

CARTAN CALCULUS AND ITS GENERALIZATIONS VIA A PATH-INTEGRAL APPROACH TO CLASSICAL MECHANICS

ENNIO GOZZI

ABSTRACT. In this paper we review the recently proposed path-integral counterpart of the Koopman-von Neumann operatorial approach to classical Hamiltonian mechanics. We identify in particular the geometrical variables entering this formulation and show that they are essentially a basis of the cotangent bundle to the tangent bundle to phase-space. In this space we introduce an extended Poisson brackets structure which allows us to re-do all the usual Cartan calculus on symplectic manifolds via these brackets. We also briefly sketch how the Schouten-Nijenhuis, the Frölicher-Nijenhuis and the Nijenhuis-Richardson brackets look in our formalism.

1. Introduction

Soon after the appearance of quantum mechanics with its intrinsic operatorial structure, Koopman and von Neumann [6][11] gave an operatorial formulation also to classical Hamiltonian mechanics.

It is well-known that any theory which exists in the operatorial formulation admits also a path-integral version as it was shown for quantum mechanics [2] long ago by R.P.Feynman. Having, thanks to the work of Koopman and von Neumann, also classical mechanics in an operatorial version, it was easy to give a path-integral for it [4]. The weight in this classical path-integral (CPI) is just a functional Dirac delta forcing all paths on the classical ones.

We will briefly review ref.[4] in section 2 showing how the simple Dirac delta mentioned above can be turned into an effective evolution classical operator. In section 3 We will illustrate the geometrical meaning of the various variables appearing in our path-integral, variables which do not parametrize only the phase space \mathcal{M} of the system but instead the $T^*(T\mathcal{M})$ which is the cotangent bundle to the tangent bundle to phase-space. In this space (being a cotangent bundle) there naturally exists an extended Poisson structure (*epb*). In the same section we will then show how one can reproduce all the operations of the usual Cartan calculus on symplectic manifolds via our *epb* and via some universal charges present in our CPI. We conclude the paper with section 4 where the Schouten-Nijenhuis (NS)[7] [10], the Frölicher-Nijenhuis (FN)[3] and the Nijenhuis- Richardson (NR)[8] brackets are built out of our *epb* and the variables of $T^*(T\mathcal{M})$.

This work has been partly supported by a grant from the Ministero della ricerca scientifica e tecnologica of Italy and by a Nato grant.

2. Classical Path-Integral

We shall briefly review in this section the path-integral formulation of classical mechanics[4]. The propagator $P(\phi_2, t_2 | \phi_1, t_1)$, which gives the *classical* probability for a particle to be at the point ϕ_2 at time t_2 , given that it was at the point ϕ_1 at time t_1 , is just a delta function

$$P(\phi_2, t_2 | \phi_1, t_1) = \delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) \quad (2-1)$$

where $\Phi_{cl}(t, \phi_0)$ is a solution of Hamilton's equation $\dot{\phi}^a(t) = \omega^{ab} \partial_b H(\phi(t))$ subject to the initial conditions $\phi^a(t_1) = \phi_1^a$. Here H is the conventional Hamiltonian of a dynamical system defined on some phase-space \mathcal{M}_{2n} with local coordinates $\phi^a, a = 1 \cdots 2n$ and a constant symplectic structure $\omega = \frac{1}{2} \omega_{ab} d\phi^a \wedge d\phi^b$.

The delta function in (2-1) can be rewritten¹ as

$$\delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) = \left\{ \prod_{i=1}^{N-1} \int d\phi_{(i)} \delta^{2n}(\phi_{(i)} - \Phi_{cl}(t_i, \phi_1)) \right\} \delta^{2n}(\phi_2 - \Phi_{cl}(t_2, \phi_1)) \quad (2-2)$$

where We have sliced the interval $[0, t]$ in N intervals and labelled the various instants as t_i and the fields at t_i as $\phi_{(i)}$. Each delta function contained in the product on the RHS of (2-2) can be written as:

$$\delta^{2n}(\phi_{(i)} - \Phi_{cl}(t_i, \phi_1)) = \prod_{a=1}^{2n} \delta(\dot{\phi}^a - \omega^{ab} \partial_b H)_{|t_i} \det[\delta_b^a \partial_t - \partial_b(\omega_{ac}(\phi) \partial_c H(\phi))]_{|t_i} \quad (2-3)$$

where the argument of the determinant is obtained from the functional derivative of the equation of motion with respect to $\phi_{(i)}$. Introducing Grassmannian variables c^a and \bar{c}_a to exponentiate the determinant[9], and an auxiliary variable λ_a to exponentiate the delta functions, one can re-write the propagator above as a path-integral.

