A Complex Hilbert Space for
Classical Electromagnetic Potentials

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Abstract

We demonstrate the existence of a complex Hilbert Space with Hermitian operators for calculations in classical electromagnetism. This approach lets us derive a variety of fundamental expressions for electromagnetism using minimal mathematics and a calculation sequence well-known for traditional quantum mechanics. The purpose of this Hilbert Space is not to calculate the expectation values of known observables, however, like in Koopman-von Neumann-Sudarshan (KvNS) mechanics (for classical point particles) and quantum mechanics (quantum waves or fields). We also demonstrate the existence of the wave commutation relationship $[\hat{x}, \hat{k}] = i$, which is the never seen before classical analogue to the canonical commutator $[\hat{x}, \hat{p}] = i\hbar$. The difference between classical and quantum mechanics lies in the presence of $\hbar$. This is the first report of noncommutativity of observables for a classical theory. Further comparisons between electromagnetism, KvNS classical mechanics, and quantum mechanics are made. Finally, supplementing the analysis presented, we additionally demonstrate for the first time a completely relativistic version of Feynman's proof of Maxwell's equations (Dyson, 1990). Unlike what Dyson (1990) indicated, there is no need for Galilean relativity for the proof to work. This fits parsimoniously with our usage of classical commutators for electromagnetism.

1 Classical and Quantum Hilbert Spaces

Quantum theory is a unique theory based on a complex Hilbert Space with Hermitian operators acting on the vector kets to produce eigenvalues. The superficially strange mathematics of quantum theory has for a long time haunted

many great physicists. There have been many attempts made to make quantum mathematics appear in form to resemble the classical functions of position, momentum, and time used in Newtonian physics, in order to avoid the complex Hilbert space with vectors living in an infinite number of dimension (Mauro, 2002, 2003). Some hoped this would make quantum theory more interpretable or less mysterious seeming, however, the Hilbert Space formalism could not be avoided.

Koopman-von Neumann-Sudarshan (KvNS) mechanics was an early approach to go the other way. Instead of representing quantum mechanics in terms of more familiar representations, it showed that one can take classical mechanics and represent it inside a complex Hilbert space as well. Koopman, von Neumann, and Sudarshan were able to take the same postulates of quantum mechanics and establish a purely classical theory (Koopman, 1931; von Neumann, 1932; Sudarshan, 1976; Mauro, 2002, 2003; Bondar et al., 2012, 2019; McCaul and Bondar, 2021; Piasecki, 2021). KvNS mechanics, like standard quantum mechanics, uses Hermitian operators acting on state kets to calculate expectation values of observables (Mauro, 2002, 2003; Bondar et al., 2012; McCaul and Bondar, 2021). It has a classical wavefunction ket $|\psi\rangle$ and a Born Rule to calculate probabilities. Also like with quantum theory, it has the curious property of collapse of the waveform once a classical measurement is made, although particular details differ (Mauro, 2003). One could say that this makes classical mechanics a hidden variable theory of quantum mechanics (Sudarshan, 1976). Bondar et al. (2012) was able to show that the only difference between classical theory and the quantum theory is in the choice of the position-momentum commutator. For relativistic and nonrelativistic classical mechanics, position and momentum commute:

$$[\hat{x}, \hat{p}] = 0$$

Using this fact and the Koopman algebra (Bondar et al., 2012; Cabrera et al., 2019; McCaul and Bondar, 2021; Piasecki, 2021), one is able to compute the Louiville equation for classical probability densities. Operational Dynamic Modeling (ODM) takes the KvNS and quantum formalism and merges them together to explore theoretical questions of interest (Bondar et al., 2012, 2013; Cabrera et al., 2019).

Here we present another classical Hilbert Space approach for electromagnetic fields. Like KvNS mechanics, it will also contain Hermitian operators acting on kets and the eigenvalue problem. However, unlike both quantum theory and KvNS, we will maintain electromagnetism's deterministic flavor, and there will not be any probabilistic interpretation for it (Born Rule). This approach will be shown to be useful, giving us a simple, quantum-like way to make relevant expansions for electromagnetism. It might also be pedagogically useful, as it provides a simple set of rules to how to develop typical expressions commonly used for electromagnetic configurations. Surprising consequences of this approach are explored, including the fact that the commutator for electromagnetism is not the classical KvNS commutator, but:

$$[\hat{x}, \hat{k}] = i$$

The implication for KvNS mechanics and quantum mechanics will be discussed.

Feynman taught us that it is important to have a variety of mathematical tools and approaches to any theoretical problem, as where one set of tools may fail, another theory might capture underlying physical processes (for instance, see the famous 1955 Feynman lecture entitled 'The Value of Science'). Heading Feynman's advice, we explore the Hilbert Space for electromagnetic potentials. This is a "foundational" paper to this approach, which we plan to develop further in future works.

2 A Complex Hilbert Space for Electromagnetism

2.1 Axioms of Theory

KvNS classical mechanics and quantum mechanics are built on the same axioms, which allows for the unified investigative framework of Operational Dynamical Modeling (Bondar et al., 2012; Cabrera et al., 2019; Bondar et al., 2019; McCaul and Bondar, 2021). We summarize the axioms for both classical KvNS and quantum mechanics as follows (Dirac, 1930; Shankar, 1988; Sakurai and Napolitano, 2021; Piasecki, 2021):

1. The wavefunction ket $|\psi\rangle$ (a vector in complex Hilbert Space) describes the state of the system

2. For observable O, there is an associated Hermitian operator $\hat{O}$ which obeys the eigenvalue problem $\hat{O}|O\rangle = o|O\rangle$, where o is the value seen by measurement. Two common observables are position $\hat{x}$ and momentum $\hat{p}$, which obey (in one-dimension)

$$\hat{x}|x\rangle = x|x\rangle$$
$$\hat{p}|p\rangle = p|p\rangle$$

3. Born Rule: The probability density for making any measurement for observable O is given by
$$\rho_o = \langle\psi|O\rangle\langle O|\psi\rangle$$

   Upon measurement, the state of the system collapses from $|\psi\rangle$ to $|O\rangle$.

4. The state space of a composite system is the tensor product of the subsystem's state spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ...$

The above lacks terminology of quantum or classical, because it is a set of postulates that encompasses both (Koopman, 1931; von Neumann, 1932; Sudarshan, 1976; Bondar et al., 2012; Cabrera et al., 2019).

For the classical electromagnetic theory, we will begin with a related set of axioms as our foundation, but for elements of the four-potential. In recent decades, physicists have realized the importance of the electromagnetic four-vector $A_\mu = (V, A_x, A_y, A_z)$. It has been argued by some that the four-potential is in ways more fundamental than the electric and magnetic fields, which act on the level of force, whereas the four-potential exists on the level of potentials and momentum (Aharonov and Bohm, 1959, 1961; Feynman et al., 1964; Calkin, 1966, 1971; Konopinski, 1978; Calkin, 1979; Gingras, 1980; Mead, 2002; Leus et al., 2013; Heras and Heras, 2020, etc.). For any square integrable components $A_\mu(\zeta, t)$, we can represent them as an abstract vector ket $|A_\mu(t)\rangle$ existing in the Hilbert Space, signifying the component state of the potential A_μ (where $\mu = 0, 1, 2, 3$). The space

$$L^2(M, d^n\zeta) = \{\psi : M \to \mathbb{C} | \int d^n\zeta \ \psi^*\psi = N < \infty\}, \tag{1}$$

completely analogous to the quantum case (Dirac, 1930; Shankar, 1988; Blank et al., 1994; Gallone, 2015), will be the space of electromagnetic potentials.

In the usual sense, linear operators $\hat{O} : \mathcal{H} \to \mathcal{H}$ are defined in the complex Hilbert space as linear mappings (Blank et al., 1994, pp. 17). The eigenvalue o of linear operator $\hat{O}$ is defined as a complex number where $\hat{O} - o\hat{I}$ is non-injective (Blank et al., 1994, pp. 25). Eigenvectors $|O\rangle$ likewise follow from the typical $\hat{O}|O\rangle = o|O\rangle$ (Blank et al., 1994, pp. 25).

For the electromagnetic Hilbert Space, we propose the following axioms:

1. The kets $|A_0\rangle$ and $|\boldsymbol{A}\rangle$ (vectors in complex Hilbert Space) describe the state of the four-potential elements V and $\boldsymbol{A}$, respectively.

