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Path Integral for a Harmonic Oscillator with Time-Dependent Mass and Frequency

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Abstract: The exact solutions to the time-dependent Schrodinger equation for a harmonic oscillator with time-dependent mass and frequency were derived in a general form. The quantum mechanical propagator was calculated by the Feynman path integral method, while the wave function was derived from the spectral representation of the obtained propagator. It was shown that the propagator and the wave function depended on the s solution of a classical oscillator, in which the amplitude and phase satisfied the auxiliary equations. To demonstrate the derivation of the solution from our auxiliary equations, exponential and periodic functions of mass with constant frequency were imposed to evaluate the propagator and wave function for the Caldirola-Kanai and pulsating mass oscillators, respectively.

Keywords: Path integral, propagator, wave function, a harmonic oscillator with time-dependent mass and frequency.

Introduction

In recent years, the study of Hamiltonian with explicitly time-dependent coefficients becomes very popular.[1-11] The mathematical challenge and important applications in various areas of physics, such as quantum optics,[12] cosmology,[13] and nanotechnology,[14] are the main reasons for intensive studies. The most common problem in this area is the harmonic oscillator with time-dependent frequency and/or mass. The harmonic oscillator with time-dependent frequency is the first exactly solved problem.[15] The standard method for solving the time-dependent problems is the Lewis-Riesenfeld (LR) invariant operator method.[2,3,5,11,15] This method is based on constructing an invariant operator and writing Schrodinger's wave function in terms of invariant operator eigenstates connecting with time-dependent phase factor.

However, in the case of a harmonic oscillator with time-dependent frequency and mass the LR-invariant operator method has some difficulty.[16-17] In 1992, Dantas and et al.[16] constructed an invariant operator from the canonical transformation variables and transformed the invariant operator to a simple harmonic oscillator operator by unitary transformation. However, their wave functions satisfied the Schrodinger's equation only in the case of constants mass, but were not applicable in general case of time-dependent mass. Later in 1997, Pedrosa[17] revised this problem by modifying the invariant operator and used another unitary operator to include the time-dependent mass parameter. His result presented the first wave function for the harmonic oscillator with time-dependent mass and frequency. Finally, Ciftja[18] proposed an alternative method by assuming the Schrodinger's wave function in terms of the Gaussian function with time-dependent coefficients and using the space-time transformation to reduced the problem to a simple harmonic oscillator. He suggested that there should be some attempt to develop an easier method than the LR-invariant operator method to tackle the time-dependent problem.

The aim of this paper is to derive the propagator and wave function for a harmonic oscillator with time-dependent mass and frequency in any function form as described by the Hamiltonian

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2}m(t)\omega^2(t)x^2, \tag{1}$$

where $m(t)$ and $\omega(t)$ are the time-dependent mass and frequency, respectively.

Our developed method is not based on the Hamiltonian and solving the differential equations as described in previously reported articles,[16-18] but base on the Lagrangian and solving the integral equation by the Feynman path integral approach.[19] In this formulation the time-dependent Schrodinger's equation is replaced by integral equation

$$\psi(x'',t'') = \int K(x'',t'';x',t')\psi(x',t')dx', \tag{2}$$

where kernel K or propagator represents the transition probability amplitude for the initial wave function $\psi(x',t')$ propagating to the final wave function $\psi(x'',t'')$ in the space-time configuration.

As first stated by Feynman,[19] the propagator can be defined by the path integral of

$$K(x'',t'';x',t') = \int e^{\frac{i}{\hbar}S[x(t)]} D[x(t)], \tag{3}$$

where the measure $D[x(t)]$ denotes the sum over all path between (x',t') and $\psi(x'',t'')$. The function $S[x(t)]$ is an action defined by

$$S[x(t)] = \int_{t'}^{t''} L(\dot{x},x,t)dt, \tag{4}$$

where $L(\dot{x},x,t)$ is the Lagrangian of the system.

