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No Lagrangian? No quantization!

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This work starts with classical equations of motion and sets very general quantization conditions (commutation relations). It is proved that these conditions imply that the equations of motion are equivalent to the Euler–Lagrange equations of a Lagrangian L. The result is a generalization of work by Feynman, recently reported by Dyson [Am. J. Phys. **58**, 209–211 (1990)]. The Lagrangian L need not be unique. Examples are given, including classical equations that do not come from a Lagrangian and therefore cannot be quantized consistently.

I. INTRODUCTION

Dyson recently reported that Feynman had derived the Maxwell equations from quantization conditions set on a classical system.[1] The idea is to start with classical equations of motion in the form

$$m\ddot{x}^i = f^i \tag{1.1}$$

and give commutation relations for the operators X^i and $\dot{X}^i$ corresponding to the classical coordinates and velocities:

$$[X^i, X^j] = 0, \tag{1.2}$$

$$m[X^i, \dot{X}^j] = i\hbar\delta^{ij}. \tag{1.3}$$

Then Feynman proved that the force f^i is of the form of the electromagnetic Lorentz force, the electromagnetic field being derivable from an electromagnetic scalar/vector potential. Feynman wanted to develop a procedure to quantize (1.1) without resort to a Lagrangian (or Hamiltonian), according to Dyson, but the result was that (1.1) is indeed the Euler–Lagrange equations of a Lagrangian L, and, moreover, L is of electromagnetic form. (This result does not imply that the equations are Lorentz invariant; see our discussion at the end of example 1, Sec. IV.)

In this paper, we prove that the existence of a Lagrangian for (1.1) essentially comes from (1.2); the explicit form of the Lagrangian is from (1.3). Dyson's paper dealt primarily with (1.3), but here we will replace those conditions with more general ones. Thus Feynman's hope to quantize without a Lagrangian was doomed when he set the very reasonable conditions that the coordinates commute.

Sections II and III contain a more explicit statement of the problem and the proof of the existence of a Lagrangian from quantization requirements. Section II deals with second-order equations of motion; Sec. III deals with first-order equations. In Sec. IV we present several examples; among others we include equations that do not come from a Lagrangian and show that any quantization of them must be inconsistent. In this paper we deal with unconstrained, regular systems, and in the last section we both remark on our results and comment on possible extensions.

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II. QUANTIZATION REQUIRES LAGRANGIAN

Consider the classical equations of motion (all masses are set to unity for convenience; Latin indices range from 1 to N):

$$\ddot{x}^i = f^i(x,\dot{x},t). \tag{2.1}$$

The inverse problem of classical mechanics is to determine whether these equations either are the Euler–Lagrange equations based on a Lagrangian L or are equivalent to such equations. In other words, what is sought is a nonsingular matrix (possibly a function of position, velocity, and time) w_{ij} and a function $L(x,\dot{x},t)$ so that

$$\begin{aligned} w_{is}(\ddot{x}^s - f^s) &= \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} \\ &= \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^s}\ddot{x}^s + \frac{\partial^2 L}{\partial \dot{x}^i \partial x^s}\dot{x}^s + \frac{\partial^2 L}{\partial \dot{x}^i \partial t} - \frac{\partial L}{\partial x^i}. \end{aligned} \tag{2.2}$$

The conditions for the existence of w_{ij} and L are called the Helmholtz conditions.[2-4] These requirements are

$$w_{ij} = w_{ji}; \tag{2.3a}$$

$$\frac{\partial w_{ij}}{\partial \dot{x}^k} = \frac{\partial w_{ik}}{\partial \dot{x}^j}; \tag{2.3b}$$

$$\frac{D}{dt} w_{ij} = -\frac{1}{2} w_{ik}\frac{\partial f^k}{\partial \dot{x}^j} - \frac{1}{2} w_{jk}\frac{\partial f^k}{\partial \dot{x}^i}; \tag{2.3c}$$

$$\frac{1}{2}\frac{D}{dt}\left(w_{ik}\frac{\partial f^k}{\partial \dot{x}^j} - w_{jk}\frac{\partial f^k}{\partial \dot{x}^i}\right) = w_{ik}\frac{\partial f^k}{\partial x^j} - w_{jk}\frac{\partial f^k}{\partial x^i}; \tag{2.3d}$$

where the on-shell time-derivative is defined by

$$\frac{D}{dt} \equiv \frac{\partial}{\partial t} + \dot{x}^s\frac{\partial}{\partial x^s} + f^s\frac{\partial}{\partial \dot{x}^s}. \tag{2.4}$$

These conditions are both necessary and sufficient: If a Lagrangian L exists, then w_{ij} is given by

$$w_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}, \tag{2.5}$$

and it satisfies the Helmholtz conditions (2.3). Conversely, if for a given f^i a solution w_{ij} to Eqs. (2.3) can be found, then not only does L exist, but it can be constructed explicitly and uniquely (up to a total time derivative and an overall multiplicative constant). Note that a solution of (2.3) may not be unique.

