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Diagrammar In Classical Scalar Field Theory

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Abstract: In this paper we analyze perturbatively a $g\phi^4$ *classical* field theory with and without temperature. In order to do that, we make use of a path-integral approach developed some time ago for *classical* theories. It turns out that the diagrams appearing at the classical level are many more than at the quantum level due to the presence of extra auxiliary fields in the classical formalism. We shall show that several of those diagrams cancel against each other due to a universal supersymmetry present in the classical path integral mentioned above. The same supersymmetry allows the introduction of super-fields and super-diagrams which considerably simplify the calculations and make the *classical* perturbative calculations almost "identical" formally to the quantum ones. Using the super-diagrams technique we develop the classical perturbation theory up to third order. We conclude the paper with a perturbative check of the fluctuation-dissipation theorem.

Keywords: Classical Field Theory, Feynman Diagrams.

Contents

1. Introduction

Before starting this paper we would like to apologize with the authors of the famous paper [1] who first made use of the ancient-italian word "*diagrammar*" for having copied their idea. We used that same word because we feel "diagrammar" express in the most complete way the skills one has to develop to draw all possible Feynman diagrams of a *classical* theory which are many more than the quantum counterpart. We also think that the word express in a nice way the feeling we had when we realized that the very many classical diagrams could be encapsulated in few *super-diagrams*.

Let us go back to the paper. In the last fifteen years the physics of heavy ion collisions has become one of the hottest topics in high energy physics [2]. The most interesting point is that this may lead to the formation of state of matter known as quark-gluon plasma

(QGP)(for a review see ref. [3]). This topic has attracted the attention of many sectors of theoretical physics, last but not least, even of strings using the duality AdS/QCD (for example see ref. [4] and references therein).

We know that in the first instants after the (QGP) formation, the gluon occupation numbers is going to be very high [5], so one could describe the system by a classical field theory [6], analogously to what happens to QED which, when the photon occupation number is high, can be described to a good degree of accuracy by the Maxwell equations. A lot of work started in the nineties in this direction [7, 8, 9, 10]. Most of these authors studied classical field theories by just solving the Hamilton's equations of motion and developing from there perturbative calculations. Only the author of ref. [10] used a different route making use of a path-integral approach for *classical* mechanics developed in the eighties [11]. This is basically the functional counterpart of the operatorial approach to classical mechanics developed in the thirties by Koopman and von Neumann [12]. We know that the path-integral is the most natural tool to use in order to develop the perturbative calculations and the associated Feynman diagrams. It was so for the quantum path-integral (QPI) [13] and we will show that it will be the same for classical systems via the classical path-integral (CPI). In this paper, like in ref. [7] and Ref. [10], we will limit ourselves to a $g\phi^4$ theory.

The paper is organized as follows. In sec. 2 we will briefly review the CPI in order to make this paper self-contained. We shall show that, besides the phase-space variables, this formalism makes use of extra auxiliary fields. These have a well know physical meaning, being associated with the Jacobi fields [14] of the theory plus its symplectic conjugates and the response fields [16]. The presence of all these fields make the diagrammatic in the CPI rather complicated. In sec. 3 we develop the formalism in the case of a field theory without temperature and present the associated diagrams which, for a point particle, have already been worked out in Ref. [17]. In sec. 4 we develop the formalism in the presence of temperature and make contact with the calculations developed in Ref. [7] and Ref. [10]. We will notice that there are cancelations among various diagrams. This cancelation is due to some hidden symmetries present in the CPI. One of these is a *supersymmetry* and this leads to the use of super-fields (for a review see Ref. [18]) and their associated super-diagrams which we will introduce in sec. 5. In the same section, we will show how the super-diagrams will allow us to reproduce the results of sec. 4, but with fewer diagrams. These super-diagrams have the same vertices as in the quantum case. This is a consequence of the relation between the CPI and the QPI which was studied in Ref. [19]. In sec. 6 we conclude the paper with a perturbative analysis of the fluctuation-dissipation theorem [16] using the super-diagrams developed in sec. 5. We confine the detailed derivation of several results to few appendices.

2. The Path-Integral for Classical Mechanics (CPI).

In the thirties Koopman and von Neumann [12] developed an Hilbert space and operatorial approach to classical statistical mechanics. Their formalism is based on four postulates:

I. A state is given by an element $|\psi\rangle$ of an Hilbert space.

II. On this Hilbert space the operators $\hat{q}$ and $\hat{p}$ associated to the classical variables q and p commutes, i.e. $[\hat{q},\hat{p}]=0$. If we indicate with $\varphi^a=(q,p)$, $a=1,2$, then the simultaneous diagonalization of $\hat{q}$ and $\hat{p}$ can be expressed as

$$\hat{\varphi}^a|\varphi_0^a\rangle=\varphi^a|\varphi_0^a\rangle.$$

III. The evolution of $\psi(q,p)$ is given by the Liouville equation

$$i\frac{\partial\psi(q,p,t)}{\partial t}=\hat{\tilde{\mathcal{H}}}\psi(q,p,t), \tag{2.1}$$

where $\hat{\tilde{\mathcal{H}}}\equiv -i\partial_pH(q,p)\partial_q+i\partial_qH(q,p)\partial_p$ is the Hamiltonian operator.

IV. The Liouville probability density $\rho(q,p)$ is given by

$$\rho(q,p)=|\psi(q,p)|^2. \tag{2.2}$$

As a consequence of the four postulates above also ρ satisfies the Liouville equation as it should be.

As this is an operatorial formalism there should be an associated path-integral and, in fact, it is easy to build one [11]. Let us start from the classical transition amplitude $\langle\varphi^a,t|\varphi_0^a,t_0\rangle$ which clearly is

$$\langle\varphi^a,t|\varphi_0^a,t_0\rangle=\tilde{\delta}[\varphi^a-\phi_{\rm cl}^a(t;\varphi_0,t_0)] \tag{2.3}$$

where $\phi_{\rm cl}$ is the solution at time t of the Hamiltonian equation of motion

$$\dot{\varphi}^a=\omega^{ab}\frac{\partial H}{\partial\varphi^b} \tag{2.4}$$

with initial configuration φ_0 at time t_0. In Eq. (2.4) ω^{ab} is the symplectic matrix [25] and H the Hamiltonian of the system.

Slicing the interval of time in Eq. (2.3) in N small intervals and replacing the functional Dirac delta on the r.h.s of Eq. (2.3) with the expression

$$\widetilde{\delta}[\varphi^a-\phi_{cl}^a(t;\varphi_0,t_0)]=\widetilde{\delta}[\dot{\varphi}^a-\omega^{ab}\partial_bH]\det(\delta_b^a\partial_t-\omega^{ad}\partial_d\partial_bH), \tag{2.5}$$

where det[ ] is a properly regularized functional determinant, we get that Eq. (2.3) can be written as

$$\langle\varphi^a,t|\varphi_0^a,t_0\rangle=\int\mathscr{D}''\varphi^a\mathscr{D}\lambda_a\mathscr{D}c^a\mathscr{D}\bar{c}_a\exp\left[i\int_{t_0}^t d\tau\widetilde{\mathcal{L}}\right], \tag{2.6}$$

where $\mathscr{D}''\varphi$ indicates the integration over all points, except the end points which are fixed, while the symbol $\mathscr{D}$ indicates that the end points are integrated over. The $\widetilde{\mathcal{L}}$ is

$$\widetilde{\mathcal{L}}=\lambda_a\dot{\varphi}^a+i\bar{c}_a\dot{c}^a-\lambda_a\omega^{ab}\partial_bH-i\bar{c}_a\omega^{ad}\partial_d\partial_bHc^b. \tag{2.7}$$

The auxiliar variables λ_a have been introduced to do the Fourier transform of the Dirac delta on the r.h.s of Eq. (2.5), while the Grassmanninan variables c^a and $\bar{c}_a$ are needed to exponentiate the determinant in Eq. (2.5).

In Ref. [11] it was erroneously written that the determinant above is one. This is not so, but it depends on the regularization used to calculate it. If we discretize the time and do the same for the matrix associated to the determinant, then this turns out to be one, or a constant, if we use the the *Itô calculus* [20] or pre-point discretization [21], while it turns out to be non-constant and dependent on the ghost fields if we use the *Stratonovich calculus* [22] or mid-point discretization. It is only with this discretization that the Feynman rules which we derive from the continuum theory are going to be the same as those in the discretized case. This was shown in details in [23]. If we had used the Itô calculus in which the determinant is constant and could be disregarded, then the Feynman rules derived from the continuous version of the non-Grassmannian part could, in principle, not be the same as those derivable from the discretized version. This is the reason we use the Stratonovich prescription in which discretized and continuum Feynman rules coincide. In this prescription, the determinant depends on the fields and so there is a true interaction between the Grassmannian variables c^a, $\bar{c}_a$ and φ^a. The vertex coming from this interaction has to be considered in order to get the right results. This is not the only reason to keep the Grassmannian variables c^a and $\bar{c}_a$. Physically, they are the Jacobi fields and their correlations give the Lyapunov exponents [24]. These quantities are related to the matrix elements of the determinant above and to get these elements we need to couple c^a, $\bar{c}_a$ with external currents as explained in ref. [11].

Let us now go back to Eq. (2.6) and build the equivalent operatorial formalism. Given two variables O_1 and O_2 from the path-integral (2.6) we can derive the commutators defined as follows [13, 14]

$$\left[\hat{O}_1, \hat{O}_2\right] = \lim_{\epsilon\to 0}\langle O_1(t-\epsilon)O_2(t) - O_2(t-\epsilon)O_1(t)\rangle. \tag{2.8}$$

The symbol $\langle\rangle$ means the average under the path-integral and sandwiched among any state. We have put a hat $\hat{}$ on the variables on the l.h.s. of Eq. (2.8) because they become operator via this definition. Applying (2.8) to the basic variables φ^a, λ_a, c^a, $\bar{c}_a$ we get:

$$\left[\hat{\varphi}^a, \hat{\lambda}_b\right] = i\delta^a_b\,, \quad \left[\hat{c}^a, \hat{\bar{c}}_b\right]_+ = \delta^a_b, \tag{2.9}$$

where by $[\ ,\ ]_+$ we indicate the anti-commutators. All the other commutators and anti-commutators are zero in Eq. (2.9). In particular, $[\hat{\varphi}^a, \hat{\varphi}^b] = 0$, which implies $[\hat{q}, \hat{p}] = 0$, and this confirms that we are doing classical mechanics. From the first equation in (2.9) we get that a representation for λ_a is

$$\hat{\lambda}_a = -i\frac{\partial}{\partial\varphi^a} \tag{2.10}$$

and this identifies $\hat{\lambda}_a$ with the response field (see for example Ref. [16]).

Let us first write down the Hamiltonian associated to the Lagrangian $\tilde{\mathcal{L}}$

$$\widetilde{\mathcal{H}} = \lambda_a\omega^{ab}\partial_b H + i\bar{c}_a\omega^{ad}\partial_d\partial_b H c^b. \tag{2.11}$$

Having obtained our operatorial formalism let us see if we can recover the KvN approach. To do that, let us consider the first piece of $\widetilde{\mathcal{H}}$, which we call Bosonic one (B)

$$\widetilde{\mathcal{H}}_B = \lambda_a\omega^{ab}\partial_b H.$$

Using Eq. (2.10) we get that $\tilde{\mathcal{H}}$ goes into the following operator

$$\begin{aligned} \tilde{\mathcal{H}}_B \to \hat{\tilde{\mathcal{H}}}_B &= -i\omega^{ab}\partial_b H \frac{\partial}{\partial\varphi^a} \\ &= -i\partial_p H\partial_q + i\partial_q H\partial_p \equiv \hat{L} \end{aligned} \tag{2.12}$$

and so we obtain the Liouville operator. This confirms that, at least for the Bosonic part, the path integral behind the KvN formalism is exactly the one presented in (2.6). As we said after the KvN postulates, both the elements of the KvN Hilbert space $\psi(\varphi)$ and the probability densities $\rho(\varphi)$ evolve via the same operator $\tilde{\mathcal{H}}$. As a consequence also the path-integral for the evolution of both ψ and ρ is the same. In this paper, as was done in the first papers [11], we will consider the averages taken with the probabilities densities $\rho(\varphi)$. The reason for this choice is due to the fact, as it has been proven in Ref. [15], the Hilbert space of KvN is actually made of just Dirac deltas on the phase-space points: $\delta(\varphi-\varphi_0)=\langle\varphi|\varphi_0\rangle$, and the superposition principle does not act on these states. This is natural in CM. So saying that we use a Hilbert space formalism in CM is a rather "formal "statement, and for this reason we prefer to use the probability densities.

Going now back to $\tilde{\mathcal{H}}$ the curious reader may ask what is the mathematical meaning of $\tilde{\mathcal{H}}$ if we keep also the Grassmannian variables as in (2.11). Actually it has been proved in ref. [11] that it becomes a generalization of the Liouville operator know in the literature [25] as the Lie derivative of the Hamiltonian flow. This is easily proved once one realizes that the c^a behaves under symplectic transformation as basis of differential forms of φ^a, while the $\bar{c}_a$ are the symplectic duals [25] to c^a or, in other words, the basis of the totally anti-symmetric tensors. All the geometrical aspects of the path-integral for classical mechanics have been analyzed in details in [11, 26] . In the second reference it has been proved that the set of variables φ^a, c^a, $\bar{c}_a$, λ_a "coordinatize "a double-bundle over the phase-space [25]. The whole Cartan calculus [25, 26] can be reproduced using these variables and a set of seven charges present in this formalism [11]. Among those charges we will mention just two which are important for the rest of the paper

$$\begin{aligned} \hat{Q}_H &= i\hat{c}^a\hat{\lambda}_a - \hat{c}^a\partial_a H \\ \hat{\bar{Q}}_H &= i\hat{\bar{c}}_a\omega^{ab}\hat{\lambda}_b + \hat{\bar{c}}_a\omega^{ab}\partial_b H. \end{aligned} \tag{2.13}$$

They make up a universal $N=2$ *supersymmetry* for any Hamiltonian system. In fact, they close on the Hamiltonian

$$\left[\hat{Q}_H, \hat{\bar{Q}}_H\right] = 2i\hat{\tilde{\mathcal{H}}}. \tag{2.14}$$

We will see that, thanks to this supersymmetry, there will be many cancelations among Feynman diagrams once we embark on the perturbation theory. The geometrical meaning of these super-charges has also been investigated: they are basically the exterior derivative on the constant-energy surfaces of the phase-space. Or, in a more mathematically technical language, they are associated to the equivariant cohomology of the Hamiltonian field [27].

In this paper we will make use of another tool typical of supersymmetry which is the one of *super-fields* (for a review see ref. [18]). This is an object which allow us to put in the same multiplet all the variables φ^a, λ_a, c^a and $\bar{c}_a$. To do that we have first to extend time t to two Grassmannian partners of time θ, $\bar{\theta}$. The super-field $\Phi^a(t,\theta,\bar{\theta})$ is a function of t, θ, $\bar{\theta}$ defined as

$$\Phi^a(t,\theta,\bar{\theta}) \equiv \varphi^a + \theta c^a + \bar{\theta}\omega^{ab}\bar{c}_b + i\theta\bar{\theta}\omega^{ab}\lambda_b. \tag{2.15}$$

It is then possible to give a much simpler expression for the Lagrangian in $\tilde{\mathcal{L}}$ (2.7). In fact, it is easy [19] to prove that

$$\tilde{\mathcal{L}} = i\int d\theta d\bar{\theta} L\left[\Phi^a\right] + (s.t), \tag{2.16}$$

where L is the standard Lagrangian associated to the Hamiltonian H of classical mechanics which we have used in the r.h.s of Eq. (2.7) and $(s.t.)$ is a surface term.

Using the super-field it is also possible to give a generalization of expression (2.6), that is the following (for its derivation see Ref. [19]),

$$\langle \Phi^a, t|\Phi_0^a, t_0\rangle = \int \mathscr{D}''\Phi^q\mathscr{D}\Phi^p \exp\left\{i\int_{t_0}^t id\tau d\theta d\bar{\theta} L\left[\Phi\right]\right\}. \tag{2.17}$$

The indeces p and q indicate the first and second of the indices of φ^a . They are called this way because we know that $\varphi^1 = q$ and $\varphi^2 = p$.

An expression similar to (2.17) can also be written for the generating functional, which instead of being the complicated object:

$$Z[J=0] = \int \mathscr{D}\phi^a\mathscr{D}\lambda_a\mathscr{D}c^a\mathscr{D}\bar{c}_a \exp\left\{i\int_{t_0}^t d\tau\tilde{\mathcal{L}}\right\} \tag{2.18}$$

can be written as:

$$Z[J=0] = \int \mathscr{D}\Phi^a \exp\left\{i\int_{t_0}^t d\tau \int id\theta d\bar{\theta} L\left[\Phi\right] + (s.t.)\right\}. \tag{2.19}$$

Note how similar is (2.19) to the quantum analogue:

$$Z_{QM}[J=0] = \int \mathscr{D}\varphi^a \exp\left\{\frac{i}{\hbar}\int_{t_0}^t d\tau L\left[\varphi\right]\right\}. \tag{2.20}$$

Basically φ^a is replaced by Φ^a and $1/\hbar$ by $i\int d\theta d\bar{\theta}$ while the functional weight is given in both cases by the standard Lagrangian L of classical mechanics.

The similarity between Eq. (2.19) and Eq. (2.20) has been studied in great details in Ref. [19]. In this paper this similarity will be used to drastically simplify the perturbation theory which, instead of making use of the complicated Lagrangian $\tilde{\mathcal{L}}$ of (2.7) appearing in Eq. (2.18), will make use of just L appearing in Eq. (2.19) borrowing many results from the quantum analog (2.20).

Before closing this section we would like to bring to the attention of the reader the physics behind the variables c^a. We do that because people may consider them useless. We have already made hints before to their mathematical meaning, being that of basis for differential forms (more details can be found in ref.[26]). Their physical meaning has been explored in Ref. [24] and it goes as follows. Let us look at the equation of motion for c^a which can be derived from the $\tilde{\mathcal{L}}$ of Eq. (2.7)

$$\dot{c}^a - \omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^d}c^d = 0. \tag{2.21}$$

This equation is the same as the one satisfied by the first variations $\delta\varphi^a$:

$$\delta\dot{\varphi}^a - \omega^{ab}\frac{\partial^2 H}{\partial\varphi^b\partial\varphi^d}\delta\varphi^d = 0. \tag{2.22}$$

These first variations are also known as Jacobi fields. (See for example Ref. [14].) So we can identify $c^a \approx \delta\varphi^a$. The correlations of the Jacobi fields are related to the Lyapunov exponents of chaotic systems:

$$\lim_{t\to\infty}\langle\delta\varphi(0)\delta\varphi(t)\rangle \sim e^{-\lambda t}$$

where (λ) is the highest Lyapunov exponent of the system. This is equivalent to saying that the correlations of $c^a, \bar{c}_a$ gives the Lyapunov exponents[24] So we see that the $c^a, \bar{c}_a$ are crucial ingredients to get a full information on the dynamical system [24] we are studying. We talk about correlations between c^a and $\bar{c}_a$ because the correlations among only c^a gives zero[33][24][34]. The $\bar{c}_a$ are just their symplectic dual and, once they are multiplied by the symplectic matrix, they can also be identified with the Jacobi fields. So the correlations among $c^a, \bar{c}_a$ are also equivalent to correlations among jacobi fields. Of course, the Lyapunov exponents could be obtained in many other ways [28] without using the $c^a, \bar{c}_a$, but we find this manner rather elegant. Actually, we know that classical mechanics could be built by just using the φ^a, and so the variables λ_a, c^a and $\bar{c}_a$ are redundant variables and this redundancy is signaled by the symmetries present in $\tilde{\mathcal{L}}$ like the supersymmetry and other invariances. But, as usual in physics, the redundancies and the symmetries make things more elegant and allows the use of all the tools that group theory put at our disposal. In our case, if we had not used all the variables, we would not be able to build the super-field and use the expression (2.19) for the perturbation theory.

Before starting the next section we should introduce for completeness a new representation of the commutator algebra (2.9). More details can be found in Ref. [19]. This new representation will be very important in order to do perturbation theory using super-fields. We had seen that $\tilde{\mathcal{L}}$ in Eq. (2.16) and $L[\Phi]$ differ by a surface term $(s.t)$ and the same will happen at the level of generating functionals (2.17) and (2.19). In order to get rid of that surface term the trick [19] is to change the representation in (2.10) that we derived from the commutator (2.9). We could, for example, represent $\hat{q}$ and $\hat{\lambda}_p$ as multiplicative operators and $\hat{p}$ and $\hat{\lambda}_q$ as derivative ones

$$\begin{aligned}\hat{p} &= i\frac{\partial}{\partial\lambda_p}\\ \hat{\lambda}_q &= -i\frac{\partial}{\partial q}.\end{aligned} \tag{2.23}$$

Analogously we can proceed for the Grassmannian variables: represent $\hat{c}^q$ and $\hat{\bar{c}}_p$ as multiplicative operators and $\hat{c}^p$ and $\hat{\bar{c}}_q$ as derivative ones:

$$\begin{aligned}\hat{c}^p &= i\frac{\partial}{\partial \bar{c}_p}\\ \hat{\bar{c}}_q &= \frac{\partial}{\partial c^q}.\end{aligned} \tag{2.24}$$

The generalized states [19] in the extended Hilbert space, including the forms, will then be $\langle q, \lambda_p, c^q, \bar{c}_p|$. Note that the variables q, λ_p, c^q, $\bar{c}_p$ are exactly those which enter the super-field Φ^q

$$\Phi^q = q + \theta c^q + \bar{\theta}\bar{c}_p + i\theta\bar{\theta}\lambda_p. \tag{2.25}$$

This is crucial in order to give a complete super-field representation of our theory without any surface terms. We invite the reader to master the details contained in ref.[19] before embarking in the rest of this paper.

3. Perturbation Theory Without Temperature.

What we will do in this section for a $g\phi^4$ scalar field theory, has been partly been done for a point particle in Ref. [17]. We use the word "partly" because those authors neglected the Grassmannian variables c^a and $\bar{c}_a$. So their analysis apply only to the zero-form sector of the theory. We called this sector this way because, as indicated before, the c^a are the basis of the differential forms [25],[26].

Let us start with a scalar field theory whose Hamiltonian is

$$H = \int d^4x \left[\frac{\pi}{2} + \frac{(\nabla\phi)^2}{2} + \frac{m^2\phi^2}{2} + g\frac{\phi^4}{4!}\right] \tag{3.1}$$

where $\pi(x)$ is the momentum conjugate to the field $\phi(x)$. We will indicate with $\mathscr{H}$ the Hamiltonian density:

$$\mathscr{H}(x) = \frac{\pi(x)}{2} + \frac{[\nabla\phi(x)]^2}{2} + \frac{m^2\phi^2(x)}{2} + g\frac{\phi^4(x)}{4!} \tag{3.2}$$

Let us next build the generating functional associated to the classical path-integral of the previous section, but we shall choose only those trajectories that start from a fixed point in phase-space $\varphi_i = (\phi_i, \pi_i)$ and we will not average over this initial configuration. The expression is:

$$\begin{aligned}&\mathcal{Z}_{\varphi_i}[J_\phi, J_{\lambda^\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}] \equiv\\ &\equiv \int \mathscr{D}'\varphi\mathscr{D}\lambda\mathscr{D}c\mathscr{D}\bar{c}\exp\left\{i\tilde{\mathcal{S}} + i\int_{t_i}^{t_f} dt\int d^3x\left[J_\phi\phi + J_{\lambda_\pi}\lambda_\pi - i\bar{J}_{c^\phi}c^\phi - i\bar{c}_\pi J_{\bar{c}_\pi}\right]\right\}\end{aligned} \tag{3.3}$$

where

$$\tilde{\mathcal{S}} = \int_{t_i}^{t_f} dt\int d^3x\left(\lambda_a\dot{\varphi}^a + i\bar{c}_a\dot{c}^a - \lambda_a\omega^{ab}\partial_b H - i\bar{c}_a\omega^{ad}\partial_d\partial_b H c^b\right) \tag{3.4}$$

with the H given by Eq. (3.1) and the symbol $\mathscr{D}'$ indicates the "sum" over all the trajectories starting from a fixed φ_i and we will not integrate over this initial configuration. In the λ, c, $\bar{c}$ we have put as indices ϕ, π to indicate either the first or the second set of indices "a" as in postulate **II** of section 2. To the currents we have coupled only the "configuration variables" in the representation (2.25) where ϕ, λ_π, c^ϕ, $\bar{c}_\pi$ are multiplicative operators. We did that because, as it is clear from (2.25), it will be straightforward to pass to the super-field formulation. The peculiar combination of "$\pm$" and "i" in the coupling of current and fields (3.3) is also related to the fact that eventually we also want to build a supercurrent to couple it to the superfield.

As it is usually done, let us divide $\tilde{\mathcal{S}}$ into a free part:

$$\begin{aligned}\tilde{\mathcal{S}}_0 \equiv \int_{t_i}^{t_f} dt \int d^3x \Big[&\lambda_a\dot{\phi}^a - \lambda_\phi\pi - \lambda_\pi\left(\nabla^2 - m^2\right)\phi \\ &+ i\bar{c}_a\dot{c}^a - i\bar{c}_\phi c^\pi - i\bar{c}_\pi\left(\nabla^2 - m^2\right)c^\phi\Big]\end{aligned} \tag{3.5}$$

and an interaction one:

$$\tilde{\mathcal{S}}_V \equiv \int d^4x \left(\frac{g}{3!}\lambda_\pi\phi^3 + \frac{iq}{2!}\bar{c}_\pi\phi^2c^\phi\right), \tag{3.6}$$

where we indicate with $\int d^4x$ the expression $\int_{t_i}^{t_f} dt \int d^3x$. To develop the perturbation theory, as it is usually done in quantum field theory, in the $\tilde{\mathcal{S}}_V$ of $\mathcal{Z}$ we shall replace the fields with the derivative operators with respect to the associated currents as follows:

$$\phi \to \frac{1}{i}\frac{\delta}{\delta J_\phi},\ \lambda_\pi \to \frac{1}{i}\frac{\delta}{\delta J_{\lambda_\pi}},\ c^\phi \to \frac{\delta}{\delta \bar{J}_{c^\phi}},\ \bar{c}_\pi \to -\frac{\delta}{\delta J_{\bar{c}_\pi}} \tag{3.7}$$

so we get:

$$\tilde{\mathcal{S}}_V\left[\frac{1}{i}\frac{\delta}{\delta J_\phi}, \frac{1}{i}\frac{\delta}{\delta J_\phi}, \frac{\delta}{\delta \bar{J}_{c^\phi}}, -\frac{\delta}{\delta J_{\bar{c}_\pi}}\right] \equiv \int d^4x \left(\frac{g}{3!}\frac{\delta}{\delta J_{\lambda_\pi}}\frac{\delta^3}{\delta J_\phi^3} + \frac{ig}{2!}\frac{\delta}{\delta J_{\bar{c}_\pi}}\frac{\delta^2}{\delta J_\phi^2}\frac{\delta}{\delta \bar{J}_{c^\phi}}\right) \tag{3.8}$$

It is then easy to write $\mathcal{Z}_{\varphi_i}[J_\phi, J_{\lambda^\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}]$ in Eq. (3.3) as follows:

$$\mathcal{Z}_{\varphi_i}[J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}] = \exp\left\{i\tilde{\mathcal{S}}_V\left[\frac{1}{i}\frac{\delta}{\delta J_\phi}, \frac{1}{i}\frac{\delta}{\delta J_\phi}, \frac{\delta}{\delta \bar{J}_{c^\phi}}, -\frac{\delta}{\delta J_{\bar{c}_\pi}}\right]\right\}\mathcal{Z}^{(0)}_{\varphi_i}[J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}] \tag{3.9}$$

where $\mathcal{Z}^{(0)}_{\varphi_i}$ is the free generating functional built out of the $\tilde{\mathcal{S}}_0$. It is easy to prove that this part $\mathcal{Z}^{(0)}_{\varphi_i}$ can be factorized into a "Bosonic" (B) part for the variables φ and λ and a "Fermionic "one (F) for the Grassmannian variables c, $\bar{c}$:

$$\mathcal{Z}^{(0)}_{\varphi_i}[J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}] = \mathcal{Z}^{(0)}_{(B)\ \varphi_i}[J_\phi, J_{\lambda_\pi}]\mathcal{Z}^{(0)}_{(F)}[\bar{J}_{c^\phi}, J_{\bar{c}_\pi}] \tag{3.10}$$

where

$$\begin{aligned}&\mathcal{Z}^{(0)}_{(B)\ \varphi_i}[J_\phi, J_{\lambda_\pi}] \\ &= \int \mathscr{D}'\varphi\mathscr{D}\lambda \exp\left\{i\int_{t_i}^{t_f} dt \int d^3x \left[\lambda_a\dot{\phi}^a - \lambda_\phi\pi - \lambda_\pi\left(\nabla^2 - m^2\right)\phi + J_\phi\phi + J_{\lambda_\pi}\lambda_\pi\right]\right\}\end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
&\mathscr{Z}^{(0)}_{(F)}[\bar{J}_{c^\phi}, J_{\bar{c}_\pi}] \\
&= \int \mathscr{D}c\mathscr{D}\bar{c} \exp\left\{ \int_{t_i}^{t_f} dt \int d^3x \left[i\bar{c}_a \dot{c}^a - i\bar{c}_\phi c^\pi - i\bar{c}_\pi \left(\nabla^2 - m^2\right) c^\phi - i\bar{J}_{c^\phi} c^\phi - i\bar{c}_\pi J_{\bar{c}_\pi} \right] \right\}
\end{aligned} \tag{3.12}
$$

All the fields enter at most in a quadratic form. It is easy to integrate them out in Eq. (3.11) as it has been done in ref. [17]. The result is:

$$
\mathscr{Z}^{(0)}_{(B)\ \varphi_i}[J_\phi, J_{\lambda_\pi}] = \exp\left[i \int d^4x J_\phi(x)\phi_0(x) + i \int d^4x d^4x' J_\phi(x) G_R(x-x') J_{\lambda_\pi}(x') \right]. \tag{3.13}
$$

where $\phi_0(x)$ is a solution to the equation

$$
\left(\Box + m^2\right)\phi = 0 \tag{3.14}
$$

and G_R is the retarded propagator satisfying

$$
\left(\Box + m^2\right) G_R = -\delta(x). \tag{3.15}
$$

Its expression is:

$$
G_R(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip.x}}{(p^0 + i\epsilon)^2 - \vec{p}^2 - m^2}. \tag{3.16}
$$

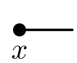
Also for the $\mathscr{Z}^{(0)}_{(F)}$ we can easily integrate out the fields (for details see Ref. [30])and obtain:

$$
\mathscr{Z}^{(0)}_{(F)}[\bar{J}_{c^\phi}, J_{\bar{c}_\pi}] = \exp\left[\int d^4x d^4x' \bar{J}_{c^\phi}(x) G_R(x-x') J_{\bar{c}_\pi}(x') \right]. \tag{3.17}
$$

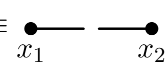
We have now all the tools to obtain all the propagators.

