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New formulation of the classical path integral with reparametrization invariance

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A classical path integral (CPI) provides a functional integral representation of the kernel which propagates phase space density distributions. In this paper a new formulation of the CPI is developed in which time and energy are promoted to dynamical variables. The reparametrization invariance, inherent in this formalism, is handled by means of the Batalin–Fradkin–Vilkovisky method. The path integral action possesses a set of $I\mathrm{Sp}(2)$ symmetries connected with reparametrization invariance and an additional set of $I\mathrm{Sp}(2)$ symmetries connected with the symplectic geometry of the extended phase space. Supersymmetry is also present in the CPI action. This formulation of the CPI allows us to study the dependence on energy of the dynamical evolution of Hamiltonian systems. It naturally incorporates the constraints onto the energy surface.
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I. INTRODUCTION

In previous work[1] a path integral formulation of classical mechanics was developed as a new tool for studying chaos in Hamiltonian systems. While it is often impractical to track individual trajectories through phase space for chaotic systems, due to sensitive dependence on initial conditions, we can obtain a great deal of information about such systems by studying the evolution of density distributions.[2] The classical path integral (CPI) provides an expression for the kernel which propagates phase space density distributions along the classical paths. The path integral representation of this kernel is constructed directly from the classical equations of motion. It can be written in terms of the exponential of an action containing, in addition to the phase space variables ϕ^a, $\underline{a}=1\cdots 2n$, the commuting variables Λ_a and the Grassmann (anticommuting) variables C^a and $\overline{C}_a$. The Grassmann variables carry information about first order deviations between neighboring trajectories; correlations among them are related to the Lyapunov exponents.[3] The variables C^a can be interpreted as one-forms $d\phi^a$ on phase space. With an appropriate choice of boundary conditions this formalism describes the propagation of phase space p-forms, as well as scalar distributions.[4]

The CPI action exhibits a set of symmetries[1] connected with the symplectic geometry of phase space.[5] An additional supersymmetry present in Hamiltonian systems[6] is connected with ergodic versus orderly behavior.[7] In Ref. 6 it was shown that unbroken supersymmetry implies ergodicity. When a system has unbroken supersymmetry, it is ergodic, and when the system behaves in an orderly manner, the supersymmetry is broken. We were not able to show, however, that ergodicity implies unbroken supersymmetry. Since the Hamiltonian constitutes a constant of motion, a conservative system can only be ergodic when confined to the energy surface. The CPI of Ref. 1 describes the Hamiltonian flow[5] on the entire phase space. What is missing is a natural way to fix the energy.

It is well known that the behavior of a Hamiltonian system depends on its energy. The Henon–Heiles model[8] is a classic example of this. At low energies this system exhibits regular behavior, following smooth orbits which are sections of tori for nearly all initial conditions. As the

[a)]Electronic mail: thacker@newton.slu.edu energy is increased, the motion becomes increasingly irregular, until most of the energy surface is covered by chaotic orbits.

We would like to have a formulation of the CPI which brings out the role of the energy. One thing to try is to perform a canonical transformation to a new set of phase space variables that includes the energy, which equals the Hamiltonian, and time as the conjugate variable. The problem with this approach is that in Hamiltonian mechanics time is just a parameter. What is needed is a formalism in which time and energy appear as dynamical variables.

This is accomplished in the formulation of classical mechanics proposed by Jacobi,[9] and recently revived within the context of gravitation theory[10] and in the context of theories with internal-reparametrization invariances.[11] In this formalism a new parameter is introduced and time is treated as an additional position variable. The momentum conjugate to time is minus the total energy. The Jacobi action is invariant under reparametrization. This symmetry is associated with a constraint which confines the motion to energy surfaces.

A modern approach to the quantization of reparametrization invariant systems uses the method of Batalin, Fradkin and Vilkovisky[12] (BFV).[13] An auxilliary field and its conjugate momentum are introduced for each constraint, as well as a pair of ghosts. These fields go into the construction of the BFV action which appears in the quantum path integral.

In this article we construct the classical path integral from the equations of motion arising out of the BFV action for Jacobi's reparametrization invariant formalism. We find that it is necessary to include the BFV ghosts even at the classical level, in order to obtain a unitary classical propagator. This new formulation of the CPI naturally incorporates the constraints onto the energy surface. When time, energy and all of the ghosts and auxilliary fields, introduced in the BFV method, are integrated out of the new CPI, the original one[1] is obtained. A different choice of boundary conditions leads to the propagator for p-forms on the enlarged phase space which includes time and energy. A further restriction on the boundary conditions leads to the propagator for p-forms confined to the energy surface.

The article is organized as follows. In Sec. II the path integral approach to classical mechanics of Ref. 1 is reviewed. In Sec. III we describe the Jacobi formulation[9–11] of classical mechanics and its path integral quantization,[13] emphasizing the role of the various constraints and gauge fixing. In Sec. IV we construct the classical path integral for the Jacobi formulation of classical mechanics and show the need to use the BFV method *even at the classical level*. In Sec. V we derive the various symmetries present in this formalism, stemming both from the constraints and from the geometry of phase space. In Sec. VI we construct the supersymmetry charges for the reparametrization invariant CPI and find the physical states that are invariant under this supersymmetry. In Sec. VII we derive propagators from the reparametrization invariant CPI for various choices of boundary conditions. Section VIII contains some concluding remarks.

II. REVIEW OF THE PATH INTEGRAL APPROACH TO CLASSICAL MECHANICS

A few years after the introduction of quantum mechanics, Koopman and von Neumann[14] proposed an *operatorial* approach to classical mechanics in order to compare it better with quantum mechanics. Let us start from the Hamiltonian equations of motion:

$$\dot{\phi}^a(t)=\omega^{ab}\partial_b H(\phi(t))\equiv h^a(\phi(t)), \tag{2.1}$$

where $\phi^a\equiv(q^1,\cdots,q^n,p_1,\cdots,p_n)$, $a=1,\cdots,2n$, is a coordinate on a $2n$-dimensional phase space $\mathcal{M}_{2n}$, H is the Hamiltonian, $\omega^{ab}=-\omega^{ba}$ is the symplectic matrix and $h^a(\phi)$ is the Hamiltonian vector field. Any density function $\varrho(\phi,t)$ on phase space evolves in time according to the Liouville equation

$$\frac{\partial}{\partial t}\varrho(\phi,t)=-\{\varrho,H\}\equiv-\hat{L}\varrho(\phi,t), \tag{2.2}$$

where the Liouville operator $\hat{L} = -h^a \partial_a$ is the central ingredient of the operatorial approach to classical mechanics.[14] The formal solution of Eq. (2.2) is

$$\varrho(\phi,t) = e^{-\hat{L}t} \varrho(\phi,0). \tag{2.3}$$

This summarizes the operatorial version of classical Hamiltonian mechanics.

As in quantum mechanics, the operatorial approach to classical mechanics should have a corresponding path integral formulation. The idea is to represent the kernel of the time evolution operator in (2.3) as a path integral. In classical mechanics this kernel is just a delta-functional

$$K[\phi,t|\phi_0,0] = \langle \phi | e^{-\hat{L}t} | \phi_0 \rangle = \int_{\phi_0}^{\phi} \mathscr{D}' \phi \tilde{\delta} [\phi - \phi_{cl}] \tag{2.4}$$

which forces the system to lie on classical solutions ϕ_{cl} of (2.1).[15] Since the classical solution $\phi_{cl}(t,\phi_0)$ is unique for a given initial condition $\phi(0) = \phi_0$, we can represent the delta-functional projecting onto the classical solutions as a delta-functional projecting onto the equations of motion times a Jacobian

$$\tilde{\delta}[\phi - \phi_{cl}] = \tilde{\delta}[\dot{\phi} - h(\phi)] \det\left(\partial_t \delta^a_b - \frac{\partial h^a}{\partial \phi^b} \right). \tag{2.5}$$

We can now Fourier transform the delta-functional on the right hand side of (2.5), using an auxiliary field Λ_a, and exponentiate the determinant via the Grassmann (anticommuting) variables C^a, and $\overline{C}_a$. Thus we arrive at

$$K[\phi,t|\phi_0,0] = \int_{\phi_0}^{\phi} \mathscr{D}' \phi \mathscr{D}\Lambda \mathscr{D}C \mathscr{D}\overline{C} \delta^{2n}(C_0) \exp i \int_0^t dt \tilde{\mathscr{L}} \tag{2.6}$$

with the CPI Lagrangian

$$\tilde{\mathscr{L}} = \Lambda_a [\dot{\phi}^a - \omega^{ab} \partial_b H(\phi)] + i \overline{C}_a \left[\partial_t \delta^a_b - \frac{\partial h^a}{\partial \phi^b} \right] C^b. \tag{2.7}$$

Note that $\tilde{\mathscr{L}}$ contains only first derivatives with respect to time, and therefore we can immediately read off the associated Hamiltonian

$$\tilde{\mathscr{H}} = \Lambda_a \omega^{ab} \partial_b H + i \overline{C}_a \frac{\partial h^a}{\partial \phi^b} C^b. \tag{2.8}$$

From the path integral (2.6) we can compute the equal-time (graded) commutators of ϕ^a, Λ_a, C^a and $\overline{C}_a$, using standard techniques,[16] to find[17]

$$\langle [\phi^a, \Lambda_b] \rangle = i \delta^a_b, \langle [\overline{C}_b, C^a] \rangle = \delta^a_b. \tag{2.9}$$

All other (graded) commutators vanish. In particular, ϕ^a and ϕ^b commute for all values of the indices a and b. In terms of the q's and p's (which were combined into ϕ^a) this means $[q^i, p_j] = 0$ for all i and j. This shows that we are doing classical mechanics and not quantum mechanics. The operator algebra (2.9) can be realized by differential operators

$$\Lambda_a = -i \frac{\partial}{\partial \phi^a}, \quad \overline{C}_a = \frac{\partial}{\partial C^a} \tag{2.10}$$

and multiplicative operators ϕ^a and C^a acting on functions $\tilde{\varrho}(\phi,C,t)$. Inserting the above operators into $\tilde{\mathscr{H}}$ we obtain

$$\tilde{\mathscr{H}} = -ih^a\partial_a + i\frac{\partial}{\partial C^a}\frac{\partial h^a}{\partial \phi^b}C^b. \tag{2.11}$$

Equation (2.11) shows that the Grassmann part of $\tilde{\mathscr{H}}$ gives zero when it acts on distributions $\varrho(\phi)$ that do not contain anticommuting variables, while the bosonic part is $(-i)$ times the Liouville operator: $\tilde{H}|_{(C=0)} = -i\hat{L}$. This confirms that the classical path integral reproduces the operatorial approach to classical mechanics of Koopman and von Neumann.[14] However, as we have seen, the determinant in (2.5) leads naturally to the introduction of the new anticommuting variables C and $\bar{C}$, not present in the formalism of Koopman and von Neumann. The interpretation of these variables was discussed at great length in Ref. 1. The Grassmann variables C^a can be brought into correspondence with phase space one-forms, i.e., $C^a \Leftrightarrow d\phi^a$, while their canonical conjugate variables $\bar{C}_a$ can be interpreted as vectors dual to the one-forms. The product of Grassmann variables corresponds to the wedge product. The whole Cartan calculus on phase space (exterior derivative, inner products, etc.) has been translated in Ref. 1 into a calculus based on these Grassmann variables. In particular, the CPI Hamiltonian $\tilde{\mathscr{H}}$ is nothing else than the Lie-derivative $l_{(dH)^\#}$ of the Hamiltonian flow acting on p-forms[18]

$$\tilde{\mathscr{H}} = -il_{(dH)^\#}.$$

This Lie-derivative generates the time-evolution not only of scalar distributions in phase space $\varrho(\phi)$, but also of general distributions

$$\tilde{\varrho}(\phi,C) = \sum_0^{2n} \frac{1}{p!}\varrho^{(p)}_{a_1\cdots a_p}C^{a_1}\cdots C^{a_p}, \tag{2.12}$$

where each term in the sum can be interpreted as a p-form on phase space. These distributions (2.12) obey the evolution equation

$$\partial_t\tilde{\varrho}(\phi,C) = -i\tilde{\mathscr{H}}\tilde{\varrho}(\phi,C) = -l_{(dH)^\#}\tilde{\varrho}(\phi,C). \tag{2.13}$$

The propagator for a general distribution is given in path integral form by

$$K(\phi,C,t|\phi_0,C_0,t_0) = \int_{\phi_0,C_0}^{\phi,C}\mathscr{D}'\phi\mathscr{D}\Lambda\mathscr{D}'C\mathscr{D}\bar{C}\exp[i\tilde{S}]. \tag{2.14}$$

Let us now consider the symmetries of $\tilde{S} = \int\tilde{\mathscr{L}}\,dt$. In Ref. 1 it was found that $\tilde{S}$ is invariant under the transformations generated by the following $I\mathrm{Sp}(2)$ charges:

$$\begin{gathered} Q = iC^a\Lambda_a, \quad \bar{Q} = i\bar{C}_a\omega^{ab}\Lambda_b, \quad Q_f = C^a\bar{C}_a, \\ K = \tfrac{1}{2}\omega_{ab}C^aC^b, \quad \bar{K} = \tfrac{1}{2}\omega^{ab}\bar{C}_a\bar{C}_b. \end{gathered} \tag{2.15}$$

These generators can be interpreted in terms of the symplectic geometry of phase space. In particular, Q, which we call a Becchi–Rouet–Stora–Tyupin (BRST) charge,[19] acts as the *exterior derivative* on phase space, while K is the symplectic 2-form, and a similar interpretation can be given for all of the generators. (See Ref. 1 for details.)

