A New Look at the Schouten-Nijenhuis, Frölicher-Nijenhuis and Nijenhuis-Richardson Brackets for Symplectic Spaces

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Abstract

In this paper we re-express the Schouten-Nijenhuis, the Frölicher-Nijenhuis and the Nijenhuis-Richardson brackets on a symplectic space using the extended Poisson brackets structure present in the path-integral

formulation of classical mechanics.

1 Introduction

Some time ago a *path-integral* formulation of classical mechanics (CM) appeared in the literature [1]. This formulation was nothing else than the path-integral counterpart of the *operatorial* version of CM provided long ago by Koopman and von Neumann [2]. From now on we will refer to the formulation contained in ref. [1] as CPI for "Classical Path Integral".

Calling the phase-space of the system as \mathcal{M} one had that in the CPI, besides the 2n phase-space variables: $\varphi^a = (q^1 \cdots q^n, p^1 \cdots p^n)$, the measure in the path-integral contained a set of 6n auxiliary variables which were indicated as $(\lambda_a, c^a, \bar{c}_a)$. All together these 8n variables $(\varphi^a, \lambda_a, c^a, \bar{c}_a)$ labelled a space whose geometrical meaning was studied in ref. [3]. It turned to be what is called the cotangent bundle to the reversed-parity tangent bundle to phase-space¹, and is indicated in brief as $T^*(\Pi T \mathcal{M})$. Being a cotangent bundle this space had a Poisson structure which was called [1] Extended Poisson Brackets (or EPB) to distinguish it from the standard Poisson brackets defined on the phase-space \mathcal{M} . Via these EPB and the 8n variables indicated above it was shown [1] that all the standard variables (forms, multivectors, etc) could be mapped into functions of our 8n variables and the standard operations (exterior derivative, interior contraction, Lie-brackets, Lie-derivative, etc.) of the Cartan calculus [4] could be obtained by inserting those functions into chains of EPB.

What had not been mapped into this formalism of the CPI and of the EPB were those generalizations of the Lie-brackets known [5][8] as the Schouten-Nijenhuis (\mathbf{SN}) brackets, the Frölicher-Nijenhuis (\mathbf{FN}) brackets and the Richardson-Nijenhuis (\mathbf{RN}) ones. In this paper we will derive the above mentioned mapping for these backets.

The paper is organized as follows. In Sec. 2 we will briefly review the path integral for classical mechanics (CPI) and explain the *EPB* structure present there. In Sec. 3 we will show how to do the Cartan calculus via our variables and the associated *EPB* structure. In Sec. 4 we will map the **SN** brackets, the **FN** brackets and the **RN** ones into operations done with only the *EPB* brackets with inserted different functions of our variables. In Appendix A we will report the main formulas of the Cartan calculus while the calculations of Sec. 4 are given in details in Appendices B, C and D.

2 Path Integral for Classical Mechanics (CPI)

The idea is to give a *path integral* for CM which will reproduce the *operatorial* version of CM as given by the *Liouville* operator [2] or by the *Lie derivative* of the Hamiltonian flow [4]. We will be brief here because more details can be found in [1].

¹For the definitions and meanings of these words we refer the reader to ref. [3].

Let us start with a 2n-dimensional phase space \mathcal{M} whose coordinates are indicated as $\varphi^a(a=1,\ldots,2n)$, i.e.: $\varphi^a=(q^1\cdots q^n,p^1\cdots,p^n)$. Let us indicate the Hamiltonian of the system as $H(\varphi)$ and the symplectic-matrix as ω^{ab} . The equations of motions are then:

$$\dot{\varphi}^a = \omega^{ab} \frac{\partial H}{\partial \varphi^b} \tag{1}$$

We shall now *suggest*, as path integral for CM, one that forces all paths in \mathcal{M} to sit on the classical ones. The *classical* analog Z_{CM} of the quantum generating functional would be:

$$Z_{CM}[J] = N \int \mathcal{D}\varphi \ \tilde{\delta}[\varphi(t) - \varphi_{cl}(t)] \ exp \int J\varphi \ dt \tag{2}$$

where φ are the $\varphi^a \in \mathcal{M}$, φ_{cl} are the solutions of eq.(1), J is an external current and $\tilde{\delta}[\]$ is a functional Dirac-delta which forces every path $\varphi(t)$ to sit on a classical one $\varphi_{cl}(t)$. Of course there are all possible initial conditions integrated over in (2). One should be very careful in properly defining the measure and the functional Dirac delta. This careful analysis has been done in the literature [6] for stochastic evolution equations and it applies to Hamiltonian deterministic equations as well.

We should now check if the path integral of eq. (2) leads to the well known operatorial formulation [2] of CM done via the Liouville operator and the Lie derivative. To do that let us first rewrite the functional Dirac delta in (2) as:

$$\tilde{\delta}[\varphi - \varphi_{cl}] = \tilde{\delta}[\dot{\varphi}^a - \omega^{ab}\partial_b H] \det[\delta^a_b\partial_t - \omega^{ac}\partial_c\partial_b H]$$
(3)

where we have used the analog of the relation

$$\delta[f(x)] = \frac{\delta[x - x_i]}{\left| \frac{\partial f}{\partial x} \right|_{x_i}} \tag{4}$$

The determinant which appears in (3) is always positive and so we can drop the modulus sign $| \cdot |$. The next step is to insert (3) in (2) and write the $\tilde{\delta}[\cdot]$ as a Fourier transform over some new variables λ_a , i.e.:

$$\tilde{\delta} \left[\dot{\varphi}^a - \omega^{ab} \frac{\partial H}{\partial \varphi^b} \right] = \int \mathcal{D}\lambda_a \exp i \int \lambda_a \left[\dot{\varphi}^a - \omega^{ab} \frac{\partial H}{\partial \varphi^b} \right] dt \tag{5}$$

and to re-write the determinant $det[\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H]$ via Grassmannian variables \bar{c}_a, c^a :

$$det[\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H] = \int \mathcal{D}c^a \mathcal{D}\bar{c}_a \ exp - \int \bar{c}_a [\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H]c^b \ dt$$
 (6)

Inserting (5), (6) and (3) in (2) we get:

$$Z_{CM}[0] = \int \mathcal{D}\varphi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \exp\left[i \int dt \,\tilde{\mathcal{L}}\right]$$
 (7)

where $\widetilde{\mathcal{L}}$ is:

$$\widetilde{\mathcal{L}} = \lambda_a [\dot{\varphi}^a - \omega^{ab} \partial_b H] + i \bar{c}_a [\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H] c^b \tag{8}$$

One can easily see that this Lagrangian gives the following equations of motion:

$$\dot{\varphi}^a - \omega^{ab} \partial_b H = 0 \tag{9}$$

$$[\delta_b^a \partial_t - \omega^{ac} \partial_c \partial_b H] c^b = 0 \tag{10}$$

$$\delta_b^a \partial_t \bar{c}_a + \bar{c}_a \omega^{ac} \partial_c \partial_b H = 0 \tag{11}$$

$$[\delta_b^a \partial_t + \omega^{ac} \partial_c \partial_b H] \lambda_a = -i \bar{c}_a \omega^{ac} \partial_c \partial_d \partial_b H c^d$$
(12)

One notices immediately the following two things:

- 1) $\widetilde{\mathcal{L}}$ leads to the same Hamiltonian equations for φ as H did;
- 2) c^b transforms under the Hamiltonian vector field $h \equiv \omega^{ab} \partial_b H \partial_a$ as a form $d\varphi^b$ does. From the above formalism one can get the equations of motion (9)-(12) also via an Hamiltonian $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i \bar{c}_a \omega^{ac} (\partial_c \partial_b H) c^b \tag{13}$$

and via some extended Poisson brackets (*EPB*) defined in the space $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. They are:

$$\{\varphi^a, \lambda_b\}_{EPB} = \delta_b^a \quad ; \quad \{\bar{c}_b, c^a\}_{EPB} = -i\delta_b^a \tag{14}$$

The equations of motion (9)-(12) are then reproduced via the formula $\frac{d}{dt}A = \{A, \widetilde{\mathcal{H}}\}_{EPB}$ where A is one of the variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$. All the other EPB are zero; in particular $\{\varphi^a, \varphi^b\}_{EPB} = 0$ and so these are not the standard Poisson brackets on \mathcal{M} which would give $\{\varphi^a, \varphi^b\}_{PB} = \omega^{ab}$.

