

Exact evolution operator on non-compact group manifolds

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Abstract

Free quantal motion on group manifolds is considered. The Hamiltonian is given by the Laplace – Beltrami operator on the group manifold, and the purpose is to get the (Feynman's) evolution kernel K_t . The spectral expansion, which produced a series of the representation characters for K_t in the compact case, does not exist for non-compact group, where the spectrum is not bounded. In this work real analytical groups are investigated, some of which are of interest for physics. An integral representation for K_t is obtained in terms of the Green function, i.e. the solution to the Helmholtz equation on the group manifold. The alternative series expressions for the evolution operator are reconstructed from the same integral representation, the spectral expansion (when exists) and the sum over classical paths. For non-compact groups, the latter can be interpreted as the (*exact*) semi-classical approximation, like in the compact case. The explicit form of K_t is obtained for a number of non-compact groups.

1 Introduction

Normal physical systems have energy spectra bounded from below, so a stable ground state exists and may be considered as a zero-temperature limit of the Gibbs thermal state. If the Hamiltonian operator \hat{H} is positive semi-definite (we shall use the notation $\hat{H} \geq 0$), the Gibbs density operator $\exp(-\beta\hat{H})$ exists and its kernel is the fundamental solution of the Bloch equation $\partial\Psi/\partial\beta = -\hat{H}\Psi$, for $\beta > 0$. This equation, however, has no stable solution if the spectrum of \hat{H} is extended to $-\infty$.

On the other hand, the Schrödinger equation and the corresponding evolution operator $\hat{U}_t \equiv \exp(-it\hat{H})$,

$$(1) \quad i\partial\Psi/\partial t = \hat{H}\Psi, \quad \Psi_t = \hat{U}_t\Psi_0,$$

may be meaningful for a regular self-adjoint operator \hat{H} even if its spectrum is not bounded at all. The evolution operator may be defined for any real t (in a properly define Hilbert space of the wave functions Ψ), even if it cannot be continued analytically to the complex (lower-half) t -plane, as in the usual case, where points on the negative imaginary axis correspond to inverse temperatures. Note that (1) is a wave equation for real t (even being first-order in t), and not a parabolic heat-transport (or Bloch) equation, where the positive definiteness of \hat{H} is essential. Actually, it may be extended to the real form $\partial^2\Psi/\partial t^2 + \hat{H}^2\Psi = 0$, which is of the hyperbolic type, since $\hat{H}^2 \geq 0$ (like $-\Delta$ in the standard wave equation), so the Cauchy (initial-value) problem has a proper solution [20].

It is remarkable that equations of the type (1) with non-definite \hat{H} were considered extensively for functions on the pseudo - Euclidean (Minkowski) space. In that case, $\hat{H} = \square$, i.e. the d'Alembertian (or a more complicated operator in presence of an external field), and t played the role of the *proper time* (the classical references are [15, 26, 28, 29]).

Analysis of the Schrödinger-type equation (1) for a (non-definite) operator \hat{H} enables one to get an insight into the properties of its spectrum and the eigen-functions, as was the case in the proper-time formalism. We shall consider the free quantal motion on group manifolds, which is described by Eq. (1) where $\hat{H} = -\Delta$ is the second-order Laplace – Beltrami operator. For compact groups, $-\Delta \geq 0$, and the complete solution is known for decades (see Section 2 for references). As to real non-compact groups, Δ is well defined but indefinite, like \square in the Minkowski space. It is an interesting class of problems which can be also solved completely, as shown in this work.

If $\hat{H} \geq 0$, the evolution operator can be represented by its spectral expansion, which is a convergent series for $t > 0$ (and $\hat{U}_0 = \hat{I}$ - the unit operator),

$$(2) \quad \hat{U}_t = \sum_{n \geq 0} e^{-i\varepsilon_n t} \psi_n \otimes \psi_n^*, \quad \hat{H}\psi_n = \varepsilon_n \psi_n.$$

In the situation concerned here, the series would not be convergent, yet one can use the Laplace representation of \hat{U}_t in terms of the resolvent \hat{G}_λ ,

$$(3) \quad \hat{U}_t = \frac{1}{2\pi i} \int_C \hat{G}_\lambda e^{-i\lambda t} d\lambda, \quad \hat{G}_\lambda \equiv (\hat{H} - \lambda)^{-1}.$$

The contour C in the complex λ -plane should be defined properly, with account of singularities of \hat{G}_λ , which take place on the real axis, since $\hat{H} = \hat{H}^\dagger$. It is assumed that the contour is in the upper half-plane, according to the principle of causality, so that $\hat{U}_t = 0$ for $t < 0$.

Now the resolvent generates the solution to the inhomogeneous Helmholtz - type equation [25], within the properly defined class of functions,

$$(4) \quad (\hat{H} - \lambda)\psi = f, \quad \psi_\lambda = \hat{G}_\lambda f.$$

(Note that for $\hat{H} = \square$ and $\lambda = -(m^2 - i\epsilon)$, $\epsilon > 0$, the integral kernel of \hat{G}_λ is just the standard causal propagator of a massive scalar particle.) For $\hat{H} \geq 0$, the singularities of \hat{G}_λ are all on the half-axis $\text{Re } \lambda \geq 0$. In that case, for $t > 0$, the contour C may be deformed to the lower half-plane and closed there at ∞ (fig. 1), so the residues of the poles at $\lambda = \varepsilon_n$ (and the imaginary part of Green's function for the continuous part of the spectrum, respectively) would reproduce the spectral expansion (2). In the case concerned here, however, singularities appear on the whole real axis, and Eq. (3) provides with a more general representation which cannot be reduced to the series in (2). Now equation (4) is of the hyperbolic type, like the Klein - Gordon equation, yet it should be solved for all (complex) values of λ which appear in the Laplace integral (3).

The problem is solved in the following way. The Helmholtz equation on the group manifold is considered, and the integral kernel of the resolvent operator is constructed explicitly. The result is the integral representation (3) for the evolution kernel, to be subject to a further analysis. Remarkably, the integral for the evolution kernel can be represented as a series of terms which correspond to contributions from "classical paths" (geodetics) connecting the points of the group manifold to its origin (the group unity). For non-compact groups, like in the compact case, the semi-classical approach (including the pre-exponential factor) leads to the *exact* result, provided that *all* classical paths are taken into account. For compact groups the series are infinite and r -fold (r is the group rank). In contrast, each non-compact group is splitted in a number of classes (like the Minkowski space, containing "space-like" and "time-like" vectors). Each class has its specific set of paths and the corresponding series for the evolution kernel.

After a description of the approach in general in Sections 2 and 3, the known results are reproduced for compact groups in section 4. In section 5 the evolution operator is built for non-compact groups, but the expression that is obtained depends on the maximal torus topology, which is different in different domains on the manifold, and corresponds to the different 'classes' of the classical paths. The mathematical tools that are needed in order to analyze non-compact groups, and in particular the different domains and the corresponding maximal tori are given in section 6. Some particular examples of real non-compact groups are presented in Sections 7 and 8. Necessary mathematical notations and results are given in Appendix.

2 Free motion on a group manifold

The Lie groups have the natural Riemannian structure given by the Cartan - Killing metrics (see Appendix A). For semi-simple Lie groups, the Riemannian metric is non-degenerate, and the invariant second-order Laplace - Beltrami operator (called here Laplacean Δ) is defined, as usual. (A general theory of invariant differential operators on the group manifolds was given by Berezin[3, 4].) Free motion on the group manifold is introduced naturally by means of the Schrödinger-type wave equation,

$$(5) \quad i\partial\Psi(g)/\partial t = -\Delta\Psi(g), \quad \int_G |\Psi(g)|^2 d\mu(g) = 1,$$

where $\Psi(g)$ is a (square-integrable) function on the group, $g \in \mathbf{G}$, and $d\mu(g)$ is the invariant measure on \mathbf{G} , given by the Riemannian structure. The solution of the wave equation (5), for any initial condition $\Psi_0(g)$, is given by the evolution kernel \mathcal{K}_t ,

$$(6) \quad \Psi_t(g_t) = \int_{\mathbf{G}} \mathcal{K}_t(g_t, g_0) \Psi_0(g_0) d\mu(g_0), \quad \mathcal{K}_t(g_t, g_0) \equiv \langle g_t | \hat{U}_t | g_0 \rangle.$$

Because of the invariance of Eq. (5) under the shifts on the group, $g \rightarrow g_1 g g_2$ for all $g_1, g_2 \in \mathbf{G}$, the evolution kernel is reduced to an invariant function on the group manifold,

$$(7) \quad \mathcal{K}_t(g_t, g_0) \equiv K(g_t g_0^{-1}); \quad K_t(g) = K_t(g_1 g g_1^{-1}), \quad \forall g_1 \in \mathbf{G}.$$

Moreover, because of the latter property, the evolution kernel $K_t(g)$ is a *central function*. Namely, it depends in fact only on the element of the Cartan subgroup, i.e. the maximal torus \mathbf{T} , $h \in \mathbf{T} \subset \mathbf{G}$, where $g = v h v^{-1}$. The element v is a representative of the coset space $\mathbf{V} = \mathbf{G}/\mathbf{T}$, and $K_t(g) \equiv K_t(h)$ is independent of v .

The wave equation on the group $SU(2)$ was considered by Bopp and Haag[5] and Schulman[27]. Schulman presented the explicit solution for $SU(2)$ as well as for $SO(3) = SU(2)/Z_2$ and showed that the semi-classical approximation is exact in that case. The heat transport equation, which may be considered as the analytical continuation of Eq. (5) to negative imaginary values of t , was solved by Eskin[14] for all compact Lie groups. The solutions of the wave equation for *compact* groups were considered in a number of works[12, 13, 22, 23]; see also Ref.[9] for a review.

For compact Lie groups, the evolution kernel has the *spectral expansion*, which is the sum over all unitary irreducible representations,

$$(8) \quad K_t(h) = \frac{1}{V_{\mathbf{G}}} \sum_{\mathbf{l} \in \Lambda} d_{\mathbf{l}} \chi_{\mathbf{l}}(h) \exp(-i \lambda_{\mathbf{l}} t).$$

Here \mathbf{l} are the representation dominant weights, Λ is the weight lattice in the r -dimensional root space ($r = \text{rank}(\mathbf{G})$), $d_{\mathbf{l}}$ is the representation dimensionality, $\chi_{\mathbf{l}}(h)$ is the representation character, $\lambda_{\mathbf{l}}$ is the eigen-value of the second-order Casimir operator, corresponding to the Laplacean $-\Delta$, and

$$(9) \quad V_{\mathbf{G}} \equiv \int_{\mathbf{G}} d\mu(g)$$

is the invariant volume of the group manifold.

On the other hand, employing the Poisson transformation for θ -functions[2], the evolution kernel can be represented as a sum over all the *classical trajectories* (geodetics) on the group manifold, connecting g_0 and g_t . The geodetics are described by means of a r -dimensional vector φ in the Euclidean space tangent to \mathbf{T} . The result is

$$(10) \quad K_t(g) = \frac{1}{(4\pi i t)^{n/2}} \sum_{\mathbf{m} \in \Gamma} F_{\mathbf{m}}(\varphi) \exp \left[i \frac{S_{\mathbf{m}}^2(\varphi)}{4t} + i t \frac{n}{24} \right].$$

Here n is the group dimensionality, \mathbf{m} is the winding-number vector on the lattice Γ , dual to Λ , $S_{\mathbf{m}}(\varphi)$ is the distance from the origin to the point $h \in \mathbf{T}$, as measured along a line winded a number of times around the torus, and $F_{\mathbf{m}}(\varphi)$ is a known function (see in Section 4), which appears as the van Vleck determinant of the semi-classical approximation and may be considered as the first quantum correction. The first term in the exponent is the classical action, and the second term is the constant (and the last) quantum correction. It was shown that for any compact Lie group the *semi-classical approximation is exact* (a discussion is given in Ref.[9]). In the other words, the sum of contributions from solutions of the classical equations of motion satisfies the wave equation (5) with the initial condition,

$$(11) \quad \lim_{t \rightarrow 0} K_t(g) = \delta(g).$$

Here the δ -function on \mathbf{G} is defined as usual with the integration measure which is employed in Eq. (5). Note that every separate term in Eq. (10) is a function on \mathbf{T}^* , the space tangent to \mathbf{T} , and does not satisfy the desired boundary conditions on \mathbf{T} , but the series as a whole is indeed a function on the group.

For compact groups the spectrum of $-\Delta$ is positive semi-definite, i.e. $\lambda_l \geq 0$, and the series in (4) is convergent in the complex t -plane below the real axis. Because of the same reason, the heat transport equation has a stable solution. For non-compact Lie groups, however, the spectrum is not positive, so evidently the spectral expansion does not exist. On one hand, the series cannot be convergent even for complex t . On the other hand, the unitary representations are all infinite-dimensional, so d_l would be infinite, as well as the volume of Eq. (9), while the characters are singular. The heat transport equation would have no stable solutions, but one may still consider the wave equation, determine the appropriate class of the wave functions and look for a valid representation of the evolution kernel. This is done in the present work.

3 Green's function

The operator equation (3) is equivalent to the Laplace transform for the corresponding integral kernels,

$$(12) \quad K_t(h) = \frac{1}{2\pi i} \int_{\mathbb{C}} G_\lambda(h) e^{-i\lambda t} d\lambda, \quad G_\lambda(g_1 g_0^{-1}) \equiv \langle g_1 | \hat{G}_\lambda | g_0 \rangle.$$

Like $K_t(g)$, $G_\lambda(g)$ is a central function on the group. It is Green's function for the (inhomogeneous) Helmholtz equation, which means that it solves the following problem,

$$(13) \quad (\Delta + \lambda)\psi(g) = -f(g), \quad \psi(g_1) = \int_{\mathbb{G}} G_\lambda(g_1 g_0^{-1}) f(g_0) d\mu(g_0),$$

where the proper boundary conditions are taken into account.

The coordinates on the group manifold are introduced by the decomposition $g = v h v^{-1}$, $\forall g \in \mathbb{G}$. The measure is factorized, and the Laplacean is splitted respectively,

$$(14) \quad \Delta = \Delta_{\mathbb{T}} + w^{-2}(h) \Delta_{\mathbb{V}}.$$

where $\Delta_{\mathbb{T}}$ is the *radial* part and $\Delta_{\mathbb{V}}$ is the *angular* part of the Laplacean; $w(h)$ is the Weyl function on \mathbb{T} ,

$$(15) \quad w(h) \equiv \prod_{\alpha > 0} \sin(\alpha \varphi / 2), \quad h(\varphi) = \exp(i \varphi^j H_j) \in \mathbb{T}.$$

H_j , $j = 1, \dots, r$ are the basis elements of the Cartan Subalgebra, φ^j are the radial group parameters, which reside in the r -dimensional space tangent to \mathbb{T} .

As was shown by Berezin[3], the radial Laplacean may be reduced to the Euclidean form, as follows,

$$(16) \quad \Delta_{\mathbb{T}} \equiv \frac{1}{\Lambda} \frac{1}{w^2} \frac{\partial}{\partial \varphi} w^2 \frac{\partial}{\partial \varphi} = \frac{1}{\Lambda} \left[\frac{1}{w} \frac{\partial^2}{\partial \varphi^2} w + \rho^2 \right].$$

(The constants Λ and ρ^2 depends on normalization, see Eq. 78. In our convention, $\rho^2/\Lambda = \frac{n}{24}$.) Thus the desired Green's function can be reduced to that for the (pseudo)- Euclidean problem

$$(17) \quad \frac{\partial^2 y}{\partial \varphi^2} + \epsilon y = -F,$$

where

$$(18) \quad y = w\psi, \quad F = \Lambda w f, \quad \epsilon = \rho^2 + \Lambda \lambda.$$

The geometry is flat, yet the group structure manifests itself in the boundary conditions. In the two following sections, the boundary conditions are analyzed for compact and non-compact groups, the Green functions are constructed explicitly and inserted in the integral representation Eq. 12 to produce the exact evolution operators.

