Time evolution of correlation functions in Non-equilibrium Field Theories

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We investigate the non-equilibrium properties of an N-component scalar field theory. The time evolution of the correlation functions for an arbitrary ensemble of initial conditions is described by an exact functional differential equation. In leading order in the $1/N$ expansion the system can be understood in terms of infinitely many conserved quantities. They forbid the approach to the canonical thermal distribution. Beyond leading order only energy conservation is apparent generically. Nevertheless, we find a large manifold of stationary distributions both for classical and quantum fields. They are the fixed points of the evolution equation. For small deviations of the correlation functions from a large range of fixed points we observe stable oscillations. These results raise the question of if and in what sense the particular fixed point corresponding to thermal equilibrium dominates the large time behavior of the system.

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The understanding of the time evolution for a quantum field theory with initial conditions away from thermal equilibrium is needed for a large variety of phenomena, ranging from inflationary cosmology in the early universe to laboratory experiments of defect formation. First insights have been gained by perturbative methods and, in particular, by an investigation of the leading order in the $1/N$ expansion, with N the number of components of the field [1–3]. This approximation has been applied to a variety of problems like decoherence of quantum states [2], heating after cosmological inflation [3], disoriented chiral condensates in heavy ion collisions [4], decoherence and photon pair creation from a homogeneous electric field [5]. In most of this work a particle number basis was used in order to describe the evolution of each field mode. From the mode evolution the two point function can in turn be constructed.

It is a severe limitation, however, that the leading $1/N$ approximation neglects scattering and cannot describe the approach of a system to a thermal equilibrium state. A systematic extension of the particle number formalism to finite N seems a difficult task. We pursue here an alternative strategy and investigate directly the time evolution of the correlation functions of the system. One starts, at $t = 0$, with an ensemble of initial values for the field variables and their time derivatives. This specifies the initial correlation functions. At any later time t the correlation functions are, in principle, computable from the microscopic field equations. In practice, we follow this time evolution by means of an exact functional differential evolution equation for the generating functional of the correlation functions [6]. Approximate solutions can be obtained by truncation.

Our main tool is the time dependent effective action $\Gamma[\phi, \pi; t]$ [6]. For field theories Γ is a functional of the fields $\phi(x)$ and $\pi(x) = \partial_t \phi(x)$ and generates the one particle irreducible equal time correlation functions. Its time evolution obeys

$$\partial_t \Gamma[\phi, \pi] = -\left(\mathcal{L}_{\rm cl} + \mathcal{L}_{\rm q}\right) \Gamma[\phi, \pi], \tag{1}$$

where the first operator, $\mathcal{L}_{\rm cl}$, generates the classical evolution, whereas $\mathcal{L}_{\rm q}$ determines the dynamics of the quantum effects. We will deal with the specific example of a $O(N)$ symmetric field theory for real scalars $\chi_a(x)$, $a = 1, ..., N$, obeying the microscopic equations of motion of a ϕ^4-model

$$\partial_t^2 \chi_a = \partial_i \partial_i \chi_a - m^2 \chi_a - \frac{\lambda}{2} \chi_b \chi_b \chi_a. \tag{2}$$

The arguments of Γ are the mean values of the field $\phi_a = \langle \chi_a \rangle$ and its time derivative $\pi_a = \langle \dot{\chi}_a \rangle$ (in the presence of sources). From [6,7], one infers

$$\begin{aligned}
\mathcal{L}_{\rm cl} = & \int d^D x \left\{ \pi_a(x) \frac{\delta}{\delta \phi_a(x)} + \phi_a(x) \left(\nabla^2 - m^2\right) \frac{\delta}{\delta \pi_a(x)} \right. \\
& - \frac{\lambda}{2} \left[\phi_b(x) \phi_b(x) \phi_a(x) + \phi_a(x) G^{\phi\phi}_{bb}(x,x) \right. \\
& + 2 \phi_b(x) G^{\phi\phi}_{ba}(x,x) \\
& - \int d^D x_1 d^D x_2 d^D x_3 G^{\phi\psi}_{ai}(x,x_1) G^{\phi\psi}_{bj}(x,x_2) \\
& \left. \left. \times\ G^{\phi\psi}_{bk}(x,x_3) \frac{\delta^3 \Gamma}{\delta \psi_i(x_1) \delta \psi_j(x_2) \delta \psi_k(x_3)} \right] \frac{\delta}{\delta \pi_a(x)} \right\},
\end{aligned} \tag{3}$$

$$\mathcal{L}_{\rm q} = \frac{\lambda}{8} \hbar^2 \int d^D x\ \phi_a(x) \frac{\delta \Gamma}{\delta \pi_b(x)} \frac{\delta \Gamma}{\delta \pi_b(x)} \frac{\delta}{\delta \pi_a(x)}, \tag{4}$$

with $\psi \in \{\phi, \pi\}$ and repeated index summations implied. The exact field dependent propagator G, interpreted as a matrix with multi-indices (ψ, i, x), is the inverse of the second functional derivative of Γ,

$$\left(G^{-1}\right)^{\psi\psi'}_{ij}(x,y) = \frac{\delta^2 \Gamma}{\delta \psi_i(x) \delta \psi'_j(y)}. \tag{5}$$

We emphasize that the minimum of $\Gamma[\phi,\pi;t]$ describes the time evolution of a cosmological scalar field like the inflaton. The true dynamical equation (1) for non-equilibrium situations differs, in general, from the variation of some equilibrium effective action or the classical action. The latter are at best an approximation to Eq. (1) in certain limiting situations not too far from equilibrium. An understanding of the structure of the dynamical equations following from (1) is of crucial importance for an understanding of cosmological models like inflation.