The generators (2.15) are not the only charges commuting with $\tilde{\mathscr{H}}$. Let us consider,[20] for example, the quantities

$$Q_H = e^{\beta H} Q e^{-\beta H} = Q - \beta[Q,H],$$
$$\overline{Q_H} = e^{-\beta H}\overline{Q} e^{\beta H} = \overline{Q} + \beta[\overline{Q},H], \tag{2.16}$$

where β is an arbitrary complex parameter. It is easy to verify[6] that

$$[Q_H, Q_H] = [\overline{Q_H}, \overline{Q_H}] = 0 \tag{2.17}$$

and

$$[Q_H, \overline{Q_H}] = 2i\beta \tilde{\mathscr{H}} \equiv 2\beta l_{(dH)^\#}. \tag{2.18}$$

Equation (2.18) shows that the anticommutator of Q_H and $\overline{Q_H}$ is proportional to the CPI Hamiltonian and therefore these operators are genuine *supersymmetry* generators.

Thus far we have only shown that the action $\tilde{S}$ is invariant under the supersymmetry, but we have not addressed the question ''*Can this symmetry be spontaneously broken?*''[21] This comes down to the question: ''How does the ground state transform? Is it invariant under the symmetries of the action?'' For simplicity, let us confine our attention to observables A which do not depend on the forms: $A = A(\phi)$. Then, in order to produce a non-zero expectation value,

$$\langle A\rangle_t = \int d^{2n}\phi d^{2n}C\; A(\phi)\tilde{\varrho}(\phi,C,t) \tag{2.19}$$

the density $\tilde{\varrho}$ must contain all $2n$ one-forms: $\tilde{\varrho}(\phi,C,t) = \varrho(\phi,t)C^1\cdots C^{2n}$. When the forms are integrated out, (2.19) yields the standard expression for the expectation value (without forms):

$$\langle A\rangle_t = \int d^{2n}\phi\; A(\phi)\varrho(\phi,t). \tag{2.20}$$

Since the above $\tilde{\varrho}$ contains the maximum number of forms, it is trivially annihilated by Q_H: $Q_H\tilde{\varrho}(\phi,C) = C^a[\partial_a - \beta(\partial H/\partial\phi^a)]\tilde{\varrho}(\phi,C) = 0$. On the other hand, the invariance under transformations generated by $\overline{Q_H}$ requires

$$\overline{Q_H}\tilde{\varrho}(\phi,C) = \frac{\partial}{\partial C^a}\,\omega^{ab}\left(\partial_b + \beta\,\frac{\partial H}{\partial \phi^b}\right)\tilde{\varrho}(\phi,C) = 0. \tag{2.21}$$

This equation can be satisfied[6] only if

$$\tilde{\varrho}(\phi,C,t) = k e^{-\beta H(\phi)} C^1\cdots C^{2n}, \tag{2.22}$$

where k is a constant

The state invariant under the supersymmetry is the Gibbs state with β interpreted as the inverse temperature. This state is ergodic since it depends only on the Hamiltonian $H(\phi)$ and is therefore constant on energy surfaces. If the supersymmetry is unbroken then (2.22) is the only ground state of the CPI Hamiltonian, satisfying $\tilde{\mathscr{H}}\tilde{\varrho} = 0$. Therefore unbroken supersymmetry implies ergodicity. It is important to note that the converse of this statement is not true. In fact, since the energy is not fixed in (2.14), any normalizable function $F(H(\phi))$, as well as $e^{-\beta H}$, can be taken as an acceptable ground state for an ergodic system. These functions $F(H)$ are constants and thus proportional to each other on the energy surface, but not outside. Since in general $\tilde{\varrho}$

$=F(H(\phi))C^1\cdots C^{2n}$ does not satisfy (2.21), we conclude that ergodicity is possible even if the supersymmetry (2.18) is spontaneously broken, i.e., unbroken supersymmetry is not *necessary* for ergodicity, but only sufficient.

Ergodicity is a feature of the motion on an energy surface $\mathscr{M}_{2n-1}(E)$. The anticommutator of the supercharges (2.16), on the other hand, generates the Lie derivative of the Hamiltonian flow on the entire phase space $\mathscr{M}_{2n}$. To resolve this issue, we need to build something analogous to the propagator given by Eq. (2.14) but with the energy fixed, or to have a formalism in which the energy is singled out as a new canonical variable. This is done by the Maupertuis–Jacobi formulation of classical mechanics which we review in the next section.

Before concluding this section we should also mention, for the reader who may be interested, that some other applications of the path integral formulation of classical mechanics have appeared in Ref. 22.

III. REVIEW OF THE MAUPERTUIS–JACOBI FORMULATION OF CLASSICAL AND QUANTUM MECHANICS

According to Hamilton's principle, the trajectories of a classical system extremize the action

$$\mathscr{A}[x(t)]\equiv\int_{t_1}^{t_2}\left\{\frac{1}{2}\,g_{ij}\dot{x}^i\dot{x}^j-U(x)\right\}dt, \tag{3.1}$$

where the x^i are the n configuration space variables, g_{ij} is a metric on configuration space, and $\dot{x}$ means dx/dt. The extremization is carried out over all paths $x[t]$ which run between two fixed points x_1 and x_2 in the *same time interval* t_2-t_1. In this formulation time is treated as a parameter.

Maupertuis and more explicitly Jacobi[9] introduced a variational principle in which time appears as a dynamical variable. The action to minimize, in order to obtain the Newtonian equations of motion, is given by

$$\mathscr{S}[x(\tau),t(\tau)]=\int_{\tau_1}^{\tau_2}d\tau\left[\frac{1}{2\dot{t}^2(\tau)}\,g_{ij}(x(\tau))\dot{x}^i(\tau)\dot{x}^j(\tau)-U(x(\tau))\right]\dot{t}(\tau), \tag{3.2}$$

where the affine parameter τ has been introduced to label points along the trajectories and the overdot indicates differentiation with respect to this parameter. Now the extremization is done over paths between x_1 and x_2 which have the *same energy*.

The action (3.2) is invariant under reparametrizations in τ, i.e.,

$$\mathscr{S}[\widetilde{x}(\tau),\widetilde{t}(\tau)]=S[x(\tau),t(\tau)], \tag{3.3}$$

where

$$\widetilde{x}^i(\tau)=x^i(f(\tau)),\quad \widetilde{t}(\tau)=t(f(\tau)) \tag{3.4}$$

with $f(\tau)$ an arbitrary function satisfying the conditions: $f(\tau_{1,2})=\tau_{1,2}$.

Let us now choose a flat metric $g_{ij}=\delta_{ij}$ and derive from (3.2) the equations of motion. Variation of x^k yields

$$\frac{1}{\dot{t}}\,\frac{d}{d\tau}\,\frac{1}{\dot{t}}\,\frac{d}{d\tau}\,x^k(\tau)=-\partial_k U(x(\tau)) \tag{3.5}$$

and variation of t yields

$$\frac{1}{2\dot{t}^2(\tau)}\,\dot{x}^i(\tau)\dot{x}_i(\tau)+U(x(\tau))=\text{const}\equiv h. \tag{3.6}$$

If we insert the t-equation of motion into the action $\mathcal{S}$ of Eq. (3.2), we would get an action $\mathcal{S}_h$ given by

$$\mathcal{S}_h[x]=\int_{\tau_1}^{\tau_2}d\tau\,[2(h-U(x(\tau)))\dot{x}^i\dot{x}_i]^{1/2}-h[t(\tau_2)-t(\tau_1)]. \tag{3.7}$$

This is the more familiar action[9] to which the reader might have associated the names of Maupertuis and Jacobi. In fact, the variation of (3.7) with respect to $x^i(\tau)$ yields precisely the geodesics of the Maupertuis metric:

$$ds_M^2=2\{h-U(x)\}dx^i\;dx_i \tag{3.8}$$

and the action (3.7) can be written as

$$\mathcal{S}_h[x]=\int ds_M\equiv\int\sqrt{ds_M^2}, \tag{3.9}$$

where in Eqs. (3.8) and (3.9) the surface term $h[t(\tau_2)-t(\tau_1)]$ has been dropped since it is irrelevant for the variational principle. The geodesics of ds_M^2 give the classical paths $x^i(\tau)$ as a function of τ. Their dependence on t is found by integrating Eq. (3.6): $\dot{t}=\sqrt{\dot{x}^2/2(h-U)}$.

Let us now turn to the canonical formalism associated with the action (3.2). The momenta conjugate to x^i and t are

$$p_i\equiv\frac{\partial L}{\partial\dot{x}^i}=\frac{1}{\dot{t}(\tau)}\,\dot{x}_i\,,\quad -E\equiv\frac{\partial L}{\partial\dot{t}}=-\frac{1}{2}\,\dot{t}^{-2}\dot{x}^2-U, \tag{3.10}$$

where L is the Lagrangian for the action (3.2): $\mathcal{S}\equiv\int_{\tau_1}^{\tau_2}d\tau\,L$. In (3.10) we have defined the momentum conjugate to t as $-E$, because we can then interpret E as the energy of the system.

Introducing the function $H(x,p)\equiv\frac{1}{2}p^ip_i+U(x)$, and inserting the first equation of (3.10) into the second one, we get the constraint[23]

$$\mathcal{H}\equiv H(x,p)-E=0. \tag{3.11}$$

This is now to be interpreted not as the definition of energy, as is usually done in the Hamiltonian formalism, but as a constraint between the independent phase space variables (x,p,E,t). This is the constraint naturally associated to the local symmetry (3.4). The Maupertuis–Jacobi formulation of classical mechanics singles out the role of the energy, promoting it, together with time, to an independent canonical variable. Let us not forget that the goal of this article is to develop a formalism with which we can study the dependence on energy of the behavior of Hamiltonian systems. We would like to slice phase space into energy surfaces in order to see what happens to the behavior of a system as the energy is changed. Since this was not possible by a simple canonical transformation to new variables including the energy, the only manner was to enlarge phase space from the $2n$ variables (x,p) to $2n+2$ variables, by including E,t. The price to pay, in order to include these variables, is the necessity of imposing the constraint (3.11). This is equivalent to introducing a local symmetry into the action (3.1) and going over to the action (3.2).

Let us now proceed to rewrite the action (3.2) using phase space variables. We note that the naive Hamiltonian is zero:

$$H_{\text{naive}}=p_i\dot{x}^i-E\dot{t}-L=\frac{1}{\dot{t}}\,\dot{x}_i\dot{x}^i-\frac{1}{2}\,\dot{t}^{-1}\dot{x}^2-U-L=0. \tag{3.12}$$

Adding the constraint (3.11) with a Lagrange multiplier λ, we obtain the canonical action

$$\mathscr{S}_{\text{can}}[x,p,E,t,\lambda]=\int_{\tau_1}^{\tau_2}d\tau\,[p_i(\tau)\dot{x}^i(\tau)-E(\tau)\dot{t}(\tau)-\lambda(\tau)\mathscr{H}(x,p,E)]. \tag{3.13}$$

Denoting,[24] as in Ref. 1, the phase space variables (x^i,p_i), $i=1\cdots n$, by the fields ϕ^a, $a=1\cdots 2n$ we can write (3.13) as

$$\mathscr{S}_{can}[\phi,E,t,\lambda]=\int_{\tau_1}^{\tau_2}d\tau\left[\frac{1}{2}\,\phi^a(\tau)\omega_{ab}\dot{\phi}^b(\tau)-E(\tau)\dot{t}(\tau)-\lambda\{H(\phi)-E\}\right]. \tag{3.14}$$

We can put this into compact form by introducing the notation: $\phi^A\equiv(x^i,p_i,t,-E)$. The index A runs between 1 and $2n+2$. Introducing the enlarged symplectic matrix

$$\hat{\omega}_{AB}=\begin{pmatrix}\omega_{ab} & & 0 \\ & 0 & -1 \\ 0 & & \\ & 1 & 0\end{pmatrix} \tag{3.15}$$

we can rewrite (3.14) as

$$\mathscr{S}_{can}[\phi^A,\lambda]=\int_{\tau_1}^{\tau_2}d\tau\left[\frac{1}{2}\,\phi^A\hat{\omega}_{AB}\dot{\phi}^B-\lambda(\tau)\mathscr{H}(\phi^A)\right] \tag{3.16}$$

up to a surface term. The enlarged Poisson brackets are defined as

$$\{F(\phi^A),G(\phi^B)\}=(\partial_A F)\,\hat{\omega}^{AB}(\partial_B G). \tag{3.17}$$

The equations of motion are

$$\frac{d}{d\tau}\,\phi^a(\tau)=\lambda(\tau)\omega^{ab}\partial_b H(\phi),\quad \frac{d}{d\tau}\,E(\tau)=0,$$
$$\frac{d}{d\tau}\,t(\tau)=\lambda(\tau),\quad H(\phi)=E, \tag{3.18}$$

which can be compactly written as

$$\dot{\phi}^A(\tau)=\lambda\,\hat{\omega}^{AB}\partial_B\mathscr{H}. \tag{3.19}$$

The third equation in (3.18) yields the relation between the parameter τ and the physical time t:

$$t(\tau_2)=\int_{\tau_1}^{\tau_2}d\tau'\lambda(\tau')+t(\tau_1).$$

The action $\mathscr{S}_{can}$ is invariant under the transformations

$$\delta\phi^A(\tau)=\epsilon(\tau)\{\phi^A,\mathscr{H}\},\quad \delta\lambda(\tau)=\dot{\epsilon}(\tau),\quad \epsilon(\tau_{1,2})=0 \tag{3.20}$$

which, in components, become

$$\delta\phi^a=\epsilon(\tau)\omega^{ab}\partial_b H,\quad \delta t=\epsilon(\tau),\quad \delta E=0, \\ \delta\lambda=\dot{\epsilon}(\tau),\quad \epsilon(\tau_{1,2})=0. \tag{3.21}$$

These are the transformations associated on shell to the reparametrization $\tau\to\tau'=f(\tau)$. These transformations are connected to the identity but, as in string theory, there are other transformations, disconnected from the identity, called the modular transformations. The analysis of modular transformations has been performed for the relativistic particle in Ref. 25, and for the nonrelativistic particle in Ref. 24.