Being (7) a path integral one could also introduce the concept of *commutator* as Feynman did in the quantum case. If we define the graded commutator of two functions $O_1(t)$ and $O_2(t)$ as the expectation value $\langle \ \rangle$ under our path integral of some timesplitting combinations of the functions themselves, as:

$$\langle [O_1(t), O_2(t)] \rangle \equiv \lim_{\epsilon \to 0} \langle O_1(t+\epsilon)O_2(t) \pm O_2(t+\epsilon)O_1(t) \rangle$$
 (15)

then we get immediately from (7) that the only expressions different from zero are:

$$\langle [\varphi^a, \lambda_b] \rangle = i\delta_b^a \; ; \; \langle [\bar{c}_b, c^a] \rangle = \delta_b^a$$
 (16)

We notice immediately two things:

- 1) there is an isomorphism between the extended Poisson structure (14) and the graded commutator structure (16): $\{\cdot,\cdot\}_{EPB} \longrightarrow -i[\cdot,\cdot];$
- 2) via the commutator structure (16) one can "realize" λ_a and \bar{c}_a as:

$$\lambda_a = -i\frac{\partial}{\partial \varphi^a} \; ; \; \bar{c}_a = \frac{\partial}{\partial c^a}$$
 (17)

Now, using (17), we can check that actually what we got as weight in (7) corresponds to the operatorial version of CM. In fact take, for the moment, only the bosonic (B) part of $\widetilde{\mathcal{H}}$ in (13):

$$\widetilde{\mathcal{H}}_{B} = \lambda_{a} \omega^{ab} \partial_{b} H \tag{18}$$

This one, via (17), goes into the operator:

$$\widehat{\widetilde{\mathcal{H}}}_B \equiv -i\omega^{ab}\partial_b H \partial_a \tag{19}$$

which is the Liouville operator of CM. If we had added the Grassmannian part to $\widetilde{\mathcal{H}}_B$ and inserted the operatorial representation of \bar{c} (17), we would have got the Lie derivative of the Hamiltonian flow as we will see in the next section. So this proves that the operatorial version of CM comes from a path-integral weight that is just a

Dirac delta on the classical paths. Somehow this is the classical anologue of what Feynman did for Quantum Mechanics where he proved that the Schroedinger operator of evolution comes from a path-integral weight of the form expiS.

3 Cartan calculus

We have seen in Sec. 2 that c^a transform as $d\varphi^a$, that is as the basis of generic forms $\alpha \equiv \alpha_a(\varphi)d\varphi^a$ or as the *components* of tangent vectors: $V^a(\varphi)\frac{\partial}{\partial \phi^a}$. The space whose coordinates are (φ^a, c^a) is called [7] reversed-parity tangent bundle, indicated as $\Pi T \mathcal{M}$. The "reversed-parity" specification is because the c^a are Grassmannian variables. As the (λ_a, \bar{c}_a) are the "momenta" of the previous variables (see eq. (19)), we conclude that the 8n variables $(\varphi^a, c^a, \lambda_a, \bar{c}_a)$ span the cotangent bundle to the reversed-parity tangent bundle [3]: $T^*(\Pi T\mathcal{M})$. For more details about this we refer the interested reader to ref. [3]. So our space is a cotangent bundle and this is the reason why it has a Poisson structure which is the one we found via the CPI and indicated in eq. (15).

In the remaining part of this section we will show how to reproduce all the abstract Cartan calculus via our EPB and the Grassmannian variables. Let us first introduce 5 charges which are conserved under the \mathcal{H} of eq. (13) and which will play an important role in the Cartan calculus. They are:

$$Q^{BRS} \equiv ic^a \lambda_a \tag{20}$$

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$$\bar{Q}^{BRS} \equiv i\bar{c}_a \omega^{ab} \lambda_b \tag{21}$$

$$Q_g \equiv c^a \bar{c}_a \tag{22}$$

$$Q_g \equiv c^a \bar{c}_a \tag{22}$$

$$K \equiv \frac{1}{2} \omega_{ab} c^a c^b \tag{23}$$

$$\bar{K} \equiv \frac{1}{2}\omega^{ab}\bar{c}_a\bar{c}_b \tag{24}$$

where ω_{ab} are the matrix elements of the inverse of ω^{ab} . The next thing we should observe is that \bar{c}_a transforms under the Hamiltonian flow as the basis of vector fields, see eq. (11), while λ_a does not seem to transform as a vector field, eq. (12), even if it can be interpreted as $\frac{\partial}{\partial \varphi^a}$. The explanation of this fact is given in ref. [3].

Now since c^a transforms as basis of forms $d\varphi^a$ and \bar{c}_a as basis of vector fields $\frac{\partial}{\partial c^a}$, let us start building the following map, called "hat" map \wedge :

²This is so not only under the Hamiltonian flow but under any diffeomorphism: see ref. [3] for details.

$$\alpha = \alpha_a d\varphi^a \quad \widehat{\longrightarrow} \quad \widehat{\alpha} \equiv \alpha_a c^a \tag{25}$$

$$V = V^a \partial_a \quad \hat{\longrightarrow} \quad \hat{V} \equiv V^a \bar{c}_a \tag{26}$$

It is actually a much more general map between forms α , antisymmetric tensors V and functions of φ, c, \bar{c} :

$$F^{(p)} = \frac{1}{p!} F_{a_1 \cdots a_p} d\varphi^{a_1} \wedge \cdots \wedge d\varphi^{a_p} \quad \hat{F}^{(p)} \equiv \frac{1}{p!} F_{a_1 \cdots a_p} c^{a_1} \cdots c^{a_p}$$
 (27)

$$V^{(p)} = \frac{1}{p!} V^{a_1 \cdots a_p} \partial_{a_1} \wedge \cdots \wedge \partial_{a_p} \quad \hat{V} \equiv \frac{1}{p!} V^{a_1 \cdots a_p} \bar{c}_{a_1} \cdots \bar{c}_{a_p}$$
 (28)

Once the correspondence (25)-(28) is extablished we can easily find out what correspond to the various Cartan operations known as the exterior derivative \mathbf{d} of a form, or interior contraction between a vector field V and a form F. It is easy to check that, see [1]:

$$dF^{(p)} \stackrel{\widehat{}}{\longrightarrow} i\{Q^{BRS}, \widehat{F}^{(p)}\}_{EPB} \tag{29}$$

$$\iota_{V}F^{(p)} \stackrel{\hat{}}{\longrightarrow} i\{\hat{V}, \hat{F}^{(p)}\}_{EPB}$$
 (30)

$$pF^{(p)} \stackrel{\widehat{}}{\longrightarrow} i\{Q_q, \widehat{F}^{(p)}\}_{EPB}$$
 (31)

where Q^{BRS} , Q_g are the charges of (20)-(22). At the same level we can translate in our language the usual mapping [4] between vector fields V and forms V^{\flat} realized by the symplectic 2-form $\omega(V,0) \equiv V^{\flat}$, or the inverse operation of building a vector field α^{\sharp} out of a form: $\alpha = (\alpha^{\sharp})^{\flat}$. These operations can be translated in our formalism as follows:

$$V^{\flat} \stackrel{\hat{}}{\longrightarrow} i\{K, \widehat{V}\}_{EPB}$$
 (32)

$$\alpha^{\sharp} \stackrel{\hat{}}{\longrightarrow} i\{\bar{K}, \hat{\alpha}\}_{EPB}$$
 (33)

where again K, \bar{K} are the charges (23)-(24). We can also translate the standard operation of building a vector field out of a function $f(\varphi)$, and also the Poisson brackets between two functions f and g:

$$(df)^{\sharp} \stackrel{\hat{}}{\longrightarrow} i\{\bar{Q}^{BRS}, f\}_{EPB} \tag{34}$$

$$\{f,g\}_{PB} = df[(dg)^{\sharp}] \quad \hat{\longrightarrow} \quad -\{\{\{f,Q^{BRS}\},\bar{K}\},\{\{\{g,Q^{BRS}\},\bar{K}\},K\}\}_{EPB} \quad (35)$$

The next thing to do is to translate the concept of Lie derivative which is defined as: $\mathcal{L}_V = d\iota_V + \iota_V d$. It is easy to prove that:

$$\mathcal{L}_{V}F^{(p)} \stackrel{\hat{}}{\longrightarrow} \{-\widetilde{\mathcal{H}}_{V}, \widehat{F}^{(p)}\}_{EPB} \tag{36}$$

where $\widetilde{\mathcal{H}}_V = \lambda_a V^a + i \bar{c}_a \partial_b V^a c^b$; note that, for $V^a = \omega^{ab} \partial_b H$, $\widetilde{\mathcal{H}}_V$ becomes the $\widetilde{\mathcal{H}}$ of (13). This confirms that the full $\widetilde{\mathcal{H}}$ of eq. (13) is the Lie derivative of the Hamiltonian flow as we said at the end of the previous section. Finally the Lie brackets between two vector fields V, W are reproduced by:

$$[V, W]_{Lie-brack.} \xrightarrow{} \{-\widetilde{\mathcal{H}}_V, \widehat{W}\}_{EPB}$$
 (37)

We will collect in Appendix A all the important formulas we mentioned in this section.

