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Quantum kinematics are considered for two classes of systems: (i) the coordinate space is a homogeneous Riemannian manifold and the kinetic energy is the Laplace–Beltrami operator, (ii) the phase-space is a homogeneous Kählerian manifold where the algebra of observables is the universal enveloping algebra of the symmetry group. In both the cases, the path integrals are derived from more fundamental principles. The derivation enables a definite description of the classes of trajectories which constitute the domain of integration in the path integrals. The specification of the classes of functions is important for consistency of the approach. The standard action functional and the variational principle appear in the steepest descent method corresponding to the semiclassical approximation. © *1995 American Institute of Physics.*

I. INTRODUCTION

Quantization of a (classical) theory of fields $\varphi(x)$, subject to dynamics determined by a Lagrangian density $\mathscr{L}(\varphi,\partial\varphi)$, where $\partial\varphi\equiv\partial\varphi/\partial x$, is usually considered as a problem to be solved straightforwardly by means of the functional (path) integral. It is claimed sometimes that one has just to investigate properties of a functional given by the integral representation

$$\mathscr{Z}=\int \mathscr{D}\varphi\, \exp\left[\frac{i}{\hbar}\int \mathscr{L}(\varphi,\partial\varphi)d^n x\right], \tag{1}$$

where $\mathscr{D}\varphi$ is an appropriate measure in the space of functions $\varphi(x)$, to be specified in some *ad hoc* way. It is also assumed that the field φ may take its values in a (super)manifold having, more or less, arbitrary geometrical properties.

The dominating role of the path integral as a principle of quantization may raise a doubt in those who believe that the canonical quantization, based upon the Heisenberg commutation relations and their realization in the Hilbert space of state vectors, is more fundamental and apparently consistent. In the canonical approach, the dynamical description starts from the Hamiltonian operator $\hat{H}$ and the corresponding unitary time evolution operator, while the functional in view is

$$\mathscr{Z}_t=\mathrm{Tr}\,\exp(-it\hat{H}/\hbar). \tag{2}$$

The path integral can be obtained by representing the evolution operator for a given t as a product of N operators describing (approximately) the evolution for times t/N. The result would be a multiple integral resulting from multiplication of the operator kernels (or symbols). The functional integral appears, formally, when $N\to\infty$, if one can prove that the approximation errors vanish in the limit. Considering $\mathscr{Z}_t$ just as a function of time, one can extract the spectrum of $\hat{H}$. In order to get more detailed information on the system, one has to probe its response to an external driving force or source fields. Ultimately, one can also set $t\to\infty$ in order to get the generating functional, which is employed in relativistic field theories.

A careful inspection of the transition from the multiple integral to the functional integral reveals certain restrictions upon the class of functions which constitute the "integration domain"

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in Eq. (1). These restrictions should be taken into account in practical applications, since they stem from the basic principles of quantum theory, like uncertainty relations and the unitarity of the evolution operator.

Section II gives a brief review of the path integral in Euclidean phase spaces. The approach from the canonical formalism enables one to determine the corresponding classes of trajectories in the path integral. Coordinate spaces having homogeneous Riemannian structures and nontrivial geometrical properties are considered in Sec. III. The Hamiltonians incorporating the Laplace–Beltrami operators on the manifolds are usually considered in this case. The classes of trajectories are much more involved here. Quantization on homogeneous phase-space manifolds and the corresponding analogs of the path integral are discussed in Sec. IV.

II. EUCLIDEAN PHASE SPACE AND INTEGRATION DOMAINS IN THE PATH INTEGRALS

A. Feynman's path integrals

In the usual quantum mechanics, the representation in Eq. (1) was *derived* from Eq. (2) within the operator approach, and it was found helpful for a number of problems. The derivation was based, mainly, upon the following assumptions:

(A) The classical phase space has the Euclidean geometry, and the canonical commutation relations (CCR) are imposed on the operators of coordinates and momenta.

(B) The Hamiltonian is smooth and has appropriate asymptotic properties in the phase space, enabling a simple exponential extrapolation of the evolution operator at small times. One must assume, for example, that

$$\frac{\langle p|\exp(-i\,\delta t\hat{H}/\hbar)|q\rangle}{\langle p|q\rangle}=\exp[-i\,\delta t H(q,p)/\hbar]+O(\delta t^2) \tag{3}$$

and the classical Hamilton function is *defined* from the Hamiltonian operator by

$$H(q,p)\equiv\frac{\langle p|\hat{H}|q\rangle}{\langle p|q\rangle}. \tag{4}$$

Here $|q\rangle$ and $|p\rangle$ are the eigenstate vectors for the canonical operators $\hat{q}$ and $\hat{p}$ (coordinate and momentum, respectively).

