Path integral approach to random neural networks

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In this work we study of the dynamics of large-size random neural networks. Different methods have been developed to analyze their behavior, and most of them rely on heuristic methods based on Gaussian assumptions regarding the fluctuations in the limit of infinite sizes. These approaches, however, do not justify the underlying assumptions systematically. Furthermore, they are incapable of deriving in general the stability of the derived mean-field equations, and they are not amenable to analysis of finite-size corrections. Here we present a systematic method based on path integrals which overcomes these limitations. We apply the method to a large nonlinear rate-based neural network with random asymmetric connectivity matrix. We derive the dynamic mean field (DMF) equations for the system and the Lyapunov exponent of the system. Although the main results are well known, here we present the detailed calculation of the spectrum of fluctuations around the mean-field equations from which we derive the general stability conditions for the DMF states. The methods presented here can be applied to neural networks with more complex dynamics and architectures. In addition, the theory can be used to compute systematic finite-size corrections to the mean-field equations.

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I. INTRODUCTION

The present paper aims to present a detailed derivation of the path integral framework for the study of the dynamical properties of neural networks. This framework can be applied to a broad spectrum of network models. For concreteness, we shall consider a class of model, extending the model first introduced in 1972 by Amari [1]. This class of model is simple enough to allow for a full analytical description yet presenting a highly nontrivial dynamical behavior. In the models considered here, the state of the ith neuron of the network at time t is represented by a continuous spin variable $S_i(t)$, which represents the firing activity of the neuron. The state of a neuron is determined by the "post-synaptic" potential $h_i(t)$ acting on it through the relationship $S_i(t) = \phi[gh_i(t)]$, where g is a gain parameter measuring the gain of the response. The function $\phi(x)$ is usually a sigmoid function which defines the input/output characteristic of the neurons. As a concrete example we shall consider $\phi(x) = \tanh(x)$ as prototype of generic odd symmetric saturated sigmoid functions satisfying: $\phi(x) = -\phi(-x)$, $\phi(\pm\infty) = \pm 1$, and $\phi(0) = d\phi(x)/dx|_{x=0} = 1$, so that g is the slope of the linear response of the neuron to small post-synaptic potential. The theory can be easily extended to transfer functions which are not odd symmetric. The case of nonsaturated transfer functions have been studied recently [2,3].

The dynamical behavior of a network of N neurons is governed by the first-order differential equations:

$$\frac{d}{dt} h_i(t) = -h_i(t) + \sum_{j=1}^{N} J_{ij}\, S_j(t), \quad i = 1, \dots N. \tag{1}$$

In electrical terms, Eq. (1) are the Kirchhoff current law of the neuron, where the current charging the membrane capacitance, the term on the left-hand side, must equal the current through the membrane resistance, the first term on the right-hand side, plus the current due to the activity of the other cells, last term on the right-hand side. For simplicity the microscopic time constant is taken equal to one.

The (real) matrix J_{ij}, with $J_{ii} = 0$, gives the properties of the synaptic coupling between the pre-synaptic jth neurons and the post-synaptic ith neuron. It defines the topology of the network: $J_{ij} = 0$ not connected $J_{ij} \neq 0$ connected; the type of the synaptic connection: $J_{ij} > 0$ excitatory $J_{ij} < 0$ inhibitory; the strength of the connection: $|J_{ij}|$.

We shall focus on the steady state of the network, that is the dynamical state in which the network settles down after a reasonable time has elapsed from the initial time t_0. Thus, we shall assume that $t_0 \to -\infty$, so that memory of the initial state at t_0 has been lost.

Clearly the dynamical behavior of the network depends on J_{ij}. Nevertheless, we can distinguish two classes. If the matrix J_{ij} is *symmetric*, i.e., $J_{ji} = J_{ij}$, then the dynamical Eq. (1) describes the relaxation

$$\frac{d}{dt} S_i(t) = -\frac{\partial}{\partial h_i} E(h_1, \dots, h_N)|_{h_i = h_i(t)} \tag{2}$$

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of the energy function:

$$E(h_1, \ldots, h_N) = \sum_i \int_0^{h_i} dh\, h \frac{dS}{dh} - \frac{1}{2} \sum_{ij} J_{ij}\, S_i S_i. \quad (3)$$

The dynamics hence converges toward stable fixed points, which correspond to the stable local minima of $E(h_1, \ldots, h_N)$. The structure of the fixed points can be complex [4,5]; nevertheless, the asymptotically long-time state is simple.

If the matrix is *nonsymmetric*, i.e., $J_{ji} \neq J_{ij}$, then an energy function cannot be defined and a richer steady-state behavior emerges: besides fixed points, limit cycles and chaotic behavior are also possible.

We shall consider here the simple case of a fully connected network with *random*, *asymmetric*, and *independent* couplings:

$$\overline{J_{ij}} = 0, \quad \overline{(J_{ij})^2} = 1/N, \quad \overline{J_{ij}\, J_{ji}} = 0, \quad i \neq j. \quad (4)$$

Here, and in the following, $\overline{(\cdots)}$ denotes averaging with the coupling probability distribution $P(\mathbf{J}) = \prod_{ij} P(J_{ij})$. The scaling of the second moment with N ensures that the second term on the right-hand side of Eq. (1) is $O(1)$ as $N \to \infty$ (thermodynamic limit).

The assumption of zero average implies that there is not a preferred type of synaptic connection. This can be relaxed by imposing a finite average J_0/N to tune preferred inhibitory ($J_0 < 0$) or excitatory ($J_0 > 0$) synaptic connections.

Provided the high order moments of $P(\mathbf{J})$ do not grow too fast with N, in the large N limit only the first two moments are needed. Thus, we can assume that J_{ij} are i.i.d. Gaussian variables.

The full solution of the model, referred to as dynamic mean-field theory (DMFT), has been presented and discussed in Ref. [6]. Since then, several variations of this model have been studied; see, e.g., Refs. [2,7–14]. This model has served also as the basis for computational modeling in recurrent networks in particular work on echo state networks, reservoir computing, force learning.

Although the DMFT can be derived by an intuitive construction of self-consistent equations for the fluctuations in the system, the *ad hoc* derivation suffers from potentially severe limitations. Most importantly, determining the stability conditions for the network dynamical state is a considerable challenge for such a naive approach. Also, computing various response and correlation functions require going beyond the DMFT themselves. Finally, extensions to more complex architecture or dynamics may be less amenable to naive approaches to the construction of the correct self consistent DMFT equations. Last but not least, it is hard to compute corrections to the theory without a more systematic formalism. Here we present a systematic approach to the study of dynamical states in random neural networks using path integral method. Path integrals have been extensively used in the study of stochastic dynamics in statistical mechanics, from the pioneering work of the Martin-Siggia-Rose [15] to work on critical phenomena and RG analysis [16–18] and to study the stochastic dynamics of spin glasses [19–21].

The study of deterministic dynamical systems with path integrals, such as in the present study, is less common. Nevertheless, in our case this application is facilitated by the presence of an asynchronous chaotic state, which generates dynamical deterministic fluctuations with stationary statistics. The present approach, which was used to derive the results reported in Ref. [6], expands on unpublished manuscript by the same authors from 1988. For a related approach, see Ref. [22]. For an alternative study of neural networks based on the analogy with conservative Newtonian dynamics, see, e.g., Ref. [23].

The plan of the paper is as follows. In Sec. II we derive the dynamical field theory (DFT) describing the dynamical behavior of the model Eq. (1). The possible different solutions of the DMFT valid in the limit $N \gg 1$ are discussed in Sec. III, and their stabilities are analyzed in Sec. IV. Finally, in Sec. V, as an illustration of how dynamical quantities can be computed using DFT, we present the calculation of the maximum Lyapunov exponent.

II. DYNAMICAL FIELD THEORY

In this section we shall show how the the dynamical behaviour of the network can be described using path integral methods. Prior to this we introduce the useful shorthand notation $h_i^a = h_i(t_a)$ and rewrite the equation of motion, Eq. (1), as

$$\partial_a h_i^a = -h_i^a + \sum_{j=1}^{N} J_{ij}\, S_j^a, \quad (5)$$

where $\partial_a = (d/dt_a) + \delta$ ($\delta \to 0^+$) to ensure causality [24], and $S_i^a = \phi(g h_i^a)$.

A. Path integral and dynamical field theory

The strategy of the path integral approach is to derive a generating functional for the relevant correlation and response functions induced by the dynamics Eq. (5). To work on a finite-dimensional space, one starts by dividing the time interval of interest $[t_0, t]$ into n segments of length δt and changing the differential equation $\partial_a h_i^a = f(h_i^a)$ into the finite-difference equation:

$$h_i^{a+1} - h_i^a = f\big(h_i^a\big)\, \delta t + b_i^a \delta t + h_i^0 \delta_{a0}^{\mathrm{Kr}}, \quad (6)$$

with the (discrete) index $a = 0, 1, \ldots, n$ indicating the time. Two terms have been added: an external field b_i^a to evaluate response functions and the initial condition $h_i^0 \delta_{a0}^{\mathrm{Kr}}$, where $\delta_{ab}^{\mathrm{Kr}}$ is the Kronecker δ, to enforce the initial condition at t_0. The continuum limit is recovered by taking $n \to \infty$ and $\delta t \to 0$ with $n\delta t$ fixed [25].

Denoting by $\tilde{h}_i^a$ the solution to Eq. (6), the generating functional of correlation and response functions of the dynamical system Eq. (6) reads

$$\begin{aligned} Z[\hat{b}, b] &= \int \prod_a \prod_i dh_i^a\, \delta\big(h_i^a - \tilde{h}_i^a\big)\, e^{-i \hat{b}_i^a h_i^a} \\ &= \int \prod_a \prod_i dh_i^a\, \delta\big(h_i^{a+1} - h_i^a - f\big(h_i^a\big)\delta t \\ &\quad - b_i^a \delta t - h_i^0 \delta_{a0}^{\mathrm{Kr}}\big)\, e^{-i \hat{b}_i^a h_i^a}. \end{aligned} \quad (7)$$

The second line is obtained by using the identity $\delta(h_i^a - \tilde{h}_i^a) = |F'(h_i^a)|\ \delta[F(h_i^a)]$, where $F(h_i^a) = h_i^{a+1} - h_i^a - f(h_i^a)\delta t -$

$b_i^a\delta t - h_i^0\delta_{a0}^{\rm Kr}$ and $F'(h_i^a)$ is the Jacobian of the transformation $h_i^a - \tilde{h}_i^a = 0 \to F(h_i^a) = 0$. The Jacobian depends on the discretization scheme used to translate the differential equation into a finite-difference equation, even in the continuum limit $n \to \infty$ [26]. The role of the Jacobian is to ensure that correlation and response functions do not depend on the discretization scheme used to construct the generating functional, apart form the initial value of the response functions [27]. The scheme adopted in Eq. (6), known as the Ito scheme in the theory of stochastic differential equations, has the advantage that the Jacobian is equal to one. Another consequence of this scheme is that $\theta(0^-) = 0$ and $\theta(0^+) = 1$, where $\theta(x)$ is the Heaviside step function; see, e.g., Ref. [28].

Equation (7) can be made more manageable by using the Fourier representation of the Dirac δ function,

$$\delta\big(z_i^a\big) = \int_{-\infty}^{+\infty} \frac{d\hat{h}_i^a}{2\pi}\, e^{-i\hat{h}_i^a z_i^a}, \tag{8}$$

to rewrite it as

$$Z[\hat{b}, b] = \int \prod_a \prod_i \frac{d\hat{h}_i^a\, dh_i^a}{2\pi} \times e^{-i\hat{h}_i^a\,(h_i^{a+1} - h_i^a - f(h_i^a)\delta t - b_i^a\delta t - h_i^0\delta_{a0}^{\rm Kr}) + i\hat{b}_i^a h_i^a}. \tag{9}$$

Taking the continuum limit $n \to \infty$ with $n\delta t = t - t_0$, $Z[\hat{b}, b]$ becomes a path integral over all possible paths $\{\hat{h}_i, h_i\}_{t_a\in[t_0,t]}$:

$$Z[\hat{b}, b] = \int \prod_i \mathcal{D}\hat{h}_i\, \mathcal{D}h_i\, e^{-S[\hat{h},h] + \sum_{ia}(i\hat{b}_i^a h_i^a + i\hat{h}_i^a b_i^a)}, \tag{10}$$

where $\mathcal{D}\hat{h}_i = \lim_{n\to\infty} \prod_a d\hat{h}_i^a/2\pi$, $\mathcal{D}h_i = \lim_{n\to\infty}\prod_a dh_i^a$, $\sum_a \equiv \int dt_a$ and $S[\hat{h}, h]$ is the dynamical action:

$$\begin{aligned} S[\hat{h}, h] &= \sum_{ia} i\hat{h}_i^a\,\big(\partial_a h_i^a - f\big(h_i^a\big) - h_i^0\delta_{a0}\big) \\ &= \sum_{ia} i\hat{h}_i^a \left(\partial_a h_i^a + h_i^a - \sum_j J_{ij} S_j^a - h_i^0\delta_{a0}\right), \end{aligned} \tag{11}$$

of the equation of motion, Eq. (5), with initial condition $h_i^0 = h_i(t_0)$. Here $\delta_{a0} \equiv \delta(t_a - t_0)$. We have not included the term b_i^a into the dynamical action because the original problem does not have an external field.

From the definition Eq. (7) it follows that $Z[0, b] = 1$, then

$$\begin{aligned} &\frac{\delta}{\delta i\hat{b}_{i_1}^{a_1}} \cdots \frac{\delta}{\delta i\hat{b}_{i_n}^{a_n}} \frac{\delta}{\delta b_{j_1}^{b_1}} \cdots \frac{\delta}{\delta b_{j_m}^{b_m}} Z[\hat{b}, b]\bigg|_{\hat{b}=b=0} \\ &\quad = \big\langle h_{i_1}^{a_1} \ldots h_{i_n}^{a_n} \hat{h}_{j_1}^{b_1} \ldots \hat{h}_{j_m}^{b_m}\big\rangle_J, \end{aligned} \tag{12}$$

are the correlation functions of h_i^a and $\hat{h}_i^a$ over the dynamics generated by the action Eq. (11) for fixed coupligs J_{ij}.