Let us now introduce the Batalin–Fradkin–Vilkovisky[12] (BFV) procedure in order to quantize the system. First of all, from the action $\mathscr{S}_{can}$, we find that the momentum conjugate to $\lambda(\tau)$ is zero: $\pi\equiv\partial L/\partial\dot{\lambda}=0$. This is a new constraint which must be added to (3.11), $\mathscr{H}=0$. Thus the full set of constraints is

$$G_1\equiv\pi=0,\quad G_2\equiv\mathscr{H}=0. \tag{3.22}$$

The BFV procedure associates a ghost and an antighost with each constraint. Let us call these ghosts and antighosts

$$\eta^\alpha\equiv(-iP,c),\quad \mathscr{P}_\alpha\equiv(i\overline{c},\overline{P}). \tag{3.23}$$

The Poisson brackets are

$$\{x^i,p_j\}=\delta^i_j,\quad \{t,-E\}=1,\quad \{\lambda,\pi\}=1,\quad \{\eta^\alpha,\mathscr{P}_\beta\}=-\delta^\alpha_\beta \tag{3.24}$$

which, upon quantization, become the following (graded) commutators:

$$[x^i,p_j]=i\delta^i_j,\quad [t,-E]=i,\quad [\lambda,\pi]=i,\quad [\eta^\alpha,\mathscr{P}_\beta]=-i\delta^\alpha_\beta. \tag{3.25}$$

Out of the constraints and the ghosts, we build the BRST charge

$$\Omega=\eta^\alpha G_\alpha=\eta^1 G_1+\eta^2 G_2=-iP\pi+c\mathscr{H} \tag{3.26}$$

which goes into the construction of the BFV action

$$\mathscr{S}_{\mathrm{BFV}}=\int_{\tau_1}^{\tau_2}d\tau\,[p_i\dot{x}^i-E\dot{t}+\pi\dot{\lambda}+\dot{\eta}^\alpha\mathscr{P}_\alpha+\{\psi,\Omega\}]. \tag{3.27}$$

Here ψ is the gauge fixing function which we choose[11] to be $\psi=\lambda\overline{P}$ in order to implement the gauge fixing $\dot{\lambda}=0$. With this choice, the action then becomes

$$\mathscr{S}_{\mathrm{BFV}}=\int_{\tau_1}^{\tau_2}d\tau\,[p_i\dot{x}^i-E\dot{t}-\lambda\dot{\pi}-P\dot{\overline{c}}+\dot{c}\overline{P}-i\overline{P}P-\lambda(H-E)]\equiv\int_{\tau_1}^{\tau_2}d\tau\,L_{\mathrm{BFV}}. \tag{3.28}$$

This is the action which goes into the quantum path integral for the transition amplitude

$$\langle(\mathbf{x}_2t_2),\tau_2|(\mathbf{x}_1t_1),\tau_1\rangle=\int\mathscr{D}'x\mathscr{D}p\mathscr{D}'t\mathscr{D}E\mathscr{D}\lambda\mathscr{D}'\pi\mathscr{D}\overline{P}\mathscr{D}P\mathscr{D}'\overline{c}\mathscr{D}'c\;e^{i\mathscr{S}_{\mathrm{BFV}}}. \tag{3.29}$$

The boundary conditions for this path integral must be BRST-invariant. Following Refs. 11 and 13, we choose

$$\pi=\overline{c}=c=0 \quad \text{at} \quad \tau=\tau_{1,2}, \quad \mathbf{x}(\tau_{1,2})=\mathbf{x}_{1,2}, \quad t(\tau_{1,2})=t_{1,2}. \tag{3.30}$$

To evaluate (3.29), we separate out the ghost part $\mathbf{Z}_{ghost}$ and the part $\mathbf{Z}_{\lambda,\pi,E,t}$

$$\langle \mathbf{x}_2, t_2, \tau_2 | \mathbf{x}_1, t_1, \tau_1 \rangle = \mathbf{Z}_{ghost} \int \mathscr{D}' x \mathscr{D} p \mathbf{Z}_{\lambda,\pi,E,t} \, e^{i\int_{\tau_1}^{\tau_2} d\tau [p_i \dot{x}^i - \lambda_0 H]}, \tag{3.31}$$

where[11]

$$\mathbf{Z}_{\text{ghost}} \equiv \int \mathscr{D}\overline{P}\mathscr{D}P\mathscr{D}'\overline{c}\mathscr{D}'c \; e^{i\int_{\tau_1}^{\tau_2} d\tau[-P\dot{\overline{c}}+\dot{c}\overline{P}-i\overline{P}P]} = -(\tau_2-\tau_1) \tag{3.32}$$

and

$$\mathbf{Z}_{\lambda,\pi E,t} \equiv \int \mathscr{D}' t \mathscr{D} E \mathscr{D}\lambda \mathscr{D}' \pi \; e^{i\int_{\tau_1}^{\tau_2} d\tau [E(-\dot{t}+\lambda)-\lambda\dot{\pi}]} = \int_{-\infty}^{\infty} d\lambda_0 \, \frac{1}{\tau_2-\tau_1} \, \delta\!\left(\frac{t_2-t_1}{\tau_2-\tau_1}-\lambda_0\right). \tag{3.33}$$

Therefore, we obtain

$$\begin{aligned}\langle (\mathbf{x}_2 t_2), \tau_2 | (\mathbf{x}_1 t_1), \tau_1 \rangle &= -\int_{-\infty}^{\infty} d\lambda_0 \, \delta\!\left(\frac{t_2-t_1}{\tau_2-\tau_1}-\lambda_0\right) \int \mathscr{D}' x(\tau) \mathscr{D} p(\tau) e^{i\int_{\tau_1}^{\tau_2} d\tau [p_i (dx^i/d\tau) - \lambda_0 H]} \\ &= -\int \mathscr{D}' x(\tau) \mathscr{D} p(\tau) e^{i\int_{\tau_1}^{\tau_2} d\tau \{p_i (dx^i/d\tau) - [(t_2-t_1)/(\tau_2-\tau_1)] H(x,p)\}}.\end{aligned} \tag{3.34}$$

Now, if the parameter τ is replaced by the physical time

$$t(\tau) = \frac{t_2-t_1}{\tau_2-\tau_1} (\tau-\tau_1) + t_1 \tag{3.35}$$

(3.34) reduces to the Feynman propagator

$$\langle x_2 t_2 | x_2 t_1 \rangle = -\int \mathscr{D}' x^i(t) \mathscr{D} p_i(t) e^{i\int_{t_1}^{t_2} dt [p_i (dx^i/dt) - H]}. \tag{3.36}$$

Notice that a factor of $(\tau_2-\tau_1)$, coming from $\mathbf{Z}_{ghost}$, has been canceled by a factor of $(\tau_2-\tau_1)^{-1}$, coming from $\mathbf{Z}_{\lambda,\pi E,t}$. If the ghost sector had not been included in the action, the theory would not have been unitary.

Since time and energy appear as conjugate variables in the action (3.28), it is not possible to fix both the time interval and the energy in the quantum path integral. In order to obtain the path integral for the transition amplitude at fixed energy we add a surface term to the action (3.28) to obtain

$$\mathscr{S}_{Jac} = \int_{\tau_1}^{\tau_2} d\tau [p_i \dot{x}^i + t\dot{E} - \lambda\dot{\pi} - P\dot{\overline{c}} + \dot{c}\overline{P} - i\overline{P}P - \lambda(H-E)]. \tag{3.37}$$

Then, with the boundary conditions

$$\pi=\overline{c}=c=0 \quad \text{at} \quad \tau=\tau_{1,2}, \quad x(\tau_{1,2})=x_{1,2}, \quad E(\tau_1)=E_0 \tag{3.38}$$

we find[10]

$$\langle(\mathbf{x}_2E_0),\tau_2|(\mathbf{x}_1E_0),\tau_1\rangle=\int \mathscr{D}'x\mathscr{D}p\mathscr{D}t\mathscr{D}E\mathscr{D}\lambda\mathscr{D}'\pi\mathscr{D}\overline{P}\mathscr{D}P\mathscr{D}'\overline{c}\mathscr{D}'c\,\delta(E(\tau_1)-E_0)e^{i\mathscr{S}_{Jac}}$$
$$=\langle\mathbf{x}_2|\delta[E_0-H]|\mathbf{x}_1\rangle. \tag{3.39}$$

IV. BFV FORMALISM IN THE PATH INTEGRAL REPRESENTATION OF CLASSICAL MECHANICS

The usual folklore about *quantizing* gauge theories is that one needs the gauge fixing in order to be able to invert the quadratic part of the Lagrangian and thus to obtain the *quantum* propagator. In some gauges the Faddeev–Popov ghosts must be introduced in order to restore the unitarity of the theory. The gauge fixing and Faddeev–Popov ghosts are also needed in order to factor out of the path integral the infinite volume of the gauge group. Recently it has been understood[26] that all of these structures (ghosts, BRST charges, etc.) have a geometric meaning also at the *classical* level, and they appear naturally in a geometrical approach to gauge theory, independent of the quantization of the system.

For the reader who may still be puzzled by the need of the BFV procedure at the classical level, we offer a pragmatic view. Starting from the equations of motion (3.18), we show that, in order to get one solution for a given set of intial conditions, we need a gauge fixing. The ghosts are needed in order to obtain a unitary classical propagator. In retrospect this is a natural thing to do because we have to start from the action which provides the *unitary* quantum theory, in order to get the correct *unitary* classical theory.

Let us start with the issue of gauge fixing. In constructing the CPI, according to the steps (2.4)–(2.6), we go from the solutions of the equations of motion (2.1) to the equations of motion themselves. This requires a one-to-one correspondence between equations of motion and solutions (once the initial conditions are given), even if we are unable to solve the equations of motion explicitly. If there were more than one solution for a given initial condition, then (2.5) would not be valid. This can be seen from the formula for the Dirac delta-function of a multi-component function $F(x)$

$$\delta(F(x))=\sum_{x_i}\frac{\delta(x-x_i)}{\left\|\dfrac{\partial F}{\partial x}\right\|_{x_i}}, \tag{4.1}$$

where the x_i are the solutions of $F(x)=0$.

Let us now consider the equations of motion (3.18):[27]

$$\frac{d}{d\tau}\phi^a(\tau)=\lambda(\tau)\omega^{ab}\partial_bH(\phi(\tau)), \tag{4.2a}$$

$$\frac{d}{d\tau}E(\tau)=0, \tag{4.2b}$$

$$\frac{d}{d\tau}t(\tau)=\lambda(\tau_i), \tag{4.2c}$$

$$H(\phi)=E. \tag{4.2d}$$

One might think that these $2n+3$ equations for the $2n+3$ variables (ϕ^a,λ,E,t) can be solved as follows:

• Solve Eq. (4.2a) by giving the initial conditions $\phi^a(0)$ and a function $\lambda(\tau)$ to be determined afterwards:

$$\phi^a(\tau)=\mathscr{F}^a(\phi(0),\lambda(\tau),\tau). \tag{4.3}$$

• Insert Eq. (4.3) into Eq. (4.2d)

$$H(\mathscr{F}(\phi(0),\lambda(\tau),\tau))=E(\tau)$$

to obtain a relation between $\lambda(\tau)$ and $E(\tau)$

$$E(\tau)=\mathscr{G}(\lambda(\tau),\phi(0),\tau). \tag{4.4}$$

• Insert now Eq. (4.4) into Eq. (4.2b) to obtain

$$\frac{d\mathscr{G}(\lambda(\tau),\cdots)}{d\tau}=0 \tag{4.5}$$

and now, given some intial condition $\lambda(0)$, one could determine $\lambda(\tau)$ from Eq. (4.5).

$$\lambda(\tau)=\mathscr{P}(\lambda(0),\tau,\cdots).$$

• The above solution can then be inserted in (4.2c) to get t as a function of τ:

$$\frac{dt}{d\tau}=\mathscr{P}(\lambda(0),\tau,\cdots).$$

Thus it appears that one obtains, out of the $2n+3$ equations in (4.2), a unique solution for (ϕ^a,λ,E,t) once initial conditions are chosen for all of the variables. Actually this is not correct. In fact, the Equation (4.2b), which we used to determine $\lambda(\tau)$, is an identity following from Equations (4.2a) and (4.2d). Therefore, the function $\lambda(\tau)$ is not determined by (4.2). This means that for a given set of initial conditions $(\phi^a(0),E(0),t(0))$ there is a whole family of solutions depending on the freely chosen $\lambda(\tau)$. This ambiguity arises from the gauge freedom (3.21). Even when the initial conditions are fixed the symmetry transformations (3.21) slide each point in phase space along its classical orbit.