As is known,[1] the step by step in time process leads to the phase-space path integral with the Hilbert-type action

$$\mathscr{A}=\int (p\,dq-H\,dt) \tag{5}$$

and the Liouville measure in Eq. (1), cf. Eq. (7). The integral for $\mathscr{A}$ is the limit of the sums[2]

$$\mathscr{A}_N\equiv\sum_{n=1}^{N}[p_n(q_n-q_{n-1})-H(q_{n-1},p_n)\,\delta t],\quad \delta t\equiv t/N, \tag{6}$$

where $q_0\equiv q_N$, as soon as the sequence of the intermediate points (and the corresponding trajectory in the limit $N\to\infty$) is considered as closed. The "symplectic one-form" $p\,dq$ in the exponent stems from the product of the functions $\langle p|q\rangle$ and $\langle q|p\rangle$, which are exponentials representing the Fourier transformation, specific for the flat phase space. One should note that no "operator ordering problem" appears here;[3,4] the order was fixed from the very beginning and manifests itself in the Hamilton function defined by Eq. (4).

In this particular case, the path integral has a clear meaning: $\varphi \equiv (q,p)$, $x \equiv t$, $\mathscr{L}$ is linear in dq/dt (and independent of dp/dt), and the measure is the (infinite) product of the dimensionless invariant phase-space measures

$$\mathscr{D}\varphi \equiv \prod \frac{dq\ dp}{2\pi\hbar}. \tag{7}$$

Moreover, if H is quadratic in p, which is the case for dynamics of a nonrelativistic particle (or a number of particles) and for standard theories of scalar fields, one can calculate all the (Gaussian) integrals in p, getting the Lagrangian from the Legendre transform of H

$$L(q,\dot{q}) = p\dot{q} - H(q,p), \quad \dot{q} = \partial H/\partial p. \tag{8}$$

Now the measure gets a more complicated form, $\mathscr{D}\varphi \equiv \Pi \rho(q)\, dq$ incorporating a dimensional (and infinitely large) factor, which is inessential if $\mathscr{Z}$ has to be normalized anyway, as is usually done in applications. It is noteworthy that the standard Lagrangian appears from the stationary phase approximation,[5] which is not exact unless the Hamiltonian is quadratic in p.

It is quite important to indicate definitely the integration domain in Eq. (1), i.e., the classes of trajectories. In the original path integral with the action (5) no conditions are imposed upon $p(t)$, which appears without its derivatives in the action functional.[6] On the other hand, the substitution of $q_n - q_{n-1}$ for $dq \equiv \dot{q}\ dt$ means that $q(t)$ is approximated by piecewise linear functions. Calculating the integrals in p, still for finite N, one gets a restriction of the functional class to compositions of segments which are solutions of (approximate) classical equations for small-time intervals. For the problems where the kinetic part of $\hat{H}$ corresponds to the Laplace operator in a Riemannian space, these segments are geodesic lines connecting q_{n-1} and q_n for given (infinitesimally small) time intervals.

The sum in Eq. (6) can be written in a different way

$$\sum_{n=1}^{N} p_n(q_n - q_{n-1}) = -\sum_{n=1}^{N} q_n(p_{n+1} - p_n), \quad p_{N+1} \equiv p_1, \tag{9}$$

which suggests another class of trajectories: with continuous $p(t)$ and arbitrary $q(t)$. Thus the phase-space path integral, having an apparent coordinate-momentum symmetry, is in fact supported by different classes of functions. As is known, the correspondence between the form of the action and the class of trajectories in the phase space results from the uncertainty relations.

B. Holomorphic coordinates

Another important class of the phase-space path integrals is associated with the coherent states. In that case (described in Refs. 4 and 7) the state vectors are holomorphic functions $\psi(z)$, where $z \in \mathbb{C}^m$ (for m degrees of freedom), and the inner product of two wave functions is given by

$$(\psi_2, \psi_1) = \int \overline{\psi_2(z)}\psi_1(z) e^{-(z\cdot\bar{z})}\, d\mu(z,\bar{z}), \quad d\mu(z,\bar{z}) \equiv \prod \frac{dz \wedge d\bar{z}}{-2\pi i}. \tag{10}$$

Operators in the Hilbert space are given by their symbols, $\hat{A} \to A(\zeta,\bar{z})$, which are functions holomorphic in the first argument and antiholomorphic in the second one, so that

$$(\hat{A}\psi)(\zeta) = \int A(\zeta,\bar{z})\psi(z) e^{(\zeta - z)\cdot\bar{z}}\, d\mu(z,\bar{z}). \tag{11}$$

If the operator has a trace, it is given by the integral of its symbol

$$\mathrm{Tr}(\hat{A}) = \int A(z,\bar{z})\, d\mu(z,\bar{z}). \tag{12}$$

Respectively, the product of operators is given by an integral of their symbols, namely,

$$\hat{A}\hat{B} \to (A*B)(\zeta,\bar{\eta}) \equiv \int A(\zeta,\bar{z})B(z,\bar{\eta})\exp[(\zeta - z)\cdot(\bar{z}-\bar{\eta})]\, d\mu(z,\bar{z}). \tag{13}$$