Correlations of only h-fields are correlation functions of the dynamics ruled by Eq. (5). Those involving both $\hat{h}$ and h fields the response functions, as can be inferred from Eq. (10) by noticing that

$$\big\langle h_{i_1}^{a_1} \ldots h_{i_n}^{a_n} \hat{h}_{j_1}^{b_1} \ldots \hat{h}_{j_m}^{b_m}\big\rangle_J = \frac{\delta}{\delta b_{j_1}^{b_1}} \cdots \frac{\delta}{\delta b_{j_m}^{b_m}} \big\langle h_{i_1}^{a_1} \ldots h_{i_n}^{a_n}\big\rangle_{Jb}\bigg|_{b=0}, \tag{13}$$

where the average $\langle(\cdots)\rangle_{Jb}$ is over all trajectory generated by the equation of motion, Eq. (5), in the presence of the external field b_i^a. For this reason hat-fields are also called *response*-fields. Note that since $(\delta/\delta b)Z[\hat{b}, b]|_{\hat{b}=b=0} = (\delta/\delta b)Z[0, b]|_{b=0}$ and $Z[0, b] = 1$ correlations involving only $\hat{h}$-fields vanish.

The correlation functions Eq. (12) depends on the coupling matrix J_{ij} and are random quantities. Since $Z[0, 0] = 1$ averaged correlation functions can be obtained by averaging $Z[\hat{b}, b]$ over the couplings J_{ij} [19]. In the case of the i.i.d. Gaussian J_{ij} Eq. (4), this leads to

$$\begin{aligned} \overline{Z}[\hat{b}, b] = \int \prod_i \mathcal{D}\hat{h}_i\, \mathcal{D}h_i\, \exp\Bigg[&-\sum_{ia} i\hat{h}_i^a\,(1 + \partial_a)h_i^a \\ &+ \frac{1}{2N}\sum_{ij}\left(\sum_a i\hat{h}_i^a S_j^a\right)^2 - h_i^0\delta_{a0} \\ &+ \sum_{ia}\big(i\hat{b}_i^a h_i^a + i\hat{h}_i^a b_i^a\big)\Bigg]. \end{aligned} \tag{14}$$

The nonlocal term can be simplified by introducing $C^{ab} = \sum_i S_i^a S_i^b/N$ [20] using the identity [29]

$$\begin{aligned} 1 &= \int dC^{ab}\, \delta\left(C^{ab} - \sum_i S_i^a S_i^b/N\right) \\ &= \int \frac{N}{4\pi} d\hat{C}^{ab}\, dC^{ab}\, \exp\left[-\frac{1}{2} i\hat{C}^{ab}\left(NC^{ab} - \sum_i S_i^a S_i^b\right)\right]. \end{aligned} \tag{15}$$

The exponent in Eq. (14) becomes diagonal in the site index i with a residual site dependence due to the auxiliary fields b_i^a and $\hat{b}_i^a$, because the system is fully connected. The averaged generating functional can then be written as the partition function

$$\overline{Z}[\hat{b}, b] = \int \mathcal{D}\hat{C}\, \mathcal{D}C\, e^{-N\mathcal{L}[\hat{C}, C; \hat{b}, b]}, \tag{16}$$

of a dynamical field theory for the fields $\{\hat{C}^{ab}, C^{ab}\}$, with $\hat{C}^{ba} = \hat{C}^{ab}$ and $C^{ba} = C^{ab}$, described by the action

$$\begin{aligned} \mathcal{L}[\hat{C}, C; \hat{b}, b] = \frac{1}{2}\sum_{ab}\left(i\hat{C}^{ab}C^{ab} + \frac{\epsilon}{2}\hat{C}^{ab}\hat{C}^{ab}\right) \\ - W[\hat{C}, C; \hat{b}, b], \end{aligned} \tag{17}$$

where

$$\begin{aligned} NW[\hat{C}, C; \hat{b}, b] = \sum_i \ln \int \mathcal{D}\hat{h}_i\, \mathcal{D}h_i \\ \times e^{-S[\hat{h}_i, h_i; C, \hat{C}] + \sum_a (i\hat{h}_i^a b_i^a + i\hat{b}_i^a h_i^a)}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} S[\hat{h}_i, h_i; \hat{C}, C] = \sum_a \big[i\hat{h}_i^a(1 + \partial_a)h_i^a - h_i^0\delta_{a0}\big] \\ - \frac{1}{2}\sum_{ab}\big[i\hat{C}^{ab}S_i^a S_i^b + C^{ab} i\hat{h}_i^a i\hat{h}_i^b\big]. \end{aligned} \tag{19}$$

We have added a small regularizing term $\epsilon \to 0^+$ in Eq. (17) to make integrals well definite [30]. Note that the functional $W[\hat{C}, C; \hat{b}, b]$ is the generating functional of connected (time) correlation functions of h_i and $\hat{h}_i$ generated by the action:

$$\mathcal{L}[\hat{h}, h; \hat{C}, C, \hat{b}, b] = \sum_i S[\hat{h}_i, h_i; C, \hat{C}] - \sum_{ia} \left(i\hat{h}_i^a b_i^a + i\hat{b}_i^a h_i^a \right). \quad (20)$$

The relevant time correlation and response functions of the field S^a along the dynamical evolution governed by Eq. (5) can be obtained from averages of $\hat{C}^{ab}$ and C^{ab} with the action $\mathcal{L}[\hat{C}, C; 0, 0]$. For example,

$$\frac{1}{N} \sum_{i=1}^{N} \overline{\langle S_i(t_a) S_i(t_b) \rangle}_J = \langle S^a S^b \rangle = \langle C^{ab} \rangle, \quad (21)$$

where $\langle (\cdots) \rangle$ denotes DFT average with action $\mathcal{L}[\hat{C}, C; 0, 0]$. Details are in Appendix B. The last equality follows because if $\hat{b} = b = 0$, or more generally if they do not depend on the site, different sites decouples and are all equivalent.

The results obtained so far are valid for any value of N. In the rest of this paper we shall consider the (thermodynamic) limit $N \gg 1$.

B. Thermodynamic limit and dynamical mean-field theory

In the limit $N \to \infty$ the integral in Eq. (16) is dominated by the largest value of the exponent. Therefore,

$$\overline{Z}[\hat{b}, b] \sim \overline{Z}_0[\hat{b}, b] = e^{-N\mathcal{L}_0[\hat{C}, C; \hat{b}, b]}, \quad N \gg 1, \quad (22)$$

where $\mathcal{L}_0[\hat{C}, C; \hat{b}, b]$ is the value of the action at the stationary point:

$$\frac{\delta}{\delta C^{ab}} \mathcal{L}[\hat{C}, C; \hat{b}, b] = 0 \Rightarrow i\hat{C}^{ab} = \frac{1}{N} \sum_i \left\langle i\hat{h}_i^a i\hat{h}_i^b \right\rangle_0, \quad (23)$$

$$\frac{\delta}{\delta i\hat{C}^{ab}} \mathcal{L}[\hat{C}, C; \hat{b}, b] = 0 \Rightarrow C^{ab} = \frac{1}{N} \sum_i \left\langle S_i^a S_i^b \right\rangle_0 + \epsilon i\hat{C}^{ab}. \quad (24)$$

The (self-consistent) average $\langle (\cdots) \rangle_0$ is over all paths of the dynamical process $\{\hat{h}, h\}_{t \in [t_0, t]}$ governed by the action $\mathcal{L}[\hat{h}, h; \hat{C}, C, \hat{b}, b]$ evaluated at the stationary point. The normalization $\overline{Z}[0, b] = 1$ implies that $\hat{C}^{ab} = 0$ is the correct self-consistent solution; see below. Then $\overline{Z}_0[\hat{b}, b] = \prod_i Z_i[\hat{b}, b]$ with

$$Z_i[\hat{b}, b] = \int \mathcal{D}h_i \, \mathcal{D}\hat{h}_i \, e^{-S[\hat{h}_i, h_i; 0, C] + \sum_a (i\hat{h}_i^a b_i^a + i\hat{b}_i^a h_i^a)}, \quad (25)$$

and the dynamical behavior of the network in the limit $N \to \infty$ is fully described by the single-site dynamical processes $\{\hat{h}_i, h_i\}$.

Using the identity

$$\exp\left[\frac{1}{2} \sum_{ab} i\hat{h}_i^a C^{ab} i\hat{h}_i^b \right] = \left\langle \exp \sum_a i\hat{h}_i^a \eta^a \right\rangle_\eta, \quad (26)$$

where η^a is Gaussian field of mean $\langle \eta^a \rangle_\eta = 0$ and

$$\langle \eta^a \eta^b \rangle_\eta = C^{ab}, \quad (27)$$

$Z_i[\hat{b}, b]$ can be written as

$$Z_i[\hat{b}, b] = \left\langle \int \mathcal{D}\hat{h}_i \, \mathcal{D}h_i \, e^{-\sum_a i\hat{h}_i^a [(1+\partial_a) h_i^a - \eta^a - h_i^0 \delta_{a0}] + \sum_a (i\hat{h}_i^a b_i^a + i\hat{b}_i^a h_i^a)} \right\rangle_\eta. \quad (28)$$

Therefore, $Z_i[\hat{b}, b]$ is the generating functional of the stochastic process described by the stochastic differential equation,

$$\partial_a h_i^a = -h_i^a + b_i^a + \eta^a, \quad (29)$$

with initial condition $h_i(t_0) = h_i^0$ averaged over the Gaussian noise η^a. This process provides the full description of the dynamics of the network in the limit $N \to \infty$. While diagonal in the site index, the process maintains memory of the other sites through the Gaussian field η^a because $\langle \eta^a \eta^b \rangle_\eta$ must be computed self-consistently through the constraint Eq. (24). Equations (24), (27), and (29) are the central equations of the DMFT.

Note that if $\hat{C}^{ab}$ were not zero then $Z_i[0, b] = \langle \exp[\sum_{ab} i\hat{C}^{ab} S_i^a S_i^b] \rangle$, where the average is over the stochastic process Eq. (29), and $\overline{Z}[0, b]$ would not be necessarily equal to 1.

III. SOLUTION OF DMFT EQUATIONS

In this section we discuss the solutions of the DMFT equations. Without loss of generality we can take uniform $\hat{b}^a$ and b^a and drop the site index. Moreover, since we are interested in the steady state, we take the initial time $t_0 = -\infty$ and we can neglect the initial state $h_i(t_0)$.

A. DMFT equations

To discuss the DMFT it is useful to rewrite the DMFT equations as follows. Using the relation $S^a = \phi(gh^a)$ the DMFT Eqs. (24) and (27) can be reduced to

$$\langle \eta^a \eta^b \rangle_\eta = C^{ab} = \langle \phi(gh^a) \phi(gh^b) \rangle_\eta, \quad (30)$$

where, from Eq. (29),

$$h^a = h(t_a) = \int_{-\infty}^{t_a} dt_b \, e^{-(t_a - t_b)} \, \eta(t_b). \quad (31)$$

We have set $b_i^a = 0$ because no external field is present in the original problem.

The synaptic field h^a is a linear functional of η^a, and hence it is a Gaussian process with zero mean and correlation

$$\langle h^a h^b \rangle_\eta = \Delta^{ab}. \quad (32)$$

Equation (30) provides a nonlinear relation between the field correlation Δ^{ab} and the activity correlation C^{ab}.

Explicitly,

$$C^{ab} = \iint \frac{d^2\mathrm{h}}{2\pi\sqrt{\det\Delta}} \exp\left[-\frac{1}{2}\mathrm{h}^T\Delta^{-1}\mathrm{h}\right]\phi(gh^a)\phi(gh^b), \tag{33}$$

where $\mathrm{h}^T = (h^a, h^b)$, and $\mathbf{\Delta}$ is the 2×2 symmetric matrix:

$$\mathbf{\Delta} = \begin{bmatrix} \Delta^{aa} & \Delta^{ab} \\ \Delta^{ab} & \Delta^{bb} \end{bmatrix}. \tag{34}$$

It is sometime convenient to write this relation as

$$C^{ab} = \int Dz \int Dx\, \phi(gx\sqrt{\Delta^{aa} - |\Delta^{ab}|} + gz\sqrt{|\Delta^{ab}|}) \times \int Dy\, \phi(gy\sqrt{\Delta^{bb} - |\Delta^{ab}|} + gz\sqrt{|\Delta^{ab}|}), \tag{35}$$

where $Dx = dx\exp(-x^2/2)/\sqrt{2\pi}$, and similarly Dy and Dz, are Gaussian measures. Alternatively, introducing the Fourier transform $\tilde{\phi}(k)$ of the function $\phi(x)$, the relation between Δ^{ab} and C^{ab} can also be written as

$$C^{ab} = \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\,\tilde{\phi}(k)\tilde{\phi}(k') \times \exp\left[-\frac{g^2}{2}(\Delta^{aa}k^2 + \Delta^{bb}k'^2) - g^2\Delta^{ab}kk'\right]. \tag{36}$$

Details are in Appendix C.

However, by multiplying Eq. (29) by itself and averaging over η, we obtain the relation

$$(1+\partial_a)(1+\partial_b)\Delta^{ab} = C^{ab}, \tag{37}$$

expressing Δ^{ab} as function of C^{ab}.

Equations (33) and (37) constitute the DMFT self-consistent equations of our system.

