Quantum Equivalence Principle for Path Integrals in Spaces with Curvature and Torsion*

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Abstract

We formulate a new quantum equivalence principle by which a path integral for a particle in a general metric-affine space is obtained from that in a flat space by a non-holonomic coordinate transformation. The new path integral is free of the ambiguities of earlier proposals and the ensuing Schrödinger equation does not contain the often-found but physically false terms proportional to the scalar curvature. There is no more quantum ordering problem. For a particle on the surface of a sphere in D dimensions, the new path integral gives the correct energy $\propto \hat{L}^2$ where $\hat{L}$ are the generators of the rotation group in $\mathbf{x}$-space. For the transformation of the Coulomb path integral to a harmonic oscillator, which passes at an intermediate stage a space with torsion, the new path integral renders the correct energy spectrum with no unwanted time-slicing corrections.

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1) In Schrödinger quantum mechanics, the dynamical properties of a point particle are governed by the differential equation $\hat{H}\psi(\mathbf{x},t) = i\hbar\partial_t\psi(\mathbf{x},t)$ where $\psi(\mathbf{x},t)$ is a square-integrable probability amplitude. If the particle moves in flat space parametrized by cartesian coordinates, the Hamilton operator $\hat{H}$ is found from the *correspondence principle*. It prescribes that $\hat{H}$ be obtained from the classical Hamiltonian $H(\mathbf{p},\mathbf{x})$ by replacing the canonical coordinates x^i and momenta p_i by the Schrödinger operators $\hat{x}^i = x^i$, $\hat{p}_i = -i\partial/\partial x^i$. If the particle moves on the surface of a sphere in D dimensions or if the system is a spinning top, things are not as simple. Then canonical quantization rules are of no help due to ordering ambiguities arising in the attempt to do the above operator replacement in the kinetic part of the Hamiltonian, which now has the general form

$$H = \frac{1}{2M} p_\mu g^{\mu\nu}(q) p_\nu$$

with no principle instructing us how to order the momentum operators $\hat{p}_\mu = -i\partial/\partial q^\mu$ with with respect to the position variables $\hat{q}^\mu$.

For a particle on the surface of a sphere the most obvious way out of this dilemma is to make use of the symmetry of the system, express the classical Hamiltonian as a square of the classical angular momentum $\mathbf{L}$,

$$H = \frac{1}{2Mr^2}\mathbf{L}^2$$

(r = radius of the sphere), and replace $\mathbf{L}$ by the differential operators $\hat{\mathbf{L}}$ which generate rotations in the space of square integrable functions on the surface. The resulting Hamilton operator turns out to coincide with

$$\hat{H} = -\frac{1}{2Mr^2}\Delta \tag{1}$$

where Δ is the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{g}}\partial_\mu g^{\mu\nu}\sqrt{g}\partial_\nu. \tag{2}$$

This agrees with what we would find from the flat-space Hamiltonian operator

$$\hat{H} = -\frac{1}{2M}{\partial_{\mathbf{x}}}^2 \tag{3}$$

by subjecting it to a local *non-holonomic* transformation $dx^i \rightarrow e^i_\mu(q)dq^\mu$ from flat cartesian coordinates to curved space. This is a nontrivial observation since initially one is allowed to transform the flat-space Hamilton operator only to new coordinates, which is always done holonomically.

If one wants to find the correct rules for path integration in curved spaces one better comes out in agreement with such group quantization rules. This has recently been achieved [2, 3].

In spaces with torsion there is no physical system which could directly be used for an experimental confirmation of the result. Fortunately, however, there is the path integral of the hydrogen atom [4]. It can be defined and solved only by a transformation to new coordinates in which it becomes harmonic. This transformation happens to be non-holonomic and leads, at an intermediate stage, through a space with torsion. Also here it turns out that the correct Hamiltonian operator at the intermediate stage is simply the non-holonomically transformed flat-space operator.

The path integral of the Coulomb system can therefore be used as a testing ground for the correct treatment of torsion.