Given the Hamiltonian operator $\hat{H}$ and its symbol $H(\zeta,\bar{z})$, one can get a representation for $\mathscr{Z}_t$, provided that at small δt

$$\exp(-i\,\delta t\hat{H}) \to G_{\delta t}(\zeta,\bar{z}) = \exp[-i\,\delta t H(\zeta,\bar{z})] + O(\delta t^2). \tag{14}$$

For finite N, the action functional is represented now by a (complex) sum

$$\mathscr{A}_N = \sum_{n=1}^{N} [-i\zeta_n\cdot(\bar{z}_{n+1}-\bar{z}_n) - H(\zeta_n,\bar{z}_{n+1})\,\delta t]. \tag{15}$$

In the continuous limit, $N \to \infty$, $\delta t \to 0$, complex extensions of the trajectories appear in the functional integral, so that $\zeta(t)$ and $\bar{z}(t)$ are independent and not considered as complex conjugated (which is true for classical trajectories). No conditions are imposed on $\zeta(t)$, while $\bar{z}(t)$ is a continuous function.

The holomorphic formalism is especially useful for bilinear Hamiltonians, representing harmonic oscillators and free fields, where

$$H(\zeta,\bar{z}) = (\zeta, K\bar{z}); \quad K = K^{\dagger}. \tag{16}$$

(K is a non-negative matrix.) In this case, the Schrödinger equation and the corresponding equation for the symbol of the evolution operator are reduced to the first-order differential equations, which follow from Eqs. (11) and (13) after the integration in $\bar{z}$

$$\epsilon\psi(z) = (z, K\partial_z\psi(z)), \tag{17}$$

$$i\partial_t G_t(\zeta,\bar{\eta}) = (\zeta, K\partial_\zeta G_t) + (\zeta, K\bar{\eta})G_t, \quad G_0 \equiv 1. \tag{18}$$

The equations have the evident solutions

$$\psi_\kappa(z) = (z\cdot\bar{v}_\kappa)^n/\sqrt{n!}, \quad \epsilon_n = n\kappa, \tag{19}$$

$$G_t(\zeta,\bar{z}) = \exp(\zeta, B_t\bar{z}), \quad B_t = \exp(-itK) - I, \tag{20}$$

where κ is an eigenvalue of K, v_κ is its corresponding eigenvector, and n is an integer occupation number. The wave functions for the standard coherent states are

$$\phi_v(z) = \exp(z, K\bar{v}). \tag{21}$$

The solution for G_t can be also calculated immediately from the path integral with the action (15), since the integral is Gaussian. The matrix B_t appears from the linear recursion

$$B_{t+\delta t} = B_t + (B_t + I)B_{\delta t}, \quad B_{\delta t} \approx -iK\,\delta t. \tag{22}$$

The holomorphic description is extended immediately to Fermi-type degrees of freedom,[8] where z are anticommuting (generators of the Grassmann algebra). This approach is employed extensively in perturbative quantum field theories, like quantum electrodynamics.

III. HOMOGENEOUS COORDINATE MANIFOLDS

A. Riemannian manifolds

The derivation described in Sec. II fails if the coordinate manifold Q has a nontrivial geometry, while the phase space is the cotangent bundle based upon Q. The Heisenberg CCR are hardly relevant to this case, and one has to look for another principle. It is assumed usually that Q has a Riemannian structure, the Hilbert space $\mathscr{H}$ of state vectors is given by square integrable functions $\psi(q)$ on Q, with the invariant measure, and the Hamiltonian has a special form: it is a sum of the kinetic energy, which is the Laplace–Beltrami operator, and a potential energy, which is just a function on Q.

$$\hat{H} = -\tfrac{1}{2}\Delta + V(q). \tag{23}$$

In order to get an analog of the path integral, one has to calculate the evolution kernel

$$\langle q_2|\exp(-it\hat{H})|q_1\rangle \equiv \mathscr{K}_t(q_1,q_2) \tag{24}$$

at small t, and to integrate the product of N functions $\mathscr{K}_{\delta t}$, $\delta t = t/N$. One may suppose that at small δt the evolution kernel is given by the semiclassical approximation, where the initial and final points are connected by the geodesic line $\Gamma \equiv \{q(\tau)\}$

$$\mathscr{K}_t^{\Gamma}(q_1,q_2) = C_{\Gamma}(q_1,q_2)\exp\left[i\frac{s^2}{2t} + i\int_{\Gamma} V(q)d\tau \right]. \tag{25}$$