B. Steady state solutions

The DMFT considerably simplifies in the steady-state regime, which is the focus of the present paper. In this regime, the dynamical correlation functions are time translation invariant and Δ^{ab} depends on t_a and t_b only through the time difference $\tau = t_a - t_b$:

$$\Delta^{ab} = \Delta(\tau) \equiv \Delta, \tag{38}$$

$$\Delta^{aa} = \Delta^{bb} = \Delta(0) \equiv \Delta_0. \tag{39}$$

In this case the DMFT Eq. (35) becomes

$$C(\Delta;\Delta_0) = \int Dz\left[\int Dx\,\phi(gx\sqrt{\Delta_0 - |\Delta|} + gz\sqrt{|\Delta|})\right]^2, \tag{40}$$

while, using the identities $\partial_a\Delta(t_a - t_b) = \partial_\tau\Delta(\tau)$ and $\partial_b\Delta(t_a - t_b) = -\partial_\tau\Delta(\tau)$, Eq. (37) reduces to

$$\Delta - \partial_\tau^2\Delta = C(\Delta;\Delta_0). \tag{41}$$

Since $\Delta(\tau)$ is a correlation function, acceptable solutions to Eq. (41) must obey

$$|\Delta(\tau)| \leqslant \Delta(0), \tag{42}$$

and in particular they must be bounded.

Equation (41) admits a two-parameter family of solutions parametrized by the *initial conditions* $\Delta(0)$ and $\partial_\tau\Delta|_{\tau=0} = 0$. This choice is rather convenient because the initial "velocity" is

$$\partial_\tau\Delta|_{\tau=0} = 0, \tag{43}$$

as follows from the explicit solution of the differential Eq. (41),

$$\Delta(\tau) = \frac{1}{2}\int_{-\infty}^{+\infty} d\tau'\, e^{-|\tau-\tau'|}\,C(\tau'), \tag{44}$$

which implies that $\Delta(\tau)$ is a differentiable even function of τ: $\Delta(-\tau) = \Delta(\tau)$. The initial "position" $\Delta(0)$ is fixed by the requirement

$$\Delta(0) = \Delta_0, \tag{45}$$

so that the steady-state DMFT solutions are a one-parameter family of curve $\Delta \equiv \Delta(\tau;\Delta_0)$ parameterized by Δ_0.

1. Fixed point solution

The simplest solution is that of a fixed point: $\Delta(\tau) = \Delta_0 = C$, leading to the self consistent equation,

$$\Delta_0 = [\phi^2]_{\Delta_0}, \tag{46}$$

where, for later use, we have introduced the notation

$$[f]_{\Delta_0} = \int Dx\ f(gx\sqrt{\Delta_0}). \tag{47}$$

For the odd-symmetric transfer function, such as $\phi(x) = \tanh(x)$, there is always a solution with $\Delta_0 = 0$, corresponding to the zero fixed point $h_i = 0$ of the original dynamics. A solution with nonzero Δ_0 appears when $g > 1$. The static fixed-point solution is, however, unstable for $g > 1$, as shown in the next section; see also Appendix A.

2. Time-dependent solution: Potential and energy

Solving the DMFT equations is greatly facilitated by noting that for a fixed Δ_0 the differential Eq. (41) can be viewed as the inertial dynamics of a particle moving under the influence of a potential $V(\Delta;\Delta_0)$, i.e.,

$$\partial_\tau^2\Delta = -\partial_\Delta V(\Delta;\Delta_0), \tag{48}$$

where

$$V(\Delta;\Delta_0) = -\frac{\Delta^2}{2} + \int_0^\Delta d\Delta'\,C(\Delta';\Delta_0). \tag{49}$$

Introducing the function $\Phi(x) = \int_0^x dy\,\phi(y)$, primitive of the gain function $\phi(x)$, the potential can be expressed as

$$V(\Delta;\Delta_0) = -\frac{\Delta^2}{2} + \frac{1}{g^2}\int Dz\left[\int Dx\,\Phi(gx\sqrt{\Delta_0 - |\Delta|} + gz\sqrt{|\Delta|})\right]^2 - \frac{1}{g^2}\left[\int Dx\,\Phi(gx\sqrt{\Delta_0})\right]^2, \tag{50}$$

The last term ensures that $V(0,\Delta_0) = 0$. Details can be found in Appendix C. All solutions to Eq. (48) conserve the energy:

$$E_c = \tfrac{1}{2}(\partial_\tau\Delta)^2 + V(\Delta;\Delta_0). \tag{51}$$

Hence, since DMFT solutions must have $\Delta(0) = \Delta_0$, $\partial_\tau \Delta|_{\tau=0} = 0$, and be bounded, all solutions to Eq. (48) with energy $E_c = V(\Delta_0; \Delta_0)$ leading to bounded orbits are possible DMFT solutions $\Delta = \Delta(\tau; \Delta_0)$. The qualitative behavior of the solutions can be inferred from the shape of the potential $V(\Delta; \Delta_0)$. Solutions with different properties are possible because $V(\Delta; \Delta_0)$ depends parametrically on the value of Δ_0, reflecting the self-consistent nature of the DMFT.

3. Phase diagram

The exact form of $V(\Delta; \Delta_0)$ depends on $\phi(x)$. However, its qualitative behavior can be determined. First, we note that for $\Delta > 0$:

$$\partial^3_\Delta V(\Delta; \Delta_0) = g^4 \int Dz \left[\int Dx\, \phi''(gx\sqrt{\Delta_0 - \Delta} + gz\sqrt{\Delta}) \right]^2 > 0. \tag{52}$$

The "prime" denotes derivative of the function with respect to its argument, hence $\phi''(x)$ is the second derivative of $\phi(x)$ with respect to x. Thus, $\partial^2_\Delta V(\Delta; \Delta_0)$ is monotonously increasing and can vanish at most once for $0 < \Delta < \Delta_0$. As a consequence, the shape of $V(\Delta; \Delta_0)$ is either a single-well or a double-well depending on the sign of $\partial^2_\Delta V(\Delta; \Delta_0)$ at $\Delta = 0$. Expanding $V(\Delta; \Delta_0)$ for $|\Delta| \ll 1$, gives, see Appendix C,

$$V(\Delta; \Delta_0) = \left(-1 + g^2[\phi']^2_{\Delta_0}\right)\frac{\Delta^2}{2} + g^6[\phi''']^2_{\Delta_o}\frac{\Delta^4}{24} + \dots . \tag{53}$$

If $-1 + g^2[\phi']_{\Delta_0} \geqslant 0$, then the potential is a single well: The energy E_c is positive and $\Delta(\tau)$ is time-periodic. It changes sign during one oscillation.

In the case $-1 + g^2[\phi']_{\Delta_0} < 0$ the potential has a double-well shape, and qualitatively different solutions appears depending on the sign of the energy

$$E_c = V(\Delta_0; \Delta_0) = -\frac{\Delta_0^2}{2} + \frac{1}{g^2}[\Phi^2]_{\Delta_0} - \frac{1}{g^2}[\Phi]^2_{\Delta_0}. \tag{54}$$

When $E_c > 0$ the solution is qualitatively similar to the previous case: $\Delta(\tau)$ is time-periodic and changes sign during one oscillation. On the contrary if $E_c < 0$ and $\partial_\Delta V(\Delta; \Delta_0)$ at $\Delta = \Delta_0$ is positive then $\Delta(\tau)$ is time-periodic but does not change sign during one oscillation. The two regimes are separated by the boundary $E_c = 0$, where $\Delta(\tau)$ decays monotonously to 0 as $\tau \to \infty$. When E_c reaches the minimum of $V(\Delta; \Delta_0)$, the solution becomes *time-independent*. This occurs for,

$$\partial_\Delta V(\Delta; \Delta_0)|_{\Delta=\Delta_0} = -\Delta_0 + C(\Delta_0; \Delta_0) = 0, \tag{55}$$

and one recovers the time-independent solution found previously. The different cases are shown in Fig. 1.

By drawing in the plane $(\Delta_0, 1/g)$ the curves separating the different type of solutions we obtain the phase diagram shown in Fig. 2.

Above the curve f there are no solutions with $\Delta_0 > 0$. Thus, for $g < 1$ only the time-independent solution $\Delta = \Delta_0 = 0$ exists. The vanishing of the equal time correlation Δ_0 in the steady state implies that the system flows to the zero fix point $h_i = 0$. The stability of this solution for $g < 1$ can be deduced by linearising Eq. (1) and noting that the maximum real part of the eigenvalues of the matrix J_{ij} is 1.

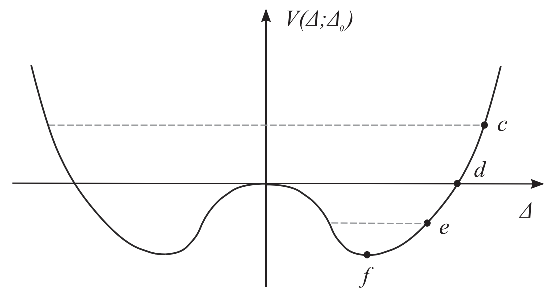

FIG. 1. Qualitative behavior of $V(\Delta; \Delta_0)$ for $-1 + g^2[\phi']_{\Delta_0} < 0$. Labels denotes the different possible behaviours of the solution. Label c: $E_c > 0$, $\Delta(\tau)$ is periodic and changes sign. Label e: $E_c < 0$, $\Delta(\tau)$ is periodic but remains positive. Label d: $E_c = 0$, $\Delta(\tau)$ decays to zero as $\tau \to \infty$. Label f: minimum allowable value of E_c, $\Delta(\tau) = \Delta_0$, static solution.

For $g > 1$ different scenarios appears. On the curve f the energy E_c attains its minimum value and the solution is time-independent: $\Delta(\tau) = \Delta_0$. On this curve the state the network flows to a nonzero fix-point characterized by a nontrivial distribution of h_i. In the region below the curve f the energy E_c is larger than the minimum of $V(\Delta; \Delta_0)$ and $\Delta(\tau)$ becomes time-dependent. Here time-periodic solutions appear, either changing sign in one period, below curve d, or not, between curves d and f. In either cases these solutions imply that the dynamical behaviour of the network in the steady state is

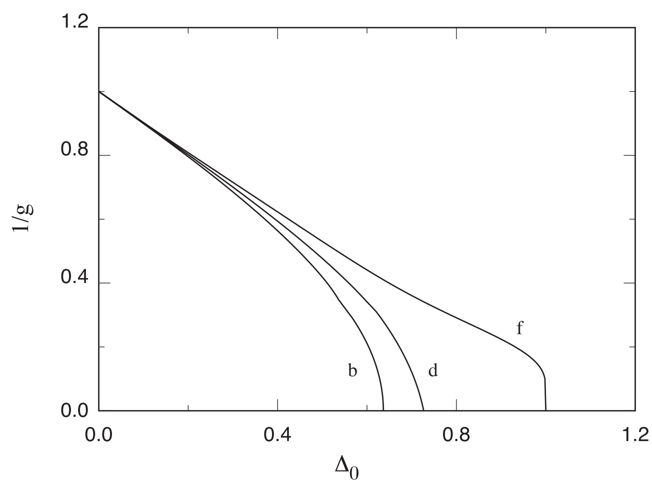

FIG. 2. Dynamical mean-field theory phase diagram. The curves delimit the regions of qualitative different behaviours. On the curve f the energy E_c is equal to the minimum of $V(\Delta; \Delta_0)$ and $\Delta(\tau)$ is time-independent. Between the curve f and the curve d the energy $E_c < 0$ and $\Delta(\tau)$ is time-periodic but positive. On the curve d the energy E_c vanishes and $\Delta(\tau)$ decays to 0 as $\tau \to \infty$. Below the curve d the energy $E_c > 0$ and $\Delta(\tau)$ is time-periodic with not definite sign. On the curve b the potential $V(\Delta; \Delta_0)$ changes from a double well shape to a single well shape. Above the curve f there are no solution to the DMFT equations. For $g > 1$ all curves collapse and only the static solution $\Delta(\tau) = \Delta_0 = 0$ survives. Numerical values are for $\phi(x) = \tanh(x)$.

a limit cycle. On the curve d, corresponding to $E_c = 0$ and separating the two types of periodic solutions, $\Delta(\tau)$ is not periodic and decays monotonously to 0 as $\tau \to \infty$. On this curve the dynamical behavior of the network is chaotic.

IV. FLUCTUATIONS AROUND THE DMFT AND SOLUTION SELECTION

The large number of solutions of the DMFT for $g > 1$ raises the question of what are the criteria of selection of one or a few of those solutions as the physically relevant ones. The problem is twofold. The DMFT follows from a saddle point calculation of the path integrals. Thus, only solutions leading to a stable saddle point, i.e., a local maximum of the action, must be retained. An analysis of the Hessian of the fluctuations reveals that all solutions are stable saddle point (see below); the steady state of the network is hence given by all the above mentioned solutions: fixed points, limit cycles and chaos.

The second problem is which of these solutions represent a stable attractor of the network dynamics. Stated differently, toward which steady state will the dynamics flow with probability one as the system size $N \to \infty$? We address this question by studying the behavior of two copies (replicas) of the network as $N \to \infty$.

A. Two replica formalism

The stability of an attractor is related to the properties of the linear response matrix $\langle \partial h_i(t+\tau)/\partial h_j(t)\rangle_J \sim \langle h_i(t+\tau)\hat{h}_j(t)\rangle_J$. Due to the quenched random couplings J_{ij} instability in this matrix may be washed out by averaging over J_{ij}, hence to uncover instability, one needs to consider quantities such as $\overline{\langle h_i(t+\tau)\hat{h}_j(t)\rangle_J^2}$. Such averages can be computed using DMFT starting from two identical copies of the system, namely, $h_i^\alpha(t)$, $\alpha = 1, 2$, obeying

$$\frac{d}{dt} h_i^\alpha(t) = -h_i^\alpha(t) + \sum_{j=1}^{N} J_{ij}\, \phi(g h_j^\alpha(t)), \quad i = 1, \dots, N, \tag{56}$$

and evaluating

$$\overline{\langle h_i(t+\tau)\hat{h}_j(t)\rangle_J^2} = \overline{\left\langle h_i^1(t+\tau) h_i^2(t'+\tau)\hat{h}_j^1(t)\hat{h}_j^2(t')\right\rangle_J} \tag{57}$$

with $t \neq t'$. Hence, the full dynamic stability can be determined from a stability analysis of the path integral formulation of the replicated system Eq. (56).