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Suppose one wants to quantize the system described by (2.1). We will show that very reasonable quantum mechanical commutation relations yield a solution of the conditions (2.3), and therefore quantum mechanics implies the existence of a Lagrangian.

To quantize (2.1) means, at the very least, that to x^i and $\dot{x}^i$ correspond operators X^i and $\dot{X}^i$ (on some Hilbert space). If a Lagrangian L did exist, one would set commutation relations among the coordinate operators X^i and operators corresponding to conjugate momenta, but our program is to prove the existence of L rather than to assume it. We therefore start simply by requiring that the coordinate operators commute:

$$[X^i, X^j] = 0. \tag{2.6}$$

The time derivative of this relation shows that the commutators of the coordinate operators and the velocity operators form a symmetric array:

$$[X^i, \dot{X}^j] \equiv i\hbar G^{ij} = i\hbar G^{ji}. \tag{2.7}$$

In general, G^{ij} will be an operator, a function of X^i, $\dot{X}^i$, and t. The classical limit of an expression is obtained by letting $\hbar$ go to zero. More precisely, a quantum quantity that depends on the basic operators X^i and $\dot{X}^i$ (and on t) will, in general, involve various powers of $\hbar$; its classical analog is the coefficient of the lowest power of $\hbar$ expressed as a classical function of $x^i, \dot{x}^i, t$. The classical analog of a quantum mechanical equation is then a relation among the coefficients of the lowest power of $\hbar$ appearing in the equation.

The classical analog of G^{ij} will be denoted g^{ij}. A very reasonable quantum mechanical condition is to require that g^{ij} be a nonsingular matrix; we will prove that its matrix inverse w_{ij} satisfies the Helmholtz conditions.

The requirement that g^{ij} be nonsingular is a natural one for unconstrained systems (regular systems). The Lagrangian for a constrained system will typically result in a mass matrix $\partial^2 L / \partial \dot{x}^i \partial \dot{x}^j$ which is singular. Then the inverse (which would correspond to g^{ij}) does not exist. If, instead, commutation relations G^{ij} are taken so that g^{ij} is singular, then it is w_{ij} which does not exist.

The inverse of g^{ij}, namely w_{ij}, certainly obeys (2.3a). The first part of the proof of the remaining conditions is to start with the Jacobi identity

$$[X^i, [X^j, \dot{X}^k]] + [\dot{X}^k, [X^i, X^j]] + [X^j, [\dot{X}^k, X^i]] = 0,$$

which is the same as

$$[X^i, G^{jk}] = [X^j, G^{ik}]. \tag{2.8}$$

This equation will lead to (2.3b), but first we examine the commutator of X^i and a general function of the basic operators $X^i, \dot{X}^i$ (this commutator is the Fréchet derivative).[5] The function will have terms that are products of the basic operators, a typical term being $\dot{X}^s A \dot{X}^t B \cdots \dot{X}^u$, where $A, B, \ldots$, are functions of the X^i's only. The commutator of this term with X^i is

$$\begin{aligned}[X^i, \dot{X}^s A \dot{X}^t B \cdots \dot{X}^u] &= i\hbar [G^{is} A \dot{X}^t B \cdots \dot{X}^u + \dot{X}^s A G^{it} B \cdots \dot{X}^u \\ &\quad + \cdots + \dot{X}^s A \dot{X}^t B \cdots G^{iu}].\end{aligned} \tag{2.9}$$

The classical analog of $\dot{X}^s A \dot{X}^t B \cdots \dot{X}^u$ is $ab \cdots \dot{x}^s \dot{x}^t \cdots \dot{x}^u$. The classical analog of the commutator (2.9) is therefore

$$[X^i, \dot{X}^s A \dot{X}^t B \cdots \dot{X}^u] \to g^{ik} \frac{\partial}{\partial \dot{x}^k} (ab \cdots \dot{x}^s \dot{x}^t \cdots \dot{x}^u). \tag{2.10}$$