Here s is the distance between q_1 and q_2 along Γ, and C_{Γ} is a known invariant function on Γ, namely,

$$C_{\Gamma}(q_1,q_2) = (2\pi it)^{-m/2}\left[\det\left(-s\frac{\partial^2 s}{\partial\kappa_1^a \partial\kappa_2^b} \right) \right]^{1/2}, \tag{26}$$

where $\kappa_{1,2}^a$, $a = 1,\ldots, m-1$, are geodesic coordinates,[9] transversal to Γ at its ends. This expression is obtained with account of the invariant measure on Q, which is $d\mu(q) \equiv \sqrt{|g|}dq$, $|g| \equiv \det g_{\alpha\beta}(q)$, and $g_{\alpha\beta}$ is the metric tensor, $\alpha = 1,\ldots,m$. The standard van Vleck matrix $-\partial^2\mathscr{A}/\partial q_1^{\alpha}\partial q_2^{\beta}$ leads to the invariant $\det(\tilde{g}_1 b \tilde{g}_2 b^T)$, where $\tilde{g}_{1,2} \equiv \tilde{g}^{\alpha\beta}(q_{1,2})$ is the tensor inverse to the metric, and

$$b_{\alpha\beta} \equiv n_{\alpha}^{(1)} n_{\beta}^{(2)} + s\sigma_{\alpha\beta}, \quad n_{\alpha}^{(1,2)} \equiv \frac{\partial s}{\partial q_{1,2}^{\alpha}}, \quad \sigma_{\alpha\beta} \equiv \frac{\partial^2 s}{\partial q_1^{\alpha}\partial q_2^{\beta}}. \tag{27}$$

As soon as $s(q_1,q_2)$ is the geodesic distance, the quantities appearing here satisfy the following identities, which follow from the classical equations of motion:

$$n_{\alpha}n_{\beta}\tilde{g}^{\alpha\beta}(q) = 1, \quad n_{\alpha}^{(1)}\tilde{g}^{\alpha\gamma}\sigma_{\gamma\beta} = 0 = \sigma_{\alpha\beta}\tilde{g}^{\beta\gamma}n_{\gamma}^{(2)}. \tag{28}$$

The pre-exponential factor in Eq. (26) is obtained after a simple algebra.

The expression in Eq. (25) has the following property for small times, where the integral can be calculated by the stationary phase method:

$$K^{\Gamma}_{t_1+t_2}(q_1,q_2)=\int_{\mathsf{Q}} K^{\Gamma_1}_{t_1}(q_1,q)K^{\Gamma_2}_{t_2}(q,q_2)d\mu(q) \tag{29}$$

and Γ is a union of Γ_1 and Γ_2. For a generic Riemannian manifold, the geodesic line connecting any pair of points is not unique, and one has to sum up all the contributions of the type (25)

$$K_t(q_1,q_2)=\sum_{\Gamma} \mathscr{K}^{\Gamma}_t(q_1,q_2)e^{i\alpha_\Gamma t}, \tag{30}$$

with appropriate phase shifts α_Γ, which must be defined in such a way that

$$\lim_{t\to 0}\int_{\mathsf{Q}} K_t(q,q_0)\psi(q_0)d\mu(q)=\psi(q), \quad \forall \psi\in\mathscr{H}. \tag{31}$$

If a single small-time interval is considered, it seems reasonable to retain only the contribution from the geodesic on which the distance is minimal, omitting all the other terms in the sum (30), since they are smaller exponentially. This is not the correct way, however, if the small-time evolution kernel must be used in the iterative procedure for the path integral. Discarding contributions from nonminimal geodesics, one would lose an important information on the geometric properties of Q, and Eq. (31) would not hold.

In general, the classification of geodesics and the corresponding representation for the δ function on the manifold (the so-called "decomposition of unity") is a complicated problem, and we restrict our interest to two classes of *homogeneous* manifolds, where the solution is known.

B. Spheres

For an m-dimensional sphere $\mathsf{Q}=S^m$, the geodesics are principal circles, and the evolution kernel for the free motion, i.e., $V(q)\equiv 0$, depends only on $\cos\theta$ where θ is the angular distance between the initial and final points q_1 and q_2. The semiclassical approximation is obtained from Eqs. (25) and (30); it is given by the sum over all the revolutions around the sphere

$$\mathscr{K}^{\mathrm{cl}}_t(\cos\theta)=(2\pi it)^{-m/2}(\sin\theta)^{-(m-1)/2}\sum_{k=-\infty}^{\infty}\theta_k^{(m-1)/2}\exp\frac{i\theta_k^2}{2t}, \tag{32}$$

where $\theta_k=\theta+2\pi k$. This solution is written for the normalized measure on S^m, $\int_{S^m} d\mu(q)=1$. Checking the accuracy of the approximation, we get[10]

