

# Can we look at the quantisation rules as constraints?

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In this paper we explore the idea of looking at the Dirac quantisation conditions as  $\hbar$ -dependent constraints on the tangent bundle to phase-space. Starting from the path-integral version of *classical* mechanics and using the natural Poisson brackets structure present in the cotangent bundle to the tangent bundle of phase-space, we handle the above constraints using the standard theory of Dirac for constrained systems. The hope is to obtain, as total Hamiltonian, the Moyal operator of time-evolution and as Dirac brackets the Moyal ones. Unfortunately the program fails indicating that something is missing. We put forward at the end some ideas for future work which may overcome this failure.

## 1. INTRODUCTION

Quantum mechanics (QM) is witnessing a revival of interest these days due to a variety of problems which range from the black-hole evaporation to the old issue of measurement, to the problem of the wave-function of the universe and, last but not least, to the corrections which gravity would induce on QM. Somehow people are slowly realizing that some of the secrets of QM are still with us and partly unsolved.

We thought that it would help if we could look at QM in a different way. In this report we will try to do that by looking[1][2][3] at the quantisation conditions as constraints on the tangent bundle to phase-space. This idea[1] springs naturally from a path-integral formulation of classical mechanics (CM)[4] we proposed in 1986-1989.

The idea was further rekindled by some work [5] in which the authors tried to rederive geometric quantisation[6] using the tools of constrained and gauge systems. Along this direction further work appeared later[7], work in which the authors obtained a global formulation of geometric quantisation via the theory of constraints.

Our work[1][2][3] instead was not aimed at obtaining geometric quantisation or at studying some global issues in it, but was aimed at obtaining the Weyl-Wigner-Moyal formulation of QM[8]. The reason for trying to obtain that par-

ticular formulation was because that one was the formulation of QM most closely related to our CM path-integral [4]. Both in Moyal QM and in CM the states are distributions in phase-space and the quantum ones (Wigner functions), being[9] both<sup>1</sup>  $S^{(1)}$  and<sup>2</sup>  $L^{(2)}$ , are a subset of properly *enlarged* classical states which are only  $S^{(1)}$ . So if the quantum ones are a subset of the *enlarged*-classical distributions, this strongly indicates the presence of a mechanism similar the one of gauge theory where the physical Hilbert space appears as a subset of the full space. Working instead with geometric quantisation[5][7] [5] one goes from classical distributions in phase-space which are only  $S^{(1)}$  to wave-functions which are  $L^{(2)}$  but not necessarily  $S^{(1)}$ . So the mechanism of reduction is less transparent.

Besides these aspects, our method[1][2][3] seems more natural in the sense that the strange auxiliary fields, which are needed[5][7], appear quite naturally in our formulation of CM[4] springing out of our path-integral as the basis of the tangent bundle to phase-space. Moreover, differently than in refs. [5][7], we never have to quantize at the end, but we should pass naturally from the classical path-integral [4] to the quan-

<sup>1</sup>By  $S^{(1)}$  we mean that the integral over all of phase-space of the distributions (not of the norm) is finite.

<sup>2</sup>At least for pure density matrices or for mixed ones with a finite number of components.

tum one of evolution of Wigner functions [10]. Anyhow the works[5][7] which lead from CM to geometric quantisation are very interesting ones and they have achieved remarkable results. Our work[1][2][3] instead, even if more natural, it still afflicted with some problems as will be explained in what follows. Problems which we hope to overcome in the near future.

## 2. QUANTISATION CONDITIONS AS CONSTRAINTS.

The Dirac quantisation conditions are the usual rules of quantisation:

$$p \longrightarrow -i\hbar \frac{\partial}{\partial q} \quad (1)$$

In technical language this condition can be interpreted as an  $\hbar$ -dependent constraint between elements  $\phi \equiv (q, p)$  of the phase-space  $\mathcal{M}$  of the system and elements  $(\frac{\partial}{\partial p}, \frac{\partial}{\partial q})$  of the tangent space  $T_\phi \mathcal{M}$  in  $\phi$  to  $\mathcal{M}$ . In order to really interpret this as a constraint one should work in a space which unifies the phase-space  $\mathcal{M}$  with all the tangent spaces  $T_\phi \mathcal{M}$  of  $\mathcal{M}$ . This space is what is known as the tangent bundle to phase space and it is indicated by  $T\mathcal{M}$ . A formulation of classical mechanics (CM) in  $T\mathcal{M}$  has been given in ref.[4]. We will skip the details here and ask the reader to consult that paper. The final result is that CM can be given a path-integral formulation whose generating functional is:

$$Z_{cl} = \int \mathcal{D}\mu \exp i \int dt \{ \tilde{\mathcal{L}} + \text{source terms} \} \quad (2)$$