4 Schouten-Nijenhuis, Frölicher-Nijenhuis and Nijenhuis-Richardson brackets done via the extended Poisson brackets

4.1 Schouten-Nijenhuis (SN) brackets

These brackets are a generalization of the Lie brackets between vector fields: in fact the SN are brackets between multivector fields and they become the usual Lie brackets in case of vector fields. As Lie brackets associate to two vector fields X and Y another vector field [X,Y], so SN brackets associate to two multivector fields of rank p $(P = X_{(1)} \wedge \cdots \wedge X_{(p)})$ and r $(R = Y_{(1)} \wedge \cdots \wedge Y_{(r)})$ a multivector field of rank p + r - 1 via the following rule [8]:

$$[\cdot, \cdot]_{SN} : \mathcal{V}^{p}(M) \times \mathcal{V}^{r}(M) \longrightarrow \mathcal{V}^{p+r-1}(M)$$

$$[P, R]_{SN} \equiv \sum_{i=1}^{p} (-1)^{i+1} X_{(1)} \wedge \cdots \wedge \widehat{\widehat{X}}_{(i)} \cdots \wedge X_{(p)} \wedge [X_{(i)}, R]$$
(38)

where the \mathcal{V}^s indicates the space of mutivector fields of rank s and the double hat $\widehat{X}_{(i)}$ indicates that we have removed the $X_{(i)}$, while $[X_{(i)}, R] = \mathcal{L}_{X_{(i)}}R$ is the Lie derivative of a multivector defined as:

$$\mathcal{L}_{X_{(i)}}R = \sum_{j=1}^{r} Y_1 \wedge \dots \wedge [X_{(i)}, Y_{(j)}] \wedge \dots \wedge Y_{(r)}$$
(39)

This Lie derivative is reproduced via our extended Poisson brackets (EPB) as:

$$\mathcal{L}_{X_{(i)}}R \stackrel{\hat{}}{\longrightarrow} \{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{R}\}_{EPB}$$
 (40)

where we have defined: $\widetilde{\mathcal{H}}_{X_{(i)}} = \{\widehat{X}_{(i)}, Q^{\scriptscriptstyle BRS}\}_{\scriptscriptstyle EPB}$. The SN brackets become then in our notation :

$$[P,R]_{SN} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_P,\widehat{R}\}_{EPB}$$
 (41)

where:

$$\widetilde{\mathcal{H}}_{P} = \{Q, \widehat{X}_{(1)} \cdots \widehat{X}_{(p)}\} = \sum_{i=1}^{p} (-1)^{i+1} \widehat{X}_{(1)} \cdots \widehat{X}_{(i)} \cdots \widehat{X}_{(p)} \widetilde{\mathcal{H}}_{X_{(i)}}$$
(42)

and:

$$\hat{R} = Y_{(1)}^{j_1} \bar{c}_{j_1} \cdots Y_{(r)}^{j_r} \bar{c}_{j_r} \tag{43}$$

The quantities which one has in the equations above are those written in terms of c^a or \bar{c}_a as explained in the previous section.

The extended Poisson brackets (*EPB*), besides allowing us to write complicated formulas in a very compact way, can also be used to prove some properties of the Schouten-Nijenhuis brackets, as we will show in Appendix B.

4.2 Frölicher-Nijenhuis (FN) Brackets

These are brackets which associate to two vector-valued forms³ $K \in \Omega^{k+1}(M;TM)$ of degree k+1 and $L \in \Omega^{l+1}(M;TM)$ of degree l+1 a vector-valued form of degree

³The notation we follow, regarding the manner to indicate the space of vector valued forms with $\Omega^k(M,TM)$, is the one of ref. [5].

k + l + 2:

$$[\cdot,\cdot]_{FN}: \Omega^{k+1}(M;TM) \times \Omega^{l+1}(M;TM) \longrightarrow \Omega^{k+l+2}(M;TM)$$
 (44)

They are defined in the following manner [5]:

a) let us first define the interior contraction ι_K not with a vector field but with a vector-valued form K of degree k+1, and apply it on a form ω of degree l. As K is a (k+1)-form, $\iota_K\omega$ can eat k+l vector fields, i.e. when we apply $\iota_K\omega$ to k+l vectors, we obtain the following number:

$$\frac{(\iota_{K}\omega)(X_{(1)},\cdots,X_{(k+l)})}{1} \equiv \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} (sign \ \sigma) \ \omega[K(X_{\sigma(1)},\cdots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\cdots,X_{\sigma(k+l)}]$$
(45)

where S_{k+l} is the set of permutations of the k+l vector fields $X_{(1)} \cdots X_{(k+l)}$. We note how the k+1 vector-valued form K, acting on k+1 vector fields, produces another vector field;

b) having defined this generalized interior product ι_K , we can also define a generalized Lie derivative as:

$$\mathcal{L}_K = [\iota_K, d] \tag{46}$$

where $[\cdot, \cdot]$ is the usual graded commutator and $K \in \Omega^{k+1}(M; TM)$.

c) Now, having done a) and b), the FN brackets are defined in the following implicit way:

$$[\mathcal{L}_K, \mathcal{L}_L] \equiv \mathcal{L}_{[K,L]_{EN}} \tag{47}$$

where $[\mathcal{L}_K, \mathcal{L}_L]$ is the usual graded commutator among Lie derivatives.

Now if K and L are written in our linguage as:

$$K \xrightarrow{\hat{}} \frac{1}{(k+1)!} K^{i}_{i_{1}i_{2}\cdots i_{k+1}} [c^{i_{1}}c^{i_{2}}\cdots c^{i_{k+1}}][\bar{c}_{i}]$$

$$L \xrightarrow{\hat{}} \frac{1}{(l+1)!} L^{j}_{j_{1}j_{2}\cdots j_{l+1}} [c^{j_{1}}c^{j_{2}}\cdots c^{j_{l+1}}][\bar{c}_{j}]$$

$$(48)$$

then the FN brackets become:

$$[K, L]_{FN} \xrightarrow{\hat{}} - \{\widetilde{\mathcal{H}}_K, \widehat{L}\}_{EPB}$$
 (49)

where:

$$\widetilde{\mathcal{H}}_{K} = \frac{1}{(k+1)!} \left(\lambda_{j} K_{j_{1} j_{2} \cdots j_{k+1}}^{j} + i \bar{c}_{j} (\partial_{d} K_{j_{1} j_{2} \cdots j_{k+1}}^{j} c^{d}) \right) c^{j_{1}} \cdots c^{j_{k+1}}$$
(50)

The calculational details are given in Appendix C.

4.3 Nijenhuis-Richardson (NR) brackets

They are brackets defined among two vector-valued forms: $K \in \Omega^{k+1}(M;TM)$ and $L \in \Omega^{l+1}(M;TM)$ and they give a k+l+1 vector-valued form defined in the following implicit way:

$$[\cdot, \cdot]_{NR}: \Omega^{k+1}(M; TM) \times \Omega^{l+1}(M; TM) \longrightarrow \Omega^{k+l+1}(M; TM)$$

$$\iota_{[K,L]_{NR}} \equiv [\iota_K, \iota_L]$$
(51)

where ι_K and ι_L are the generalized interior contractions defined in the previous section. The definition (51) can also be expressed in a more explicit way as:

$$[K, L]_{NR} = \iota_K L - (-1)^{kl} \iota_L K \tag{52}$$

Now with our calculus and \widehat{K} and \widehat{L} defined as in (48) we can show that :

$$[K, L]_{NR} \stackrel{\hat{}}{\longrightarrow} i\{\widehat{K}, \widehat{L}\}_{ePb}$$
 (53)

The calculational details are provided in Appendix D.