Since there is no unique solution for (4.2) it is not possible to go over from the classical solutions to the equations of motion as in steps (2.4) and (2.5) in the construction of the CPI. For this reason it is necessary to introduce a gauge fixing which determines the functional form of $\lambda(\tau)$. We have chosen $\dot{\lambda}=0$. In Sec. VII we show that it is also necessary to include the BFV ghosts in order to obtain the correct (unitary) classical propagator.

So our strategy is to write down all the classical equations of motion arising from $\mathscr{S}_{\mathrm{BFV}}$ and then to exponentiate them into the classical path integral as in (2.4)–(2.7). The action $\mathscr{S}_{\mathrm{BFV}}$ leads to the classical equations of motion

$$\frac{d}{d\tau}\phi^a(\tau)=\lambda(\tau)\omega^{ab}\partial_b H(\phi), \tag{4.6a}$$

$$\frac{d}{d\tau}E(\tau)=0, \tag{4.6b}$$

$$\frac{d}{d\tau}t(\tau)=\lambda(\tau), \tag{4.6c}$$

$$\dot{\pi}+H(\phi)-E=0, \tag{4.6d}$$

$$\dot{\lambda}=0, \tag{4.6e}$$

$$\dot{\bar{P}}=0, \tag{4.6f}$$

$$\dot{P}=0, \tag{4.6g}$$

$$\dot{\bar{c}}-i\bar{P}=0, \tag{4.6h}$$

$$\dot{c}+iP=0. \tag{4.6i}$$

The above equations can be written in Hamiltonian form

$$\dot{\phi}^{\mu}=\hat{\omega}^{\mu\nu}\frac{\partial\hat{H}}{\partial\phi^{\nu}}\equiv h^{\mu}, \tag{4.7}$$

where ϕ^{μ} includes all of the canonical variables $(\phi^a,t,E,\lambda,\pi,c,\bar{P},\bar{c},P)$ in (4.6). The effective Hamiltonian is

$$\hat{H}=\lambda(H(\phi)-E)+i\bar{P}P \tag{4.8}$$

and

$$\hat{\omega}^{\mu\nu}=\begin{pmatrix} \omega^{ab} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & \omega^{tE} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \omega^{Et} & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & 0 & \omega^{\lambda\pi} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega^{\pi\lambda} & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \omega^{c\bar{P}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \omega^{\bar{P}c} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \omega^{P\bar{c}} & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \omega^{\bar{c}P} & 0 & \end{pmatrix} \tag{4.9}$$

with

$$\begin{gathered} \omega^{tE}=-1=-\omega^{Et}, \quad \omega^{\lambda\pi}=1=-\omega^{\pi\lambda}, \\ \omega^{c\bar{P}}=-1=\omega^{\bar{P}c}, \quad \omega^{\bar{c}P}=-1=\omega^{P\bar{c}}, \end{gathered} \tag{4.10}$$

In following the steps from (2.4)–(2.7), we need to take into account the fact that the BFV ghosts are Grassmann-odd. Consequently, Eq. (2.5) must be modified. For a set of Grassmann-odd variables ξ^{α} satisfying the equations of motion $\dot{\xi}^{\alpha}-h^{\alpha}=0$, we have

$$\tilde{\delta}[\xi-\xi_{cl}]=\tilde{\delta}[\dot{\xi}-h]\left(\det\left[\delta^{\alpha}_{\beta}\partial_t-\frac{\partial}{\partial\xi^{\beta}}h^{\alpha}\right]\right)^{-1}.$$

The reciprocal of the determinant can be exponentiated by means of bosonic variables C^{α} and $\bar{C}_{\alpha}$, according to the standard rule for Gaussian integration

$$\left(\det\left[\delta^{\alpha}_{\beta}\partial_t - \frac{\partial}{\partial \xi^{\beta}}\, h^{\alpha}\right]\right)^{-1} = \int \mathscr{D}C\mathscr{D}\overline{C}\, \exp\left[-\int d\tau\, C_{\alpha}\left[\delta^{\alpha}_{\beta}\partial_{\tau} - \frac{\partial}{\xi^{\beta}}\, h^{\alpha}\right]C^{\beta}\right]. \tag{4.11}$$

We exponentiate the delta-functional by introducing the Grassmann-odd auxilliary fields Λ_{α}:

$$\tilde{\delta}[\dot{\xi} - h] = \int \mathscr{D}\Lambda\, \exp\left[i\int d\tau \Lambda_{\alpha}[\xi^{\alpha} - h^{\alpha}]\right]. \tag{4.12}$$

The factor of i in (4.12) is correct since the number of Grassmann-odd variables (BFV-ghosts) ξ^{α} is even $(2m)$. In fact, at each lattice point in the discretization of (4.12) there is a Grassmann integral of the form

$$\int d^{2m}\Lambda\, \exp[i\Lambda_{\alpha}\eta^{\alpha}] = \int d\Lambda_{2m}\cdots d\Lambda_1(-1)^m(\Lambda_1\eta^1)\cdots(\Lambda_{2m}\eta^{2m}) = \eta^1\cdots\eta^{2m} = \delta^{2m}(\eta).$$

Therefore, from all of the equations of motion in (4.7), both bosonic and fermionic, we can construct a classical path integral with the CPI action[28]

$$\tilde{\mathscr{S}}_J = \int \tilde{\mathscr{L}}_J d\tau,$$

with

$$\tilde{\mathscr{L}}_J = \Lambda_{\mu}[\phi^{\mu} - h^{\mu}] + i\overline{C_{\mu}}\left[\delta^{\mu}_{\nu}\, \frac{d}{d\tau} - \frac{\partial h^{\mu}}{\partial \phi^{\nu}}\right]C^{\nu}, \tag{4.13}$$

where the auxiliary fields Λ_{μ} and the ghosts $C^{\nu}, \overline{C_{\mu}}$ are generalizations of the fields Λ_a, C^a, $\overline{C_a}$. The Grassmann character of all the fields in (4.13) is given as follows. The Grassmann-even fields $(\phi^a, t, E, \lambda, \pi)$ have Grassmann-even auxiliary fields $(\Lambda_a, \Lambda_t, \Lambda_E, \Lambda_{\lambda}, \Lambda_{\pi})$ and Grassmann-odd forms and vectors

$$(\overline{C_a}, C^a, \overline{C_t}, C^t, \overline{C_E}, C^E, \overline{C_{\lambda}}, C^{\lambda}, \overline{C_{\pi}}, C^{\pi}).$$

The Grassmann-odd fields $(c, \overline{P}, \overline{c}, \overline{P})$ have Grassmann-odd auxiliary fields $(\Lambda_c, \Lambda_P, \Lambda_{\overline{c}}, \Lambda_{\overline{P}})$ and Grassmann-even forms and vectors

$$(\overline{C_c}, C^c, \overline{C_P}, C^P, \overline{C_{\overline{c}}}, C^{\overline{c}}, \overline{C_{\overline{P}}}, C^{\overline{P}}).$$

The Lagrangian in (4.13) can be written, in components, as

$$\begin{aligned}\tilde{\mathscr{L}}_J = {} & \Lambda_a(\dot{\phi}^a - \lambda h^a) + i\overline{C_a}\dot{C}^a - i\lambda\overline{C_a}\, \frac{\partial h^a}{\partial \phi^b}\, C^b + \Lambda_E\dot{E} + i\overline{C_E}\dot{C}^E + \Lambda_t(\dot{t} - \lambda) + i\overline{C_t}\dot{C}^t + \Lambda_{\pi}\dot{\pi} + i\overline{C_{\pi}}\dot{C}^{\pi} \\ & + \Lambda_{\lambda}\dot{\lambda} + i\overline{C_{\lambda}}\dot{C}^{\lambda} - i(\overline{C_a}h^a + \overline{C_t})C^{\lambda} + \Lambda_{\pi}(H - E) + i\overline{C_{\pi}}\left(\frac{\partial H}{\partial \phi^a}\, C^a - C^E\right) + \Lambda_P\dot{P} + i\overline{C_P}\dot{C}^P \\ & + \Lambda_{\overline{P}}\dot{\overline{P}} + i\overline{C_{\overline{P}}}\dot{C}^{\overline{P}} + \Lambda_c(\dot{c} + iP) + i\overline{C_c}(\dot{C}^c + iC^P) + \Lambda_{\overline{c}}(\dot{\overline{c}} - i\overline{P}) + i\overline{C_{\overline{c}}}(\dot{C}^{\overline{c}} - iC^{\overline{P}}).\end{aligned} \tag{4.14}$$

From the kinetic terms in the $\tilde{\mathscr{L}}_J$ of (4.14), one immediately derives[16] the graded commutators

$$[\phi^a, \Lambda_b]_- = i\delta^a_b \Rightarrow \Lambda_a = -i\, \frac{\partial}{\partial \phi^a}, \quad [t, \Lambda_t]_- = i \Rightarrow \Lambda_t = -i\, \frac{\partial}{\partial t},$$

$$[E,\Lambda_E]_-=i\Rightarrow\Lambda_E=-i\,\frac{\partial}{\partial E},\quad [\lambda,\Lambda_\lambda]_-=i\Rightarrow\Lambda_\lambda=-i\,\frac{\partial}{\partial\lambda},$$

$$[\pi,\Lambda_\pi]_-=i\Rightarrow\Lambda_\pi=-i\,\frac{\partial}{\partial\pi},\quad [c,\Lambda_c]_+=i\Rightarrow\Lambda_c=i\,\frac{\partial}{\partial c},$$

$$[\bar{c},\Lambda_{\bar{c}}]_+=i\Rightarrow\Lambda_{\bar{c}}=i\,\frac{\partial}{\partial\bar{c}},\quad [P,\Lambda_P]_+=i\Rightarrow\Lambda_P=i\,\frac{\partial}{\partial P},$$

$$[\bar{P},\Lambda_{\bar{P}}]_+=i\Rightarrow\Lambda_{\bar{P}}=i\,\frac{\partial}{\partial\bar{P}},\quad [C^a,\bar{C}_b]_+=\delta^a_b\Rightarrow\bar{C}_b=\frac{\partial}{\partial C^b},$$

$$[C^t,\bar{C}_t]_+=1\Rightarrow\bar{C}_t=\frac{\partial}{\partial C^t},\quad [C^E,\bar{C}_E]_+=1\Rightarrow\bar{C}_E=\frac{\partial}{\partial C^E},\tag{4.15}$$

$$[C^\lambda,\bar{C}_\lambda]_+=1\Rightarrow\bar{C}_\lambda=\frac{\partial}{\partial C^\lambda},\quad [C^\pi,\bar{C}_\pi]_+=1\Rightarrow\bar{C}_\pi=\frac{\partial}{\partial C^\pi},$$

$$[C^c,\bar{C}_c]_-=1\Rightarrow\bar{C}_c=-\,\frac{\partial}{\partial C^c},\quad [C^P,\bar{C}_P]_-=1\Rightarrow\bar{C}_P=-\,\frac{\partial}{\partial C^P},$$

$$[C^{\bar{c}},\bar{C}_{\bar{c}}]_-=1\Rightarrow\bar{C}_{\bar{c}}=-\,\frac{\partial}{\partial C^{\bar{c}}},\quad [C^{\bar{P}},\bar{C}_{\bar{P}}]_-=1\Rightarrow\bar{C}_{\bar{P}}=-\,\frac{\partial}{\partial C^{\bar{P}}},$$

where the suffix ‘‘$-$’’ or ‘‘$+$’’ distinguishes commutators from anticommutators. The differential representation of the various fields indicated on the right hand side of (4.15) is meant as derivation to the right.