Let us start at the zero order in perturbation theory and we will draw next to them the diagrams:

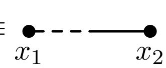
$$
\begin{aligned}
\langle \phi(x) \rangle^0 &= \frac{1}{i}\frac{\delta}{\delta J_\phi} \mathscr{Z}^{(0)}_{\varphi_i}[J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}]|_{J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}=0} \\
&= \tfrac{1}{i}\tfrac{\delta}{\delta J_\phi} \mathscr{Z}^{(0)}_{B\ \varphi_i}[J_\phi, J_{\lambda_\pi}]|_{J_\phi, J_{\lambda_\pi}=0} \\
&= \phi_0(x) \equiv
\end{aligned} \tag{3.18}
$$



$$
\langle \phi(x_2)\phi(x_1) \rangle^0 = \phi_0(x_2)\phi_0(x_1) \equiv \tag{3.19}
$$



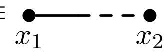
$$
\begin{aligned}
\langle \lambda_\pi(x_2)\phi(x_1) \rangle^0 &= \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J_{\lambda_\pi}(x_2)} \frac{\delta}{\delta J_\phi(x_1)} \mathscr{Z}^{(0)}_{\varphi_i}[J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}]|_{J_\phi, J_{\lambda_\pi}, \bar{J}_{c^\phi}, J_{\bar{c}_\pi}=0} \\
&= -iG_R(x_1 - x_2) = G_{\lambda_\pi \phi} \equiv
\end{aligned} \tag{3.20}
$$



In Eq. (3.20) we borrow from ref. [7]. the notation of full-dashed propagator . In Eq. (3.19) we have drawn a diagram made of two pieces not linked to each other, because they are actually the product of two separated fields and not a propagator. We will have them soldered to each other once we use the temperature to average over the initial conditions.

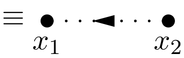
Analogously to Eq. (3.20) we can calculate

$$\langle \phi(x_2)\lambda_\pi(x_1)\rangle^0 = -iG_R(x_2-x_1) = G_{\phi\lambda_\pi} \equiv \tag{3.21}$$



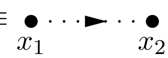
The correlations among Grassmannian variables give:

$$\begin{aligned}\langle \bar{c}_\pi(x_2)c^\phi(x_1)\rangle^0 &= -\frac{\delta}{\delta J_{\bar{c}_\pi}(x_2)}\frac{\delta}{\delta \bar{J}_{c^\phi}(x_1)}\mathcal{Z}_F^{(0)}[\bar{J}_{c^\phi},J_{\bar{c}_\pi}]|_{\bar{J}_{c^\phi},J_{\bar{c}_\pi}=0} \\ &= G_R(x_1-x_2) = G_{\bar{c}_\pi c^\phi} \equiv\end{aligned} \tag{3.22}$$



and

$$\begin{aligned}\langle c^\phi(x_2)\bar{c}_\pi(x_1)\rangle^0 &= \frac{\delta}{\delta \bar{J}_{c^\phi}(x_2)}\left[-\frac{\delta}{\delta J_{\bar{c}_\pi}(x_1)}\right]\mathcal{Z}_F^{(0)}[\bar{J}_{c^\phi},J_{\bar{c}_\pi}]|_{\bar{J}_{c^\phi},J_{\bar{c}_\pi}=0} \\ &= -G_R(x_2-x_1) = G_{c^\phi\bar{c}_\pi} \equiv\end{aligned} \tag{3.23}$$



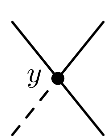
Above we have adopted the convention of putting an arrow which points from c to $\bar{c}$ (see Ref. [30]).

As both the propagators, of the $\phi\phi$ and $c\bar{c}$, are related to the G_R we will sometimes put an index "(F)" or "(B)" to indicate if it comes from the Bosonic fields or "Fermionic" (Grassmannian) ones. The fact that the two propagators are equal (modulo i) to each other is due to the supersymmetry present in this formalism [11]. We will see other manifestation of it later on in several cancelations among diagrams.

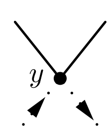
Let us now derive the rule for the vertices. Expanding $e^{i\tilde{\mathcal{S}}_V}$ in Eq. (3.9) to the first order in g, we get :

$$i\tilde{\mathcal{S}}_V\left[\frac{1}{i}\frac{\delta}{\delta J_\phi},\frac{1}{i}\frac{\delta}{\delta J_\phi},\frac{\delta}{\delta \bar{J}_{c^\phi}},-\frac{\delta}{\delta J_{\bar{c}_\pi}}\right] \equiv \int d^4x\left[ig\left(\frac{1}{3!}\frac{\delta}{\delta J_{\lambda_\pi}}\frac{\delta^3}{\delta J_\phi^3}\right) - g\left(\frac{1}{2!}\frac{\delta}{\delta J_{\bar{c}_\pi}}\frac{\delta^2}{\delta J_\phi^2}\frac{\delta}{\delta \bar{J}_{c^\phi}}\right)\right]. \tag{3.24}$$

Keeping account of the symmetry factors we get the following rules for the vertices :



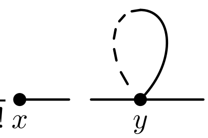
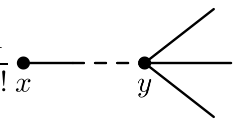
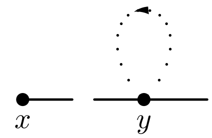
$$= i\,g\int d^4y, \qquad$$



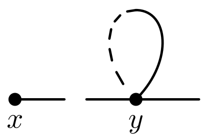
$$= -g\int d^4y \tag{3.25}$$

where the continuos line refer to the field ϕ, the dashed one to the field λ and the dotted one to c or $\bar{c}$.

Let us calculate the first order correction to the expectation value of the field $\langle\phi_0(x)\rangle$, i.e. $\langle\phi_0(x)\rangle$ (the super-index (1) on $\langle\phi\rangle$ indicates the first order correction). Using the vertices and the diagrams we get (where y is integrated over)

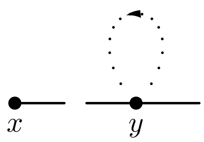
$$\langle\phi(x)\rangle^{(1)}_{\varphi_i} = \Bigg[\frac{1}{2!}$$  $$+ \frac{1}{3!}$$  $$+ \frac{1}{2!}$$  $$\Bigg] \tag{3.26}$$

The analytic expression of the first diagram above is



$$= ig \int d^4y \left[\phi_0^2(y) G_{\lambda_\pi \phi}(0)\phi_0(x)\right] \tag{3.27}$$

while the expression for the third one is

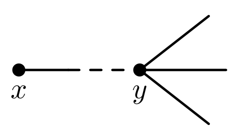


$$= g \int d^4y \left[\phi_0^2(y) G_{c^\phi \bar{c}_\pi}(0)\phi_0(x)\right]. \tag{3.28}$$

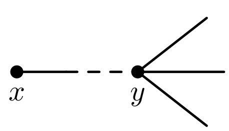
Let us remember from (3.21) and (3.23) that

$$G_{c^\phi \bar{c}_\pi} = -G_R = -iG_{\lambda_\pi \phi} \tag{3.29}$$

so (3.27) and (3.28) cancel each other and the only correction left in (3.26) is

$$\langle\phi\rangle^{(1)}_{\varphi_i} =$$  $$\tag{3.30}$$

whose analytic expression is



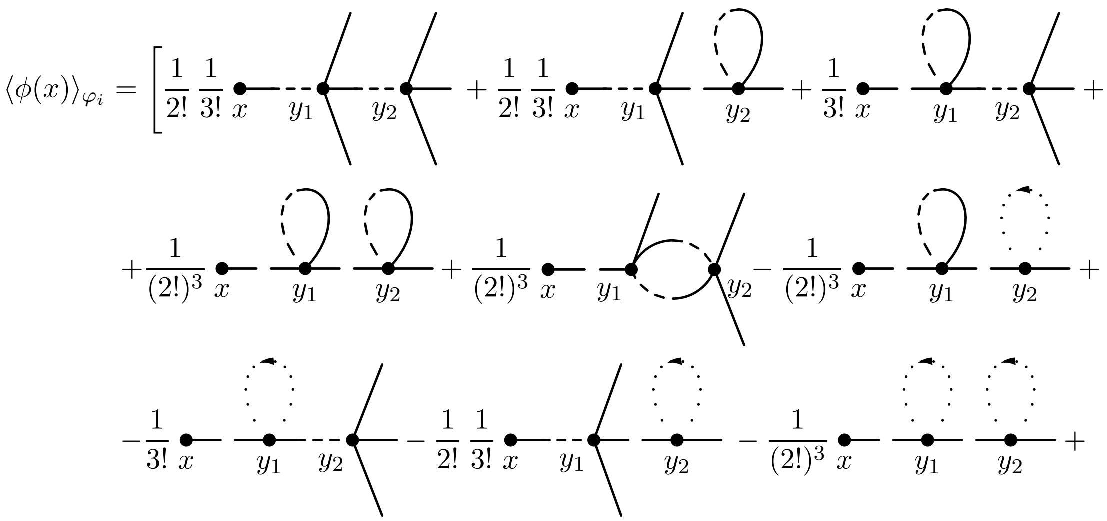
$$= i\,g \int d^4y \left[\phi_0^3(y) G_{\phi\lambda_\pi}(x-y)\right]. \tag{3.31}$$

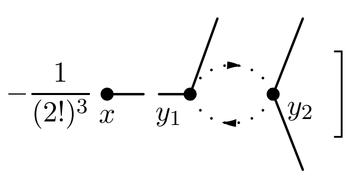
The reader could, at this point, claim that the loops of the retarded propagators appearing in Eq. (3.26) are both zero, because of the $\theta(t)$ which appears in all retarded propagators. For a loop to be zero we have anyhow to assume that $\theta(0) = 0$. This regularization of the θ corresponds to a prepoint choice in a discretized form of t [21]. The prepoint choice has some problems once it is used in the path-integral. It has in fact been proved in Ref. [23] that only if one uses the midpoint choice then the Feynman rules which can be read off the continuum version are the same as the one of the discretized version. So, for this reason, we have preferred the midpoint discretization which gives $\theta(0) = 1/2$

and this implies that the retarded loops are not zero. Let us remember also that it is with the midpoint rule that the determinant is not a constant and the Grassmannian variables are needed. So, our formalism has an overall logical coherence.

Going now back to our diagrams the reader may be puzzled by diagrams like (3.30) where there are just three lines at the end going nowhere or like the single line in (3.18) also going nowhere. This is due to the fact that our generating functional $\mathcal{Z}_{\varphi_i}$ depends on the initial configuration which is not averaged over. These kind of diagrams, but without Grassmannian variables, were first developed in [17] and are somehow similar to those developed for fluid dynamics in [31]. Perturbation theory and something similar to Feynman diagrams where also developed long ago in [32]. We have not tried to compare our formalism with this other one. For sure our formalism is different from these other ones because it contains extra ingredients like the Grassmaniann variables. Moreover, because of these extra ingredients, we will be able to develop the super-diagrams which will simplify the diagrammatics.

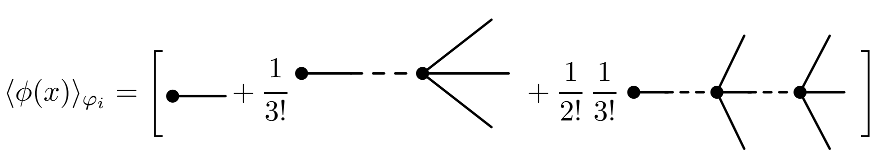
We will continue now to the second order for the $\langle\phi\rangle$. The result that can easily be obtained(where y_1 and y_2 are integrated over) is the following:





(3.32)

As before, it is easy to see that there are cancelations. The second diagram is cancelled by the eighth one, the third by the seventh, the fourth is cancelled by the sum of the sixth and ninth, the fifth one by the tenth. So the final result is



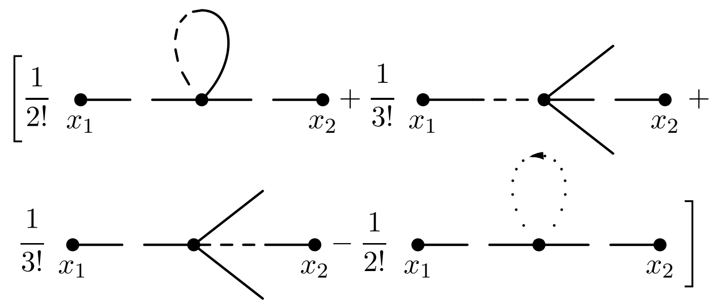
(3.33)

notice that no *loop* is left.

The various symmetry factors have being obtained by working out the analytical details from the $\mathcal{Z}_{\varphi_i}$ in (3.9) up to second order.

Let us now go back to the two-point functions which, to zero order, was given in Eq. (3.19). After some tedious calculations with the second derivative of $\mathcal{Z}_{\varphi_i}[J]$, we get:

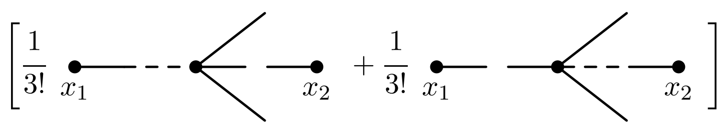
$$\langle\phi(x_2)\phi(x_1)\rangle^{(1)}_{\varphi_i} =$$



(3.34)

The first and the fourth diagram cancel each other and so the final result is

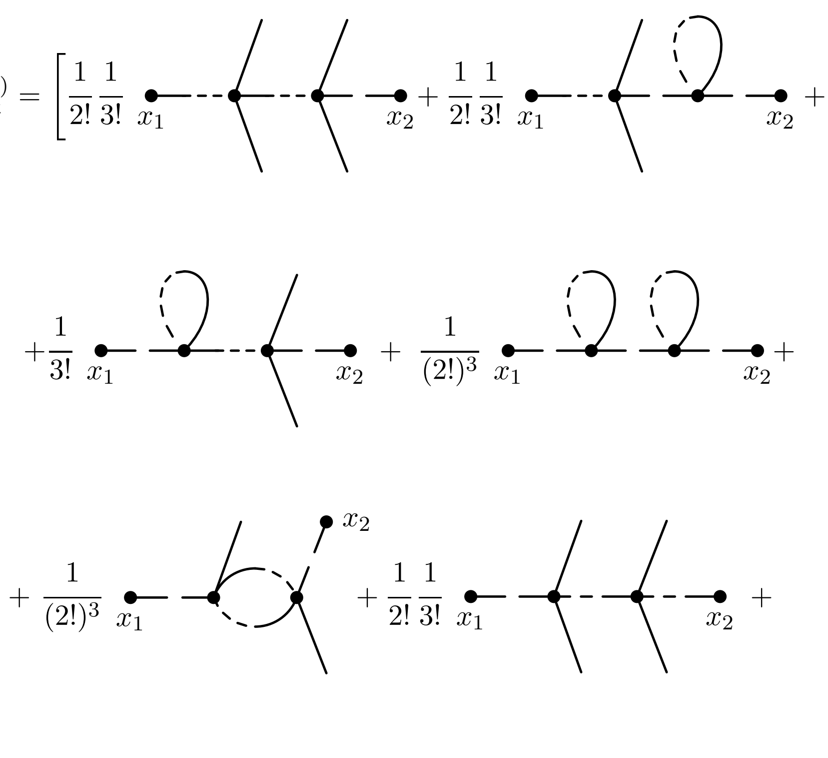
$$\langle\phi(x_2)\phi(x_1)\rangle^{(1)}_{\varphi_i} =$$

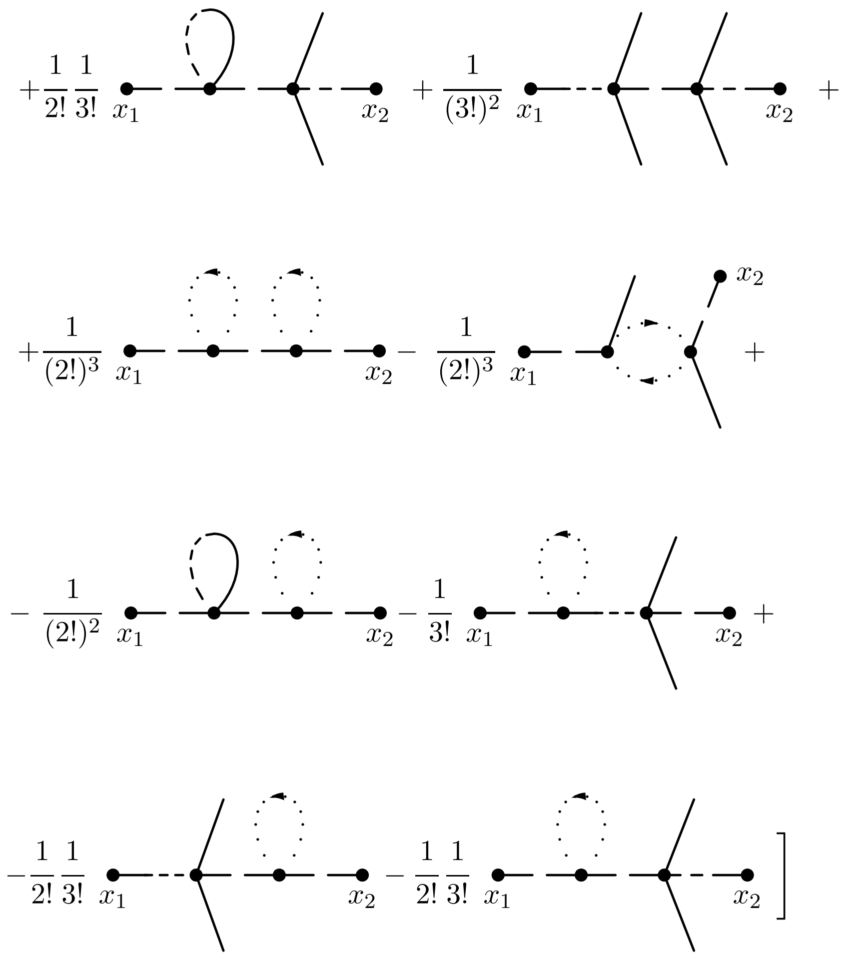


(3.35)

Going to the second order, after some long but straightforward calculations, we obtain:

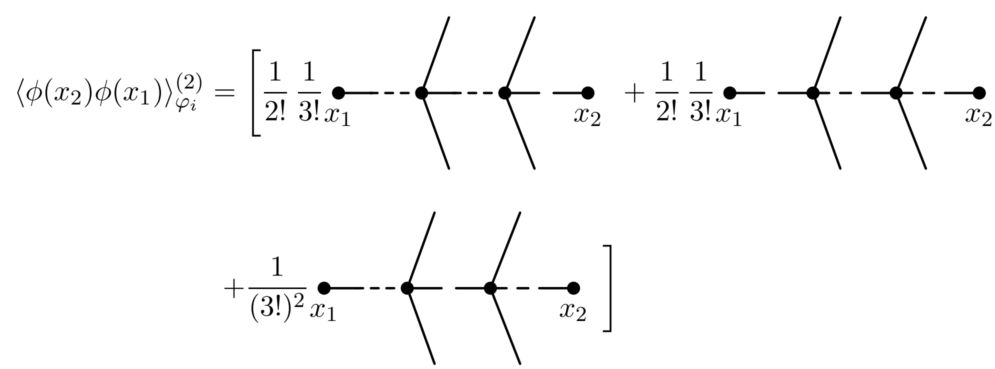
$$\langle\phi(x_2)\phi(x_1)\rangle^{(2)}_{\varphi_i} =$$





(3.36)

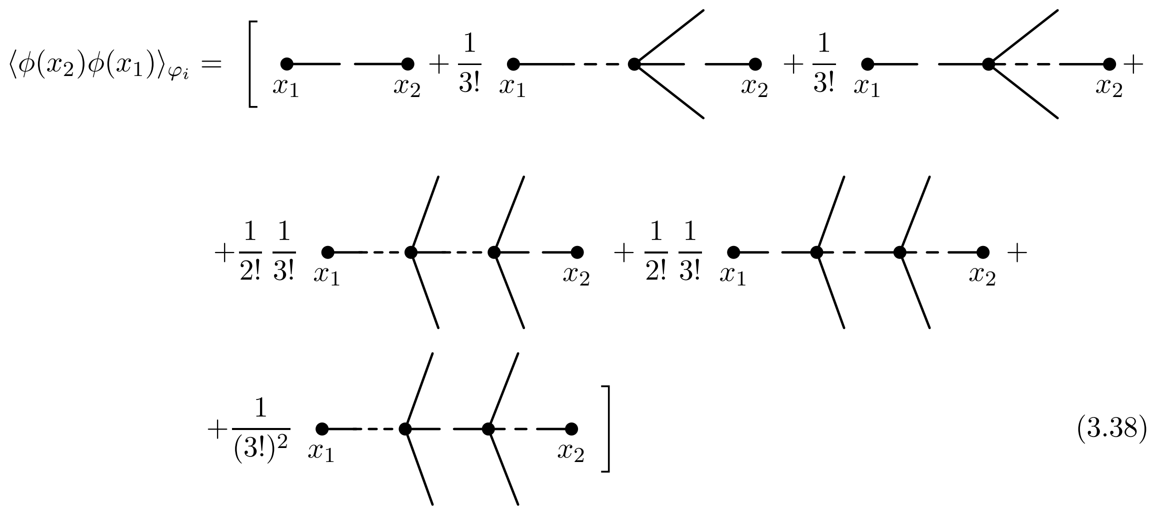
There are the usual cancelations between various diagrams and the final result is:



(3.37)

Note that also here no loop is left. This is the real sign that we are doing a *classical* perturbation theory. When we will introduce the temperature later on loops will appear, but they are due to temperature and not quantum effects. Actually, loops would appear any time we do an average over the initial conditions so we can not strictly say that no loops is a sign of classicality, we can say that no loop is the sign of a generating functional which has no average over the initial configurations.

Collecting now all diagrams developed before for the two-point functions ,up to second order, we get:



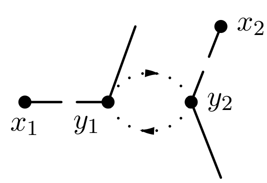
(3.38)

The rules for the symmetry factors are the following:

1) put a factor of $1/n!$ where n are the number of "free" legs, i.e. those not starting or ending at x_1 or x_2;

2) add a factor of $1/2!$ for any exchange of propagators which would leave the diagram invariant.

Before concluding this section we would like to answer to a question that for sure many readers may have. The question is why all the diagrams in this section are disconnected or have legs going nowhere. The reason is not related to the fact that we may have used the wrong generating functional, but to the presence of the diagram (3.18). To convince the reader, let us give the analytical expression for some of the most "strange" diagrams. For example, the 10th diagram in (3.36) where there are both disconnected pieces and lines going nowhere:



$$\sim \left[\int d^4y_1\, d^4y_2\, \phi_0(y_1)^2\, \phi_0(z_2)^2\, G_R^{(F)}(x_1 - x_2)\right.$$

$$\left. G_R^{(F)}(x_2 - x_1)\right]\, \phi_0(x_1)\, \phi_0(x_2)$$

We see that we have at x_1 and x_2 the fields $\phi_0(x_1)$ and $\phi_0(x_2)$ and their diagram is the one of (3.18). In y_1 and y_2 we also have two lines going nowhere and, according to (3.18) there must be two fields $\phi_0(y)$. The rest, between y_1 and y_2, are two "Fermionic" retarded propagators.

The reason we can not connect x_1 with y_1 or y_2 with x_2 with a continuous line is because we do not have this kind of propagator in the formalism with fixed initial condition. The only propagators we have are those in (3.20), (3.21), (3.23) and (3.22), which are the dash-full propagator or the Fermionic dot propagators. For the $\phi\phi$ we have only the disconnected propagators (3.19). The reader may think that by coupling in some way various dash-full propagators one could get a $\phi\phi$ propagator. If one tries, it is easy to convince himself that this is not possible.

In this chapter we have studied the "average" of a single field ϕ or correlations of two fields $\phi\phi$. It is possible to do the same for the λ_π fields and the c^ϕ, $\bar{c}_\pi$ ones. Instead of doing them explicitly we will work out the super-field perturbation theory for this $\mathcal{Z}_{\varphi_i}[J]$ in appendix A. We can then project out the components of the super-fields and get all the correlations mentioned above. We advise the reader not to jump immediately to appendix A, but to wait till he has read sec. 4.

4. Perturbation Theory With Temperature.

Let us now suppose that, instead of working with some fixed initial configuration, as in section 3, we do a thermal average over the initial configuration. Let us use the following notation: $\vec{x}$ for the 3-dim vector and x for the 4-dimensional one. Let us remember that in section 3 we choose a solution $\phi_0(\vec{x},t)$ of the Klein-Gordon equation:

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi_0(\vec{x},t) = 0. \tag{4.1}$$

Its Fourier transform $\tilde{\phi}_0(\vec{p},t)$ is :

$$\phi_0(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\tilde{\phi}_0(\vec{p},t)e^{i\vec{p}.\vec{x}} \tag{4.2}$$

and satisfies the equation:

$$\left(\partial_t^2 - E_{\vec{p}}^2\right)\tilde{\phi}_0(\vec{p},t) = 0, \tag{4.3}$$

whose solution has the form

$$\tilde{\phi}_0(\vec{p},t) = \phi_i(\vec{p})\cos\left[E_{\vec{p}}(t-t_i)\right] + \frac{\pi_i(\vec{p})}{E_{\vec{p}}}\sin\left[E_{\vec{p}}(t-t_i)\right] \tag{4.4}$$

with t_i, ϕ_i, π_i the initial time and field configurations. So the final expression for the field is:

$$\phi_0(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\left\{\phi_i(\vec{p})\cos\left[E_{\vec{p}}(t-t_i)\right] + \frac{\pi_i(\vec{p})}{E_{\vec{p}}}\sin\left[E_{\vec{p}}(t-t_i)\right]e^{i\vec{p}.\vec{x}}\right\}. \tag{4.5}$$

The thermal-averaged correlation- functions that we want to evaluate are defined, for example for a two-point function , as follows:

$$\langle\phi\phi\rangle_\beta = \frac{\int \mathscr{D}\phi_i(\vec{x})\mathscr{D}\pi_i(\vec{x})\langle\phi\phi\rangle_{\varphi_i\equiv(\phi_i,\pi_i)}e^{-\beta H(\phi_i,\pi_i)}}{\int \mathscr{D}\phi_i(\vec{x})\mathscr{D}\pi_i(\vec{x})e^{-\beta H(\phi_i,\pi_i)}} \tag{4.6}$$

where $\phi_i(\vec{x})$, $\pi_i(\vec{x})$ are the Fourier transform of the $\phi_i(\vec{p})$ and $\pi(\vec{p})$ appearing in Eq. (4.4).

On the correlation $\langle\phi\phi\rangle_\beta$ we have not indicated the argument of the fields. We did that in order to simplify the notation for the moment.

