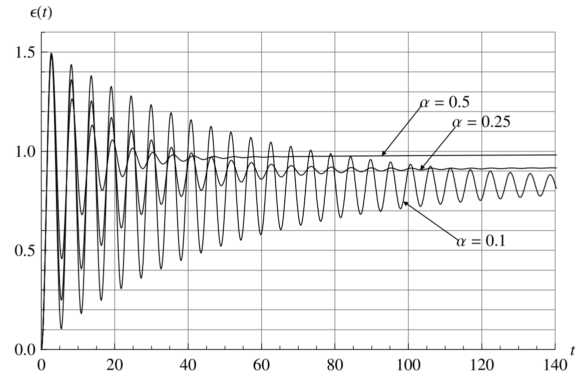

Figure 4.23. *Strain $\varepsilon(t)$ in the case $F = H$ as a function of time $t \in (0, 140)$*

We increase the damping effect by choosing $a = 0.55$, $b = 0.6$. The rest of the parameters are: $\alpha = 0.45$, $\omega = 1.1$. The curve of ε, presented in Figure 4.27 for smaller times, resembles the curve of the pulsation (the elastic properties of the material prevail), while later it resembles the curve of the forcing function (the viscous properties of the material prevail).

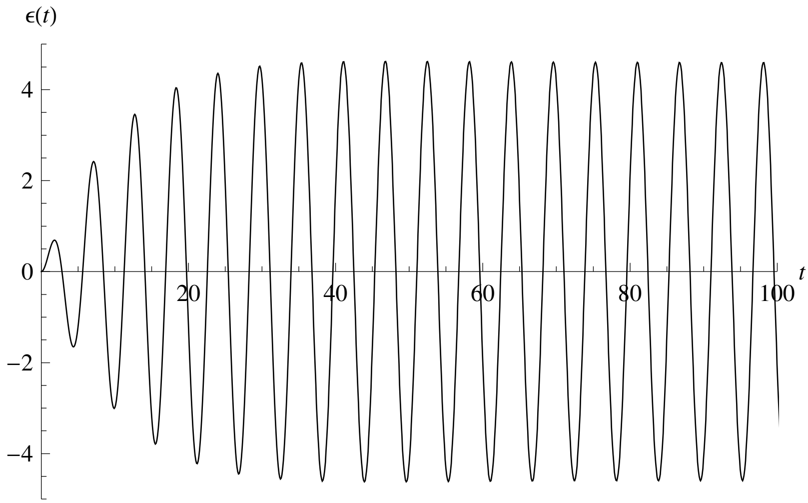

Figure 4.24. *Strain $\varepsilon(t)$ in the case $F(t) = \cos(\omega t)$ as a function of time $t \in (0, 100)$*

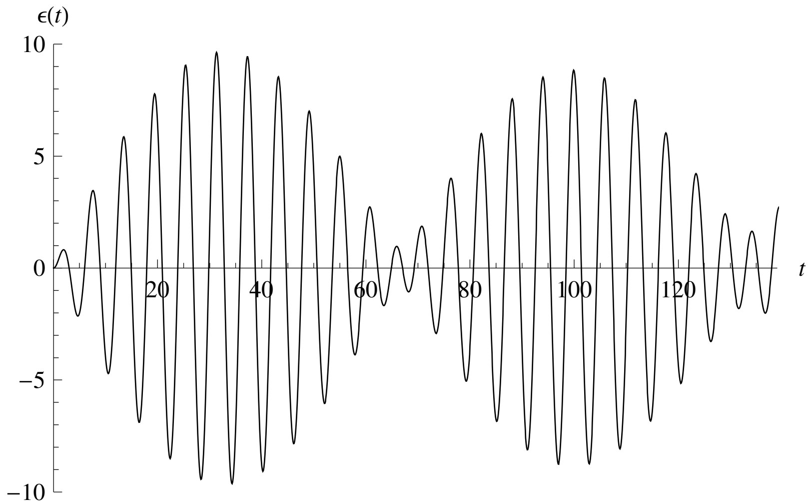

Figure 4.25. *Strain $\varepsilon(t)$ in the case $F(t) = \cos(\omega t)$ as a function of time $t \in (0, 139)$*

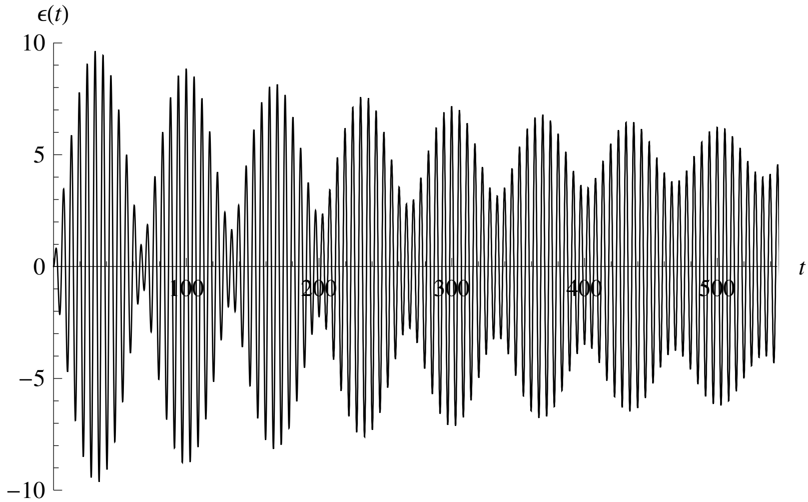

Figure 4.26. *Strain $\varepsilon(t)$ in the case $F(t) = \cos(\omega t)$ as a function of time $t \in (0, 545)$*

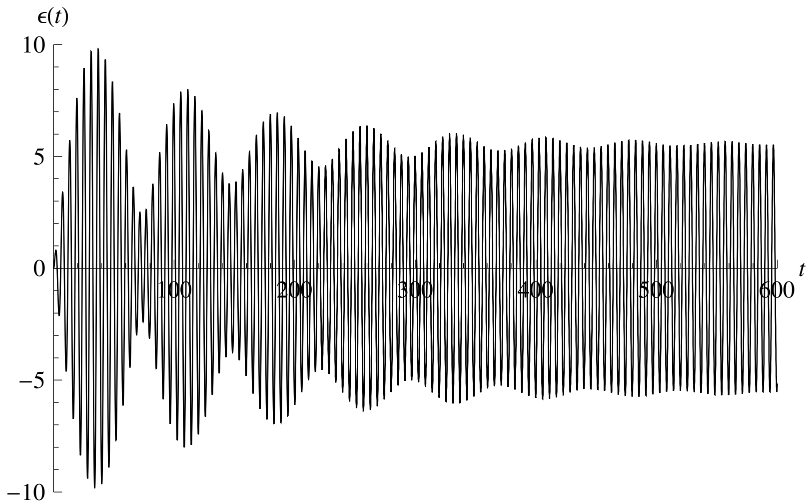

Figure 4.27. *Strain $\varepsilon(t)$ in the case $F(t) = \cos(\omega t)$ as a function of time $t \in (0, 600)$*

4.2.2. *The case of the parallel connection of spring and viscoelastic rod*

Following the lines of [ATA 05a], we treat systems [4.134] and [4.135] in the case when the body is attached to a parallel connection of a solid-like viscoelastic rod of distributed-order type and a spring. In this case, (dimensionless) σ appearing in [4.135] is of the form $\sigma = \sigma_s + \sigma_r$, where $\sigma_s = \omega^2 \varepsilon$, with ω being positive constant. In the constitutive equation [4.134], we assume that constitutive functions are of the form

$$\phi_\sigma(\gamma) = a^\gamma, \quad \phi_\varepsilon(\gamma) = b^\gamma, \quad \gamma \in (0,1).$$

The second law of thermodynamics requires (see [3.23] of [ATA 14b]) that $0 < a \leq b$. Thus, systems [4.134], [4.135] become

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t) + \omega^2\varepsilon + \sigma_r(t) = F(t), \quad t > 0, \qquad [4.180]$$

$$\int_0^1 a^\gamma \, {}_0\mathrm{D}_t^\gamma \sigma_r(x,t)\,\mathrm{d}\gamma = \int_0^1 b^\gamma \, {}_0\mathrm{D}_t^\gamma \varepsilon(x,t)\,\mathrm{d}\gamma, \quad t > 0. \qquad [4.181]$$

Systems [4.180], [4.181] are subject to initial conditions

$$\varepsilon(0) = \varepsilon_0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon(0) = \dot{\varepsilon}_0. \qquad [4.182]$$

Systems [4.180]–[4.182] correspond to the generalization of the fractional-order oscillator. Different types of the fractional-order oscillators were treated by many authors and we mention only a few works: [NAR 02, NAR 01, ATA 02c, BEY 95, GOR 97b, MAI 96].

The formal use of the Laplace transform method applied to systems [4.180]–[4.182] shows that the framework for solving this system is the space $\mathcal{K}'_+$. It will be shown that $\sigma_r \in \mathcal{K}'_+$ and $\varepsilon \in \mathcal{S}'_+$. Formally applying the Laplace transform to [4.181] we obtain

$$\frac{as-1}{\ln(as)}\tilde{\sigma}_r(s) = \frac{bs-1}{\ln(bs)}\tilde{\varepsilon}(s), \quad \operatorname{Re} s > 0, \qquad [4.183]$$

We have to prove that ε and σ_r are elements of $\mathcal{K}'_+$ and to interpret the above integral form of the Laplace transform in the sense of exponential distributions. We will do this later and this implies that the procedure to be followed is legitimate. From [4.183], we obtain

$$\tilde{\sigma}_r\left(s\right) = \frac{\ln\left(as\right)}{\ln\left(bs\right)}\frac{bs-1}{as-1}\tilde{\varepsilon}\left(s\right), \quad \operatorname{Re} s > 0. \qquad [4.184]$$

Now applying the Laplace transform to [4.180] and using [4.184], we obtain

$$\tilde{\varepsilon}\left(s\right) = \frac{\varepsilon_0 s + \dot{\varepsilon}_0 + \tilde{F}\left(s\right)}{\Phi\left(s\right)}, \qquad [4.185]$$

where

$$\Phi\left(s\right) = s^2 + \omega^2 + \frac{\ln\left(as\right)}{\ln\left(bs\right)}\frac{bs-1}{as-1}. \qquad [4.186]$$

PROPOSITION 4.10.– For systems [4.180]–[4.181] we have $\varepsilon \in \mathcal{S}'_+$ and $\sigma_r \in \mathcal{K}'_+$.

PROOF.– In order to prove that $\varepsilon \in \mathcal{S}'_+$, we need an estimate of J where

$$J(s) = (s^2+\omega^2)(as-1)\ln(bs) + (bs-1)\ln(as), \quad \operatorname{Re} s > 0. \qquad [4.187]$$

There exists $C > 0$ such that

$$|J(s)| > C, \quad \operatorname{Re} s > 0. \qquad [4.188]$$

We prove [4.188]. Note that if $\varepsilon > 0$ is small enough then there exists $d > 0$ such that

$$(|s| \leq \varepsilon)(\operatorname{Re} s > 0) \Rightarrow (|J\left(s\right)| > d). \qquad [4.189]$$

This is a consequence of the fact that $\lim_{|s|\to 0} J\left(s\right) = |\omega^2 + 1|$.

Further more, for any $R > 0$, there exists $d_1 > 0$ such that

$$(|s| \in [\varepsilon, R])(\operatorname{Re} s > 0) \Rightarrow \left(|J\left(s\right)| > d_1\right). \qquad [4.190]$$

We can enlarge R so that

$$\left|s^2 - \omega^2\right| \geq R^2 - |\omega^2|, \ |as-1|\,|\ln bs| \geq \frac{1}{2}|as||\ln bs|.$$

With this and $|\ln(bs)| = |\ln R + \ln b + \mathrm{i} \arg s| \geq \ln R + \ln b - \frac{\pi}{2}$, we have

$$\begin{aligned} |J(s)| &\geq \frac{aR}{2}(R^2 - |\omega^2|)(\ln b + \ln R - \frac{\pi}{2}) \\ &\quad - \left(bR\left(\ln a + \ln R + \frac{\pi}{2}\right) + 1\right) \geq R. \end{aligned} \tag{4.191}$$

Now from [4.189], [4.190] and [4.191], the estimate [4.188] follows.

For the next estimate, we use the elementary inequality for a polynomial of order 2 and $\ln|s| \leq |s| + \frac{1}{|s|}$, $s \in \mathbb{C}$. From [4.188], it follows that there exists (another) $C > 0$ such that

$$\begin{aligned} |\tilde{\varepsilon}(s)| &= \left| \frac{(\varepsilon_0 s + \dot{\varepsilon}_0 + \tilde{F}(s))(as-1)\ln(bs)}{J(s)} \right| \\ &\leq C\,(1+|s|)^2 \left(\left|\tilde{F}(s)\right| + 1\right)\left(\frac{1}{|s|} + |s|\right). \end{aligned} \tag{4.192}$$

(Note, $F \in \mathcal{S}'_+$.). Recall the characterization of $\mathcal{S}'_+$ (see [VLA 73]):

$$g \in \mathcal{S}'_+ \Leftrightarrow |\tilde{g}(s)| \leq C\frac{|\mathrm{Re}\, s|^p}{(1+|\mathrm{Im}\, s|^q)} \quad \text{for some} \quad C > 0 \quad \text{and} \quad p, q \in \mathbb{R}. \tag{4.193}$$

For F, [4.191] and [4.192], it follows that $\tilde{\varepsilon}$ satisfies [4.193] with another constant, and thus, $\varepsilon \in \mathcal{S}'_+$.

We will show that $\sigma_r \in \mathcal{K}'_+$ by explicit calculation. Let ϕ_1 and ϕ_2 denote

$$\phi_1 = \mathcal{L}^{-1}\left[\frac{\ln as}{\ln bs}\right], \quad \phi_2 = \mathcal{L}^{-1}\left[\frac{bs-1}{as-1}\right],$$

where the Laplace transform is taken in the sense of tempered distributions. We have

$$\phi_1 = \mathcal{L}^{-1}\left[1 + \frac{\ln a - \ln b}{\ln b + \ln s}\right] = \delta + \ln\frac{a}{b}\mathcal{L}^{-1}\left[\frac{1}{\ln b + \ln s}\right],$$

where δ is the delta distribution. Note that $\mathcal{L}\left[b^{-t}\right](s) = (s + \ln b)^{-1}$, $\operatorname{Re} s > 0$. This and [ERD 54, equation (29), p. 132] imply

$$\mathcal{L}^{-1}\left[\frac{1}{\ln b + \ln s}\right](t) = \int_0^\infty \frac{t^{u-1}\mathrm{e}^{-u\ln b}}{\Gamma(u)}\mathrm{d}u, \quad t > 0.$$

Here, we note that for some $k > 0$ and $C > 0$, which depend on b,

$$\left|\int_0^\infty \frac{t^{u-1}\mathrm{e}^{-u\ln b}}{\Gamma(u)}\mathrm{d}u\right| \leq C\mathrm{e}^{kt}, \quad t > 0.$$

and that this integral is not polynomially bounded. Thus, this integral defines an element of $\mathcal{K}'_+ \setminus \mathcal{S}'_+$. We have

$$\phi_1(t) = \delta(t) + \ln\frac{a}{b}\int_0^\infty \frac{t^{u-1}b^{-u}}{\Gamma(u)}\mathrm{d}u, \quad \phi_2(t) = \frac{b}{a}\delta(t) + \frac{b-a}{a^2}\mathrm{e}^{\frac{t}{a}}, \quad t > 0, \quad [4.194]$$

and both distributions are equal to zero on $(-\infty, 0)$. Clearly, both distributions are elements of $\mathcal{K}'_+$. The same holds for

$$\sigma_r(t) = \lambda\phi_1(t) * \phi_2(t) * \varepsilon(t) \quad [4.195]$$

since $\mathcal{K}'_+$ is a commutative and associative algebra under convolution. ■

For the function Φ, we have the following proposition.

PROPOSITION 4.11.– Let Φ_0 be the principal branch of Φ, i.e. the branch for which $\ln z = \ln|z| + \mathrm{i}\arg z$, where $|\arg z| < \pi$. If $b > a > 0$, then Φ_0 has exactly two zeros which are simple, conjugate and placed in the open left half-plane.

PROOF.– Clearly,

$$\Phi_0(\bar{s}) = \overline{\Phi_0(s)}, \quad s \in \mathbb{C}, \quad [4.196]$$

and this implies that $\Phi_0(\bar{s}_0) = 0$ if $\Phi_0(s_0) = 0$.

Let us prove that Φ_0 has exactly two zeros. We will prove this with the argument principle. Consider the domain

$$\Omega = \{s \in \mathbb{C} \mid 0 < r < |s| < R, \ |\arg s| < \pi\}$$

and let $C_{r,R}$ be its boundary.

Put $s = R\mathrm{e}^{\mathrm{i}t},\ 0 \leq t \leq \pi$. Then, $\Phi_0(R) > 0$ and

$$\Phi_0(R\mathrm{e}^{\mathrm{i}t}) = R^2\mathrm{e}^{2\mathrm{i}t}\left(1 + \frac{\omega^2}{R^2\mathrm{e}^{2\mathrm{i}t}} + \frac{1}{R^2\mathrm{e}^{2\mathrm{i}t}}\frac{\ln aR + \mathrm{i}t}{\ln bR + \mathrm{i}t}\frac{bR\mathrm{e}^{\mathrm{i}t} - 1}{aR\mathrm{e}^{\mathrm{i}t} - 1}\right).$$

As the expression in the parentheses tends to 1 as $R \to \infty$, we conclude that on the semicircle $s = R\mathrm{e}^{\mathrm{i}t}$, $0 \leq t \leq \pi$.

$$\lim_{R\to\infty} \Delta \arg \Phi_0(s) = 2\pi, \qquad [4.197]$$

where Δ denotes the variation.

Put $s = t\mathrm{e}^{\mathrm{i}\pi},\ r \leq t \leq R$. The imaginary part of Φ_0 satisfies

$$\operatorname{Im} \Phi_0(t\mathrm{e}^{\mathrm{i}\pi}) = \frac{bt + 1}{at + 1}\frac{\pi(\ln b - \ln a)}{\pi^2 + \ln^2 bt} > 0$$

and tends to 0 as $t \to 0$ or $t \to \infty$. Thus, we conclude that on this part of the ray (if Δ denotes a variation)

$$\lim_{r\to 0, R\to\infty} \Delta \arg \Phi_0(s) = 0. \qquad [4.198]$$

Put $s = r\mathrm{e}^{\mathrm{i}t},\ 0 \leq t \leq \pi$ and observe that

$$\lim_{r\to 0} \Phi_0(r\mathrm{e}^{\mathrm{i}t}) = \omega^2 + 1,$$

which implies that on this semicircle

$$\lim_{r\to 0} \Delta \arg \Phi_0(s) = 0. \qquad [4.199]$$

Now, according to the argument principle, relations [4.196]–[4.199] imply

$$N = \frac{1}{2\pi}\Delta \arg \Phi_0(s) = \frac{1}{2\pi} \cdot 2 \cdot 2\pi = 2,$$

where N denotes the number of zeros of Φ_0 in the domain.

Now we will prove that Φ_0 does not have zeros in the right half-plane. Consider the domain

$$\Omega^* = \left\{ s \in \mathbb{C} \mid 0 < r < |s| < R,\ |\arg s| < \frac{\pi}{2} \right\}$$

and let $C^*_{r,R}$ be its boundary. Put $s = Re^{\mathrm{i}t}$, $0 \leq t \leq \frac{\pi}{2}$. Similarly to the first part, we conclude that

$$\lim_{R\to\infty} \Delta \arg \Phi_0(s) = \pi. \qquad [4.200]$$

Put $s = te^{\mathrm{i}\frac{\pi}{2}},\ r \leq t \leq R$. The imaginary part of Φ_0 satisfies

$$\operatorname{Im} \Phi_0(te^{\mathrm{i}\frac{\pi}{2}}) = \operatorname{Im}\left(\frac{\ln(\mathrm{i}at)}{\ln(\mathrm{i}bt)} \frac{\mathrm{i}bt - 1}{\mathrm{i}at - 1} \right) = \operatorname{Im}\left[\frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1} \right]_{s=\mathrm{i}t} > 0. [4.201]$$

We identify the right-hand side of [4.200] as the imaginary part of the complex modulus see [4.184]. Thus, we conclude that on this interval

$$\lim_{r\to 0, R\to\infty} \Delta \arg \Phi_0(s) = -\pi. \qquad [4.202]$$

Using the argument principle, relation [4.196] and relations [4.197], [4.198] and [4.200] it follows

$$N = \frac{1}{2\pi} \cdot 2 \cdot (\pi - \pi) = 0.$$

This completes the proof. ■

Now we give the integral representation of the solutions to systems [4.180]–[4.182]. Recall,

$$\tilde{\varepsilon}(s) = \frac{(\varepsilon_0 s + \dot{\varepsilon}_0 + \tilde{F}(s))(as-1)\ln(bs)}{(s^2+\omega^2)(as-1)\ln(bs) + (bs-1)\ln(as)}, \quad \operatorname{Re} s > 0, \qquad [4.203]$$

$$\tilde{\sigma}_r(s) = \frac{\ln(as)}{\ln(bs)} \frac{bs-1}{as-1} \tilde{\varepsilon}(s). \qquad [4.204]$$

Applying the inverse Laplace transform to [4.203] we have

$$\varepsilon(t) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \tilde{\varepsilon}(s) e^{st} \mathrm{d}s, \quad t \geq 0, \quad \text{where } \gamma = \{s \mid \operatorname{Re} s = \sigma,\ \sigma > \sigma_0 = 0\}.$$

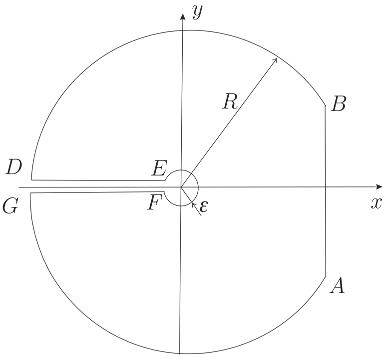

Figure 4.28. *Integration contour* γ_0

Let $\gamma_0 = \widehat{AB} \cup \widehat{BD} \cup \widehat{DE} \cup \widehat{EF} \cup \widehat{FG} \cup \widehat{GA}$ be the closed contour shown in Figure 4.28. Cauchy's formula gives

$$\int_{\gamma_0} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\sum_{i=1}^{N} \mathrm{Res}\left(\tilde{\varepsilon}(s)\mathrm{e}^{st}, s_i\right). \qquad [4.205]$$

Integrals

$$\int_{\widehat{BD}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s, \quad \int_{\widehat{GA}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s \;\text{ and }\; \int_{\widehat{EF}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s,$$

tend to 0 when $R \to \infty$ and $\varepsilon \to 0$; so they do not contribute to the left-hand side of [4.205]. What contributes to the left-hand side of [4.205] are integrals

$$\int_{\widehat{AB}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s, \quad \int_{\widehat{DE}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s, \quad \int_{\widehat{FG}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s.$$

We will show that, when $r \to 0$ and $R \to \infty$, the contribution of the latter two integrals is provided by

$$2\pi\mathrm{i}F_\alpha(t) = \int_0^\infty K_\alpha(r)\mathrm{e}^{-rt}\mathrm{d}r, \quad t > 0,$$

where K_α will be defined later, while the first integral tends to $\int_\gamma \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s$. Denote $G_\alpha(t) = \sum_{i=1}^{N} \mathrm{Res}\left(\tilde{\varepsilon}(s)\mathrm{e}^{st}, s_i\right)$, $t \geq 0$. Then

$$\int_\gamma \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s + 2\pi\mathrm{i}F_\alpha(t) = 2\pi\mathrm{i}G_\alpha(t), \;\text{ i.e., }\; \varepsilon(t) = G_\alpha(t) - F_\alpha(t), \qquad [4.206]$$

and we need to calculate F_α and G_α.

First we calculate F_α. Let $s = r\mathrm{e}^{\mathrm{i}\pi}$ on $\overset{\frown}{DE}$ and $s = r\mathrm{e}^{-\mathrm{i}\pi}$ on $\overset{\frown}{FG}$, $r > 0$. Then

$$\begin{aligned}
&\int_{\overset{\frown}{DE}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s + \int_{\overset{\frown}{FG}} \tilde{\varepsilon}(s)\mathrm{e}^{st}\mathrm{d}s \\
&\quad = \int_{\varepsilon}^{R} \frac{(\ln br + \mathrm{i}\pi)(-ar-1)(\varepsilon_0 \cdot (-r) + \dot{\varepsilon}_0 + \tilde{F}(-r))}{(r^2+\omega^2)(\ln br - \mathrm{i}\pi)(-ar-1) + (\ln ar - \mathrm{i}\pi)(-br-1)} \mathrm{e}^{-tr}\mathrm{d}r \\
&\qquad - \int_{\varepsilon}^{R} \frac{(\ln br - \mathrm{i}\pi)(-ar-1)(\varepsilon_0 \cdot (-r) + \dot{\varepsilon}_0 + \tilde{F}(-r))}{(r^2+\omega^2)(\ln br + \mathrm{i}\pi)(-ar-1) + (\ln ar + \mathrm{i}\pi)(-br-1)} \mathrm{e}^{-tr}\mathrm{d}r.
\end{aligned}$$

Letting $R \to \infty$ and $\varepsilon \to 0$, it follows that

$$F_\alpha(t) = \ln\frac{a}{b}\int_0^\infty \frac{A(r)}{B(r)}\mathrm{e}^{-tr}\mathrm{d}r, \qquad [4.207]$$

where

$$\begin{aligned}
A(r) &= (ar+1)(br+1)\left[-\varepsilon_0 r + \dot{\varepsilon}_0 + \tilde{F}(-r)\right], \\
B(r) &= \left[\left(r^2+\omega^2\right)(ar+1)\ln(br) + (br+1)\ln(ar)\right]^2 \\
&\qquad + \pi^2\left[\left(r^2+\omega^2\right)(ar+1) + (br+1)\right]^2.
\end{aligned}$$

Furthermore, let $\Phi_1(s) = \varepsilon_0 s + \dot{\varepsilon}_0 + \tilde{F}(s)$. Then from [4.185] $\tilde{\varepsilon}(s) = \frac{\Phi_1(s)}{\Phi(s)}$ and two simple zeros of $\Phi(s)$ are the only singularities of $\tilde{\varepsilon}(s)\,\mathrm{e}^{st}$. Thus,

$$\sum_{i=1}^{2} \operatorname{Res}\left(\tilde{\varepsilon}(s)\mathrm{e}^{st}, s_i\right) = \sum_{i=1}^{2} \frac{\Phi_1(s_i)}{\Phi'(s_i)}\mathrm{e}^{s_i t}. \qquad [4.208]$$

Since $\Phi(s) = s^2 + \omega^2 + \frac{\ln(as)}{\ln(bs)}\frac{bs-1}{as-1}$, it follows that $\Phi'(\bar{s}) = \overline{\Phi'(s)}$. To write [4.208] in a more explicit form, suppose that $\overline{\tilde{F}(s)} = \tilde{F}(\bar{s})$. Then, $\Phi_1(\bar{s}) = \overline{\Phi_1(s)}$ and [4.208] may be written as

$$\begin{aligned}
\sum_{i=1}^{2} \operatorname{Res}\left(\tilde{\varepsilon}(s)\mathrm{e}^{st}, s_i\right) &= \frac{2\mathrm{e}^{-\Sigma t}\cos(\Omega t)}{\left[\operatorname{Re}\left(\Phi'(s_1)\right)\right]^2 + \left[\operatorname{Im}\left(\Phi'(s_1)\right)\right]^2} \\
&\quad \times \left(\operatorname{Re}\left[\overline{\Phi_1(s_1)}\Phi'(s_1)\right] + \operatorname{Im}\left[\overline{\Phi_1(s_1)}\Phi'(s_1)\right]\right), \qquad [4.209]
\end{aligned}$$

where $s_{1,2} = -\Sigma \pm \mathrm{i}\Omega$, $\Sigma > 0$, $\Omega > 0$. Therefore, the solution has the form

$$\varepsilon(t) = \frac{2\mathrm{e}^{-\Sigma t}\cos(\Omega t)}{\left[\mathrm{Re}\left(\Phi'(s_1)\right)\right]^2 + \left[\mathrm{Im}\left(\Phi'(s_1)\right)\right]^2} \times \left(\mathrm{Re}\left[\overline{\Phi_1(s_1)}\Phi'(s_1)\right] + \mathrm{Im}\left[\overline{\Phi_1(s_1)}\Phi'(s_1)\right]\right) - \ln\frac{a}{b}\int_0^\infty \frac{A(r)}{B(r)}\mathrm{e}^{-tr}\mathrm{d}r. \qquad [4.210]$$

REMARK 4.5.– Equation [4.181] could be generalized by changing the integration limit so that

$$\int_0^2 \phi_\sigma(\gamma)\, {}_0\mathrm{D}_t^\gamma \sigma_r(x,t)\,\mathrm{d}\gamma = \int_0^2 \phi_\varepsilon(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(x,t)\,\mathrm{d}\gamma. \qquad [4.211]$$

The body described by [4.211] may be viewed as consisting of viscoelastic and viscoinertial elements (see [HAR 03]). If we take $\phi_\sigma(\gamma) = a^\gamma$, $\phi_\varepsilon(\gamma) = b^\gamma$, then by applying the Laplace transform, instead of [4.186], we obtain

$$\Phi(s) = s^2 + \omega^2 + \frac{\ln(as)}{\ln(bs)}\frac{(bs)^2 - 1}{(as)^2 - 1}.$$

Also the explicit form of the constitutive equation, i.e. the solution of [4.211], with respect to σ_r becomes

$$\sigma_r(t) = \lambda\hat{\phi}_1(t) * \hat{\phi}_2(t) * \varepsilon(t),$$

where

$$\hat{\phi}_1(t) = \mathcal{L}^{-1}\left[\frac{\ln as}{\ln bs}\right](t) = \delta(t) + \ln\frac{a}{b}\int_0^\infty \frac{t^{u-1}b^{-u}}{\Gamma(u)}\mathrm{d}u,$$

$$\hat{\phi}_2(t) = \mathcal{L}^{-1}\left[\frac{(bs)^2 - 1}{(as)^2 - 1}\right](t) = \left(\frac{b}{a}\right)^2\delta(t) + \frac{b^2 - a^2}{a^4}\sinh\frac{t}{a}.$$

The constitutive equation [4.211] is a generalization of relation [4.181] since [4.211] takes both viscoelastic and viscoinertial effects.

REMARK 4.6.– The stress–strain relation following from [4.181] is given by [4.195] and reads

$$
\begin{aligned}
\sigma_r(t) &= \phi_1(t) * \phi_2(t) * \varepsilon(t) \\
&= \left[\frac{b}{a}\varepsilon(t) + \frac{b-a}{a^2}\int_0^t \mathrm{e}^{\frac{t-\xi}{a}}\varepsilon(\xi)\,\mathrm{d}\xi\right. \\
&\quad + \frac{b}{a}\ln\frac{a}{b}\int_0^t \left(\int_0^\infty \frac{(t-\tau)^{u-1}\,b^{-u}}{\Gamma(u)}\mathrm{d}u\right)\varepsilon(\tau)\,\mathrm{d}\tau \\
&\left. + \frac{b-a}{a^2}\ln\frac{a}{b}\int_0^t \left(\int_0^\infty \frac{(t-\tau)^{u-1}\,b^{-u}}{\Gamma(u)}\mathrm{d}u\right)\varepsilon(\xi)\,\mathrm{e}^{\frac{t-\xi}{a}}\,\mathrm{d}\tau\right].
\end{aligned}
\tag{4.212}
$$

In the special case when $a = b$, the constitutive equation [4.181] describes an elastic body (see [ATA 03, p. 690]). By substituting $a = b$ into [4.212], we obtain $\sigma_r = \varepsilon$, i.e. the viscoelastic rod becomes an elastic rod.

REMARK 4.7.– The impulse response of the oscillator is obtained if we take $\varepsilon_0 = \dot{\varepsilon}_0 = 0$, $F = \delta$ so that $\tilde{F}(s) = 1$. The function ε in this case becomes

$$\varepsilon(t) = G_\alpha(t) - F_\alpha(t),$$

where

$$
\begin{aligned}
G_\alpha(t) &= \frac{2\mathrm{e}^{-\Sigma t}\cos(\Omega t)}{(\mathrm{Re}(\Phi'(s_1)))^2 + (\mathrm{Im}(\Phi'(s_1)))^2}\left(\mathrm{Re}\left(\overline{\Phi_1(s_1)}\Phi'(s_1)\right)\right. \\
&\quad\left. + \mathrm{Im}\left(\overline{\Phi_1(s_1)}\Phi'(s_1)\right)\right), \\
F_\alpha(t) &= -\ln\frac{a}{b}\int_0^\infty \frac{(ar+1)(br+1)}{B(r)}\mathrm{e}^{-tr}\mathrm{d}r.
\end{aligned}
$$

The function ε could be used to obtain solution ε_F for arbitrary F by forming the convolution $\varepsilon * F$.

4.2.3. *The case of the rod of distributed-order Kelvin–Voigt type*

We treat systems [4.134], [4.135] within the space of tempered distributions, along the lines of [ATA 05b]. We specify the constitutive equation [4.134] to be of the distributed-order Kelvin–Voigt type. This is done by choosing the constitutive functions (distributions) ϕ_σ and ϕ_ε in [4.134] to be of the form

$$\phi_\sigma(\gamma) = \delta(\gamma), \quad \phi_\varepsilon(\gamma) = \phi(\gamma) > 0, \ \ \gamma \in (a,b), \tag{4.213}$$

where ϕ is a known positive function. We will specify its further properties later. Note that in the case when ϕ_σ and (or) ϕ_ε have terms containing the Dirac distribution, the integrals in [4.213] are not the Riemann, but the Stieltjes integrals. Thus, for example, $\int_0^1 \delta(\gamma-\alpha)\,\mathrm{d}\gamma$ should be interpreted as

$$\int_0^1 \delta(\gamma-\alpha)\,\mathrm{d}\gamma \equiv \int_0^1 \mathrm{d}\psi_\sigma(\gamma), \ \text{ where } \ \psi_\sigma(\gamma) = \begin{cases} 0, & 0 < \gamma < \alpha, \\ 1, & \gamma \geq \alpha. \end{cases}$$

In general, it is assumed that ψ is a Borel-measurable function of normalized bounded variation.

With the choice of constitutive functions (distributions) given by [4.213], systems [4.134] and [4.135] become

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t) + \sigma(t) = F(t), \ \ t > 0, \tag{4.214}$$

$$\sigma(t) = \int_a^b \phi(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(t)\,\mathrm{d}\gamma, \ \ t > 0. \tag{4.215}$$

Note that we could have started with the constitutive equation in the form [4.215], because it can be treated as a generalization of the Kelvin–Voigt model of a viscoelastic body. The function ϕ characterizes the material under consideration. The mechanical interpretation of the constitutive law [4.215] is that the stress σ depends on all derivatives of the strain ε with respect to time with the weighting function $\phi_1(\gamma)$ for $a \leq \gamma \leq b$. For the mechanical application, important cases satisfy $0 \leq a < b \leq 2$. The material is called viscoelastic if $b \leq 1$ and viscoinertial if $1 < b \leq 2$, (see [HAR 03]).

The single equation, corresponding to systems [4.214] and [4.215] is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon(t) + \int_a^b \phi(\gamma)\, {}_0\mathrm{D}_t^\gamma \varepsilon(t)\,\mathrm{d}\gamma = F(t), \ \ t > 0. \tag{4.216}$$

4.2.3.1. *Study of the generalization of [4.216]*

Equation [4.216] is a special case of a more general equation

$$\sum_{i=0}^{m} a_i \frac{\mathrm{d}^i}{\mathrm{d}t^i} y(t) + \int_a^b \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} y(t)\, \mathrm{d}\alpha = h(t), \qquad [4.217]$$

where $m > 1$ and $b > a > 0$. We will analyze the solvability and uniqueness of the solution of [4.217] within the space of tempered distributions $\mathcal{S}'_+$. It is known, see [VLA 88, p. 50], that for given $g \in \mathcal{S}'_+$, the equation $g * y = h$, where $g * y$ is a convolution of distributions in S'_+ is uniquely solvable in $\mathcal{S}'_+$ for any $h \in \mathcal{S}'_+$ if and only if there exist $p, q \in \mathbb{R}$, $C > 0$ such that

$$\frac{1}{|\tilde{g}(s)|} \leq C \frac{(1+|s|)^p}{x^q}, \quad s = x + \mathrm{i}y \in \mathbb{C}_+. \qquad [4.218]$$

Next, we rewrite [4.217] in a slightly more general form

$$\sum_{i=0}^{m} a_i y * f_{-j_i} + \int_a^b \phi(\alpha)\,(y * f_{-\alpha})\, \mathrm{d}\alpha = h,$$

where $0 \leq j_0 < j_1 < \ldots j_m$ are real numbers and f_α is given by [1.6].

Assume that

$$[a,b] \ni \alpha \mapsto \phi(\alpha) \in [0,\infty),$$

is a piecewise continuous bounded function with a finite number of points of discontinuity, where ϕ has one side limits (the discontinuities are of the first order). With this assumption on ϕ, we have that the mapping $[a,b] \to \mathcal{S}'_+$ defined by

$$[a,b] \ni \alpha \mapsto \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} y(\cdot) \qquad [4.219]$$

is Bochner integrable and that

$$\sum_{i=1}^{r} \phi(\alpha_i)\, {}_0\mathrm{D}_t^{\alpha_i} y(\cdot)\, \Delta\alpha_i \xrightarrow{\mathcal{S}'_+} \int_a^b \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} y(\cdot)\, \mathrm{d}\alpha, \qquad [4.220]$$

where on the left-hand side we have the Riemann sum corresponding to partitions of the interval $[a, b]$ but in $\mathcal{S}'_+$, as usual. $\mathcal{L}\left[f_\alpha\left(t\right)\right]\left(s\right) = \frac{1}{s^\alpha}$, $\alpha \in \mathbb{R}$, $s \in \mathbb{C}_+$, [4.220] and mapping [4.219] is Bochner–Riemann integrable. It follows that

$$\mathcal{L}\left[\int_a^b \phi\left(\alpha\right) {}_0\mathrm{D}_t^\alpha y\left(t\right) \mathrm{d}\alpha\right]\left(s\right) = \tilde{y}\left(s\right)\left(\int_a^b \phi\left(\alpha\right) s^\alpha \mathrm{d}\alpha\right), \quad s \in \mathbb{C}_+.$$

With [4.220] and applying the Laplace transform to [4.217], we obtain

$$\left(\sum_{i=0}^{m} a_i s^i + \int_a^b \phi\left(\alpha\right) s^\alpha \mathrm{d}\alpha\right) \tilde{y}\left(s\right) = \tilde{h}\left(s\right), \quad s \in \mathbb{C}_+.$$

Thus, by using [4.218], we state the following theorem:

THEOREM 4.9.– Equation [4.217] is uniquely solvable in $\mathcal{S}'_+$ for any $h \in \mathcal{S}'_+$ if and only if there exist $p, q \in \mathbb{R}$ and $C > 0$ such that

$$\left|\sum_{i=0}^{m} a_i s^i + \int_a^b \phi\left(\alpha\right) s^\alpha \mathrm{d}\alpha\right| \geq C \frac{x^q}{\left(1 + |s|\right)^p}, \quad s \in \mathbb{C}_+. \qquad [4.221]$$

Theorem 4.9 is rather theoretical and we will describe some special cases which are, in fact, the most important in applications.

4.2.3.2. *Two special cases of [4.217]*

–*Case I.*

Consider the equation of [4.216], i.e.

$$v \frac{\mathrm{d}^m}{\mathrm{d}t^m} y\left(t\right) + \int_a^b \phi\left(\alpha\right) {}_0\mathrm{D}_t^\alpha y\left(t\right) \mathrm{d}\alpha = h\left(t\right), \qquad [4.222]$$

where $m \in [0, \infty)$, $v > 0$. Assumptions on ϕ, a, b are already given. For [4.222] we state the following theorem:

THEOREM 4.10.– Equation [4.222] is uniquely solvable for any $h \in \mathcal{S}'_+$ if

$$m - 1 \leq a < b \leq m. \qquad [4.223]$$

PROOF.– Applying the Laplace transform to [4.222],we have

$$\tilde{y}(s)\int_a^b (v_0 s^m + s^\alpha)\,\phi(\alpha)\,\mathrm{d}\alpha = \tilde{h}(s), \quad s \in \mathbb{C}_+, \qquad [4.224]$$

where

$$v_0 = \frac{v}{\int_a^b \phi(\alpha)\,\mathrm{d}\alpha}.$$

Note, $\int_a^b \phi(\alpha)\,\mathrm{d}\alpha > 0$. Let $s \in \mathbb{C}_+$. Denote

$$\theta_s(\alpha) = \arg(v_0 s^m + s^\alpha), \quad \alpha \in [a,b].$$

We will show that for given $s = r\mathrm{e}^{\mathrm{i}\varphi}$, $|\varphi| < \frac{\pi}{2}$

$$\theta_s(\alpha) \in [0,\varphi], \quad \alpha \in [a,b] \ \text{ if } \ \varphi > 0,$$
$$\theta_s(\alpha) \in [\varphi,0], \quad \alpha \in [a,b] \ \text{ if } \ \varphi < 0.$$

In fact, the change of the argument, call it the opening of $\theta_s(\alpha)$, is given by $\left|\arg \frac{v_0 s^m + s^b}{v_0 s^m + s^a}\right|$. We also have

$$\arg \frac{v_0 s^m + s^b}{v_0 s^m + s^a} = \arg \frac{v_0 s^{m-a} + s^{b-a}}{v_0 s^{m-a} + 1},$$

where $0 < m - a \leq 1$, $0 < b - a \leq m - a$. The opening is maximal in the case when $a = m - 1$ and $b = m$, and then we have that the opening in this case equals $\arg \frac{(v_0+1)s}{v_0 s + 1}$. From elementary arguments, we obtain

$$0 \leq \arg \frac{(v_0+1)\,s}{v_0 s + 1} \leq \arg s, \ \text{ if } \ \varphi > 0,$$
$$\arg s \leq \arg \frac{(v_0+1)\,s}{v_0 s + 1} \leq 0, \ \text{ if } \ \varphi < 0, \ \ |\varphi| < \frac{\pi}{2}.$$

Also

$$v_0 s^m + s^\alpha = |v_0 s^m + s^\alpha|\, e^{i\theta_s(\alpha)}, \quad \alpha \in [a,b],$$

and

$$\cos\theta_s\left(\alpha\right) \geq \cos\left(\arg s\right) = \frac{x}{|s|}, \quad \alpha \in [a,b] \,.$$

We now turn back to [4.224]

$$\begin{aligned}
&\left|\int_a^b \left(v_0 s^m + s^\alpha\right)\phi\left(\alpha\right)\mathrm{d}\alpha\right| \\
&\qquad = \left|\int_a^b \left|v_0 s^m + s^\alpha\right|\left(\cos\left(\theta_s\left(\alpha\right)\right) + i\sin\left(\theta_s\left(\alpha\right)\right)\right)\phi\left(\alpha\right)\mathrm{d}\alpha\right| \\
&\qquad \geq \left|\mathrm{Re}\int_a^b \left(v_0 s^m + s^\alpha\right)\phi\left(\alpha\right)\mathrm{d}\alpha\right| \\
&= \int_a^b \left|v_0 s^m + s^\alpha\right|\cos\left(\theta_s\left(\alpha\right)\right)\phi\left(\alpha\right)\mathrm{d}\alpha \\
&\qquad \geq \frac{x}{|s|}\int_a^b \left|v_0 s^m + s^\alpha\right|\phi\left(\alpha\right)\mathrm{d}\alpha.
\end{aligned}$$

Now from the mean value theorem, there exists $\alpha_0 = \alpha_0\left(s\right)$ such that

$$\int_a^b \left|v_0 s^m + s^\alpha\right|\phi\left(\alpha\right)\mathrm{d}\alpha = \left(b-a\right)\left|v_0 s^m + s^{\alpha_0}\right|\phi\left(\alpha_0\right).$$

Since, from [4.213], we have $\phi\left(\alpha_0\right) > 0$; further, it follows that

$$\left|v_0 s^m + s^{\alpha_0}\right| \geq r^{\alpha_0}\left|v_0 r^{m-\alpha_0}\cos\left(\left(m-\alpha_0\right)\phi\right) + 1\right|,$$

where $r = |s|$. Since $0 \leq m - \alpha_0 \leq 1$ (and $\cos\left(\left(m-\alpha_0\right)\right)\varphi > 0$), we have

$$\left|v_0 s^m + s^{\alpha_0}\right| \geq |s|^m \frac{x}{|s|} = |x|\,|s|^{m-1},$$

and this implies [4.221]. ■

– *Case II.*

Consider the equation of the type

$$v\frac{\mathrm{d}^m}{\mathrm{d}t^m}y\left(t\right)+w\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}y\left(t\right)+\int_a^b\phi\left(\alpha\right){}_0\mathrm{D}_t^{\alpha}y\left(t\right)\mathrm{d}\alpha=h\left(t\right), \qquad [4.225]$$

where $m\in[0,\infty)$, and v, w are positive real numbers. Equation [4.225] for the case $m=2$ describes a viscoelastic oscillator with the additional viscous damping. The same procedure as above leads to the following theorem.

THEOREM 4.11.– Equation [4.225] is uniquely solvable for any $h\in\mathcal{S}'_+$ *if* [4.223] holds.

PROOF.– We have to note that now with $v_0s^m+w_0s^{m+1}$, where $w_0=\frac{w}{\int_a^b\phi(\alpha)\mathrm{d}\alpha}$ instead of v_0s^m, the opening $\theta_s\left(\alpha\right)$, given by

$$\left|\arg\frac{v_0s^m+w_0s^{m-1}+s^b}{v_0s^m+w_0s^{m-1}+s^a}\right|,$$

is maximal if $a=m-1$ and $b=m$. Then, $|\theta_s(\alpha)|\leq|\phi|,\ |\phi|<\frac{\pi}{2}$. This leads to the estimation

$$\begin{aligned}\left|v_0s^m+w_0s^{m-1}+s^{\alpha_0}\right| &\geq r^{\alpha_0}\left|v_0r^{m-\alpha_0}\cos\left(\left(m-\alpha_0\right)\phi\right)\right.\\ &\qquad\left.+v_0r^{m-1-\alpha_0}\cos\left(\left(m-1-\alpha_0\right)\phi\right)+1\right|\\ &\geq |s|^m\frac{x}{|s|}=|x|\,|s|^{m-1}.\end{aligned}$$

Now from [4.221], the assertion of the theorem follows. ■

REMARK 4.8.– If $m=2$, then [4.222] corresponds to the mechanical system. Restriction [4.223] implies that the conclusion of theorem 4.10 applies to the equation of the type

$$v\frac{\mathrm{d}^2}{\mathrm{d}t^2}y\left(t\right)+\int_1^2\phi\left(\gamma\right){}_0\mathrm{D}_t^{\alpha}y\left(t\right)\mathrm{d}\gamma=h\left(t\right), \qquad [4.226]$$

that represents a linear oscillator with the distributed-order viscoinertial damping (see [HAR 03, p. 2289]).

REMARK 4.9.– If $m \geq 2$ and $b \leq 1$, then arguments of the proof given above could not be used because in this case the opening $\theta_s(\phi)$ can be bigger than $\frac{\pi}{2}$ and the mean value theorem cannot be used. In fact, in this case, the assertions in theorems 4.10 and 4.11 are false. More precisely, in order to have a solution of equation

$$v\frac{\mathrm{d}^2}{\mathrm{d}t^2}y\left(t\right) + \int_a^b \phi\left(\alpha\right){}_0\mathrm{D}_t^{\alpha}y\left(t\right)\mathrm{d}\alpha = h\left(t\right), \qquad [4.227]$$

with $b \leq 1$, we need additional assumptions for a (generalized) function h.

If in [4.227] we take $\phi = \delta(\alpha - \alpha_0)$, $\alpha_0 \in (0,1)$, (we can accommodate this case to our theorems), then the equation

$$v\frac{\mathrm{d}^2}{\mathrm{d}t^2}y\left(t\right) + {}_0\mathrm{D}_t^{\alpha_0}y\left(t\right) = h\left(t\right)$$

has a solution if the Laplace transform of h (not in $\mathcal{S}'_+$) is of the form

$$\tilde{h}\left(s\right) = (vs^{2-\alpha_0} + 1)\tilde{g}(s),$$

where $g \in \mathcal{S}'_+$. Then, the solution is given by

$$y(t) = \mathcal{L}^{-1}\left[\frac{\tilde{g}(s)}{s^{\alpha_0}}\right](t).$$

Actually, the problem of solvability, in this case is transferred to the right-hand side of equation, to h. Clearly, this equation does not have a solution in $\mathcal{S}'_+$ if h is an element of $\mathcal{S}'_+$ since in this case, we could not have the above factorization of $\tilde{h}$.

Results obtained here could be interpreted in the theory of constitutive equations for viscoelastic bodies giving the necessary conditions of the unique solvability for the strain when the stress is prescribed. Thus, suppose that in [4.215] we choose ϕ so that the constitutive equation reads

$$\sigma\left(t\right) = v\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varepsilon\left(t\right) + \int_0^1 \phi_1\left(\gamma\right){}_0\mathrm{D}_t^{\gamma}\varepsilon\left(t\right)\mathrm{d}\gamma \qquad [4.228]$$

with $v > 0$ and $m \in [0, \infty)$. Suppose further that we prescribe the stress $\sigma = h$. In order that [4.228] guarantees a unique strain ε for a prescribed stress, theorem 4.10 requires that

$$0 \leq m \leq 1.$$

Thus, in a particular equation

$$\sigma(t) = v\varepsilon(t) + \int_0^1 \phi_1(\gamma)\, {}_0\mathrm{D}_t^{\gamma} \varepsilon(t)\, \mathrm{d}\gamma,$$

for each prescribed $\sigma \in \mathcal{S}'_+$ determines a unique $\varepsilon \in \mathcal{S}'_+$.

Chapter 5

Impact of Viscoelastic Body Against the Rigid Wall

Consider a body consisting of a light viscoelastic rod of length L in the undeformed state that is attached to a rigid block of a mass m. By the light rod, we mean that its mass is negligible in comparison to the mass of the body, i.e. the density of the rod equals zero. The body moves in translatory motion by sliding on the horizontal surface along the x-axis that coincides with the axis of the rod. The origin of a coordinate system is placed at distance L from the wall (see Figure 5.1). At $t = 0$, the initial velocity of the body is v_0, the rod is undeformed and it comes into contact with the wall. The impact is achieved by the contact between the viscoelastic rod and the wall. We refer to [BRO 99, FRÉ 02, GLO 99, PFE 96] for a general approach to the motion with unilateral constraints and impact.

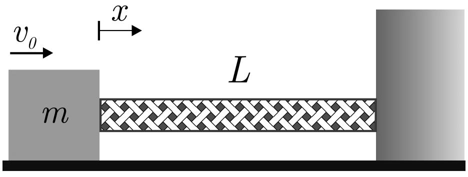

Figure 5.1. *Viscoelastic body: rigid block and light viscoelastic rod*

We refer to [ATA 06, ATA 04a] for the results presented in this chapter. Namely, the problem described above, without the presence of dry friction, was analyzed analytically in [ATA 06] and numerically in [ATA 04a]. The same problem, with the friction force taken into account, was treated in [GRA 12a]. Results presented in sections 5.2.1. and 5.2.2. are obtained in a joint work with N. Grahovac and M. Žigić.

5.1. Rigid block with viscoelastic rod attached slides without friction

Following [ATA 06], we treat the impact of a viscoelastic body, which slides without the presence of friction, against the rigid wall. We use the deformation model of impact (see [BRO 99]), so that the time during which the rod is in contact with the wall could be divided into two phases. In the first phase, called the approach phase, the rod shortens in length until the velocity of the attached block (relative velocity between the attached block and wall) becomes zero. In the second phase, called the rebound phase, the rod moves in the opposite direction with increasing velocity until it separates from the wall. We say that the rod is separated from the wall when the contact force between the rod and the wall becomes equal to zero.