We can now summarize all SN, FN, NR brackets in the following very compact way:

$$[P,R]_{SN} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_{P},\widehat{R}\}_{EPB}$$

$$[K,L]_{FN} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_{K},\widehat{L}\}_{EPB}$$

$$[K,L]_{NR} \stackrel{\hat{}}{\longrightarrow} i\{\widehat{K},\widehat{L}\}_{EPB}$$
(54)

where:

$$R = Y_{(1)} \wedge \cdots \wedge Y_{(r)} \quad \stackrel{\hat{}}{\longrightarrow} \quad Y_{(1)}^{j_1} \bar{c}_{j_1} \cdots Y_{(r)}^{j_r} \bar{c}_{j_r}$$

$$\widetilde{\mathcal{H}}_P = \{Q, \widehat{X}_{(1)} \cdots \widehat{X}_{(p)}\} = \sum_{i=1}^p (-1)^{i+1} \widehat{X}_{(1)} \cdots \widehat{X}_{(i)} \cdots \widehat{X}_{(p)} \widetilde{\mathcal{H}}_{X_{(i)}}$$

$$\widetilde{\mathcal{H}}_K = \frac{1}{(k+1)!} \left(\lambda_j K_{j_1 j_2 \cdots j_{k+1}}^j + i \bar{c}_j (\partial_d K_{j_1 j_2 \cdots j_{k+1}}^j c^d) \right) c^{j_1} \cdots c^{j_{k+1}}$$

$$K \in \Omega^{k+1}(M; TM) \quad \stackrel{\hat{}}{\longrightarrow} \quad \frac{1}{(k+1)!} K_{i_1 i_2 \cdots i_{k+1}}^i [c^{i_1} c^{i_2} \cdots c^{i_{k+1}}] [\bar{c}_i]$$

$$L \in \Omega^{l+1}(M; TM) \quad \stackrel{\hat{}}{\longrightarrow} \quad \frac{1}{(l+1)!} L_{j_1 j_2 \cdots j_{l+1}}^j [c^{j_1} c^{j_2} \cdots c^{j_{l+1}}] [\bar{c}_j]$$
(55)

5 Conclusions

The reader may ask which is the use of all this. Our answer is that by looking at eq. (54) one immediately realizes that we have reduced *three* different and complicated brackets, like the SN, FN and NR brackets, to only *one* bracket which is our Extended Poisson Bracket (or *EPB*) in which the entries are different functions of our variables. So instead of changing the brackets we just have to change the entries to reproduce all the three SN, FN and NR brackets. This unifying structure is not only appealing but it may also indicate something more profound which may be worth to investigate in the future.

APPENDIX A

CARTAN CALCULUS

The correspondence between the standard Cartan calculus [4] and our formulation is provided below:

$$Q^{BRS} \equiv ic^a \lambda_a \quad ; \quad \bar{Q}^{BRS} \equiv i\bar{c}_a \omega^{ab} \lambda_b \tag{56}$$

$$Q_a \equiv c^a \bar{c}_a \tag{57}$$

$$Q_g \equiv c^a \bar{c}_a$$

$$K \equiv \frac{1}{2} \omega_{ab} c^a c^b ; \quad \bar{K} \equiv \frac{1}{2} \omega^{ab} \bar{c}_a \bar{c}_b$$

$$(57)$$

$$\{\varphi^a, \lambda_b\}_{EPB} = \delta^a_b \quad ; \quad \{\bar{c}_b, c^a\}_{EPB} = -i\delta^a_b$$
 (59)

$$\alpha = \alpha_a d\varphi^a \quad \widehat{\longrightarrow} \quad \widehat{\alpha} \equiv \alpha_a c^a \tag{60}$$

$$V = V^a \partial_a \quad \hat{V} \equiv V^a \bar{c}_a \tag{61}$$

$$F^{(p)} = \frac{1}{p!} F_{a_1 \cdots a_p} d\varphi^{a_1} \wedge \cdots \wedge d\varphi^{a_p} \quad \hat{F}^{(p)} \equiv \frac{1}{p!} F_{a_1 \cdots a_p} c^{a_1} \cdots c^{a_p}$$

$$(62)$$

$$V^{(p)} = \frac{1}{p!} V^{a_1 \cdots a_p} \partial_{a_1} \wedge \cdots \wedge \partial_{a_p} \quad \hat{V} \equiv \frac{1}{p!} V^{a_1 \cdots a_p} \bar{c}_{a_1} \cdots \bar{c}_{a_p}$$

$$(63)$$

$$dF^{(p)} \stackrel{\widehat{}}{\longrightarrow} i\{Q^{^{BRS}}, \widehat{F}^{(p)}\}_{^{EPB}} \tag{64}$$

$$\iota_{V}F^{(p)} \stackrel{\hat{}}{\longrightarrow} i\{\hat{V}, \hat{F}^{(p)}\}_{EPB}$$
 (65)

$$pF^{(p)} \stackrel{\hat{}}{\longrightarrow} i\{Q_q, \hat{F}^{(p)}\}_{EPB}$$
 (66)

$$V^{\flat} \stackrel{\hat{}}{\longrightarrow} i\{K, \hat{V}\}_{EPB}$$
 (67)

$$\alpha^{\sharp} \stackrel{\hat{}}{\longrightarrow} i\{\bar{K}, \hat{\alpha}\}_{EPB}$$
 (68)

$$(df)^{\sharp} \stackrel{\hat{}}{\longrightarrow} i\{\bar{Q}^{BRS}, f\}_{EPB} \tag{69}$$

$$\{f,g\}_{PB} = df[(dg)^{\sharp}] \stackrel{\hat{}}{\longrightarrow} -\underbrace{\{\{\{\}}_{EPB}f,Q^{BRS}\},\bar{K}\},\underbrace{\{\{\{\}}_{EPB}g,Q^{BRS}\},\bar{K}\},K\}\}}_{EPB}$$

(70)

$$\mathcal{L}_{V}F^{(p)} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_{V}, \widehat{F}^{(p)}\}_{EPB}$$
 (71)

$$[V, W]_{Lie-brack.} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_V, \widehat{W}\}_{EPB}$$
 (72)

APPENDIX B

CALCULATIONAL DETAILS REGARDING THE SN BRACKETS

We report here some more detailed calculations about the SN brackets and we follow the book [8]. From now on all the curly brackets mean EPB-brackets and Q indicates the BRS charge that previously we indicated as Q^{BRS} . Since we will use it widely, we want first to return to the formula regarding the Lie brackets between two vector fields, of which the SN brackets are only a generalization. As we have seen in (72) the correct translation in our language of the Lie brackets is

$$[V, W]_{Lie-brack.} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_V, \widehat{W}\}$$
 (73)

In fact:

$$-\{\widetilde{\mathcal{H}}_{V}, \widehat{W}\} = -\{\lambda_{a}V^{a} + i\bar{c}_{a}(\partial_{b}V^{a})c^{b}, W^{c}\bar{c}_{c}\} =$$

$$= V^{a}\bar{c}_{c}\partial_{a}W^{c} - \bar{c}_{a}(\partial_{b}V^{a})W^{b} = [V^{b}(\partial_{b}W^{a}) - W^{b}(\partial_{b}V^{a})]\bar{c}_{a}$$

$$(74)$$

So we have obtained, correctly, a vector field whose components are just the components of the Lie brackets between $V^a \bar{c}_a$ and $W^a \bar{c}_a$, see [5].

Next we extend the concept of *interior contraction*: the interior product between a p-multivector field $P = X_{(1)} \wedge \cdots \wedge X_{(p)}$ and an *l*-form ω is defined as:

$$\iota_P \omega(\cdots) = w(X_{(1)}, X_{(2)}, \cdots, X_{(p)}, \cdots)$$
 (75)

Now from the definition itself of interior product we have:

$$\iota_{X_{(p)}}\iota_{X_{(p-1)}}\dots\iota_{X_{(1)}}\omega(\dots) = \iota_{X_{(p-1)}}\dots\iota_{X_{(1)}}\omega(X_{(p)},\dots) = = \iota_{X_{(p-2)}}\dots\iota_{X_{(1)}}\omega(X_{(p-1)},X_{(p)},\dots) = \dots = \omega(X_{(1)},X_{(2)},\dots,X_{(p)},\dots) = \iota_{P}\omega(\dots)$$
(76)

In this way we can transform the interior product with a multivector into a set of interior contractions with normal vector fields:

$$\iota_P \omega = \iota_{X_{(p)}} \iota_{X_{(p-1)}} \cdots \iota_{X_{(2)}} \iota_{X_{(1)}} \omega \tag{77}$$

Let us remember that we know which is the \land -map of an interior contraction between a form and a vector:

$$\iota_{V}\omega \stackrel{\hat{}}{\longrightarrow} i\{\widehat{V},\widehat{\omega}\}$$
 (78)