We assume an $O(N)$-symmetric initial distribution for which all terms with odd powers of the fields in Γ vanish. This property remains conserved by the evolution. Similarly, we consider a translational and rotational invariant situation. The most general form of Γ consistent with these symmetries can be expanded in terms of the Fourier components of the fields

$$
\begin{aligned}
\Gamma = \frac{1}{2}\int \frac{d^Dq}{(2\pi)^D}&\Big\{A(q)\phi_a^*(q)\phi_a(q) + B(q)\pi_a^*(q)\pi_a(q)\\
&+2C(q)\pi_a^*(q)\phi_a(q)\Big\} + \frac{1}{8}\int \frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{d^Dq_3}{(2\pi)^D}\\
&\Big\{u(q_1,q_2,q_3)\phi_a(q_1)\phi_a(q_2)\phi_b(q_3)\phi_b(-q_1-q_2-q_3)\\
&+v(q_1,q_2,q_3)\pi_a(q_1)\phi_a(q_2)\phi_b(q_3)\phi_b(-q_1-q_2-q_3)\\
&+w(q_1,q_2,q_3)\pi_a(q_1)\pi_a(q_2)\phi_b(q_3)\phi_b(-q_1-q_2-q_3)\\
&+s(q_1,q_2,q_3)\left[\pi_a(q_1)\pi_b(q_2)\phi_a(q_3)\phi_b(-q_1-q_2-q_3)\right.\\
&\qquad\left.-\pi_a(q_1)\pi_a(q_2)\phi_b(q_3)\phi_b(-q_1-q_2-q_3)\right] \qquad (6)\\
&+y(q_1,q_2,q_3)\pi_a(q_1)\pi_a(q_2)\pi_b(q_3)\phi_b(-q_1-q_2-q_3)\\
&+z(q_1,q_2,q_3)\pi_a(q_1)\pi_a(q_2)\pi_b(q_3)\pi_b(-q_1-q_2-q_3)\Big\}\\
&+\ldots
\end{aligned}
$$

Here A, B and C are functions of q^2 and the quartic couplings u, v, *etc.*, depend on the invariants that can be constructed from the three independent momenta q_1, q_2, q_3.

The evolution equation can also be expanded in powers of the fields and the exact equation for the inverse propagator reads

$$
\begin{aligned}
&\partial_t A(q) = 2\omega_q^2 C(q),\\
&\partial_t B(q) = -2C(q) - 2\gamma_q B(q),\\
&\partial_t C(q) = \omega_q^2 B(q) - A(q) - \gamma_q C(q).
\end{aligned}
\tag{7}
$$

Here, the time dependent frequency, ω_q, and the coefficient γ_q are:

$$
\begin{aligned}
\omega_q^2 &= q^2 + m^2 + \frac{N+2}{2}\lambda\int\frac{d^Dp}{(2\pi)^D}G(p) \qquad (8)\\
&-\frac{N+2}{8}\lambda\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dp'}{(2\pi)^D}G(p)G(p')G(p+p'+q)\\
&\Big[4u(p,p',-p-p'-q) - 3v(p,p',-p-p'-q)c(p)\\
&+2w(p,p',-p-p'-q)c(p)c(p')\\
&-y(p,p',-p-p'-q)c(p)c(p')c(p+p'+q)\Big],\\
\gamma_q &= \frac{N+2}{8}\lambda\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dp'}{(2\pi)^D}G(p)G(p')G(p+p'+q)\\
&\Big[v(q,p,p') - 2w(q,p,p')c(p) + 3y(q,p,p')c(p)c(p')\\
&-4z(q,p,p')c(p)c(p')c(p+p'+q)\Big], \qquad (9)
\end{aligned}
$$

with

$$
c(q) = \frac{C(q)}{B(q)} \tag{10}
$$

and $G(q)$ the exact $\phi-\phi$ propagator for vanishing fields

$$
G(q) = \frac{B(q)}{A(q)B(q) - C^2(q)} \equiv \frac{B(q)}{\alpha^2(q)}. \tag{11}
$$

We suppose that all momentum integrals are ultraviolet regularized by a cutoff Λ, ($q^2 < \Lambda^2$). For the simple system of N coupled anharmonic oscillators ($D = 0$) the momentum labels and integrals should be omitted. From (7) one obtains the flow equations for $G(q)$, $c(q)$ and $\alpha^2(q)$,

$$
\begin{aligned}
&\partial_t\alpha^2(q) = -2\gamma_q\alpha^2(q),\\
&\partial_t G(q) = -2c(q)G(q),\\
&\partial_t c(q) = \omega_q^2 - \frac{1}{\alpha^2(q)G^2(q)} + \gamma_q c(q) + c^2(q).
\end{aligned}
\tag{12}
$$

For large N, with $\lambda, u, v, w, s, y, z$ scaling like $1/N$, one may expand in $1/N$ as a small parameter. We observe that $\gamma_q \sim 1/N$ and ω_q becomes independent of $u, v, \ldots$, in leading order. In the limit $N \to \infty$ the system of evolution equations for the inverse two-point functions A, B and C is closed.