   Notation:
   $$|A_\mu\rangle = \begin{cases} |A_0\rangle & for \ scalar \ potential \\ |\boldsymbol{A}\rangle & for \ vector \ potential \end{cases}$$

2. For classical position and wavenumber, there exists an associated Hermitian operator $\hat{x}$ for position and $\hat{k}$ for wavenumber which obey the one-dimensional eigenvalue problems

   $$\hat{x}|x\rangle = x|x\rangle$$
   $$\hat{k}|k\rangle = k|k\rangle$$

   We consider the wavenumber k a classical observable since the wavelength is in principle classically measurable.

3. The state space of a composite system is the tensor product of the sub-system's state spaces, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ...$

Postulate 1 is in direct comparison to quantum mechanics, where the abstract ket $|\psi\rangle$ describes the state of the quantum system (Dirac, 1930; Shankar,

1988; Sakurai and Napolitano, 2021). For example, since $|A_0\rangle$ is the abstract representation of the electric potential V, whose position basis would give us the familiar scalar valued field utilizing the Hilbert Space inner product on a continuous basis:

$$V(\boldsymbol{x}, t) = \langle \boldsymbol{x} | A_0(t) \rangle, \tag{2}$$

and likewise

$$\boldsymbol{A}(\boldsymbol{x}, t) = \langle \boldsymbol{x} | \boldsymbol{A}(t) \rangle. \tag{3}$$

Since this is a theory on the level of V and $\boldsymbol{A}$, we still impose the usual gauge conditions (Jackson, 1975).

An observant reader will notice the Born Rule has been removed from set of electromagnetic postulates. Unlike the Hilbert Space for quantum mechanics, we do not impose any probabilistic interpretation on the mathematics. Any square integrable function can have a representation in a complex Hilbert space (Blank et al., 1994; Gallone, 2015). In the context of quantum mechanics, the property of square integrability is exploited to normalize a wavefunction so that sensible probabilities can be extracted (eq. 1).

For the electromagnetic case, we are merely interested in the property that the integral in eq. 1 is finite for different objects of interest, and will not be mapping the finite amplitude into a probability density. Therefore, a large difference between the classical wave and quantum theory is that the quantum theory has a probabilistic interpretation, but the classical wave theory is deterministic due to lack of such an imposition. The mathematical structure, however, is otherwise identical.

We will see that the second postulate of the electromagnetic Hilbert Space theory leads to a commutator relationship analogous to that of the canonical commutator. It only leads to a "classical" Heisenberg Uncertainty Principle when the electromagnetic amplitude is normalizable (Torre, 2005; Mansuripur, 2009, etc.), as we will see. The third postulate, borrowed from the other Hilbert Space theories, will be necessary in carrying out certain calculations, as we will demonstrate. A Venn diagram summarizes the relationship between KvNS classical point mass mechanics, quantum theory, and this electromagnetic theory (Figure 1).

2.2 Further identities and orthogonal expansions

Based on the axioms of the theory, we introduce some further concepts and notation. We will give continuous vectors a continuous orthonormal basis, just like in both classical and quantum theories. The classical Hilbert Space would contain:

$$\langle \zeta' | \zeta \rangle = \delta(\zeta' - \zeta) \quad \begin{cases} \langle x' | x \rangle = \delta(x' - x) \\ \langle k' | k \rangle = \delta(k' - k) \end{cases} \tag{4}$$

5

KvNS Mechanics   Quantum Mechanics

Born Rule

Collapse of Waveform

$[\hat{x}, \hat{k}] = 0$

3N coordinate space

Indeterministic

Hilbert Space $\mathcal{H}$

Deterministic

$[\hat{x}, \hat{k}] = i$

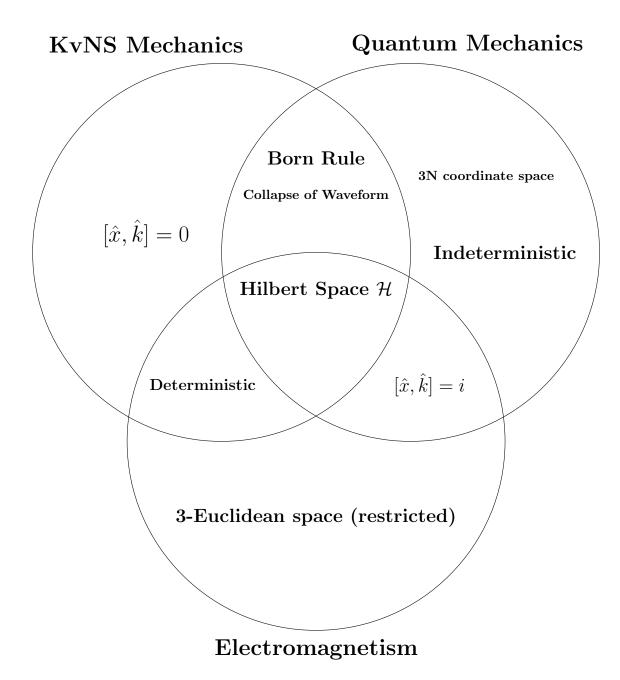3-Euclidean space (restricted)

Electromagnetism

Figure 1: Venn Diagram of relationship between Koopman-von Neumann-Sudarshan mechanics, standard quantum mechanics, and the electromagnteic potential Hilbert Space theory.

where ζ is a continuous variable, x represents position, and k is the wavenumber of electromagnetism. Identifying an orthonormal, denumberable (instead of continuous) basis set $\{|U_n\rangle\}$ denoted with the Kronecker delta:

$$\langle U_n|U_m\rangle = \delta_{nm} \tag{5}$$

We can also identify closure for the classical Hilbert Space, for both continuous and discrete states:

$$\int d\zeta \, |\zeta\rangle \, \langle\zeta| = \hat{\mathbb{1}} \quad \begin{cases} \int dx \, |x\rangle \, \langle x| = \hat{\mathbb{1}} \\ \int dk \, |k\rangle \, \langle k| = \hat{\mathbb{1}} \end{cases} \tag{6}$$

$$\sum_n |U_n\rangle \, \langle U_n| = \hat{\mathbb{1}} \tag{7}$$

The same expressions exist for both quantum and KvNS classical mechanics, as these are all Hilbert Space theories obeying the spectral theorem (Blank et al., 1994; Gallone, 2015).

From these simple quantum-like relationships follow many very commonly known electromagnetic relationships. A common practice in a graduate electromagnetism course is to take the potential, for instance, and expand it along an orthogonal set of functions (Jackson, 1975, section 2.8). Any square integratable potential $|A_\mu\rangle$ is expandable in the following fashion:

$$\langle\zeta|A_\mu\rangle = \sum_n \langle\zeta|U_n\rangle \, \langle U_n|A_\mu\rangle \tag{8}$$

(Jackson, 1975, eq. 2.33)

where $\langle\zeta|U_n\rangle$ represents the orthogonal functions of continuous variable ζ and $a_n = \langle U_n|A_\mu\rangle$ are the coefficients for each term, given by:

$$a_n = \int d\zeta \, \langle U_n|\zeta\rangle \, \langle\zeta|A_\mu\rangle \tag{9}$$

(Jackson, 1975, eq. 2.32)

Orthogonality of $\langle\zeta|U_n\rangle$ is easily demonstrated from Eq. 5:

$$\int d\zeta \, U_n^*(\zeta)U_m(\zeta) = \delta_{nm}$$

(Jackson, 1975, eq. 2.29)

$$\sum_n \langle\zeta'|U_n\rangle \, \langle U_n|\zeta\rangle = \delta(\zeta' - \zeta)$$

(Jackson, 1975, eq. 2.35)

Like in quantum mechanics, $\langle \zeta | U_n \rangle$ is defined in electromagnetism through an eigenvalue problem, i.e., through the Sturm–Liouville theory (see Appendix A).