Let l denote the length of the rod in the deformed state. Then, the strain is given as

$$x(t) = \frac{L - l\,(t)}{L}, \quad t > 0. \tag{5.1}$$

Since we assume that there are no adhesive forces between the rod and the wall, function x is non-negative. The strain x determines the force (stress) F in the rod through the constitutive equation. We assume that F and x are related through the generalized Zener model:

$$F\,(t) + \tau_F\, {}_0\mathrm{D}_t^\alpha F\,(t) = EA\,(x\,(t) + \tau_x\, {}_0\mathrm{D}_t^\alpha x\,(t))\,, \quad t > 0, \tag{5.2}$$

where τ_F and τ_x are generalized relaxation times, E is the generalized Young's modulus and A is the cross-sectional area of the rod. The second law of thermodynamics requires: $E > 0$, $0 < \tau_F \leq \tau_x$ (see [3.34] of [ATA 14b]). The equation of motion of the rigid block reads

$$mL\frac{\mathrm{d}^2}{\mathrm{d}t^2}x\,(t) = -F\,(t)\,, \quad t > 0. \tag{5.3}$$

Initial conditions are assumed as

$$x(0) = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}x(0) = v_0, \qquad F\,(0) = 0. \tag{5.4}$$

The contact force F could be positive only. If F becomes equal to zero (separation), it remains zero all the time.

We rewrite systems [5.2]–[5.4] in a dimensionless form by introducing the following quantities

$$f = \frac{F}{AE}, \quad \bar{t} = t\sqrt{\frac{AE}{mL}}, \quad a = \tau_F\left(\sqrt{\frac{AE}{mL}}\right)^\alpha, \quad b = \tau_x\left(\sqrt{\frac{AE}{mL}}\right)^\alpha \tag{5.5}$$

so that systems [5.2]–[5.4], after omitting the bar in $\bar{t}$, become

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x\,(t) = -f\,(t)\,, \quad f\,(t) + a\,{}_0\mathrm{D}_t^{\alpha} f\,(t) = x\,(t) + b\,{}_0\mathrm{D}_t^{\alpha} x\,(t)\,, \ \ t > 0, \tag{5.6}$$

$$x(0) = 0, \ \ \frac{\mathrm{d}}{\mathrm{d}t}x(0) = 1, \ \ f\,(0) = 0, \tag{5.7}$$

where, without loss of generality, we set $\frac{\mathrm{d}}{\mathrm{d}t}x(0) = v_0\sqrt{\frac{m}{AEL}} = 1$. Note that $a \leq b$.

5.1.1. *Analytical solution to [5.6] and [5.7]*

5.1.1.1. *Existence and uniqueness results*

Initial data [5.7] and an additional assumption that x and f are elements of $\mathcal{K}'_+$ enable us to rewrite [5.6] and [5.7] as

$$x\,(t) = t - t * f\,(t)\,, \quad T_{a,\alpha} f\,(t) = T_{b,\alpha} x\,(t)\,, \ \ t > 0, \tag{5.8}$$

where the operator $T_{a,\alpha}$ is defined as

$$T_{a,\alpha} u\,(t) = u\,(t) + a\,{}_0\mathrm{D}_t^{\alpha} u\,(t)\,, \ \ u \in \mathcal{K}'_+.$$

Recall, for the functions of exponential growth, we deal with the classical Laplace transform.

We need some results related to a function $\varphi_\alpha(s)$ of the complex variable s

$$\varphi_\alpha(s) = as^{\alpha+2} + s^2 + bs^\alpha + 1, \ \ s \in D_{r,R} \subset \mathbb{C},$$

where $\alpha \in (0,1)$ and $D_{r,R}$ is domain bounded by the contour $ABCD$, where $AB = \{s = x\mathrm{e}^{-\mathrm{i}\pi} \mid r < x < R\}$, $BC = \{s = R^{\mathrm{i}\theta} \mid -\pi < \theta < \pi\}$, $CD = \{s = x\mathrm{e}^{\mathrm{i}\pi} \mid r < x < R\}$, $DA = \{s = r\mathrm{e}^{\mathrm{i}\theta} \mid -\pi < \theta < \pi\}$. Function φ_α is analytic in $D_{r,R}$ and its properties are summarized in the following lemma. Note that, if $\varphi_\alpha(s_0) = 0$, $s_0 \in D_{r,R}$, then $\varphi_\alpha(\bar{s}_0) = 0$.

LEMMA 5.1.–

1) Assume $b > a$ and $\varphi_\alpha(s_0) = 0$, $s_0 \in D_{r,R}$, then $\operatorname{Re} s_0 < 0$.

2) There exists at most one point $s_0 \in D_{r,R}$ so that $\varphi_\alpha(s_0) = \varphi_\alpha(\bar{s}_0) = 0$.

3) If $\alpha \leq \frac{1}{2}$, then there exists two zeros for φ_α in $D_{r,R}$.

4) If $\alpha \in \left(\frac{1}{2},1\right)$ and $\cos^2(\alpha\pi) \leq \frac{(2b-a)a}{b^2}$ then there exist two zeros for φ_α in $D_{r,R}$.

PROOF.–

1) Let $s_0 = \rho_0 \mathrm{e}^{\mathrm{i}\varphi_0}$ be the zero of φ_α and $(s_0^2 + \frac{b}{a}) = \rho \mathrm{e}^{\mathrm{i}\theta}$. Then, $(1 + a s_0^\alpha) = \frac{(\frac{b}{a}-1)}{\rho} \mathrm{e}^{-\mathrm{i}(\theta+2k\pi)}$. From $\mathrm{Im}(s^2 + \frac{b}{a}) = \rho^2 \sin(2\varphi) = r\sin\theta$ and $\mathrm{Im}(as^\alpha + 1) = a\rho^\alpha \sin\varphi = \frac{(\frac{b}{a}-1)}{r}\sin(-\theta)$, it follows

$$\frac{\rho_0^{2-\alpha}}{a}\frac{\sin(2\varphi_0)}{\sin(\alpha\varphi_0)} = -\frac{\rho}{\frac{b}{a}-1}.$$

Now, $b > a$ implies $\frac{\sin(2\varphi_0)}{\sin(\alpha\varphi_0)} < 0$ so that $|\varphi_0| > \frac{\pi}{2}$. It means that s_0 lies in the left half plain.

2) Let $\alpha = \frac{p}{q}$ be a rational number in $(0,1)$, where p, q are mutually simple. Then by the change of variables $s = w^q$, we come to a polynomial

$$\begin{aligned} P(w) &= aw^{2q+p} + w^{2q} + bw^p + 1, \\ w \in \mathbb{C}_w &= \left\{ w = \rho \mathrm{e}^{\mathrm{i}\theta} \mid \rho \in \left(\sqrt[q]{r}, \sqrt[q]{R}\right),\ \theta \in (-\pi\alpha, \pi\alpha) \right\}. \end{aligned}$$

Let $f(s_0) = 0$, $s_0 \in D_{r,R}$. If we had three different zeros, i.e. if $s_1 \neq s_0$, $s_1 \neq \bar{s}_0$, $s_1 \in D_{r,R}$ and $\varphi(s_1) = 0$, then points s_0, $\bar{s}_0$, s_1 as well as $\bar{s}_1$ would generate $4q$ different zeros of P in $\mathbb{C}_w$. But this is impossible because P can have exactly $2q + p$ zeros. Let α be the irrational number in $(0,1)$. Assume again that we have three different points in $D_{r,R}$: s_0, $\bar{s}_0$, s_1 (as well as $\bar{s}_1$) so that $\varphi_\alpha(s_0) = \varphi_\alpha(\bar{s}_0) = \varphi_\alpha(s_1) = \varphi_\alpha(\bar{s}_1) = 0$. Put $\varphi(t,s) = \varphi_t(s)$, $t \in (0,1)$, $s \in D_{r,R}$. Because $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,s) = as^t(as^2+b)\ln s \neq 0$ at points (α, s_0), $(\alpha, \bar{s}_0)$ and (α, s_1). From the implicit function theorem related to points (s_0,α), $(\bar{s}_0,\alpha)$ and (s_1,α), it follows that there exists a neighborhood of α: $(\alpha-\varepsilon,\alpha+\varepsilon)$ and balls around s_0, $\bar{s}_0$, s_1: $B(s_0,\delta)$, $B(\bar{s}_0,\delta)$ and $B(s_1,\delta)$, respectively, without intersection so that if α varies in $(\alpha-\varepsilon,\alpha+\varepsilon)$, then s varies in $B(s_0,\delta)$ (respectively, $B(\bar{s}_0,\delta)$) and $B(s_1,\delta)$. Thus, if we let $\alpha^* = \frac{p}{q} \in (\alpha-\varepsilon,\alpha+\varepsilon)$, then we have three points $\tilde{s}_0 \in B(s_0,\delta)$, $\widetilde{\bar{s}}_0 \in B(\bar{s}_0,\delta)$ and $\tilde{s}_1 \in B(s_1,\delta)$ so that $\varphi_{\frac{p}{q}}(\tilde{s}_0) = \varphi_{\frac{p}{q}}(\widetilde{\bar{s}}_0) = \varphi_{\frac{p}{q}}(\tilde{s}_1) = 0$. This is impossible according to the previous case.

3) We will use the Rouché theorem. Recall that if f and g are analytic functions inside the simple Jordan contour C and satisfy $|f| > |g|$ on C, then f and $f+g$ have the same number of zeros. We have

$$\varphi_\alpha(s) = as^{\alpha+2} + s^2 + bs^\alpha + 1 = \left(s^2 + \frac{b}{a}\right)(1 + as^\alpha) - \left(\frac{b}{a} - 1\right).$$

Take

$$f(s)=\left(s^2+\frac{b}{a}\right)(1+as^\alpha) \text{ and } g(s)=-\left(\frac{b}{a}-1\right)$$

and for C we take the contour $D_{r,R}$ described above. Then on the circle BC

$$\begin{aligned} |f(s)| &= \left|as^{\alpha+2}+s^2+bs^\alpha+\frac{b}{a}\right| \geq a\left|s^{\alpha+2}\right|-\left|s^2\right|-b\left|s^\alpha\right|-\frac{b}{a} \\ &= aR^{\alpha+2}-R^2-bR^\alpha-\frac{b}{a} \geq R^{\alpha+2}\left(a-\frac{1}{R^\alpha}-\frac{b}{R^2}\right)-\frac{b}{a} \end{aligned}$$

and for R large enough

$$|f(s)|>\left|-\left(\frac{b}{a}-1\right)\right|=|g(s)|, \quad s\in BC.$$

On the circle AD, we have similar situation:

$$|f(s)|=\left|as^{\alpha+2}+s^2+bs^\alpha+\frac{b}{a}\right| \geq ar-r^2-br^\alpha-\frac{b}{a}.$$

Thus, for r small enough, we obtain

$$|f(s)|\geq\left|-\frac{b}{a}+1\right|=|g(s)|.$$

On the line $s=x\mathrm{e}^{\mathrm{i}\pi}$ or $s=x\mathrm{e}^{-\mathrm{i}\pi}$, $0<x<\infty$, we have

$$|f(s)|=\left|\left(s^2+\frac{b}{a}\right)(1+as^\alpha)\right|\geq|\mathrm{Re}\, f(s)|=\left|\left(x^2+\frac{b}{a}\right)(1+ax^\alpha\cos(\alpha\pi))\right|.$$

For $0<\alpha\leq\frac{1}{2}$ $\cos(\alpha\pi)>0$, $1+ax^\alpha\cos(\alpha\pi)>1$ and this implies that for $s\in AB$, $s\in CD$

$$|f(s)|\geq\left|x^2+\frac{b}{a}\right|>|g(s)|.$$

Therefore, f and φ_α have the same number of zeros. Because $f(s)=\left(s^2+\frac{b}{a}\right)(1+as^\alpha)$ has two zeros in C, φ_α has two zeros.

4) For $s = x\mathrm{e}^{-\mathrm{i}\pi}$, we have

$$\begin{aligned} |f(s)|^2 &= \left|(s^2 + \frac{b}{a})(1 + as^\alpha)\right|^2 = (x^2 + \frac{b}{a})^2 \left(1 + 2ax^\alpha \cos\alpha\pi + a^2 x^{2\alpha}\right) \\ &\geq \left(\frac{b}{a}\right)^2 \left(1 - \cos^2(\alpha\pi)\right). \end{aligned}$$

Now if $\left(\frac{b}{a}\right)^2 (1 - \cos^2(\alpha\pi)) > \left(\frac{b}{a} - 1\right)^2$, then φ_α has two zeros. Thus, if $\cos^2(\alpha\pi) \leq \frac{a}{b}\left(2 - \frac{a}{b}\right)$, the function φ_α has two zeros. ■

THEOREM 5.1.– Let $a, b \in \mathbb{R}$, $a > 0$, $b > 0$, $\alpha \in (0, 1)$. Then equation [5.8] is uniquely solvable in $\mathcal{S}'_+$.

If, moreover, $b > a$ and $\varphi_\alpha(s) = as^{\alpha+2} + s^2 + bs^\alpha + 1$ has two different zeros s_1 and $s_2 = \bar{s}_1$, $s_1 \in \mathbb{C}\backslash(-\infty, 0]$, then

$$\begin{aligned} x(t) &= \frac{\sin(\alpha\pi)}{\pi} \\ &\times \int_0^\infty \frac{(b-a)\, r^\alpha}{(1+r^2)^2 + 2(1+r^2)\, r^\alpha (b + r^2 a)\cos\alpha\pi + (r^\alpha(b + r^2 a))^2} \mathrm{e}^{-rt} \mathrm{d}r \\ &+ \sum_{i=1}^{2} \frac{(1 + as_i^\alpha)\mathrm{e}^{s_i t}}{a(\alpha+2)s_i^{\alpha+1} + 2s_i + b\alpha s_i^{\alpha-1}}, \quad t \geq 0, \end{aligned} \qquad [5.9]$$

$$\begin{aligned} f(t) &= \frac{\sin(\alpha\pi)}{\pi} \\ &\times \int_0^\infty \frac{(a-b)\, r^\alpha}{(1+r^2)^2 + 2(1+r^2)\, r^\alpha (b + r^2 a)\cos\alpha\pi + (r^\alpha(b + r^2 a))^2} \mathrm{e}^{-rt} \mathrm{d}r \\ &+ \sum_{i=1}^{2} \frac{(1 + bs_i^\alpha)\mathrm{e}^{s_i t}}{a(\alpha+2)s_i^{\alpha+1} + 2s_i + b\alpha s_i^{\alpha-1}}, \quad t \geq 0. \end{aligned} \qquad [5.10]$$

REMARK 5.1.– Initial conditions $x(0) = 0$ and $f(0) = 0$ imply

$$\begin{aligned}
&\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{(b-a)\,r^\alpha}{(1+r^2)^2+2\,(1+r^2)\,r^\alpha(b+r^2a)\cos\alpha\pi+(r^\alpha(b+r^2a))^2}\mathrm{d}r\\
&\qquad = -\sum_{i=1}^{2}\frac{1+as_i^\alpha}{a(\alpha+2)s_i^{\alpha+1}+2s_i+b\alpha s_i^{\alpha-1}},\\
&\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \frac{(a-b)\,r^\alpha}{(1+r^2)^2+2\,(1+r^2)\,r^\alpha(b+r^2a)\cos\alpha\pi+(r^\alpha(b+r^2a))^2}\mathrm{d}r\\
&\qquad = -\sum_{i=1}^{2}\frac{1+bs_i^\alpha}{a(\alpha+2)s_i^{\alpha+1}+2s_i+b\alpha s_i^{\alpha-1}}.
\end{aligned}$$

REMARK 5.2.– From equations [5.9] and [5.10] we conclude that x and f are continuous. Also, we can differentiate [5.9] and [5.10] two times and get continuous functions, i.e. f and x belong to $C^2([0,\infty))$.

PROOF.– Applying the Laplace transform to [5.8], we obtain

$$(as^\alpha+1)\,\tilde{f}(s) = (bs^\alpha+1)\tilde{x}(s), \quad \operatorname{Re} s > 0, \tag{5.11}$$

$$\tilde{x}(s) = \frac{1}{s^2}(1-\tilde{f}(s)), \quad \operatorname{Re} s > 0, \tag{5.12}$$

which leads to

$$\tilde{x}(s) = \frac{as^\alpha+1}{as^{\alpha+2}+s^2+bs^\alpha+1}, \quad \operatorname{Re} s > 0, \tag{5.13}$$

$$\tilde{f}(s) = \frac{bs^\alpha+1}{as^{\alpha+2}+s^2+bs^\alpha+1}, \quad \operatorname{Re} s > 0. \tag{5.14}$$

Note that [5.13] and [5.14] are the Laplace transforms of real-valued functions of real variable (see [DOE 55, volume I, theorem 2, p. 293]). Now by applying the inverse Laplace transform to [5.13], we have

$$x(t) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s, \quad t \geq 0,$$

where $\gamma = \{s \mid \operatorname{Re} s = \sigma,\ \sigma > \sigma_0 = 0\}$. Let $\gamma_0 = \widehat{AB} \cup \widehat{BD} \cup \widehat{DE} \cup \widehat{EF} \cup \widehat{FG} \cup \widehat{GA}$ be the contour shown in Figure 5.2. Cauchy's formula gives

$$\int_{\gamma_0} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s = 2\pi\mathrm{i}\left(\operatorname{Res}\left(\tilde{x}(s)\mathrm{e}^{st}, s_1\right) + \operatorname{Res}\left(\tilde{x}(s)\mathrm{e}^{st}, \bar{s}_1\right)\right). \qquad [5.15]$$

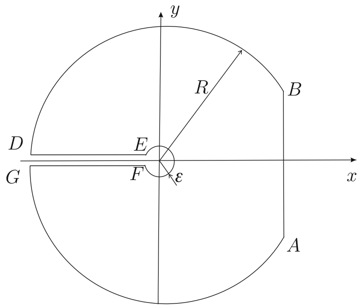

Figure 5.2. *Integration contour* γ

Integrals

$$\int_{\widehat{BD}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s, \quad \int_{\widehat{GA}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s \ \text{ and } \ \int_{\widehat{EF}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s,$$

tend to zero when $R \to \infty$ and $\varepsilon \to 0$; so they do not contribute to the left-hand side of [5.15]. Those that contribute to the left-hand side of [5.15] are integrals

$$\int_{\widehat{AB}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s, \quad \int_{\widehat{DE}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s, \quad \int_{\widehat{FG}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s.$$

We will show that when $R \to \infty$ and $\varepsilon \to 0$, the contribution of latter two integrals is provided by

$$F_\alpha(t) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty K_\alpha(r)\mathrm{e}^{-rt}\mathrm{d}r,$$

where K_α will be defined later, while the first integral tends to $\int_\gamma \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s$. Denote

$$G_\alpha(t) = \operatorname{Res}\left(\tilde{x}(s)\mathrm{e}^{st}, s_1\right) + \operatorname{Res}\left(\tilde{x}(s)\mathrm{e}^{st}, \bar{s}_1\right), \quad t \geq 0.$$

Then

$$\int_{\gamma} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s + 2\pi\mathrm{i}F_{\alpha}(t) = 2\pi\mathrm{i}G_{\alpha}(t), \quad \text{i.e.} \;\; x = G_{\alpha} - F_{\alpha} \qquad [5.16]$$

and we need to calculate F_{α} and G_{α} to get [5.9]. First, we calculate F_{α}. Let $s = r\mathrm{e}^{\mathrm{i}\pi}$ on $\widehat{DE}$ and $s = r\mathrm{e}^{-\mathrm{i}\pi}$ on $\widehat{FG}$. Then

$$\begin{aligned}&\int_{\widehat{DE}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s + \int_{\widehat{FG}} \tilde{x}(s)\mathrm{e}^{st}\mathrm{d}s \\ &\quad = \int_{\varepsilon}^{R} \left(\frac{1 + ar^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi}}{r^2\mathrm{e}^{2\pi\mathrm{i}}(ar^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi} + 1) + br^{\alpha}\mathrm{e}^{\mathrm{i}\alpha\pi} + 1} \right. \\ &\qquad \left. - \frac{1 + ar^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi}}{r^2\mathrm{e}^{-2\pi\mathrm{i}}(ar^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi} + 1) + br^{\alpha}\mathrm{e}^{-\mathrm{i}\alpha\pi} + 1} \right) \mathrm{e}^{-rt}\mathrm{d}r.\end{aligned}$$

Letting $R \to \infty$ and $\varepsilon \to 0$, it follows

$$F_{\alpha}(t) = -\frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} K_{\alpha}(r)\mathrm{e}^{-rt}\mathrm{d}r,$$

where

$$K_{\alpha}(r) = \int_0^{\infty} \frac{(b-a)\, r^{\alpha}}{(1+r^2)^2 + 2\,(1+r^2)\, r^{\alpha}(b + r^2 a)\cos\alpha\pi + (r^{\alpha}(b + r^2 a))^2} \mathrm{e}^{-rt}\mathrm{d}r.$$

Assume now that $b > a$ and that $F_2(s) = as^{\alpha+2} + s^2 + bs^{\alpha} + 1$ has two zeros s_1 and $s_2 = \bar{s}_1$. Put $F_1(s) = (1 + as^{\alpha})\mathrm{e}^{st}$. We have

$$\begin{aligned}G_{\alpha}(t) &= \frac{F_1(s_1)}{F_2'(s_1)} + \frac{F_1(s_2)}{F_2'(s_2)} \\ &= \frac{(1 + as_1^{\alpha})\mathrm{e}^{s_1 t}}{a(\alpha+2)s_1^{\alpha+1} + 2s_1 + b\alpha s_1^{\alpha-1}} + \frac{(1 + as_2^{\alpha})\mathrm{e}^{s_2 t}}{a(\alpha+2)s_2^{\alpha+1} + 2s_2 + b\alpha s_2^{\alpha-1}}.\end{aligned}$$

From [5.16], we obtain [5.9]. Similarly, we obtain $f = H_{\alpha} + F_{\alpha}$. ■

5.1.1.2. *Qualitative properties of solutions*

Our next theorem is related to the thermodynamical consequence of the model.

THEOREM 5.2.– Let f and x be the solution of [5.8]. Then, $b > a$ if and only if $f(t) > x(t)$ in a neighborhood of zero.

PROOF.– Repeating the procedure with the Laplace transform to $[5.8]_2$, we obtain

$$\tilde{f}(s) = \frac{\mathcal{L}[T_{b,\alpha}x](s)}{1 + as^{\alpha}}, \quad \mathrm{Re}\, s > 0,$$

i.e.

$$\tilde{f}(s) = \tilde{x}(s) + \left(\frac{b}{a} - 1\right) \frac{s^{\alpha}}{s^{\alpha} + \frac{1}{a}} \tilde{x}(s), \quad \mathrm{Re}\, s > 0. \qquad [5.17]$$

Now we apply the inverse Laplace transform to [5.17] to obtain

$$f(t) = x(t) + \left(\frac{b}{a} - 1\right) (\delta(t) + \dot{e}_{\alpha}(t)) * x(t), \qquad [5.18]$$

where $\dot{e}_{\alpha} = \frac{\mathrm{d}}{\mathrm{d}t} e_{\alpha}$, $e_{\alpha}(t) = E_{\alpha}\left(-\frac{t^{\alpha}}{a}\right)$ and E_{α} is the one-parameter Mittag-Leffler function (see [MAI 00]). Integration by parts with initial condition gives

$$(\delta(t) + \dot{e}_{\alpha}(t)) * x(t) = e_{\alpha}(t) * \frac{\mathrm{d}}{\mathrm{d}t} x(t).$$

This equation leads to

$$f(t) = x(t) + \left(\frac{b}{a} - 1\right) e_{\alpha}(t) * \frac{\mathrm{d}}{\mathrm{d}t} x(t). \qquad [5.19]$$

Function e_{α} is completely monotonic[1] for $\lambda > 0$ and $0 < \alpha < 1$, see [GOR 97b]. Therefore, $e_{\alpha} > 0$. Because $\frac{\mathrm{d}}{\mathrm{d}t} x(0) = 1$, there is $\varepsilon > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t} x(t) > 0$ for $t \in [0, \varepsilon)$. Hence from [5.19], $f > x$ if and only if $\frac{b}{a} - 1 > 0$ which we wanted to prove. ■

1 Function $f(t)$ is completely monotone for $t > 0$ if $(-1)^n f^{(n)}(t) \geq 0$ for all $n = 0, 1, 2...$ and all $t > 0$.

REMARK 5.3.– Initial conditions [5.7] imply that there exists $\varepsilon > 0$ such that $x(t) > 0$ for $t \in (0, \varepsilon)$. The statement of theorem 5.2 leads to the estimate

$$f(t) > x(t) > 0, \quad t \in (0, \varepsilon). \tag{5.20}$$

Let $T, T^*, T_1 \in \mathbb{R}$ be time instants when the velocity, force and deformation become zero, respectively. Thus, we have $\frac{\mathrm{d}}{\mathrm{d}t} x(T) = 0$, $f(T^*) = 0$ and $x(T_1) = 0$. Note that T^* is the time instant when the rod separates from the wall because $f(T^*) = 0$ is the definition of the separation instant. Therefore, $f(t) = 0$, $t \in [T^*, \infty)$. Then, we have the following proposition.

PROPOSITION 5.1.– If there exists $T_1 > 0$ such that $x(T_1) = 0$ and $x(t) > 0$, for $t \in (0, T_1)$, then there exists T^* such that $0 < T^* < T_1$ and $f(T^*) = 0$.

PROOF.– From [5.18] we obtain

$$f(t) = \frac{b}{a} x(t) + \left(\frac{b}{a} - 1\right) \dot{e}_\alpha(t) * x(t), \tag{5.21}$$

which when evaluated at $t = T_1$, leads to

$$f(T_1) = \frac{b}{a} x(T_1) + \left(\frac{b}{a} - 1\right) \int_0^{T_1} \dot{e}_\alpha(\tau)\, x(T_1 - \tau)\, \mathrm{d}\tau.$$

Since $x(T_1) = 0$, by assumption and $\dot{e}_\alpha(t) < 0$, it follows that

$$f(T_1) < 0. \tag{5.22}$$

From [5.20], [5.22] and the continuity of f, the result of the proposition follows. ■

PROPOSITION 5.2.– The following inequality holds

$$T^* > T > 0. \tag{5.23}$$

PROOF.– Suppose that [5.23] does not hold. Since $\frac{\mathrm{d}}{\mathrm{d}t} x$ is continuous and $\frac{\mathrm{d}}{\mathrm{d}t} x(0) = 1$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t} x(t) > 0$, $t \in [0, T^*]$. After T^*, $f(t) = 0$ so that from $[5.6]_1$, we conclude that $\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = 0$, $t \in [T^*, T]$. Therefore, it follows that $\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \frac{\mathrm{d}}{\mathrm{d}t} x(T^*) > 0$. Thus, x is increasing. Since x is continuous, $x(T^*)$ is positive, we conclude that $x(t)$ cannot become zero for $t = T_1 > T^*$. ■

THEOREM 5.3.– There exist $\varepsilon_1, \varepsilon_2 > 0$, such that, for $\alpha \in [0, \varepsilon_1)$, or $\alpha \in (1-\varepsilon_2, 1]$, the solution x satisfies $\left|\frac{\mathrm{d}}{\mathrm{d}t}x(T^*)\right| < 1$.

PROOF.– By [5.19], we have

$$f(t) = x(t) + \left(\frac{b}{a} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e_\alpha\,(t-\tau)\,x(\tau)\mathrm{d}\tau, \quad t \geq 0. \qquad [5.24]$$

Let

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e_\alpha\,(t-\tau)\,x(\tau)\mathrm{d}\tau, \quad t \geq 0.$$

Then, u is the solution of the Abel equation (see [MAI 00])

$$u(t) + \frac{1}{a}\frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau = x(t), \quad u(0) = 0. \qquad [5.25]$$

Differentiating [5.25] it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}u\,(t) + \frac{1}{a}{}_0\mathrm{D}_t^{1-\alpha}u\,(t) = \frac{\mathrm{d}}{\mathrm{d}t}x\,(t)\,, \quad u(0) = 0. \qquad [5.26]$$

Using [5.24] in $[5.6]_1$, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x\,(t) + x\,(t) = -\left(\frac{b}{a} - 1\right)\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e_\alpha\,(t-\tau)\,x(\tau)\mathrm{d}\tau = -\left(\frac{b}{a} - 1\right)u\,(t)\,. \qquad [5.27]$$

Multiplying [5.27] by $\frac{\mathrm{d}}{\mathrm{d}t}x$, using [5.26] and integrating from 0 to T^*, we have

$$\begin{aligned}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}x(T^*)\right)^2}{2} + \frac{x^2(T^*)}{2} - \frac{1}{2} \quad = \quad & -\left(\frac{b}{a} - 1\right)\left(\int_0^{T^*} u\,(\tau)\,\frac{\mathrm{d}}{\mathrm{d}\tau}u\,(\tau)\,\mathrm{d}\tau\right. \\ & \left. + \frac{1}{a}\int_0^{T^*} u\,(\tau)\,{}_0\mathrm{D}_\tau^{1-\alpha}u\,(\tau)\,\mathrm{d}\tau\right). \qquad [5.28]\end{aligned}$$

Because $I(1) = \int_0^{T^*} u\,(\tau)\,\frac{\mathrm{d}}{\mathrm{d}\tau}u\,(\tau)\,\mathrm{d}\tau = \frac{u^2(T^*)}{2} > 0$, $I(0) = \int_0^{T^*} u^2\,(\tau)\,\mathrm{d}\tau > 0$ and

$$I(\alpha) = \int_0^{T^*} u\,(\tau)\,{}_0\mathrm{D}_\tau^{1-\alpha}u\,(\tau)\,\mathrm{d}\tau, \quad \alpha \in (0,1)\,,$$

is continuous, it follows that there exist ε_1 and ε_2 such that $I(\alpha) \geq 0$ for $\alpha \in [0, \varepsilon_1)$ or $\alpha \in (1-\varepsilon_2, 1]$. Therefore, the assumption $b \geq a$ implies that the right-hand side of [5.28] is negative and the result of the theorem 5.3 follows. Note also that

$$\int_0^{T^*} u(\tau)\, {}_0\mathrm{D}_\tau^{1-\alpha} u(\tau) > 0, \quad \alpha \in (0,1),$$

see [4.61 of volume 1]. Therefore, the energy is dissipated for every $\alpha \in (0,1)$. ■

5.1.1.3. *Iterative procedure*

Let us denote

$$B(s) = \frac{bs^\alpha + 1}{as^\alpha + 1}.$$

If we substitute B and $\tilde{x}$, given by [5.12], into [5.11], we obtain

$$\tilde{f}(s) = \frac{1}{s^2} B(s)\,(1 - \tilde{f}(s)), \quad \operatorname{Re} s > 0.$$

We calculate $\tilde{x}$ from [5.12] and obtain

$$\tilde{x}(s) = \frac{1}{s^2} \sum_{k=0}^{n} \left(-\frac{B(s)}{s^2} \right)^k + (-1)^{n+1} \left(\frac{B(s)}{s^2} \right)^n \tilde{f}(s).$$

One can show that for $\operatorname{Re} s > \frac{b}{a}$, we have $\left| \left(\frac{B(s)}{s^2} \right)^n \tilde{f}(s) \right| \to 0$ as $n \to \infty$. More precisely, for $\operatorname{Re} s > \frac{b}{a}$, we have

$$\left| \left(\frac{B(s)}{s^2} \right)^n \tilde{f}(s) \right| \leq \left(\frac{b}{a} \right)^n \frac{\left| \tilde{f}(s) \right|}{s^{2n}}.$$

Thus, $\tilde{x}$ is an analytic function so that

$$\tilde{x}(s) = \frac{1}{s^2} \sum_{k=0}^{\infty} \left(-\frac{B(s)}{s^2} \right)^k, \quad \operatorname{Re} s > \frac{b}{a}.$$

Since $\tilde{x}$ belongs to D_+ and has the given form for $\operatorname{Re} s > \frac{b}{a}$, its inverse Laplace transform is given by

$$x(t) = t * \sum_{k=0}^{\infty} \left(\frac{b}{a} t + \left(\frac{b}{a} - 1 \right) \dot{e}_\alpha (t) * t \right)^{*k}, \quad t > 0, \qquad [5.29]$$

where $u^{*k} = \underbrace{u * u * ... * u}_{k\text{-times}}$ and $u^{*1} = u$. Formula [5.29] represents another form of the solution for system [5.8]. With x known, the function f is given by

$$f(t) = x(t) * \left(\frac{b}{a} t + \left(\frac{b}{a} - 1 \right) \dot{e}_\alpha (t) * t \right), \quad t \geq 0. \qquad [5.30]$$

5.1.2. *Numerical solution to [5.6] and [5.7]*

In order to compute the solution to [5.6], [5.7], we apply arguments presented in [POD 99, p. 223] and follow [ATA 04a]. We eliminate f and the remove non-homogeneous initial condition $[5.7]_2$. Namely, by introducing the variable

$$z(t) = x(t) - t, \quad t > 0, \qquad [5.31]$$

we obtain the following differential equation of real order

$$a\, {}_0\mathrm{D}_t^{2+\alpha} z(t) + \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + z(t) + b\, {}_0\mathrm{D}_t^{\alpha} z(t) = -t - b \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha}, \qquad [5.32]$$

with homogeneous initial conditions

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} z(0) = 0, \quad k = 0, 1, 2. \qquad [5.33]$$

Using the first-order approximation of problems [5.32] and [5.33], we derive the following algorithm for obtaining the numerical solution

$$\begin{aligned} z_0 &= 0, \quad z_1 = 0, \quad z_2 = 0, & [5.34] \\ z_m &= \frac{1}{1 + h^{-2} + a h^{-\alpha} + b h^{-2-\alpha}} \left(\frac{2 z_{m-1} - z_{m-2}}{h^2} \right. \\ &\quad - \frac{a}{h^\alpha} \sum_{j=1}^{m} \omega_{j,\alpha} z_{m-j} - \frac{b}{h^{2+\alpha}} \sum_{j=1}^{m} \omega_{j,2+\alpha} z_{m-j} \\ &\quad \left. - mh - a \frac{\Gamma(2)}{\Gamma(2-\alpha)} (mh)^{1-\alpha} \right), \quad m = 3, 4, \ldots, & [5.35] \end{aligned}$$

where h is the time step and the coefficients $\omega_{j,\kappa}$, $\kappa \in \{\alpha, 2+\alpha\}$, are calculated by using the recurrence relations $\omega_{0,\kappa} = 1$, $\omega_{j,\kappa} = \left(1 - \frac{\kappa+1}{j}\right)\omega_{j-1,\kappa}$, $j = 1, 2, 3, \ldots$ (see [POD 99]). Noting that $z_m = z(t_m) = z(mh)$, from [5.31], we obtain

$$x(t_m) = mh + z_m. \qquad [5.36]$$

Finally, by using the second-order backward differences, we find

$$f(t_m) = -\frac{z_m - 2z_{m-1} + z_{m-2}}{h^2}. \qquad [5.37]$$

Note that in writing [5.35], we used the fact that the Riemann–Liouville and Grünwald–Letnikov fractional derivatives are equivalent. The described numerical method was experimentally verified on a number of test problems by comparing it (when it was possible) with analytical solutions (see [POD 99]).

In the case of equation [5.32], and therefore equations [5.36] and [5.37], the concept of the Post inversion formula, although less accurate for small n, could be very useful. The procedure of applying the Post inversion formula to [5.13] and [5.14] leads to

$$\begin{aligned} x(t) &= \lim_{n\to\infty}\left(\frac{(-1)^n\left(\frac{n}{t}\right)^{n+1}}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n}\tilde{x}(s)\right]_{s=\frac{n}{t}}\right), \\ f(t) &= \lim_{n\to\infty}\left(\frac{(-1)^n\left(\frac{n}{t}\right)^{n+1}}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n}\tilde{f}(s)\right]_{s=\frac{n}{t}}\right). \end{aligned} \qquad [5.38]$$

The solution [5.38] could be used for the error estimation of the numerical solution.

In Table 5.1 we present the duration of impact T, determined by the condition $f(T) = 0$, maximal values of x and f for several values of dimensionless relaxation times in the case of the standard linear viscoelastic solid and for several values of constants α, a and b. The values of dimensionless time corresponding to these maximums are given in parenthesis. The values x and $\frac{\mathrm{d}}{\mathrm{d}t}x$ at T ($\frac{\mathrm{d}}{\mathrm{d}t}x(T)$ determines the restitution coefficient) are also presented. Numerical values of constants α, a and b were taken from [FEN 98], where the rail pad models were investigated. For $\alpha = 1$, we apply the inverse Laplace transform to [5.13]. For $\alpha < 1$, we apply numerical procedure [5.35] and then [5.36], [5.37]. In all calculations, the time step was $h = 10^{-3}$.

In Figure 5.3, we present some solutions to [5.6] and [5.7]. Values obtained by applying the Post inversion formula [5.38] for $\alpha = 1$, $a = 0.04$ and $b = 0.2$ are also presented (squares in Figure 5.3). The difference between the exact solution and the solution obtained by the Post inversion formula for $n = 70$ is less than 5×10^{-2}. Also, values calculated by the Post inversion formula [5.38] for $\alpha = 0.23$, $a = 0.004$, $b = 1.183$ for $n = 40$ are also marked by circles in Figure 5.3. In this case, the difference between the numerical solution and these values is less than 5×10^{-2}.

Material description	Impact description	
$\alpha = 1$ $a = 0.01$ $b = 0.2$	$f_{\max}(1.289) = 0.887$ $x_{\max}(1.480) = 0.869$	$T = 2.962$ $x(T) = 0.144$ $\frac{\mathrm{d}}{\mathrm{d}t}x(T) = -0.756$
$\alpha = 1$ $a = 0.04$ $b = 0.2$	$f_{\max}(1.330) = 0.907$ $x_{\max}(1.490) = 0.890$	$T = 2.981$ $x(T) = 0.126$ $\frac{\mathrm{d}}{\mathrm{d}t}x(T) = -0.792$
$\alpha = 1$ $a = 0.08$ $b = 0.2$	$f_{\max}(1.388) = 0.933$ $x_{\max}(1.507) = 0.918$	$T = 3.013$ $x(T) = 0.100$ $\frac{\mathrm{d}}{\mathrm{d}t}x(T) = -0.842$
$\alpha = 0.95$ $a = 0.01$ $b = 0.2$	$f_{\max}(1.289) = 0.899$ $x_{\max}(1.475) = 0.870$	$T = 2.945$ $x(T) = 0.144$ $x^{(1)} = -0.762$
$\alpha = 0.95$ $a = 0.01$ $b = 0.011$	$f_{\max}(1.570) = 0.999$ $x_{\max}(1.571) = 0.998$	$T = 3.141$ $x = 0$ $x^{(1)} = -0.997$
$\alpha = 0.49$ $a = 5 \times 10^{-8}$ $\tau_{x\alpha} = 0.886$	$f_{\max}(0.832) = 1.140$ $x_{\max}(1.110) = 0.642$	$T = 2.151$ $x = 0.232$ $x^{(1)} = -0.596$
$\alpha = 0.23$ $\tau_{f\alpha} = 0.004$ $b = 1.183$	$f_{\max}(0.910) = 1.356$ $x_{\max}(1.034) = 0.632$	$T = 2.025$ $x = 0.135$ $\frac{\mathrm{d}}{\mathrm{d}t}x(T) = -0.771$

Table 5.1. *Duration of impact T, maximal values of x and f*

Finally, hysteresis diagrams corresponding to solutions presented in Figure 5.3 are shown in Figure 5.4.

Note that when compared to the standard linear viscoelastic solid, the solid described by fractional derivatives exhibits shorter duration of the impact, smaller maximal deformation and larger maximal force.

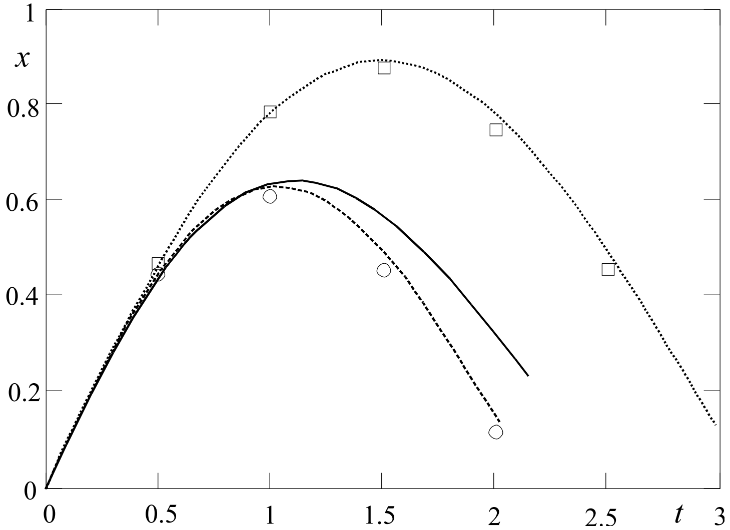

Figure 5.3. *Curves x for the standard linear solid $\alpha = 1$, $a = 0.04$, $b = 0.2$ (dotted line), fractional standard solid $\alpha = 0.49$, $a = 5 \times 10^{-8}$, $b = 0.886$ (solid line) and for $\alpha = 0.23$, $a = 0.004$, $b = 1.183$ (dashed line)*

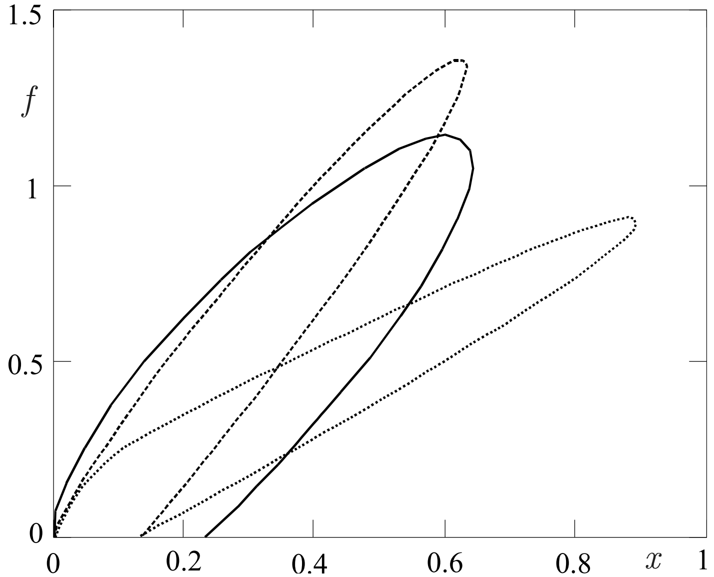

Figure 5.4. *Hysteresis diagrams for the standard linear solid $\alpha = 1$, $a = 0.04$, $b = 0.2$ (dotted line), fractional standard solid $\alpha = 0.49$, $a = 5 \times 10^{-8}$, $b = 0.886$ (solid line) and for $\alpha = 0.23$, $a = 0.004$, $b = 1.183$ (dashed line)*

From the values presented in Table 5.1, we conclude that for the case when $\alpha \to 1$, the solutions for the fractional standard linear solid are very close to the ones that describe the standard linear solid (for $\alpha = 1$ and the same values of the constants describing material). In other words, the solution is continuous with respect to the order of derivative. Also, when $b \to a$, there is no damping as expected, i.e. velocity after the rebound is almost of the same intensity as before the impact. In such a case, the presented values of T and $x_{\max}$ could be compared with the case of the nonlinear spring, as presented in [KIL 76]. Namely, in [KIL 76], the impact is modeled by the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} x(t) = -\rho x^{\frac{3}{2}}(t), \quad x(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} x(0) = 1.$$

For example, if we take $\rho = 1$, we obtain $T = 3.218$, $\frac{\mathrm{d}}{\mathrm{d}t} x(T) = -1$ and $x_{\max} = 1.093$. However, it should be noted that in handling the compliant impact problem by using nonlinear elastic spring, one methodological anomaly is encountered. For example, if the body is released from rest in a vertical plane and impinges the horizontal surface, it will bounce forever. In reality, this is not the case due to damping.

Our final remark concerns the thermodynamical restrictions whose violation could pose severe problems. Roughly speaking, putting $a > b$ will lead to the rebound velocity which is higher than the approaching velocity, i.e. $\left|\frac{\mathrm{d}}{\mathrm{d}t} x(T)\right| > \frac{\mathrm{d}}{\mathrm{d}t} x(0) = 1$. In the former example, if the body falls a distance h, it will bounce to the distances higher than h and also bounce forever.

5.2. Rigid block with viscoelastic rod attached slides in the presence of dry friction

Experiments show that after impact, the block will either separate from or remain in contact with the wall. Both cases are realized in a finite time. The problem discussed in the previous section describes only the rebound script. However, during collisions followed by a capture of the colliding bodies, large amounts of energy may be dissipated in a very short period of time; so, the dissipation in the viscoelastic rod is not enough to cease the motion. Therefore, extra constitutive laws are necessary to account for that energy loss as a whole. In order to predict the capture behavior, problem [5.6], [5.7] can be complemented by a set valued force law describing dry friction between the block and the horizontal surface (see [GLO 01]). This leads to an approach that will merge fractional calculus with non-smooth dynamics (see [GRA 12b, GRA 12a]).

Consider a rigid block of mass m, with a light rod attached to it. The rod is described by the fractional Zener model of a viscoelastic body

$$\sigma(t) + \tau_\sigma \, {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha \sigma(t) = E\left(\varepsilon(t) + \tau_\varepsilon \, {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha \varepsilon(t)\right), \quad t > 0, \qquad [5.39]$$

where σ and ε describe the stress and strain in the uniaxial, isothermal deformation process of the rod, $\alpha \in (0,1]$ is the order of the derivative, $E > 0$ is the modulus of elasticity, while τ_ε and $\tau_\sigma$are the relaxation constants satisfying the restrictions $\tau_\varepsilon > \tau_\sigma > 0$. Having only four parameters $(E, \tau_\sigma, \tau_\varepsilon, \alpha)$, the fractional Zener model is often used in various applications because of its good prediction of viscoelastic behavior for a large class of viscoelastic materials (see [BAG 86, GRA 10]). Thus, the first type of energy dissipation is due to the deformation of the rod and its material properties, and is given in form [5.39].

Sliding in the presence of dry friction represents the second strictly dissipative process. A very tractable model of dry friction is the Coulomb friction law given in a set-valued (or multivalued) form

$$F_{fx}(\dot{x}) \in -\mu N \,\mathrm{Sgn}(\dot{x}), \qquad [5.40]$$

where F_{fx} and $\dot{x}$ are projections on the x-axis of the friction force and velocity, respectively, while μ and N denote the friction coefficient and the normal projection of the contact force. In the set valued force law [5.40] $\mathrm{Sign}(\cdot)$ stands for a maximal monotone set-valued map (the filled-in relay function) defined as

$$\mathrm{Sign}(\dot{x}) = \begin{cases} \{1\}, & \dot{x} > 0, \\ [-1,1], & \dot{x} = 0, \\ \{-1\}, & \dot{x} < 0. \end{cases} \qquad [5.41]$$

Note that the set valued function or multifunction is a concept taken from non-smooth mathematical analysis representing a function almost everywhere except at a finite number of isolated points where it forms a subset of $\mathbb{R}$. Also, note that coefficients of friction for impact phenomena cannot be accurately determined, and consequently, their specification rests upon either pure hypothesis or corresponding values for non-collision processes.

In [5.40] three different scenarios that can occur for planar contact problems are recognized: sticking $(\dot{x} = 0 \Longrightarrow |F_{fx}| < \mu N)$, positive sliding $(\dot{x} > 0 \Rightarrow F_{fx} = -\mu N)$ and negative sliding $(\dot{x} < 0 \Rightarrow F_{fx} = \mu N)$. Small friction leads to a sequence of states of sliding with alternating sign of the velocity $\dot{x}$. As expected, by increasing μ the system tends to the state of rest. The sticking case when motion ceases after the impact motivates the adopted model with the friction force F_{fx} taken as the multifunction. Note that F_{fx} may jump to any point within the set $[-\mu N, \mu N]$, when transition to stiction occurs, while there can still be a friction force that balances some other external forces acting on the block (see [GLO 01]). We intuitively understand that, for the case when the block comes to rest in a finite time after the impact against the rigid wall, the viscoelastic rod is captured between the wall and the asperities of the dry surface below the block, and thus, the impact is followed by some kind of stress relaxation due to the compression of the rod. We do not discuss this relaxation process here because the viscoelastic rod is assumed to be light.

As an extension to [5.3], the equation of motion of the block with friction model [5.40] in contact with the rigid wall reads as

$$m\ddot{x}(t) = F_{fx}(\dot{x}(t)) - P(t), \quad t > 0,$$

where $P = P(t)$ denotes the contact force between the block and the wall. This equation of motion together with the friction model [5.40] can be regarded as a differential inclusion

$$m\ddot{x}(t) + P(t) \in -\mu mg\,\mathrm{Sgn}(\dot{x}(t)), \quad t > 0. \qquad [5.42]$$

The concept of differential inclusion has arisen by considering differential equations with discontinuous right-hand sides (see [LEI 04, SMI 02]). In fractional systems, it also possesses an essential mathematical interest because we have to deal with non-local operators and non-smooth functions simultaneously.

Assuming that the viscoelastic rod is of length L and denoting its cross-section by A, we note that $P = \sigma A$ and $x = \varepsilon L$; so, from [5.39] the constitutive equation for the light rod, i.e. the fractional Zener model, is the fractional differential equation

$$P(t) + \tau_\sigma\, {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha P(t) = \frac{EA}{L}\left(x(t) + \tau_\varepsilon\, {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha x(t)\right), \quad t > 0. \qquad [5.43]$$

Systems [5.42] and [5.43] are subject to initial conditions

$$x(0) = 0, \quad \dot{x}(0) = v_0, \quad P(0) = 0. \qquad [5.44]$$

The model of the impact problem given by differential inclusion [5.42], and constitutive equation [5.43] belongs to a class of set-valued (or multivalued) fractional differential equations, see [EL 95, EL 01, GRA 12a]. By use of the Laplace transform, the problem can be transformed into the equivalent form in terms of the Cauchy problem for integro-differential inclusions at either coordinate or velocity level (see [GRA 12a, GRA 12b]). The latter form is more tractable for mathematical analysis because the result ensuring the contractible solution set exists, see [NAR 01]. In order to find the solution the problem was treated numerically *ab initio*; so, the algorithm similar to [5.34], [5.35] was applied together with a slack variable algorithm used for handling discontinuous model motion phases, see [GRA 12a, GRA 12b] for details. The slack variable algorithm is used due to its advantages with respect to other methods by which the time instant when the block changes the motion phase can be determined, see [TUR 01]. In a system treated in [GRA 12a], three different impact scripts, rebound after the impact, capture in the approaching phase, and capture in the rebound phase, were confirmed. It should be noted that increasing the degree of freedom of the system will significantly increase the number of possible impact scripts. For example, as shown in [GRA 12b], by use of the combinatorial analysis in the system with two degrees of freedom with both fractional and dry friction type of dissipation, 11 imaginable scripts were identified while only 10 feasible scripts were confirmed numerically by a procedure that combines non-local and non-smooth effects. It was shown that one of the scripts contradicts restrictions that follow from the second law of thermodynamics and has to be excluded.

In the following, we intend to re-examine the problem of a rigid block impinging the rigid wall in the presence of dry friction.

5.2.1. *Solution of system [5.42]–[5.44]*

Non-smooth system [5.42]–[5.44] includes different phases of the motion (approach and rebound phases) which are described by different dynamical models (see [TUR 01]). This is due to the fact that the friction force changes its direction, which is opposite to the direction of a velocity during the motion, as described by [5.40]. At the same time, systems [5.42]–[5.44] contain fractional derivatives as non-local (in time) operators. In general, systems which combine both non-smooth and non-local effects require special treatment (see [GRA 12a]). The dynamical models originating from non-smooth system [5.42]–[5.44] are as follows:

1) For the approach phase, systems [5.42]–[5.44] read

$$\begin{gathered}\ddot{x}_a(t) = -p_a(t) - \mu, \quad t \in (0,T], \\ p_a(t) + a\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} p_a(t) = x_a(t) + b\,{}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_a(t), \quad t \in (0,T], \\ x_a(0) = 0, \quad \dot{x}_a(0) = \xi, \quad p_a(0) = 0.\end{gathered} \qquad [5.45]$$

2) For the rebound phase, systems [5.42]–[5.44] read

$$\begin{gathered}\ddot{x}_r(t) = -p_r(t) + \mu, \quad t \in (T, T_f], \\ p_r(t) + a\, {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} p_r(t) = x_r(t) + b\, {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r(t), \quad t \in (T, T_f], \\ x_r(T) = x_T, \quad \dot{x}_r(T) = 0, \quad p_r(T) = p_T.\end{gathered} \qquad [5.46]$$

Note that in system [5.46], the history is taken into account from the initial time instant $t = 0$ due to the terms containing fractional derivatives, while the system itself has to be solved for $t \in (T, T_f]$. This fact represents the additional difficulty in the analysis, which is the consequence of the combination of both non-smooth and non-local effects.

