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Dynamics on the Group Manifold and Path Integral

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Abstract

Classical and quantal dynamics on the compact simple Lie group and on a sphere of arbitrary dimensionality are considered. The accuracy of the semiclassical approximation for Green's functions is discussed. Various path integral representations of Green's functions are presented. The special features of these representations due to the compactness and curvature are analysed. Basic results of the theory of Lie algebras and Lie groups used in the main text are presented in the Appendix.

Contents

1. Introduction

Dynamics of any quantal system is described by the evolution operator, and its coordinate-space matrix element is the Green's function

$$\mathcal{K}(\boldsymbol{q}_t, \boldsymbol{q}_0; t) = \langle \boldsymbol{q}_k | \exp\left(-\frac{i}{\hbar} tH\right) | \boldsymbol{q}_0 \rangle \tag{1.1}$$

[1]) On leave from ITEP since June 1979

where $\boldsymbol{q}$ is the coordinate vector and H is the Hamiltonian operator. The Green's function has a fundamental property

$$\mathcal{K}(\boldsymbol{q}_t, \boldsymbol{q}_0; t_2 + t_1) = \int \mathcal{K}(\boldsymbol{q}_t, \boldsymbol{q}; t_2)\, \mathcal{K}(\boldsymbol{q}, \boldsymbol{q}_0; t_1)\, d\boldsymbol{q} \tag{1.2}$$

where $d\boldsymbol{q}$ is an integration measure. In particular, it follows from (1.2) that

$$\lim_{t\to 0} \mathcal{K}(\boldsymbol{q}_t, \boldsymbol{q}_0; t) = \delta(\boldsymbol{q}_t - \boldsymbol{q}_0). \tag{1.3}$$

A consequence of the definition (1.1) is that Green's function may be represented by the spectral expansion

$$\mathcal{K}(\boldsymbol{q}_t, \boldsymbol{q}_0; t) = \sum_n \psi_n(\boldsymbol{q}_t)\, \psi_n{}^*(\boldsymbol{q}_0) \exp\left(-\frac{i}{\hbar}\, tE_n\right) \tag{1.4}$$

where $\psi_n(\boldsymbol{q})$ is an eigen-function of the operator H, $H\psi_n = E_n\psi_n$. The ψ_n functions are assumed to be orthonormal with the measure $d\boldsymbol{q}$, while the Hamiltonian H is hermitian with this measure. The function (1.1) satisfies the Schroedinger equation, in both of its arguments, with the initial condition (1.3).

A representation of Green's function in the closed form of the path integral is useful for physical applications. This representation is especially appropriate for the perturbative expansion and for the semiclassical approximation. However, the Feynman original form [*1, 2*]

$$\mathcal{K}(\boldsymbol{q}_t, \boldsymbol{q}_0; t) = \int \mathcal{D}\boldsymbol{q} \exp\left[\frac{i}{\hbar} \int_0^t \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, d\tau\right] \tag{1.5}$$

where $\mathcal{L}$ is the classical Lagrangian, and the functional measure $\mathcal{D}q$ is defined as a limit of correspondingly normalized multiple integral, is known to be directly applicable only to the simplest (though the most interesting) case, when the coordinate space is infinite, and the Lagrangian is of the for, $\mathcal{L} = 1/2 g_{kl}\dot{q}^k\dot{q}^l - V(\boldsymbol{q})$ with g_{kl} independent of q. For a more complicated Lagrangian with, for instance, the metric tensor g_{kl} depending on q (the motion in a Riemannian space) Eq. (1.5) needs a modification involving an additional term in the Lagrangian, proportional to $\hbar^2$. This approach was developed by De Witt [*3*], the corrections to $\mathcal{L}$ were also discussed by McLaughlin and Schulman [*4*]. The reasons for modification of the path integral in case of curvilinear coordinates were also considered by Edwards and Gulyaev [*5*], Arthurs [*6*] (polar coordinates) and in more recent works by Gervais and Jevicki [*7*] and Salomonson [*8*] (in view of the problem of quantization in nonlinear field theories).