4 Evolution operator on compact groups

The boundary conditions are naively determined by requiring the vanishing of y on the hyper-surfaces of vanishing w . The smallest domain enclosed by these hyper-surfaces in the r -dimensional root space is called the Weyl alcove[6]. (For example, the Weyl alcove of $SU(3)$ is shown in fig. 2). The Green function is found using the image method[25]

$$(19) \quad G^y(\varphi, \varphi_0) = \sum_{\sigma} \epsilon_{\sigma} G^r(\sigma\varphi, \varphi_0).$$

G^r is the known Green function for Helmholtz Eq. in r -dimensional flat infinite space \mathbb{R}^r . The superscript y is used to remind us that this is the Green function for Eq. 17. The summation is over all reflections of the point φ inside the Weyl alcove, $\epsilon_{\sigma} = 1(-1)$ if the reflection is even (odd).

Since we are interested in the Green function itself, the boundary conditions can be imposed on it directly. The boundary conditions, for which the Green function on the group manifold should account, are periodicity in the radial parameters, and symmetry under Weyl reflections (central functions are invariant under Weyl reflections, which permute the eigenvalues of the group element). The multiplication by the Weyl function 18 which is antisymmetric under Weyl reflections imposes antisymmetry on the Green function in the flat space, G^y ,

$$(20) \quad G^y(\varphi, \varphi_0) = \sum_{\mathbf{m}} \sum_{\sigma \in W} \epsilon_{\sigma} G^r[\sigma(\varphi + 2\pi\mathbf{m}), \varphi_0]$$

where W is the Weyl group, $\sigma \in W$ is a Weyl reflection, $\epsilon_{\sigma} = +1(-1)$ for σ even (odd), and \mathbf{m} is the winding numbers vector

$$(21) \quad \mathbf{m} = \sum_{i=1}^r m_i \frac{2\gamma_i}{\gamma_i^2}, \quad m_i = 0, \pm 1, \pm 2, \dots$$

γ_i , ($i = 1, \dots, r$) are the simple roots. The Lattice of images that is created by reflections of a point inside the Weyl alcove of $SU(3)$ is shown in fig. 3. It is important to note (for future calculations) that there are two equivalent ways to perform the summation

$$(22) \quad \sum_{\mathbf{m}} \sum_{\sigma \in W} \epsilon_{\sigma} f[\sigma(\varphi + 2\pi\mathbf{m})] = \sum_{\mathbf{m}} \sum_{\sigma \in W} \epsilon_{\sigma} f[\sigma\varphi + 2\pi\mathbf{m}].$$

$G^y(\varphi, \varphi_0)$ is the Green function for Helmholtz equation in flat space with nontrivial boundary conditions, and not the resolvent that appears in the integral representation Eq. 12. The resolvent $G_{\lambda}(\varphi)$, where φ are the radial parameters that correspond to the group element $g_1 g_0^{-1}$, is obtained by taking into account the substitutions 18, the different integration measures on the group manifold and the flat root space, and the fact that the radial parameters already represent the ‘distance’ between two points on the manifold so the point φ_0 should be set to zero (the procedure is explained in more details in appendix D). On the other hand, it may be more convenient to continue the computation with G^y , insert G^y into the integral representation 12 and get the evolution operator in flat r dimensional space. The evolution operator on the group manifold is then obtained by the same procedure

$$(23) \quad G_{\lambda}(\varphi) = \frac{\Lambda}{V_{G/T} w(\varphi)} \left[\frac{G^y(\varphi, \varphi_0)}{\Lambda^{r/2} 2^{n-r} w(\varphi_0)} \right]_{\varphi_0=0}, \quad K_t(\varphi) = \frac{\Lambda}{V_{G/T} w(\varphi)} \left[\frac{K^y(\varphi, \varphi_0)}{\Lambda^{r/2} 2^{n-r} w(\varphi_0)} \right]_{\varphi_0=0}$$

$V_{G/T}$ is the volume of the quotient space G/T (Eq. 106) in the compact case, and a normalization constant in the non-compact case.

The integration 12 can be performed in two alternative ways leading to the known expressions 8,10 for the evolution operator. Using the residue method, the spectral expansion is reconstructed. Integrating the infinite sum 20 term by term we get an exact expression for the evolution operator which is interpreted as the sum over classical paths. Up till now, the sum over classical paths was built using the semi-classical approximation, and its exact equivalence to the spectral expansion was proven by using multidimensional Theta function theorems [2, 23]. The fact that this expression can be computed directly using no approximations is a new result that will be useful in the non-compact case.

4.1 Sum over classical paths

We shall use the following integral representation for the Green function for Helmholtz equation in flat r dimensional infinite space

$$(24) \quad G_k^r(\varphi, \varphi_0) = \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} d^r \mathbf{p} \frac{1}{p^2 - k^2} \exp[i\mathbf{p} \cdot (\varphi - \varphi_0)],$$

$k^2 = \rho^2 + \Lambda\lambda$ in our case, and \mathbf{p} is an r -dimensional vector. Inserting one term of the infinite sum 20 into the integral representation for the evolution operator and changing the order of integration yields

$$(25) \quad \begin{aligned} K^r(\varphi, \varphi_0) &= \frac{1}{2\pi i} \int_{\mathcal{C}} e^{-i\lambda t} G_{\rho^2 + \Lambda\lambda}^r(\varphi, \varphi_0) d\lambda = \\ &= \frac{1}{2\pi i} \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} d^r \mathbf{p} \exp[i\mathbf{p} \cdot (\varphi - \varphi_0)] \int_{\mathcal{C}} e^{-i\lambda t} \frac{1}{p^2 - \rho^2 - \Lambda\lambda} d\lambda. \end{aligned}$$

To account for the singularities of the resolvent which lie on the real axis in the complex λ plane, the integration contour \mathcal{C} passes infinitesimally above the real axis, and it closes underneath at infinity where the integrand vanishes (for $t > 0$). Performing the integration in λ by the residue method we get

$$(26) \quad \begin{aligned} K^r(\varphi, \varphi_0) &= \frac{1}{(2\pi)^r} \int_{\mathbb{R}^r} d^r \mathbf{p} \exp \left[i\mathbf{p} \cdot (\varphi - \varphi_0) - i(p^2 - \rho^2) \frac{t}{\Lambda} \right] = \\ &= \left(\frac{\Lambda}{4\pi i t} \right)^{r/2} \frac{1}{\Lambda} e^{i \frac{\rho^2}{\Lambda} t} \exp \left[\frac{i\Lambda}{4t} (\varphi - \varphi_0)^2 \right]. \end{aligned}$$

Thus, the infinite sum (Eq. 20) becomes

$$(27) \quad K^y(\varphi, \varphi_0) = \left(\frac{\Lambda}{4\pi i t} \right)^{r/2} \frac{1}{\Lambda} e^{i \frac{\rho^2}{\Lambda} t} \sum_{\mathbf{m}} \left\{ \sum_{\sigma \in W} \epsilon_{\sigma} \exp \left[\frac{i\Lambda}{4t} (\sigma(\varphi + 2\pi\mathbf{m}) - \varphi_0)^2 \right] \right\}.$$

To get the exact evolution operator on the group manifold, K^y is inserted into Eq. 23. However, both the numerator and denominator vanish when $\varphi_0 = 0$. The distance from the origin of all the Weyl reflections of the point $\varphi + 2\pi\mathbf{m}$ is equal, and due to the factor ϵ_{σ} the term in curled brackets in Eq. 27 vanishes. The Weyl function in the denominator is a product of p (the number of positive roots) sines which result in a pole of order p when $\varphi_0 = 0$. There are two ways (which are actually the same) to resolve the problem. The first way is to set $\varphi_0 = t\zeta$ where ζ is an *arbitrary* vector in the root space, and let $t \rightarrow 0$. Then the usual L'Hopital rule can be used

$$(28) \quad \lim_{\varphi \rightarrow 0} \left[\frac{K^y(\varphi, \varphi_0)}{w(\varphi_0)} \right] = \lim_{t \rightarrow 0} \frac{\frac{\partial^p}{\partial t^p} K^y(\varphi, t\zeta)}{\frac{\partial^p}{\partial t^p} w(t\zeta)}$$

This method is used in obtaining the celebrated Weyl dimension formula (see e.g. Ref. [8]). It is important to stress that the actual choice of ζ does not change the result since we are dealing with functions on the torus that possess the special symmetry of the root space. A convenient choice is to take $\zeta = \rho$ (convenient in the sense that it is easier to work with this choice analytically).

An alternative method is to act directly on the functions in the numerator and the denominator by any p -order differential operator on the torus. Again, the actual choice of the operator is not important. We can differentiate p times along a specific direction, which is the same as the previous method, or we can take an arbitrary combination of p directional derivatives and to obtain the same result. We chose to work with the p -order operator \mathcal{D} (see appendix C) that has some special features

$$(29) \quad \mathcal{D}(\varphi) = \prod_{\alpha > 0} \left[\frac{2\alpha_j}{\alpha_j^2} \cdot \sum_{i=1}^r \mathbf{w}_i \frac{\partial}{\partial \varphi_i} \right],$$

where \mathbf{w}_i are the fundamental weights, and φ_i are the components of the vector φ in the natural basis

$$(30) \quad \varphi = \sum_{i=1}^r \varphi_i \gamma_i,$$

and γ_i are the primitive roots (see appendix B.1). When \mathcal{D} operates on a function with a definite symmetry under Weyl reflections (i.e. a function which is either symmetric or anti-symmetric under Weyl reflections), it changes the symmetry. Both the Weyl function and $K^y(\varphi, \varphi_0)$ are antisymmetric under Weyl reflections, and therefore they vanish when on the hyper-surfaces of the Weyl reflections, and in particular when $\varphi_0 \rightarrow 0$. Acting on them with \mathcal{D} turn them into symmetric functions, and therefore different from zero as $\varphi_0 \rightarrow 0$. Acting with $\mathcal{D}(\varphi_0)$ on the Weyl function and taking the limit $\varphi_0 = 0$ we get

$$(31) \quad \mathcal{D}(\varphi_0)w(\varphi_0)|_{\varphi_0=0} = \frac{1}{2^p} N(W) \prod_{\alpha > 0} (\alpha \cdot \rho),$$

where $N(W)$ is the order of the Weyl group. Acting on K^y with \mathcal{D} and setting $\varphi_0 = 0$ we get

$$(32) \quad \begin{aligned} \mathcal{D}(\varphi_0)K^y(\varphi, \varphi_0)|_{\varphi_0=0} &= \left(\frac{\Lambda}{4\pi i t} \right)^{r/2} \frac{1}{\Lambda} e^{i \frac{\rho^2}{\Lambda} t} \sum_{\mathbf{m}} N(W) \left(\frac{-i\Lambda}{2t} \right)^p \times \\ &\times \prod_{\alpha > 0} [\alpha \cdot (\varphi + 2\pi \mathbf{m})] \exp \left[\frac{i\Lambda}{4t} (\varphi + 2\pi \mathbf{m})^2 \right]. \end{aligned}$$

Inserting all the factors in the final expression for the evolution operator Eq. 23 together with the volume of the quotient space $V_{G/T}$ (Eq. 106), we get

$$(33) \quad K_t(\varphi) = \left(\frac{1}{4\pi i t} \right)^{n/2} \sum_{\mathbf{m}} \left\{ \prod_{\alpha > 0} \frac{\alpha \cdot (\varphi + 2\pi \mathbf{m})}{2 \sin \frac{\alpha \cdot \varphi}{2}} \right\} \exp \left[i \frac{\Lambda}{4t} (\varphi + 2\pi \mathbf{m})^2 + i \frac{\rho^2}{\Lambda} t \right].$$

This is an exact expression for the evolution operator since it is calculated directly using no approximations, yet it can be interpreted as the sum over classical paths (Eq. 10). The first term in the exponent is the classical action, the pre-exponential factor (in curled brackets) is ‘the first quantum correction’ (Van-Vleck determinant), and the second term in the exponent is ‘the second quantum correction’ which is proportional to the scalar curvature on the manifold ($\rho^2/\Lambda = n/24 = R/6$).

4.2 Spectral expansion

We shall use again the integral representation for the Green function in flat r dimensional infinite space Eq. 24. However, in this case choosing the appropriate basis for the vector \mathbf{p} becomes important. To simplify the computation, it is beneficial to use a basis which is dual to the natural basis of φ (Eq. 30), i.e. to use the fundamental weights \mathbf{w}_j as the basis vectors for \mathbf{p} : $\mathbf{p} = \sum_{j=1}^r i p_j \mathbf{w}_j$. Thus, the scalar product in the exponential is $i \sum_j p_j (\varphi_j \gamma_j^2/2 + 2\pi m_j)$. The integration measure is $d^r \mathbf{p} = [\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2} \prod_i dp_i$. The evolution operator is

$$(34) \quad K^y(\varphi, \varphi_0) = \frac{[\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2}}{i(2\pi)^{r+1}} \int_{-\infty}^{\infty} d\lambda e^{-i\lambda t} \sum_{\mathbf{m}} \sum_{\sigma \in W} \epsilon_{\sigma} \int_{-\infty}^{\infty} \prod_i dp_i \frac{\exp[i\mathbf{p} \cdot (\sigma \varphi + 2\pi \mathbf{m} - \varphi_0)]}{p^2 - \rho^2 - \Lambda \lambda}$$

Changing the order of integration and performing the integral in λ by the residue method one gets

$$(35) \quad \begin{aligned} K^y(\varphi, \varphi_0) &= \frac{[\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2}}{\Lambda(2\pi)^r} \sum_{\sigma \in W} \epsilon_{\sigma} \int_{-\infty}^{\infty} \prod_i dp_i \exp[i(p^2 - \rho^2)t/\Lambda + i\mathbf{p} \cdot (\sigma \varphi - \varphi_0)] \times \\ &\times \prod_{i=1}^r \left[\sum_{m_i=-\infty}^{\infty} \exp(2\pi i m_i p_i) \right] \end{aligned}$$

The integration in p_i can be easily performed by using the following identity

$$(36) \quad \int_{-\infty}^{\infty} dp_i \sum_{m_i=-\infty}^{\infty} \exp(2\pi i m_i p_i) f(p_i) = \sum_{n_i=-\infty}^{\infty} f(p_i = n_i)$$

where f is an arbitrary smooth function and n_i are integers. Thus, the evolution operator becomes

$$(37) \quad K^y(\varphi, \varphi_0) = \frac{[\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2}}{\Lambda(2\pi)^r} \sum_{\mathbf{n}} \sum_{\sigma \in W} \epsilon_\sigma \exp[-i(n^2 - \rho^2)t/\Lambda] \exp[i\mathbf{n} \cdot (\sigma\varphi - \varphi_0)].$$