Systems [5.45] and [5.46] are obtained by introducing dimensionless variables

$$\begin{gathered}\bar{x}_i = x_i \frac{EA}{mgL}, \quad \bar{t} = t\sqrt{\frac{EA}{mL}}, \quad p_i = \frac{P_{a,r}}{mg}, \quad \xi = \frac{v_0}{g}\sqrt{\frac{EA}{mL}}, \\ a = \tau_\sigma \left(\sqrt{\frac{EA}{mL}}\right)^{\alpha}, \quad b = \tau_\varepsilon \left(\sqrt{\frac{EA}{mL}}\right)^{\alpha},\end{gathered}$$

where $i = a, r$ (a (r) stands for approach (rebound)) and omitting the bar for the sake of simplicity. We should note that the friction coefficient μ is equivalent to the (dimensionless) friction force during the motion phases. Dimensionless time instant T (T_f) denotes the moment when approach (rebound) phase ends. The approach phase ends when the body stops, i.e. when $\dot{x}_a(T) = 0$. The initial conditions for the rebound phase x_T, p_T are determined from the approach phase as $x_T = x_a(T), p_T = p_a(T)$. For $t > T$, the body may stay at rest (if $p_T \leq \mu$) or it may move in the opposite direction ($p_T > \mu$). The rebound phase ends (at time instant T_f) either when the body separates from the wall or when it stops while still being in contact with the wall. Mathematically, this reads either as $p_r(T_f) = 0$, or $\dot{x}_r(T_f) = 0$, i.e. $T_f = \min\{T_1, T_2\}$, where T_1 and T_2 satisfy $p_r(T_1) = 0$ and $\dot{x}_r(T_2) = 0$.

5.2.1.1. *Approach phase*

We define x_a and p_a to be equal zero for $t > T$ in order to be able to apply the classical Laplace transform to [5.45]. Thus, we obtain

$$s^2 \tilde{x}_a(s) - \xi = -\tilde{p}_a(s) - \frac{\mu}{s}, \quad \operatorname{Re} s > 0, \qquad [5.47]$$

$$\tilde{p}_a(s) + a s^{\alpha} \tilde{p}_a(s) = \tilde{x}_a(s) + b s^{\alpha} \tilde{x}_a(s), \quad \operatorname{Re} s > 0, \qquad [5.48]$$

where we use the notation $\tilde{f}(s) = \mathcal{L}[f(t)](s)$. The velocity is $v_{ax} = \dot{x}_a$, so that

$$\tilde{v}_{ax}(s) = s\tilde{x}_a(s) - x_a(0) = s\tilde{x}_a(s), \quad \operatorname{Re} s > 0. \tag{5.49}$$

The solution of [5.47]–[5.49] with respect to $\tilde{x}_a$, $\tilde{v}_{ax}$ and $\tilde{p}_a$ reads

$$\begin{aligned} \tilde{x}_a(s) &= \tilde{X}(s)\left(\xi - \frac{\mu}{s}\right), \quad \tilde{v}_{ax}(s) = \xi\tilde{V}(s) - \mu\tilde{X}(s), \\ \tilde{p}_a(s) &= \tilde{P}(s)\left(\xi - \frac{\mu}{s}\right), \quad \operatorname{Re} s > 0, \end{aligned} \tag{5.50}$$

where

$$\begin{aligned} \tilde{X}(s) &= \frac{1}{s^2 + \frac{1+bs^\alpha}{1+as^\alpha}}, \\ \tilde{V}(s) &= s\tilde{X}(s), \quad \tilde{P}(s) = \frac{1+bs^\alpha}{1+as^\alpha}\tilde{X}(s), \quad \operatorname{Re} s > 0. \end{aligned} \tag{5.51}$$

Inverting the Laplace transform in [5.50] with [5.51], the solution for the approach phase, i.e. for the system [5.45], is

$$x_a(t) = \xi X(t) - \mu[H(t) * X(t)], \quad t \in [0, T], \tag{5.52}$$

$$v_{ax}(t) = \xi\dot{X}(t) - \mu X(t), \quad t \in [0, T], \tag{5.53}$$

$$p_a(t) = \xi P(t) - \mu[H(t) * P(t)], \quad t \in [0, T], \tag{5.54}$$

where H denotes the Heaviside function.

In order to obtain function X, we use the definition of the inverse Laplace transform

$$X(t) = \frac{1}{2\pi \mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \tilde{X}(s)\mathrm{e}^{st}\mathrm{d}s, \quad a \geq 0, \ t \geq 0, \tag{5.55}$$

where $\tilde{X}$ is given by $[5.51]_1$ and the integration in the complex plane along the contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_\varepsilon \cup \Gamma_3 \cup \Gamma_4 \cup \gamma_0$, shown in Figure 5.5. In order to be able to apply the Cauchy residues theorem, we establish the number of poles of $\tilde{X}$, i.e. the number of zeros of the function $\Psi_\alpha : \mathbb{C} \to \mathbb{C}$, defined by

$$\Psi_\alpha(s) = s^2 + \frac{1+bs^\alpha}{1+as^\alpha}. \tag{5.56}$$

Using the argument principle, one shows that Ψ_α admits exactly two zeros s_0 and $\bar{s}_0$. They are complex-conjugate numbers located in the left complex half-plane and of multiplicity one. If the zero s_0 of [5.56] is written as $s_0 = r\mathrm{e}^{\mathrm{i}\varphi}$, then r and φ are solutions of the system

$$r^2 \cos(2\varphi) = -\frac{bar^{2\alpha} + (a+b)\, r^\alpha \cos(\alpha\varphi) + 1}{(ar^\alpha)^2 + 2ar^\alpha \cos(\alpha\varphi) + 1},$$

$$r^2 \sin(2\varphi) = -\frac{(b-a)\, r^\alpha \sin(\alpha\varphi)}{(ar^\alpha)^2 + 2ar^\alpha \cos(\alpha\varphi) + 1}.$$

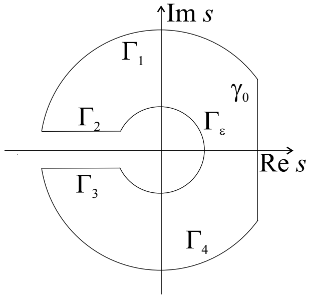

Figure 5.5. *Integration contour* Γ

The Cauchy residues theorem yields

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \tilde{X}(s)\mathrm{e}^{st}\mathrm{d}s = \mathrm{Res}(\tilde{X}, (s)\mathrm{e}^{st}, s_0) + \mathrm{Res}(\tilde{X}, (s)\mathrm{e}^{st}, \bar{s}_0). \qquad [5.57]$$

It can be proved (see, for example, [KON 10]), that in [5.57], when r tends to infinity and ε tends to zero, integrals along contours Γ_1, Γ_4 and Γ_ε vanish, while the integrals along contours Γ_2 and Γ_3 are non-vanishing and integral along γ_0 gives the inverse Laplace transform of $\tilde{X}$, [5.55]. Summarizing all the integral contributions and residues, we finally obtain

$$\begin{aligned} X(t) &= \frac{1}{2\pi\mathrm{i}} \int_0^\infty \left(\frac{1}{q^2 + \frac{1+bq^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+aq^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}} - \frac{1}{q^2 + \frac{1+bq^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+aq^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}} \right) \mathrm{e}^{-qt}\mathrm{d}q \\ &\quad + \left.\frac{\mathrm{e}^{st}}{2s + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+bs^\alpha}{1+as^\alpha}\right)}\right|_{s=s_0} + \left.\frac{\mathrm{e}^{st}}{2s + \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+bs^\alpha}{1+as^\alpha}\right)}\right|_{s=\bar{s}_0}, \end{aligned} \qquad [5.58]$$

Similar arguments and calculations are valid for the inversion of $\tilde{P}$, given by $[5.51]_3$, so that we have

$$
\begin{aligned}
P(t) &= \frac{1}{2\pi\mathrm{i}}\int_0^\infty \left(\frac{\frac{1+bq^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+aq^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}}{q^2+\frac{1+bq^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}{1+aq^\alpha \mathrm{e}^{-\mathrm{i}\alpha\pi}}} - \frac{\frac{1+bq^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+aq^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}}{q^2+\frac{1+bq^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}{1+aq^\alpha \mathrm{e}^{\mathrm{i}\alpha\pi}}}\right)\mathrm{e}^{-qt}\mathrm{d}q \\
&\quad + \left.\frac{\frac{1+bs^\alpha}{1+as^\alpha}\mathrm{e}^{st}}{2s+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+bs^\alpha}{1+as^\alpha}\right)}\right|_{s=s_0} + \left.\frac{\frac{1+bs^\alpha}{1+as^\alpha}\mathrm{e}^{st}}{2s+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1+bs^\alpha}{1+as^\alpha}\right)}\right|_{s=\bar{s}_0}. \qquad [5.59]
\end{aligned}
$$

We note that X and P are also (in a different form) given in [ATA 06].

Inverting the Laplace transform in $[5.51]_2$, we obtain $(\mathrm{Re}\, s > 0)$

$$\tilde{V}(s) = s\tilde{X}(s) = \mathcal{L}\left[\dot{X}(t)\right](s) + X(0) = \mathcal{L}\left[\dot{X}(t)\right](s),$$

i.e.

$$\dot{X} = V = \mathcal{L}^{-1}\left[\tilde{V}\right], \qquad [5.60]$$

where we, assuming that the stated limit exists, used the Tauberian theorem (see [DEB 07, theorem 3.8.1]) to obtain

$$X(0) = \lim_{s\to\infty}\left(s\tilde{X}(s)\right) = \lim_{s\to\infty}\frac{s}{s^2+\frac{1+bs^\alpha}{1+as^\alpha}} = 0.$$

Note that function P, given by [5.59], can be rewritten as

$$
\begin{aligned}
P(t) &= \mathcal{L}^{-1}\left[\frac{1+bs^\alpha}{1+as^\alpha}\right](t) * X(t) \\
&= \left(\frac{b}{a}\delta(t) + \left(\frac{b}{a}-1\right)\dot{e}_\alpha(t)\right) * X(t), \quad t\in[0,T],
\end{aligned}
$$

where δ is the Dirac delta distribution and

$$e_\alpha(t) = E_\alpha\left(-\frac{t^\alpha}{a}\right), \quad t \geq 0, \qquad [5.61]$$

with E_α being the one-parameter Mittag-Leffler function.

We note that in the case without the dry friction, functions X, $\dot{X}$ and P in [5.52]–[5.54] represent (up to the multiplication with the constant ξ) the coordinate, velocity of the body and contact force, respectively (see [ATA 06]).

5.2.1.2. *Rebound phase*

If $p_T \leq \mu$, then there will be no rebound phase at all, i.e. the body will maintain the state of rest reached at the end of the approach phase. On the contrary, if $p_T > \mu$, then rebound phase starts at the time instant $t = T$ and ends at $t = T_f$. The coordinate and the velocity at the end of the approach phase represent the initial conditions for the rebound phase, i.e. $x_a(T) = x_r(T)$, $v_{ax}(T) = v_{rx}(T)$. The material is viscoelastic and its response to the action of the force depends on the history of deformation, which should be taken into account from the initial time instant $t = 0$. Hence, the term ${}_0^{\mathrm{C}}\mathrm{D}_t^\alpha p_r$ is written as

$$\begin{aligned} {}_0^{\mathrm{C}}\mathrm{D}_t^\alpha p_r(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^T \frac{\dot{p}_a(u)}{(t-u)^\alpha}\mathrm{d}u + \frac{1}{\Gamma(1-\alpha)}\int_T^t \frac{\dot{p}_r(u)}{(t-u)^\alpha}\mathrm{d}u, \\ & \quad t \in [T, T_f]. \end{aligned} \qquad [5.62]$$

Introducing the memory function $\mathcal{P}$ as

$$\mathcal{P}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^T \frac{\dot{p}_a(u)}{(t-u)^\alpha}\mathrm{d}u, \quad t \in [T, T_f]$$

and noting that

$${}_T^{\mathrm{C}}\mathrm{D}_t^\alpha p_r(t) = \frac{1}{\Gamma(1-\alpha)}\int_T^t \frac{\dot{p}_r(u)}{(t-u)^\alpha}\mathrm{d}u, \quad t \in [T, T_f],$$

[5.62] becomes

$${}_0^{\mathrm{C}}\mathrm{D}_t^\alpha p_r(t) = \mathcal{P}(t) + {}_T^{\mathrm{C}}\mathrm{D}_t^\alpha p_r(t), \quad t \in [T, T_f]. \qquad [5.63]$$

Memory function $\mathcal{P}$ takes into account the influence of the history of the contact force from the approach phase to its values in the rebound phase. Furthermore, this history is not represented by a constant: at each time instant of the rebound phase, it affects the value of the contact force p_r differently. Note that the history of the contact force from the rebound phase is included by the term ${}_T^{\mathrm{C}}\mathrm{D}_t^\alpha p_r$ in [5.63].

Decomposition of the Caputo fractional derivative as in [5.63] is known as the time-varying initialization of the Caputo derivative, see [LOR 13] and references therein. In [LOR 13], the history function is referred to as the initialization function. Similarly as in [5.63], ${}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r$ reads

$$ {}_0^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r(t) = \mathcal{X}(t) + {}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r(t), \quad t \in [T, T_f], \tag{5.64} $$

where

$$ \mathcal{X}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{\dot{x}_a(u)}{(t-u)^{\alpha}} \mathrm{d}u, \quad t \in [T, T_f], $$
$$ {}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r(t) = \frac{1}{\Gamma(1-\alpha)} \int_T^t \frac{\dot{x}_r(u)}{(t-u)^{\alpha}} \mathrm{d}u, \quad t \in [T, T_f]. $$

Similar considerations as in the cases of $\mathcal{P}$ and ${}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} p_r$ are valid for $\mathcal{X}$ and ${}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r$.

Thus, system [5.46] becomes

$$ \begin{gathered} \ddot{x}_r(t) = -p_r(t) + \mu, \quad t \in (T, T_f], \\ p_r(t) + a\,{}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} p_r(t) = H_f(t) + x_r(t) + b\,{}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} x_r(t), \quad t \in (T, T_f], \\ x_r(T) = x_T, \quad \dot{x}_r(T) = 0, \quad p_r(T) = p_T, \end{gathered} \tag{5.65} $$

where

$$ H_f(t) = b\,\mathcal{X}(t) - a\,\mathcal{P}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{b\dot{x}_a(u) - a\dot{p}_a(u)}{(t-u)^{\alpha}} \mathrm{d}u, \quad t \in [T, T_f], $$

is the history function. According to the fractional Zener model, the function H_f combines the individual histories $\mathcal{P}$ and $\mathcal{X}$ of force and coordinate into a single (model dependent) history function.

In order to use the Laplace transform method to solve [5.46], i.e. [5.65], we introduce $\bar{t} = t - T$ so that $\bar{t} \in \left(0, \bar{T}_f\right)$. Transformations: $\bar{t} = t - T$ and $\bar{y} = y$ imply $\bar{y}(\bar{t}) = y(t) = y(\bar{t} + T) = \bar{y}(t - T)$. Under these transformations, we have $\dot{y}(t) = \dot{\bar{y}}(\bar{t})$, $\ddot{y}(t) = \ddot{\bar{y}}(\bar{t})$ and

$$ \begin{aligned} {}_T^{\mathrm{C}}\mathrm{D}_t^{\alpha} y(t) &= \frac{1}{\Gamma(1-\alpha)} \int_T^t \frac{\dot{y}(u)}{(t-u)^{\alpha}} \mathrm{d}u = \frac{1}{\Gamma(1-\alpha)} \int_T^{\bar{t}+T} \frac{\dot{\bar{y}}(u-T)}{(\bar{t}+T-u)^{\alpha}} \mathrm{d}u \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\bar{t}} \frac{\dot{\bar{y}}(\bar{u})}{(\bar{t}-\bar{u})^{\alpha}} \mathrm{d}\bar{u} = {}_0^{\mathrm{C}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{y}(\bar{t}). \end{aligned} \tag{5.66} $$

Now, [5.65] becomes

$$\ddot{\bar{x}}_r(\bar{t}) = -\bar{p}_r(\bar{t}) + \mu, \quad \bar{t} \in \left(0, \bar{T}_f\right],$$
$$\bar{p}_r(\bar{t}) + a\, {}_0^{\mathrm{C}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{p}_r(\bar{t}) = \bar{H}_f(\bar{t}) + \bar{x}_r(\bar{t}) + b\, {}_0^{\mathrm{C}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{x}_r(\bar{t}), \quad \bar{t} \in \left(0, \bar{T}_f\right], \tag{5.67}$$
$$\bar{x}_r(0) = x_T, \quad \dot{\bar{x}}_r(0) = 0, \quad \bar{p}_r(0) = p_T,$$

where

$$\bar{H}_f(\bar{t}) = H_f(\bar{t} + T) = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{b\dot{x}_a(u) - a\dot{p}_a(u)}{(\bar{t} + T - u)^{\alpha}} \mathrm{d}u, \quad \bar{t} \in \left[0, \bar{T}_f\right].$$

We put $\bar{x}_r$, $\bar{p}_r$, $\bar{H}_f$ to be equal to zero for $\bar{t} > \bar{T}_f$ in order to be able to apply the classical Laplace transform to [5.67]. Thus, we obtain ($\mathrm{Re}\, s > 0$)

$$s^2 \tilde{x}_r(s) - s x_T = -\tilde{p}_r(s) + \frac{\mu}{s},$$
$$\tilde{p}_r(s) + a\left(s^{\alpha} \tilde{p}_r(s) - s^{\alpha-1} p_T\right) = \tilde{H}_f(s) + \tilde{x}_r(s) + b\left(s^{\alpha} \tilde{x}_r(s) - s^{\alpha-1} x_T\right), \tag{5.68}$$

where we use the notation $\tilde{f}(s) = \mathcal{L}\left[\bar{f}(\bar{t})\right](s)$. Introducing

$$K_T = \frac{b}{a} x_T - p_T,$$

and knowing $\tilde{v}_{rx}(s) = s\tilde{x}_r(s) - x_T$, the solution of system [5.68] is ($\mathrm{Re}\, s > 0$)

$$\tilde{x}_r(s) = x_T \tilde{V}(s) + \mu \frac{1}{s} \tilde{X}(s) + \left(K_T \frac{s^{\alpha-1}}{s^{\alpha} + \frac{1}{a}} - \frac{\tilde{H}_f(s)}{1 + a s^{\alpha}}\right) \tilde{X}(s), \tag{5.69}$$

$$\tilde{v}_{rx}(s) = -x_T \tilde{P}(s) + \mu \tilde{X}(s) + \left(K_T \frac{s^{\alpha-1}}{s^{\alpha} + \frac{1}{a}} - \frac{\tilde{H}_f(s)}{1 + a s^{\alpha}}\right) \tilde{V}(s), \tag{5.70}$$

$$\tilde{p}_r(s) = x_T \tilde{Q}(s) + \mu \frac{1}{s} \tilde{P}(s) - \left(K_T \frac{s^{\alpha-1}}{s^{\alpha} + \frac{1}{a}} - \frac{\tilde{H}_f(s)}{1 + a s^{\alpha}}\right) \tilde{W}(s), \tag{5.71}$$

where $\tilde{X}$, $\tilde{V}$ and $\tilde{P}$ are given by [5.51] and

$$\tilde{Q}(s) = s\tilde{P}(s), \quad \tilde{W}(s) = s\tilde{V}(s). \tag{5.72}$$

Inverting the Laplace transform in [5.69]–[5.71], we obtain the solution to [5.67] in the form

$$\bar{x}_r(\bar{t}) = x_T \dot{X}(\bar{t}) + \mu [H(\bar{t}) * X(\bar{t})] \qquad [5.73]$$
$$+ \left[K_T e_\alpha(\bar{t}) + \bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t})\right] * X(\bar{t}), \quad \bar{t} \in \left[0, \bar{T}_f\right],$$
$$\bar{v}_{rx}(\bar{t}) = -x_T P(\bar{t}) + \mu X(\bar{t}) \qquad [5.74]$$
$$+ \left[K_T e_\alpha(\bar{t}) + \bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t})\right] * \dot{X}(\bar{t}), \quad \bar{t} \in \left[0, \bar{T}_f\right],$$
$$\bar{p}_r(\bar{t}) = x_T \dot{P}(\bar{t}) + \mu [H(\bar{t}) * P(\bar{t})] - K_T e_\alpha(\bar{t}) - \bar{H}_f(\bar{t})$$
$$- \left[K_T e_\alpha(\bar{t}) + \bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t})\right] * \ddot{X}(\bar{t}), \quad \bar{t} \in \left[0, \bar{T}_f\right], \qquad [5.75]$$

where X, P, $\dot{X}$ and e_α are given by [5.58]–[5.61], respectively. In [5.73]–[5.75], we have

$$\ddot{X} = \mathcal{L}^{-1}\left[\tilde{W}\right] - \delta \text{ and } \dot{P} = \mathcal{L}^{-1}\left[\tilde{Q}\right],$$

since from [5.60],

$$\tilde{W}(s) = s\tilde{V}(s) = \mathcal{L}\left[\dot{V}(t)\right](s) + V(0) = \mathcal{L}\left[\delta(t) + \ddot{X}(t)\right](s),$$

and, from [5.72],

$$\tilde{Q}(s) = s\tilde{P}(s) = \mathcal{L}\left[\dot{P}(t)\right](s) + P(0) = \mathcal{L}\left[\dot{P}(t)\right](s),$$

where we used the Tauberian theorem in order to obtain

$$V(0) = \lim_{s\to\infty}\left(s\tilde{V}(s)\right) = \lim_{s\to\infty}\frac{s^2}{s^2 + \frac{1+bs^\alpha}{1+as^\alpha}} = 1,$$
$$P(0) = \lim_{s\to\infty}\left(s\tilde{P}(s)\right) = \lim_{s\to\infty}\frac{s\frac{1+bs^\alpha}{1+as^\alpha}}{s^2 + \frac{1+bs^\alpha}{1+as^\alpha}} = 0.$$

Thus, the solution to system [5.46], according to [5.73]–[5.75] and $\bar{y}(\bar{t}) = y(t) = \bar{y}(t-T)$, yields

$$\begin{aligned}
x_r(t) &= x_T \dot{X}(t-T) + \mu\left[H(\bar{t}) * X(\bar{t})\right]_{\bar{t}=t-T} \\
&\quad + K_T\left[e_\alpha(\bar{t}) * X(\bar{t})\right]_{\bar{t}=t-T} + \left[\bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t}) * X(\bar{t})\right]_{\bar{t}=t-T}, \\
&\qquad t \in [T, T_f],
\end{aligned} \qquad [5.76]$$

$$\begin{aligned}
v_{rx}(t) &= -x_T P(t-T) + \mu X(t-T) \\
&\quad + K_T\left[e_\alpha(\bar{t}) * \dot{X}(\bar{t})\right]_{\bar{t}=t-T} + \left[\bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t}) * \dot{X}(\bar{t})\right]_{\bar{t}=t-T}, \\
&\qquad t \in [T, T_f],
\end{aligned} \qquad [5.77]$$

$$\begin{aligned}
p_r(t) &= x_T \dot{P}(t-T) + \mu\left[H(\bar{t}) * P(\bar{t})\right]_{\bar{t}=t-T} - K_T e_\alpha(t-T) \\
&\quad - \left[\bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t})\right]_{\bar{t}=t-T} - K_T\left[e_\alpha(\bar{t}) * \ddot{X}(\bar{t})\right]_{\bar{t}=t-T} \\
&\quad + \left[\bar{H}_f(\bar{t}) * \dot{e}_\alpha(\bar{t}) * \ddot{X}(\bar{t})\right]_{\bar{t}=t-T}, \quad t \in [T, T_f].
\end{aligned} \qquad [5.78]$$

5.2.2. *Numerical examples*

We compare x_a, v_{ax}, p_a as solutions to system [5.45], given by [5.52]–[5.54], and x_r, v_{rx}, p_r, as solutions to system [5.46], given by [5.76]–[5.78] with the ones obtained by using the numerical procedure *ab initio*, proposed in [GRA 12a]. The comparison of results is presented in Tables 5.2–5.4, where x, v_x, p stand for position, velocity and force, respectively, either in the approach or in rebound phase obtained by [5.52]–[5.54] and [5.76]–[5.78], while $x^{(n)}$, $v_x^{(n)}$, $p^{(n)}$ denote the same quantities calculated by the numerical procedure from [GRA 12a]. Also, the plots of x_a, v_{ax}, p_a, x_r, v_{rx}, p_r, obtained by evaluating analytical solutions [5.52]–[5.54] and [5.76]–[5.78] are shown in Figures 5.6–5.10. The plots for the approach phase and rebound phase (if it exists), are presented together in the same figure, because only jointly do they represent the plots of solutions to the dimensionless form of the original problem [5.42]–[5.44]. In order to do so, we fix parameters of the model: $\xi = 1$, $\alpha = 0.25$, $b = 1.2$, $a = 0.6$. By changing the values of the friction coefficient μ, we are able to recover all three possible scripts.

Script of separation is obtained for $\mu = 0.2$. From Table 5.2 and Figure 5.6, we see that at the time instant $T = 1.1852$, when the approach phase ends (velocity equals zero), the force $p(T) = 0.9625$ is greater than the (dimensionless) friction force $\mu = 0.2$; so, the body enters into the rebound phase. The rebound phase ends with separation of the body from the wall because at $T_f = 1.5443$, the force equals zero and velocity $v_x(T_f) = -0.5872$. Thus, the body separates from the wall with velocity $|v_x(T_f)|$, which is less than the initial velocity ξ due to the energy dissipation. Good agreement between the values of position, velocity and force obtained by the two mentioned methods can be seen in Table 5.2.

t	x	$x^{(n)}$	v_x	$v_x^{(n)}$	p	$p^{(n)}$
0.2	0.19396	0.19398	0.9300	0.9298	0.2906	0.2907
0.4	0.36844	0.36847	0.8067	0.8063	0.5351	0.5352
0.6	0.51367	0.51371	0.6392	0.6387	0.7312	0.7313
0.8	0.62185	0.62189	0.4379	0.4374	0.8710	0.8711
1.0	0.68739	0.68744	0.2149	0.2143	0.9486	0.9487
$T = 1.1852$	0.70734		2.50×10^{-16}		0.9625	
$T^{(n)} = 1.1847$		0.70738		0.58×10^{-16}		0.9627
1.5	0.67033	0.67039	-0.2313	-0.2315	0.889	0.890
2	0.47860	0.47867	-0.5114	-0.5116	0.599	0.600
2.5	0.1893	0.1894	-0.6113	-0.6114	0.190	0.191
$T_f = 2.7296$	0.05096		-0.5872		3.75×10^{-16}	
$T_f^{(n)} = 2.7309$		0.05022		-0.5869		1.36×10^{-16}

Table 5.2. *Script of separation. Comparison of x, v_x, p and $x^{(n)}$, $v_x^{(n)}$, $p^{(n)}$*

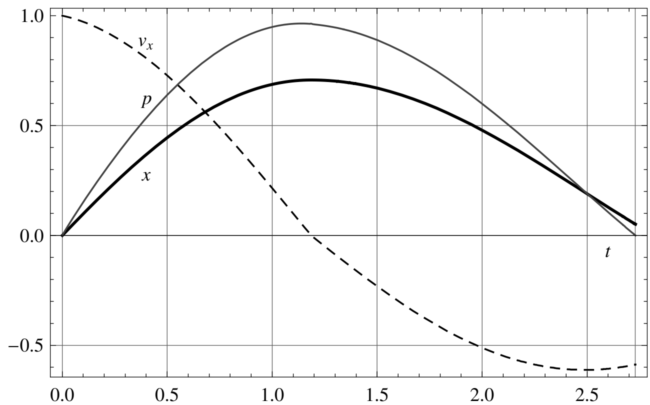

Figure 5.6. *Script of separation: position, velocity and force as functions of time*

Script of capture in the rebound phase is obtained for $\mu = 0.5$. Table 5.3 and Figure 5.7 correspond to the case of capture in the rebound phase because at $T = 0.9859$, as in the previous case, the approach phase ends with the force $p(T) = 0.7608 > \mu = 0.5$; so, the body rebounds, and at $T_f = 3.5849$, velocity $v_x(T_f) = 0.35 \times 10^{-17} \approx 0$, i.e. the body stops while still being in contact with the wall. This is observed by the fact that $p(T_f) = 0.2762 < \mu = 0.5$, which implies that the friction force equals $p(T_f)$ keeping the body at rest. This supports the fact that the presence of dry friction ensures that the body may stop at finite time. One also may note, see Table 5.3, the good agreement between x, v_x, p and $x^{(n)}$, $v_x^{(n)}$, $p^{(n)}$.

t	x	$x^{(n)}$	v_x	$v_x^{(n)}$	p	$p^{(n)}$
0.2	0.18799	0.18804	0.8706	0.8704	0.2815	0.2816
0.4	0.3449	0.3450	0.6914	0.6911	0.5001	0.5003
0.6	0.4621	0.4622	0.4747	0.4743	0.6558	0.6561
0.8	0.5332	0.5333	0.2337	0.2331	0.7432	0.7435
$T = 0.9859$	0.55495		4.44×10^{-16}		0.7608	
$T^{(n)} = 0.9854$		0.55511		0.22×10^{-16}		0.7611
1.0	0.5549	0.5551	−0.00375	−0.00381	0.759	0.760
1.5	0.5227	0.5229	−0.1194	−0.1195	0.691	0.692
2	0.4438	0.4439	−0.1862	−0.1864	0.568	0.569
2.5	0.34783	0.34786	−0.1860	−0.1861	0.430	0.431
3.0	0.26850	0.26845	−0.1222	−0.1223	0.3215	0.3221
3.5	0.2323	0.2321	−0.01894	−0.01893	0.2763	0.2769
$T_f = 3.5849$	0.23147		0.35×10^{-17}		0.2762	
$T_f^{(n)} = 3.5847$		0.23133		-1.23×10^{-17}		0.2767

Table 5.3. *Script of capture in rebound phase. Comparison of x, v_x, p and $x^{(n)}$, $v_x^{(n)}$, $p^{(n)}$*

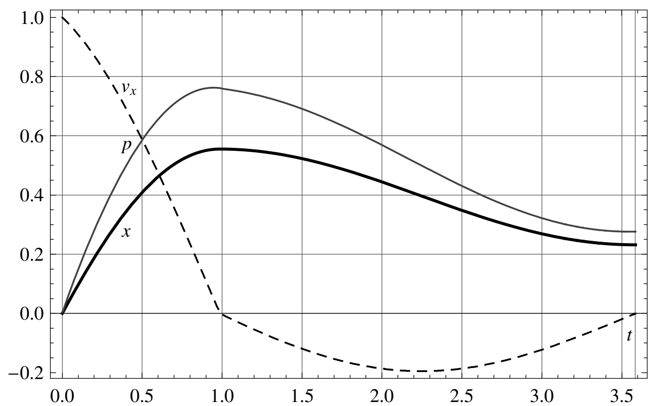

Figure 5.7. *Script of capture in rebound phase: position, velocity and force as functions of time*

Script of capture in approach phase is obtained for $\mu = 0.8$. Table 5.4 and Figure 5.8 show that at $T = 0.8208$, the velocity $v_x(T) = -0.11 \times 10^{-15} \approx 0$ and the force $p(T) = 0.6145 < \mu = 0.8$. This implies that the rebound phase will not occur since the friction force, being equal to $p(T)$, will keep the body at rest. The duration of impact T_f is, in this case, equal to the duration of the approach phase T. Again, from Table 5.4, we observe the good agreement between the results of the analytical and the numerical method.

t	x	$x^{(n)}$	v_x	$v_x^{(n)}$	p	$p^{(n)}$
0.2	0.1820	0.1821	0.8112	0.8110	0.2723	0.2725
0.4	0.3214	0.3215	0.5762	0.5759	0.4652	0.4654
0.6	0.4104	0.4106	0.3103	0.3099	0.5804	0.5808
0.8	0.4445	0.4448	0.0294	0.0288	0.6154	0.6159
$T_f = 0.8208$	0.4448		-0.11×10^{-15}		0.6145	
$T_f^{(n)} = 0.8204$		0.4451		2.85×10^{-15}		0.6149

Table 5.4. *Script of capture in approach phase. Comparison of x, v_x, p and $x^{(n)}$, $v_x^{(n)}$, $p^{(n)}$*

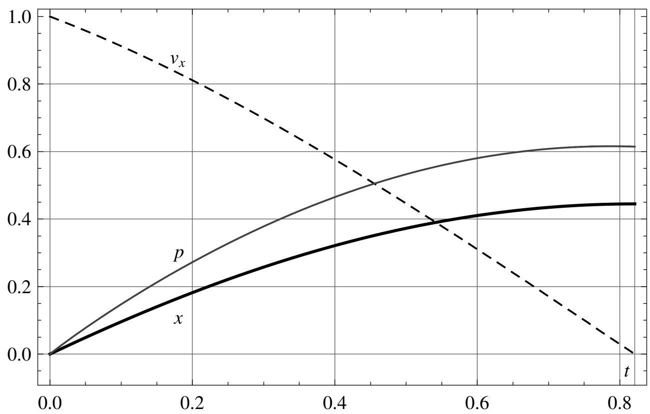

Figure 5.8. *Script of capture in the approach phase: position, velocity and force as functions of time*

By changing the value of friction coefficient (and keeping all other parameters of the model constant), all three possible scripts are recovered. Now, we investigate the influence of the order of the fractional derivative α on the motion of the body, i.e. we are interested whether, by the change of $\alpha \in (0, 1)$, we can obtain all three possible scripts. In order to do so, we fix $\xi = 1$, $b = 1.2$, $a = 0.6$, $\mu = 0.4$ and change α. For $\alpha = 0.1$, we obtain the script of separation as shown in Figure 5.9, while for $\alpha = 0.9$, the script of capture in rebound was recovered, as shown in Figure 5.10. Thus, we conclude that for the given set of parameters, it is not possible to obtain the script of capture in the approach phase by changing the value of α. We may conclude that the fractional derivative order α does not influence the energy dissipation as much as the friction coefficient μ does; so, the dry friction might be considered as the dominant mechanism of energy dissipation.

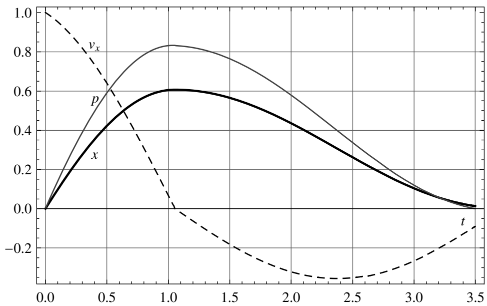

Figure 5.9. *Script of separation in the rebound phase: position, velocity and force as functions of time*

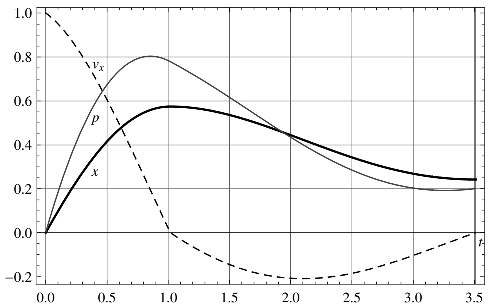

Figure 5.10. *Script of capture in the rebound phase: position, velocity and force as functions of time*

In order to recover the script of capture in the approach phase, we set $\xi = 0.9$, $b = 1.2$, $a = 0.6$, $\mu = 0.6$ and again change α. For $\alpha = 0.1$, we obtain the script of capture in the rebound phase, as shown in Figure 5.11, while for $\alpha = 0.9$, the script of capture in the approach was recovered, as shown in Figure 5.12. Again, for the given set of parameters, we could not obtain all three possible scripts. One may see in Figures 5.11 and 5.12, that for change of α in almost all of its range, the change of the value of contact force at the end of the approach phase is small: from $p(T) = 0.61$ (for $\alpha = 0.1$) to $p(T) = 0.59$ (for $\alpha = 0.9$). These values are very close to $\mu = 0.6$.

Thus, the large change in α (compared to its range) succeeded in changing the script, although the change in p is minor.

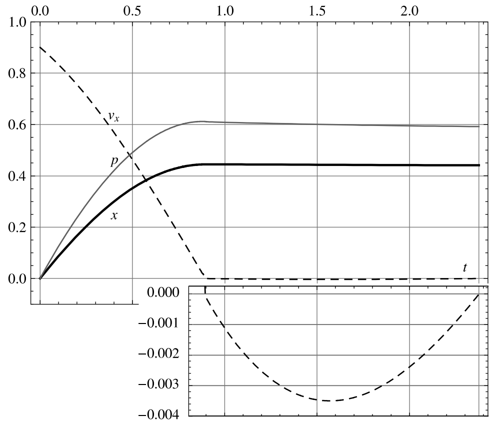

Figure 5.11. *Script of capture in the rebound phase: position, velocity and force as functions of time*

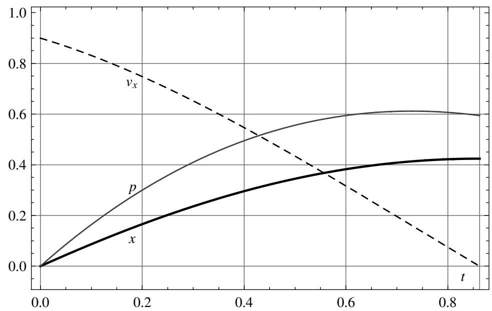

Figure 5.12. *Script of capture in the approach phase: position, velocity and force as functions of time*

Chapter 6

Variational Problems with Fractional Derivatives

In this chapter, we will present some results for variational problems in which the Lagrangian density involves derivatives of real (fractional) order, as well as a generalization of the classical Hamilton principle of mechanics in which Hamilton action integral is minimized within the specified set of functions and with respect to the order of the derivative, constant or variable. Hamilton's principle is one of the basic principles of physics. Anthony in [ANT 86] states: "In theoretical physics a theory is often considered to be complete if its variational principle in the sense of Hamilton is known". When Hamilton's principle is known, the whole information concerning processes of a particular system is included into its Lagrangian. It could be used to find conservation laws for the corresponding Euler–Lagrange equation and solutions via so-called direct methods of the variational calculus (see [VUJ 04], for example).

Fractional calculus of variations unifies the calculus of variations and the fractional calculus, by inserting fractional derivatives into variational problems. Riewe in [RIE 96, RIE 97] was the first who tackled the question of necessary conditions in the calculus of variations with fractional derivatives. We refer to [BAL 12a, MAL 12] for development in fractional variational principles.

Our presentation is mostly based on [ATA 14a, ATA 12a, ATA 10b, ATA 08a, ATA 09a, ATA 11c, STA 06, STA 09].

6.1. Euler–Lagrange equations

The classical variational problem consists of finding necessary and sufficient conditions for a minimum (maximum) of a functional of the type

$$J[u] = \int_A^B L\left(t, u(t), \frac{\mathrm{d}}{\mathrm{d}t}u\,(t)\right)\mathrm{d}t, \qquad [6.1]$$

where L is specified function and A and B are given and u belongs to a prescribed set of functions $\mathcal{U}$. Here, we will consider the Lagrangian L as a function of t, u and ${}_a\mathrm{D}_t^\alpha u$, i.e.

$$L = L(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t)).$$

The partial derivatives of L will be denoted by $\frac{\partial L}{\partial t}$, $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}$, or by $\partial_1 L$, $\partial_2 L$ and $\partial_3 L$, respectively. The first (or Lagrangian) variation will be denoted by δ, as usual. The case when the Lagrangian L is function dependent on ${}_t\mathrm{D}_b^\alpha u$, for some b, may be treated similarly.

Let (A, B) be a subinterval of (a, b). Consider the functional

$$J[u] = \int_A^B L\left(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t)\right)\mathrm{d}t, \quad 0 \le \alpha < 1, \ u \in \mathcal{U} \subset AC^1\left([a,b]\right), \qquad [6.2]$$

where L is a function in $(a, b) \times \mathbb{R} \times \mathbb{R}$ such that

$$\left.\begin{array}{l} L \in C^1((a,b) \times \mathbb{R} \times \mathbb{R}) \text{ and} \\ t \mapsto \partial_2 L(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t)) \text{ is integrable in } (a,b) \text{ and} \\ t \mapsto \partial_3 L(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t)) \in AC([a,b]), \text{ for every } u \in AC([a,b]). \end{array}\right\} \qquad [6.3]$$

Consider the problem of finding extremal values of functional [6.2] in the set of admissible functions $\mathcal{U}$, which we define later. Note that in [6.2], the constants a and A are assumed to be different in general and $a \le A$. In standard fractional variational problems, it is assumed that $A = a$ and $B = b$. The physical meaning of a and A is also different. Although the interval (A, B) defines Hamilton's action, the value a is related to a memory of the system. In the special case, which was treated previously, (see. [AGR 02a, AGR 06, BAL 04, FRE 07, MUS 05]), it was assumed that $a = A$.

The functional J is to be minimized (or maximized) over the set of admissible functions. Hence, we have to specify where we look for a minimum of [6.2]: the admissible set $\mathcal{U}$ will consist of all absolutely continuous functions u in $[a, b]$, which pass through a fixed point at a and b, i.e. $u(a) = a_0$, $u\,(b) = b_0$ for a fixed $a_0, b_0 \in \mathbb{R}$.

Let us discuss the constraints on $\mathcal{U}$ and L related to the Euler–Lagrange equations [6.2].

The case $A = a$ and $B = b$ was treated by Agrawal in [AGR 02a]. It was proved there that if we want to minimize [6.2] among all functions u, which have continuous left αth Riemann-Liouville fractional derivative and which satisfy the Dirichlet boundary conditions $u(a) = a_0$ and $u(b) = b_0$, for some real constant values a_0 and b_0, then a minimizer should be sought among all solutions of the Euler–Lagrange equation

$$\frac{\partial L}{\partial u} + {}_t\mathrm{D}_b^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) = 0. \qquad [6.4]$$

This result was modified in [AGR 06], where again $A = a$ was used, and the boundary condition was specified at $t = a$ only, which allowed the natural boundary conditions to be developed. The corresponding Euler–Lagrange equation was obtained as

$$\frac{\partial L}{\partial u} + {}_t^C\mathrm{D}_b^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) = 0, \qquad [6.5]$$

with the transversality (natural) condition

$$\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{I}_t^{1-\alpha} \delta u = 0, \ \text{ at } \ t = b. \qquad [6.6]$$

It is clear that [6.5] and [6.6] imply [6.4]. But the converse does not hold in general. This depends on assumptions on L and on the set of admissible functions. The following examples confirm it.

Consider a fractional variational problem of the form

$$J[u] = \int_0^1 L\left(t, u(t), {}_0\mathrm{D}_t^\alpha u\,(t)\right) \mathrm{d}t \to \min, \qquad [6.7]$$

with $u \in AC([0, 1])$ and L to be specified as follows.

1) Let the Lagrangian L satisfy [6.3] and have the form

$$L = f(t)_a\mathrm{D}_t^\alpha u\,(t) + F(t, u\,(t)).$$

Then $\partial_3 L = f(t)$, $t \in (0,1)$, and (see [AGR 06], equation [13])

$$\int_0^1 \left(\frac{\partial F}{\partial u} + {}_t^C\mathrm{D}_1^\alpha f \right) \delta u \, \mathrm{d}t + \left[f {}_0\mathrm{D}_t^{\alpha-1} \delta u \right]_0^1 = 0,$$

$$\int_0^1 \left(\frac{\partial F}{\partial u} + \frac{1}{\Gamma(1-\alpha)} \int_t^1 \frac{f^{(1)}(\theta)}{(\theta - t)^\alpha} \mathrm{d}\theta + \frac{f(1)}{\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha} \right) \delta u \, \mathrm{d}t = 0,$$

where we used [2.47]. Now, take, for example, $F(t,u) = \frac{u^2}{2\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha}$ and $f(t) = -1$, $t \in (0,1)$. Then, the Euler–Lagrange equation [6.4] gives

$$\frac{u}{\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{1}{(1-t)^\alpha},$$

thus $u \equiv 1$ in $[0,1]$. Hence, if we want to formulate [6.7] so that [6.4] holds if and only if [6.5] and [6.6] hold, then some additional assumptions on F have to be supposed. For instance, the condition $f(1) = 0$ provides the desired equivalence of [6.4] with [6.5] and [6.6].

2) Consider the Lagrangian

$$L = ({}_a\mathrm{D}_t^\alpha u - u)^2.$$

Solutions of the corresponding fractional variational problem [6.7] can be found directly. It is clear that the functional J achieves its minimum (which is zero) when ${}_a\mathrm{D}_t^\alpha u - u = 0$. Hence, the problem reduces to solving the equation ${}_a\mathrm{D}_t^\alpha u = u$. It was shown in [KIL 06, p. 222] that this equation has no solution, which is bounded at $t = 0$. In other words, we cannot solve the fractional variational problem with the Lagrangian given previously among the functions with the prescribed, finite boundary condition $u(0) = a_0$ at 0. If instead of the left Riemann–Liouville fractional derivative, we consider the Lagrangian L as a function of the left Caputo fractional derivative, then the equation ${}_a^C\mathrm{D}_t^\alpha u = u$ has a solution bounded at zero, which is also a solution of the corresponding fractional variational problem [6.7]. In [AGR 07], the symmetrized Caputo fractional derivative (Riesz-Caputo fractional derivative), defined as ${}^{R-C}_{\ \ a}\mathrm{D}_t^\alpha u := \frac{1}{2}({}_a^C\mathrm{D}_t^\alpha u - {}_t^C\mathrm{D}_b^\alpha u)$, was used in L. Similar kinds of fractional derivatives were used earlier in [KLI 01, LAZ 06].

These examples suggest that the Euler–Lagrange equation for the fractional variational problem [6.2] with $A = a$ and with the boundary condition specified at $t = a$ should be reformulated as follows:

$$\frac{\partial L}{\partial u} + {}_t^C\mathrm{D}_b^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) + \left. \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right|_{t=b} \frac{1}{\Gamma(1-\alpha)} \frac{1}{(b-t)^\alpha} = 0, \qquad [6.8]$$

instead of [6.5] and [6.6].

We present, in the following theorem, the Euler–Lagrange equation for [6.2].

THEOREM 6.1.– [ATA 08a]– Let $u^* \in AC((a,b))$ be an extremal of the functional J in [6.2], whose Lagrangian L satisfies [6.3]. Then, u^* satisfies the following Euler–Lagrange equations

$$\frac{\partial L}{\partial u} + {}_t^C\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) + \left. \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right|_{t=B} \frac{1}{\Gamma(1-\alpha)} \frac{1}{(B-t)^\alpha} = 0, t \in (A,B), \quad [6.9]$$

$${}_t\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) - {}_t\mathrm{D}_A^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) = 0, \quad t \in (a,A). \quad [6.10]$$

PROOF.– The first variation of [6.2] reads

$$\delta J[u] = \int_A^B \left(\frac{\partial L}{\partial u} \delta u + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \right) \mathrm{d}t. \quad [6.11]$$

From integration by parts formula

$$\int_a^B \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \, \mathrm{d}t = \int_a^B {}_t\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) \delta u \, \mathrm{d}t,$$

we obtain

$$\begin{aligned} \int_a^B \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \, \mathrm{d}t &= \int_A^B \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \, \mathrm{d}t + \int_a^A \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \, \mathrm{d}t \\ &= \int_A^B {}_t\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) \delta u \, \mathrm{d}t \\ &\quad + \int_a^A {}_t\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) \delta u \, \mathrm{d}t, \end{aligned}$$

thus

$$\begin{aligned} \int_A^B \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \, \mathrm{d}t &= \int_A^B {}_t\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) \delta u \, \mathrm{d}t \\ &\quad + \int_a^A {}_t\mathrm{D}_B^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) \delta u \, \mathrm{d}t \\ &\quad - \int_a^A \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} {}_a\mathrm{D}_t^\alpha \delta u \, \mathrm{d}t. \end{aligned}$$

By using this in [6.11], we obtain

$$\frac{\partial L}{\partial u} + {}_t\mathrm{D}_B^{\alpha}\left(\frac{\partial L}{\partial\ {}_a\mathrm{D}_t^{\alpha} u}\right) = 0, \quad t \in (A, B)$$

and

$$\int_a^A \left({}_t\mathrm{D}_B^{\alpha}\left(\frac{\partial L}{\partial\ {}_a\mathrm{D}_t^{\alpha} u}\right) - {}_t\mathrm{D}_A^{\alpha}\left(\frac{\partial L}{\partial\ {}_a\mathrm{D}_t^{\alpha} u}\right)\right)\delta u \,\mathrm{d}t = 0.$$

The claim now follows if we replace the right Riemann–Liouville by the right Caputo fractional derivative according to ${}_t\mathrm{D}_b^{\alpha} u\,(t) = {}_t^C\mathrm{D}_b^{\alpha} u\,(t) + \frac{1}{\Gamma(1-\alpha)}\frac{u(b)}{(b-t)^{\alpha}}$ in the first equation, and from the arbitrariness of δu in the second equation. ■

It is interesting to compare the Euler–Lagrange equations [6.8] and [6.9], [6.10] in the case $B = b$, when $A \to a$. Thus, if we let $A \to a$ in [6.9] and [6.10], we obtain the Euler–Lagrange equation [6.8] plus an additional condition

$${}_A\mathrm{I}_B^{1-\alpha}\left(\frac{\partial L}{\partial\ {}_a\mathrm{D}_t^{\alpha} u}\right) \equiv const.$$

Indeed, since

$$\begin{aligned}
&{}_t\mathrm{D}_B^{\alpha}(\partial_3 L) - {}_t\mathrm{D}_A^{\alpha}(\partial_3 L) = 0\\
&-\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_A^B \frac{\partial_3 L(\theta, u(\theta), {}_a\mathrm{D}_\theta^{\alpha} u\,(\theta))}{(\theta - t)^{\alpha}}\mathrm{d}\theta = 0, \quad t \in (a, A),
\end{aligned}$$

we obtain that ${}_A\mathrm{I}_B^{1-\alpha}(\partial_3 L) \equiv const.$

REMARK 6.1.– We note that [6.9] is equivalent to

$$\frac{\partial L}{\partial u} + {}_t\mathrm{D}_B^{\alpha}\left(\frac{\partial L}{\partial\ {}_a\mathrm{D}_t^{\alpha} u}\right) = 0, \quad t \in (A, B). \qquad [6.12]$$

If instead of [6.2] we have

$$J[u] = \int_A^B L\left(t, u(t), {}_a\mathrm{D}_t^{\alpha} u\,(t)\,, {}_t\mathrm{D}_b^{\alpha} u\,(t)\right)\mathrm{d}t,$$

with $a < A$ and $B < b$, then the optimality conditions read

$$\begin{aligned}
&\frac{\partial L}{\partial u} + {}_t\mathrm{D}_B^{\alpha}\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u}\right) + {}_A\mathrm{D}_t^{\alpha}\left(\frac{\partial L}{\partial\, {}_t\mathrm{D}_b^{\alpha} u}\right) = 0, \ \ t \in (A, B),\\
&{}_t\mathrm{D}_B^{\alpha}\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u}\right) - {}_t\mathrm{D}_A^{\alpha}\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u}\right) = 0, \ \ t \in (a, A),\\
&{}_A\mathrm{D}_t^{\alpha}\left(\frac{\partial L}{\partial\, {}_t\mathrm{D}_b^{\alpha} u}\right) - {}_B\mathrm{D}_t^{\alpha}\left(\frac{\partial L}{\partial\, {}_t\mathrm{D}_b^{\alpha} u}\right) = 0, \ \ t \in (B, b).
\end{aligned}$$

The sufficient condition for an extremum in the fractional calculus of variations may be formulated by using an appropriate convexity assumption on the Lagrangian. For such an analysis in the case when L depends on Caputo fractional derivatives, see [ALM 12, ALM 11]. Another important issue is the problem of finding necessary and sufficient conditions that guarantee that a given fractional-order differential equation is derivable from a (fractional) variational principle. This problem is still open. It is shown, however, in [DRE 03a, DRE 03b] that a necessary condition for constructing a fractional Lagrangian from the given Euler–Lagrange equation is that the Euler–Lagrange equation involves both left and right Riemann–Liouville fractional derivatives. If only one of them appears, then such an equation cannot be the Euler–Lagrange equation for some Lagrangian. For example, according to [DRE 03a, DRE 03b], it is not possible to construct a fractional Lagrangian for a linear oscillator with fractional derivative damping $u^{(2)} + u + {}_aD_t^{\alpha} u = 0$. Next, we present an example of construction of the variational principle for the given fractional differential equation.