$$P(\phi_2, t_2 | \phi_1, t_1) = \int_{\phi_1}^{\phi_2} \mathcal{D}\lambda \mathcal{D}c \mathcal{D}\bar{c} \exp i\tilde{S} \quad (2-4)$$

where $\tilde{S} = \int_{t_1}^{t_2} dt \tilde{\mathcal{L}}$ with

$$\tilde{\mathcal{L}} \equiv \lambda_a [\dot{\phi}^a - \omega^{ab} \partial_b H(\phi)] + i\bar{c}_a (\delta_b^a \partial_t - \partial_b[\omega^{ac} \partial_c H(\phi)]) c^b \quad (2-5)$$

In the path-integral (2-4) we have used the slicing (2-2) and then taken the limit of $N \rightarrow \infty$. Holding ϕ and c both fixed at the endpoints of the path-integral, one obtains the kernel, $K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1)$, which propagates distributions in the space (ϕ, c)

$$\tilde{\varrho}(\phi_2, c_2, t_2) = \int d^{2n}\phi_1 d^{2n}c_1 K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) \tilde{\varrho}(\phi_1, c_1, t_1) \quad (2-6)$$

The distributions $\tilde{\varrho}(\phi, c)$ are finite sums of monomials of the type

$$\tilde{\varrho}(\phi, c) = \frac{1}{p!} \varrho_{a_1 \dots a_p}^{(p)}(\phi) c^{a_1} \dots c^{a_p} \quad (2-7)$$

¹We will often write ϕ without putting the upper indices a . The lower indices will instead indicate if they are the first or last point of a trajectory.

The kernel $K(\cdot|\cdot)$ is represented by the path-integral

$$K(\phi_2, c_2, t_2 | \phi_1, c_1, t_1) = \int \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp i \int_{t_1}^{t_2} dt \tilde{\mathcal{L}} \quad (2-8)$$

with the boundary conditions $\phi^a(t_{1,2}) = \phi_{1,2}^a$ and $c^a(t_{1,2}) = c_{1,2}^a$. It is also easy from here to build a classical generating functional Z_{cl} from which all correlation-functions can be derived. It is given by

$$Z_{cl} = \int \mathcal{D}\phi^a(t) \mathcal{D}\lambda_a(t) \mathcal{D}c^a(t) \mathcal{D}\bar{c}_a \exp i \int dt \{ \tilde{\mathcal{L}} + \text{source terms} \} \quad (2-9)$$

where the Lagrangian can be written as $\tilde{\mathcal{L}} = \lambda_a \dot{\phi}^a + i \bar{c}_a \dot{c}^b - \tilde{\mathcal{H}}$ with the "Hamiltonian" given by

$$\tilde{\mathcal{H}} = \lambda_a h^a + i \bar{c}_a \partial_b h^a c^b \quad (2-10)$$

and where h^a are the components of the Hamiltonian vector field[1] $h^a(\phi) \equiv \omega^{ab} \partial_b H(\phi)$. From the path-integral (2-8) and (2-9) one can easily check[4] that the variables (ϕ, λ) and (c, \bar{c}) form conjugate pairs satisfying the $(Z_2\text{-graded})$ commutation relations:

$$\begin{aligned} [\phi^a, \lambda_b] &= i \delta_b^a \\ [c^a, \bar{c}_b] &= \delta_b^a \end{aligned} \quad (2-11)$$

The commutators above are the usual split-time commutators which one can define in any path-integral[2]. For more details see ref.[4]. Because of these commutators, the variables λ_a and \bar{c}_a can be represented, in a sort of "Schroedinger-like" picture, as

$$\lambda_a = -i \frac{\partial}{\partial \phi^a} \equiv -i \partial_a ; \quad \bar{c}_a = \frac{\partial}{\partial c^a} \quad (2-12)$$

So one sees immediately that the λ_a represent a basis in the tangent space $T_\phi \mathcal{M}$. Inserting (2-12) in (2-10) the Hamiltonian becomes an operator²

$$\hat{\mathcal{H}} = -i l_h \equiv h^a \partial_a + c^b (\partial_b h^a) \frac{\partial}{\partial c^a} \quad (2-13)$$

The non-Grassmannian part of this operator coincides with the Liouvillian $\hat{L} = h^a \partial_a$, which gives the evolution of standard distributions $\varrho^{(0)}(\phi)$ in phase-space:

$$\partial_t \varrho^{(0)}(\phi, t) = -l_h \varrho^{(0)}(\phi, t) = -\hat{L} \varrho^{(0)}(\phi, t) \quad (2-14)$$

This is the standard operatorial version of CM of Koopman and von Neumann[6] [11]. This proves that our path-integral is really what is behind this operatorial formulation.