This generalizes well to the multivariable case, where like in standard quantum mechanics we use the tensor product to bind together different vector spaces (third postulate from electromagnetic axioms), and from that the same familiar electromagnetic relations emerge (demonstrated for continuous variables ζ and η):

$$|\boldsymbol{x}\rangle = |x_1, x_2, ..., x_N\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes ... \otimes |x_N\rangle$$

$$A_\mu(\zeta, \eta) = \langle \zeta, \eta | A_\mu \rangle = \sum_{mn} \langle \zeta | U_n \rangle \langle \eta | V_m \rangle \langle U_n, V_m | A_\mu \rangle$$

(Jackson, 1975, eq. 2.38)

where

$$a_{mn} = \langle U_n, V_m | A_\mu \rangle = \int d\zeta \int d\eta \; U_n^*(\zeta) V_m^*(\eta) A_\mu(\zeta, \eta)$$

(Jackson, 1975, eq. 2.39)

An example of an important orthogonal function for electromagnetism Jackson (1975, pp. 67) discusses is the complex exponential:

$$\langle x | U_n \rangle = \frac{1}{\sqrt{a}} e^{i(2\pi n x/a)} \tag{10}$$

(Jackson 1975 eq. 2.40)

where $n = 0, \pm 1, \pm 2, ... \in \mathbb{Z}$, is defined on interval $(-a/2, a/2)$. Jackson (1975, pp.67) discusses how taking the limit of a goes to infinity causes the set of orthogonal functions $\langle x | U_n \rangle$ to transform into a set of continuous functions. Essentially, the denumerable kets $|U_n\rangle$ transform to a continuum ket $|k\rangle$, related to the classical wavenumber. We would have:

$$\begin{cases} |U_n\rangle \to |k\rangle \\ \frac{2\pi n}{a} \to k \\ \sum_n \to \int_{-\infty}^{\infty} dn = \frac{a}{2\pi} \int_{-\infty}^{\infty} dk \\ a_n \to \sqrt{\frac{2\pi}{a}} a(k) \end{cases} \tag{11}$$

(Based on Jackson 1975, eq. 2.43)

The Kronecker delta will be replaced with the Dirac delta functional in all relevant expressions. Using eqs. 8 and 9 with 11 gives us the famous Fourier Integral:

$$\langle x | A_\mu \rangle = \frac{1}{\sqrt{2\pi}} \int dk \; a(k) \; e^{ikx} = \int dk \; \langle x | k \rangle \langle k | A_\mu \rangle \tag{12}$$

with

$$a(k) = \langle k|A_\mu \rangle = \frac{1}{\sqrt{2\pi}} \int dx \; e^{-ikx} \; A_\mu(x) = \int dx \; \langle k|x \rangle \, \langle x|A_\mu \rangle \qquad (13)$$

We have utilized closure (eq. 6) for the above two expressions. From eqs. 12 and 13, we can produce an expression for $\langle x|k \rangle$, which implies a surprising quantum-like momentum position commutator for a purely classical wave. The interpretation of this will be explored in the next section.

Another important orthogonal function is the spherical harmonics. The spherical harmonics can be identically computed and used in both the classical Hilbert Space and the quantum Hilbert Space. They play a central role in both fields, but are used to different ends. The identical quantum/classical Hilbert relations for $Y_l^m(\theta, \phi)$ are as follows:

$$\langle \theta, \phi|lm \rangle := Y_l^m(\theta, \phi)$$

$$\sum |lm \rangle \, \langle lm| = \hat{\mathbb{I}}, \langle l'm'|lm \rangle = \delta_{mm'}\delta_{ll'}$$

$$\int d\Omega \, |\theta, \phi \rangle \, \langle \theta, \phi| = \hat{\mathbb{I}}$$

The spherical harmonics for classical potentials also obey the usual

$$\langle \theta, \phi| \, \hat{L}^2 \, |lm \rangle = [-\frac{1}{\sin^2(\theta)}\frac{\partial}{\partial \theta}(\sin(\theta)\frac{\partial}{\partial \theta}) - \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}] \, \langle \theta, \phi|lm \rangle = l(l+1) \, \langle \theta, \phi|lm \rangle$$

$$\langle \theta, \phi| \, \hat{L}_z \, |lm \rangle = -i\frac{\partial}{\partial \phi} \, \langle \theta, \phi|lm \rangle = m \, \langle \theta, \phi|lm \rangle$$

These eigenvalue problems are, in fact, how the spherical harmonics are defined. The spherical harmonics form a complete set of orthogonal functions, so they can be used as a basis for expansion. For a configuration of spherical symmetry, a typical expansion for the electric potential (in a region with no charge singularities) is:

$$V(r, \theta, \phi) = \langle r, \theta, \phi|A_0 \rangle = \sum_{l=0}^{n} \sum_{m=-l}^{m=l} \langle \theta, \phi|lm \rangle \, \langle lm| \otimes \langle r|A_0 \rangle = \sum_{l=0}^{n} \sum_{m=-l}^{m=l} V_{lm}(r)Y_l^m(\theta, \phi)$$

where our weighted prefactor V_{lm} is

$$V_{lm}(r) = \int d\Omega \, \langle lm|\theta, \phi \rangle \, \langle \theta, \phi| \otimes \langle r|A_0 \rangle = \int d\Omega \; Y_l^{m*}(\theta, \phi)V(r, \theta, \phi)$$

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In the quantum Hilbert Space, the $\langle \theta, \phi | l, m \rangle$ famously takes central stage in understanding the spectrum of the Hydrogen atom (Shankar, 1988; Sakurai and Napolitano, 2021). Other eigenfunctions of note useful for Hilbert Space electromagnetism can be found summarized in Appendix A.

3   A Classical Position-Momentum Commutator and Feynman's Proof of Maxwell's Equations

In both quantum mechanics and KvNS classical mechanics, we start with the position and momentum eigenvalue expressions as postulates for our theory (Dirac, 1930; Shankar, 1988; Bondar et al., 2012; Sakurai and Napolitano, 2021; Piasecki, 2021).

$$\hat{x} \left| x \right\rangle = x \left| x \right\rangle$$

$$\hat{p} \left| p \right\rangle = p \left| p \right\rangle \leftrightarrow \hat{k} \left| k \right\rangle = k \left| k \right\rangle$$

Here, we will start with the same axioms in the same spirit. Since both a quantum mechanics and a classical theory (KvNS mechanics) start with these, we begin in the same way.

From eqs. 12 and 13, which are pulled from Jackson (1975), we can see that the wavenumber-position inner product in one dimension is

$$\langle x | k \rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{14}$$

We provide two derivations that for the position and wavenumber eigenvalue problems, this gives us the commutation relationship $[\hat{x}, \hat{k}] = i$.

3.1   Derivation I: Utilization of properties of Dirac delta functional

We first establish that

$$[\hat{x}, \hat{k}] = i \leftrightarrow \langle x | k \rangle = \frac{1}{\sqrt{2\pi}} e^{ikx}$$

To do, we begin with

$$\langle x' | [\hat{x}, \hat{k}] | x \rangle = (x' - x) \langle x' | \hat{k} | x \rangle = i\delta(x' - x) \tag{15}$$

We utilize the fact that a distribution A that is zero everywhere except at one point x_0 can be expanded in terms of derivatives of Dirac delta functionals (for example, see Treves 2006, pp. 264–266), similar in form to a Taylor expansion:

$$A(x) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x - x_0) \tag{16}$$

With the identity $x\delta'(x) = -\delta(x)$, we obtain

$$\langle x' | \hat{k} | x \rangle = -i\delta'(x' - x)$$

$$k \langle x|k \rangle = \langle x| \hat{k} |k \rangle = \int dx' \ \langle x| \hat{k} |x' \rangle \langle x'|k \rangle = -i \int dx' \ \delta'(x' - x) \langle x'|k \rangle \quad (17)$$

The definition of the rightmost integral above gives us:

$$k \langle x|k \rangle = -i\frac{\partial}{\partial x} \langle x|k \rangle \quad (18)$$

This, of course, has the solution

$$\langle x|k \rangle = Ce^{ikx}$$

where C is the constant of integration. To identify the constant, we continue with eq. 4 and the definition of the Dirac functional:

$$\delta(x' - x) = \langle x'|x \rangle = \int dk \ \langle x'|k \rangle \langle k|x \rangle = C^2(2\pi)\delta(x' - x)$$

Ergo, we recover for one dimension

$$\langle x|k \rangle = \frac{1}{\sqrt{2\pi}}e^{ikx}$$

We went from $[\hat{x}, \hat{k}] = i$ to the Fourier expression for $\langle x|k \rangle$, but could have just as easily went in reverse. The result is completely general.