For a solution of Eq. (7) at finite N, however, one needs the time dependent four point functions. Their evolution equations involve, in turn the six point functions. The exact system is not closed and for any practical purpose we have to proceed to some approximation. In the present note we simply omit all contributions from 1 PI six point vertices. Furthermore for $D \geq 1$, we neglect the momentum dependence of the quartic couplings. With this truncation the evolution equations for the quartic couplings are given by

$$
\begin{aligned}
&\partial_t u = \bar\omega^2 v - \lambda\hbar^2\bar B^3\bar c^3\\
&+4\lambda\bar B\bar c\left[1 - u\mathcal{S}_0 + \frac{v}{2}\mathcal{S}_1 - \frac{(N+2)w-(N-1)s}{2(N+8)}\mathcal{S}_2\right],\\
&\partial_t v = 2\bar\omega^2 w - 4u - \bar\gamma v - 3\lambda\hbar^2\bar B^3\bar c^2\\
&+4\lambda\bar B\left[1 - u\mathcal{S}_0 + \frac{v}{2}\mathcal{S}_1 - \frac{(N+2)w-(N-1)s}{2(N+8)}\mathcal{S}_2\right]\\
&\qquad -2\lambda\bar B\bar c\left[v\mathcal{S}_0 - \frac{12w+2(N-1)s}{N+8}\mathcal{S}_1 + y\mathcal{S}_2\right], \qquad (13)\\
&\partial_t w = 3\bar\omega^2 y - 3v - 2\bar\gamma w - 3\lambda\hbar^2\bar B^3\bar c
\end{aligned}
$$

$$-2\lambda\bar{B}\left[v\mathcal{S}_0 - \frac{12w + 2(N-1)s}{N+8}\mathcal{S}_1 + y\mathcal{S}_2\right]$$
$$-2\lambda\bar{B}\bar{c}\left[\frac{(N+2)w-(N-1)s}{N+8}\mathcal{S}_0 - y\mathcal{S}_1 + 2z\mathcal{S}_2\right],$$
$$\partial_t s = 2\bar{\omega}^2 y - 2v - 2\bar{\gamma}s - 2\lambda\hbar^2\bar{B}^3\bar{c}$$
$$-\lambda\frac{2\bar{B}}{N+8}\Big[(N+6)v\mathcal{S}_0 - 2(4w+Ns)\mathcal{S}_1 + (N+6)y\mathcal{S}_2\Big]$$
$$-\frac{4\lambda\bar{B}\bar{c}}{N+8}\Big[s\mathcal{S}_0 - 2y\mathcal{S}_1 + 4z\mathcal{S}_2\Big],$$
$$\partial_t y = 4\bar{\omega}^2 z - 2w - 3\bar{\gamma}y - \lambda\hbar^2\bar{B}^3$$
$$-2\lambda\bar{B}\left[\frac{(N+2)w-(N-1)s}{N+8}\mathcal{S}_0 - y\mathcal{S}_1 + 2z\mathcal{S}_2\right],$$
$$\partial_t z = -y - 4\bar{\gamma}z,$$

with

$$\mathcal{S}_{\mathrm{k}} = \frac{N+8}{2}\int\frac{d^Dp}{(2\pi)^D}G^2(p)c^{\mathrm{k}}(p). \tag{14}$$

For $D \geq 1$ there remains at this stage an ambiguity at which momentum the couplings u, v, etc. and therefore the barred quantities $\bar{\omega}^2, \bar{\gamma}, \bar{c}, \bar{B}$, should be evaluated.(For $D = 0$ the bars should be omitted.) One may choose some appropriate momentum average to be discussed below. From the truncated evolution equation of the quartic couplings we have omitted some terms $\sim 1/N^2$ which involve two-loop integrals. The neglected six point functions can also be treated consistently as being $\sim 1/N^2$. On the other hand, the difference $u(q,p,p') - u(0,0,0)$ appears in the contributions $\sim 1/N$. Nevertheless, the effects of this momentum dependence are partly averaged out by the momentum integrations. Thus our set of evolution equations (13) contains for $D = 0$ all contributions in the next to leading order $1/N$. This extends to arbitrary D only if the momentum dependence of the quartic couplings can be neglected.

The microscopic field equations (2) conserve the total energy E for every choice of initial state

$$E = \int d^Dx\left[\frac{1}{2}\partial_t\chi_a\partial_t\chi_a + \frac{1}{2}\partial_i\chi_a\partial_i\chi_a + \frac{1}{2}m^2\chi_a\chi_a + \frac{1}{8}\lambda\chi_a\chi_a\chi_b\chi_b\right]. \tag{15}$$

Therefore the average energy per volume

$$\epsilon = \frac{\langle E\rangle}{\Omega} = \epsilon_1 + \epsilon_2,$$
$$\epsilon_1 = \frac{N}{2}\int\frac{d^Dq}{(2\pi)^D}\Bigg\{B^{-1}(q) + \Big[q^2 + m^2 + c^2(q) + \frac{N+2}{4}\lambda\int\frac{d^Dp}{(2\pi)^D}G(p)\Big]G(q)\Bigg\}, \tag{16}$$
$$\epsilon_2 = -\frac{N(N+2)}{8}\lambda\int\frac{d^Dq_1}{(2\pi)^D}\frac{d^Dq_2}{(2\pi)^D}\frac{d^Dq_3}{(2\pi)^D}G(q_1)G(q_2)$$
$$\times G(q_3)G(-q_1-q_2-q_3)\Big[u(q_1,q_2,q_3) - v(q_1,q_2,q_3)c(q_1)$$
$$+w(q_1,q_2,q_3)c(q_1)c(q_2) - y(q_1,q_2,q_3)c(q_1)c(q_2)c(q_3)$$
$$+z(q_1,q_2,q_3)c(q_1)c(q_2)c(q_3)c(-q_1-q_2-q_3)\Big],$$