The propagator can be related to the time-dependent Schrodinger's wave function by

$$K(x'',t'';x',t') = \sum_{n=0}^{\infty} \psi_n^*(x',t')\psi_n(x'',t''). \tag{5}$$

To solve this formulation, the propagator from the Feynman definition in Eq. (3) is calculated and then the spectral representation of propagator in Eq. (5) is applied to derive the time-dependent Schrodinger's wave function. It is well known that the quantum solutions for the time-dependent problems[16-18] depend on the undetermined auxiliary equation. Hence, to demonstrate the derivation of the explicit form of the propagator and wave function, the exponential and periodic function of mass with constant frequency is employed as an example in the cases of a Caldirola-Kanai oscillator and a harmonic oscillator with strongly pulsating mass respectively.

In section 2, our derivation of the path integral for a harmonic oscillator with time-dependent mass and frequency is described. In section 3, the wave functions for a harmonic oscillator with time-dependent mass and frequency are derived in a general form. To demonstrate particular cases, our general solutions are applied to derive the propagator and wave functions for the well-known Caldirola-Kanai oscillator and the harmonic oscillator with strongly pulsating mass in section 4 and 5, respectively, and the conclusion is given in section 6.

The Path Integral For a Harmonic Oscillator with Time-Dependent Mass and Frequency

The Lagrangian associated to the Hamiltonian in Eq. (1) is

$$L = \frac{1}{2}m(t)\dot{x}^2 - \frac{1}{2}m(t)\omega^2(t)x^2. \tag{6}$$

The quadratic Lagrangian propagator can be separated into a pure function of time $F(t'',t')$ and the exponential function of classical action $S_{cl}(x'',t'';x',t')$ as suggested in Ref. 19

$$K(x'',t'';x',t') = F(t'',t')e^{iS_{cl}(x'',t'';x',t)/\hbar}. \tag{7}$$

Calculation of the function $F(t'',t')$ presented by Pauli, Morette, or Jones and Papadoupoulos[20-22] can be performed by the semiclassical approximation of path integral formula and then applied to Eq. (3)

$$F(t'',t') = \left[\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x'\partial x''}S_{cl}\left(x'',t'';x',t'\right)\right]^{\frac{1}{2}}. \tag{8}$$

Therefore, the crucial issue in propagator calculation in this system is to obtain the classical action $S_{cl}(x'',t'';x',t')$. By using the Euler-Lagrange equation for the Lagrangian in Eq. (6), the equation of motion can be written as

$$\ddot{x} + 2\frac{\dot{\eta}}{\eta}\dot{x} + \omega^2(t)x = 0, \tag{9}$$

where we define $\eta(t) = \sqrt{m(t)}$.

For physical reasons, let consider the solution to the equation of the motion in the form

$$x(t) = \rho(t)\,[A\cos\gamma(t) + B\sin\gamma(t)], \tag{10}$$

where $\rho(t) = \alpha(t)/\eta(t)$, $\gamma(t)$ refers to the amplitude and phase of the classical oscillatos and A and B are constants. The functions $\alpha(t)$, $\gamma(t)$ and $\eta(t)$ can be determined by substituting Eq. (10) into Eq. (9) and

$$\left[\frac{\ddot{\alpha}}{\eta} - \frac{\alpha\ddot{\eta}}{\eta^2} - \frac{\alpha\dot{\gamma}^2}{\eta} + \frac{\alpha\omega^2(t)}{\eta}\right]\{A\cos\gamma + B\sin\gamma\} - \left[\frac{\alpha\ddot{\gamma}}{\eta} + \frac{2\dot{\alpha}\dot{\gamma}}{\eta}\right]\{A\sin\gamma - B\cos\gamma\} = 0. \tag{11}$$

Since constants A and B can not vanish simultaneously, the functions $\alpha(t)$, $\gamma(t)$ and $\eta(t)$ have to satisfy the following auxiliary equations

$$\ddot{\alpha} - \dot{\gamma}^2\alpha + \left[\omega^2(t) - \frac{\ddot{\eta}(t)}{\eta(t)}\right]\alpha = 0 \tag{12}$$

and $$\ddot{\gamma} + \frac{2\dot{\alpha}\dot{\gamma}}{\alpha} = 0. \tag{13}$$

The constants A and B in Eq. (10) can be determined by imposing the boundary conditions of $x(t') = x'$ and $x(t'') = x''$. The classical path that connects the point of (x',t') and (x'',t'') can be written as