So, applying the previous formula over and over again, we obtain from (77) and (78):

$$\iota_{P}\omega \xrightarrow{\hat{}} i\{\widehat{X}_{(p)}, (\iota_{X_{(p-1)}} \cdots \iota_{X_{(2)}} \iota_{X_{(1)}}\omega)^{\wedge}\} =$$

$$= i^{2}\{X_{(p)}^{i_{p}} \bar{c}_{i_{p}}, \{X_{(p-1)}^{i_{p-1}} \bar{c}_{i_{p-1}}, (\iota_{X_{(p-2)}} \cdots \iota_{X_{(2)}} \iota_{X_{(1)}}\omega)^{\wedge}\}\} = \cdots =$$

$$= \frac{i^{p}}{l!}\{X_{(p)}^{i_{p}} \bar{c}_{i_{p}}, \{X_{(p-1)}^{i_{p-1}} \bar{c}_{i_{p-1}}, \{\cdots, \{X_{(1)}^{i_{1}} \bar{c}_{i_{1}}, \omega_{ij\cdots l} c^{i} c^{j} \cdots c^{l}\}\} \underbrace{\cdots}_{p \ brackets}}$$
(79)

One could have been tempted to make the \land -correspondence not with the RHS of (79), but with something proportional to:

$$\{X_{(1)}^{i_1}\bar{c}_{i_1}X_{(2)}^{i_2}\bar{c}_{i_2}\cdots X_{(p)}^{i_p}\bar{c}_{i_p}, \omega_{ij\cdots l}c^ic^j\cdots c^l\}$$
(80)

but it would be wrong. In fact, while (79) is an (l-p)-form, that is a (l-p)-string of c, (80) is not a string of only c.

Now we notice that the interior contraction with a 2-vector can be expressed as a combination of well-known objects, such as Lie-brackets, exterior derivatives and interior contractions with normal vector fields. In fact, according to our formalism, we have:

$$\iota_{V \wedge W} dw \stackrel{\hat{}}{\longrightarrow} -i\{\hat{W}, \{\hat{V}, \{Q, \hat{\omega}\}\}\}$$
 (81)

where V and W are vector fields, ω is a 1-form and Q is the usual BRS-charge. Using the Jacobi identity, we can write:

$$-i\{\hat{W}, \{\hat{V}, \{Q, \hat{\omega}\}\}\} = i\{\hat{W}, \{Q, \{\hat{V}, \hat{\omega}\}\}\} + i\{\hat{W}, \{\hat{\omega}, \{Q, \hat{V}\}\}\}$$
(82)

The last term of (82) can be rewritten using again the Jacobi identity:

$$+i\{\hat{W},\{\hat{\omega},\{Q,\hat{V}\}\}\} = +i\{\hat{\omega},\{\{Q,\hat{V}\},\hat{W}\}\} - i\{\{Q,\hat{V}\},\{\hat{W},\hat{\omega}\}\}$$
(83)

Let us now manipulate the last term of (83):

$$-i\{\{Q, \hat{V}\}, \{\hat{W}, \hat{\omega}\}\} = +i\{\{\hat{W}, \hat{\omega}\}, \{\hat{V}, Q\}\} =$$
$$-i\{\hat{V}, \{Q, \{\hat{W}, \hat{\omega}\}\}\} - i\{Q, \{\{\hat{W}, \hat{\omega}\}, \hat{V}\}\}$$
(84)

The last term of (84) is identically zero, in fact:

$$\{\{\hat{W}, \hat{\omega}\}, \hat{V}\} = \{\{W^a \bar{c}_a, \omega_b c^b\}, V^d c_d\} = \{-iW^a \omega_a, V^d c_d\} = 0$$
(85)

Collecting all the previous results in (81) we obtain:

$$\iota_{V \wedge W} dw \xrightarrow{} i\{\hat{W}, \{Q, \{\hat{V}, \hat{\omega}\}\}\} +$$

$$+i\{\{\{Q, \hat{V}\}, \hat{W}\}, \hat{\omega}\} - i\{\hat{V}, \{Q, \{\hat{W}, \hat{\omega}\}\}\} =$$

$$= \left(-\iota_{W} d(\iota_{V}\omega) - \iota_{[V,W]}\omega + \iota_{V} d(\iota_{W}\omega)\right)^{\wedge}$$
(86)

So we have obtained the formula:

$$\iota_{V \wedge W} dw = -\iota_W d(\iota_V \omega) - \iota_{[V,W]} \omega + \iota_V d(\iota_W \omega)$$
(87)

that can be generalized to the case of multivector fields; we can in fact express interior contraction with a multivector field using exterior derivatives, interior contractions with multivectors of lower rank and the SN brackets according to the formula:

$$\iota_{P \wedge R} dw = -\iota_R d(\iota_P \omega) - \iota_{[P,R]_{SN}} \omega + \iota_P d(\iota_R \omega)$$
(88)

where P and R are multivector fields of rank p and r respectively and ω is a p+r-1 form. A proof of (88), that is the natural generalization of (87), can be found in [9].

Following [8], we can define the Schouten-Nijenhuis brackets between two multivector fields $P = X_{(1)} \wedge X_{(2)} \wedge \cdots \wedge X_{(p)}$ and $R = Y_{(1)} \wedge Y_{(2)} \wedge \cdots \wedge Y_{(r)}$ as the (p + r - 1)-multivector field given by:

$$[P,R]_{SN} = \sum_{i=1}^{p} (-1)^{i+1} X_{(1)} \wedge \dots \wedge \widehat{\widehat{X}}_{(i)} \wedge \dots \wedge X_{(p)} \wedge [X_{(i)}, R]$$
 (89)

where $\widehat{\widehat{X}}_{(i)}$ means $X_{(i)}$ is missing and where:

$$[X_{(i)}, R] = \mathcal{L}_{X_{(i)}} R = \sum_{j=1}^{r} Y_{(1)} \wedge \dots \wedge [X_{(i)}, Y_{(j)}] \wedge \dots \wedge Y_{(r)}$$
(90)

In the previous formula $[X_{(i)}, Y_{(j)}]$ are the usual Lie brackets between vector fields. Now (90), that is the Lie derivative along a vector field of a multivector, can be translated in our language as:

$$\mathcal{L}_{X_{(i)}}R \xrightarrow{} \{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{R}\} = -\{\{\widehat{X}_{(i)}, Q\}, \widehat{R}\}$$

$$\tag{91}$$

In fact:

$$\{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{R}\} = \{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{Y}_{(1)}\widehat{Y}_{(2)}\cdots\widehat{Y}_{(r)}\} = \{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{Y}_{(1)}\}\widehat{Y}_{(2)}\cdots\widehat{Y}_{(r)} + \\ +\widehat{Y}_{(1)}\{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{Y}_{(2)}\}\widehat{Y}_{(3)}\cdots\widehat{Y}_{(r)} + \cdots + \widehat{Y}_{(1)}\widehat{Y}_{(2)}\cdots\widehat{Y}_{(r-1)}\{-\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{Y}_{(r)}\} = \\ = \sum_{i=1}^{r} \widehat{Y}_{(1)}\widehat{Y}_{(2)}\cdots([X_{(i)}, Y_{(j)}])^{\wedge}\cdots\widehat{Y}_{(r)} = (\mathcal{L}_{X_{(i)}}R)^{\wedge}$$

$$(92)$$

We note that the extended Poisson brackets take automatically into account the sum over j which appears in the definition of Lie derivative of a multivector.

Now we can cosider the SN brackets. According to their definition we have

$$[P,R]_{SN} \xrightarrow{\hat{}} \sum_{i=1}^{p} (-1)^{i+1} \widehat{X}_{(1)} \cdots \widehat{\widehat{X}}_{(i)} \cdots \widehat{X}_{(p)} \{ -\{\widehat{X}_{(i)}, Q\}, \widehat{R} \}$$
 (93)

The previous formula can be written in a very compact way as:

$$[P,R]_{SN} \stackrel{\hat{}}{\longrightarrow} -\{\{Q,\hat{P}\},\hat{R}\} \tag{94}$$

In fact:

$$-\{\{Q, \widehat{P}\}, \widehat{R}\} = -\{\{Q, \widehat{X}_{(1)} \cdots \widehat{X}_{(p)}\}, \widehat{R}\} =$$

$$= -\{\{Q, \widehat{X}_{(1)}\}\widehat{X}_{(2)} \cdots \widehat{X}_{(p)}, \widehat{R}\} + \{\widehat{X}_{(1)}\{Q, \widehat{X}_{(2)}\}\widehat{X}_{(3)} \cdots \widehat{X}_{(p)}, \widehat{R}\} - \cdots =$$

$$= -\{\widehat{\widehat{X}}_{(1)}\widehat{X}_{(2)} \cdots \widehat{X}_{(p)}\{Q, \widehat{X}_{(1)}\}, \widehat{R}\} + \{\widehat{X}_{(1)}\widehat{\widehat{X}}_{(2)} \cdots \widehat{X}_{(p)}\{Q, \widehat{X}_{(2)}\}, \widehat{R}\} - \cdots =$$

$$= \sum_{i=1}^{p} (-)^{i+1}\widehat{X}_{(1)} \cdots \widehat{\widehat{X}}_{(i)} \cdots \widehat{X}_{(p)}\{-\{Q, \widehat{X}_{(i)}\}, \widehat{R}\} = [P, R]_{SN}^{\wedge}$$
(95)

So we don't need any sum or any strange factor if we use the EPB brackets to represent the SN brackets. On the RHS of (94) we have the images, via the \land -map, of the multivectors P and R, which appear on the LHS of the same equation, and the usual BRS charge Q that appears naturally also in this context.

