Functional Techniques in Classical Mechanics

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In 1931 Koopman and von Neumann extended previous work of Liouville and provided an operatorial version of Classical Mechanics (CM). In this talk we will review a path-integral formulation of this operatorial version of CM. In particular we will study the geometrical nature of the many auxiliary variables present and of the unexpected universal symmetries generated by the functional technique.

1. INTRODUCTION

I usually do not go to conferences which have in the title the standard fashionable words "*Quantum Gravity*" (QG). I made an exception this time because my dear friend Giampiero had the good taste of creating two sessions: one dedicated to QG and another to "*Foundations of Quantization*"(FQ). In fact I belong to that minority which thinks that we should not only try to attack the second of the two horns of the problem of *Quantum Gravity*, that is *Gravity*, but also the first one that is the *Quantum*. Gravity is the queen of *geometrical* theories, and I feel we should better understand QM from a more *geometrical* point of view before putting the two theories together. Attempts in this direction already exist like for example the method called "*Geometric Quantization*"(see ref. [1] for a review). We feel anyhow that method should be made less cumbersome and understood from a more physical point of view. With this goal in mind we proved [2] that, by formulating classical mechanics (CM) via functional methods [3], the standard geometric quantization rules become equivalent to freezing to zero some Grassmannian partners of time. This may throw some light on the geometrical aspects of quantization and work is in progress on it [2]. In this paper we will limit ourselves to reviewing the classical mechanics part of this project that means the geometrical structures which enter the functional approach to CM.

2. FUNCTIONAL APPROACH TO CLASSICAL MECHANICS

The functional formulation of CM mentioned above is a path-integral approach to the *operatorial* version of CM proposed by Koopman and von Neumann [4] in 1931. These authors, instead of using the Hamiltonian and the Poisson brackets for the classical evolution of a system, used the well-known Liouville operator, $\hat{L} \equiv \dfrac{\partial H}{\partial p}\dfrac{\partial}{\partial q} - \dfrac{\partial H}{\partial q}\dfrac{\partial}{\partial p}$, and the associated commutators. One can generalize their formalism to higher forms and get what is known as the Lie derivative of the Hamiltonian flow, [5]. It was shown in [3] that the operatorial formalism mentioned above could have a functional or path integral counterpart. The procedure goes as follows. Let us denote by $\mathcal{M}$ our phase space with $2n$ phase-space coordinates $\varphi^a = (q^j, p^j)$ (the index "a" spans both q's and p's) and by $H(\varphi)$ the Hamiltonian of the system. The classical trajectories are solutions of the Hamilton equations of motion: $\dot{\varphi}^a = \omega^{ab}\frac{\partial H}{\partial \phi^b}$ where ω^{ab} is the standard symplectic matrix. A natural object to introduce is the *classical* analog, $Z_{CM}[j]$, of the quantum generating functional:

$$Z_{CM}[j] = N \int \mathcal{D}\varphi \tilde{\delta}[\varphi(t) - \varphi_{cl}(t)] \exp \int j\varphi dt \quad (1)$$

where φ are the $\varphi^a \in \mathcal{M}$, φ_{cl} are the solutions of the equations of motion, j is an external current and $\widetilde{\delta}[\ \ ]$ is a functional Dirac-delta which forces every path $\varphi(t)$ to sit on a classical one $\varphi_{cl}(t)$. There are all possible initial conditions integrated over in (1) and, because of this, one should be very careful in properly defining the measure of integration and the functional Dirac delta. We should now check whether the path integral of eq. (1) leads to the well known operatorial formulation [4] of CM . To do that let us first rewrite the functional Dirac delta in (1) as:

$$\tilde{\delta}[\varphi(t)-\varphi_{cl}(t)] = \\ = \tilde{\delta}[\dot{\varphi}^a - \omega^{ab}\partial_b H] |\det[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]| \tag{2}$$

The determinant which appears in (2) is always positive and so we can drop the modulus sign $|\ \ |$. The next step is to insert (2) in (1) and write the $\tilde{\delta}[\ \ ]$ as a Fourier transform over some new variables λ_a, i.e.:

$$\tilde{\delta}\left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial\varphi^b}\right] = \\ = \int \mathcal{D}\lambda_a \ \exp\ i\int \lambda_a\left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial\varphi^b}\right]dt \tag{3}$$

Next we express the determinant in eq.(2) via grassmannian variables $\bar{c}_a, c^a$:

$$\int \mathcal{D}c^a\mathcal{D}\bar{c}_a \exp\left[-\int \bar{c}_a[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]c^b\ dt\right] \tag{4}$$