The recently proposed rule [2, 3] for writing down a path integral in spaces with curvature and torsion treats the above systems correctly. It can therefore be considered as a reliable *quantum equivalence principle* at the level of path integrals telling us how to generalize the Feynman path integral formula in cartesian coordinates to non-euclidean spaces. Hopefully, it will eventually lead to the correct measure for the functional integral of quantum gravity. Earlier proposals for the path integral given by DeWitt [5] and others [6, 7] produced various additional constants to the Schrödinger operator (1) which were proportional to the Riemannian scalar curvature $\bar{R}$. For a sphere of radius r this is $\bar{R} = (D-1)(D-2)/r^2$ and for the top $3/2I$, where I is the moment of inertia {for the asymmetric top with three moments of inertia it is $[(I_1+I_2+I_3)^2-2(I_1^2+I_2^2+I_3^2)]/2I_1I_2I_3\}$. Apart from contradicting the above natural quantization via the rotational Lie algebra such constants, if really present, would change the gravitational properties of interstellar gases of rotating molecules and will be rejected.

The first important success of the new quantum equivalence principle was the closing of an outstanding gap in the solution of the path integral of the $D=3$ Coulomb system [2]. The previous solution had been only structural in character [4] and the proper treatment of the time sliced expression had been limited only to the unphysical case of $D=2$ dimensions [8], the reason being that the combined coordinate and time transformations which make the system harmonic and integrable are holonomic in $D=2$ and do not produce curvature nor torsion. In $D=3$ where this happens it was the new quantum equivalence principle which finally led to the solution [9].

The purpose of this lecture is to review the essence of this new path integral approach.

2) Our starting point is the certainly valid path integral in a flat space parametrized with euclidean coordinates $\mathbf{x}$. It has the time-sliced form:

$$(\mathbf{x},t|\mathbf{x}',t')=\frac{1}{\sqrt{2\pi\epsilon\hbar/M}^D}\prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty}d\Delta x_n\right]\prod_{n=1}^{N+1}K_0^\epsilon(\Delta\mathbf{x}_n), \tag{4}$$

with the short-time amplitudes

$$K_0^\epsilon(\Delta\mathbf{x}_n)=\langle\mathbf{x}_n|\exp\left\{-\frac{1}{\hbar}\epsilon\hat{H}\right\}|\mathbf{x}_{n-1}\rangle\equiv\frac{1}{\sqrt{2\pi\epsilon\hbar/M}^D}\exp\left\{-\frac{1}{\hbar}\frac{M}{2}\frac{(\Delta\mathbf{x}_n)^2}{\epsilon}\right\} \tag{5}$$

where $\Delta\mathbf{x}_n\equiv\mathbf{x}_n-\mathbf{x}_{n-1}, \mathbf{x}\equiv\mathbf{x}_{N+1}, \mathbf{x}'\equiv\mathbf{x}_0$ (we may omit a possible extra potential which would enter trivially). We now transform $(\Delta\mathbf{x}_n)^2$ to a space with curvature and torsion by a *non-holonomic* mapping [10], parametrized with coordinates q^μ. For infinitesimal $\Delta\mathbf{x}_n\approx d\mathbf{x}_n$, the transformation would simply yield $(d\mathbf{x})^2 = g_{\mu\nu} dq^\mu dq^\nu$. For finite $\Delta\mathbf{x}_n$, however, we must expand $(\Delta\mathbf{x}_n)^2$ up to forth order in $\Delta q_n{}^\mu = q_n{}^\mu - q_{n-1}{}^\mu$ since only this yields all terms that will eventually contribute to order ϵ [11, 12]. We expand around the final point q_n^μ (omitting for brevity the argument q_n in the $e^i{}_\mu$'s as well a the subscripts n of Δq^μ):

$$x^i(q_{n-1}) \equiv x^i(q_n - \Delta q_n) = \\ x^i(q_n) - e^i{}_\mu \Delta q^\mu + \frac{1}{2} e^i{}_{\mu,\nu} \Delta q^\mu \Delta q^\nu - \frac{1}{3!} e^i{}_{\mu,\nu\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda + \ldots \; . \tag{6}$$