The idea to construct the path integral for a general dynamical system originated from Dirac [*9*] and developed by Feynman [*1*] and Pauli (the lecture notes cited in [*3*]) is formulated as follows. At small times the quantum fluctuations are inessential and Green's function is given by the semiclassical approximation

$$\mathcal{K}^{\mathrm{cl}}(\boldsymbol{q}_t, \boldsymbol{q}_0; t) = (2\pi i\hbar)^{-n/2}\, [\varrho^{-1}(\boldsymbol{q}_t)\, D(\boldsymbol{q}_t, \boldsymbol{q}_0; t)\, \varrho^{-1}(\boldsymbol{q}_0)]^{1/2} \cdot \exp\frac{i}{\hbar}\, \mathcal{A}(\boldsymbol{q}_t, \boldsymbol{q}_0; t). \tag{1.6}$$

Here $\mathcal{A} = \int_0^t \mathcal{L}\, d\tau$ is the classical action (the Hamilton principal function), and

$$D = \det\left[-\frac{\partial^2\mathcal{A}}{\partial q_0{}^j\, \partial q_t{}^k}\right] \tag{1.7}$$

is the Van-Vleck determinant; the integration measure is written as $d\boldsymbol{q} = \varrho(\boldsymbol{q}) \prod_j dq^j$. In case of the Riemannian space the invariant measure is given by $\varrho(\boldsymbol{q}) = C_R \sqrt{g}$, where $g = \det g_{kl}(\boldsymbol{q})$ and C_R is a constant. Green's function at finite times t is obtained by integration of a product of N Green's functions for the small time intervals, i.e. by a multiple application of Eq. (1.2). The limit $N \to \infty$ leads to the functional integral over all the intermediate coordinates, that is interpreted as the path integral. The result is a representation of the type (1.5) where $\mathcal{L}$ is substituted by an effective Lagrangian $\mathcal{L}_{\text{eff}}$ obtained by the expansion of the action $\mathcal{A}(\boldsymbol{q}_t, \boldsymbol{q}_0; \tau)$ up to the first-order terms in τ. The expansion of the preexponential factor (1.6) also contributes to $\mathcal{L}_{\text{eff}}$. De Witt [3] has found that in the case of Riemannian space besides the term $\hbar^2(R/12)$ arising from expansion of the preexponential factor one has to add ad hoc a supplementary term $\hbar^2(R/12)$ to the Lagrangian (R is the scalar curvature). This correction is necessary for the Green's function to satisfy the Schroedinger equation determined by means of the invariant Laplace operator. This aspect was discussed later by Cheng [10] and Ringwood [11].

If the coordinate space is infinite, any two points $\boldsymbol{q}_0$ and $\boldsymbol{q}_t$ are connected by a unique classical trajectory at sufficiently small times. However, if the space has a nontrivial topology, the classical trajectory is not fixed by its boundary points even at $t \to 0$. The simplest examples of such situation are the motion of a particle on a circle (this system is considered in some detail in Section 2) or the motion of a particle in a box with elastic walls. In these cases the classical trajectories are fixed by a number of evolutions of the point around the circle or a number of shocks against the walls. It seems natural to generalize the principle of construction of the path integral in such a way, that to sum up the terms (1.6) over all possible classical trajectories. Simple examples provide with an evidence in favour of such a generalization. However, with this prescription the Green's function at finite times is no longer an integral (1.5), because at any step one has not only to integrate over q, but also to sum over possible trajectories. Note also that the contributions (1.6) from different trajectories may enter with different phases. This fact is well known in the context of the semiclassical approach (see e.g. the review by Berry and Mount [12]). Thus the quantization by means of the path integral is essentially more complicated if the configuration space is a manifold with a nontrivial topology. Note also that the canonical quantization based on construction of the momentum and energy operators also deserves a special treatment in such a situation (see e.g. the work by Wan and Viazminsky [13]).

The subject of the present work is an analysis of problems arising at quantization and construction of the path integrals for the dynamics on the compact Lie groups. This example is interesting in view of a number of reasons. First, the dynamics on the group manifold is an object of some modern quantum field theories, such as the chiral models or theories of gauge fields on a space lattice. Second, the dynamics on groups have the mentioned features: the kinetic energy depends not only on velocites but also on the coordinates, and the compactness of the group manifold is the reason of appearance of an infinite set of classical trajectories with fixed end points. Third, the group structure of the dynamics results in an unambiguous canonical quantization (see e.g. the paper [13] and the work by Charap [14], as well as the books by Mackey [15] and Segal [16]) so that the path integral construction may be independently checked. Moreover, for the free motion on the group manifold the semiclassical approximation is found to coincide with the exact solution (with an appropriate constant shift of the energy spectrum). Needless to say, the presence of external forces spoils this coincidence, but at small times the potential does not change qualitatively the trajectories and the shape of the Green's function used in construction of the path integral. In fact, modifications necessary for quantization in curved spaces are always due to the kinetic energy. For instance, in the operator approach the ordering problem arises just in the kinetic term (see [14]).