where  $\mathcal{D}\mu \equiv \mathcal{D}\phi^a(t) \mathcal{D}\lambda_a(t) \mathcal{D}c^a(t) \mathcal{D}\bar{c}_a$ . The variables  $\phi^a$  are the phase-space coordinates while the  $\lambda_a$   $c^a$   $\bar{c}_a$  are respectively commuting and anticommuting auxiliary variables. The Lagrangian is  $\tilde{\mathcal{L}} = \lambda_a \dot{\phi}^a + i\bar{c}_a \dot{c}^b - \tilde{\mathcal{H}}$  with the "Hamiltonian" given by

$$\tilde{\mathcal{H}} = \lambda_a h^a + i\bar{c}_a \partial_b h^a c^b \quad (3)$$

and where  $h^a$  are the components of the Hamiltonian vector field[11]  $h^a(\phi) \equiv \omega^{ab} \partial_b H(\phi)$ . The various auxiliary variables introduced above have all a precise geometrical meaning[4]. For example the  $c^a$  are a basis for the forms[11],

so  $c^a \equiv d\phi^a$ . This means that the coordinates  $(\phi^a, c^a)$  make up the cotangent bundle to phase-space, i.e  $T^*\mathcal{M}$ . We refer the reader for details to ref.[4]. To find the geometrical meaning of the  $\lambda_a$  and the  $\bar{c}_a$ , we should first note[4] that from the path-integral of eq.(2), one could define and calculate graded commutators among the various variables. The result is the following  $[\phi^a, \lambda_b] = i\delta_b^a$  and  $[c^a, \bar{c}_b] = \delta_b^a$ . This implies that we can realize  $\lambda_a$  and  $\bar{c}_a$  operatorially as  $\lambda_a = -i\frac{\partial}{\partial \phi^a} \equiv -i\partial_a$  ;  $\bar{c}_a = \frac{\partial}{\partial c^a}$ . Noticing this, we can say that  $\lambda_a$  and  $\bar{c}_a$  make a basis of the tangent space to the  $(\phi^a, c^a)$  space. So the  $8n$ -coordinates  $(\phi^a, c^a, \lambda_a, \bar{c}_a)$  are a basis for the  $T(T^*\mathcal{M})$  which is the tangent bundle to the cotangent bundle to phase-space. Inserting the operatorial realization of  $\lambda_a$  and  $\bar{c}_a$  into  $\tilde{\mathcal{H}}$  we would get what is known[11] as the Lie-derivative of the Hamiltonian flow[4]. The first term of which is the Liouville operator of classical mechanics:  $\hat{L} = \frac{\partial H}{\partial p} \frac{\partial}{\partial q} - \frac{\partial H}{\partial q} \frac{\partial}{\partial p}$ . The usual quantisation rule of eq.(1), if inserted in the Liouville operator above, would not make sense because we would have to say what is  $\frac{\partial}{\partial p}$  at the *quantum* level where  $p$  has become an operator. This means knowing what it is the derivative of an operator. This belongs to the realm of non-commutative geometry. We will adopt a less sophisticated strategy. As both CM[4] and QM[10] are now formulated via-path-integrals we will try to go from one to the other via the Dirac[12] theory of constraints. The reader may object that the constraints usually act on a classical phase-space and not in an operatorial space. Well, in ref.[4] we proved that our path-integral, besides providing an operatorial version of CM, is also naturally endowed with a classical Poisson brackets structure just because the space  $T(T^*\mathcal{M})$  is isomorphic to  $T^*(T\mathcal{M})$  which is a cotangent bundle. We will call this Poisson structure an extended Poisson structure (*epb*). The  $\{\cdot, \cdot\}_{epb}$  among the basic fields are  $\{\phi^a, \lambda_b\}_{epb} = \delta_b^a$ ,  $\{c^a, \bar{c}_b\}_{epb} = -i\delta_b^a$  and they reproduce the classical equations of motion via the Hamiltonian of eq.(3) :  $\{\phi^a, H\}_{pb} = \{\phi^a, \tilde{\mathcal{H}}\}_{epb}$  where the  $\{\cdot, \cdot\}_{pb}$  are the old standard Poisson brackets on  $\mathcal{M}$ , which are  $\{\phi^a, \phi^b\}_{pb} = \omega^{ab}$ . Basically this formulation

of classical mechanics provides a manner ( if we disregard the anticommuting variables) to generate the dynamics on the full  $T\mathcal{M}$ . In this space now, remembering the operatorial meaning of  $\lambda$ , the quantisation conditions can be written as the following constraints:

$$\Phi_{(0)}^a \equiv \theta(a-n)(\phi^a + \hbar\omega^{ab}\lambda_b) \quad (4)$$

where the  $\theta(a-n)$  is the standard step function which tells us that the first  $n$  indices  $a$ , which are indicating the q-variables, are put to zero. Now that we have a CM in this enlarged space, we should treat the above constraint as it is usually done in constrained theory.