The analog of the generating functional (3.9) will now be:

$$\mathcal{Z}_\beta[J_\phi] = \frac{\int \mathscr{D}\phi_i(\vec{x})\mathscr{D}\pi_i(\vec{x})e^{-\beta H(\phi_i,\pi_i)}\mathcal{Z}_{\phi_i}[J]}{\int \mathscr{D}\phi_i(\vec{x})\mathscr{D}\pi_i(\vec{x})e^{-\beta H(\phi_i,\pi_i)}} \tag{4.7}$$

In order to do the integration over the initial configuration in (4.7), let us first do the anti-Fourier transform of the $\phi(\vec{p})$ and $\pi(\vec{p})$:

$$\phi_i(\vec{p}) = \int d^3\vec{x}e^{-i\vec{p}.\vec{x}}\phi_i(\vec{x}) \tag{4.8}$$

$$\pi_i(\vec{p}) = \int d^3\vec{x}e^{-i\vec{p}.\vec{x}}\pi_i(\vec{x}). \tag{4.9}$$

So we can write $\phi_0(\vec{x},t)$ as:

$$\phi_0(\vec{x},t) = \int d^3\vec{x}'\phi_i(\vec{x}')a_\phi(\vec{x}-\vec{x}',t) + \int d^3\vec{x}'\pi_i(\vec{x}')a_\pi(\vec{x}-\vec{x}',t) \tag{4.10}$$

where

$$a_\phi(\vec{x}-\vec{x}',t) = \int d^3\vec{x}e^{-i\vec{p}.(\vec{x}-\vec{x}')}\cos\left[E_{\vec{p}}(t-t_i)\right] \tag{4.11}$$

$$a_\pi(\vec{x}-\vec{x}',t) = \int d^3\vec{x}e^{-i\vec{p}.(\vec{x}-\vec{x}')}\frac{\sin\left[E_{\vec{p}}(t-t_i)\right]}{E_{\vec{p}}}. \tag{4.12}$$

Next, let us extract from (4.7) the part which depends only on the initial field configurations which is:

$$\tilde{\mathcal{Z}}_\beta[J_\phi] \equiv \frac{\displaystyle\int \mathscr{D}\phi_i(\vec{x})\mathscr{D}\pi_i(\vec{x})e^{-\beta H_0(\phi_i,\pi_i)+\int d^4x J_\phi(x)\phi_0(x)}}{\int \mathscr{D}\phi_i(\vec{x})\mathscr{D}\pi_i(\vec{x})e^{-\beta H_0(\phi_i,\pi_i)}} \tag{4.13}$$

where we have switched off [10] the interaction in H so that H_0 is just:

$$H_0(\phi_i,\pi_i) = \int d^4x\left[\frac{\pi_i}{2} + \frac{(\nabla\phi_i)^2}{2} + \frac{m^2\phi_i^2}{2}\right] \tag{4.14}$$

Inserting (4.10), (4.11) and (4.12) in (4.13) it is easy to perform the integration over the initial condition (see Ref. [10] for details or appendix A of our paper) and the result is:

$$\tilde{\mathcal{Z}}_\beta[J_\phi] = \exp\left[-\frac{1}{2}\int d^4xd^4x'J_\phi(x)\Delta_\beta(x-x')J_\phi(x')\right] \tag{4.15}$$

where:

$$\Delta_\beta(x-x') = \int \frac{d^4p}{(2\pi)^4}\frac{2\pi}{\beta|p^0|}\delta(p^2-m^2)e^{-ip.(x-x')}. \tag{4.16}$$

It is easy to prove that

$$\Delta_\beta(x-x') = \Delta_\beta(x'-x). \tag{4.17}$$

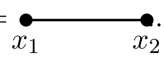
Going now back to the full expression (4.7) of the generating functional, the full Bosonic part is:

$$\mathcal{Z}_{(B)\,\beta}[J_\phi] = \Big[-\tfrac{1}{2}\int d^4x d^4x' J_\phi(x)\Delta_\beta(x-x')J_\phi(x') \\ +i\int d^4x d^4x' J_\phi(x) G_R^{(B)}(x-x')J_{\lambda_\pi}(x')\Big]. \tag{4.18}$$

Let us calculate the thermal two-point function

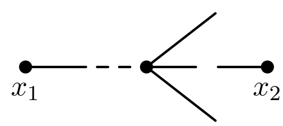
$$\langle \phi(x_2)\phi(x_1)\rangle_\beta = \left(\frac{1}{i}\right)^2 \frac{\delta}{\delta J_\phi(x_2)}\frac{\delta}{\delta J_\phi(x_1)} Z_\beta[J_\phi]|_{J_\phi=0} = \Delta(x_2-x_1). \tag{4.19}$$

We can see that this correlation is not anymore the product of the two fields at x_2 and x_1 like in (3.19), but it is a function $\Delta_\beta(x_2-x_1)$ which links x_2 with x_1. We will use the full line to indicate the Feynman diagram associated to $\Delta_\beta(x_2-x_1)$

$$\Delta_\beta(x_2-x_1) =$$ 

(4.20)

Let us now see how to get the first order correction to the two-point function. Let us, for example, look at the first diagram in (3.35), and let us take its thermal average indicated by $\langle\ \rangle_\beta$:

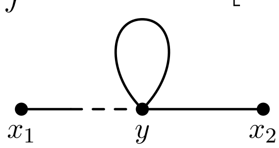
$\frac{1}{3!}\Big\langle$ 

$\Big\rangle_\beta$

$$= \frac{g}{3!}\int dy\, G_R(x_1-y)\left\langle \phi_0(y)^3\,\phi_0(x_2)\right\rangle_\beta$$

$$= \frac{g}{3!}\int dy\, G_R(x_1-y)\,\frac{1}{i^4}\,\frac{\delta^3}{\delta J_\phi(y)^3}\frac{\delta}{\delta J_\phi(x_2)} Z_\beta[J_\phi]\Big|_{J_\phi=0}$$

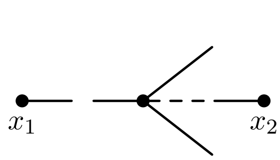
$$= \frac{g}{3!}\int dy\, G_R(x_1-y)\Big[3\,\Delta_\beta(y,y)\,\Delta_\beta(y-x_2)\Big]$$

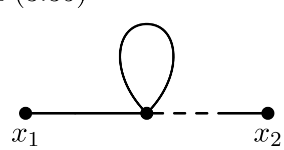
$= \frac{1}{2!}$ 

(4.21)

Let us notice that we do not have anymore disconnected diagrams. The tree legs in y going nowhere and the one in x_2 get soldered to each other in all possible combination producing the loops in (4.21) in y and the propagator between y and x_2.

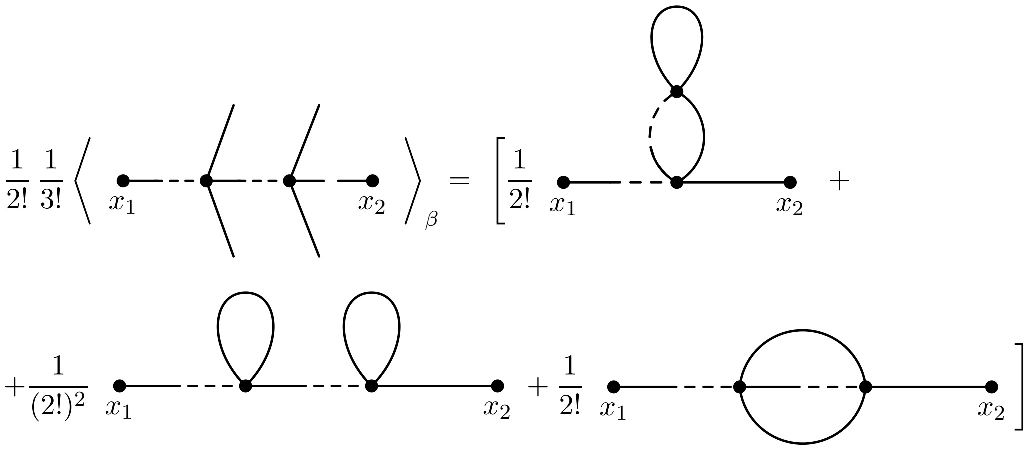
The same can be done for the second diagram in (3.35)

$\frac{1}{3!}\Big\langle$ 

$\Big\rangle_\beta = \frac{1}{2!}$ 

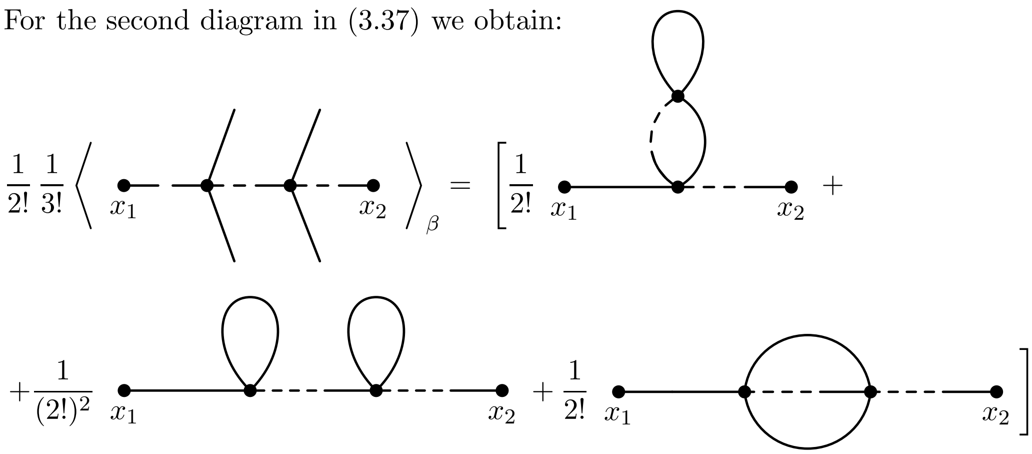
(4.22)

The same for the second order contribution in (3.37). After some long but straightforward calculations, we get from the first diagram in (3.37):



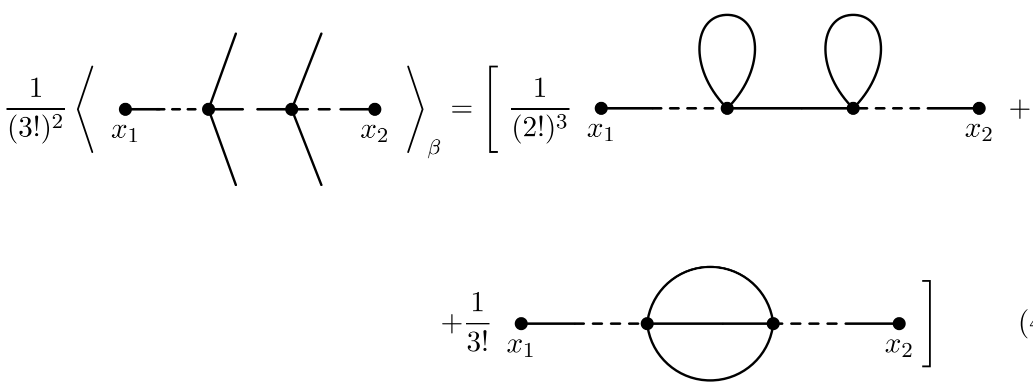
(4.23)

For the second diagram in (3.37) we obtain:



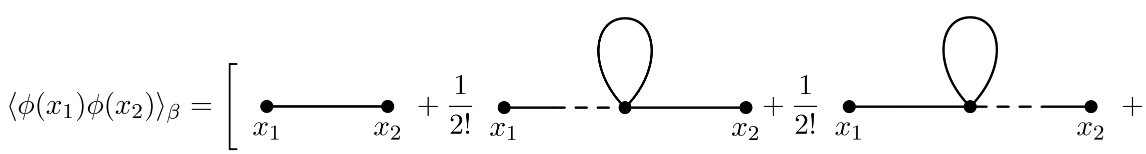
(4.24)

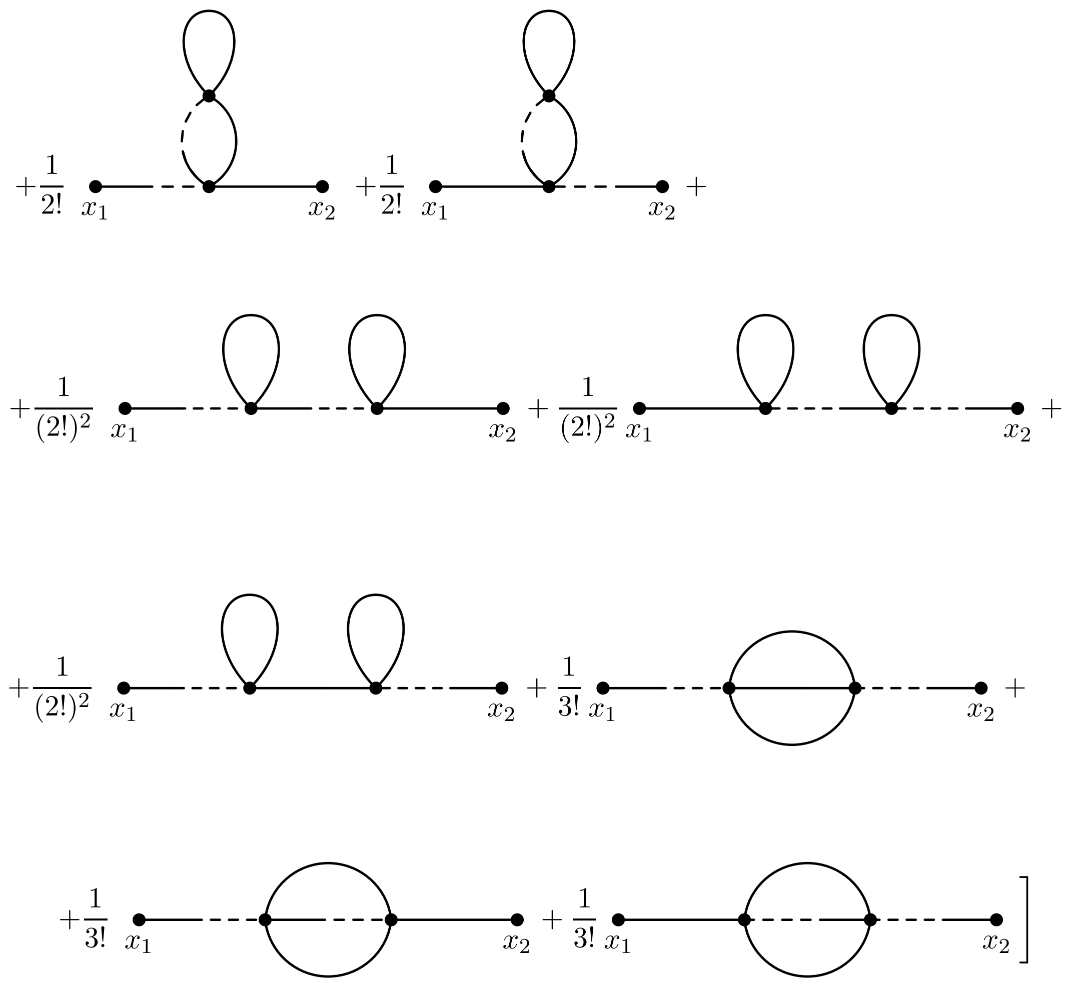
and the third term in (3.37) gives:



(4.25)

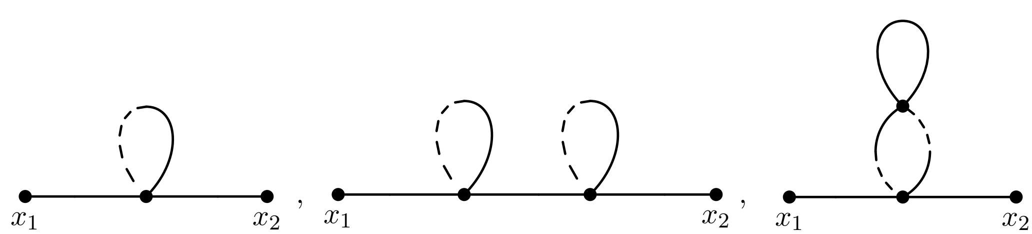
Summing up all the terms up to 2nd order we get:



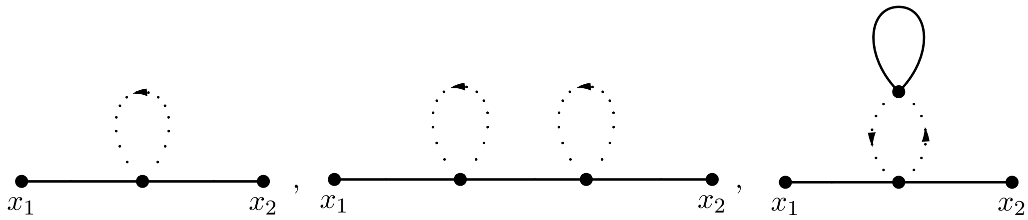


(4.26)

The reader may wonder that there could be some extra diagram coming from the Fermionic part of the generating functional that we have not considered. This is not so because in this calculation we started from the result we had got for the analog diagrams without thermal averages (3.38) and for those diagrams we had already taken account of the Fermionic diagrams cancelation. For example, just using topological considerations we could have for the two-point functions also the following diagrams besides the (4.26)



but all these would be canceled by the diagrams with Fermionic loops like:

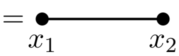


(4.27)

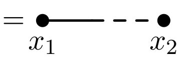
while no one of those in (4.26) could be canceled by diagrams with Fermionic loops.

The diagrams we get by averaging over the initial conditions with the Boltzmann weight are basically based on the following propagators and vertices

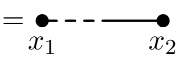
$$\Delta_\beta(p) = \frac{2\pi}{\beta|p^0|}\delta(p^2-m^2) = \tag{4.28}$$



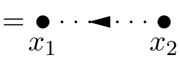
$$-\,iG_R^{(B)}(p) = -\frac{i}{p^2-m^2+i\epsilon p^0} = \tag{4.29}$$



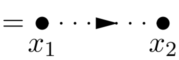
$$-\,i\,G_A^{(F)}(p) = -i\,G_R^{(B)}(-p) = \tag{4.30}$$



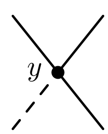
$$-\,G_R^{(F)}(p) = -G_R^{(B)}(p) = \tag{4.31}$$



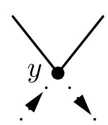
$$-\,G_A^{(F)}(p) = -G_R^{(B)}(-p) = \tag{4.32}$$



$$i\,g = \tag{4.33}$$



$$-\,g = \tag{4.34}$$



5. Perturbation Theory With Superfields.

If we add to the $\mathcal{Z}_\beta^{(B)}$ of (4.18) the Grassmannian piece (3.12) (which is not affected by the thermal average) we get the full generating functional which, for the free theory, is :

$$\begin{aligned}\mathcal{Z}_\beta^0[J] = \Bigg[&-\tfrac{1}{2}\int d^4x d^4x' J_\phi(x)\Delta_\beta(x-x')J_\phi(x') + \\ &+i\int d^4x d^4x' J_\phi(x) G_R^{(B)}(x-x')J_{\lambda_\pi}(x') + \\ &-i\int d^4x d^4x' \bar{J}_{c^\phi}(x) G_R^{(F)}(x-x')J_{\bar{c}_\pi}(x')\Bigg].\end{aligned}\tag{5.1}$$

Let us now move quickly to the superfield formalism. The super-field analog of (2.25) is the following one:

$$\Phi = \phi + \theta c^\phi + \bar{\theta}\bar{c}_\pi + i\bar{\theta}\theta\lambda_\pi \tag{5.2}$$

and in order to have the current-field coupling that we have in (3.11) and (3.12), we need to introduce a super-current of the form:

$$\mathbb{J}_\phi = -J_{\lambda_\pi} + \theta J_{\bar{c}_\pi} + \bar{\theta}\bar{J}_{c^\phi} - i\bar{\theta}\theta J_\phi. \tag{5.3}$$

The term:

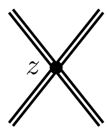
$$\exp\left[i\int d^4x d\theta d\bar{\theta}\,\mathbb{J}_\phi(x,\theta,\bar{\theta})\Phi(x,\theta,\bar{\theta})\right] \tag{5.4}$$

will then give the four couplings of currents and fields present in (3.11) and (3.12). If we want to get not the free generating functional but the full one, its formal expression will be

$$\mathcal{Z}[\mathbb{J}_\phi] = \exp\left[\int dz V\left(-\frac{\delta}{\delta\mathbb{J}_\phi(z)}\right)\right]\mathcal{Z}^0_\beta[\mathbb{J}_\phi] \tag{5.5}$$

where V is the potential of the $g\phi^4/4!$ theory, so

$$V\left(-\frac{\delta}{\delta\mathbb{J}_\phi(z)}\right) = \frac{g}{4!}\frac{\delta^4}{\delta\mathbb{J}^4_\phi(z)} \Longrightarrow$$



$$= g\int d^4y\, d\theta\, d\bar{\theta} = g\int dz, \tag{5.6}$$

where the collective variable $z = (x, \theta, \bar{\theta})$ has been introduced. The reason we can use this potential and not the $\tilde{\mathcal{S}}_V$ of Eq. (3.10) is because, as we proved in Ref. [19], the $\tilde{\mathcal{S}}_V$ or any $\tilde{\mathcal{S}}$ can be obtained via the usual $\mathcal{S}$ with ϕ replaced by Φ as in formula (2.18) and (2.19). A super-field formalism and the associated Feynman diagrams had been developed for the Langevin stochastic equation in ref.[36]. This has inspired us to do an analogous formalism for deterministic systems which evolve via Hamilton's equation.

Going now back to the coupling between supercurrents and superfields in (5.4), it is straightforward to prove that:

$$J_\phi = i\int d\theta d\bar{\theta}\,\mathbb{J}_\phi(x,\theta,\bar{\theta}) \tag{5.7}$$

$$J_{\lambda_\pi} = -\int d\theta d\bar{\theta}\bar{\theta}\theta\,\mathbb{J}_\phi(x,\theta,\bar{\theta}) \tag{5.8}$$

$$\bar{J}_{c^\phi} = -\int d\theta d\bar{\theta}\,\mathbb{J}_\phi(x,\theta,\bar{\theta})\theta \tag{5.9}$$

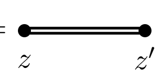
$$J_{\bar{c}_\pi} = \int d\theta d\bar{\theta}\bar{\theta}\,\mathbb{J}_\phi(x,\theta,\bar{\theta}). \tag{5.10}$$

Using the rules (5.7), (5.8), (5.9), (5.10) and the super-field we can write the free generating functional (5.1) as

$$\mathcal{Z}^0_\beta[\mathbb{J}_\phi] = \exp\left[\frac{1}{2}\int dzdz'\,\mathbb{J}_\phi(z)\mathbb{G}(z,z')\mathbb{J}_\phi(z')\right] \tag{5.11}$$

where $z = (x, \theta, \bar{\theta})$ and $\mathbb{G}(z,z')$ is defined as:

$$\mathbb{G}(z,z') = \Delta_\beta(x-x') + \mathcal{G}(z,z') = \tag{5.12}$$



with

$$\mathcal{G}(z,z') \equiv G_R^{(B)}(x-x')\bar{\theta}'\theta' + G_R^{(B)}(x'-x)\bar{\theta}\theta + G_R^{(F)}(x-x')\theta\bar{\theta}' + G_R^{(F)}(x'-x)\theta'\bar{\theta}. \quad (5.13)$$

The $\mathbb{G}(z,z')$ and $\mathcal{G}(z,z')$ have several nice properties that are the following (with $m \geq 1$) :

1.

$$\mathbb{G}(z,z') = \mathbb{G}(z',z). \quad (5.14)$$

2.

$$\mathbb{G}(z,z) = \Delta_\beta(x-x). \quad (5.15)$$

3.

$$\mathcal{G}^n(z,z') = \delta_{n1}\mathcal{G}^n(z,z')[1-\delta(z-z')]\,, n \geq 1 \quad (5.16)$$

4.

$$\mathbb{G}^m(z,z') = \Delta_\beta^m(x-x') + m\,\Delta_\beta^{m-1}(x-x')\mathcal{G}(z,z'), \quad (5.17)$$

While **1**) and **2**) are trivial to prove from the symmetry properties of Δ_β and $\mathcal{G}$, **4**) can be proved as follows:

$$\begin{aligned}
\mathbb{G}^m(z,z') &= \sum_{0\leq n\leq m} \binom{m}{n} \Delta_\beta^{m-n}(x-x')\mathbb{G}^n(z,z') \\
&= \binom{m}{0} \Delta_\beta^m(x-x') + \binom{m}{1} \Delta_\beta^{m-1}(x-x')\mathcal{G}(z,z') \\
&= \Delta_\beta^m(x-x') + m\,\Delta_\beta^{m-1}(x-x')\mathcal{G}(z,z').
\end{aligned} \quad (5.18)$$

In the last step we have used $\mathcal{G}^n(z,z') = 0 (n \geq 2)$ which is a consequence of the property number **3**). Let us now prove property number **3**). We have, for $n \geq 1$:

$$\begin{aligned}
\mathcal{G}^n(z,z') &= \mathcal{G}^n(z,z')\left\{\delta(z-z') + [1-\delta(z-z')]\right\} \\
&= \mathcal{G}^n(z,z)\delta(z-z') + \mathcal{G}^n(z,z')[1-\delta(z-z')] \\
&= [\delta_{n1} + (1-\delta_{n1})]\,\mathcal{G}^n(z,z')[1-\delta(z-z')] \\
&= \delta_{n1}\mathcal{G}^n(z,z')[1-\delta(z-z')] \\
&\quad +(1-\delta_{n1})\mathcal{G}^n(z,z')[1-\delta(z-z')] \\
&= \delta_{n1}\mathcal{G}^n(z,z')[1-\delta(z-z')].
\end{aligned} \quad (5.19)$$

The property no. **3**) is at the root of the fact that loops made of dash-full line cancel against Grassmannian loops. So it is at the root of "*classicality*", i.e. that without temperature we would not have any loops.

Going back to the expressions (5.5) and (5.11) we have seen that the Feynman diagrams are the same as those of a $g\phi^4/4!$ theory, but with the field replaced by *super-field*, and the propagator replaced by the super $\mathbb{G}(z,z')$. We call the reader's attention that, in contrast to what happens in standard quantum field theory, where, due to translational invariance we have $G_{\tilde{F}}(z,z') \equiv G_{\tilde{F}}(z-z')$, where $G_{\tilde{F}}$ stays for the Feynman propagator, here we have

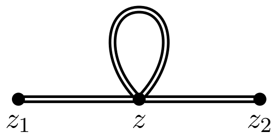
$\mathbb{G}(z,z') \neq \mathbb{G}(z-z')$. That is why we use the notation $\mathbb{G}(z,z')$ in the perturbative analytic expressions.

Let us start deriving the two-point function:

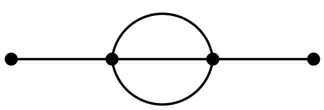
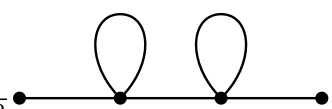
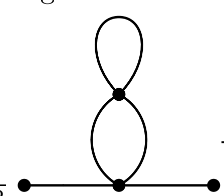
$$\mathbb{G}_\beta(z_1,z_2) = \langle \Phi(z_1)\Phi(z_2)\rangle_\beta = (-i)^2 \frac{\delta_{\mathbb{J}_{\phi(z_1)}}\delta_{\mathbb{J}_{\phi(z_2)}}\mathcal{Z}[\mathbb{J}_\phi]}{\mathcal{Z}[\mathbb{J}_\phi]}|_{\mathbb{J}_\phi=0}. \tag{5.20}$$

Doing the analog of standard $g\phi^4/4!$ QFT perturbation theory, we have that the first order correction to the two-point function is

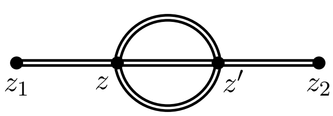
$$\frac{1}{2}\int dz \mathbb{G}(z_1,z)\mathbb{G}(z,z)\mathbb{G}(z,z_2) = \frac{1}{2} \tag{5.21}$$



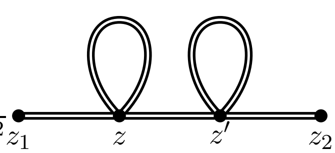
which is the super-analog of the QFT diagram with the same symmetry factor. The second order correction, following the analogy with QFT, which has the three diagrams:

$\Bigg[\frac{1}{3!}$  $+\frac{1}{(2!)^2}$  $+\frac{1}{(2!)^2}$  $\Bigg]$

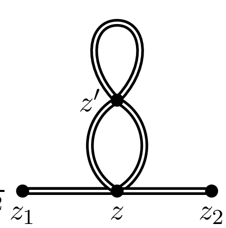
is made of the following three super-diagrams:

$\frac{1}{3!}$ 

$$= \frac{1}{3!}\int dz\, dz'\, \mathbb{G}(z_1,z)\ \left[\mathbb{G}(z,z')\right]^3\ \mathbb{G}(z',z_2) \tag{5.22}$$

$\frac{1}{(2!)^2}$ 

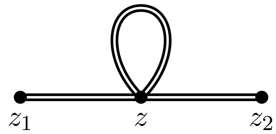
$$= \frac{1}{4}\int dz\, dz' \mathbb{G}(z_1,z)\, \mathbb{G}(z,z)\, \mathbb{G}(z,z')\mathbb{G}(z',z')\, \mathbb{G}(z',z_2) \tag{5.23}$$

$\frac{1}{(2!)^2}$ 

$$= \frac{1}{4}\int dz\, dz'\, \mathbb{G}(z_1,z)\, \mathbb{G}(z,z')\, \mathbb{G}(z',z')\, \mathbb{G}(z',z)\, \mathbb{G}(z,z_2). \tag{5.24}$$

Let us now check if from the super-diagram (5.21) we can derive, for example, the $\langle\phi\phi\rangle$ correlation at the first order which were the sum of (4.21) and (4.22). The manner to do that is to extract from the external super-fields of Eq. (5.21) the components φ^a. Remembering the formula of the super-field (5.2), this can be done via the following integration

$$\int d\theta_1 d\bar{\theta}_1 \left(\bar{\theta}_1\theta_1\right) d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_2\theta_2\right) \left[\frac{1}{2!} \right.$$



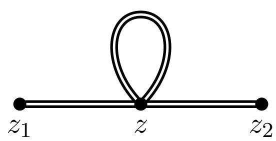
$$\left. \right]. \tag{5.25}$$

The variables inside the brackets $\left(\bar{\theta}_1\theta_1\right)$ and $\left(\bar{\theta}_2\theta_2\right)$ basically act as projectors from the super-field Φ to the field ϕ.