$$\left(i\frac{\partial}{\partial t}-\hat{H}\right)\mathscr{K}^{\mathrm{cl}}_t(\cos\theta)=\left[\frac{1}{12}R+\frac{1}{120}(m-1)(m-3)\hat{\Theta}\right]\mathscr{K}^{\mathrm{cl}}_t(\cos\theta), \tag{33}$$

where $R=m(m-1)$ is the scalar Riemannian curvature, and $\hat{\Theta}=\theta^2+O(\theta^4)$ is an operator acting in the space of functions of $\cos\theta$. (Its explicit form is given in Ref. 10.) Thus, Eq. (32) provides with an approximate solution of the Schrödinger equation. As to the initial condition, Eq. (31), it holds, as soon as the evolution kernel satisfies Eq. (29) approximately for small t_2. Another way of checking it is to apply the Poisson summation formula to the series of Eq. (32). Namely, if a function $f(x)$ has its Fourier transform $\tilde{f}(\xi)$, one has for the series

$$\sum_{k=-\infty}^{\infty} f(\theta_k)=\sum_{k=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\xi)e^{i\xi\theta_k}\,d\xi=\frac{1}{2\pi i}\int_C \tilde{f}(\xi)e^{i\xi\theta+i\xi\pi}\frac{d\xi}{\sin\xi\pi}, \tag{34}$$

where C is a closed contour in the complex ξ plane running below the real axis in the positive direction and coming back above the axis. The integrand has poles at integer points $\xi=m$, and the series resulting from the residua leads to an approximation for the spectral expansion of the evolution kernel. The exact spectral expansion in tensor representations of the group $SO(m+1)$ acting in S^m is obtained by the separation of variables in the Schrödinger equation and is written in terms of the Gegenbauer polynomials.[10]

For small t, the term with $k=0$ dominates in the series (32). Other terms are exponentially smaller, but they should not be discarded, because they bear an important information: actually, the evolution kernel is a function of $\cos\theta$, not just of θ, and the energy spectrum is discrete. Therefore one cannot set only one term into the multiple product of the evolution kernels which would generate the path integral with the Lagrangian corresponding to the classical free motion on S^m. The quantum dynamics on spheres was also considered by Fukutaka and Kashiwa.[11]

C. Compact simple groups

Let the coordinate manifold be a compact simple group G, its rank being r and its dimensionality n. The Riemannian structure is introduced in G by means of the Cartan–Maurer one-form $\lambda=g^{-1}\,dg$, taking its values in the Lie algebra $\mathbf{g}$ of G. The interval is defined by

$$ds^2=\mathrm{Tr}(\check{\lambda}\cdot\check{\lambda}^{\dagger}), \tag{35}$$

where $\check{\lambda}$ stands for the adjoint representation of $\lambda\in\mathbf{g}$. The Laplace–Beltrami operator in the quantum Hamiltonian is defined, respectively. For the free motion, the classical trajectories are straight lines in the Weyl alcove H of the group, which is the coset space of the r-dimensional maximal torus T of G by the Weyl group W, H=T/W. This fact stems from the following decomposition:

$$g_2g_1^{-1}=vhv^{-1},\quad h\in\mathsf{T},\quad \forall g_1,g_2\in\mathsf{G}. \tag{36}$$

The elements v can be parametrized by the coset space G/H, and the geodesic distance between g_1 and g_2 is independent of v. The semiclassical approximation is given by the sum over the lattice K, dual to the lattice Γ in the r-dimensional Euclidean root space, generated by the primitive roots of $\mathbf{g}$. The evolution kernel is a function of the r-dimensional vector $\varphi\in\mathsf{H}$

$$\mathscr{K}_t^{\mathrm{cl}}(\varphi)=\frac{C}{(2\pi it)^{n/2}}\sum_{k\in\mathsf{K}}\prod_{\alpha\in\Delta_+}\frac{\alpha\cdot(\varphi+2\pi\mathbf{k})}{2\sin[\alpha\cdot(\varphi+2\pi\mathbf{k})/2]}\exp\frac{i(\varphi+2\pi\mathbf{k})^2}{2t}, \tag{37}$$

where Δ_+ is the set of positive roots of $\mathbf{g}$, and C is a normalization constant. In this case, the semiclassical approximation is (almost) exact; the direct check shows that

$$\left(i\frac{\partial}{\partial t}-\hat{H}\right)\mathscr{K}_t^{\mathrm{cl}}(\varphi)=\frac{1}{12}R\mathscr{K}_t^{\mathrm{cl}}(\varphi). \tag{38}$$

The Riemannian curvature is $R=n/4$, and the constant may be added to the Hamiltonian. Two compact groups are spheres, $U(1)=S^1$, $SU(2)=S^3$, so Eq. (33) is reduced to Eq. (38), and the semiclassical approximation is exact. The exactness of the semiclassical approximation for heat transport on compact group manifolds was discovered by Eskin.[12] Schulmann[13] obtained that result for quantum motion on SU(2). Other references can be found in Ref. 10.