²We will see in section 3 that this is the Lie-derivative of the Hamiltonian flow[1].

3. Cartan Calculus

The reader at this point may start wondering what is the full $\tilde{\mathcal{H}}$ of eq.(2-10) with the Grassmannian part included. In order to answer that question we have to understand the geometrical meaning of the Grassmannian variables c^a appearing in our path-integral (2-9). It is easy to see[4] that under the time evolution the variables c^a transform as

$$c'^a = c^a - c^b \partial_b h^a \Delta t \quad (3-1)$$

i.e. they transform as forms. So we can say that each $c^a(\phi)$ belongs to the cotangent fiber in ϕ to \mathcal{M} , i.e. to $T^*_{|\phi} \mathcal{M}$. So the whole set of c^a and ϕ^a make up the cotangent bundle $T^* \mathcal{M}$ to \mathcal{M} . Having realized that, let us now see what the other variables λ_a, \bar{c}_a , entering our path-integral, are. Looking at eq.(2-12), we notice that they are a basis of the tangent space (T) to the (ϕ, c) -space which is $T^* \mathcal{M}$. We can then say that the $8n$ variables $(\phi^a, c^a, \lambda_a, \bar{c}_a)$ are the coordinates of $T(T^* \mathcal{M})$ which the tangent bundle to the cotangent bundle to phase-space.

As we know[1] that $T(T^* \mathcal{M}) \sim T^*(T\mathcal{M})$ and this last one is a cotangent bundle, we expect that there is an extended Poisson structure (epb) on $T(T^* \mathcal{M})$. There is in fact one which is

$$\{\phi^a, \lambda_b\}_{epb} = \delta_b^a, \quad \{c^a, \bar{c}_b\} = -i\delta_b^a, \quad \text{all others} = 0 \quad (3-2)$$

Note that these are different from the normal Poisson brackets on \mathcal{M} which were $\{\phi^a, \phi^b\}_{pb} = \omega^{ab}$. Via the extended Poisson brackets(3-2) we obtain from $\tilde{\mathcal{H}}$ of (2-10) the same equations of motion as those which one would obtain from the lagrangian $\tilde{\mathcal{L}}$ of (2-5). For ϕ^a in particular we have that the same equations provided by H via the normal Poisson brackets are also provided by $\tilde{\mathcal{H}}$ via the extended Poisson Brackets:

$$\{\phi^a, H\}_{pb} = \{\phi^a, \tilde{\mathcal{H}}\}_{epb} \quad (3-3)$$

Before proceeding further we should also point out that the $\tilde{\mathcal{H}}$ presents some universal invariance whose charges are the following[4]:

$$Q \equiv i c^a \lambda_a, \quad \bar{Q} \equiv i \bar{c}_a \omega^{ab} \lambda_b, \quad Q_g \equiv c^a \bar{c}_a, \quad K \equiv \frac{1}{2} \omega_{ab} c^a c^b, \quad \bar{K} \equiv \frac{1}{2} \omega^{ab} \bar{c}_a \bar{c}_b \quad (3-4)$$

The last thing to notice is that the variables \bar{c}_a transform under time-evolution as:

$$\bar{c}'_a = \bar{c}_a + \bar{c}_b \partial_a h^b \Delta t \quad (3-5)$$

which is exactly how a basis of the vector-fields transform. Notice that λ_a , even if it is $-i\partial_a$, as indicated in eq. (2-12), does not transform under time evolution as a basis of the vector fields. Its transformation is in fact:

$$\lambda'_a = [\lambda_a + \lambda_b \partial_a h^b \Delta t] + i \bar{c}_i \partial_b \partial_a h^i c^b \Delta t \quad (3-6)$$

This is not in contradiction with (2-12) because (3-6) is exactly how the derivatives $\frac{\partial}{\partial \phi^a}$ transform but when they are applied on functions of both ϕ and c . More work on this issue will appear in another paper.