Let us now work out (5.25) in details using (5.21) and the analytic expression of the super-propagators presented on the l.h.s. of (5.13). Below we will go pedantically through the details of the calculations. We do that in order to get the reader familiar with the formalism.

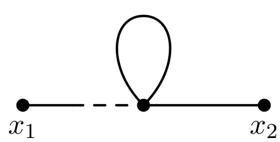
The explicit expression of (5.25) turns out to be

$$\int d\theta_1 d\bar{\theta}_1 \left(\bar{\theta}_1\theta_1\right) d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_2\theta_2\right) \left[\frac{1}{2!} \right.$$

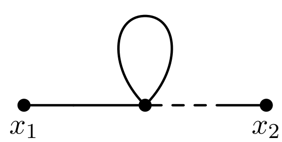


$$\begin{aligned}
&\left. \right] \\
&= \tfrac{1}{2} \int d\theta_1 d\bar{\theta}_1 \left(\bar{\theta}_1\theta_1\right) d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_2\theta_2\right) \int d^4x d\theta d\bar{\theta} \\
&\times \Big[\Delta_\beta(x_1 - x) + G_R^{(B)}(x - x_1)\bar{\theta}_1\theta_1 + G_R^{(B)}(x_1 - x)\bar{\theta}\theta + G_R^{(F)}(x - x_1)\theta\bar{\theta}_1 + G_R^{(F)}(x_1 - x)\theta_1\bar{\theta}\Big] \\
&\times \left[\Delta_\beta(x - x) + \mathcal{G}(z, z)\right] \\
&\times \Big[\Delta_\beta(x - x_2) + G_R^{(B)}(x - x_2)\bar{\theta}_2\theta_2 + G_R^{(B)}(x_2 - x)\bar{\theta}\theta + G_R^{(F)}(x - x_2)\theta\bar{\theta}_2 + G_R^{(F)}(x_2 - x)\theta_2\bar{\theta}\Big] \\
&= \tfrac{1}{2} \int d^4x d\theta d\bar{\theta} \Big[\Delta_\beta(x_1 - x) + G_R^{(B)}(x_1 - x)\bar{\theta}\theta\Big] \Delta_\beta(x - x') \Big[\Delta_\beta(x - x_2) + G_R^{(B)}(x_2 - x)\bar{\theta}\theta\Big] \\
&= \tfrac{1}{2} \int d^4x d\theta d\bar{\theta} \Big[\Delta_\beta(x_1 - x) + G_R^{(B)}(x_1 - x)\bar{\theta}\theta\Big] \Delta_\beta(x - x') \\
&\times \Big[\Delta_\beta(x - x_2) + G_R^{(B)}(x_2 - x)\bar{\theta}\theta\Big]
\end{aligned} \tag{5.26}$$

$$= \left[\frac{1}{2!} \right.$$



$$+ \frac{1}{2!}$$

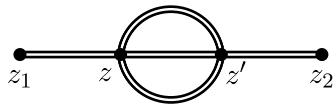


$$\left. \right]$$

and these last diagrams are exactly the sum of (4.21) and (4.22) which were the first order correction to the $\langle\phi\phi\rangle$ propagator. So the lesson we learn from here is the following: it is enough to project the external super-legs of the super-diagrams on the fields we want, in order to get also the correct internal part of the super-diagrams.

Let us now check the second order. That means let us check if the three super-diagrams in Eqs. (5.22), (5.23) and (5.24), once the external legs are projected on ϕ, reproduce the eight second order diagrams contained in (4.26).

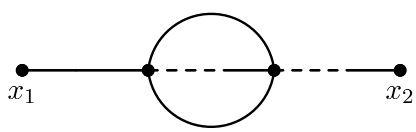
Again, in order to get the reader familiar with this formalism, let us calculate explicitly here the first of the three super-diagrams in (5.22), while the calculation of the other two will be confined in appendix B

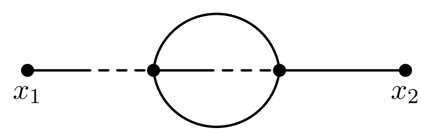
$$\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\left[\ \frac{1}{3!}\right.$$ 

$$\left.\vphantom{\frac{1}{3!}}\right] = \qquad (5.27)$$

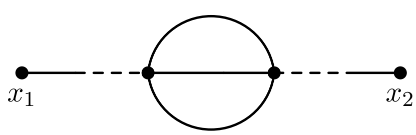
$$= \int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\int d^4x d\theta d\bar\theta \int d^4x' d\theta' d\bar\theta'$$
$$\left[\Delta_\beta(x_1-x) + G_R^{(B)}(x_1-x)\bar\theta\theta + G_R^{(B)}(x-x_1)\bar\theta_1\theta_1 + G_R^{(F)}(x_1-x)\theta_1\bar\theta + G_R^{(F)}(x-x_1)\theta\bar\theta_1\right]\times$$
$$\left[\Delta_\beta(x-x') + \mathcal{G}(z,z')\right]^3\times$$
$$\left[\Delta_\beta(x'-x_2) + G_R^{(B)}(x'-x_2)\bar\theta_2\theta_2 + G_R^{(B)}(x_2-x')\bar\theta'\theta' + G_R^{(F)}(x'-x_2)\theta'\bar\theta_2 + G_R^{(F)}(x_2-x')\theta_2\bar\theta'\right]$$
$$= \int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\int d^4x d\theta d\bar\theta \int d^4x' d\theta' d\bar\theta'\times$$

$$\left[\Delta_\beta(x_1-x) + G_R^{(B)}(x_1-x)\bar\theta\theta + G_R^{(B)}(x-x_1)\bar\theta_1\theta_1 + G_R^{(F)}(x_1-x)\theta_1\bar\theta + G_R^{(F)}(x-x_1)\theta\bar\theta_1\right]$$
$$\times\left[\Delta_\beta^3(x-x') + 3\Delta_\beta^2(x-x')\mathcal{G}(z,z')\right]^3$$
$$= \frac{1}{2}\int d^4x d^4x' \Delta_\beta(x_1-x)\Delta_\beta^2(x-x')G_R^{(B)}(x'-x)G_R^{(B)}(x_2-x')$$
$$+\frac{1}{2}\int d^4x d^4x' G_R^{(B)}(x_1-x)\Delta_\beta^2(x-x')G_R^{(B)}(x-x')\Delta_\beta(x'-x_2)$$
$$+\frac{1}{6}\int d^4x d^4x' G_R^{(B)}(x_1-x)\Delta_\beta^3(x-x')G_R^{(B)}(x_2-x') =$$

$$= \left[\frac{1}{2!}\right.$$ 

$$+\frac{1}{2!}$$ 

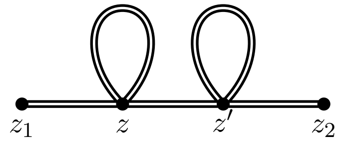
$$+ \qquad (5.28)$$

$$+\frac{1}{3!}$$ 

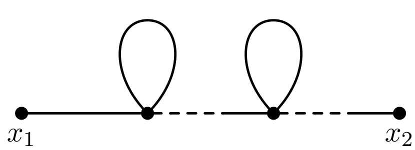
$$\left.\vphantom{\frac{1}{3!}}\right].$$

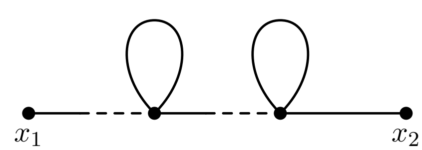
These are exactly the last three diagrams of (4.26) and exactly with the same coefficients.

Analogously we can prove (see appendix B) that

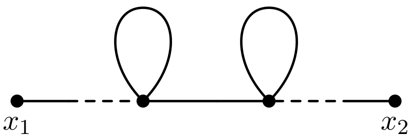
$$\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\left[\ \frac{1}{(2!)^2}\right.$$ 

$$\left.\vphantom{\frac{1}{(2!)^2}}\right] = \qquad (5.29)$$

$$= \left[\frac{1}{(2!)^2}\right.$$ 

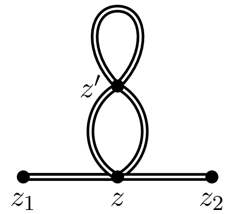
$$+\frac{1}{(2!)^2}$$ 

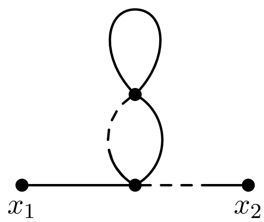
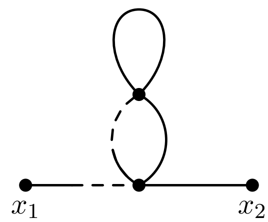
$$+$$

$+ \quad \frac{1}{(2!)^2}$  $\Bigg]$,

which are the 6th, 7th, 8th diagrams presented in (4.26).

Along the same lines we can easily get (see appendix B)

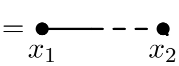
$$\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \, (\bar{\theta}_1\theta_1) \, (\bar{\theta}_2\theta_2) \left[\frac{1}{(2!)^2} \right.$$  $$\left. \vphantom{\frac{1}{(2!)^2}} \right] = \tag{5.30}$$

$$= \left[\frac{1}{2!} \right.$$  $+ \frac{1}{2!}$  $\Bigg]$.

Summing up (5.29), (5.30) and (5.27) we get all the second order diagrams contained in Eq. (4.26). These last ones were eight in numbers and we got the from just the three super-diagrams in Eqs. (5.22), (5.23) and (5.24). This decrease in the number of diagrams is only one of the virtue of the super-diagram technique. A second one is that we only have to perform integrations over the Grassmaniann variables (which are easy to do) to get all the old diagrams of $\tilde{\mathcal{L}}$ without bothering with their complicated symmetry factors, without bothering with soldering in the correct manner the legs and vertices of $\tilde{\mathcal{L}}$ and so on. Above all, the main advantage of the super-diagrams is that they can be derived from the analog *quantum* theory whose Lagrangian L is in general simpler than $\tilde{\mathcal{L}}$. So, once we have the *quantum* theory and its associated Feynman diagrams and symmetry factors, we are already more than halfway also with the perturbation theory of the associated *classical* theory. We have just to substitute the fields with the super-fields and the propagators with the super-propagators and perform, as we said above, some very simple Grassmaniann integrations. Moreover, the super-diagram automatically perform for us the cancelation between Fermion and dash-full loops.

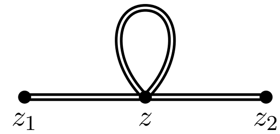
Last, but not least, by changing the projectors on the external legs, we can get not only $\langle\varphi\varphi\rangle$ correlations, but also $\langle\lambda\varphi\rangle$, the $\langle\bar{c}c\rangle$, the $\langle cc\rangle$, the $\langle\bar{c}\bar{c}\rangle$ and the $\langle\lambda\lambda\rangle$ correlations. So just three second order super-diagrams will produce tens of standard diagrams.

We will start now with the $\langle\phi\lambda_\pi\rangle$ correlations. Looking at the super-field expression in (5.2), we realize immediately that, differently than the $\langle\phi\phi\rangle$ correlation, we have to project out only the $\phi(x_1)$ field because the $\lambda_\pi(x_2)$ is already equipped with its own $\bar{\theta}_2\theta_2$ and so it naturally makes in appearance once we integrate over the final points. From the vertices and propagators that we can build from the formalism in components, it is easy to see that $\langle\phi\lambda_\pi\rangle$ correlation will have the zero-order component of the form:

$$\langle\phi(x_1)\lambda_\pi(x_2)\rangle_0 = \tag{5.31}$$ 

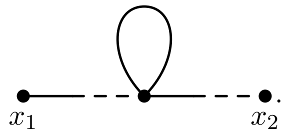
While the first order correction is:

$$\langle\phi(x_1)\lambda_\pi(x_2)\rangle_1 = \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(-i\bar{\theta}_1\theta_1\right) \Bigg[\frac{1}{2!}$$



$$\Bigg] = \qquad (5.32)$$

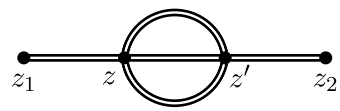
$$= \frac{1}{2!}$$



Here we have used the projector $\left(-i\bar{\theta}_1\theta_1\right)$ in order to project $\phi(x_1)$ out of the super-field.

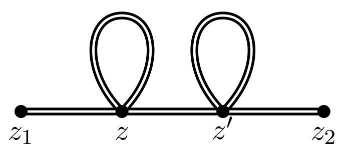
The second order corrections have the form:

$$\langle\phi(x_1)\lambda_\pi(x_2)\rangle_2 = \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(-i\bar{\theta}_1\theta_1\right) \Bigg[\frac{1}{3!}$$

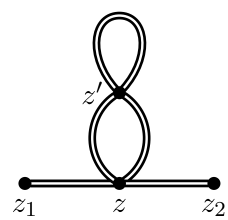


$$+ \qquad (5.33)$$

$$+\frac{1}{(2!)^2}$$



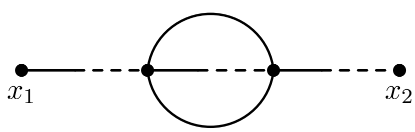
$$+\frac{1}{(2!)^2}$$



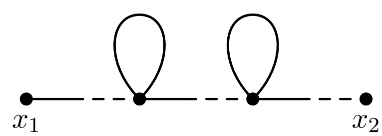
$$\Bigg].$$

By long but straightforward calculations similar to those performed in appendix B, the result that we get for the diagrams in (5.33) is the following one:

$$\Bigg[\frac{1}{3!}$$

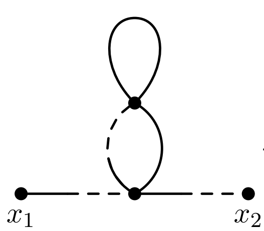


$$+\frac{1}{(2!)^2}$$



$$+ \qquad (5.34)$$

$$+\frac{1}{2!}$$



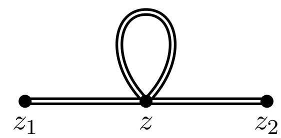
$$\Bigg].$$

We checked and found out that the diagrams on the r.h.s. of (5.34) are exactly those that we would have obtained using the component formalism with the rules given in Eqs. (4.28),(4.29),(4.30),(4.31),(4.32),(4.33),(4.34) and taking into account of the cancelations between dash-full propagators and Fermionic ones.

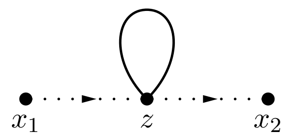
From now on we will trust the results given by super-fields once they are properly projected on the components fields we are interested in. We will trust them because they seem to give the same results as if we have done the calculations in components using the rules in Eqs. (4.28) (4.29),(4.30),(4.31),(4.32), (4.33),(4.34).

Let us now calculate the Grassmaniann variables correlations. That means we consider them as physical fields and not like "ghosts", as it is done in gauge theories, where they contribute only to internal loops and are never physical (i.e. external) fields. The reason we consider them as physical fields is because, as we said before, they are the Jacobi fields and they enter in the calculations of the Lyapunov exponents[24].

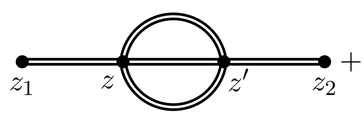
Let us start from the correlation $\langle c^{\phi}(x_1)\bar{c}_{\pi}(x_2)\rangle$. The zero order is given in Eq. (4.30). The first order, via super-fields , will be obtained from the super-diagram of Eq. (5.21) by projecting out the $c^{\phi}(x_1)$ and $\bar{c}_{\pi}(x_2)$. This is achieved with the projector $\theta_2\bar{\theta}_1$. The first one extracts from the super-field the field $\bar{c}_{\pi}(x_2)$, while the second one extracts the $c^{\phi}(x_1)$. So

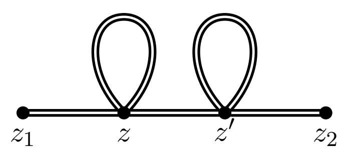
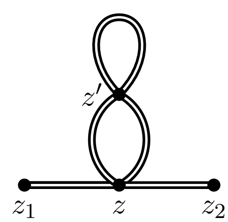
what we have to do is: $\displaystyle\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2\,(\theta_2\bar{\theta}_1)\left[\frac{1}{2!}\right.$  $\left.\right].$

Performing the long but straightforward Grassmannian integration above , we get:

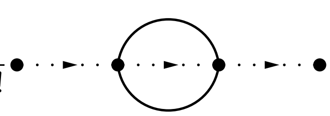
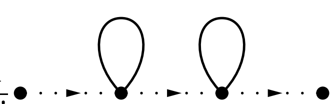
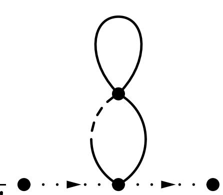
$\dfrac{1}{2!}$  (5.36)

For the second order we just have to perform the following calculations:

$\displaystyle\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2\,(\theta_2\bar{\theta}_1)\Big[\frac{1}{3!}$  $+$ (5.37)

$+\dfrac{1}{(2!)^2}$  $+\dfrac{1}{(2!)^2}$  $\Big]$

and the result is:

$\Big[-\dfrac{1}{2!}$  $-\dfrac{1}{2!}$  $-\dfrac{1}{2!}$  $\Big].$

We leave this calculation to the reader and we let him compare it with the calculations in components which we can get from the rules stated in Eqs. (4.28), (4.29), (4.30), (4.32), (4.33).

Next, let us calculate

$$\langle c^{\phi}(x_1)c^{\phi}(x_2)\rangle_{\beta}. \tag{5.38}$$

The projector in this case is $\left(\bar{\theta}_1\bar{\theta}_2\right)$:

$$\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_1\bar{\theta}_2\right) \langle\Phi(z_1)\Phi(z_2)\rangle_\beta = \langle c^\phi(x_1)c^\phi(x_2)\rangle_\beta. \tag{5.39}$$

By an argument presented in appendix D we can prove ,to all orders in perturbation theory, that:

$$\langle c^\phi(x_1)c^\phi(x_2)\rangle_\beta = 0. \tag{5.40}$$

With similar arguments we can also prove that

$$\langle\phi(x_1)c^\phi(x_2)\rangle_\beta = \langle\phi(x_1)\bar{c}_\pi(x_2)\rangle_\beta = \langle\lambda(x_1)c^\phi(x_2)\rangle_\beta = \langle\lambda(x_1)\bar{c}_\pi(x_2)\rangle_\beta = 0 \tag{5.41}$$

Let us just work out the first one of the relations in (5.41). The projector needed to derive things from the super-fields is $\left(\bar{\theta}_2\bar{\theta}_1\theta_1\right)$, so:

$$\begin{aligned}
&\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_2\bar{\theta}_1\theta_1\right) \langle\Phi(z_1)\Phi(z_2)\rangle_\beta \\
&= \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_2\bar{\theta}_1\theta_1\right) \int dzdz' \mathbb{G}(z_1,z)\mathbb{F}^{(n)}(z,z')\mathbb{G}(z',z_2) \\
&= \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_2\bar{\theta}_1\theta_1\right) \int d^4x d\theta d\bar{\theta} \int d^4x' d\theta' d\bar{\theta}' \Big[\Delta_\beta(x_1-x) + G_R^{(B)}(x_1-x)\bar{\theta}\theta \\
&+G_R^{(B)}(x-x_1)\bar{\theta}_1\theta_1 + G_R^{(F)}(x_1-x)\theta_1\bar{\theta} + G_R^{(F)}(x-x_1)\theta\bar{\theta}_1\Big] \mathbb{F}^{(n)}(z,z') \\
&\times \Big[\Delta_\beta(x'-x_2) + G_R^{(B)}(x'-x_2)\bar{\theta}_2\theta_2 + G_R^{(B)}(x_2-x')\bar{\theta}'\theta' \\
&+G_R^{(F)}(x'-x_2)\theta'\bar{\theta}_2 + G_R^{(F)}(x_2-x')\theta_2\bar{\theta}'\Big] \\
&\equiv \mathbb{A} + \mathbb{B}.
\end{aligned} \tag{5.42}$$

where $\mathbb{F}^{(n)}$ is defined in appendix C, while $\mathbb{A}$ and $\mathbb{B}$ are:

$$\mathbb{A} \equiv \int dzdz'\bar{\theta}'\Delta_\beta(x_1-x)\mathbb{F}^{(n)}(z,z')G_R^{(F)}(x_2-x') \tag{5.43}$$

and

$$\mathbb{B} \equiv \int dzdz'\bar{\theta}\theta\bar{\theta}'G_R^{(F)}(x_1-x)\mathbb{F}^{(n)}(z,z')G_R^{(F)}(x_2-x'). \tag{5.44}$$

In appendix C we proved that the Grassmannian coefficients of $\mathbb{F}^{(n)}$ can only be one of the following six forms: 1) $\theta\bar{\theta}\theta'\bar{\theta}'$, 2) $\theta\bar{\theta}$, 3) $\theta'\bar{\theta}'$, 4) $\theta\bar{\theta}'$, 5) $\bar{\theta}\theta'$ and 6) $\mathbb{I}$. Inserting any of them in $\mathbb{A}$ or $\mathbb{B}$ we get that $\mathbb{A} = 0 = \mathbb{B}$ in all cases and this proves the first of the Eq. (5.41). The proof of the other relations in Eq. (5.41) are analogous and we leave them to the reader.

Let us now prove that, to all orders in perturbation theory, we have:

$$\langle\lambda_\pi(x_1)\lambda_\pi(x_2)\rangle_\beta = 0, \tag{5.45}$$

To simplify the calculations let us use the notation $d\mu \equiv d\theta d\bar{\theta}$. In order to project out the λ field from the super-field we need, as projector, the operator $-\mathbb{I}$, so

$$\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2(-\mathbb{I})\langle\Phi(z_1)\Phi(z_2)\rangle_\beta \equiv \langle\lambda_\pi(x_1)\lambda_\pi(x_2)\rangle_\beta. \tag{5.46}$$

Let us now consider (5.46) at n-th order in perturbation theory

$$
\begin{aligned}
&\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 (-\mathbb{I}) \langle \Phi(z_1)\Phi(z_2)\rangle_\beta^{(n)} \\
&= -\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \int dz dz' \mathbb{G}(z_1,z)\mathbb{F}^{(n)}(z,z')\mathbb{G}(z',z_2) \\
&= -\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \int d^4x d\theta d\bar{\theta} \int d^4x' d\theta' d\bar{\theta}' \Big[\Delta_\beta(x_1-x) + G_R^{(B)}(x_1-x)\bar{\theta}\theta \\
&+G_R^{(B)}(x-x_1)\bar{\theta}_1\theta_1 + G_R^{(F)}(x_1-x)\theta_1\bar{\theta} + G_R^{(F)}(x-x_1)\theta\bar{\theta}_1\Big] \mathbb{F}^{(n)}(z,z') \\
&\times \Big[\Delta_\beta(x'-x_2) + G_R^{(B)}(x'-x_2)\bar{\theta}_2\theta_2 + G_R^{(B)}(x_2-x')\bar{\theta}'\theta' \\
&+G_R^{(F)}(x'-x_2)\theta'\bar{\theta}_2 + G_R^{(F)}(x_2-x')\theta_2\bar{\theta}'\Big] \\
&= -\int dz dz' \int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \Big[G_R^{(B)}(x-x_1)\bar{\theta}_1\theta_1\Big] \mathbb{F}^{(n)}(z,z') \Big[G_R^{(B)}(x'-x_2)\bar{\theta}_2\theta_2\Big] \\
&= -\int d^4x d^4x' G_R^{(B)}(x-x_1) G_R^{(B)}(x'-x_2) \left[\int d\mu d\mu' \mathbb{F}^{(n)}(z,z')\right]
\end{aligned}
\tag{5.47}
$$

In appendix C we will prove several identities which will lead to

$$
\int d\mu d\mu' \mathbb{F}^{(n)}(z,z') = 0. \tag{5.48}
$$

Because of this, from (5.47) we obtain, to all orders, that :

$$
\langle \lambda_\pi(x_1)\lambda_\pi(x_2)\rangle_\beta = 0, \tag{5.49}
$$

The correlation above is zero not only because of the identies that we proved but also because we somehow are using here the closed-time-path (CTP) formalism for classical thermal field theory[30]. We will not expand on this here but advice the interested reader to read ref.[30].

For the correlation $\langle \phi(x_2)\phi(x_1)\rangle_\beta$ we have calculated things up to second order which is what has been done in the literature up to now. We will show in appendix E that with our super-field technique it is not difficult to reach the *third* order.

6. Fluctuation-Dissipation Theorem.

This is a well known theorem that holds for many systems like those obeying the Langevin equation and similar equations. It basically relates the two-point function (fluctuation) of the system to the manner the system responds to external perturbation (dissipation). It was proved to hold [16] also for deterministic systems whose initial distribution was the Boltzmann one, so it should hold also for the approach to classical field theory with temperature that we have explored in this paper.

Several years ago it was showed [33] that the fluctuation-dissipation theorem (FDT) for Langevin equation could be proved non-perturbatively as a "Ward identity" of a hidden supersymmetry present in the Langevin equation. As that supersymmetry is very similar to the one we have in the classical formalism presented here, also in the classical case the FDT can be derived as a Ward identity of the supersymmetry. In appendix F we will present that non-perturbative derivation generously provided to us by Martin Reuter. Of course, the FDT for deterministic systems can be proved in a much simpler way [16]than going through the Ward identities of susy, nevertheless, we like to look at it as a Ward identity because it actually relates different correlation functions like Ward identities do.

For the moment , anyhow, let us leave aside its relations to Susy and Ward identities and let us start verifying its validity at the perturbative level, so that we can test the tools developed in this paper. Let us follow the notation of Ref. [16], that from now on we will indicate with the initial letters "DH", while we will indicate ours by the acronym CPI. In the table below we write down the correspondence between the notation of DH and the one of the CPI. We urge the reader to read Ref. [16] in order to understand the symbols contained in the table below:

$[DH]$	$[CPI]$
D_{ij}	ω^{ab}
$R_{ij}(t-t')$	$-i\theta(t-t')\langle\varphi^a(t)\lambda_b(t')\rangle$
$C_{ij}(t-t')$	$\langle\varphi^a(t)\varphi^b(t')\rangle$
ψ_i	φ^a
$\hat{\psi}_i \equiv -\frac{\partial}{\partial\psi_i}$	$-i\lambda_a$

The φ^a indicates all the phase-space coordinates. According to Eq. (2.34) of DH the FDT is

$$R_{ij}(t) = -\beta\theta(t)\frac{\partial}{\partial t}C_{i\bar{k}}D_{\bar{k}j}, \tag{6.1}$$

where we follow the convention adopted in DH: the bar over an index, like $\bar{k}$, means they are summed over.

Eq. (6.1) can be written in an alternative form as

$$\frac{\partial}{\partial t}C_{ij}(t) = \frac{1}{\beta}R_{i\bar{k}}(t)D_{\bar{k}j} - \frac{1}{\beta}D_{\bar{k}i}R_{j\bar{k}}(-t). \tag{6.2}$$

We can prove it in the following way. For $t > 0$, Eq. (6.1) becomes

$$R_{ij}(t) = -\beta\frac{\partial}{\partial t}C_{i\bar{k}}D_{\bar{k}j}. \tag{6.3}$$

If we multiply the relation above by D we get:

$$\frac{1}{\beta}R_{i\bar{\ell}}(t)D_{\bar{\ell}j} = -\frac{\partial}{\partial t}C_{i\bar{k}}D_{\bar{k}\bar{\ell}}D_{\bar{\ell}j}. \tag{6.4}$$

Next, using the fact that $D_{k\bar{\ell}}D_{\bar{\ell}j} = -\delta_{kj}$, we obtain:

$$\frac{\partial}{\partial t}C_{ij}(t) = \frac{1}{\beta}R_{i\bar{k}}(t)D_{\bar{k}j}. \tag{6.5}$$

For $t < 0$ Eq. (6.1) gives

$$R_{ji}(-t) = -\beta\theta(-t)\frac{\partial}{\partial(-t)}C_{j\bar{k}}(-t)D_{\bar{k}i}$$
$$= \beta\theta(-t)\frac{\partial}{\partial t}C_{j\bar{k}}(t)D_{\bar{k}i} \tag{6.6}$$

where we have used the fact that

$$\frac{\partial}{\partial t}C_{jk}(t) = -\frac{\partial}{\partial t}C_{kj}(-t). \tag{6.7}$$

Multiplying (6.7) by D and operating as before, we get for $t < 0$

$$\frac{\partial}{\partial t}C_{ij}(t) = -\frac{1}{\beta}D_{\bar{k}i}(t)R_{j\bar{k}}(-t). \tag{6.8}$$

Collecting (6.5) and (6.8) for $t \gtrless 0$ we get exactly (6.2).