Squaring $\Delta\mathbf{x}_n$ and expressing everything in terms of the affine connection leads to the short-time sliced action expressed entirely in terms of intrinsic quantities (omitting again all sub n's),

$$\mathcal{A}^\epsilon_>(q, q - \Delta q) = \frac{M}{2\epsilon} \{ g_{\mu\nu} \Delta q^\mu \Delta q^\nu - \Gamma_{\mu\nu\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda \\ + \left[\frac{1}{3} g_{\mu\tau} (\partial_\kappa \Gamma_{\lambda\nu}{}^\tau + \Gamma_{\lambda\nu}{}^\delta \Gamma_{\kappa\delta}{}^\tau) + \frac{1}{4} \Gamma_{\kappa\lambda}{}^\sigma \Gamma_{\nu\kappa\sigma} \right] \Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\kappa + \ldots \} \tag{7}$$

with $g_{\mu\nu} \equiv e^i{}_\mu e^i{}_\nu$ and $\Gamma_{\mu\nu\lambda}$ evaluated at the final point q. The measure of path integration in (4) is transformed to q^μ-space with a Jacobian following from (6),

$$J = \frac{\partial(\Delta x)}{\partial(\Delta q)} = \det(e^i{}_\kappa) \det(\delta^\kappa{}_\mu - e_i{}^\kappa e^i_{\{\mu,\nu\}} \Delta q^\nu + \frac{1}{2} e^\kappa_i e^i_{\{\mu,\nu\lambda\}} \Delta q^\nu \Delta q^\lambda + \ldots), \tag{8}$$

where the curly brackets around the indices denote their symmetrization. Expanding the second determinant in powers of Δq^μ, writing $\det(e^i{}_\mu) \equiv e(q) = \sqrt{\det g_{\mu\nu}(q)} \equiv \sqrt{g(q)}$, and expressing the series in terms of a "Jacobian effective action" $\mathcal{A}_J$ with the definition $J = \sqrt{g(q)} \exp\{i\mathcal{A}_J/\hbar\}$, we find

$$\begin{aligned} \frac{i}{\hbar} \mathcal{A}_J &= -\Gamma_{\{\nu\mu\}}{}^\mu \Delta q^\nu \\ &+ \frac{1}{2} [\partial_{\{\mu} \Gamma_{\nu,\kappa\}}{}^\kappa + \Gamma_{\{\nu,\kappa}{}^\sigma \Gamma_{\mu\},\sigma}{}^\kappa - \Gamma_{\{\nu\kappa\}}{}^\sigma \Gamma_{\{\mu,\sigma\}}{}^\kappa] \Delta q^\nu \Delta q^\mu + \ldots \end{aligned} \tag{9}$$

and arrive at the time-sliced path integral in q-space

$$\langle q|\exp\{-\frac{1}{\hbar}(t-t')\hat{H}\}|q'\rangle \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}^D}\prod_{n=2}^{N+1}\left[\int d^D\Delta q_n \frac{\sqrt{g(q_n)}}{\sqrt{2\pi\epsilon\hbar/M}^D}\right]$$
$$\times\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}[\mathcal{A}_>^\epsilon(q_n, q_n-\Delta q_n)+\mathcal{A}_J]\right\} \tag{10}$$

The integrals over Δq_n are to be performed successively from $n=N$ down to $n=1$.