3.2 Derivation II: Wave Operator $\hat{k}$ as Generator of Electromagnetic Motion

Starting from the commutator $[\hat{x}, \hat{k}] = i$, we begin by constructing an anti-adjoint operator, defined by:

$$\hat{K} := -i\hat{k}$$

Since $[\hat{x}, \hat{K}] = \hat{\mathbb{I}}$, this implies

$$[\hat{x}, e^{\delta x \hat{K}}] = \delta x \cdot e^{\delta x \hat{K}}$$

Acting on a position ket $|x\rangle$, the above defined operator $e^{\delta x \hat{K}}$ would displace the state from $|x\rangle$ to $|x + \delta x\rangle$ as a generator of wave motion. Because $e^{\delta x \hat{K}}$ is unitary, we can show that the norm of the phase factor α is 1:

$$e^{\delta x \hat{K}} |x\rangle = \alpha |x + \delta x\rangle \quad (19)$$

11

$$\delta(x'-x) = \langle x'|x\rangle = \langle x'|\, e^{-\delta x \hat{K}} e^{\delta x \hat{K}}\, |x\rangle = \alpha^* \alpha \, \langle x' + \delta x | x + \delta x \rangle = \alpha^* \alpha \cdot \delta(x'-x)$$

Using the expression 19 with $\alpha = 1$, it is a simple task to Taylor expand each side:

$$\hat{\mathbb{1}}\,|x\rangle + \delta x \hat{K}\,|x\rangle + \mathcal{O}(\delta x^2)\,|x\rangle = \hat{\mathbb{1}}\,|x\rangle + \delta x \frac{\partial}{\partial x}\,|x\rangle + \mathcal{O}(\delta x^2)\,|x\rangle$$

Comparing like terms and sandwiching from the left with a wavenumber bra $\langle k|$, it is trivial that:

$$\hat{K}\,\langle k|x\rangle = -i\hat{k}\,\langle k|x\rangle = \frac{\partial}{\partial x}\,\langle k|x\rangle$$

$$k\,\langle x|k\rangle = -i\frac{\partial}{\partial x}\,\langle x|k\rangle \tag{20}$$

From here, the derivation proceeds exactly as in the previous section, with the same normalization used to derive expression 14. Quantum textbooks such as Sakurai use the same sequence of steps to derive the standard quantum generators of motion (Sakurai and Napolitano, 2021, pp.40-64, 152-175, etc.). The wave commutator $[\hat{x}, \hat{k}] = i$ therefore automatically implies expression 14 (and for a Hilbert Space of operators and commutators, vice versa).

3.3   Feynman's Derivation of Maxwell's equations

Feynman gave an interesting proof of Maxwell's equations based on the quantum position-momentum commutator and Newton's second law (Dyson, 1990). This strange usage of a quantum mathematical object for a classical field is not strange from the perspective of this work. The presence of this commutator in this purely classical Hilbert Space is a consequence of wave behavior, and Feynman applying the commutator to Newton's second law is arguably imposing wave behavior on Newtonian mechanics. Maxwell's equations as a consequence reflect the internal self-consistency of physical law, manifesting as the electric and magnetic force fields (Dyson, 1990).

As Dyson (1990) points out, the commutator relationship alternatively implies the existence of a vector potential $\boldsymbol{A}$ which obeys

$$[\hat{x}_j, \hat{A}_k] = 0$$

Even though Feynman derives Maxwell's Laws (which depend on the concept of the force field), it is very parsimonious with the $A_\mu = (V, \boldsymbol{A})$ approach we adopt here, as a more fundamental aspect of nature, and has been argued by many (Aharonov and Bohm, 1959, 1961; Feynman et al., 1964; Calkin, 1966; Konopinski, 1978; Mead, 2002, etc.).

The most unusual aspect of Feynman's derivation is the fact that it starts with a nonrelativistic version of Newton's second law and arrives at relativistic Maxwell equations (Dyson, 1990). A fully relativistic version of Feynman's

proof is presented in Appendix B, using the language of quantum/classical commutators.

Some may wonder if a Hilbert Space representation is appropriate for relativistic systems at all. Cabrera et al. (2019) were able to utilize Operational Dynamic Modelling and the KvNS formalism to derive the relativistic Dirac equation and nonrelativistic Spohn equation for spin 1/2 particles based on the Hilbert Space. In their work, they used the canonical commutator to derive the Dirac equation, with the only difference being that the momentum was now the relativistic momentum.

4   Hermitian Operator and State Ket Representations of Four-Potential Elements

In quantum mechanics, one sees the electromagnetic potentials represented as Hermitian operators $\hat{V}$ and $\hat{\boldsymbol{A}}$ acting on position kets. One may naturally wonder if there is a relationship between these operators and statements presented in this paper, such as those in eqs. 2 and 3. There is, in fact, a very natural way one can interpolate between the operatorial forms and bra-ket representations.

One can quite straightforwardly demonstrate that the two representations are related by

$$|A_0\rangle = \int d\boldsymbol{x} \ \hat{V} |\boldsymbol{x}\rangle \tag{21}$$

$$|\boldsymbol{A}\rangle = \int d\boldsymbol{x} \ \hat{\boldsymbol{A}} |\boldsymbol{x}\rangle \tag{22}$$

or, equivalently,

$$\hat{V} = |A_0\rangle \langle \boldsymbol{x}| \tag{23}$$

$$\hat{\boldsymbol{A}} = |\boldsymbol{A}\rangle \langle \boldsymbol{x}| \tag{24}$$

This signifies that $|A_0\rangle$, for example, is constructed out of a continuous sum of scaled weighted positions $|x\rangle$, as one might expect.

This construction guarantees that the inner product for the potentials (eqs. 2 and 3) will always be real-valued. Due to the prevalence of complex numbers inside the representation of electromagnetic potentials, however, we might wonder what would happen if the Hermitivity condition of operators $\hat{V}$ and $\hat{\boldsymbol{A}}$ were relaxed, and the real part of the resulting eigenvalue were subsequently taken (as is standard practice in electromagnetism). However, this line of reasoning will not be pursued further in the present paper.

5   Fourier Analysis of the Four-Potential Elements

We can generalize the one dimensional $\langle x|k\rangle$ into multiple dimensions using the tensor product familiar to quantum mechanics:

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$$\langle \boldsymbol{x}|\boldsymbol{k}\rangle = \langle xyz|k_x k_y k_z\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \tag{25}$$

This Hilbert Space represents a 3-Euclidean space, as is also apparent in Feynman's derivation of Maxwell's equations using the commutator (where $i, j, k = 1, 2, 3$).

In this section, we will also write the vector potential ket as the tensor product of a Euclidean vector portion and scalar amplitude portion:

$$|\boldsymbol{A}\rangle = |\boldsymbol{n}\rangle |\Phi\rangle \tag{26}$$

where $|\boldsymbol{n}\rangle$ lives in $\mathbb{R}^3$ and the magnitude of the vector potential $|\Phi\rangle$ lives in infinite dimensional Hilbert Space, so that $|\boldsymbol{A}\rangle$ lives in the tensor product of the two spaces, or $\mathcal{H}_1 \otimes \mathcal{H}_2$. We can always expand $|\boldsymbol{A}\rangle$ across any Euclidean basis $\{|e_i\rangle\}$, for example:

$$|\boldsymbol{A}\rangle = |e_1\rangle \langle e_1|\boldsymbol{n}\rangle |\Phi\rangle + |e_2\rangle \langle e_2|\boldsymbol{n}\rangle |\Phi\rangle + |e_3\rangle \langle e_3|\boldsymbol{n}\rangle |\Phi\rangle$$

where $\langle e_i|\boldsymbol{A}\rangle$ is our notation for the 3-Euclidean dot product in Hilbert Space. These concepts may provide useful for certain calculations related to what follows.

Two common Fourier analysis cases in electromagnetism are when we restrict the potentials to a finite volume vs analyze the potentials over all spaces. In the case of a restricted volume of interest for the vector potential, it is expanded in the discrete basis $|U_m V_n W_p\rangle$:

$$\boldsymbol{A}(\boldsymbol{x}, t) = \sum_{mnp} \langle x|U_m\rangle \langle y|V_n\rangle \langle z|W_p\rangle \langle U_m V_n W_p|\boldsymbol{A}(t)\rangle$$

where $\langle \boldsymbol{x}|U_m V_n W_p\rangle$ would represent the three-dimensional version of equation 10 with

$$\boldsymbol{k} = \frac{2\pi}{a}[m \quad n \quad p]$$

where $m, n, p = 0, \pm 1, \pm 2, \dots$. The standard coefficients of this expansion are given by $\boldsymbol{a_k}(t) = \langle U_m V_n W_p|\boldsymbol{A}(t)\rangle$. This is what we expect for the magnetic potential (Mandel and Wolf, 1995, eq. 10.2-9 and pp. 467-468). Likewise, for the scalar potential, we expect similarly

$$V(\boldsymbol{x}, t) = \sum_{mnp} \langle x|U_m\rangle \langle y|V_n\rangle \langle z|W_p\rangle \langle U_m V_n W_p|A_0(t)\rangle,$$

which is the correct result (Jackson, 1975, sections 2.8-2.9,etc).

