Constraints and gauge transformations: Dirac's theorem is not always valid

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Abstract

A standard tenet of canonical quantum gravity is that evolution generated by a Hamiltonian constraint is just a gauge transformation on the phase space and therefore does not change the physical state. The basis for this belief is a theorem of Dirac that identifies primary first-class constraints as generators of physically irrelevant motions. We point out that certain assumptions on which Dirac based his argument do not hold for reparametrization invariant systems, and show that the primary Hamiltonian constraint of these systems does generate physical motion. We show explicitly how the argument fails for systems described by Jacobi's principle, which has a structure closely resembling that of general relativity. We defer discussion of general relativity and the implications for quantum gravity to a later paper.

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I. INTRODUCTION

A standard belief in the canonical quantum gravity community is that the evolution generated by any Hamiltonian constraint H "is just the unfolding of a gauge transformation" [1]. This suggests that if observables in quantum gravity are gauge invariant, they must commute with H—they must be 'perennials', to use Kuchař's coining [2]. Kuchař argues persuasively, however, that this feature runs counter to intuition; he therefore makes the valuable distinction between what he calls observables, which need only commute with the ADM momentum constraint, and the perennials, which commute with both it and H. He also notes that perennials are extraordinarily difficult to find in any dynamical system, let alone one as intricate as general relativity (GR). Kuchař [2] and one of us [3] have also argued that the physical state is manifestly changed by constrained Hamiltonian evolution. In this paper, we will sharpen the challenge to the standard belief by showing that the premise on which it is based is false.

The standard designation of the GR Hamiltonian as a gauge generator is based on a simple argument of Dirac [4], from which he concluded that primary first-class constraints treated as generating functions of infinitesimal contact transformations "lead to changes in the [canonical data] that do not affect the physical state". Thus, one concludes that all points in the phase space on an orbit of such a constraint represent the same physical state but in different gauges; the constraint implements a gauge transformation in carrying the state from one point to another along such an orbit. This interpretation is universally accepted for, say, the Gauss constraint in electrodynamics. H is a primary first-class constraint in GR, so it has been widely concluded that it too is a gauge generator of this kind.

The symmetry connected to H in GR is invariance of the action under a change of foliation of a single spacetime, together with the freedom to change the curve parameter that labels the leaves of a fixed foliation. It is not widely recognized that a refoliation and a reparametrization are distinct operations, but the difference is indicated by the fact that the former changes the curve in phase space, while the latter does not. A theory is reparametrization invariant (RI) if and only if its Lagrangian is homogeneous of degree one in its velocities. Such a theory necessarily has a primary first-class constraint: its Hamiltonian H must vanish.

In this paper, we shall not consider fully foliation invariant theories, but just those that are simply RI and, for simplicity, do not have any other constraints apart from H=0. We will not include theories with ' $1 \times \infty$ ' constraints—that is, field theories with a single local constraint. Parametrized particle dynamics is often invoked as a simple RI analogue of GR. Jacobi's principle in classical dynamics, however, leads to an RI theory that is a much better analogue [3, 5]. We shall use it here to show that Dirac's argument does not have general validity. Our argument will apply to all theories of the class that we consider, including parametrized dynamics.

The essential flaw in Dirac's argument is this: His proof, given in Chp. 1 of his Lectures on Quantum Mechanics [4], assumes that the independent variable of the Hamiltonian evolution is an absolute time. In Chp. 3, Dirac then considers RI theories and shows that in them there is no absolute time, but he does not consider the implications for his earlier proof. In particular, he does not repeat the explicit calculation that is the key to his result. We shall show, by properly completing this calculation, that the difference invalidates the proof for RI theories. In their standard reference on constrained systems, Henneaux and Teitelboim [1] do note that the issue of the status of time affects the argument but still adopt the standard

conclusion. We will review the discussions of [4] and [1] below.

We emphasize that our claim is much more significant than a refutation of the 'Dirac conjecture' that all *secondary* first-class constraints generate motions that do not change the physical state. Dirac makes this conjecture in Chp. 1 [4] just after giving his proof for *primary* first-class constraints. The Hamiltonian constraint in GR and RI theories is a primary constraint, so it is Dirac's apparent proof that we are concerned with. See [1] for more discussion of the conjecture, including a simple counterexample.