Conveniently, our above results incorporate readily the replicated system, if we replace the index a in Eq. (7) by $\boldsymbol{a} = (\alpha, a) = (\alpha, t_a)$ representing both replica index and time, e.g., $h_i^{\boldsymbol{a}} = h_i^{\alpha,a} = h_i^\alpha(t_a)$. For example, the averaged generating functional $\overline{Z}[\hat{b}, b]$ of the replicated system can be read directly from Eqs. (16)–(19). In particular, the action can be written as

$$\mathcal{L}[\hat{C}, C; \hat{b}, b] = \frac{1}{2}\sum_{\boldsymbol{ab}}\left(i\hat{C}^{\boldsymbol{ab}} C^{\boldsymbol{ab}} + \frac{\epsilon}{2}\hat{C}^{\boldsymbol{ab}}\hat{C}^{\boldsymbol{ab}}\right) - W[\hat{C}, C; \hat{b}, b], \tag{58}$$

where

$$N W[\hat{C}, C; \hat{b}, b] = \sum_i \ln \int \mathcal{D}\hat{h}_i\, \mathcal{D}h_i \times e^{-S[\hat{h}_i, h_i; C, \hat{C}] + \sum_{\boldsymbol{a}}(i\hat{h}_i^{\boldsymbol{a}} b_i^{\boldsymbol{a}} + i\hat{b}_i^{\boldsymbol{a}} h_i^{\boldsymbol{a}})}, \tag{59}$$

and

$$S[\hat{h}_i, h_i; \hat{C}, C] = \sum_{\boldsymbol{a}}\left[i\hat{h}_i^{\boldsymbol{a}}(1+\partial_a)h_i^{\boldsymbol{a}} - h_i^{0,\boldsymbol{a}}\delta_{a0}\right] - \frac{1}{2}\sum_{\boldsymbol{ab}}\left[i\hat{C}^{\boldsymbol{ab}} S_i^{\boldsymbol{a}} S_i^{\boldsymbol{b}} + C^{\boldsymbol{ab}} i\hat{h}_i^{\boldsymbol{a}} i\hat{h}_i^{\boldsymbol{b}}\right], \tag{60}$$

where $\sum_{\boldsymbol{a}} \equiv \sum_{\alpha=1,2}\int dt_a$, $C^{\boldsymbol{ab}} = C^{\alpha\beta,ab} = C^{\alpha\beta}(t_a, t_b)$ and similarly, $\hat{C}^{\boldsymbol{ab}} = \hat{C}^{\alpha\beta,ab} = \hat{C}^{\alpha\beta}(t_a, t_b)$. As a consequence, the results of Sec. II can be immediately extended to the replicated system. Hence, in the limit $N \to \infty$ the behavior of the replicated system is described by the saddle point of the functional Eq. (58), leading to the 2-replica DMFT. At the saddle point $\hat{C}^{\boldsymbol{ab}} = 0$ while $C^{\boldsymbol{ab}}$, solution of the stationary point equations, may in general depend on both replica and time indexes. However, since the two copies of the system are identical (including their initial value), the DMFT order parameters cannot depend on the replica index and hence $C^{\boldsymbol{ab}} = C^{ab} = C(t_a, t_b)$ for all α, β.

1. Fluctuations of the replicated DMFT

The stability of the 2-replica DMFT solutions can be inferred from the analysis of the Gaussian fluctuations about the stationary point of the action $\mathcal{L}[\hat{C}, C; \hat{b}; b]$ of the replicated system. Denoting by $Q^{\boldsymbol{ab}}$ and $\hat{Q}^{\boldsymbol{ab}}$ the fluctuations and expanding the action Eq. (58) to second order in Q and $\hat{Q}$ leads to

$$\overline{Z}[\hat{b}, b] \sim \overline{Z}_0[\hat{b}, b]\int \mathcal{D}\hat{Q}\,\mathcal{D}Q\, e^{-N\mathcal{L}_2[\hat{Q}, Q; \hat{b}, b]}, \quad N \gg 1, \tag{61}$$

where

$$\mathcal{L}_2[\hat{Q}, Q; \hat{b}, b] = \frac{1}{2}\sum_{\boldsymbol{ab}} i\hat{Q}^{\boldsymbol{ab}} Q^{\boldsymbol{ab}} - \frac{1}{8}\sum_{\boldsymbol{ab},\boldsymbol{cd}} i\hat{Q}^{\boldsymbol{ab}}\mathcal{M}^{\boldsymbol{ab};\boldsymbol{cd}} i\hat{Q}^{\boldsymbol{cd}} - \frac{1}{4}\sum_{\boldsymbol{ab},\boldsymbol{cd}} i\hat{Q}^{\boldsymbol{ab}}\langle S^{\boldsymbol{a}} S^{\boldsymbol{b}} i\hat{h}^{\boldsymbol{c}} i\hat{h}^{\boldsymbol{d}}\rangle_0 Q^{\boldsymbol{cd}}, \tag{62}$$

with

$$\mathcal{M}^{\boldsymbol{ab};\boldsymbol{cd}} = \epsilon\,\delta_{\boldsymbol{ab},\boldsymbol{cd}} + \langle S^{\boldsymbol{a}} S^{\boldsymbol{b}} S^{\boldsymbol{c}} S^{\boldsymbol{d}}\rangle_0 - \langle S^{\boldsymbol{a}} S^{\boldsymbol{b}}\rangle_0\langle S^{\boldsymbol{c}} S^{\boldsymbol{d}}\rangle_0, \tag{63}$$

and

$$\delta_{\boldsymbol{ab},\boldsymbol{cd}} = \delta_{\boldsymbol{ac}}\delta_{\boldsymbol{bd}} + \delta_{\boldsymbol{ad}}\delta_{\boldsymbol{bc}} \tag{64}$$

is the symmetrized δ function. The average $\langle(\cdots)\rangle_0$ is over the dynamical process governed by the action Eq. (60) with $\hat{C}^{ab}$ and C^{ab} evaluated at the stationary point of $\mathcal{L}[\hat{C}, C; \hat{b}; b]$.

Using the identity

$$e^{-S[\hat{h}, h; \hat{C}, C]}\, i\hat{h}^{\boldsymbol{a}} i\hat{h}^{\boldsymbol{b}} = \frac{\delta}{\delta C^{\boldsymbol{ab}}}\, e^{-S[\hat{h}, h; \hat{C}, C]}, \tag{65}$$

the average $\langle S^{\boldsymbol{a}} S^{\boldsymbol{b}} i\hat{h}^{\boldsymbol{c}} i\hat{h}^{\boldsymbol{d}}\rangle_0$ is equal to

$$\langle S^{\boldsymbol{a}} S^{\boldsymbol{b}} i\hat{h}^{\boldsymbol{c}} i\hat{h}^{\boldsymbol{d}}\rangle_0 = \frac{\delta}{\delta C^{\boldsymbol{cd}}}\,\langle S^{\boldsymbol{a}} S^{\boldsymbol{b}}\rangle_0. \tag{66}$$

The derivative is evaluated by recalling that $\langle S^a S^b\rangle_0 = \langle\phi(gh^a)\phi(gh^b)\rangle_0$, where h^a is the solution of the DMFT stochastic differential equation

$$\partial_a h^a = -h^a + \eta^a, \tag{67}$$

with η^a Gaussian field of zero mean and variance $\langle\eta^a\eta^b\rangle_\eta = C^{ab}$, cf. Sec. II B. The average $\langle S^a S^b\rangle_0$ is thus a function of the field-field correlation function $\Delta^{ab} = \langle h^a h^b\rangle_\eta$, so that using the chain rule $\langle S^a S^b i\hat{h}^c i\hat{h}^d\rangle_0$ is given by

$$\langle S^a S^b i\hat{h}^c i\hat{h}^d\rangle_0 = \frac{\partial}{\partial\Delta^{cd}}\langle S^a S^b\rangle_0 \frac{\delta\Delta^{cd}}{\delta C^{cd}}, \tag{68}$$

where, from the DMFT equations, $\delta\Delta^{ab}/\delta C^{cd}$ is solution of

$$(1+\partial_a)(1+\partial_b)\frac{\delta\Delta^{ab}}{\delta C^{cd}} = \delta_{ac,bd}. \tag{69}$$

To further proceed, it is then more convenient to transform $\mathcal{L}_2$ to the equivalent quadratic form

$$\mathcal{L}_2[\hat{Q},\Psi;\hat{b},b] = -\frac{1}{8}\sum_{ab,cd} i\hat{Q}^{ab}\mathcal{M}^{ab;cd} i\hat{Q}^{cd} + \frac{1}{4}\sum_{ab,cd} i\hat{Q}^{ab}\mathcal{A}^{ab;cd}\Psi^{cd}, \tag{70}$$

where Ψ^{ab} is defined through

$$(1+\partial_a)(1+\partial_b)\Psi^{ab} = Q^{ab}, \tag{71}$$

and the operator $\mathcal{A}$ is acting on Ψ via

$$\mathcal{A}^{ab;cd} := (1+\partial_a)(1+\partial_b)\delta_{ac,bd} - \frac{\partial}{\partial\Delta^{cd}}\langle S^a S^b\rangle_0. \tag{72}$$

The (Gaussian) integration over $\hat{Q}$ in Eq. (61) is well defined and can be performed. It leads to a term of the form $\exp[-(\text{const})\Psi\mathcal{A}^\dagger\mathcal{M}^{-1}\mathcal{A}\Psi]$, where $\mathcal{A}^\dagger$ is the adjoint of $\mathcal{A}$. Stability of the stationary point requires that the operator $\mathcal{A}$ has no zero eigenvalue.

Making use of the explicit form of $\partial\langle S^a S^b\rangle_0/\partial\Delta^{cd}$:

$$\frac{\partial}{\partial\Delta^{cd}}\langle S^a S^b\rangle_0 = \frac{\partial}{\partial\Delta^{ab}}\langle S^a S^b\rangle_0\delta_{ac}\delta_{bd} + \frac{\partial}{\partial\Delta^{aa}}\langle S^a S^b\rangle_0\delta_{ca}\delta_{da} + \frac{\partial}{\partial\Delta^{bb}}\langle S^a S^b\rangle_0\delta_{cb}\delta_{db}, \tag{73}$$

the eigenvalue equation for the operator $\mathcal{A}$ reads:

$$(1+\partial_a)(1+\partial_b)\Psi^{ab} - \frac{\partial}{\partial\Delta^{ab}}\langle S^a S^b\rangle_0\Psi^{ab} - \frac{\partial}{\partial\Delta^{aa}}\langle S^a S^b\rangle_0\Psi^{aa} - \frac{\partial}{\partial\Delta^{bb}}\langle S^a S^b\rangle_0\Psi^{bb} = \Lambda\Psi^{ab}. \tag{74}$$

The stability condition requires that this equation must *not* admit a solution with $\Lambda = 0$. The stability criterion does not require an evaluation of $\mathcal{M}$.

Note that the intra-replica fluctuations $\alpha = \beta$ are decoupled and independent of the inter-replica fluctuations $\alpha \neq \beta$.

Note also that since $\mathcal{A}$ is a symmetric operator, the solutions to the eigenvalue Eq. (74) can be classified as either symmetric eigenvectors $\Psi^{ab} = \Psi^{ba}$ or antisymmetric eigenvectors $\Psi^{ab} = -\Psi^{ab}$, where the symmetry operation is the simultaneous exchange of both replica and time indices.

B. Stability of the time-independent solution

The general expression of time-independent DMFT solution $\Delta^{\alpha\beta} = C^{\alpha\beta} = \langle\phi(gh^\alpha)\phi(gh^\beta)\rangle_0$ can be written as in Eq. (36):

$$\Delta^{\alpha\beta} = \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\tilde{\phi}(k)\tilde{\phi}(k') \times \exp\left[-\frac{g^2}{2}(\Delta^{\alpha\alpha}k^2 + \Delta^{\beta\beta}k'^2) - g^2\Delta^{\alpha\beta}kk'\right], \tag{75}$$

where $\tilde{\phi}(k)$ is the Fourier transform of the function $\phi(x)$.

The relevant solution to these equations is $\Delta^{\alpha\beta} = \Delta$, where Δ is obtained from the self-consistent equation:

$$\Delta = \int\frac{d\eta}{\sqrt{2\pi}}e^{-\eta^2/2}\phi(g\sqrt{\Delta}\,h)^2 = [\phi^2]_\Delta, \tag{76}$$

as in the single replica time-independent solution. For this solution, using Eqs. (C4) and (C5), $\partial\langle S^a S^b\rangle_0/\partial\Delta^{ab} = g^2[(\phi')^2]_\Delta$ and $\partial\langle S^a S^b\rangle_0/\partial\Delta^{aa} = (g^2/2)[\phi\phi'']_\Delta$, and the eigenvalue Eq. (74) becomes

$$[(1+\partial_a)(1+\partial_b) - g^2[(\phi')^2]_\Delta]\Psi^{ab} - \frac{g^2}{2}[\phi\phi'']_\Delta[\Psi^{aa}+\Psi^{bb}] = \Lambda\Psi^{ab}. \tag{77}$$

Since $\phi(0) = 0$ Eq. (76) admits the trivial solution $\Delta = 0$ for all g. In this case, recalling that by assumption $\phi'(0) = 1$, the eigenvalue Eq. (77) reduces to

$$[(1+\partial_a)(1+\partial_b) - g^2]\Psi^{ab} = \Lambda\Psi^{ab}. \tag{78}$$

Taking the Fourier transform with respect to t_a and t_b, we find

$$\Lambda = (1 - i\omega_a + \delta)(1 - i\omega_b + \delta) - g^2, \tag{79}$$

where $\delta \to 0^+$ to ensure causality. A null eigenvalue can only occur if $\omega_b = -\omega_a$; otherwise, Λ would be complex. Since $\omega_a^2 \geqslant 0$, the equation

$$(1+\delta)^2 + \omega_a^2 - g^2 = 0 \tag{80}$$

does not have solution for $g < 1$. The time-independent solution $\Delta = 0$ is hence stable for $g < 1$; however, it becomes unstable for $g > 1$.