The Feynman integral (1.5) displays the reason why the systems with quadratic Hamiltonians like the oscillator or the free motion in Euclidean space are described exactly in the semiclassical approach. In these cases one has a multiple Gaussian integral, that is calculated exactly by the stationary phase method. We have no such a simple explanation why the free quantal motion in group manifolds is also quasiclassical. It is clear that the high symmetry of the system is involved but the reason is not too manifest. For instance, the motion on the spheres S^n is also highly symmetric but the semiclassical formulas are exact only at $n = 1$ or 3 (when the sphere is a group).

It is remarkable that the group manifold has a natural Riemannian metrics while the curvature R is constant and simply related to the group dimension n, $R = n/4$. A clear example is the $SU(2)$ group that is isomorphic to a three-dimensional sphere. Actually, a unitary 2×2 matrix may be written as $u = a_0 + i\boldsymbol{\sigma a}$, while $u \in SU(2)$, if $\det u = a_0^2 + \boldsymbol{a}^2 = 1$. For other groups the geometry is more complicated.

The construction of Green's function for a manifold may be reduced to the problem with a trivial topology (the same as for the infinite Euclidean space) in the case when the manifold may be represented as a coset space of an infinite space by a transformation group. For example, the unit circle is a coset of a line by the group of translations, multiple of 2π. Therefore Green's function for the circle may be represented as a sum of Green's functions for a line having the initial (or final) points shifted by $2\pi n$ (see further Eq. (2.4)). These points are not distinguishible on the circle. In general, let $\mathfrak{M}$ be a coset space, i.e. $\mathfrak{M} = \mathfrak{R}/\Gamma$. If the Hamiltonian in the space $\mathfrak{R}$ is invariant under transformations belonging to the group Γ, and Green's function for $\mathfrak{R}$ is known, the Green's function for $\mathfrak{M}$ may be constructed as a sum over the group Γ:

$$\mathcal{K}_{\mathfrak{M}}(\boldsymbol{q}_t, \boldsymbol{q}_0; t) = \sum_{\gamma \in \Gamma} \mathcal{K}_{\mathfrak{R}}(\boldsymbol{q}_t, \gamma \cdot \boldsymbol{q}_0; t) = \sum_{\gamma \in \Gamma} \mathcal{K}_{\mathfrak{R}}(\gamma \cdot \boldsymbol{q}_t, \boldsymbol{q}_0; t) \tag{1.8}$$

If $\mathfrak{R}$ is a space with trivial topology, then $\mathcal{K}_{\mathfrak{R}}$ may be written as a path integral, while the sum for $\mathcal{K}_{\mathfrak{M}}$ arises only at the last step. We show that this approach is valid for any compact Lee group. In this case $\mathfrak{R}$ is the corresponding Lie algebra and Γ is the characteristic lattice of the group. As for the Lie algebra, it is a usual infinite Riemannian space and the Green's function may be written in the Feynman form.

The quantal motion in group manifolds was considered previously in a number of works. Evidently, the first construction of Green's function for general compact Lie group was elaborated by Eskin [*17*]. This author considered the heat transfer equation, but this results may be immediately applied to the Schroedinger equation after the substitution $t \to it$. Motion in the $SU(2)$ and $SO(3)$ groups, as well as the path integrals, was studied by Schulman [*18*]. He has shown, in particular, that the exact solution in these cases is given by the semiclassical series. However, this author has presented no simple path integral. Dowker [*19, 20*] expanded the work by Schulman to any group but explicit formulas and proofs were given by him only for the $SU(N)$ groups. Evidently, neither Schulman, nor Dowker were aware of Eskin's work that overlaps most of their results. The path integral for $SU(N)$ was considered in a work by the present authors [*21*]. The path integral discussed in that work is now presented in a more simple form using the relation (1.8).