$$x_{cl}(t) = \frac{\alpha(t)}{\eta(t)\sin(\gamma''-\gamma')}\left\{\frac{\eta'x'}{\alpha'}\sin(\gamma''-\gamma) - \frac{\eta''x''}{\alpha''}\sin\left(\gamma'-\gamma\right)\right\}, \tag{14}$$

where the notations γ', α' and η' refer to $\gamma(t')$, $\alpha(t')$ and $\eta(t')$ respectively. The action can be calculated from the time integration of the Lagrangian from t' to t''.

$$S(x'',t'';x',t') = \int_{t'}^{t''} L(\dot{x},x,t)dt. \tag{15}$$

For the action of our general system, the Lagrangian in Eq. (6) is substituted into Eq. (15), and then integrated by parts of the first term on the right hand side of Eq. (6) using the equation of motion in Eq. (9).

The classical action can be written as

$$S_{cl}(x'',t'';x',t') = \frac{m''}{2} x_{cl}'' \dot{x}_{cl}'' - \frac{m'}{2} x_{cl}' \dot{x}_{cl}'. \tag{16}$$

Substituting the classical paths of Eq. (14) into Eq. (16), the classical action becomes

$$S_{cl}(x'',t'';x',t') = \frac{m''x''^2}{2}\left(\frac{\dot{\alpha}''\eta''-\alpha''\dot{\eta}''}{\alpha''\eta''}\right) - \frac{m'x'^2}{2}\left(\frac{\dot{\alpha}'\eta'-\alpha'\dot{\eta}'}{\alpha'\eta'}\right)$$
$$+\frac{1}{2}\left(m''\dot{\gamma}''x''^2 + m'\dot{\gamma}'x'^2\right)\cot\left(\gamma''-\gamma'\right) - \eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}x'x''\csc\left(\gamma''-\gamma'\right). \tag{17}$$

By substituting the above classical action into Eq. (8), the pre-exponential factor can be obtained as

$$F(t'',t') = \left(\frac{\eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}}{2\pi i\hbar\sin\left(\gamma''-\gamma'\right)}\right)^{\frac{1}{2}}. \tag{18}$$

From Eqs. (7), (17) and (18), the propagator for the harmonic oscillator with a time-dependent mass and frequency can be expressed by

$$K(x'',t'';x',t') = \left(\frac{\eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}}{2\pi i\hbar\sin\left(\gamma''-\gamma'\right)}\right)^{\frac{1}{2}}$$
$$\exp\left[\frac{i}{2\hbar}\left(\begin{array}{l} m''x''^2\left(\dfrac{\dot{\alpha}''\eta''-\alpha''\dot{\eta}''}{\alpha''\eta''}\right) \\ -m'x'^2\left(\dfrac{\dot{\alpha}'\eta'-\alpha'\dot{\eta}'}{\alpha'\eta'}\right)\end{array}\right)\right]$$
$$\exp\left(\frac{i}{2\hbar\sin\left(\gamma''-\gamma'\right)}\left[\begin{array}{l}\left(m''\dot{\gamma}''x''^2 + m'\dot{\gamma}'x'^2\right) \\ \times\cos\left(\gamma''-\gamma'\right) \\ -2\eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}x'x''\end{array}\right]\right) \tag{19}$$

This result can be verified by assuming $\eta' = \eta'' = \sqrt{m} = const$. This reduces to the propagator for a harmonic oscillator with time-dependent frequency and constant mass as

$$K(x'',t'';x',t') = \left(\frac{m\sqrt{\dot{\gamma}'\dot{\gamma}''}}{2\pi i\hbar\sin(\gamma''-\gamma')}\right)^{\frac{1}{2}}$$
$$\exp\left[\frac{im}{2\hbar}\left(\frac{\dot{\alpha}''x''^2}{\alpha''} - \frac{\dot{\alpha}'x'^2}{\alpha'}\right)\right]$$
$$\exp\left(\frac{im}{2\hbar\sin\left(\gamma''-\gamma'\right)}\left[\begin{array}{l}\left(\dot{\gamma}''x''^2 + \dot{\gamma}'x'^2\right) \\ \times\cos\left(\gamma''-\gamma'\right) \\ -2x'x''\sqrt{\dot{\gamma}'\dot{\gamma}''}\end{array}\right]\right) \tag{20}$$