Like in the case of vector fields, where $\widetilde{\mathcal{H}}_X = \{\widehat{X}, Q\} = \{Q, \widehat{X}\}$, we can also define a sort of Hamiltonian associated with a multivector in the following way:

$$\widetilde{\mathcal{H}}_{P} = \{Q, \widehat{X}_{(1)} \cdots \widehat{X}_{(p)}\} = \sum_{i=1}^{p} (-1)^{i+1} \widehat{X}_{(1)} \cdots \widehat{\widehat{X}}_{(i)} \cdots \widehat{X}_{(p)} \widetilde{\mathcal{H}}_{X_{(i)}}$$
(96)

In this way we can finally write:

$$[P,R]_{SN} \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_P,\widehat{R}\}$$
 (97)

From the expressions (96) and (97) we also notice how the SN brackets become the usual Lie brackets in case of vector fields.

Besides this, we can use the properties of the EPB and of Grassmannian variables to demonstrate immediately some other properties, or alternative definitions, of the Schouten-Nijenhuis brackets. If we start from (93) and we take into account the definition of Lie derivative of a multivector, we obtain:

$$[P,R]_{SN} \xrightarrow{\hat{}} \sum_{i=1}^{p} (-1)^{i+1} \widehat{X}_{(1)} \cdots \widehat{\hat{X}}_{(i)} \cdots \widehat{X}_{(p)} \{ -\widetilde{\mathcal{H}}_{X_{(i)}}, \widehat{R} \}$$

$$= \sum_{i=1}^{p} \sum_{j=1}^{r} (-1)^{i+1} \widehat{X}_{(1)} \cdots \widehat{\hat{X}}_{(i)} \cdots \widehat{X}_{(p)} \widehat{Y}_{(1)} \cdots \widehat{Y}_{(j-1)} ([X_{(i)}, Y_{(j)}])^{\wedge} \cdots \widehat{Y}_{(r)}$$
(98)

Remembering that the Lie bracket of two vector fields is a vector field (and so it is Grassmannian odd in our language) we can write:

$$[P,R]_{SN} \xrightarrow{\hat{}} \sum_{i=1}^{p} \sum_{j=1}^{r} (-1)^{i+j+p+1} ([X_{(i)},Y_{(j)}])^{\hat{}} \widehat{X}_{(1)} \cdots \widehat{X}_{(i)} \cdots \widehat{X}_{(p)} \widehat{Y}_{(1)} \cdots \widehat{Y}_{(j)} \cdots \widehat{Y}_{(r)}$$

$$= \left((-1)^{p+1} \sum_{i=1}^{p} \sum_{j=1}^{r} (-1)^{i+j} [X_{(i)}, Y_{(j)}] \wedge X_{(1)} \wedge \dots \wedge \widehat{\widehat{X}}_{(i)} \wedge \dots \wedge X_{(p)} \wedge Y_{(1)} \wedge \dots \wedge \widehat{\widehat{Y}}_{(j)} \wedge \dots \wedge \widehat{\widehat{Y}}_{(j)} \wedge \dots \wedge Y_{(r)} \right)^{\wedge}$$

$$(99)$$

In the same way we can start from the last equation and we can use properties of Grassmannian variables to obtain the formula:

$$\stackrel{\widehat{}}{-} (-1)^{pr} \sum_{i=1}^{p} \sum_{j=1}^{r} (-1)^{i+j} \widehat{Y}_{(1)} \cdots \widehat{\widehat{Y}}_{(j)} \cdots \widehat{Y}_{(r)} [Y_{(j)}, X_{(i)}]^{\wedge} \widehat{X}_{(1)} \cdots \widehat{\widehat{X}}_{(i)} \cdots \widehat{X}_{(p)} =
= (-1)^{pr} \sum_{j=1}^{r} (-1)^{i+j} \widehat{Y}_{(1)} \cdots \widehat{\widehat{Y}}_{(j)} \cdots \widehat{Y}_{(r)} \sum_{i=1}^{p} (-1)^{i-1} \widehat{X}_{(1)} \cdots ([Y_{(j)}, X_{(i)}])^{\wedge} \cdots \widehat{X}_{(p)} =
= \left((-1)^{pr} \sum_{j=1}^{r} (-1)^{j+1} Y_{(1)} \wedge \cdots \wedge \widehat{\widehat{Y}}_{(j)} \wedge \cdots \wedge Y_{(r)} (\mathcal{L}_{Y_j} P) \right)^{\wedge}$$
(100)

In this way we have obtained two other properties of the SN brackets that may be considered as alternative definitions of the brackets themselves, as one can see from [8].

APPENDIX C

CALCULATIONAL DETAILS REGARDING THE FN BRACKETS

In this section we will handle vector-valued forms $K \in \Omega^{k+1}(M;TM)$. Usually we indicate (k+1)-forms with $\Omega^{k+1}(M)$, but when we indicate in Ω also TM we mean vector-valued forms. Via our \wedge -map K becomes:

$$K \xrightarrow{\hat{}} \frac{1}{(k+1)!} K^{i}_{i_1 i_2 \cdots i_{k+1}} [c^{i_1} c^{i_2} \cdots c^{i_{k+1}}] [\bar{c}_i]$$
 (101)

Following [5] we can introduce the interior product between vector-valued forms and usual forms ω . If $K \in \Omega^{k+1}(M;TM)$ and $\omega \in \Omega^l(M)$, then $\iota_K \omega$ is a (k+l)-form, so it can eat multivectors of degree k+l and it is defined as:

$$(\iota_{K}\omega)[X_{(1)},\cdots,X_{(k+l)}] \equiv$$

$$\frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, \omega[K(X_{\sigma(1)},\cdots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\cdots,X_{\sigma(k+l)}]$$

$$(102)$$

Which is the \land -map of $\iota_K \omega$? We expect, as in the case of interior contraction with vector fields, that:

$$\iota_{\kappa}\omega \xrightarrow{\hat{}} i\{\widehat{K}, \widehat{\omega}\}$$
 (103)

but we have to control that (103) is in accordance with the general definition (102). If $K = \alpha \otimes X$ then we can rewrite (102) in the following way:

$$(\iota_{K}\omega)[X_{(1)}, \cdots, X_{(k+l)}] \equiv \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, \omega[\alpha(X_{\sigma(1)}, \cdots, X_{\sigma(k+1)})X, X_{\sigma(k+2)}, \cdots, X_{\sigma(k+l)}] = \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, \alpha(X_{\sigma(1)}, \cdots, X_{\sigma(k+1)})\omega[X, X_{\sigma(k+2)}, \cdots, X_{\sigma(k+l)}] = \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, \alpha(X_{\sigma(1)}, \cdots, X_{\sigma(k+1)})\iota_{X}[X_{\sigma(k+2)}, \cdots, X_{\sigma(k+l)}] = \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, \alpha(X_{\sigma(1)}, \cdots, X_{\sigma(k+1)})\iota_{X}[X_{\sigma(k+2)}, \cdots, X_{\sigma(k+l)}] = \alpha \wedge \iota_{X}\omega(X_{(1)}, \cdots, X_{(k+l)})$$

$$(104)$$

where $\alpha \in \Omega^{k+1}(M)$ and $\iota_X \omega \in \Omega^{l-1}(M)$. From the previous equalities we deduce that we can translate the interior contraction between a form and a vector-valued form as the exterior product between two forms:

$$\iota_{\kappa}\omega = \alpha \wedge \iota_{\chi}\omega \tag{105}$$

Now we want to prove that, if $\alpha \in \Omega^{k+1}(M)$ and $\beta \in \Omega^{l-1}(M)$ are two differential forms, we can represent their exterior product as:

$$(\alpha \wedge \beta) \ \widehat{\longrightarrow} \ \widehat{\alpha}\widehat{\beta} \tag{106}$$