Nearly all quantum gravity reviews invoke the standard tenet with no discussion of the details. The only serious challenge to the 'orthodoxy' that we have seen is from Kuchař [2], although neither [2] nor [3] makes clear how the problem is resolved by identification of the weak points in Dirac's argument. In his recent book, Kiefer [6] does present Dirac's argument, though without identifying the assumption of absolute time; he therefore concurs with the standard conclusion but emphasizes the importance of Kuchař's distinction between observables and perennials, thus avoiding the counterintuitive implication of the standard conclusion.

Our result casts strong doubt on the standard quantum gravity belief. We speculate here that the Hamiltonian constraint in GR is partly a gauge generator and partly a generator of true physical change—and moreover more the latter than the former. Because full foliation invariance is a much more complex issue than simple reparametrization invariance, we defer further discussion of it to a later paper. We will also consider there the difficult question of whether observables in quantum GR must be perennials. We are however confident that they need not be for simple RI theories, as we shall also argue in a later paper.

We should also emphasize that we follow closely Dirac's method of investigation and consider the infinitesimal change of the state generated by an infinitesimal action of the Hamiltonian constraint in the case of RI theories. So far as we know, no one has done this hitherto when considering the meaning of constraints in RI theories; Dirac certainly did not and therefore failed to follow the logic of his own method. As a consequence, his result, which is correct for a large class of theories with other kinds of constraints, has been widely applied without question to RI theories.

In this connection, some argue—incorrectly in our view—that the entire history in phase space, rather than the individual points on it, defines the physical state [7]. It has also been suggested to us in correspondence that the Hamiltonian constraint could be considered to act on this complete history and thereby generate a reparametrization of it. But reparametrization of an entire history is quite different from the infinitesimal action of the constraint on the instantaneous state, which is what we need to consider in following the logic of Dirac's approach. We believe that faithful adherence to that logic will at last bring clarity to this vexed issue.

We will now go into more detail, beginning with a discussion of the nature of time itself and its relation to Jacobi's principle and parametrized dynamics. We then present Dirac's argument, identify its fault, and summarize what was actually said by Dirac, and Henneaux and Teitelboim. It is convenient to first study Jacobi's principle, as it makes the arbitrariness of the curve parameter and the definiteness of the Hamiltonian evolution clear. This preparation demonstrates beyond doubt that Dirac's result does not have general validity and that the Hamiltonian constraint in this class of theories generates physical motion. Throughout, we follow Dirac's conventions [4].

II. TIME AND JACOBI'S PRINCIPLE

A. Time

We begin with comments about time itself. Newtonian dynamics treats it as absolute: an external independent variable unrelated to what happens in the universe. In the Newtonian n-body problem, a dynamical history is a curve in a 3n-dimensional configuration space Q traversed at a speed measured by that time t. But we could never say that time passes if we did not observe change in general and motion in particular. In reality, time is measured by a motion; for millennia the rotation of the earth provided the hand of the clock and the fixed stars the clock face. In a universe consisting of n particles with no nearly rigid earth and fixed stars, all that is given objectively is the curve in Q. Deprived of an easy convenient measure of time, we must think about these things clearly. What is time? How do we express the objective content of Newtonian theory without absolute time?

In fact, we shall show that the curve in Q contains all that is essential: the succession of configurations that the Newtonian t labels. What is lost is the metric of t, which becomes a mere label better called λ , and the direction of time. But neither are essential. We shall see shortly that the true content of Newtonian dynamics is that it determines such curves given an initial point q in Q and an initial direction at q, which we denote d_q . The physical state is defined by the pair (q, d_q) .

Before we present the theory of geodesics on Q, let us note that it will meet not only the epistemological insight that time must be derived from change but also the requirement that a dynamical curve be determined by a pair (q, d_q) . Such a pair constitutes the initial condition for a geodesic. Note also that specification of q requires 3n numbers but of d_q one less. We shall see that this is the natural and necessary origin of Hamiltonian constraints.

Because the geodesic theory we shall obtain is a representation of Newtonian theory defined on the same space Q, distinct points in Q must represent physically distinct states just as they do in Newtonian theory. Indeed, we can assert that the same is true for points in the configuration space for all RI theories with just the single Hamiltonian constraint. Thus all the considered dynamical variables are observable. As we explain below, this includes the time in parametrized dynamics; only the unphysical label need be assumed to be unobservable.