Our path integral is to be contrasted with that of earlier works. Expressed in our language, they start out from the time-sliced flat-space path integral

$$(\mathbf{x},t|\mathbf{x}',t') = \frac{1}{\sqrt{2\pi\epsilon\hbar/M}^D}\prod_{n=2}^{N+1}\left[\int_{-\infty}^{\infty}dx_n\right]\prod_{n=1}^{N+1}K_0^\epsilon(\Delta\mathbf{x}_n). \tag{11}$$

In flat space, this is the same thing as (4). Under a non-holonomic transformation, however, it is mapped into a *different* final expression. Here the measure would go over into the naive group invariant measure and the amplitude would read

$$\langle q|\exp\{-\frac{1}{\hbar}(t-t')\hat{H}\}|q'\rangle \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}^D}\prod_{n=2}^{N+1}\left[\int d^D q_{n-1} \frac{\sqrt{g(q_{n-1})}}{\sqrt{2\pi\epsilon\hbar/M}^D}\right]$$
$$\times\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}[\mathcal{A}_>^\epsilon(q_n, q_n-\Delta q_n)\right\}. \tag{12}$$

Expressing the correct amplitude (10) in terms of this naively expected measure we see that it reads

$$\langle q|\exp\{-\frac{1}{\hbar}(t-t')\hat{H}\}|q'\rangle \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/M}^D}\prod_{n=2}^{N+1}\left[\int d^D q_{n-1} \frac{\sqrt{g(q_{n-1})}}{\sqrt{2\pi\epsilon\hbar/M}^D}\right]$$
$$\times\exp\left\{-\frac{1}{\hbar}\sum_{n=1}^{N+1}[\mathcal{A}_>^\epsilon(q_n, q_n-\Delta q_n)+\Delta_{meas}\mathcal{A}_J]\right\} \tag{13}$$

with a correction term $\Delta_{meas}\mathcal{A}_J$ which is the difference

$$\Delta_{meas}\mathcal{A}_J = \mathcal{A}_J - \mathcal{A}_{J_0} \tag{14}$$

between $\mathcal{A}_J$ of (9) and $\mathcal{A}_{J_0}$ that arises when bringing the measure in (10)

$$\prod_{n=2}^{N+1}\left[\int d^D\Delta q_n \frac{\sqrt{g(q_n)}}{\sqrt{2\pi\epsilon\hbar/M}^D}\right]$$

to the naively expected form in (13), i.e.

$$\exp\{i\mathcal{A}_{J_0}/\hbar\} \equiv \sqrt{g(q_n)}/\sqrt{g(q_{n-1})} = e(q_n)/e(q_{n-1}). \tag{15}$$

Expanding the determinant $e(q_n) = e(q_{n-1} - \Delta q_n)$ in powers of Δq_n we see that $\mathcal{A}_{J_0}$ is the same as $\mathcal{A}_J$ in (9) except that symmetrization symbols are absent:

$$\begin{aligned} \frac{i}{\hbar}\mathcal{A}_{J_0} &= -\Gamma_{\{\nu\mu\}}{}^{\mu}\Delta q^{\nu} \\ &+ \frac{1}{2}[\partial_\mu \Gamma_{\nu,\kappa}{}^{\kappa} + \Gamma_{\nu,\kappa}{}^{\sigma}\Gamma_{\mu,\sigma}{}^{\kappa} - \Gamma_{\nu\kappa}{}^{\sigma}\Gamma_{\mu,\sigma}{}^{\kappa}]\Delta q^{\nu}\Delta q^{\mu} + \ldots, \end{aligned} \tag{16}$$

Either (10) or (13) may be used as the correct path integral formulas in spaces with curvature and torsion.

3) As an application consider now the path integral for the above discussed point particle on the surface of a sphere in D dimensions. First we solve an auxiliary problem *near* the surface of the sphere. Its imaginary-time-sliced form reads