Having now established that c^a are forms and \bar{c}_a a basis for the vector fields, it is then natural to build the following correspondence between forms and polinomials in c (which, due to the Grassmannian nature of the c , do not need the use of the wedge product \wedge), and between antisymmetric multivectors fields and polinomials in \bar{c} . We will indicate this correspondence via a $\widehat{(\cdot)}$ symbol:

$$\begin{aligned} F^{(p)} &= \frac{1}{p!} F_{a_1 \dots a_p} d\phi^{a_1} \wedge \dots \wedge d\phi^{a_p} \implies \widehat{F}^{(p)} = \frac{1}{p!} F_{a_1 \dots a_p} c^{a_1} \dots c^{a_p} \\ v^{(p)} &= \frac{1}{p!} V^{a_1 \dots a_p} \partial_{a_1} \wedge \dots \wedge \partial_{a_p} \implies \widehat{V}^{(p)} = \frac{1}{p!} \bar{c}_{a_1} \dots \bar{c}_{a_p} \end{aligned} \quad (3-7)$$

Using this correspondence it is then possible to rewrite all the normal operations of the Cartan calculus[1], like doing an exterior derivative on forms dF , or doing an interior product between a vector field and a form $i_v F$, or building the Lie-derivative of a vector field l_h , by just using polinomials in c and \bar{c} together with the extended Poisson brackets structure and the charges built in (3-4). These rules, which we called $\{\cdot, \cdot\}_{epb}$ -rules, are summarized below :

$$\begin{aligned} dF^{(p)} &\implies i\{Q, \widehat{F}\}_{epb} \\ i_v F^{(p)} &\implies i\{\widehat{v}, \widehat{F}^{(p)}\}_{epb} \\ l_h F = di_h F + i_h dF &\implies -\{\widetilde{\mathcal{H}}, \widehat{F}\}_{epb} \\ pF^{(p)} &\implies i\{Q_g, \widehat{F}\}_{epb} \\ \omega(v, \cdot) \equiv v^\flat &\implies i\{\bar{K}, \widehat{V}\}_{epb} \\ (df)^\sharp &\implies i\{\bar{Q}, f\}_{epb} \end{aligned} \quad (3-8)$$

where the last three operations indicated in (3-8) above are, respectively, multiplying a form $F^{(p)}$ by its degree p , mapping a vector field V into its associated one form V^\flat via the symplectic form, and building the associated Hamiltonian vector field $(df)^\sharp$ out of a function f . One sees from above that the various abstract derivations of the Cartan calculus are all implemented by some charges acting via the epb-brackets. From the third relation in (3-8) one also can notice that the Lie-derivative of the Hamiltonian vector field of time evolution becomes nothing else than the $\widetilde{\mathcal{H}}$ of (2-10), thus confirming that the weight-function of our classical path-integral, generated by just a simple Dirac delta, is the right *geometrical* object associated to the time-evolution.

The last question which our reader may be interested in getting an answer is what becomes of the Lie-bracket[1] of two vector fields V, W . The answer is the following:

$$[V, W]_{lie} \implies \propto \{\widetilde{\mathcal{H}}_V, \widehat{W}\}_{epb} \propto \{\widetilde{\mathcal{H}}_W, \widehat{V}\}_{epb} \quad (3-9)$$

where $\widehat{W} = W^a \bar{c}_a$ and $\widehat{V} = V^a \bar{c}_a$ while $\widetilde{\mathcal{H}}_V = \lambda_a V^a + i\bar{c}_a \partial_b V^a c^b$ and $\widetilde{\mathcal{H}}_W = \lambda_a W^a + i\bar{c}_a \partial_b W^a c^b$ are the analog of the Lie-derivatives associated respectively to the vector field V and W .

4. Generalized Cartan Calculus.

What we called "*Generalized Cartan Calculus*" is basically the following set of brackets: the Schouten-Nijenhuis ones(NS)[10][7] between antisymmetric multivector fields, the Frölicher-Nijenhuis brackets(FN)[8] and the Nijenhuis-Richardson ones(NR)[8] among vector-valued forms.

