Supersymmetry in Classical Mechanics

E. Deotto, E. Gozzi and D. Mauro

Dipartimento di Fisica Teorica, Università di Trieste
Strada Costiera 11, P.O. Box 586, Trieste, Italy
and INFN, Sezione di Trieste

In 1931 Koopman and von Neumann [1] proposed an *operatorial* formulation of Classical Mechanics (CM) expanding earlier work of Liouville. Their approach is basically the following: given a dynamical system with a phase space $\mathcal{M}$ labelled by coordinates $\varphi^a = (q^i, p^i)$; $a = 1, \ldots, 2n$; $i = 1, \ldots, n$, with Hamiltonian H and symplectic matrix ω^{ab}, the evolution of a probability density $\rho(\varphi)$ can be given either via the Poisson brackets $\{\ ,\ \}$ or via the Liouville operator:

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} = -\hat{L}\rho; \quad \widehat{L} = \omega^{ab}\partial_b H \partial_a \tag{1}$$

The evolution via the Liouville operator is basically what is called the operatorial approach to CM. The natural question to ask is whether we can associate to the *operatorial* formalism of CM a *path integral* one, like it is done in quantum mechanics. The answer is yes [2]. In fact we can describe the transition probability $P(\varphi^a_{(2)}t_2|\varphi^a_{(1)}t_1)$ of being in configuration $\varphi_{(2)}$ at time t_2 if we were at time t_1 in configuration $\varphi_{(1)}$ via a functional integral of the form

$$\begin{aligned} P(\varphi^a_{(2)}t_2|\varphi^a_{(1)}t_1) &= \int \mathcal{D}\varphi^a \, \widetilde{\delta}[\varphi^a(t_2) - \varphi^a_{cl}(t_2; \varphi_{(1)}t_1)] \\ &= \int \mathcal{D}\varphi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \; e^{i\int \widetilde{\mathcal{L}}dt} \end{aligned} \tag{2}$$

where $\varphi^a_{cl}(t; \varphi_{(1)}t_1)$ is the solution of the *classical* equations of motion $\dot{\varphi}^a = \omega^{ab}\frac{\partial H}{\partial \varphi^b}$ and $\widetilde{\delta}$ is a functional Dirac delta which gives weight one to the classical paths and zero to the others. In the second line of (2), via some manipulations [2], we have turned the Dirac $\widetilde{\delta}$ into a more standard looking weight where

$$\widetilde{\mathcal{L}} = \lambda_a \dot{\varphi}^a + i\bar{c}_a \dot{c}^a - \widetilde{\mathcal{H}}; \qquad \widetilde{\mathcal{H}} = \lambda_a \omega^{ab}\partial_b H + i\bar{c}_a \omega^{ac}\partial_c \partial_b H c^b \tag{3}$$

The $\lambda_a, c^a, \bar{c}_a$ are auxiliary variables with c^a and $\bar{c}_a$ of grassmannian character. The geometrical meaning of these variables has been studied in [2] and [3].

It is natural at this point to make contact with the operatorial formalism of eq. (1). It is easy to prove [2] that the first piece of $\widetilde{\mathcal{H}}$ in (3) is nothing else than the Liouville operator of (1). To understand the meaning of the full $\widetilde{\mathcal{H}}$ is important to notice [2] that the c^a are nothing else than the basis $d\varphi^a$ of the forms [4] while the $\bar{c}_a$ are the basis of the vector fields. So we can create a correspondence between forms (tensor fields) and polynomials of c ($\bar{c}$):

$$\begin{aligned} F^{(p)} &= \frac{1}{p!}F_{a_1\ldots a_p}d\varphi^{a_1}\wedge\cdots\wedge d\varphi^{a_p} \quad\longrightarrow\quad \widehat{F}^{(p)} \equiv \frac{1}{p!}F_{a_1\ldots a_p}c^{a_1}\cdots c^{a_p} \\ V^{(p)} &= \frac{1}{p!}V^{a_1\ldots a_p}\partial\varphi^{a_1}\wedge\cdots\wedge \partial\varphi^{a_p} \quad\longrightarrow\quad \widehat{V}^{(p)} \equiv \frac{1}{p!}V^{a_1\ldots a_p}\bar{c}_{a_1}\cdots \bar{c}_{a_p} \end{aligned} \tag{4}$$