must be conserved by the exact evolution equations. We have verified that the truncated evolution equations (7), (13) conserve ϵ for $D = 0$ and arbitrary N. For $D \geq 1$ this holds automatically in leading order in the $1/N$ expansion. For finite N our truncation conserves energy only provided the averages $\bar{\omega}^2$, $\bar{B}$, $\bar{c}$, $\bar{\gamma}$, are chosen appropriately such that they obey the relation

$$\partial_t\epsilon = -\frac{N(N+2)}{8}\lambda\int\frac{d^Dq}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\frac{d^Dp'}{(2\pi)^D}G(q)G(p)$$
$$\times G(p')G(p+p'+q)\Bigg\{\Big[\left(\bar{\omega}^2 - \omega_q^2\right) + \left(\bar{\gamma} - \gamma_q\right)c(q)\Big]$$
$$[v - 2wc(p) + 3yc(p)c(p') - 4zc(p)c(p')c(p+p'+q)]$$
$$+4\lambda\bar{B}\left(\bar{c} - c(q)\right)\Big[1 - u\mathcal{S}_0 + \frac{v}{2}\left(\mathcal{S}_1 + c(p)\mathcal{S}_2\right)$$
$$-\frac{(N+2)\,w - (N-1)\,s}{2\,(N+8)}\left[\mathcal{S}_2 + c(p)c(p')\mathcal{S}_0\right]$$
$$-\frac{6w + (N-1)\,s}{(N+8)}c(p)\mathcal{S}_1 + \frac{y}{2}\left[c(p)\mathcal{S}_2 + c(p)c(p')\mathcal{S}_1\right]$$
$$-z\mathcal{S}_2 - \frac{1}{4}\hbar^2\bar{B}^2\left(\bar{c} - c(p)\right)\left(\bar{c} - c(p')\right)\Big]\Bigg\} = 0. \tag{17}$$

For practical purposes it may be sufficient that this relation holds in a time averaged sense. Whereas ϵ_1 is separately conserved in the leading $1/N$ approximation the flow of energy between ϵ_1 and ϵ_2 can be attributed to scattering.

The fixed points of the system Eqs. (7), (13) correspond to stationary probability distributions. They obey, for $\omega_q^2 > 0$,

$$\begin{aligned}
&A_*(q) = \omega_q^2 B_*(q), \qquad C_*(q) = 0,\\
&v_* = 0, \qquad y_* = 0,\\
&u_* = \frac{\lambda\bar{B}_* + \left(\bar{\omega}^2/2\right)w_*}{1 + \lambda\bar{B}_*\mathcal{S}_0},\\
&w_* = \frac{2\bar{\omega}^2 z_* + \frac{N-1}{N+8}\lambda\bar{B}_*\mathcal{S}_0 s_* - \frac{\lambda}{2}\hbar^2\bar{B}_*^3}{1 + \frac{N+2}{N+8}\lambda\bar{B}_*\mathcal{S}_0}.
\end{aligned} \tag{18}$$

We observe a large manifold of fixed points since Eqs. (18) have solutions for arbitrary $B_*(q)$, s_* and z_* ! This property seems not to be an artifact of the truncation. It persists for momentum dependent four point functions. The fixed point manifold becomes even larger if one includes the six point functions and seems to characterize any higher (finite) polynomial truncation. Classical thermodynamic equilibrium, at temperature T (for $\hbar = 0$) corresponds to the particular point in this manifold:

$$
\begin{aligned}
&B_{\rm eq}(q) = \beta = 1/T, \quad C_{\rm eq}(q) = 0, \\
&A_{\rm eq}(q) = \beta(\omega_q^2)_{\rm eq} = G_{\rm eq}^{-1}(q), \\
&v_{\rm eq} = w_{\rm eq} = s_{\rm eq} = y_{\rm eq} = z_{\rm eq} = 0, \\
&u_{\rm eq} = \lambda\beta\left(1 - \mathcal{S}_0 u_{\rm eq}\right).
\end{aligned} \tag{19}
$$

Here the fixed point conditions for $A_{\rm eq}$ and $u_{\rm eq}$ constitute classical Schwinger-Dyson equations (corresponding to a Euclidean D-dimensional quantum field theory at zero temperature) for the effective propagator (mass and kinetic terms) and the effective quartic coupling, respectively. In quantum field theory ($\hbar = 1$) the thermal equilibrium fixed point does not correspond to $w_{\rm eq} = s_{\rm eq} = z_{\rm eq} = 0$, anymore. The Schwinger-Dyson equations have now to be evaluated in a $D+1$ dimensional quantum field theory with periodic boundary conditions in 'Euclidean time'. The difference between the quantum and classical fixed points is due to the nonzero Matsubara frequencies. We observe, nevertheless, a rather similar structure of the classical and quantum dynamical systems. In particular, we find in the quantum system a fixed point, which differs from the classical equilibrium fixed point only by $z_* = \lambda\hbar^2\bar{B}_*^3/(4\bar{\omega}^2)$, instead of $z_{\rm eq} = 0$. For this fixed point, all correlation functions involving only ϕ are the same as for classical equilibrium. Inversely, for every fixed point of the quantum system there is a corresponding one for the classical system, differing only by a shift in z_*. Up to this shift, the quantum equilibrium could be reached by the classical system !