We could now proceed to construct a timeless theory of geodesics in Q *ab initio* and show how Newtonian theory can be recovered from it. It will be more useful for the purposes of our discussion to work in the opposite direction, however, which we do now.

B. Jacobi's principle

Consider an action I, evaluated along a fixed curve in Q with end point configurations $(A, B) \in Q$:

$$I = \int_{A}^{B} d\lambda L[q_i, \dot{q}_i], \tag{1}$$

where $\dot{q}_i = dq_i/d\lambda$. Consider the case where L is homogeneous of degree one in the velocities \dot{q}_i . Under a reparametrization of the curve label λ ,

$$\lambda \to \lambda'(\lambda),$$
 (2)

L transforms as

$$L[q_i, \frac{dq_i}{d\lambda}] \to L'[q_i, \frac{dq_i}{d\lambda'}] = \frac{d\lambda}{d\lambda'} L[q_i, \frac{dq_i}{d\lambda}].$$
 (3)

Thus, the action (1) is invariant under the reparametrization (2); λ is a mere label without physical significance. Note that it is not necessary to leave fixed the value of λ at the end points of integration. To express the invariance of the action properly requires stipulating fixed configurations A and B; it is meaningless to fix the end points by giving values of λ .

The general form of L also implies that the canonical momenta $p_i = \delta L/\delta \dot{q}_i$ are homogeneous of degree zero and thus invariant under a reparametrization. Furthermore, the canonical Hamiltonian vanishes identically:

$$\sum_{i} p_i \dot{q}_i - L = 0. \tag{4}$$

This 'Hamiltonian constraint', expressed as a function of the canonical data (q_i, p_i) , is a primary first-class constraint; it is thus the type to which Dirac's argument will apply. A primary constraint is a relation between the canonical variables that holds by definition of the momenta. Obviously, (4) is of this type. A first-class function of the canonical variables is one such that its Poisson bracket with every constraint of the system is a linear combination of the constraints. Since we will only consider systems with just this Hamiltonian constraint, the constraint is first-class.

Let us consider the explicit example of a system of Newtonian particles described by Cartesian coordinates \mathbf{q}_i . We can begin with a standard, non-RI action I, defined along a curve in Q with fixed end point configurations A at time t_0 and B at t_f :

$$I = \int_{A(t_0)}^{B(t_f)} dt \left(\frac{1}{2} \sum_{i} \frac{d\mathbf{q}_i}{dt} \cdot \frac{d\mathbf{q}_i}{dt} - V(\mathbf{q}_i)\right), \tag{5}$$

where all the particle masses are set to unity for simplicity. We render this action RI by promoting the time coordinate t to a dynamical variable and introducing an evolution parameter λ that labels the curve in the new configuration space QT, Q augmented by the space T of absolute times. The action becomes

$$I = \int_{(A,t_0)}^{(B,t_f)} d\lambda (\frac{T}{\dot{t}} - \dot{t}V), \tag{6}$$

where $T = \frac{1}{2} \sum_{i} \dot{\mathbf{q}}_{i} \cdot \dot{\mathbf{q}}_{i}$. The canonical momentum p_{t} of t is

$$p_t = -\left(\frac{T}{\dot{t}^2} + V\right),\tag{7}$$

while the canonical momentum \mathbf{p}_i of particle i is

$$\mathbf{p}_i = \frac{\dot{\mathbf{q}}_i}{\dot{t}}.\tag{8}$$

The identity (4) can be expressed as

$$\frac{1}{2} \sum_{i} \mathbf{p}_i \cdot \mathbf{p}_i + p_t + V = 0. \tag{9}$$

Since $T/(\dot{t})^2$ is the unparametrized kinetic energy, p_t is just minus the total energy E. Because t is cyclic (L depends only on its velocity \dot{t}), p_t is conserved. We can then eliminate \dot{t} from the action via Routhian reduction. Following standard procedure [8], we do this by solving $p_t = -E$ for \dot{t} and substituting the result into a reduced Lagrangian \bar{L} :

$$\bar{L} = L + E\dot{t}.\tag{10}$$

This leads to the reduced, 'Jacobi' action I_J :

$$I_J = 2 \int_A^B d\lambda \sqrt{(E - V)T},\tag{11}$$

with a characteristic square root form. I_J defines a geodesic principle, Jacobi's principle, on the original configuration space Q.