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) \approx \frac{1}{\sqrt{2\pi\hbar\epsilon/Mr^2}^{D-1}}\prod_{n=1}^{N}\left[\int \frac{d^{D-1}\mathbf{u}_n}{\sqrt{2\pi\hbar\epsilon/Mr^2}^{D-1}}\right]\exp\left\{-\frac{1}{\hbar}\mathcal{A}^N\right\}, \tag{17}$$

with the sliced action

$$\mathcal{A}_0^N = \frac{M}{2\epsilon}r^2\sum_{n=1}^{N+1}(\mathbf{u}_n - \mathbf{u}_{n-1})^2 = \frac{M}{\epsilon}r^2\sum_{n=1}^{N+1}(1-\cos\Delta\vartheta_n), \tag{18}$$

where $\Delta\vartheta$ is the small angle between $\mathbf{u}_n$ and $\mathbf{u}_{n-1}$ (the $\approx$ sign becomes an equality for $N \to \infty$). There are two reasons for using the term *near* rather than *on* the sphere.

i) The sliced action of the solvable path integral (17) involves the shortest distances between the points in the *embedding* euclidean space rather than the intrinsic geodesic distances on the sphere. This will be easy to correct [13, 14].

ii) There is an additional action associated with the measure of path integration given by (14) [2].

The exact solution of the auxiliary path integral (17) goes as follows: For each time interval ϵ, the exponential $\exp\{-Mr^2(1-\cos\Delta\vartheta_n)/\hbar\epsilon\}$ is expanded into spherical harmonics according to formula

$$\begin{aligned}\exp&\left\{-\frac{Mr^2}{2\hbar\epsilon}(\mathbf{u}_n-\mathbf{u}_{n-1})^2\right\}\\ &=\sum_{l=0}^{\infty}a_l(h)\frac{l+D/2-1}{D/2-1}\frac{1}{S_D}C_l^{(D/2-1)}(\cos\Delta\vartheta_n)\\ &\qquad=\sum_{l=0}^{\infty}a_l(h)\sum_{\mathbf{m}}Y_{l\mathbf{m}}(\mathbf{u}_n)Y^*_{l\mathbf{m}}(\mathbf{u}_{n-1})\end{aligned}\tag{19}$$

where $a_l(h)\equiv(2\pi/h)^{(D-1)/2}\,\tilde{I}_{l+D/2-1}(h)$, $\tilde{I}_\nu(z)\equiv\sqrt{2\pi z}e^{-z}I_\nu(z)$ with $I_\nu(z)=$ Bessel functions, and $h\equiv Mr^2/\hbar\epsilon$. The functions $C_l^{(\nu)}(z)$ are the Gegenbauer polynomials and $Y_{l\mathbf{m}}(\mathbf{u})$ the hyperspherical harmonics in D dimensions [15]. For each adjacent pair of such factors $(n+1,n),(n,n-1)$, the integration over the intermediate $\mathbf{u}_n$ variable can be done using the well-known orthonormality relation for the hyperspherical harmonics. The combined two-step amplitude has the same expansion as (19) with $a_l(h)$ replaced by $(h/2\pi)a_l(h)^2$. By successive integration in (17) we obtain the total time sliced amplitude

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a)\approx\left(\frac{h}{2\pi}\right)^{(N+1)(D-1)/2}\sum_{l=0}^{\infty}a_l(h)^{N+1}\sum_{\mathbf{m}}Y_{l\mathbf{m}}(\mathbf{u}_b)Y^*_{l\mathbf{m}}(\mathbf{u}_a).\tag{20}$$

We now go to the limit $N \to \infty, \epsilon = (\tau_b - \tau_a)/(N+1) \to 0$, where

$$\left(\frac{h}{2\pi}\right)^{(N+1)(D-1)/2} a_l(h)^{N+1} = \left[\tilde{I}_{l+D/2-1}\left(\frac{Mr^2}{\hbar\epsilon}\right)\right]^{N+1}$$
$$\longrightarrow \exp\left\{-(\tau_b - \tau_a)\hbar\frac{(l+D/2-1)^2 - 1/4}{2Mr^2}\right\}, \tag{21}$$

and obtain the time displacement amplitude for the motion *near* the sphere as the spectral expansion