The vector \mathbf{n} is defined on the weights lattice $\mathbf{n} = \sum_i n_i \mathbf{w}_i$. The Weyl reflections permute the weights as well as the roots, so it is possible to sum on the vectors \mathbf{n} in the Weyl chamber (where the components n_i are positive integers, see appendix B.7), and act on each vector with the Weyl group to reproduce the rest of the weight lattice

$$(38) \quad K^y(\varphi, \varphi_0) = \frac{[\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2}}{\Lambda(2\pi)^r} \sum_{\mathbf{n} > 0} e^{-i(n^2 - \rho^2)t/\Lambda} \left\{ \sum_{\sigma_1 \in W} \epsilon_{\sigma_1} \sum_{\sigma_2 \in W} \epsilon_{\sigma_2} e^{i\sigma_1 \mathbf{n} \cdot (\sigma_2 \varphi - \varphi_0)} \right\}.$$

To obtain the evolution operator on the group manifold K^y should be inserted into Eq. 23. When $\varphi_0 = 0$ the term in curled brackets vanishes. Thus, we have to act on it with the operator \mathcal{D} (Eq. 29) before taking the limit

$$(39) \quad \mathcal{D}K^y(\varphi, \varphi_0)|_{\varphi_0=0} = \frac{N(W)(-i)^p}{\Lambda(2\pi)^r} [\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2} \sum_{\mathbf{n} > 0} e^{-i(n^2 - \rho^2)t/\Lambda} \prod_{\alpha > 0} (\alpha \cdot \mathbf{n}) \sum_{\sigma \in W} \epsilon_\sigma e^{i(\mathbf{n} \cdot \sigma \varphi)}.$$

The final result is

$$(40) \quad K_t(\varphi) = \frac{[\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2}}{V_{G/T}(2\pi)^r \Lambda^{r/2}} \sum_{\mathbf{n} > 0} \left[\prod_{\alpha > 0} \frac{(\alpha \cdot \mathbf{n})}{(\alpha \cdot \rho)} \right] \left[\frac{1}{(2i)^p w(\varphi)} \sum_{\sigma \in W} \epsilon_\sigma \exp[i(\mathbf{n} \cdot \sigma \varphi)] \right] e^{-i(n^2 - \rho^2)t/\Lambda}$$

The factor in the exponent coincides with the known spectrum of the Casimir operator $(-\Delta)$. Identifying $\mathbf{n} = \mathbf{l} + \rho$ where \mathbf{l} is the highest weight of the representation, $\lambda_{\mathbf{l}} = n^2 - \rho^2$. The two factors in square brackets are the dimensionality and the character of each representation (compare with Eqs. 81,109). Since \mathbf{n} is defined over the weight lattice, and the summation is limited to vectors with positive coefficients n_i , the summation is easily translated to a summation over the highest weights $l_i = n_i - 1$. The numerical factor in front of the first summation is the inverse of the group volume (see eqs. 106,105).

This expression coincide with the known expression for the spectral expansion 8. Thus, the Weyl formulas for the characters and dimensionalities of the UIR are obtained here by direct computation of the evolution operator.

5 Evolution operator on non-compact group manifolds

The method for obtaining the evolution operator using an integral representation, described in section 4 for compact groups, works equally well for non-compact groups. The difference between a compact group and its non-compact partners is that for the compact group, the killing metrics (eq. 87) is positive definite, while for the non-compact groups it is not (a method for obtaining all the noncompact groups having the same complex extension from a compact group is given in section 6.1). It is natural to expect that the indefinite metric on the group manifold, and in the corresponding algebra space, induces indefinite metric in the tangent space \mathbb{T}^* where the radial parameters reside. Therefore, the flat r -dimensional space of the radial parameters is no longer Euclidean but pseudo-Euclidean $\mathbb{R}^r \rightarrow \mathbb{R}^{a,b}$, where there are a ‘space-like’ radial parameters and b ‘time-like’ radial parameters, and $a + b = r$. The vector φ that is built of the radial parameters can be rearranged in two sub-vectors

$$\varphi = (\phi, i\theta) = (\phi_1, \dots, \phi_a, i\theta_1, \dots, i\theta_b).$$

It should be noted that the two subspaces are orthogonal, so the natural coordinate system for the radial parameters 30, where the radial parameters are defined along the primitive roots, is not appropriate in this case. The maximal torus cease to be a closed torus and becomes an open torus $\mathbb{T}^r \rightarrow \mathbb{T}^{a,b} = \mathbb{T}^a \otimes \mathbb{R}^b$.

The periodicity of the group element in the radial parameters is altered due to the fact that b of the radial parameters are no longer real

$$(41) \quad h(\varphi) = \exp \left[i \sum_{j=1}^r (\varphi_j H_j) \right] = \exp \left[i \sum_{j=1}^a (\phi_j H_j) - \sum_{k=1}^b (\theta_k H_k) \right].$$

H_j are the basis elements of the Cartan subalgebra. Thus, the winding number vector in the non-compact case is altered with respect to the compact case (eq. 21) $\mathbf{m} \rightarrow \tilde{\mathbf{m}}$. $\tilde{\mathbf{m}}$ is obtained from \mathbf{m} by requiring that it vanishes in the subspace \mathbb{R}^b , and therefore it is an a -dimensional vector instead of the r -dimensional vector in the compact case

$$\varphi + 2\pi\tilde{\mathbf{m}} = (\phi + 2\pi\tilde{\mathbf{m}}, \theta).$$

Now we can proceed along the lines of the previous section. The first step is to find the Green function for Helmholtz equation in r -dimensional flat (pseudo-Euclidean) space (eq. 17) with nontrivial boundary conditions. The boundary conditions, that are imposed on the evolution operator, are that it should be symmetric under Weyl reflections, that it should account for the periodicity *in the periodic radial parameters*, and that it should decrease properly in the open domains. Thus, we can use the appropriate Green function for Helmholtz equation in the flat infinite space $\mathbb{R}^{a,b}$, and sum over the equivalent points, as we did in the compact case

$$(42) \quad G^y(\varphi, \varphi_0) = \sum_{\sigma \in W} \epsilon_\sigma \sum_{\tilde{\mathbf{m}}} G^{a,b}[\sigma(\varphi + 2\pi\tilde{\mathbf{m}}), \varphi_0].$$

This expression is inserted into the integral representation (Eq. 12). Because of the incompleteness of the winding number vector, the integration method that reconstruct the spectral expansion in the compact case is not applicable, which is to be expected since we know that the spectral expansion does not exist in the non-compact case. Yet, we can integrate term by term and reconstruct the ‘sum over classical paths’.

The Green function for Helmholtz equation in infinite pseudo-Euclidean space that appears in Eq. 42 has the general form

$$(43) \quad G_\epsilon^{a,b}(\varphi, \varphi_0) = \frac{1}{(2\pi)^{(a+b)}} \int_{\mathbb{R}^{a,b}} d^r \mathbf{p} \frac{1}{p^2 - \epsilon} \exp[i\mathbf{p} \cdot (\varphi - \varphi_0)], \quad \epsilon = \rho^2 + \Lambda\lambda,$$

where the ‘momentum’ vector \mathbf{p} has a ‘space like’ coordinates and b ‘time-like’ coordinates,

$$\mathbf{p} = (\mathbf{q}, i\mathbf{k}) = (q_1, \dots, q_a, ik_1, \dots, ik_b).$$

Inserting one term in the sum 42 into the integral representation for the evolution operator and changing the order of integration we get

$$(44) \quad K^{a,b}(\varphi, \varphi_0) = \frac{1}{2\pi i} \frac{1}{(2\pi)^r} \int_{\mathbb{R}^{a,b}} d^r \mathbf{p} \exp[i\mathbf{p} \cdot (\varphi - \varphi_0)] \int_{\mathcal{C}} e^{-i\lambda t} \frac{1}{p^2 - \rho^2 - \Lambda\lambda} d\lambda.$$

Unlike the compact case, p^2 is not positive definite, but the single pole at $\lambda = (p^2 - \rho^2)/\Lambda$ is located on the real axis of the complex λ plane, and the integral in λ is solved by the residue method (the integration contour \mathcal{C} is defined as in the compact case). Thus, we are left with the integral over $\mathbb{R}^{a,b}$ which can be divided into two separate integrals over the subspaces $\mathbb{R}^a, \mathbb{R}^b$

$$\begin{aligned} K^{a,b}(\varphi, \varphi_0) &= \frac{1}{(2\pi)^r} \frac{1}{\Lambda} e^{i\rho^2 \frac{t}{\Lambda}} \int_{\mathbb{R}^{a,b}} d^r \mathbf{p} \exp[i\mathbf{p} \cdot (\varphi - \varphi_0) - itp^2/\Lambda] \\ &= \frac{1}{\Lambda} e^{i\rho^2 \frac{t}{\Lambda}} \left[\frac{1}{(2\pi)^a} \int_{\mathbb{R}^a} d^a \mathbf{q} e^{i\mathbf{q} \cdot (\phi - \phi_0) - itq^2/\Lambda} \right] \left[\frac{1}{(2\pi)^b} \int_{\mathbb{R}^b} d^b \mathbf{k} e^{-i\mathbf{k} \cdot (\theta - \theta_0) + itk^2/\Lambda} \right] \\ &= \frac{1}{\Lambda} e^{i\rho^2 \frac{t}{\Lambda}} \left[\left(\frac{\Lambda}{4\pi it} \right)^{a/2} \exp \left[\frac{i\Lambda}{4t} (\phi - \phi_0)^2 \right] \right] \left[\left(\frac{\Lambda}{4\pi it} \right)^{b/2} \exp \left[-\frac{i\Lambda}{4t} (\theta - \theta_0)^2 \right] \right] = \\ (45) \quad &= \left(\frac{\Lambda}{4\pi it} \right)^{r/2} \frac{1}{\Lambda} e^{i\frac{\rho^2}{\Lambda} t} \exp \left[\frac{i\Lambda}{4t} (\varphi - \varphi_0)^2 \right]. \end{aligned}$$

Integrating over Eq. 42 term by term, the evolution operator in the flat space is obtained

$$(46) \quad K^y(\varphi, \varphi_0) = \left(\frac{\Lambda}{4\pi i t} \right)^{r/2} \frac{1}{\Lambda} e^{i \frac{\rho^2}{\Lambda} t} \sum_{\tilde{\mathbf{m}}} \left\{ \sum_{\sigma \in W} \epsilon_{\sigma} \exp \left[\frac{i\Lambda}{4t} (\sigma(\phi + 2\pi\tilde{\mathbf{m}}, i\theta) - (\phi_0, i\theta_0))^2 \right] \right\}.$$

To get the evolution operator on the group manifold it should be inserted into Eq. 23. Due to the anti-symmetry under Weyl reflections of both the numerator and denominator, we have to reverse the symmetry when taking the limit $\varphi_0 = 0$ with the help of the operator \mathcal{D} (Eq. 29). The operator is unchanged except for the fact that the parameters φ_i are not real in general, and it should be re-expressed in terms of the real parameters ϕ_i, θ_i . To produce the correct normalization of the evolution operator, the factor $V_{G/T}$ should remain as in the compact case. The final expression is

$$(47) \quad \begin{aligned} K_t(\varphi) &= \left(\frac{1}{4\pi i t} \right)^{n/2} \sum_{\tilde{\mathbf{m}}} \left\{ \prod_{\alpha > 0} \frac{\alpha \cdot (\varphi + 2\pi\tilde{\mathbf{m}})}{2 \sin \frac{\alpha \cdot \varphi}{2}} \right\} \exp \left[i \frac{\Lambda}{4t} (\varphi + 2\pi\tilde{\mathbf{m}})^2 + i \frac{\rho^2}{\Lambda} t \right] \\ &= \left(\frac{1}{4\pi i t} \right)^{n/2} \sum_{\tilde{\mathbf{m}}} \left\{ \prod_{\alpha > 0} \frac{\alpha \cdot (\phi + 2\pi\tilde{\mathbf{m}}, i\theta)}{2 \sin \frac{\alpha \cdot (\phi, i\theta)}{2}} \right\} \exp \left[i \frac{\Lambda}{4t} ((\phi + 2\pi\tilde{\mathbf{m}})^2 - \theta^2) + i \frac{\rho^2}{\Lambda} t \right]. \end{aligned}$$

Once more we emphasize that although this expression has a semi-classical interpretation, it is an exact expression that is obtained using no approximations.

Up to this point, the discussion was limited to a specific configuration of the radial parameters. However, when considering the evolution operator on non-compact groups a complication arises. The manifold of a non-compact group is splitted, in most cases, into several domains, and in each domain the maximal torus topology is different. Therefore, a global spherical coordinate system on the manifold of non-compact groups does not exist in these cases. There are several coordinate patches on the manifold, and they differ by the number of radial parameters in which the group element is periodic, i.e. the number of the real radial parameters. Thus, the first step in finding the evolution operator on a non-compact group manifold is to determine the different coordinate patches (we shall call them *evolution domains*) which differ by the decomposition of the radial parameters vector φ into the sub-vectors $\phi, i\theta$. In figs. 4,5 we can see for example the lattices of equivalent points (that have to be summed upon, see Eq. 19) for $SU(2,1)$ and $SL(3, \mathbb{R})$, respectively. Each of these non-compact groups has two evolution domains, which differs from each other by the winding numbers vector $\tilde{\mathbf{m}}$ (see the results section 8.1). The different topology of the torus in each domain affects the winding numbers vector, i.e. the periodicity in the radial group parameters, hence the lattices are different. A full analysis of the different coordinate patches of certain ‘families’ (e.g. $SU(p,q)$, $SO(p,q)$, etc.) of non-compact real groups is given in section 6.2.

The different periodicity in the radial group parameters affects the boundary conditions that are imposed on the evolution operator. Thus, the evolution operator should be determined separately in each domain. The evolution of a state $\Psi(g)$ on the manifold, given an initial state $\Psi(g)|_{t=0} = \psi(g)$, is predicted by the following integral

$$(48) \quad \Psi(g_t) = \sum_a \int_{D_a} K_a(g_t g_0^{-1}) \psi(g_0) dg_0.$$

D_a represent the domain to which the group element $g = g_t g_0^{-1}$ belongs, and the index a is the number of real radial parameters in which the group element is periodic. The element $h \in T$, that corresponds to $g = v h v^{-1} \in D_a$, has the general form that is given in Eq. 41.

6 Non-compact real groups and evolution domains analysis

Three subjects, that are required to better understanding of the the structure of non-compact real groups, are discussed in this section. The first subject is a short summary of the general method of finding all the non-compact groups having the same complex extension from a compact group. The second subject concerns the coordinate patches on the noncompact group manifold when spherical coordinate system is

used (evolution domains). The third subject concerns the use of the evolution operator. The evolution operator depends only on the radial parameters of $g = g_t g_0^{-1}$. A simple method for connecting the general group element $g = g_t g_0^{-1}$ and the evolution domain to which it belongs is explained.

6.1 Classification of non-compact real groups

The method of finding all the real groups associated with a compact group and having the same complex extension (and the same dimensionality) is equivalent to finding all the involutive automorphisms of the compact form. An involutive automorphism is a linear one-to-one transformation of the compact group on itself conserving the Lie multiplication

$$S[x, y] = [Sx, Sy] \quad (x, y \in \mathcal{G})$$

and fulfilling

$$S^2 = 1.$$

Choosing a basis in the algebra space, which is composed of eigenvectors of the matrix S , multiplying the eigenvectors that correspond to eigenvalue -1 by i and leaving the rest of the eigenvectors unchanged, a non-compact real group is obtained [16, 17, 1]. An equivalent way of finding the real non-compact forms associated with a compact Lie algebra is to look for the direct-sum decompositions of the compact Lie Algebra [21]. \mathcal{G} has a direct sum decomposition of the form $\mathcal{G} = K \oplus P$ if the following commutation relations hold:

$$(49) \quad [K, K] \subset K, \quad [K, P] \subset P, \quad [P, P] \subset K.$$

A non-compact real form is obtained by multiplying the generators that constitute P by an i : $P \rightarrow iP$. The form of the commutation relations 49 ensures that the structure constants remain real. This method may be easier to grasp than the more general method given above, however it is basis dependent and thus may not give the full list of possible direct-sum decompositions and the corresponding non-compact real forms.