The timeless nature of Jacobi's principle is especially clear when expressed without an evolution parameter, using only the \mathbf{q}_i and their displacements $\delta \mathbf{q}_i$:

$$I_J = \int_A^B \sqrt{2(E - V) \sum_i \delta \mathbf{q}_i \cdot \delta \mathbf{q}_i}.$$
 (12)

This is the timeless geodesic theory that we could have derived *ab initio* based on the inescapable fact that time as such is not observable. Here, E is regarded as part of the potential—that is, part of the law of the 'island universe' described by (12); Einstein's cosmological constant Λ occurs in the Baierlein–Sharp–Wheeler action for general relativity in exactly the same way [5]. The form (12) of the action makes the physical irrelevance of any label λ evident and shows that the curve end points for the calculation of the action are defined by the configurations A and B.

C. The Lapse

In conventional applications, Jacobi's principle is used to split dynamical problems into the determination of a timeless orbit in the configuration space; energy conservation is then used to determine the speed in orbit. However, if the system under consideration models the universe, it is more illuminating to see how a physical and directly observable time along the orbit can be obtained [3]. Define the 'lapse' function N:

$$N = \sqrt{\frac{T}{E - V}}. (13)$$

Then the canonical momenta \mathbf{p}_i corresponding to Jacobi's principle (11) are

$$\mathbf{p}_i = \frac{\dot{\mathbf{q}}_i}{N},\tag{14}$$

and the Euler-Lagrange equations are

$$\dot{\mathbf{p}}_i = -N \frac{\partial V}{\partial \mathbf{q}_i}.\tag{15}$$

The canonical Hamiltonian H vanishes identically, but we can formally express it:

$$H = \sum_{i} \mathbf{p}_{i} \cdot \dot{\mathbf{q}}_{i} - \bar{L} = Nh, \tag{16}$$

where

$$h \equiv \frac{1}{2} \sum_{i} \mathbf{p}_{i} \cdot \mathbf{p}_{i} + V - E, \tag{17}$$

vanishes by the definitions (13) and (14). The Hamiltonian constraint h = 0 arises because the initial condition for a geodesic principle involves a direction and not a direction with a speed along it. Directions are most conveniently specified by direction cosines, which satisfy a constraint: the sum of their squares is unity. The Jacobi momenta are essentially direction cosines multiplied by $\sqrt{(E-V)}$, so their squares sum to give the quadratic identity.

The definitions (13) and (14) also show that a given set of canonical data $(\mathbf{q}_i, \mathbf{p}_i)$ such that the \mathbf{p}_i satisfy the Hamiltonian constraint are uniquely determined by the physical state (q, d_q) , and vice versa. Thus, H generates a unique curve of the evolution in phase space, and each point along it corresponds to a unique physical state. The only 'gauge' aspect of this description is the value of the unphysical label at which a particular state is reached.

A crucial property of the lapse is that it transforms $N \to N'$ under a reparametrization such that

$$N'(\lambda')d\lambda' = N(\lambda)d\lambda. \tag{18}$$

This result is obvious from the definition (13), but it can also be derived in the Hamiltonian picture—that is, without invoking the definitions of lapse (13) or momenta (14)—by requiring invariance of the canonical action $I = \int d\lambda (\sum_i \mathbf{p}_i \dot{\mathbf{q}}_i - H[\mathbf{q}, \mathbf{p}])$. The transformation of I must again be made with fixed curve end points for the invariance to be physically meaningful. The lapse is an arbitrary function, in the sense that its value is not fixed by the dynamical equations; however, given a choice of lapse and parametrization of a dynamical curve, changing the lapse implies changing the parameter in the manner of (18).

The lapse also enables us to recover Newtonian time and to see how it arises from the genuinely observable \mathbf{q}_i and $\delta \mathbf{q}_i$. Requiring that the Euler-Lagrange equations take the form of Newton's laws leads to the condition N=1, equivalent to a choice of distinguished parameter t. The increment δt of this Newtonian time corresponds to

$$\delta t = N\delta\lambda = \sqrt{\frac{\sum_{i} \delta \mathbf{q}_{i} \cdot \delta \mathbf{q}_{i}}{2(E - V)}}.$$
(19)

Thus, the increment of physical time δt is determined by the shape of the geodesic.