In process of this work we have to do with various results of the Lie group theory. They are contained in a number of sources but rather dispersely and not always in a form available for a physicist. Therefore we suppose it would be helpful to supply this article with an Appendix containing a brief survey of the results in the Lie group theory that are most important for our purposes. We present commentaries to the cited statements without a pretence to be strict or absolutely general and avoid a terminology alien to the physics. As for the literature, we used a paper by Cartan [*22*], books by Eisenhart [*23*], Weyl [*24*], Zelobenko [*25*], and Gilmore [*26*], the reviews by Racah [*27*], Bere-

ZIN and GELFAND [28], DYNKIN and ONISHCHIK [29], BEREZIN [30]. In concepts of Riemannian geometry we follow the book by LANDAU and LIFSHITS [31].
The composition of our paper is as follows. In Section 2 we discuss the motion on a circle. This simple example is of interest because the circle is the compact Abelian group $U(1)$ as well as the one-dimensional sphere. The motion on a circle was discussed previously in various contexts but we return to this subject in order to illustrate methods applicable to less trivial situations. In Section 3 the classical dynamics of a point in space of the group parameters is described. Solutions of equations of motion are presented as well as the classical action. In Section 4 the quantal dynamics on a compact simple Lie group is considered. The Hamiltonian, energy spectrum, and the eigenfunctions are presented. The $SU(2)$ group is considered in some detail. In Section 5 the semiclassical approximation for Green's function is constructed. In particular, the preexponential factor is calculated. It is shown that the semiclassical approximation satisfies the Schroedinger equation. In Section 6 two variants of the path integral for the Lie group are constructed. In Section 7 the motion on a sphere of any dimension is considered. Two path integral representations are constructed. The first treats the sphere as a Riemannian space, the second is based on embedding of the sphere S^n into the Euclidean space of $(n + 1)$ dimensions.
In Appendix A necessary results on the Lie algebras, the Lie groups, and their matrix representations are compiled. In Appendix B the geometry of the group manifold is described. Appendix C deals with the structure of the maximal torus. The Θ-functions on the maximal torus are also considered.

2. Material Point on a Circle: the $U(1)$ Group

Consider a free motion on a circle. The coordinate is the angle φ, $0 \leqq \varphi < 2\pi$, the Lagrangian is $\mathscr{L} = 1/2\dot{\varphi}^2$, the trajectory is $\varphi_t = \varphi_0 + \omega t$ where φ_0 and ω are constant. The classical action depends not only on the boundary points of the trajectory but also on the number of evolutions in process of the motion. Evidently, the action is

$$\mathcal{A}_n(\varphi_t, \varphi_0; t) = (2t)^{-1} (\varphi_t - \varphi_0 + 2\pi n)^2 \tag{2.1}$$

where $n = 0, \pm 1, \pm 2, \ldots$ represents the number of evolutions in positive or negative directions.
The quantal Hamiltonian is $H = -(\hbar^2/2)\,(\partial^2/\partial\varphi^2)$, its eigen-functions and eigen-values are

$$\psi_m(\varphi) = \exp(im\varphi), \qquad E_m = \frac{1}{2}(\hbar m)^2 \tag{2.2}$$

where m is an integer and the ψ-functions are normalized $\int\limits_0^{2\pi} \psi_m^*(\varphi)\, \psi_n(\varphi)\, d\varphi/2\pi = \delta_{mn}$. In this case the spectral expansion is:

$$\mathscr{K}(\varphi_t, \varphi_0; t) \equiv K(\varphi_t - \varphi_0; t) = \sum_{m=-\infty}^{\infty} \exp\left[im(\varphi_t - \varphi_0) - \frac{i\hbar}{2} m^2 t\right]. \tag{2.3}$$

Using the transformation of the Θ-function (Eq. (C.7)) the function $K(\varphi_t - \varphi_0; t)$ may also be written as a sum over classical trajectories

$$\begin{aligned} K(\varphi; t) &= \sum_{n=-\infty}^{\infty} \tilde{K}(\varphi + 2\pi n; t) \\ \tilde{K}(\Phi; t) &= \left(\frac{2\pi}{i\hbar t}\right)^{1/2} \exp\left[i\,\frac{\Phi^2}{2\hbar t}\right]. \end{aligned} \tag{2.4}$$

Separate terms in this series are Green's functions on a line while the series as a whole is periodical, $K(\varphi; t) = K(\varphi + 2\pi; t)$. The sum in Eq. (2.4) is a particular example of (1.8).