For $g < 1$ only the solution $\Delta = 0$ exists. When $g > 1$ a nontrivial $\Delta > 0$ solution to Eq. (76) exists. The fluctuations around this solution consists of two different branches.

The first are diagonal, *within replica*, fluctuations $\Psi^{ab} = \Psi(t_a,t_b)\,\delta^{\text{Kr}}_{\alpha\beta}$. The eigenvalue Eq. (77) then becomes

$$[(1+\partial_a)(1+\partial_b) - g^2[(\phi')^2]_\Delta]\Psi(t_a,t_b) - \frac{g^2}{2}[\phi\phi'']_\Delta[\Psi(t_a,t_a)+\Psi(t_b,t_b)] = \Lambda\Psi_S(t_a,t_b). \tag{81}$$

The eigenfunctions are of the form $\Psi_S(t_a,t_b) = \Psi(t_a) + \Psi(t_b)$. Taking the Fourier transform with respect to time we find

$$\Lambda = 1 + \delta - i\omega - g^2[[(\phi')^2]_\Delta + [\phi\phi'']_\Delta]. \tag{82}$$

For $\omega = 0$ and $\phi(x) = \tanh(x)$ it is well known that $\Lambda > 0$ for all g. In the theory of spin glasses this is equal to the second eigenvalue of the Hessian of the fluctuations of the replica symmetric solution of the Sherrington-Kirkpatrick

(SK) model; see Ref. [31]. Thus, the static solution is stable against *within* replica fluctuations. Note that this implies that in a one-replica system the time-independent solution is stable for all g.

Equation (77) admits also off-diagonal, *between replica*, fluctuations $\Psi^{\boldsymbol{ab}} = \Psi(t_a, t_b)\,(1-\delta^{\mathrm{Kr}}_{\alpha\beta})$. For these fluctuations the eigenvalue Eq. (77) reduces to

$$[(1+\partial_a)(1+\partial_b)-g^2[(\phi')^2]_\Delta]\Psi(t_a,t_b) = \Lambda\Psi(t_a,t_b). \tag{83}$$

Fourier transforming we find as before

$$\Lambda = (1-i\omega_a+\delta)(1-i\omega_b+\delta) - g^2[(\phi')^2]_\Delta, \tag{84}$$

which for $\omega_b = -\omega_a$ gives

$$\Lambda = (1+\delta)^2 + \omega_a^2 - g^2[(\phi')^2]_\Delta. \tag{85}$$

The quantity $1-g^2[(\phi')^2]_\Delta$ with $\phi(x)=\tanh(x)$ appears also in the mean-field theory of spin glasses. There it is the relevant eigenvalue of the Hessian of the fluctuations of the replica symmetric solution of the SK model, see Ref. [31], and it is known to be negative for $g>1$. Thus, Λ can vanish for some ω_a and the time-independent solution $\Delta^{\alpha\beta} = C^{\alpha\beta} = \Delta$ is unstable for $g>1$.

C. Stability of time-dependent solutions

The stability analysis of the steady-state solutions follows the same path as the time-independent solutions and it shall not be repeated in details.

As for the time-independent case, the relevant self-consistent solution of the DMFT equations is replica independent: $\Delta^{\boldsymbol{ab}} = \Delta^{ab} = \Delta(\tau)$, $\tau = t_a - t_b$, where $\Delta(\tau)$ is solution of Eq. (48).

By denoting with $\Delta = \Delta(\tau)$ and $\Delta_0 = \Delta(\tau=0)$ the derivatives occurring in the eigenvalue Eq. (74) can be written as

$$\frac{\partial}{\partial\Delta^{ab}}\langle S^a S^b\rangle_0 = 1 + \partial_\Delta^2 V(\Delta;\Delta_0), \tag{86}$$

$$\frac{\partial}{\partial\Delta^{aa}}\langle S^a S^b\rangle_0 = \frac{1}{2}\partial_{\Delta_0}\partial_\Delta V(\Delta;\Delta_0), \tag{87}$$

where $V(\Delta;\Delta_0)$ is the potential Eq. (50) function of Δ and Δ_0. Details are in Appendix C. The eigenvalue Eq. (74) then reads

$$\begin{aligned}&\left[\partial_a + \partial_b + \partial_a\partial_b - \partial_\Delta^2 V(\Delta;\Delta_0)\right]\Psi^{\boldsymbol{ab}}\\ &\quad - \frac{1}{2}\partial_{\Delta_0}\partial_\Delta V(\Delta;\Delta_0)[\Psi^{\boldsymbol{aa}} + \Psi^{\boldsymbol{bb}}] = \Lambda\Psi^{\boldsymbol{ab}}.\end{aligned} \tag{88}$$

Since Λ would be complex if $\Psi^{\alpha\beta}(t_a,t_b)$ does not depend on $\tau = t_a - t_b$, we will restrict to fluctuations depending only on τ. Hence, making explicit the time dependence of $\Psi^{\boldsymbol{ab}}$, we have the equation

$$\begin{aligned}&\left[2\delta - \partial_\tau^2 - \partial_\Delta^2 V(\Delta;\Delta_0)\right]\Psi^{\alpha\beta}(\tau)\\ &- \frac{1}{2}\partial_{\Delta_0}\partial_\Delta V(\Delta;\Delta_0)[\Psi^{\alpha\alpha}(0)+\Psi^{\beta\beta}(0)] = \Lambda\Psi^{\alpha\beta}(\tau).\end{aligned} \tag{89}$$

The term δ^2 has been neglected because $\delta\to 0^+$.

Again the critical fluctuations are off-diagonal: $\Psi^{\alpha\beta}(\tau) = \Psi(\tau)\,(1-\delta^{\mathrm{Kr}}_{\alpha\beta})$. The second term in Eq. (89) then vanishes and, defining $\epsilon = \Lambda - 2\delta$, the eigenvalue equation reduces to

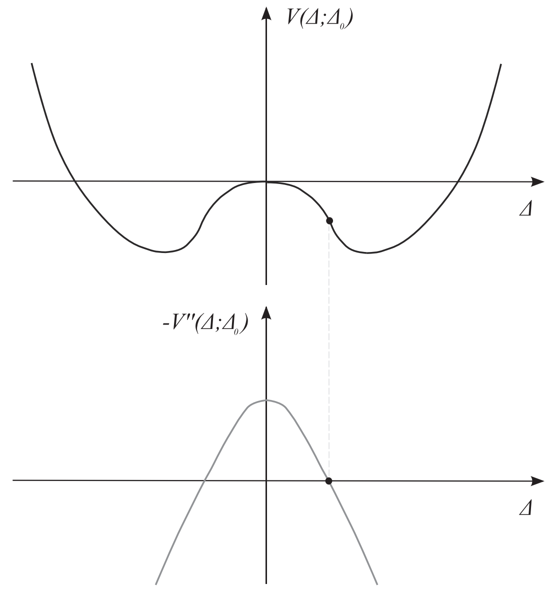

FIG. 3. Qualitative construction of the quantum potential Eq. (91) from the potential Eq. (50).

a one-dimensional time-independent Schrödinger equation in the variable τ:

$$\mathcal{H}_\tau\Psi(\tau) := \left[-\partial_\tau^2 + V_{QM}(\tau)\right]\Psi(\tau) = \varepsilon\Psi(\tau), \tag{90}$$

with the quantum mechanical potential

$$\begin{aligned}V_{\mathrm{QM}}(\tau) &= -\partial_\Delta^2 V(\Delta;\Delta_0)\Big|_{\Delta=\Delta(\tau)}\\ &= 1 - g^2\int Dz\bigg[\int Dx\,\phi'(gx\sqrt{\Delta_0-|\Delta|}\\ &\quad + gz\sqrt{|\Delta|})\bigg]^2\bigg|_{\Delta=\Delta(\tau)},\end{aligned} \tag{91}$$

where $\Delta(\tau)\equiv\Delta(\tau;\Delta_0)$ is the solution to Eq. (48).

Equation (90) always admits the eigenvalue $\varepsilon=0$ with the eigenfunction $\Psi^0(\tau) = \partial_\tau\Delta(\tau)$, as follows by differentiating Eq. (48) with respect to τ. This corresponds to eigenvalue $\Lambda = 2\delta$, which is marginally positive for $\delta\to 0^+$.

The structure of the other eigenvalues depends on the form of the quantum potential which ultimately depends on the solution $\Delta(\tau;\Delta_0)$; see Fig. 3.

1. Time-periodic solutions

For time-periodic solutions the quantum potential $V_{\mathrm{QM}}(\tau)$ is periodic with the qualitative form shown in Fig. 4. The eigenfunction $\Psi^0(\tau)$ is also periodic and changes sign once within one period T, vanishing at $\tau = 0, T$. Thus, there is *exactly* one periodic eigenfunction of $\mathcal{H}_\tau$ with eigenvalue $\varepsilon_0 < 0$ which vanishes only at $\tau = 0, T$. However, since the potential is periodic there are bands of solutions where ϵ_0 is the bottom of the lowest band and $\varepsilon = 0$ the top of the next band. Therefore, the eigenvalue $\Lambda = \varepsilon + 2\delta$ passes continuously through zero whatever small δ is, and hence the time-periodic solutions are unstable.

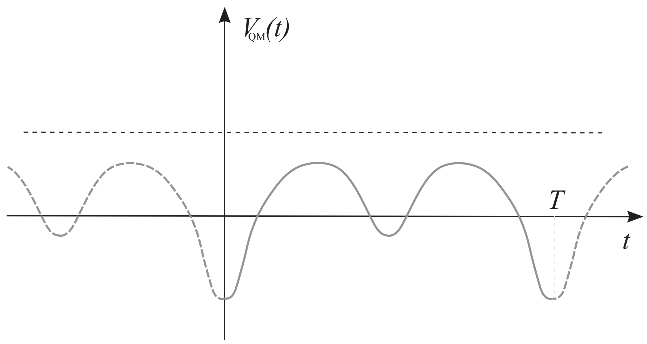

FIG. 4. Qualitative behavior of the quantum potential Eq. (91) for time-periodic solutions.

2. *Time-decaying solution*

If $\Delta(\tau)$ is the time-decaying solution, then $V_{\rm QM}(\tau)$ has the qualitative form shown in Fig. 5. Again the eigenfunction $\Psi^0(\tau)$ has exactly one node and from elementary quantum mechanics we know that there is *exactly* one eigenfunction of $\mathcal{H}_\tau$ with no nodes and eigenvalue $\varepsilon_0 < 0$. However, in this case the eigenvalues of $V_{\rm QM}(\tau)$ are isolated and $\Lambda = \varepsilon_0 + 2\delta$, $\delta \to 0^+$, cannot be zero and the solution is stable.

Summarizing, when $g > 1$ only the time-decaying solution represents a stable attractor of the dynamics.

We conclude this section by noticing that the Lagrangian Eq. (70) can be used also to evaluate correlation functions of fluctuations around the mean field as well as response functions. In the next section we are using it to calculate the Lyapunov exponent of the time-decaying solution and prove that it represents a chaotic state.

V. MAXIMUM LYAPUNOV EXPONENT

The decay of the time-dependent h-correlation function $\Delta(\tau)$ suggests that the underlying neural dynamics is chaotic. A chaotic dynamics exhibits an exponential sensitivity to initial conditions. A measure of the extent to which the dynamics is chaotic is provided by the maximal Lyapunov exponent which measures the sensitivity of the dynamics to small changes in the initial condition. To evaluate this exponent, we consider a small (infinitesimal) change in the state of the system at time t_0 by $\delta h_i(t_0)$, $i = 1, ..., N$. After a time t the perturbation has grown as

$$|\delta h(t)| \sim |\delta h(t_0)|\, e^{\lambda(t-t_0)}, \tag{92}$$

where λ is the maximal Lyapunov exponent. Positive λ implies that the dynamic is chaotic.

As long as the perturbation is small, the perturbed trajectory $h_i(t) + \delta h_i(t)$ is close to the unperturbed trajectory, and the time evolution of $\delta h_i(t)$ is ruled by the differential equation,

$$\partial_t \delta h_i(t) = -\delta h_i(t) + g \sum_{j=1}^{N} J_{ij}\, \phi'[g h_j(t)]\, \delta h_j(t), \tag{93}$$

obtained linearizing the dynamical Eq. (1) about the unperturbed trajectory $h_i(t)$. The solution to this linear equation with initial condition $\delta h_i(t_0)$ can be written as

$$\delta h_i(t) = \sum_{j=1}^{N} \chi_{ij}(t, t_0)\, \delta h_j(t_0), \tag{94}$$

where

$$\chi_{ij}(t, t') = \delta h_i(t)/\delta b_j(t'), \quad t \geqslant t', \tag{95}$$

is the linear response of $h_i(t)$ to an infinitesimal perturbation in the form of a small external field $\delta b_j(t')$ added to the right-hand side of the dynamical Eq. (1) at earlier time t'. From the multiplicative ergodic theorem of Oseledec, the Lyapunov of the stationary dynamics is [32]

$$\lambda = \lim_{t-t_0\to\infty} \frac{1}{2(t-t_0)} \ln\left[\frac{1}{N}\sum_{ij} \chi_{ij}(t, t_0)^2\right], \tag{96}$$

and it gives the dominant exponential growing rate of the linear response as $t - t_0 \gg 1$.

For finite systems the dynamics depends on the couplings J_{ij}. Hence, for finite N the exponent λ is a random quantity. However, in the limit $N \to \infty$ the dynamics converges to a nonrandom behaviour, as described by the DMFT, $\sum_{ij} \chi_{ij}(t, t_0)^2/N$ becomes self-averaging and λ converges to the nonrandom value:

$$\lambda = \lim_{t-t_0\to\infty} \frac{1}{2(t-t_0)} \ln\left[\frac{1}{N}\sum_{ij} \overline{\chi_{ij}(t, t_0)^2}\right]. \tag{97}$$

The maximum Lyapunov exponent can be computed using the DFT developed so far. However, to illustrate the power, and limitations, of the intuitive construction of self-consistent equations for fluctuations for $N \gg 1$, we first present the intuitive calculation of λ. This uses some results discussed in the previous sections supplemented by some reasonable assumptions. The systematic approach using the DFT will be present next.