We can also now see the conceptual relationship between Jacobi's principle and parametrized dynamics. Suppose a system of n point particles and an effectively isolated subsystem of m particles within it $(n \gg m)$. The time t with increment $\delta t = Nd\lambda$ can be obtained for the large system as we have done for Jacobi theory and then serve as the independent variable for the small subsystem, which can then be parametrized. This makes it clear that the time t in parametrized dynamics is in principle as observable as the q's and p's of the subsystem to which it is adjoined for the purposes of parametrization. It may be worth mentioning in this connection, however, that the time obtained in this manner is not a function on the phase space. This is because the many-to-one relationship between the velocities and the momenta expressed by the Hamiltonian constraint precludes conversion of the $\delta \mathbf{q}_i$ in (19) into momentum-type variables.

Finally, we note that Dirac's methods [4] can be used to show that the Hamiltonian for the general RI theories that we consider has the form (16) as in Jacobi theory. N will again be an arbitrary function, and invariance of the canonical action will imply the invariance of $Nd\lambda$. The function h(p,q) will be a primary first-class constraint, although potentially with a different form. In parametrized particle dynamics, for example, h is given by (9).

D. Evolution

Evolution in the case of RI theories is quite unlike ordinary unconstrained evolution. We will focus on evolution associated with an infinitesimal amount of action, because this is the case Dirac considers. In standard Hamiltonian theory in which absolute time t is the integration variable, one can generate a physical, infinitesimal motion by letting H act for infinitesimal δt . In RI theories, however, an infinitesimal $\delta \lambda$ has no meaning by itself. In fact, we have seen in (12) that Jacobi theory can be defined with no λ whatsoever. The only meaningful increment is of the emergent Newtonian time δt (19). If we use a parameter, the increment of evolution only becomes well-defined when we specify both $\delta \lambda$ and $N(\lambda)$. It should also be noted that a finite evolution is obtained by specifying $N(\lambda)$ over some finite interval $[\lambda_1, \lambda_2]$ of the independent variable. The action of H then generates the curve in phase space and the parametrization on it. Therefore, one cannot say that H generates a reparametrization since no parametrization exists before H acts.

The infinitesimal change of any dynamical variable g along the evolution curve is

$$\delta g \approx \delta \lambda[g, H] = \delta t[g, h],$$
 (20)

where the brackets denote the Poisson bracket and \approx denotes Dirac's weak equality [4], meaning that the relation holds provided the Poisson bracket is calculated before h=0 is imposed. For a chosen N and λ , we see that δg is proportional to $N\delta\lambda=\delta t$, so that it is a meaningful difference. In particular, δg is generally nonvanishing and is unchanged under a reparametrization.

We see that although the evolution is driven by a Hamiltonian constraint, there is no doubt that the evolution carries us to different configurations $q \in \mathbb{Q}$. As remarked in Sec. II A, these configurations must represent physically distinct states. Of course, if we ask what state is reached at a particular value of the curve label λ , the answer will be changed by reparametrization. But it is the product $N\delta\lambda$ which determines an infinitesimal Hamiltonian evolution. This, and with it the physical evolution, is invariant under a reparametrization that respects the symmetry of the canonical action.

III. THE STANDARD ARGUMENT

We have just argued that, counter to the standard belief, the Hamiltonian constraint in single-constraint RI theories generates physical motion. We will now present Dirac's 'proof' that primary first-class constraints generate transformations that do not change the physical state, but demonstrate precisely how it fails for these theories.

A. Dirac's first lecture

We begin by noting that nowhere in his *Lectures* does Dirac use the terms 'gauge transformation' or 'gauge generator', presumably because gauge theory was not the dominant paradigm in theoretical physics at the time. Instead, he considers various kinds of constraints and their effects on a dynamical system. Thus, for Dirac, the precise question is not whether a constraint is a 'gauge generator' but whether it generates motions in the phase space "that do not affect the physical state."