In this example the multiplicative property (1.2) is of the form

$$K(\varphi_t - \varphi_0; t_2 + t_1) = \int_0^{2\pi} K(\varphi_t - \varphi; t_2)\, K(\varphi - \varphi_0; t_1)\, \frac{d\varphi}{2\pi}. \tag{2.5}$$

This is a direct consequence of Eq. (2.3). The semiclassical series (2.4) also enables one to check (2.5) as with the sum over n the intermediate integral over φ may be expanded to the whole line, while the function K satisfies (1.2) when integrated within infinite limits. Thus

$$K(\varphi; t_2 + t_1) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{K}(\varphi - \Phi + 2\pi n; t_2)\, \tilde{K}(\Phi; t_1)\, \frac{d\Phi}{2\pi}$$

and Eq. (2.4) follows directly when Eq. (1.2) is applied to K.

Repeated use of (2.5) leads to a path integral

$$K(\varphi; t) = \int_0^{2\pi} \prod_{j=1}^{N-1} \frac{d\varphi_j}{2\pi}\, K(\varphi_t - \varphi_{N-1}; \delta t) \cdots K(\varphi_2 - \varphi_1; \delta t)\, K(\varphi_1 - \varphi_0; \delta t) \tag{2.6}$$

where $\varphi = \varphi_t - \varphi_0$; $\delta t = t/N \equiv \tau/\hbar$. Substitution of $K(\varphi_{j+1} - \varphi_j; \delta t)$ in form of Eq. (2.4) results in a representation of $K(\varphi; t)$ by means of the N-fold sum over $n_1, \ldots, n_N$ combined with the $(N - 1)$-fold integral from 0 up to 2π. However, shifting the integration variable one substitutes at any step

$$\sum_{n_j=-\infty}^{\infty} \int_0^{2\pi} d\varphi_j \to \int_{-\infty}^{\infty} d\Phi_j$$

(cf. a commentary to Eq. (2.5)). The result is

$$K(\varphi; t) = \sum_{n=-\infty}^{\infty} \int \mathcal{D}\Phi \exp\left[\frac{i}{\hbar} \int_0^t \mathcal{L}\, d\tau\right]$$

$$\mathcal{D}\Phi = \left(\frac{2\pi}{i\tau}\right)^{1/2} \prod_{j=1}^{N-1} \left[(2\pi i\tau)^{-1/2}\, d\Phi_j\right] \tag{2.7}$$

$$\Phi_0 = \varphi_0; \Phi_N = \varphi_t + 2\pi n; -\infty < \Phi_j < \infty, \qquad j = 1, 2, \ldots N - 1.$$

The integral of the Lagrangian is approximated by a sum over time intervals $\delta t = \tau/\hbar$, and at any interval $\dot{\varphi} \approx (\Phi_{j+1} - \Phi_j)/\delta t$. Thus the path integral for a circle is reduced to the path integral on a line summed over the equivalent points whose difference is a multiple of 2π.

We have considered just the case of the free motion. However, the presence of a potential changes nothing in principle, though in that case Green's function depends essentially on both arguments φ_t and φ_0 (not only on the difference $\varphi_t - \varphi_0$), and the sum over classical trajectories is not, in general, the exact solution. If $\mathcal{L} = 1/2\, \dot{\varphi}^2 - V(\varphi)$, one may consider an associated problem for a line with $\mathcal{L} = 1/2\, \dot{\Phi}^2 - \tilde{V}(\Phi)$ where $\tilde{V}(\Phi)$ is the periodical continuation of $V(\varphi)$ from the interval $(0,2\pi)$. The Hamiltonian commutes with the translation of the line by 2π, so Green's function for the line is invariant under simultaneous shifts of Φ_0 and Φ_t, while Green's function on the circle is periodical in any of its arguments separately

$$\begin{aligned}\tilde{\mathscr{K}}(\Phi_t, \Phi_0 + 2\pi; t) &= \tilde{\mathscr{K}}(\Phi_t - 2\pi, \Phi_0; t) \\ \mathscr{K}(\varphi_t, \varphi_0; t) = \mathscr{K}(\varphi_t, \varphi_0 + 2\pi; t) &= \mathscr{K}(\varphi_t + 2\pi, \varphi_0; t).\end{aligned} \tag{2.8}$$

Both functions $\mathscr{K}$ and $\tilde{\mathscr{K}}$ satisfy the same differential equation but with different boundary conditions. Therefore in general one may write $\mathscr{K}$ as a series similar to (2.4). The corresponding path integral (2.7) is also valid. Any of the points, Φ_0 or Φ_t, may be shifted by $2\pi n$, the result is the same.