Making use of the table of comparison between DH and CPI, we can turn (6.2) into the following relation

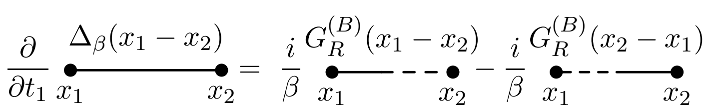
$$\frac{\partial}{\partial t_1}\langle\varphi^a(t_1)\varphi^b(t_2)\rangle = \frac{1}{\beta}\theta(t_1-t_2)\langle\varphi^a(t_1)(-i)\lambda_d(t_2)\rangle\omega^{db} - \frac{1}{\beta}\theta(t_2-t_1)\omega^{da}\langle\varphi^b(t_2)(-i)\lambda_d(t_1)\rangle. \tag{6.9}$$

This is the full fluctuation-dissipation theorem.

Let us choose $a = b = 1$ in Eq. (6.9). We will then obtain for a field theory, where $\varphi^1 = \phi$ and $\varphi^2 = \pi$, the relation :

$$\frac{\partial}{\partial t_1}\langle\varphi^a(t_1)\varphi^b(t_2)\rangle = i\frac{1}{\beta}\left[\theta(t_1-t_2)\langle\phi(t_1)\lambda_\pi(t_2)\rangle - \theta(t_2-t_1)\langle\phi(t_2)\lambda_\pi(t_1)\rangle\right]. \tag{6.10}$$

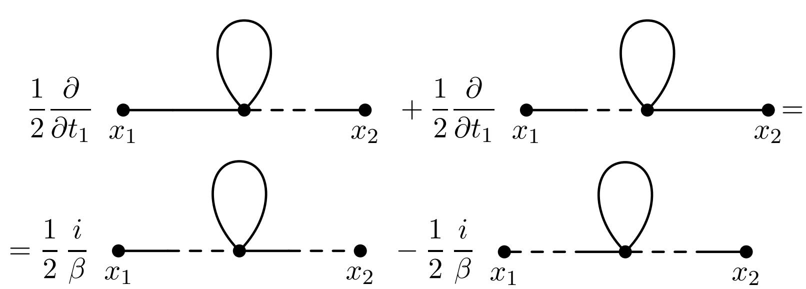
At *order zero* in perturbation theory, in terms of Feynman diagrams, Eq. (6.10) can be written as follows:



(6.11)

We will provide a proof of this in appendix G.

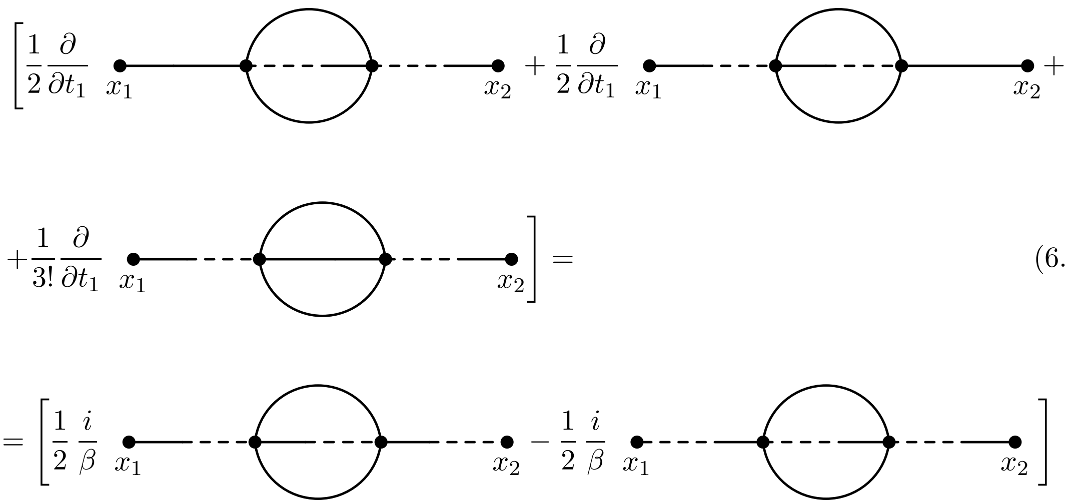
At the *first order* in perturbation theory the diagrammatic form of the FDT (6.10) is:



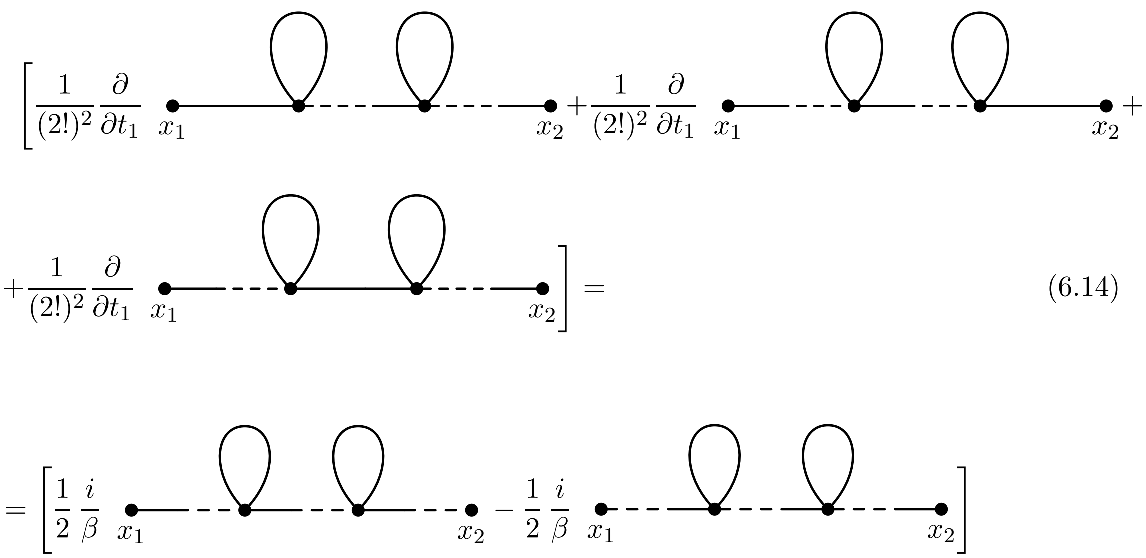
(6.12)

and also this will be proved in appendix G.

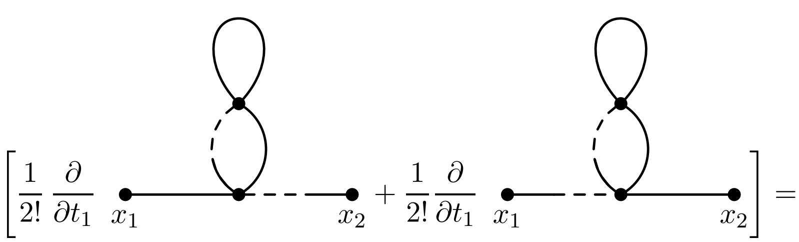
The *second order* result is given by the following three diagrammatic relations.



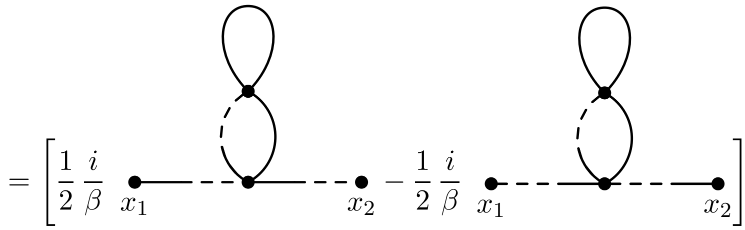
(6.13)



(6.14)



(6.15)



We feel that all these diagrammatic identities will turn out to be very very usefull in simplyfing long sum of diagrams which may appear in phenomenological applications of our formalism.

7. Conclusions and Outlooks.

In this paper we have developed the Feynman diagrams for a *classical* $g\frac{\phi^4(x)}{4!}$ field theory either with and without temperature. This topic is of high interest in the field of heavy ions scattering where classical field theory with temperature seems to be playing a central role. Besides this application, the Feynman diagrams for a classical theory can be useful in many fields from planetary motion, to fluid dynamics, to chaotic motion, etc. We have been able do develop the Feynman diagrams in a very easy way because a path integral for classical systems had been previously developed [11]. With respect to other approaches to perturbation theory for classical systems, we have been able to give the Feynman diagrams not only for the Bosonic variables of the system and for the response fields, but also for the Jacobi fields associated to the dynamics. Actually are these last fields (the Jacobi ones) which can provide us with indications on the cahotic behavior of the system. Our Feynman diagrams could for example be used to calculate perturbatively the Lyapunov exponents of a dynamical system.

Besides developing this diagrammatics for all these fields and their interactions, we have showed that the many different *classical* diagrams of the various different fields mentioned above can be unified in few super-diagrams which have the same kind of vertices and kinetic term as the *quantum* one associated to φ^a.

We hope that this super-diagram formalism can be of some help not only in simplifying the notation, but also in understanding the subtle interplay between the quantum high temperature behavior and the classical one. Interplay which is very important in the heavy ions scattering field. We did not address this last issue here because this paper is only aimed at giving the formal diagrammatics necessary in this field.

What we will do in a forthcoming paper [35] is to develop a formalism analog to the one of the "effective action" and try to give something like a "renormalization group" approach. All this is done in the spirit of provinding the formal tools which could later be used by physicists for more phenomenological applications.

Acknowledgments

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A. A Fundamental Identity.

In this appendix we will show the details on how to go from (4.13) to (4.15) If we insert (4.5) into (4.13), we get:

$$
\begin{aligned}
\tilde{\mathscr{Z}}_\beta[J_\phi] = & \int \mathscr{D}\phi_i(\vec{x}) \exp\left\{-\beta\left[\int d^3\vec{x}\frac{(\nabla\phi_i)^2}{2} + \frac{m^2\phi_i^2}{2}\right.\right. \\
& \left.\left.\times i\int d^4x J_\phi(x)\int d^3\vec{x}'\phi_i(x')a_\phi(\vec{x}-\vec{x}',t)\right\}\right. \\
& \times\int \mathscr{D}\pi_i(\vec{x}) \exp\left[-\beta\int d^3\vec{x}\frac{(\nabla\pi_i)^2}{2}\right. \\
& \left.\times i\int d^4x J_\phi(x)\int d^3\vec{x}'\pi_i(\vec{x}')a_\pi(\vec{x}-\vec{x}',t)\right].
\end{aligned} \tag{A.1}
$$

Let us now define the following quantities:

$$
J_\phi^a(\vec{x}') = i\int dt d^3\vec{x} J_\phi(\vec{x},t)a_\phi(\vec{x}-\vec{x}',t), \tag{A.2}
$$

$$
J_\pi^a(\vec{x}') = i\int dt d^3\vec{x} J_\phi(\vec{x},t)a_\pi(\vec{x}-\vec{x}',t) \tag{A.3}
$$

and let us rewrite the normalized $\tilde{\mathscr{Z}}_\beta[J_\phi]$ using (A.2) and (A.3):

$$
\begin{aligned}
& \tilde{\mathscr{Z}}_\beta[J_\phi] \\
& = \frac{1}{\tilde{\mathscr{Z}}_\beta[0]^\phi}\int \mathscr{D}\phi_i(\vec{x}) \exp\left\{-\beta\int d^3\vec{x}\left[\frac{(\nabla\phi_i)^2}{2} + \frac{m^2\phi_i^2}{2}\right]\right. \\
& \left.+\int d^3\vec{x}\phi_i(\vec{x})J_\phi^a(\vec{x})\right\}
\end{aligned} \tag{A.4}
$$

$$
\times\frac{1}{\tilde{\mathscr{Z}}_\beta[0]^\pi}\int \mathscr{D}\pi_i(\vec{x}) \exp\left\{-\beta\int d^3\vec{x}\frac{(\nabla\pi_i)^2}{2} + \int d^3\vec{x}\pi_i(\vec{x})J_\pi^a(\vec{x})\right\}. \tag{A.5}
$$

where

$$
\tilde{\mathscr{Z}}_\beta[0]^\phi = \int \mathscr{D}\phi_i(\vec{x}) \exp\left\{-\beta\int d^3\vec{x}\left[\frac{(\nabla\phi_i)^2}{2} + \frac{m^2\phi_i^2}{2}\right]\right\} \tag{A.6}
$$

$$
\tilde{\mathscr{Z}}_\beta[0]^\pi = \int \mathscr{D}\pi_i(\vec{x}) \exp\left\{-\beta\left[\int d^3\vec{x}\frac{(\nabla\pi_i)^2}{2}\right]\right\}. \tag{A.7}
$$

Performing a partial integration for the kinetic piece in Eq. (A.4) we get :

$$
\int \mathscr{D}\phi_i(\vec{x}) \exp\left\{-\frac{1}{2}\int d^3\vec{x}d^3\vec{x}'\phi_i(\vec{x})\left[\beta\delta(\vec{x}-\vec{x}')\left(-\nabla^{2'}+m^2\right)\right]\phi_i(\vec{x}') + \int d^3\vec{x}\phi_i(\vec{x})J_\phi^a(\vec{x})\right\}. \tag{A.8}
$$

As the weight is quadratic, the integration can be formally done for all (A.4) and the numerator of (A.4) turns out to be:

$$
\exp\left[\frac{1}{2}\int d^3\vec{x}d^3\vec{x}' J_\phi^a(\vec{x})A^{-1}(\vec{x},\vec{x}')J_\phi^a(\vec{x}')\right]\tilde{Z}_\beta[0]^\phi \tag{A.9}
$$

where $A(\vec{x},\vec{x}') = \beta\delta(\vec{x}-\vec{x}')(-\nabla^{2'}+m^2)$ and it obeys the equation:

$$\begin{aligned}&\int d^3\vec{x}' A(\vec{x},\vec{x}')A^{-1}(\vec{x}',\vec{x}'') = \delta(\vec{x}-\vec{x}'')\\ &= \int d^3\vec{x}'\left[\beta\delta(\vec{x}-\vec{x}')(-\nabla^{2'}+m^2)\right]A^{-1}(\vec{x}',\vec{x}'') = \delta(\vec{x}-\vec{x}'')\end{aligned} \tag{A.10}$$

so

$$\left[\beta(-\nabla^2+m^2)A^{-1}\right](\vec{x},\vec{x}'') = \delta(\vec{x}-\vec{x}''). \tag{A.11}$$

Performing the Fourier transform of A^{-1} we get:

$$A^{-1}(\vec{x},\vec{x}'') = \frac{1}{(2\pi)^3}\int d^3\vec{p}e^{-i\vec{p}.(\vec{x}''-\vec{x})}\tilde{A}^{-1}(\vec{p}). \tag{A.12}$$

It follows from (A.10) that

$$\tilde{A}^{-1}(\vec{p}) = \frac{1}{\beta(\vec{p}^2+m^2)}. \tag{A.13}$$

Using this and the expressions (A.2), (A.3), (4.11), (4.12) we obtain, for the argument in the exponent of (A.9,) the following expression:

$$\begin{aligned}&\int d^3\vec{x}d^3\vec{x}'\left\{i\int d^3\vec{x}_1dt_1J_\phi(\vec{x}_1,t_1)\int\frac{d^3\vec{p}_1}{(2\pi)^3}e^{i\vec{p}_1.(\vec{x}_1-\vec{x})}\cos\left[E_{\vec{p}_1}(t_1-t_i)\right]\right\}\\ &\times\left[\int\frac{d^3\vec{p}}{(2\pi)^3}e^{i\vec{p}.(\vec{x}'-\vec{x})}\tilde{A}^{-1}(\vec{p})\right]\left\{i\int d^3x_2dt_2J_\phi(\vec{x}_2,t_2)\int\frac{d^3p_1}{(2\pi)^3}e^{i\vec{p}_2.(\vec{x}_2-\vec{x}')}\cos\left[E_{\vec{p}_2}(t_2-t_i)\right]\right\}\end{aligned} \tag{A.14}$$

where t_i is the initial time appearing in (4.4). Performing, in the expression above, the integration over $\vec{x}$ and $\vec{x}'$ we get:

$$-\int d^4x_1d^4x_2J_\phi(x_1)J_\phi(x_2)\int\frac{d^3\vec{p}}{(2\pi)^3}e^{-i\vec{p}.(\vec{x}_1-\vec{x}_2)}\tilde{A}^{-1}(\vec{p})\cos\left[E_{\vec{p}}(t_1-t_i)\right]\cos\left[E_{\vec{p}}(t_2-t_i)\right]. \tag{A.15}$$

So (A.4) becomes:

$$\begin{aligned}&\frac{1}{\tilde{\mathscr{Z}}_\beta[0]^\phi}\int\mathscr{D}\phi_i(\vec{x})\exp\left\{-\beta\int d^3\vec{x}\left[\frac{(\nabla\phi_i)^2}{2}+\frac{m^2\phi_i^2}{2}\right]+\int d^3\vec{x}\phi_i(\vec{x})J^a_\phi(\vec{x})\right\}\\ &=\exp\left\{-\tfrac{1}{2}\int d^3\vec{x}d^3\vec{x}'J_\phi(\vec{x})\left\{\int\frac{d^3\vec{p}}{(2\pi)^3}\tilde{A}^{-1}(\vec{p})\cos\left[E_{\vec{p}}(t-t_i)\right]\right.\right.\\ &\left.\left.\times\cos\left[E_{\vec{p}}(t'-t_i)\right]e^{i\vec{p}.(\vec{x}'-\vec{x})}\right\}J_\phi(\vec{x}')\right\}\end{aligned} \tag{A.16}$$

Let us now do an analog set of manipulations for (A.5)

$$\begin{aligned}&\frac{1}{\tilde{\mathscr{Z}}_\beta[0]^\pi}\int\mathscr{D}\pi_i(\vec{x})\exp\left[-\beta\int d^3\vec{x}\frac{(\nabla\pi_i)^2}{2}+\int d^3\vec{x}\pi_i(\vec{x})J^a_\pi(\vec{x})\right]\\ &=\exp\left[\int d^3\vec{x}d^3\vec{x}'J^a_\pi(\vec{x})B^{-1}(\vec{x},\vec{x}')J^a_\pi(\vec{x}')\right]\\ &=\exp\left\{\frac{1}{2\beta}\int d^3\vec{x}\,[J^a_\pi(\vec{x})]^2\right\}\end{aligned} \tag{A.17}$$

where $B(\vec{x},\vec{x}') \equiv \beta\delta(\vec{x}-\vec{x}')$. Using the expression (A.2), (A.3), (4.11), (4.12) the exponent in (A.17) becomes:

$$\begin{aligned}
&\int d^3\vec{x}\,[J^a_\pi(\vec{x})]^2 \\
&= \int d^3\vec{x}\left[i\int dt_1 d^3\vec{x}_1 J_\phi(\vec{x}_1,t_1)a_\pi(\vec{x}_1-\vec{x},t)\right]\left[i\int dt_2 d^3\vec{x}_2 J_\phi(\vec{x}_2,t_2)a_\pi(\vec{x}_2-\vec{x},t)\right] \\
&= -\int dx_1 dx_2 J_\phi(x_1)J_\phi(x_2)\left\{\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\sin\left[E_{\vec{p}}(t_1-t_i)\right]}{E^2_{\vec{p}}}\frac{\sin\left[E_{\vec{p}}(t_2-t_i)\right]}{E^2_{\vec{p}}}e^{-i\vec{p}.(\vec{x}_2-\vec{x}_1)}\right\}
\end{aligned} \tag{A.18}$$

Inserting (A.18) and (A.16) in (A.4) and (A.5) we obtain:

$$\tilde{Z}_\beta[J_\phi] = \exp\left\{-\frac{1}{2}\int d^4x d^4x' J_\phi(x)\left\{\frac{d^3\vec{p}}{(2\pi)^3}\frac{\cos\left[E_{\vec{p}}(t-t')\right]}{\beta\left(\vec{p}^2+m^2\right)}e^{i\vec{p}.(\vec{x}-\vec{x}')}\right\}J_\phi(x')\right\}. \tag{A.19}$$

Via the identity

$$\int dp^0 \frac{1}{|p^0|}e^{-ip^0(t-t')}\delta(p^2-m^2) = \frac{\cos\left[E_{\vec{p}}(t-t')\right]}{E^2_{\vec{p}}} \tag{A.20}$$

we can rewrite $\tilde{Z}_\beta[J_\phi]$ as

$$\tilde{Z}_\beta[J_\phi] = \exp\left[-\frac{1}{2}\int d^4x d^4x' J_\phi(x)\Delta_\beta(x-x')J_\phi(x')\right] \tag{A.21}$$

where

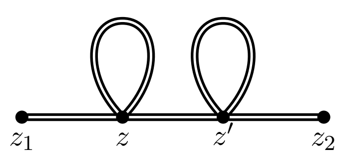
$$\Delta_\beta(x) = \int \frac{d^4p}{(2\pi)^4}\frac{2\pi}{\beta|p^0|}\delta(p^2-m^2)e^{-ip.x}. \tag{A.22}$$

This is what we have used in (4.15).

B. Fields-Correlations Via Super-Field Projections.

In this appendix we will present some of the detailed calculations that we skipped in the text of the paper. Here, in particular, we will report the calculations which support the results in (5.29) and (5.30). Let us start from (5.29):

$$\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_1\theta_1\right)\left(\bar{\theta}_2\theta_2\right)\left[\frac{1}{(2!)^2}\right.$$

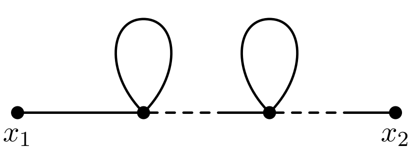


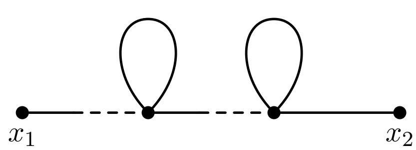
$$\left.\right].$$

Using the fact that $\mathcal{G}(z,z)=0$ and the rules of Grassmannian integrations, we get for the expression above:

$$
\begin{aligned}
&\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_1\theta_1\right)\left(\bar{\theta}_2\theta_2\right) \int d^4x d\theta d\bar{\theta} \int d^4x' d\theta' d\bar{\theta}' \Big[\Delta_\beta(x_1-x) + G_R^{(B)}(x_1-x)\bar{\theta}\theta \\
&+G_R^{(B)}(x-x_1)\bar{\theta}_1\theta_1 + G_R^{(F)}(x_1-x)\theta_1\bar{\theta} + G_R^{(F)}(x-x_1)\theta\bar{\theta}_1\Big]\left[\Delta_\beta(x-x)\right]\left[\Delta_\beta(x-x')\right. \\
&\left.+G_R^{(B)}(x-x')\bar{\theta}'\theta' + G_R^{(B)}(x'-x)\bar{\theta}\theta + G_R^{(F)}(x-x')\theta\bar{\theta}' + G_R^{(F)}(x'-x)\theta'\bar{\theta}\right]\,\Delta_\beta(x'-x') \\
&\Big[\Delta_\beta(x'-x_2) + G_R^{(B)}(x'-x_2)\bar{\theta}_2\theta_2 + G_R^{(B)}(x_2-x')\bar{\theta}'\theta' + G_R^{(F)}(x'-x_2)\theta'\bar{\theta}_2 + G_R^{(F)}(x_2-x')\theta_2\bar{\theta}'\Big] \\
&= \tfrac{1}{4}\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_1\theta_1\right)\left(\bar{\theta}_2\theta_2\right) \int d^4x d\theta d\bar{\theta} \int d^4x' d\theta' d\bar{\theta}' \left[\Delta_\beta(x_1-x)\Delta_\beta(x'-x_2)\right. \\
&+\Delta_\beta(x_1-x)G_R^{(B)}(x_2-x')\bar{\theta}'\theta' + G_R^{(B)}(x_1-x)\bar{\theta}\theta\Delta_\beta(x'-x_2) + G_R^{(B)}(x_1-x)\bar{\theta}\theta \\
&\left.G_R^{(B)}(x_2-x')\bar{\theta}'\theta'\right]\Delta_\beta(x-x) \times \Big[\Delta_\beta(x-x') + G_R^{(B)}(x-x')\bar{\theta}'\theta' + G_R^{(B)}(x'-x)\bar{\theta}\theta \\
&+G_R^{(F)}(x-x')\theta\bar{\theta}' + G_R^{(F)}(x'-x)\theta'\bar{\theta}\Big]\,\Delta_\beta(x'-x') \\
&= \tfrac{1}{4}\int d\theta_1 d\bar{\theta}_1 d\theta_2 d\bar{\theta}_2 \left(\bar{\theta}_1\theta_1\right)\left(\bar{\theta}_2\theta_2\right) \int d^4x d\theta d\bar{\theta} \int d^4x' d\theta' d\bar{\theta}' \\
&\{\Delta_\beta(x_1-x)\Delta_\beta(x'-x_2)\Delta_\beta(x-x)\left[\Delta_\beta(x-x') + \mathcal{G}(z,z')\right]\Delta_\beta(x'-x') \\
&+\Big[\Delta_\beta(x_1-x)G_R^{(B)}(x_2-x')\bar{\theta}'\theta' G_R^{(B)}(x'-x)\bar{\theta}\theta + G_R^{(B)}(x_1-x)\bar{\theta}\theta\Delta_\beta(x'-x_2)G_R^{(B)}(x-x')\bar{\theta}'\theta' \\
&+G_R^{(B)}(x_1-x)\bar{\theta}\theta G_R^{(B)}(x_2-x')\bar{\theta}'\theta'\Delta_\beta(x-x')\Big]\,\Delta_\beta(x-x)\Delta_\beta(x'-x')\Big\} \\
&= \tfrac{1}{4}\int d^4x \int d^4x' \Delta_\beta(x_1-x)\Delta_\beta(x-x)G_R^{(B)}(x'-x)\Delta_\beta(x'-x')G_R^{(B)}(x_2-x') \\
&+\tfrac{1}{4}\int d^4x \int d^4x' \Delta_\beta(x_2-x)\Delta_\beta(x-x)G_R^{(B)}(x'-x)\Delta_\beta(x'-x')G_R^{(B)}(x_1-x') \\
&+\tfrac{1}{4}\int d^4x \int d^4x' G_R^{(B)}(x_1-x)\Delta_\beta(x-x)\Delta_\beta(x-x')\Delta_\beta(x'-x')G_R^{(B)}(x_2-x')
\end{aligned}
\tag{B.1}
$$

The three diagrams presented in the last expression are precisely

$\Bigg[\dfrac{1}{(2!)^2}$ 

$+\dfrac{1}{(2!)^2}$ 

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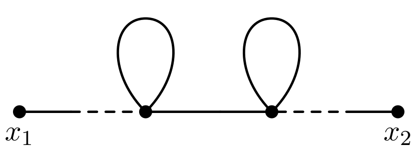
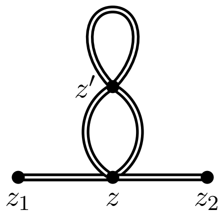
$+\dfrac{1}{(2!)^2}$ 

$\Bigg]$

which is exactly the result we claimed in (5.29).

Let us now give the proof of (5.30). Again we will make use of $\mathcal{G}(z,z)=0$ and of the standard rules of integration over Grassmannian coordinates.