An important distinction should be made between inner and outer automorphisms of the compact group (an inner automorphism is an isomorphic mapping of the group \mathbf{G} into itself by a fixed element x of the group: $z' = xzx^{-1}$, for all $z \in \mathbf{G}$. It induces an inner automorphism of the corresponding Lie algebra. All other automorphisms are outer automorphisms). Inner automorphisms lead to real Lie groups with similar structures, while the outer automorphisms lead in general to different real Lie groups. The existence of outer automorphisms is related to invariance of the corresponding Dynkin diagrams under transformations other than the identity transformation [1, 30]. As we shall see in section 8, in the case of non-compact groups that correspond to inner automorphisms some of the evolution operators in the different domains coincide, and in particular there is always one evolution domain where the operator is identical to that in the compact case. Yet, such coincidences do not exist in non-compact groups that correspond to outer automorphisms, although they may occur when comparing groups that both correspond to outer automorphisms. A non-compact group correspond to an inner (outer) automorphism if the determinant of S is equal to $+1(-1)$, or if the number of generators that belong to the subset P is even (odd). The algebras that have outer as well as inner automorphisms are A_n , D_n and E_6 [1, 17, 30].

6.2 Characteristic evolution domains

As was mentioned in the previous section, a global spherical coordinate system does not exist in general for noncompact groups. Thus, the group manifold is divided into several domains, and the evolution operator is found separately in each domain. Analyzing the eigenvalue system of the group element, which depends solely on the radial parameters, shows the allowed values of the radial parameters in the non-compact case, and each range of values corresponds to a different evolution domain. In this section, an analysis of the evolution domains of several families of groups is given.

6.2.1 Quasi-Unitary groups $SU(p,q)$

The real group $SU(p,q)$ ($p+q=n$) corresponds to an inner automorphism of the compact group $SU(n)$. It is obtained by leaving the generators of the subgroup $SU(p) \otimes SU(q) \otimes U(1)$ invariant (these generators constitute the maximal compact subalgebra of $SU(p,q)$ denoted by K) and multiplying the rest of the generators by an i .

A quasi unitary group element g (in the fundamental representation) satisfies the relation

$$(50) \quad g\eta g^\dagger = \eta$$

where η is a constant $n \times n$ matrix (in the compact case $\eta = I$) with p eigenvalues equal to $+1$ and q eigenvalues equal to -1 . An understanding of the structure of the evolution domains is achieved by contemplating the eigenvalue system of the group element g , which depends on the $n-1$ radial parameters. The characteristic polynomial of a unitary group element is $P_\lambda(g) = \det(\lambda I - g) = \prod_i (\lambda - \lambda_i)$. The complex conjugate of the characteristic polynomial can be written in two alternative forms

$$(51) \quad \begin{aligned} P_\lambda^*(g) &= \prod_i (\lambda - \lambda_i^*) \\ &= \det(\lambda I - g^\dagger) = \det(\lambda I - \eta^{-1} g^{-1} \eta) = \det(\lambda I - g^{-1}) = \prod_j (\lambda - \lambda_j^{-1}), \end{aligned}$$

which leads to the following relation

$$(52) \quad \lambda_i^* = \lambda_j^{-1},$$

i.e. for any eigenvalue λ_i there must be another eigenvalue λ_j that satisfies relation 52. In the compact case, where all the radial parameters are real, this relation is satisfied for $i = j$. For non-compact real groups, there are several possibilities. In general, there are $\min(p,q)+1$ evolution domains corresponding to $0, 1, \dots, \min(p,q)$ imaginary radial parameters. The eigenvalues should be written in the following way:

$$\exp[i(\varphi_1 + \varphi_2)], \exp[i(\varphi_1 - \varphi_2)], \exp[i(-\varphi_1 + \varphi_3)], \exp[i(-\varphi_1 - \varphi_3)], \dots$$

where $\varphi_1, \varphi_2, \varphi_3, \dots$ are the radial parameters. It is clear that φ_1 must remain real, while φ_2, φ_3 can be either pure real or imaginary in the above example (for an odd number of eigenvalues the last one must have a real radial parameter, i.e. a pure imaginary argument). This choice of radial parameters corresponds to an orthogonal coordinate system in T^* .

6.2.2 Real Unimodular groups $SL(n, \mathbb{R})$

The real group $SL(n, \mathbb{R})$ corresponds to an outer automorphism of the compact group $SU(n)$. The compact subgroup is $SO(n)$, and the generators that remain unchanged (i.e. belong to K) are the generators of $SO(n)$ in the vector representation (the spinor representation of $SU(n)$ is of the same dimension of the vector representation of $SO(n)$ and among its generators there are n generators which are pure imaginary), and the rest of the generators are multiplied by an i . The fundamental group element is a n -dimensional real matrix whose determinant is equal to one. The invariants of this group element (e.g $\text{Tr}(g)$, $\text{Tr}(g^2)$, etc.) must be real, and this fact imposes conditions on the radial parameters of the group. Thus, the eigenvalues of the group element must be either complex conjugate to each other, or real:

$$(53) \quad \lambda_i = \lambda_j^*$$

The first domain is the domain in which all eigenvalues are real so the radial parameters are pure imaginary. The rest of the domains correspond to taking pairs of eigenvalues to be complex conjugate. The maximal number of allowed domains is $[n/2]+1$. It is important to note that since $SL(n, \mathbb{R})$ (for $n > 2$) correspond to an outer automorphism of $SU(n)$ [1], while the different $SU(p,q)$ correspond to inner ones, none of the domains of the radial parameters in $SL(n, \mathbb{R})$ coincide with the domains of $SU(p,q)$.

6.2.3 Quasi-Orthogonal groups $SO(p,q)$, $p+q=2n+1$

The real group $SO(p,q)$ corresponds to an inner automorphism of the compact group $SO(2n+1)$. The maximal compact subgroup is $SO(p) \otimes SO(q)$, and the rest of the generators are multiplied by an i to form the group $SO(p,q)$ (all these groups correspond to the complex algebra B_n which has no outer automorphisms). Thus, the number of non-compact groups that can be formed out of $SO(2n+1)$ equal the number of possible divisions of $2n+1$ into p and q .

A pseudo-orthogonal group element (in the vector representation which is $p+q$ dimensional) satisfies the relation

$$(54) \quad g\eta g^T = \eta$$

where η is a constant matrix. In the compact case η is simply the unit matrix, so $g^T = g^{-1}$, and the eigenvalues of g are the same as those of g^{-1} . This means that for every eigenvalue λ_i of g , there exists another eigenvalue of g , λ_j , that satisfies the relation

$$(55) \quad \lambda_i = \lambda_j^{-1}.$$

However, this is also true in the pseudo-orthogonal case, since g^T and g^{-1} are connected by a similarity transformation. Since there are n radial parameters, the eigenvalues (in the compact as well as non-compact case) are divided into pairs of $e^{i\varphi_i}$, $e^{-i\varphi_i}$ so the condition $\lambda_i = \lambda_j^{-1}$ is satisfied by each pair. Since the number of eigenvalues is odd ($p+q=2n+1$), the last eigenvalue is simply $\lambda_{2n+1} = 1$. The radial parameters have a simple physical interpretation in this case. $SO(p,q)$ is the rotation group in $p+q$ Minkowski space. Each pair of eigenvalues corresponds to an independent rotation plane (in the $2n+1$ dimensional space there are n independent rotation planes). Therefore a pure rotation can be performed in a plane that is consisted of two ‘space-like’ or two ‘time-like’ axes, or an hyperbolic rotation can be performed in a plane which is consisted of one ‘time-like’ axis and one ‘space-like’ axis. Therefore, a radial parameter that appears in the exponential $\lambda_i = \exp(i\varphi_i)$ can be either real or pure imaginary, depending on the type of the rotation. It is important to note that condition 55 holds for complex parameters also, however a complex parameter can be treated as two real parameters, so the next pair of eigenvalues will also depend on the same two real parameters (the total number of real radial parameters is fixed). A rotation of the coordinate system of the radial parameters in the root space transforms the eigenvalue system to the desired form.

There can be a maximum of $d = \min(p,q)$ independent hyperbolic rotations, and therefore there will be $d + 1$ domains on the group manifold, the first corresponds to pure rotations only, the second corresponds to a hyperbolic rotation in one plane and pure rotations in the rest of the independent planes, etc., while the last domain corresponds to the maximal number of hyperbolic rotations with d imaginary radial parameters. It should be noted that it does not matter in this case which of the radial parameters is imaginary and which is real, since the action of the Weyl group permutes the radial parameters. Since the evolution operator is symmetric under the action of the Weyl group, the information that is needed to determine the evolution domain is the total number of imaginary parameters.

6.2.4 Quasi-Orthogonal groups $SO(p,q)$, $p+q=2n$

The real groups $SO(p,q)$ correspond to inner as well as outer automorphisms of the compact group, and this fact complicates the determination of the evolution domains. Even p and q correspond to an inner automorphism, while odd p and q correspond to an outer automorphism. Condition 55 holds in this case, and we can relate each pair of eigenvalues in the $2n$ dimensional vector representation to a rotation in an independent plane (there are n independent rotation planes and n radial parameters), however this case is more complicated then the previous one. An intuitive understanding of the profound difference between the groups that correspond to inner and outer automorphisms of the compact group is gained when considering the types of possible rotations in $p+q$ Minkowski space. When p and q are even, we can divide separately the time like part and the space like part into independent rotation planes, so that all the n radial parameters correspond to pure rotations. All the group elements of this kind belong to a domain where all the radial parameters are real. In this domain the evolution operator coincide with

the evolution operator in the compact case, and this kind of non-compact group correspond to an inner automorphism of the compact group. On the other hand, when p and q are odd, the maximal number of pure rotation planes is $n - 1$, and at least one radial parameter in each domain must correspond to a hyperbolic rotation. None of the domains is similar to the compact case, and the group correspond to an outer automorphism.

Another subtlety concerns the general subspace with metrics $++--$ in the $p+q$ flat vector space where the group acts. There are three possible evolution domains; the first correspond to two pure rotations in the $++$ and $--$ planes (with e.v. $e^{i\phi_1}, e^{-i\phi_1}, e^{i\phi_2}, e^{-i\phi_2}$), the second correspond to two hyperbolic rotations in the $+-$ planes (with e.v. $e^{\phi_1}, e^{-\phi_1}, e^{\phi_2}, e^{-\phi_2}$), but there is also the possibility of one real parameter and one imaginary parameters in the rotated coordinate system (with e.v. $e^{i\phi_1+\phi_2}, e^{-i\phi_1-\phi_2}, e^{-i\phi_1+\phi_2}, e^{i\phi_1-\phi_2}$). Determination of the appropriate coordinate system of the radial parameters in the root space can be made using the method that is described in section 6.3. From observation, none of the evolution domains of groups that correspond to outer automorphisms coincide with those of groups that correspond to inner automorphisms.

6.2.5 Quasi-Unitary Symplectic groups $\text{USp}(2p, 2q)$

In order to leave the two bilinear forms invariant, the group element of $\text{USp}(2p, 2q)$, where $2p+2q=2n$, has to satisfy the following conditions

$$(56) \quad g^\dagger \eta g = \eta, \quad g^T \zeta g = \zeta.$$

η is a diagonal constant matrix of the form

$$\eta = \text{diag}(I_p, -I_q, I_p, -I_q),$$

(in the compact case $\eta = I_{2n}$), and ζ is of the ‘metric tensor’ for the symplectic bilinear form

$$\zeta = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The two conditions should be satisfied simultaneously, and in particular the eigenvalues of the group element in the $2n$ -dimensional fundamental representation should satisfy the conditions

$$(57) \quad g^\dagger = \eta g^{-1} \eta^{-1} : \quad \lambda_i^* = \lambda_j^{-1}$$

$$(58) \quad g^T = \zeta g^{-1} \zeta^{-1} : \quad \lambda_i = \lambda_k^{-1}.$$

In the compact case, the first condition is satisfied automatically, $\lambda_i^* = \lambda_i^{-1}$. To satisfy the second condition, the eigenvalues are divided into pairs of the form e^{ia}, e^{-ia} (where a is real).

In the non-compact cases, the radial parameters may assume complex values. The eigenvalues are divided into quartets of the form

$$e^{i(a+b)}, e^{i(a-b)}, e^{i(-a+b)}, e^{i(-a-b)},$$

where either a or b may assume pure imaginary values, but not both. For each quartet the two conditions are satisfied when one of the parameters is real and the other is imaginary. On the manifold of $\text{USp}(2p, 2q)$ there are $|p-q| + 1$ domains, in the first domain all the radial parameters are real and divided into pairs as in the compact case, in the second domain there is one quartet which includes one real parameter and one imaginary parameter, and the rest of the parameters are real, etc.

6.2.6 Real Symplectic groups $\text{Sp}(2n, \mathbb{R})$

The group element of $\text{Sp}(2n, \mathbb{R})$ in the fundamental representation is a real matrix that satisfies the relation

$$g^T \zeta g = \zeta.$$

The eigenvalues satisfy the conditions

$$\lambda_i = \lambda_j^*, \quad \lambda_i = \lambda_k^{-1}.$$

Dividing the eigenvalues into pairs $\lambda_i = e^{ia}$, $\lambda_j = e^{-ia}$, we see that the two conditions are satisfied by each pair when the radial parameter is either pure real or pure imaginary. Therefore, there are $n + 1$ domains on the manifold of $\text{Sp}(2n, \mathbb{R})$ that correspond to $0, 1, \dots, n$ imaginary parameters.

6.3 Evolution domains on the group manifold

The connection between the general group element given in the ‘Cartesian-like’ coordinate system and the radial parameters is easier to investigate in the algebra space. The general algebra element in Cartesian coordinates is

$$x = \xi_a X_a, \quad a = 1, \dots, n \quad x \in \mathcal{G}$$

where X_a are the group generators, ξ_a are the group parameters and n is the group dimensionality. The algebra element invariants such as $\text{Tr}x, \text{Tr}x^2 \dots$ depend only on the radial parameters, for example

$$\text{Tr}X^2 = \text{Tr}(\xi_a X_a)^2 = \text{Tr}(\varphi_j H_j)^2 \quad j = 1, \dots, r$$

where H_j are the generators of the Cartan subalgebra. The number of independent invariants equals the rank of the algebra r , thus an equation system for finding the radial parameters is obtained. The independent invariants that constitute the equation system should be chosen with care. The representation that should be used is the fundamental representation (the representation that correspond to the *covering group*). This representation has in general more invariants than are needed to obtain the radial parameters (since only r of them are independent), and choosing invariants of higher order would result in higher order equations and more solutions to the equation system while there are only $N(W)$ equivalent solutions for the radial parameters, i.e. there should be only $N(W)$ solutions to the equation system ($N(W)$ is the order of the Weyl group). For the groups that correspond to the series A_r , the fundamental (spinor) representation is $r + 1$ dimensional, and the r invariants that should be chosen are $\text{Tr}x^2, \dots, \text{Tr}x^{r+1}$. In other cases the set of lowest order independent invariants should be chosen. It is important to note that all the solutions correspond to the same group element, and they can be obtained from each other by the action of the Weyl group. This is the origin of the requirement that the evolution operator is symmetric under Weyl reflections, i.e. invariant under the action of the Weyl group.