Consider [4.80] in [ATA 14b] with the boundary conditions

$$y(0) = y_0, \quad y(b) = y_1.$$

We formulate a variational principle for [4.80] in volume 1. Let $Y = AC^1([0, b])$ and let $Z = L^r(0, b)$, where $1 \leq r < \frac{1}{\alpha}$ and $0 < \alpha < 1$. Let $(y, \tilde{y}) \in Y \times Y$. Consider a functional $J : Y \times Y \to \mathbb{R}$ defined by

$$\begin{aligned}
J[y, \tilde{y}] \ \ = \ \ &\int_0^b \left(-g(t)\frac{\mathrm{d}}{\mathrm{d}t} y(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{y}(t)\right.\\
&\left. + \lambda\tilde{y}(t)\left(\int_0^2 \phi(\gamma)\ {}_0\mathrm{D}_t^{\gamma} y(t)\,\mathrm{d}\gamma - f(t, y(t))\right)\right)\mathrm{d}t. \qquad [6.13]
\end{aligned}$$

Then, we have the following proposition.

PROPOSITION 6.1.– Functional [6.13] is stationary at the point $(y^*, \tilde{y}^*) \in \mathcal{U} = Y \times Y$, where y^* satisfies [4.80] [ATA 14b] and $\tilde{y}^*$ satisfies an additional equation, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\frac{\mathrm{d}}{\mathrm{d}t}y^*(t)\right) + \lambda\int_0^2 \phi(\gamma)\ {}_a\mathrm{D}_t^\gamma y^*(t)\,\mathrm{d}\gamma = f(t, y^*(t)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{y}^*(t)\right) + \lambda\int_0^2 \phi(\gamma)\ {}_t\mathrm{D}_b^\gamma \tilde{y}^*(t)\,\mathrm{d}\gamma = \frac{\partial}{\partial y^*}f(t, y^*(t)). \qquad [6.14]$$

PROOF.– To obtain stationarity conditions for [6.13], we assume that point $(y^*, \tilde{y}^*) \in Y \times Y$ is a critical point of functional [6.13]. Let $y(t) = y^*(t) + \varepsilon_1\eta(t)$, $\tilde{y}(t) = \tilde{y}^*(t) + \varepsilon_2\xi(t)$, $t \in [0, b]$. By using $\left.\frac{\partial J}{\partial \varepsilon_1}\right|_{\varepsilon_1=0,\varepsilon_2=0} = 0$, $\left.\frac{\partial J}{\partial \varepsilon_2}\right|_{\varepsilon_1=0,\varepsilon_2=0} = 0$, and the boundary conditions $\eta(0) = \eta(b) = 0$, $\xi(0) = \xi(b) = 0$, we obtain

$$\int_0^b \Bigg(g(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{y}^*(t)\frac{\mathrm{d}}{\mathrm{d}t}\eta(t)$$
$$+ \lambda\tilde{y}^*(t)\left(\int_0^2 \phi(\gamma)\ {}_a\mathrm{D}_t^\gamma \eta(t)\,\mathrm{d}\gamma - \eta(t)\frac{\partial}{\partial y^*}f(t, y^*(t))\right)\Bigg)\mathrm{d}t = 0,$$
$$\int_0^b \Bigg(g(t)\frac{\mathrm{d}}{\mathrm{d}t}y^*(t)\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)$$
$$+ \lambda\xi(t)\int_0^2 \phi(\gamma)\ {}_a\mathrm{D}_t^\gamma y^*(t)\,\mathrm{d}\gamma - \xi(t) f(t, y(t))\Bigg)\mathrm{d}t = 0. \qquad [6.15]$$

Using the integration by parts formula [2.22] and the fact that η and ξ are arbitrary and satisfy $\eta(0) = \xi(0) = \eta(b) = \xi(b) = 0$, we conclude that [6.14] follows from [6.15]. ■

REMARK 6.2.– In the special case when $\phi(t) = \delta(t - \gamma)$, $t \in (0, b)$, functional [6.13] becomes

$$J[y, \tilde{y}] = \int_0^b \left(-g(t)\frac{\mathrm{d}}{\mathrm{d}t}y(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{y}(t) + \tilde{\lambda}(t)\left({}_0\mathrm{D}_t^\gamma y(t) - f(t, y(t))\right)\right)\mathrm{d}t, \quad [6.16]$$

so that stationarity conditions [6.14] read as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\frac{\mathrm{d}}{\mathrm{d}t}y(t)\right) + \lambda\,{}_0\mathrm{D}_t^\gamma y(t) = f(t, y(t)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(g(t)\frac{\mathrm{d}}{\mathrm{d}t}\tilde{y}(t)\right) + \lambda\,{}_t\mathrm{D}_b^\gamma \tilde{y}(t) = \frac{\partial}{\partial y}f(t, y(t)), \qquad [6.17]$$

subject to $y(0) = y(b) = \tilde{y}(0) = \tilde{y}(b) = 0$. Functional [6.16] is different from that used in [DRE 03b, equation (58)]. As a result, our equation $[6.17]_2$ is different from equation [63] in [DRE 03b].

6.2. Linear fractional differential equations as Euler–Lagrange equations

For a motion of a particle in a fractal medium, the Lagrangian L has been proposed in [REK 04] as

$$L = \frac{m}{2}\left({}_a\mathrm{D}_t^\alpha y(t)\right)^2 - U(t, y(t)), \qquad [6.18]$$

where m is the mass (usually assumed to be a constant) and U is the potential energy of the particle. With [6.18], equation [6.4] becomes

$$m\,{}_t\mathrm{D}_b^\alpha\left({}_a\mathrm{D}_t^\alpha y\right) - \frac{\partial U}{\partial y} = 0. \qquad [6.19]$$

If we take $m = 1$ and $U(t, y) = \frac{\lambda y^2}{2}$, then [6.19] becomes differential equation of a fractional oscillator, treated in [BAL 08].

Another special case is obtained if we assume that the Lagrangian has the form

$$L = \frac{a_0}{2}\left({}_a\mathrm{D}_t^\alpha y(t)\right)^2 + a_1 y(t)\,{}_a\mathrm{D}_t^\alpha y(t) + a_2 y(t)\,{}_t\mathrm{D}_b^\alpha y(t) + \frac{a_3}{2}y^2(t) + y(t)\,f(t),$$

where a_0, a_1, a_2 and a_3 are constants and f is a function with appropriate properties. Then, the Euler–Lagrange equation [6.4] reads

$$a_0\,{}_t\mathrm{D}_b^\alpha\left({}_a\mathrm{D}_t^\alpha y(t)\right) + a_1\,{}_t\mathrm{D}_b^\alpha y(t) + a_2\,{}_a\mathrm{D}_t^\alpha y(t) + a_3 y(t) + f(t) = 0. \qquad [6.20]$$

Equation [6.20] with prescribed boundary conditions must be solved in order to obtain the minimizing function. Next,we review next some results concerning [6.20].

–Case I

Let $a_0 = 0$, $a_1 = \mu$, $a_2 = 1$, $a_3 = \eta$, $a = -\infty$, $b = \infty$, $0 < \alpha < 1$, so that [6.20] becomes

$${}_{-\infty}\mathrm{D}_t^\alpha y(t) + \mu\,{}_t\mathrm{D}_\infty^\alpha y(t) + \eta y(t) + f(t) = 0, \quad t \in \mathbb{R}. \qquad [6.21]$$

Equation [6.21] with $\eta = 0$ may be connected with the generalized Abel integral equation treated and solved in a different way in [SAM 93, p. 625–626].

We follow [STA 09] and present several results for [6.21]. Suppose that $Y, f \in L^p(\mathbb{R})$, $1 \leq p < \frac{1}{1-\alpha}$. We associate with equation [6.21] the following equation in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$:

$$
{}_{-\infty}\mathrm{D}_t^{\alpha} Y(t) + \mu\, {}_t\mathrm{D}_{\infty}^{\alpha} Y(t) + \eta Y(t) + f(t) = 0. \qquad [6.22]
$$

Using the Fourier transform, we can find solutions to [6.22].

PROPOSITION 6.2.– [STA 09]–

1) Let $\frac{1}{2} < \alpha < 1$, $\eta \neq 0$ and let $\mu \neq 1$, or $\mu = 1$ and $\eta > 0$. If $f \in L^1(\mathbb{R})$, then equation [6.22] has a solution $Y \in L^2(\mathbb{R})$. This solution has the Fourier transform

$$
\hat{Y}(\omega) = -\frac{1}{|\omega|^{\alpha}\left((\mu+1)\cos\frac{\alpha\pi}{2} + \mathrm{i}(\mu-1)\mathrm{sgn}\omega\sin\frac{\alpha\pi}{2}\right) + \eta}\hat{f}(\omega), \quad \omega \in \mathbb{R}, \qquad [6.23]
$$

and it can be obtained by finding the inverse Fourier transform of [6.23].

2) Let $0 < \alpha < 1$, $\eta \neq 0$ and let $\mu \neq 0$, or $\mu = 1$ and $\eta > 0$. If $f \in L^2(\mathbb{R}$, then equation [6.22]) has a solution $Y \in L^2(\mathbb{R})$ of the form [6.23].

3) Let $0 < \alpha < \frac{1}{2}$, $\eta = 0$ and let $\mu \neq 1$, or $\mu = 1$ and $\eta > 0$.

a) If $f \in L^p(\mathbb{R})$, $p = \frac{2}{1+2\alpha}$, then equation [6.22] has a solution $Y \in L^2(\mathbb{R})$.

b) If $f \in L^p(\mathbb{R})$ and $1 < p < \frac{1}{\alpha}$, then equation [6.22] has a solution $Y \in L^q(\mathbb{R})$, $q = \frac{p}{1-\alpha p} > 1$.

4) Let $\frac{1}{2} < \alpha < 1$, $\eta \neq 0$ and let $\mu \neq 1$ or $\mu = 1$ and $\eta > 0$. If $f \in L^1(\mathbb{R})$, then equation [6.22] has a solution $y \in L^2(\mathbb{R})$. This solution can be calculated from [6.23].

REMARK 6.3.– Note that for the case $f = 0$, we have $\hat{Y} = 0$ and consequently $Y = 0$, almost everywhere (a.e).

Consider the special case of equation [6.21] with $\mu = -1$, $\eta = 0$, $f = 0$ and $0 < \alpha < 1$

$$
{}_{-\infty}\mathrm{D}_t^{\alpha} y(t) - {}_t\mathrm{D}_{\infty}^{\alpha} y(t) = 0, \quad t \in \mathbb{R}. \qquad [6.24]
$$

This is an equation of the type [6.22] with $\mu = -1, \eta = 0, f = 0$. Therefore, from [6.23] we conclude that $y = 0$ is the only solution to [6.24]. Thus, if $u(x,t)$, $x \in \mathbb{R}$, $t > 0$, is a displacement vector at a point x and time y of an infinite rod, then

$$\mathcal{E}^{\alpha} u(x,t) = \frac{1}{2}\left({}_{-\infty}\mathrm{D}_x^{\alpha} u(x,t) - {}_x\mathrm{D}_{\infty}^{\alpha} u(x,t)\right), \quad x \in \mathbb{R},\ t > 0,$$

may be used as a measure of deformation since $\mathcal{E}^{\alpha} u(x,t) = 0$ implies that $u(x,t) = g(t)$ with g being an arbitrary function of time t.

REMARK 6.4.– The solution to equation [6.24] with finite a and b, that is

$$ {}_a\mathrm{D}_t^{\alpha} u(t) - {}_t\mathrm{D}_b^{\alpha} u(t) = 0, \quad t \in \mathbb{R}, \qquad [6.25]$$

may be connected with Riesz fractional derivative. Namely, [6.25] is equivalent to

$$\frac{1}{\Gamma(1-\alpha)}\left(\int_a^t \frac{u(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau + \int_t^b \frac{u(\tau)}{(\tau-t)^{\alpha}}\mathrm{d}\tau\right) = C = const., \qquad [6.26]$$

or

$$\frac{1}{\Gamma(1-\alpha)}\int_a^b \frac{u(\tau)}{|t-\tau|^{\alpha}}\mathrm{d}\tau = C, \qquad [6.27]$$

which is a special case of [2.62]. If $n = 0$ in [2.63], we obtain that the solution to [6.25] is $u \neq 0$.

If, however, we use the Caputo derivative in [6.25], we have

$$\frac{1}{\Gamma(1-\alpha)}\int_a^b \frac{u^{(1)}(\tau)}{|t-\tau|^{\alpha}}\mathrm{d}\tau = 0. \qquad [6.28]$$

The only solution to [6.28] is $u(t) = const.$ (see [SAM 93], equation [30.84]).

–Case II

Let $a_0 = 1$, $a_1 = a_2 = 0$, $a_3 = \lambda$, $a = -\infty$, $b = \infty$, $0 < \alpha < 1$. Then, [6.20] reduces to

$$ {}_t\mathrm{D}_{\infty}^{\alpha}\left({}_{-\infty}\mathrm{D}_t^{\alpha} y(t)\right) + \lambda y(t) + f(t) = 0, \quad t \in \mathbb{R}. \qquad [6.29]$$

Equation [6.29] is treated in [BAL 08, KLI 09]. It may be interpreted as a model of fractional forced oscillator.

We need the following result.

PROPOSITION 6.3.– [STA 09]– Let $0 < \alpha < 1$. If ${}_t\mathrm{D}_{\infty}^{\alpha} y(t) = 0$, then $y(t) = 0$ a.e. on $\mathbb{R}$. Also, if ${}_t\mathrm{D}_{\infty}^{\alpha} y = F$, where $F \in L^p(\mathbb{R}), p \geq 1$, then

$$y(t) = {}_t\mathrm{I}_{\infty}^{\alpha} F(t), \text{ a.e.} \qquad [6.30]$$

We now state the proposition regarding equation [6.29].

PROPOSITION 6.4.– [STA 09]

1) Let $\lambda < 0$. If $f \in L^2(\mathbb{R})$, then equation [6.29] has a solution in $L^2(\mathbb{R})$. If $f \in L^1(\mathbb{R})$, and $\frac{1}{4} < \alpha < 1$, then equation [6.29] has a solution in $L^2(\mathbb{R})$.

2) Let $\lambda > 0$, $\frac{1}{4} < \alpha < 1$ and

$$f(t) = h(t) - \sqrt{\lambda}\left(\frac{t^{\alpha-1}}{2\Gamma(1-\alpha)\sin\frac{\alpha\pi}{2}} * h(t)\right), \quad t \in \mathbb{R},\ h \in L^2(\mathbb{R}).$$

Then

$$y(t) = \phi(t) * h(t), \quad \text{where } \phi(t) = \mathcal{F}^{-1}\left[\frac{1}{|\omega|^{\alpha} + \sqrt{\lambda}}\right](t), \quad t \in \mathbb{R}.$$

–*Case III*

Let $a_0 = 0$, $a_1 = a_2 = 1$, $a_3 = 0$, $f = -F$, $a = -\infty$, $b = \infty$, $0 < \alpha < 1$. Then, [6.20] becomes

$$_{-\infty}\mathrm{D}_t^{\alpha} y(t) + {}_t\mathrm{D}_{\infty}^{\alpha} y(t) = F(t), \quad t \in \mathbb{R}. \qquad [6.31]$$

From [6.31], we have

$$F(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{-\infty}^{t} \frac{y(\tau)}{(t-\tau)^{\alpha}}\mathrm{d}\tau - \int_{t}^{\infty} \frac{y(\tau)}{(\tau-t)^{\alpha}}\mathrm{d}\tau\right)$$

$$F(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \frac{y(\tau)}{|t-\tau|^{\alpha}\operatorname{sgn}(t-\tau)}\mathrm{d}\tau$$

$$F(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(u(t) * y(t)\right), \quad \text{with } u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\operatorname{sgn}(t)}{|t|^{\alpha}}, \quad t \in \mathbb{R}. \qquad [6.32]$$

Applying the Fourier transform $\mathcal{F}\left[F\left(t\right)\right]\left(\omega\right)=\hat{F}\left(\omega\right)=\int_{-\infty}^{\infty}F\left(t\right)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$, $\omega\in\mathbb{R}$, to [6.32], we obtain

$$\hat{y}\left(\omega\right)=\frac{\hat{F}\left(\omega\right)}{\mathrm{i}\omega\hat{u}\left(\omega\right)}=\frac{\left|\omega\right|^{-\alpha}}{2\cos\frac{\alpha\pi}{2}}\hat{F}\left(\omega\right),\quad\omega\in\mathbb{R},\tag{6.33}$$

since for $\alpha\in(0,1)$, we have $\hat{u}\left(\omega\right)=-2\mathrm{i}\cos\frac{\alpha\pi}{2}\frac{\mathrm{sgn}(\omega)}{|\omega|^{1-\alpha}}$, $\omega\in\mathbb{R}$.

Let $F=0$ in [6.31]. Then, it becomes

$${}_{-\infty}\mathrm{D}_{t}^{\alpha}y\left(t\right)+{}_{t}\mathrm{D}_{\infty}^{\alpha}y\left(t\right)=0,\quad t\in\mathbb{R}.\tag{6.34}$$

Equation [6.34] is the Euler–Lagrange equation for the Lagrangian $L=y_{-\infty}\mathrm{D}_{t}^{\alpha}y$. From [6.33], we conclude

$$\hat{y}=0,\quad\text{for}\quad0\leq\alpha<1,\quad\text{and}\quad\hat{y}\text{ arbitrary for, }\alpha=1.$$

–Case IV

Consider equation [6.20] with $a_0=1$, $a_1=a_2=0$, $a_3=-\lambda$, $f=0$, $0<\alpha<1$:

$${}_{t}\mathrm{D}_{b}^{\alpha}\left({}_{a}\mathrm{D}_{t}^{\alpha}y\left(t\right)\right)-\lambda y\left(t\right)=0,\quad t\in\left[a,b\right],\tag{6.35}$$

and assume $\frac{1}{2}<\alpha<1$. This case is extensively treated in [KLI 09].

PROPOSITION 6.5.– [KLI 09, theorem 8.11]– Equation [6.35] has a unique solution belonging to $C_{1-\alpha}\left([a,b]\right)$ [1] fulfilling the following boundary conditions

$$\left[{}_{a}\mathrm{I}_{t}^{1-\alpha}y\left(t\right)\right]_{t=a}=A_{-1},\qquad\left[{}_{t}\mathrm{I}_{b}^{1-\alpha}y\left(t\right)\right]_{t=b}=A_{1},$$

where

1) in the case when $A_{-1}\neq0$, we should add the condition

$$\left|\lambda\right|<\frac{\Gamma\left(\alpha+1\right)\Gamma\left(2\alpha\right)}{\left(b-a\right)^{2\alpha}\Gamma\left(\alpha\right)};$$

1 A function y belongs to $C_{\gamma}\left([a,b]\right)$ if $\left(t-a\right)^{\gamma}f\left(t\right)\in C\left([a,b]\right)$.

2) in the case when $A_{-1} = 0$, we should add the assumption

$$|\lambda| < \frac{1}{\|{}_a\mathrm{I}_t^\alpha\left({}_t\mathrm{I}_b^\alpha 1\right)\|} = \left((b-a)^{2\alpha}\left|\sum\nolimits_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(\alpha-k+1)\,\Gamma(\alpha+k+1)}\right|\right)^{-1}.$$

Then, the solution to [6.35] reads

$$y(t) = \frac{A_{-1}}{\Gamma(\alpha)}\mathcal{F}_\lambda^{a,-1}(t) + \frac{A_1}{\Gamma(\alpha)}\mathcal{F}_\lambda^{a,1}(t), \quad t \in [a,b], \qquad [6.36]$$

where

$$\mathcal{F}_\lambda^{a,-1}(t) = \lambda\sum_{m=0}^{\infty}\left({}_a\mathrm{I}_t^\alpha\ {}_t\mathrm{I}_b^\alpha\right)^m (t-a)^{\alpha-1} \quad \text{and}$$

$$\mathcal{F}_\lambda^{a,1}(t) = \lambda\sum_{m=0}^{\infty}\left({}_a\mathrm{I}_t^\alpha\ \ {}_t\mathrm{I}_b^\alpha\right)^m\left({}_a\mathrm{I}_t^\alpha (t-a)^{\alpha-1}\right).$$

Note that y given by [6.36] is a minimizer of the functional

$$J[y] = \int_a^b\left(\frac{1}{2}\left({}_a\mathrm{D}_t^\alpha y(t)\right)^2 - \frac{\lambda}{2}y^2(t)\right)\mathrm{d}t.$$

Let $\lambda = 0$ in [6.35], so that this equation becomes

$${}_t\mathrm{D}_b^\alpha\left({}_a\mathrm{D}_t^\alpha y(t)\right) = 0, \quad t \in [a,b].$$

For $0 < \alpha < 1$, its solution satisfying $y(0) = 0$, $y(1) = 1$ is given as

$$y(t) = \frac{1}{2\alpha-1}\int_0^t\frac{\mathrm{d}\tau}{\left[(1-\tau)(t-\tau)\right]^{1-\alpha}},$$

see [AGR 02a]. It is the minimizer of the functional

$$J[y] = \frac{1}{2}\int_a^b\left({}_a\mathrm{D}_t^\alpha y(t)\right)^2\mathrm{d}t, \quad y(0) = 0, \ \ y(1) = 1.$$

–Case V

Consider equation [6.20] with $a_0 = -1$, $a = a_1 = a_2 = a_3 = 0$, $0 < \alpha < 1$:

$$ {}_t\mathrm{D}_b^{\alpha}\left({}_0\mathrm{D}_t^{\alpha} y\left(t\right)\right) = f\left(t\right), \quad t \in \left(0, b\right). \qquad [6.37] $$

Regarding [6.37], we state the following result from [STA 06].

PROPOSITION 6.6.– Let $\alpha \in (0,1)$ and $f \in L^1(0,b)$. Then, the family of functions

$$ y\left(t\right) = {}_0\mathrm{I}_t^{\alpha}\left({}_t\mathrm{I}_b^{\alpha} f\left(t\right)\right) + C_1\, {}_0\mathrm{I}_t^{\alpha}\left(b-t\right)^{\alpha-1} + C_2 t^{\alpha-1}, \quad t \in \left(0, b\right), \qquad [6.38] $$

satisfies the equation [6.37] and belongs to $L^1(0,b)$.

If, in addition, $\alpha \in \left(\frac{1}{2}, 1\right)$ and function f has the properties:

1) $f\left(t\right) = t^{\gamma} l_1\left(t\right),\ t \in \left(0, \varepsilon\right),\ \varepsilon > 0$;

2) $f\left(b-t\right) = t^{\beta} l_2\left(t\right),\ t \in \left(0, \varepsilon\right),\ \beta > -1$;

where $l_2\left(\frac{1}{t}\right)$ is quasi-monotone slowly varying at infinity, then there exists y_0 belonging to the family y, given by [6.38], which satisfies boundary conditions $y_0(0) = y_0(b) = 0$, with the properties:

1) $y_0 \in L^1(0,b)$;

2) $y_0\left(t\right) = Bt^{\alpha} + o\left(1\right),\ t \to 0^+;\ B = \frac{1}{\Gamma(\alpha)}\int_0^b \frac{f(\tau)}{\tau^{1-\alpha}}\mathrm{d}\tau + \frac{b^{\alpha-1}}{\Gamma(\alpha+1)}$;

3) $\lim_{t\to 0^+} y_0\left(b-t\right) = {}_0\mathrm{I}_t^{\alpha}\left({}_t\mathrm{I}_b^{\alpha} f\left(b\right)\right) + \frac{b^{2\alpha-1}}{\Gamma(\alpha)\Gamma(2\alpha-1)} = y_0\left(b\right)$;

4) $y_0\left(t\right) = {}_0\mathrm{I}_t^{\alpha}\left({}_t\mathrm{I}_b^{\alpha} f\left(t\right)\right) + C_1\, {}_0\mathrm{I}_t^{\alpha}\left(b-t\right)^{\alpha-1}$, where $C_1 = \frac{\left[{}_0\mathrm{I}_t^{\alpha}\left({}_t\mathrm{I}_b^{\alpha} f(t)\right)\right]_{t=b}}{\left[{}_0\mathrm{I}_t^{\alpha}(b-t)^{\alpha-1}\right]_{t=b}}$.

REMARK 6.5.– We note that [6.37] is the Euler–Lagrange equation for the Lagrangian

$$ L = \frac{1}{2}\left({}_0\mathrm{D}_t^{\alpha} y\left(t\right)\right)^2 - y\left(t\right) f\left(t\right). $$

6.3. Constrained variational principles

We consider the fractional isoperimetric problem. Suppose that we want to minimize the functional [6.2] with $a = A$, $b = B$ subject to an integral constraint

$$ \mathrm{Co}[u] = \int_a^b C(t, u(t), {}_a\mathrm{D}_t^{\alpha} u\left(t\right))\mathrm{d}t = c, \quad 0 \leq \alpha < 1,\ u \in \mathcal{U}, \qquad [6.39] $$

where $c \in \mathbb{R}$ is prescribed and $\mathcal{U}$ is an appropriate set of admissible functions. From [AGR 02a, ALM 09, ITO 08], we have the following theorem.

THEOREM 6.2.– Let $u^* \in AC([a,b])$ be an extremal of the functional J in [6.2], whose Lagrangian L satisfies [6.3] subject to constraint [6.39]. Then, there exist non-zero constants λ_0 and λ, such that u^* satisfies the Euler–Lagrange equation

$$\frac{\partial K}{\partial u} + {}_t^C\mathrm{D}_B^{\alpha}\left(\frac{\partial K}{\partial\, {}_a\mathrm{D}_t^{\alpha}u}\right) = 0, \quad t \in (a,b), \tag{6.40}$$

where $K = \lambda_0 L + \lambda C$. If u^* is not an extremal of [6.39], then we can choose $\lambda_0 = 1$.

This theorem can be easily generalized so as to include both left and right fractional derivatives in L and C.

Next, we consider the problem of minimizing [6.2] with differential constraints. Thus, we treat fractional optimal control problems. Following [AGR 04], we present a heuristic derivation of the optimality conditions. Our aim is to minimize

$$J[u] = \int_a^b L\left(t, x(t), u\left(t\right)\right) \mathrm{d}t, \quad x \in \mathcal{U}, \; u \in \mathcal{U}_0, \tag{6.41}$$

with the constraint

$${}_a\mathrm{D}_t^{\alpha}x\left(t\right) = G\left(t, x(t), u\left(t\right)\right), \qquad x\left(a\right) = x_0,$$

where $\alpha \in (0,1)$, x is the state variable in appropriately chosen $\mathcal{U}$ and u is the control variable. We assume that $u \in \mathcal{U}_0$, where $\mathcal{U}_0$ is the set of admissible controls. Formally, by using the method of Lagrange multipliers, we modify the optimality criteria [6.41] and consider the problem of minimizing

$$\bar{J}[x,u] = \int_a^b \left(L\left(t, x(t), u\left(t\right)\right) + \lambda\left(G\left(t, x(t), u\left(t\right)\right) - {}_a\mathrm{D}_t^{\alpha}x\left(t\right)\right)\right) \mathrm{d}t, \tag{6.42}$$

where λ is the Lagrange multiplier. By requiring that $\delta\bar{J}[u] = 0$, we obtain

$${}_a\mathrm{D}_t^{\alpha}x\left(t\right) = G\left(t, x(t), u\left(t\right)\right), \quad {}_t\mathrm{D}_b^{\alpha}\lambda\left(t\right) = \frac{\partial L}{\partial x} + \lambda\frac{\partial G}{\partial x}, \quad \frac{\partial L}{\partial u} + \lambda\frac{\partial G}{\partial u} = 0, \tag{6.43}$$

as the necessary conditions for optimality. If we introduce the Pontryagin function $\mathcal{H} = L + \lambda G$, then $[6.43]_{2,3}$ become

$${}_t\mathrm{D}_b^{\alpha}\lambda\left(t\right) = \frac{\partial \mathcal{H}}{\partial x}, \quad \frac{\partial \mathcal{H}}{\partial u} = 0.$$

The boundary condition on Lagrange multipliers is

$$\lambda(b) = 0. \qquad [6.44]$$

Now, we treat the problem of finding a minimum of a functional

$$J[x,u] = \int_0^1 F(t, x(t), u(t))\,\mathrm{d}t, \quad x \in \mathcal{U},\ u \in \mathcal{U}_0, \qquad [6.45]$$

with the differential constraint

$$x^{(1)}(t) + \int_0^1 \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} x(t)\,\mathrm{d}\alpha = G(t, x(t), u(t)), \quad x(0) = x_0, \qquad [6.46]$$

where ϕ is the known function. Thus, we assume that the highest derivative in the dynamics of the problem is of integer order. This assumption makes the problem of satisfying the initial condition $[6.46]_2$ by x possible. To obtain the optimality conditions for problems [6.45] and [6.46], we follow [AGR 04]. The modified performance criterion becomes

$$\begin{aligned}\bar{J}[x,u] \;=\; & \int_0^1 \Big(F(t,x,u) + \lambda(t)\Big(G(t,x,u) - x^{(1)}(t) \\ & - \int_0^1 \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} x(t)\,\mathrm{d}\alpha\Big)\Big)\,\mathrm{d}t, \end{aligned} \qquad [6.47]$$

where λ is the Lagrange multiplier. Setting the first variation of [6.47] to zero, we obtain

$$\begin{aligned}\delta\bar{J}[x,u] \;=\; & \int_0^1 \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial x}\delta x + \delta\lambda\Big(G(t,x,u)\right. \\ & -x^{(1)}(t) - \int_0^1 \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} x(t)\,\mathrm{d}\alpha\Big) \\ & +\lambda(t)\left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial x}\delta x - \delta x^{(1)}(t)\right. \\ & \left.\left. - \int_0^1 \phi(\alpha)\, {}_0\mathrm{D}_t^{\alpha} \delta x\,\mathrm{d}\alpha\right)\right)\mathrm{d}t \;=\; 0, \end{aligned} \qquad [6.48]$$

where δu, $\delta\lambda$ and δx are Lagrange variations of dependent variables. From $[6.46]_2$, we conclude that $\delta x\left(0\right) = 0$. Exchanging the order of integration and using the integration by parts formula [2.22], we obtain

$$\int_0^1 \lambda\left(t\right)\left({}_0\mathrm{D}_t^{\alpha}\delta x\left(t\right)\right)\mathrm{d}t = \int_0^1 \left({}_t\mathrm{D}_1^{\alpha}\lambda\left(t\right)\right)\delta x\left(t\right)\mathrm{d}t. \tag{6.49}$$

From [6.49] in [6.48] and by invoking the fundamental lemma of the variational calculus, we finally obtain

$$\begin{aligned}
&x^{(1)}\left(t\right) = -\int_0^1 \phi\left(\alpha\right){}_0\mathrm{D}_t^{\alpha}x\left(t\right)\mathrm{d}\alpha + G\left(t,x,u\right),\\
&\frac{\partial F}{\partial u} + \lambda\left(t\right)\frac{\partial G}{\partial u} = 0,\\
&\lambda^{(1)}\left(t\right) = \int_0^1 \phi\left(\alpha\right){}_t\mathrm{D}_1^{\alpha}\lambda\left(t\right)\mathrm{d}\alpha - \lambda\left(t\right)\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x},
\end{aligned} \tag{6.50}$$

with

$$x\left(0\right) = x_0, \quad \lambda\left(1\right) = 0. \tag{6.51}$$

In general, the system of equations [6.50] and [6.51] solves the optimization problem in appropriate domains. However, the presence of both left and right fractional derivatives in [6.50] makes the process of analytical solutions difficult. Therefore, numerical procedures are often used to solve [6.50] and [6.51] (see [AGR 04, AGR 08]). As an example, consider the problem with the optimality criteria

$$J\left[u\right] = \frac{1}{2}\int_0^1 \left(x^2\left(t\right) + u^2\left(t\right)\right)\mathrm{d}t, \tag{6.52}$$

and the constraint

$$x^{(1)}\left(t\right) + \int_0^1 \phi\left(\alpha\right){}_0\mathrm{D}_t^{\alpha}x\left(t\right)\mathrm{d}\alpha = -x\left(t\right) + u\left(t\right), \quad x\left(0\right) = 1. \tag{6.53}$$

Note that in this case, $F(t,x,u) = \frac{1}{2}\left(x^2+u^2\right)$, $G(t,x,u) = -x+u$, so that [6.50] becomes

$$\begin{aligned}
&x^{(1)}(t) = -\int_0^1 \phi(\alpha)\,{}_0\mathrm{D}_t^\alpha x(t)\,\mathrm{d}\alpha - x(t) + u(t)\,,\\
&u(t) + \lambda(t) = 0,\\
&\lambda^{(1)}(t) = \int_0^1 \phi(\alpha)\,{}_t\mathrm{D}_1^\alpha \lambda(t)\,\mathrm{d}\alpha + \lambda(t) - x(t)\,,
\end{aligned} \qquad [6.54]$$

with

$$x(0) = x_0, \quad \lambda(1) = 0 \qquad [6.55]$$

or, with $u = -\lambda$, we have

$$\begin{aligned}
x^{(1)}(t) &= -\int_0^1 \phi(\alpha)\,{}_0\mathrm{D}_t^\alpha x(t)\,\mathrm{d}\alpha - x(t) - \lambda(t)\,,\\
\lambda^{(1)}(t) &= \int_0^1 \phi(\alpha)\,{}_t\mathrm{D}_1^\alpha \lambda(t)\,\mathrm{d}\alpha + \lambda(t) - x(t)\,.
\end{aligned} \qquad [6.56]$$

Application of expansion formula [2.29] for the solution of the optimal control problem of the type analyzed here, for the case when $\phi(\alpha) = \delta(\alpha-\alpha_0)$, with α_0 given, is presented in [RAP 10].

6.4. Approximation of Euler–Lagrange equations

We present two procedures of approximation of the Euler–Lagrange equation.

6.4.1. *Approximation 1*

We follow [ATA 08a] where the approximation of the Riemann–Liouville fractional derivative is performed by the finite sum of integer-order derivatives. In this way, the fractional variational problem is transformed into the problem involving integer-order derivatives only.

We assume in the following that $L \in C^N([a,b]\times\mathbb{R}\times\mathbb{R})$. Let (c,d), $-\infty < c < d < \infty$, be an open interval in $\mathbb{R}$ that contains $[a,b]$, such that for each $t \in [a,b]$, the closed ball $L(t, b-a)$, with the center at t and radius $b-a$, lies in (c,d). Recall the expansion formula [2.23] of f real and analytic in (c,d)

$$ {}_a\mathrm{D}_t^{\alpha} f\,(t) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} f^{(i)}(t), \quad t \in L(t, b-a) \subset (c,d), \qquad [6.57] $$

where $\binom{\alpha}{i} = \frac{(-1)^{i-1}\alpha\Gamma(i-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)}$. Actually, condition $L(t, b-a) \subset (c,d)$ comes from the Taylor expansion of $f(t-\tau)$ at t, for $\tau \in (a,t)$ and $t \in (a,b)$.

Consider the fractional variational problem [6.2]. Assume that we are looking for a minimizer $u \in C^{2N}([a,b])$, for some $N \in \mathbb{N}$. We replace, in the Lagrangian, the left Riemann–Liouville fractional derivative ${}_a\mathrm{D}_t^{\alpha} u$ with the finite sum of integer-valued derivatives as in [6.57]:

$$ \int_A^B L\left(t, u(t), \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(t)\right) \mathrm{d}t $$

$$ = \int_A^B \bar{L}\left(t, u(t), u^{(1)}(t), u^{(2)}(t), \ldots, u^{(N)}(t)\right) \mathrm{d}t. \qquad [6.58] $$

Now the Lagrangian $\bar{L}$ depends on t, u and all (integer-order) derivatives of u up to order N. Moreover, $\partial_3 \bar{L}, \ldots, \partial_{N+2}\bar{L} \in C^{N-1}([a,b] \times \mathbb{R} \times \mathbb{R})$, since $\partial_i \bar{L} = \partial_3 L \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}$, $i = 3, ..., N+2$.

The Euler–Lagrange equation for [6.58] has the following form:

$$ \sum_{i=0}^{N} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \frac{\partial \bar{L}}{\partial u^{(i)}} = 0. $$

This is equivalent to

$$ \frac{\partial L}{\partial u} + \sum_{i=0}^{N} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\partial_3 L \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) = 0. \qquad [6.59] $$

Let $\mathcal{A}((c,d))$ be the space of real analytic functions in (c,d) with the family of semi-norms

$$ p_{[m,n]}(\varphi) = \sup_{t \in [m,n]} |\varphi(t)|, \quad \varphi \in \mathcal{A}((c,d)), $$

where $[m, n]$ are subintervals of (c, d). Every function $f \in C([a, b])$, which we extend to be zero in $(c, d)\backslash[a, b]$, defines an element of the dual $\mathcal{A}'((c, d))$ via

$$\varphi \mapsto \langle f, \varphi \rangle = \int_a^b f(t)\varphi(t)\mathrm{d}t, \quad \varphi \in \mathcal{A}((c, d)).$$

As usual, we say that f and g from $\mathcal{A}'((c, d))$ are equal in the weak sense if $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for every $\varphi \in \mathcal{A}((c, d))$. In the proposition and theorem that follow, we will assume, as we did in [6.57], that $L(t, b - a) \subset (c, d)$, for all $t \in [a, b]$.

Our aim is to show that the left-hand side in [6.59] converges to the left-hand side of

$$\frac{\partial L}{\partial u} + {}_t\mathrm{D}_b^\alpha \left(\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^\alpha u} \right) = 0, \qquad [6.60]$$

as $N \to \infty$, in the weak sense in $\mathcal{A}'((c, d))$. We present theorem from [ATA 08a].

THEOREM 6.3.– Let $J[u]$ be a fractional variational problem [6.2], which is solved in case 1) or 2) given below.

a) Let $u \in C^\infty([a, b])$ such that $u(a) = a_0$, $u(b) = b_0$, for fixed $a_0, b_0 \in \mathbb{R}$, and L_3 (where L_3 stands for $\partial_3 L$) be a function in $[a, b]$ defined by $t \mapsto L_3(t) = L_3(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t))$, $t \in [a, b]$. Let $L_3^{(i)}(b, b_0, p) = 0$, for all $i \in \mathbb{N}$, meaning that for $(t, s, p) \mapsto L_3(t, s, p)$, $t \in [a, b]$, $s, p \in \mathbb{R}$, the following holds:

(i) $\frac{\partial^i L_3}{\partial t^i}(b, b_0, p) = 0, \forall p \in \mathbb{R}$;

(ii) $\frac{\partial^i L_3}{\partial s^i}(b, b_0, p) = 0, \forall p \in \mathbb{R}$;

(iii) $\frac{\partial^i L_3}{\partial p^i}(b, b_0, p) = 0, \forall p \in \mathbb{R}$.

b) Let $u \in C^\infty([a, b])$ such that $u^{(i)}(b) = 0$, for all $i \in \mathbb{N}_0$, and $u(a) = a_0$, for fixed $a_0 \in \mathbb{R}$. Let $L_3^{(i)}(b) = L_3^{(i)}(b, 0, {}_a\mathrm{D}_b^\alpha u) = 0$, for all $i \in \mathbb{N}$ and for every fixed u, meaning that for $(t, s, p) \mapsto L_3(t, s, p)$, $t \in [a, b]$, $s, p \in \mathbb{R}$, the following holds:

(i) $\frac{\partial^i L_3}{\partial t^i}(b, 0, p) = 0, \forall p \in \mathbb{R}$;

(ii) $\frac{\partial^i L_3}{\partial p^i}(b, 0, p) = 0, \forall p \in \mathbb{R}$.

Denote by P the fractional Euler–Lagrange equation [6.60], and by P_N the Euler–Lagrange equations [6.59], which correspond to the variational problem [6.58], in which the left Riemann–Liouville fractional derivative is approximated according to [6.57] by the finite sum. Then, in both cases 1) and 2)

$$P_N \to P \text{ in the weak sense in } \mathcal{A}'((c,d)), \text{ as } N \to \infty.$$

The proof of this theorem and more details on approximation of the Euler–Lagrange equations are given in [ATA 08a].

6.4.2. *Approximation 2*

Another way to approximate the Euler–Lagrange equation is based on the expansion formula developed in [2.24]. Following [ATA 14a], we suppose that $y \in C^1([0,T])$, and write ${}_0\mathrm{D}_t^\alpha u$ as

$$ {}_0\mathrm{D}_t^\alpha u(t) = \widehat{{}_0^N\mathrm{D}_t^\alpha} u(t) + Q_{N+1}(u)(t), \quad t \in (0,T]\,, \qquad [6.61]$$

where

$$\widehat{{}_0^N\mathrm{D}_t^\alpha} u(t) = \frac{u(t)}{t^\alpha} A(N,\alpha) - \sum_{p=1}^{N} C_{p-1}(\alpha) \frac{V_{p-1}(u)(t)}{t^{p+\alpha}}, \quad t \in (0,T]\,, \qquad [6.62]$$

and $Q_{N+1}(y)$ is the remainder term. Now, we insert [6.61] into [6.2]. We consider the problem of minimizing the following functional:

$$J\,[u] = \int_0^T L(t, u(t), \widehat{{}_0^N\mathrm{D}_t^\alpha} u(t) + Q_{N+1}(u)(t))\,\mathrm{d}t.$$

Applying the mean value theorem on L, we obtain

$$J\,[u] = \int_0^T L(t, u(t), \widehat{{}_0^N\mathrm{D}_t^\alpha} u(t))\,\mathrm{d}t + \int_0^T Q_{N+1}(u)(t) \partial_3 L(t, u(t), \xi(t))\,\mathrm{d}t,$$

where $\xi(t)$ lies between $\widehat{{}_0^N\mathrm{D}_t^\alpha} u(t)$ and $\widehat{{}_0^N\mathrm{D}_t^\alpha} u(t) + Q_{N+1}(u)(t)$, $t \in [0,T]$. Set

$$J_N\,[u] = \int_0^T L(t, u(t), \widehat{{}_0^N\mathrm{D}_t^\alpha} u(t))\,\mathrm{d}t.$$

Then

$$J[u] = J_N[u] + \int_0^T Q_{N+1}(u)(t)\partial_3 L(t, u(t), \xi(t))\, \mathrm{d}t. \qquad [6.63]$$

The relation between the functionals J and J_N is established in the following theorem from [ATA 14a].

THEOREM 6.4.– Let $L \in C^1([0,T] \times \mathbb{R} \times \mathbb{R})$ and $t \mapsto \partial_3 L(t, u(t), {}_0\mathrm{D}_t^\alpha u(t)) \in AC([0,T])$, $u \in AC([0,T])$. Assume that $\mathcal{U}$ is a subspace of $AC([0,T])$, and that there exist $u^* \in \mathcal{U}$ and $u_N^* \in \mathcal{U}$ that minimize functionals J and J_N, respectively. Then, $\lim_{N\to\infty} J_N[u_N^*] = J[u^*]$. In addition, if $\mathcal{U}$ (with a suitable Hausdorff topology) is compact, and if u^* and u_N^* are unique solutions of the minimization problems $J \to \min$ and $J_N \to \min$, respectively, then $u_N^* \to u^*$, as $N \to \infty$.

PROOF.– Let $u^* \in \mathcal{U}$ be a minimizer of [6.2], i.e.

$$\min_u J[u] = J[u^*] = \int_0^T L(t, u^*(t), {}_0\mathrm{D}_t^\alpha u^*(t))\, \mathrm{d}t,$$

and let $u_N^* \in \mathcal{U}$ be a minimizer of J_N, i.e.

$$\min_u J_N[u] = J_N[u_N^*] = \int_0^T L(t, u_N^*(t), \widehat{{}_0^N\mathrm{D}_t^\alpha} u_N^*(t))\, \mathrm{d}t.$$

Consider [6.63]. The assumptions on L and u, and the estimate

$$|Q_{N+1}(f,g)| \leq \frac{C \cdot M_f M_g}{\Gamma(1-\alpha)|\Gamma(\alpha)|} \frac{T^{2-\alpha}}{N^{\alpha_1}}, \qquad [6.64]$$

see [2.28], where $0 < \alpha_1 < 1 - \alpha$, $C > 0$ is a constant, $M_f = \max_{t\in[0,T]} |f^{(1)}(\tau)|$, and $M_g = \max_{t\in[0,T]} |g(\tau)|$ (see [2.28]) imply that for any $\varepsilon > 0$ there is $N_0 \in \mathbb{N}$ such that $\left|\int_0^T Q_{N+1}(u)(t)\partial_3 L(t, u(t), \xi(t))\, \mathrm{d}t\right| < \varepsilon$, for all $N \geq N_0$. Thus

$$J[u^*] = J_N[u^*] - \delta \geq J_N[u_N^*] - \delta,$$

as well as

$$J[u^*] \leq J[u_N^*] = J_N[u_N^*] + \delta,$$

for arbitrarily small $\delta \in \mathbb{R}$. Writing this together, we obtain $J_N\left[u_N^*\right] - \delta \leq J\left[u^*\right] \leq J_N\left[u_N^*\right] + \delta$, which implies

$$J\left[u^*\right] = \lim_{N\to\infty} J_N\left[u_N^*\right].$$

The second part of the claim follows from the uniqueness of minimizers, and the fact that every subsequence of $\{u_N^*\}_{N\in\mathbb{N}}$ has a convergent subsequence. ■

We present the announced approximation scheme for the Euler–Lagrange equations based on [6.62]. We consider the problem of determining the minimum of J_N, i.e.

$$\min_{u\in\mathcal{U}} J_N\left[u\right] = \min_{u\in\mathcal{U}} \int_0^T L(t, u(t), \widehat{{}_0^N\mathrm{D}_t^\alpha} u(t))\,\mathrm{d}t, \qquad [6.65]$$

where $\mathcal{U} \subset C^1([0,T])$ is the set of admissible functions, and $\widehat{{}_0^N\mathrm{D}_t^\alpha} u$ is given by [6.62]. We derive necessary conditions for minimizers of [6.65].

Substituting [6.62] into [6.65], we have

$$J_N\left[u\right] = \int_0^T L\left(t, u(t), \frac{u(t)}{t^\alpha} A(N,\alpha) - \sum_{p=1}^{N} C_{p-1}(\alpha) \frac{V_{p-1}(u)(t)}{t^{p+\alpha}}\right) \mathrm{d}t, \qquad [6.66]$$

where $V_{p-1}(u)$, $p = 1, \ldots, N$, satisfy (see [2.27])

$$V_p^{(1)}(u)(t) = t^p u(t), \quad p \in \mathbb{N}. \qquad [6.67]$$

Minimization of [6.66] and [6.67] represents the classical problem of minimization of functional with constraints in the form of differential equations of integer order. Set

$$\bar{J}\left[t, u, V_0, \ldots, V_{N-1}\right] = \int_0^T \bar{L}\left(t, u, V_0, \ldots, V_{N-1}\right) \mathrm{d}t,$$

where

$$\bar{L}\left(t, u, V_0, \ldots, V_{N-1}\right) = L\left(t, u(t), \frac{u(t)}{t^\alpha} A(N,\alpha) - \sum_{p=1}^{N} C_{p-1}(\alpha) \frac{V_{p-1}(u)(t)}{t^{p+\alpha}}\right)$$

Thus, the new Lagrangian $\bar{L}$ is a function depending on one independent variable t, and N dependent variables (functions of t): $u, V_0, \ldots, V_{N-1}$. Following the well-known procedure for deriving the Euler–Lagrange equations for the variational problem with constraints, we construct the modified functional

$$\begin{aligned}\Phi(t, u, V_0, \ldots, V_{N-1}) &= \bar{L}(t, u, V_0, \ldots, V_{N-1}) \\ &\quad + \sum_{p=1}^{N} \lambda_{p-1}(t) \left(V_{p-1}^{(1)}(u)(t) - t^{p-1} u(t) \right),\end{aligned}$$

and obtain the system

$$\partial_2 L + \partial_3 L \frac{A(N,\alpha)}{t^\alpha} - \sum_{p=1}^{N} \lambda_{p-1}(t) t^{p-1} = 0, \qquad [6.68]$$

$$\lambda_{p-1}^{(1)}(t) = -\partial_3 L \frac{C_{p-1}(\alpha)}{t^{p+\alpha}}, \;\; p = 1, 2, \ldots, N, \; \lambda_{p-1}(T) = 0, \qquad [6.69]$$

$$V_{p-1}^{(1)}(u)(t) = t^{p-1} u(t), \;\; V_{p-1}(u)(0) = 0, \; p = 1, 2, \ldots, N. \qquad [6.70]$$

Here, we used $\partial_2 L$ and $\partial_3 L$ to denote partial derivatives of the Lagrangian L with respect to the second and third variable, respectively.

The question that now arises is how the optimality conditions [6.68]–[6.70] are related to the necessary condition for optimality condition [6.60]. More precisely, we want to show that solutions to [6.68]–[6.70] converge to [6.60], as $N \to \infty$, in a weak sense.

We explain the precise meaning of the weak sense of convergence in this context. Set $\mathcal{A} = \{\varphi : \varphi \in C^1([0,T])\}$. Every function $f \in C([0,T])$ defines an element of the dual space $\mathcal{A}'$ via

$$\varphi \mapsto \langle f, \varphi \rangle = \int_0^T f(t)\, \varphi(t) \mathrm{d}t, \;\; \varphi \in \mathcal{A}.$$

We say that f and g from $\mathcal{A}'$ are equal in the weak sense if $\langle f, \varphi \rangle = \langle g, \varphi \rangle$, $\forall \varphi \in \mathcal{A}$. Therefore, the weak form of [6.60] reads

$$\langle \partial_3 L + {}_t\mathrm{D}_T^\alpha (\partial_3 L), \varphi \rangle = 0, \;\; \forall \varphi \in \mathcal{A}.$$

THEOREM 6.5.– Let J be the fractional variational problem [6.2] with the Lagrangian $L \in C^1([0,T] \times \mathbb{R} \times \mathbb{R})$ and $t \mapsto \partial_3 L(t, u(t), {}_0\mathrm{D}_t^\alpha u(t)) \in AC([0,T])$. Denote by P the left hand-side of the fractional Euler–Lagrange equation [6.60], and by P_N the left-hand side of [6.68] together with the constraints [6.69]–[6.70], that correspond to the variational problem [6.65]–[6.67] in which the left Riemann–Liouville fractional derivative of $u \in \mathcal{U}$ is approximated by [6.62]. Then

$$P_N \to P \ \text{ in the weak sense, as } \ N \to \infty.$$

PROOF.– Using the assumption that $\partial_3 L \in AC([0,T])$, we have

$$\langle {}_t\mathrm{D}_T^\alpha(\partial_3 L), \varphi \rangle = \int_0^T {}_t\mathrm{D}_T^\alpha(\partial_3 L)\varphi(t)\,\mathrm{d}t = \int_0^T \partial_3 L\, {}_0\mathrm{D}_t^\alpha \varphi(t)\mathrm{d}t. \qquad [6.71]$$

Now, expansion of formula [2.24] for ${}_0\mathrm{D}_t^\alpha \varphi$ yields

$${}_0\mathrm{D}_t^\alpha \varphi(t) = \lim_{N\to\infty} \left(\frac{\varphi(t)}{t^\alpha} A(N,\alpha) - \sum_{p=1}^{N} C_{p-1}(\alpha) \frac{V_{p-1}(\varphi)(t)}{t^{p+\alpha}} \right).$$

Substituting this into [6.71], we obtain

$$\begin{aligned} \int_0^T \partial_3 L\, {}_0\mathrm{D}_t^\alpha \varphi(t)\,\mathrm{d}t &= \int_0^T \partial_3 L \lim_{N\to\infty} \left(\frac{\varphi(t)}{t^\alpha} A(N,\alpha) \right. \\ &\quad \left. - \sum_{p=1}^{N} C_{p-1}(\alpha) \frac{V_{p-1}(\varphi)(t)}{t^{p+\alpha}} \right) \mathrm{d}t \\ &= \lim_{N\to\infty} \int_0^T \partial_3 L \left(\frac{\varphi(t)}{t^\alpha} A(N,\alpha) \right. \\ &\quad \left. - \sum_{p=1}^{N} C_{p-1}(\alpha) \frac{V_{p-1}(\varphi)(t)}{t^{p+\alpha}} \right) \mathrm{d}t, \end{aligned}$$

or

$$\begin{aligned} \int_0^T \partial_3 L \cdot {}_0\mathrm{D}_t^\alpha \varphi(t)\,\mathrm{d}t &= \lim_{N\to\infty} \int_0^T \left(\partial_3 L\, \frac{\varphi(t)}{t^\alpha} A(N,\alpha) \right. \\ &\quad \left. - \sum_{p=1}^{N} \partial_3 L\, C_{p-1}(\alpha) \frac{V_{p-1}(\varphi)(t)}{t^{p+\alpha}} \right) \mathrm{d}t. \qquad [6.72] \end{aligned}$$

Inserting [6.69] into the last term in [6.72], we have

$$\int_0^T \sum_{p=1}^N \partial_3 L\, C_{p-1}(\alpha) \frac{V_{p-1}(\varphi)(t)}{t^{p+\alpha}}\,\mathrm{d}t = -\sum_{p=1}^N \int_0^T \lambda_{p-1}^{(1)}(t) V_{p-1}(\varphi)(t)\mathrm{d}t$$

$$= \sum_{p=1}^N \int_0^T \lambda_{p-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} V_{p-1}(\varphi)(t)\mathrm{d}t = \sum_{p=1}^N \int_0^T \lambda_{p-1}(t) t^{p-1} \varphi(t)\mathrm{d}t,$$

where we used [6.67]. Collecting all the above results, we eventually obtain

$$\begin{aligned}&\langle \partial_2 L + {}_t\mathrm{D}_T^\alpha(\partial_3 L), \varphi\rangle \\ &\quad = \lim_{N\to\infty} \int_0^T \left(\partial_2 L + \frac{\partial_3 L}{t^\alpha} A(N,\alpha) - \sum_{p=1}^N \lambda_{p-1}(t) t^{p-1}\right) \varphi(t)\mathrm{d}t,\end{aligned}$$

which proves the claim. ■

REMARK 6.6.– It is an open problem to find conditions that will transfer the approximation procedure in a weak sense to the approximation procedure in some function space with the norm structure.