Equation (2.8) therefore implies that the classical g^{ij} obeys

$$g^{is} \frac{\partial g^{jk}}{\partial \dot{x}^s} = g^{js} \frac{\partial g^{ik}}{\partial \dot{x}^s}. \tag{2.11}$$

Derivatives of the inverse w_{ij} of g^{ij} are related to those of g^{ij} (∂ being a general derivative operator) by

$$\partial w_{ij} = - w_{is} \partial g^{st} w_{tj}, \tag{2.12}$$

and so (2.11) implies that w_{ij} obeys (2.3b).

The second time derivative of (2.6) yields

$$i\hbar S^{ij} \equiv [\dot{X}^i, \dot{X}^j] = -\tfrac{1}{2}[X^i, F^j] - \tfrac{1}{2}[F^i, X^j], \tag{2.13}$$

where F^i is the operator analog of the classical force f^i. The time derivative of G^{ij} yields

$$i\hbar \frac{D}{dt} G^{ij} = [\dot{X}^i, \dot{X}^j] + [X^i, F^j]. \tag{2.14}$$

These equations imply

$$i\hbar \frac{D}{dt} G^{ij} = \frac{1}{2} [X^i, F^j] - \frac{1}{2} [F^i, X^j]. \tag{2.15}$$

The classical analog of this equation involves $\partial f^i / \partial \dot{x}^j$:

$$\frac{D}{dt} g^{ij} = \frac{1}{2} \frac{\partial f^j}{\partial \dot{x}^s} g^{si} + \frac{1}{2} \frac{\partial f^i}{\partial \dot{x}^s} g^{sj}, \tag{2.16}$$

and (2.3c) follows when (2.12) is used. The classical analog of (2.14) will itself be useful:

$$\frac{D}{dt} g^{ij} = s^{ij} + \frac{\partial f^j}{\partial \dot{x}^s} g^{si}, \tag{2.17}$$

where s^{ij} is the classical counterpart of S^{ij}; according to (2.13), it is

$$S^{ij} \to s^{ij} = -\frac{1}{2} g^{ik} \frac{\partial f^j}{\partial \dot{x}^k} + \frac{1}{2} g^{jk} \frac{\partial f^i}{\partial \dot{x}^k}. \tag{2.18}$$

The time derivative of S^{ij} yields

$$i\hbar \frac{D}{dt} S^{ij} = [\dot{X}^i, F^j] + [F^i, \dot{X}^j]. \tag{2.19}$$

The right side involves derivatives $\partial f^i / \partial x^j$ and $\partial f^i / \partial \dot{x}^j$ in the classical analog; compare (2.10):

$$[\dot{X}^i, F^j] \to -\frac{\partial f^j}{\partial x^k} g^{ik} + \frac{\partial f^j}{\partial \dot{x}^k} s^{ik}. \tag{2.20}$$

Note that

$$t_{ij} \equiv w_{ia} w_{bj} s^{ab} = \frac{1}{2} w_{ik} \frac{\partial f^k}{\partial \dot{x}^j} - \frac{1}{2} w_{jk} \frac{\partial f^k}{\partial \dot{x}^i} \tag{2.21}$$

is the quantity that appears in the derivative on the left side of (2.3d). The time derivative of S^{ij} therefore implies

$$\begin{aligned}\frac{D}{dt}(t_{ab} g^{ia} g^{bj}) &= -\frac{\partial f^j}{\partial x^k} g^{ik} - \frac{\partial f^j}{\partial \dot{x}^k} s^{ki} + \frac{\partial f^i}{\partial x^k} g^{jk} + \frac{\partial f^i}{\partial \dot{x}^k} s^{kj} \\ &= g^{ia} g^{bj} \frac{D}{dt} t_{ab} + t_{ab} g^{bj} \frac{D}{dt} g^{ia} + t_{ab} g^{ia} \frac{D}{dt} g^{bj}.\end{aligned} \tag{2.22}$$

Use of (2.17) allows this equation to be written

$$g^{ia}g^{bj}\frac{D}{dt}t_{ab}=\frac{\partial f^i}{\partial x^k}g^{jk}-\frac{\partial f^j}{\partial x^k}g^{ik}, \tag{2.23}$$

and it is immediately seen that (2.3d) follows.