Choosing an appropriate coordinate system for the radial parameters in the root space according to section 6.2, we can make sure that at least in one solution all the radial parameters are pure real and imaginary. Thus, to each domain on the group manifold D_a (a is the number of the real parameters) there corresponds a certain range of the invariants, and we can relate directly the group element to the domain to which it belongs.

7 Evolution operator on $\text{SU}(2)$ and $\text{SU}(1,1)$

The fundamental group element of $\text{SU}(2)$ in the spherical coordinate system is

$$g = \cos \frac{\varphi}{2} I_2 + i \sin \frac{\varphi}{2} (\vec{\sigma} \cdot \hat{n}) = v h(\varphi) v^{-1}$$

where \hat{n} is a unit vector in three dimensional Euclidean space, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector composed of the three Pauli matrices (which are also the generators of $\text{SU}(2)$). This group element correspond to a rotation at an angle φ around the unit vector \hat{n} . The eigenvalues of the group element depend only on the radial parameter, φ ,

$$\lambda_1 = e^{i\varphi/2}, \quad \lambda_2 = e^{-i\varphi/2}.$$

Note that the trace of the group element in the fundamental representation lies in the interval $[-2, 2]$.

$\text{SU}(1,1)$ is obtained from the generators of $\text{SU}(2)$ by substituting $\sigma_1, \sigma_2 \rightarrow i\sigma_1, i\sigma_2$. The maximal compact subgroup is $U(1)$, and it is generated by σ_3 . Since the group is non-compact, the radial parameter

φ may assume non-real values. However, condition 52 implies that either $\lambda_1^{-1} = \lambda_1^*$, $\lambda_2^{-1} = \lambda_2^*$, or $\lambda_1^{-1} = \lambda_2^*$. In the first case, φ is real as in the compact case, and in the second case φ is pure imaginary. Thus, two evolution domains are obtained; D_1 where $\varphi = \phi$ is real, and D_0 where $\varphi = i\theta$, and θ is real. The group elements that belong to D_1 are periodic in φ , while the group elements that belong to D_0 are not. The evolution operator has to be computed separately in each domain, since the boundary conditions that should be satisfied are different. The evolution of a state on $SU(1,1)$ is given by the following integral

$$(59) \quad \Psi_t(g_t) = \int_{g_t g_0^{-1} \in D_1} K_1(\phi) \psi(g_0) d\mu(g_0) + \int_{g_t g_0^{-1} \in D_0} K_0(i\phi) \psi(g_0) d\mu(g_0).$$

Note that on D_1 , ϕ is given explicitly by the relation

$$\phi = \cos^{-1} \left[\frac{1}{2} \text{Tr}(g_t g_0^{-1}) \right],$$

and on D_0 , θ is given by the relation

$$\theta = \cosh^{-1} \left[\frac{1}{2} \text{Tr}(g_t g_0^{-1}) \right].$$

7.1 Evolution operator on $SU(2)$

The root space of $SU(2)$ is one dimensional. Normalizing the root length to $|\alpha| = 1$, the constants Λ, ρ^2 that appear in the general expression for the radial Laplacean (Eq. 16) are $\Lambda = 2$, $\rho^2 = 1/4$, and the Weyl function is

$$w(\varphi) = \sin \frac{\varphi}{2}.$$

We are interested in finding the Green function for the inhomogeneous equation

$$(60) \quad (\Delta_{\mathbb{T}} + \lambda) \psi(\varphi) = \frac{1}{2} \left[\frac{1}{\sin \frac{\varphi}{2}} \frac{\partial^2}{\partial \varphi^2} \sin \frac{\varphi}{2} + \frac{1}{4} + 2\lambda \right] \psi(\varphi) = -f(\varphi)$$

with appropriate boundary conditions. Substituting $y(\varphi) = w(\varphi) \psi(\varphi)$ we arrive at the one dimensional Helmholtz equation

$$(61) \quad \left[\frac{\partial^2}{\partial \varphi^2} + k^2 \right] y(\varphi) = -F(\varphi), \quad k^2 = \frac{1}{4} + 2\lambda, \quad F = 2 \sin \frac{\varphi}{2} f$$

with boundary conditions $y(\varphi = 0) = y(\varphi = 2\pi) = 0$. The Green function can be found by the image method; two ‘conducting walls’ are placed at $\varphi = 0$, $\varphi = 2\pi$, and a unit charge, that is placed at φ_0 inside the domain $[0, 2\pi]$, is reflected with respect to the walls, and an infinite series of images is created.

The appropriate Green function is

$$(62) \quad G^y(\varphi, \varphi_0) = \sum_{n=-\infty}^{\infty} [G^1(\varphi, \varphi_0 + 4\pi n) - G^1(\varphi, -\varphi_0 + 4\pi n)]$$

where $G^1(\varphi, \varphi_0)$ is the Green function for Helmholtz equation in \mathbb{R}^1

$$(63) \quad G^1(\varphi, \varphi_0) = \frac{i}{2k} e^{ik|\varphi - \varphi_0|}.$$

The same expression for the Green function is obtain if we impose the more general boundary conditions discussed in section 4, that the green function should account for the periodicity of the group elements in the radial parameter φ and that it should be antisymmetric (so the evolution operator would be symmetric) under Weyl reflections. Thus, we should sum over all the points that differ from each other by a period of 4π , and subtract all the points that are created by the action of the Weyl group (i.e. reflection through the point $\varphi = 0$) on the infinite series of periodic points.

Inserting G^1 into the infinite sum 62 and summing, the following expression is obtained

$$(64) \quad G^y(\varphi, \varphi_0) = \frac{\sin(k\varphi_<) \sin[k(2\pi - \varphi_>)]}{k \sin(2k\pi)}, \quad \varphi_< = \min(\varphi, \varphi_0), \quad \varphi_> = \max(\varphi, \varphi_0)$$

The desired Green function for Helmholtz equation on the group manifold is obtained by inserting G^y into Eq. 23, substituting $\varphi_> = \varphi$, $\varphi_< = \varphi_0$ and taking the limit $\varphi_0 = 0$. The invariant volume of the angular parameters $V_{G/T} = 8\pi$ (see Eq. 106). The final expression is

$$(65) \quad G_\lambda(\varphi) = \frac{\sin k(2\pi - \varphi)}{8\sqrt{2}\pi \sin 2k\pi \sin \frac{\varphi}{2}}$$

This expression is inserted into the integral representation for the evolution operator (Eq. 12). The integration contour C can be contracted around the poles of the integrand at $\lambda = \frac{1}{8}(n^2 - 1)$ where n is an integer. Changing the integration variable from λ to k , the integration contour ‘opens’ and we obtain the following integral representation for the evolution operator

$$(66) \quad K_t(\varphi) = \frac{1}{2\pi i} \int_{-\infty+i\delta}^{\infty+i\delta} e^{-\frac{i}{2}(k^2 - \frac{1}{4})t} \frac{\sin k(2\pi - \varphi)}{8\sqrt{2}\pi \sin 2k\pi \sin \frac{\varphi}{2}} k dk.$$

The integral can be performed using two alternative methods, which lead to the two expressions found by Schulman[27] for the evolution operator, the spectral expansion and the sum over classical paths.

To obtain the known expression for the spectral expansion, the integration contour can be closed by going from infinity to minus infinity below the real axis, and then dividing the result by a factor of two. The integral is solved by the residue method, and the spectral expansion is

$$(67) \quad K_t(\varphi) = \frac{1}{32\sqrt{2}\pi^2} \sum_{n=1}^{\infty} n \frac{\sin(n\varphi)}{\sin \varphi} e^{-\frac{i}{8}(n^2-1)t}.$$

Returning to the integral 66, expanding the denominator

$$\frac{\sin k(2\pi - \varphi)}{\sin(2k\pi)} = (e^{ik(4\pi-\varphi)} - e^{ik\varphi})e^{-4ik\pi} \sum_{m=0}^{\infty} e^{-4ik\pi m}$$

and integrating term by term, an alternative expression is obtained

$$(68) \quad K_t(\varphi) = \frac{1}{(4\pi it)^{3/2}} \sum_{m=-\infty}^{\infty} \frac{\varphi + 4\pi m}{2 \sin \frac{\varphi}{2}} \exp \left[\frac{i}{2t}(\varphi + 4\pi m)^2 + \frac{it}{8} \right],$$

and this expression coincide with the sum over classical paths. Note, that this expression, that was formerly obtained only by using the semi-classical approximation (and was proved to be exact using Θ -function theorems) is obtained here by a direct computation, and in fact it stems from the same integral representation that produces the spectral expansion.

7.2 Evolution operator for SU(1,1)

7.2.1 The evolution operator on D_1 :

The radial Laplacean in the first coordinate patch, where the radial parameter $\varphi = \phi$ is real, is identical to that of SU(2) since the constants that depend on the root system do not change, and the Weyl function $w(\phi) = \sin \frac{\phi}{2}$ is the same

$$(69) \quad \Delta_1 = \Delta_{\text{SU}(2)}.$$

The boundary conditions are also unchanged since the Green function G_λ has to account for the unchanged periodicity of the group elements in ϕ , and to be symmetric under Weyl reflections. Thus, the evolution operator on D_1 is identical to that of SU(2)

$$(70) \quad K_1(\phi) = K_{\text{SU}(2)}(\phi) = \frac{1}{(4\pi it)^{3/2}} \sum_{m=-\infty}^{\infty} \frac{\phi + 4\pi m}{2 \sin \frac{\phi}{2}} \exp \left[\frac{i}{2t}(\phi + 4\pi m)^2 + \frac{it}{8} \right].$$

The ‘sum over classical paths’ form was chosen here out of the two equivalent expressions, since this expression can be ‘analytically continued’ to the expression that is obtained in the second evolution domain.

7.2.2 The evolution operator on D_0 :

In this domain φ is pure imaginary. Setting $\varphi = i\theta$, the Weyl function becomes $w(\varphi = i\theta) = \sinh \frac{\theta}{2}$. The inhomogeneous equation for which we have to find the Green function is

$$(71) \quad (\Delta_0 + \lambda)\psi = \frac{1}{2} \left[-\frac{1}{\sinh \frac{\theta}{2}} \frac{\partial^2}{\partial \theta^2} \sinh \frac{\theta}{2} + \frac{1}{4} + 2\lambda \right] \psi(\theta) = -f(\theta)$$

Substituting $y(\theta) = w(\theta)\psi(\theta)$, we get the inhomogeneous Helmholtz equation in \mathbb{R}^1

$$(72) \quad \left[\frac{\partial^2}{\partial \varphi^2} - \left(\frac{1}{4} + 2\lambda \right) \right] y(\theta) = F(\theta).$$

Since the periodicity of the group element in $\varphi = i\theta$ is ruined, we are left only with the boundary condition that the Green function G^y is antisymmetric under Weyl reflections. The Green function in the infinite space should be chosen with great care due to the minus sign before the factor $\frac{1}{4} + 2\lambda$ to avoid singularities when performing the integration in the complex λ plane (in the integral representation Eq. 12). Therefore, we must distinguish between the two cases where $k^2 = -(\frac{1}{4} + 2\lambda) > 0$ and $k^2 < 0$

$$(73) \quad \begin{aligned} G_k^1(\theta, \theta_0)|_{k^2 = -(\frac{1}{4} + 2\lambda) < 0} &= \frac{i}{2\sqrt{\frac{1}{4} + 2\lambda}} e^{-\sqrt{\frac{1}{4} + 2\lambda}|\theta - \theta_0|}, \\ G_k^1(\theta, \theta_0)|_{k^2 = -(\frac{1}{4} + 2\lambda) > 0} &= \frac{i}{2\sqrt{\frac{1}{4} + 2\lambda}} e^{\sqrt{\frac{1}{4} + 2\lambda}|\theta - \theta_0|} \end{aligned}$$

Summing over the two Weyl reflections

$$(74) \quad G^y(\theta, \theta_0) = G^1(\theta, \theta_0) - G^1(\theta, -\theta_0)$$

and inserting G^y into Eq. 23 we get the final expression for the resolvent $G_\lambda(\varphi)$ in this domain

$$(75) \quad G_\lambda(\theta)|_{\frac{1}{4} + 2\lambda > 0} = \frac{1}{8\sqrt{2}\pi \sinh \frac{\theta}{2}} e^{-\sqrt{\frac{1}{4} + 2\lambda}\theta}, \quad G_\lambda(\theta)|_{\frac{1}{4} + 2\lambda < 0} = \frac{1}{8\sqrt{2}\pi \sinh \frac{\theta}{2}} e^{\sqrt{\frac{1}{4} + 2\lambda}\theta}$$

Inserting into the resolvent into the integral representation Eq. 12 we get

$$(76) \quad K_0(\theta) = \frac{1}{2\pi i} \frac{1}{8\sqrt{2}\pi \sinh \frac{\theta}{2}} \int_C \left[e^{-i\lambda t} e^{\sqrt{\frac{1}{4} + 2\lambda}\theta} \Theta(-\frac{1}{4} - 2\lambda) + e^{-i\lambda t} e^{-\sqrt{\frac{1}{4} + 2\lambda}\theta} \Theta(\frac{1}{4} + 2\lambda) \right] d\lambda.$$

Θ is the usual step function, $\Theta(x < 0) = 0$, $\Theta(x > 0) = 1$. The second term does not contain neither cuts nor poles, so it does not contribute to the integral. There is a cut in the first term, and contracting the integration contour around the cut the final result is obtained.

The exact evolution operator in the domain D_0 is

$$(77) \quad K_0(\varphi = i\theta) = \frac{1}{(4\pi i t)^{3/2}} \frac{\theta}{2 \sinh \frac{\theta}{2}} \exp \left[-\frac{i}{2t} \theta^2 + \frac{it}{8} \right].$$

Together with $K_1(\varphi)$ (Eq. 70) in the evolution domain D_1 where φ is real, the evolution operator on the entire group manifold is found.

8 Results

Finding the evolution operator on a non-compact group according to section 5 involves the following steps. First, the evolution domains have to be established according to section 6.2. Then the appropriate coordinate system for the radial parameters, where the parameters are either real or imaginary is determined in each domain. The next step is to determine the winding number vector $\tilde{\mathbf{m}}$, which is inserted in the expression for the evolution operator in each domain (eq. 47). In this section the method is demonstrated on the real groups associated with the four simple algebras, A_2, B_2, A_3 and C_3 .

We shall use the following notations: G is the noncompact group under consideration, K its maximal compact subgroup, K_j the generators of K , iP_j the generators of the coset space G/K , D_a the evolution domain that correspond to a real radial parameters and $r - a$ imaginary parameters. The imaginary parameters φ_j are given in terms of the real parameters θ_j , $\varphi_j = i\theta_j$. The rotation matrices L_{ab} in the 4-dim. spinor representation of $SO(5)$ and $SO(6) \sim SU(4)$ are given in Appendix E.1, and the basis for C_3 is given in Appendix E.2.