A. Intuitive calculation of the Lyapunov exponent

The quantity $\sum_{ij} \overline{\chi_{ij}(t, t_0)^2}/N$ appearing in Eq. (97) can be computed from the Green function

$$G(t_a, t_b, t_c, t_d) = \frac{1}{N}\sum_{i,j}^{N} \overline{\chi_{ij}(t_a, t_c)\chi_{ij}(t_b, t_d)}, \tag{98}$$

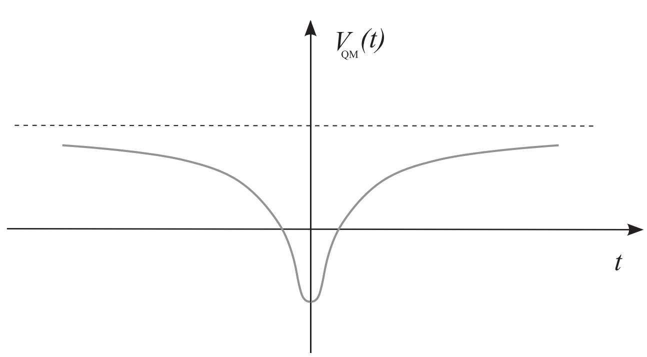

FIG. 5. Qualitative behavior of the quantum potential Eq. (91) for the time-decaying solution.

by taking $t_a = t_b = t$ and $t_c = t_d = t_0$. An equation for $G(t_a, t_b, t_c, t_d)$ can be constructed noticing that from the dynamical Eq. (1) it follows that the linear response $\chi_{ij}(t,t')$ obeys the differential equation

$$\left(1+\frac{d}{dt}\right)\chi_{ij}(t,t') = g\sum_{k=1}^{N} J_{ik}\phi'[gh_k(t)]\,\chi_{kj}(t,t') + \delta(t-t')\,\delta_{ij}^{\rm Kr}, \tag{99}$$

where the Kronecker and Dirac δ functions denote a local spatiotemporal perturbation. Thus, by multiplying Eq. (99) by itself and taking the spatial average, we find

$$\left(1+\frac{\partial}{\partial t_a}\right)\left(1+\frac{\partial}{\partial t_b}\right)G(t_a,t_b,t_c,t_d) - \frac{\partial C(t_a-t_b)}{\partial\Delta(t_a-t_b)}G(t_a,t_b,t_c,t_d) = \delta(t_a-t_b-t_c+t_d)\delta(t_a+t_b-t_c-t_d). \tag{100}$$

To arrive at this equation we have used the fact that under averaging $\phi'(t)\phi'(t')$ can be replaced by $\partial C(t-t')/\partial\Delta(t-t')$ and assumed that the cross term in squaring Eq. (99) vanishes or, equivalently, vanishes in the large N limit due to the summation in Eq. (98).

Defining the new time variables

$$s = t_a + t_b, \qquad s' = t_c + t_d, \tag{101}$$

$$\tau = t_a - t_b, \qquad \tau' = t_c - t_d, \tag{102}$$

Eq. (100) can be written as

$$\left[2\partial_s + \partial_s^2 + \mathcal{H}_\tau\right] G(s,s',\tau,\tau') = 2\,\delta(s-s')\,\delta(\tau-\tau'), \tag{103}$$

where $\mathcal{H}_\tau = -\partial_\tau^2 - \partial_\Delta^2 V(\Delta;\Delta_0)$ is the quantum mechanical Hamiltonian acting on variable τ introduced in Sec. IV C. The solution to this equation can be written as

$$G(s,s',\tau,\tau') = 2\sum_n g_n(s,s')\,\varphi_n(\tau)\,\varphi_n^*(\tau'), \tag{104}$$

where $\varphi_n(\tau)$ is the set of orthonormal eigenfunctions of $\mathcal{H}_\tau$: $\mathcal{H}_\tau\,\varphi_n = \epsilon_n\,\varphi_n$, and the function $g(s,s')$ is solution of the differential equation:

$$\left[2\partial_s + \partial_s^2 + \varepsilon_n\right] g_n(s,s') = \delta(s-s'). \tag{105}$$

The sum in Eq. (104) is over all eigenfunctions of $\mathcal{H}_\tau$ and may include a continuum part of the spectrum of $\mathcal{H}_\tau$. Taking now $t_a = t_b = t$ and $t_c = t_d = t_0$ we finally arrive at

$$\frac{1}{N}\sum_{ij}\overline{\chi_{ij}(t,t_0)^2} = 2\sum_n g_n(2t,2t_0)\,\varphi_n(0)\,\varphi_n^*(0). \tag{106}$$

The maximum Lyapunov exponent is related to the asymptotic behavior of $g_n(t,t')$ as $t-t'\to\infty$. While this equation leads to the correct λ, its derivation is clearly not systematic. It is difficult to have control on the approximations and, moreover, in more complex cases it can be difficult to be constructed. Therefore, before discussing the Lyapunov exponent, we show how equations like Eq. (103) can systematically derived within the DFT.

B. DFT calculation of the Lyapunov exponent

Within the replica formalism introduced in Sec. IV A, it is more convenient to calculate Lyapunov exponent via the spin susceptibility,

$$\tilde{\chi}_{ij}(t,t_0) = \left.\frac{\delta S_i(t)}{\delta b_j(t_0)}\right|_{b=0} = g\phi'(gh_i)\,\chi_{ij}(t,t_0), \tag{107}$$

the linear response of $S_i(t)$ to an infinitesimal external field $\delta b_j(t_0)$ at the earlier time t_0.

The Lyapunov exponent λ is related to the fluctuations $\sum_{ij}\overline{\tilde{\chi}_{ij}(t,t_0)^2}/N$ of the spin susceptibility. Introducing two identical replicas of the system these can be computed in using the DFT as

$$\frac{1}{N}\sum_{ij}\overline{\left(\left.\frac{\delta\langle S_i(t)\rangle_{Jb}}{\delta b_j(t_0)}\right|_{b=0}\right)^2} = \frac{1}{N}\sum_{ij}\overline{\langle S_i(t)i\hat{h}_j(t_0)\rangle_J\,\langle S_i(t)i\hat{h}_j(t_0)\rangle_J} = \frac{1}{N}\sum_{ij}\langle S_i^{\boldsymbol a}S_i^{\boldsymbol b}i\hat{h}_j^{\boldsymbol c}i\hat{h}_j^{\boldsymbol d}\rangle, \tag{108}$$

with replica indexes $\alpha=\gamma\neq\beta=\delta$, α and β being the replica indices of $\boldsymbol a$ and $\boldsymbol b$ while γ and δ those of $\boldsymbol c$ and $\boldsymbol d$, and time arguments $t_a=t_b=t$ and $t_c=t_d=t_0$.

Evaluating the average leads to

$$\frac{1}{N}\sum_{ij}\langle S_i^{\boldsymbol a}S_i^{\boldsymbol b}i\hat{h}_j^{\boldsymbol c}i\hat{h}_j^{\boldsymbol d}\rangle = N\langle C^{\boldsymbol{ab}}i\hat{C}^{\boldsymbol{cd}}\rangle - \frac{1}{2}\,\delta_{\boldsymbol{ab},\boldsymbol{cd}}. \tag{109}$$

Details can be found in Appendix D. This relation is an exact relation valid for any N. In the limit $N\to\infty$ the two-point correlation function $\langle C^{\boldsymbol{ab}}i\hat{C}^{\boldsymbol{cd}}\rangle$ reduces to the two-point correlation function $\langle Q^{\boldsymbol{ab}}i\,\hat{Q}^{\boldsymbol{cd}}\rangle$ of the Gaussian fluctuations about the saddle point. This can be evaluated from the quadratic action $\mathcal{L}_2[\hat{Q},\Psi;\hat{b},b]$ as

$$(1+\partial_a)(1+\partial_b)\langle\Psi^{\boldsymbol{ab}}i\,\hat{Q}^{\boldsymbol{cd}}\rangle = \langle Q^{\boldsymbol{ab}}i\,\hat{Q}^{\boldsymbol{cd}}\rangle, \tag{110}$$

where, from Eq. (70), $\langle\Psi^{\boldsymbol{ab}}i\,\hat{Q}^{\boldsymbol{cd}}\rangle$ satisfies the equation

$$\sum_{\boldsymbol{ef}}\mathcal{A}^{\boldsymbol{ab};\boldsymbol{ef}}\langle\Psi^{\boldsymbol{ef}}i\,\hat{Q}^{\boldsymbol{cd}}\rangle = \frac{1}{2N}\delta_{\boldsymbol{ab},\boldsymbol{cd}}, \tag{111}$$

with the operator $\mathcal{A}$ defined in Eq. (72).

For the particular choice of replica indexes $\gamma=\alpha$ and $\delta=\beta$ but $\alpha\neq\beta$, and changing time variables as in Eqs. (101) and (102), Eq. (110) becomes

$$\left[1+2\partial_s+\partial_s^2-\partial_\tau^2\right]\langle\Psi^{\alpha\beta}(s,\tau)\,i\,\hat{Q}^{\alpha\beta}(s',\tau')\rangle = \langle Q^{\alpha\beta}(s,\tau)\,i\,\hat{Q}^{\alpha\beta}(s',\tau')\rangle. \tag{112}$$

Similarly, working out the explicit form of $\mathcal{A}$ as done in Sec. IV C, Eq. (111) becomes

$$\left[2\partial_s+\partial_s^2+\mathcal{H}_\tau\right]\langle\Psi^{\alpha\beta}(s,\tau)\,i\,\hat{Q}^{\alpha\beta}(s',\tau')\rangle = \frac{1}{N}\delta(s-s')\,\delta(\tau-\tau'), \tag{113}$$

with the quantum mechanical Hamiltonian $\mathcal{H}_\tau$ acting on the (time) variable τ.

The solution to Eq. (113) reads

$$\langle \Psi^{\alpha\beta}(s,\tau)\, i\hat{Q}^{\alpha\beta}(s',\tau')\rangle = \frac{1}{N}\sum_n g_n(s,s')\,\varphi_n(\tau)\,\varphi_n^*(\tau'), \tag{114}$$

where $\varphi_n(\tau)$ are the orthonormal eigenfunctions of $\mathcal{H}_\tau$ and $g_n(s,s')$ is the solution of the differential Eq. (105). However, subtracting Eq. (113) from Eq. (112) leads to

$$\begin{aligned}&\langle Q^{\alpha\beta}(s,\tau)\, i\hat{Q}^{\alpha\beta}(s',\tau')\rangle\\ &\quad= [1 - V_{\rm QM}(\tau)]\langle \Psi^{\alpha\beta}(s,\tau)\, i\hat{Q}^{\alpha\beta}(s',\tau')\rangle\\ &\qquad+\frac{1}{N}\delta(s-s')\,\delta(\tau-\tau').\end{aligned} \tag{115}$$

Inserting these expressions into Eq. (109) gives

$$\begin{aligned}&\frac{1}{N}\sum_{ij}\big\langle S_i^\alpha(t_a) S_i^\beta(t_b) i\hat{h}_j^\alpha(t_c) i\hat{h}_j^\beta(t_d)\big\rangle\\ &\quad= [1 - V_{\rm QM}(\tau)]\sum_n g_n(s,s')\,\varphi_n(\tau)\,\varphi_n^*(\tau'),\end{aligned} \tag{116}$$

and taking $t_a = t_b = t$ and $t_c = t_d = t_0$ finally leads to

$$\frac{1}{N}\sum_{ij}\overline{\tilde{\chi}_{ij}(t,t_0)^2} = [1 - V_{\rm QM}(0)]\sum_n g_n(2t, 2t_0)\,\varphi_n(0)\,\varphi_n^*(0), \tag{117}$$

which is identical to Eq. (106) apart from a constant, related to the transformation Eq. (107) between the two susceptibilities.

The solution to Eq. (105) which vanishes for $s < s'$ is

$$g_n(s,s') = \frac{\theta(s-s')}{\sqrt{1-\varepsilon_n}}\, e^{-(s-s')}\sinh[\sqrt{(1-\varepsilon_n)}(s-s')]. \tag{118}$$

Thus, in the limit $t - t_0 \gg 1$,

$$\frac{1}{N}\sum_{ij}\overline{\tilde{\chi}_{ij}(t,t_0)^2} \sim [1 - V_{\rm QM}(0)]\sum_n \frac{\varphi_n(0)\,\varphi_n^*(0)}{\sqrt{1-\epsilon_n}}\, e^{2\lambda_n(t-t_0)}, \tag{119}$$

with $\lambda_n = -1 + \sqrt{1-\epsilon_n}$, and hence the maximal Lyapunov exponent is

$$\lambda = \max_n \lambda_n = -1 + \sqrt{1-\epsilon_0}, \tag{120}$$

where ϵ_0 is the lowest eigenvalue of $\mathcal{H}_\tau$.

For $g < 1$ we have seen that only the time-independent solution $\Delta = 0$ exists. In this case $V_{\rm QM} = 1 - g^2$, see Eq. (91), therefore $\epsilon_0 = 1 - g^2$ and $\lambda = -1 + g < 0$, showing that $\Delta = 0$ is a stable fix point for $g < 1$. When $g > 1$ the stable solution is the time-dependent decaying solution which leads to a negative ϵ_0. The Lyapunov exponent is then positive and the solution is *chaotic*.