The first thing to do is to calculate the *secondary* constraints which are those obtained by the evolution of the primary ones  $\Phi_{(0)}^a$ . The sub-index "(0)" is to indicate that it is primary. We will use the sub-indices "(1), (2), (3),..." to indicate the *secondary* and *tertiary*, etc. constraints generated this way. This procedure starts by using what is called[12] the total Hamiltonian  $\tilde{\mathcal{H}}_T$  defined as  $\tilde{\mathcal{H}}_T \equiv \tilde{\mathcal{H}} + \sum_a u_a \Phi_{(0)}^a$  where  $u_a$  are Lagrange multipliers. The secondary constraint  $\Phi_{(1)}^a$  are then generated by imposing that the primary ones are left invariant by the time evolution under  $\tilde{\mathcal{H}}_T$ . In this case only secondary ones are generated because the next step starts determining the Lagrange multiplier. The secondary constraints turn out to be:  $\Phi_{(1)}^a = \theta(a-n)(\omega^{ab}\frac{\partial H}{\partial \phi^b} - \hbar\omega^{ab}\lambda_e\omega^{ef}\frac{\partial^2 H}{\partial \phi^b \partial \phi^f})$ . With the same procedure it is possible to determine the Lagrange multipliers and we refer the interested reader to ref.[3]. The constraints are second class and the next step is to build the associated Dirac brackets[12](which we will call extended Dirac brackets (*edb*) because they are based on the extended Poisson brackets) between two observables  $F$  and  $G$ . These brackets are defined as  $\{F, G\}_{edb} \equiv \{F, G\}_{epb} - \{F, \Psi_\alpha\}_{epb} C^{\alpha\beta} \{\Psi_\beta, G\}_{epb}$  where we have indicated collectively with  $\Psi_\alpha$  the set of primary  $\Phi_{(0)}^a$  and secondary  $\Phi_{(1)}^a$  constraints and  $C^{\alpha\beta}$  is the inverse of the matrix  $C_{\alpha\beta}$  defined as  $C_{\alpha\beta} \equiv \{\Psi_\alpha, \Psi_\beta\}_{epb}$ . Applying all this to an harmonic oscillator  $H = 1/2(p^2 + q^2)$  we obtain[3], for the constrained evolution of an observable  $F$ , the following:  $\{F, \tilde{\mathcal{H}}_T\}_{edb} = p\frac{\partial F}{\partial q} - q\frac{\partial F}{\partial p}$ . For an Hamil-

tonian with quartic potential:  $H = (1/2)p^2 + (1/4)q^4$  the analog extended Dirac bracket evolution would give:  $\{F, \tilde{\mathcal{H}}_T\}_{edb} = 3/2p\frac{\partial F}{\partial q} - q^3\frac{\partial F}{\partial p}$ .

Now that we have this formulation, which we could call an  *$\hbar$ -constrained CM* ( $\hbar$ -CCM), we want to compare it with real QM. Of course the formulation of QM, with which we want to compare our  $\hbar$ -CCM, must be a formulation in phase-space (as our  $\hbar$ -CCM is) and moreover it must handle everything not with operators but with c-numbers as our  $\hbar$ -CCM does. This formulation of QM exists and it was provided by Weyl, Wigner and Moyal[8]. We will not review here all the Moyal formalism. Let us say that it is a procedure which replaces all operators of standard QM with functions on phase-space (called symbols of the operator) and the commutators with brackets called Moyal brackets (*mb*). So the Heisenberg evolution of the density matrix  $i\hbar \partial_t \hat{\rho} = -[\hat{\rho}, \hat{H}]$  goes into  $\partial_t \varrho(\phi^a, t) = -\{\varrho, H\}_{mb}$  where the  $\{\cdot, \cdot\}_{mb}$  are the Moyal brackets which are defined as:

$\{A, B\}_{mb} = A(\phi) \frac{2}{\hbar} \sin[\frac{\hbar}{2} \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b] B(\phi) = \{A, B\}_{pb} + O(\hbar^2)$ . In the classical limit ( $\hbar \rightarrow 0$ ) the Moyal brackets reduces to the classical Poisson brackets. The next thing to ask is if, as we did in CM, we can lift the action of the Moyal Bracket on the  $T\mathcal{M}$ . The answer is yes[10] and the result looks as follows: It is possible to build an  $\tilde{\mathcal{H}}^{\hbar}$  and a set of extended Moyal brackets  $\{\cdot, \cdot\}_{emb}$  such that:  $\{\phi^a, H\}_{mb} = \{\phi^a, \tilde{\mathcal{H}}^{\hbar}\}_{emb}$ . For the interested reader the details are written in ref.[10]. For the purpose of this report it is enough to mention here the explicit form of  $\tilde{\mathcal{H}}^{\hbar}$ , which is:

$$\tilde{\mathcal{H}}^{\hbar} = \tilde{\mathcal{H}} + \frac{(\hbar)^2}{3!} \mathcal{M}_{(1)} + \frac{(\hbar)^4}{5!} \mathcal{M}_{(2)} + \dots \text{ where}$$

$$\mathcal{M}_{(j)} = \overbrace{[\lambda_a \omega^{ab}][\lambda_c \omega^{cd}] \dots [\lambda_e \omega^{ef}]}^{j+2} [\partial_b \partial_d \dots \partial_f H] \quad (5)$$