8.1 Groups associated with A_2

Compact group: $SU(3)$

Generators: Gell-Mann matrices[19] $\lambda_1, \dots, \lambda_8$ in 3-dim. spinor representation

Primitive roots (in \mathbb{R}^2):

$$\gamma_1 = \hat{x}, \quad \gamma_2 = -\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}$$

Radial parameters vector

$$\varphi = \varphi_1\gamma_1 + \varphi_2\gamma_2 = \phi_1\hat{x} + \phi_2\hat{y}, \quad \phi_1 = \varphi_1 - \frac{\varphi_2}{2}, \quad \phi_2 = \frac{\sqrt{3}}{2}\varphi_2$$

Winding number vector:

$$\mathbf{m} = 2m_1\gamma_1 + 2m_2\gamma_2 = (2m_1 - m_2)\hat{x} + \sqrt{3}m_2\hat{y}$$

eigenvalues of the group element in the spinor representation

$$e^{\frac{i}{2}(\phi_1 + \phi_2/\sqrt{3})}, \quad e^{\frac{i}{2}(-\phi_1 + \phi_2/\sqrt{3})}, \quad e^{-\frac{i}{\sqrt{3}}\phi_2}$$

Noncompact groups:

G	$SU(2,1)$	$SL(3, \mathbb{R})$
K	$SU(2) \otimes U(1)$	$SO(3)$
K_j	$\lambda_1, \lambda_2, \lambda_3, \lambda_8$	$\lambda_2, \lambda_5, \lambda_7$
iP_j	$i\lambda_4, i\lambda_5, i\lambda_6, i\lambda_7$	$i\lambda_1, i\lambda_3, i\lambda_4, i\lambda_6, i\lambda_8$
D_a	$D_2: \quad \varphi = \phi_1\hat{x} + \phi_2\hat{y}$ $\quad \quad \tilde{\mathbf{m}} = (2m_1 - m_2)\hat{x} + \sqrt{3}m_2\hat{y}$	$D_1: \quad \varphi = \phi_1\hat{x} + i\theta_2\hat{y}$ $\quad \quad \tilde{\mathbf{m}} = 2m_1\hat{x}$
	$D_1: \quad \varphi = i\theta_1\hat{x} + \phi_2\hat{y}$ $\quad \quad \tilde{\mathbf{m}} = 2\sqrt{3}m\hat{y}$	$D_0: \quad \varphi = i\theta_1\hat{x} + i\theta_2\hat{y}$ $\quad \quad \tilde{\mathbf{m}} = 0$

8.2 Groups associated with B_2

Compact group: $SO(5)$

Generators: 10 rotation matrices L_{ab} , $a, b = 1, \dots, 5$ in the 4-dim. spinor representation (see appendix E.1)

Primitive roots (in \mathbb{R}^2):

$$\gamma_1 = \hat{x} - \hat{y}, \quad \gamma_2 = \hat{y}$$

Radial parameters vector:

$$\varphi = \varphi_1 \gamma_1 + \varphi_2 \gamma_2 = \phi_1 \hat{x} + \phi_2 \hat{y}, \quad \varphi_1 = \phi_1, \quad \varphi_2 = \phi_1 + \phi_2$$

Winding number vector:

$$\mathbf{m} = m_1 \gamma_1 + 2m_2 \gamma_2 = m_1 \hat{x} + (2m_2 - m_1) \hat{y}$$

eigenvalues of the group element in the 4-dim. spinor representation

$$e^{\frac{i}{2}(\phi_1 + \phi_2)}, \quad e^{\frac{i}{2}(-\phi_1 + \phi_2)}, \quad e^{\frac{i}{2}(\phi_1 - \phi_2)}, \quad e^{\frac{i}{2}(-\phi_1 - \phi_2)}$$

eigenvalues of the group element in the 5-dim. vector representation

$$e^{i\phi_1}, \quad e^{-i\phi_1}, \quad e^{i\phi_2}, \quad e^{-i\phi_2}, \quad 1$$

Noncompact groups

G	SO(4,1)	SO(3,2)
K	SO(4)	SO(3) \otimes SO(2)
K_j	$L_{ab}, \quad a, b = 1, \dots, 4$	L_{ab}, L_{45}
iP_j	$iL_{a5}, \quad a = 1, \dots, 4$	$iL_{a4}, iL_{a5}, \quad a, b = 1, 2, 3$
D_a	$D_2: \quad \varphi = \phi_1 \hat{x} + \phi_2 \hat{y}$ $\quad \quad \quad \tilde{\mathbf{m}} = m_1 \hat{x} + (2m_2 - m_1) \hat{y}$	$D_2: \quad \varphi = \phi_1 \hat{x} + \phi_2 \hat{y}$ $\quad \quad \quad \tilde{\mathbf{m}} = m_1 \hat{x} + (2m_2 - m_1) \hat{y}$
	$D_1: \quad \varphi = \phi_1 \hat{x} + i\theta_2 \hat{y}$ $\quad \quad \quad \tilde{\mathbf{m}} = 2m \hat{x}$	$D_1: \quad \varphi = \phi_1 \hat{x} + i\theta_2 \hat{y}$ $\quad \quad \quad \tilde{\mathbf{m}} = 2m \hat{x}$
		$D_0: \quad \varphi = i\theta_1 \hat{x} + i\theta_2 \hat{y}$ $\quad \quad \quad \tilde{\mathbf{m}} = 0$

8.3 Groups associated with $A_3 \sim D_3$

Compact group: SU(4)

Generators: Two alternative bases in the 4-dim. spinor representation

- Gell-Mann type[19] 4×4 matrices $\lambda_1, \dots, \lambda_{15}$
- 15 rotation matrices $L_{ab}, \quad a, b = 1, \dots, 6$ in spinor 4-dim. representation (see appendix E.1)

Primitive roots (in \mathbb{R}^3):

$$\gamma_1 = \hat{x}, \quad \gamma_2 = -\frac{1}{2}\hat{x} + \frac{1}{\sqrt{2}}\hat{y} - \frac{1}{2}\hat{z}, \quad \gamma_3 = \hat{z}$$

Radial parameters vector

$$\varphi = \varphi_1 \gamma_1 + \varphi_2 \gamma_2 + \varphi_3 \gamma_3 = \phi_1 \hat{x} + \phi_2 \hat{y} + \phi_3 \hat{z},$$

$$\varphi_1 = \phi_1 + \frac{\phi_2}{\sqrt{2}}, \quad \phi_2 = \sqrt{2}\phi_2, \quad \varphi_3 = \phi_3 + \frac{\phi_2}{\sqrt{2}}$$

Winding number vector:

$$\mathbf{m} = 2m_1 \gamma_1 + 2m_2 \gamma_2 + 2m_3 \gamma_3 = (2m_1 - m_2) \hat{x} + \sqrt{2}m_2 \hat{y} + (2m_3 - m_2) \hat{z}$$

eigenvalues of the group element in the spinor representation

$$e^{\frac{i}{2}(\phi_1 + \phi_2/\sqrt{2})}, \quad e^{\frac{i}{2}(-\phi_1 + \phi_2/\sqrt{2})}, \quad e^{\frac{i}{2}(\phi_3 - \phi_2/\sqrt{2})}, \quad e^{\frac{i}{2}(-\phi_3 - \phi_2/\sqrt{2})}$$

eigenvalues of the group element in the 6-dim. vector representation

$$e^{\frac{i}{\sqrt{2}}(\phi_1+\phi_3)}, e^{-\frac{i}{\sqrt{2}}(\phi_1+\phi_3)}, e^{\frac{i}{\sqrt{2}}(\phi_1-\phi_3)}, e^{-\frac{i}{\sqrt{2}}(\phi_1-\phi_3)}, e^{i\phi_2}, e^{-i\phi_2},$$

Noncompact groups

The non-compact groups that correspond to inner automorphisms of SU(4) are SU(3,1) and SU(2,2).

G	SU(3,1)~SO*(6)	SU(2,2)~SO(4,2)
K	SU(3)~U(1)	SU(2)~SU(2)~U(1)
K_j	$\lambda_1, \dots, \lambda_8, \lambda_{15}$	$\lambda_a, a = 1, 2, 3, 8, 13, 14, 15$
iP_j	$i\lambda_9, \dots, i\lambda_{14}$	$i\lambda_b, b = 4, 5, 6, 7, 9, 10, 11, 12$
D_a	$D_3: \quad \varphi = \phi_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = (2m_1 - m_2)\hat{x} + \sqrt{2}m_2\hat{y} + (2m_3 - m_2)\hat{z}$	$D_3: \quad \varphi = \phi_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = (2m_1 - m_2)\hat{x} + \sqrt{2}m_2\hat{y} + (2m_3 - m_2)\hat{z}$
	$D_2: \quad \varphi = i\theta_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = 2\sqrt{2}m_1\hat{y} + (2m_3 - 2m_1)\hat{z}$	$D_2: \quad \varphi = i\theta_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = 2\sqrt{2}m_1\hat{y} + (2m_3 - 2m_1)\hat{z}$
		$D_1: \quad \varphi = i\theta_1\hat{x} + \phi_2\hat{y} + i\theta_3\hat{z}$ $\tilde{\mathbf{m}} = 2\sqrt{2}m\hat{y}$

The noncompact groups that correspond to outer automorphisms of SU(4) are SL(4,ℝ) and Q(2)~SO(5,1). Q(2) is a group that acts in 2-dim. quaternionic space[1, 10].

G	SO(3,3)~SL(4,ℝ)	SO(5,1)~Q(2)
K	SO(3)~SO(3)	SO(5)
K_j	$L_{12}, L_{23}, L_{13}, L_{45}, L_{46}, L_{56}$	$L_{ab}, a, b = 1, \dots, 5$
iP_j	$iL_{a4}, iL_{a5}, iL_{a6}, a = 1, 2, 3$	$iL_{a6}, a = 1, \dots, 5$
D_a	$D_2: \quad \varphi = \phi_1\hat{x} + i\theta_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = 2m_1\hat{x} + 2m_3\hat{z}$	$D_2: \quad \varphi = \phi_1\hat{x} + i\theta_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = 2m_1\hat{x} + 2m_3\hat{z}$
	$D_1: \quad \varphi = i\theta_1\hat{x} + i\theta_2\hat{y} + \phi_3\hat{z}$ $\tilde{\mathbf{m}} = 2m_3\hat{z}$	
	$D_0: \quad \varphi = i\theta_1\hat{x} + i\theta_2\hat{y} + i\theta_3\hat{z}$ $\tilde{\mathbf{m}} = 0$	

8.4 Groups associated with C_3

Compact group: USp(6)

Generators: X_1, \dots, X_{21} (see Appendix E.2)

Primitive roots (in \mathbb{R}^3):

$$\gamma_1 = \hat{x}, \quad \gamma_2 = -\frac{1}{2}\hat{x} + \frac{1}{2}\hat{y} - \frac{1}{\sqrt{2}}\hat{z}, \quad \gamma_3 = \sqrt{2}\hat{z}$$

Radial parameters vector

$$\varphi = \varphi_1\gamma_1 + \varphi_2\gamma_2 + \varphi_3\gamma_3 = \phi_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$$

Winding number vector:

$$\mathbf{m} = 2m_1\gamma_1 + 2m_2\gamma_2 + m_3\gamma_3 = (2m_1 - m_2)\hat{x} + m_2\hat{y} + \sqrt{2}(m_3 - m_2)\hat{z}$$

Eigenvalues of the group element in the fundamental representation

$$e^{\frac{i}{2}(\phi_1+\phi_2)}, e^{\frac{i}{2}(-\phi_1+\phi_2)}, e^{\frac{i}{2}(\phi_1-\phi_2)}, e^{\frac{i}{2}(-\phi_1-\phi_2)}, e^{\frac{i}{\sqrt{2}}\phi_3}, e^{-\frac{i}{\sqrt{2}}\phi_3}$$

Noncompact groups:

G	USp(4,2)	Sp(6,ℝ)
K	USp(4)⊗USp(2)	SU(3)⊗U(1)
K_j	$X_1, X_2, X_3, X_4, X_5, X_{10}, X_{11}, X_{12}, X_{13},$ $X_{16}, X_{17}, X_{20}, X_{21}$	$X_{2n+1}, n = 2, 3, \dots, 10$
iP_j	$iX_6, iX_7, iX_8, iX_9, iX_{14}, iX_{15}, iX_{18}, iX_{19}$	$iX_1, iX_3, iX_{2n} n = 1, \dots, 10$
D_a	$D_3: \quad \varphi = \phi_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\quad \tilde{\mathbf{m}} = (2m_1 - m_2)\hat{x} + m_2\hat{y} +$ $\quad \quad \quad + \sqrt{2}(m_3 - m_2)\hat{z}$	$D_3: \quad \varphi = \phi_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\quad \tilde{\mathbf{m}} = (2m_1 - m_2)\hat{x} + m_2\hat{y} +$ $\quad \quad \quad + \sqrt{2}(m_3 - m_2)\hat{z}$
	$D_2: \quad \varphi = i\theta_1\hat{x} + \phi_2\hat{y} + \phi_3\hat{z}$ $\quad \tilde{\mathbf{m}} = 2m_1\hat{y} + \sqrt{2}(m_3 - 2m_1)\hat{z}$	$D_2: \quad \varphi = \phi_1\hat{x} + \phi_2\hat{y} + i\theta_3\hat{z}$ $\quad \tilde{\mathbf{m}} = (2m_1 - m_2)\hat{x} + m_2\hat{y}$
		$D_1: \quad \varphi = i\theta_1\hat{x} + i\theta_2\hat{y} + \phi_3\hat{z}$ $\quad \tilde{\mathbf{m}} = \sqrt{2}m_3\hat{z}$
		$D_0: \quad \varphi = i\theta_1\hat{x} + i\theta_2\hat{y} + i\theta_3\hat{z}$ $\quad \tilde{\mathbf{m}} = 0$

9 Conclusion

We have shown that using the integral representation (eq. 12) for the evolution operator is a powerful method, that produces exact evolution operators of free motion on the manifolds of both compact and non-compact groups.

For compact groups, the two complimentary representations for the evolution operator, the spectral expansion and the sum over classical paths, are reproduced from the same integral representation by using two different integration methods.

For non-compact groups, this method enables us to find the exact evolution operator. A general expression for the evolution operator is given in eq. 47. This expression depends on the root system of the specific group, on the radial parameters vector and on the winding numbers vector of the radial parameters around the maximal torus.

A complication, that arise in the non-compact case, is that the maximal torus topology is not unique on the entire group manifold. This is very different from the compact case, where the maximal torus is a closed torus, and the winding number vector is determined solely by the primitive roots (eq. 21). In the non-compact case, the torus becomes an open torus, and the manifold is splitted into several domains which differ from each other by the maximal torus topology. The radial parameters, which reside in a space tangent to the torus, are no longer real, however a coordinate system where the radial parameters are either pure real or pure imaginary can always be found. Thus, the general expression for the evolution operator is different in each domain, and using it requires the knowledge of the winding numbers vector in the domain.

The various domains, that correspond to each non-compact group, are derived from the group type (quasi-unitary, quasi-orthogonal, etc.). In particular, the maximal torus depends only on the eigenvalue system of the group element (since we are dealing with matrix groups, the notion of eigenvalues usually corresponds to the fundamental representation). The allowed values of the radial parameters are derived from the conditions that are imposed on the eigenvalues according to the group type. This analysis was carried out in section 6.2 for most of the simple groups that are generated from the classical algebras A_n , B_n , C_n and D_n .