VI. EXPLICIT EXPRESSIONS FOR λ IN TIME-DEPENDENT STATE

To find the expression of λ we first have to solve the DMFT Eq. (48) and then find the lowest eigenvalue of the associated quantum mechanical problem:

$$\mathcal{H}_\tau\Psi(\tau) := \left[-\partial_\tau^2 - \partial_\Delta^2 V(\Delta;\Delta_0)\right]\Psi(\tau) = \varepsilon\Psi(\tau). \tag{121}$$

This is not an easy task for an arbitrary $g > 1$. However, in the limit $g \to 1^+$ and $g \to \infty$ the leading behavior of $\lambda(g)$ can be determined, as shown below.

A. Limit $g \to 1^+$

The energy Eq. (51) of the decaying DMFT is $E_c = 0$. The solution to the DMFT Eq. (48) can then be written in the implicit form as

$$\tau = -\int_{\Delta_0}^{\Delta}\frac{d\Delta}{\sqrt{-2V(\Delta;\Delta_0)}}. \tag{122}$$

In the limit $g \to 1^+$ the equal-time field correlation Δ_0 vanishes, thus $|\Delta| \leqslant \Delta_0 \ll 1$ for all t as $\sigma = g - 1 \ll 1$. Expanding the potential $V(\Delta;\Delta_0)$ in powers of Δ and Δ_0 to the leading nontrivial (fourth) order gives

$$V(\Delta;\Delta_0) \sim \left(-1 + g^2 - 2g^4\Delta_0 + 5g^6\Delta_0^2\right)\frac{\Delta^2}{2} + g^6\frac{\Delta^4}{6}. \tag{123}$$

The value of Δ_0 is found from the condition $V(\Delta_0;\Delta_0) = 0$, and reads $\Delta_0 \sim \sigma - 4\sigma^2/3 + O(\sigma^3)$ as $\sigma \to 0^+$. Thus,

$$V(\Delta;\Delta_0) \sim -\frac{\sigma^2}{6}\Delta^2 + \frac{1}{6}\Delta^4, \quad \sigma \to 0^+. \tag{124}$$

Substituting this expression into Eq. (122) leads to

$$\tau \sim \frac{\sqrt{3}}{\sigma}\int_{\Delta_0/\sigma}^{\Delta/\sigma}\frac{dx}{x\sqrt{1-x^2}}, \quad \sigma \to 0^+, \tag{125}$$

which to the leading term in σ gives

$$\Delta(\tau) = \sigma\cosh^{-1}\left(\frac{\sigma\tau}{\sqrt{3}}\right) + O(\sigma^{3/2}), \quad \sigma \to 0^+. \tag{126}$$

Note that as $\sigma \to 0^+$ the amplitude of $\Delta(\tau)$ vanishes linearly with σ while the characteristic decaying time diverges as σ^{-1}. Thus, as $g \to 1^+$ the dynamics slows down and the chaotic attractor goes continuously to the fix point $\Delta = 0$ at the critical point $g = 1$.

Evaluating the quantum potential $V_{\rm QM}(\tau) = -\partial_\Delta^2 V(\Delta;\Delta_0)|_{\Delta=\Delta(\tau)}$ relative to the solution Eq. (126) the associated quantum mechanical problem becomes

$$\left[\partial_\tau^2 - 2\sigma^2\left[\cosh^{-1}\left(\frac{\sigma\tau}{\sqrt{3}}\right)\right]^2\right]\varphi_n = \left(\epsilon_n - \frac{\sigma^2}{3}\right)\varphi_n. \tag{127}$$

The solution to this differential equation are the generalized Legendre functions with eigenvalues $\epsilon_n = -\sigma^2[(2-n)^2 - 1]/3$; see, e.g., Ref. [33]. Thus, $\epsilon_0 = -\sigma^2$ and

$$\lambda = -1 + \sqrt{1-\epsilon_0} \sim \frac{1}{2}(g-1)^2, \quad g \to 1^+. \tag{128}$$

Notice that near the onset of chaos the rate λ^{-1} of the exponential divergence of close-by trajectories scales as the square of the rates of the decay of memory along the chaotic trajectory.

B. Limit $g \to \infty$

The quantum potential behaves as $V_{\rm QM}(\tau) \sim -g$ for $\tau = O(1/g)$, while it converges to a finite values independent of the value of g as $\tau \to \pm\infty$. Thus, in the limit $g \gg 1$ the potential $V_{\rm QM}(\tau)$ becomes a very deep and narrow potential well close to $\tau = 0$; see Fig. 5. The ground state eigenfunction $\varphi_0(\tau)$ is localized in a region of width $O(1/g) \ll 1$ at $\tau = 0$ and decays exponentially fast outside this region.

In this scenario the leading behavior of the lowest eigenvalue ϵ_0 of $\mathcal{H}_t$ as $g \to \infty$ can be obtained replacing the original quantum mechanical problem by

$$\left[-\partial_\tau^2 - V_0\,\delta(t)\right]\varphi_0(\tau) = \epsilon_0\,\varphi_0(\tau), \tag{129}$$

where

$$-V_0 = \int_{-\Lambda}^{+\Lambda} d\tau\, V_{\rm QM}(\tau) = 2\int_0^{+\Lambda} d\tau\, V_{\rm QM}(\tau), \tag{130}$$

because the quantum potential is an even function of τ. The parameter $\Lambda = O(1)$ is an arbitrary cut-off whose precise value is irrelevant as long as we are interested in the leading behavior as $g \gg 1$. The solution to this equation is $\varphi_0(\tau) \propto \exp(-\sqrt{-\epsilon_0}|\tau|)$, with $\epsilon_0 = -(V_0/2)^2$.

To compute V_0 we introduce a point a/g, where a is an arbitrary positive constant, and split the integration as

$$-\frac{V_0}{2} = \int_0^{a/g} d\tau\, V_{\rm QM}(\tau) + \int_{a/g}^{+\Lambda} d\tau\, V_{\rm QM}(\tau). \tag{131}$$

The first integral is $O(1)$ as $g \gg 1$ because $V_{\rm QM}(\tau) = O(g)$ is this region. Thus, the leading behavior of V_0 as $g \gg 1$ is fully determined by the behavior of $V_{\rm QM}(\tau)$ as $\tau = O(1/g)$.

Expanding $\Delta(\tau)$ about $\tau = 0$ we find to the leading order in τ:

$$\Delta(\tau) = \Delta_0 + (\Delta_0 - 1)\frac{\tau^2}{2} + O(\tau^3), \qquad \tau \ll 1. \tag{132}$$

To obtain this expression we have used the initial condition $\partial_\tau \Delta(\tau)|_{\tau=0} = 0$, the DMFT Eq. (48) to evaluate $\partial_\tau^2 \Delta(\tau)|_{\tau=0}$ and

$$\partial_\Delta V(\Delta;\Delta_0)|_{\Delta=\Delta_0} = -\,\Delta_0 + [\phi^2]_{\Delta_0} \sim -\Delta_0 + 1 + O(1/g), \quad g \gg 1. \tag{133}$$

The value of Δ_0 is again fixed by the requirement $V(\Delta_0;\Delta_0) = 0$, which as $g \to \infty$ gives $\Delta_0 = 2(1 - 2/\pi)$ to the leading order.

Using Eq. (132) leads the following asymptotic expansion of $V_{\rm QM}(\tau)$ valid for $g \gg 1$ and τ to $O(1/g)$:

$$V_{\rm QM}(\tau) \sim -\frac{C}{\tau} + 1 + O(1/g^2), \tag{134}$$

where $C = \frac{2}{\pi}/\sqrt{\Delta_0(1-\Delta_0)}$. Thus, from Eq. (131) it follows

$$-\frac{V_0}{2} \sim C \ln q + O(1), \quad g \gg 1, \tag{135}$$

so that

$$\lambda = -1 + \sqrt{1-\epsilon_0} \sim C\,\ln g, \quad g \gg 1. \tag{136}$$

Notice that while the rate of exponential divergence of close-by trajectories diverges in the large g limit, the decay rate of memory along a trajectory remains finite. Indeed, in the limit $g \to \infty$ the DMFT Eq. (48) becomes

$$\partial_\tau^2 \Delta = \Delta - \frac{2}{\pi}\sin^{-1}\left(\frac{\Delta}{\Delta_0}\right) \underset{\Delta \ll 1}{\sim} \left(1 - \frac{2}{\pi\,\Delta_0}\right)\Delta, \tag{137}$$

so that $\Delta(\tau)$ decay exponentially for $\tau \gg 1$ with a characteristic time $\sqrt{1 - 2/\pi\,\Delta_0}$, cf. Ref. [34].

VII. DISCUSSION

In this paper we have described a systematic approach to the dynamics of randomly connected neural networks based on the Path Integral Formalism originally introduced to study the stochastic dynamics in statistical mechanics. The problem of studying the dynamical behavior of the networks is formulated in terms of a dynamical field theory. For the sake of simplicity, we focused on a class of network models with simple architecture and odd-symmetric sigmoidal nonlinearity, as in model Eqs. (1) and (4). Using the path integral formalism, we have shown how the DMF equations can be derived as a saddle point of the path integrals, which becomes exact in the large N limit. Next, we studied the fluctuations around the saddle point and derived expressions for the multiple response and correlation functions. This fluctuation analysis yielded stability conditions for the stability of the DMF solutions. Finally, using the well-known relations between the maximal Lyapunov exponent of a dynamical system to an appropriate linear response function, we derived equations for the Lyapunov exponent of the random network. Interestingly, in this simple network, the DMF equations for the order parameter bear a mechanical analog of a conservative Newtonian dynamics, whereas the susceptibility associated with the Lyapunov exponent have a quantum mechanical analog in the form of one dimensional (which is time) Schrodinger equation. In the simplest network architectures and dynamics, such as model Eqs. (1)–(4), the DMF equations can be derived by an intuitive construction of the self-consistent equation governing the fluctuations in the system, using Gaussianity ansatz of the fluctuating synaptic fields. Likewise, heuristic assumptions about the statistics of response functions can be used to calculate the maximal Lyapunov exponent, as we have shown here. However, this heuristic approach suffers from considerable limitations. First, it is hard to control the underlying ad-hoc assumptions. Notably, the extension to more complex connectivity or dynamics may be difficult to derive by heuristic methods, as for example the case of connections which are not fully asymmetric, or dynamics involving non-Gaussian stochasticity (e.g., Poisson neurons). An additional difficulty is deriving stability conditions for the DMF solutions. As shown in Ref. [2], even the derivation of stability conditions for fixed points in random networks with more complex architecture may be quite challenging. Finally, in principle, the path integral method can be used to study systematic perturbations analysis to a finite-dimensional systems as well as systematic finite-size corrections. Such applications of the path integral methods have been extensively developed for nonrandom stochastic dynamics in statistical mechanics, as well as in spin glasses. It will be very interesting to explore these directions in deterministic dynamics of random neural networks.

ACKNOWLEDGMENTS

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APPENDIX A: CHAOTIC BEHAVIOR IN THE ISING LIMIT

In the Ising limit $g \to \infty$ the spin (or field) autocorrelation function decays exponentially in time with a finite characteristic time τ_a [34]. Thus, if we discretize the time in step $\delta t \sim \tau_a$ the evolution of the model is described by the N-dimensional map,

$$h_i(n+1) = (1-\delta t)\, h_i(n) + \delta t \sum_{j=1}^{N} J_{ij} S[h_j(n)]. \tag{A1}$$

The maximal Lyapunov exponent λ is obtained from the time-evolution of the tangent vector [32,35,36],

$$\boldsymbol{\xi}(n+1) = \mathcal{A}(n)\boldsymbol{\xi}(n), \tag{A2}$$

$$\mathcal{A}_{ij}(n) = (1-\delta t)\,\delta_{ij} + g\delta t\, J_{ij}\, \phi'[g h_j(n)]. \tag{A3}$$

Since $\delta t \sim \tau_a$, we can assume that $h_i(n)$ and $h_i(n')$ are uncorrelated if $n \neq n'$. Moreover, if $g \gg 1$, then the leading contribution to $\mathcal{A}(n)$ comes from $|h_j| < 1/g$. Thus, we can replace in Eq. (A3) ϕ' by a constant, so that $\boldsymbol{\xi}(n)$ is given by a product of independent $N \times N$ random matrices. In the limit $N \gg 1$ the diagonal part of $\mathcal{A}(n)$ does not contribute and one has [37,38]

$$\lambda \sim \begin{cases} \ln \overline{A_{ij}}, & \text{if } \overline{A_{ij}} \neq 0, \\ \ln \overline{A_{ij}^2}, & \text{if } \overline{A_{ij}} = 0, \end{cases} \tag{A4}$$

where $\overline{(\cdot)}$ means averaging over the different realizations of J_{ij}'s. Therefore,

$$\lambda \sim \ln g, \quad g \gg 1, \tag{A5}$$

and the dynamics is chaotic.