All the standard operations of the Cartan calculus [4], like the exterior derivative d, the interior contraction ι_v, the symplectic correspondence between forms and Hamiltonian vector fields $\alpha = (\alpha^\sharp)^\flat$ can be reproduced via the graded-commutators associated to the path integral (2) in the following way:

$$\begin{aligned} &dF^{(p)} \rightarrow [Q_{BRS}\widehat{F}^{(p)}], \qquad \iota_V F^{(p)} \rightarrow [\widehat{V}, \widehat{F}^{(p)}] \\ &pF^{(p)} \rightarrow [Q_g, \widehat{F}^{(p)}], \qquad V^\flat \rightarrow [K, \widehat{V}] \\ &\alpha^\sharp \rightarrow [\overline{K}, \widehat{\alpha}], \qquad (df)^\sharp \rightarrow [\overline{Q}_{BRS}, f] \end{aligned} \tag{5}$$

where

$$\begin{aligned} &Q_{BRS} \equiv ic^a\lambda_a; \qquad \overline{Q}_{BRS} \equiv i\bar{c}_a\omega^{ab}\lambda_b \\ &Q_g \equiv c^a\bar{c}_a; \quad K \equiv \frac{1}{2}\omega_{ab}c^ac^b; \quad \overline{K} \equiv \frac{1}{2}\omega^{ab}\bar{c}_a\bar{c}_b \end{aligned} \tag{6}$$

are universally conserved charges under our $\widetilde{\mathcal{H}}$. Equipped with this formalism it is then easy to prove [2] that the full $\widetilde{\mathcal{H}}$ is nothing else than the Lie-derivative $\mathcal{L}_{(dH)^\sharp} = \iota_{(dH)^\sharp}d + d\iota_{(dH)^\sharp}$ of the Hamiltonian flow and the correspondence is the following:

$$\mathcal{L}_{(dH)^\sharp}F^{(p)} \quad\rightarrow\quad i[\widetilde{\mathcal{H}}, \widehat{F}^{(p)}] \tag{7}$$

Beside the five charges in (6) also $N_H = c^a\partial_a H$ and $\overline{N}_H = \bar{c}_a\omega^{ab}\partial_b H$ are conserved under $\widetilde{\mathcal{H}}$ and as a consequence these other charges are also conserved:

$$Q_H \equiv Q_{BRS} - N_H; \qquad \overline{Q}_H \equiv \overline{Q}_{BRS} + \overline{N}_H \tag{8}$$

They are two supersymmetry charges. In fact, while the Q_{BRS} and $\overline{Q}_{BRS}$ anti-commute, Q_H and $\overline{Q}_H$ close on $\widetilde{\mathcal{H}}$:

$$[Q_H, \overline{Q}_H] = 2i\widetilde{\mathcal{H}} \tag{9}$$

If we enlarge the base space, including two grassmannian partners of time, θ and $\bar{\theta}$, we can put together all the variables that appear in (3) into the following superfield: $\Phi^a = \varphi^a + \theta c^a + \bar{\theta}\omega^{ab}\bar{c}_b + i\bar{\theta}\theta\omega^{ab}\lambda_b$. This superfield allows us to connect the two hamiltonians H and $\widetilde{\mathcal{H}}$ via the relation $\widetilde{\mathcal{H}} = i\int d\theta d\bar{\theta} H[\Phi]$ and to represent the supersymmetry charges as the following operators acting on the superspace $(t,\theta,\bar{\theta})$: $\widehat{Q}_{H} = -\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial t}$, $\widehat{\overline{Q}}_{H} = \frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial t}$. This is an N=2 supersymmetry. In fact we could combine the $Q_{BRS}, \overline{Q}_{BRS}, N_H, \overline{N}_H$ into the following two charges $Q_{(1)}, Q_{(2)}$:

$$\begin{aligned} &Q_{(1)} \equiv Q_{BRS} - \overline{N}_H, \quad Q_{(2)} \equiv \overline{Q}_{BRS} + N_H \\ &[Q_{(1)}, Q_{(2)}] = 0 \end{aligned} \tag{10}$$