The central issue of this work is the question of whether and how the fixed points are approached for $t \to \infty$. We start with the leading behavior for $N \to \infty$, where

$$
\gamma_q = 0, \quad \omega_q^2 = q^2 + m^2 + \Delta m^2, \tag{20}
$$

with

$$
\Delta m^2 = \frac{\lambda N}{2}\int \frac{d^D p}{(2\pi)^D} G(p) \tag{21}
$$

and

$$
\partial_t \omega_q^2 = \partial_t \Delta m^2 = -\lambda N \int \frac{d^D p}{(2\pi)^D}\,\frac{C(p)}{\alpha^2(p)} \equiv \beta_\omega. \tag{22}
$$

In this limit the Fourier components of the determinant $\alpha^2(q)$ become an infinite set of conserved quantities ! (cf. Eqs. (18), (22).) Obviously, the system can reach thermal equilibrium only for a very particular set of initial conditions, namely if $\alpha^2(q) = G_{\rm eq}^{-1}(q)$, for every q.

Nevertheless, for large times and $D \geq 1$, Δm^2 (or ω_q^2) tends to a constant value as can be seen in Fig. 1, for $D = 3$. This phenomenon, called 'dephasing' in [2], is due to the superposition of rapidly oscillatory functions $C(q)$ on the right hand side of Eq. (22). In fact, for $C(q) = \psi_R(q)\cos(2\omega_q t) + \psi_I(q)\sin(2\omega_q t)$, $\alpha^2(q) = \omega_q^2/\zeta^2(q)$ and using $y = \omega_q t = t\sqrt{q^2 + m_R^2}$ as integration variable ($m_R^2 = m^2 + \Delta m^2$) one has, for $D = 3$,

$$
\beta_\omega(t) = -\frac{\lambda N}{2\pi^2 t}\int_{m_R t}^{\infty} dy\ \zeta^2(y)\ \sqrt{1 - \frac{m_R^2 t^2}{y^2}} \times \left(\psi_R(y)\cos(2y) + \psi_I(y)\sin(2y)\right). \tag{23}
$$

For large $m_R t$ the amplitudes ψ_R and ψ_I vary only slowly with y since the leading oscillations have been factorized out. This also should hold for $\zeta^2(y)$. Typically these quantities are smooth functions of $y/m_R t$ such that their y-derivatives are suppressed by $1/m_R t$. (For $q^2 \to \infty$ we will assume that $\psi_R(q), \psi_I(q)$ and $\zeta^2(q)$ remain finite.) The oscillatory behavior of the integrand prevents the y-integral from increasing like $m_R t$ and β_ω therefore vanishes for $t \to \infty$. We emphasize that in general the asymptotic constant value Δm^2 depends on the initial conditions and is *not equal* to the temperature dependent mass renormalization in thermal equilibrium.

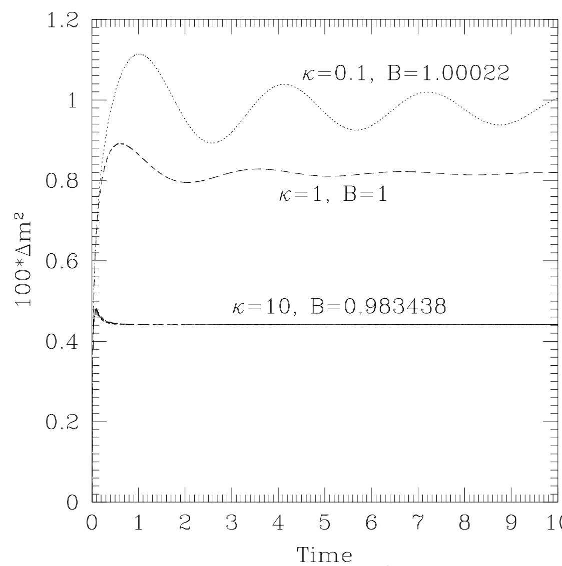

FIG. 1. The time evolution of Δm^2 for initial conditions with $G(q) = 1/(q^2 + \kappa^2 m^2)$ and $\alpha^2(q) = B(q)/G(q)$, $m^2 = 1$ and $\lambda N = 2$. $B(q) = B$ is chosen so that the energy ϵ_1 is the same in all three cases. The ultraviolet cutoff was chosen to be $\Lambda = 100$ and the total number of modes is 2000. For long times Δm^2 tends to different constant values for initial conditions with the same ϵ_1 but different $G(q)$ and $\alpha^2(q)$.

For a constant Δm^2 the dynamical system simplifies considerably and we obtain a free oscillator for each momentum q, with frequency ω_q. The amplitude of the oscillations determines in turn the distance to the fixed point $C_* = 0$ and $A_* - (\omega_q^2)_* B_* = 0$. Thus for generic initial conditions the $N \to \infty$ system does not reach asymptotically the fixed point manifold Eqs. (18). If there is a 'dynamical fixed point' for Δm^2 the correlation functions of the system end up oscillating. In addition to $\alpha(q)$ the system with constant Δm^2 has another infinite set of conserved quantities, namely $A(q) + \omega_q^2 B(q)$. The late time behavior of the system is clearly not ergodic. Ergodicity is usually thought to be a sufficient, albeit not necessary, condition for subsystems of a large closed system to approach the canonical distribution [8].