4.A Schouten-Nijenhuis Brackets. These brackets are a generalization on multivector fields of the Lie-brackets between vector fields. Following ref. [5] and given two multivector fields $P \equiv X_{(1)} \wedge \cdots \wedge X_{(p)}$ and $Q \equiv Y_{(1)} \wedge \cdots \wedge Y_{(q)}$ of rank respectively p and q , the NS-brackets among them is a multivector of rank $(p+q-1)$ given by:

$$[P, Q]_{(NS)} \equiv (-1)^{pq} \sum_{J=1}^q (-1)^{J+1} Y_{(1)} \wedge \cdots \wedge \hat{Y}_{(j)} \cdots \wedge Y_q \wedge [Y_{(j)}, P] \quad (4.A-1)$$

where the \hat{Y}_J means that that vector-field has been taken away, and $[Y_j, P] = l_{Y_j} P$ is the Lie-derivative of the vector field Y_j applied to the multivector P .

The NS-brackets can easily be translated into our (epb)-formalism via the rules established in section 3. The details of the calculations will be presented elsewhere, but the final result is the following:

$$[P, Q]_{(NS)} \implies \propto \{L_{Y_{(1)} \cdots Y_{(q)}}, X_{(1)}^a \bar{c}_a \cdots X_{(p)}^l \bar{c}_l\}_{epb} \quad (4.A-2)$$

where $L_{Y_{(1)} \cdots Y_{(q)}}$ is defined as:

$$L_{Y_{(1)} \wedge \cdots \wedge Y_{(q)}} \equiv \sum_{j=1}^q (-1)^{j-1} Y_{(1)}^a \bar{c}_a \cdots \widehat{Y_{(j)}^b \bar{c}_b} \cdots Y_{(q)}^l \bar{c}_l \tilde{\mathcal{H}}_{Y_j} \quad (4.A-3)$$

with $\tilde{\mathcal{H}}_{Y_{(j)}} = \lambda_a Y_{(j)}^a + i \bar{c}_a \partial_b Y_{(j)}^a c^b$

The $L_{Y_{(1)} \cdots Y_{(q)}}$ above is a generalization of the standard Lie-derivative.

4.B Frölicher-Nijenhuis Brackets. This is a bracket which associates³ to two *vector-valued forms*, $K \in \Omega^{k+1}(\mathcal{M}, T\mathcal{M})$ and $V \in \Omega^{l+1}(\mathcal{M}, T\mathcal{M})$ of rank respectively $(k+1)$ and $(l+1)$, a $(k+l+2)$ vector-valued form

$$[K, V]_{(FN)} \in \Omega^{k+l+2}(\mathcal{M}, T\mathcal{M})$$

Before proceeding we need to introduce some new notation[5]. First we have to generalize the notion[1] of interior contraction $i_v \Theta$ of a form Θ with a vector field v . The generalization is the contraction of an l -form Θ with a vector-valued $(k+1)$ -form K , the result will be a $(k+1)$ -form which can be contracted with $(k+1)$ -vectors $X_1 \cdots X_{k+1}$. Its precise definition is:

$$\begin{aligned} (i_K \Theta)(X_1, \cdots, X_{k+l}) &\equiv \\ &\equiv \frac{1}{(k+1)!(l-1)!} \sum_{\sigma \in S_{k+l}} \text{sign } \sigma \Theta \left[K(X_{\sigma_1} \cdots X_{\sigma_{k+1}}), X_{\sigma_{k+2}}, \cdots, X_{\sigma_{k+l}} \right] \end{aligned} \quad (4.B-1)$$

where σ is the set of permutation S_{k+l} of the $(k+1)$ vector fields $X_1 \cdots X_{k+l}$. Having now the generalized interior contraction defined above, we can then define a generalized Lie-derivative with respect to a vector-valued $(k+1)$ -form K :

$$L_K \equiv i_K d + di_K \quad (4.B-2)$$

³We use the notation of ref.5.

Using now (4.B-1) and (4.B-2), the FN brackets are defined[5] in the following implicit way:

$$[L_K, L_V]\Theta \equiv L_{[K,V]_{(FN)}}\Theta \quad (4.B-3)$$

where $[L_K, L_V]$ is the usual commutators among Lie-derivative and Θ is a form on which they act.