This propagator is in agreement with the result of Khandekar and Lawande[23] by using the Feynman polygonal method and more recently with the result of Yeon[24] et al. by expanding the wave function obtained from the LR-invariant method. Furthermore, this result can be reduced to the case of the simple harmonic oscillator propagator, by setting α, ω and $\eta' = \eta'' = \sqrt{m}$ to be constants. The auxiliary equations become

$$-\dot{\gamma}^2\alpha + \omega^2\alpha = 0 \text{ or } \gamma''-\gamma' = \omega(t''-t'). \tag{21}$$

Substituting these parameters into Eq. (19), the well-known propagator for a simple harmonic oscillator, as appearing in Feynman and Hibbs,[19] can be obtained as

$$K\left(x'',t'';x',t'\right) = \left(\frac{m\omega}{2\pi i\hbar\sin\omega\left(t''-t'\right)}\right)^{\frac{1}{2}}$$
$$\times\exp\left(\frac{im\omega}{2\hbar\sin\omega\left(t''-t'\right)}\left[\left(x''^2+x'^2\right)\cos\omega\left(t''-t'\right) - 2x'x''\right]\right). \tag{22}$$

Wave Function For a Harmonic Oscillator with Time-Dependent Mass and Frequency

In this section, the time-dependent Schrodinger's wave function is calculated from the spectral representation of the propagator in Eq. (5) by defining

$$z = e^{-i\phi}, \quad \phi = \gamma''-\gamma', \tag{23}$$

$$\sin\phi = \frac{1}{2i}\frac{1-z^2}{z}, \tag{24}$$

$$\cos\phi = \frac{1+z^2}{2z}, \tag{25}$$

$$a = \sqrt{\frac{m''\dot{\gamma}''}{\hbar}}x'', \quad b = \sqrt{\frac{m'\dot{\gamma}'}{\hbar}}x'. \tag{26}$$

The general propagator in Eq. (19) can be written as

$$K(x'',t'';x',t')=\left(\frac{\eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}z}{\pi\hbar}\right)^{\frac{1}{2}}\left(1-z^2\right)^{-\frac{1}{2}}$$
$$\times\exp\left(\frac{i}{2\hbar}\left[m''x''^2\left(\frac{\dot{\alpha}''\eta''-\alpha''\dot{\eta}''}{\alpha''\eta''}\right)-m'x'^2\left(\frac{\dot{\alpha}'\eta'-\alpha'\dot{\eta}'}{\alpha'\eta'}\right)\right]\right)$$
$$\times\exp\left\{\frac{1}{1-z^2}\left[2abz-\left(a^2+b^2\right)\left(\frac{1+z^2}{z}\right)\right]\right\}. \qquad (27)$$

Now using the identity

$$\frac{1+z^2}{2\left(1-z^2\right)}=\frac{1}{2}+\frac{z^2}{1-z^2}, \qquad (28)$$

the propagator can be rewritten as

$$K(x'',t'';x',t')=\left(\frac{\eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}z}{\pi\hbar}\right)^{\frac{1}{2}}\left(1-z^2\right)^{-\frac{1}{2}}$$
$$\times\exp\left(\frac{i}{2\hbar}\left[m''x''^2\left(\frac{\dot{\alpha}''\eta''-\alpha''\dot{\eta}''}{\alpha''\eta''}\right)-m'x'^2\left(\frac{\dot{\alpha}'\eta'-\alpha'\dot{\eta}'}{\alpha'\eta'}\right)\right]\right)$$
$$\times\exp\left[-\frac{1}{2}\left(a^2+b^2\right)\right]\times\exp\left[\frac{2abz-\left(a^2+b^2\right)z^2}{1-z^2}\right]. \qquad (29)$$