In Chp. 1 of his *Lectures* [4], Dirac considers systems with a 'true' Hamiltonian H, which is not constrained to vanish, but when some constraints are present. He shows that then the system is governed by a total Hamiltonian H_T of the form

$$H_T = H + v_a \phi_a, \tag{21}$$

where ϕ_a are primary first-class constraints (as defined below our Eqn. (4)) and v_a are arbitrary functions of the time t. (We have here ignored the difference in Dirac's Eq. (1-33) between H and his H', which is unimportant in this context.) The increment in a dynamical variable g in the time δt is [Dirac's (1-37)]:

$$\delta g = \delta t[g, H_T] = \delta t\{[g, H] + v_a[g, \phi_a]\}. \tag{22}$$

Dirac shows that the arbitrariness of v_a allows one to change δg freely via a new choice v'_a , with a difference of:

$$\Delta g(\delta t) = \varepsilon_a[g, \phi_a],\tag{23}$$

where

$$\varepsilon_a = \delta t(v_a - v_a'), \tag{24}$$

"is a small arbitrary number, small because of the coefficient δt and arbitrary because the v's and v's are arbitrary." Dirac assumes that the theory is such that "the initial state must determine the state at later times"; thus, the ambiguity (23) in the evolution (22) implies that the ϕ_a are "generating functions [that] lead to changes in the q's and the p's that do not affect the physical state" (Dirac's italics).

Dirac's ability to draw this conclusion depends crucially on the presence of the two terms of very different nature present in (22). They are present simultaneously because there is not only the 'true' Hamiltonian H, which unquestionably generates physical motion, but also the constraints. The presence of the absolute time t is also vital: Dirac assumes that evolution with different v_a but over an identical increment δt gives the same physical state. Subject to these important qualifications, we can see no flaw in Dirac's argument as presented. Henneaux and Teitelboim [1] give essentially the same argument.

B. Dirac's third lecture

In his third lecture, Dirac considers how his theory of generalized Hamiltonian dynamics can be modified to take into account relativity. For this, he says that we cannot simply use the "one absolute time" assumed in the first lecture. We must now "go back to first principles. It is no longer sufficient to consider just one time variable referring to one particular observer... We want to have the possibility of various times which are all on the same footing."

To indicate how this can be done, he considers the special case in which the Lagrangian is homogeneous of degree one in the velocities. The total Hamiltonian is now built up entirely from primary first-class constraints with arbitrary coefficients:

$$H_T = v_a \phi_a, \tag{25}$$

"showing that there must be at least one primary first-class constraint if we are to have any motion at all". We shall attempt to interpret the quoted words in a moment. Note that while Dirac wishes ultimately to consider theories that are relativistic in the usual sense, his general analysis includes the nonrelativistic theories that we have discussed above.

Dirac notes that the equations of motion are now

$$\frac{dg}{d\lambda} \approx v_a[g, \phi_a]. \tag{26}$$

He notes that multiplying the arbitrary functions v_a by a common factor is equivalent to introducing another time variable λ' and gives the new equations of motion

$$\frac{dg}{d\lambda'} \approx v_a'[g, \phi_a],\tag{27}$$

noting that now "there is no absolute time variable."

Now comes the crux. Having made this correct observation, Dirac then moves on to another issue without doing the calculation analogous to (22) for the increment δg in the case of theories with homogeneous velocities. Had he done so, he would have obtained the expression (20)

$$\delta g = \delta \lambda (dg/d\lambda) = \delta t[g, h],$$
 (28)

where we have specialized to H of single-constraint RI theories (16) in the second line.

Whereas the expression for evolution in the first lecture (22) contains two terms, the first corresponding to physical change with an unknown admixture of unphysical change, and the second to pure unphysical change, (28) contains only one term. This expression must represent a pure physical change to the state, since the right-hand side is nonvanishing for general g, and distinct sets of canonical data in Jacobi theory (as in all theories of the class we consider) represent physically distinct states, as we have noted in Sec. II C.

Of course, as we noted earlier, one can get different values for δg by changing N while fixing $\delta \lambda$, but one always gets physical change. Moreover, it is clearly confusing the issue to not change $\delta \lambda$ and N simultaneously. Transforming them both so as to respect the symmetries of the action means leaving δt invariant, and thus also δg .

C. Dirac's commentary

To be clear, Dirac does not explicitly state anywhere in his *Lectures* that the Hamiltonian constraint in the theories he considers generates only physically irrelevant motions, despite frequent mentions that it is a primary first-class constraint. In the light of the emphasis placed in the first lecture on his proof, however, we find it remarkable that he nowhere discusses the apparent discrepancy between that result and the implication that (25) does

generate real change even if the 'speed' depends on an arbitrary label. What else can Dirac's words "if we are to have any motion at all" mean?