In the continuum limit, we once more switch the $|U_m V_n W_p\rangle$ with $|k_x k_y k_z\rangle$ (eq. 11 but in three dimensions). The Hilbert Space representation provided here gives a simple and easy method to derive all Fourier expressions by simple expansions utilized in quantum calculations:

$$\langle \boldsymbol{x}|A_\mu\rangle = \int d^3k \, \langle \boldsymbol{x}|\boldsymbol{k}\rangle \langle \boldsymbol{k}|A_\mu\rangle$$

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with coefficients

$$\langle \boldsymbol{k} | A_\mu \rangle = \int d^3x \, \langle \boldsymbol{k} | \boldsymbol{x} \rangle \, \langle \boldsymbol{x} | A_\mu \rangle$$

where equation 25 may be utilized.

6 Treatment of Green's Function

6.1 Time-Independent Green's Operator

The Hilbert Space formalism of electromagnetism lets us utilize Green's functions in the same fashion as in quantum mechanics. For the time-independent Green's function, we begin with the same standard definitions (Economou, 2006, section 1.1):

$$[z - \hat{L}]\hat{G}(z) = \hat{\mathbb{1}} \tag{27}$$

(Economou, 2006, eq. 1.1)

where $\hat{G}$ is the Green's operator, $z \in \mathbb{C}$ exists as a parameter, and $\hat{L}$ is a Hermitian operator with a complete set of eigenkets $\{|L\rangle\}$ obeying the eigenvalue problem

$$\hat{L} |L\rangle = L |L\rangle \tag{28}$$

(Economou, 2006, eq. 1.2)

In general, the form of $\hat{L}$ follows from the differential problem we are trying to solve (Economou, 2006).

The operators are defined in such a manner that

$$\langle \boldsymbol{x} | \hat{G}(z) | \boldsymbol{x'} \rangle \equiv G(\boldsymbol{x}, \boldsymbol{x'}; z) \tag{29}$$

$$\langle \boldsymbol{x} | \hat{L} | \boldsymbol{x'} \rangle \equiv \delta(\boldsymbol{x} - \boldsymbol{x'}) L(\boldsymbol{x}) \tag{30}$$

resulting in the usual Green's function expression:

$$[z - L(\boldsymbol{x})]G(\boldsymbol{x}, \boldsymbol{x'}; z) = \delta(\boldsymbol{x} - \boldsymbol{x'})$$

As a consequence of the spectral theorem, we also know that

$$\langle L | L' \rangle = \delta_{LL'} \tag{31}$$

(Economou 2006, eq. 1.3; Jackson 1975, eq. 3.155)

$$\sum |L\rangle \langle L| = \hat{\mathbb{1}} \tag{32}$$

(Economou 2006, eq. 1.3; utilized in Jackson 1975, eqs. 3.157,8)

We can begin by solving eq. 27 to get the expression for the Green's operator

$$\hat{G}(z) = \frac{\hat{\mathbb{I}}}{z - \hat{L}}$$

With eq. 32, it can be quickly seen that (in a continuous or denumberable basis) one can write

$$\hat{G}(z) = \sum_n \frac{|L\rangle \langle L|}{z - L} + \int dc \, \frac{|L_c\rangle \langle L_c|}{z - L_c} \tag{33}$$

(Economou, 2006, eq. 1.11)

where the subscript c represents continuous states.

With the position basis, one finds the standard electromagnetic expansions of the Green's function

$$G(\boldsymbol{x}, \boldsymbol{x}'; z) = -4\pi \sum_n \frac{\psi_n^*(\boldsymbol{x}')\psi_n(\boldsymbol{x})}{z - L} - 4\pi \int dc \frac{\psi_{nc}^*(\boldsymbol{x}')\psi_{nc}(\boldsymbol{x})}{z - L_c} \tag{34}$$

(Jackson (1975) eq. 3.160)

where $\psi_n(\boldsymbol{x}) = \langle \boldsymbol{x}|L\rangle$ and $\psi_{nc}(\boldsymbol{x}) = \langle \boldsymbol{x}|L_c\rangle$ (prefactor of 4π is conventional for cgs units). One can therefore equivalently represent the Green's function in terms of orthonormal eigenkets and eigenvalues in both electromagentism (Jackson, 1975) and quantum mechanics (Economou, 2006).

An example of a commonly used differential is the Laplace operator for the electrostatic potential:

$$\boldsymbol{\nabla}^2 V = -4\pi \rho_e$$

(in cgs units). With $L(\boldsymbol{x}) = -\boldsymbol{\nabla}^2$ over all 3-Euclidean space, we identify $|L\rangle$ with $|k\rangle$ (Economou 2006 pp. 9; also see eq. 1.34) from the fact that

$$-\boldsymbol{\nabla}^2 \langle \boldsymbol{x}| = \langle \boldsymbol{x}| \, \hat{\boldsymbol{k}}^2$$

Then:

$$G(\boldsymbol{x}, \boldsymbol{x}'; z) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\langle \boldsymbol{x}|\boldsymbol{k}\rangle \langle \boldsymbol{k}|\boldsymbol{x}'\rangle}{z - k^2} \tag{35}$$

Carrying out integrations and setting $z = 0$ for the electric point source (Economou, 2006), we retrieve as an illustration the well established fact from electrostatics

$$V(\boldsymbol{x}) = -4\pi \int d\boldsymbol{x}' \, G(\boldsymbol{x}, \boldsymbol{x}'; 0)\rho_e(\boldsymbol{x}') = \int d\boldsymbol{x} \frac{\rho_e(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

(Based on Economou 2006 eq. 1.44 and Jackson 1975 eq. 3.164)

In the same fashion, we can compute from eq. 35 the frequently used Helmoltz equation Green's function (Economou 2006 eq. 1.40):

$$G(\boldsymbol{x}, \boldsymbol{x}; k) = \frac{e^{\pm i \boldsymbol{k} \cdot |\boldsymbol{x} - \boldsymbol{x'}|}}{4\pi |\boldsymbol{x} - \boldsymbol{x'}|}$$

Typical electromagnetic calculations can be carried out with Dirac notation in Hilbert Space.

6.2 Unitary Time Evolution and Time-Dependent Green's Operator

In the same manner Stone's theorem is developed in quantum mechanics (Blank et al., 1994; Gallone, 2015), we can develop the theorem for the time evolution of classical potentials. Stone's theorem describes the properties of strongly continuous one-parameter unitary groups such as utilized for the time parameter (Gallone, 2015, ch. 19):

$$|A_\mu(t)\rangle = \hat{U}(t - t') |A_\mu(t')\rangle \tag{36}$$

where

$$\hat{U} = \exp(-i\hat{\Omega}(t - t')) \tag{37}$$

The operator $\hat{\Omega}$ is Hermitian and here the infinitesimal generator of time evolution of classical fields with respect to a fixed reference frame. The above expression leads to a Schrödinger-like equation for electromagnetic potentials:

$$i\frac{\partial}{\partial t} |A_\mu(t)\rangle = \hat{\Omega} |A_\mu(t)\rangle \tag{38}$$

where the angular frequency operator $\hat{\Omega}$ takes the place of the Hamiltonian in the Schrödinger equation. The constant $\hbar$ does not appear. The factor $e^{-i\omega(k)t}$ (eq. 37) is universally recognized as the propagator of a monochromatic plane wave in electromagnetic theory (e.g., Jackson 1975, Ishimaru 2017 eq. 2.15 and details therein, etc.), so the Schrödinger-like equation is consistent with this. It is trivial to see that this conclusion is also consistent with the kernel $\langle \boldsymbol{x}| \hat{U} |\boldsymbol{x'}\rangle$, but utilized for electromagnetic propagation relative to a fixed frame:

$$\langle \boldsymbol{x}|A_\mu(t)\rangle = \int_{-\infty}^{\infty} d\boldsymbol{x'} \ \langle \boldsymbol{x}| \hat{U}(t - t') |\boldsymbol{x'}\rangle \langle \boldsymbol{x'}|A_\mu(t')\rangle$$

These facts can be used in the construction of time-dependent Green's operators for electromagnetism.