The circle is the simplest compact group $U(1)$ with elements $\exp i\varphi$. The line is the abelian one-dimensional translation group T_1. Translations by $2\pi n$, $n = 0, \pm 1 \ldots$ is a subgroup Z, while the unit circle is the coset space $U(1) = T_1/Z$.

So one may consider the representation (2.7) as an application of the general statement (1.8).

It is instructive to obtain another path integral for motion on a circle treating it as a motion on a plane with a constraint $\boldsymbol{q}^2 = 1$, where $\boldsymbol{q} = (\cos\varphi, \sin\varphi)$ is a two-dimensional vector. Rewrite, the integral in Eq. (2.6) by means of a Lagrange multiplier:

$$\int_0^{2\pi} d\varphi = 2\int_{-\infty}^{\infty} d^2\boldsymbol{q}\,\delta(q^2 - 1) = \int\!\!\int\!\!\int_{-\infty}^{\infty} d^2\boldsymbol{q}\, d\lambda \exp\frac{i}{2}\,\lambda(\boldsymbol{q}^2 - 1). \tag{2.9}$$

At small τ Green's function (2.4) is approximated by

$$K(\varphi, \delta t) \simeq \left(\frac{2\pi}{i\tau}\right)^{1/2}\left[1 - \frac{i\tau}{4} - \frac{i}{6\tau}\,(1 - \cos\varphi)^2\right]\exp\left[i\,\frac{1 - \cos\varphi}{\tau}\right]. \tag{2.10}$$

To prove this expansion one may expand the r.h.s. in the Fourier series. The coefficients are proportional to the Bessel functions. Retaining two main terms in their asymptotic expansion at small τ (this is the accuracy sufficient to construct the path integral) and comparing the result with the spectral expansion (2.3) one can see that the two forms are equal with the necessary accuracy (more details are given in Section 7).

Taking into account the identity

$$1 - \cos(\varphi_1 - \varphi_2) = \frac{1}{2}\,(\boldsymbol{q}_1 - \boldsymbol{q}_2)^2$$

at $\boldsymbol{q}_1^2 = \boldsymbol{q}_2^2 = 1$, substitute (2.9) and (2.10) into (2.6). In the limit $\tau \to 0$ one has

$$\begin{aligned}K(\varphi, t) &= \int \mathscr{D}\boldsymbol{q}\,\tilde{\mathscr{D}}\lambda \exp\frac{i}{\hbar}\int_0^t \mathscr{L}_{\text{eff}}\, d\tau \\ \mathscr{L}_{\text{eff}} &= \frac{1}{2}\,\dot{\boldsymbol{q}}^2 + \frac{1}{2}\,\lambda\,(\boldsymbol{q}^2 - 1) + \frac{\hbar^2}{12} \\ \mathscr{D}\boldsymbol{q} &= \left(\frac{2\pi}{i\tau}\right)\prod_{j=1}^{N-1}\left[\frac{d^2\boldsymbol{q}_j}{2\pi i\tau}\right] \\ \tilde{\mathscr{D}}\lambda &= (2\pi i\tau)^{1/2}\prod_{j=1}^{N-1}(2\pi i\tau)^{1/2}\, d\lambda_j.\end{aligned} \tag{2.11}$$

The integration over $\boldsymbol{q}$ and λ is within infinite limits, as in the Feynman integral. Note that besides a term due to the constraint, in the effective Lagrangian a constant "potential" $-\hbar^2/12$ is also present, that is of purely quantal origin. (In Eq. (66) of the paper [21] a wrong number was given). In Eq. (2.11) the trajectories are approximated by piecewise linear chain on the plain, while in Eq. (2.7) the links of the chain are arcs of the circle. This difference results in a correction to the Lagrangian. Such an effect was also detected in [5] when changing the Cartesian coordinates by the polar coordinates in the path integral. Such corrections are specific for quantum motion on curved surfaces. Probably, they account for the quantum fluctuations deviating the point out of the manifold determined by the constraints.