We have thus proved that a Lagrangian L exists. The form of L is uniquely determined by G^{ij} up to an overall multiplicative constant and an addition of a total time derivative.

III. FIRST-ORDER SYSTEMS

Consider the first-order system (Greek indices range from 1 to $2N$):

$$\dot{y}^\alpha=\phi^\alpha(y,t). \tag{3.1}$$

The second-order system (2.1) may of course be put in this form; one way is to define

$$y^i\equiv x^i \quad \text{and} \quad y^{i+N}\equiv\dot{x}^i. \tag{3.2}$$

(If a Lagrangian L exists for 2.1, an alternative method is to define $y^i=x^i$ and $y^{i+N}=p_i$, where $p_i\equiv\partial L/\partial\dot{x}^i$ are the components of the momentum.)

If a Lagrangian $M(\dot{y},y,t)$ exists for (3.1), it must have the form[6]

$$M=m_\alpha\dot{y}^\alpha+m_0, \tag{3.3}$$

where m_α and m_0 are functions of y^α,t only. The Euler–Lagrange equations of motion for this Lagrangian are ($_{,\mu}\equiv\partial/\partial y^\mu$ and $_{,0}\equiv\partial/\partial t$)

$$(m_{\mu,\alpha}-m_{\alpha,\mu})\dot{y}^\mu-m_{\alpha,0}+m_{0,\alpha}=0. \tag{3.4}$$

The original system thus comes from a Lagrangian if a $\sigma_{\alpha\mu}$, an m_α, and an m_0 can be found such that

$$\sigma_{\alpha\mu}(\dot{y}^\mu-\phi^\mu)=(m_{\mu,\alpha}-m_{\alpha,\mu})\dot{y}^\mu-m_{\alpha,0}+m_{0,\alpha}=0.$$

It is clear that $\sigma_{\alpha\mu}$ must be antisymmetric, nonsingular, and curl-free:

$$\sigma_{\alpha\mu}=-\sigma_{\mu\alpha}, \tag{3.5a}$$

$$\det(\sigma_{\alpha\mu})\neq 0, \tag{3.5b}$$

$$\sigma_{\alpha\mu,\rho}+\sigma_{\mu\rho,\alpha}+\sigma_{\rho\alpha,\mu}=0. \tag{3.5c}$$

Furthermore, these conditions are sufficient for the existence of m_μ. Since

$$m_{0,\alpha}=-\sigma_{\alpha\mu}\phi^\mu+m_{\alpha,0},$$

the existence of m_0 requires that

$$(\sigma_{\alpha\mu}\phi^\mu-m_{\alpha,0})_{,\beta}-(\sigma_{\beta\mu}\phi^\mu-m_{\beta,0})_{,\alpha}=0.$$

This condition is more conveniently written [using (3.5c)] as

$$\frac{D}{dt}\sigma_{\alpha\beta}=\sigma_{\beta\mu}\phi^\mu_{,\alpha}-\sigma_{\alpha\mu}\phi^\mu_{,\beta}, \tag{3.5d}$$

where the on-shell derivative is defined as

$$\frac{D}{dt}\equiv\frac{\partial}{\partial t}+\phi^\sigma\frac{\partial}{\partial y^\sigma}. \tag{3.6}$$

The conditions (3.5) guarantee the existence of a Lagrangian for the general first-order system. They can always be satisfied, at least locally.[6] Nevertheless, if (3.2) has been used to convert the second-order system (2.1) into a first-order system, the existence of M does not mean that a Lagrangian L for (2.1) can be found. It may be thought that starting with (3.1), one simply uses (3.2) to define variables $x^i,\dot{x}^i$. The substitution of these variables into M produces a Lagrangian of the form[7]

$$\widetilde{M}=m_i\dot{x}^i+m_{i+N}\ddot{x}^i+m_0. \tag{3.7}$$

This Lagrangian yields Euler–Lagrange equations of third order, since it is linear in $\ddot{x}^i$. The condition that $\widetilde{M}$ give Euler–Lagrange equations of second order is that the coefficient of $\dddot{x}^i$ vanish:[7–9]

$$\sigma_{i+N,j+N}=\frac{\partial m_{j+N}}{\partial\dot{x}^i}-\frac{\partial m_{i+N}}{\partial\dot{x}^j}=0. \tag{3.8}$$