V. SYMMETRIES OF $\mathscr{L}_J$

In this section we investigate the symmetries of the CPI Lagrangian $\widetilde{\mathscr{L}}_J$. First, let us consider the symmetry generators analogous to those in (2.15), which are connected with the geometry of the extended phase space $\{\phi^a,t,E,\lambda,\pi,c,P,\bar{c},\bar{P}\}$. The *exterior derivative* on this phase space is generated by the BRST charge

$$\hat{Q}=C^\mu\,\frac{\partial}{\partial\phi^\mu}=C^a\,\frac{\partial}{\partial\phi^a}+C^t\,\frac{\partial}{\partial t}+C^E\,\frac{\partial}{\partial E}+C^\lambda\,\frac{\partial}{\partial\lambda}+C^\pi\,\frac{\partial}{\partial\pi}+C^c\,\frac{\partial}{\partial c}+C^P\,\frac{\partial}{\partial P}+C^{\bar{c}}\,\frac{\partial}{\partial\bar{c}}+C^{\bar{P}}\,\frac{\partial}{\partial\bar{P}}$$
$$=i(C^a\Lambda_a+C^t\Lambda_t+C^E\Lambda_E+C^\lambda\Lambda_\lambda+C^\pi\Lambda_\pi-C^c\Lambda_c-C^P\Lambda_P-C^{\bar{c}}\Lambda_{\bar{c}}-C^{\bar{P}}\Lambda_{\bar{P}}).\tag{5.1}$$

BRST transformations leave the CPI Lagrangian invariant

$$\delta_{\hat{Q}}\widetilde{\mathscr{L}}_J=[\bar{\epsilon}\hat{Q},\widetilde{\mathscr{L}}_J]_-=0,\tag{5.2}$$

where $\bar{\epsilon}$ is an anticommuting parameter. In addition to the BRST charge there is the anti-BRST charge

$$\hat{\hat{Q}} = \frac{\partial}{\partial C^{\mu}}\,\hat{\omega}^{\mu\nu}\,\frac{\partial}{\partial \phi^{\nu}} = \frac{\partial}{\partial C^{a}}\,\omega^{ab}\,\frac{\partial}{\partial \phi^{b}} - \frac{\partial}{\partial C^{t}}\,\frac{\partial}{\partial E} + \frac{\partial}{\partial C^{E}}\,\frac{\partial}{\partial t} + \frac{\partial}{\partial C^{\lambda}}\,\frac{\partial}{\partial \pi} - \frac{\partial}{\partial C^{\pi}}\,\frac{\partial}{\partial \lambda}$$
$$- \frac{\partial}{\partial C^{c}}\,\frac{\partial}{\partial \bar{P}} - \frac{\partial}{\partial C^{\bar{P}}}\,\frac{\partial}{\partial c} - \frac{\partial}{\partial C^{\bar{c}}}\,\frac{\partial}{\partial P} - \frac{\partial}{\partial C^{P}}\,\frac{\partial}{\partial \bar{c}}$$
$$= i(\bar{C}_a\omega^{ab}\Lambda_b - \bar{C}_t\Lambda_E + \bar{C}_E\Lambda_t + \bar{C}_\lambda\Lambda_\pi - \bar{C}_\pi\Lambda_\lambda$$
$$- \bar{C}_c\Lambda_{\bar{P}} - \bar{C}_{\bar{P}}\Lambda_c - \bar{C}_{\bar{c}}\Lambda_P - \bar{C}_P\Lambda_{\bar{c}}) \tag{5.3}$$

which acts as the symplectic dual to the exterior derivative, lowering the form number by one. While the BRST variation of $\tilde{\mathscr{L}}_J$ is zero, the anti-BRST variation is the surface term

$$\delta_{\hat{\hat{Q}}}\tilde{\mathscr{L}}_J = -\epsilon\,\frac{d}{d\tau}\,\hat{\hat{Q}} \tag{5.4}$$

with the anticommuting parameter ϵ.

The charges K and $\bar{K}$ in (2.15) are also connected with the symplectic geometry of phase space. We can construct analogs of these charges on the extended phase space from the matrix $\hat{\omega}^{\mu\nu}$, given by (4.9), and its inverse $\hat{\omega}_{\mu\nu}$, i.e., $\hat{\omega}_{\mu\alpha}\hat{\omega}^{\alpha\nu} = \delta_\mu^\nu$. The symplectic two-form on the extended phase space is given by

$$\hat{K} \equiv \tfrac{1}{2}\hat{\omega}_{\mu\nu}C^\mu C^\nu = \tfrac{1}{2}\omega_{ab}C^aC^b + C^tC^E + C^\pi C^\lambda - C^cC^{\bar{P}} - C^PC^{\bar{c}} \tag{5.5}$$

and its symplectic dual is

$$\hat{\hat{K}} \equiv \tfrac{1}{2}\hat{\omega}^{\mu\nu}\bar{C}_\mu\bar{C}_\nu = \tfrac{1}{2}\omega^{ab}\bar{C}_a\bar{C}_b + \bar{C}_E\bar{C}_t + \bar{C}_\lambda\bar{C}_\pi - \bar{C}_c\bar{C}_{\bar{P}} - \bar{C}_P\bar{C}_{\bar{c}}. \tag{5.6}$$

The transformations generated by $\hat{K}$ and $\hat{\hat{K}}$ generate surface terms in the CPI action

$$\delta_{\hat{K}}\tilde{\mathscr{L}}_J = i\epsilon\,\frac{d}{d\tau}\,\hat{K}, \quad \delta_{\hat{\hat{K}}}\tilde{\mathscr{L}}_J = -i\bar{\epsilon}\,\frac{d}{d\tau}\,\hat{\hat{K}}. \tag{5.7}$$

The commutator of the symplectic form with its dual gives rise to the charge

$$Q_f = C^a\,\frac{\partial}{\partial C^a} + C^t\,\frac{\partial}{\partial C^t} + C^E\,\frac{\partial}{\partial C^E} + C^\lambda\,\frac{\partial}{\partial C^\lambda} + C^\pi\,\frac{\partial}{\partial C^\pi}$$
$$+ C^c\,\frac{\partial}{\partial C^c} + C^P\,\frac{\partial}{\partial C^P} + C^{\bar{c}}\,\frac{\partial}{\partial C^{\bar{c}}} + C^{\bar{P}}\,\frac{\partial}{\partial C^{\bar{P}}}. \tag{5.8}$$

This charge counts *form* number, giving a weight of $+1$ to each one-form from the set

$$\{C^a, C^t, C^E, C^\lambda, C^\pi, C^c, C^P, C^{\bar{c}}, C^{\bar{P}}\}$$

and a weight of -1 to each vector

$$\{\bar{C}_a, \bar{C}_t, \bar{C}_E, \bar{C}_\lambda, \bar{C}_\pi, \bar{C}_c, \bar{C}_P, \bar{C}_{\bar{c}}, \bar{C}_{\bar{P}}\}.$$

Every term in the CPI Lagrangian has form number zero, and therefore

$$\delta_{Q_f}\tilde{\mathscr{L}}_J = 0. \tag{5.9}$$

The set of charges

$$\{\hat{Q},\hat{\bar{Q}},Q_f,\hat{K},\hat{\bar{K}}\} \tag{5.10}$$

generate the algebra $I\mathrm{Sp}(2)$ with graded commutators

$$\begin{aligned}
&[\hat{Q},\hat{Q}]=[\hat{\bar{Q}},\hat{\bar{Q}}]=[\hat{Q},\hat{\bar{Q}}]=0, \quad [Q_f,\hat{Q}]=\hat{Q}, \quad [Q_f,\hat{\bar{Q}}]=-\hat{\bar{Q}},\\
&[\hat{Q},\hat{K}]=[\hat{\bar{Q}},\hat{\bar{K}}]=0, \quad [\hat{\bar{K}},\hat{Q}]=\hat{\bar{Q}}, \quad [\hat{K},\hat{\bar{Q}}]=\hat{Q},\\
&[Q_f,\hat{K}]=2\hat{K}, \quad [Q_f,\hat{\bar{K}}]=-2\hat{\bar{K}}, \quad [\hat{K},\hat{\bar{K}}]=Q_f.
\end{aligned} \tag{5.11}$$

The effective action S_{BFV} given by Eq. (3.28) has a BRST symmetry arising from the original reparametrization invariance present in (3.2). This symmetry should be inherited by the CPI action $\mathscr{S}_J=\int d\tau\, \mathscr{L}_J$. We shall call it the BFV-BRST invariance. In order to construct the BFV-BRST charges, we find it convenient to employ the superfield formalism introduced in Ref. 1.

Let us enlarge the 1-dimensional base space τ to a superspace, including the two anticommuting variables θ and $\bar{\theta}$. The superspace then consists of $\{\tau,\theta,\bar{\theta}\}$. Following Ref. 1, we define the superfield

$$\Phi^{\mu}\equiv \phi^{\mu}+\theta C^{\mu}+\bar{\theta}\hat{\omega}^{\mu\nu}\bar{C}_{\nu}+i\bar{\theta}\theta\hat{\omega}^{\mu\nu}\Lambda_{\nu}, \tag{5.12}$$

where ϕ^{μ} again stands for the set of fields $(\phi^a,t,E,\lambda,\pi,c,\bar{c},P,\bar{P})$. The components of Φ^{μ} are

$$\begin{aligned}
&\Phi^{a}=\phi^{a}+\theta C^{a}+\bar{\theta}\omega^{ab}\bar{C}_b+i\bar{\theta}\theta\omega^{ab}\Lambda_b,\\
&\hat{t}\equiv t+\theta C^{t}-\bar{\theta}\bar{C}_E-i\bar{\theta}\theta\Lambda_E, \quad \hat{E}\equiv E+\theta C^{E}+\bar{\theta}\bar{C}_t+i\bar{\theta}\theta\Lambda_t,\\
&\hat{\lambda}\equiv\lambda+\theta C^{\lambda}+\bar{\theta}\bar{C}_{\pi}+i\bar{\theta}\theta\Lambda_{\pi}, \quad \hat{\pi}\equiv\pi+\theta C^{\pi}-\bar{\theta}\bar{C}_{\lambda}-i\bar{\theta}\theta\Lambda_{\lambda},\\
&\hat{c}\equiv c+\theta C^{c}-\bar{\theta}\bar{C}_{\bar{P}}+i\bar{\theta}\theta\Lambda_{\bar{P}}, \quad \hat{P}\equiv P+\theta C^{P}-\bar{\theta}\bar{C}_{\bar{c}}+i\bar{\theta}\theta\Lambda_{\bar{c}},\\
&\hat{\bar{c}}\equiv\bar{c}+\theta C^{\bar{c}}-\bar{\theta}\bar{C}_{P}+i\bar{\theta}\theta\Lambda_{P}, \quad \hat{\bar{P}}\equiv\bar{P}+\theta C^{\bar{P}}-\bar{\theta}\bar{C}_{c}+i\bar{\theta}\theta\Lambda_{c}.
\end{aligned} \tag{5.13}$$

When we replace the fields by superfields in the Lagrangian L_{BFV} and integrate using the Grassmann measure $i\ d\theta\ d\bar{\theta}$, we obtain

$$\begin{aligned}
i\int d\theta\ d\bar{\theta} L_{\mathrm{BFV}}(\Phi)=\tilde{\mathscr{L}}_J-\frac{d}{d\tau}\Big[&\frac{1}{2}\,(\Lambda_a\phi^a+iC_aC^a)+\Lambda_E E+i\bar{C}_E C^E+\Lambda_{\pi}\pi\\
&+i\bar{C}_{\pi}C^{\pi}+\Lambda_P P+iC_P C^P+\Lambda_{\bar{P}}\bar{P}+i\bar{C}_{\bar{P}}C^{\bar{P}}\Big].
\end{aligned} \tag{5.14}$$

which differs from the CPI Lagrangian (4.14) by a surface term.

Applying the same construction to Ω of (3.26), we obtain

$$\begin{aligned}
\hat{\Omega}&=i\int d\theta\ d\bar{\theta}[-i\hat{P}\hat{\pi}+\hat{c}(H(\Phi)-\hat{E})]\\
&=-iP\Lambda_{\lambda}+\bar{C}_{\lambda}C^{P}-\bar{C}_{\bar{c}}C^{\pi}+i\Lambda_{\bar{c}}\pi+c(\tilde{\mathscr{H}}+\Lambda_t)\\
&\quad+i(\bar{C}_a h^a+\bar{C}_t)C^c-i\bar{C}_{\bar{P}}\left(C^a\,\frac{\partial H}{\partial\phi^a}-C^E\right)-\Lambda_{\bar{P}}(H-E).
\end{aligned} \tag{5.15}$$

The CPI Lagrangian is invariant under the transformations generated by this charge

$$\delta_{\hat{\Omega}}\tilde{\mathscr{L}}_J=0. \tag{5.16}$$

The action (3.27) also possesses an anti-BRST symmetry generated by the charge

$$\bar{\Omega}=i\bar{P}\pi+\bar{c}(H-E). \tag{5.17}$$

The corresponding symmetry of $\tilde{L}_J$ is generated by

$$\begin{aligned}\hat{\bar{\Omega}}&\equiv i\int d\theta d\bar{\theta}[i\hat{\bar{P}}\hat{\pi}+\hat{\bar{c}}(H(\Phi^a)-\hat{E})]\\&=i\bar{P}\Lambda_\lambda-\bar{C}_\lambda C^{\bar{P}}+\bar{C}_c C^\pi-i\pi\Lambda_c+\bar{c}(\tilde{\mathscr{H}}+\Lambda_t)\\&\quad+i(\bar{C}_a h^a+\bar{C}_t)C^{\bar{c}}-\bar{C}_P\left(C^a\frac{\partial H}{\partial\phi^a}-C^E\right)-\Lambda_P(H-E)\end{aligned} \tag{5.18}$$

and we have

$$\delta_{\hat{\bar{\Omega}}}\tilde{\mathscr{L}}_J=0. \tag{5.19}$$

The BFV-BRST and anti-BRST charges are nilpotent and anticommute with each other

$$[\hat{\Omega},\hat{\Omega}]=[\hat{\bar{\Omega}},\hat{\bar{\Omega}}]=[\hat{\Omega},\hat{\bar{\Omega}}]=0. \tag{5.20}$$

In addition to the above two charges it is easy to prove that the following charge is conserved:

$$Q_g=c\frac{\partial}{\partial c}+P\frac{\partial}{\partial P}-\bar{c}\frac{\partial}{\partial\bar{c}}-\bar{P}\frac{\partial}{\partial\bar{P}}+C^c\frac{\partial}{\partial C^c}+C^P\frac{\partial}{\partial C^P}-C^{\bar{c}}\frac{\partial}{\partial C^{\bar{c}}}-C^{\bar{P}}\frac{\partial}{\partial C^{\bar{P}}}, \tag{5.21}$$

$$\delta_{Q_g}\tilde{\mathscr{L}}_J=0. \tag{5.22}$$

This charge gives the BFV-ghost number, attributing a weight of $+1$ to the BFV-ghosts c, P and their forms, and a weight of -1 to the anti-ghosts $\bar{c}$, $\bar{P}$ and their forms. The auxilliary fields Λ and vectors C have the opposite ghost number. The ghost number charge is analogous to the form number charge Q_f. The BFV ghost number and the form number of the various fields are summarized in Table I. The ghost number and form number charges commute with each other

$$[Q_g,Q_f]=0. \tag{5.23}$$

A variable is Grassman even if its form number plus ghost number is even and odd if its form number plus ghost number is odd.