By exploying the Mehler's formula[25]

$$\left(1-z^2\right)^{-\frac{1}{2}}\exp\left[\frac{2abz-\left(a^2+b^2\right)z^2}{1-z^2}\right]$$
$$=\sum_{n=0}^{\infty}H_n(a)H_n(b)\frac{z^n}{2^n n!}, \qquad (30)$$

where $H_n(x)$ is the Hermite polynomial, the propagator becomes

$$K(x'',t'';x',t')=\left(\frac{\eta'\eta''\sqrt{\dot{\gamma}'\dot{\gamma}''}}{\pi\hbar}\right)^{\frac{1}{2}}$$
$$\times\exp\left(\frac{i}{2\hbar}\left[\begin{array}{l}m''x''^2\left(\dfrac{\dot{\alpha}''\eta''-\alpha''\dot{\eta}''}{\alpha''\eta''}\right)\\ -m'x'^2\left(\dfrac{\dot{\alpha}'\eta'-\alpha'\dot{\eta}'}{\alpha'\eta'}\right)\end{array}\right]\right)$$
$$\times\exp\left[-\frac{1}{2\hbar}\left(m''\dot{\gamma}''x''^2+m'\dot{\gamma}'x'^2\right)\right]$$
$$\times\sum_{n=0}^{\infty}H_n\left(\sqrt{\frac{m''\dot{\gamma}''}{\hbar}}x''\right)H_n\left(\sqrt{\frac{m'\dot{\gamma}'}{\hbar}}x'\right)\frac{e^{-i\phi\left(n+\frac{1}{2}\right)}}{2^n n!}$$
$$(31)$$

Comparing the propagator in Eq. (31) with Eq. (5), the wave function for a harmonic oscillator with time-dependent mass and frequency can be expressed as

$$\psi_n(x,t)=\left[\frac{1}{2^n n!}\left(\frac{m(t)\dot{\gamma}(t)}{\pi\hbar}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\exp\left[-i\left(n+\frac{1}{2}\right)\gamma(t)\right]$$
$$\times\exp\left[\frac{im(t)}{2\hbar}\left\{\left(\frac{\dot{\alpha}(t)\eta(t)-\alpha(t)\dot{\eta}(t)}{\alpha(t)\eta(t)}\right)+i\dot{\gamma}(t)\right\}x^2\right]$$
$$\times H_n\left(\sqrt{\frac{m(t)\dot{\gamma}(t)}{\hbar}}x\right).$$
$$(32)$$

Since each $\psi_n(x,t)$ satisfies the time-dependent Schrodinger's equation, the general solution can be written as

$$\Psi(x,t)=\sum_{n=0}^{\infty}C_n\psi_n(x,t), \qquad (33)$$

where C_n are constants.

The general wave function in Eq. (32) can be verified by setting $\alpha,\omega,\eta'=\eta''=\sqrt{m}$ to be constant in the auxiliary equation as mentioned in Eq. (21).

The wave function can be reduced to

$$\psi_n(x,t)=\left[\frac{1}{2^n n!}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}\exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right]$$
$$\times\exp\left[-\frac{m\omega}{2\hbar}x^2\right]H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \qquad (34)$$

which is the wave function for a simple harmonic oscillator appearing in the text-book on quantum mechanics.[26-27]

It should be noted that the general wave function in Eq. (32) is slightly different from reported by the result of Pedrosa[17] and Ciftja[18] because of different notations. In order to comparing, let $\frac{\alpha(t)}{\eta(t)}=\rho(t)$ and $\gamma(t)=\int_0^t\frac{1}{m(t')\rho^2(t')}dt'$ then the auxiliary equation in Eq. (12) becomes

$$\ddot{\rho}(t)+\frac{\dot{m}(t)}{m(t)}\dot{\rho}(t)+\omega^2(t)\rho(t)=\frac{1}{m^2(t)\rho^3(t)}. \qquad (35)$$

and the wave function can be written as

$$\psi_n(x,t)=\frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\pi\hbar\rho^2(t)}\right)^{\frac{1}{4}}$$
$$\times\exp\left[-i\left(n+\frac{1}{2}\right)\int_0^t\frac{dt'}{m(t')\rho^2(t')}\right]$$
$$\times\exp\left[\frac{im(t)}{2\hbar}\left(\frac{\dot\rho(t)}{\rho(t)}+\frac{i}{m(t)\rho^2(t)}\right)x^2\right]$$
$$\times H_n\left(\sqrt{\frac{1}{\hbar}}\frac{x}{\rho(t)}\right),\tag{36}$$

which agrees with their results.[17-18]