The definition of the exterior product of differential forms is:

$$\alpha \wedge \beta(X_{(1)}, \dots, X_{(k+l)}) \equiv$$

$$\equiv \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, \alpha(X_{\sigma(1)}, \dots, X_{\sigma(k+1)}) \beta(X_{\sigma(k+2)}, \dots, X_{\sigma(k+l)})$$
(107)

We can start translating via the \land -map the RHS of (107):

$$\alpha(X_{\sigma(1)}, \dots, X_{\sigma(k+1)}) = \iota_{X_{\sigma(k+1)}} \dots \iota_{X_{\sigma(1)}} \alpha \xrightarrow{\hat{}}$$

$$\stackrel{\hat{}}{\longrightarrow} i^{k+1} \{ X_{\sigma(k+1)}^{j_{k+1}} \bar{c}_{j_{k+1}} \{ \dots, \{ X_{\sigma(1)}^{j_1} \bar{c}_{j_1}, \hat{\alpha} \underbrace{\} \} \dots \}}_{k+1 \text{ brackets}} =$$

$$= \frac{1}{(k+1)!} \sum_{\{\tau \in S_{k+1}\}} sign \, \tau \, X_{\sigma(k+1)}^{j_{k+1}} \dots X_{\sigma(1)}^{j_1} \alpha_{\tau(j_1) \dots \tau(j_{k+1})} = X_{\sigma(k+1)}^{j_{k+1}} \dots X_{\sigma(1)}^{j_1} \alpha_{j_1 \dots j_{k+1}}$$

In the last passage we have employed the fact that $\alpha_{\tau(j_1)\cdots\tau(j_{k+1})}=sign\,\tau\,\alpha_{j_1\cdots j_{k+1}}$ and that the number of permutations in S_{k+1} is just (k+1)!. In the same way we have:

$$\beta(X_{\sigma(k+2)}, \cdots, X_{\sigma(k+l)}) \xrightarrow{\hat{}} X_{\sigma(k+l)}^{j_{k+l}} \cdots X_{\sigma(k+2)}^{j_{k+2}} \beta_{j_{k+2} \cdots j_{k+l}}$$

$$\tag{109}$$

So, from the definition itself of exterior product, we must have:

$$\alpha \wedge \beta(X_{(1)}, \dots, X_{(k+l)}) \stackrel{\hat{}}{\longrightarrow} \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, X_{\sigma(k+l)}^{j_{k+l}} \dots X_{\sigma(k+2)}^{j_{k+2}} \cdot X_{\sigma(k+1)}^{j_{k+1}} \dots X_{\sigma(1)}^{j_1} \alpha_{j_1 \dots j_{k+1}} \beta_{j_{k+2} \dots j_{k+l}}$$

$$(110)$$

Now we can say that (106) is correct if it reproduces (110). So we have to evaluate:

$$\alpha \wedge \beta(X_{(1)}, \dots, X_{(k+l)}) \xrightarrow{\hat{}} \frac{1}{(k+1)!(l-1)!} i^{k+l} \{ X_{(k+l)}^{i_{k+l}} \bar{c}_{i_{k+l}}, \{ \dots, \{ X_{(1)}^{i_1} \bar{c}_{i_1}, \alpha_{j_1 \dots j_{k+1}} c^{j_1} \dots c^{j_{k+1}} \beta_{j_{k+2} \dots j_{k+l}} c^{j_{k+2}} \dots c^{j_{k+l}} \} \} \dots \} =$$

$$= \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, X_{(k+l)}^{\sigma(j_{k+l})} \dots X_{(1)}^{\sigma(j_1)} \alpha_{j_1 \dots j_{k+1}} \beta_{j_{k+2} \dots j_{k+l}} =$$

$$= \frac{1}{(k+1)!(l-1)!} \sum_{\{\sigma \in S_{k+l}\}} sign \, \sigma \, X_{\sigma(k+l)}^{j_{k+l}} \dots X_{\sigma(1)}^{j_1} \alpha_{j_1 \dots j_{k+1}} \beta_{j_{k+2} \dots j_{k+l}}$$

$$(111)$$

The last terms of (110) and (111) are equal; so we can conclude that the correct rappresentation via \wedge -map of $\alpha \wedge \beta$ is just $\widehat{\alpha}\widehat{\beta}$.

At this point we have all the elements to translate in our language the operation $\iota_{\kappa}\omega$. In fact:

$$\iota_{K}\omega = \alpha \wedge \iota_{X}\omega \xrightarrow{\hat{}} \widehat{\alpha}(\iota_{X}\omega)^{\wedge} \tag{112}$$

Using (65) and the fact that $\{\hat{\alpha}, \hat{\omega}\} = 0$ we can go on writing:

$$\widehat{\alpha}(\iota_X \omega)^{\wedge} = i\widehat{\alpha}\{\widehat{X}, \widehat{\omega}\} = i\{\widehat{\alpha}\widehat{X}, \widehat{\omega}\} = i\{\widehat{K}, \widehat{\omega}\}$$
(113)

So, as we had expected, we have proved that:

$$\iota_{\scriptscriptstyle K}\omega \stackrel{\hat{}}{\longrightarrow} i\{\widehat{K},\widehat{\omega}\}$$
 (114)

At this point, having defined the concept of interior contraction with a vector-valued form, we can go on introducing the Lie derivative associated with a vector valued form K:

$$\mathcal{L}_{\kappa} = \left[\iota_{\kappa}, d\right] = \iota_{\kappa} d + (-1)^{k+1} d\iota_{\kappa} \tag{115}$$

Since we know how to translate in our language both the interior contraction and the exterior derivative, we can write:

$$\mathcal{L}_{K}\omega \stackrel{\hat{}}{\longrightarrow} i\{\widehat{K}, (d\omega)^{\hat{}}\} + (-1)^{k+1}i\{Q, (\iota_{K}\omega)^{\hat{}}\} =$$

$$= -\{\widehat{K}, \{Q, \widehat{\omega}\}\} + (-1)^{k}\{Q, \{\widehat{K}, \widehat{\omega}\}\} = -\{\{\widehat{K}, Q\}, \widehat{\omega}\}$$
(116)

where, in the last step, we have used the Jacobi identity. So we have:

$$\mathcal{L}_{\kappa}\omega \stackrel{\wedge}{\longrightarrow} -\{\widetilde{\mathcal{H}}_{\kappa}, \widetilde{\omega}\} \tag{117}$$

where we have defined, as usual, $\widetilde{\mathcal{H}}_K = \{\widehat{K}, Q\}$. From the definition itself and making use of (101) it follows that the explicit expression of $\widetilde{\mathcal{H}}_K$ is:

$$\widetilde{\mathcal{H}}_{K} = \frac{1}{(k+1)!} \left(\lambda_{j} K_{j_{1} j_{2} \cdots j_{k+1}}^{j} + i \bar{c}_{j} (\partial_{d} K_{j_{1} j_{2} \cdots j_{k+1}}^{j} c^{d}) \right) c^{j_{1}} \cdots c^{j_{k+1}}$$
(118)

From the previous expression we note that if k is even then $\widetilde{\mathcal{H}}_K$ is Grassmannian odd and if k is odd $\widetilde{\mathcal{H}}_K$ is even. Moreover, from (117), the Grassmannian parity of $\widetilde{\mathcal{H}}_K$ coincides with that of the correspondent Lie derivative \mathcal{L}_K .