The Poisson summation formula leads to the spectral expansion of the evolution kernel, which is a series over the unitary representations of G and the terms in the expansion are the representation characters $\chi(\varphi)$

$$\mathscr{K}_t(\varphi)=\sum_{l\in\Gamma} d_l\chi_l(\varphi)\exp\left(-\frac{i}{2}\epsilon_l t\right), \tag{39}$$

where ϵ_l is the eigenvalue of the second-order Casimir operator, and d_l is the representation dimensionality.

Like in the case of spheres, no Lagrangian path integral is generated by iteration of small-t evolution kernel, since the latter contains infinitely many terms, and all of them must be retained to get the correct result.

IV. HOMOGENEOUS PHASE-SPACE MANIFOLDS

A. Classical dynamics

Classical dynamics in a generic phase-space manifold X is based upon an action functional on trajectories $x(t)\in\mathsf{X}$

$$\mathscr{A}\equiv\int [f_a(x)dx^a-H(x)dt], \tag{40}$$

where $H(x)$ is a Hamilton function, and the symplectic form is given in (local) coordinates on X. The kinematical term in the action is transformed nonhomogeneously under the change of coordinates

$$x\to\tilde{x}\rightsquigarrow f_a\,dx^a\to\tilde{f}_a\,d\tilde{x}^a+d\Lambda,\quad \tilde{f}_a=f_b\partial x^b/\partial\tilde{x}^a, \tag{41}$$

where $\Lambda(x)$ is a function on X. The total differential $d\Lambda$ does not change the equations of motion which are derived from the variational principle

$$\delta\mathscr{A}=0\rightsquigarrow\omega_{ab}\,dx^b/dt=\partial_a H,\quad \omega_{ab}\equiv\partial_a f_b-\partial_b f_a, \tag{42}$$

where $\partial_a\equiv\partial/\partial x^a$. If the two-form $\omega\equiv\omega_{ab}dx^a\wedge dx^b$ is nondegenerate, i.e., $\det\omega_{ab}\neq 0$, the Poisson brackets can be introduced with the field ϖ^{ab}, dual to ω

$$\{F_1,F_2\}_{\mathrm{PB}}\equiv\varpi^{ab}\partial_a F_1\partial_b F_2. \tag{43}$$

As a result, the equations of motion are written in terms of the Poisson brackets, $dF/dt=\{H,F\}_{\mathrm{PB}}$. If ω is degenerate, some of the equations in (42) should be considered as constraints. The dynamical scheme described above is sometimes called the *first-order formalism*; it has been applied to a number of problems, in particular, in general relativity and supergravity.[14]

The naive approach to quantization would be an attempt to find a representation to the commutations relations for operators x_a, as given by the Poisson brackets, $\{x^a,x^b\}_{\mathrm{PB}}=\varpi^{ab}(x)$. The problem is that, unlike the Heisenberg CCR, the Lie algebra generated by this procedure is not necessarily finite-dimensional, so the representation problem has no straightforward solution. One might assume also that the action formalism in classical theory may indicate another quantization procedure if the action (40) would be put into a path integral in X. Both the approaches are hopelessly complicated and hardly consistent, in general.

A regular way to quantum theory on *homogeneous* phase-space manifolds is to start the quantization from the Lie algebra **g** of its transformation group G. If the group acts transitively, the phase space manifold is a coset space, $\mathsf{X}=\mathsf{G}/\mathsf{H}$, where H is a subgroup of G. In this approach, the algebra of observables is constructed as the universal enveloping algebra of **g**, which acts in the space of representations of G. The problem is rather a *dequantization*, i.e., one has to represent the observables by functions in X, called symbols of the operators, and to construct appropriate Poisson brackets. This approach is known as the Kirillov–Kostant *geometric quantization*.[15] In the

case of the standard quantum theory, $\mathsf{X}=\mathbb{R}^{2m}$, G is the Heisenberg–Weyl group, and $\mathsf{H}=\mathrm{U}(1)$. The complete solution was indicated by Berezin[16] for the case where X is a homogeneous Kähler manifold.