For instance, the Green's function will be the solution of

$$(\frac{i}{c}\frac{\partial}{\partial t} - L(\boldsymbol{x}))G(\boldsymbol{x}, \boldsymbol{x'}; t - t') = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x'})\delta(t - t')$$

for first-order diffusion-type or Schrödinger-type equations (Economou, 2006). It can be shown that the propagator equation (eq. 37) is itself a Green's function

with the above definition (i.e., see Economou 2006 eq. 2.15 and description). For second order (wave-like) equations, the Green's function will be the solution of (Economou, 2006)

$$(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - L(\boldsymbol{x}))G(\boldsymbol{x}, \boldsymbol{x}'; t - t') = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t')$$

which leads to well-known solutions (using eq. 35 and time-frequency Fourier transform) of the Laplacian operator:

$$G(\boldsymbol{x}, \boldsymbol{x}'; t - t') = \frac{\delta(t' - (t - |\boldsymbol{x} - \boldsymbol{x}'|/c))}{|\boldsymbol{x} - \boldsymbol{x}'|}$$

The above simply demonstrates a very natural common ground in Hilbert Space for both electromagnetic potentials and quantum theory.

7   Maxwell Fields in Hilbert Space

So far in this paper, we have avoided the important force description of electric and magnetic fields. Based on time evolution and the potential kets, one can very simply define these fields as:

$$|B_l\rangle \equiv -i\epsilon_{jkl}\hat{k}_j |A_k(t)\rangle$$

$$|E_l\rangle \equiv -\frac{i}{c}\hat{\Omega} |A_l(t)\rangle - i\hat{k}_l |A_0(t)\rangle$$

where we utilize the Einstein summation convention. The above definitions of electric and magnetic force fields are gauge-invariant (Jackson, 1975).

8   Comparison of Quantum and Classical Wave Hilbert Space Theories

In the absence of current and charges, the four-potential propagates as a wave equation:

$$\boldsymbol{\nabla}^2 A_\mu - \frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2} = 0$$

Because of this, it is not surprising that we can propose a Hilbert Space with wave behavior for A_μ. Just like the wavefunction, for example, the point charge electric potential V and its derivative go to zero at infinity. By Quantum Electrodynamics, we know that the $1/r$ law for point potential breaks down at small values of r, avoiding the singularity encountered at $r = 0$. It is not, therefore, surprising that elements of the four-potential can be square normalizable in general and therefore are elements of a Hilbert Space (Gallone, 2015).

Unlike the Hilbert Space in quantum mechanics, we do not impose any probabilistic interpretation on the mathematics. Electromagnetism is still presented

as a deterministic theory. Therefore, a large difference between the classical wave and quantum theory is that the quantum theory has a probabilistic interpretation, but the classical wave theory is deterministic due to lack of such an imposition. The mathematical structure, however, is identical.

A purely classical commutator relationship for wavenumber (or momentum) and position operators was derived for the electromagnetic Hilbert Space, unambiguously highlighting that particle-wave uncertainty emerges from the wave-like nature of the quantum. The commutator for electromagnetism and the canonical commutator are identical in shape. Since de Broglie has shown that $\boldsymbol{p} = \hbar\boldsymbol{k}$, it is not difficult to see that one goes from the electromagnetic commutator $[\hat{x}, \hat{k}] = i$ to the quantum canonical commutator $[\hat{x}, \hat{p}] = i\hbar$ by simply multiplying both sides by $\hbar$. The difference between classical and quantum mechanics is not in the existence between a commutator of position and momentum, but in the presence of the reduced Planck unit $\hbar$, i.e., in the existence of the Planck-Einstein and de Broglie relations.

One might wonder if therefore one might pen a "classical" Heisenberg Uncertainty Principle. From introductory quantum mechanics (Shankar, 1988; Sakurai and Napolitano, 2021), it is possible to show that for any two Hermitian operators $\hat{A}$ and $\hat{B}$, the uncertainty principle would be (Sakurai and Napolitano, 2021):

$$\sigma_A \sigma_B \geq \frac{|\langle\psi| [\hat{A}, \hat{B}] |\psi\rangle|}{2} \tag{39}$$

Since we have a Hilbert Space theory for classical electromagnetism with Hermitian operators, what is stopping us from proposing a classical uncertainty principle? In general, electromagnetic waves are not considered in terms of normalizability in the same manner the quantum waveket $|\psi\rangle$ is, and therefore the amplitude squared is not mapped into a type of classical probability density. Under the condition that the wave amplitude is normalizable, a classical version of the Uncertainty Principle has already been discovered for electromagnetism and reported on (Jackson, 1975, section 7.8), although not widely recognized as such. For instance, Jackson (1975) demonstrates that amplitude normalizable electromagnetic waves have the property

$$\sigma_x \sigma_k \geq \frac{1}{2}, \tag{40}$$

(Jackson, 1975, eq. 7.82)

exactly what one derives if the electromagnetic commutator is plugged into eq. 39. A similar relationship exists for time and frequency of the electromagnetic wave (Jackson, 1975, pp. 301). Torre (2005) and Mansuripur (2009) go into great detail on how eq. 40 represents a classical analogue to the Uncertainty Principle.

The KvNS formalism of classical mechanics has a commutator of $[\hat{x}, \hat{p}] = 0$, since we are no longer dealing with waves or fields, but specifying infinitely precise point particles in the typical style of Newtonian mechanics. The "fuzzy tra-

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jectories" of quantum theory are really just waves spread throughout space (as argued by Hobson 2013 and many others). The KvNS formalism allows for point particles by placing the wave uncertainty (spread) into other operators of the Koopman algebra (Sudarshan, 1976; Bondar et al., 2012; McCaul and Bondar, 2021; Piasecki, 2021). The amplitude and phase of the classical wavefunction in KvNS theory are completely separate, unlike the amplitude and phase in quantum theory (Mauro, 2003).

In a similar vein, it is perhaps not too surprising that the same quantum like operators are seen in both electromagnetism and quantum mechanics (equations 18 and 20). Schrödinger's original inspiration for the form of the quantum operators was likely from electromagnetism. In his original papers proposing matter has a wave-like structure, de Broglie attempted to give electromagnetism and matter an equal footing in treatment through the lens of the then new Relativity theory (de Broglie, 1923a,b,c, 1924). Schrödinger, motivated by de Broglie's work (Schrödinger, 1926), penned his now famous equation, likely by deducing the wave operators by ansatz from the known classical wave equations. It turns out, in the electromagnetic Hilbert Space you also have identical x and k (or p) representations for your position and wavenumber (momentum) operators as you do in quantum mechanics. Starting from $[\hat{x}, \hat{k}] = i$, we find in eqs. 18 and 20 that we can write the position representation as

$$\hat{x} \doteq x \quad and \quad \hat{k} \doteq -i\frac{\partial}{\partial x}$$

However, if we began from the left side of eq. 15 with $\langle k' | [\hat{x}, \hat{k}] | k \rangle$ instead of $\langle x' | [\hat{x}, \hat{k}] | x \rangle$, we would have produced the wavenumber (momentum) representation of our operators:

$$\hat{x} \doteq i\frac{\partial}{\partial k} \quad and \quad \hat{k} \doteq k$$

The difference between the quantum and classical lies once again in the presence of $\hbar$.

These are the same mathematical objects in both electromagnetism, KvNS classical mechanics, and quantum mechanics. What differs is how we interpret and utilize the mathematical objects in electromagnetic verses quantum Hilbert Space. Both include superpositions of weighted orthonormal functions (eq. 8), the usage of an eigenvalue problem to identify the orthogonal functions, an inner product between vectors in a dual space, Dirac delta and Kronecker delta representations of closure (in continuous and discrete bases), etc. Many mathematical objects, like the spherical harmonics $\langle \theta, \phi | lm \rangle$, are identical in form in both classical and quantum spaces.

The principle difference between the electromagnetic approach and the quantum approach is that the electromagnetic approach contains no probabilistic interpretation (e.g., a Born Rule), and therefore we do not utilize the Hilbert Space to make probabilistic predictions of the outcomes of systems, unlike in both quantum (for waves) and KvNS (for classical point masses). Collapse of the waveform is completely absent, unlike in both quantum and classical KvNS mechanics.