Inserting (2),(3) and (4) in (1) we get:

$$Z_{CM}[0] = \int \mathcal{D}\varphi^a\mathcal{D}\lambda_a\mathcal{D}c^a\mathcal{D}\bar{c}_a \exp\left[i\int\ dt\widetilde{\mathcal{L}}\right] \tag{5}$$

where $\widetilde{\mathcal{L}}$ is:

$$\widetilde{\mathcal{L}} = \lambda_a[\dot{\varphi}^a - \omega^{ab}\partial_b H] + i\bar{c}_a[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]c^b \tag{6}$$

Varying this Lagrangian with respect to λ_a, one can easily obtain the standard equation of motion for φ^a, while varying it with respect to $\bar{c}_a$ gives the equations for the c^a. The overall set of equations of motion is:

$$\begin{aligned} \dot{\varphi}^a - \omega^{ab}\partial_b H &= 0 \\ \left[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H\right]c^b &= 0 \\ \delta^a_b\partial_t\bar{c}_a + \bar{c}_a\omega^{ac}\partial_c\partial_b H &= 0 \\ \left[\delta^a_b\partial_t + \omega^{ac}\partial_c\partial_b H\right]\lambda_a &= -i\bar{c}_a\omega^{ac}\partial_c\partial_d\partial_b H c^d \end{aligned} \tag{7}$$

The Hamiltonian associated to the $\widetilde{\mathcal{L}}$ of eq.(6)is:

$$\widetilde{\mathcal{H}} = \lambda_a\omega^{ab}\partial_b H + i\bar{c}_a\omega^{ac}(\partial_c\partial_b H)c^b \tag{8}$$

The equations of motion of eqs.(7) can be obtained also from this Hamiltonian using some extended Poisson brackets (*EPB*) defined in the space (φ^a, c^a, λ_a, $\bar{c}_a$) as follows: $\{\varphi^a, \lambda_b\}_{EPB} = \delta^a_b$; $\{\bar{c}_b, c^a\}_{EPB} = -i\delta^a_b$. All the other brackets are zero. Having a path-integral we can also define the concept of commutators [3] and realize the various variables as operators. It is then easy to prove [3] that the bosonic part of $\widetilde{\mathcal{H}}$ turns into the Liouville operator of Koopman and von Neumann. This confirms that our path-integral (1) is the correct counterpart of the operatorial approach to CM [4]. We skip here these details which can be found in the first paper of ref. [3]. The reader may not like the pletora of variables (φ^a, c^a, λ_a, $\bar{c}_a$)that we had to introduce. It is actually possible to simplify things considerably by introducing the concept of superfield. If we enlarge the base space, which is up to now just the time t, by including two grassmannian partners of time, θ and $\bar{\theta}$, we can put together all the variables (φ^a, c^a, λ_a, $\bar{c}_a$) into the following superfield:

$$\Phi^a = \varphi^a + \theta c^a + \bar{\theta}\omega^{ab}\bar{c}_b + i\bar{\theta}\theta\omega^{ab}\lambda_b \tag{9}$$

Via this superfield the complicated expression of $\widetilde{\mathcal{H}}$ can be written as $\widetilde{\mathcal{H}} = i\int d\theta d\bar{\theta} H[\Phi]$. This formula is the starting point for the quantization procedure we presented in ref. [2]. Besides this unification , we shall also show in the next section that all the variables (φ^a, c^a, λ_a, $\bar{c}_a$)have a clear geometrical meaning and they are all needed to shed light on our construction.

3. GEOMETRICAL CONTENT

¿From the equations of motion (7) one notice immediately that c^b transforms under the Hamiltonian vector field [5] $h \equiv \omega^{ab}\partial_b H\partial_a$ as the *basis* $d\varphi^b$ of generic forms $\alpha \equiv \alpha_a(\varphi)d\varphi^a$. It is possible to show that this happens not only under the Hamiltonian flow but also under any diffeomorphism of the phase-space $\mathcal{M}$ whose coordinates are φ^a. We can look at the c^a in a manner dual to the previous one because we can say that the c^a also transform as *components* of tangent vectors: $V^a(\varphi)\frac{\partial}{\partial\phi^a}$. Because of this the space whose coordinates are (φ^a, c^a) is called in [6] *reversed-parity tangent bundle* and is indicated as $\Pi T\mathcal{M}$. The "*reversed-parity*" specification is because the c^a are Grassmannian variables. From the Lagrangian (6) we notice that the $(\lambda_a, \bar{c}_a)$ are the "momenta" of the variables (φ^a, c^a), so we can conclude that the 8n variables $(\varphi^a,\, c^a,\, \lambda_a,\, \bar{c}_a)$ span the cotangent bundle to the reversed-parity tangent bundle which is indicated as $T^\star(\Pi T\mathcal{M})$. For more details about this we refer the interested reader to the sixth paper contained in ref. [3]. In the remaining part of this section we will show how to reproduce all the abstract Cartan calculus via our *EPB* and the Grassmannian variables. Let us first introduce five charges which are conserved under the $\widetilde{\mathcal{H}}$ of eq. (8) and which will play an important role in the Cartan calculus. They are:

$$Q^{BRS} \equiv ic^a\lambda_a \quad , \quad \bar{Q}^{BRS} \equiv i\bar{c}_a\omega^{ab}\lambda_b \tag{10}$$

$$Q_g \equiv c^a\bar{c}_a \tag{11}$$

$$K \equiv \frac{1}{2}\omega_{ab}c^ac^b \quad , \quad \bar{K} \equiv \frac{1}{2}\omega^{ab}\bar{c}_a\bar{c}_b \tag{12}$$

where ω_{ab} are the matrix elements of the inverse of ω^{ab}. Next we should note, from the eqs. of motion (7), that $\bar{c}_a$ transform under the Hamiltonian flow as the basis $\dfrac{\partial}{\partial\varphi^a}$ of vector fields. This happens also under any diffeomorphism of $\mathcal{M}$. Now since c^a transform as basis of forms $d\varphi^a$ and $\bar{c}_a$ as basis of vector fields $\frac{\partial}{\partial\varphi^a}$, let us start building the following map, called [3] "hat" map :

$$\alpha = \alpha_a d\varphi^a \quad \hat{\longrightarrow} \quad \widehat{\alpha} \equiv \alpha_a c^a \tag{13}$$

$$V = V^a\partial_a \quad \hat{\longrightarrow} \quad \widehat{V} \equiv V^a\bar{c}_a \tag{14}$$

It is actually a much more general map between forms α, antisymmetric tensors V and functions of $\varphi, c, \bar{c}$:

$$F^{(p)} = \frac{1}{p!} \quad F_{a_1\cdots a_p} \quad d\varphi^{a_1}\wedge\cdots\wedge d\varphi^{a_p} \hat{\longrightarrow} \tag{15}$$

$$\hat{\longrightarrow} \quad \widehat{F}^{(p)} \equiv \frac{1}{p!}F_{a_1\cdots a_p}c^{a_1}\cdots c^{a_p}$$

$$V^{(p)} = \frac{1}{p!} \quad V^{a_1\cdots a_p} \quad \partial_{a_1}\wedge\cdots\wedge\partial_{a_p} \hat{\longrightarrow} \tag{16}$$

$$\hat{\longrightarrow} \quad \widehat{V} \equiv \frac{1}{p!}V^{a_1\cdots a_p}\bar{c}_{a_1}\cdots\bar{c}_{a_p}$$

Once the correspondence (15-16) is extablished we can easily find out what corresponds in our formalism to the various operations of the so called *Cartan calculus* [5]. They are for example the exterior derivative **d** of a form, or the interior contraction between a vector field V and a form F and other similar operations. It is easy [3] to check that:

$$dF^{(p)} \quad \hat{\longrightarrow} \quad i\{Q^{BRS}, \widehat{F}^{(p)}\}_{EPB} \tag{17}$$

$$\iota_V F^{(p)} \quad \hat{\longrightarrow} \quad i\{\widehat{V}, \widehat{F}^{(p)}\}_{EPB} \tag{18}$$

$$pF^{(p)} \quad \hat{\longrightarrow} \quad i\{Q_g, \widehat{F}^{(p)}\}_{EPB} \tag{19}$$

where Q^{BRS}, Q_g are the charges of (10-11). In a similar manner we can implement in our language the usual mapping [5] between vector fields V and forms $V^\flat$ realized by the symplectic 2-form $\omega(V,0) \equiv V^\flat$, or the inverse operation of building a vector field $\alpha^\sharp$ out of a form: $\alpha = (\alpha^\sharp)^\flat$. These operations can be turned in our formalism as follows:

$$V^\flat \;\hat{\longrightarrow}\; i\{K, \widehat{V}\}_{EPB} \;\;;\;\; \alpha^\sharp \;\hat{\longrightarrow}\; i\{\bar{K}, \widehat{\alpha}\}_{EPB} \tag{20}$$