and prove that

$$Q_{(1)}^2 = Q_{(2)}^2 = -i\widetilde{\mathcal{H}} \tag{11}$$

As the Q_{BRS} of (6) is basically the exterior derivative on phase space, it would be nice to understand the geometrical meaning also of the susy charges like $Q_{(1)}$ or $Q_{(2)}$. This was done in ref. [5]. The strategy used there was of making local the susy associated to $Q_{(1)}$. The Lagrangian with this local invariance is

$$\widetilde{\mathcal{L}}_{EQ} := \widetilde{\mathcal{L}} + \alpha(t)Q_{(1)} + g(t)\widetilde{\mathcal{H}} \tag{12}$$

where $\alpha(t)$ and $g(t)$ are gauge fields. The physical-state conditions associated to this gauge invariance turns out to be:

$$\begin{aligned} &\widetilde{\mathcal{H}}|\mathrm{phys}\rangle = 0 \\ &Q_{(1)}|\mathrm{phys}\rangle = 0 \\ &\Pi_g|\mathrm{phys}\rangle = 0 \\ &\Pi_\alpha|\mathrm{phys}\rangle = 0 \end{aligned} \tag{13}$$

where Π_g and Π_α are the momenta associated to the gauge variables α, g. Using the correspondence (4)-(7) it is easy to translate (13) into a differential-geometric language and prove that the states selected by (13) are in one-to-one correspondence with the states of the so-called equivariant cohomology [6] with respect to the Hamiltonian vector field. The equivariant cohomology w.r.t. a vector field V is defined as the set of forms $|\rho\rangle$ which satisfy the following conditions:

$$\begin{aligned} &(d - \iota_V)|\rho\rangle = 0 \\ &\mathcal{L}_V|\rho\rangle = 0 \\ &|\rho\rangle \neq (d - \iota_V)|\chi\rangle \\ &\mathcal{L}_V|\chi\rangle = 0 \end{aligned} \tag{14}$$

This is the geometrical light we could throw on the susy charge $Q_{(1)}$. Our universal symmetries, besides having a nice geometrical interpretation, should also have a *dynamical* meaning. This is the case for the susy invariance which seems to have some interplay with the concept of ergodicity [5]-[7]. We will not expand on it here but turn to another aspect of this susy.

Supersymmetry has found its most important applications in field theory where it has produced theories which have a better ultraviolet behaviour than non supersymmetric ones. With that in mind in ref. [9] an attempt was made to build the analog of $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{L}}$ of eq. (3) also for field theory. Starting for example from the Hamiltonian of a φ^4 theory $\mathcal{H}_{\varphi^4} = \int d^3x\ \{\frac{1}{2}\Pi_\varphi^2 + \frac{1}{2}(\partial_k\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4!}g\varphi^4\}$ the associated $\widetilde{\mathcal{L}}$ is

$$\widetilde{\mathcal{L}}_{\varphi^4} = \int \{\Lambda_a(\dot{\xi}^a - \omega^{ab}\delta_b H_{\varphi^4}) + i\overline{\Gamma}_a(\partial_t\delta_b^a - \omega^{ac}\delta_c\delta_b H_{\varphi^4})\Gamma^b\}d^3x \tag{15}$$

where ξ^a are made of (φ, Π_φ) and $\Lambda_a(\vec{x},t), \Gamma^a(\vec{x},t), \overline{\Gamma}_a(\vec{x},t)$ are the fields analogous to the point-particle variables $\lambda_a, c^a, \bar{c}_a$ but, differently from these ones, they do not depend only on t but also on $\vec{x}$. Like for the point particle, $\widetilde{\mathcal{L}}_{\varphi^4}$ has an N=2 susy whose charges are:

$$\begin{aligned} Q_{H} &= \int d^3x(i\Gamma^a\Lambda_a - \Gamma^a\delta_a H_{\varphi^4}) \\ \overline{Q}_{H} &= \int d^3x(i\overline{\Gamma}_a\omega^{ab}\Lambda_b + \overline{\Gamma}_a\omega^{ab}\delta_b H_{\varphi^4}) \end{aligned} \tag{16}$$