6.5. Invariance properties of J: Nöther's theorem

Nöther's theorem plays an important role in mechanics. It provides constants of motions (first integrals) for dynamical systems from the study of invariance properties of variational principles. Following [ATA 09a], we derive Nöther's theorem for the variational principle with fractional derivatives [6.2]. We refer to [FRE 07] for Nöther's theorem for the fractional variational principle. Let G be a local one-parameter group of transformations acting on a space of independent and dependent variables as follows: $(\bar{t}, \bar{u}) = g_\eta \cdot (t,u) = (\Xi_\eta(t,u), \Psi_\eta(t,u))$, for smooth functions Ξ_η and Ψ_η, and $g_\eta \in G$. Let

$$\mathbf{v} = \tau(t,u)\frac{\partial}{\partial t} + \xi(t,u)\frac{\partial}{\partial u},$$

be the infinitesimal generator of G. Then, we also have

$$\begin{aligned}\bar{t} &= t + \eta\tau(t,u) + o(\eta),\\ \bar{u} &= u + \eta\xi(t,u) + o(\eta).\end{aligned} \qquad [6.73]$$

We introduce the following notation (see, e.g., [VUJ 89]): $\Delta t = \frac{\mathrm{d}}{\mathrm{d}\eta}|_{\eta=0}(\bar{t} - t)$ and $\Delta u = \frac{\mathrm{d}}{\mathrm{d}\eta}|_{\eta=0}(\bar{u}(\bar{t}) - u(t))$. More precisely, the notations Δt and Δu denote $\lim_{\eta\to 0} \frac{\bar{t}(\eta)-t}{\eta}$ and $\lim_{\eta\to 0} \frac{\bar{u}(\bar{t},\eta)-u(t)}{\eta}$, respectively. It follows from [6.73] that $\Delta t = \tau$ and writing the Taylor expansion of $\bar{u}(\bar{t}) = \bar{u}(t + \eta\tau(t,u) + o(\eta))$ at $\eta = 0$ yields that $\Delta u = \xi$. If we write the Taylor expansion of $\bar{u}(\bar{t})$ at $\bar{t} = t$, we get

$$\Delta u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} (\bar{u}(t) - u(t)) + \dot{u}\Delta t,$$

where $\dot{u}\,(t) = \frac{\mathrm{d}}{\mathrm{d}t}u\,(t)$, or, by introducing the Lagrangian variation

$$\delta u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} (\bar{u}(t) - u(t)), \qquad [6.74]$$

we obtain the total variation as

$$\Delta u = \delta u + \dot{u}\Delta t.$$

Thus, we have

$$\delta u = \xi - \tau\dot{u}.$$

Note that $\delta t = 0$, if $\bar{t} = t$.

In the same way, we can define ΔF and δF of an arbitrary absolutely continuous function $F = F(t, u(t), \dot{u}(t))$

$$\begin{aligned} \Delta F &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} (F(\bar{t}, \bar{u}(\bar{t}), \dot{\bar{u}}(\bar{t})) - F(t, u(t), \dot{u}(t))) \\ &= \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial u}\Delta u + \frac{\partial F}{\partial \dot{u}}\Delta \dot{u}, \end{aligned}$$

$$\delta F = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} (F(t, \bar{u}(t), \dot{\bar{u}}(t)) - F(t, u(t), \dot{u}(t))) = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial \dot{u}}\delta \dot{u},$$

and

$$\Delta F = \delta F + \dot{F}\Delta t.$$

If we want to find an infinitesimal criterion, we need to know $\Delta {}_a\mathrm{D}_t^\alpha u$ and ΔJ, where

$$J[u] = \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t))\mathrm{d}t, \quad 0 < \alpha < 1. \qquad [6.75]$$

Therefore, we have to transform the left Riemann–Liouville fractional derivative of u under the action of a local one-parameter group of transformations [6.73]. There are two different cases, which will be considered separately. In the first case the lower bound a in ${}_a\mathrm{D}_t^\alpha u$ is not transformed, while in the second case, a is transformed in the same way as the independent variable t. Physically, the first case is important when ${}_a\mathrm{D}_t^\alpha u$ represents memory effects, and the second case is important when action on a distance is involved.

6.5.1. *The case when a in ${}_a\mathrm{D}_t^\alpha u$ is not transformed*

In this section, we consider a local group of transformations G which transforms $t \in (A, B)$ into $\bar{t} \in (\bar{A}, \bar{B})$ so that both intervals remain subintervals of (a, b), but the action of G has no effect on the lower bound a in ${}_a\mathrm{D}_t^\alpha u$, i.e. $\tau(a, u(a)) = 0$. So, suppose that G acts on t, u and ${}_a\mathrm{D}_t^\alpha u$ in the following way:

$$g_\eta \cdot (t, u, {}_a\mathrm{D}_t^\alpha u) = (\bar{t}, \bar{u}, {}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u}),$$

where $\bar{t}$ and $\bar{u}$ are defined by [6.73]. In this case, we have:

LEMMA 6.1.– Let $u \in AC([a, b])$ and let G be a local one-parameter group of transformations given by [6.73]. Then

$$\Delta {}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^\alpha u \cdot \tau(t, u),$$

where

$$\delta {}_a\mathrm{D}_t^\alpha u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} ({}_a\mathrm{D}_t^\alpha \bar{u} - {}_a\mathrm{D}_t^\alpha u) = {}_a\mathrm{D}_t^\alpha \delta u.$$

PROOF.– To prove that $\delta {}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha \delta u$, it is enough to apply the definition of the Lagrangian variation [6.74]. Also by definition, we have

$$\Delta {}_a\mathrm{D}_t^\alpha u = \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u} - {}_a\mathrm{D}_t^\alpha u\right).$$

Thus,

$$\begin{aligned}
\Delta {}_a\mathrm{D}_t^\alpha u &= \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_a^{\bar{t}} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha} \mathrm{d}\theta \pm \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{\bar{u}(\theta)}{(t-\theta)^\alpha} \mathrm{d}\theta \right. \\
&\quad \left. - \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{u(\theta)}{(t-\theta)^\alpha} \mathrm{d}\theta \right) \\
&= {}_a\mathrm{D}_t^\alpha \delta u + \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u} - {}_a\mathrm{D}_t^\alpha \bar{u} \right) \\
&= {}_a\mathrm{D}_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \left.\frac{\mathrm{d}\bar{t}}{\mathrm{d}\eta}\right|_{\eta=0} \left({}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u} - {}_a\mathrm{D}_t^\alpha \bar{u} \right) \\
&= {}_a\mathrm{D}_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^\alpha u \cdot \tau(t,u).
\end{aligned}$$

Next, we determine ΔJ under the action of [6.73]. ■

LEMMA 6.2.– Let J be a functional of the form $J[u] = \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t))\mathrm{d}t$, where u is an absolutely continuous function in $[a,b]$, $(A,B) \subseteq (a,b)$ and L satisfies [6.3]. Let G be a local one-parameter group of transformations given by [6.73]. Then

$$\Delta J = \delta J + \left.(L\Delta t)\right|_A^B,$$

where $\delta J = \int_A^B \delta L\,\mathrm{d}t$.

PROOF.– It is clear that $\delta J = \int_a^b \delta L\,\mathrm{d}t$. For ΔJ, we have

$$\begin{aligned}
\Delta J &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_{\bar{A}}^{\bar{B}} L(\bar{t}, \bar{u}(\bar{t}), {}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u}(\bar{t}))\mathrm{d}\bar{t} - \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u(t))\mathrm{d}t \right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_A^B L(\bar{t}, \bar{u}(\bar{t}), {}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u}(\bar{t}))(1+\eta\dot{\tau}(t,u(t)))\mathrm{d}t \right. \\
&\quad \left. - \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u(t))\mathrm{d}t \right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_A^B L(\bar{t}, \bar{u}(\bar{t}), {}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u}(\bar{t}))\mathrm{d}t - \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u(t))\mathrm{d}t \right) \\
&\quad + \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u(t))\, \dot{\tau}(t, u(t))\,\mathrm{d}t.
\end{aligned}$$

This yields

$$\begin{aligned}\Delta J &= \int_A^B \Delta L(t,u(t),{}_a\mathrm{D}_t^\alpha u(t))\mathrm{d}t + \int_A^B L(t,u(t),{}_a\mathrm{D}_t^\alpha u(t))\dot{\tau}(t,u(t))\mathrm{d}t \\ &= \int_A^B \delta L(t,u(t),{}_a\mathrm{D}_t^\alpha u(t))\mathrm{d}t + \int_A^B \frac{\mathrm{d}}{\mathrm{d}t}L(t,u(t),{}_a\mathrm{D}_t^\alpha u(t))\tau(t,u(t))\mathrm{d}t \\ &\quad + \int_A^B L(t,u(t),{}_a\mathrm{D}_t^\alpha u(t))\dot{\tau}(t,u(t))\mathrm{d}t \\ &= \int_A^B \delta L\,\mathrm{d}t + \left.(L\tau)\right|_A^B\end{aligned}$$

and the claim is proved. ■

We define a variational symmetry group of the fractional variational problem [6.2]. We call it the fractional variational symmetry group.

DEFINITION 6.1.– *A local one-parameter group of transformations G, given by* [6.73]*, is a variational symmetry group of the fractional variational problem* [6.2] *if the following conditions holds: for every $[A',B'] \subset (A,B)$, $u = u(t) \in AC([A',B'])$ and $g_\eta \in G$ such that $\bar{u}(\bar{t}) = g_\eta \cdot u(\bar{t})$ is in $AC([\bar{A}',\bar{B}'])$, we have*

$$\int_{\bar{A}'}^{\bar{B}'} L(\bar{t},\bar{u}(\bar{t}),{}_a\mathrm{D}_{\bar{t}}^\alpha \bar{u}(\bar{t}))\mathrm{d}\bar{t} = \int_{A'}^{B'} L(t,u(t),{}_a\mathrm{D}_t^\alpha u(t))\mathrm{d}t. \qquad [6.76]$$

We are now able to prove the following infinitesimal criterion.

THEOREM 6.6.– Let J be a fractional variational problem [6.2] and let G be a local one-parameter transformation group [6.73] with the infinitesimal generator $\mathbf{v} = \tau(t,u)\partial_t + \xi(t,u)\partial_u$. Then G is a variational symmetry group of J if and only if

$$\tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial u} + \left({}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^\alpha u\right)\tau\right)\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} + L\dot{\tau} = 0. \quad [6.77]$$

PROOF.– Suppose that G is a variational symmetry group of J. Then, [6.76] holds for all subintervals (A', B') of (A, B) with closure $[A', B'] \subset (A, B)$. We have

$$\begin{aligned}
\Delta J &= \int_{A'}^{B'} \delta L \, \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}\delta({}_a\mathrm{D}_t^{\alpha} u) \right) \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}(\Delta u - \dot{u}\Delta t) + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}{}_a\mathrm{D}_t^{\alpha}(\Delta u - \dot{u}\Delta t) \right) \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'} \\
&= \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}\Delta u + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}{}_a\mathrm{D}_t^{\alpha}\Delta u \right) \mathrm{d}t \\
&\quad - \int_{A'}^{B'} \left(\frac{\partial L}{\partial u}(\dot{u}\Delta t) + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}{}_a\mathrm{D}_t^{\alpha}(\dot{u}\Delta t) \pm \frac{\partial L}{\partial t}\Delta t \right) \mathrm{d}t + (L\Delta t)\Big|_{A'}^{B'}.
\end{aligned}$$

Applying the Leibnitz rule for $\frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t)$, we replace $\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}\dot{u}\Delta t$ by $\frac{\mathrm{d}}{\mathrm{d}t}(L\Delta t) - \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}\left(\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^{\alpha} u\right)\Delta t - L\,(\Delta t)^{\cdot}$. Then

$$\begin{aligned}
\Delta J &= \int_{A'}^{B'} \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}\Delta u + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}{}_a\mathrm{D}_t^{\alpha}\Delta u \right. \\
&\quad \left. + \frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u}\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^{\alpha} u \right)\Delta t - {}_a\mathrm{D}_t^{\alpha}(\dot{u}\Delta t) \right) + L\,(\Delta t)^{\cdot} \right) \mathrm{d}t.
\end{aligned}$$

Since ΔJ has to be zero in all (A', B') with $[A', B'] \subset (A, B)$, the above integrand has also to be equal zero, i.e.

$$\tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial u} + \left({}_a\mathrm{D}_t^{\alpha}(\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^{\alpha} u \right)\tau \right)\frac{\partial L}{\partial \, {}_a\mathrm{D}_t^{\alpha} u} + L\dot{\tau} = 0.$$

where we have used that $\Delta u = \xi$ and $\Delta t = \tau$. Hence, necessity of the statement is proved. To prove that condition [6.77] is also sufficient, note that if [6.77] holds on every $[A', B'] \subset (A, B)$, then $\Delta J = 0$ in every $[A', B'] \subset (A, B)$. Thus, integrating ΔJ from 0 to η, we obtain [6.76], for η near the identity. The proof is now complete. ■

As an example we calculate the transformation of the fractional derivative ${}_a\mathrm{D}_t^{\alpha} u$ under the group of translations. Here, a in ${}_a\mathrm{D}_t^{\alpha} u$ is not transformed.

EXAMPLE 6.1.– Let G be a local one-parameter translation group: $(\bar{t}, \bar{u}) = (t+\eta, u+\eta)$, with the infinitesimal generator $v = \partial_t + \partial_u$. Then, $\bar{u}(\bar{t}) = u(\bar{t} - \eta) + \eta$ and by a straightforward calculation it can be shown that

$$ {}_a\mathrm{D}_{\bar{t}}^{\alpha} \bar{u}(\bar{t}) = {}_a\mathrm{D}_t^{\alpha}(u\,(t) + \eta) + \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{a-\eta}^{a} \frac{u(s) + \eta}{(t-s)^{\alpha}}\, \mathrm{d}s. $$

6.5.2. *The case when a in ${}_a\mathrm{D}_t^{\alpha} u$ is transformed*

Now let the action of a local one-parameter group of transformations [6.73] be of the form

$$ g_\eta \cdot (t, u, {}_a\mathrm{D}_t^{\alpha} u) = (\bar{t}, \bar{u}, {}_{\bar{a}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{u}). $$

Thus, the one-parameter group also acts on a and transforms it to $\bar{a}$, where $\bar{a} = g_\eta \cdot t|_{t=a}$. This will also influence the calculation of $\Delta {}_a\mathrm{D}_t^{\alpha} u$ and ΔJ.

LEMMA 6.3.– Let $u \in AC([a, b])$ and let G be a local one-parameter group of transformations [6.73]. Then

$$ \Delta {}_a\mathrm{D}_t^{\alpha} u = {}_a\mathrm{D}_t^{\alpha} \delta u + \frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^{\alpha} u \cdot \tau(t, u(t)) + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a, u(a)), \quad [6.78] $$

where

$$ \delta {}_a\mathrm{D}_t^{\alpha} u = \left. \frac{\mathrm{d}}{\mathrm{d}\eta} \right|_{\eta=0} ({}_a\mathrm{D}_t^{\alpha} \bar{u} - {}_a\mathrm{D}_t^{\alpha} u) = {}_a\mathrm{D}_t^{\alpha} \delta u. $$

PROOF.– By using [6.74], we check that $\delta {}_a\mathrm{D}_t^{\alpha} u = {}_a\mathrm{D}_t^{\alpha} \delta u$. To prove [6.78], we start with the definition of $\Delta {}_a\mathrm{D}_t^{\alpha} u$

$$ \Delta {}_a\mathrm{D}_t^{\alpha} u = \left. \frac{\mathrm{d}}{\mathrm{d}\eta} \right|_{\eta=0} \left({}_{\bar{a}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{u} - {}_a\mathrm{D}_t^{\alpha} u \right). $$

Thus

$$\begin{aligned}\Delta {}_a\mathrm{D}_t^\alpha u &= \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_{\bar{a}}^{\bar{t}} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta \pm \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{\bar{u}(\theta)}{(t-\theta)^\alpha}\mathrm{d}\theta \right.\\ &\left. -\frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{u(\theta)}{(t-\theta)^\alpha}\mathrm{d}\theta \right)\\ &= {}_a\mathrm{D}_t^\alpha \delta u + \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_{\bar{a}}^{a} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta + \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_a^{\bar{t}} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta \right.\\ &\left. -\frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{\bar{u}(\theta)}{(t-\theta)^\alpha}\mathrm{d}\theta \right)\\ &= {}_a\mathrm{D}_t^\alpha \delta u + \frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^\alpha u \cdot \tau(t,u(t)) + \frac{1}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \frac{\mathrm{d}}{\mathrm{d}\bar{t}} \int_{\bar{a}}^{a} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta.\end{aligned}$$

In the last term, we can interchange the order of integration and differentiation with respect to $\bar{t}$. This gives

$$-\frac{\alpha}{\Gamma(1-\alpha)} \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \int_{\bar{a}}^{a} \frac{\bar{u}(\theta)}{(\bar{t}-\theta)^\alpha}\mathrm{d}\theta.$$

Differentiation of this integral with respect to η at $\eta = 0$, we obtain [6.78]. ■

We determine now ΔJ when a in ${}_a\mathrm{D}_t^\alpha u$ is transformed.

LEMMA 6.4.– Let J be a functional of the form $J[u] = \int_A^B L(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t))\mathrm{d}t$, where u is an absolutely continuous function in (a,b), $(A,B) \subseteq (a,b)$ and L satisfies [6.3]. Let G be a local one-parameter group of transformations given by [6.73]. Then

$$\Delta J = \delta J + \left. (L\Delta t)\right|_A^B + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}} \tau(a,u(a)) \int_A^B \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\,\mathrm{d}t, \qquad [6.79]$$

where $\delta J = \int_A^B \delta L\,\mathrm{d}t$.

PROOF.– Clearly, $\delta J = \int_a^b \delta L \,\mathrm{d}t$. On the other hand, we have

$$\begin{aligned}
\Delta J &= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_{\bar{A}}^{\bar{B}} L(\bar{t}, \bar{u}(\bar{t}), {}_{\bar{a}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{u}(\bar{t}))\mathrm{d}\bar{t} - \int_A^B L(t, u(t), {}_a\mathrm{D}_t^{\alpha} u\,(t))\mathrm{d}t \right) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\eta}\right|_{\eta=0} \left(\int_A^B L(\bar{t}, \bar{u}(\bar{t}), {}_{\bar{a}}\mathrm{D}_{\bar{t}}^{\alpha} \bar{u}(\bar{t}))(1 + \eta\dot{\tau}(t, u(t)))\,\mathrm{d}t \right. \\
&\quad \left. - \int_A^B L(t, u(t), {}_a\mathrm{D}_t^{\alpha} u\,(t))\mathrm{d}t \right) \\
&= \int_A^B \Delta L \mathrm{d}t + \int_A^B L(t, u(t), {}_a\mathrm{D}_t^{\alpha} u\,(t))\,\dot{\tau}(t, u(t))\mathrm{d}t \\
&= \int_A^B \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}\Delta u + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u}\Delta_a\mathrm{D}_t^{\alpha} u \right) \mathrm{d}t \\
&\quad + \int_A^B L(t, u(t), {}_a\mathrm{D}_t^{\alpha} u\,(t))\dot{\tau}(t, u(t))\mathrm{d}t.
\end{aligned}$$

Using [6.78], we obtain

$$\begin{aligned}
\Delta J &= \int_A^B \left(\frac{\partial L}{\partial t}\Delta t + \frac{\partial L}{\partial u}(\delta u + \dot{u}\Delta t) \right. \\
&\quad + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u} \left({}_a\mathrm{D}_t^{\alpha}\delta u + \frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^{\alpha} u \cdot \tau(t, u(t)) \right. \\
&\quad \left.\left. + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}}\tau(a, u(a)) \right) \right) \mathrm{d}t \\
&\quad + \int_A^B L(t, u(t), {}_a\mathrm{D}_t^{\alpha} u(t))\dot{\tau}(t, u(t))\mathrm{d}t \\
&= \int_A^B \delta L \mathrm{d}t + \left.(L\Delta t)\right|_A^B + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}}\tau(a, u(a)) \int_A^B \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u}\mathrm{d}t, \\
&= \delta J + \left.(L\Delta t)\right|_A^B + \frac{\alpha}{\Gamma(1-\alpha)} \frac{u(a)}{(t-a)^{\alpha+1}}\tau(a, u(a)) \int_A^B \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha} u}\mathrm{d}t. \quad \blacksquare
\end{aligned}$$

REMARK 6.7.– It should be emphasized that the last summand on the right-hand side of [6.78], as well as of [6.79], appears only as a consequence of the fact that the action of a transformation group also affects a, on which the left Riemann–Liouville fractional derivative depends. In the case when $u(a) = 0$ or $\alpha = 1$, that term is equal to zero (since $\lim_{\alpha\to 1^-}\Gamma(1-\alpha)=\infty$) and $\Delta L=\delta L+(L\Delta t)|_A^B$, which is the classical result.

We define a variational symmetry group of the fractional variational problem [6.2] as follows.

DEFINITION 6.2.– *A local one-parameter group of transformations G* [6.73] *is a variational symmetry group of the fractional variational problem* [6.2] *if the following conditions holds: for every $[A',B']\subset(A,B)$, $u=u(t)\in AC([A',B'])$ and $g_\eta\in G$ such that $\bar{u}(\bar{t})=g_\eta\cdot u(\bar{t})$ is in $AC([\bar{A}',\bar{B}'])$, we have*

$$\int_{\bar{A}'}^{\bar{B}'} L(\bar{t},\bar{u}(\bar{t}),{}_{\bar{a}}\mathrm{D}_{\bar{t}}^{\alpha}\bar{u}(\bar{t}))\mathrm{d}\bar{t}=\int_{A'}^{B'} L(t,u(t),{}_a\mathrm{D}_t^{\alpha}u(t))\mathrm{d}t.$$

We are solving the fractional variational problem [6.2] among all absolutely continuous functions in $[a,b]$, which additionally satisfy $u(a)=0$. Therefore, the last term in both [6.78] and [6.79] vanishes, and the infinitesimal criterion reads the same as it does in theorem 6.6.

THEOREM 6.7.– Let J be a fractional variational problem defined by [6.2] and let G be a local one-parameter transformation group defined by [6.73] with the infinitesimal generator $\mathbf{v}=\tau(t,u)\partial_t+\xi(t,u)\partial_u$. Assume that $u(a)=0$, for all admissible functions u. Then, G is a variational symmetry group of J if and only if

$$\tau\frac{\partial L}{\partial t}+\xi\frac{\partial L}{\partial u}+\left({}_a\mathrm{D}_t^{\alpha}(\xi-\dot{u}\tau)+\left(\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^{\alpha}u\right)\tau\right)\frac{\partial L}{\partial\,{}_a\mathrm{D}_t^{\alpha}u}+L\dot{\tau}=0. \quad [6.80]$$

PROOF.– The proof is the same as in the case of theorem 6.6. ■

As an example, we calculate the transformation of the fractional derivative ${}_a\mathrm{D}_t^{\alpha}u$ under the group of translations. Here, a in ${}_a\mathrm{D}_t^{\alpha}u$ is transformed.

EXAMPLE 6.2.– Let $v=\partial_t+\partial_u$ be the infinitesimal generator of the translation group $(\bar{t},\bar{u})=(t+\eta,u+\eta)$. Then, $\bar{u}(\bar{t})=u(\bar{t}-\eta)+\eta$ and

$${}_{\bar{a}}\mathrm{D}_{\bar{t}}^{\alpha}\bar{u}={}_a\mathrm{D}_t^{\alpha}(u+\eta).$$

6.5.3. *Nöther's theorem*

We will need a generalization of the fractional integration by parts. We state it as the following lemma.

LEMMA 6.5.– Let $f, g \in AC([a,b])$. Then, for all $t \in [a,b]$,

$$\begin{aligned}\int_a^t f(s)\left({}_s\mathrm{D}_b^\alpha g(s)\right)\mathrm{d}s &= \int_a^t \left({}_a\mathrm{D}_s^\alpha f(s)\right) g(s)\mathrm{d}s + \frac{1}{\Gamma(1-\alpha)}\\ &\times \int_a^t f(s)\left(\frac{g(b)}{(b-s)^\alpha} - \frac{g(t)}{(t-s)^\alpha}\right.\\ &\left.-\int_t^b \frac{\dot g(\sigma)}{(\sigma-s)^\alpha}\mathrm{d}\sigma\right)\mathrm{d}s. \qquad [6.81]\end{aligned}$$

PROOF.– To derive [6.81], we will use [2.11], as well as [2.22]:

$$\begin{aligned}&\int_a^t f(s)\left({}_s\mathrm{D}_b^\alpha g(s)\right)\mathrm{d}s\\ &\quad= \int_a^t f(s)\frac{1}{\Gamma(1-\alpha)}\left(\frac{g(b)}{(b-s)^\alpha} - \int_s^b \frac{\dot g(\sigma)}{(\sigma-s)^\alpha}\mathrm{d}\sigma\right)\mathrm{d}s\\ &\quad= \frac{1}{\Gamma(1-\alpha)}\int_a^t f(s)\left(\frac{g(b)}{(b-s)^\alpha} - \int_s^t \frac{\dot g(\sigma)}{(\sigma-s)^\alpha}\mathrm{d}\sigma\right.\\ &\left.-\int_t^b \frac{\dot g(\sigma)}{(\sigma-s)^\alpha}\mathrm{d}\sigma \pm \frac{g(t)}{(t-s)^\alpha}\right)\mathrm{d}s\\ &\quad= \int_a^t \left(f(s)\left({}_s\mathrm{D}_t^\alpha g(s)\right) + \frac{1}{\Gamma(1-\alpha)}f(s)\right.\\ &\left.\times\left(\frac{g(b)}{(b-s)^\alpha} - \frac{g(t)}{(t-s)^\alpha} - \int_t^b \frac{\dot g(\sigma)}{(\sigma-s)^\alpha}\mathrm{d}\sigma\right)\right)\mathrm{d}s\\ &\quad= \int_a^t \left(\left({}_a\mathrm{D}_s^\alpha f(s)\right) g(s) + \frac{1}{\Gamma(1-\alpha)}f(s)\right.\\ &\left.\times\left(\frac{g(b)}{(b-s)^\alpha} - \frac{g(t)}{(t-s)^\alpha} - \int_t^b \frac{\dot g(\sigma)}{(\sigma-s)^\alpha}\mathrm{d}\sigma\right)\right)\mathrm{d}s.\end{aligned}$$

■

REMARK 6.8.– If we put $t=b$ in [6.81], we obtain [2.22].

The goal of the symmetry group analysis in the calculus of variations are the first integrals of Euler–Lagrange equations of a variational problem, that is a Nöther-type result.

Analogously to the classical case, the expression of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}P(t,u(t),{}_a\mathrm{D}_t^{\alpha}u\,(t))=0$$

is called the fractional first integral (or a fractional conservation law) for a fractional differential equation

$$F(t,u(t),{}_a\mathrm{D}_t^{\alpha}u\,(t))=0,$$

if it vanishes along all solutions u of F. In general, we could define a "conserved" quantity as the expression of the form $Q(t,u(t),{}_a\mathrm{D}_t^{\alpha}u\,(t))$ if ${}_a\mathrm{D}_t^{\beta}Q(t,u(t),$ ${}_a\mathrm{D}_t^{\alpha}u\,(t))=0$, for some $0<\beta\leq 1$. Then, our definition corresponds to the case $\beta=1$.

In the statement that is to follow, we prove a version of the fractional Nöther theorem. As we will show, a fractional conserved quantity will also contain integral terms, which is unavoidable, due to the presence of fractional derivatives. If $\alpha=1$, then our results reduce to the well-known classical conservation laws for a first-order variational problem.

THEOREM 6.8.– Nöther's theorem– Let G be a local fractional variational symmetry group defined by [6.73] of the fractional variational problem [6.2], and let $\mathbf{v}=\tau(t,u(t))\partial_t+\xi(t,u(t))\partial_u$ be the infinitesimal generator of G. Then, for $t\in[A',B']\,,[A',B']\subset(A,B)\,,$

$$L\tau+\int_a^t\left({}_a\mathrm{D}_s^{\alpha}(\xi-\dot{u}\tau)\frac{\partial L}{\partial_a\mathrm{D}_s^{\alpha}u}-(\xi-\dot{u}\tau)_s\mathrm{D}_B^{\alpha}\left(\frac{\partial L}{\partial\,{}_a\mathrm{D}_s^{\alpha}u}\right)\right)\mathrm{d}s=const. \quad [6.82]$$

is a fractional first integral (or fractional conservation law) for the Euler–Lagrange equation [6.9].

The fractional conservation law [6.82] can equivalently be written in the form

$$L\tau-\frac{1}{\Gamma(1-\alpha)}\int_a^t(\xi-\dot{u}\tau)\left(\frac{\partial_3L(B,u(B),{}_a\mathrm{D}_B^{\alpha}u\,(B))}{(B-s)^{\alpha}}-\frac{\partial_3L(t,u(t),{}_a\mathrm{D}_t^{\alpha}u\,(t))}{(t-s)^{\alpha}}\right.$$
$$\left.-\int_t^b\frac{\frac{\mathrm{d}}{\mathrm{d}\sigma}\partial_3L(\sigma,u(\sigma),{}_a\mathrm{D}_{\sigma}^{\alpha}u\,(\sigma))}{(\sigma-s)^{\alpha}}\mathrm{d}\sigma\right)\mathrm{d}s=const. \quad [6.83]$$

REMARK 6.9.– This theorem is valid in the case when the lower bound a in ${}_a\mathrm{D}_t^\alpha u$ is not transformed, as well as in the case when a is transformed. In the second case we also have to suppose that $u(a) = 0$, for all admissible functions u, which provides the same form of the infinitesimal criterion in both cases.

PROOF.– We want to insert the Euler–Lagrange equation [6.12] into the infinitesimal criterion [6.77] or [6.80]. Hence, we will, for $t \in [A', B']$, write the latter in a suitable form

$$
\begin{aligned}
0 &= \tau\frac{\partial L}{\partial t} + \xi\frac{\partial L}{\partial u} + \left({}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^\alpha u\right)\tau\right)\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} + L\dot{\tau} \pm \dot{u}\tau\frac{\partial L}{\partial u} \\
&= \tau\left(\frac{\partial L}{\partial t} + \frac{\partial L}{\partial u}\cdot\dot{u} + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\cdot\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^\alpha u\right) + (\xi - \dot{u}\tau)\frac{\partial L}{\partial u} \\
&\quad + {}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} + L\dot{\tau} \\
&= \tau\frac{\mathrm{d}}{\mathrm{d}t}L + L\frac{\mathrm{d}}{\mathrm{d}t}\tau + (\xi - \dot{u}\tau)\frac{\partial L}{\partial u} + {}_a\mathrm{D}_t^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial_a\mathrm{D}_t^\alpha u} \\
&\quad \pm(\xi - \dot{u}\tau)_t\mathrm{D}_B^\alpha\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(L\tau) + (\xi - \dot{u}\tau)\left(\frac{\partial L}{\partial u} + {}_t\mathrm{D}_B^\alpha\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\right)\right) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t\left({}_a\mathrm{D}_s^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial\, {}_a\mathrm{D}_s^\alpha u} - (\xi - \dot{u}\tau)_s\mathrm{D}_B^\alpha\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_s^\alpha u}\right)\right)\mathrm{d}s.
\end{aligned}
$$

The middle term in the last equation is the Euler–Lagrange equation [6.12] multiplied by $\xi - \dot{u}\tau$. Therefore

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(L\tau + \int_a^t\left({}_a\mathrm{D}_s^\alpha(\xi - \dot{u}\tau)\frac{\partial L}{\partial\, {}_a\mathrm{D}_s^\alpha u} - (\xi - \dot{u}\tau)_s\mathrm{D}_B^\alpha\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_s^\alpha u}\right)\right)\mathrm{d}s\right) = 0
$$

for all solutions of the Euler–Lagrange equation, and [6.82] holds. If we now apply lemma 6.5, we can transform [6.82] into [6.83]. ∎

In the next example, we consider the case in which the Lagrangian of a fractional variational problem does not depend on t explicitly.

EXAMPLE 6.3.– Consider a variational problem whose Lagrangian does not depend on t explicitly

$$J[u] = \int_A^B L(u(t), {}_a\mathrm{D}_t^\alpha u\,(t))\mathrm{d}t. \qquad [6.84]$$

As in the classical case, the one-parameter group of translations of time is a variational symmetry group of [6.84]. Indeed, $\bar{t} = t + \eta, \bar{u} = u$ (hence $\bar{u}(\bar{t} = u(\bar{t} - \eta))$ and ${}_{\bar{a}}\mathrm{D}_{\bar{t}}^\alpha \bar{u} = {}_a\mathrm{D}_t^\alpha u$ (see example 6.2). Therefore, with $\bar{A} = A + \eta$ and $\bar{B} = B + \eta$,

$$\int_{\bar{A}}^{\bar{B}} L(\bar{u}(\bar{t}), {}_{\bar{a}}\mathrm{D}_{\bar{t}}^\alpha \bar{u}(\bar{t}))\mathrm{d}\bar{t} = \int_A^B L(u(t), {}_a\mathrm{D}_t^\alpha u\,(t))\mathrm{d}t.$$

This fact can be also confirmed by the infinitesimal criterion

$$\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\left(-{}_a\mathrm{D}_t^\alpha\left(\frac{\mathrm{d}}{\mathrm{d}t}u\right) + \frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^\alpha u\right) = 0,$$

where the last equality holds since $u(a) = 0$. Thus, $\frac{\mathrm{d}}{\mathrm{d}t}{}_a\mathrm{D}_t^\alpha u = {}_a\mathrm{D}_t^\alpha \frac{\mathrm{d}}{\mathrm{d}t}u$.

Using Nöther's theorem, we may write a fractional conservation law that comes from this translation group ($t \in [A', B']$, $[A', B'] \subset (A, B)$)

$$L + \int_a^t \left(-{}_a\mathrm{D}_t^\alpha \dot{u} \cdot \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u} + \dot{u} \cdot {}_t\mathrm{D}_B^\alpha\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^\alpha u}\right)\right)\mathrm{d}s = const.$$

or equivalently

$$\begin{aligned} L + \frac{1}{\Gamma(1-\alpha)}\int_a^t \dot{u}(s)\Bigg(&\frac{\partial_3 L(u(B), {}_a\mathrm{D}_B^\alpha u\,(B))}{(B-s)^\alpha} - \frac{\partial_3 L(u(t), {}_a\mathrm{D}_t^\alpha u\,(t))}{(t-s)^\alpha} \\ &- \int_t^b \frac{\frac{\mathrm{d}}{\mathrm{d}\sigma}\partial_3 L(u(\sigma), {}_a\mathrm{D}_\sigma^\alpha u\,(\sigma))}{(\sigma - s)^\alpha}\mathrm{d}\sigma\Bigg)\mathrm{d}s = const. \end{aligned} \qquad [6.85]$$

Note that [6.85] tends to

$$L - \dot{u}(t) \cdot \frac{\partial L(u(t), \dot{u}(t))}{\partial \dot{u}} = const.$$

when $\alpha \to 1^-$, which is the classical energy integral (see [VUJ 89]).

EXAMPLE 6.4.– As a concrete example related to example 6.3, let us consider a fractional oscillator for which the Lagrangian reads

$$L = \frac{1}{2}({}_a\mathrm{D}_t^\alpha u)^2 - \frac{1}{2}\omega^2 u^2, \tag{6.86}$$

where ω is given constant. The fractional variational problem consists of minimizing the functional $\int_0^1 L\mathrm{d}t$ among all smooth functions that satisfy the initial conditions $u(0) = 0$ and $u^{(1)}(0) = 1$. The Euler–Lagrange equation for such an L is

$$\omega^2 u - {}_t\mathrm{D}_1^\alpha \left({}_0\mathrm{D}_t^\alpha u\right) = 0. \tag{6.87}$$

Since L does not depend on t explicitly and $u(0) = 0$, the translation group $(\bar{t}, \bar{u}) = (t + \eta, u)$ is a fractional variational symmetry group for [6.86]. It generates the following fractional conservation law for [6.87]

$$\frac{1}{2}({}_a\mathrm{D}_t^\alpha u)^2 - \frac{1}{2}\omega^2 u^2 + \int_0^t \left(-{}_0\mathrm{D}_s^\alpha \dot{u} \cdot {}_0\mathrm{D}_s^\alpha u + \dot{u} \cdot {}_s\mathrm{D}_1^\alpha ({}_0\mathrm{D}_s^\alpha u)\right) \mathrm{d}s = const.$$

In the case when $\alpha \to 1^-$, we obtain $\frac{1}{2}\left(\dot{u}^2 + \omega^2 u^2\right) = const.$

6.5.4. *Approximations of fractional derivatives in Nöther's theorem*

We use approximations of the Riemann–Liouville fractional derivatives by finite sums of derivatives of integer order, which leads to variational problems involving only classical derivatives. We examine the relation between the Euler–Lagrange equations, infinitesimal criterion and Nöther's theorem obtained in the process of approximation and the fractional Euler–Lagrange equations, infinitesimal criterion and Nöther's theorem derived in previous sections.

In the following, we make the subsequent assumptions:

1) $L \in C^N([a, b] \times \mathbb{R} \times \mathbb{R})$, at least, for some $N \in \mathbb{N}$.

2) We will simplify following calculations by considering the case $A = a$ and $B = b$.

3) Let (c, d), $-\infty < c < d < +\infty$, be an open interval in $\mathbb{R}$ that contains $[a, b]$, such that for each $t \in [a, b]$, the closed ball $B(t, b - a)$, with the center at t and radius $b - a$, lies in (c, d).

In addition, we will separately consider two cases:

(a) Let $u \in C^\infty([a,b])$ such that $u(a) = a_0$, $u(b) = b_0$, for fixed $a_0, b_0 \in \mathbb{R}$, and L_3 (where L_3 stands for $\partial_3 L$) be a function in $[a,b]$ defined by $t \mapsto L_3(t) = L_3(t, u(t), {}_a\mathrm{D}_t^\alpha u\,(t))$, $t \in [a,b]$. Let $L_3^{(i)}(b, b_0, p) = 0$, for all $i \in \mathbb{N}$, meaning that for $(t,s,p) \mapsto L_3(t,s,p)$, $t \in [a,b]$, $s, p \in \mathbb{R}$, the following holds

$$\frac{\partial^i L_3}{\partial t^i}(b, b_0, p) = 0, \quad \frac{\partial^i L_3}{\partial s^i}(b, b_0, p) = 0, \quad \frac{\partial^i L_3}{\partial p^i}(b, b_0, p) = 0, \ \ \forall p \in \mathbb{R}.$$

(b) Let $u \in C^\infty([a,b])$ such that $u^{(i)}(b) = 0$, for all $i \in \mathbb{N}_0$, and $u(a) = a_0$, for fixed $a_0 \in \mathbb{R}$. Let $L_3^{(i)}(b) = L_3^{(i)}(b, 0, {}_a\mathrm{D}_b^\alpha u\,(b)) = 0$, for all $i \in \mathbb{N}$ and for every fixed u, meaning that for $(t,s,p) \mapsto L_3(t,s,p)$, $t \in [a,b]$, $s, p \in \mathbb{R}$, the following holds:

$$\frac{\partial^i L_3}{\partial t^i}(b, 0, p) = 0, \quad \frac{\partial^i L_3}{\partial p^i}(b, 0, p) = 0, \ \ \forall p \in \mathbb{R}.$$

Let f be a real analytic function in (c,d). Then, according to [2.23], we have

$${}_a\mathrm{D}_t^\alpha f\,(t) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} f^{(i)}(t), \ \ t \in B(t, b-a) \subset (c,d), \qquad [6.88]$$

where $\binom{\alpha}{i} = \frac{(-1)^{i-1}\alpha\Gamma(i-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)}$.

Consider again the fractional variational problem [6.2]. Assume that we are looking for a minimizer $u \in C^{2N}([a,b])$, for some $N \in \mathbb{N}$. We replace in the Lagrangian the left Riemann–Liouville fractional derivative ${}_a\mathrm{D}_t^\alpha u$ with the finite sum of integer-valued derivatives as in [6.88]:

$$\int_a^b L\left(t, u(t), \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)}(t))\right) \mathrm{d}t$$
$$= \int_a^b \bar{L}(t, u(t), u^{(1)}(t), u^{(2)}(t), \ldots, u^{(N)}(t))\mathrm{d}t. \qquad [6.89]$$

Now the Lagrangian $\bar{L}$ depends on t, u and all (classical) derivatives of u up to order N. Moreover, $\partial_3 \bar{L}, \ldots, \partial_{N+2}\bar{L} \in C^{N-1}([a,b] \times \mathbb{R} \times \mathbb{R})$, since $\partial_i \bar{L} = \partial_3 L \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}$, $i = 3, ..., N+2$. For problem [6.89], we can calculate the Euler–Lagrange equations, infinitesimal criterion and conservation laws (via Nöther's theorem).

The Euler–Lagrange equation for [6.89] is of the following form:

$$\sum_{i=0}^{N}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\frac{\partial \bar{L}}{\partial u^{(i)}} = 0.$$

This is equivalent to

$$\frac{\partial L}{\partial u} + \sum_{i=0}^{N}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\partial_3 L \cdot \binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right) = 0. \qquad [6.90]$$

Now we will consider the fractional variational problem [6.2] in case (1) and in case (2).

THEOREM 6.9.– Let J be a fractional variational problem [6.2], which is solved in case (1) or (2). Denote by CL the fractional conservation law [6.82], and by CL_N the fractional conservation law for the Euler–Lagrange equation [6.90], which corresponds to the variational problem [6.89]. Then

$$CL_N \to CL \text{ in the weak sense, as } N \to \infty.$$

Convergence in the weak sense means that as a test function space, we use the space of real analytic functions as in section 6.4.1. Note that analytic functions are chosen in order to provide convergence of the sum as in [6.88].

Before we prove theorem 6.9, we prove several auxiliary results.

First we recall a result from [ATA 08a], which provides an expression for the right Riemann–Liouville fractional derivative in terms of the lower bound a, which figures in the left Riemann–Liouville fractional derivative.

PROPOSITION 6.7.– Let $F \in C^\infty([a,b])$, such that $F^{(i)}(b) = 0$, for all $i \in \mathbb{N}_0$, and $F \equiv 0$ in $(c,d)\backslash[a,b]$. Let ${}_t\mathrm{D}_b^\alpha F$ be extended by zero in $(c,d)\backslash[a,b]$. Then, the following holds:

1) For every $i \in \mathbb{N}$, the $(i-1)$th derivative of $t \mapsto F(t)(t-a)^{i-\alpha}$ is continuous at $t=a$ and $t=b$ and the ith derivative of this function, for $i \in \mathbb{N}_0$, is integrable in (c,d) and supported in $[a,b]$.

2) The partial sums $S_N, N \in \mathbb{N}_0$,

$$t \mapsto S_N(t) = \begin{cases} \sum\limits_{i=0}^{N}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(F(t)\cdot\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\right), & t \in [a,b], \\ 0, & t \in (c,d)\backslash[a,b], \end{cases}$$

are integrable functions in (c,d) supported in $[a,b]$.

3) $$ {}_t\mathrm{D}_b^{\alpha} F(t) = \sum_{i=0}^{\infty} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(F(t) \cdot \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \right) $$

in the weak sense.

PROOF.– See [ATA 08a, proposition 4.2]. ■

REMARK 6.10.– The Euler–Lagrange equation [6.90] represents a necessary condition when we solve the variational problem [6.89] in the class $C^{2N}([a,b])$, with the prescribed boundary conditions at a and b, i.e. $u(a) = a_0$ and $u(b) = b_0$, a_0, b_0 are fixed real numbers.

The following theorem shows that the Euler–Lagrange equation [6.90] converges to [6.8], as $N \to \infty$, in the weak sense.

THEOREM 6.10.– Let J be the fractional variational problem [6.2] to be solved in case (1) or (2). Denote by P the fractional Euler–Lagrange equations [6.9], and by P_N the Euler–Lagrange equations [6.90], which correspond to the variational problem [6.89], in which the left Riemann–Liouville fractional derivative is approximated according to [6.88] by a finite sum. Then, in both cases (1) and (2)

$$ P_N \to P \text{ in the weak sense, as } N \to 0. $$

PROOF.– See [ATA 08a, theorem 4.3]. ■

Let

$$ \mathbf{v}_N = \tau_N \frac{\partial}{\partial t} + \xi_N \frac{\partial}{\partial u}, \qquad [6.91] $$

be the infinitesimal generator of a local one-parameter variational symmetry group of [6.89]. The Nth prolongation of $\mathbf{v}_N$ is given by

$$ \mathrm{pr}^{(N)}\mathbf{v}_N = \mathbf{v}_N + \sum_{i=1}^{N} \xi_N^i \frac{\partial}{\partial u^{(i)}}, $$

with

$$ \xi_N^i = \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\xi_N - \tau_N u^{(1)} \right) + \tau_N u^{(i+1)}. $$

REMARK 6.11.– It should be emphasized that the vector field v_N in [6.91] differs (in general) from the one given by [6.73].

THEOREM 6.11.– Let J and $\bar{J}$ be fractional variational problems [6.2] and [6.89] respectively. Denote by IC and IC_N the corresponding infinitesimal criteria. If $\tau_N \to \tau$ and $\xi_N \to \xi$ as $N \to \infty$, uniformly on compact sets, then

$$IC_N \to IC \text{ uniformly on compact sets, as } N \to \infty.$$

PROOF.– The infinitesimal criterion for the variational problem [6.89] says that a vector field $\mathbf{v}_N$ generates a local one-parameter variational symmetry group of functional [6.89] if and only if

$$\mathrm{pr}^{(N)}\mathbf{v}_N(\bar{L}) + \bar{L}\dot{\tau}_N = 0.$$

This is equivalent to

$$\begin{aligned}
0 &= \tau_N \frac{\partial \bar{L}}{\partial t} + \xi_N \frac{\partial \bar{L}}{\partial u} + \sum_{i=1}^{N} \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\xi_N - \tau_N u^{(1)} \right) + \tau_N u^{(i+1)} \right) \frac{\partial \bar{L}}{\partial u^{(i)}} + \bar{L}\dot{\tau}_N \\
&= \tau_N \left(\partial_1 L + \partial_3 L \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(i-\alpha)(t-a)^{i-\alpha-1}}{\Gamma(i+1-\alpha)} u^{(i)} \right) \\
&\quad + \xi_N \left(\partial_2 L + \partial_3 L \binom{\alpha}{0} \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \right) \\
&\quad + \sum_{i=1}^{N} \left(\frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\xi_N - \tau_N u^{(1)} \right) + \tau_N u^{(i+1)} \right) \partial_3 L \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} + L\dot{\tau}_N \\
&= \tau_N \partial_1 L + \xi_N \partial_2 L v \\
&\quad + \left(\sum_{i=1}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} + \binom{\alpha}{0} \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \xi_N \right) \partial_3 L \\
&\quad \pm \binom{\alpha}{0} \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \tau_N u^{(1)} + \tau_N \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(i-\alpha)(t-a)^{i-\alpha-1}}{\Gamma(i+1-\alpha)} u^{(i)} \right. \\
&\quad \left. + \sum_{i=1}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i+1)} \right) \partial_3 L + L\dot{\tau}_N
\end{aligned}$$

$$
\begin{aligned}
&= \tau_N \partial_1 L + \xi_N \partial_2 L + \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} \right. \\
&\quad \left. + \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)} \right) \tau_N \right) \partial_3 L + L \dot{\tau}_N.
\end{aligned} \qquad [6.92]
$$

So, if $\tau_N \to \tau$ and $\xi_N \to \xi$ as $N \to \infty$, then the last expression on the right-hand side tends to

$$
\tau \partial_1 L + \xi \partial_2 L + \left({}_a\mathrm{D}_t^{\alpha} (\xi - \dot{u}\tau) + \left(\frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{D}_t^{\alpha} u \right) \tau \right) \partial_3 L + L\dot{\tau},
$$

and that is exactly the infinitesimal criterion [6.77] for [6.2]. ■

Now, we give the proof of theorem 6.9.

PROOF OF THEOREM 6.9.– First, we will derive the form of a conservation law for the Euler–Lagrange equation [6.90] of the approximated variational problem [6.89], which comes from a variational symmetry group with the infinitesimal generator [6.91]. For this purpose, we need to insert the Euler–Lagrange equations [6.90] into the infinitesimal criterion [6.92]:

$$
\begin{aligned}
0 &= \tau_N \partial_1 L + \xi_N \partial_2 L + \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} \right. \\
&\quad \left. + \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)} \right) \tau_N \right) \partial_3 L + L \dot{\tau}_N \pm \tau_N u^{(1)} \partial_2 L \\
&= \tau_N \left(\partial_1 L + u^{(1)} \partial_2 L + \partial_3 L \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} u^{(i)} \right) \right) + L \dot{\tau}_N \\
&\quad + \left(\xi_N - \tau_N u^{(1)} \right) \partial_2 L + \sum_{i=0}^{N} \binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} (\xi_N - \tau_N u^{(1)})^{(i)} \partial_3 L \\
&\quad \pm (\xi_N - \tau_N u^{(1)}) \sum_{i=0}^{N} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i} \left(\binom{\alpha}{i} \frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)} \partial_3 L \right)
\end{aligned}
$$

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}(L\tau_N) + (\xi_N - \tau_N u^{(1)}) \\
&\quad \times \left(\partial_2 L + \sum_{i=0}^{N} (-1)^i \frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\binom{\alpha}{i}\frac{(t-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\partial_3 L\right)\right) \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \left(\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N - \tau_N u^{(1)})^{(i)}\partial_3 L\right. \\
&\quad \left. - (\xi_N - \tau_N u^{(1)})\sum_{i=0}^{N}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}s^i}\left(\binom{\alpha}{i}\frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\partial_3 L\right)\right)\mathrm{d}s.
\end{aligned}$$

We recognize that the expression in brackets in the middle term in this sum is the Euler–Lagrange equation [6.90], and hence vanishes. Therefore, we obtain that the following quantity is conserved:

$$\begin{aligned}
&L\tau_N + \int_a^t \left(\sum_{i=0}^{N}\binom{\alpha}{i}\frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}(\xi_N - \tau_N u^{(1)})^{(i)}\partial_3 L\right. \\
&\quad \left. - (\xi_N - \tau_N u^{(1)})\sum_{i=0}^{N}(-1)^i \frac{\mathrm{d}^i}{\mathrm{d}s^i}\left(\binom{\alpha}{i}\frac{(s-a)^{i-\alpha}}{\Gamma(i+1-\alpha)}\partial_3 L\right)\right)\mathrm{d}s = const.
\end{aligned} \qquad [6.93]$$

This of course is one way to write the first integral of the Euler–Lagrange equations [6.90], which corresponds to the local one-parameter variational symmetry group generated by [6.91]. Applying proposition 6.7 (ii), we obtain that [6.93] converges to [6.82] in the weak sense, provided that $(\tau_N, \xi_N) \to (\tau, \xi)$, as $N \to \infty$. ■

The invariance and first integrals when approximation [2.24] is used in [6.2] is given in [ATA 14a].