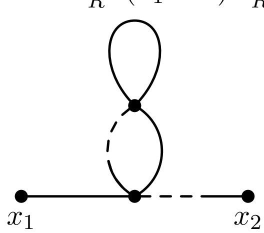
$$\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\left[\frac{1}{(2!)^2}\right. \tag{B.2}$$



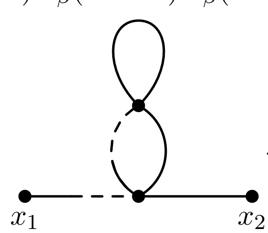
$$\left.\right] =$$

$$= \frac{1}{4}\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\int d^4x d\theta d\bar\theta \int d^4x' d\theta' d\bar\theta'$$
$$\times\left[\Delta_\beta(x_1-x) + G_R^{(B)}(x_1-x)\bar\theta\theta + G_R^{(B)}(x-x_1)\bar\theta_1\theta_1 + G_R^{(F)}(x_1-x)\theta_1\bar\theta + G_R^{(F)}(x-x_1)\theta\bar\theta_1\right]$$
$$\times\left[\Delta_\beta(x-x') + \mathcal{G}(z,z')\right]\left[\Delta_\beta(x'-x') + \mathcal{G}(z',z')\right]\left[\Delta_\beta(x'-x) + \mathcal{G}(z',z)\right]$$
$$\times\left[\Delta_\beta(x-x_2) + G_R^{(B)}(x-x_2)\bar\theta_2\theta_2 + G_R^{(B)}(x_2-x)\bar\theta\theta + G_R^{(F)}(x-x_2)\theta\bar\theta_2 + G_R^{(F)}(x_2-x)\theta_2\bar\theta\right]$$
$$= \frac{1}{4}\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\int d^4x d\theta d\bar\theta \int d^4x' d\theta' d\bar\theta'$$
$$\left[\Delta_\beta(x_1-x)\Delta_\beta(x-x_2) + \Delta_\beta(x_1-x)G_R^{(B)}(x_2-x)\bar\theta\theta + G_R^{(B)}(x_1-x)\bar\theta\theta\Delta_\beta(x-x_2)\right.$$
$$\left.+G_R^{(B)}(x_1-x)\bar\theta\theta G_R^{(B)}(x_2-x)\bar\theta\theta\right]\left[\Delta_\beta(x-x') + \mathcal{G}(z,z')\right]\Delta_\beta(x'-x')\left[\Delta_\beta(x'-x) + \mathcal{G}(z',z)\right]$$
$$= \frac{1}{4}\int d\theta_1 d\bar\theta_1 d\theta_2 d\bar\theta_2 \left(\bar\theta_1\theta_1\right)\left(\bar\theta_2\theta_2\right)\int d^4x d\theta d\bar\theta \int d^4x' d\theta' d\bar\theta'$$
$$\left\{\left[\Delta_\beta(x_1-x)G_R^{(B)}(x_2-x)\bar\theta\theta\right]\left[\Delta_\beta(x-x')G_R^{(B)}(x-x')\bar\theta'\theta'\right]\Delta_\beta(x'-x')\right.$$
$$\left.+\left[G_R^{(B)}(x_1-x)\bar\theta\theta\Delta_\beta(x-x_2)\right]\left[2\Delta_\beta(x-x')G_R^{(B)}(x-x')\bar\theta'\theta'\right]\Delta_\beta(x'-x')\right\}$$
$$= \frac{1}{2}\int d^4x d^4x' \Delta_\beta(x_1-x)G_R^{(B)}(x-x')\Delta_\beta(x'-x')\Delta_\beta(x'-x)G_R^{(B)}(x_2-x)$$
$$+\frac{1}{2}\int d^4x d^4x' G_R^{(B)}(x_1-x)G_R^{(B)}(x-x')\Delta_\beta(x'-x')\Delta_\beta(x'-x)\Delta_\beta(x-x_2). \tag{B.3}$$

$$= \left[\frac{1}{2!}\right.$$



$$+\frac{1}{2!}$$



$$\left.\right].$$

This proves (5.30).

In both of these proofs we have been rather pedantic with the details of the calculations. We did that because we wanted to help the reader not familiar with our formalism in going through the details of the calculations.

C. Loop Identities.

In this appendix we will prove various identities which, among other things, will help us in "proving" Eq. (5.48). We have put quotation marks around "proving", because ours will not be an analytic proof. Anyhow the identities that we will get will turn out to be rather usefull. Let us use the following notatio: $d\mu \equiv d\theta d\bar\theta$, $d\mu_1 \equiv d\theta_1 d\bar\theta_1$, etc. The first identity we want to prove is:

$$\int d\mu d\mu_1 \dots d\mu_n \mathbb{G}(z, z_1)\mathbb{G}(z_1, z_2)\dots\mathbb{G}(z_n, z) = 0. \tag{C.1}$$

We shall call it the "first *loop* identity", because the strings of $\mathbb{G}$ makes a super-loop. Let us first observe that

$$\begin{aligned}\int dz \mathbb{G}(z_1, z)\mathbb{G}(z, z_2) = & \int d^4x \left[G_R^{(B)}(x_1 - x)\Delta_\beta(x - x_2) + G_R^{(B)}(x_2 - x)\Delta_\beta(x_1 - x)\right] \\ & + \int d^4x \left[G_R^{(B)}(x_1 - x)G_R^{(B)}(x - x_2)\right]\bar{\theta}_2\theta_2 \\ & + \int d^4x \left[G_R^{(B)}(x_2 - x)G_R^{(B)}(x - x_1)\right]\bar{\theta}_1\theta_1 \\ & + \int d^4x \left[G_R^{(F)}(x_1 - x)G_R^{(F)}(x - x_2)\right]\theta_1\bar{\theta}_2 \\ & + \int d^4x \left[G_R^{(F)}(x_2 - x)G_R^{(F)}(x - x_1)\right]\theta_2\bar{\theta}_1.\end{aligned} \tag{C.2}$$

If we add to the l.h.s. of Eq. (C.2) one more $\mathbb{G}$ and build the expression

$$\int dz dz' \mathbb{G}(z_1, z)\mathbb{G}(z, z')\mathbb{G}(z', z_2), \tag{C.3}$$

we can then do the integration over the Grassmannian variables and easily prove that:

$$\begin{aligned}& \int dz dz' \mathbb{G}(z_1, z)\mathbb{G}(z, z')\mathbb{G}(z', z_2) \\ & = \int d^4x \left[G_R^{(B)}(x_1 - x)\Delta_\beta(x - x')G_R^{(B)}(x_2 - x') + \Delta_\beta(x_1 - x)G_R^{(B)}(x' - x)G_R^{(B)}(x' - x_2)\right. \\ & \left. + G_R^{(B)}(x_1 - x)G_R^{(B)}(x - x')\Delta_\beta(x' - x_2)\right] + \int d^4x \left[G_R^{(B)}(x_1 - x)G_R^{(B)}(x - x')G_R^{(B)}(x' - x_2)\right]\bar{\theta}_2\theta_2 \\ & + \int d^4x \left[G_R^{(B)}(x_2 - x')G_R^{(B)}(x' - x)G_R^{(B)}(x - x_1)\right]\bar{\theta}_1\theta_1 + \int d^4x \left[G_R^{(F)}(x_1 - x)G_R^{(F)}(x - x')\right. \\ & \left. \times G_R^{(F)}(x' - x_2)\right]\theta_1\bar{\theta}_2 + \int d^4x \left[G_R^{(F)}(x_2 - x')G_R^{(F)}(x' - x)G_R^{(F)}(x - x_1)\right]\theta_2\bar{\theta}_1.\end{aligned} \tag{C.4}$$

In general, with a string of all $\mathbb{G}$, we can prove by induction that the following relation holds (where $z \equiv z_0$, $z' \equiv z_{n+1}$):

$$
\begin{aligned}
&\int dz_1 dz_2 \dots dz_n \mathbb{G}(z,z_1)\mathbb{G}(z_1,z_2)\dots\mathbb{G}(z_n,z') \\
&= \left(G_R^{(B)} \star \dots \star \Delta_{\beta\, i,i+1} \star \dots \star G_R^{(B)}\right)(x_0,x_{n+1}) \\
&+ \left(G_R^{(B)} * \dots * G_R^{(B)}\right)(x_0,x_{n+1})\bar{\theta}_{n+1}\theta_{n+1} \\
&+ \left(G_R^{(B)} * \dots * G_R^{(B)}\right)(x_{n+1},x_0)\bar{\theta}_0\theta_0 \\
&+ \left(G_R^{(F)} * \dots * G_R^{(F)}\right)(x_0,x_{n+1})\theta_0\bar{\theta}_{n+1} \\
&+ \left(G_R^{(F)} * \dots * G_R^{(F)}\right)(x_{n+1},x_0)\theta_{n+1}\bar{\theta}_0.
\end{aligned}
\tag{C.5}
$$

The various symbols appearing above are defined as follows:

$$
\left(G_R^{(B)} * \dots * G_R^{(B)}\right)(x_0,x_{n+1}) = \int d^4x_1 \dots d^4x_n G_R^{(B)}(x_0-x_1)G_R^{(B)}(x_1-x_2)\dots G_R^{(B)}(x_n-x_{n+1}).
\tag{C.6}
$$

(we call this *"convolution".*) While the first expression on the r.h.s. of Eq. (C.5) is defined as:

$$
\begin{aligned}
&\left(G_R^{(B)} \star \dots \star \Delta_{\beta\, i,i+1} \star \dots \star G_R^{(B)}\right)(x_0,x_{n+1}) \equiv \\
&\equiv \int d^4x_1 \dots d^4x_n G_R^{(B)}(x_0-x_1)G_R^{(B)}(x_1-x_2)\dots G_R^{(B)}(x_{i-2}-x_{i-1})G_R^{(B)}(x_{i-1}-x_i) \\
&\times \Delta_\beta(x_i-x_{i+1})G_R^{(B)}(x_{i+2}-x_{i+1})G_R^{(B)}(x_{i+3}-x_{i+2})\dots G_R^{(B)}(x_n-x_{n+1})
\end{aligned}
\tag{C.7}
$$

(we call this "*oriented convolution*"). Eq. (C.5) can be proved by induction as follows. Let us take $P(n) =$ (Proposition of order n) to be given by Eq. (C.5). We already know, from the beginning of this appendix, that $P(1)$ holds true [see Eq. (C.2)]. Next, we need to show that $P(n) \Rightarrow P(n+1)$. Indeed, take $z' \equiv z_{n+1}$ in Eq. (C.5), multiply it by $\mathbb{G}(z_{n+1},z')$ and integrate over z_{n+1}. Using the definitions in Eq. (C.6) and Eq. (C.7) the results follows in a straightforward manner.

Let bus now go back to (C.5), take $z = z'$ and integrate over z, using the definitions (C.6) and (C.7), we easily get:

$$
\int dz dz_1 \dots dz_n \mathbb{G}(z,z_1)\mathbb{G}(z_1,z_2)\dots\mathbb{G}(z_n,z) = 0.
\tag{C.8}
$$

A second loop equality which can be proved is

$$
\int d\mu d\mu_1 \dots d\mu_n \mathbb{G}^m(z,z_1)\mathbb{G}^{m_1}(z_1,z_2)\dots\mathbb{G}^{m_n}(z_n,z) = 0.
\tag{C.9}
$$

This is a generalization of (C.8) which is recovered when $m = m_1 = \ldots = m_n = 1$.

Using the property **4)** of Eq. (5.17) we get that the l.h.s. of Eq. (C.9) is equal to:

$$
\begin{aligned}
&\int d\mu d\mu_1 \dots d\mu_n \mathbb{G}^m(z,z_1)\mathbb{G}^{m_1}(z_1,z_2)\dots\mathbb{G}^{m_n}(z_n,z) \\
&= \int d\mu d\mu_1 \dots d\mu_n \left[\Delta_\beta^m(x-x_1) + m\Delta_\beta^{m-1}(x-x_1)\mathcal{G}(z,z_1)\right] \\
&\times \left[\Delta_\beta^{m_1}(x_1-x_2) + m_1\Delta_\beta^{m_1-1}(x_1-x_2)\mathcal{G}(z_1,z_2)\right] \\
&\times \dots \times \left[\Delta_\beta^{m_n-1}(x_n-x) + m_n\Delta_\beta^{m_n-1}(x_n-x)\mathcal{G}(z_n,z)\right] \\
&= m m_1 \dots m_n \Delta_\beta^{m-1}(x-x_1)\Delta_\beta^{m_1-1}(x_1-x_2)\dots\Delta_\beta^{m_n-1}(x_n-x) \\
&\times \int d\mu d\mu_1 \dots d\mu_n \left[\mathcal{G}(z,z_1)\mathcal{G}(z_1,z_2)\dots\mathcal{G}(z_n,z)\right] = 0.
\end{aligned}
\tag{C.10}
$$

In the steps above we have made use of (C.8), and of the relations:

$$
\int d\mu d\mu_1 \dots d\mu_n \mathcal{G}(z,z_1)\mathcal{G}(z_1,z_2)\dots\mathcal{G}(z_{i-1},z_i)\Delta_\beta^{m_i}(x_i-x_{i+1})\mathcal{G}(z_{i+1},z_{i+2})\dots\mathcal{G}(z_n,z) = 0 \tag{C.11}
$$

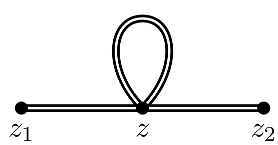
and

$$
\int d\mu d\mu_1 \dots d\mu_n \Delta_\beta^m(z,z_1)\Delta_\beta^{m_1}(z_1,z_2)\dots\Delta_\beta^{m_n}(z_n,z) = 0. \tag{C.12}
$$

Relations (C.11) and (C.12) can be easily proved considering the mismatch in the number of integration variables with respect to the variables present in the argument.

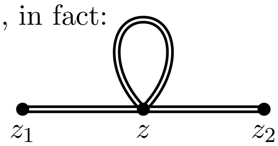
We should not forget that we have to prove relation (5.48). The $\mathbb{F}^{(n)}(z,z')$ will be any combination of two-point functions with any number of loops inserted in order to reach the order n-th in perturbation theory.

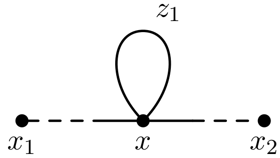
Let us start from $n=1$ which is:

$\frac{1}{2!}$ 

$= \mathbb{F}^{(1)}(z,z').$ (C.13)

It is easy to prove (5.48) for this $\mathbb{F}^{(1)}$, in fact:

$\int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2$ 

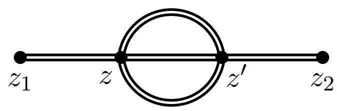
$= -$ 

(C.14)

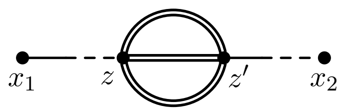
$$
= -\int dz G_R^{(B)}(x-x_1)\Delta_\beta(x-x)G_R^{(B)}(x-x_2) = 0
$$

This result comes from the fact that the integration contains $d\theta d\bar{\theta}$, but the integrant does not contain any θ, $\bar{\theta}$.

Let us pass to the $\mathbb{F}^{(2)}(z,z')$ functions. We could choose , for example, the following one:

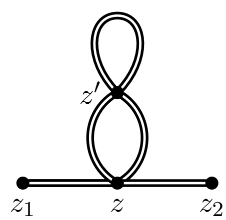
$\int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2$ 

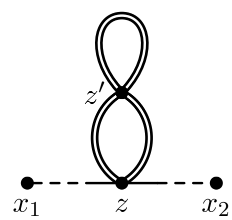
$=$

$= -$ 

$$\begin{aligned}
&= -\int dz dz' G_R^{(B)}(x-x_1)\left[\mathbb{G}(z,z')\right]^3 G_R^{(B)}(x'-x_2) \\
&= -\int d^4x d^4x' G_R^{(B)}(x-x_1) G_R^{(B)}(x'-x_2)\left[\int d\mu d\mu' \mathbb{G}^3(z,z')\right] \\
&= -d^4x d^4x' G_R^{(B)}(x-x_1) G_R^{(B)}(x'-x_2)\left[\int d\mu d\mu' \mathbb{G}^2(z,z')\mathbb{G}(z',z)\right]
\end{aligned} \tag{C.15}$$

In the last step we have used property **1)** of Eq. (5.14). The piece in the last equality of Eq. (C.15) contained in square brackets is zero because of the loop identity in Eq. (C.8). So, everything turns out to be zero. Another example of $\mathbb{F}^{(2)}(z,z')$ function is

$\int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2$ 

$= -$ 

$$\begin{aligned}
&= -\int dz dz' G_R^{(B)}(x-x_1)\mathbb{G}(z,z')\mathbb{G}(z',z')\mathbb{G}(z',z) G_R^{(B)}(x-x_2) \\
&= -d^4x d^4x' G_R^{(B)}(x-x_1) G_R^{(B)}(x'-x_2)\left[\int d\mu d\mu' \mathbb{G}(z,z')\mathbb{G}(z',z')\mathbb{G}(z',z)\right].
\end{aligned} \tag{C.16}$$

Once again the piece above contained in square brackets is zero because of the loop identity of Eq. (C.8).

We checked many two-point functions of the third order in perturbation theory and we got zero because of the loop identity (C.8). We do not have an analytical proof to all orders, but we believe, from the many examples we worked out, that (5.48) is true.

D. Jacobi Fields Correlations.

In this appendix , to all orders in perturbation theory, we will prove thart:

$$\langle c^\phi(x_1) c^\phi(x_2)\rangle_\beta = 0 \tag{D.1}$$

to all orders in perturbation theory. The reader could think that in $\tilde{\mathcal{L}}$ the c^{ϕ} and c^{ϕ} never interact and this could be the real reason behind (D.1). But, actually, the c^{ϕ} interact with ϕ and so, trough the ϕ, they could interact among themselves.

We know from (5.39) that

$$\int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2 \bar{\theta}_2 \theta_1 \langle \Phi(x_1)\Phi(x_2)\rangle_\beta = \langle c^{\phi}(x_1)c^{\phi}(x_2)\rangle_\beta. \tag{D.2}$$

The l.h.s at the n-th order of perturbation theory will have the general form:

$$\int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2 \int dz dz' \mathbb{G}(z_1, z)\mathbb{F}^{(n)}(z, z')\mathbb{G}(z', z_2) \tag{D.3}$$

where $\mathbb{F}^{(n)}$ is a function $\mathcal{O}(g^n)$ that we do not write down explicitly. For the first and second order its explicit form is

$$\mathbb{F}^{(1)} = \frac{1}{2}\mathbb{G}(z, z)\delta(z - z') \tag{D.4}$$

$$\mathbb{F}^{(2)} = \begin{cases} \frac{1}{6}\mathbb{G}^3(z, z') \\ \frac{1}{4}\mathbb{G}(z, z)\mathbb{G}(z, z')\mathbb{G}(z', z') \\ \frac{1}{4}\int dz''\mathbb{G}(z, z'')\mathbb{G}(z'', z'')\mathbb{G}(z'', z)\delta(z - z') \end{cases} \tag{D.5}$$

The explicit form of (D.3) turns out to be:

$$\begin{aligned}
(D.3) = & \int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2 \bar{\theta}_2 \theta_1 \int d^4x d\theta d\bar{\theta} \int d^4x' d\theta' d\bar{\theta}' \\
& \times \Big[\Delta_\beta(x_1 - x) + G_R^{(B)}(x_1 - x)\bar{\theta}\theta + G_R^{(B)}(x - x_1)\bar{\theta}_1\theta_1 \\
& + G_R^{(F)}(x_1 - x)\theta_1\bar{\theta} + G_R^{(F)}(x - x_1)\theta\bar{\theta}_1\Big]\, \mathbb{F}^{(n)}(z, z') \\
& \times \Big[\Delta_\beta(x' - x_2) + G_R^{(B)}(x' - x_2)\bar{\theta}_2\theta_2 + G_R^{(B)}(x_2 - x')\bar{\theta}'\theta' \\
& + G_R^{(F)}(x' - x_2)\theta'\bar{\theta}_2 + G_R^{(F)}(x_2 - x')\theta_2\bar{\theta}'\Big] \\
= & \int dz dz' \int d\theta_1 d\theta_2 d\bar{\theta}_1 d\bar{\theta}_2 \bar{\theta}_2 \theta_1 \Big[G_R^{(F)}(x_1 - x)\theta_1\bar{\theta}\Big]\, \mathbb{F}^{(n)}(z, z') \Big[G_R^{(F)}(x_2 - x')\theta_2\bar{\theta}'\Big] \\
= & \int dz dz' \theta\bar{\theta}' G_R^{(F)}(x_1 - x)\mathbb{F}^{(n)}(z, z')G_R^{(F)}(x_2 - x')
\end{aligned} \tag{D.6}$$

The last integral above is zero. The reason is the following: $\mathbb{F}^{(n)}$ comes from the integration of products of various G_R. As each G_R comes with an even number of $\hat{\theta}$ (where $\hat{\theta}$ indicates either θ or $\bar{\theta}$) attached , also $\mathbb{F}^{(n)}$ will have an even number of $\hat{\theta}$ because the integrations take away always an even number of $\hat{\theta}$. As $\mathbb{F}^{(n)}$ depends only on z and z' and has an even number of $\hat{\theta}$, the only possible contributions must have the following strings of $\hat{\theta}$:

1) $\theta\bar{\theta}\theta'\bar{\theta}'$ which gives zero because θ' is attached also to the final $G_R^{(B)}$ in the second equality in (D.6).

2) $\theta\bar{\theta}$ which gives zero because of the initial $G_R^{(B)}$ (the term $\bar{\theta}_1\theta$ in $G_R^{(B)}$ gives zero because of the projector $\bar{\theta}_1\theta_2$ and the same for θ_2).

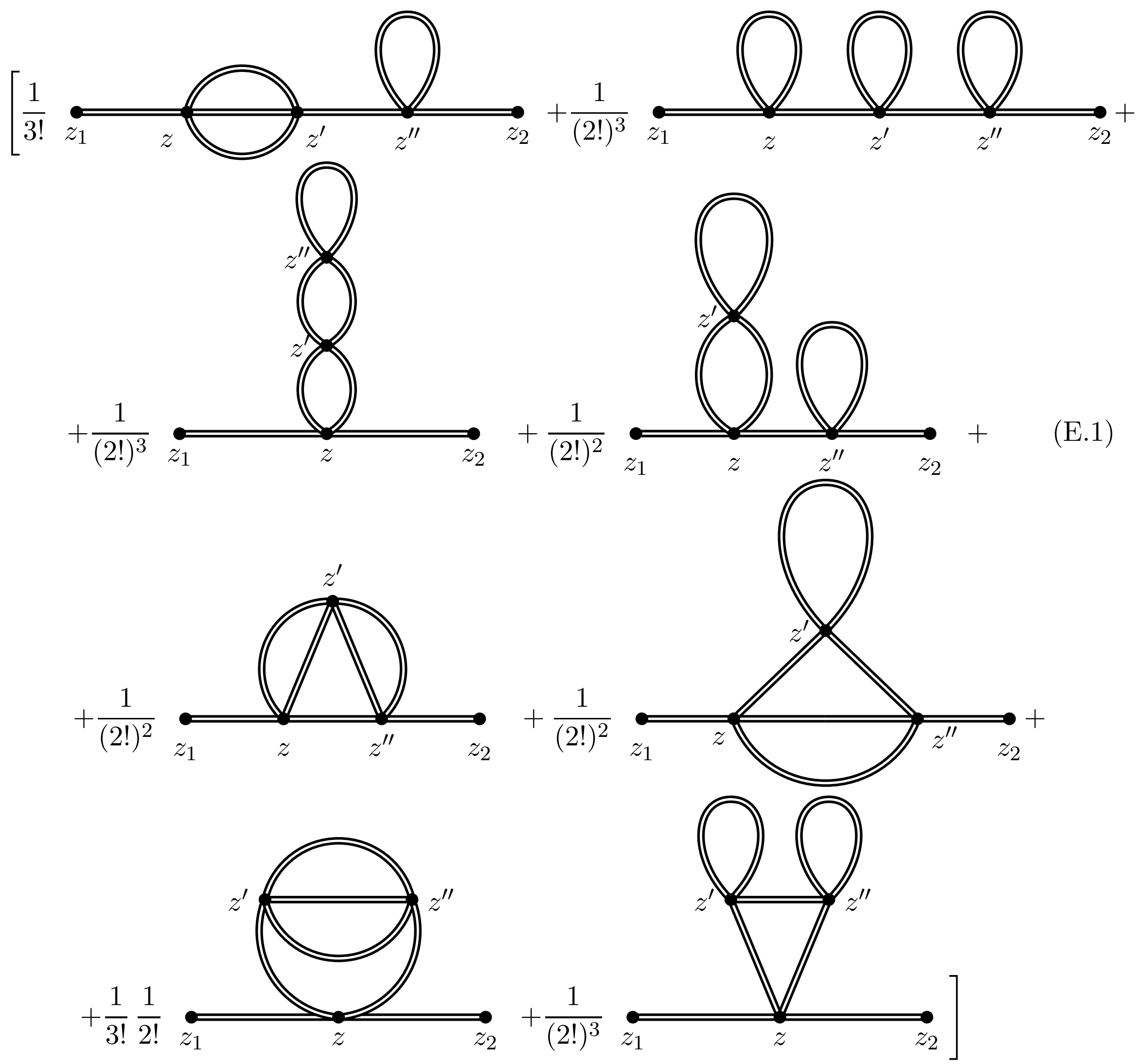
3) $\theta'\bar{\theta}'$ same as in 2) but because of the final $G_R^{(B)}$.

4) $\theta\bar{\theta}'$ same as in 3).

5) $\bar{\theta}\theta'$ same as in 2)

E. Third Order Results For The Two-point Correlation.

In this appendix we will provide the third order perturbative calculations for the $\langle\phi(x_1)\phi(x_2)\rangle_\beta$. correlation using the super-field technique. We will list below the super-diagrams that are needed together with the corresponding symmetry factors. These we calculate using the algebraic rules given in [29]. The set of super-diagrams needed are

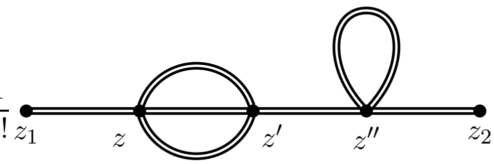


(E.1)

Also, by considering the first and fourth diagrams in eq. (E.1), we note that the diagrams obtained by the interchange ($z_1 \leftrightarrow z_2$) gives a contribution. So all together we end up with ten superdiagrams at the third order (In order to get the $\langle\phi\phi\rangle$ components of these graphs we have to integrate them with the projector $\bar{\theta}_1\theta_1\bar{\theta}_2\theta_2$ like we did in (5.21)

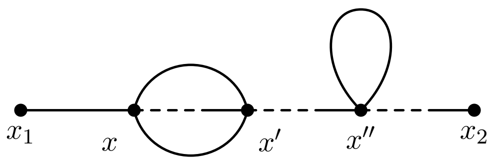
The calculations are long but straightforward and the result is

$$\int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 \quad \Bigg[\frac{1}{3!}$$



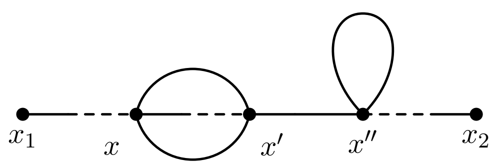
$$\Bigg] =$$

$$= \Bigg[\frac{1}{2!}$$



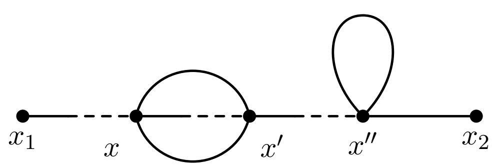
$$+$$

$$+\frac{1}{2!}$$



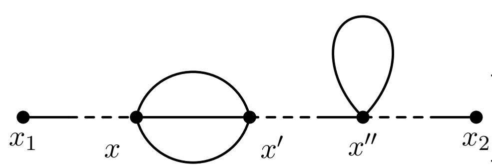
$$+$$

$$+\frac{1}{2!}$$



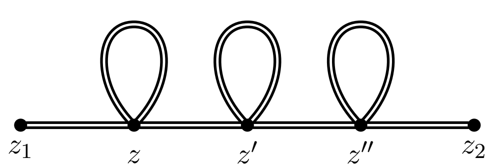
$$+$$

$$+\frac{1}{3!}$$

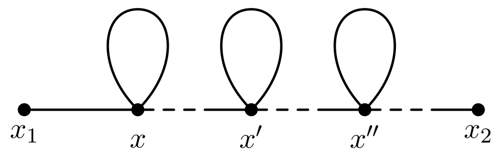


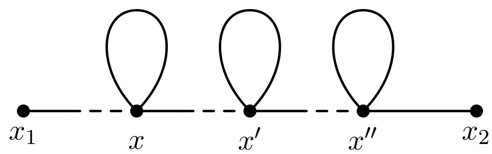
$$\Bigg]$$

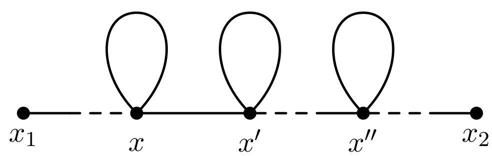
$$\int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \bar{\theta}_1\theta_1\bar{\theta}_2\theta_2 \quad \Bigg[\frac{1}{(2!)^3}$$

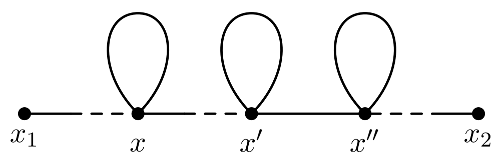


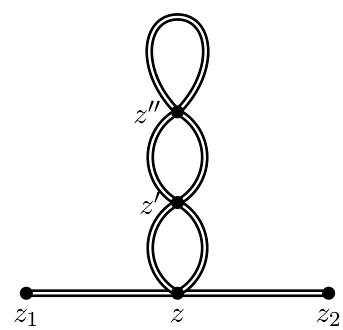
$$\Bigg] =$$

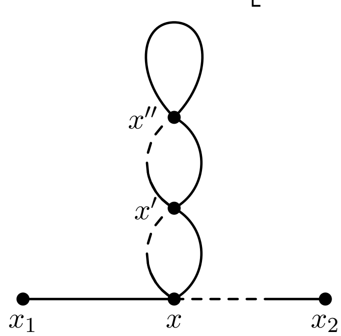
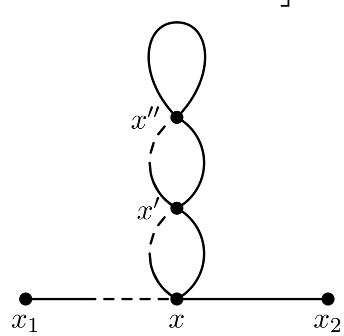
$= \Bigg[\dfrac{1}{(2!)^3}$  $+$