The method was demonstrated on SU(2) and SU(1,1) by a straightforward computation of the evolution operators. For the larger groups, that are generated by the algebras A_2 , B_2 , A_3 and C_3 , only the final results are given. For each group, an appropriate coordinate system for the radial parameters was chosen, and the different domains were found. The radial parameters vector and winding number vector were written explicitly for each domain. These vectors can be inserted into expression 47 to produce the exact expression for the evolution operator in the specific domain.

The expression for the evolution operator on non-compact groups can be interpreted as a sum over classical paths. The different evolution domains correspond to different classes of geodetics. The group

manifold is open in some dimensions, and compact in the others. Thus, the winding number vector depends on the ‘direction’ of the classical trajectory.

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Appendix

A Some notations

We shall use the following notations: r is the rank of the Algebra; n is the dimensionality; $p = (n - r)/2$ is the number of positive roots. There is an ordered system of positive roots α_ν , $\nu = 1, \dots, p$; among them are simple roots (a basis in the root space) γ_j , $j = 1, \dots, r$, and the highest root $\alpha_1 = \sum_{j=1}^r a_j \gamma_j$ where a_j are positive integers. The scale factor Λ and the vector ρ are involved in sum important formulae. They are expressed in terms of the roots

$$(78) \quad \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha, \quad \Lambda = 2r^{-1} \sum_{\alpha > 0} \alpha^2;$$

the sums are over all positive roots. The Cartan matrix has elements

$$(79) \quad M_{jk} = 2\gamma_j \cdot \gamma_k / \gamma_j^2$$

that are integers: $0, \pm 1, \pm 2, -3$. The fundamental weights satisfy the following relation with the simple roots

$$(80) \quad \gamma_i w_j = \gamma_i^2 / 2\delta_{ij}.$$

The unitary representations of compact groups are represented by the highest weight $\mathbf{l} = \sum_i l_i w_i$, l_i are integers. The eigenvalues of the Laplacean (Casimir operator) as well as the dimensionality of the representations are given in terms of highest weight

$$(81) \quad \lambda_{\mathbf{l}} = \frac{1}{\Lambda} (n^2 - \rho^2), \quad d_{\mathbf{l}} = \prod_{\alpha > 0} \frac{\alpha \cdot \mathbf{n}}{\alpha \cdot \rho},$$

where $\mathbf{n} = \mathbf{l} + \rho$.

B Lie groups: summary

B.1 The Lie group and its Lie algebra

Let G be a Lie group and \mathcal{G} its corresponding Lie algebra,

$$(82) \quad \mathcal{G} \rightarrow G : g(x) = \exp x, \quad x \in \mathcal{G}, g \in G.$$

A basis $\{e_a\}$, where $1 \leq a \leq n$, is introduced in \mathcal{G} , so that

$$(83) \quad x = \xi^a e_a, \quad [e_a, e_b] = C_{ab}^c e_c,$$

where C_{ab}^c are the structure constants, and $[\cdot, \cdot]$ stands for the Lie product in \mathcal{G} . It is assumed that \mathcal{G} is real, i.e. there exists a basis where the structure constants are real. The adjoint matrix representation of G acting in \mathcal{G} is defined by

$$(84) \quad g^{-1} \exp(\zeta^a e_a) g = \exp[\zeta^a A_a^b(g) e_b],$$

where the $n \times n$ matrix $\mathbf{A}(g)$ with elements $A_a^b(g)$ is given by the exponential of the structure constants,

$$(85) \quad \mathbf{A}(g) = \exp \mathbf{X}, \quad \mathbf{X} \equiv \xi^a \mathbf{E}_a, \quad (\mathbf{E}_a)_b^c \equiv C_{ba}^c = -C_{ab}^c.$$

(In this section, bold-face capitals are used for matrices in the adjoint representation). The commutators of matrices \mathbf{E}_a provide with the basis for the regular representation of \mathcal{G} , as follows from the Jacobi identity,

$$(86) \quad [\mathbf{E}_a, \mathbf{E}_b] = C_{ab}^c \mathbf{E}_c.$$

The scalar product is defined in \mathcal{G} by means of the adjoint representation,

$$(87) \quad (x, y) \equiv -\text{Tr}(\mathbf{X}\mathbf{Y}) = \xi^a \eta^b \Gamma_{ab}, \quad \Gamma_{ab} \equiv -\text{Tr}(\mathbf{E}_a \mathbf{E}_b) = C_{ad}^c C_{cb}^d.$$

The Cartan – Killing matrix Γ may be reduced by a proper linear transformation to a diagonal form $\Gamma_{ab} = \Lambda \eta_{ab}$, where Λ is a scaling factor and η_{ab} has n_+ eigen-values $+1$ and n_- eigen-values -1 . It is assumed that \mathcal{G} is (semi-)simple, so Γ is non-degenerate, and $n_+ + n_- = n$. If $n_+ = n$, i.e. Γ is positive definite, the real group \mathbf{G} is compact. In general, n_+ is the dimensionality of the maximal compact subgroup $\mathbf{G}_+ \subset \mathbf{G}$.

B.2 The Cartan – Maurer form and the Lie derivatives.

The (left-invariant) Cartan – Maurer one-form on \mathcal{G} is defined by

$$(88) \quad \delta x \equiv g(x)^{-1} dg(x) \equiv \delta \xi^a e_a = d\xi^b B_b^a(x) e_a,$$

where the matrix $\mathbf{B}(x)$ is expressed in terms of the adjoint group representation,

$$(89) \quad \mathbf{B}(x) = \int_0^1 \mathbf{A}(g_\tau) d\tau, \quad g_\tau \equiv \exp(\tau x).$$

The Killing field $\nabla_a(x)$, dual to the form $\delta \xi^a$, is defined by

$$(90) \quad \delta \xi^a \nabla_a(x) \equiv d\xi^a \partial_a : \quad \nabla_a(x) = L_a^b(x) \partial_b,$$

where $\partial_a \equiv \partial / \partial \xi^a$, $\mathbf{L}(x) = [\mathbf{B}(x)]^{-1}$. The Lie derivatives are introduced in terms of the Killing fields; namely for $z = \zeta^a e_a$ and $\forall x, z \in \mathcal{G}$,

$$(91) \quad \mathcal{L}_z(x) \equiv \zeta^a \nabla_a(x) = \zeta^a L_a^b(x) \partial / \partial \xi^b.$$

By virtue of the Cartan – Maurer equation, at any given $x \in \mathcal{G}$, the commutators of the Lie derivatives provide with a representation of \mathcal{G} ,

$$(92) \quad d\delta x + \delta x \wedge \delta x = 0 \rightarrow [\mathcal{L}_y(x), \mathcal{L}_z(x)] = \mathcal{L}_{[y,z]}(x).$$

Writing that in the components one has, respectively,

$$(93) \quad \begin{aligned} & \partial_a B_b^c - \partial_b B_a^c + B_a^{a'} B_b^{b'} C_{a'b'}^c = 0, \\ & [\nabla_a(x), \nabla_b(x)] = C_{ab}^c \nabla_c(x); \quad L_a^{a'} \partial_{a'} L_b^c - L_b^{b'} \partial_{b'} L_a^c = C_{ab}^{c'} L_{c'}^c. \end{aligned}$$

Note that as follows from the definition of the Lie derivative,

$$(94) \quad \nabla_a(x) \mathbf{A}(g) = \mathbf{E}_a \mathbf{A}(g).$$

B.3 Group as a Riemannian manifold.

The Riemannian metrics in \mathbf{G} is induced naturally by the (pseudo-) Euclidean metrics in its tangent linear space \mathcal{G} , as given by Eq. (87). The invariant length element is defined by

$$(95) \quad dx^2 \equiv (\delta x, \delta x) = d\xi^a d\xi^b \gamma_{ab}(x),$$

where $\gamma_{ab}(x) = B_a^c(x) B_b^d(x) \Gamma_{cd}$. The Riemannian structure is invariant under general coordinate transformations, and under the shifts of the group elements, $g \rightarrow gg_1$, in particular. The corresponding invariant measure on the group manifold is

$$(96) \quad d\mu(g) = |\gamma(\xi)|^{1/2} d^n \xi,$$

where $|\gamma(\xi)| \equiv \det(\gamma_{ab}) = [\det \mathbf{B}(x)]^2 \det(\Gamma)$. This measure is proportional to Weyl's invariant measure on the group. For compact groups, the total invariant volume exists, and the explicit expression was given in Ref.[24] (see also appendix B.6).

The invariant Laplace operator Δ is defined as usual since the metric is non-degenerate. It can be also considered as a realization of the invariant Casimir operator in the universal enveloping algebra of \mathcal{G} , represented by the Killing fields,

$$(97) \quad \Delta \equiv \frac{1}{|\gamma|} \frac{\partial}{\partial \xi^a} |\gamma| \gamma^{ab} \frac{\partial}{\partial \xi^b} = \Gamma^{ab} \nabla_a \nabla_b,$$

The operator Δ is elliptic for compact groups and hyperbolic for (real) non-compact groups. (Higher-order invariant differential operators may be constructed by means of Γ^{ab} , structure constants and ∇ , see also appendix C.)

B.4 Group as a fiber bundle.

Let \mathbf{G}_1 be a subgroup of \mathbf{G} . Introduce a basis in the subalgebra \mathcal{G}_1 , $\{e_j\}$ and a basis in $\mathcal{Z} \equiv \mathcal{G} \setminus \mathcal{G}_1$, $\{e_\alpha\}$ where $1 \leq j \leq n_1$, $1 \leq \alpha \leq (n - n_1)$. If \mathbf{G}_1 is semi-simple, the basis can be chosen to satisfy $(e_j, e_\alpha) = 0$. Now the basic relations are

$$(98) \quad \begin{aligned} [e_j, e_k] &= C_{jk}^l e_l, \quad [e_j, e_\alpha] = C_{j\alpha}^\beta e_\beta, \\ [e_\alpha, e_\beta] &= C_{\alpha\beta}^j e_j + C_{\alpha\beta}^\gamma e_\gamma. \end{aligned}$$

The resulting regular representation of \mathbf{G}_1 has a block structure, since $(\mathbf{E}_j)_\alpha^k = 0 = (\mathbf{E}_j)_k^\alpha$. The same is true also for the adjoint group representation: $\mathbf{A}(h) = \exp(\sum_{j=1}^{n_1} \mathbf{E}_j \eta^j)$, $\forall h \in \mathbf{G}_1$.

The group elements can be decomposed as follows,

$$(99) \quad g(x) = v(z) h(y) v(z)^{-1},$$

where $h \in \mathbf{G}_1$, $y \in \mathcal{H} \subset \mathcal{G}$, and $x(z)$ is a representative of the equivalence class $v(z) \sim v(z)h_1$, $\forall h_1 \in \mathbf{G}_1$. The Cartan – Maurer one-form and the metric form are decomposed respectively,

$$(100) \quad \delta x = v (\delta y + h^{-1} \delta z - \delta z) v^{-1},$$

$$(101) \quad \begin{aligned} (\delta x, \delta x) &= (\delta y, \delta y) + 2 [(\delta z, \delta z) - (\delta z, h \delta z h^{-1})] \\ &\equiv (\delta y, \delta y) + 2 \delta \zeta^\alpha \delta \zeta^\beta \tilde{\Gamma}_{\alpha\beta}(y), \end{aligned}$$

where $\tilde{\Gamma}_{\alpha\beta}(y) = \Gamma_{\alpha\beta} - A_\alpha^{\alpha'}(h) \Gamma_{\alpha'\beta}$. The first term is the metric induced in \mathbf{G}_1 , while the metric induced in \mathcal{Z} depends on h .

B.5 Radial coordinates

The case of a particular interest is $\mathcal{G}_1 = \mathcal{H}$ – the Cartan subalgebra, $\mathbf{G}_1 = \mathbf{H}$ – the maximal Abelian subgroup, and $n_1 = r$. Now the metric induced in \mathbf{H} is flat. The h -dependence of the measure on the group is given in terms of the adjoint representation, given by a central function which may be presented in a manifestly invariant form,

$$(102) \quad d\mu(g) = W^2(h) d\mu(h) d\mu(v),$$

$$W^2(h) = \det' [\mathbf{I} - \mathbf{A}(h)] = \lim_{\omega \rightarrow 1} \left(\frac{\det [\omega \mathbf{I} - \mathbf{A}(g)]}{(\omega - 1)^r} \right).$$

Here \det' is the determinant for the block corresponding to \mathcal{Z} , and we note that, $\forall g$, the matrix $\mathbf{A}(g)$ has at least r eigen-values which are equal to 1, corresponding to the subgroup \mathbf{H} . Other eigenvalues of $\mathbf{A}(g)$ are denoted by $\lambda_\alpha = \exp [\pm(\boldsymbol{\alpha} \cdot \boldsymbol{\varphi})]$, where $\boldsymbol{\alpha}$ are positive roots of \mathcal{G} , i.e. real r -dimensional vectors, and $\boldsymbol{\varphi}$ is a vector in the dual space. For compact groups, $\mathbf{A}(g)$ is unitary, $|\lambda_\alpha| = 1$, so $\boldsymbol{\varphi}$ is a real vector, so that

$$(103) \quad W(h) = \prod_{\alpha} \left[2 \sin \frac{(\boldsymbol{\alpha} \cdot \boldsymbol{\varphi})}{2} \right].$$

Note that $W(h)$ is proportional to the Weyl function $w(h)$ (Eq. 15). For non-compact real groups, some of the roots $(\boldsymbol{\alpha} \cdot \boldsymbol{\varphi})$ are complex conjugate pairs.

B.6 Invariant volumes of compact groups

The invariant volume of a compact group is [24]

$$(104) \quad V_{\mathbf{G}} = \int_{\mathbf{G}} d\mu(g) = \frac{\Lambda^{n/2} (2\pi)^{p+r} [\det(M_{jk})]^{1/2}}{\prod_{\gamma} (\gamma^2/2)^{1/2} \prod_{\boldsymbol{\alpha} > 0} (\boldsymbol{\alpha} \cdot \boldsymbol{\rho})}.$$

The invariant volume is a product of the volume of the maximal torus \mathbf{T} and the volume of the coset space \mathbf{G}/\mathbf{T} , $V_{\mathbf{G}} = V_{\mathbf{T}} V_{\mathbf{G}/\mathbf{T}}$, and each factor can be computed separately

$$(105) \quad V_{\mathbf{T}} = \Lambda^{r/2} \int_{\mathbf{T}} 2^{n-r} [w(\boldsymbol{\varphi})]^2 d\boldsymbol{\varphi} = \Lambda^{r/2} \frac{(2\pi)^r [\det(M_{jk})]^{1/2}}{\prod (\gamma^2/2)^{1/2}} = \frac{\Lambda^{r/2} (2\pi)^r}{[\det(\mathbf{w}_i \mathbf{w}_j)]^{1/2}},$$

where $d\boldsymbol{\varphi} = \prod d\varphi$ is the integration measure in the flat space, and

$$(106) \quad V_{\mathbf{G}/\mathbf{T}} = \frac{(2\pi)^p \Lambda^p}{\prod_{\boldsymbol{\alpha} > 0} (\boldsymbol{\alpha} \cdot \boldsymbol{\rho})}.$$

Note that all these volumes are invariant under a change of the root normalization.