Let us now find out how the FN-brackets appear in our epb-formalism. The vector-valued forms K and V become

$$\begin{aligned} K &\Longrightarrow \hat{K} \equiv K_{a\dots k+1}^i \bar{c}_i [c^a \dots c^{k+1}] \\ V &\Longrightarrow \hat{V} \equiv V_{a\dots l+1}^j \bar{c}_j [c^a \dots c^{l+1}] \end{aligned} \quad (4.B-4)$$

Using this notation and the formulas of section 3, it is not difficult to prove that

$$[K, V]_{(FN)} \Longrightarrow \propto \left\{ \hat{K}, \{\hat{V}, Q\}_{epb} \right\}_{epb} \quad (4.B-5)$$

Also the details of the above calculations will be presented in a forthcoming paper.

4.C Nijenhuis-Richardson brackets. These are brackets also defined, as the FN ones, among $(k+1)$ and $(l+1)$ vector-valued forms $K \in \Omega^{k+1}(\mathcal{M}; T\mathcal{M})$ $V \in \Omega^{l+1}(\mathcal{M}; T\mathcal{M})$ but whose result is a $(k+l+1)$ vector valued-form. Their exact definition[5] is

$$[K, V]_{(NR)} \equiv i_K V - (-1)^{kl} i_V K \quad (4.C-1)$$

Here the i_K and i_V are the generalized interior contraction defined in (4.B-1).

In the language of the epb-brackets the NR-brackets have a simple expression:

$$[K, V]_{NR} \Longrightarrow \propto \{\hat{K}, \hat{V}\}_{epb} \quad (4.C-2)$$

where the \hat{K} and \hat{V} are given in (4.B-4). The calculational details of this derivations will be presented elsewhere.

5. Conclusions

The reader may wonder of what is the need of the dictionary we have created between Cartan (and generalized) calculus and our epb-formalism. The answer is in the fact that with our formalism we do not have to take care of all the various numerical factors and signs and permutations (see 4.B-1) which one has to remember by heart in doing the standard abstract Cartan calculus. In our case everything is taken care automatically by the Grassmannian natures of the c^a and \bar{c}_a and the graded structure of the epb-brackets. These, together with the five charges (3-4), seem to be the central and only ingredients needed to build all these operations. This reduction to these simple ingredients seemed to me a thing to bring to the attention of the mathematics and physics community in order to stimulate further investigations.

ACKNOWLEDGMENTS:

I wish to thank G.Marmo for suggesting the study of the NS, FN, NR brackets and for guidance and help throughout this work. A crucial discussion in january 1996 with G.Landi helped me in clearing up my ideas on the space $T^*(T\mathcal{M})$. Last, but not least, a special thank to M.Reuter and M.Regini for help on some technical points.

REFERENCES

1. Abraham, R., and Marsden, J., *Foundations of Mechanics*, Benjamin, 1978.
2. Feynman, R.P., Hibbs, R.A., *Quantum mechanics and path-integrals*, McGraw-Hill, Inc., 1965.
3. Frölicher, A. and Nijenhuis, A., *Theory of vector valued differential forms. Part I*, Indag. Math. **18** (1956), 338–359.
4. Gozzi, E., Reuter, M. and Thacker, W.D., *Hidden BRS invariance in classical mechanics. II*, Phys. Rev.D **40** (1989), 3363–3377.
5. Kolar, I. Michor, P.W., Slovák, J., *Natural Operations in differential Geometry*, Springer-Verlag, 1993.
6. Koopman, B.O., *Hamiltonian systems and transformations in Hilbert space*, Proc. Nat. Acad. Sci. USA **17** (1931), 315–318.
7. Nijenhuis, A., *Jacobi-type identities for bilinear differential concomitants of certain tensor fields I*, Indag. Math. **17** (1955), 390–403.
8. Nijenhuis, A., Richardson, R., *Deformation of Lie algebra structures*, J. Math. Mech. **17** (1967), 89–105.
9. Sakita, B., *Quantum theory of many variables systems and fields*, World.Scient. Publ, 1985.
10. Schouten, J.A., *Über differentialkomitanten zweier kontravarianter grossen*, Proc.Kon. Ned Akad. Wet. Amsterdam **43** (1940), 449–452.
11. von Neumann, J., *Zur operatorenmethode in der klassischen mechanik*, Ann.Math. **33** (1932), 587–648.

DIPARTIMENTO DI FISICA TEORICA, UNIVERSITA' DI TRIESTE, STRADA COSTIERA 11,
MIRAMARARE-GRIGNANO 34014 TRIESTE AND INFN, SEZIONE DI TRIESTE, ITALY