Another difference between the three Hilbert Space theories is the spaces they represent. Quantum mechanics represents a 3N-coordinate space, KvNS is in 3N-phase space, and this electromagnetic theory represents a 3-Euclidean space. The wavefunction of quantum mechanics living in 3N-coordinate space has famously perplexed physicists such as Einstein and Schrödinger. Einstein expressed frustration when he famously said "Schrödinger's works are wonderful – but even so one nevertheless hardly comes closer to a real understanding. The field in a many-dimensional coordinate space does not smell like something real" (Howard, 1990). Also: "Schrödinger is, in the beginning, very captivating. But the waves in n-dimensional coordinate space are indigestible..." (Howard, 1990). Schrödinger, Lorentz, Heisenberg, Bohm, Bell, and others struggled with the same property of quantum fields (Howard, 1990; Norsen et al., 2015). For the electromagnetic Hilbert Space, we do not face the same issues, since, for example, a scalar potential of the form of $\langle x_1, x_2, ..., x_{3N} | A_0 \rangle$ would still be interpreted as living in $\mathbb{R}^3$, even though it is a function on a configuration space. The interpretation of the wavefunction $\langle x_1, x_2, ..., x_{3N} | \psi \rangle$ is not so staightfoward, however, and has been hotly debated for over a century (Howard, 1990; Norsen et al., 2015).

There are many ways the preliminary groundwork of this paper may be extended. It has been argued, for example, that there exist classical entanglement states in classical optics and that it is related to a Hilbert Space structure (Spreeuw, 2001; Ghose and Mukherjee, 2014a,b; Rajagopal and Ghose, 2016). This brings up a very interesting question how Bell's Inequality relates to classical optics (Ghose and Mukherjee, 2014a,b; Rajagopal and Ghose, 2016). Perhaps this formulation with potentials can also be further extended into typically quantum-specific algorithms, as has been recently proposed for KvNS classical mechanics (Joseph, 2020). The boundary of classical vs quantum can be further explored as well as questions of spin, angular momentum, etc. We plan on further developing these concepts in future works.

Acknowledgments

Appendix A: Brief Summary of Orthonormal Expansions for Hilbert Space Electromagnetism

In section 2.2, the spherical harmonics were summarized for expansions in electromagnetism. Here, we cover some other useful functions for electromagnetic theory. These functions are defined through the standard Sturm–Liouville theory:

$$[\frac{d}{dx}(p(x)\frac{d}{dx}) + q(x)]\Lambda(x) = -\lambda r(x)\Lambda(x)$$

where λ is the eigenvalue and Λ is the eigenfunction.

Bessel function of the first kind:
Sturm–Liouville problem: $\frac{d}{d\rho}(\rho\frac{dJ_\nu}{d\rho}) + (\rho - \frac{\nu^2}{\rho})J_\nu = 0$
$J_\nu(x_{\nu n}\rho/a) = \langle x_{\nu n}\rho/a|J_\nu\rangle$ in Hilbert Space.
Orthogonal function: $\langle x_{\nu n}\rho/a|U_n\rangle = \sqrt{\rho}J_\nu(x_{\nu n}\rho/a)$
Orthogonality: $\langle U_{\nu n}|U_{\nu m}\rangle = \int_0^a d\rho\, \rho J_\nu(x_{\nu n}\rho/a)J_\nu(x_{\nu m}\rho/a) = \frac{a^2}{2}[J_{\nu+1}(x_{\nu n})]^2\delta_{nm}$

Legendre Polynomial:
Sturm–Liouville problem: $\frac{d}{dx}((1-x^2)\frac{dP_\nu}{dx}) + \nu(\nu-1)P_\nu = 0$
$P_\nu(x) = \langle x|P_\nu\rangle$ in Hilbert Space.
Orthonormal function: $\langle x|U_\nu\rangle = \sqrt{\frac{2\nu+1}{2}}\langle x|P_\nu\rangle$
Orthogonality: $\langle P_n|P_m\rangle = \int_{-1}^1 dx\, P_n(x)P_m(x) = \frac{2}{2n+1}\delta_{nm}$
Resolution of identity: $\hat{\mathbb{I}} = \sum_{n=0}^\infty \frac{2n+1}{2}|P_n\rangle\langle P_n|$

Associated Legendre Polynomial:
Sturm–Liouville problem: $\frac{d}{dx}((1-x^2)\frac{dP_{\nu m}}{dx}) + [\nu(\nu-1) - \frac{m^2}{1-x^2}]P_{\nu m} = 0$
$P_{\nu m}(x) = \langle x|P_{\nu m}\rangle$ in Hilbert Space.
Orthonormal function: $\langle x|U_\nu\rangle = \sqrt{\frac{2\nu+1}{2}\frac{(\nu-m)!}{(\nu+m)!}}\langle x|P_{\nu m}\rangle$
Orthogonality: $\langle P_{\nu m}|P_{\nu'm}\rangle = \int_{-1}^1 dx\, P_{\nu m}(x)P_{\nu'm}(x) = \frac{2}{2n+1}\frac{(\nu-m)!}{(\nu+m)!}\delta_{\nu\nu'}$
Resolution of identity: $\hat{\mathbb{I}} = \sum_{m=-\nu}^{m=+\nu}\sum_{n=0}^\infty \frac{2n+1}{2}\frac{(\nu+m)!}{(\nu-m)!}|P_{\nu m}\rangle\langle P_{\nu m}|$

Hermite Polynomial:
Sturm–Liouville problem: $\frac{d}{dx}(e^{-x^2}\frac{dH_\nu}{dx}) + 2\nu e^{-x^2}H_\nu = 0$
$H_\nu(x) = \langle x|H_\nu\rangle$ in Hilbert Space.
Orthogonal functions: $\langle x|U_\nu\rangle = \sqrt{e^{-x^2}}\langle x|H_\nu\rangle$
Orthogonality: $\langle U_n|U_m\rangle = \int_{-\infty}^\infty dx\, e^{-x^2}H_n(x)H_m(x) = \sqrt{\pi}2^n n!\delta_{nm}$
Resolution of identity: $\hat{\mathbb{I}} = \sum_{\nu=0}^\infty \frac{e^{-x^2}}{\sqrt{\pi}2^\nu(\nu!)}|H_\nu\rangle\langle H_\nu|$

Appendix B: Relativistic Feynman's Proof of Maxwell's Equations

Feynman's proof (Dyson, 1990) captures an underlying physical structure, being able to reproduce the homogeneous Maxwell equations, although we believe it captures the physical structure imperfectly due to certain assumptions. This is the consensus view (Lee, 1990; Hughes, 1992; Tanimura, 1993; Land et al., 1995; Bracken, 1996; Montesinos and Pérez-Lorenzana, 1999; Hokkyo, 2004; Berard et al., 2007; Swamy, 2009; Prykarpatsky and Bogolubov Jr, 2012, etc.). It can be shown that it is possible to construct a relativistically consistent version of Feynman's proof, while at the same time avoiding its unsavory elements.

Feynman's motivation was nobler than my own. Feynman was trying to establish a theory for the quantum based on the least number of assumptions as possible; we, however, are interested in the continuity of ideas between different branches of physics. We establish continuity, whereas Feynman was attempting (and ultimately failed in this regard) in building new physics.

Almost all work following up on Feynman's curious proof relies on the Poisson bracket structure or Lagrange formalism. In the spirit of this Hilbert Space formalism (similar to the KvNS approach), we will do all classical calculations using quantum-like commutators, like Dyson (1990) used in the original paper. As Cabrera et al. (2019) shows, it is possible in principle to utilize the commutator structure and mathematics of Hilbert Space to describe relativistic systems. Following the original proof, we use classical commutators.

Feynman's proof (Dyson, 1990) begins with

$$\dot{\hat{p}}_j = \hat{F}_j(\hat{x}, \dot{\hat{x}}, t) \tag{41}$$

$$[\hat{x}_i, \hat{x}_j] = 0 \tag{42}$$

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{43}$$

and ends in

$$\hat{F}_j = \hat{E}_j + \epsilon_{jkl}\dot{\hat{x}}_k\hat{H}_l \tag{44}$$

$$div\ \hat{H} = 0 \tag{45}$$

$$\frac{\partial}{\partial t}\hat{H} + curl\ \hat{E} = 0 \tag{46}$$

with the remaining two Maxwell equations left as definitions of charge density and current. Although unstated throughout the proof, Feyman sneaks in a fourth assumption (other than eqs. 41 - 43), and that is that the form of the momentum is

$$\hat{p}_j = m\dot{\hat{x}}_j \tag{47}$$

which is where the issue with the Feynman proof lies. This assumption is what leads to several contradictions. For one, it leads to the fact that the velocity components do not commute. Feynman defines the magnetic field to be:

$$\hat{H}_l = -\frac{im^2}{2\hbar}\epsilon_{jkl}[\dot{\hat{x}}_j, \dot{\hat{x}}_k] \tag{48}$$

However, a glaring issue with this is that it implies, with Feynman's tacit assumption (eq. 47) , that $[\hat{p}_i, \hat{p}_j] \neq 0$, which contradicts a basic principle of quantum mechanics (Sakurai and Napolitano, 2021, eq. 1.224). Although Feynman's proof adopts one quantum commutator (eq. 43), it neglects another commutator principle of quantum mechanics: $[\hat{p}_i, \hat{p}_j] = 0$. This can all be remedied, however, with a more appropriate definition of momentum. Another issue one might also notice is that Feynman's defined magnetic fields also depend on the mass. If we take the limit of $m \to 0$ for the photon, we would essentially eliminate the magnetic field, which we know is not possible in a classical electromagnetic theory.