B. Quantization on homogeneous Kähler manifolds

A complex Kählerian structure[17] is introduced in the phase space X, with local coordinates $(z,\bar{z})$ and a potential function $K(\zeta,\bar{z})$. A group G acts in X, $z \to gz$, $\forall g \in \mathsf{G}$, and the potential is transformed by

$$K(z,\bar{z}) \to K(gz,\overline{gz}) = K(z,\bar{z}) + \Phi_g(z) + \overline{\Phi_g(z)}, \tag{44}$$

where $\Phi_g(z)$ is a holomorphic cocycle function. The invariant (1,1)-form is expressed in terms of the potential, $\omega \equiv \partial_\alpha \partial_{\bar{\beta}} K \, dz^\alpha \wedge d\bar{z}^\beta$. Following the suggestion by Berezin,[16,18] the Hilbert space of state vectors $\mathscr{H}$ is introduced by the holomorphic line bundle, i.e., (locally) holomorphic functions $\psi(z)$, with the scalar product

$$(\psi_2,\psi_1) = \int_{\mathsf{X}} \overline{\psi_2(z)} \psi_1(z) \exp[-K(z,\bar{z})] d\mu(z,\bar{z}), \tag{45}$$

where the invariant measure on X is defined in a standard way, with a constant normalization factor C, fixed by normalization of the inner product

$$d\mu(z,\bar{z}) \equiv C\omega \wedge \cdots \wedge \omega, \quad \int_{\mathsf{X}} \exp[-K(z,\bar{z})] d\mu(z,\bar{z}) = 1. \tag{46}$$

The construction above is Berezin's extension of the holomorphic formalism in quantum mechanics,[7] described in Sec. II B, cf. Eq. (10). The action of any linear operator $\hat{A}$ is defined in terms of its symbol $A(\zeta,\bar{z})$

$$(\hat{A}\psi)\dot{}(\zeta) = \int_{\mathsf{X}} A(\zeta,\bar{z}) \psi(z) \exp[K(\zeta,\bar{z}) - K(z,\bar{z})] d\mu(z,\bar{z}). \tag{47}$$

The symbol of a product of two operators is given by the $*$-product

$$\hat{A}\hat{B} \to (A*B)(\zeta,\bar{\eta}) \equiv \int_{\mathsf{X}} A(\zeta,\bar{z}) B(z,\bar{\eta}) \exp[K(\zeta,\bar{z}) - K(z,\bar{z}) + K(z,\bar{\eta}) - K(\zeta,\bar{\eta})] d\mu(z,\bar{z}). \tag{48}$$

The last two equalities are analogs of Eqs. (11) and (13), while Eq. (12) for traces holds without any modification. Equalities (47) and (48) provide with a construction of the algebra of observables for the dynamical on the phase-space manifold X. The Lie algebra $\mathbf{g}$ of the transformation group G is represented in $\mathscr{H}$ by means of the first-order differential operators. Namely, if τ_a constitute a basis in $\mathbf{g}$, one has the following representation in terms of the first-order differential operators:

$$\tau_a \to \hat{T}_a = \nabla_a - \varphi_a(z), \quad [\hat{T}_a, \hat{T}_b] = f_{ab}^c \hat{T}_c. \tag{49}$$

Here f_{ab}^c are structure constants in $\mathbf{g}$, ∇_a are the Lie derivatives in X, and the holomorphic functions φ_a are expressed in terms of the Lie derivatives $\pounds_a(g)$ acting upon the cocycle functions [Eq. (44)] in the group elements $g \in \mathsf{G}$

$$\varphi_a(z) = \pounds_a \Phi_g(z)\big|_{g=e}. \tag{50}$$

The explicit formulas may be found in Ref. 19. Symbols of $\hat{T}_a$, which are functions $T_a(z,\bar{z})$ in the phase space X, have been also obtained explicitly,[19] and it was shown that the Poisson brackets given by the field ϖ dual to ω provide with the representation of the Lie algebra **g**

$$\{T_a,T_b\}_{\mathrm{PB}}=\nabla_a T_b-\nabla_b T_a=f_{ab}^c T_c\,. \tag{51}$$

The construction is based upon the Kähler potentials, which have been constructed in a closed form[20] for all compact simple groups G and all the subgroups H containing at least one centralizer U(1). A Kähler manifold is constructed for any unitary representation of G. The algebra of observables, including the Hamiltonian, is constructed as the universal enveloping algebra of **g** in terms of $\hat{T}_a$, which are the basis elements of **g**.

For noncompact manifolds X, corresponding to noncompact transformation groups, a similar construction has been found[20] for discrete representations. An extension of the method to other noncompact group manifolds is the purpose of a future work.

Dynamics of homogeneous phase-space manifolds is determined by an operator $\hat{H}$, the Hamiltonian, which is an element of the universal enveloping algebra of **g**. The symbol of $\hat{H}$ is an analog of the classical Hamilton function. The classical dynamics can be formulated in terms of the variational principle for an action functional, and an analog of the phase-space path integral is also considered below.

C. Semiclassical approximation and the path integral

The semiclassical approximation usually works in the situations where large parameters are present in exponential integrands, and the integrals are evaluated by means of the steepest descent method. This is true as well for dynamics on homogeneous Kähler manifolds. The large parameter appears at the Kähler potential if the representation highest weight is large. In that case, the total number of states, which is proportional to the volume of X, is also large. The large parameter must be present as well in the Hamiltonian $\hat{H}$, otherwise the classical dynamics would be trivial.