The Caldirola-Kanai Oscillator

In this section, the application of the solution of our auxiliary Eqs. (12) and (13) is demonstrated for deriving the explicit form of the propagator and wave function. The system selected as an example is the quantum damped harmonic oscillator or the Caldirola-Kanai oscillator.[28-29] By introducing the mass law, the time-dependent mass can be written as

$$m(t)=me^{rt}\ ,\tag{37}$$

where m is the constant mass and r is the constant damping coefficient.

The Caldirola-Kanai Hamiltonian can be obtained by the Hamiltonian in Eq. (1) with constants ω

$$H(t)=\left(\frac{p^2}{2m}\right)e^{-rt}+\frac{1}{2}m\omega^2e^{rt}x^2.\tag{38}$$

In order to obtain the propagator of this system, the explicit forms of the function $\alpha(t)$ and $\gamma(t)$ in Eq. (19) have to be derived. By substituting the time-dependent mass in Eq. (37) into the auxiliary Eqs. (12) and (13), it can be derived that

$$\alpha=\frac{1}{\sqrt{\Omega}}\quad\text{and}\tag{39}$$

$$\gamma(t)=\Omega t\ ,\tag{40}$$

where Ω is the reduce frequency defined by

$$\Omega^2=\omega^2-\frac{r^2}{4}.\tag{41}$$

By substituting Eqs. (37), (39), (40) and $\eta(t)=\sqrt{m(t)}$ into Eq. (19), the propagator for the Caldirola-Kanai oscillator can be obtained as

$$K(x'',t'';x',t')=\left(\frac{m\Omega e^{r(t''+t')/2}}{2\pi i\hbar\sin\Omega(t''-t')}\right)^{\frac{1}{2}}$$

$$\times\exp\left\{\frac{im\Omega}{2\hbar}\left[\cot\Omega(t''-t')\left(e^{rt''}x''^2+e^{rt'}x'^2\right)-\frac{2x'x''e^{r(t''+t')/2}}{\sin\Omega(t''-t')}\right]\right\}$$

$$\times\exp\left\{\frac{irm}{4\hbar}\left(e^{rt''}x''^2-e^{rt'}x'^2\right)\right\}.\tag{42}$$

The wave function for this system can be calculated by Eq. (32) as

$$\psi_n(x,t)=\frac{1}{\sqrt{2^n n!}}\left(\frac{m\Omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[\left\{\frac{r}{4}-i\Omega\left(n+\frac{1}{2}\right)\right\}t\right]$$

$$\times\exp\left[-\frac{m}{2\hbar}\left(\Omega+\frac{ir}{4}\right)e^{rt}x^2\right]H_n\left[\left(\frac{m\Omega}{\hbar}\right)^{\frac{1}{2}}e^{\frac{rt}{2}}x\right].\tag{43}$$

The obtained propagator and wave function in Eq. (42) and (43) are in the same form as that reported by Jannusis et al.[30]

The Harmonic Oscillator with Strongly Pulsating Mass

The other well known example of a time-dependent mass oscillator is a harmonic oscillator with strongly pulsating mass.[31] This oscillator can be applied in connection with the electromagnetic field in a Fabry-Perot cavity in contact with a reservoir of resonant two-level atoms. The periodic release and reabsorption of photon can be represented by an oscillator of periodically fluctuating energy. In other words, it can be represented by a periodically varying mass as

$$m(t)=m\cos^2\nu t\ ,\tag{44}$$

where ν is the frequency of mass. In this case the Hamiltonian becomes

$$H(t)=\frac{p^2}{2m}\sec^2\nu t+\frac{1}{2}m\cos^2\nu t\,\omega^2x^2.\tag{45}$$

By substituting the mass law into the auxiliary Eqs. (12) and (13), we can get

$$\alpha=\frac{1}{\sqrt{\Omega}}\quad\text{and}\tag{46}$$

$$\gamma(t)=\Omega t\ ,\tag{47}$$

where the augmented frequency Ω is defined by

$$\Omega^2=\omega^2+\nu^2.\tag{48}$$

By substituting these Eqs. (44), (46) and (47) into the general propagator in Eq. (19), the propagator for a harmonic oscillator with strongly pulsating mass can be derived as