Thus far we have found the charges $\hat{\Omega}$, $\hat{\bar{\Omega}}$ and Q_g, connected with reparameterization invariance and analogous to the charges Q, $\bar{Q}$ and Q_f, respectively. It would be interesting to find anologues of $\hat{K}$ and $\hat{\bar{K}}$, for which the commutator closes on Q_g. These charges are given by

$$\begin{aligned}\bar{\mathscr{K}}&\equiv-\bar{P}\frac{\partial}{\partial P}-C^{\bar{P}}\frac{\partial}{\partial C^P}+\bar{c}\frac{\partial}{\partial c}+C^{\bar{c}}\frac{\partial}{\partial C^c},\\\mathscr{K}&\equiv-P\frac{\partial}{\partial\bar{P}}-C^P\frac{\partial}{\partial C^{\bar{P}}}+c\frac{\partial}{\partial\bar{c}}+C^c\frac{\partial}{\partial C^{\bar{c}}}.\end{aligned} \tag{5.24}$$

TABLE I. Form and ghost numbers of the fields in $\tilde{\mathscr{L}}_J$.

Fields	Form number	Ghost number
$\phi^a, t, E, \lambda, \pi$	0	0
$C^a, C^t, C^E, C^\lambda, C^\pi$	1	0
$\bar{C}_a, \bar{C}_t, \bar{C}_E, \bar{C}_\lambda, \bar{C}_\pi$	-1	0
c, P	0	1
Λ_c, Λ_P	0	-1
C^c, C^P	1	1
$\bar{C}_c, \bar{C}_P$	-1	-1
$\bar{c}, \bar{P}$	0	-1
$\Lambda_{\bar{c}}, \Lambda_{\bar{P}}$	0	1
$C^{\bar{c}}, C^{\bar{P}}$	1	-1
$\bar{C}_{\bar{c}}, \bar{C}_{\bar{P}}$	-1	$+1$

They both generate new symmetries of the CPI action

$$\delta_{\mathscr{K}} \tilde{\mathscr{L}}_J = 0, \quad \delta_{\bar{\mathscr{K}}} \tilde{\mathscr{L}}_J = 0.$$

The commutator of these two charges is given by

$$[\mathscr{K}, \bar{\mathscr{K}}] = Q_g \tag{5.25}$$

and their commutators with the BFV-BRST and anti-BRST charges are

$$[\bar{\mathscr{K}}, \hat{\Omega}] = \hat{\bar{\Omega}}, \quad [\mathscr{K}, \hat{\bar{\Omega}}] = \hat{\Omega}, \quad [\mathscr{K}, \hat{\Omega}] = [\bar{\mathscr{K}}, \hat{\bar{\Omega}}] = 0. \tag{5.26}$$

The charges $\mathscr{K}$ and $\bar{\mathscr{K}}$ have ghost number $+2$ and -2, respectively,

$$[Q_g, \mathscr{K}] = 2\mathscr{K}, \quad [Q_g, \bar{\mathscr{K}}] = -2\bar{\mathscr{K}}, \tag{5.27}$$

while the charges $\hat{\Omega}$ and $\hat{\bar{\Omega}}$ have ghost number $+1$ and -1

$$[Q_g, \hat{\Omega}] = \hat{\Omega}, \quad [Q_g, \hat{\bar{\Omega}}] = -\hat{\bar{\Omega}}. \tag{5.28}$$

From the above graded commutators we conclude that the charges

$$\{\hat{\Omega}, \hat{\bar{\Omega}}, Q_g, \mathscr{K}, \bar{\mathscr{K}}\} \tag{5.29}$$

generate a second $I\mathrm{Sp}(2)$ algebra, in addition to that generated by the charges (5.10), which are a generalization of those found in Ref. 1. The charges of the $I\mathrm{Sp}(2)$ algebra (5.29) commute with the charges of the other $I\mathrm{Sp}(2)$ algebra (5.10).

VI. SUPERSYMMETRY IN $\tilde{\mathscr{L}}_J$

In addition to the pair of $I\mathrm{Sp}(2)$ symmetry algebras described in the previous section, the action (4.13) possesses a supersymmetry analogous to (2.16). We can construct the supersymmetry charges

$$\hat{\Omega}_{\hat{H}}=\hat{Q}-\beta[\hat{Q},\hat{H}]=C^a\left(\partial_a-\beta\lambda\,\frac{\partial H}{\partial\phi^a}\right)+C^t\partial_t+C^E(\partial_E+\beta\lambda)+C^\pi\partial_\pi$$
$$+C^\lambda(\partial_\lambda-\beta(H-E))+C^c\partial_c+C^{\bar{c}}\partial_{\bar{c}}+C^P(\partial_P+i\beta\bar{P})+C^{\bar{P}}(\partial_{\bar{P}}-i\beta P), \tag{6.1}$$

and

$$\hat{\bar{Q}}_{\hat{H}}=\hat{\bar{Q}}+\beta[\hat{\bar{Q}},\hat{H}]=\bar{C}_a\omega^{ab}\left(\frac{\partial}{\partial\phi^b}+\beta\lambda\,\frac{\partial H}{\partial\phi^b}\right)+\bar{C}_t(-\partial_E+\beta\lambda)+\bar{C}_E\partial_t+\bar{C}_\lambda\partial_\pi$$
$$+\bar{C}_\pi(-\partial_\lambda-\beta(H-E))+\bar{C}_P\partial_c+\bar{C}_{\bar{P}}\partial_{\bar{c}}+\bar{C}_c(\partial_{\bar{P}}+i\beta P)+\bar{C}_{\bar{c}}(\partial_P-i\beta\bar{P}), \tag{6.2}$$

where β is an arbitrary parameter and $\hat{H}$ is the effective Hamiltonian (4.8). We call $\hat{Q}_{\hat{H}}$ and $\hat{\bar{Q}}_{\hat{H}}$ supersymmetry charges, since they satisfy

$$[\hat{Q}_{\hat{H}},\hat{\bar{Q}}_{\hat{H}}]=2i\beta\widetilde{\mathscr{H}}_J, \tag{6.3}$$

where $\widetilde{\mathscr{H}}_J$ is the CPI Hamiltonian, associated to $\widetilde{\mathscr{L}}_J$ of (4.14), and is given explicitly by

$$\widetilde{\mathscr{H}}_J=\lambda\left[\Lambda_a h^a+i\bar{C}_a\,\frac{\partial h^a}{\partial\phi^b}\,C^b+\Lambda_t\right]-\Lambda_\pi(H-E)+i(\bar{C}_a h^a+\bar{C}_t)C^\lambda$$
$$-i\bar{C}_\pi\left(\frac{\partial H}{\partial\phi^a}\,C^a-C^E\right)-i\Lambda_c P+\bar{C}_c C^P+i\Lambda_{\bar{c}}\bar{P}-\bar{C}_{\bar{c}}C^{\bar{P}}. \tag{6.4}$$

The variations of $\widetilde{\mathscr{L}}_J$ generated by the two supersymmetry charges are given by the surface terms

$$\delta_{\hat{Q}_{\hat{H}}}\widetilde{\mathscr{L}}_J=-i\bar{\epsilon}\beta\,\frac{d}{d\tau}\,([\hat{Q},\hat{H}]),\quad \delta_{\hat{\bar{Q}}_{\hat{H}}}\widetilde{\mathscr{L}}_J=-i\epsilon\,\frac{d}{d\tau}\,\hat{\bar{Q}}. \tag{6.5}$$

Let us now consider which physical states $\tilde{\varrho}$ are invariant under the supersymmetry generated by (6.1) and (6.2). If we confine our attention to scalar observables, as in Sec. II, then $\tilde{\varrho}$ must have maximal form number

$$\tilde{\varrho}=\varrho C^1\ldots C^{2n}C^tC^EC^\lambda C^\pi\delta(C^c)\,\delta(C^P)\,\delta(C^{\bar{c}})\,\delta(C^{\bar{P}}). \tag{6.6}$$

Physical states must have ghost number zero[13] and be annihalated by the BFV-BRST and anti-BRST charges $\hat{\Omega}$ and $\hat{\bar{\Omega}}$. The most general state (6.6) with ghost number zero has the form

$$\varrho=\varrho_{(0)}+\varrho_{(c\bar{c})}c\bar{c}+\varrho_{(c\bar{P})}c\bar{P}+\varrho_{(P\bar{c})}P\bar{c}+\varrho_{(P\bar{P})}P\bar{P}+\varrho_{(cP\bar{c}\bar{P})}cP\bar{c}\bar{P}. \tag{6.7}$$

The conditions

$$\hat{\Omega}\tilde{\varrho}=0,\quad \hat{\bar{\Omega}}\tilde{\varrho}=0 \tag{6.8}$$

imply the following relations among the components in (6.7):

$$\varrho_{(P\bar{c})}=-\varrho_{(c\bar{P})},$$
$$(h^a\partial_a+\partial_t)\varrho_{(0)}=(H-E)\varrho_{(c\bar{P})}-i\pi\varrho_{(c\bar{c})},$$

$$\frac{\partial \varrho_{(0)}}{\partial \lambda} = -\pi \varrho_{(c\overline{P})} + i(H-E)\varrho_{(P\overline{P})},$$

$$\frac{\partial \varrho_{(c\overline{c})}}{\partial \lambda} = -i(h^a\partial_a + \partial_t)\varrho_{(c\overline{P})} - i(H-E)\varrho_{(cP\overline{c}\overline{P})}, \tag{6.9}$$

$$\frac{\partial \varrho_{(c\overline{P})}}{\partial \lambda} = i(h^a\partial_a + \partial_t)\varrho_{(P\overline{P})} + \pi \varrho_{(cP\overline{c}\overline{P})}.$$

If a physical state is supersymmetric it is also annihilated by $\hat{Q}_{\hat{H}}$ and $\hat{\overline{Q}}_{\hat{H}}$. States of the form (6.6) are automatically annihilated by $\hat{Q}_{\hat{H}}$. The condition $\hat{\overline{Q}}_{\hat{H}}\widetilde{\varrho} = 0$ implies that

$$\left(\partial_b + \beta\lambda \frac{\partial H}{\partial \phi^b}\right)\varrho = 0, \quad (-\partial_E + \beta\lambda)\varrho = 0,$$

$$\partial_t \varrho = \partial_\pi \varrho = 0, \quad (\partial_\lambda + \beta(H-E))\varrho = 0, \tag{6.10}$$

$$\partial_c \varrho = \partial_{\overline{c}} \varrho = 0, \quad (\partial_{\overline{P}} + i\beta P)\varrho = 0, \quad (\partial_P - i\beta \overline{P})\varrho = 0.$$

Substituting (6.7) into (6.10), we find immediately that

$$\varrho_{(c\overline{c})} = \varrho_{(c\overline{P})} = \varrho_{(P\overline{c})} = \varrho_{(cP\overline{c}\overline{P})} = 0, \quad \varrho_{(P\overline{P})} = i\beta \varrho_{(0)}.$$

The equations

$$\left(\partial_b + \beta\lambda \frac{\partial H}{\partial \phi^b}\right)\varrho_{(0)} = 0, \quad (-\partial_E + \beta\lambda)\varrho_{(0)} = 0,$$

$$\partial_t \varrho_{(0)} = \partial_\pi \varrho_{(0)} = 0, \quad (\partial_\lambda + \beta(H-E))\varrho_{(0)} = 0,$$

then have the solution

$$\varrho_{(0)} = k \exp[-\beta\lambda(H-E)],$$

where k is a constant. Therefore, we have found that a supersymmetric invariant state with maximal form number is given by (6.6) with

$$\varrho = k \exp[-\beta\lambda(H-E)](1 + i\beta P\overline{P}) = k \exp[-\beta(\lambda(H-E) + iP\overline{P})] = k \exp[-\beta\hat{H}]. \tag{6.11}$$

This is just the Gibbs state with the effective Hamiltonian $\hat{H}$ (4.8). It is easy to check that ϱ in (6.11) satisfies the physical state conditions (6.9).

