TRANSFORMATION OF THE FREE PROPAGATOR TO THE QUADRATIC PROPAGATOR

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A space and time transformation is found, which changes the classical action for a quadratic lagrangian into that for a free particle. It is shown that the propagator for a time-dependent damped oscillator can be obtained from the free propagator.

As is well known, the propagator of a quantum system is in principle obtainable from its classical lagrangian by path integration [1]. In particular, if the lagrangian is quadratic, the propagator can be evaluated directly from the classical action S_c via the van Vleck–Pauli formula [1,2],

$$K(x'', t''; x', t') = \left(\frac{i}{2\pi\hbar}\frac{\partial^2 S_c}{\partial x'\partial x''}\right)^{1/2} \exp[(i/\hbar)S_c(x'', t''; x', t')] . \tag{1}$$

Although this procedure for a quadratic system is exact and unambiguous, the actual calculation of the classical action is not always simple. Therefore, calculations of the propagators for various quadratic systems have been tirelessly appearing in the literature [3–8].

Recently, the technique of changing "space" and "time" variables in path integration has been proven useful for non-quadratic systems such as the hydrogen atom [9,10], the Morse oscillator [11], and the Dirac–Coulomb problem [12], to which formula (1) is not immediately applicable. It is certainly interesting to explore a similar transformation technique for quadratic systems. For non-quadratic systems used are local transformations of short time intervals which are usually non-integrable. In contrast, a transformation that relates the classical equation of motion for one quadratic system to that for another quadratic system, if available, is globally meaningful in quantization, since the propagator given by (1) depends only on the classical solutions. In this paper, we propose to utilize such a global transformation of "space" and "time" variables, say, $y = y(x, t)$ and $s = s(t)$, for finding the propagator of the second quadratic system $K_2(x'', t''; x', t')$ from that of the first $K_1(x'', t''; x', t')$ as

$$K_2(x'', t''; x', t') = [(\partial y'/\partial x')(\partial y''/\partial x'')]^{1/2} K_1(y'', s''; y', s') . \tag{2}$$

First, we present a space and time transformation which changes a quadratic action into a free particle action. Then we show that the propagator for a quadratic lagrangian can be obtained from the propagator for a free particle by the transformation.

The most general quadratic lagrangian is [1]

$$L(\dot{q}, q, t) = a(t)\dot{q}^2 + b(t)\dot{q}q + c(t)q^2 + d(t)\dot{q} + e(t)q + f(t) , \tag{3}$$

where a, b, c, d, e and f are all well-behaved functions of time and $a \neq 0$. As a physical lagrangian, (3) is a little more general than necessary. The equation of motion for a system described by (3) is given by

$$\ddot{q} + \dot{\lambda}(t)\dot{q} + \omega^2(t)q = g(t) , \tag{4}$$

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where $\dot{\lambda}(t) = \dot{a}/a$, $\omega^2(t) = (\dot{b} - 2c)/2a$ and $g(t) = (e - \dot{d})/2a$. It is easy to see that (4) can also be obtained from the lagrangian of the form

$$L = \tfrac{1}{2} m\, \mathrm{e}^{\lambda(t)} [\dot{q}^2 - \omega^2(t) q^2 + 2g(t) q] + \dot{F}(q, t)\,, \tag{5}$$

where $F(q, t)$ is an arbitrary function of q and t. Thus we may take (5) rather than (3) as a general quadratic lagrangian. Furthermore, if a special solution $h(t)$ of (4) can be found for a given external force $g(t)$, then the simple translation of the variable,

$$x = q - h(t)\,, \tag{6}$$

reduces (4) into

$$\ddot{x} + \dot{\lambda}(t)\dot{x} + \omega^2(t) x = 0\,. \tag{7}$$

The corresponding lagrangian is

$$L = \tfrac{1}{2} m\, \mathrm{e}^{\lambda(t)} [\dot{x}^2 - \omega^2(t) x^2] + \dot{G}(x, t)\,, \tag{8}$$

where $G(x, t)$ is another arbitrary function. In this paper, we limit ourselves to the quadratic systems whose lagrangians are reducible to those of the form (8). In doing so, we are apparently dealing with a linear oscillator with a time-dependent damping parameter $\lambda(t)$ and a time-dependent frequency $\omega(t)$.