6.6. Complementary fractional variational principles

In this section we construct two variational principles: one that is on the solution of Euler–Lagrange equations in the minimum and the other that is on the solution of an adjoint variational problem in the maximum. Values of both functionals on the exact solution of the corresponding Euler–Lagrange equations are equal. In the analysis that follows we will use the Ritz approach adjusted to fractional order equations, see [ART 73, ART 80] for integer-order equations, to obtain approximate solution to the corresponding Euler–Lagrange equations. Also, we will propose an error estimate procedure. We follow the presentation of [ATA 12a]. Here, we are not concerned with questions of the existence of solutions to Euler–Lagrange equations, i.e. with minimizers of the fractional variational problems. We simply assume that the solutions exist. We will study the fractional complementary variational principles when the Lagrangians are of the form:

1) $L_1\left(t, \boldsymbol{u}, {}_a\mathrm{D}_t^\alpha \boldsymbol{u}\right) = \frac{1}{2}\left({}_a\mathrm{D}_t^\alpha \boldsymbol{u}\right)^2 + \Pi\left(t, \boldsymbol{u}\right), \ \boldsymbol{u} = (u_1, \ldots, u_n)$ is a vector function with the scalar argument t, and ${}_a\mathrm{D}_t^\alpha \boldsymbol{u} = \left({}_a\mathrm{D}_t^\alpha u_1, \ldots, {}_a\mathrm{D}_t^\alpha u_n\right)$ (see [6.94]);

2) $L_2(u, {}^\alpha\mathrm{grad}\, u) = \Pi(u) + R({}^\alpha\mathrm{grad}\, u)$, u is a scalar function with the vector argument $\boldsymbol{x} = (x_1, \ldots, x_n)$ and ${}^\alpha\mathrm{grad}\, u = ({}^\alpha\mathrm{D}_{x_1} u, \ldots, {}^\alpha\mathrm{D}_{x_n} u)$ (see [6.97]).

The practical aim of our analysis is to obtain an error estimate of an approximate solution to the Euler–Lagrange equations, which correspond to the mentioned Lagrangians, without knowing their exact solutions. The form of the Lagrangian L_1 is motivated by applications in mechanics, as well as in the study of the first Painlevé equation, which we also formulate in terms of the fractional derivatives. The form of the Lagrangian L_2 is motivated by the problems of image regularization.

6.6.1. *Notation*

Let $\boldsymbol{u}\left(t\right) = \left(u_1\left(t\right), \ldots, u_n\left(t\right)\right), t \in [a, b], n \in \mathbb{N}$. We assume $\boldsymbol{u} \in \left(AC^1\left[a, b\right]\right)^n$ and define the left and right Riemann–Liouville fractional derivatives of $\boldsymbol{u}$ of order $\alpha \in (0, 1)$ as follows:

$${}_a\mathrm{D}_t^\alpha \boldsymbol{u} = \left({}_a\mathrm{D}_t^\alpha u_1, \ldots, {}_a\mathrm{D}_t^\alpha u_n\right), \quad {}_t\mathrm{D}_b^\alpha \boldsymbol{u} = \left({}_t\mathrm{D}_b^\alpha u_1, \ldots, {}_t\mathrm{D}_b^\alpha u_n\right), \tag{6.94}$$

where ${}_a\mathrm{D}_t^\alpha u_j$, and ${}_t\mathrm{D}_b^\alpha u_j$, $j \in \{1, \ldots, n\}$, are the left and right Riemann–Liouville fractional derivatives.

For the case of a function of a vector-valued argument, we will need a generalization of a standard nabla operator $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. Let $f \in C^1\left(\mathbb{R}^n\right)$. The (standard) gradient of a function is

$$\mathrm{grad}\, f = \nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Also in the case of the composition of functions $f\left(\boldsymbol{x}\right) = f\left(\boldsymbol{g}\left(\boldsymbol{x}\right)\right), \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{g} = \left(g_1, \ldots, g_n\right)$, we use notation

$$\mathrm{grad}_{\boldsymbol{g}}\, f = \nabla_{\boldsymbol{g}} f = \left(\frac{\partial f}{\partial g_1}, \ldots, \frac{\partial f}{\partial g_n}\right).$$

To "fractionalize" the nabla operator ∇, we recall the Fourier transform of $u \in L^1(\mathbb{R}^n)$

$$\hat{u}(\boldsymbol{\xi}) \equiv \mathcal{F}[u(\boldsymbol{x})](\boldsymbol{\xi}) = \int_{\mathbb{R}^n} u(\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}\mathrm{d}^n\boldsymbol{x}, \;\; \boldsymbol{\xi} \in \mathbb{R}^n,$$

where the dot in $\boldsymbol{\xi} \cdot \boldsymbol{x}$ denotes the scalar product. The fractional Sobolev space $H^s(\mathbb{R}^n)$, $s > 0$, is defined by

$$\begin{aligned} H^s(\mathbb{R}^n) &= \left\{u \in L^2(\mathbb{R}^n) \mid \|u\|_{H^s(\mathbb{R}^n)} < \infty\right\}, \;\; \text{where} \\ \|u\|_{H^s(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \left(1 + |\boldsymbol{\xi}|^2\right)^s |\hat{u}(\boldsymbol{\xi})|^2 \,\mathrm{d}^n\boldsymbol{\xi}\right)^{1/2}. \end{aligned} \tag{6.95}$$

The left and right fractional nabla operators are defined as

$${}^{\alpha}\nabla = \left({}^{\alpha}\mathrm{D}_{x_1}, \ldots, {}^{\alpha}\mathrm{D}_{x_n}\right), \;\; \nabla^{\alpha} = \left(\mathrm{D}^{\alpha}_{x_1}, \ldots, \mathrm{D}^{\alpha}_{x_n}\right), \tag{6.96}$$

where

$${}^{\alpha}\mathrm{D}_{x_j} u = \mathcal{F}^{-1}\left[\left(\mathrm{i}\xi_j\right)^{\alpha} \hat{u}(\boldsymbol{\xi})\right], \;\; \mathrm{D}^{\alpha}_{x_j} u = \mathcal{F}^{-1}\left[\left(-\mathrm{i}\xi_j\right)^{\alpha} \hat{u}(\boldsymbol{\xi})\right], \;\; j = 1, \ldots, n,$$

and we used notation ${}^{\alpha}\mathrm{D}_{x_j} u = {}_{-\infty}\mathrm{D}^{\alpha}_{x_j} u$ and $\mathrm{D}^{\alpha}_{x_j} u = {}_{x_j}\mathrm{D}^{\alpha}_{\infty} u$. We note that, like in the classical setting, we have the fractional gradients if the fractional nabla operators [6.96] are applied to $f \in H^s(\mathbb{R}^n)$ as

$$\begin{aligned} &{}^{\alpha}\mathrm{grad}\, f = {}^{\alpha}\nabla f = \left({}^{\alpha}\mathrm{D}_{x_1} f, \ldots, {}^{\alpha}\mathrm{D}_{x_n} f\right), \\ &\mathrm{grad}^{\alpha} f = \nabla^{\alpha} f = \left(\mathrm{D}^{\alpha}_{x_1} f, \ldots, \mathrm{D}^{\alpha}_{x_n} f\right), \end{aligned} \tag{6.97}$$

and the fractional divergences of a vector function $\boldsymbol{g} = (g_1, \ldots, g_n)$, $\boldsymbol{g} \in \left(H^s(\mathbb{R}^n)\right)^n$, that are defined as

$${}^{\alpha}\mathrm{div}\,\boldsymbol{g} = {}^{\alpha}\nabla \cdot \boldsymbol{g} = \sum_{j=1}^{n} {}^{\alpha}\mathrm{D}_{x_j} g_j, \;\; \mathrm{div}^{\alpha}\,\boldsymbol{g} = \nabla^{\alpha} \cdot \boldsymbol{g} = \sum_{j=1}^{n} \mathrm{D}^{\alpha}_{x_j} g_j.$$

Note that in the previous expressions, we used a dot to denote the scalar product $\boldsymbol{g} \cdot \boldsymbol{h} = \sum_{j=1}^{n} g_j h_j$. Also, we use the notation $g^2 \equiv \boldsymbol{g}^2 = \boldsymbol{g} \cdot \boldsymbol{g} = \sum\limits_{j=1}^{n} g_j^2$. The integration by parts also holds, i.e. for $f \in H^s(\mathbb{R}^n)$ and $\boldsymbol{g} \in \left(H^s\left(\mathbb{R}^n\right)\right)^n$, $s \geq 1$

$$\int_{\mathbb{R}^n} \left({}^{\alpha}\mathrm{grad}\, f\left(\boldsymbol{x}\right)\right) \cdot \boldsymbol{g}\left(\boldsymbol{x}\right) \mathrm{d}^n \boldsymbol{x} = \int_{\mathbb{R}^n} f\left(\boldsymbol{x}\right)\, \mathrm{div}^{\alpha}\, \boldsymbol{g}\left(\boldsymbol{x}\right) \mathrm{d}^n \boldsymbol{x}. \qquad [6.98]$$

6.6.2. *Lagrangian depending on a vector function of scalar argument*

We assume $\boldsymbol{u} = (u_1, \ldots, u_n) \in \left(AC^2\left[a,b\right]\right)^n$. Let us consider the Lagrangian

$$L\left(t, \boldsymbol{u}, {}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right) = \frac{1}{2}\left({}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right)^2 + \Pi\left(t, \boldsymbol{u}\right), \qquad [6.99]$$

where $\left({}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right)^2 = \sum\limits_{j=1}^{n}\left({}_a\mathrm{D}_t^{\alpha}u_j\right)^2$. We assume that $[a,b] \ni t \mapsto \Pi\left(t,\cdot\right)$ is a C^1 mapping from $[a,b]$ into the space of function with the continuous derivatives up to order two, denoted by $C^2\left(\mathbb{R}^n\right)$, i.e. $\Pi\left(t,\cdot\right) : \boldsymbol{x} \mapsto \Pi\left(t, \boldsymbol{x}\right)$, $\Pi\left(t,\cdot\right) \in C^2\left(\mathbb{R}^n\right)$. We will denote this space by $C^1\left(\left[a,b\right]; C^2\left(\mathbb{R}^n\right)\right)$. In formulating the Hamilton principle, we find a minimum of a functional

$$\begin{aligned} J\left[\boldsymbol{u}\right] \equiv J\left[\boldsymbol{u}, {}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right] &= \int_a^b L\left(t, \boldsymbol{u}, {}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right) \mathrm{d}t \\ &= \int_a^b \left(\frac{1}{2}\left({}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right)^2 + \Pi\left(t, \boldsymbol{u}\right)\right) \mathrm{d}t, \end{aligned} \qquad [6.100]$$

where $\boldsymbol{u}$ belongs to a space of admissible functions $\mathcal{U}^n$, defined as

$$\mathcal{U}^n = \left\{\boldsymbol{u} \mid \boldsymbol{u} \in \left(AC^2\left[a,b\right]\right)^n,\ \boldsymbol{u} \text{ satisfies prescribed boundary conditions}\right\}.$$

We refer to [AGR 02a, AGR 06, ATA 08a] for the formulation of the Hamilton principle and Euler–Lagrange equations for the Lagrangian depending on a scalar function, as well as to [KLI 09] for the Lagrangian depending on a vector function.

In the following, we also refer to Hamilton's principle of least action as the primal variational principle. Requiring that [6.100] attains a minimum over $\mathcal{U}^n$, we obtain the Euler–Lagrange equations

$${}_t\mathrm{D}_b^{\alpha}\left(\mathrm{grad}_{{}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}}\, L\right) + \mathrm{grad}_{\boldsymbol{u}}\, L = 0.$$

Since L has the form [6.99], we have $\left(\operatorname{grad}_{{}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}} L\right)_j = {}_a\mathrm{D}_t^{\alpha} u_j,\, j \in \{1, \ldots, n\}$, and

$$ {}_t\mathrm{D}_b^{\alpha}\left({}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right) + \operatorname{grad}_{\boldsymbol{u}} \Pi = 0. \quad [6.101] $$

Each of the Euler–Lagrange equations can be written as a system of two equations with independent functions $\boldsymbol{u}$ and $\boldsymbol{p}$. These equations are called the canonical (or the Hamilton) equations (for the case of a scalar function see [RAB 07]). We define the generalized momentum and the corresponding Hamiltonian as

$$ \boldsymbol{p} = \operatorname{grad}_{{}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}} L = {}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}, \quad [6.102] $$

$$ \mathcal{H}\left(t, \boldsymbol{u}, \boldsymbol{p}\right) = \boldsymbol{p} \cdot \left({}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right) - L\left(t, \boldsymbol{u}, {}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right) = \frac{1}{2}p^2 - \Pi\left(t, \boldsymbol{u}\right), \quad [6.103] $$

where $\boldsymbol{p} \in \mathcal{P}^n = (AC^1[a,b])^n$ at least, since ${}_a\mathrm{D}_t^{\alpha} u_j = \frac{\mathrm{d}}{\mathrm{d}t} {}_a\mathrm{I}_t^{1-\alpha} u_j$ and ${}_a\mathrm{I}_t^{1-\alpha} u_j \in AC^2\left[a, b\right],\ j = 1, \ldots, n$. Next, we write [6.100] in terms of the Hamiltonian as $\int_a^b \left[\boldsymbol{p} \cdot \left({}_a\mathrm{D}_t^{\alpha}\boldsymbol{u}\right) - \mathcal{H}\left(t, \boldsymbol{u}, \boldsymbol{p}\right)\right] \mathrm{d}t$, and use the integration by parts formula [2.22] to obtain

$$ \begin{aligned} \mathcal{I}\left(\boldsymbol{u}, \boldsymbol{p}\right) &= \int_a^b \left(\left({}_t\mathrm{D}_b^{\alpha}\boldsymbol{p}\right) \cdot \boldsymbol{u} - \mathcal{H}\left(t, \boldsymbol{u}, \boldsymbol{p}\right)\right) \mathrm{d}t \\ &= \int_a^b \left(\left({}_t\mathrm{D}_b^{\alpha}\boldsymbol{p}\right) \cdot \boldsymbol{u} - \frac{1}{2}p^2 + \Pi\left(t, \boldsymbol{u}\right)\right) \mathrm{d}t. \end{aligned} \quad [6.104] $$

Note that $\mathcal{I}$ is a functional with two independent functions $\boldsymbol{u}$ and $\boldsymbol{p}$. By requiring that $\mathcal{I}$ attains its minimum at $(\boldsymbol{u}, \boldsymbol{p}) \in \mathcal{U}^n \times \mathcal{P}^n$, we obtain the Hamilton equations

$$ {}_a\mathrm{D}_t^{\alpha}\boldsymbol{u} = \operatorname{grad}_{\boldsymbol{p}} \mathcal{H} = \boldsymbol{p}, \quad {}_t\mathrm{D}_b^{\alpha}\boldsymbol{p} = \operatorname{grad}_{\boldsymbol{u}} \mathcal{H} = -\operatorname{grad}_{\boldsymbol{u}} \Pi. \quad [6.105] $$

To formulate the dual functional, we assume that the following conditions are satisfied. These conditions enable us to solve uniquely the second Hamilton equation $[6.105]_2$ with respect to $\boldsymbol{u}$.

CONDITION 6.1.–

a) $\Pi \in C^1\left(\left[a, b\right]; C^2\left(\mathbb{R}^n\right)\right)$;

b) $\operatorname{grad} \Pi\left(t, \cdot\right) : \mathbb{R}^n \mapsto D$ is a bijection of $\mathbb{R}^n$ to an open subset D of $\mathbb{R}^n$ for every $t \in \left[a, b\right]$;

c) $\det \left|\frac{\partial^2 \Pi(t, \boldsymbol{x})}{\partial x_i \partial x_j}\right| \neq 0$ for every $t \in [a, b]$ and every $\boldsymbol{x} \in \mathbb{R}^n$.

If condition 6.1 is satisfied, then there exists $\mathbf{\Phi} : [a, b] \times D \mapsto \mathbb{R}^n$, of class C^1, such that

$$\boldsymbol{u}(t) = \mathbf{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}(t)\right), \quad t \in [a, b], \qquad [6.106]$$

and

$${}_t\mathrm{D}_b^\alpha \boldsymbol{p} \equiv -\mathrm{grad}_{\mathbf{\Phi}} \Pi\left(t, \mathbf{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)\right), \quad t \in [a, b], \qquad [6.107]$$

i.e. the second Hamilton equation $[6.105]_2$ is identically satisfied with $\mathbf{\Phi}$.

Now, the dual functional G is obtained from [6.104] as ($\boldsymbol{p} \in \mathcal{P}^n$)

$$\begin{aligned} G[\boldsymbol{p}] &\equiv \mathcal{I}\left(\mathbf{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right), \boldsymbol{p}\right) \\ &= \int_a^b \left(\left({}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right) \cdot \mathbf{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right) - \frac{1}{2} p^2 + \Pi\left(t, \mathbf{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)\right) \right) \mathrm{d}t. \end{aligned} \qquad [6.108]$$

We say that the complementary principle holds if for the primal functional J, [6.100], there exists the dual functional G, [6.108], so that if J attains a minimum at a function $\boldsymbol{u} \in \mathcal{U}^n$, then G attains a maximum at a function $\boldsymbol{p} \in \mathcal{P}^n$, where the connection between $\boldsymbol{u}$ and $\boldsymbol{p}$ is given by [6.102].

THEOREM 6.12.– Assume that Π satisfies condition 6.1. Additionally, assume that there exists a constant $c > 0$, such that

$$\sum_{i,j=1}^{n} \frac{\partial^2 \Pi(t, \boldsymbol{x})}{\partial x_i \partial x_j} \mathrm{d}x_i \mathrm{d}x_j \geq c (\mathrm{d}x)^2, \quad t \in [a, b], \ \boldsymbol{x} \in \mathbb{R}^n. \qquad [6.109]$$

Then, the complementary principle holds for functionals [6.100] and [6.108], i.e. it holds that

$$J[\boldsymbol{u}] \leq J[\boldsymbol{U}], \quad G[\boldsymbol{p}] \geq G[\boldsymbol{P}], \qquad [6.110]$$

where $\boldsymbol{U} = \boldsymbol{u} + \delta\boldsymbol{u}$ and $\boldsymbol{P} = \boldsymbol{p} + \delta\boldsymbol{p}$ for $\boldsymbol{u}, \boldsymbol{U} \in \mathcal{U}^n$ and $\boldsymbol{p}, \boldsymbol{P} \in \mathcal{P}^n$. Moreover

$$\|\boldsymbol{U} - \boldsymbol{u}\|_{L^2[a,b]} \leq \sqrt{\frac{2}{c}\left(J[\boldsymbol{U}] - G[\boldsymbol{P}]\right)}. \qquad [6.111]$$

PROOF.– By the Taylor expansion formula for Π and $\boldsymbol{U} = \boldsymbol{u} + \delta\boldsymbol{u}$, we have

$$\begin{aligned}
J[\boldsymbol{U}] - J[\boldsymbol{u}] &= \int_a^b \left(\frac{1}{2}\left(({}_a\mathrm{D}_t^\alpha \boldsymbol{U})^2 - ({}_a\mathrm{D}_t^\alpha \boldsymbol{u})^2\right) + \Pi(t,\boldsymbol{U}) - \Pi(t,\boldsymbol{u})\right)\mathrm{d}t \\
&= \int_a^b \left(({}_a\mathrm{D}_t^\alpha \boldsymbol{u})\cdot({}_a\mathrm{D}_t^\alpha \delta\boldsymbol{u}) + \delta\boldsymbol{u}\cdot \mathrm{grad}_{\boldsymbol{u}}\,\Pi(t,\boldsymbol{u})\right)\mathrm{d}t \\
&\quad + \frac{1}{2}\int_a^b \Big[({}_a\mathrm{D}_t^\alpha \delta\boldsymbol{u})^2 + \delta\boldsymbol{u}\cdot\left([\delta\boldsymbol{u}\cdot\nabla_{\boldsymbol{u}}]\,\mathrm{grad}_{\boldsymbol{u}}\,\Pi(t,\boldsymbol{u})\right)\Big]_{\boldsymbol{u}=\boldsymbol{\xi}}\mathrm{d}t \\
&= \frac{1}{2}\int_a^b \left[({}_a\mathrm{D}_t^\alpha \delta\boldsymbol{u})^2 + \sum_{i,j=1}^{n}\frac{\partial^2\Pi(t,\boldsymbol{u})}{\partial u_i \partial u_j}(\delta u_i)(\delta u_j)\right]_{\boldsymbol{u}=\boldsymbol{\xi}}\mathrm{d}t \\
&\geq \frac{c}{2}\int_a^b (\delta\boldsymbol{u})^2\,\mathrm{d}t = \frac{c}{2}\left\|\delta\boldsymbol{u}\right\|^2_{L^2[a,b]},
\end{aligned} \qquad [6.112]$$

where $\boldsymbol{\xi} = \boldsymbol{u} + \eta(\boldsymbol{U} - \boldsymbol{u})$, $\eta \in (0,1)$. In obtaining [6.112] we used the integration by parts formula [2.22] in order to use the Euler–Lagrange equations [6.101]. We also used [6.109]. Therefore, $J[\boldsymbol{U}] \geq J[\boldsymbol{u}]$.

Further, consider $G(\boldsymbol{p})$ and assume that $\boldsymbol{p} \in \mathcal{P}^n$ is a solution to [6.105]. Let $\boldsymbol{P} = \boldsymbol{p} + \delta\boldsymbol{p}$ be an arbitrary element of $\mathcal{P}^n$. We use the Taylor expansion formula for Π and $\boldsymbol{\Phi}$ and consider the difference

$$\begin{aligned}
&G[\boldsymbol{P}] - G[\boldsymbol{p}] \\
&\quad= \int_a^b \Big[({}_t\mathrm{D}_b^\alpha \boldsymbol{P})\cdot\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{P}) - ({}_t\mathrm{D}_b^\alpha \boldsymbol{p})\cdot\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p}) \\
&\qquad -\frac{1}{2}\left[P^2 - p^2\right] + \Pi\left(t,\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{P})\right) - \Pi\left(t,\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p})\right)\Big]\mathrm{d}t \\
&\quad= \int_a^b \Big[({}_t\mathrm{D}_b^\alpha \delta\boldsymbol{p})\cdot\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p}) - \boldsymbol{p}\cdot\delta\boldsymbol{p} \\
&\qquad + \left(\left[({}_t\mathrm{D}_b^\alpha \delta\boldsymbol{p})\cdot\nabla_{{}_t\mathrm{D}_b^\alpha \boldsymbol{p}}\right]\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p})\right)\cdot\left(({}_t\mathrm{D}_b^\alpha \boldsymbol{p}) + \mathrm{grad}_{\boldsymbol{\Phi}}\,\Pi\left(t,\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p})\right)\right)\Big]\mathrm{d}t \\
&\qquad + \frac{1}{2}\int_a^b \Big[2\,({}_t\mathrm{D}_b^\alpha \delta\boldsymbol{p})\cdot\left(\left[({}_t\mathrm{D}_b^\alpha \delta\boldsymbol{p})\cdot\nabla_{{}_t\mathrm{D}_b^\alpha \boldsymbol{p}}\right]\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p})\right) - (\delta\boldsymbol{p})^2 \\
&\qquad + \left(\left[({}_t\mathrm{D}_b^\alpha \delta\boldsymbol{p})\cdot\nabla_{{}_t\mathrm{D}_b^\alpha \boldsymbol{p}}\right]^2\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p})\right)\cdot\left(({}_t\mathrm{D}_b^\alpha \boldsymbol{p}) + \mathrm{grad}_{\boldsymbol{\Phi}}\,\Pi\left(t,\boldsymbol{\Phi}(t,{}_t\mathrm{D}_b^\alpha \boldsymbol{p})\right)\right)
\end{aligned}$$

$$+\left(\left[\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\nabla_{{}_t\mathrm{D}_b^\alpha\boldsymbol{p}}\right]\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)$$
$$\cdot\Big[\left[\left(\left[\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\nabla_{{}_t\mathrm{D}_b^\alpha\boldsymbol{p}}\right]\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)\cdot\nabla_{\boldsymbol{\Phi}}\right]\mathrm{grad}_{\boldsymbol{\Phi}}\,\Pi\left(t,\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)\Big]\Big]_{\boldsymbol{p}=\boldsymbol{\zeta}}\mathrm{d}t,$$

where $\boldsymbol{\zeta}=\boldsymbol{p}+\eta\left(\boldsymbol{P}-\boldsymbol{p}\right),\ \eta\in(0,1)$. By [6.107], we have

$${}_t\mathrm{D}_b^\alpha\boldsymbol{p}+\mathrm{grad}_{\boldsymbol{\Phi}}\,\Pi\left(t,\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)=0,$$

while, according to [2.22], [6.106], and $[6.105]_1$, the following holds

$$\int_a^b\left[\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)-\boldsymbol{p}\cdot\delta\boldsymbol{p}\right]=\int_a^b\left[{}_a\mathrm{D}_t^\alpha\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)-\boldsymbol{p}\right]\cdot\delta\boldsymbol{p}=0.$$

Thus, we obtain

$$G\left[\boldsymbol{P}\right]-G\left[\boldsymbol{p}\right]=-\frac{1}{2}\int_a^b\left[\left(\delta\boldsymbol{p}\right)^2-\left(2w_1+w_2\right)\right]_{\boldsymbol{p}=\boldsymbol{\zeta}}\mathrm{d}t,\qquad[6.113]$$

where

$$\begin{aligned}w_1&=\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\left(\left[\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\nabla_{{}_t\mathrm{D}_b^\alpha\boldsymbol{p}}\right]\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)\\&=\sum_{i,j=1}^n\frac{\partial\Phi_j\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)}{\partial\,{}_t\mathrm{D}_b^\alpha p_i}\delta\left({}_t\mathrm{D}_b^\alpha p_i\right)\delta\left({}_t\mathrm{D}_b^\alpha p_j\right),\\w_2&=\left(\left[\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\nabla_{{}_t\mathrm{D}_b^\alpha\boldsymbol{p}}\right]\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)\\&\quad\cdot\left(\left[\left(\left[\left({}_t\mathrm{D}_b^\alpha\delta\boldsymbol{p}\right)\cdot\nabla_{{}_t\mathrm{D}_b^\alpha\boldsymbol{p}}\right]\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)\cdot\nabla_{\boldsymbol{\Phi}}\right]\mathrm{grad}_{\boldsymbol{\Phi}}\,\Pi\left(t,\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)\right)\\&=\sum_{i,j,k,l=1}^n\frac{\partial^2\Pi\left(t,\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)}{\partial\Phi_i\partial\Phi_j}\frac{\partial\Phi_i\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)}{\partial\,{}_t\mathrm{D}_b^\alpha p_k}\\&\quad\cdot\frac{\partial\Phi_j\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)}{\partial\,{}_t\mathrm{D}_b^\alpha p_l}\delta\left({}_t\mathrm{D}_b^\alpha p_k\right)\delta\left({}_t\mathrm{D}_b^\alpha p_l\right).\end{aligned}\qquad[6.114]$$

We write [6.107] as

$${}_t\mathrm{D}_b^\alpha p_j=-\frac{\partial\Pi\left(t,\boldsymbol{\Phi}\left(t,{}_t\mathrm{D}_b^\alpha\boldsymbol{p}\right)\right)}{\partial\Phi_j},\quad j=1,\ldots,n,$$

and differentiate it with respect to ${}_t\mathrm{D}_b^\alpha p_k$. Then, we obtain

$$\delta_{j,k} = -\sum_{i=1}^{n} \frac{\partial^2 \Pi\left(t, \boldsymbol{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)\right)}{\partial \Phi_j \partial \Phi_i} \frac{\partial \Phi_i\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)}{\partial\, {}_t\mathrm{D}_b^\alpha p_k}, \qquad [6.115]$$

where $\delta_{j,k}$ denotes the Kronecker delta. According to [6.114] and by the use of [6.115] in $[6.114]_2$, we have

$$w_2 = -\sum_{k,l=1}^{n} \frac{\partial \Phi_k\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)}{\partial\, {}_t\mathrm{D}_b^\alpha p_l} \delta\left({}_t\mathrm{D}_b^\alpha p_k\right) \delta\left({}_t\mathrm{D}_b^\alpha p_l\right) = -w_1.$$

Now, [6.113] becomes

$$\begin{aligned} G[\boldsymbol{P}] - G[\boldsymbol{p}] \;=\; & -\frac{1}{2}\int_a^b \left[(\delta \boldsymbol{p})^2 + \sum_{i,j,k,l=1}^{n} \frac{\partial^2 \Pi\left(t, \boldsymbol{\Phi}\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)\right)}{\partial \Phi_i \partial \Phi_j} \right. \\ & \left. \times \frac{\partial \Phi_i\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)}{\partial\, {}_t\mathrm{D}_b^\alpha p_k} \frac{\partial \Phi_j\left(t, {}_t\mathrm{D}_b^\alpha \boldsymbol{p}\right)}{\partial\, {}_t\mathrm{D}_b^\alpha p_l} \delta\left({}_t\mathrm{D}_b^\alpha p_k\right) \delta\left({}_t\mathrm{D}_b^\alpha p_l\right) \right]_{\boldsymbol{p}=\boldsymbol{\zeta}} \mathrm{d}t \geq 0. \end{aligned}$$

Therefore, $G[\boldsymbol{p}] \geq G[\boldsymbol{P}]$.

Let $\boldsymbol{U}$ and $\boldsymbol{P}$ be two approximate solutions to [6.105], obtained, for example, by the Ritz method. Since $J[\boldsymbol{U}] \geq J[\boldsymbol{u}]$ and $G[\boldsymbol{p}] \geq G[\boldsymbol{P}]$, [6.110] holds. Moreover

$$J[\boldsymbol{U}] - G[\boldsymbol{P}] \geq J[\boldsymbol{U}] - G[\boldsymbol{p}] = J[\boldsymbol{U}] - J[\boldsymbol{u}],$$

so that from [6.112], it holds that

$$J[\boldsymbol{U}] - G[\boldsymbol{P}] \geq \frac{c}{2}\left\|\delta \boldsymbol{u}\right\|_{L^2[a,b]}^2. \quad \blacksquare$$

For the estimates of type [6.111], where the Lagrangians in primal and dual functionals contain derivatives of integer order, see, for example, [ART 80, ATA 83].

REMARK 6.12.–

1) Recall [DEM 67] that the necessary and sufficient condition for the positive definiteness of $V(t,\mathrm{d}\boldsymbol{x}) = \sum_{i,j=1}^{n} \frac{\partial^2 \Pi(t,\boldsymbol{x})}{\partial x_i \partial x_j} \mathrm{d}x_i \mathrm{d}x_j$ is that there exists a continuous function $V^*(\mathrm{d}\boldsymbol{x}) > 0$, $V^*(\mathbf{0}) = 0$, such that $V(t,\mathrm{d}\boldsymbol{x}) \geq V^*(\mathrm{d}\boldsymbol{x})$ for all $(t,\mathrm{d}\boldsymbol{x}) \in [a,b] \times \mathbb{R}^n$. Thus, [6.109] is the sufficient condition for the positive definiteness.

2) In certain cases, we can find $k > 0$, such that $\int_a^b ({}_a\mathrm{D}_t^\alpha \delta\boldsymbol{u})^2 \,\mathrm{d}t \geq k \int_a^b (\delta\boldsymbol{u})^2 \,\mathrm{d}t$. Then, we have the sharper form of [6.111]

$$\|\boldsymbol{U} - \boldsymbol{u}\|_{L^2[a,b]} \leq \sqrt{\frac{2}{c+k} \left(J[\boldsymbol{U}] - G[\boldsymbol{P}]\right)}. \qquad [6.116]$$

We will show this in proposition 6.8.

6.6.2.1. *An example with the quadratic potential* Π

In the following, we treat the Lagrangian [6.99] for the case of the one-dimensional function $u = u(t)$, $t \in [0,1]$ and the quadratic potential $\Pi(t,u) = \frac{1}{2} w(t) u^2(t) - q(t) u(t)$, $t \in [0,1]$, so that the Lagrangian [6.99] takes the form

$$L(t, u, {}_0\mathrm{D}_t^\alpha u) = \frac{1}{2} ({}_0\mathrm{D}_t^\alpha u)^2 + \frac{1}{2} w u^2 - q u, \qquad [6.117]$$

where $\alpha \in (0,1)$, $w \in C^1[0,1]$, $\min_{t\in[0,1]} w(t) = w_0 > 0$, $q \in C^1[0,1]$. We set the space of admissible functions to be

$$\mathcal{U} = \left\{ u \mid u \in C^1[0,1],\ u(0) = 0,\ {}_0\mathrm{D}_t^\alpha u(t)\big|_{t=1} = 0 \right\}.$$

According to [MIS 08], if $u \in C^1[a,b]$ and $\alpha \in (0,1)$, then ${}_a\mathrm{D}_t^\alpha u \in L^r[a,b]$ (${}_t\mathrm{D}_b^\alpha u \in L^r[a,b]$) for $1 \leq r < \frac{1}{\alpha}$. Since $\alpha \in (0,1)$, and $u \in C^1[0,1]$, $u(0) = 0$, then ${}_0\mathrm{D}_t^\alpha u$ is continuous on $[0,1]$. Note that [6.117] represents the fractionalized Lagrangian considered in [ART 80, p. 8].

The Euler–Lagrange equation [6.101], subject to specified boundary conditions, reads

$${}_t\mathrm{D}_1^\alpha ({}_0\mathrm{D}_t^\alpha u) + wu - q = 0, \quad u(0) = 0, \ {}_0\mathrm{D}_t^\alpha u(t)\big|_{t=1} = 0, \qquad [6.118]$$

while the generalized momentum, according to [6.102], is $p = {}_0\mathrm{D}_t^\alpha u$. The canonical equations are

$${}_0\mathrm{D}_t^\alpha u = p, \quad {}_t\mathrm{D}_1^\alpha p = -wu + q.$$

Since the function Π satisfies condition 6.1, the second canonical equation $[6.105]_2$ solved with respect to u yields

$$u = \Phi\left(t, {}_t\mathrm{D}_1^\alpha p\right) = -\frac{{}_t\mathrm{D}_1^\alpha p - q}{w}.$$

The complementary functionals J and G, given by [6.100] and [6.108], respectively, become

$$J[u] = \int_0^1 \left[\frac{1}{2}\left({}_0\mathrm{D}_t^\alpha u\right)^2 + \frac{1}{2}wu^2 - qu\right] \mathrm{d}t, \qquad [6.119]$$

$$G[p] = -\frac{1}{2}\int_0^1 \left[\frac{\left({}_t\mathrm{D}_1^\alpha p - q\right)^2}{w} + p^2\right] \mathrm{d}t. \qquad [6.120]$$

Error estimate [6.111], given in theorem 6.12, yields

$$\|U - u\|_{L^2[0,1]} \leq \sqrt{\frac{2}{w_0}\left(J[U] - G[P]\right)}, \qquad [6.121]$$

where U and P are approximate solutions to [6.118], since [6.109] yields

$$\frac{\partial^2 \Pi(t,u)}{\partial u^2} = w(t) \geq w_0, \quad t \in [0,1].$$

We can improve error estimate [6.121] in the sense of remark 6.12 (2). For that, we need the following proposition.

PROPOSITION 6.8.– Let $f \in L^1(0,1)$ have an integrable fractional derivative ${}_0\mathrm{D}_t^\alpha f \in L^\infty(0,1)$ of order $\alpha \in \left(\frac{1}{2}, 1\right)$, such that ${}_0\mathrm{I}_t^{1-\alpha} f(t)\big|_{t=0} = 0$. Let $s_1, s_2 > 1$, and $s_2 < \frac{2}{3-2\alpha}$. Then it holds that

$$\int_0^1 |f(\tau)|^2 \mathrm{d}\tau \leq K \int_0^1 \left|{}_0\mathrm{D}_t^\alpha f(\tau)\right|^2 \mathrm{d}\tau, \qquad [6.122]$$

where

$$\begin{aligned} K &= \frac{\sigma^{2\sigma}}{(\Gamma(\alpha))^2 (\sigma + \alpha - 1)^{2\sigma} (\rho s_1 + 1)^{\frac{1}{s_1}}}, \\ \sigma &= \frac{1}{s_2} - \frac{1}{2}, \quad \rho = 2\left(\alpha + \frac{1}{s_2}\right) - 3. \end{aligned} \qquad [6.123]$$

PROOF.– We use [ANA 09, theorem 5.5, p. 57] and assume: $p = 2$, $r = 2$, $\omega_1 = \omega_2 = 1$, $l = 1$, $\mu_1 = 0$, $r_1 = 2$. ■

Therefore, by [6.122], [6.116] becomes

$$\|U - u\|_{L^2[a,b]} \leq \sqrt{\frac{2}{w_0 + \frac{1}{K}} \left(J\left[U\right] - G\left[P\right]\right)}. \quad [6.124]$$

We will use the Ritz method to obtain an approximate solution to [6.118]. Thus, we assume

$$U\left(t\right) = \sum_{i=1}^{N} c_i U_i\left(t\right), \quad t \in \left[0, 1\right], \quad [6.125]$$

where c_i are arbitrary constants, and U_i, $i = 1, \ldots, N$, are trial functions satisfying boundary conditions $[6.118]_{2,3}$. This leads to

$$\begin{aligned} {}_0\mathrm{D}_t^{\alpha} U_i\left(t\right) &= \frac{U_i\left(0\right)}{\Gamma\left(1-\alpha\right) t^{\alpha}} + \frac{1}{\Gamma\left(1-\alpha\right)} \int_0^t \frac{U_i'\left(\tau\right)}{\left(t-\tau\right)^{\alpha}} \mathrm{d}\tau \\ &= \frac{1}{\Gamma\left(1-\alpha\right)} \int_0^t \frac{U_i'\left(\tau\right)}{\left(t-\tau\right)^{\alpha}} \mathrm{d}\tau. \end{aligned}$$

Put

$$\int_0^t \frac{U_i'\left(\tau\right)}{\left(t-\tau\right)^{\alpha}} \mathrm{d}\tau = g_i\left(t\right), \quad t \in \left[0, 1\right], \quad [6.126]$$

where g_i, $i = 1, \ldots, N$, are functions that will be given bellow. Boundary condition $[6.118]_3$ implies $g_i\left(1\right) = 0, i = 1, \ldots, N$. From [GAH 77, p. 572], the solution of [6.126] reads

$$U_i'\left(t\right) = \frac{\sin\left(\alpha\pi\right)}{\pi} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{g_i\left(\tau\right)}{\left(t-\tau\right)^{1-\alpha}} \mathrm{d}\tau, \quad t \in \left[0, 1\right], \; i = 1, \ldots, N. \quad [6.127]$$

Now, since $g_i\left(1\right) = 0$, we chose g_i in the form $g_i\left(t\right) = \left(1-t\right)^i$, $t \in \left[0, 1\right]$, $i = 1, \ldots, N$. Substituting this into [6.127] we obtain, up to an arbitrary constant C_i,

$$U_i\left(t\right) = C_i \int_0^t \frac{\left(1-\tau\right)^i}{\left(t-\tau\right)^{1-\alpha}} \mathrm{d}\tau, \quad t \in \left[0, 1\right], \; i = 1, \ldots, N.$$

We put $C_i = 1$, $i = 1, \ldots, N$. The first three functions, defined for $t \in [0,1]$, are

$$\begin{aligned} U_1(t) &= \frac{t^\alpha}{\alpha} - \frac{t^{\alpha+1}}{\alpha(\alpha+1)}, \\ U_2(t) &= \frac{t^\alpha}{\alpha} - 2\frac{t^{\alpha+1}}{\alpha(\alpha+1)} + 2\frac{t^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}, \\ U_3(t) &= \frac{t^\alpha}{\alpha} - 3\frac{t^{\alpha+1}}{\alpha(\alpha+1)} + 6\frac{t^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)} - 6\frac{t^{\alpha+3}}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}. \end{aligned}$$

Let $P_i = \Gamma(\alpha)(1-t)^i$, $i = 1, \ldots, N$, and let k_i, $i = 1, \ldots, N$, be arbitrary constants. We will assume P in the form

$$P(t) = \sum_{i=1}^{N} k_i P_i(t), \quad t \in [0,1]. \qquad [6.128]$$

For the numerical purposes we take $w = w_0 = q = 1$, $\alpha = 0.6$, and $N = 9$. By substituting [6.125] into [6.119] and minimizing with respect to c_i, $i = 1, \ldots, 9$, and by substituting [6.128] into [6.120] and maximizing with respect to k_i, $i = 1, \ldots, 9$, we obtain

$$J[U] = -0.176499, \quad G[P] = -0.176850.$$

By applying proposition 6.8, we obtain $s_2 < \frac{2}{3-2\alpha} = 1.1111$. We chose $s_1 = 2$ and $s_2 = 1.1$ so that from [6.123] we obtain $K = 9.9771$. The error estimate [6.124], yields

$$\|U - u\|_{L^2[0,1]} = \sqrt{\frac{2}{w_0 + \frac{1}{K}}(J[U] - G[P])} = 0.0252692,$$

while from [6.121], we have

$$\|U - u\|_{L^2[0,1]} \leq \sqrt{2(J[U] - G[P])} = 0.0265053,$$

so that we improved the error estimate by 4.7%.

Multiplying $[6.118]_1$ by u, we obtain

$$[{}_t\mathrm{D}_1^\alpha({}_0\mathrm{D}_t^\alpha u)]\, u + u^2 - u = 0. \qquad [6.129]$$

Integration of [6.129] and the use of [2.22] leads to

$$\int_0^1 \left[({}_0\mathrm{D}_t^\alpha u)^2 u + u^2 - u\right] \mathrm{d}t = 2J[u] + \int_0^1 u(t)\,\mathrm{d}t = 0.$$

Therefore

$$J[u] = -\frac{1}{2}\int_0^1 u(t)\,\mathrm{d}t$$

and

$$-0.176499 \leq -\frac{1}{2}\int_0^1 u(t)\,\mathrm{d}t \leq -0.176850.$$

Thus, complementary variational principles give an estimate of the integral of the solution.

In Figure 6.1, we show the graph of an approximate solution U for [6.118], obtained as described above.

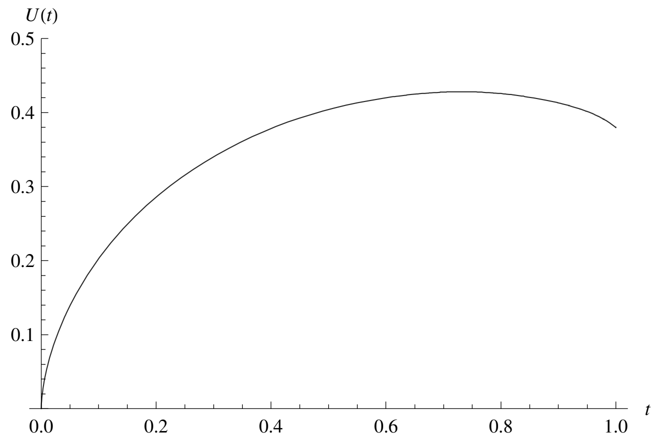

Figure 6.1. *Approximate solution U for [6.118]*

6.6.2.2. *Complementary principles for the fractional and classical Painlevé equations*

In this section, we formulate the complementary variational principles for the fractional and classical first Painlevé equation. In the case of the fractional equation, we formulate the complementary principle on a specific domain of admissible functions. In the case of the classical equation, we know the existence result for the Cauchy problem (see [HOL 84]) and define the space of admissible functions in accordance with the initial data for which solutions exist. In this way we construct, in this set of admissible functions, an analytical approximate solution and give an error estimate. Our analysis opens several questions and indicates the possibility of the use of the complementary principles in both cases.

In both fractional and classical Painlevé equations, we set the space of admissible functions to be

$$\mathcal{U} = \left\{ u \mid u \in C^2\left[0,1\right],\ u\left(0\right) = 0,\ u\left(1\right) = 1,\ u\left(t\right) \geq \sqrt{\frac{t}{6}},\ t \in \left[0,1\right] \right\}. \quad [6.130]$$

6.6.2.2.1. The fractional first Painlevé equation

Consider the Lagrangian of the form [6.99]

$$L\left(t, u, {}_0\mathrm{D}_t^\alpha u\right) = \frac{1}{2}\left({}_0\mathrm{D}_t^\alpha u\right)^2 + \Pi\left(t,u\right),\ \ \Pi\left(t,u\right) = 2u^3 - tu.$$

The Euler–Lagrange equation [6.101] corresponding to this Lagrangian is

$${}_t\mathrm{D}_1^\alpha\left({}_0\mathrm{D}_t^\alpha u\left(t\right)\right) = -6u^2\left(t\right) + t,\ \ t \in \left[0,1\right], \quad [6.131]$$

where $u \in \mathcal{U}$, given by [6.130]. Actually, we can assume $u \in AC^2\left(\left[0,1\right]\right)$. If $\alpha = 1$, then [6.131] is the first Painlevé equation on the finite time interval $t \in \left[0,1\right]$.

The primal functional [6.100] reads

$$J\left[u\right] = \int_0^1 \left(\frac{1}{2}\left({}_0\mathrm{D}_t^\alpha u\right)^2 + 2u^3 - tu\right)\mathrm{d}t. \quad [6.132]$$

The generalized momentum [6.102] is $p = {}_0\mathrm{D}_t^\alpha u$, so that the Hamiltonian [6.102] takes form

$$\mathcal{H}\left(t,u,p\right) = \frac{1}{2}p^2 - 2u^3 + tu,\ \ t \in \left[0,1\right],\ u,p \in \mathbb{R}.$$

The functional $\mathcal{I}$, given by [6.104], becomes

$$\mathcal{I}\left(u,p\right) = \int_0^1 \left(\left({}_t\mathrm{D}_1^\alpha p\right)u - \frac{1}{2}p^2 + 2u^3 - tu\right)\mathrm{d}t. \quad [6.133]$$

Canonical equations (also given by [6.105]) are obtained by the minimization of [6.133] as

$$ {}_0\mathrm{D}_t^\alpha u = p, \;\; {}_t\mathrm{D}_1^\alpha p = -6u^2 + t, \;\; t \in [0,1], \; u, p \in \mathbb{R}. \qquad [6.134] $$

Since Π satisfies condition 6.1 and $u \in \mathcal{U}$, the second canonical equation $[6.134]_2$ solved with respect to u gives

$$ u = \Phi\left(t, {}_t\mathrm{D}_1^\alpha p\right) = \sqrt{\frac{-{}_t\mathrm{D}_1^\alpha p + t}{6}}, \;\; t \in [0,1], \; p \in \mathbb{R}. \qquad [6.135] $$

Therefore, the dual functional takes the form

$$ G[p] = -\int_0^1 \left(\frac{\sqrt{6}}{9} \left(\sqrt{-{}_t\mathrm{D}_1^\alpha p + t} \right)^3 + \frac{1}{2} p^2 \right) \mathrm{d}t. \qquad [6.136] $$

The error estimate follows from theorem 6.12 and it is given by [6.111].

6.6.2.2.2. The classical first Painlevé equation

It is obtained from [6.131] as

$$ \ddot{u}(t) = 6u^2 - t, \;\; t \in [0,1], \qquad [6.137] $$

since $\lim\limits_{\alpha \to 1} {}_0\mathrm{D}_t^\alpha u(t) = \dot{u}(t) \equiv \frac{\mathrm{d}u(t)}{\mathrm{d}t}$, $\lim\limits_{\alpha \to 1} {}_t\mathrm{D}_1^\alpha u(t) = -\dot{u}(t)$ and therefore, $\lim\limits_{\alpha \to 1} {}_t\mathrm{D}_1^\alpha \left({}_0\mathrm{D}_t^\alpha u(t) \right) = -\ddot{u}(t) \equiv \frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2}$. We recall that the space of admissible functions $\mathcal{U}$ is given by [6.130].

REMARK 6.13.– From [HOL 84, lemma 1], we know that the Cauchy problem

$$ \ddot{u}(t) = 6u^2 - t, \;\; u(t)|_{t=0} = u_0, \; \dot{u}(t)|_{t=0} = v_0, \; t > 0, \qquad [6.138] $$

has three sets of solutions denoted by A, B and C. Set A consists of a one-parameter family solutions that have $\sqrt{t}$ as asymptotics as $t \to \infty$. Set B consists of a two-parameter family of solutions oscillating around the asymptotics $-\sqrt{t}$ as $t \to \infty$. Set C is also a two-parameter family; it consists of solutions (as long as they exist) that remain above the asymptotics $\sqrt{t}$, or cross it from below (and therefore remain above). Also, from [HOL 84, remark 4], there exists $\gamma > 0$, such that all solutions to [6.138] with $|u_0| > \gamma$ and $v_0 = 0$ in the set C blow up at some finite t_0. Numerical experiments in [QIN 08, Table 1] show that the solutions to [6.138] belong to set B if initial data u_0 and v_0 have values in certain domains.

Thus, bearing in mind the previous remark and since we consider the variational problem, we consider the solutions belonging to either set A or set C (as long as the solution exists). Thus, we assume that initial and boundary data [6.130] are satisfied so that our solution coincides with the solution to Cauchy problem [6.138] for suitably chosen initial data u_0 and v_0. This is the main hypothesis and it will be tested in the subsequent numerical analysis.

The primal functional, according to [6.132], reads

$$J[u] = \int_0^1 \left(\frac{1}{2}\dot{u}^2 + 2u^3 - tu \right) \mathrm{d}t. \qquad [6.139]$$

Canonical equations and the function Φ, by [6.134] and [6.135], are

$$\dot{u} = p, \quad \dot{p} = 6u^2 - t, \quad u = \Phi\left(t, {}_t\mathrm{D}_1^\alpha p\right) = \sqrt{\frac{\dot{p} + t}{6}}. \qquad [6.140]$$

The dual functional is obtained from [6.136] as

$$G[p] = p(1) - \int_0^1 \left(\frac{\sqrt{6}}{9} \left(\sqrt{\dot{p} + t} \right)^3 + \frac{1}{2}p^2 \right) \mathrm{d}t. \qquad [6.141]$$

The first term in [6.141] is the consequence of the integration by parts in [6.133], and boundary conditions in $[6.137]_{2,3}$.

Let $U \geq 0$ and P be approximate solutions to $[6.140]_{1,2}$. According to the complementary principle, we have

$$J[U] - G[P] \geq J[U] - G[p] = J[U] - J[u],$$

so that

$$J[U] - G[P] \geq \frac{1}{2} \int_0^1 \left[(\delta\dot{u})^2 + 12\Psi(\delta u)^2 \right] \mathrm{d}t \geq \frac{1}{2} \|\delta\dot{u}\|^2_{L^2[0,1]}, \qquad [6.142]$$

where $\Psi = (1 - \varepsilon) u + \varepsilon U > 0$, $\varepsilon \in (0,1)$. Let $t^* \in [0,1]$ be such that

$$|M| = \|\delta u\|_{L^\infty[0,1]} = \sup_{t \in [0,1]} |\delta u(t)| = |\delta u(t^*)|.$$

Then

$$M = \int_0^{t^*} \delta \dot{u}(t)\, \mathrm{d}t, \quad \text{i.e.} \quad |M| \le \int_0^{t^*} |\delta \dot{u}(t)|\, \mathrm{d}t. \tag{6.143}$$

Also

$$M = -\int_{t^*}^{1} \delta \dot{u}(t)\, \mathrm{d}t, \quad \text{i.e.} \quad |M| \le \int_{t^*}^{1} |\delta \dot{u}(t)|\, \mathrm{d}t. \tag{6.144}$$

Thus, [6.143] and [6.144], imply

$$|M| \le \frac{1}{2} \int_0^1 |\delta \dot{u}(t)|\, \mathrm{d}t.$$

From the Cauchy–Schwartz inequality

$$|M| \le \frac{1}{2} \left(\int_0^1 \mathrm{d}t \right)^{1/2} \left(\int_0^1 |\delta \dot{u}(t)|^2\, \mathrm{d}t \right)^{1/2}, \quad \text{i.e.} \quad \|\delta u\|_{L^\infty[0,1]} \le \frac{1}{2} \|\delta \dot{u}\|_{L^2[0,1]}.$$

The previous inequality and [6.142] give

$$\|\delta u\|_{L^\infty[0,1]} = \|U - u\|_{L^\infty[0,1]} \le \frac{\sqrt{2}}{2} \sqrt{J[U] - G[P]}, \tag{6.145}$$

if the solution to [6.137] exists.