The supersymmetry we have found in this section does not have a simple relationship with ergodicity. The CPI Hamiltonian $\widetilde{\mathscr{H}}_J$ of (6.4) generates the Lie derivative of the Hamiltonian flow in the extended phase space introduced by the BFV procedure. In order to establish a strict correspondence between supersymmetry and ergodicity, we would need to construct a pair of supercharges for which the anticommutator (6.3) produces the Lie derivative of the Hamiltonian flow on a surface of fixed energy. Although we have not succeeded in doing this, we shall find in the next section that with a suitable choice of boundary conditions the CPI with Lagrangian $\widetilde{\mathscr{L}}_J$ does yield the propagator on energy surfaces.

VII. PROPAGATORS FOR THE REPARAMETRIZATION INVARIANT CPI

In this section we derive some propagators from the classical path integrals with the action $\widetilde{\mathscr{S}}_J$ given by (4.13) and (4.14). First, we show that, with a suitable choice of boundary conditions, integrating out all of the fields associated to the BFV ghosts

$$c,P,\Lambda_c,\Lambda_P,\bar{c},\bar{P},\Lambda_{\bar{c}},\Lambda_{\bar{P}},C^{\bar{c}},\overline{C_{\bar{c}}},C^{\bar{P}},\overline{C_{\bar{P}}},C^c,\overline{C_c},C^P,\overline{C_P}$$

produces a factor inverse to the one obtained by integrating out the fields associated to the gauge fixing and constraint parts of the action

$$\Lambda_t,t,\Lambda_\lambda,\lambda,\Lambda_E,E,\Lambda_\pi,\pi,C^t,\overline{C_t},C^\lambda,\overline{C_\lambda},C^E,\overline{C_E},C^\pi,\overline{C_\pi}.$$

When all forms except for the C^a's are saturated at the initial point; through the insertion of delta-functions, the result is the propagator (2.14) for p-forms on the original phase space. This means that the CPI arising from the BFV formalism is unitary in the same sense as the quantum theory described in Sec. III. Again, this would not have been the case if we had excluded the BFV ghosts, because a factor coming from the ghost integration was needed to cancel a factor coming from the gauge fixing and constraints.

Next, we use a different choice of boundary conditions for the forms to derive the propagator for p-forms on the enlarged phase space including (ϕ^a,t,E). Finally, we use this result to obtain the propagator for p-forms on the energy surface.

As in (3.31), we can factor the path integral with action $\tilde{\mathscr{L}}_J$ as follows:

$$\int D\mu \exp i\int d\tau \tilde{\mathscr{L}}_J = \mathbf{Z}_{\text{ghost}} \int \mathscr{D}\phi \mathscr{D}\Lambda \mathscr{D}C \mathscr{D}\overline{C} \mathbf{Z}_{gfc} \times \exp i\int d\tau \left[\Lambda_a(\dot{\phi}^a - \lambda h^a) + i\overline{C_a}\left(\dot{C}^a - \lambda \frac{\partial h^a}{\partial \phi^b} C^b \right) \right], \tag{7.1}$$

where $\mathscr{D}\mu$ is the CPI measure including all fields in $\tilde{\mathscr{L}}_J$, the boundary conditions have not been specified, and

$$\mathbf{Z}_{\text{ghost}} \equiv \int \mathscr{D}\bar{c}\mathscr{D}\bar{P}\mathscr{D}\Lambda_{\bar{c}}\mathscr{D}\Lambda_{\bar{P}}\mathscr{D}c\mathscr{D}P\mathscr{D}\Lambda_c\mathscr{D}\Lambda_P\mathscr{D}C^{\bar{c}}\mathscr{D}\overline{C_{\bar{c}}}\mathscr{D}C^{\bar{P}}\mathscr{D}\overline{C_{\bar{P}}}\mathscr{D}C^c\mathscr{D}\overline{C_c}\mathscr{D}C^P\mathscr{D}\overline{C_P}$$
$$\times \exp \int_{\tau_1}^{\tau_2} d\tau [i\Lambda_P\dot{P} + i\Lambda_c(\dot{c}+iP) - \overline{C_P}\dot{C}^P - \overline{C_c}(\dot{C}^c + iC^P) + i\Lambda_{\bar{P}}\dot{\bar{P}} + i\Lambda_{\bar{c}}(\dot{\bar{c}} - i\bar{P})$$
$$- \overline{C_{\bar{P}}}\dot{C}^{\bar{P}} - \overline{C_{\bar{c}}}(\dot{C}^{\bar{c}} - iC^{\bar{P}})] \tag{7.2}$$

while[29]

$$\mathbf{Z}_{gfc} \equiv \int \mathscr{D}t\mathscr{D}\lambda\mathscr{D}\Lambda_t\mathscr{D}\Lambda_\lambda\mathscr{D}E\mathscr{D}\pi\mathscr{D}\Lambda_E\mathscr{D}\Lambda_\pi\mathscr{D}C^t\mathscr{D}\overline{C_t}\mathscr{D}C^\lambda\mathscr{D}\overline{C_\lambda}\mathscr{D}C^E\mathscr{D}\overline{C_E}\mathscr{D}C^\pi\mathscr{D}\overline{C_\pi}$$
$$\times \exp \int_{\tau_1}^{\tau_2} [i\Lambda_t(\dot{t}-\lambda) + i\Lambda_\lambda\dot{\lambda} + i\Lambda_E\dot{E} + i\Lambda_\pi(\dot{\pi} + H - E) - \overline{C_\lambda}\dot{C}^\lambda - \overline{C_t}\dot{C}^t + (\overline{C_t} + \overline{N})C^\lambda$$
$$- \overline{C_E}\dot{C}^E - \overline{C_\pi}\dot{C}^\pi - \overline{C_\pi}(N - C^E)] \tag{7.3}$$

with $N \equiv C^a(\partial H/\partial \phi^a)$ and $\overline{N} \equiv \overline{C_a}\omega^{ab}(\partial H/\partial \phi^b)$. If we are only interested in the propagation of the original phase space variables ϕ^a and their forms C^a, then all of the other forms should be set equal to zero at the initial point τ_1. Integrating these other forms out of the CPI will then yield a Jacobian which equals one.[1]

Let us first evaluate $\mathbf{Z}_{\text{ghost}}$ with the boundary conditions[11]

$$c(\tau_1)=c(\tau_2)=0=\overline{c(}\tau_1)=\overline{c(}\tau_2),$$

$$C^c(\tau_1)=C^{\bar{c}}(\tau_1)=0=C^P(\tau_1)=C^{\bar{P}}(\tau_1)$$

as in (3.30). Then we find

$$\int \mathscr{D}'c\mathscr{D}P\mathscr{D}\Lambda_c\mathscr{D}\Lambda_P \exp \int_{\tau_1}^{\tau_2}(i\Lambda_P\dot{P}+i\Lambda_c(\dot{c}+iP))d\tau=i(\tau_2-\tau_1), \tag{7.4}$$

and

$$\int \mathscr{D}'\bar{c}\mathscr{D}\bar{P}\mathscr{D}\Lambda_{\bar{c}}\mathscr{D}\Lambda_{\bar{P}} \exp \int_{\tau_1}^{\tau_2}(i\Lambda_{\bar{c}}(\dot{\bar{c}}-i\bar{P})+i\Lambda_{\bar{P}}\dot{\bar{P}})d\tau=-i(\tau_2-\tau_1) \tag{7.5}$$

while the form integration, over the bosonic variables, yields

$$\int \mathscr{D}C^c\mathscr{D}\overline{C}_c\mathscr{D}C^P\mathscr{D}\overline{C}_P\mathscr{D}C^{\bar{c}}\mathscr{D}\overline{C}_{\bar{c}}\mathscr{D}C^{\bar{P}}\mathscr{D}\overline{C}_{\bar{P}}\delta(C^c(\tau_1))\,\delta(C^P(\tau_1))\,\delta(C^{\bar{c}}(\tau_1))\,\delta(C^{\bar{P}}(\tau_1))$$

$$\times\exp \int_{\tau_1}^{\tau_2}d\tau[-\overline{C}_P\dot{C}^P-\overline{C}_c(\dot{C}^c+iC^P)-\overline{C}_{\bar{P}}\dot{C}^{\bar{P}}-\overline{C}_{\bar{c}}(\dot{C}^{\bar{c}}-iC^{\bar{P}})]=1. \tag{7.6}$$

Combining the above results we obtain

$$\mathbf{Z}_{\text{ghost}}=(\tau_2-\tau_1)^2. \tag{7.7}$$

Let us now proceed to evaluate $\mathbf{Z}_{gfc}$. The functional integral over forms

$$C^t,\overline{C}_t,C^\lambda,\overline{C}_\lambda,C^E,\overline{C}_E,C^\pi,\overline{C}_\pi$$

with the boundary conditions

$$C^\pi(\tau_1)=C^E(\tau_1)=0=C^t(\tau_1)=C^\lambda(\tau_1)$$

again equals one. The remaining part of $\mathbf{Z}_{gfc}$ can be factored into an integral over $\Lambda_t,t,\Lambda^\lambda,\lambda$ times an integral over $\Lambda_E,E,\Lambda_\pi,\pi$. In accordance with (3.30), we use the boundary conditions

$$t(\tau_{1,2})=t_{1,2},\quad \pi(\tau_{1,2})=0$$

while the other variables are integrated at the endpoints. Then we obtain

$$\int \mathscr{D}'t\mathscr{D}\lambda\mathscr{D}\Lambda_t\mathscr{D}\Lambda_\lambda \exp \int_{\tau_1}^{\tau_2}[i\Lambda_t(\dot{t}-\lambda)+i\Lambda_\lambda\dot{\lambda}]d\tau=\int d\lambda_0(\tau_2-\tau_1)^{-1}\delta\left[\frac{t_2-t_1}{\tau_2-\tau_1}-\lambda_0\right] \tag{7.8}$$

and

$$\int \mathscr{D}E\mathscr{D}'\pi\mathscr{D}\Lambda_E\mathscr{D}\Lambda_\pi \exp \int_{\tau_1}^{\tau_2}d\tau[i\Lambda_E\dot{E}+i\Lambda_\pi(\dot{\pi}+H-E)]=(\tau_2-\tau_1)^{-1}\int dE_0\delta(\langle H\rangle-E_0), \tag{7.9}$$

where $\langle H\rangle=(\tau_2-\tau_1)^{-1}\int_{\tau_1}^{\tau_2}H(\phi(\tau))d\tau$ which equals $H(\phi_1)$ on a classical trajectory through ϕ_1. $\mathbf{Z}_{gfc}$ is the product of (7.8) and (7.9). It contains the factor $(\tau_2-\tau_1)^{-2}$ which gets exactly cancelled by $\mathbf{Z}_{\text{ghost}}$. This confirms the need to also exponentiate the BFV ghost equations of motion in constructing the CPI. Inserting (7.7), (7.8), and (7.9) into (7.1), we find that for the boundary conditions

$$\begin{gathered}
t(\tau_1)=t_1, \quad t(\tau_2)=t_2, \quad \phi^a(\tau_1)=\phi_1^a, \quad \phi^a(\tau_2)=\phi_2^a, \\
\pi(\tau_1)=\pi(\tau_2)=0, \quad c(\tau_1)=c(\tau_2)=0, \\
\overline{c(}\tau_1)=\overline{c(}\tau_2)=0, \quad C^a(\tau_1)=C_1^a, \quad C^a(\tau_2)=C_2^a, \\
\text{all other forms}=0 \quad \text{at } \tau_1
\end{gathered} \tag{7.10}$$

we get

$$\begin{aligned}
&\int D'\mu \exp i\int d\tau \widetilde{\mathscr{L}}_J \\
&\quad =\int d\lambda_0 \delta\left(\frac{t_2-t_1}{\tau_2-\tau_1}-\lambda_0\right) dE_0 \delta(\langle H\rangle - E_0)\mathscr{D}'\phi \mathscr{D}\Lambda \mathscr{D}' C \mathscr{D}\overline{C} \\
&\qquad \times \exp \int_{\tau_1}^{\tau_2}\left[i\Lambda_a(\dot{\phi}^a - h^a\lambda_0) - \overline{C}_\alpha \dot{C}^a + \lambda_0 \overline{C}_a \frac{\partial h^a}{\partial \phi^b} C^b\right] d\tau \\
&\quad =\int \mathscr{D}'\phi \mathscr{D}\Lambda \mathscr{D}' C \mathscr{D}\overline{C} \exp \int_{\tau_1}^{\tau_2}\left[i\Lambda_a\left(\frac{d\phi^a}{d\tau} - \frac{t_2-t_1}{\tau_2-\tau_1} h^a\right)\right. \\
&\qquad \left. - \overline{C}_a \frac{dC^a}{d\tau} + \frac{t_2-t_1}{\tau_2-\tau_1}\overline{C}_a \frac{\partial h^a}{\partial\phi^b} C^b\right] d\tau \\
&\quad =\int \mathscr{D}'\phi \mathscr{D}\Lambda \mathscr{D}' C \mathscr{D}\overline{C} \exp \int_{t_1}^{t_2}\left[i\Lambda_a\left(\frac{d\phi^a}{dt} - h^a\right) - \overline{C}_a \frac{dC^a}{dt} + \overline{C}_a \frac{\partial h^a}{\partial\phi^b} C^b\right] dt \\
&\quad = K(\phi_2, C_2, t_2 | \phi_1, C_1, t_1),
\end{aligned} \tag{7.11}$$

where we have made the identification (3.35). Since, for the boundary conditions (7.10), the CPI with action $\widetilde{\mathscr{S}}_J$ reduces to the propagator (2.14) for p-forms on the original phase space, the theory is unitary.