Quantization of (3.1) requires that operators Y^α be defined, along with their commutation relations. We define $H^{\alpha\beta}$ to be

$$i\hbar H^{\alpha\beta}\equiv[Y^\alpha,Y^\beta]. \tag{3.9}$$

Let the classical analog of $H^{\alpha\beta}$ be $\eta^{\alpha\beta}$; a reasonable quantum mechanical requirement is that it be a nonsingular matrix. The matrix $\eta^{\alpha\beta}$ must satisfy the classical equation which comes from the time derivative of (3.9):

$$\frac{D}{dt}\eta^{\alpha\beta}=\eta^{\alpha\sigma}\phi^\beta_{,\sigma}-\eta^{\beta\sigma}\phi^\alpha_{,\sigma}. \tag{3.10}$$

Define the inverse of $\eta^{\alpha\beta}$ to be a matrix $\sigma_{\alpha\beta}$ which satisfies

$$\sigma_{\beta\alpha}\eta^{\alpha\gamma}=\delta^\gamma_\beta. \tag{3.11}$$

The derivatives of $\sigma_{\alpha\beta}$ are related to those of $\eta^{\alpha\beta}$ by

$$\eta^{\mu\alpha}\partial\sigma_{\alpha\beta}\eta^{\beta\gamma}=-\partial\eta^{\mu\gamma}. \tag{3.12}$$

It is therefore clear that (3.10) implies that $\sigma_{\alpha\beta}$ obeys (3.5d); it certainly obeys (3.5a) and (3.5b). The proof of (3.5c) for $\sigma_{\alpha\beta}$ follows from the Jacobi identity

$$[Y^\alpha,[Y^\beta,Y^\gamma]]+[Y^\beta,[Y^\gamma,Y^\alpha]]+[Y^\gamma,[Y^\alpha,Y^\beta]]=0. \tag{3.13}$$

Thus the quantum mechanical assignment of operators Y^α implies the existence of a Lagrangian of the form (3.3). The further requirement (3.8) means $H^{\alpha\beta}$ obeys

$$H^{ij}=0, \tag{3.14}$$

namely, that the coordinate operators $X^i=Y^i$ commute; compare (2.6).

The Lagrangian (3.3) can be used to give a Hamiltonian, and by Darboux's theorem, canonical coordinates $\{q^i,p_i\}$ may be chosen to put the equations of motion in the usual Hamiltonian form. If the first-order system does come from a second-order system, it is the second-order system which determines the coordinates $\{x^i\}$ of configuration space, and the canonical coordinates $\{q^i\}$ need not be directly related to the $\{x^i\}$.

IV. EXAMPLES

We present several examples, including the electromagnetic case treated by Feynman[1] and an example which has equations of motion not derivable from a Lagrangian. The above discussion shows that one way to a quantum theory is to give the force f^i and to define the matrix G^{ij} consistently, all other quantities, such as w_{ij}, s^{ij}, t_{ij}, being then derivable. An alternative approach is to give a form for G^{ij} and to see what forces f^i are compatible with it.

Example 1: Electromagnetic case. In this case, G^{ij} is given by (1.3). The classical matrix w_{ij} is proportional to δ_{ij}, and so the Lagrangian obeys

$$\frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} = \delta_{ij}, \tag{4.1}$$

if L exists (for convenience we take $m = 1$). If L exists, it must have the form

$$L = \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + a_i\dot{x}^i + a_0. \tag{4.2}$$

The existence of L, however, does follow from the Helmholtz conditions: Equations (2.3a) and (2.3b) are trivially satisfied. Equation (2.3c) implies that

$$\frac{\partial f^i}{\partial \dot{x}^j} = -\frac{\partial f^j}{\partial \dot{x}^i}, \tag{4.3}$$

but this equation follows from the above form for L (the space metric tensor is δ_{ij}, and so superscript and subscript indices are the same):

$$f^i = (a_{s,i} - a_{i,s})\dot{x}^s + a_{0,i} - a_{i,0} \tag{4.4}$$

(where $_{,i} \equiv \partial/\partial x^i$ and $_{,0} \equiv \partial/\partial t$). The last condition (2.3d) also immediately follows from the forms of w_{ij} and f^i.