The generic oscillatory behavior is also apparent from the Hamiltonian structure which governs the dynamical evolution of the two point functions of the $N \to \infty$ system. In terms of canonical coordinates $Q(q)$ and $P(q)$,

$$A(q) = \alpha^2(q)P^2(q) + 1/Q^2(q), \quad B(q) = \alpha^2(q)Q^2(q),$$
$$C(q) = -\alpha^2(q)P(q)Q(q), \qquad G(q) = Q^2(q) \tag{24}$$

the equations of motion become

$$\partial_t Q(q) = P(q),$$
$$\partial_t P(q) = -\omega_q^2 Q(q) + \frac{1}{\alpha^2(q)Q^3(q)}, \tag{25}$$

$$\omega_q^2 = q^2 + m^2 + \frac{\lambda N}{2}\int \frac{d^Dp}{(2\pi)^D} Q^2(p). \tag{26}$$

The effective Hamiltonian

$$H_{\text{eff}} = \epsilon_1 = N\int \frac{d^Dq}{(2\pi)^D}\left\{\frac{P^2(q)}{2} + V_q[Q]\right\}, \tag{27}$$
$$V_q[Q] = \frac{1}{2}(q^2+m^2)Q^2(q) + \frac{1}{2}\frac{1}{\alpha^2(q)Q^2(q)}$$
$$+\lambda\frac{N}{8}\int\frac{d^Dp}{(2\pi)^D}Q^2(p)Q^2(q),$$

is conserved in time*. The potential is bounded from below and the minimum corresponds to the fixed point solution Eq. (18), *i.e.* $P(q) = 0$, $\omega_q^2 Q^4(q) = 1/\alpha^2(q)$. For any initial value of H_{eff} larger than the minimum, the system cannot approach the fixed point and must remain oscillatory.

Let us next turn to the system for finite N and investigate the behavior in the vicinity of the fixed points. For this purpose we identify u, v, *etc.*, with the couplings at zero momentum and therefore $\bar\omega^2 = \omega_0^2$, $\bar B = B(0)$, and so on. In the following we concentrate on classical statistics ($\hbar = 0$). We consider the linear equations for small deviations from one of the fixed points (18), with $\delta G(q) = G(q) - G_*(q)$, *etc.*. For simplicity we choose $w_* = s_* = 0$, whereas $B_*(q)$ remains arbitrary such that the fixed point does not necessarily coincide with thermal equilibrium (19). In terms of directions orthogonal to the fixed point manifold

$$\hat a(q) = \delta\alpha^2(q) + \frac{2}{\omega_q^2 G^3(q)}\delta G(q) + \frac{1}{\omega_q^4 G^2(q)}\delta\omega_q^2,$$
$$\hat u = \delta u - \frac{1}{2}\frac{\omega_0^4 G(0)}{\omega_0^2 G(0) + \lambda\mathcal{S}_0}w \tag{28}$$
$$+\frac{\lambda}{\left(\omega_0^2 G(0) + \lambda\mathcal{S}_0\right)^2}\left(\omega_0^2\delta G(0) + G(0)\delta\omega_0^2 + \lambda\delta\mathcal{S}_0\right),$$
$$\hat z = z - \frac{1}{2\omega_0^2}\left[\left(\frac{6}{N+8} + \frac{N+2}{N+8}\frac{\lambda}{u}\frac{1}{\omega_0^2 G(0)}\right)w\right.$$
$$\left.+\frac{(N-1)}{N+8}\left(1 - \frac{\lambda}{u}\frac{1}{\omega_0^2 G(0)}\right)s\right],$$

the linear dynamical equations read

$$\partial_t\delta G(q) = -2G(q)c(q),$$
$$\partial_t\hat a(q) = -\frac{\lambda\mathcal{T}_0(q)}{2\omega_q^2 G^2(q)}v - \frac{4}{\omega_q^2 G^2(q)}c(q) + \frac{1}{\omega_q^4 G^2(q)}\partial_t\omega_q^2,$$
$$\partial_t c(q) = \omega_q^4 G^2(q)\hat a(q),$$
$$\partial_t\hat u = \frac{2+4\sigma}{1+\sigma}uc(0) + \frac{5+4\sigma}{2(1+\sigma)}\omega_0^2 v$$
$$-\frac{3}{2(1+\sigma)}\omega_0^4 y + \frac{1}{1+\sigma}\frac{u}{\omega_0^2}\partial_t\omega_0^2 + u^2\partial_t S_0,$$
$$\partial_t v = -4(1+\sigma)\hat u + 4u\omega_0^2 G^2(0)\hat a(0), \tag{29}$$
$$\partial_t w = 3\omega_0^2 y - (3+2\sigma)v,$$
$$\partial_t s = 2\omega_0^2 y - 2\left(1 + \frac{N+6}{N+8}\sigma\right)v,$$
$$\partial_t y = 4\omega_0^2\hat z,$$
$$\partial_t\hat z = -\frac{(5+\sigma)}{2}y + \frac{3}{2\omega_0^2}\left[1 + \sigma + \frac{10N+44}{3(N+8)^2}\sigma^2\right]v,$$

with

$$\sigma = \frac{\lambda\mathcal{S}_0}{\omega_0^2 G(0)} = \frac{\lambda}{u\omega_0^2 G(0)} - 1,$$
$$\mathcal{T}_0(q) = \frac{N+2}{2}\int\frac{d^Dp}{(2\pi)^D}\frac{d^Dp'}{(2\pi)^D}G(p) \tag{30}$$
$$\times G(p')G(p+p'+q).$$

All coefficients multiplying the linear excitations should be evaluated at the fixed point. The excitations $\delta G(q)$, w and s are considered coordinates on the fixed point manifold (18) and do not appear on the right-hand side of Eq. (29). (Note that $\partial_t\omega_q^2$ and $\partial_t S_0$ do not involve these quantities.) We can therefore discuss the subsystem $\{\hat a(q), c(q), \hat u, v, y, \hat z\}$ separately. We first consider a scenario, for $D \geq 1$, where ω_0^2 and S_0 are almost independent of time due to the averaging out of fluctuations with different momenta. In the limit $\partial_t\omega_0^2 = 0$, $\partial_t S_0 = 0$, the system for $\chi = \{\hat a(0), c(0), \hat u, v, y, \hat z\}$ becomes closed, $\partial_t\chi_i = A_{ij}\chi_j$. On the other hand, the anharmonic oscillator for $D = 0$ obeys

$$\partial_t\omega_0^2 = -\kappa_\omega\lambda(N+2)G(0)c(0)\left(1 - \frac{6}{N+8}\frac{\sigma}{1+\sigma}\right),$$
$$\partial_t\mathcal{S}_0 = -2\kappa_{\mathcal{S}}(N+8)G^2(0)c(0), \tag{31}$$

with $\kappa_\omega = \kappa_{\mathcal{S}} \equiv 1$. In order to interpolate between both situations we keep κ_ω and $\kappa_{\mathcal{S}}$ as free parameters. The matrix A_{ij} determines the behavior near the fixed point.