A general solution of the time-dependent damped oscillator equation (7) may be given by

$$x(t) = A\rho(t)\, \mathrm{e}^{\mathrm{i}\bar{\omega}\tau(t)} + B\rho(t)\, \mathrm{e}^{-\mathrm{i}\bar{\omega}\tau(t)}\,, \tag{9}$$

where the two functions $\rho(t)$ and $\tau(t)$ satisfy the relations

$$\ddot{\rho} + \dot{\lambda}\dot{\rho} + (\omega^2 - \bar{\omega}^2\dot{\tau}^2)\rho = 0 \tag{10}$$

and

$$\ddot{\tau}/\dot{\tau} + \dot{\lambda} + 2\dot{\rho}/\rho = 0\,, \tag{11}$$

or $\bar{\omega}\dot{\tau}\rho^2 \mathrm{e}^{\lambda} = C$. Here A, B, C and $\bar{\omega}$ are real constants. Now we wish to show that the action for the time-dependent damped oscillator (7) can be converted into that for a free particle by the following transformation of "space" and "time" variables;

$$y = x\, \mathrm{e}^{\lambda(t)/2} \dot{\tau}(t)^{1/2} \sec[\bar{\omega}\tau(t)]\,, \qquad s = \tan[\bar{\omega}\tau(t)]/\bar{\omega}\,. \tag{12}$$

Substitution of (12) into (8) yields

$$L = \tfrac{1}{2}(m/\dot{s})\dot{y}^2 - \bar{G}(y, t) + \dot{G}(y, t)\,, \tag{13}$$

where

$$\bar{G} = \tfrac{1}{2} m y^2 (\bar{\omega} \cos\bar{\omega}\tau \sin\bar{\omega}\tau - \dot{\rho}/\rho\dot{s})\,. \tag{14}$$

Thus we see that the classical action for the quadratic lagrangian (8),

$$S_\mathrm{c} = \int_{t'}^{t''} L(\dot{x}, x, t)\, \mathrm{d}t \tag{15}$$

reduces to the action for a free particle,

$$S_\mathrm{c} = \int_{s'}^{s''} \tfrac{1}{2} m \mathring{y}^2\, \mathrm{d}s + [G(y, t) - \bar{G}(y, t)]_{t'}^{t''}\,, \tag{16}$$

where $\bar{\omega}s' = \tan[\bar{\omega}\tau(t')]$, $\bar{\omega}s'' = \tan[\bar{\omega}\tau(t'')]$ and $\mathring{y} = dy/ds = \dot{y}/\dot{s}$. The last term in (16) is physically unimportant since it does not contribute to the equation of motion. Indeed, it can always be removed by setting $G = \bar{G}$ without altering physics. Strange though it may sound, a time-dependent damped oscillator in one set of space and time variables is a free particle in another choice of variables. This interesting property, which seems to have never been noticed before, may be employed to derive the propagator for the time-dependent damped oscillator from the free propagator.

The free propagator for (16) is given if $G = \bar{G}$ by

$$K_0(y'', s''; y', s') = [m/2\pi i\hbar(s'' - s')]^{1/2} \exp[im(y'' - y')^2/2\hbar(s'' - s')] . \tag{17}$$

If $G \neq \bar{G}$, then (17) must be multiplied by a phase factor,

$$f(y'', s''; y', s') = \exp\{(i/\hbar)[G(y, s) - \bar{G}(y, s)]_{s'}^{s''}\} . \tag{18}$$

By the space and time transformation (12), therefore, the propagator for the time-dependent damped oscillator (8) can readily be obtained from (2) as

$$K(x'', t''; x', t') = [(\partial y'/\partial x')(\partial y''/\partial x'')]^{1/2} K_0(y'', s''; y', s') f(y'', s''; y', s') . \tag{19}$$