(Although L is of electromagnetic form, and f^i has the form of the Lorentz force, the relation between f^i and the pair a_i, a_0 implies only half of the Maxwell equations. The other Maxwell equations serve to define an electromagnetic charge density ρ and current J_i, but they involve a spacetime metric. Here there is only a space metric, proportional to δ_{ij}; a parameter with units of velocity must be given to define the spacetime metric; and this parameter is arbitrary. Consequently, Lorentz invariance does not appear in these equations.)

Example 2. Suppose the system is one dimensional, with equations of motion

$$\ddot{x} - f = 0. \tag{4.5}$$

The quantum condition $[X,X] = 0$ is trivial. The various time derivatives of this condition do not restrict G, defined by $[X,\dot{X}] = i\hbar G$, in any way. The time derivative of G must satisfy

$$i\hbar \frac{D}{dt} G = [X,F] \tag{4.6}$$

where F is the quantum equivalent of the force f. This relation yields (2.3c):

$$\frac{D}{dt} w = -w \frac{\partial f}{\partial \dot{x}}. \tag{4.7}$$

The requirements (2.3a), (2.3b), and (2.3d) are trivial. It is always possible to find infinitely many solutions to (4.7).[2] The corresponding quantum mechanical commutation relations may give different quantum theories.[2]

Example 3. This example is a system which comes from a Lagrangian, but the Lagrangian is not always unique (up to multiplication by a constant and addition of a total time derivative):[8] Consider a system that is derivable from a rotationally symmetric potential v, depending on $r \equiv (\delta_{ij}\dot{x}^i\dot{x}^j)^{1/2}$:

$$\ddot{x}^i = -(v'/r)x^i, \tag{4.8}$$

where $v' \equiv dv/dr$. One Lagrangian is

$$L = \tfrac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j - v. \tag{4.9}$$

The derivatives of the commutation relations $[x^i,x^j] = 0$, imply that $[\dot{x}^i,\dot{x}^j] = 0$, and that

$$[\dot{x}^i,(v'/r)x^j] - [\dot{x}^j,(v'/r)x^i] = 0. \tag{4.10}$$

This last relationship implies that

$$(g^{is}x^s x^j - g^{js}x^s x^i)(v''r - v') = 0. \tag{4.11}$$

Of course $g^{ij} = \delta^{ij}$ is one solution of this equation for any function v. If $v''r - v' = 0$, many solutions are possible; the general form of v which satisfies this relation is

$$v = ar^2 + b, \tag{4.12}$$

with a and b constants.

Suppose, however, that v has a general form, so that $v''r - v' \neq 0$. If $N = 1$, (4.11) is identically zero; this fact corresponds to our Example 2. If $N = 2$, then (4.11) and its derivatives determine a unique solution $g^{ij} = \delta^{ij}$ up to a multiplicative constant of the motion. As a consequence of (2.11), this constant of the motion is a numerical constant; the Lagrangian L is thus essentially unique.[8,10] If $N \geqslant 3$, the solution for g^{ij} is not unique, and other Lagrangians exist which are equivalent to L but not obtainable from L by addition of a total time derivative or by multiplication by a constant.[8]

Example 4. Our final example will illustrate the difference between second-order and first-order systems. Consider the following equations of motion in two variables x,y:[6]

$$\ddot{x} = -\dot{y}; \quad \ddot{y} = -y. \tag{4.13}$$

These equations are not derivable from a Lagrangian.[6,10] Assume that $[x,y] = 0$. Then the second time derivative of this requirement implies that

$$[y,\dot{y}] = -2[\dot{x},\dot{y}], \tag{4.14}$$

and the derivative of this requirement yields

$$[y,\dot{x}] = [x,\dot{y}] = 0. \tag{4.15}$$

One final derivative yields the equation $[y,\dot{y}] = [\dot{x},\dot{y}] = 0$. (No information can be obtained about $[x,\dot{x}]$.) It is therefore impossible to quantize the system by demanding that the matrix of $[x^i,\dot{x}^j]$ be nonsingular.