*This effective Hamiltonian has been found in a different approach in [2]. We emphasize that in leading order in the $1/N$ expansion there is no difference between classical and quantum statistical systems and H_{eff} appears in both of them.

For the free system, with $\lambda = 0$, $\sigma = 0$, the six eigenvalues ξ of A are $\pm 2i\omega_0, \pm 2i\omega_0, \pm 4i\omega_0$, as expected for the harmonic oscillations around the fixed point. For the interacting system the eigenvalues are determined by a cubic equation, $\eta^3 + b\eta^2 + c\eta + d = 0$, in $\eta = \xi^2/4\omega_0^2$, where

$$
\begin{aligned}
b &= 6\left\{1 + \frac{5\sigma}{12} + \frac{\tau}{3} + \Sigma_1\right\}, \\
c &= 9\left\{1 + \frac{13}{18}\sigma + \frac{1}{18}\left[2 - \frac{3\,(5N+22)}{(N+8)^2}\right]\sigma^2 \right. \\
&\quad \left. + \frac{1}{9}\,(4-\sigma)\,\tau + \frac{5(2+\sigma)}{3}\Sigma_1 + \frac{1}{9}\Sigma_2\right\}, \qquad (32)\\
d &= 4\left\{\left[1 + \sigma + \frac{1}{8}\left[2 - \frac{3\,(5N+22)}{(N+8)^2}\right]\sigma^2\right](1+6\Sigma_1) \right. \\
&\quad \left. - \frac{1}{8}\left(5 + 11\sigma + 2\sigma^2\right)\tau + \frac{5+\sigma}{8}\Sigma_2\right\},
\end{aligned}
$$

with

$$
\begin{aligned}
\tau &= \lambda u \mathcal{T}_0 \left(4\omega_0^2\right)^{-1}, \\
\Sigma_1 &= \frac{\kappa_\omega \rho \sigma}{12}\frac{N+2}{N+8}\left(1 - \frac{6}{N+8}\frac{\sigma}{1+\sigma}\right), \\
\Sigma_2 &= \rho\sigma\tau\left[2\kappa_{\mathcal{S}} \right. \qquad (33)\\
&\quad \left. + \kappa_\omega\left(\frac{N+2}{N+8}\right)\left(1 - \frac{6}{N+8}\frac{\sigma}{1+\sigma}\right)\right], \\
\rho &= \frac{N+8}{2}\frac{1+\sigma}{\sigma}uG^2(0).
\end{aligned}
$$

The behavior of the linearized system depends on the sign of the discriminant, $K = \frac{1}{27}b^2c^2 - \frac{4}{27}c^3 - d^2 - \frac{4}{27}b^3 d + \frac{2}{3}bcd$. For $K \geq 0$, the three roots η_i are all real and negative, for positive b, c and d. The linearized system is oscillatory with eigenvalues $\xi_{i\pm} = \pm 2i\omega_0\sqrt{|\eta_i|}$. For negative K two roots η are complex and conjugate to each other. Since the real part of ξ does not vanish in this case one concludes that two eigenvalues ξ must have a positive real part. They dominate the evolution for $t \to \infty$. The fixed point is unstable for $K < 0$.

The fact that the eigenvalues of A_{ij} occur in pairs with opposite sign is a direct consequence of time reflection symmetry of the microscopic equations of motion. The correlation functions are eigenstates of time reflection $\hat{T}$- in our case c, v and y have odd $\hat{T}$-parity. One infers that the linear system obeys $\hat{T}A\hat{T} = -A$ with $\hat{T} = \mathrm{diag}(+1, -1, +1, -1, -1, +1)$. This implies that for any eigenvalue ξ of A there also exists an eigenvalue $-\xi$. One concludes that an approach to the equilibrium fixed point cannot be seen in linear order. For time reflection invariant systems the only way an equilibrium fixed point could be approached is a purely oscillatory behavior in linear order with vanishing real parts of all eigenvalues of the 'stability matrix' A. Beyond linear order it is then conceivable that the oscillations are damped by the nonlinear flow - at least for a large class of trajectories. Of course, $\hat{T}$-invariance implies that for any trajectory with flow towards the fixed point there must also exist a trajectory with flow away from it. Nevertheless, for a nonlinear flow it seems possible that only a very special class of excitations around the fixed point leads to unstable behavior. The relative volume of the space of unstable excitations (as compared to the stable excitations) could shrink to zero as the number of degrees of freedom goes to infinity.

We should point out that the general oscillatory behavior found in our linear analysis is not in contradiction with the damping rates computed in linear response theory for classical or quantum field theories in thermodynamic equilibrium [9]. Indeed, a linear superposition of a continuum of frequencies with suitable amplitudes can easily lead to a damped behavior, as illustrated by the identity $\int_{-\infty}^{+\infty}\frac{d\omega}{\pi}\frac{\gamma}{\omega^2+\gamma^2}e^{i\omega t} = e^{-\gamma|t|}$. The damping of Δm^2 shown in Fig. 1 also constitutes a good example. It is therefore conceivable that on a microscopic level the oscillations are never damped and the system never reaches the thermodynamic equilibrium fixed point in a strict sense. Nevertheless, macroscopic quantities may well approach equilibrium values for $t \to \infty$.