We assume

$$U(t) = c_1 t^{c_2} + (1 - c_1 - c_4)\, t^{c_3} + c_4 t^{c_5}, \tag{6.146}$$

where $c_1, \ldots, c_5$ are arbitrary constants. By substituting [6.146] into [6.139] and minimizing with respect to c_i, $i = 1, \ldots, 5$, it follows that

$$U(t) = 0.858624\, t^{5.3256} - 0.657\,24\, t^{5.3256} + 0.798615\, t^{0.994154},$$
$$J[U] = 0.609162.$$

Let

$$P(t) = d_1 + d_2 t^{d_3} + d_4 t^{d_5}, \tag{6.147}$$

where $d_1, \ldots, d_5$ are arbitrary constants. By substituting [6.147] into [6.141] and maximizing with respect to d_i, $i = 1, \ldots, 5$, we obtain

$$P(t) = 0.802055 + 0.504117\, t^{3.96437} + 0.571215\, t^{4.92265},$$
$$G[P] = 0.608744.$$

According to the formula for P, i.e. the approximate solution for $\dot{u}$ (see $[6.140]_1$), we have $P(0) = 0.802055$. Since $u_0 = 0$ and (approximately) $v_0 = 0.802055$, we see, according to [QIN 08, Table 1], that our solution belongs to either set A or to set C.

From [6.145], if the solution to [6.137] exists, we have

$$\|U - u\|_{L^\infty[0,1]} \leq \frac{\sqrt{2}}{2}\sqrt{J[U] - G[P]} = 0.014457.$$

In Figure 6.2, we show the plot of an approximate solution U for [6.137], obtained as described above.

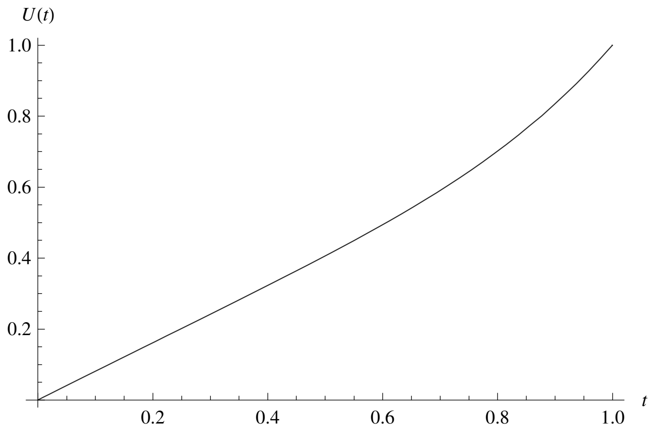

Figure 6.2. *Approximate solution U for [6.137]*

6.6.3. *Lagrangian depending on a scalar function of vector argument*

We consider a scalar function $u(\boldsymbol{x}) = u(x_1, \ldots, x_n)$, such that $u \in \mathcal{U} = H^{s+\alpha}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\alpha > 0$, $s \geq \max\{1, \alpha\}$, where H^s is the Sobolev space defined in section 6.6.1. This implies that $\boldsymbol{u}^{(\alpha)} = {}^{\alpha}\mathrm{grad}\, u \in \left(H^s\left(\mathbb{R}^n\right) \cap L^{\infty}\left(\mathbb{R}^n\right)\right)^n$. We consider the Lagrangian

$$L(u, \boldsymbol{u}^{(\alpha)}) = \Pi(u) + R(\boldsymbol{u}^{(\alpha)}), \qquad [6.148]$$

where $\Pi \in C^2\left(\mathbb{R}\right)$ satisfies $\Pi''\left(x\right) = \frac{\mathrm{d}^2\Pi}{\mathrm{d}x^2} \geq c$, $c > 0$, for every $x \in \mathbb{R}$ and $R \in C^{\infty}\left(\mathbb{R}^n\right)$ satisfies $\left.\mathrm{grad}\, R\left(\boldsymbol{x}\right)\right|_{\boldsymbol{x}=\boldsymbol{0}} = \boldsymbol{0}$. This form of the Lagrangian is used in the image denoising, which will be demonstrated by an example. The primal functional is

$$J\left[u\right] \equiv J\left[u, \boldsymbol{u}^{(\alpha)}\right] = \int_{\mathbb{R}^n} L(u, \boldsymbol{u}^{(\alpha)}) \mathrm{d}^n \boldsymbol{x} = \int_{\mathbb{R}^n} \left(\Pi(u) + R(\boldsymbol{u}^{(\alpha)})\right) \mathrm{d}^n \boldsymbol{x}. \quad [6.149]$$

The Euler–Lagrange equation, obtained by the minimization of the primal functional J, is[2]

$$\mathrm{div}^{\alpha}\left(\mathrm{grad}_{\boldsymbol{u}^{(\alpha)}} L\right) + \frac{\partial L}{\partial u} = 0, \quad \text{i.e.}$$

$$\Pi' + \mathrm{div}^{\alpha}\left(\mathrm{grad}_{\boldsymbol{u}^{(\alpha)}} R\right) = 0, \quad \text{or} \quad \Pi'(u) + \sum_{j=1}^{n} \mathrm{D}_{x_j}^{\alpha} \frac{\partial R\left({}^{\alpha}\mathrm{D}_{x_1} u, \ldots, {}^{\alpha}\mathrm{D}_{x_n} u\right)}{\partial\, {}^{\alpha}\mathrm{D}_{x_j} u} = 0. \qquad [6.150]$$

The generalized momentum is defined by

$$\boldsymbol{p} = \mathrm{grad}_{\boldsymbol{u}^{(\alpha)}} L = \mathrm{grad}_{\boldsymbol{u}^{(\alpha)}} R. \qquad [6.151]$$

To show that $\boldsymbol{p} \in \mathcal{P}^n = \left(H^s\left(\mathbb{R}^n\right)\right)^n$, $s \geq \max\left\{1, \alpha\right\}$, we use:

THEOREM 6.13.– [RUN 96, theorem 2, p. 368]. Let $G \in C^{\infty}\left(\mathbb{R}^n\right)$ and let $G\left(\boldsymbol{0}\right) = 0$. Suppose $s \geq 1$ and $\sigma_p < s < m \in \mathbb{N}$, where $\sigma_p = n \cdot \max\left\{0, \frac{1}{p} - 1\right\}$. Then, there exists a constant c, such that

$$\left\|G\left(f_1, \ldots, f_n\right)\right\|_{F_{p,q}^s(\mathbb{R})} \leq c \cdot \max_{j\in[1,n]} \left(\left\|f_j\right\|_{F_{p,q}^s(\mathbb{R})}\right) \cdot \max_{j\in[1,n]} \left(1 + \left\|f_j\right\|_{L^{\infty}(\mathbb{R})}^{m-1}\right)$$

holds for all $\left(f_1, \ldots, f_n\right) \in \left(F_{p,q}^s\left(\mathbb{R}\right) \cap L^{\infty}\left(\mathbb{R}\right)\right)^n$.

2 The procedure of deriving [6.150] is standard one (see [AGR 02a]) and it is omitted.

According to [RUN 96] (proposition (vi) on p. 13 and proposition (vii) on p. 14) and [6.95], $H^s = H_2^s = F_{2,2}^s$, so that $\sigma_2 = n \cdot \max\left\{0, -\frac{1}{2}\right\} = 0$. We also have $p_j = \frac{\partial R\left({}^{\alpha}\mathrm{D}_{x_1}u,\ldots,{}^{\alpha}\mathrm{D}_{x_n}u\right)}{\partial\, {}^{\alpha}\mathrm{D}_{x_j}u}$ which is such that $p_j \in H^s\left(\mathbb{R}^n\right) \cap L^\infty\left(\mathbb{R}^n\right), j = 1, \ldots, n$, since $\left({}^{\alpha}\mathrm{D}_{x_1}u, \ldots, {}^{\alpha}\mathrm{D}_{x_n}u\right) \in \left(H^s\left(\mathbb{R}^n\right) \cap L^\infty\left(\mathbb{R}^n\right)\right)^n$, $\frac{\partial R\left({}^{\alpha}\mathrm{D}_{x_1}u,\ldots,{}^{\alpha}\mathrm{D}_{x_n}u\right)}{\partial\, {}^{\alpha}\mathrm{D}_{x_j}u} \in C^\infty\left(\mathbb{R}^n\right)$ and $\left.\frac{\partial R(x_1,\ldots,x_n)}{\partial x_j}\right|_{(x_1,\ldots,x_n)=(0,\ldots,0)} = 0$.

Note that we assumed $s \geq \max\{1, \alpha\}$ because in [6.150], we have $\operatorname{div}^\alpha\left(\operatorname{grad}_{\boldsymbol{u}^{(\alpha)}} R\right) = \operatorname{div}^\alpha \boldsymbol{p}$.

We define the Hamiltonian as

$$\mathcal{H}\left(u, \boldsymbol{p}\right) = \boldsymbol{p} \cdot \boldsymbol{u}^{(\alpha)} - L(u, \boldsymbol{u}^{(\alpha)}) = \boldsymbol{p} \cdot \boldsymbol{u}^{(\alpha)} - \Pi(u) - R(\boldsymbol{u}^{(\alpha)}). \qquad [6.152]$$

The Hamiltonian depends on both canonical variables $(u, \boldsymbol{p})$; hence, we use [6.151] and the following condition in order to obtain $\boldsymbol{u}^{(\alpha)}$ as a function of $\boldsymbol{p}$.

CONDITION 6.2.–

a) $R \in C^\infty\left(\mathbb{R}^n\right)$;

b) $\operatorname{grad} R : \mathbb{R}^n \mapsto D$ is a bijection on open set $D \subset \mathbb{R}^n$;

c) $\det\left|\frac{\partial^2 R(\boldsymbol{x})}{\partial x_i \partial x_j}\right| \neq 0$ for every $\boldsymbol{x} \in \mathbb{R}^n$.

Then, by the implicit function theorem, there exists $\boldsymbol{\Upsilon} : D \mapsto \mathbb{R}^n$ of class $\left(C^\infty\right)^n$, such that [6.151] can be inverted and solved for $\boldsymbol{u}^{(\alpha)}$, i.e.

$$\boldsymbol{u}^{(\alpha)} = \boldsymbol{\Upsilon}\left(\boldsymbol{p}\right).$$

Therefore, [6.152] becomes

$$\mathcal{H}\left(u, \boldsymbol{p}\right) = \boldsymbol{p} \cdot \boldsymbol{\Upsilon}\left(\boldsymbol{p}\right) - \Pi(u) - R(\boldsymbol{\Upsilon}\left(\boldsymbol{p}\right)). \qquad [6.153]$$

We write the Lagrangian in [6.149] by using [6.152], i.e. we express the Lagrangian in terms of the Hamiltonian, so that

$$J\left[u, \boldsymbol{u}^{(\alpha)}\right] = \int_{\mathbb{R}^n} \left[\boldsymbol{p} \cdot \boldsymbol{u}^{(\alpha)} - \mathcal{H}\left(u, \boldsymbol{p}\right)\right] \mathrm{d}^n \boldsymbol{x}.$$

Using the integration by parts formula [2.22], we obtain

$$\begin{aligned}\mathcal{I}(u,\boldsymbol{p}) &= \int_{\mathbb{R}^n}\left[u\operatorname{div}^{\alpha}\boldsymbol{p}-\mathcal{H}(u,\boldsymbol{p})\right]\mathrm{d}^n\boldsymbol{x}\\ &= \int_{\mathbb{R}^n}\left[u\operatorname{div}^{\alpha}\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{\Upsilon}(\boldsymbol{p})+\Pi(u)+R(\boldsymbol{\Upsilon}(\boldsymbol{p}))\right]\mathrm{d}^n\boldsymbol{x}. \qquad [6.154]\end{aligned}$$

We treat [6.154] as a variational problem with independent functions u and $\boldsymbol{p}$. By minimizing [6.154] with respect to $(u,\boldsymbol{p})\in\mathcal{U}\times\mathcal{P}^n$, we obtain the Hamilton equations

$$\boldsymbol{u}^{(\alpha)}=\operatorname{grad}_{\boldsymbol{p}}\mathcal{H}=\boldsymbol{\Upsilon}(\boldsymbol{p}), \quad \operatorname{div}^{\alpha}\boldsymbol{p}=\frac{\partial\mathcal{H}}{\partial u}=-\Pi'. \qquad [6.155]$$

To formulate the dual functional, we solve the second Hamilton equation $[6.155]_2$ with respect to $\boldsymbol{u}$. Since $\Pi\in C^2(\mathbb{R})$ satisfies $\Pi''(x)\geq c$, $c>0$, for every $x\in\mathbb{R}$, there exists $\theta:\mathbb{R}\mapsto\mathbb{R}$ of class C^1, such that $[6.155]_2$ can be inverted and solved for u, i.e.

$$u=\theta\left(\operatorname{div}^{\alpha}\boldsymbol{p}\right). \qquad [6.156]$$

Finally, by [6.154] and [6.156], we obtain the dual functional as

$$G[\boldsymbol{p}]=\int_{\mathbb{R}^n}\left[\theta\left(\operatorname{div}^{\alpha}\boldsymbol{p}\right)\operatorname{div}^{\alpha}\boldsymbol{p}-\boldsymbol{p}\cdot\boldsymbol{\Upsilon}(\boldsymbol{p})+\Pi\left(\theta\left(\operatorname{div}^{\alpha}\boldsymbol{p}\right)\right)+R(\boldsymbol{\Upsilon}(\boldsymbol{p}))\right]\mathrm{d}^n\boldsymbol{x}. \qquad [6.157]$$

Next, we state the main result regarding this problem as the following theorem.

THEOREM 6.14.– Assume that $\Pi\in C^2(\mathbb{R})$ satisfies $\Pi''(x)\geq c$, $c>0$, for every $x\in\mathbb{R}$. Also assume that $R\in C^{\infty}(\mathbb{R}^n)$ satisfies $\operatorname{grad}R(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{0}}=\boldsymbol{0}$ and condition 6.2. In addition assume

$$\sum_{i,j=1}^{n}\frac{\partial^2 R(\boldsymbol{x})}{\partial x_i\partial x_j}\mathrm{d}x_i\mathrm{d}x_j\geq 0, \quad \boldsymbol{x}\in\mathbb{R}^n. \qquad [6.158]$$

Then, the complementary principle holds for functionals [6.149] and [6.157], i.e. it holds

$$J[u]\leq J[U], \quad G[\boldsymbol{p}]\geq G[\boldsymbol{P}], \quad J[u]=G[\boldsymbol{p}], \qquad [6.159]$$

where $U = u + \delta u$ and $\boldsymbol{P} = \boldsymbol{p} + \delta \boldsymbol{p}$ for $u, U \in \mathcal{U}$ and $\boldsymbol{p}, \boldsymbol{P} \in \mathcal{P}^n$. Moreover

$$\|U - u\|_{L^2(\mathbb{R}^n)} \leq \sqrt{\frac{2}{c}\left(J[U] - G[\boldsymbol{P}]\right)}. \qquad [6.160]$$

The corresponding results for the Lagrangians containing integer-order derivatives are given in [ART 80].

PROOF.– By the use of the Taylor formula, $U = u + \delta u$ and ${}^{\alpha}\mathrm{grad}\,(\delta u) = \delta\left({}^{\alpha}\mathrm{grad}\, u\right) = \delta \boldsymbol{u}^{(\alpha)}$, we have

$$\begin{aligned} &J[U] - J[u] \\ &= \int_{\mathbb{R}^n} \left[\Pi(U) - \Pi(u) + R(\boldsymbol{U}^{(\alpha)}) + R(\boldsymbol{u}^{(\alpha)})\right] \mathrm{d}^n \boldsymbol{x} \\ &= \int_{\mathbb{R}^n} \left[\Pi'(u)\delta u + \delta \boldsymbol{u}^{(\alpha)} \cdot \mathrm{grad}_{\boldsymbol{u}^{(\alpha)}} R(\boldsymbol{u}^{(\alpha)})\right] \mathrm{d}^n \boldsymbol{x} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \left[\Pi''(u)(\delta u)^2 + \delta \boldsymbol{u}^{(\alpha)} \cdot \left(\left[\delta \boldsymbol{u}^{(\alpha)} \cdot \nabla_{\boldsymbol{u}^{(\alpha)}}\right] \mathrm{grad}_{\boldsymbol{u}^{(\alpha)}} R(\boldsymbol{u}^{(\alpha)})\right)\right]_{u=\xi} \mathrm{d}^n \boldsymbol{x} \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \left[\Pi''(u)(\delta u)^2 + \sum_{i,j=1}^{n} \frac{\partial^2 R(\boldsymbol{u}^{(\alpha)})}{\partial\left({}^{\alpha}\mathrm{D}_{x_i} u\right) \partial\left({}^{\alpha}\mathrm{D}_{x_j} u\right)} \delta\left({}^{\alpha}\mathrm{D}_{x_i} u\right) \delta\left({}^{\alpha}\mathrm{D}_{x_j} u\right)\right]_{u=\xi} \mathrm{d}^n \boldsymbol{x} \\ &\geq \frac{c}{2} \int_{\mathbb{R}^n} (\delta u)^2 \mathrm{d}^n \boldsymbol{x} = \frac{c}{2} \|\delta u\|^2_{L^2(\mathbb{R}^n)}, \end{aligned} \qquad [6.161]$$

where $\xi = u + \eta\,(U - u)$, $\eta \in (0,1)$. In obtaining [6.161], we used integration by parts formula [2.22] and after that we substituted the Euler–Lagrange equations [6.150] and [6.158]. Therefore, $J[U] \geq J[u]$.

Now consider [6.157]. It follows that

$$G[\boldsymbol{P}] - G[\boldsymbol{p}] = \Delta_1 + \Delta_2, \qquad [6.162]$$

where

$$\Delta_1 = \int_{\mathbb{R}^n} \left[P^{(\alpha)}\theta(P^{(\alpha)} - p^{(\alpha)}\theta(p^{(\alpha)}) + \Pi(\theta(P^{(\alpha)})) - \Pi(\theta(p^{(\alpha)}))\right] \mathrm{d}^n \boldsymbol{x}, \qquad [6.163]$$

$$\Delta_2 = \int_{\mathbb{R}^n} \left[R\left(\boldsymbol{\Upsilon}\left(\boldsymbol{P}\right)\right) - R\left(\boldsymbol{\Upsilon}\left(\boldsymbol{p}\right)\right) - \boldsymbol{P} \cdot \boldsymbol{\Upsilon}(\boldsymbol{P}) + \boldsymbol{p} \cdot \boldsymbol{\Upsilon}(\boldsymbol{p})\right] \mathrm{d}^n \boldsymbol{x}. \qquad [6.164]$$

In [6.163], we used the notation $P^{(\alpha)} = \operatorname{div}^{\alpha} \boldsymbol{P}$ and $p^{(\alpha)} = \operatorname{div}^{\alpha} \boldsymbol{p}$. By the use of the Taylor formula, and $\boldsymbol{P} = \boldsymbol{p} + \delta\boldsymbol{p}$, [6.163] becomes

$$\begin{aligned} \Delta_1 &= \int_{\mathbb{R}^n} \left[\theta(p^{(\alpha)}) + \frac{\mathrm{d}\theta(p^{(\alpha)})}{\mathrm{d}p^{(\alpha)}} \left(p^{(\alpha)} + \frac{\mathrm{d}\Pi(\theta)}{\mathrm{d}\theta} \right) \right] \delta p^{(\alpha)} \mathrm{d}^n \boldsymbol{x} \\ &+ \frac{1}{2} \int_{\mathbb{R}^n} \left[\left(2\frac{\mathrm{d}\theta(p^{(\alpha)})}{\mathrm{d}p^{(\alpha)}} + \frac{\mathrm{d}^2\Pi(\theta)}{\mathrm{d}\theta^2} \left(\frac{\mathrm{d}\theta(p^{(\alpha)})}{\mathrm{d}p^{(\alpha)}} \right)^2 \right. \right. \\ &+ \left. \left. \frac{\mathrm{d}^2\theta(p^{(\alpha)})}{\mathrm{d}(p^{(\alpha)})^2} \left(p^{(\alpha)} + \frac{\mathrm{d}\Pi(\theta)}{\mathrm{d}\theta} \right) \right) (\delta p^{(\alpha)})^2 \right]_{p^{(\alpha)} = \operatorname{div}^{\alpha} \boldsymbol{\xi}} \mathrm{d}^n \boldsymbol{x}, \quad [6.165] \end{aligned}$$

where $\boldsymbol{\xi} = \boldsymbol{p} + \epsilon(\boldsymbol{P} - \boldsymbol{p})$, for some $\epsilon \in (0,1)$. According to $[6.155]_2$, we have

$$p^{(\alpha)} + \frac{\mathrm{d}\Pi(\theta)}{\mathrm{d}\theta} = 0.$$

Since $\theta = \theta(p^{(\alpha)})$ (see [6.156]), the differentiation of the previous expression with respect to $p^{(\alpha)}$ yields

$$1 + \frac{\mathrm{d}^2\Pi(\theta)}{\mathrm{d}\theta^2} \frac{\mathrm{d}\theta(p^{(\alpha)})}{\mathrm{d}p^{(\alpha)}} = 0,$$

so that [6.165] becomes

$$\Delta_1 = \int_{\mathbb{R}^n} \theta(p^{(\alpha)} \delta p^{(\alpha)} \mathrm{d}^n \boldsymbol{x} - \frac{1}{2} \int_{\mathbb{R}^n} \left[\left(\frac{\mathrm{d}^2\Pi(\theta)}{\mathrm{d}\theta^2} \right)^{-1} (\delta p^{(\alpha)})^2 \right]_{p^{(\alpha)} = \operatorname{div}^{\alpha} \boldsymbol{\xi}} \mathrm{d}^n \boldsymbol{x}. \quad [6.166]$$

By the use of the Taylor formula in [6.164], we obtain

$$\begin{aligned} \Delta_2 &= \int_{\mathbb{R}^n} \Big[[(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}) \boldsymbol{\Upsilon}(\boldsymbol{p})] \cdot (\operatorname{grad}_{\boldsymbol{\Upsilon}} R(\boldsymbol{\Upsilon}) - \boldsymbol{p}) - \boldsymbol{\Upsilon}(\boldsymbol{p}) \cdot \delta\boldsymbol{p} \Big] \mathrm{d}^n \boldsymbol{x} \\ &+ \frac{1}{2} \int_{\mathbb{R}^n} \left[\Big[(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}})^2 \boldsymbol{\Upsilon}(\boldsymbol{p}) \Big] \cdot (\operatorname{grad}_{\boldsymbol{\Upsilon}} R(\boldsymbol{\Upsilon}) - \boldsymbol{p}) - 2\delta\boldsymbol{p} \right. \\ &\cdot [(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}) \boldsymbol{\Upsilon}(\boldsymbol{p})] + [(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}) \boldsymbol{\Upsilon}(\boldsymbol{p})] \\ &\cdot \left. [([(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}) \boldsymbol{\Upsilon}(\boldsymbol{p})] \cdot \nabla_{\boldsymbol{\Upsilon}}) \operatorname{grad}_{\boldsymbol{\Upsilon}} R(\boldsymbol{\Upsilon})] \right]_{\boldsymbol{p} = \boldsymbol{\xi}} \mathrm{d}^n \boldsymbol{x}, \quad [6.167] \end{aligned}$$

where $\boldsymbol{\xi} = \boldsymbol{p} + \epsilon(\boldsymbol{P} - \boldsymbol{p})$, for some $\epsilon \in (0, 1)$. According to [6.151] and [6.155], we have

$$\boldsymbol{p} = \operatorname{grad}_{\boldsymbol{\Upsilon}} R\left(\boldsymbol{\Upsilon}\right), \tag{6.168}$$

so that [6.167] yields

$$\Delta_2 = -\int_{\mathbb{R}^n} \boldsymbol{\Upsilon}(\boldsymbol{p}) \cdot \delta\boldsymbol{p}\, \mathrm{d}^n\boldsymbol{x} - \frac{1}{2}\int_{\mathbb{R}^n} \left[2q_1 - q_2\right]_{\boldsymbol{p}=\boldsymbol{\xi}} \mathrm{d}^n\boldsymbol{x}, \tag{6.169}$$

where

$$\begin{aligned} q_1 &= \delta\boldsymbol{p} \cdot \left[\left(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}\right) \boldsymbol{\Upsilon}(\boldsymbol{p})\right] = \sum_{k,l=1}^{n} \frac{\partial \Upsilon_k}{\partial p_l} \delta p_k \delta p_l, & [6.170] \\ q_2 &= \left[\left(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}\right) \boldsymbol{\Upsilon}(\boldsymbol{p})\right] \cdot \left[\left(\left[\left(\delta\boldsymbol{p} \cdot \nabla_{\boldsymbol{p}}\right) \boldsymbol{\Upsilon}(\boldsymbol{p})\right] \cdot \nabla_{\boldsymbol{\Upsilon}}\right) \operatorname{grad}_{\boldsymbol{\Upsilon}} R\left(\boldsymbol{\Upsilon}\right)\right] \\ &= \sum_{i,j,k,l=1}^{n} \frac{\partial^2 R\left(\boldsymbol{\Upsilon}\right)}{\partial \Upsilon_i\, \partial \Upsilon_j} \frac{\partial \Upsilon_i}{\partial p_k} \frac{\partial \Upsilon_j}{\partial p_l} \delta p_k \delta p_l. & [6.171] \end{aligned}$$

We write [6.168] as

$$p_j = \frac{\partial R\left(\boldsymbol{\Upsilon}\left(\boldsymbol{p}\right)\right)}{\partial \Upsilon_j}, \quad j = 1, \ldots, n,$$

and differentiate it with respect to p_k. Then, we obtain

$$\delta_{j,k} = \sum_{i=1}^{n} \frac{\partial^2 R\left(\boldsymbol{\Upsilon}\left(\boldsymbol{p}\right)\right)}{\partial \Upsilon_j\, \partial \Upsilon_i} \frac{\partial \Upsilon_i}{\partial p_k}. \tag{6.172}$$

According to [6.170] and by the use of [6.172] in [6.171], we have

$$q_2 = \sum_{k,l=1}^{n} \frac{\partial \Upsilon_k}{\partial p_l} \delta p_k \delta p_l = q_1.$$

Now, [6.169] becomes

$$\Delta_2 = -\int_{\mathbb{R}^n} \boldsymbol{\Upsilon}(\boldsymbol{p}) \cdot \delta\boldsymbol{p}\, \mathrm{d}^n\boldsymbol{x} - \frac{1}{2}\int_{\mathbb{R}^n} \left[\sum_{i,j,k,l=1}^{n} \frac{\partial^2 R\left(\boldsymbol{\Upsilon}\right)}{\partial \Upsilon_i\, \partial \Upsilon_j} \frac{\partial \Upsilon_i}{\partial p_k} \frac{\partial \Upsilon_j}{\partial p_l} \delta p_k \delta p_l\right]_{\boldsymbol{p}=\boldsymbol{\xi}} \mathrm{d}^n\boldsymbol{x},$$

Using the previous expression and [6.166] in [6.162], we obtain

$$\begin{aligned} G\left[\boldsymbol{P}\right]-G\left[\boldsymbol{p}\right] &= \int_{\mathbb{R}^n}\left[\theta(\operatorname{div}^\alpha \boldsymbol{p})\left(\operatorname{div}^\alpha \delta\boldsymbol{p}\right)-\boldsymbol{\Upsilon}(\boldsymbol{p})\cdot\delta\boldsymbol{p}\right]\mathrm{d}^n\boldsymbol{x} \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^n}\left[\left(\frac{\mathrm{d}^2\Pi(\theta)}{\mathrm{d}\theta^2}\right)^{-1}\left(\operatorname{div}^\alpha \delta\boldsymbol{p}\right)^2\right. \\ &\quad \left. +\sum_{i,j,k,l=1}^{n}\frac{\partial^2 R\left(\boldsymbol{\Upsilon}\right)}{\partial\Upsilon_i\,\partial\Upsilon_j}\frac{\partial\Upsilon_i}{\partial p_k}\frac{\partial\Upsilon_j}{\partial p_l}\delta p_k\delta p_l\right]_{\boldsymbol{p}=\boldsymbol{\xi}}\mathrm{d}^n\boldsymbol{x}. \qquad [6.173] \end{aligned}$$

From [6.98], $[6.155]_1$, [6.156], and $\boldsymbol{u}^{(\alpha)}={}^{\alpha}\mathrm{grad}\,u$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n}\left[\theta\left(\operatorname{div}^\alpha \boldsymbol{p}\right)\left(\operatorname{div}^\alpha \delta\boldsymbol{p}\right)-\boldsymbol{\Upsilon}\left(\boldsymbol{p}\right)\cdot\delta\boldsymbol{p}\right]\mathrm{d}^n\boldsymbol{x} \\ &\qquad =\int_{\mathbb{R}^n}\left[{}^{\alpha}\mathrm{grad}\,\left(\theta\left(\operatorname{div}^\alpha \boldsymbol{p}\right)\right)-\boldsymbol{\Upsilon}\left(\boldsymbol{p}\right)\right]\cdot\delta\boldsymbol{p}\,\mathrm{d}^n\boldsymbol{x}=0, \end{aligned}$$

so that [6.173] becomes

$$\begin{aligned} G\left[\boldsymbol{P}\right]-G\left[\boldsymbol{p}\right] &= -\frac{1}{2}\int_{\mathbb{R}^n}\left[\left(\frac{\mathrm{d}^2\Pi(\theta)}{\mathrm{d}\theta^2}\right)^{-1}\left(\operatorname{div}^\alpha \delta\boldsymbol{p}\right)^2\right. \\ &\quad \left. +\sum_{i,j,k,l=1}^{n}\frac{\partial^2 R\left(\boldsymbol{\Upsilon}\right)}{\partial\Upsilon_i\,\partial\Upsilon_j}\frac{\partial\Upsilon_i}{\partial p_k}\frac{\partial\Upsilon_j}{\partial p_l}\delta p_k\delta p_l\right]_{\boldsymbol{p}=\boldsymbol{\xi}}\mathrm{d}^n\boldsymbol{x}\le 0. \end{aligned}$$

Thus, $G\left[\boldsymbol{p}\right]\ge G\left[\boldsymbol{P}\right].$

Let U and $\boldsymbol{P}$ be two approximate solutions to [6.155]. Since $J\left[U\right]\ge J\left[u\right]$ and $G\left[\boldsymbol{p}\right]\ge G\left[\boldsymbol{P}\right]$, then [6.159] holds. Moreover, we have

$$J\left[U\right]-G\left[\boldsymbol{P}\right]\ge J\left[U\right]-G\left[\boldsymbol{p}\right]=J\left[U\right]-J\left[u\right].$$

Therefore, from [6.161],

$$J\left[U\right]-G\left[\boldsymbol{P}\right]\ge\frac{c}{2}\left\|\delta u\right\|^2_{L^2(\mathbb{R}^n)}. \quad \blacksquare$$

6.6.3.1. *Example in the image regularization*

We will specify the Lagrangian, given by [6.148], in the form that is used the image denoising and regularization tasks (see [AUB 06, BAI 07, PER 90]). We chose $\Pi(u) = \frac{1}{2\lambda}(u - u_0)^2$, $u_0 \in H^{s+\alpha}(\mathbb{R}^2)$, $\alpha > 0$, $s \geq \max\{1, \alpha\}$, $\lambda > 0$, and $R(\boldsymbol{u}^{(\alpha)}) = \sqrt{1 + (\boldsymbol{u}^{(\alpha)})^2}$, $\boldsymbol{u}^{(\alpha)} = {}^{\alpha}\mathrm{grad}\, u$. This form of the function R is called the "minimal surface" edge stopping function (see [AUB 06]). Therefore, the Lagrangian takes the form

$$L(u, \boldsymbol{u}^{(\alpha)}) = \frac{1}{2\lambda}(u - u_0)^2 + \sqrt{1 + (\boldsymbol{u}^{(\alpha)})^2}, \quad u \in H^{s+\alpha}(\mathbb{R}^2).$$

The primal functional [6.149] becomes

$$J[u] = \int_{\mathbb{R}^2} \left(\frac{1}{2\lambda}(u - u_0)^2 + \sqrt{1 + (\boldsymbol{u}^{(\alpha)})^2} \right) \mathrm{d}^2\boldsymbol{x}, \qquad [6.174]$$

while the Euler–Lagrange equation, calculated by [6.150], is

$$\mathrm{div}^{\alpha} \frac{\boldsymbol{u}^{(\alpha)}}{\sqrt{1 + (\boldsymbol{u}^{(\alpha)})^2}} + \frac{u - u_0}{\lambda} = 0.$$

The generalized momentum $\boldsymbol{p} \in \left(H^s\left(\mathbb{R}^2\right)\right)^2$, given by [6.151], becomes

$$\boldsymbol{p} = \frac{\boldsymbol{u}^{(\alpha)}}{\sqrt{1 + (\boldsymbol{u}^{(\alpha)})^2}}. \qquad [6.175]$$

Since $R(\boldsymbol{x}) = \sqrt{1 + x^2}$ satisfies condition 6.2, then [6.175] can be solved with respect to $\boldsymbol{u}^{(\alpha)}$ as

$$\boldsymbol{u}^{(\alpha)} = \boldsymbol{\Upsilon}(\boldsymbol{p}) = \frac{\boldsymbol{p}}{\sqrt{1 - p^2}}. \qquad [6.176]$$

From [6.176] it follows that $p^2 < 1$. The Hamiltonian [6.153] takes the form

$$\mathcal{H}(u, \boldsymbol{p}) = -\frac{1}{2\lambda}(u - u_0)^2 - \sqrt{1 - p^2},$$

while the canonical equations are given by [6.155]

$$\boldsymbol{u}^{(\alpha)} = \frac{\boldsymbol{p}}{\sqrt{1-p^2}}, \quad \operatorname{div}^{\alpha} \boldsymbol{p} = \frac{u_0 - u}{\lambda}. \qquad [6.177]$$

Since $\Pi(u) = \frac{1}{2\lambda}(u-u_0)^2$ satisfies $\Pi''(u) = \frac{1}{\lambda} = c \geq 0$, the second canonical equation $[6.177]_2$ can be solved with respect to u as

$$u = \theta(\operatorname{div}^{\alpha} \boldsymbol{p}) = u_0 - \lambda \operatorname{div}^{\alpha} \boldsymbol{p}. \qquad [6.178]$$

The dual functional, according to [6.157], with [6.176] and [6.178], reads

$$G[\boldsymbol{p}] = \int_{\mathbb{R}^2} \left(u_0 \operatorname{div}^{\alpha} \boldsymbol{p} - \frac{1}{2}\lambda (\operatorname{div}^{\alpha} \boldsymbol{p})^2 + \sqrt{1-p^2} \right) \mathrm{d}^2 \boldsymbol{x}. \qquad [6.179]$$

Form theorem 6.14 the primal functional [6.174] attains its minimum, while the dual functional [6.179] attains its maximum. Thus, by [6.160], we have

$$\|\delta u\|_{L^2(\mathbb{R}^n)} \leq \sqrt{2\lambda (J[U] - G[\boldsymbol{P}])}, \qquad [6.180]$$

since [6.158] is also satisfied. In [JAN 11], function u is determined numerically, and an approximate solution is obtained together with the error estimate.

6.7. Generalizations of Hamilton's principle

In this section, we present two generalizations of the classical Hamilton's principle. Namely, we allow minimization of the action integral with respect to a function (from the set of admissible functions) and with respect to the order of derivatives appearing in the Lagrangian. Thus, our stationarity conditions will be more general than those obtained earlier, since we will be able to determine the minimizing function and the order of derivative in the resulting equations describing the dynamics of the process. Two separate cases will be considered. One in which the order of the derivative is constant and one in which the order of the derivative changes with time.

In the standard variational formulation of mechanical problems with fractional derivatives, the following minimization problem is studied: find a minimum of a functional

$$J[y, \alpha] = \int_0^b L(t, y(t), {}_0\mathrm{D}_t^{\alpha} y(t), \alpha) \mathrm{d}t \qquad [6.181]$$

where $t \in [0, b], \ b > 0$, y is an admissible function, i.e. a function having certain regularity properties and satisfying specified boundary conditions, and α is the order of derivative of u. In [6.181] the order of the fractional derivative $0 \leq \alpha < 1$ is given in advance. Also, $L(t, y(t), {}_0\mathrm{D}_t^{\alpha} y\,(t)\,, \alpha)$ is a known function (in physics, Lagrangian density), which satisfies conditions [6.3], i.e.

$$\left.\begin{array}{l} L \in C^1((a,b) \times \mathbb{R} \times \mathbb{R}), \text{ and} \\ t \mapsto \partial_2 L(t, y(t), {}_a\mathrm{D}_t^{\alpha} y\,(t)) \text{ is integrable in } (a,b) \text{ and} \\ t \mapsto \partial_3 L(t, y(t), {}_a\mathrm{D}_t^{\alpha} y\,(t)) \in AC([a,b]), \text{ for every } y \in AC([a,b]), \end{array}\right\} \qquad [6.182]$$

where $AC([0, b])$ is the space of absolutely continuous functions on $[0, b]$, with the norm $\|f\| = \sup_{x \in [0,b]} |f(x)|$.

Let $\mathcal{U}$, the set of admissible functions, be defined as

$$\mathcal{U} = \{y \mid y \in AC([0,b]), \ y \text{ satisfies specified boundary conditions}\}.$$

Since $y \in AC([0, b])$, it follows that ${}_0\mathrm{D}_t^{\alpha} \in L^1([0, b])$. Let $A = [0, 1]$. We consider the following minimization problem (see [ATA 10b]):

6.7.1. *Generalized Hamilton's principle with constant-order fractional derivatives*

If we want to minimize $J\,[y, \alpha]$ in [6.181], there are three possibilities.

1) Given $\mathcal{U}$ and $A = [0, 1]$, find

$$\min_{(y,\alpha) \in \mathcal{U} \times A} J[y, \alpha]. \qquad [6.183]$$

2) Given $\mathcal{U}$ and $A = [0, 1]$, find

$$\min_{\alpha \in A} \left(\min_{y \in \mathcal{U}} J[y, \alpha] \right). \qquad [6.184]$$

3) Given $\mathcal{U}$ and $A = [0, 1]$, find

$$\min_{y \in \mathcal{U}} \left(\min_{\alpha \in A} J[y, \alpha] \right). \qquad [6.185]$$

We will analyze the stationarity conditions for [6.183] and [6.184].

There are two special cases of [6.183]. The first one is obtained when $A = \{1\}$. Then, since ${}_0\mathrm{D}_t^{\alpha} y\,(t)\big|_{\alpha=1} = y^{(1)}(t)$, for $y \in C^1([0,b])$, the solution y^* of [6.183] satisfies the classical Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\,(y^*)^{(1)}} - \frac{\partial L}{\partial y^*} = 0,$$

and the condition $\min_{(y,\alpha)\in\mathcal{U}\times\{1\}} J[y,1]$ is known as classical Hamilton's principle. If again A has single element $A = \{\alpha\}$, given in advance, $0 < \alpha < 1$, then $\min_{(y,\alpha)\in\mathcal{U}\times\{\alpha\}} J[y,\alpha]$ leads to

$${}_t\mathrm{D}_b^{\alpha}\left(\frac{\partial L}{\partial\,{}_0\mathrm{D}_t^{\alpha} y^*}\right) + \frac{\partial L}{\partial y^*} = 0.$$

We analyze the problem when both y and α are varied in $J[y,\alpha]$. Thus, we proceed with the optimality conditions for the minimization problem defined by [6.183]. The generalized Hamilton principle is now in the form

$$\min_{(y,\alpha)\in\mathcal{U}\times A}\int_0^b L(t, y(t), {}_0\mathrm{D}_t^{\alpha} y\,(t)\,, \alpha)\mathrm{d}t = \int_0^b L(t, y^*(t), {}_0\mathrm{D}_t^{\alpha^*} y^*\,(t)\,, \alpha^*)\mathrm{d}t. \quad [6.186]$$

For the optimization problem defined by [6.186], we have the following result. For details, we refer the order to [ATA 10b].

PROPOSITION 6.9.– Let the Lagrangian L satisfy [6.182]. Suppose that $J[y,\alpha]$ mapping $\mathcal{U}\times A$ into $\mathbb{R}$ is continuous. Also suppose that for every $\alpha\in A$, $J[\cdot,\alpha]$ is differentiable, as well as that for every $y\in\mathcal{U}$, $J[y,\cdot]$ is differentiable in $[0,1]$. Then, a necessary condition that functional [6.181] has an extremal point at $(y^*,\alpha^*)\in\mathcal{U}\times A$ is that (y^*,α^*) is a solution of the system of equations

$$\frac{\partial L}{\partial y^*} + {}_t\mathrm{D}_b^{\alpha^*}\left(\frac{\partial L}{\partial\,{}_0\mathrm{D}_t^{\alpha^*} y^*}\right) = 0, \quad [6.187]$$

$$\int_0^b\left(\frac{\partial L}{\partial\,{}_0\mathrm{D}_t^{\alpha^*} y^*}G(y^*,\alpha^*) + \frac{\partial L}{\partial\alpha^*}\right)\mathrm{d}t = 0, \quad [6.188]$$

where

$$G(y^*,\alpha^*) = \frac{\partial\,{}_0\mathrm{D}_t^{\alpha} y^*}{\partial\alpha^*} = \frac{\mathrm{d}}{\mathrm{d}t}(f_1 * y^*)(t,\alpha^*),$$

$$f_1(t,\alpha^*) = \frac{1}{t^{\alpha^*}\Gamma(1-\alpha^*)}[\psi(1-\alpha^*) - \ln t], \quad t > 0,$$

with the Euler function $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \Gamma(z)$ and $(f_1 * y)(t, \alpha) = \int_0^t f_1(\tau, \alpha) y(t-\tau) d\tau$.

PROOF.– Let (y^*, α^*) be an element of $\mathcal{U} \times A$ for which $J[y, \alpha]$ has an extremal value. Let

$$y(t) = y^*(t) + \varepsilon_1 f(t), \quad \alpha = \alpha^* + \varepsilon_2, \;\; \varepsilon_1, \varepsilon_2 \in \mathbb{R},$$

with $f \in AC([0, b])$, $\alpha^* \in (0, 1)$ and the boundary conditions on f are specified so that the varied path $y^* + \varepsilon_1 f$ is an element of $\mathcal{U}$. Then, $J[y, \alpha] = J[y^* + \varepsilon_1 f, \alpha^* + \varepsilon_2] = J(\varepsilon_1, \varepsilon_2)$. A necessary condition for an extremal value of $J[y, \alpha]$ is

$$\left.\frac{\partial J(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_1}\right|_{\varepsilon_1=0,\varepsilon_2=0} = 0, \qquad \left.\frac{\partial J(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_2}\right|_{\varepsilon_1=0,\varepsilon_2=0} = 0. \qquad [6.189]$$

Using [6.189], we obtain

$$\int_a^b \left(\frac{\partial L}{\partial y} f + \frac{\partial L}{\partial \, {}_0\mathrm{D}_t^\alpha y} \, {}_0\mathrm{D}_t^\alpha f \right) \mathrm{d}t = 0, \qquad [6.190]$$

$$\int_0^b \left(\frac{\partial L}{\partial \, {}_0\mathrm{D}_t^\alpha y} \frac{\partial \, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha} + \frac{\partial L}{\partial \alpha} \right) \mathrm{d}t = 0. \qquad [6.191]$$

Applying the fractional integration by parts formula [2.22] to [6.190], the latter is transformed to

$$\int_0^b \left(\frac{\partial L}{\partial y} + {}_t\mathrm{D}_b^\alpha \left(\frac{\partial L}{\partial \, {}_0\mathrm{D}_t^\alpha y} \right) \right) f \, \mathrm{d}t = 0.$$

From this equation, using the fundamental lemma of the calculus of variations (see [DAC 08, p. 115]), we conclude that condition [6.187] holds, for the optimal values y^* and α^*. In [6.188] the term $\frac{\partial \, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}$ may be transformed by the use of expression

$$\begin{aligned} \frac{\partial \, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha} &= \psi(1-\alpha) \, {}_0\mathrm{D}_t^\alpha y - \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\ln(t-\tau) y(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau \\ &= \frac{\mathrm{d}}{\mathrm{d}t} (f_1 * y)(t, \alpha) = G(y, \alpha), \end{aligned} \qquad [6.192]$$

obtained in [FIL 89, p. 592]. Substituting [6.192] into [6.191], we obtain [6.188].

Finally, if [6.188] has no solution for $\alpha \in (0,1)$, then, since $J[y,\cdot]$ is a continuous function in $[0,1]$, the extremal value is attained at the boundaries of the interval, i.e. at $\alpha = 0$ or $\alpha = 1$. ■

REMARK 6.14.– The variational principle we have considered so far is a special case of a class of functionals depending on linear operators (see [FIL 89, p. 51]). Indeed, suppose that Lagrangian L in [6.181] depends on t, y and $\mathcal{L}y$ (instead of ${}_0\mathrm{D}_t^\alpha y$) where $\mathcal{L} : M \to L^p([0,b])$, $p \in [1,\infty)$ is a linear operator defined on a set of admissible functions M which is linear, open and dense in $L^p([0,b])$ (i.e., $\mathcal{L}$ belongs to $\mathrm{Lin}(M, L^p([0,b]))$, the space of continuous, linear functions with the uniform norm). Also suppose that L is continuously differentiable with respect to t and y and twice continuously differentiable with respect to $\mathcal{L}y$. Moreover, suppose that the function $t \longmapsto L(t, y(t), \mathcal{L}y(t))$ is continuous, $t \in [0,b]$. Then, the Euler–Lagrange equation reads

$$\frac{\partial L}{\partial y} + \mathcal{L}^* \frac{\partial L}{\partial(\mathcal{L}y)} = 0,$$

where $\mathcal{L}^*$ denotes the adjoint operator of $\mathcal{L}$. In our case, $\mathcal{L}$ is the left Riemann–Liouville operator ${}_0\mathrm{D}_t^\alpha$, with the adjoint ${}_t\mathrm{D}_b^\alpha$. If instead of $\mathcal{L}$ we consider a family $\{\mathcal{L}_\alpha\}_{\alpha \in A}$, where $A = [0,1]$ (or some other interval), and the mapping $A \to \mathrm{Lin}(M, Lp([0,b]))$, $\alpha \to \mathcal{L}_\alpha$, is differentiable, then a more general problem of finding stationary points with respect to y and α can be formulated. In this case, we can derive the second stationarity condition similar to [6.188]

$$\int_0^b \left(\frac{\partial L}{\partial\left(\mathcal{L}_\alpha y\right)} \frac{\partial\left(\mathcal{L}_\alpha y\right)}{\partial \alpha} + \frac{\partial L}{\partial \alpha} \right) \mathrm{d}t = 0.$$

REMARK 6.15.– Now we discuss the form of the operator $\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}$ at $\alpha = 0$, and $\alpha = 1$. As could be seen from [6.188], it is necessary to calculate the operator $\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}$. In [ATA 07a], it is shown that for $y \in AC([0,b])$, at the boundary point $\alpha = 0$, we have

$$\begin{aligned} \left.\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}\right|_{\alpha=0^+} &= -(\gamma + \ln t) y(0) - \int_0^t (\gamma + \ln \tau) y(t-\tau)\, \mathrm{d}\tau \\ &= -(\gamma + \ln t) y(t) + \int_0^t \frac{y(t) - y(t-\tau)}{\tau}\, \mathrm{d}\tau, \end{aligned} \qquad [6.193]$$

where $\gamma = 0.5772156...$ is the Euler constant. Another form of $\left.\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}\right|_{\alpha=0^+}$ is given in [WES 03, p. 111].

To obtain $\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}$ at $\alpha = 1^-$, we use the method proposed in [TAR 06]. For the sake of completeness, first we give the expansion of $\frac{(t-\tau)^\varepsilon}{\Gamma(1+\varepsilon)}$ with respect to ε, at $\varepsilon = 0$, with $\tau < t$ (see [TAR 06, p. 401]), which will be used in the following:

$$\frac{(t-\tau)^\varepsilon}{\Gamma(1+\varepsilon)} = \frac{e^{\varepsilon \ln(t-\tau)}}{\Gamma(1+\varepsilon)} = 1 + \varepsilon(\gamma + \ln(t-\tau)) + o(\varepsilon).$$

Next, assume that $y \in C^2([0,b])$. Then

$$\begin{aligned} {}_0\mathrm{D}_t^\alpha y\,(t) &= \frac{y(0)}{\Gamma(1-\alpha)t^\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y^{(1)}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau \\ &= \frac{y(0)}{\Gamma(1-\alpha)t^\alpha} + \frac{y^{(1)}(0)}{\Gamma(2-\alpha)t^{\alpha-1}} + \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} y^{(2)}(\tau)\mathrm{d}\tau. \end{aligned} \quad [6.194]$$

Let $\alpha = 1 - \varepsilon$. Then [6.194] becomes

$${}_0\mathrm{D}_t^{1-\varepsilon} y\,(t) = \frac{y(0)}{\Gamma(\varepsilon)t^{1-\varepsilon}} + \frac{y^{(1)}(0)t^\varepsilon}{\Gamma(1+\varepsilon)} + \frac{1}{\Gamma(1+\varepsilon)} \int_0^t (t-\tau)^\varepsilon y^{(2)}(\tau)\mathrm{d}\tau. \quad [6.195]$$

From [6.195] it follows that

$$\begin{aligned} \left.\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}\right|_{\alpha=1^-} &= -\left.\frac{\partial\, {}_0\mathrm{D}_t^{1-\varepsilon} y}{\partial \varepsilon}\right|_{\varepsilon=0^+} \\ &= -\frac{y(0)}{t} - y^{(1)}(0)\ln t - \gamma y^{(1)}(t) - \int_0^t y^{(2)}(\tau)\ln(t-\tau)\,\mathrm{d}\tau. \end{aligned} \quad [6.196]$$

Setting $y(0) = 0$ in [6.196], we recover the results presented in [TAR 06, TOF 08] for the Caputo fractional derivative. In fact, since ${}_0\mathrm{D}_t^\alpha y = {}_0^C\mathrm{D}_t^\alpha y$, $0 < \alpha < 1$, when $y(0) = 0$ by setting $y(0) = 0$ in [6.196], we obtain

$$\left.\frac{\partial\, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha}\right|_{\alpha=1^-} = \left.\frac{\partial\, {}_0^C\mathrm{D}_t^\alpha y}{\partial \alpha}\right|_{\alpha=1^-} = y^{(1)}(0)\ln t + \gamma y^{(1)}(t) + \int_0^t y^{(2)}(\tau)\ln(t-\tau)\,\mathrm{d}\tau.$$

Under some additional conditions, we also have the following proposition. We endow $AC^1([0,b])$ with the usual $C^1([0,b])$ norm. We give conditions which show that problems [6.184] and [6.185] coincide.

PROPOSITION 6.10.– Let the Lagrangian L satisfy assumptions of proposition 6.9. Assume that for every $\alpha \in [0,1]$, there is a unique $y^*(t,\alpha) \in \mathcal{U}$ (a solution to [6.187], and that the mapping $\alpha \mapsto y^*(t,\alpha)$ is differentiable as a mapping from $[0,1]$ to $AC^1([0,b])$. Then, the problem $\min_{(y,\alpha)\in\mathcal{U}\times A} J[y,\alpha]$ is equivalent to the problem $\min_{\alpha\in A}(\min_{y\in\mathcal{U}} J[y,\alpha])$.

PROOF.– We have shown in proposition 6.9, that any solution to the problem $\min_{(y,\alpha)\in\mathcal{U}\times A} J[y,\alpha]$ satisfies systems [6.187] and [6.188]. It can be solved as follows. We first solve [6.187] and the corresponding boundary conditions to obtain $y^* = y^*(t,\alpha)$. According to the assumption, the solution y^* is unique. Then we insert y^* in [6.188] to obtain α^*. In this case, functional $J[y,\alpha]$ becomes a functional depending only on α, $\alpha \mapsto J[y^*(t,\alpha),\alpha] = J[\alpha]$, and therefore, [6.188] transforms into the total derivative of $J[\alpha]$ since

$$\begin{aligned}
\frac{\mathrm{d}J[\alpha]}{\mathrm{d}\alpha} &= \left.\frac{\mathrm{d}J[\alpha+\varepsilon]}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \\
&= \int_0^b \left(\frac{\partial L}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial L}{\partial\, {}_0\mathrm{D}_t^\alpha y}\left({}_0\mathrm{D}_t^\alpha\left(\frac{\partial y}{\partial\alpha}\right) + \frac{\partial\, {}_0\mathrm{D}_t^\alpha}{\partial\alpha} y\right) + \frac{\partial L}{\partial\alpha}\right)\mathrm{d}t \\
&= \int_0^b \left(\frac{\partial y}{\partial\alpha}\left(\frac{\partial L}{\partial y} + {}_t\mathrm{D}_b^\alpha\left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^\alpha y}\right)\right) + \frac{\partial L}{\partial\, {}_0\mathrm{D}_t^\alpha y}\frac{\partial\, {}_0\mathrm{D}_t^\alpha}{\partial\alpha} y + \frac{\partial L}{\partial\alpha}\right)\mathrm{d}t \\
&= \int_0^b \left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^\alpha y}\frac{\partial\, {}_0\mathrm{D}_t^\alpha}{\partial\alpha} y + \frac{\partial L}{\partial\alpha}\right)\mathrm{d}t, \qquad [6.197]
\end{aligned}$$

where we used fractional integration by parts formula [2.22] in the third equality, and equation [6.187] in the last equality. This proves the claim. ∎

One more case is of particular interest.