$$K(x'',t'';x',t')=\left(\frac{m\Omega\cos\nu t'\cos\nu t''}{2\pi i\hbar\sin\Omega(t''-t')}\right)^{\frac{1}{2}}$$

$$\times \exp\left[\frac{im\nu}{2\hbar}\left(\cos^2 \nu t'' \tan \nu t'' x''^2 - \cos^2 \nu t' \tan \nu t' x'^2\right)\right]$$

$$\times \exp\left(\frac{im\Omega}{2\hbar \sin\Omega(t''-t')}\left[\begin{array}{l}\left(\cos^2 \nu t'' x''^2 + \cos^2 \nu t' x'^2\right) \\ \cos\Omega(t''-t') \\ -2\cos\nu t' \cos\nu t'' x' x''\end{array}\right]\right). \quad (49)$$

This propagator can be simplified by setting $\nu = 0$, and $\Omega = \omega$. The result is reduced to the simple harmonic oscillator propagator.

By substituting Eqs. (46) and (47) into Eq.(32), the wave function for a harmonic oscillator with strongly pulsating mass can be obtained as

$$\psi_n(x,t) = \frac{1}{\sqrt{2^n n!}}\left[\frac{m(t)\Omega}{\pi\hbar}\right]^{\frac{1}{4}} \exp\left[-i\left(n+\frac{1}{2}\right)\Omega t\right]$$

$$\times \exp\left[\frac{im(t)}{2\hbar}\left(\nu \tan \nu t + i\Omega\right)x^2\right] H_n\left(\sqrt{\frac{m(t)\Omega}{\hbar}}x\right). \quad (50)$$

This result agrees with the wave function of Colegrave and Abdalla.[31]

Conclusion

In this paper we successfully calculated the exact propagator and wave function for a harmonic oscillator with time-dependent mass and frequency by the Feynman path integral formulation. The resulting propagator in Eq. (19) can be reduced to the propagator for a harmonic oscillator with time-dependent frequency and constant mass which agrees with the result of Khandekar and Lawande[23] and Yeon et al.[24] as shown in Eq. (20). Moreover, our propagator in Eq. (19) can be reduced to the simple harmonic oscillator propagator[19] by setting the mass and frequency to be constants as shown in Eq. (22). The resulting wave functions in Eq. (32) can be also reduced to the simple harmonic oscillator wave functions[26-27] by setting the time-dependent mass and frequency function to be constants.

The crucial result in our calculation is to express the general solution of a time-dependent mass and frequency oscillator as mentioned in Eq. (10). This solution in Eq. (10) has the time-dependent phase and amplitude which satisfy the auxiliary Eqs. (12) and (13). By modifying the notations, the auxiliary Eq. (12) become the well known Pinney equation[32] as shown in Eq. (35). In sections 4 and 5, we have shown the usefulness of auxiliary equations for deriving the explicit form of the propagator and wave function in the case of the Caldirola-Kanai and strongly pulsating mass oscillator.

In order to compare our approach with other works, the amplitude $\rho(t)$ in Eq. (10) corresponds to the space transformation of $x = \rho(t)y$, and the phase factor $\gamma(t) = \int_0^t \frac{dt'}{m(t')\rho^2(t')}$ is the time transformation, both of the Ciftja approach.[18] In the Pedrosa[17] approach the amplitude $\rho(t)$ is the crucial factor to construct an invariant and unitary operator. Surprisingly, by the different formulations of mathematics, the $\rho(t)$ function in the three approaches can satisfy the same nonlinear differential equation or the Pinney equation. Finally, it can be concluded here that the path integral is the effective and straight forward method for solving the time-dependent problems because it requires the same procedure in solving time-independent problems without employing any transformation compared with the other methods.

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