Let us first consider $D = 0$ where $\kappa_\omega = \kappa_{\mathcal{S}} = \rho = 1, \tau = \frac{N+2}{2(N+8)^2}\frac{\sigma^2}{1+\sigma}$, such that the cubic equation only depends on one parameter, σ. In an expansion in powers of $1/N$ we find in lowest order $K = 0$, with roots $\eta_i = -(1+\frac{\sigma}{2}), -(1+\frac{\sigma}{2}), -4(1+\frac{\sigma}{2})$. This is very similar to the oscillations of the free system. However, the relevant frequency is shifted to

$$\omega\sqrt{1+\sigma/2} = m^2 + \frac{3}{4}N\lambda G, \qquad (34)$$

with the temperature dependent 'propagator' ($T = B_*^{-1}$)

$$G = 2T\left(\sqrt{m^4 + 2N\lambda T} + m^2\right)^{-1}. \qquad (35)$$

In order to discuss the stability of the equilibrium fixed point we have to proceed to next to leading order $1/N$ where

$$K = \frac{\sigma^2}{N}\left(1+\frac{\sigma}{2}\right)^3\left(\frac{47 + 23\sigma - 13\sigma^2}{1+\sigma}\right) + O\left(\frac{1}{N^2}\right). \qquad (36)$$

This is positive for $\sigma < \sigma_c \approx 2.982$. Using the leading order relation

$$\sigma = \left(\sqrt{1+\nu^2} + \nu\right)^{-2}, \quad \nu = \frac{m^2}{\sqrt{2N\lambda T}}, \qquad (37)$$

one has $\sigma < 1$ for positive m^2 and no unstable mode is present. On the other hand, for negative m^2 there is a critical value $\nu_c \approx -4.39$, beyond which σ exceeds σ_c. This leads to the most interesting conclusion that for negative m^2 and $\lambda N < 0.026\frac{m^4}{T}$, the system is repelled from thermodynamic equilibrium asymptotically, despite the fact that the number of coupled degrees of freedom can be arbitrarily large ($N \to \infty$) ! For a given initial average energy per mode ($\sim T$) the stability of thermodynamic equilibrium requires a minimal coupling strength !

In contrast, for $\kappa_\omega = \kappa_{\mathcal{S}} = 0$ ($D > 0$), we find already in lowest order in an $1/N$ expansion the roots $\eta = -(1+\sigma/2), -1, (-4+2\sigma)$, with

$$K = \frac{3}{4}\sigma^2 \left(1+\frac{\sigma}{2}\right)^2 \left(1+\frac{2}{3}\sigma\right)^2 > 0. \tag{38}$$

The linear system is purely oscillatory. Since K is a continuous function of $1/N$ and τ (note that $\tau \sim 1/N$) this remains true for small $1/N$. For example in next to leading order we obtain

$$\Delta K_{1/N} = \frac{\tau}{4}\left(1+\frac{2}{3}\sigma\right)\left(6\sigma^4 + 38\sigma^3 + 111\sigma^2 \right. \tag{39}$$
$$\left. +150\sigma + 72\right) - \frac{5}{24N}\sigma^3\left(1+\frac{2}{3}\sigma\right)\left(\sigma^2 + 24\sigma + 36\right).$$

This conclusion also extends to non-vanishing κ_ω and $\kappa_{\mathcal{S}}$, as long as $\kappa_\omega \sim 1/N$. We conclude that the equilibrium fixed point has no unstable direction for linear fluctuations. By simple continuity considerations this stability can be extended to a whole region in the fixed point manifold around the thermal equilibrium point.

The discussion of the nonlinear behavior for large 'distances' from the fixed point manifold remains outside the scope of this note. We only report here briefly some results for $D = 0$ where Eqs. (7), (22) reduce to nine coupled ordinary differential equations, which we have solved numerically. The stable small oscillations around the fixed points near classical thermal equilibrium can easily be observed, being determined by three relevant frequencies. In this case we have not observed any nonlinear damping. As the amplitudes of the oscillations increase the stable behavior continues for a while. The time averaged mean value corresponds now to a 'dynamical fixed point', which is close to but not part of the fixed point manifold. For a Gaussian initial distribution not too far from the fixed point manifold (and for $m^2 > 0$) this behavior is approached after a certain time. Finally, for large distances from the fixed point manifold the behavior becomes much more complicated. Typically a very large range of frequencies becomes important after a certain time interval and the problem becomes stiff. Analogous behavior has been observed in other anharmonic oscillator systems treated in the $1/N$ expansion [10]. We will address these interesting phenomena elsewhere.

In conclusion, the step beyond the leading order $1/N$ expansion seems crucial for the understanding of nonequilibrium field theories. One finds in leading order infinitely many unphysical conserved quantities which prevent the system from approaching thermal equilibrium. They do not survive the inclusion of scattering, beyond leading order. New mysteries have appeared, however. This concerns, in particular, the existence of a continuous manifold of stationary distributions within which the thermal equilibrium distribution is just a particular point. It has also become clear that small deviations from the fixed points are not damped in the context of linear fluctuation analysis. The understanding of the thermalization of isolated time reversible systems described by field theories remains an unsolved challenge. As long as such basic questions are not fully understood the use of equilibrium field equations in cosmology should be regarded with caution.

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