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_{l=0}^{\infty} \exp\left\{-\frac{\hbar L_2}{2Mr^2}(\tau_b - \tau_a)\right\} \sum_{\mathbf{m}} Y_{l\mathbf{m}}(\mathbf{u}_b)Y^*_{l\mathbf{m}}(\mathbf{u}_a), \tag{22}$$

with

$$L_2 \equiv (l + D/2 - 1)^2 - 1/4, \tag{23}$$

and the energy eigenvalues $E_l^{near} = \hbar^2 L_2/2Mr^2$. For $D = 4$, the most convenient expansion is in terms of the representation functions $\mathcal{D}^l_{mm'}(\varphi, \theta, \gamma)$ of the rotation group, involving the Euler angle parametrization of the vectors on the unit sphere

$$\begin{aligned}
\hat{x}^1 &= \cos(\theta/2)\cos[(\varphi+\gamma)/2] \\
\hat{x}^2 &= -\sin(\theta/2)\sin[(\varphi+\gamma)/2] \\
\hat{x}^3 &= \sin(\theta/2)\cos[(\varphi-\gamma)/2] \\
\hat{x}^4 &= \sin(\theta/2)\sin[(\varphi-\gamma)/2].
\end{aligned} \tag{24}$$

In terms of these, (22) reads

$$(\mathbf{u}_b\tau_b|\mathbf{u}_a\tau_a) = \sum_{l=0}^{\infty} \exp\left\{-\frac{\hbar L_2}{2Mr^2}(\tau_b - \tau_a)\right\}$$
$$\times \sum_{m_1,m_2=-l}^{l} \frac{l+1}{2\pi^2}\mathcal{D}^{l/2}_{m_1 m_2}(\varphi_b, \theta_b, \gamma_b)\mathcal{D}^{l/2\,*}_{m_1 m_2}(\varphi_a, \theta_a, \gamma_a). \tag{25}$$

These amplitudes display the correct wave functions for the movement *on* the surface of the sphere, as we know from Schrödinger theory. They do not, however, carry the correct energy eigenvalues which should be $E_l = \hbar^2 \hat{L}^2/2Mr^2$ with the eigenvalue of the squared angular momentum operator $\hat{L}^2 = l(l+D-2)$ rather than E_l^{near} with $L_2 = (l+D/2-1)^2 - 1/4$.

To have the correct energies, the path integral needs the two changes announced above. First, the time-sliced action must measure the proper geodesic distance rather than the euclidean distance in the embedding space and should thus read

$$\mathcal{A}^N = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \frac{(\Delta\vartheta_n)^2}{2}, \tag{26}$$

rather than (18). Since the time-sliced path integral is solved exactly with the action (18) it is convenient to expand the true action around the soluble one as [16]

$$\mathcal{A}^N = \mathcal{A}_0^N + \Delta_4 \mathcal{A}^N = \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \left[(1 - \cos \Delta\vartheta_n) + \frac{1}{24} \Delta\vartheta_n^4 + \dots \right], \tag{27}$$

and treat the correction perturbatively to lowest order. There is no need to go higher than quartic order since only the quartic term contributes to the relevant order ϵ in the limit $N \to \infty$. In $D = 2$ dimensions, the quartic correction is sufficient to bring the path integral from *near* to *on* the sphere (here a circle). Indeed, with the measure of the path integration being

$$\frac{1}{\sqrt{2\pi\hbar\epsilon/Mr^2}} \prod_{n=1}^{N+1} \int_{-\pi/2}^{\pi/2} \frac{d\varphi_n}{\sqrt{2\pi\hbar\epsilon/Mr^2}} \tag{28}$$

and the leading action (26), the quartic term $\Delta\vartheta_n^4 = (\varphi_n - \varphi_{n-1})^4$ can be replaced by its expectation

$$\langle \Delta\vartheta_n^4 \rangle_0 = 3 \frac{\epsilon\hbar}{Mr^2}, \tag{29}$$

so that the correction term of the action is given by

$$\langle \Delta_4 \mathcal{A}^N \rangle_0 = (\tau_b - \tau_a) \frac{\hbar^2/4}{2Mr^2}. \tag{30}$$

where we have replaced $(N+1)\epsilon$ by $\tau_b - \tau_a$. For $D = 2$, this supplies precisely the missing energy to raise E_l^{near} up to E_l.