3. Classical Dynamics on the Group Manifold

Before studying the quantal dynamics for an arbitrary group manifold, it is instructive to consider the corresponding problem in the framework of classical mechanics. Consider a dynamical system with the coordinates ξ^a that are parameters of a compact Lie group, $a = 1, \ldots, n$. The Lagrangian describing the free (inertial) motion is invariant under arbitrary shifts of the group elements: $\hat{g}(\xi) \to \hat{g}_1\hat{g}(\xi)\,\hat{g}_2$ where $\hat{g}_1$ and $\hat{g}_2$ are any fixed elements. A Lagrangian that has this property is

$$\begin{aligned} \mathscr{L}_0(\xi, \dot{\xi}) &= \frac{1}{2}\,\mu\,\mathrm{Sp}\,[\dot{D}(\hat{g})\,\dot{D}^\dagger(\hat{g})] = \frac{1}{2}\,g_{ab}\dot{\xi}^a\dot{\xi}^b \\ g_{ab}(\xi) &= \mu\,\mathrm{Sp}\left[\frac{\partial D}{\partial \xi^a}\,\frac{\partial D^\dagger}{\partial \xi^b}\right]. \end{aligned} \tag{3.1}$$

Here $D(\hat{g}) = \exp X_a\xi^a$ is an *irreducible unitary matrix representation of the group*, $X_a^\dagger = -X_a$, μ is a constant. If we put $\mu = n/\lambda d$ where n is the group dimension, d is the representation dimension, and λ is the eigenvalue of the second-order-Casimir operator, then $g_{ab}(\xi)$ is *the metric tensor in the group manifold*, see (B.10). To check this fact, it is sufficient to use Eqs. (B.15) and (A.13):

$$g_{ab}(\xi) = \frac{n}{\lambda d}\,\mathrm{Sp}\,[X_pDD^+X_q]\,M_a{}^pM_b{}^q = G_{pq}M_a{}^p(\xi)\,M_b{}^q(\xi) \tag{3.2}$$

where G_{pq} is the Killing tensor (A.2). Evidently, the Lagrangian (3.1) is actually independent on the choice of the representation and has a clear geometric meaning, $\mathscr{L}_0 = (1/2) \times (ds/dt)^2$, where ds is the element of length in the Riemannian space, that is the group manifold.

The free motion equation $\ddot{\xi}^a + \Gamma^a_{bc}(\xi)\,\dot{\xi}^b\dot{\xi}^c = 0$ (see e.g. [31]), in view of Eq. (B.11) may be written as follows

$$M_a{}^k(\xi)\,\ddot{\xi}^a + \partial_bM_c{}^k(\xi)\,\dot{\xi}^b\dot{\xi}^c = 0, \qquad k = 1, 2, \ldots n. \tag{3.3}$$

Thus, the quantities $\zeta^a = M_b{}^a(\xi)\,\dot{\xi}^b$ as well as the Lagrangian $\mathscr{L} = 1/2\,G_{ab}\zeta^a\zeta^b$, are constant along the trajectory, and the equations of motion are reduced to a first-order system $\dot{\xi}^a = L_b{}^a(\xi)\,\zeta^b$, where $L = M^{-1}$. In particular, if $\xi^a(0) = 0$, then $\zeta^a = \dot{\xi}^a(0)$, and using (B.8) one may find the solution at once, $\xi^a(t) = \zeta^a \cdot t$.

The time dependence of the group element is given by

$$\hat{g}(t) = \exp\,[\xi^a(t)\,\hat{X}_a] \tag{3.4}$$

where $\hat{X}_a$ is the group generator. If $\hat{g}(0) \equiv \hat{e}$ is the unit element, then

$$\hat{g}(t) = \hat{v}^{-1} \exp(\hat{\boldsymbol{H}}\boldsymbol{\varphi})\, \hat{v}, \qquad \boldsymbol{\varphi} = \boldsymbol{\varkappa} t \tag{3.5}$$