If instead we assume a relativistic momentum with minimal coupling (Montesinos and Pérez-Lorenzana, 1999), we will still be able to carry out Feynman's proof in similar manner, achieving in the end the same conclusions (eqs. 44 - 46), but free of contradictions and other unappealing features of the original proof. The price of this is not high, as we are just switching one unappealing assumption (eq. 47) with a better definition for the momentum:

$$\hat{P}_j = \hat{p}_j + \hat{A}_j(\hat{x}, t) \tag{49}$$

where relativistic operators are assumed (like in Cabrera et al. 2019). At the same time, we also change eq. 41 to

$$\dot{\hat{P}}_j = \hat{F}_j(\hat{x}, \dot{\hat{x}}, t) \tag{50}$$

and eq. 43 with

$$[\hat{x}_i, \hat{P}_j] = i\hbar\delta_{ij} \tag{51}$$

Montesinos and Pérez-Lorenzana (1999) very convincingly argue that Feynman's proof is capturing minimal coupling behavior. They argue this using the Poisson bracket, but here we will carry out the demonstration with classical commutators (like in Operational Dynamic Modeling - see Bondar et al. 2012) in the same spirit as the original paper (Dyson, 1990). If there is any question if classical/quantum commutators can handle a relativistic set up, Cabrera et al. (2019) derives the Dirac equation within the framework of ODM using the relativistic version of the canonical commutator (i.e., momentum is now relativistic momentum, but the commutator expression is still the same).

First, we will prove that the commutation of two components of eq. 49 must be the magnetic field. Whereas Feynman left it as a simple definition (eq. 48), by eq. 49 it must necessarily follow. The commutation of the two gives us:

$$[\hat{P}_i, \hat{P}_j] = [\hat{A}_i, \hat{p}_j] + [\hat{p}_i, \hat{A}_j] = -i\hbar(\frac{\partial \hat{A}_i}{\partial x_j} - \frac{\partial \hat{A}_j}{\partial x_i}) = i\hbar\epsilon_{ijk}\hat{H}_k \tag{52}$$

In the above, we have $\hat{A}$ be a function of $\hat{x}$, which ensures commutation of the vector potential with itself (eq. 42). Our magnetic field can therefore be shown to be

$$\hat{H}_l = -\frac{i}{2\hbar}\epsilon_{jkl}[\hat{P}_j, \hat{P}_k] \tag{53}$$

which avoids all the before-mentioned pitfalls. A similar derivation of the magnetic field from such a commutator expression can be found in Sakurai and Napolitano (2021).

Next, we demonstrate that $\hat{H}_l$ is a function of $\hat{x}$ and not $\hat{P}$:

$$[\hat{x}_i, \hat{H}_l] = 0 \tag{54}$$

By substituting in eq. 53 into the expression $[\hat{x}_i, \hat{H}_l]$, and then utilizing eq. 51 and the Jacobi identity, we derive the above result. $\hat{H}$ would therefore only be a function of $\hat{x}$ and t (Dyson, 1990), and all components of $\hat{H}$ would commute with themselves (this is also obvious when we plug eq. 53 into $[\hat{H}_k, \hat{H}_l]$ and observe from symmetry that it must equal zero).

Next, using the Jacobi identity again, we prove no magnetic monopoles. The Jacobi identity for different $\hat{P}_l$ components:

$$[\hat{P}_l, \epsilon_{jkl}[\hat{P}_j, \hat{P}_k]] + [\hat{P}_j, \epsilon_{jkl}[\hat{P}_k, \hat{P}_l]] + [\hat{P}_k, \epsilon_{jkl}[\hat{P}_l, \hat{P}_j]] = 0 \tag{55}$$

which immediately implies:

$$[\hat{P}_l, \hat{H}_l] + [\hat{P}_j, \hat{H}_j] + [\hat{P}_k, \hat{H}_k] = 0 \tag{56}$$

which is equivalent to $div \ \hat{H} = 0$, using the fact that $\hat{H}$ is a function of $\hat{x}$.

To prove the next Maxwell equation (Dyson, 1990), we take the time derivative of eq. 53:

$$\frac{\partial \hat{H}_l}{\partial t} + \frac{\partial \hat{H}_l}{\partial x_m}\dot{\hat{x}}_m = -\frac{i}{\hbar}\epsilon_{jkl}[\dot{\hat{P}}_j, \hat{P}_k] \tag{57}$$

Substituting in eqs. 50 and 44 in the same manner as Dyson (1990), we get:

$$\frac{\partial \hat{H}_l}{\partial t} + \frac{\partial \hat{H}_l}{\partial x_m}\dot{\hat{x}}_m = -\frac{i}{\hbar}\epsilon_{jkl}[\hat{E}_j, \hat{P}_k] - \frac{i}{\hbar}[\dot{\hat{x}}_k\hat{H}_l, \hat{P}_k] + \frac{i}{\hbar}[\dot{\hat{x}}_l\hat{H}_j, \hat{P}_j] \tag{58}$$

Since eq. 51 implies that $[\dot{\hat{x}}_i, \hat{P}_j] = -[\hat{x}_i, \dot{\hat{P}}_j]$, we evaluate the last two terms in the above expression to be:

$$[\dot{\hat{x}}_k\hat{H}_l, \hat{P}_k] - [\dot{\hat{x}}_l\hat{H}_j, \hat{P}_j] = -[\hat{x}_k, \dot{\hat{P}}_k]\hat{H}_l - [\dot{\hat{P}}_j, \hat{x}_l]\hat{H}_j - \dot{\hat{x}}_k[\hat{P}_k, \hat{H}_l] + \dot{\hat{x}}_l[\hat{P}_j, \hat{H}_j]$$

The third term on the left cancels with the second term on the right-hand side of eq. 58. The last term must be zero by no magnetic monopoles. Using eqs. 50 and 44 again, we evaluate the remaining terms to be:

$$-[\hat{x}_k, \dot{\hat{P}}_k]\hat{H}_l - [\dot{\hat{P}}_j, \hat{x}_l]\hat{H}_j = \epsilon_{juv}[\hat{x}_l, \dot{\hat{x}}_u]\hat{H}_v\hat{H}_j + \epsilon_{juv}\dot{\hat{x}}_u[\hat{x}_l, \hat{H}_v]\hat{H}_j - \epsilon_{kyz}[\hat{x}_k, \dot{\hat{x}}_y]\hat{H}_z\hat{H}_l - \epsilon_{kyz}\dot{\hat{x}}_y[\hat{x}_k, \hat{H}_z]\hat{H}_l$$

Using eq. 54 and symmetry, we can see that all the terms in the above right-hand side expression must be equivalent to zero. If we finally put everything together, we achieve our second Maxwell equation (Dyson, 1990):

$$\frac{\partial \hat{H}_l}{\partial t} = \epsilon_{jkl}\frac{\partial \hat{E}_j}{\partial x_k} \tag{59}$$

This concludes the relativistic Feynman proof with classical commutators. It is fully relativistic, as we utilize the relativistic momentum operator $\hat{p}_j$ with minimal coupling. No nonrelativistic assumptions were utilized, unlike the original proof. No Galilean assumptions appear in the proof, contrary to the usual understanding of Feynman's proof (as presented by Dyson 1990).

One interesting aspect of Feynman's derivation is that charge and charge current seem to be afterthoughts, and the fields appear primary. It is interesting to consider the perspective of charge and current as being emergent from physically real fields of potential and momentum, instead of the fields being byproducts of charge. The usage of a wave commutator on relativistic Newton's Laws to derive wave equations of force appears less of an anomaly given the main text of this paper.

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