Finally we have all the elements to translate in our language the Frölicher-Nijenhuis brackets. They are defined [5] in implicit way from the equation:

$$[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]_{FN}} \tag{119}$$

Now if we think of LHS of (119) as applied on a generic form ω we have:

$$[\mathcal{L}_K, \mathcal{L}_L]\omega = (\mathcal{L}_K \mathcal{L}_L)\omega - (-1)^{[\widetilde{\mathcal{H}}_L][\widetilde{\mathcal{H}}_K]}(\mathcal{L}_L \mathcal{L}_K)\omega$$
(120)

where we indicate with $[(\cdot)]$ the Grassmannian parity of (\cdot) . Via our mapping we have:

$$[\mathcal{L}_{K}, \mathcal{L}_{L}]\omega \xrightarrow{} \{\widetilde{\mathcal{H}}_{K}, \{\widetilde{\mathcal{H}}_{L}, \widehat{\omega}\}\} - (-1)^{[\widetilde{\mathcal{H}}_{L}][\widetilde{\mathcal{H}}_{K}]} \{\widetilde{\mathcal{H}}_{L}, \{\widetilde{\mathcal{H}}_{K}, \widehat{\omega}\}\} =$$

$$= \{\widetilde{\mathcal{H}}_{K}, \{\widetilde{\mathcal{H}}_{L}, \widehat{\omega}\}\} + (-1)^{[\widetilde{\mathcal{H}}_{K}]([\widetilde{\mathcal{H}}_{L}] + [\omega])} \{\widetilde{\mathcal{H}}_{L}, \{\omega, \widetilde{\mathcal{H}}_{K}\}\} = \{\{\widetilde{\mathcal{H}}_{K}, \widetilde{\mathcal{H}}_{L}\}, \widehat{\omega}\} \quad (121)$$

where in the last step we have used, as usual, the Jacobi identity. The RHS of (119) can be translated as:

$$\mathcal{L}_{[K,L]_{FN}}\omega \stackrel{\hat{}}{\longrightarrow} -\{\widetilde{\mathcal{H}}_{[K,L]_{FN}}, \hat{\omega}\}$$
 (122)

so, from the comparison of (121) and (122), we have the following important relation:

$$\widetilde{\mathcal{H}}_{[K,L]_{FN}} = \{ ([K,L]_{FN})^{\wedge}, Q \} = -\{ \widetilde{\mathcal{H}}_K, \widetilde{\mathcal{H}}_L \}$$
(123)

Now, if we want to have the correct representation of the FN brackets, we have to write $\{\widetilde{\mathcal{H}}_{\scriptscriptstyle K}, \widetilde{\mathcal{H}}_{\scriptscriptstyle L}\}$ as $\{(\cdot), Q\}$. This is not difficult to do, in fact:

$$\{\widetilde{\mathcal{H}}_K, \widetilde{\mathcal{H}}_L\} = \{\{\{\widehat{K}, Q\}, \widehat{L}\}, Q\}$$
 (124)

To demonstrate (124) we can start from its RHS and employ the Jacobi identity:

$$\{\{\{\widehat{K}, Q\}, \widehat{L}\}, Q\} = \{\{\widetilde{\mathcal{H}}_{K}, \widehat{L}\}, Q\} = \{\widetilde{\mathcal{H}}_{K}, \widetilde{\mathcal{H}}_{L}\} - (-1)^{k+l}\{\{Q, \widetilde{\mathcal{H}}_{K}\}, L\}$$
 (125)

So (124) is proved if $\{Q, \widetilde{\mathcal{H}}_K\} = 0$. But this is easy to demonstrate, since every BRS exact term has zero EPB with Q. In fact:

$$\{Q, \widetilde{\mathcal{H}}_{\scriptscriptstyle K}\} = \{Q, \{\widehat{K}, Q\}\} \tag{126}$$

The Jacobi identity in this case is:

$${Q, {\widehat{K}, Q}} + {Q, {\widehat{K}, Q}} + {\widehat{K}, {Q, Q}} = 0$$
 (127)

From the nilpotency of Q we can conclude that:

$$\{Q, \{\widehat{K}, Q\}\} = \{Q, \widetilde{\mathcal{H}}_K\} = 0$$
 (128)

and so (124) is proved. Substituting (124) into (123) we obtain finally:

$$[K, L]_{FN} \stackrel{\hat{}}{\longrightarrow} = -\{\widetilde{\mathcal{H}}_K, \widehat{L}\}$$
 (129)

From the previous expression we notice how, if K and L are zero vector-valued forms, i.e. if they are vector fields, then the FN brackets reduce to the usual Lie brackets. In a certain sense we can say that, as the SN brackets generalize Lie brackets in the case of multivector fields, so the FN brackets generalize the Lie brackets in the case of vector-valued forms.

APPENDIX D

CALCULATIONAL DETAILS REGARDING THE NR BRACKETS

The Nijenhuis-Richardson brackets are defined between two vector-valued forms: $K \in \Omega^{k+1}(M;TM)$ and $L \in \Omega^{l+1}(M;TM)$ and they give a vector-valued form of degree k+l+1 defined [5] in an implicit way as:

$$\iota_{[K,L]_{NR}} \equiv [\iota_K, \iota_L] \tag{130}$$

If we apply a generic form $\omega \in \Omega^m(M)$ on the LHS of (130) then, via our \wedge -map, it becomes:

$$\iota_{[K,L]_{NR}}\omega \stackrel{\hat{}}{\longrightarrow} i\{([K,L]_{NR})^{\wedge}, \widehat{\omega}\}$$
 (131)

while the RHS of (130) becomes:

$$[\iota_{K}, \iota_{L}]\omega = \iota_{K}(\iota_{L}\omega) - (-1)^{kl}\iota_{L}(\iota_{K}\omega) \xrightarrow{\hat{}} i\{\widehat{K}, (\iota_{L}\omega)^{\wedge}\} - (-1)^{kl}i\{\widehat{L}, (\iota_{K}\omega)^{\wedge}\} = -\{\widehat{K}, \{\widehat{L}, \widehat{\omega}\}\} + (-1)^{kl}\{\widehat{L}, \{\widehat{K}, \widehat{\omega}\}\}$$

$$(132)$$

Using the Jacobi identity we obtain:

$$[\iota_K, \iota_L]\omega \stackrel{\hat{}}{\longrightarrow} -\{\{\widehat{K}, \widehat{L}\}, \widehat{\omega}\}$$
 (133)

We can write, from the comparison of (131) with (133):

$$[K,L]_{NR} \stackrel{\hat{}}{\longrightarrow} i\{\widehat{K},\widehat{L}\}$$
 (134)

So the NR brackets between two vector-valued forms are just proportional to the extended Poisson brackets of the vector-valued forms themselves.

Now we can use properties of the extended Poisson brackets to find a more explicit definition of NR brackets, in fact:

$$i\{\widehat{K},\widehat{L}\} = \frac{i}{(l+1)!(k+1)!} \{K^i_{i_1\cdots i_{k+1}} c^{i_1} \cdots c^{i_{k+1}} \bar{c}_i, L^j_{j_1\cdots j_{l+1}} c^{j_1} \cdots c^{j_{l+1}} \bar{c}_j\} =$$

$$= \frac{i}{(l+1)!} \{\widehat{K}, L^{j}_{j_{1} \cdots j_{l+1}} c^{j_{1}} \cdots c^{j_{l+1}} \} \bar{c}_{j} + \frac{i}{(l+1)!(k+1)!} L^{j}_{j_{1} \cdots j_{l+1}} c^{j_{1}} \cdots c^{j_{l+1}} } \{K^{i}_{i_{1} \cdots i_{k+1}} c^{i_{1}} \cdots c^{i_{k+1}} \bar{c}_{i}, \bar{c}_{j} \} (-1)^{(l+1)k} = \frac{i}{(l+1)!} \{\widehat{K}, L^{j}_{j_{1} \cdots j_{l+1}} c^{j_{1}} \cdots c^{j_{l+1}} \} \bar{c}_{j} + \frac{i}{(l+1)!(k+1)!} L^{j}_{j_{1} \cdots j_{l+1}} c^{j_{1}} \cdots c^{j_{l+1}} \{\bar{c}_{j}, K^{i}_{i_{1} \cdots i_{k+1}} c^{i_{1}} \cdots c^{i_{k+1}} \} \bar{c}_{i} (-1)^{lk} = \frac{i}{(l+1)!} \{\widehat{K}, L^{j}_{j_{1} \cdots j_{l+1}} c^{j_{1}} \cdots c^{j_{l+1}} \} \bar{c}_{j} - (-1)^{lk} \frac{i}{(k+1)!} \{\widehat{L}, K^{i}_{i_{1} \cdots i_{k+1}} c^{i_{1}} \cdots c^{i_{k+1}} \} \bar{c}_{i}$$

$$(135)$$

Since $\iota_{\scriptscriptstyle K}(\omega \otimes X) \equiv \iota_{\scriptscriptstyle K}(\omega) \otimes X$ we can write:

$$i\{\widehat{K}, \widehat{L}\} = (\iota_{\kappa} L)^{\wedge} - (-1)^{lk} (\iota_{L} K)^{\wedge}$$
(136)

From the comparison of (134) with (136) we obtain:

$$[K, L]_{NR} = \iota_K L - (-1)^{kl} \iota_L K \tag{137}$$

that can be interpreted as a more explicit definition of NR brackets (see also [5]).

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