APPENDIX B: $\langle SS \rangle$ CORRELATION FUNCTION

The average $\langle \mathcal{F}[h^a] \rangle_h$ of any functional of h_i^a over the solutions of the dynamical Eq. (5) can be written as a path integral over all trajectories $\{\hat{h}_i, h_i\}_{t\in[t_0,t]}$ weighted with the dynamical action $S[\hat{h}, h]$ (11). Thus,

$$\sum_{i=1}^{N} \big\langle S_i(t_a) S_i(t_b) \big\rangle_J = \int \prod_i \mathcal{D}\hat{h}_i\, \mathcal{D}h_i\, e^{-S[\hat{h},h]} \sum_{i=1}^{N} S_i(t_a) S_i(t_b). \tag{B1}$$

Averaging over the couplings J_{ij}, and introducing the auxiliary fields C^{ab} and $\hat{C}^{ab}$, a straightforward calculation leads to

$$\begin{aligned} \sum_{i=1}^{N} \overline{\langle S_i(t_a) S_i(t_b)\rangle_J} &= \int \mathcal{D}\hat{C}\, \mathcal{D}C\, e^{-N\sum_{(ab)} i\hat{C}^{ab} C^{ab}} \\ &\quad \times \int \prod_i \mathcal{D}\hat{h}_i\, \mathcal{D}h_i\, e^{-S[\hat{h}_i,h_i;\hat{C},C]} \sum_{i=1}^{N} S_i(t_a) S_i(t_b), \end{aligned} \tag{B2}$$

where $S[\hat{h}_i, h_i; \hat{C}, C]$ is defined in Eq. (19). Using the identity

$$e^{-\sum_i S[\hat{h}_i,h_i;\hat{C},C]} \sum_{i=1}^{N} S_i(t_a) S_i(t_b) = \frac{\delta}{\delta i \hat{C}^{ab}}\, e^{-\sum_i S[\hat{h}_i,h_i;\hat{C},C]}, \tag{B3}$$

and Eqs. (17) and (18), the average Eq. (B2) can be written as,

$$\begin{aligned} \sum_{i=1}^{N} \overline{\langle S_i(t_a) S_i(t_b)\rangle_J} &= \int \mathcal{D}\hat{C}\, \mathcal{D}C\, e^{-\frac{N}{2}\sum_{ab} i\hat{C}^{ab} C^{ab}} \frac{\delta}{\delta i \hat{C}^{ab}}\, e^{-NW[\hat{C},C;0,0]}, \\ &= \int \mathcal{D}\hat{C}\, \mathcal{D}C \left[\frac{\delta}{\delta i \hat{C}^{ab}} + N C^{ab}\right] e^{-N\mathcal{L}[\hat{C},C;0,0]}. \end{aligned} \tag{B4}$$

The first terms in the square brackets leads to surface terms and gives no contribution. Thus,

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \overline{\langle S_i(t_a) S_i(t_b)\rangle_J} &= \int \mathcal{D}\hat{C}\, \mathcal{D}C\, e^{-N\mathcal{L}[\hat{C},C;0,0]}\, C^{ab} \\ &= \langle C^{ab} \rangle. \end{aligned} \tag{B5}$$

APPENDIX C: AVERAGES IN THE DMFT

This Appendix shows how the basic relations used in the main text to express averages over the solution of the DMFT are obtained. These are then used to derive the explicit expression of the potential $V(\Delta; \Delta_0)$ and its derivatives.

Given two generic functions $\phi(x)$ and $\psi(x)$, and their Fourier representation

$$\begin{aligned} \phi(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \tilde{\phi}(k)\, e^{-ikx}, \\ \psi(x) &= \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \tilde{\psi}(k)\, e^{-ikx}, \end{aligned} \tag{C1}$$

then

$$\begin{aligned} \langle \phi(h^a)\, \psi(h^b) \rangle_\eta &= \int \frac{dk}{2\pi} \frac{dk'}{2\pi}\, \tilde{\phi}(k)\, \tilde{\psi}(k') \big\langle e^{-ikh^a - ik'h^b} \big\rangle_\eta \\ &= \int \frac{dk}{2\pi} \frac{dk'}{2\pi}\, \tilde{\phi}(k)\, \tilde{\psi}(k')\, \exp\left[-\frac{1}{2}(\Delta^{aa} k^2 + \Delta^{bb} k'^2) - \Delta^{ab} k k'\right] \\ &= \int \frac{dk}{2\pi} \frac{dk'}{2\pi}\, \tilde{\phi}(k)\, \tilde{\psi}(k')\, \exp\left[-\frac{\Delta_0}{2}(k^2 + k'^2) - \Delta k k'\right]. \end{aligned} \tag{C2}$$

In the last line we use $\Delta_0 = \Delta^{aa} = \Delta^{bb}$ and $\Delta = \Delta^{ab}$ for $a \neq b$. The integral is well defined because $|\Delta| \leqslant \Delta_0$. Taking the derivative with respect to Δ^{ab} brings down a factor $-kk'$, thus

$$\frac{\partial}{\partial \Delta^{ab}} \langle \phi(h^a)\, \psi(h^b) \rangle_\eta = \langle \phi'(h^a)\, \psi'(h^b) \rangle_\eta, \tag{C3}$$

while the derivative with respect to Δ^{aa} and Δ^{bb} gives

$$\frac{\partial}{\partial \Delta^{aa}} \langle \phi(h^a)\, \psi(h^b) \rangle_\eta = \langle \phi''(h^a)\, \psi(h^b) \rangle_\eta, \tag{C4}$$

$$\frac{\partial}{\partial \Delta^{bb}} \langle \phi(h^a)\, \psi(h^b) \rangle_\eta = \langle \phi(h^a)\, \psi''(h^b) \rangle_\eta. \tag{C5}$$

The "prime" stands for the derivative of the function with respect its argument, e.g., $\phi'(x) = (d/dx)\phi(x)$.

Using the above relations it follows that

$$\begin{aligned}\frac{\partial V(\Delta;\Delta_0)}{\partial \Delta} &= -\Delta + C(\Delta;\Delta_0) \\ &= -\Delta + \langle \phi(gh^a)\phi(gh^b) \rangle_\eta \\ &= \frac{\partial}{\partial \Delta}\left[-\frac{\Delta^2}{2} + \frac{1}{g^2}\langle \Phi(gh^a)\, \Phi(gh^b) \rangle_\eta\right],\end{aligned} \tag{C6}$$

where $\Phi(x) = \int_0^x dy\, \phi(y)$ is the primitive of the gain function $\phi(x)$. Integrating now over Δ leads to

$$V(\Delta;\Delta_0) = -\frac{\Delta^2}{2} + \frac{1}{g^2}\langle \Phi(gh^a)\, \Phi(gh^b) \rangle_\eta + \text{Constant}, \tag{C7}$$

while taking successive derivatives,

$$\begin{aligned}\frac{\partial^n V(\Delta;\Delta_0)}{\partial \Delta^n} &= -\frac{\partial^n}{\partial \Delta^n}\left(\frac{\Delta^2}{2}\right) \\ &\quad + g^{2n-2}\langle \phi^{(n-1)}(gh^a)\, \phi^{(n-1)}(gh^b) \rangle_\eta,\end{aligned} \tag{C8}$$

where $\phi^{(n)}(x) = (d/dx)^n \phi(x)$.

The expressions in the main text are obtained by substituting

$$\begin{aligned}\tilde{\phi}(k) &= \int_{-\infty}^{+\infty} dx\, \phi(x)\, e^{ikx}, \\ \tilde{\psi}(k) &= \int_{-\infty}^{+\infty} dx\, \psi(x)\, e^{ikx},\end{aligned} \tag{C9}$$

into Eq. (C2) and performing the resulting Gaussian integrals over the wave number:

$$\begin{aligned}\langle \phi(h^a)\, \psi(h^b) \rangle_\eta &= \int \frac{dx\, dy}{2\pi\sqrt{\Delta_0^2 - \Delta^2}}\, \phi(x)\, \psi(y) \\ &\quad \times \exp\left\{-\frac{1}{2(\Delta_0^2 - \Delta^2)}[\Delta_0(x^2 + y^2) \right. \\ &\quad \left. - 2\Delta\, xy]\right\}.\end{aligned} \tag{C10}$$

Introducing an auxiliary Gaussian variable z the average can be further written as the integral over independent Gaussian variables:

$$\begin{aligned}\langle \phi(h^a)\, \psi(h^b) \rangle_\eta &= \int Dz \int Dx\, \phi(\xi) \int Dy\, \psi(\epsilon_\Delta \zeta) \\ &= \int Dz \int Dx\, \phi(\epsilon_\Delta \xi) \int Dy\, \psi(\zeta),\end{aligned} \tag{C11}$$

where $\epsilon_\Delta = \text{sign}(\Delta)$, $\xi = \sqrt{\Delta_0 - |\Delta|}\, x + \sqrt{|\Delta|}\, z$, $\zeta = \sqrt{\Delta_0 - |\Delta|}\, y + \sqrt{|\Delta|}\, z$, and $Dz = dz \exp(-z^2/2)/\sqrt{2\pi}$ is the Gaussian measure. Notice that if the functions $\phi(x)$ and $\psi(x)$ have a definite parity then the average vanishes unless they have the same parity. Taking $\psi(x) = \phi(x)$ we recover Eq. (40) of $C(\Delta, \Delta_0)$ given in the main text.

If the gain function is odd, as the case discussed in the main text, then from the above expressions it easily follows that

$$\left.\frac{\partial^n}{\partial \Delta^n} V(\Delta;\Delta_0)\right|_{\Delta=0} = 0, \quad n = \text{odd}. \tag{C12}$$

This also implies that $\Phi(x)$ is even, and hence the potential $V(\Delta;\Delta_0)$ reads

$$V(\Delta;\Delta_0) = -\frac{\Delta^2}{2} + \frac{1}{g^2}\int Dz\left[\int Dx\, \Phi(g\xi)\right]^2 + \text{Constant}. \tag{C13}$$

APPENDIX D: PROOF OF EQ. (109)

The four-point correlation in Eq. (109) can be evaluated following the same procedure as in Appendix B using the identity

$$e^{-S[\hat{C},C,\hat{h},h]} \sum_{i,j} S_i^{\boldsymbol{a}} S_i^{\boldsymbol{b}}\, i\hat{h}_j^{\boldsymbol{c}}\, i\hat{h}_j^{\boldsymbol{d}} = \frac{\delta}{\delta i\hat{C}^{\boldsymbol{ab}}}\, \frac{\delta}{\delta C^{\boldsymbol{cd}}}\, e^{-S[\hat{C},C,\hat{h},h]}. \tag{D1}$$

Then,

$$\begin{aligned}\frac{1}{N}\sum_{i,j}\langle S_i^{\boldsymbol{a}} S_i^{\boldsymbol{b}}\, i\hat{h}_j^{\boldsymbol{c}}\, i\hat{h}_j^{\boldsymbol{d}} \rangle &= \frac{1}{N}\int \mathcal{D}\hat{C}\, \mathcal{D}C\, e^{-\frac{N}{2}\sum_{\boldsymbol{ab}} i\hat{C}^{\boldsymbol{ab}} C^{\boldsymbol{ab}}} \\ &\quad \times \frac{\delta}{\delta i\hat{C}^{\boldsymbol{ab}}}\, \frac{\delta}{\delta C^{\boldsymbol{cd}}}\, e^{-NW[\hat{C},C,0,0]}.\end{aligned} \tag{D2}$$

Integrating by parts, since the surface terms do not contribute,

$$\begin{aligned}&\frac{1}{N}\sum_{i,j}\langle S_i^{\boldsymbol{a}} S_i^{\boldsymbol{b}}\, i\hat{h}_j^{\boldsymbol{c}}\, i\hat{h}_j^{\boldsymbol{d}} \rangle \\ &= \int \mathcal{D}\hat{C}\, \mathcal{D}C\, C^{\boldsymbol{ab}}\, e^{-\frac{N}{2}\sum_{\boldsymbol{ab}} i\hat{C}^{\boldsymbol{ab}} C^{\boldsymbol{ab}}}\, \frac{\delta}{\delta C^{\boldsymbol{cd}}}\, e^{-NW(\hat{C},C,0,0)} \\ &= \int \mathcal{D}\hat{C}\, \mathcal{D}C\left[N C^{\boldsymbol{ab}}\, i\hat{C}^{\boldsymbol{cd}} - \frac{1}{2}\delta_{\boldsymbol{ab},\boldsymbol{cd}},\right] e^{-N\mathcal{L}[\hat{C},C,0,0]} \\ &= N\langle C^{\boldsymbol{ab}}\, i\hat{C}^{\boldsymbol{cd}} \rangle - \frac{1}{2}\delta_{\boldsymbol{ab},\boldsymbol{cd}},\end{aligned} \tag{D3}$$

where $\delta_{\boldsymbol{ab},\boldsymbol{cd}} = \delta_{\boldsymbol{ac}}\delta_{\boldsymbol{bd}} + \delta_{\boldsymbol{ad}}\delta_{\boldsymbol{bc}}$ is the symmetrized δ and $\delta_{\boldsymbol{ab}} = \delta_{\alpha\beta}^{\text{Kr}}\, \delta(t_a - t_b)$.

Alternatively, one may notice that

$$\int \mathcal{D}\hat{C}\, \mathcal{D}C\, \frac{\delta}{\delta i\hat{C}^{\boldsymbol{ab}}}\, \frac{\delta}{\delta C^{\boldsymbol{cd}}}\, e^{-N\mathcal{L}[\hat{C},C;0,0]} = 0, \tag{D4}$$

because this is a surface term and vanishes. However, from the form Eq. (58) of $\mathcal{L}[\hat{C}, C; 0, 0]$ we obtain

$$\int \mathcal{D}\hat{C}\,\mathcal{D}C \frac{\delta}{\delta i\hat{C}^{ab}} \frac{\delta}{\delta C^{cd}} e^{-N\mathcal{L}[\hat{C},C;0,0]} = \int \mathcal{D}\hat{C}\,\mathcal{D}C \frac{\delta}{\delta i\hat{C}^{ab}} \left[\sum_j i\hat{h}_j^c i\hat{h}_j^d - Ni\hat{C}^{ab}\right] e^{-N\mathcal{L}[\hat{C},C;0,0]}$$

$$= \int \mathcal{D}\hat{C}\,\mathcal{D}C \left[\sum_j i\hat{h}_j^c i\hat{h}_j^d - Ni\hat{C}^{cd}\right]\left[\sum_j S_j^a S_j^b - NC^{ab}\right] e^{-N\mathcal{L}[\hat{C},C;0,0]} - \frac{N}{2}\delta_{ab,cd}.$$

Thus, since cross terms vanishes,

$$\frac{N}{2}\delta_{ab,cd} = \sum_{ij} \langle S_j^a S_j^b i\hat{h}_j^c i\hat{h}_j^d\rangle - N^2\langle C^{ab}\, i\hat{C}^{cd}\rangle, \tag{D5}$$

i.e.,

$$\frac{1}{N}\sum_{i,j} \langle S_i^a S_i^b i\hat{h}_j^c i\hat{h}_j^d\rangle = N\langle C^{ab}\, i\hat{C}^{cd}\rangle - \frac{1}{2}\delta_{ab,cd}. \tag{D6}$$

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