In order to get the semiclassical approximation, let us construct an analog of the path integral. The standard method is to use the identity

$$\hat{U}_t\equiv e^{-i\hat{H}t}=(e^{-i\hat{H}t/\mathcal{N}})^{\mathcal{N}},\quad \forall\mathcal{N} \tag{52}$$

and to write an approximate expression for the symbol of the evolution operator at small times, $\tau=t/\mathcal{N}$, where $\mathcal{N}\to\infty$

$$e^{-i\tau\hat{H}}\to e^{-i\tau H(z,\bar{z})}+O(\tau^2). \tag{53}$$

In principle, this is not correct for our manifolds, because each symbol must be a rational function, i.e., a polynomial divided by $\exp(K)$, and this form does not survive the exponentiation. In the limit of large representation dimensionalities, however, the polynomials are of a very high degree, and this condition is not too restrictive. Multiplying $\mathcal{N}$ operators (53), using the $*$-product (48), and calculating the trace, one gets the partition function

$$Z(t)=\lim_{\mathcal{N}\to\infty}\int\cdots\int\prod_{n=1}^{\mathcal{N}}d\mu(z_n,\bar{z}_n)\exp\left\{\sum_{n=1}^{\mathcal{N}}[K(z_n,\bar{z}_{n+1})-K(z_n,\bar{z}_n)-i\tau H(z_n,\bar{z}_{n+1})]\right\}. \tag{54}$$

The boundary condition is $\bar{z}_{\mathcal{N}+1}\equiv\bar{z}_1$, so the sequence of points on X can be considered as closed. The exponent has an extremum under the following conditions, for $n=1,\ldots,\mathcal{N}$:

$$\Lambda_\alpha(z_n,\bar{z}_{n+1})-\Lambda_\alpha(z_n,\bar{z}_n)-i\tau\partial_\alpha H(z_n,\bar{z}_{n+1})=0,$$
$$\bar{\Lambda}_\alpha(z_{n-1},\bar{z}_n)-\bar{\Lambda}_\alpha(z_n,\bar{z}_n)-i\tau\partial_{\bar{\alpha}}H(z_{n-1},\bar{z}_n)=0, \tag{55}$$

where $\Lambda=\partial K$ and $\bar{\Lambda}=\bar{\partial}K$. Under reasonable conditions on the Hamiltonian, in the limit $\tau\to 0$, the region $z_{n+1}\to z_n$ contributes predominantly to the integral, so that

$$\Lambda_\alpha(z_n,\bar{z}_{n+1})-\Lambda_\alpha(z_n,\bar{z}_n)\approx\omega_{\alpha\bar{\beta}}(z_n,\bar{z}_n)(\bar{z}^{\bar{\beta}}_{n+1}-\bar{z}^{\bar{\beta}}_n) \tag{56}$$

and Eqs. (55) become the usual Hamilton equations of motion, which can be set to the familiar form with the Poisson brackets defined as in Eq. (51)

$$\frac{dz^\alpha}{dt}=\{H,z^\alpha\}_{\mathrm{PB}},\quad \frac{d\bar{z}^\alpha}{dt}=\{H,\bar{z}^\alpha\}_{\mathrm{PB}}. \tag{57}$$

These equations of motion can also be derived from the variational principle applied to the action which results from the sum in Eq. (54) if the difference of two Kähler potentials is replaced by the differential

$$\mathscr{A}_t=\int_0^t[-i\bar{\Lambda}_\alpha(z,\bar{z})d\bar{z}^\alpha-H(z,\bar{z})d\tau]. \tag{58}$$

This action is an analog of that derived from Eq. (15) in the continuous limit. The action functional is calculated on closed trajectories, $z(0)=z(t)$.

The arguments presented above support the meaning of the construction as a method of quantization: the quantum theory has its classical limit described by the action functional with the Kählerian symplectic form.

It is clear from the derivation that the functional to be averaged in the functional space has not, in general, the form of the exponential of the classical action. It is reduced to the latter only in the limit where a parameter playing the role of $\hbar^{-1}$ is large. In the other words, the semiclassical approximation is done in two steps: first, the partition function takes the form of the usual path integral (1), second, this integral is evaluated by means of the steepest descent method.

V. CONCLUSION

Three types of quantum theories have been considered above: (i) the conventional quantum theory in flat phase space with different forms of the action and the corresponding requirements on the trajectories, (ii) homogeneous Riemannian manifolds, spheres and group manifolds in particular, (iii) homogeneous Kählerian manifolds. It is evident that the classes of trajectories in each case must be described in an appropriate way, with account of the small-time behavior of the evolution kernel, which determines an individual "segment" of the trajectory. This segment is that of a straight line only in the simplest version of the path integral. If the geometry of the dynamical system is nontrivial, one has an infinite number of trajectories even for small t, and one has to sum them all up, which makes the path integral method rather involved.

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