B.7 Weyl group, Weyl chamber, Weyl alcove

The Weyl group W is the group of permutations of the root system. The elements of the Weyl group are called Weyl reflections. Its action also permutes the diagonal elements of $\varphi_j H_j$, where H_j are the basis elements of the Cartan subalgebra and φ_j are the radial parameters.. To avoid this sort of ambiguity, the values of the radial parameters are restricted to the Weyl chamber by the condition $\boldsymbol{\gamma} \cdot \boldsymbol{\varphi} \geq 0$. The root space in which $\boldsymbol{\varphi}$ reside is divided by the hyper-planes $\boldsymbol{\alpha} \cdot \boldsymbol{\varphi} = 0$ into $N(W)$ regions congruent to the Weyl chamber ($N(W)$ is the order of the Weyl group). The Weyl transformations permute these regions.

Any function on a compact group is periodical in the radial coordinates

$$(107) \quad f(\boldsymbol{\varphi} + 2\pi \mathbf{m}) = f(\boldsymbol{\varphi}), \quad \mathbf{m} = \sum_{j=1}^r m_j \hat{\boldsymbol{\gamma}}_j, \quad \hat{\boldsymbol{\gamma}}_j = 2\boldsymbol{\gamma}_j / \gamma_j^2$$

where m_j are integers, so the maximal torus for the group reside inside a larger torus defined by

$$(108) \quad \varphi = \sum_{j=1}^r \varphi_j \hat{\gamma}_j, \quad -\pi < \varphi_j \leq \pi.$$

The weyl reflection hyper-surfaces $\alpha \cdot \varphi = 0$ divide the torus into $N(W)$ regions. The region enclosed by the hyper-surfaces $\gamma \cdot \varphi \geq 0$, $\alpha^1 \cdot \varphi \leq 2\pi$, where α^1 is the highest root, is called the Weyl alcove. The Weyl alcove coincide with the maximal torus.

The characters of the unitary irreducible representations, represented by the highest weight \mathbf{l} , are given by the Weyl formula

$$(109) \quad \chi_{\mathbf{l}}(\varphi) = \frac{1}{(2i)^{pw}(\varphi)} \sum_{\sigma \in W} \epsilon_{\sigma} \exp[i(\sigma \mathbf{n}, \varphi)]$$

where the summation is over Weyl reflections, $\epsilon_{\sigma} = +1(-1)$ for even (odd) reflection.

C Invariant operators and the symmetry operator \mathcal{D}

An invariant operator is built according to the following theorem (Berezin [3]):

Let $P(\varphi_1, \dots, \varphi_r)$ be any polynomial on the Cartan subalgebra H , invariant with respect to the Weyl group W . Consider the differential operator $P(\partial/\partial\varphi^1, \dots, \partial/\partial\varphi^r)$ which is obtained by formal substitution of the operators $\partial/\partial\varphi^i$ in place of the coordinates φ_i in the polynomial $P(\varphi_1, \dots, \varphi_r)$. The operator

$$(110) \quad \tilde{\Delta}(P) = \frac{1}{w(\varphi)} \left[P \left(\frac{\partial}{\partial\varphi^1}, \dots, \frac{\partial}{\partial\varphi^r} \right) \right] w(\varphi)$$

is the radial part of some Laplace operator on the group.

The converse of this theorem is also true. The coordinates φ_i and φ^i are dual coordinate systems (if $\varphi = \varphi_i \gamma_i$ they are connected by the Cartan matrix $\varphi^i = M_{ij} \varphi_j$).

We shall use the operator

$$(111) \quad \mathcal{D}(\varphi) = \prod_{\alpha > 0} \left[\frac{2\alpha_j}{\alpha_j^2} \sum_{i=1}^r w_i \frac{\partial}{\partial\varphi_i} \right],$$

which is consisted of a product of directional derivatives along the positive roots. This operator is called an intertwining operator since it intertwines the radial δ -function on the group manifold with the δ -function on the torus \mathbb{T} (see Ref. [9]). Substituting the square of the operator, \mathcal{D}^2 , for the polynomial P in Eq. 110, an invariant operator on the group is obtained. When acting on the character of an UIR with \mathcal{D}^2 , its eigenvalue is proportional to the square of the representation's dimensionality

$$(112) \quad \begin{aligned} \frac{1}{w(\varphi)} \mathcal{D}^2 w(\varphi) \chi_{\mathbf{l}}(\varphi) &= \frac{1}{(2i)^{pw}(\varphi)} \mathcal{D}^2 \sum_{\sigma \in W} \epsilon_{\sigma} \exp[i(\sigma \mathbf{n} \cdot \varphi)] \\ &= \left[\prod_{\alpha > 0} i(\mathbf{n} \cdot \alpha) \right]^2 \chi_{\mathbf{l}}(\varphi) = \left[\prod_{\alpha > 0} i(\rho \cdot \alpha) \right]^2 d_{\mathbf{l}}^2 \chi_{\mathbf{l}}(\varphi) \end{aligned}$$

The operator \mathcal{D} is not an invariant operator on the group manifold, but it corresponds to the one-dimensional antisymmetric representation of the Weyl group. Therefore it is used to change the symmetry of radial functions under Weyl reflections. In particular

$$(113) \quad \begin{aligned} \frac{1}{w(\varphi)} \mathcal{D} w(\varphi) \chi_{\mathbf{l}}(\varphi) &= \frac{1}{(2i)^{pw}(\varphi)} \mathcal{D} \sum_{\sigma \in W} \epsilon_{\sigma} \exp[i(\sigma \mathbf{n} \cdot \varphi)] = \\ &= \prod_{\alpha > 0} i(\mathbf{n} \cdot \alpha) \sum_{\sigma \in W} \exp[i(\sigma \mathbf{n} \cdot \varphi)] = \prod_{\alpha > 0} i(\rho \cdot \alpha) d_{\mathbf{l}} \sum_{\sigma \in W} \exp[i(\sigma \mathbf{n} \cdot \varphi)]. \end{aligned}$$

Taking the limit $\varphi = 0$ reproduces the celebrated Weyl dimensions formula (up to known factors).

Another interesting feature is that when working on the Weyl function itself, the operator can be used to determine the order of the Weyl group

$$(114) \frac{2^p}{(\alpha \cdot \rho)} \mathcal{D}w(\varphi_0)|_{\varphi_0=0} = N(W).$$

D The connection between Green's function on \mathbb{T} and on \mathbb{G}

The resolvent for Helmholtz equation in flat r dimensional space $G^y(\varphi, \varphi_0)$ is used to find the state $y(\varphi)$ that is described by Eq. 17

$$(115) y(\varphi) = \int G_\lambda^y(\varphi, \varphi_0) F(\varphi_0) d\varphi_0.$$

$d\varphi = \prod d\varphi$ is the integration measure in the flat space. Since our original equation for the state $\psi(\varphi)$ is

$$(116) \frac{1}{\Lambda} \left[\frac{1}{w} \frac{\partial^2}{\partial \varphi^2} w + \rho^2 + \Lambda \lambda \right] \psi(\varphi) = -f(\varphi),$$

the Green function for ψ can be obtained from G^y by inserting the expressions for y and F (Eq. 18) into Eq. 115 and dividing by $w(\varphi)$

$$(117) \psi(\varphi) = \frac{1}{w(\varphi)} \int G_\lambda^\psi(\varphi, \varphi_0) f(\varphi_0) d\varphi_0 = \frac{1}{w(\varphi)} \int G_\lambda^y(\varphi, \varphi_0) \Lambda w(\varphi_0) f(\varphi_0) d\varphi_0.$$

Collecting all the factors, the resolvent for $\psi(\varphi)$ becomes

$$(118) G_\lambda^\psi(\varphi, \varphi_0) = \Lambda \frac{w(\varphi_0)}{w(\varphi)} G_\lambda^y(\varphi, \varphi_0).$$

To get the resolvent $G_\lambda(g_1 g_0^{-1})$ that is used to obtain the state $\psi(g_1)$ on the group manifold

$$(119) \psi(g_1) = \int_{\mathbb{G}} G_\lambda(g_1 g_0^{-1}) f(g_0) d\mu(g_0)$$

two additional steps are necessary. First, we must take into account the different integration measures in the flat r -dimensional space (the torus) and on the group manifold. According to appendix B.6 the relation between the two integration measures is

$$(120) V_{\mathbb{G}} = \int_{\mathbb{G}} d\mu(g) = V_{\mathbb{G}/\mathbb{T}} \int_{\mathbb{T}} 2^{n-r} [w(\varphi)]^2 d\varphi.$$

Thus, we have to divide the Green function G^ψ by $V_{\mathbb{G}/\mathbb{T}} 2^{n-r} [w(\varphi_0)]^2$. Second, the evolution depends on the radial coordinates φ of the group element $g = g_1 g_0^{-1}$ that appears in Eq. 13, i.e. on the 'distance' from the origin to the point $h \in \mathbb{T}$. Therefore φ_0 does not have any meaning and should be put to zero, while $\varphi = \varphi(g_1 g_0^{-1})$. The final expression for the resolvent is

$$(121) G_\lambda(\varphi(g_1 g_0^{-1})) = \frac{\Lambda}{V_{\mathbb{G}/\mathbb{T}} 2^{n-r} [w(\varphi_0)]^2} G_\lambda^\psi(\varphi, \varphi_0) \Big|_{\varphi_0=0} = \frac{\Lambda}{V_{\mathbb{G}/\mathbb{T}} w(\varphi)} \left[\frac{G^y(\varphi, \varphi_0)}{2^{n-r} w(\varphi_0)} \right] \Big|_{\varphi_0=0}.$$

(One should note that exactly the same procedure is used when looking for the Green function in flat space which depends only on the distance between two points

$$G(\mathbf{q}, \mathbf{q}_0) = G(r), \quad r = \sqrt{\mathbf{q}^2 - \mathbf{q}_0^2}.$$

$G(r)$ is found by finding the resolvent for the one dimensional radial Laplacean, $G^1(r, r_0)$. $w(\varphi)^2$ plays the role of the part of the integration measure that depends on r . To get $G(r)$ it is necessary to divide G^1 by the angular volume and by r_0^{n-1} for an n dimensional space, and then set $r_0 = 0$. The only difference between this case and the group manifold is that there are several radial parameters instead of one).

E Bases for the generators of several groups

E.1 Rotation generators in spinor representation

The generators of the rotation group $SO(p,q)$ satisfy the following commutation relations[11]

$$(122) [L_{ab}, L_{cd}] = \delta_{ac}L_{bd} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad} + \delta_{bd}L_{ac},$$

where δ_{ab} is the metric tensor in the $p + q$ flat space. The rotation matrices in the 4-dim. spinor representation of the groups $SO(4)$, $SO(5)$ and $SO(6)$ are built from the Euclidean γ matrices:

$$(123) \quad \begin{aligned} L_{ij} &= \frac{i}{2} [\gamma_j \gamma_i - \gamma_i \gamma_j], \quad i, j = 1, \dots, 4 \\ L_{i5} &= \gamma_i \quad i = 1, \dots, 4 \\ L_{i6} &= i\gamma_i \gamma_5 \quad L_{56} = \gamma_5, \quad i = 1, \dots, 4 \end{aligned}$$

For example, the following basis for the γ matrices can be chosen:

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} \quad \gamma_4 = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

σ_i , $i = 1, 2, 3$ are the Pauli matrices.

The general method of building the rotation matrices in spinor representation for any rotation group is explained in a paper by Brauer[7].

E.2 The generators of C_3

Choosing the explicit basis of C_3 is based on Refs. [8, 18]. The general algebra element in this basis is

$$(124) x = \xi^a X_a = \begin{pmatrix} \xi_1 & \xi_4 + i\xi_5 & \xi_6 + i\xi_7 & \xi_{10} + i\xi_{11} & \xi_{12} + i\xi_{13} & \xi_{14} + i\xi_{15} \\ \xi_4 - i\xi_5 & \xi_2 & \xi_8 + i\xi_9 & \xi_{12} + i\xi_{13} & \xi_{16} + i\xi_{17} & \xi_{18} + i\xi_{19} \\ \xi_6 - i\xi_7 & \xi_8 - i\xi_9 & \xi_3 & \xi_{14} + i\xi_{15} & \xi_{18} + i\xi_{19} & \xi_{20} + i\xi_{21} \\ \xi_{10} - i\xi_{11} & \xi_{12} - i\xi_{13} & \xi_{14} - i\xi_{15} & -\xi_1 & -\xi_4 + i\xi_5 & -\xi_6 + i\xi_7 \\ \xi_{12} - i\xi_{13} & \xi_{16} - i\xi_{17} & \xi_{18} - i\xi_{19} & -\xi_4 - i\xi_5 & -\xi_2 & -\xi_8 + i\xi_9 \\ \xi_{14} - i\xi_{15} & \xi_{18} - i\xi_{19} & \xi_{20} - i\xi_{21} & -\xi_6 - i\xi_7 & -\xi_8 - i\xi_9 & -\xi_3 \end{pmatrix}.$$

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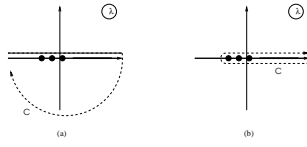


Figure 1: The integration contour C in the complex λ plane

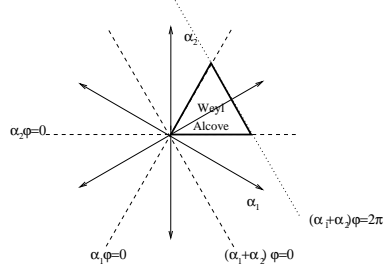


Figure 2: Root diagram and the Weyl alcove of $SU(3)$

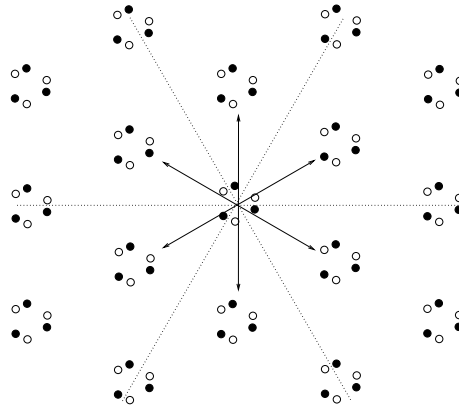


Figure 3: Images that are created by reflections of a point in the Weyl alcove of $SU(3)$. Full(empty) circles correspond to odd(even) reflections

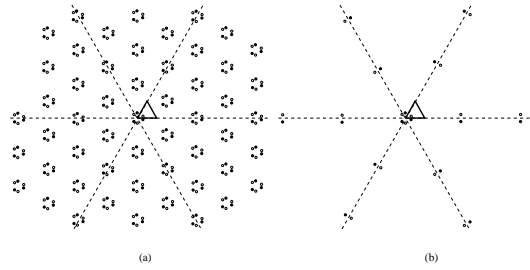


Figure 4: $SU(2,1)$ has two evolution domains. The equivalent points for each domain are shown. Full(empty) circles correspond to odd(even) reflections

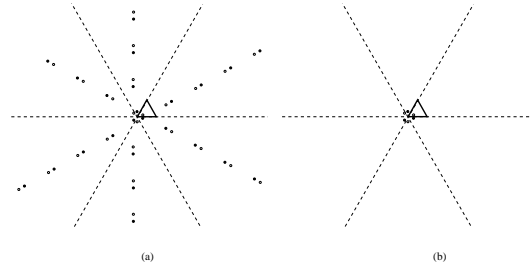


Figure 5: $SL(3, \mathbb{R})$ has two evolution domains. The equivalent points for each domain are shown. Full(empty) circles correspond to odd(even) reflections