Several passages in Dirac's third lecture suggest that he instinctively took (26) and other analogous equations to represent real change. Perhaps the most indicative comment comes in the midst of his discussion of the Hamiltonian structure of a parametrized field theory on flat spacetime. Dirac has extended the parametrization to include the four components of the Minkowskian coordinate system. This allows him to describe evolution along a sequence of curved surfaces, rather than just the customary flat ones. As a result, the Hamiltonian is just the sum of four local constraints. Dirac demonstrates that three "tangential" constraints generate coordinate transformations within a given surface while the fourth "normal" constraint generates deformations of the surface normal to itself.

He then states that the tangential components "[are] not of real physical importance." By contrast, "the quantity which is of real physical importance is the normal component." The normal motion that it generates "is something which is of dynamical importance." For someone who typically chose his words very carefully, "physically important" must surely be distinct from "not physically important".

We have also examined Dirac's earlier paper [9] in which he first developed his generalized Hamiltonian dynamics. This paper puts even more emphasis than [4] on Lagrangians that are homogeneous of degree one in the velocities, "because they are specially convenient for a relativistic treatment." First- and second-class constraints are defined explicitly, but the distinction between primary and secondary constraints is not highlighted and Bergmann's terminology for them is not introduced. There is nothing analogous to the proof in the *Lectures* that primary first-class constraints generate transformations of the canonical data that do not change the physical state.

We can only conclude that, in a remarkable oversight and because his main interest in Chp. 3 lay in the applications of RI theories, Dirac failed to see the contradiction between the action of Hamiltonian constraints and his theorem. He may also have been misled by the situation that arises in relativistic (foliation-invariant) systems. Here arbitrary elements certainly do come into play, and it is not easy to separate the genuine change from irrelevant change. This is why we considered it important to begin with the presentation of non-relativistic Jacobi theory, for which the case is much clearer.

D. Henneaux and Teitelboim's commentary

The book [1] of Henneaux and Teitelboim (HT) is perhaps the best known and most thorough modern discussion of Dirac's methods for constrained systems. It is thus an influential text, and we should examine how it has helped to strengthen the widely held view that Hamiltonian constraints are to be regarded as gauge generators. We shall see that HT state conditions that indicate a possible invalidity of Dirac's argument for Hamiltonian constraints but still argue for its general validity.

HT present Dirac's argument in their Chp. 1, at the end of which they add a word of caution, commenting that the argument implicitly assumes "that the time t ... is observable." They say that this is "information brought in from outside" and suggest one "may also take the point of view that some of the gauge arbitrariness indicates that the time itself is not observable". The whole of their Chp. 4 is then devoted to this issue.

There, we note that HT, like Dirac, make no attempt to pose or answer the question that ought to be addressed first of all: What is time? There is no suggestion that change

is primary and time as a measurable quantity can emerge through a Jacobi-like timeless law that describes change. Instead, they comment that normally "time is assumed to have a direct physical significance but is not itself a dynamical variable". They then mention "a different formulation ... in which the physical time and the dynamical variables ... are treated more symmetrically". This is just parametrized dynamics, which they present later in the chapter. They then note that parametrized dynamics can be readily deparametrized, but not GR.

Then comes the decisive passage, whose conclusion we question. HT say that because attempts to deparametrize GR "have not been quite successful", it seems [our italics]

preferable to aim at both formulating and answering questions while treating all variables on the same footing. This coincides with the point of view about gauge invariance that has been taken throughout this book. It is not accidental that this should be so, since, as we shall see below, in a generally covariant theory the motion is just the unfolding of a gauge transformation.

The 'proof' of this statement appears to consist of three lines in the discussion of parametrized dynamics, when they note that its Hamiltonian consists solely of a constraint and "therefore, the motion is the unfolding of a gauge transformation".

However, this is not a proof—it is an assertion. The issue is not whether the Hamiltonian is a constraint, which of course it is, but what the constraint does. As we have seen, it generates real change in all theories of the class we consider, including parametrized dynamics. HT's conclusion would only follow if they had already proved that all primary first-class constraints generate gauge transformations. Thus, their Chp. 4 purports to extend the correct proof when time is absolute to all cases, but when it comes to the critical point the result is simply asserted without foundation.