The idea now is to compare the quantum evolution, provided by the  $\tilde{\mathcal{H}}^{\hbar}$  under the extended Moyal brackets, with the  $\hbar$ -CCM time evolution provided by the  $\tilde{\mathcal{H}}_T$ . It is easy to see[3] that the two evolution are the same for the harmonic oscillator, but they are different for the quartic potential. Even for the harmonic oscillator the correspondence of QM with our  $\hbar$ -CCM is not

perfect: the extended Moyal bracket between two observables is not the same as the extended Dirac bracket between the same observables unless one of the two is quadratic in  $\phi^a$ . So our program of generating QM from CM via a constraint has failed. Let us see which ones may be the reasons. One which is immediately evident is the fact that we applied the Dirac procedure to an Hamiltonian  $\tilde{\mathcal{H}}$  which is not singular. What we should have done is to start from an Hamiltonian which already contains the quantisation constraints  $\mathcal{H} \equiv \tilde{\mathcal{H}} + \xi_a \Phi_{(0)}^a$  and consider the Lagrange multipliers  $\xi_a$  as dynamical variables. Anyhow even doing this does not help us going from  $\tilde{\mathcal{H}}$  to  $\tilde{\mathcal{H}}^h$ . The reason is that  $\tilde{\mathcal{H}}^h$  contains an infinite number of extra pieces (see eq.(5)) beyond the first one which is  $\tilde{\mathcal{H}}$ . The idea put forward in ref.[3] was that one should have generated an infinite set of secondary and tertiary constraints. This of course does not happen. What is more likely instead is that the constraints  $\Phi_{(0)}^a$  are not the full story. They may just be the first term of a constraint with infinite terms (a formal power serie in  $\hbar$ ). If this were so, then it would be natural to expect that the total Hamiltonian itself,  $\tilde{\mathcal{H}}_T$ , become a formal power serie in  $\hbar$ . Having got to this point, the reader expert in constraints must have realized that actually, even if we manage to build the  $\tilde{\mathcal{H}}_T$ , its effect via the (edb) is the same as the one of  $\tilde{\mathcal{H}}$ :  $\{(\cdot), \tilde{\mathcal{H}}_T\}_{edb} = \{(\cdot), \tilde{\mathcal{H}}\}_{edb}$ . So what we have to check is if there are (edb) such that  $\{(\cdot), \tilde{\mathcal{H}}\}_{edb} = \{(\cdot), \tilde{\mathcal{H}}^h\}_{emb}$ . Again the expert reader may immediately object that Dirac-Poisson brackets can never be Moyal brackets for the different way they behave with respect to derivation [13]. We feel that theorem can actually be bypassed if the constraint is not a polynomial but a formal infinite series as in our case. The fact that it is such an infinite series may puzzle the reader. Actually it is easy to check that eq.(1) is the correct quantisation rule if we work on wavefunctions and Hilbert space. It is not the correct one if we work with Wigner functions which is the space of states we want to obtain. The Wigner function is defined as follows  $\varrho(p, q) \equiv \frac{1}{2\pi} \int \psi^*(q - 1/2\hbar s) e^{-ips} \psi(q + 1/2\hbar s) ds$ . Applying on it the LHS of eq.(1) we get something dif-

ferent than applying the RHS. What we get is:  $\propto \int [\mp \frac{\partial \psi^*}{\partial q} \psi + \psi^* \frac{\partial \psi}{\partial q}] e^{-isp} ds$  where the  $\mp$  signs refer respectively to the action of the LHS or the RHS of eq.(1). This indicates that the constraint  $\Phi_{(0)}^a$  has to be modified. One last thing before concluding. If we think that there is a constraint operating here, then we must envision that the space of states (Wigner-functions) be obtainable as a subset of the classical set of states. In order to do that, we have to allow the classical states to be *non necessarily* positive functions in phase-space and even discontinuous but still  $S^{(1)}$  and to be also distributions depending eventually even on external parameters. If this enlargement (which is allowed by our CM path-integral) is done, then it is easy to prove that the Wigner functions, which are both  $S^{(1)}$  and  $L^{(2)}$ , are really a subset of the *enlarged* classical ones.

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