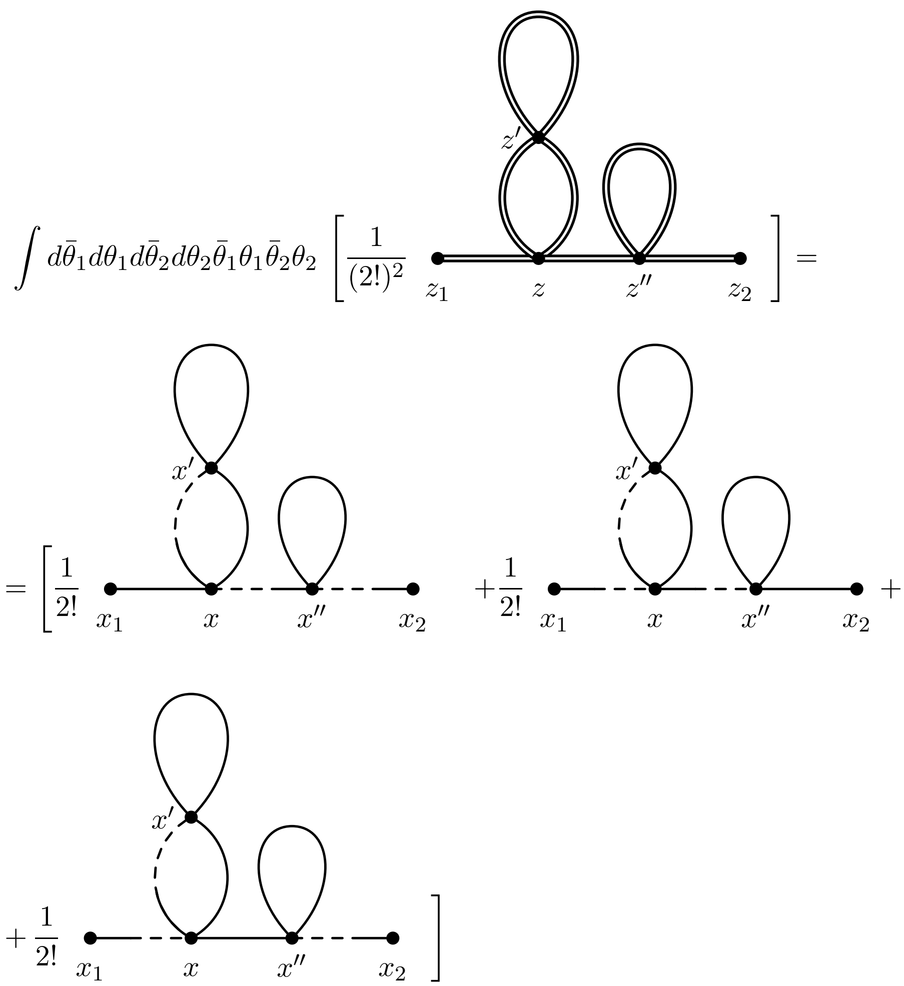
$+ \dfrac{1}{(2!)^3}$  $+$

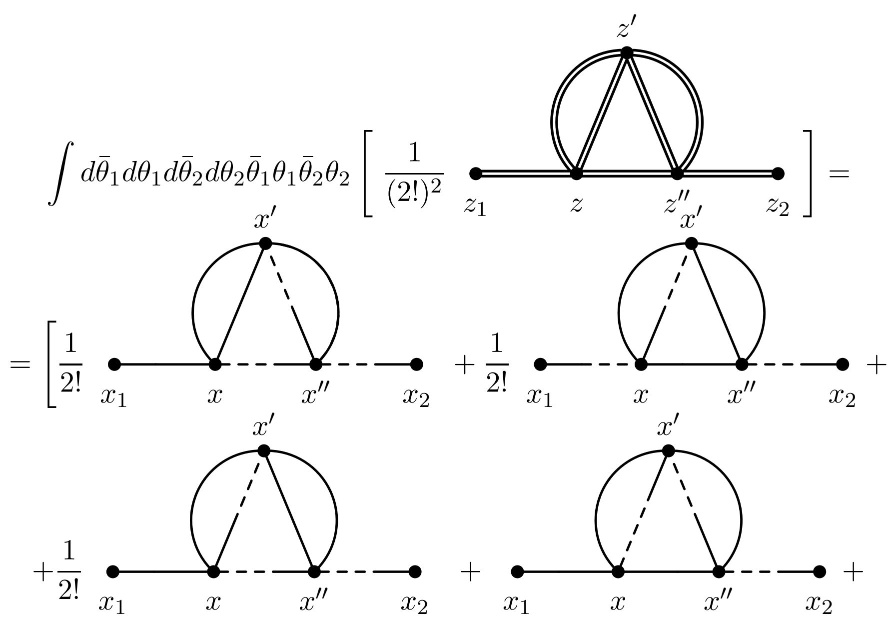
$+ \dfrac{1}{(2!)^3}$  $+$

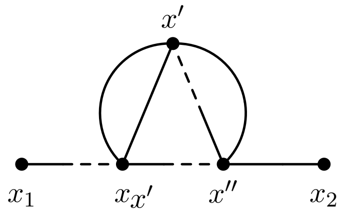
$+ \dfrac{1}{(2!)^3}$  $\Bigg]$

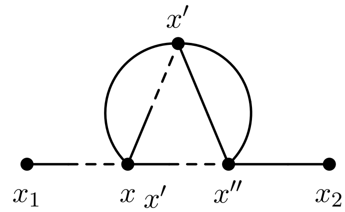
$\displaystyle\int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2 \Bigg[\frac{1}{(2!)^3}$  $\Bigg] =$

$= \Bigg[\dfrac{1}{2!}$  $+ \dfrac{1}{2!}$  $\Bigg]$


$\int d\bar\theta_1 d\theta_1 d\bar\theta_2 d\theta_2 \bar\theta_1\theta_1\bar\theta_2\theta_2 \left[\frac{1}{(2!)^2} \right] =$
z'
z_1 z z'' z_2
$= \left[\frac{1}{2!} \right.$ $+\frac{1}{2!}$ $+$
x'
x_1 x x'' x_2
$+\frac{1}{2!}$ $\left. \right]$


$\int d\bar\theta_1 d\theta_1 d\bar\theta_2 d\theta_2 \bar\theta_1\theta_1\bar\theta_2\theta_2 \left[\frac{1}{(2!)^2} \right] =$
z'
z_1 z z'' z_2
$= \left[\frac{1}{2!} \right.$ $+\frac{1}{2!}$ $+$
x'
x_1 x x'' x_2
$+\frac{1}{2!}$ $+$ $+$

$+\frac{1}{2!}$ 

$+\frac{1}{2!}$ 

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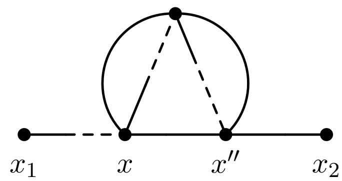
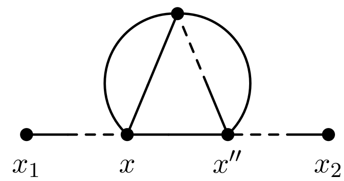
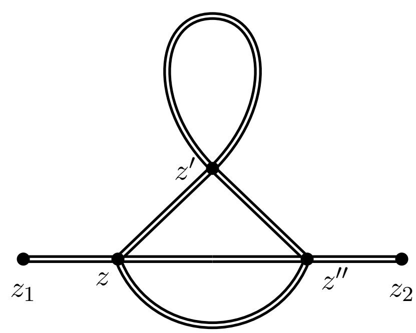
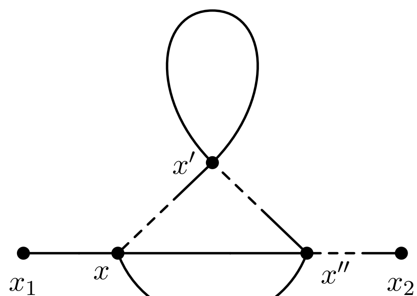
$+$ 

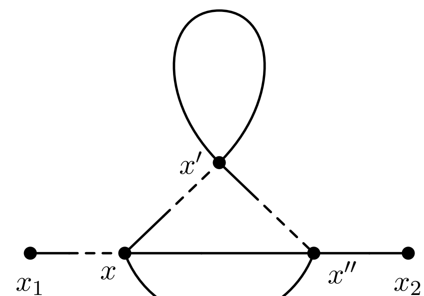
$+\frac{1}{2!}$ 

$\Big]$

$\int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2 \left[\frac{1}{(2!)^2} \right.$ 

$\Big] =$

$= \Big[\frac{1}{(2!)^2}$ 

$+\frac{1}{(2!)^2}$ 

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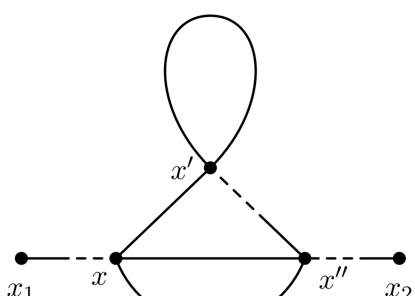
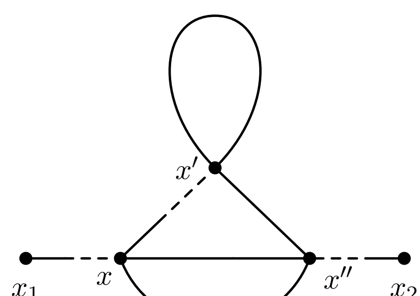
$+\frac{1}{(2!)^2}$ 

$+\frac{1}{(2!)^2}$ 

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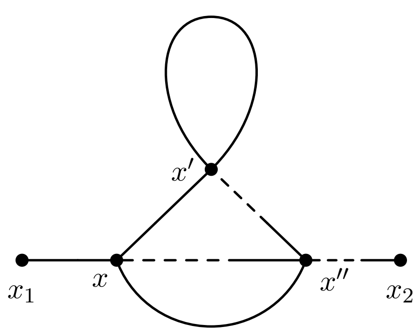
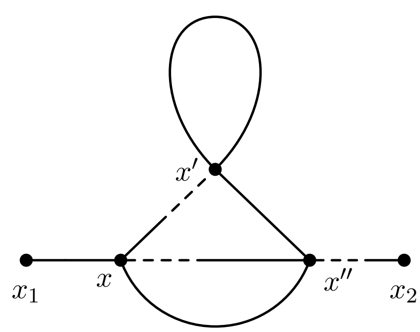
$+\frac{1}{2!}$ 

$+\frac{1}{2!}$ 

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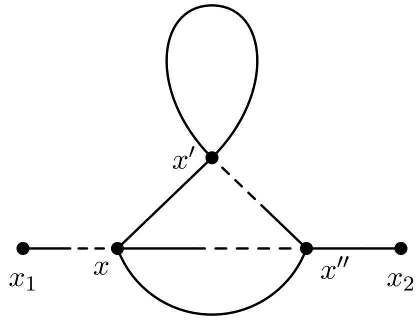
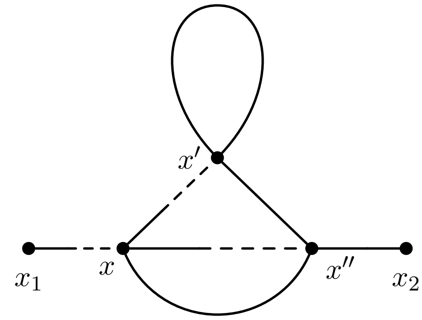
$+\frac{1}{2!}$

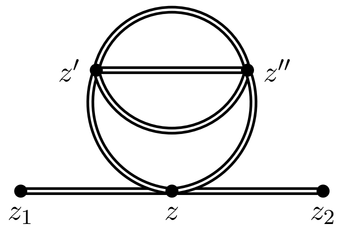


$+\frac{1}{2!}$



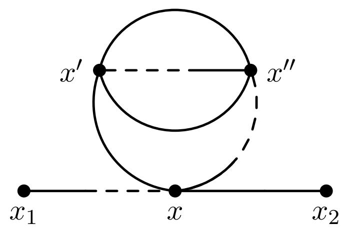
$\Big]$

$\int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2 \Bigg[\frac{1}{3!}\frac{1}{2!}$

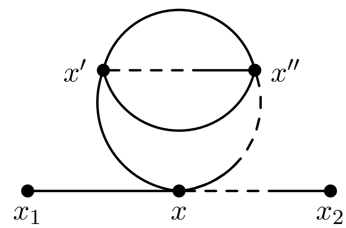


$\Bigg] =$

$\Bigg[\frac{1}{2!}$

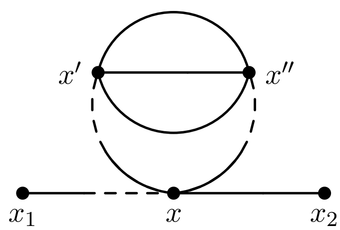


$+\frac{1}{2!}$

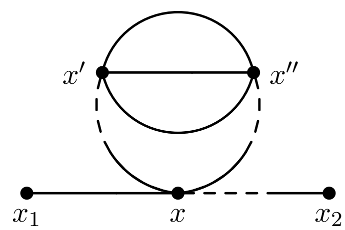


$+$

$\frac{1}{3!}\frac{1}{2!}$

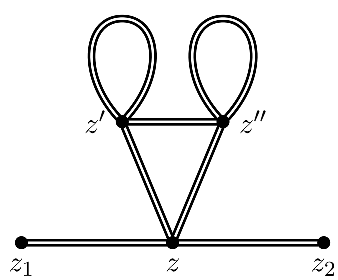


$+\frac{1}{3!}\frac{1}{2!}$



$\Bigg]$

$\int d\bar{\theta}_1 d\theta_1 d\bar{\theta}_2 d\theta_2 \bar{\theta}_1 \theta_1 \bar{\theta}_2 \theta_2 \Bigg[\frac{1}{(2!)^3}$



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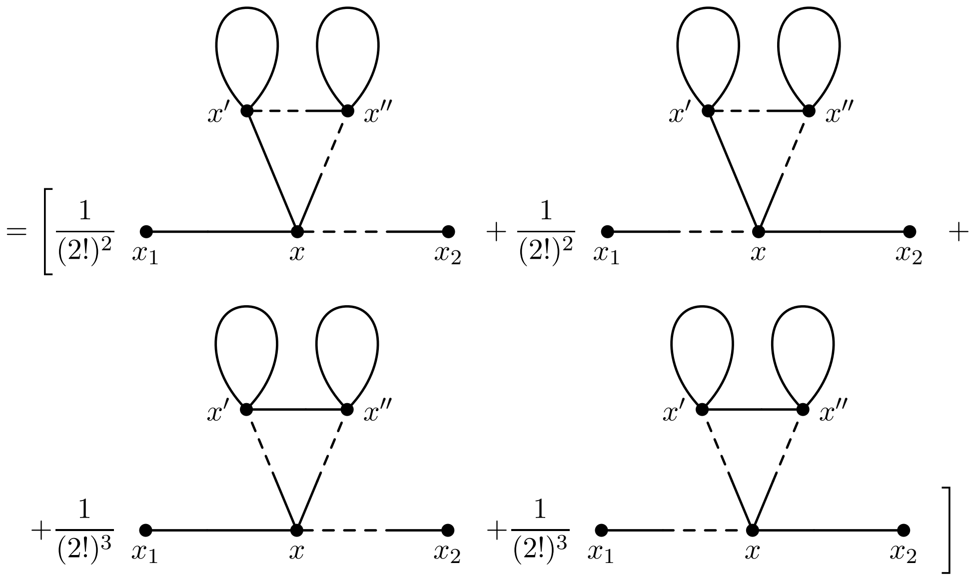


Of course using the set of super-diagrams (E.1) and different projectors we could get also the $\langle\phi\lambda\rangle$ and $\langle c\bar{c}\rangle$ correlations up to the third order.

F. Fluctuation-Dissipation Theorem Via Ward Identities.

In this appendix we will present a non-perturbative derivation of the fluctuation-dissipation theorem and of other relations that we proved perturbatively in the paper. All these relations are basically Ward identities associated to the various symmetries present in the CPI, i.e., BRS, $\overline{\text{BRS}}$, SUSY, etc. Most of the calculations presented in this appendix were done long ago by Martin Reuter whom we wish to warmly thank for his generosity.

In this appendix we will not work in the proper KvN formalism with an Hilbert space,etc,etc, but in the Liouville one where a probability density $\tilde{\rho}(\varphi)$ is used.

Let us suppose we give the initial probability density $\tilde{\rho}(-\infty)$ at time $t=-\infty$ and that we want to evaluate the average of a quantity $\mathcal{O}$. The path-integral expression for this average is:

$$\langle\mathcal{O}\rangle_{\tilde{\rho}(-\infty)} = \int \mathscr{D}\varphi\mathscr{D}\lambda\mathscr{D}c\mathscr{D}\bar{c}\,\mathcal{O}\,\tilde{\rho}\left(\varphi(-\infty),c(-\infty)\right)\exp\left(i\int_{-\infty}^{\infty}dt\tilde{\mathcal{L}}\right) \tag{F.1}$$

Let us ask for which symmetry transformation (whose infinitesimal transformation we indicate with δ) and for which $\tilde{\rho}$ we have

$$\langle\delta\mathcal{O}\rangle_{\tilde{\rho}(-\infty)} = 0. \tag{F.2}$$

Let us assume that the integration measure $\mathscr{D}\varphi\mathscr{D}\lambda\mathscr{D}c\mathscr{D}\bar{c}$ is invariant under the transformation δ and that the action $\tilde{\mathcal{S}}=\int\tilde{\mathcal{L}}dt$ changes only by a surface term

$$\delta\tilde{\mathcal{S}} = \epsilon\sigma(+\infty)-\epsilon\sigma(-\infty), \tag{F.3}$$

where ϵ is defined by $\varphi_\epsilon=\varphi+\delta\epsilon$.

Let us change the variables of integration in (F.1) going over to those obtained via a symmertry transformation. We obtain:

$$\begin{aligned}\langle\mathcal{O}\rangle_{\tilde{\rho}(-\infty)} &= \int \mathscr{D}\varphi\ldots(\mathcal{O}+\delta\mathcal{O})\left[\tilde{\rho}\,(-\infty)+\delta\tilde{\rho}\,(-\infty)\right]\exp\left(i\tilde{\mathcal{S}}+i\epsilon\sigma(+\infty)-i\epsilon\sigma(-\infty)\right)\\ &= \int \mathscr{D}\varphi\ldots\left\{\mathcal{O}\tilde{\rho}+\mathcal{O}\delta\tilde{\rho}+\tilde{\rho}\delta\mathcal{O}+\mathcal{O}\tilde{\rho}\left[i\epsilon\sigma(+\infty)-i\epsilon\sigma(-\infty)\right]\right\}e^{i\tilde{\mathcal{S}}}.\end{aligned} \tag{F.4}$$

The first term inside the integral fives $\langle\mathcal{O}\rangle_{\tilde{\rho}(-\infty)}$ which is equal to the LHS because we did only a change in integration variables and so these two terms cancell. In this manner we get for the third term in the integral the following expression:

$$\langle\delta\mathcal{O}\rangle_{\tilde{\rho}(-\infty)} = -\int \mathscr{D}\varphi\ldots\mathcal{O}\left\{\delta\tilde{\rho}(-\infty)+\tilde{\rho}(-\infty)i\epsilon\left[\sigma(+\infty)-\sigma(-\infty)\right]\right\}e^{i\tilde{\mathcal{S}}}. \tag{F.5}$$

For δ let us use the transformation generated by the super-symmetric charges [11]

$$Q_H = c^a\left(i\lambda_a-\beta\partial_a H\right) \tag{F.6}$$

$$\bar{Q}_H = \bar{c}^a\left(i\lambda_a+\beta\partial_b H\right) \tag{F.7}$$

where β is a real parameter which, when is set to zero, gives the BRS, $\overline{\mathrm{BRS}}$ [11] charges . Let us now proceed to find the surface terms associated to this symmetry transformations:

$$\delta_{Q_H}\tilde{\mathcal{L}} = -i\beta\frac{d}{dt}\epsilon N \tag{F.8}$$

$$\delta_{\bar{Q}_H}\tilde{\mathcal{L}} = -i\beta\frac{d}{dt}\bar{\epsilon}\bar{Q} \tag{F.9}$$

where $N\equiv c^a\partial_a H$ and $\bar{Q}\equiv\bar{c}_a\omega^{ab}i\lambda_b$. Let us now calculate the first term inside the brackets on the r.h.s of Eq. (F.5) for the transformation generated by Q_H

$$\delta\tilde{\rho}(-\infty) = \left(\delta\varphi^a+\delta c^a\frac{\partial}{\partial c^a}\tilde{\rho}\right)(t=-\infty) = \epsilon c^a(-\infty)\tilde{\rho}\left(\varphi(-\infty),c(-\infty)\right). \tag{F.10}$$

The surface term inside the brackets of Eq. (F.5) gives

$$i\epsilon\sigma(+\infty)-i\epsilon\sigma(-\infty) = \beta\epsilon\left(c^a\partial_a H\right)(+\infty)-\beta\epsilon\left(c^a\partial_a H\right)(-\infty). \tag{F.11}$$

Inserting all this in (F.5) we get:

$$\begin{aligned}\langle\delta_{Q_H}\mathcal{O}\rangle_{\tilde{\rho}} &= -\int \mathscr{D}\varphi\ldots\mathcal{O}\epsilon\left[c^a\,(-\infty)\,\frac{\partial}{\partial\varphi^a(-\infty)}-\beta c^a\,(-\infty)\,\partial_a H\,(\varphi(-\infty))\right]\\ &\quad\times\tilde{\rho}\left(\varphi(-\infty),c(-\infty)\right)e^{i\tilde{\mathcal{S}}}-\int\mathscr{D}\varphi\mathcal{O}\rho\epsilon c^a(+\infty)\partial_a H(+\infty)\tilde{\rho}(-\infty)e^{i\tilde{\mathcal{S}}} \qquad (\mathrm{F.12})\\ &= -\int\mathscr{D}\varphi\mathcal{O}\epsilon\hat{Q}_H\tilde{\rho}e^{i\tilde{\mathcal{S}}}-\beta\int\mathscr{D}\varphi\ldots\epsilon c^a\partial_a H\mathcal{O}\tilde{\rho}e^{i\tilde{\mathcal{S}}}\end{aligned}$$

For $\beta=0$ the second term vanish and the first is zero only if

$$\hat{Q}\tilde{\rho}=0 \tag{F.13}$$

where $\hat{Q} = \hat{Q}_H|_{\beta=0} = 0$. The solution of (F.13) is

$$\tilde{\rho} = \rho(\varphi)\delta(c). \tag{F.14}$$

Let us do the analog for the transformation generated by $\bar{Q}_H$. Calculating the various pieces we get

$$\langle\delta_{\bar{Q}_H}\mathcal{O}\rangle_{\tilde{\rho}} \equiv -\int \mathscr{D}\varphi \ldots \mathcal{O}\left(\epsilon\bar{Q}_H\tilde{\rho}\right)(-\infty)e^{i\tilde{\mathcal{S}}}. \tag{F.15}$$

This implies that $\langle\delta_{\bar{Q}_H}\mathcal{O}\rangle_{\tilde{\rho}} = 0$ if

$$\bar{Q}_H\tilde{\rho} = 0. \tag{F.16}$$

The solution of this equation was calculated in the second of Ref. [11] and is the canonical distribution

$$\tilde{\rho}(\varphi, c) = e^{-\beta H}\delta(c). \tag{F.17}$$

So the averages taken with this $\tilde{\rho}$ are exactly the thermal averages we have calculated all-along in the core of the paper.

Let us now choose as operator $\mathcal{O}$ the following one

$$\mathcal{O} \equiv \varphi^a(t)c^b(0) \quad , t > 0. \tag{F.18}$$

Doing the Q_H transformation of $\mathcal{O}$ we get:

$$\delta_{Q_H}\mathcal{O} = -\bar{\epsilon}\omega^{ad}\bar{c}_d(t)c^b(0) + \varphi^a(t)\epsilon\omega^{bd}\left[i\lambda_d(0) + \beta\partial_d H(0)\right]. \tag{F.19}$$

It was proved in [33][34][24] that

$$\langle\bar{c}_d(t)c^b(0)\rangle = 0 \tag{F.20}$$

if $t > 0$, where the average is taken with the canonical average (F.17). So we get from (F.19)

$$\langle\delta_{Q_H}\mathcal{O}\rangle = 0 = \epsilon\omega^{bd}\langle\varphi^a(t)\left[i\lambda_d(0) + \beta\partial_d H(0)\right]\rangle \tag{F.21}$$

which implies:

$$\langle\varphi^a(t)i\lambda_d(0)\rangle = -\beta\langle\varphi^a(t)\partial_d H(0)\rangle. \tag{F.22}$$

From the equation of motion we have

$$\partial_d H(0) = \omega_{de}\dot{\varphi}^e(0). \tag{F.23}$$

So (F.22) will become

$$\langle\varphi^a(t)i\lambda_d(0)\rangle = -\beta\omega_{de}\langle\varphi^a(t)\dot{\varphi}^e(0)\rangle. \tag{F.24}$$

Now in general for time-independent Hamiltonian we have, for two observables $A(t)$ and $B(t)$, that

$$\langle A(t_1)B(t_2)\rangle = F(t_1 - t_2) \tag{F.25}$$

so

$$\frac{\partial}{\partial t_1}\langle A(t_1)B(t_2)\rangle = -\frac{\partial}{\partial t_2}\langle A(t_1)B(t_2)\rangle \tag{F.26}$$

or

$$\langle \dot{A}(t_1)B(t_2)\rangle = -\langle A(t_1)\dot{B}(t_2)\rangle. \tag{F.27}$$

Applying this last relation to (F.24) we get:

$$\langle \varphi^a(t) i\lambda_d(0)\rangle = -\beta\omega_{de}\langle \varphi^a(t)\dot{\varphi}^e(0)\rangle. \tag{F.28}$$

and this is the *fluctuation-dissipation* theorem.

With respect to the easier proof presented in [16] our proof has the virtue that it can be seen as a Ward-identity of Susy and that the canonical distribution is not put by hand, but it is derived [11] as a super-symmetric invariant state.

Let us now prove few more identities. Let us choose $\tilde{\rho}(\varphi, c) = \delta^{2n}(c(-\infty))$. This distribution is BRS invariant [11] so:

$$\langle \delta_{\rm BRS}\mathcal{O}\rangle = \int \mathscr{D}\varphi \ldots (\delta_{\rm BRS}\mathcal{O})\, \delta^{2n} c(-\infty) e^{i\tilde{S}} = 0. \tag{F.29}$$

where the BRS transformation [11] is

$$\begin{cases} \delta\varphi^a = \epsilon c^a \\ \delta c^a = 0 \\ \delta\bar{c}_a = i\epsilon\lambda_a \\ \delta\lambda_a = 0. \end{cases}$$

Let us take $\hat{\mathcal{O}} = \varphi^a(t)\bar{c}_b(0)$, then

$$\delta\hat{\mathcal{O}} = -i\epsilon\left[ic^a(t)\bar{c}_b(0) - \varphi^a(t)\lambda_b(0)\right]. \tag{F.30}$$

So $\langle\delta\hat{\mathcal{O}}\rangle = 0$ implies

$$\langle \varphi^a(t)\lambda_b(0)\rangle = i\langle c^a(t)\bar{c}_b(0)\rangle. \tag{F.31}$$

This relation tell us that the dash-full propagator is equal to the Fermion one and to all order in perturbation theory. This confirms the statement that we made all-along in our paper that this identity is related to the Susy invariance (better say to the BRS invariance which is the susy with $\beta = 0$).

Let us now choose as operator

$$\hat{\mathcal{O}} = \lambda_a(t)\bar{c}_b(0), \tag{F.32}$$

then

$$\delta\hat{\mathcal{O}} = i\epsilon\lambda_a(t)\lambda_a(0), \tag{F.33}$$

and from $\langle\delta\hat{\mathcal{O}}\rangle = 0$ we obtain:

$$\langle \lambda_a(t)\lambda_a(0)\rangle = 0. \tag{F.34}$$

This is "only" similar to the one we proved in the paper perturbatively to all orders, because there we proved it with the weight $\rho = e^{-\beta H}\delta^n(c)$, while here we proved it only for $\beta = 0$, but we think the proof given here should go through also for $\beta \neq 0$.

Let us now choose as operator

$$\hat{\mathcal{O}} = c^a(t)\varphi^b(0), \tag{F.35}$$

then

$$\delta\hat{\mathcal{O}} = -\epsilon c^a(t)c^b(0), \tag{F.36}$$

so we get

$$\langle c^a(t)c^b(0)\rangle = 0. \tag{F.37}$$

This relation for $a = b = 1$ it was proved to all orders of perturbation theory in the paper and for the canonical distribution.Here we have given a non-perturbative proof.

We like to conclude this appendix by saying that both the simplification in the diagrammatics that we got via the introduction of super-diagrams and the various identites we found that will further simplify the diagrammatics, have their roots in the various symmetries of the CPI.

G. Perturbative Check of The Fluctuation-Dissipation Theorem.

In this appendix we will prove the relations (6.11) and (6.12).

Let us remember that

$$\Delta_\beta(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4}\frac{2\pi}{\beta|p^0|}\delta(p^2 - m^2)e^{-ip.x} \tag{G.1}$$

where $p.x = p^0 t - \vec{p}.\vec{x}$ and $p^2 = (p^0)^2 - \vec{p}^2$.