We have interpreted HT's words, "the motion is the unfolding of a gauge transformation," to mean that H carries any state on which it acts into other states that are physically identical. This is by analogy with the familiar Gauss constraint and in accordance with the framework of Dirac's argument. However, immediately after the above assertion, HT discuss an infinitesimal reparametrization of a finite segment of the dynamical orbit in a manner suggestive of the alternative interpretation that H is supposed to perform a transformation of an existing parametrization of part or all of the system history. But as we pointed out at the start of Sec. IID, H generates the physical evolution and simultaneously some parametrization for it. It cannot therefore implement a reparametrization. In any case, as we have already noted, the label λ has no physical significance whatsoever. All the difficulties surrounding the status of H, in RI theories at least, are immediately resolved by the simple observation that Dirac's theorem does not hold for them.

We also comment that "the motion is the unfolding of a gauge transformation" is a categoric statement, the like of which is not found in Dirac's *Lectures*. As we have noted, for some reason or other, Dirac never uses such language or anything equivalent to it when discussing Hamiltonian constraints even though, as primary first-class constraints, one might have expected him to apply his 'proof' to them.

IV. CONCLUSION

It is our conviction that much confusion has entered the discussion of the meaning of constraints because of the manner in which Dirac chose to develop his theory of them. The

great strength of his theory is its wonderful generality: the mathematical formalism applies with complete rigor to any Lagrangian with a many-to-one relationship between the velocities and the canonical momenta. This leads inevitably to the appearance of constraints in the Hamiltonian representation of the theory. The mathematical results that follow from this are not in doubt, and their utility has been proved again and again, especially in connection with the property of constraints being first or second class, which has come to be seen as the distinction par excellence between constraints.

It is however striking that Dirac, followed by HT, makes virtually no attempt to explain why the constraints occur in the first place. Speaking of the arbitrary functions that appear in the solutions of the equations of motion due to the presence of the constraints, Dirac says: "These arbitrary functions of the time must mean that we are using a mathematical framework containing arbitrary features, for example a coordinate system which we can choose arbitrarily, or the gauge in electrodynamics." The example of Jacobi's principle shows that this is only the superficial explanation for the appearance of constraints and associated arbitrary functions. The real reason for the Hamiltonian constraint has nothing whatever to do with the use of a mathematical framework containing arbitrary features. The underlying physical origin of the constraint is the fact that the initial condition for a geodesic is a point in some space and a direction at that point. We have made this clear in our discussion of Jacobi's principle. Once this has been understood, it is evident that the corresponding Hamiltonian constraint generates real physical change. This conclusion does not at all mean that all constraints generate real change. Dirac's proof is clearly valid in many familiar cases. What this example does show is that each constraint needs to be examined on its merits and the underlying reason for its occurrence established.

This will be the guiding principle of our next paper, but it may be helpful to say something here already about the relativistic particle, which is also governed by a quadratic constraint. Given any geodesic principle in a space of n dimensions, one can always take one of the variables (at least for a finite part of the evolution) as the independent variable. One then obtains an unconstrained system with n-1 dependent variables. Looked at from this point of view, one may say that the Hamiltonian constraint tells us that our system has one fewer 'true degrees of freedom' than the original n. This is true for both the relativistic particle and Jacobi's principle. In this respect, quadratic first-class constraints are no different from other first-class constraints: they tell us that the considered system can (generally with considerable difficulty) be described as a system with a reduced number of degrees of freedom.

In special relativity it is also true that at any two points on a particle worldline the particle states can be transformed into each other exactly by means of translation and boost symmetries. There is therefore a sense in which the state of a relativistic particle is physically equivalent to a single 'fiducial' state: $\mathbf{q} = (0,0,0,0)$, $\mathbf{p} = (1,0,0,0)$. In fact, this emphasizes how untypical, in being too simple, the relativistic particle is for the purposes of discussing the significance of Hamiltonian constraints. Even though the Jacobi action has similar symmetries, it is quite impossible to reduce the possible Jacobi states to one in this manner.

Moreover, this does not address the issue of what the Hamiltonian constraint *does*. If we are realistic and consider a particle in a nonempty Minkowski space, it certainly occupies observationally different positions at different points along its worldline irrespective of the inertial frame in which it is viewed. Further, translation along the geodesic is driven by a Hamiltonian constraint. Indeed, a relativistic particle interacting with an external scalar

field has essentially the same form of the action as for Jacobi's principle. We therefore conclude that, although all constraints indicate that we can in principle describe the system with fewer degrees of freedom, there are constraints that generate a physical change of the state of the system whereas other constraints – the ones for which Dirac's theorem holds – do not.

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