In higher dimensions, we must change also the measure of path integration is necessary according to [2]. What we have to explain in any D is the difference

$$\Delta L_2 = \hat{L}^2 - L_2 = 1/4 - (D/2 - 1)^2. \tag{31}$$

This vanishes at $D = 3$ where it changes sign. Note that the expectation of the quartic correction term $\Delta_4 \mathcal{A}^N$ in (27) being always positive cannot account for the discrepancy by itself. Let us calculate its contribution in D dimensions. For very small ϵ, the fluctuations near the sphere will lie close to the $D-1$ dimensional tangent space. Let $\Delta \mathbf{x}_n$ be the coordinates in this space. Then we can write

$$\Delta_4 \mathcal{A}^N \approx \frac{M}{\epsilon} r^2 \sum_{n=1}^{N+1} \frac{1}{24} (\frac{\Delta \mathbf{x}_n}{r})^4. \tag{32}$$

The $\Delta \mathbf{x}_n$'s have the lowest order correlation $\langle \Delta x_i \Delta x_j \rangle_0 = (\hbar\epsilon/M)\delta_{ij}$. This shows that $\Delta_4 \mathcal{A}^N$ has the expectation

$$\langle \Delta \mathcal{A}^N \rangle_0 = (\tau_b - \tau_a) \frac{\hbar^2}{2Mr^2} \Delta_4 L_2. \tag{33}$$

where $\Delta_4 L_2$ is the contribution of the quartic term to the value L_2,

$$\Delta_4 L_2 = \frac{D^2 - 1}{12}. \tag{34}$$

This result is obtained using the Wick contraction rules for the tensor

$$\langle \Delta x_i \Delta x_j \Delta x_k \Delta x_l \rangle_0 = (\epsilon\hbar/M)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

Thus we remain with a final discrepancy in D dimensions,

$$\Delta_{meas} L_2 = \Delta L_2 - \Delta_4 L_2 = -\frac{1}{3}(D-1)(D-2), \tag{35}$$

to be explained now.

The present problem involves no torsion. Then a simple algebra shows that $\Delta_{meas}\mathcal{A}_J$ defined by (14) with (9), (16) reduces to

$$\Delta_{meas}\mathcal{A}_J = -\frac{\hbar}{6}\bar{R}_{\mu\nu}\Delta q^\mu \Delta q^\nu, \tag{36}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor, which for a sphere of radius r is $(D-2)g_{\mu\nu}/r^2$. The perturbative treatment of (36) gives the only relevant contribution to the energy,

$$\langle \Delta_{meas}\mathcal{A}_J \rangle_0 = -\epsilon \frac{\hbar^2}{6M}\frac{(D-1)(D-2)}{r^2}, \tag{37}$$

thus producing precisely the missing energy required by (35).

4) The sphere in four dimensions is equivalent to the covering group of rotations in three dimensions, the group $SU(2)$. Knowing now how to solve the time-sliced path integral near and on the surface of the sphere, we can obtain the same quantities *near* and *on* the group space of $SU(2)$ [17]. This puts us in a position to solve the time sliced path integral of a spinning spherical top by reduction to the $SU(2)$ problem. We only have to go from $SU(2)$, which is the covering group of the rotation group, down to the rotation group itself [18]. The angular positions with Euler angles γ and $\gamma + 2\pi$ are physically indistinguishable. The physical states must be a representation of this operation and the time-displacement amplitude must reflect this. The simplest possibility is the trivial even representation where one adds the amplitudes to go from the initial configuration $\varphi_a, \theta_a, \gamma_a$ to the identical final ones $\varphi_b, \theta_b, \gamma_b$ and $\varphi_b, \theta_b, \gamma_b + 2\pi$ and forms the amplitude

$$\begin{aligned} &(\varphi_b, \theta_b, \gamma_b\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a)_{top} = \\ &\qquad (\varphi_b, \theta_b, \gamma_b\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a) + (\varphi_b, \theta_b, \gamma_b + 2\pi\ \tau_b|\varphi_b, \theta_b, \gamma_b\ \tau_a). \end{aligned} \tag{38}$$

The sum eliminates all half-integer representation functions $d^{l/2}_{mm'}(\theta)$ in the expansion (17) of the amplitude.