where again $K, \bar{K}$ are the charges (12). We can also implement in our formalism the standard operation [5] of building a vector field out of a function $f(\varphi)$. It is: $(df)^\sharp \hat{\longrightarrow} i\{\bar{Q}^{BRS}, f\}_{EPB}$ The next thing to do is to reproduce in our formalism the concept of Lie derivative [5] which is defined as: $\mathcal{L}_V = d\iota_V + \iota_V d$. It is easy to prove that

$$\mathcal{L}_V F^{(p)} \;\hat{\longrightarrow}\; \{-\widetilde{\mathcal{H}}_V, \widehat{F}^{(p)}\}_{EPB} \tag{21}$$

where $\widetilde{\mathcal{H}}_V = \lambda_a V^a + i\bar{c}_a\partial_b V^a c^b$; note that, with $V^a = \omega^{ab}\partial_b H$, $\widetilde{\mathcal{H}}_V$ becomes the one of eq.(8). This confirms that the full $\widetilde{\mathcal{H}}$ of eq. (8) is the Lie derivative of the Hamiltonian flow. Finally the Lie brackets between two vector fields V, W are reproduced as: $[V,W]_{Lie-brack.} \hat{\rightarrow} \{-\widetilde{\mathcal{H}}_V, \widehat{W}\}_{EPB}$. In the literature the mathematicians have introduced generalizations of the Lie-brackets [7]. These are brackets which act on form-valued tensor fields and on other similar objects. Also these brackets can be written in our formalism, for details see the seventh paper quoted in ref. [3]. ¿From all the previous formalism it is now easy to understand why the charges of eq.(10) commute with the Hamiltonian. The first charge is just the exterior derivative, (see eq.(17)), and it is known from differential geometry [5] that the exterior derivative commutes with the Lie-derivative. We have called the charges of eq.(10) as BRS charges because they anticommute among themselves like the BRS charges of gauge theories do and moreover, like these ones, they are basically exterior derivatives on some particular spaces. The five charges of eqs.(10-12) are not the only ones universally conserved under our Hamiltonian $\widetilde{\mathcal{H}}$. There are also two other ones which are: $N_{H} = c^a\partial_a H$ and $\overline{N}_{H} = \bar{c}_a\omega^{ab}\partial_b H$. They can be combined with the BRS and antiBRS charges in the following manner: $Q_{(1)} \equiv Q^{BRS} - \overline{N}_{H}$; $Q_{(2)} \equiv \overline{Q}^{BRS} + N_{H}$. It is easy to prove that, once the charges and the Hamiltonian are turned into operators, we get : $Q^2_{(1)} = Q^2_{(2)} = -i\widetilde{\mathcal{H}}$. This means that these charges implement a universal N=2 supersymmetry. As the Q^{BRS} of (10) was basically the exterior derivative on phase space, it would be nice to understand the geometrical meaning also of the susy charges like $Q_{(1)}$ or $Q_{(2)}$. This was done in the last of references [3]. The strategy used there was to make local the global susy invariance and to analyze the associated physical state condition. Once this physical state condition is turned, via the "hat" map of eqs.(13)-(21), into a Cartan calculus sort of operation, it tells us that the *physical* states are in one to one correspondence with the states of the so called *equivariant cohomology* [8] associated to the Hamiltonian vector field. The equivariant cohomology w.r.t. a vector field V is defined as the set of forms $|\rho\rangle$ which satisfy the following conditions:

$$\begin{aligned} &(d-\iota_{V})|\rho\rangle = 0 \quad ; \quad \mathcal{L}_{V}|\rho\rangle = 0 \\ &|\rho\rangle \neq (d-\iota_{V})|\chi\rangle \quad ; \quad \mathcal{L}_{V}|\chi\rangle = 0 \end{aligned} \tag{22}$$

This is basically the geometrical light we could throw on the susy charge $Q_{(1)}$. More details can be found in the last paper of ref. [3]. We found amazing that out of a simple Dirac delta, like the one in (1), we have managed to extract all these geometrical structures. Actually, if we impose proper boundary conditions, this kind of path-integral becomes very similar to the one of Topological Field Theories [9]. It is in fact easy to prove that it helps in calculating various topological invariants associated to the phase-space of the system. This was actually done in the fifth paper of ref. [3]. All this formalism can be generalized to YM theories [10] and there it may help in studying the geometrical features of the space of gauge orbits.

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