The same construction can be done for any field theory and even for gauge theories. For example the standard BFV Hamiltonian for Yang-Mills theories is [8]:

$$\begin{aligned} H_{BFV} = \int d^3x\bigg\{ & \frac{1}{2}\pi_a^k\pi_k^a + \frac{1}{4}F_a^{ij}F_{ij}^a + \pi_a\partial^k A_k^a - \lambda^a\partial_k\pi_a^k + \lambda^a C^b_{\ ac}\pi_b^k A_k^c \\ & + i\overline{P}_a P^a - \lambda^a\overline{P}_b C^b_{\ ac}C^c - i\overline{C}_a\partial^k(\partial_k C^a + C^a_{\ bc}A_k^c C^b)\bigg\} \end{aligned} \tag{17}$$

where A_a^k are the gauge fields, π_k^a are their conjugate momenta, F_{ij}^a is the antysimmetric tensor, λ^a is a Lagrange multiplier and π_a its conjugate momentum, $(-iP^a, C^a)$ are the BFV ghosts of the theory and $(i\overline{C}_a, \overline{P}_a)$ the BFV anti-ghosts. If we indicate with ξ^A all the fields of the theory above, including the BFV ghosts, we have that the associated $\widetilde{\mathcal{H}}$ is [9]:

$$\widetilde{\mathcal{H}}_{BFV} = \int d^3x\{\Lambda_A\omega^{AB}\vec{\delta}_B H_{BFV}(\xi) + i\bar{\Gamma}_A\omega^{AC}\ \overrightarrow{\delta}_C\ H_{BFV}(\xi)\ \overleftarrow{\delta}_B\ \Gamma^B\} \tag{18}$$

where $\Lambda_A, \Gamma^A, \overline{\Gamma}_A$ are auxiliary fields. Λ has the same grassmannian parity as the field ξ to which it refers, while Γ and $\overline{\Gamma}$ have opposite grassmannian parity. It is easy to show that there are conserved BRS and anti-BRS charges of the form: $Q = i\int d^3x\Gamma^A\Lambda_A$ and $\overline{Q} = -i\int d^3x\Lambda_A\omega^{AB}\overline{\Gamma}_B$. The Hamiltonian of eq. (18) can be written as a pure BRS variation in the following way:

$$\widetilde{\mathcal{H}}_{BFV} = -i[Q, [\overline{Q}, H_{BFV}]] \tag{19}$$

This is not a surprise because it is a property of any Lie-derivative. It makes this $\widetilde{\mathcal{H}}_{BFV}$ strongly similar to the hamiltonians of Topological Field Theories

[8]. The susy charges $Q_{_H}$ and $\overline{Q}_{_H}$ have the same form as the ones of the φ^4 theory, except for the presence of some grading factors. They are:

$$
\begin{aligned}
Q_{_H} &= \int d^3x(i\Gamma^A\Lambda_A - (-)^{[\xi^A]}\Gamma^A \overrightarrow{\delta}_A H_{_{BFV}}) \\
\overline{Q}_{_H} &= \int d^3x(-i\Lambda_A\omega^{AB}\overline{\Gamma}_B + \overline{\Gamma}_A\omega^{AB} \overrightarrow{\delta}_B H_{_{BFV}})
\end{aligned}
\tag{20}
$$

where we indicate with $[\xi^A]$ the grassmannian parity of the field ξ^A. Their anticommutator produces the Hamiltonian (18): $[Q_{_H}, \overline{Q}_{_H}] = 2i\int d^3x\widetilde{\mathcal{H}}_{_{BFV}}$. The shortcoming of all this is that we have obtained a *non-relativistic* susy even from a *relativistic* field theory. We feel anyhow that it should be possible to get a relativistic one. The strategy should be to start *not* from the Hamiltonian formalism but from an explicitly Lorentz covariant one like the DeDonder-Weyl approach [10]. The Hamiltonian formalism gives a special role to time and spoils the manifest Lorentz covariance. This special role of time is what produces a non-relativistic susy in our formalism. If we succeed in getting a relativistic susy with our mechanism we can say that somehow *susy is everywhere*, even associated to a non-susy theory like a φ^4-theory. We haven't seen this susy before because we haven't considered all the other geometrical fields (forms and vector fields) which are naturally associated with the basic fields φ. The susy appeared only when we did things in a coordinate indipendent fashion as the Lie-derivative does.

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