In going over to the reparametrization invariant formalism and utilizing the BFV procedure, we have introduced the bosonic variables t,E,λ,π and ghosts $c,P,\overline{c},\overline{P}$. We have found that if we saturate the forms for these variables at the initial point, by inserting delta-functions and integrating them out, we get back the CPI on the original phase space. However, the purpose of the reparametrization invariant formalism was to bring in the energy, and with it time, as a dynamical variable. Now, we would like to impose different boundary conditions on the CPI, in order to obtain the propagator for p-forms on the enlarged phase space including (ϕ^a,t,E). This is a prelude to obtaining the propagator for p-forms restricted to the energy surface.

Recall that in the quantum path integral, discussed in Sec. III, it was not possible to fix both time and energy at the endpoints, as a result of the uncertainty principle. In the CPI action $\widetilde{\mathscr{S}}_J$, on the other hand, time and energy are no longer canonically conjugate variables, but rather Λ_t is conjugate to t and Λ_E is conjugate to E. Therefore, in the CPI we are allowed to fix both time and energy at the endpoints

$$t(\tau_1)=t_1, \quad t(\tau_2)=t_2, \quad E(\tau_1)=E_1, \quad E(\tau_2)=E_2.$$

Then (7.9) gets replaced by

$$\int_{\pi_1=0,E_1}^{\pi_2=0,E_2} \mathscr{D}' E \mathscr{D}' \pi \mathscr{D}\Lambda_E \mathscr{D}\Lambda_\pi \exp \int_{\tau_1}^{\tau_2} d\tau [i\Lambda_E \dot{E} + i\Lambda_\pi (\dot{\pi} + H - E)]$$

$$= (\tau_2 - \tau_1)^{-1} \delta(E_2 - E_1) \delta(\langle H \rangle - E_1). \tag{7.12}$$

Since we want to propagate the forms for these variables, we impose the analogous boundary conditions

$$C^t(\tau_1) = C_1^t, \quad C^t(\tau_2) = C_2^t, \quad C^E(\tau_1) = C_1^E, \quad C^E(\tau_2) = C_2^E.$$

The boundary conditions

$$\pi(\tau_1) = \pi(\tau_2) = 0, \quad c(\tau_1) = c(\tau_2) = 0, \quad \overline{c(}\tau_1) = \overline{c(}\tau_2) = 0$$

impose the constraints (7.8), which fixes the relationship between physical time and the parameter, and (7.12), which confines the motion to energy surfaces. To impose similar constraints on the forms, we choose the analogous boundary conditions

$$C^\pi(\tau_1) = C^\pi(\tau_2) = 0, \quad C^c(\tau_1) = C^c(\tau_2) = 0, \quad C^{\bar{c}}(\tau_1) = C^{\bar{c}}(\tau_2) = 0.$$

Now the form integrations will no longer yield one. Instead, we find for the ghost forms

$$\int \mathscr{D}C^c \mathscr{D}\overline{C}_c \mathscr{D}C^P \mathscr{D}\overline{C}_P \mathscr{D}C^{\bar{c}} \mathscr{D}\overline{C}_{\bar{c}} \mathscr{D}C^{\bar{P}} \mathscr{D}\overline{C}_{\bar{P}} \delta(C^c(\tau_2)) \delta(C^{\bar{c}}(\tau_2)) \delta(C^c(\tau_1)) \delta(C^{\bar{c}}(\tau_1))$$

$$\times \exp \int_{\tau_1}^{\tau_2} d\tau [-\overline{C}_P \dot{C}^P - \overline{C}_c (\dot{C}^c + iC^P) - \overline{C}_{\bar{P}} \dot{C}^{\bar{P}} - \overline{C}_{\bar{c}} (\dot{C}^{\bar{c}} - iC^{\bar{P}})] = (\tau_2 - \tau_1)^{-2}. \tag{7.13}$$

To evaluate the remaining form integrals, we use the general formula for Grassmann functional integrals

$$\int \mathscr{D}C \mathscr{D}\overline{C} \delta^N(C(\tau_2) - C_2) \delta^N(C(\tau_1) - C_1) \exp \int_{\tau_1}^{\tau_2} d\tau [-\overline{C}_\mu \dot{C}^\mu + \overline{C}_\mu \mathscr{M}^\mu_\nu C^\nu]$$

$$= \delta^N(-C_2 + C_{cl}(\tau_2; \tau_1, C_1)), \tag{7.14}$$

where

$$C_{cl}^\mu(\tau_2; \tau_1, C_1) = \left[T \exp \int_{\tau_1}^{\tau_2} ds \mathscr{M}(s) \right]^\mu_\nu C_1^\nu \tag{7.15}$$

is the solution of the equation of motion for the components of C, with initial condition $C(\tau_1) = C_1$.

The equations of motion for the forms can be read off from (4.14). Their solutions are

$$C_{cl}^a(\tau) = S_b^a(\lambda_0(\tau - \tau_1), \phi_1) C_1^b + (\tau - \tau_1) h^a(\phi_{cl}(\lambda_0(\tau - \tau_1), \phi_1)) C_1^\lambda,$$

$$C^t(\tau) = C_1^t + (\tau - \tau_1) C_1^\lambda, \quad C^E(\tau) = C_1^E,$$

$$C^\lambda(\tau) = C_1^\lambda, \quad C^\pi(\tau) = C_1^\pi + (\tau - \tau_1) \left(C_1^E - C_1^a \frac{\partial H}{\partial \phi^a} \right), \tag{7.16}$$

where

$$S_b^a(\lambda_0(\tau-\tau_1),\phi_1)=\left[T\ \exp\int_{\tau_1}^{\tau}d\tau'\lambda_0\ \frac{\partial h}{\partial\phi}\ (\phi_{cl}(\lambda_0(\tau'-\tau_1),\phi_1))\right]_b^a$$

and $\phi_{cl}(\lambda_0(\tau-\tau_1),\phi_1)$ is the solution of $\dot{\phi}=\lambda_0 h(\phi)$ with initial condition $\phi(\tau_1)=\phi_1$. Combining (7.16), (7.14), (7.13), (7.12), and (7.8), we obtain

$$\begin{aligned}
&K(\phi_2,C_2,t_2,C_2^t,E_2,C_2^E|\phi_1,C_1,t_1,C_1^t,E_1,C_1^E)\\
&=\int D'\mu\ \exp\ i\int_{\tau_1}^{\tau_2}d\tau\tilde{\mathscr{L}}_J\\
&=(\tau_2-\tau_1)^{-2}\delta(E_2-E_1)\int_{\phi_1}^{\phi_2}\mathscr{D}'\phi\int\ d\lambda_0 dC_1^\lambda\delta\left(\frac{t_2-t_1}{\tau_2-\tau_1}-\lambda_0\right)\delta(\langle H\rangle-E_1)\\
&\quad\times\delta^{2n}(-C_2+S(\lambda_0(\tau_2-\tau_1),\phi_1)C_1+(\tau_2-\tau_1)h(\phi_{cl}(\lambda_0(\tau_2-\tau_1),\phi_1))C_1^\lambda)\\
&\quad\times(-C_2^t+C_1^t+(\tau_2-\tau_1)C_1^\lambda)(-C_2^E+C_1^E)(\tau_2-\tau_1)\left(C_1^E-C_1^a\ \frac{\partial H}{\partial\phi^a}\ (\phi_1)\right)\\
&\quad\times\exp\ i\int_{\tau_1}^{\tau_2}d\tau\Lambda_a(\dot{\phi}^a-\lambda_0 h^a)\\
&=\sigma\delta(E_2-E_1)\delta(H(\phi_1)-E_1)\delta^{2n}(\phi_2-\phi_{cl}(t_2-t_1,\phi_1))\\
&\quad\times\left[\delta^{2n}(-S^{-1}C_2+C_1)+h^a(\phi_1)\ \frac{\partial}{\partial C_1^a}\ \delta^{2n}(-S^{-1}C_2+C_1)(-C_2^t+C_1^t)\right]\\
&\quad\times(-C_2^E+C_1^E)\left(C_1^E-C_1^a\ \frac{\partial H}{\partial\phi^a}\ (\phi_1)\right),
\end{aligned}\tag{7.17}$$

where $\sigma=\pm1$ is an overall factor coming from the ordering of Grassmann variables, S^{-1} is the inverse of $S((t_2-t_1),\phi_1)$ and we have made use of the fact[3] that $\det[S]=1$.

Equation (7.17) gives an expression for the propagator of p-forms on the enlarged phase space which includes time and energy. We would like to find the propagator for forms confined to a phase space surface at fixed energy. In order to do this we need to impose additional restrictions at the boundary of the path integral. To propagate scalar distributions on the energy surface $H(\phi)=E_0$, we simply insert $\delta(E_1-E_0)$ into the initial state and then integrate out the energy at both endpoints. Since the form C^E can be thought of as a first-order variation in E, the corresponding restriction on forms entails the insertion of $\delta(C_1^E)=C_1^E$ into the initial state and the integration over C^E at both endpoints. We have also fixed the time at each endpoint. The corresponding boundary condition for forms is

$$C^t(\tau_1)=0,\quad C^t(\tau_2)=0.$$

Imposing these additional boundary conditions, we obtain the result

$$\begin{aligned}
K[\phi_2,C_2|\phi_1,C_1]|_{E_0}&=\int\ dE_2dE_1dC_2^EdC_1^EdC_2^tdC_1^tC_2^t\\
&\quad\times K(\phi_2,C_2,t_2,C_2^t,E_2,C_2^E|\phi_1,C_1,t_1,C_1^t,E_1,C_1^E)C_1^tC_1^E\delta(E_1-E_0)\\
&=\sigma\delta(H(\phi_1)-E_0)\delta^{2n}(\phi_2-\phi_{cl}(t_2-t_1,\phi_1))
\end{aligned}$$

$$\delta^{2n}(-S^{-1}C_2+C_1)C_1^a\,\frac{\partial H}{\partial\phi^a}\,(\phi_1). \tag{7.18}$$

This propagator only makes sense for p-forms with $p\geqslant 1$. It contains the two constraining delta-functions $\delta(H(\phi_1)-E_0)$ and $C_1^a(\partial H/\partial\phi^a)(\phi_1)=N_1=\delta(N_1)$. The first constraint simply ensures that the support of any state $\tilde{\varrho}(\phi,C)$ propagated by (7.18) lies on the energy surface $\mathscr{M}_{2n-1}(E_0)$. The delta-function $\delta(N_1)$ ensures that all p-forms in the expansion (2.12) of $\tilde{\varrho}(\phi,C)$ are confined to the energy surface. The one-form N_1 points out of the energy surface in the sense that the interior product of N_1 with any vector in the tangent space $T_{\phi}\mathscr{M}_{2n-1}(E_0)$, to the energy surface, is zero. We can imagine performing a transformation on the cotangent space $T_{\phi}^*\mathscr{M}_{2n}$ of the form

$$\{C^a;a=1\cdots 2n\}\rightarrow\{N,C'^2\cdots C'^{2n}\}.$$

Then any p-form $\tilde{\varrho}$ confined to the energy surface may not contain N as a factor. In terms of the wedge product, this means that

$$N\tilde{\varrho}\neq 0. \tag{7.19}$$

We see from (7.18) that only p-forms which satisfy (7.19) can be propagated by $K[\phi_2,C_2|\phi_1,C_1]|_{E_0}$.

VIII. CONCLUSIONS

In this paper we have developed the classical path integral for a reparametrization invariant formulation of classical mechanics in which time and energy appear as dynamical variables. Following the BFV procedure, we built an effective Lagrangian containing, in addition to time, energy and the original phase space variables, ghosts and auxilliary fields. The Euler–Lagrange equations for this effective Lagrangian are then the classical equations of motion. We constructed the CPI by introducing a set of fields $\Lambda,C,\bar{C}$, which appear in the exponential representation of a delta-functional that projects onto the equations of motion and a Jacobian. We also obtained the CPI action by replacing each field in the BFV effective Lagrangian by a superfield and then integrating over superspace.

The same procedure can be applied to any theory with reparametrization invariance, or with gauge invariance. Gozzi[30] constructed the classical path integral for the relativistic particle, starting from the BFV Lagrangian in Ref. 25. Carta[31] developed the classical path integral for Yang–Mills theory.

Now that we have a classical path integral formalism in which the energy is singled out, the next thing to do is to use it to study the behavior of Hamiltonian systems as a function of energy. It would be interesting to calculate Lyapunov exponents and partition functions[3] using the classical path integral (7.18) which sits on the energy surface. Another important problem is to find a necessary and sufficient condition for ergodicity based on the supersymmetry inherent in the reparametrization invariant formulation of the CPI. We hope to come back to these issues in the future.

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