To be more explicit, we substitute (12) into (19) and use the identities,

$$\frac{A \sec^2\alpha + B \sec^2\beta}{\tan\alpha - \tan\beta} = (A + B)\cot(\alpha - \beta) + A \tan\alpha - B \tan\beta , \tag{20}$$

$$\frac{\sec\alpha \sec\beta}{\tan\alpha - \tan\beta} = \frac{1}{\sin(\alpha - \beta)} , \tag{21}$$

the result being

$$K(x'', t''; x', t') = \left(\frac{m\bar{\omega}(\dot{\tau}'\dot{\tau}'' e^{\lambda' + \lambda''})^{1/2}}{2\pi i\hbar \sin[\bar{\omega}(\tau'' - \tau')]}\right)^{1/2} f(x'', t''; x', t')$$

$$\times \exp\{\{im\bar{\omega}/2\hbar \sin[\bar{\omega}(\tau'' - \tau')]\} \{(x''^2 e^{\lambda''}\dot{\tau}'' + x'^2 e^{\lambda'}\dot{\tau}') \cos[\bar{\omega}(\tau'' - \tau')] - 2x'x''(\dot{\tau}'\dot{\tau}'' e^{\lambda' + \lambda''})^{1/2}\}\} , \tag{22}$$

where

$$f(x'', t''; x', t') = \exp\{(im/2\hbar)[x^2 e^{\lambda}\dot{\rho}/\rho + (2/m) G(x, t)]_{t'}^{t''}\} . \tag{23}$$

The propagator (22) contains the results previously obtained for various quadratic systems after lengthy calculations [3–8].

Next we examine some of the special cases contained in (22). For the time-dependent oscillator with a constant damping parameter λ_0, i.e., for $\ddot{x} + \lambda_0\dot{x} + \omega^2(t)x = 0$, we simply replace $\lambda(t)$ in (12) and (22) by $\lambda_0 t$. In the limit $\lambda_0 \to 0$, we get from (22) the propagator for the oscillator with a time-dependent frequency $\omega(t)$ [3],

$$K(x'', t''; x', t') = \left(\frac{m\bar{\omega}(\dot{\tau}'\dot{\tau}'')^{1/2}}{2\pi i\hbar \sin[\bar{\omega}(\tau'' - \tau')]}\right)^{1/2} f(x'', t''; x', t')$$

$$\times \exp\{\{im\bar{\omega}/2\hbar \sin[\bar{\omega}(\tau'' - \tau')]\}\{(x''^2\dot{\tau}'' + x'^2\dot{\tau}')\cos[\bar{\omega}(\tau'' - \tau')] - 2x'x''(\dot{\tau}'\dot{\tau}'')^{1/2}\}\} . \tag{24}$$

The relations (10) and (11) satisfied by $\tau(t)$ and $\rho(t)$ become Pinney's equation,

$$\ddot{\rho} + \omega^2(t)\rho - C^2\rho^{-3} = 0 , \tag{25}$$

with $\bar{\omega}\dot{\tau}\rho^2 = C$. In the case of the damped harmonic oscillator, $\ddot{x} + \lambda_0\dot{x} + \omega_0^2 x = 0$, with a constant frequency ω_0, (10) and (11) give us $\rho = \exp(-\frac{1}{2}\lambda_0 t)$, $\bar{\omega}^2 = \omega_\lambda^2 = \omega_0^2 - \frac{1}{4}\lambda_0^2$ and $\tau = t$ when $\lambda(t) = \lambda_0 t$. Accordingly, the transformation (12) reads

$$y = x \exp(\tfrac{1}{2}\lambda_0 t)\sec(\omega_\lambda t), \quad s = \tan(\omega_\lambda t)/\omega_\lambda . \tag{26}$$