Instead of the sum we could also have formed another representation of the operation $\gamma \to \gamma + 2\pi$, the antisymmetric combination

$$
\begin{aligned}
&(\varphi_b, \theta_b, \gamma_b\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a)_{fermions} = \\
&\qquad (\varphi_b, \theta_b, \gamma_b\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a) - (\varphi_b, \theta_b, \gamma_b + 2\pi\ \tau_b | \varphi_b, \theta_b, \gamma_b\ \tau_a).
\end{aligned} \tag{39}
$$

Here the expansion (25) retains only the half-integer angular momenta $l/2$. In nature such spins are associated with fermions such as electrons, protons, muons, or neutrinos, which carry only one specific value of $l/2$.

In principle, there is no problem in treating also a non-spherical top. While the spherical top has "near the group space" a time-sliced action

$$
\frac{1}{\epsilon^2} I \left[1 - \frac{1}{2}\mathrm{tr}(g_n g_{n-1}^{-1})\right], \tag{40}
$$

the asymmetric top with three moments of inertia I_{123} requires separating the three components of the angular velocities

$$
\omega_a = i\mathrm{tr}(g \sigma_a \dot{g}^{-1}), \quad a = 1, 2, 3. \tag{41}
$$

(σ_a= Pauli matrices) on the time lattice so that the action reads

$$
\begin{aligned}
\frac{1}{\epsilon^2} \Big\{ & I_1[1 - \frac{1}{2}\mathrm{tr}(g_n \sigma_1 g_{n-1}^{-1})] + I_2[1 - \frac{1}{2}\mathrm{tr}(g_n \sigma_2 g_{n-1}^{-1})] \\
& + I_3[1 - \frac{1}{2}\mathrm{tr}(g_n \sigma_3 g_{n-1}^{-1})] \Big\}.
\end{aligned} \tag{42}
$$

rather than (40). The amplitude "near the top" is then an appropriate generalization of (25). The calculation of the correction term ΔE, however, is more complicated than before and is left to the reader, following the rules explained above.

5) As mentioned above, the correctness of the torsional aspects of the proposed path integral (10) or (13) can be tested by applying it to the path integral of the Coulomb system. The procedure is too lengthy to be presented here and we refer the reader to the textbook [3] for a detailed discussion.

References

[1] We call a coordinate transformation $dx^i \to e^i_\mu(q)dq^\mu$ *non-holonomic* if either $\partial_\mu e^i_\nu - \partial_\mu e^i_\nu \neq 0$ or $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)e^i_\lambda \neq 0$. In the first case, the mapping carries a flat space into one with torsion, in the second case to one with curvature.

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[7] For other discussions with results different from the above and ours see
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N.H. Vilenkin, *Special Functions and the Theory of Group Representations*, Am. Math. Soc., Providence, R I, 1968.

[16] This step was still done in [13]. Note, however, that the (known) correct final result stated in that paper is impossible to obtain from their calculation since the measure problem, which is the main issue of the present paper, was not solved at that time.

[17] For path integrals near group spaces (although claimed to work on group spaces) see M. Böhm, A. Junker, J. Math. Phys. *30*, 1195 (1989).

[18] This point was discussed by L. Schulman, Phys. Rev. *174*, 1558 (1968). The author also gives a correct path integral but he does so a posteriori, by reconstructing it from the known spectral representation of the Schrödinger result.