PROPOSITION 6.11.– Let the Lagrangian L satisfy [6.182]. Assume that for every $\alpha \in [0,1]$ there exists a unique $y^*(t,\alpha) \in \mathcal{U}$, a solution to the fractional variational problem [6.186], and that $J[y_\alpha,\alpha]$ is the corresponding minimal values of the functional J. Assume additionally that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha} J[y,\alpha]\right|_{y=y_\alpha} > 0, \quad y_\alpha \in \mathcal{U}.$$

Then, the minimal, respectively maximal, value of the functional $J[y,\alpha]$ is attained at $\alpha = 0$, respectively at $\alpha = 1$.

PROOF.– Under the previously given assumptions, we have

$$J[y_0, 0] \leq J[y_\alpha, 0] \leq J[y_\alpha, \alpha] \leq J[y_1, \alpha] \leq J[y_1, 1], \quad \forall \alpha \in [0, 1],$$

which proves the claim. ■

REMARK 6.16.– Since $\min(F) = -\max(-F)$, the same argument can be applied to the case when $\frac{\mathrm{d}J[y,\alpha]}{\mathrm{d}\alpha} < 0$, for any fixed $y_\alpha \in \mathcal{U}$, i.e. when J is an decreasing function of α, for any fixed $y_\alpha \in \mathcal{U}$. In this case, the minimal, respectively maximal, value of J is at $\alpha = 1$, respectively $\alpha = 0$. For more details see [ATA 10b].

We present several examples where minimization is performed with respect to both y and α.

6.7.1.1. *Examples with Lagrangians linear in y*

We define the space of admissible functions. Let for $\alpha \in [0, \alpha_0]$. We will distinguish two cases: α_0 strictly less than 1 and $\alpha_0 = 1$. In the case $\alpha_0 < 1$, set

$$\mathcal{U}_l = \{y \in L^1([0, b]) \mid {}_0\mathrm{D}_t^\alpha y \in L^1([0, b])\}.$$

Obviously, $AC([0, b])$ is a subset of $\mathcal{U}_l$. In the case $\alpha_0 = 1$, we assume that $y \in \mathcal{U}_l$ and that, in addition, ${}_0\mathrm{D}_t^1 y$ exists, and ${}_0\mathrm{D}_t^1 y = y^{(1)}$ is an integrable function. Let us note that we can consider $\mathcal{U}_l$ defined with $L^p([0, b])$ (or their subspaces) instead of $L^1([0, b])$. In general, we will use the notation

$$\mathcal{U} = \{y \in \mathcal{U}_l \mid y \text{ satisfies specified boundary conditions}\}. \qquad [6.198]$$

We will sometimes write $\mathcal{U}$ also for $\mathcal{U}_l$ as well (then the set of specified boundary conditions is empty).

Consider the action integral for the inertial motion (no force acting) of a material point of the form

$$J[y, \alpha] = \int_0^1 ({}_0\mathrm{D}_t^\alpha y)^2 \mathrm{d}t, \quad (y, \alpha) \in \mathcal{U} \times A, \qquad [6.199]$$

where $\mathcal{U} = \{y \in U_l \mid y(0) = 0,\ y(1) = 1\}$ and $A = [0, 1]$.

Obviously, the minimal value of $J[y, \alpha]$ is zero, and it is attained whenever ${}_0\mathrm{D}_t^\alpha y = 0$. Solutions to equation ${}_0\mathrm{D}_t^\alpha y = 0$ are of the form $y(t) = C \cdot t^{1-\alpha}$, $t \in [0, 1]$, $C \in \mathbb{R}$. All solutions satisfy the Euler–Lagrange equation

$$
{}_t\mathrm{D}_1^\alpha({}_0\mathrm{D}_t^\alpha y) = 0.
$$

Stationarity condition [6.188] reads

$$
\int_0^1 {}_0\mathrm{D}_t^\alpha y \left(\psi(1-\alpha) {}_0\mathrm{D}_t^\alpha y - \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{\ln(t-\tau) y(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau \right) \mathrm{d}t = 0
$$

and is automatically satisfied.

Note that $C \cdot t^{1-\alpha} \in U_l$, for all $C \in \mathbb{R}$, but only $t^{1-\alpha} \in \mathcal{U}$. Hence, we conclude that $(y^*, \alpha^*) = (t^{1-\alpha}, \alpha)$, $\alpha \in [0, 1]$, are solutions to the variational problem $J[y, \alpha] \to \min$, for J defined by [6.199].

REMARK 6.17.– If $L(t, y(t), {}_0\mathrm{D}_t^\alpha y, \alpha) = ({}_0\mathrm{D}_t^\alpha y)^2 + (\alpha - \alpha_0)^2$, for a fixed $\alpha_0 \in (0, 1)$, then the problem $\int_0^1 L(t, y(t), {}_0\mathrm{D}_t^\alpha y(t), \alpha)\mathrm{d}t \to \min$ has a unique minimizer $(y^*, \alpha^*) = (t^{1-\alpha_0}, \alpha_0)$.

Let the Lagrangian L be of the form

$$
L(t, y(t), {}_0\mathrm{D}_t^\alpha y(t), \alpha) = ({}_0\mathrm{D}_t^\alpha y(t))^2 - cy(t), \quad t \in [0, 1], \ c \in \mathbb{R},
$$

and let $U = \{y \in U_l \,|\, y(0) = 0\}$, $A = [0, 1]$, for the variational problem

$$
\min_{(y,\alpha) \in \mathcal{U} \times A} J[y, \alpha] = \min_{(y,\alpha) \in \mathcal{U} \times A} \int_0^1 (({}_0\mathrm{D}_t^\alpha y)^2 - cy)\mathrm{d}t.
$$

Equations [6.187] and [6.188] become

$$
{}_t\mathrm{D}_1^\alpha \left({}_0\mathrm{D}_t^\alpha y\right) = c \ \text{ and } \ \int_0^1 {}_0\mathrm{D}_t^\alpha y \frac{\partial \, {}_0\mathrm{D}_t^\alpha y}{\partial \alpha} \mathrm{d}t = 0. \qquad [6.200]
$$

Equation $[6.200]_1$ could be solved as follows. Introducing a substitution $z(t) = {}_0\mathrm{D}_t^{\alpha} y$, $t \in [0,1]$, and solving ${}_t\mathrm{D}_1^{\alpha} z = c$ we get

$$z(t) = {}_0\mathrm{D}_t^{\alpha} y(t) = c\frac{(1-t)^{\alpha}}{\Gamma(1-\alpha)}, \ \ t \in [0,1], \ \alpha \in A. \qquad [6.201]$$

We apply ${}_0\mathrm{I}_t^{\alpha}$ on both sides of [6.201]. Using condition $y(0) = 0$ and the fact that ${}_0\mathrm{I}_t^{\alpha}({}_0\mathrm{D}_t^{\alpha} y) = y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\left[{}_0\mathrm{I}_t^{1-\alpha} y\,(t)\right]_{t=0}$ (see [2.18]), we obtain

$$\begin{aligned} y(t,\alpha) &= \frac{c}{\Gamma(\alpha)\Gamma(1+\alpha)}\int_0^t (t-\tau)^{\alpha-1}(1-\tau)^{\alpha}\mathrm{d}\tau \\ &= \frac{c}{\Gamma(1+\alpha)}\sum_{p=0}^{\infty}\frac{\Gamma(p-\alpha)\Gamma(1+p)}{\Gamma(-\alpha)p\,!\Gamma(1+p+\alpha)}t^{p+\alpha}, \ \ t \in [0,1], \ \alpha \in A. \end{aligned}$$

This solution is unique and belongs to U. Since $\alpha \mapsto y(t,\alpha)$ is differentiable, proposition 6.10 holds.

We substitute obtained $y(t,\alpha)$ into $J[y,\alpha]$, which yields

$$\begin{aligned} J[y,\alpha] &= \int_0^1 \left(\left(c\frac{(1-t)^{\alpha}}{\Gamma(1-\alpha)}\right)^2 - \frac{c^2}{\Gamma(\alpha)\Gamma(1+\alpha)}\int_0^t (t-\tau)^{\alpha-1}(1-\tau)^{\alpha}\mathrm{d}\tau\right)\mathrm{d}t \\ &= \int_0^1 \left(\left(c\frac{(1-t)^{\alpha}}{\Gamma(1-\alpha)}\right))^2 - \frac{c^2}{\Gamma(1+\alpha)}\sum_{p=0}^{\infty}\frac{\Gamma(p-\alpha)\Gamma(1+p)}{\Gamma(-\alpha)p\,!\Gamma(1+p+\alpha)}t^{p+\alpha}\right)\mathrm{d}t. \end{aligned}$$

Simple numerical calculations show that $J[\alpha]$ is an increasing function and attains extremal values at the boundaries.

REMARK 6.18.– Equation [6.200] represents a fractional generalization of the equation of motion for a material point (with unit mass) under the action of constant force equal to c. Our result shows that an optimal value of Hamilton's action is attained for $\alpha = 1$, that is for integer-order dynamics. We note that different generalizations of classical equation of motion can be found in [KWO 05], where the problem ${}_0\mathrm{D}_t^{\alpha} y = c$, $1 < \alpha \leq 2$, was analyzed.

6.7.1.2. *Examples with Lagrangians linear in* ${}_0\mathrm{D}_t^{\alpha} y$

Let

$$L(t,y,{}_0\mathrm{D}_t^{\alpha} y,\alpha) = \Gamma(1-\alpha){}_0\mathrm{D}_t^{\alpha} y - \frac{1}{2}cy^2, \ \ c > 0, \ c \neq 1, \qquad [6.202]$$

and consider the problem of finding stationary points for the functional [6.181], where $U = \{y \in U_l \mid y(0) = \frac{1}{c}\}$ and $\alpha \in [0, \alpha_0]$, $\alpha_0 < \frac{1}{2}$. Note that L satisfies the so-called primary constraint in Dirac's classification of systems with constraints (see [HEN 92]). In the setting of fractional derivatives, such systems have been recently treated in [BAL 04, MUS 05].

Equations [6.187] and [6.188] become

$$\Gamma(1-\alpha)\,{}_t\mathrm{D}_1^{\alpha}1 - cy = 0, \qquad [6.203]$$

and

$$\int_0^1 \left(\Gamma(1-\alpha)\frac{\partial\,{}_0\mathrm{D}_t^{\alpha}y}{\partial\alpha} + \frac{\partial\Gamma(1-\alpha)}{\partial\alpha}\right)\mathrm{d}t = 0. \qquad [6.204]$$

Equation [6.203] has a unique solution $y^* = \frac{1}{c(1-t)^{\alpha}}$, $t \in [0,1]$, $\alpha \in [0, \alpha_0]$. This implies

$$\begin{aligned} J[y^*, \alpha] &= \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{1}{c(1-\tau)^{\alpha}(t-\tau)^{\alpha}}\mathrm{d}\tau - \frac{1}{2c(1-t)^{2\alpha}}\right)\mathrm{d}t \\ &= \left[\int_0^t \frac{1}{c(1-\tau)^{\alpha}(t-\tau)^{\alpha}}\mathrm{d}\tau\right]_0^1 - \int_0^1 \frac{1}{2c(1-t)^{2\alpha}}\mathrm{d}t \\ &= \int_0^1 \frac{1}{c(1-\tau)^{2\alpha}}\mathrm{d}\tau - \int_0^1 \frac{1}{2c(1-t)^{2\alpha}}\mathrm{d}t \\ &= \frac{1}{2c}\int_0^1 \frac{1}{(1-t)^{2\alpha}}\mathrm{d}t. \end{aligned}$$

Since $\alpha_0 < \frac{1}{2}$, then $J[y^*, \alpha]$ exists and is an increasing function with respect to α. Hence, $J[y^*, \alpha]$ attains its minimal value at $\alpha = 0$, and it equals $\frac{1}{2c}$. Also, that the maximal value of $J[y^*, \alpha]$ is attained at α_0.

Let $U = \{y \in U_l \mid y(0) = 0\}$, $c \neq 0$ and let L be of the form

$$L(t, y(t), {}_0\mathrm{D}_t^{\alpha}y(t), \alpha) = c \cdot {}_0\mathrm{D}_t^{\alpha}y(t) + f(y(t)), \quad t \in [0,1], \qquad [6.205]$$

where the properties of f are going to be specified. In this example, we are dealing with integrable functions that can take values ∞ or $-\infty$ at some points. We are going to analyze stationary points of

$$J[y, \alpha] = \int_0^1 (c \cdot {}_0\mathrm{D}_t^{\alpha}y(t) + f(y(t)))\mathrm{d}t.$$

Equations [6.187] and [6.188] become

$$ {}_t\mathrm{D}_1^{\alpha} c + \frac{\partial f}{\partial y} = 0, \qquad [6.206]$$

$$ c \cdot \int_0^1 \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha} \mathrm{d}t = 0. \qquad [6.207]$$

Since ${}_t\mathrm{D}_1^{\alpha} c = \frac{c}{\Gamma(1-\alpha)(1-t)^{\alpha}}$, $t \in [0,1]$, we see that in order to solve [6.206]–[6.207], we have to assume that $f \in C^1(R)$, and that $f^{(1)}$ is invertible so that $t \mapsto (f^{(1)})^{-1}(\frac{c}{\Gamma(1-\alpha)(1-t)^{\alpha}}) \in U_l$. Then, equation [6.206] is solvable with respect to y

$$ y_c(t,\alpha) = \left(\frac{f}{y}\right)^{-1} \left(\frac{c}{\Gamma(1-\alpha)(1-t)^{\alpha}}\right), \quad t \in [0,1]. \qquad [6.208]$$

Since $c \neq 0$, [6.207] implies $\int_0^1 \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha} \mathrm{d}t = 0$. Thus

$$\begin{aligned} 0 &= \int_0^1 \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha} \mathrm{d}t = \int_0^1 G(y,\alpha)(t)\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f_1 * y)(t,\alpha)\mathrm{d}t \\ &= f_1 * y(t,\alpha)|_{t=1} - f_1 * y(t,\alpha)|_{t=0} = f_1 * y(t,\alpha)|_{t=1}, \end{aligned} \qquad [6.209]$$

where we have used that $f_1 \in L^1([0,1])$ and that $y \in U$. Substitution of [6.208] into [6.209] gives $(f_1 * y(t,\alpha))(t)|_{t=1} = 0$ or

$$ \int_0^1 \frac{\psi(1-\alpha) - \ln(1-\tau)}{\Gamma(1-\alpha)(1-\tau)^{\alpha}} \left(\frac{f}{y}\right)^{-1} \left(\frac{c}{\Gamma(1-\alpha)(\tau-1)^{\alpha}}\right) \mathrm{d}\tau = 0. \qquad [6.210]$$

Solving this equation is, in general, difficult. Hence, we consider some special cases.

(a) $f(y(t)) = d \cdot \frac{y(t)^2}{2}$, $t \in [0,1]$, $d \in R$. Then, the Lagrangian is

$$ L(t, y, {}_0\mathrm{D}_t^{\alpha} y, \alpha) = c \cdot {}_0\mathrm{D}_t^{\alpha} y + \frac{1}{2} d y^2, $$

and

$$ y_c(t,\alpha) = -\frac{1}{d} \frac{c}{\Gamma(1-\alpha)(1-t)^{\alpha}}, \quad t \in [0,1]. $$

Also, [6.210] becomes

$$\int_0^1 \frac{\psi(1-\alpha) - \ln(1-\tau)}{\Gamma(1-\alpha)^2(1-\tau)^{2\alpha}} \mathrm{d}\tau = 0.$$

By a simple numerical calculation it could be shown that this equation does not have any solution for $\alpha \in (0,1)$. Hence, in this case, there does not exist any point (y, α), which is an extremal of functional $J[y, \alpha]$.

(b) $f(y(t)) = \ln y(t)$, $t \in [0,1]$. Then [6.206] becomes

$$\frac{c}{\Gamma(1-\alpha)(1-t)^{\alpha}} = \frac{1}{y},$$

and therefore

$$y = \frac{\Gamma(1-\alpha)}{c}(1-t)^{\alpha} \in AC([0,1]).$$

In this particular case, we take the set of admissible functions to be $U = \{y \in U_l \mid y(1) = 0\}$. Using [6.209], equation [6.207] reads

$$\int_0^1 \left(\psi(1-\alpha) - \ln(1-\tau)\right) \mathrm{d}\tau = 0,$$

which, after integration, yields $\psi(\alpha - 1) = 1$. A unique solution of this equation in $(0,1)$ is $\alpha = 0.604...$. Therefore, a unique stationary point of

$$J[y, \alpha] = \int_0^1 (c \cdot {}_0\mathrm{D}_t^{\alpha} y + \ln y(t)) \mathrm{d}t$$

is the point $(y, \alpha) = \left(\frac{\Gamma(0{,}396)}{c}(1-t)^{\alpha}; 0.604\right)$.

REMARK 6.19.– So far, we have considered variational problems defined via functionals of type [6.181]. In fact, we have allowed fractional derivatives of functions to appear in Lagrangians. The natural generalization of such problems consists of replacing the Lebesgue integral in [6.181] by the Riemann–Liouville fractional integral. More precisely, for $\beta > 0$ set

$$\begin{aligned} J_{\beta}[y, \alpha] &= {}_0\mathrm{I}_b^{\beta} L(t, y(t), {}_0\mathrm{D}_t^{\alpha} y(t), \alpha) \\ &= \frac{1}{\Gamma(\beta)} \int_0^b (b-t)^{\beta-1} L(t, y(t), {}_0\mathrm{D}_t^{\alpha} y(t), \alpha) \mathrm{d}t, \quad t \in (0, b). \end{aligned}$$

Then, the fractional variational problem consists of finding extremal values of the functional $J_\beta[y,\alpha]$. In the above construction, we have used the left Riemann–Liouville fractional integral of order β (which, in general, differs from the order of fractional differentiation α), evaluated at $t=b$. The choice $\beta=1$ turns us back to problem [6.181].

The study of such fractional variational problems is reduced to the case we have already considered in the following way. It suffices to redefine the Lagrangian as

$$L_1(t,y(t),{}_0\mathrm{D}_t^\alpha y\left(t\right),\alpha,\beta)=\frac{1}{\Gamma(\beta)}(b-t)^{\beta-1}L(t,y(t),{}_0\mathrm{D}_t^\alpha y\left(t\right),\alpha).$$

Then, we have to consider the functional

$$J_\beta[y,\alpha]=\int_0^b L_1(t,y(t),{}_0\mathrm{D}_t^\alpha y\left(t\right),\alpha,\beta)\mathrm{d}t. \qquad [6.211]$$

In case $\beta>1$, L_1 is of the same regularity as L, the straightforward application of the results derived in previous sections to the Lagrangian L_1 leads to the optimality conditions for the variational problem defined via the functional [6.211]. However, when $0<\beta<1$, continuity as well as differentiability of L_1 with respect to t may be violated (which depends of the explicit form of L), and hence, it may be not possible to use the theory developed so far.

6.7.2. *Generalized Hamilton's principle with variable-order fractional derivatives*

Here, we generalize the fractional Hamilton's principle by introducing the variable-order fractional derivative into the Lagrangian. Variable-order fractional derivatives are presented in several publications, see for example [COI 03, DAS 08, LOR 02, RAM 07, ROS 93, SOO 05]. We will replace the integer-order derivative in the Lagrangian density with the variable-order fractional derivatives so that the action integral has the form

$$J\left[y,\alpha\right]=\int_0^T L\left(t,y\left(t\right),{}_0\mathrm{D}_t^{\alpha(t)}y\left(t\right),\alpha\left(t\right)\right)\mathrm{d}t, \qquad [6.212]$$

where L is the Lagrangian density, y is a generalized coordinate, ${}_0\mathrm{D}_t^{\alpha(t)}y\left(t\right)$ denotes the left Riemann–Liouville fractional derivative of the order $\alpha=\alpha\left(t\right),\ t\in\left[0,T\right]$, of y and T is given time instant. Our main assumption is that *the minimization in* [6.212] *should be performed with respect to both* y *and* α. This minimization leads to a new type of variational problem. To obtain the necessary conditions for the optimality in the case of the Lagrangian having variable-order fractional derivatives we need the corresponding integration by parts formula. We derive this formula for one (of several existing) type of fractional derivatives of variable order. We will present results of [ATA 11c]. Note that in [ODZ 13] a similar problem was treated for the Caputo type of variable-order fractional derivatives (with a different definition from the one used here) and for the case when the Lagrangian depends on classical integer-order derivatives and on variable-order fractional derivatives and integrals.

In Lagrangian density $L\left(t, y\left(t\right), {}_a\mathrm{D}_t^{\alpha(t)} y\left(t\right), \alpha\left(t\right)\right)$, we assume that y and α belong to specified spaces, i.e. $y \in \mathcal{U}$, where $\mathcal{U}$ is the set of admissible state functions and $\alpha \in \mathcal{V}$, where $\mathcal{V}$ is the set of admissible order functions. In each specific example $\mathcal{U}$ and $\mathcal{V}$ are specified in advance. Generally, as before, $\mathcal{U}$ will be the set of functions satisfying some regularity properties (for example $y \in AC\left([0,T]\right)$) and specified boundary conditions. For simplicity, we assume that $0 < \alpha(t) < 1$, $t \in [0,T]$.

6.7.2.1. *Various definitions of variable-order fractional derivatives*

We stress the fact that our results depend on the definition of fractional derivative of variable-order. In publications that have appeared lately several definitions of variable order fractional derivatives have been introduced. The questions raised in these publications have a deep meaning from both mathematical and physical points of view (see [COI 03, RAM 07]).

Suppose that α is not a constant but a given function of t, satisfying $0 \leq \alpha\left(t\right) < 1$, $t \in [0,T]$. The problem is how to define ${}_0\mathrm{I}_t^{\alpha(t)} y\left(t\right)$, ${}_0\mathrm{D}_t^{\alpha(t)} y\left(t\right)$, ${}_t\mathrm{I}_T^{\alpha(t)} y\left(t\right)$ and ${}_t\mathrm{D}_T^{\alpha(t)} y\left(t\right)$. In all results that follow we assume $y(t) = 0$ for $t < 0$.

In [ROS 93], the left fractional integral of variable order is proposed as

$$ {}_0\mathrm{I}_t^{\alpha(t)} y\left(t\right) = \frac{1}{\Gamma\left(\alpha\left(t\right)\right)} \int_0^t \left(t-\tau\right)^{\alpha(t)-1} y\left(\tau\right) \mathrm{d}\tau, \quad 0 \leq t \leq T. \qquad [6.213] $$

In [LOR 02], several definitions are introduced. The first is identical to [6.213]. The next one is

$$ {}_0\mathrm{I}_t^{\alpha(t)} y\left(t\right) = \int_0^t \frac{1}{\Gamma\left(\alpha\left(\tau\right)\right)} \left(t-\tau\right)^{\alpha(\tau)-1} y\left(\tau\right) \mathrm{d}\tau, \quad 0 \leq t \leq T. \qquad [6.214] $$

Also, in [LOR 02] the following definition, in which it is assumed that α is a function of $\left(t-\tau\right)$, i.e.

$$ {}_0\mathrm{I}_t^{\alpha(t)} y\left(t\right) = \int_0^t \frac{1}{\Gamma\left(\alpha\left(t-\tau\right)\right)} \left(t-\tau\right)^{\alpha(t-\tau)-1} y\left(\tau\right) \mathrm{d}\tau, \quad 0 \leq t \leq T. \qquad [6.215] $$

is proposed.

The fractional derivative of variable order could now be defined simply (as in the case of constant order) by taking the first derivative of the $(1-\alpha(t))$ integral given by any of the expressions [6.213]–[6.215]. This leads to the definition of the form

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\Gamma(1-q(t,\tau))} (t-\tau)^{-q(t,\tau)} y(\tau)\,\mathrm{d}\tau, $$

where $q(t,\tau) = \alpha(t)$, $q(t,\tau) = \alpha(\tau)$ and $q(t,\tau) = \alpha(t-\tau)$, in cases [6.213]–[6.215]. Thus, we obtain, respectively

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\Gamma(1-\alpha(t))} (t-\tau)^{-\alpha(t)} y(\tau)\,\mathrm{d}\tau, \quad 0 \le t \le T, \qquad [6.216] $$

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\Gamma(1-\alpha(\tau))} (t-\tau)^{-\alpha(\tau)} y(\tau)\,\mathrm{d}\tau, \quad 0 \le t \le T, \qquad [6.217] $$

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} y(\tau)\,\mathrm{d}\tau, \; 0 \le t \le T. \qquad [6.218] $$

There are other definitions of variable-order fractional integrals and derivatives. As will be seen, all definitions have, as a starting point, certain form of fractional derivative of constant order. Then, this form is generalized by allowing the constant α to become a function.

In [ROS 93] the Marchaud fractional derivative is used as a starting point, so that the variable-order fractional derivative is defined as $(0 < \alpha(t) < 1)$

$$ \begin{aligned} {}_0\mathrm{D}_t^{\alpha(t)} y(t) &= \frac{y(t)}{\Gamma(1-\alpha(t))\, t^{\alpha(t)}} \\ &\quad + \frac{\alpha(t)}{\Gamma(1-\alpha(t))} \int_0^t \frac{y(t)-y(\tau)}{(t-\tau)^{1+\alpha(t)}} \mathrm{d}\tau, \quad 0 \le t \le T. \end{aligned} \qquad [6.219] $$

In [ING 04], the following definition is introduced

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\Gamma(1-\alpha(\tau))} \frac{y(t-\tau)}{\tau^{\alpha(\tau)}} \mathrm{d}\tau, \quad 0 \le t \le T, \qquad [6.220] $$

or alternatively

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} \frac{y(\tau)}{(t-\tau)^{\alpha(t-\tau)}} \mathrm{d}\tau, \quad 0 \le t \le T, \quad [6.221] $$

which is [6.218].

We present now the definition given in [COI 03]. As a motivation, assuming that $y \in AC([0,t])$ and that $\alpha = const$, we can write

$$ {}_0\mathrm{D}_t^{\alpha} y(t) = \frac{y(0^+)}{\Gamma(1-\alpha)\, t^{\alpha}} + \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y^{(1)}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau, \quad 0 \le t \le T, \quad [6.222] $$

By replacing $\alpha = const.$ with the function $\alpha(t)$ in [6.222] it follows that

$$ \begin{aligned} {}_0\mathrm{D}_t^{\alpha(t)} y(t) &= \frac{y(0)}{\Gamma(1-\alpha(t))\, t^{\alpha(t)}} \\ &\quad + \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{y^{(1)}(\tau)}{(t-\tau)^{\alpha(t)}} \mathrm{d}\tau, \quad 0 \le t \le T. \end{aligned} \quad [6.223] $$

The definition proposed in [COI 03] (see also [RAM 10]) reads

$$ \begin{aligned} {}_0\mathrm{D}_t^{\alpha(t)} y(t) &= \frac{y(0^+) - y(0^-)}{\Gamma(1-\alpha(t))\, t^{\alpha(t)}} \\ &\quad + \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{y^{(1)}(\tau)}{(t-\tau)^{\alpha(t)}} \mathrm{d}\tau, \quad 0 \le t \le T. \end{aligned} \quad [6.224] $$

For functions that satisfy $y(0) = 0$ (since we assumed $y(t) = 0$ for $t < 0$), definitions [6.223] and [6.224] agree.

Next, we present the definition of variable-order fractional derivative based on the Grünwald–Letnikov definition for derivative of constant order. This approach is especially useful in numerical treatment of differential equations with variable-order derivatives. The left Grünwald–Letnikov fractional derivative of variable order (see [ZHU 09]), is defined as

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \lim_{h \to 0, nh = t} h^{-\alpha(t)} \sum_{j=0}^{n} (-1)^j \binom{\alpha(t)}{j} y(t - jh). \quad [6.225] $$

It can be shown that in the case when y is continuously differentiable (see [ZHU 09]), then definitions [6.216] and [6.225] agree.

The definitions of variable-order fractional derivatives stated above are special cases of a general fractional variable-order derivative proposed in [UMA 09] as

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t K_{\mu,\nu}^{\alpha(t)}(t,\tau)\, y(\tau)\, \mathrm{d}\tau, \quad 0 \le t \le T, \qquad [6.226] $$

where μ and ν are real parameters and

$$ K_{\mu,\nu}^{\alpha(t)}(t,\tau) = \frac{1}{\Gamma(1-\alpha(\mu t+\nu\tau))(t-\tau)^{\alpha(\mu t+\nu\tau)}}, \quad 0 < \tau < t. \qquad [6.227] $$

Parameters μ and ν belong to the causality parallelogram of the Lorenzo-Hartley $(\mu,\nu) \in \Pi$, where

$$ \Pi = \left\{(\mu,\nu) \in \mathbb{R}^2 \mid 0 \le \mu \le 1,\ -1 \le \nu \le 1,\ 0 \le \mu+\nu \le 1\right\}. \qquad [6.228] $$

Thus, for example, with $\mu = 0$, $\nu = 1$, we recover [6.216], while with $\mu = 1$, $\nu = -1$, [6.226] becomes [6.218].

As stated in [DAS 08, RAM 10], the problem of choosing a "proper" definition of the variable-order fractional derivative is open. In the next section, we present an analysis that supports the definition given in [6.218].

6.7.2.2. *Distributional setting of variable-order derivatives*

We refer to section 1.3 for the framework of the generalized functions used in this section, as well as section 2.3 for the fractional derivatives in the distributional setting. Suppose that $\alpha \in C^1([0,\infty))$, $0 < \alpha(t) < 1$. In accordance with [1.6], we consider two families in $\mathcal{S}'_+$ given by

$$ f_{-\alpha(t)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(H(t)\frac{t^{-\alpha(t)}}{\Gamma(1-\alpha(t))}\right), \quad t \in \mathbb{R}_+, \qquad [6.229] $$

$$ g_{\alpha(t)}(t) = H(t)\frac{t^{\alpha(t)-1}}{\Gamma(\alpha(t))}, \quad t \in \mathbb{R}_+, \qquad [6.230] $$

see [DAS 08]. Let $\phi \in \mathcal{S}$. Recall that the action of $f_{-\alpha(t)}(t)$ on ϕ is $\langle f_{-\alpha(t)}(t), \phi\rangle = -\left\langle f_{1-\alpha(t)}(t), \phi^{(1)}\right\rangle$. Thus, $f_{-\alpha(t)}(t)$ is a tempered distribution.

With [2.82] and [6.229], we define the left variable-order fractional Riemann–Liouville integral and derivative as follows.

DEFINITION 6.3.– *Suppose that* $\alpha \in C[0,\infty)$, $0 < \alpha(t) < 1$ *and* $y(t) = 0$ *for* $t < 0$. *The left Riemann–Liouville fractional integral of variable-order* $\alpha(t)$ *is defined as*

$$ {}_0\mathrm{I}_t^{\alpha(t)} y(t) = g_{\alpha(t)}(t) * y(t) = \int_0^t \frac{y(\tau)}{\Gamma(\alpha(t-\tau))(t-\tau)^{1-\alpha(t-\tau)}} \mathrm{d}\tau, \quad t \in \mathbb{R}, \qquad [6.231]$$

and the left Riemann–Liouville fractional derivative of variable fractional order $\alpha(t)$ *is*

$$ {}_0\mathrm{D}_t^{\alpha(t)} y(t) = f_{-\alpha(t)}(t) * y(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{(t-\tau)^{-\alpha(t-\tau)}}{\Gamma(1-\alpha(t-\tau))} y(\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}, \qquad [6.232]$$

while the right Riemann–Liouville fractional derivative of variable fractional order $\alpha(t)$ *is*

$$ {}_t\mathrm{D}_T^{\alpha(t)} y(t) = -\frac{\mathrm{d}}{\mathrm{d}t} \int_t^T \frac{(t-\tau)^{-\alpha(t-\tau)}}{\Gamma(1-\alpha(t-\tau))} y(\tau)\,\mathrm{d}\tau, \quad 0 \le t \le T. \qquad [6.233]$$

In [6.232] and [6.233], the derivative is taken in the sense of distributions. Note that ${}_0\mathrm{D}_t^{\alpha(t)} = \frac{\mathrm{d}}{\mathrm{d}t} {}_0\mathrm{I}_t^{1-\alpha(t)}$. Also if $y \in \mathcal{S}'_+(\mathbb{R})$, then ${}_0\mathrm{I}_t^{\alpha(t)} y \in \mathcal{S}'_+(\mathbb{R})$ and $\mathrm{D}_t^{\alpha(t)} y \in \mathcal{S}'_+(\mathbb{R})$ (since $f_{-\alpha}(t)$, $0 \le \alpha(t) < 1$ is a locally integrable function). The definitions reduce to the classical ones when $\alpha = const$, since $f_{-\alpha(t)}$ and $g_{\alpha(t)}$ are connected by relation $f_{-\alpha} = \frac{\mathrm{d}}{\mathrm{d}t} g_{1-\alpha}$, see [1.6]. From [6.232] the assumption of [LOR 02] that the arguments of α in the exponent of $(t-\tau)$ and in the gamma function, of equation [6.218] are the same, here follows naturally. Physically, definition [6.231] and [6.232] imply that a memory of the value of order of integration α is taken into account in the evaluation of a fractional derivative. Based on [6.232] and [6.233] we define for absolutely continuous functions fractional derivatives as follows.

DEFINITION 6.4.– *Let* $\alpha \in C([0,T))$, $0 < \alpha(t) < 1$ *and* $y \in AC([0,T])$. *Then left and right Riemann–Liouville fractional derivatives of variable-order* α *are defined as:*

$$ \begin{aligned} {}_0\mathrm{D}_t^{\alpha(t)} y(t) &= \frac{y(0)}{\Gamma(1-\alpha(t))\,t^{\alpha(t)}} \\ &\quad + \int_0^t \frac{y^{(1)}(\tau)}{\Gamma(1-\alpha(t-\tau))(t-\tau)^{\alpha(t-\tau)}} \mathrm{d}\tau, \quad t \in [0,T] \end{aligned} \qquad [6.234]$$

and

$$
\begin{aligned}
{}_t\mathrm{D}_T^{\alpha(t)} y(t) \quad = \quad & \frac{y(T)}{\Gamma(1-\alpha(T-t))\, t^{\alpha(T-t)}} \\
& - \int_t^T \frac{y^{(1)}(\tau)}{\Gamma(1-\alpha(\tau-t))\,(\tau-t)^{\alpha(\tau-t)}} \mathrm{d}\tau, \quad t \in [0,T], \qquad [6.235]
\end{aligned}
$$

respectively.

6.7.2.3. *Integration by parts formula*

For variable-order fractional derivatives, given by [6.234] and [6.235], we have the following integration by parts formula.

PROPOSITION 6.12.– Suppose that $y \in AC([0,T])$, $0 < \alpha(t) < 1$, $t \in [0,T]$. Then, the following integration by parts formula holds:

$$
\int_0^T f(t) \left({}_a\mathrm{D}_t^{\alpha(t)} y(t) \right) \mathrm{d}t = \int_0^T \left({}_t\mathrm{D}_T^{\alpha(t)} f(t) \right) y(t)\, \mathrm{d}t. \qquad [6.236]
$$

PROOF.– To prove this proposition we transform the left-hand side term $\int_0^T f(t) \left({}_0\mathrm{D}_t^{\alpha(t)} y(t) \right) \mathrm{d}t$ as follows:

$$
\begin{aligned}
& \int_0^T f(t)_0\, \mathrm{D}_t^{\alpha(t)} y(t)\, \mathrm{d}t \\
& \quad = \int_0^T f(t) \left(\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{y(\tau)}{\Gamma(1-\alpha(t-\tau))\,(t-\tau)^{\alpha(t-\tau)}} \mathrm{d}\tau \right) \mathrm{d}t \\
& \quad = \left[f(t) \int_0^t \frac{y(\tau)}{\Gamma(1-\alpha(t-\tau))\,(t-\tau)^{\alpha(t-\tau)}} \mathrm{d}\tau \right]_{t=0}^{t=T} \\
& \qquad - \int_0^T f^{(1)}(t) \left(\int_0^t \frac{y(\tau)}{\Gamma(1-\alpha(t-\tau))\,(t-\tau)^{\alpha(t-\tau)}} \mathrm{d}\tau \right) \mathrm{d}t \\
& \quad = f(T) \left(\int_0^T \frac{y(\tau)}{\Gamma(1-\alpha(T-\tau))\,(T-\tau)^{\alpha(T-\tau)}} \mathrm{d}\tau \right) \\
& \qquad - \int_0^T y(\tau) \left(\int_\tau^T \frac{f^{(1)}(t)}{\Gamma(1-\alpha(t-\tau))\,(t-\tau)^{\alpha(t-\tau)}} \mathrm{d}t \right) \mathrm{d}\tau
\end{aligned}
$$

$$= \int_0^T y(\tau)\left(\frac{f(T)}{\Gamma(1-\alpha(T-\tau))(T-\tau)^{\alpha(T-\tau)}}\right.$$

$$\left. - \int_\tau^T \frac{f^{(1)}(t)}{\Gamma(1-\alpha(t-\tau))(t-\tau)^{\alpha(t-\tau)}}\mathrm{d}t\right)\mathrm{d}\tau$$

$$= \int_0^T y(t)\,{}_t\mathrm{D}_T^{\alpha(t)} f(t)\,\mathrm{d}t. \qquad [6.237]$$

■

Therefore, definitions [6.234] and [6.235] preserve the form of the integration by parts formula [2.22]. Expression [6.236] is the basis for the variational principle with the Lagrangian having derivatives of variable order that we treat in the next section.

6.7.2.4. *Stationarity conditions when function α is given in advance*

The problem then becomes: given α, find $y = y^*$ such that

$$\begin{aligned}\min_{y\in\mathcal{U}} J[y,\alpha] &= \min_{y\in\mathcal{U}} \int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha(t)} y(t), \alpha(t)\right)\mathrm{d}t \\ &= \int_0^T L\left(t, y^*(t), {}_0\mathrm{D}_t^{\alpha(t)} y^*(t), \alpha(t)\right)\mathrm{d}t. \qquad [6.238]\end{aligned}$$

To obtain necessary conditions for the minimum, we assume that $y = y^* + \varepsilon h$ where $\varepsilon \in \mathbb{R}$, and h is such that $y \in \mathcal{U}$ for all ε. Since α is given, it is not subject to variation. By substituting $y = y^* + \varepsilon h$ into [6.238], we find that for each h

$$J[y^* + \varepsilon h, \alpha] = \int_0^T L\left(t, y^*(t) + \varepsilon h(t), {}_0\mathrm{D}_t^{\alpha(t)}(y^*(t) + \varepsilon h(t)), \alpha(t)\right)\mathrm{d}t,$$

is a function of ε only. Thus, the condition $\left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon} J[y^* + \varepsilon h, \alpha]\right)_{\varepsilon=0} = 0$ leads to

$$\int_0^T \left(\frac{\partial L}{\partial y} h + \frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha(t)} y}\, {}_a\mathrm{D}_t^{\alpha(t)} h\right)\mathrm{d}t = 0. \qquad [6.239]$$

Using [6.236], we obtain the necessary condition for the minimum of [6.238] in the form of the Euler–Lagrange equation

$$\frac{\partial L}{\partial y} + {}_t\mathrm{D}_T^{\alpha(t)}\left(\frac{\partial L}{\partial\, {}_a\mathrm{D}_t^{\alpha(t)} y}\right) = 0. \qquad [6.240]$$

Equation [6.240] is a generalization of the classical Euler–Lagrange equations as well as generalization of the Euler–Lagrange equations given in [AGR 02a] for $\alpha = const.$

6.7.2.5. *Stationarity conditions when function α is a constant, not given in advance*

Suppose that α is a constant, i.e. $\alpha = const.$ such that $\alpha \in (0,1)$ but it is not given in advance. Then minimization of [6.212] becomes

$$\min_{y\in\mathcal{U},\alpha\in(0,1)} J[y,\alpha] = \min_{y\in\mathcal{U},\alpha\in(0,1)} \int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha} y(t), \alpha)\right)\mathrm{d}t, \qquad [6.241]$$

see section 6.7.1. The necessary conditions for optimality of [6.241] read, if optimal $\alpha = \alpha^* \in (0,1)$ (see [ATA 10b]),

$$\frac{\partial L}{\partial y} + {}_t\mathrm{D}_T^{\alpha}\left(\frac{\partial L}{\partial\, \mathrm{D}_t^{\alpha} y}\right) = 0, \quad \int_0^T \left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha} y} G(y,\alpha) + \frac{\partial L}{\partial \alpha}\right)\mathrm{d}t = 0, \quad [6.242]$$

where

$$\begin{aligned} G(y,\alpha) &= \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha} = \frac{\mathrm{d}}{\mathrm{d}t}(f_1 * y)(t,\alpha), \\ f_1(t,\alpha) &= \frac{1}{t^{\alpha}\Gamma(1-\alpha)}\left(\psi(1-\alpha) - \ln t\right), \quad t > 0, \end{aligned} \qquad [6.243]$$

with the Euler function $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}(\ln\Gamma(z))$, and $(f_1 * y)(t,\alpha) = \int_0^t f_1(\tau,\alpha) y(t-\tau)\,\mathrm{d}\tau$. Expression [6.243] holds for y being locally integrable function, i.e. $y \in L^1_{loc}(\mathbb{R})$ with $y(t) = 0$ for $t < 0$. In [ATA 07a] the expression for $\frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha}$ is determined also for the case when $y \in \mathcal{S}'_+(\mathbb{R})$ and consequently ${}_0\mathrm{D}_t^{\alpha} y \in \mathcal{S}'_+(\mathbb{R})$. In this case (see [ATA 07a, p. 604]),

$$\begin{aligned} \left\langle \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha}, \phi \right\rangle &= \frac{\partial}{\partial \alpha}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}{}_0\mathrm{I}_t^{1-\alpha} y, \phi \right\rangle = -\frac{\partial}{\partial \alpha}\left\langle {}_0\mathrm{I}_t^{1-\alpha} y, \phi^{(1)} \right\rangle = \\ &= -\left\langle \frac{\partial}{\partial \alpha}{}_0\mathrm{I}_t^{1-\alpha} y, \phi^{(1)} \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \alpha}{}_0\mathrm{I}_t^{1-\alpha} y, \phi \right\rangle \\ &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(f_1 * y)(t,\alpha)\right), \phi \right\rangle, \end{aligned} \qquad [6.244]$$

where $\phi \in \mathcal{S}(\mathbb{R})$ and where we used the fact that $\frac{\partial}{\partial \alpha}{}_0\mathrm{I}_t^{1-\alpha} y = (f_1 * y)(t,\alpha)$ with f_1 given by $[6.243]_2$.

6.7.2.6. *Stationarity conditions when function α is not given in advance*

Function $0 < \alpha(t) < 1, t \in [0,T]$, is not given and must be determined from the minimization of [6.212]. Thus, we have

$$\min_{y\in\mathcal{U},\alpha\in\mathcal{V}} J[y,\alpha] = \min_{y\in\mathcal{U},\alpha\in\mathcal{V}} \int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha(t)} y(t), \alpha(t)\right) \mathrm{d}t. \qquad [6.245]$$

Let $y = y^*$ and $\alpha = \alpha^*$ be functions such that

$$\min_{y\in\mathcal{U},\alpha\in\mathcal{V}} \int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha(t)} y(t), \alpha(t)\right) \mathrm{d}t$$
$$= \int_0^T L\left(t, y^*(t), {}_0\mathrm{D}_t^{\alpha^*(t)} y^*(t), \alpha^*(t)\right) \mathrm{d}t. \qquad [6.246]$$

To obtain the necessary conditions for the optimality in problem [6.246], we let $y = y^* + \varepsilon_1 h$, $\alpha = \alpha^* + \varepsilon_2\theta$ and θ such that $y \in \mathcal{U}$, $\alpha \in \mathcal{V}$ for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. Again substituting this into $\int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha(t)} y(t), \alpha(t)\right) \mathrm{d}t$ and by observing that there is no relation between y and α, we obtain the following necessary condition for optimality (see [GEL 63]),

$$\int_0^T \left(\frac{\partial L}{\partial y} + {}_t\mathrm{D}_T^{\alpha(t)}\left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y}\right)\right) h(t)\, \mathrm{d}t = 0,$$
$$\int_0^T \left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y} \frac{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)} + \frac{\partial L}{\partial \alpha(t)}\right) \theta(t)\, \mathrm{d}t = 0. \qquad [6.247]$$

Under suitable conditions (see [GEL 63, P. 9–11]), we obtain from [6.247]

$$\frac{\partial L}{\partial y} + {}_t\mathrm{D}_T^{\alpha}\left(\frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha} y}\right) = 0, \qquad \frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha} y} \frac{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)} + \frac{\partial L}{\partial \alpha} = 0. \qquad [6.248]$$

To determine $\frac{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)}$, we proceed as in cases [6.243] and [6.244] so that

$$\frac{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)} = \frac{\mathrm{d}}{\mathrm{d}t}(g_1 * y)(t,\alpha),$$
$$g_1(t,\alpha) = \frac{1}{t^{\alpha}\Gamma(1-\alpha(t))}\left(\psi(1-\alpha(t)) - \ln t\right), \quad t > 0, \qquad [6.249]$$

and

$$\left\langle \frac{\partial\, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)}, \phi(t) \right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(g_1 * y\right)(t,\alpha)), \phi(t) \right\rangle, \qquad [6.250]$$

for $y \in L^1_{loc}(\mathbb{R})$ and $y \in \mathcal{S}'_+(\mathbb{R})$, respectively.

6.7.2.7. *Stationarity conditions when function α is given constitutively*

Function α is not given explicitly, but is given in terms of t, α and y by an additional differential equation. Thus, the optimization problem is formulated as: minimize

$$\min_{y\in\mathcal{U}} J[y,\alpha] = \int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha(t)} y(t), \alpha(t)\right) \mathrm{d}t, \qquad [6.251]$$

subject to

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \Psi(t, \alpha(t), y(t)), \qquad [6.252]$$

with Ψ given function. This situation is typical for internal variable theories. Namely, α is treated in [6.251] as an internal variable that is determined through *constitutive equation* [6.252]. Thus, we have to solve

$$\min_{y\in\mathcal{U}} J[y,\alpha] = \min_{y\in\mathcal{U}} \int_0^T L\left(t, y(t), {}_0\mathrm{D}_t^{\alpha(t)} y(t), \alpha(t)\right) \mathrm{d}t, \qquad [6.253]$$

subjected to constraint [6.252]. We rewrite [6.253] and [6.252] as: find a minimum of

$$\min_{y\in\mathcal{U}, z\in\mathcal{V}} J[y,z,\alpha] = \min_{y\in\mathcal{U}, z\in\mathcal{V}} \int_0^T L(t, y(t), z(t), \alpha(t))\, \mathrm{d}t, \qquad [6.254]$$

where $\mathcal{U}$ and $\mathcal{V}$ are given, subjected to the constraints

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = \Psi(\tau, \alpha(t), y(t)), \quad {}_0\mathrm{D}_t^{\alpha(t)} y(t) = z(t). \qquad [6.255]$$

Using the Lagrange multiplier rule (see [AGR 02a, AGR 04, ITO 08]), we consider the modified functional

$$\begin{aligned} \min_{y\in\mathcal{U}} \bar{J}[y,z,\alpha] &= \min_{y\in\mathcal{U}} \int_0^T \Big(L(t,y,z,\alpha) + p\left(z - {}_0\mathrm{D}_t^{\alpha(t)} y\right) \\ &\quad + q\left(\Psi(\tau,\alpha,y) - \alpha^{(1)}\right)\Big)\, \mathrm{d}t, \end{aligned} \qquad [6.256]$$

where p and q are Lagrange multipliers. The necessary condition for the optimality $\delta \bar{J} = 0$, after partial integration and the use of [6.236], becomes

$$\delta \bar{J} = \int_0^T \left\{ \left(\frac{\partial L}{\partial y} - {}_t\mathrm{D}_T^{\alpha(t)} p + q \frac{\partial \Psi}{\partial y} \right) \delta y \right.$$
$$+ \left(z - {}_0\mathrm{D}_t^{\alpha(t)} y \right) \delta p + \left(\Psi(\tau, \alpha, y) - \alpha^{(1)} \right) \delta q$$
$$\left. + \left(\frac{\partial L}{\partial z} + p \right) \delta z + \left(\frac{\partial L}{\partial \alpha} - p \frac{\partial \, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)} + q^{(1)} \right) \delta \alpha \right) \mathrm{d}t. \qquad [6.257]$$

Now, from [6.257], we obtain

$$\begin{aligned} \alpha^{(1)}(t) &= \Psi(t, \alpha(t), y(t)), \quad {}_0\mathrm{D}_t^{\alpha(t)} y(t) = z(t), \\ {}_t\mathrm{D}_T^{\alpha(t)} p(t) &= \frac{\partial L}{\partial y} + q \frac{\partial \Psi}{\partial y}, \quad q^{(1)}(t) = -\frac{\partial L}{\partial \alpha} + p \frac{\partial \, {}_0\mathrm{D}_t^{\alpha(t)} y}{\partial \alpha(t)}, \\ & p(t) = -\frac{\partial L}{\partial z}, \end{aligned} \qquad [6.258]$$

with boundary conditions on p and q depending on boundary conditions on α and z. Thus, for example, if $\alpha(0)$ and $\alpha(T)$ are arbitrary, we have $q(0) = q(T) = 0$ (see [VUJ 04]).

REMARK 6.20.– Condition [6.258] implies the following special cases.

1) If $\alpha = const.$ and prescribed, then $\Psi(\tau, \alpha(t), y(t)) = 0$ and since $\delta\alpha(0) = \delta\alpha(T) = 0$, we have $\delta q(0)$ and $\delta q(T)$ arbitrary. From [6.258], we then obtain

$${}_0\mathrm{D}_t^{\alpha(t)} y(t) = z(t), \quad {}_t\mathrm{D}_T^{\alpha(t)} p(t) = \frac{\partial L}{\partial y}, \quad p = -\frac{\partial L}{\partial z},$$

or

$$\frac{\partial L}{\partial y} + {}_t\mathrm{D}_T^{\alpha} \left(\frac{\partial L}{\partial \, {}_0\mathrm{D}_t^{\alpha} y} \right) = 0, \qquad [6.259]$$

result [6.259] was obtained in [AGR 02a].

2) If we assume that $\alpha = const.$ but not given in advance, we have $\Psi(\tau, \alpha(t), y(t)) = 0$ and $\delta\alpha(0)$ and $\delta\alpha(T)$ as arbitrary. This implies

$q(0) = q(T) = 0$. From [6.258], we obtain

$$\begin{aligned} \alpha^{(1)}(t) &= 0, \quad {}_0\mathrm{D}_t^{\alpha(t)} y(t) = z(t), \quad {}_t\mathrm{D}_T^{\alpha(t)} p(t) = \frac{\partial L}{\partial y}, \\ q(t) &= \int_0^t \left(-\frac{\partial L}{\partial \alpha} - \frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha} y} \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha} \right) \mathrm{d}t, \quad p(t) = -\frac{\partial L}{\partial z}, \end{aligned}$$

or

$$\frac{\partial L}{\partial y} + {}_t\mathrm{D}_b^{\alpha} \left(\frac{\partial L}{\partial\, \mathrm{D}_t^{\alpha} y} \right) = 0, \quad \int_0^T \left(-\frac{\partial L}{\partial \alpha} - \frac{\partial L}{\partial\, {}_0\mathrm{D}_t^{\alpha} y} \frac{\partial\, {}_0\mathrm{D}_t^{\alpha} y}{\partial \alpha} \right) \mathrm{d}t = 0, \qquad [6.260]$$

where we used the condition $q(T) = 0$. Result [6.260] was obtained in [ATA 10b].

The variational problem [6.254] and [6.255] may be used to give a rational basis for obtaining equations of process when the order of the derivative depends on time and value of the dependent (state) variable. An example of this type is presented in [DIA 09] where the order of the derivative was taken to be

$$\alpha(t) = \frac{1 + y^2(t)}{2}. \qquad [6.261]$$

Our analysis (equation [6.252]) can be easily modified to include [6.261] as a constraint. We believe that in the process of fractionalization of equations of the mathematical physics, not necessarily coming from variational principles, the order of fractional derivative must be treated as the constitutive quantity, determined by the state variables. It is an open question as to what the restrictions are, coming from thermodynamics, which the constitutive equation $[6.255]_1$ has to satisfy.

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