We rewrite this system in first-order form as

$$\dot{x} = z; \quad \dot{y} = w; \quad \dot{z} = -w; \quad \dot{w} = -y. \tag{4.16}$$

Commutation relations for this system involve the antisymmetric matrix $\sigma_{\alpha\beta}$, and one simple assignment is

$$\sigma_{\alpha\beta} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.17}$$

The corresponding Lagrangian is

$$L = (y + z)\dot{x} + w\dot{z} + \tfrac{1}{2}(w^2 - 2yz - z^2). \tag{4.18}$$

This system may be put into a normal form by defining new coordinates $\{q^1,q^2,p_1,p_2\}$ by

$$q^1 \equiv x; \quad p_1 \equiv y+z; \quad q^2 \equiv z; \quad p_2 \equiv w. \tag{4.19}$$

The Lagrangian in these coordinates is

$$L = p_1 \dot{q}^1 + p_2 \dot{q}^2 + \tfrac{1}{2}((p_2)^2 + (q^2)^2 - 2q^2 p_1). \tag{4.20}$$

Needless to say, the corresponding Hamiltonian is

$$H = -\tfrac{1}{2}((p_2)^2 + (q^2)^2 - 2q^2 p_1). \tag{4.21}$$

It is easily verified that the Hamilton equations $\dot{p}_i = -\partial H/\partial q^i, \dot{q}^i = \partial H/\partial p_i$ are equivalent to the original system. The canonical quantum mechanical commutation relations may be set for the system in these phase space coordinates. Quantization, however, is achieved at the cost of taking as "coordinates" (which correspond to commuting operators) q^1 and q^2, namely x and $\dot{x}$ of the original second-order system. (Of course, p_1 and p_2 correspond to commuting operators; these variables correspond to $y + \dot{x}$ and $\dot{y}$ of the original system.)

The first-order system does not have a unique Lagrangian and therefore not a unique Hamiltonian.[6] However, any quantization conditions, when expressed in terms of the variables of the original second-order system, will produce nonstandard results because the original system does not have a Lagrangian. It is worth noting that this result arises because the original system (4.13) defined which coordinates are the coordinates of configuration space.

V. CONCLUSIONS

Why on earth would anyone want to quantize a system without reference to a Lagrangian or a Hamiltonian? One answer is that it may be that a classical system is most easily defined in terms of its equations of motion. It may be difficult to determine whether a Lagrangian exists, whether it is unique if it does exist, and what its form is even if it exists and is unique. Quantization that bypasses direct reference to a Lagrangian may have practical advantages. A second answer is that more general methods than usual quantization may lead to new ways of looking at physics. This was apparently Feynman's motivation.[1]

Feynman employed a scheme that led to equations of Lorentz-force form. Our original motivation for generalizing Feynman's technique was to see how other equations of motion could be handled. In the process, we realized that the power of the commutator is immense: Because it is an antisymmetric product of operators, the Jacobi identity follows immediately, and the time derivative of a commutator $[A,B]$ involves the operators themselves and their time derivatives:

$$[A,B]^{\cdot} = [\dot{A},B] + [A,\dot{B}].$$

These are the ingredients that enter the Helmholtz conditions for the existence of a Lagrangian.

Many people have viewed our result as surprising, but it is not miraculous, of course. One (who meets Dyson's[1] criterion of "young ... educated in the 1980s") was at first "disparaging," but our result is not trivial, either. To express equations of motion in first-order form (3.1) as the Euler–Lagrange equations for a Lagrangian M (3.3) involves the straightforward integrability conditions (3.5). These conditions are for the existence of a nonsingular symplectic form $\sigma_{\mu\nu}$ that must be compatible with the force ϕ^α. The nonsingularity of $\sigma_{\mu\nu}$ is equivalent to the natural requirement that the commutators $[Y^\alpha, Y^\beta]$ form a nonsingular matrix. The integrability of $\sigma_{\mu\nu}$ means that its curl must vanish; this requirement comes from the Jacobi identity, which as we said follows automatically for a commutator. The compatibility of $\sigma_{\mu\nu}$ with the forces comes from the time derivative of the commutators.

A Lagrangian for a second-order system (2.1) involves something more than in the first-order case: A second-order system has the explicit coordinates that describe configuration space built in. If these coordinates are used in the first-order formalism, the result is the Lagrangian $\widetilde{M}$ (3.7). Since there are only half the variables to vary in the second-order case as in the first-order case, the resulting Euler–Lagrange equations are of third order unless the requirement (3.8) is met.[7–9] This requirement translates to a requirement on the commutation relations, namely (3.14).

It will be interesting to extend our results to singular (constrained or gauge) systems as well as to field theory. Mechanical singular systems may include relativistic test point (or spinning) particles moving on gravitational backgrounds as well as a ball rolling on a table. The main examples to be considered in field theory are the scalar field, the electromagnetic field, the sigma model, the Dirac field, the Korteweg–de Vries equation, and the sine-Gordon field.

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