Let us take the derivative with respect to t_1, like on the l.h.s of Eq. (6.11)

$$\frac{\partial}{\partial t_1}\Delta_\beta(x_1 - x_2) = -i\int \frac{d^4p}{(2\pi)^4}\frac{2\pi}{\beta}\frac{p^0}{|p^0|}\delta(p^2 - m^2)e^{-ip.x}. \tag{G.2}$$

Let us remember that:

$$\frac{p^0}{|p^0|} = \theta(p^0) - \theta(-p^0) \equiv \Theta(p^0), \tag{G.3}$$

where we have defined the new symbol $\Theta(p^0)$. Let us also notice that:

$$\delta\left((p^0)^2 - \vec{p}^2 - m^2\right) = \frac{1}{2E_{\vec{p}}}\delta\left(p^0 - E_{\vec{p}}\right) - \delta\left(p^0 + E_{\vec{p}}\right), \tag{G.4}$$

and so we get:

$$\Theta(p^0)\delta\left((p^0)^2 - \vec{p}^2 - m^2\right) = \frac{1}{2E_{\vec{p}}}\delta\left(p^0 - E_{\vec{p}}\right) - \delta\left(p^0 + E_{\vec{p}}\right). \tag{G.5}$$

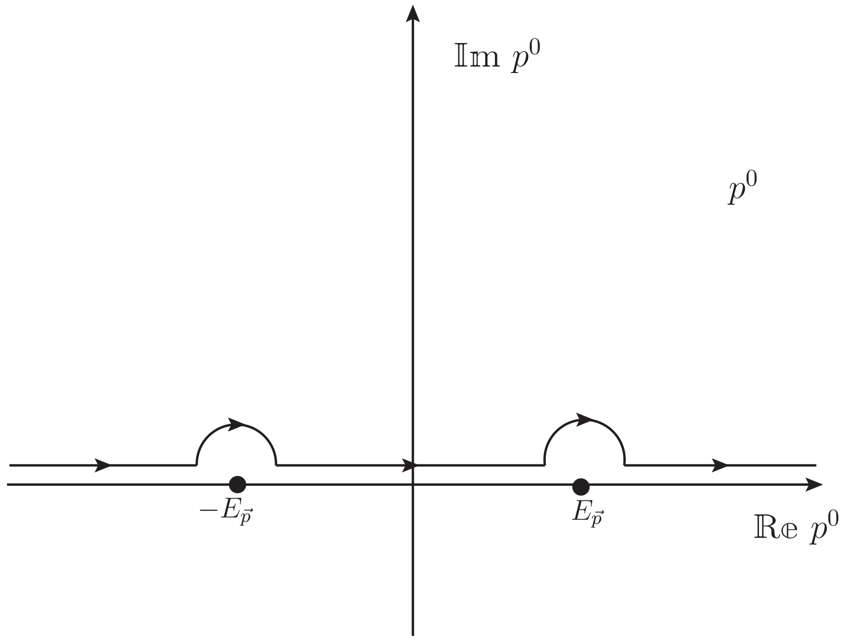
We can now rewrite (G.2) as

$$\begin{aligned}\frac{\partial}{\partial t_1}\Delta_\beta(x_1 - x_2) = &-i\frac{1}{\beta}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{p^0}{2E_{\vec{p}}}e^{-iE_{\vec{p}}(t_2 - t_1)}e^{i\vec{p}(\vec{x}_1 - \vec{x}_2)}\\ &-i\frac{1}{\beta}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{p^0}{2E_{\vec{p}}}e^{iE_{\vec{p}}(t_1 - t_2)}e^{-i\vec{p}.(\vec{x}_1 - \vec{x}_2)}\end{aligned} \tag{G.6}$$

Let us now turn to the r.h.s. of Eq. (6.11) and let us remember that G_R solves the equation:

$$\left(\Box + m^2\right) G_R = -\delta(x) \tag{G.7}$$

with $G_R(x) = 0$ for $t < 0$. Let us then choose the contour C_R of integration along the real axis in the complex p plane in such a manner that the two poles are moved below the real



axis.
For $t > 0$ we close the contour in the lower half plane:

$$G_R(x) = \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{(p^0 + i\epsilon)^2 - E_{\vec{p}}^2} = \int_{C_R} \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{(p^0 + i\epsilon)^2 - E_{\vec{p}}^2}. \tag{G.8}$$

So by the residue theorem we get:

$$G_R(x) = i \int_{-\infty}^{\infty} \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \frac{e^{-ipx}}{(p^0 + i\epsilon)^2 - E_{\vec{p}}^2} - i \int_{-\infty}^{\infty} \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \frac{e^{-ipx}}{(p^0 + i\epsilon)^2 - E_{\vec{p}}^2} \tag{G.9}$$

and comparing with Eq. (G.6) we obtain:

$$G_R(x) = G_R(x_1 - x_2) = \beta \frac{\partial}{\partial t_1} \Delta_\beta(x_1 - x_2). \tag{G.10}$$

For $t < 0$ we get that $G_R(x)$ is zero, so:

$$G_R(x) = \beta\theta(t) \frac{\partial}{\partial t} \Delta_\beta(x). \tag{G.11}$$

In this relation let us now turn $x \to -x$ and we get:

$$G_R(-x) = \beta\theta(-t) \frac{\partial}{\partial(-t)} \Delta_\beta(-x) = \beta\theta(t) \frac{\partial}{\partial(-t)} \Delta_\beta(x) = -\beta\theta(t) \frac{\partial}{\partial t} \Delta_\beta(x) \tag{G.12}$$

So for $t < 0$ we have:

$$\frac{\partial}{\partial t} \Delta_\beta(x) = -\frac{1}{\beta} G_R(-x) \tag{G.13}$$

Combining (G.11) with (G.13) we can write:

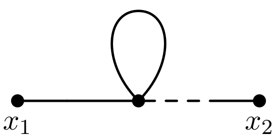
$$\frac{\partial}{\partial t}\Delta_\beta(x) = \frac{1}{\beta}G_R(x) - \frac{1}{\beta}G_R(-x) \tag{G.14}$$

which is exactly the (6.11) or the FDT theorem at the zero order that is what we wanted to prove.

We will now move on to prove Eq. (6.12). Let us first note that from (G.14) we obtain:

$$G_R(x_1 - x) = \beta\theta(x_1 - x)\frac{\partial}{\partial t_1}\Delta_\beta(x_1 - x). \tag{G.15}$$

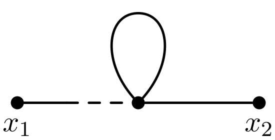
The first term on the l.h.s. of (6.12) has the following analytical expression:

$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

$$= \frac{1}{2}\frac{\partial}{\partial t_1}\int d^4x\Delta_\beta(x_1 - x)\Delta_\beta(x - x)G_R^{(B)}(x_2 - x)$$
$$= \frac{1}{2}\int d^4x\frac{1}{\beta}\left[G_R^{(B)}(x_1 - x) - G_R^{(B)}(x - x_1)\right]\Delta_\beta(x - x)G_R^{(B)}(x_2 - x) \tag{G.16}$$

where we have made use of the relation (G.14).

The second term on the l.h.s. of (6.12) has the following analytical expression:

$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

$$= \frac{1}{2}\int d^4x\frac{\partial}{\partial t_1}G_R^{(B)}(x_1 - x)\Delta_\beta(x - x)\Delta_\beta(x - x_2). \tag{G.17}$$

Let us now take the time derivative of (G.15):

$$\frac{\partial}{\partial t_1}G_R^{(B)}(x_1 - x) = \beta\delta(t_1 - t)\frac{\partial}{\partial t_1}\Delta_\beta(x_1 - x) + \beta\theta(t_1 - t)\frac{\partial^2}{\partial t_1^2}\Delta_\beta(x_1 - x). \tag{G.18}$$

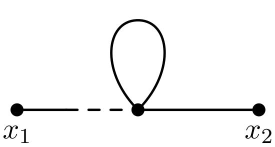
Moreover it is easy to prove that

$$\delta(t_1 - t)\frac{\partial}{\partial t_1}\Delta_\beta(x_1 - x) = 0, \tag{G.19}$$

and therefore, (G.18) becomes

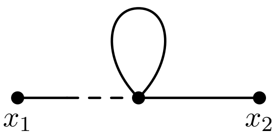
$$\frac{\partial}{\partial t_1}G_R^{(B)}(x_1 - x) = \beta\theta(t_1 - t)\frac{\partial^2}{\partial t_1^2}\Delta_\beta(x_1 - x). \tag{G.20}$$

As a consequence the (G.17) is turned into:

$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

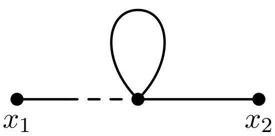
$$= \frac{1}{2}\beta\int d^4x\theta(t_1 - t)\frac{\partial^2}{\partial t_1^2}\Delta_\beta(x_1 - x)\Delta_\beta(x - x)\Delta_\beta(x - x_2). \tag{G.21}$$

Performing now an integration by parts and disregarding surface terms Eq. (G.21) can be rewritten as

$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

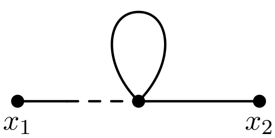
$$= -\frac{1}{2}\beta \int d^4x \delta(t_1 - t)\Delta_{\beta\, t_1}(x_1 - x)\Delta_\beta(x - x)\Delta_\beta(x_2 - x)$$
$$-\frac{1}{2}\beta \int d^4x \theta(t_1 - t)\Delta_{\beta\, t_1}(x_1 - x)\Delta_\beta(x - x)\Delta_{\beta\, t_2}(x_2 - x) \tag{G.22}$$

where $\Delta_{\beta\, t_i}(x_i - x) \equiv \partial\Delta_\beta(x_i - x)/\partial t_i$. Using (G.19) we get:

$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

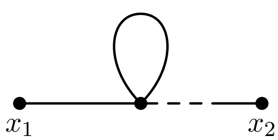
$$= -\frac{1}{2}\beta \int d^4x \theta(t_1 - t)\Delta_{\beta\, t_1}(x_1 - x)\Delta_\beta(x - x)\Delta_{\beta\, t_2}(x_2 - x). \tag{G.23}$$

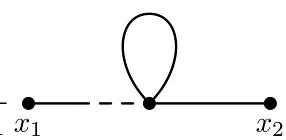
If on the r.h.s. of the equation above we use (G.14), we obtain:

$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

$$= -\frac{1}{2}\frac{1}{\beta} \int d^4x G_R^{(B)}(x_1 - x)\Delta_\beta(x - x)\left[G_R^{(B)}(x_2 - x) - G_R^{(B)}(x - x_2)\right]. \tag{G.24}$$

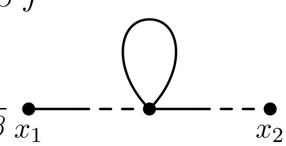
Combining (G.16) with (G.24) we get:

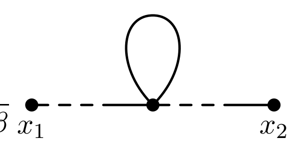
$\frac{1}{2}\frac{\partial}{\partial t_1}$ 

$+ \frac{1}{2}\frac{\partial}{\partial t_1}$ 

$=$

$$= \frac{1}{2}\frac{1}{\beta} \int d^4x G_R^{(B)}(x_1 - x)\Delta_\beta(x - x)G_R^{(B)}(x - x_2)$$
$$-\frac{1}{2}\frac{1}{\beta} \int d^4x G_R^{(B)}(x - x_1)\Delta_\beta(x - x)G_R^{(B)}(x_2 - x)$$

$= \Bigg[\frac{i}{2\,\beta}$ 

$- \frac{i}{2\,\beta}$ 

$\Bigg].$ (G.25)

This is exactly the relation (6.12) which is the FDT of the first order in perturbation theory.

Long but tedious calculations allows us to derive the FDT also at the second order in perturbation theory which is diagrammatically represented by the the three formulas (6.13), (6.14) and (6.15).

For the first order contribution to the FDT an alternative and simpler derivation than the one presented here can be given either using the momentum representation or the super-field formalism. We will present them below:

Momentum space derivation: Eq. (6.11), which is the FDT at zero order, is:

$$\frac{\partial}{\partial t_1}\Delta_\beta(x_1 - x_2) = \frac{i}{\beta}\left[-iG_R(x_1 - x_2) + iG_R(x_2 - x_1)\right]. \tag{G.26}$$

Going into momentum space we get:

$$
\begin{aligned}
&\frac{\partial}{\partial t_1}\int \frac{d^4p}{(2\pi)^4} e^{-ip.(x_1-x_2)}\Delta_\beta(p) \\
&= \frac{i}{\beta}\left\{\int \frac{d^4p}{(2\pi)^4} e^{-ip.(x_1-x_2)}\left[-iG_R(p)\right] + \int \frac{d^4p}{(2\pi)^4} e^{-ip.(x_1-x_2)}\left[iG_R(-p)\right]\right\},
\end{aligned} \tag{G.27}
$$

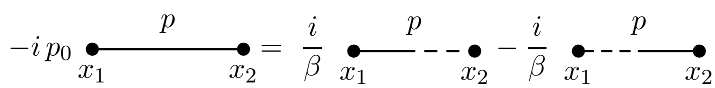
or equivalently:

$$
\begin{aligned}
&\int \frac{d^4p}{(2\pi)^4}\left[-ip^0 e^{-ip.(x_1-x_2)}\right]\Delta_\beta(p) \\
&= \frac{i}{\beta}\left\{\int \frac{d^4p}{(2\pi)^4} e^{-ip.(x_1-x_2)}\left[-iG_R(p)\right] + \int \frac{d^4p}{(2\pi)^4} e^{-ip.(x_1-x_2)}\left[iG_A(p)\right]\right\},
\end{aligned} \tag{G.28}
$$

where G_A is the advanced propagator. Formally we shall write the relation above as:

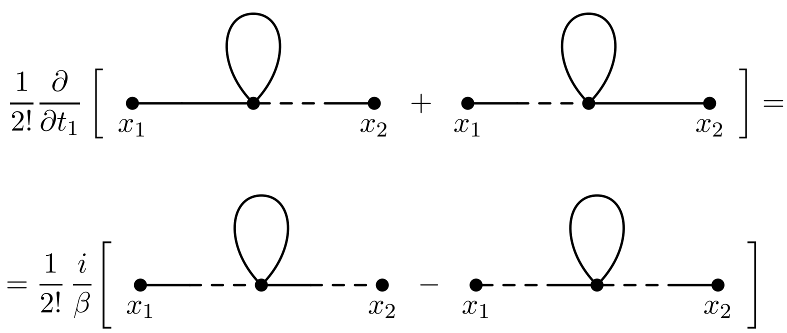
$$
-ip^0\Delta_\beta(p) = \frac{i}{\beta}\left[-iG_R(p) + iG_A(p)\right], \tag{G.29}
$$

which diagrammatically is:



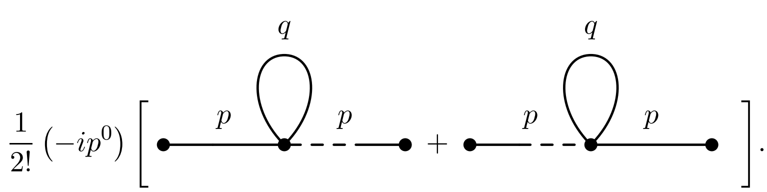
(G.30)

The FDT at first order (6.12) is given by:



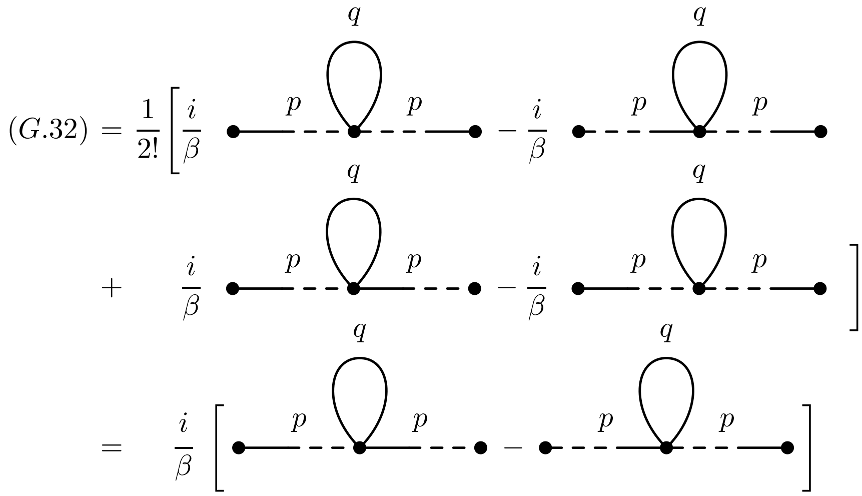
(G.31)

The l.h.s. of (G.31) in momentum space is:



(G.32)

Let us now make use of (G.29) in the following manner: let us replace the first leg of the first diagram in (G.32) using the relation (G.29) and the same for the last leg of the second diagram in (G.32). Therefore, we get:



(G.33)

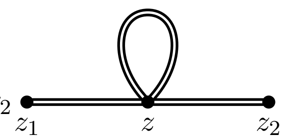
and this is exactly the momentum space representation of (G.31).

One can notice how much simpler is this derivation than the one in configuration space that we presented earlier.

Super-Diagram Derivation:

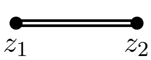
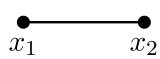
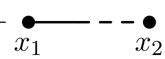
Let us remember that:

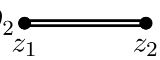
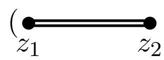
$$\langle\phi(x_1)\phi(x_2)\rangle^{(1)} = \frac{1}{2}\int d\nu_1 d\nu_2 \tag{G.34}$$



where we have used the notation $d\nu_i \equiv d\theta_i d\bar{\theta}_i \bar{\theta}_i \theta_i$ $(i = 1, 2)$ and the upper index (1) on the l.h.s. stays for first order in perturbation theory.

It is then easy to prove that:

$\left\langle \int d\nu_1\right.$  $\left.\right\rangle_\beta =$  $+$  $\bar{\theta}_2\theta_2$

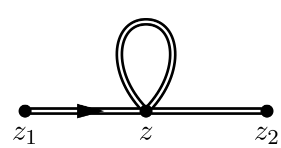
$= \int d\nu_1 \left[\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\bar{\theta}_2}\left(\bar{\theta}_2\theta_2\right.\right.$  $\left.\right) + \bar{\theta}_2\theta_2\frac{\partial}{\partial\theta_2}\frac{\partial}{\partial\bar{\theta}_2}\left(\right.$  $\left.\left.\right)\right].$

(G.35)

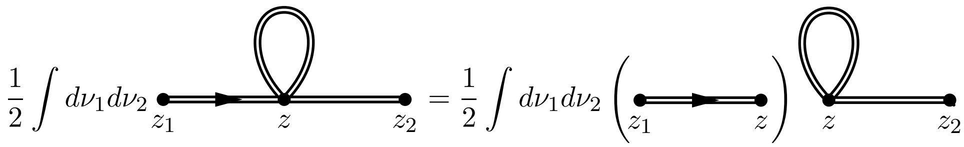
The first equality above derives from the fact that in x_1 only the field ϕ survives the integration over $d\nu_1$. At the point z_2 in principle there could be any of the fields ϕ, λ, c, $\bar{c}$, but only the correlations $\langle\phi\phi\rangle$ and $\langle\phi\lambda\rangle$ survive because the $\langle\phi c\rangle$, $\langle\phi\bar{c}\rangle$ are zero as we proved before in the paper. The second equality in (G.35) is easy to prove: the first term is exactly equal to the first term of the first equality. In fact, in z_2 only ϕ survives which is the only term which does not carry extra θ_2 and in z_1 only ϕ survives because of the integration $d\nu_1$. An analogous reasoning applies to the second term.

Let us now go back to (G.34) and do the time derivative of both, the l.h.s. and r.h.s.

$$\frac{\partial}{\partial t_1}\langle\phi(x_1)\phi(x_2)\rangle^{(1)} = \frac{1}{2}\int d\nu_1 d\nu_2 \tag{G.36}$$

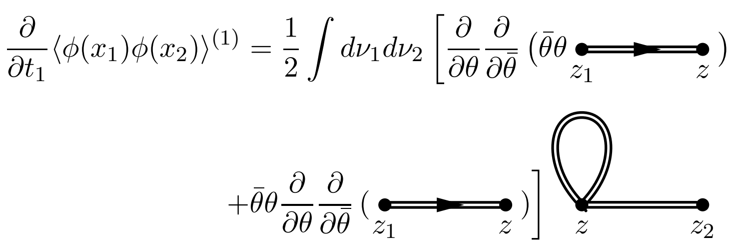


where with the arrow  we indicate the time derivative. We can formally write the r.h.s. of (G.36) as



(G.37)

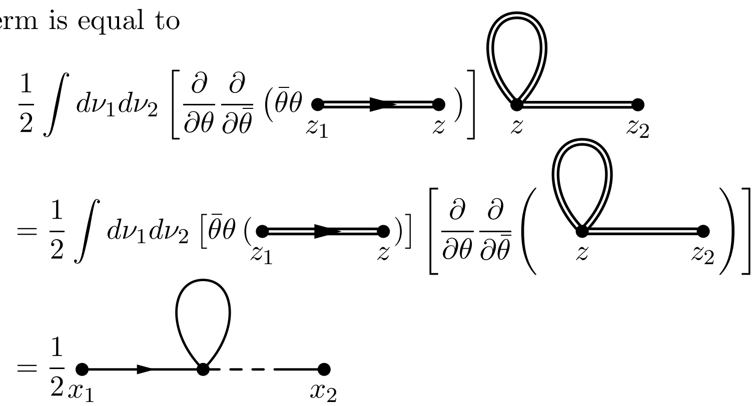
Using the relation (G.35) on the first term of the r.h.s. above, we get:



(G.38)

Let us analyze the two terms in the square brackets on the r.h.s. of (G.38).

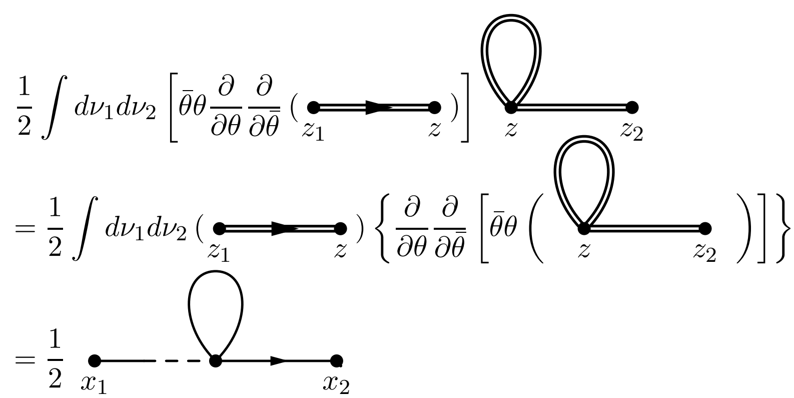
The first term is equal to



(G.39)

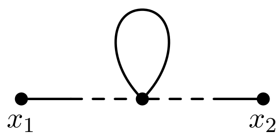
where we have used integration by parts over Grassmann variables to obtain the first equality and we have performed the integration to obtain the second one.

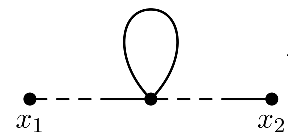
Similarly, the second term in the square brackets on the r.h.s. of (G.38) gives



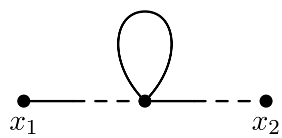
(G.40)

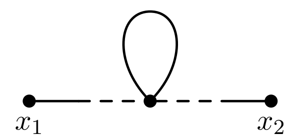
Replacing now in (G.39) and (G.40) the continuous line with the arrow (which stays for the time derivative) with the difference of the dash-full line, like it is indicated in the FDT at zero order (6.11) , we get that:

$(G.39) = \Big[\frac{i}{2\,\beta}$ 

$-\ \frac{i}{2\,\beta}$ 

$\Big]$ (G.41)

$(G.40) = \Big[\frac{i}{2\,\beta}$ 

$-\ \frac{i}{2\,\beta}$ 

$\Big].$ (G.42)

Summing up the two relations above we get (6.12) which is the FDT to first order in perturbation theory.

Both, the momentum and the super-field techniques can be used to go to higher order in the FDT. The calculations turn out to be simpler than in standard space, but still rather tedious, so we will avoid to report them here.

This last part of this appendix has been somehow a demonstration of the many things one can prove by "manipulating" the Feynman diagrams without resorting to their analytical expressions. These manupulations are "analogous" to what was done in the original "*diagrammar*" paper [1] for Quantum Field Theory, so we hope the use we made of the same word "*diagrammar*" is forgiven.

References

[1] G. 't Hooft and M. Veltman, "Diagrammar" CERN 73-9-REPORT.

[2] D. D'enterria, Nucl. Phys. A 782 (2007) 215; M.J. Tannenbaum Int. Jour. Mod. Phys.E 17 (2008) 771.

[3] K. Yagi, T. Hatsuda and Y. Miake, "*Quark-gluon plasma: from big bang to little bang*", Cambridge monographs on particle physics, nuclear physics and cosmology, vol 23 pag 446 2005, Cambridge University Press U.K.

[4] S.Galow et al.Nucl.Phys.B834,330 (2010), 0911.0627.

[5] L.V. Gribov, E.R. Levin and M.G. Riskin, Phys. Rep. 100 (1983) 1; J.P. Blaizot and A.H. Mueller, Nucl. Phys. B 239 (1997) 847.

[6] L. McLerran and R. Venugopalan, Phys. Rev. D 49 (1994) 3352; ibid D 49 (1994) 2233.

[7] G. Aarts and J. Smit, Phys. Lett. B 393 (1997); Nucl. Phys. B 511 (1998) 478; G. Aarts et al, Phys. Rev. D 61 (2000) 105002; G. Aarts, Phys. Lett. B 518 (2001) 315.

[8] W. Buckmuller and A. Jakovac, Nucl. Phys. B 521 (1998) 219.

[9] A.H. Mueller and D.J. Son, Phys. Rev. B 582 (2004) 279.

[10] S. Jeon, Phys. Rev. C 72 (2005) 014907, [hep-ph-04112121].

[11] E. Gozzi, M. Reuter and W.D. Thacker, Phys. Rev D **40** (1989), 3363; E. Gozzi and M. Reuter, Phys. Lett. B **233** (1989), 383; **240** (1990), 137

[12] B.O. Koopman, Proc. Nat. Acad. Sci. USA **17** (1931) 315; J. von Neumann, Ann. Math. **33** (1932) 587.

[13] R.P. Feynman and A.R. Hibbs, "*Quantum Mechanics and path-integrals*" (New York, McGraw-Hill book company, 1965).

[14] L.S. Schulman, "*Techniques and applications of path integrations*" (New York, Wiley, 1981).

[15] E.Gozzi, C.Pagani, arXiv: 1006.3029, *Phys.Rev.Lett.* in press.

[16] H. Deker and F. Haake, Phys. Rev. A 11 (1975) 2043.

[17] R. Penco and D. Mauro, Eur. J. Phys. 27 (2006) 1241, [hep-th-0605061].

[18] P.C. West, "*Introduction to supersymmetry and supergravity*" (Singapore, World Scientific, 1990).

[19] A.A. Abrikosov (jr.), E. Gozzi and D. Mauro, Ann. Phys. 317 (2005) 24, [quant-ph-040628].

[20] H. Ezawa and J.R. Klauder, Prog. Theor. Phys. 74 (1989) 904.

[21] B. Sakita, "*Quantum theory of many-variable systems and fields*" (Singapore, World Scientific, 1985).

[22] H. Nakazato et al, Nucl. Phys. B 346 (1990) 611.

[23] M. Sato, Prog. Theor. Phys. 58 (1977) 1262.

[24] E.Gozzi, M.Reuter, Chaos Solitons and Frasctals vol.4, no.7 (1994) 1117 (arxiv:nlin-0306042); S.Tanase-Nicola and J.Kurchan, J.Phys.A: Math.Gen.,36:10299-10334,2003.

[25] R. Abraham and J. Marsden, "*Foundations of Mechanics*" (Benjamin, New York, 1978).

[26] E. Gozzi and M. Regini, Phys. Rev. D **62** (2000), 067702 [hep-th/9903136]; E. Gozzi and D. Mauro, J. Math. Phys. **41** (2000) 1916 [hep-th/9907065].

[27] E. Witten, Comm. Math. Phys. 117 (1988) 353.

[28] L. Arnold, V. Wihstuz, Lyapunov Exponents, Lecture Notes in Mathematics Vol 1186, Springer, New York 1996.

[29] C.D. Palmer and M.E. Carrington, A general expression for symmetry factors of Feynman diagrams, Can. J. Phys. 80 (2002) 847-852.

[30] E. Gozzi and R. Penco, arXiv: 1008.5135.

[31] H. Wyld, Ann. Phys. 14 (1961) 163.

[32] C. Lenz and L. Garrido, Phys. Rev. A 143 (1976) 1258; L. Garrido and M. San Miguel, Prog. of Theor. Phys. vol. 59, no 1 (1978) 55.

[33] E. Gozzi, Phys. Rev. D 30 (6) (1984) 1218.

[34] M. Reuter, private communication;

[35] E. Cattaruzza et al, "Diagrammar in Classical Scalar Field Theory II".

[36] S. Marculescu et al, Nucl. Phys. B 349 (1991) 463.