The propagator found from (19) with (26) coincides with that in the literature [3–7]. Moreover, in the limit $\lambda_0 \to 0$, we have ρ = constant, $\bar{\omega} = \omega_0, \tau = t$ and $\lambda = 0$. The transformation (26) takes the form

$$y = x \sec(\omega_0 t), \quad s = \tan(\omega_0 t)/\omega_0 , \tag{27}$$

which is identical to what has earlier been referred to as the Jackiw transformation [13,14]. When (27) is applied, (19) leads to the well-known propagator for the harmonic oscillator [1]. If $\lambda_0 \neq 0$ and $\omega_0 = 0$, then (26) turns out to be

$$y = 2x/(1 + \mathrm{e}^{-\lambda_0 t}), \quad s = \frac{2}{\lambda_0}\,\frac{1-\mathrm{e}^{-\lambda_0 t}}{1+\mathrm{e}^{-\lambda_0 t}}, \tag{28}$$

which is similar to but different by a multiple factor $2(1 + \mathrm{e}^{-\lambda_0 t})^{-1}$ from the Levi-Civita transformation [15]. With (28), the expression (19) yields for $G = 0$

$$K(x'', t''; x', t') = (\lambda_0 m/2\pi i\hbar)^{1/2}\,(\mathrm{e}^{-\lambda_0 t'} - \mathrm{e}^{-\lambda_0 t''})^{-1/2} \exp[(i\lambda_0 m/2\hbar)(\mathrm{e}^{-\lambda_0 t'} - \mathrm{e}^{-\lambda_0 t''})^{-1}(x'' - x')^2] , \tag{29}$$

which is the propagator for a particle moving in a frictional medium [8]. For a falling particle in a frictional medium under the gravitational acceleration g_0, obeying $\ddot{q} + \lambda_0\dot{q} + g_0 = 0$, we apply the translation (6) with $h(t) = g_0 t$ directly to (19) to find

$$K(x'', t''; x', t') = (\lambda_0 m/2\pi i\hbar)^{1/2}\,(\mathrm{e}^{-\lambda_0 t'} - \mathrm{e}^{-\lambda_0 t''})^{-1/2}$$

$$\times \exp\{(i\lambda_0 m/2\hbar)(\mathrm{e}^{-\lambda_0 t'} - \mathrm{e}^{-\lambda_0 t''})^{-1}[(q'' - q')^2 - 2(g_0/\lambda_0)(t'' - t')(q'' - q') + (g_0/\lambda_0)^2(t'' - t')^2]\} . \tag{30}$$

Finally, we wish to remark that since $\mathrm{e}^{\lambda}\mathrm{d}t/x^2 = \mathrm{d}s/y^2$ under (12), our transformation technique works in the presence of a singular potential $V(x) = \frac{1}{2}k\mathrm{e}^{\lambda}/x^2$ in (8). This extension will be discussed elsewhere.

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References

[1] R.P. Feynman and A.R. Hibbs, Quantum mechanics and path integrals (McGraw-Hill, New York, 1965).
[2] L.S. Schulman, Techniques and applications of path integration (Wiley, New York, 1981).
[3] D.C. Khandekar and S.V. Lawande, J. Math. Phys. 16 (1975) 384; 20 (1979) 1870.
[4] V.V. Dodonov and V.I. Man'ko, Nuovo Cimento 44B (178) 265.
[5] A.D. Jannussis, G.N. Brodimas and A. Streclas, Phys. Lett. 74A (1979) 6.
[6] C.C. Gerry, J. Math. Phys. 25 (1984) 1820.
[7] B.K. Cheng, J. Phys. A17 (1984) 2475.
[8] I.C. Moreira, Lett. Nuovo Cimento 23 (1978) 294.
[9] I.H. Duru and H. Kleinert, Phys. Lett. 84B (1979) 185.
[10] R. Ho and A. Inomata, Phys. Rev. Lett. 48 (1982) 231;
A. Inomata, Phys. Lett. 101A (1984) 253.
[11] P.Y. Cai, A. Inomata and R. Wilson, Phys. Lett. 96A (1983) 117.
[12] M.A. Kayed and A. Inomata, Phys. Rev. Lett. 53 (1984) 105.
[13] P.Y. Cai, A. Inomata and P. Wang, Phys. Lett. 91A (1982) 331.
[14] V. de Alfano, S. Fubini and G. Furlan, Nuovo Cimento 34 (1976) 569;
R. Jackiw, Ann. Phys. (NY) 129 (1980) 183.
[15] P. Caldirola, Nuovo Cimento 77B (1983) 241.