where $\hat{v}$ and $\boldsymbol{\varkappa}$ are independent on t, and $\hat{H}_j$ are the commuting generators of the group (the basis of the Cartan subalgebra). At an arbitrary initial condition, $\hat{g}(0) = \hat{g}_0$, the trajectory is given by $\hat{g}_t = \hat{g}(t)\, \hat{g}_0$ with $\hat{g}(t)$ of the form (3.5). The classical action $\mathcal{A}$, as well as the Lagrangian, is invariant with respect to left or right shifts of the group elements. The identity $\mathcal{A}(\hat{g}_t, \hat{g}_0; t) = \mathcal{A}(\hat{g}_t\hat{g}_1, \hat{g}_0\hat{g}_1; t)$ where $\hat{g}_1$ is a fixed element, results in that the action depends on the element $\hat{g}(t) = \hat{g}_t\hat{g}_0^{-1}$:

$$\mathcal{A}(\hat{g}_t, \hat{g}_0; t) = A(\hat{g}_t\hat{g}_0^{-1}; t). \tag{3.6}$$

In view of the condition $\mathcal{A}(\hat{g}_t, \hat{g}_0; t) = \mathcal{A}(\hat{v}\hat{g}_t, \hat{v}\hat{g}_0; t)$, the action is a function only of the "radial" coordinates $\boldsymbol{\varphi}$ in Eq. (3.5). Integrating the Lagrangian (3.1) along the trajectory, one has

$$A(\hat{g}; t) = (2t)^{-1} \Lambda \boldsymbol{\varphi}^2 = (2t)^{-1}\, G_{ab}\xi^a\xi^b \tag{3.7}$$

where Λ is a scale factor, see (A.6).

The Hamilton formalism is constructed in the usual way, while the momenta p_a and the Hamiltonian H are expressed by means of conserved "generalized momenta" P_a:

$$\begin{aligned} p_a &= \frac{\partial \mathcal{L}}{\partial \dot{\xi}^a} = g_{ab}\dot{\xi}^b, \qquad P_a = L_a{}^b p_b = G_{ab}\dot{\zeta}^b \\ H(\xi, p) &= \frac{1}{2}\, g_{ab}(\xi)\, p^a p^b = \frac{1}{2}\, G^{ab} P_a P_b. \end{aligned} \tag{3.8}$$

Note that the Poisson brackets for the generalized momenta do not vanish, but in view of Eq. (B.1) they reconstruct the original Lie algebra:

$$\{P_a, P_b\}_{P.B.} = \frac{\partial P_a}{\partial p_k}\frac{\partial P_b}{\partial \xi^k} - \frac{\partial P_a}{\partial \xi^k}\frac{\partial P_b}{\partial p_k} = C^k_{ab} P_k. \tag{3.9}$$

The action $\mathcal{A}(\hat{g}_t, \hat{g}_0; t)$ satisfies the Hamilton-Jacobi equation in $\hat{g}_t$ (as well as in $\hat{g}_0$) for the Hamiltonian H given in (3.8). Using the differential operator (B.1) and the relations (B.10) and (B.14), this equation may be written as follows

$$\frac{\partial A}{\partial t} + \frac{1}{2}\, G^{ab}\big(\nabla_a(\hat{g})\, A\big)\big(\nabla_b(\hat{g})\, A\big) = 0. \tag{3.10}$$

To check this equation for the function A given in (3.8) it is sufficient to note that in view of Eq. (B.8)

$$t^{-1} G_{ab}\xi^b = \nabla_a(\hat{g})\, A(\hat{g}; t).$$

Writing the solution as an inertial motion, $\xi^a = \zeta^a t$ (or $\boldsymbol{\varphi} = \boldsymbol{\varkappa} t$) we actually describe the trajectory on the Lie algebra, where the coordinates are not limited. At fixed t any two points of the Lie algebra are connected by a single trajectory, so the action is a one-valued function. The transition from the algebra to the group is an exponential mapping, and the parameters in (3.4) are known for a given group element only up to a shift. This ambiguity is expressed in terms of the "radial" variables: the vectors $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi} + 2\pi\boldsymbol{\nu}$

correspond to the same group element, if

$$v = \sum_{j=1}^{r} n_j \tilde{\gamma}^j, \qquad \tilde{\gamma}^j = \frac{2\gamma^j}{(\gamma^j)^2} \tag{3.11}$$

where γ^j are simple roots of the algebra, r is the rank, and n_j are arbitrary integers (the discussion see in Appendix C). Correspondingly, two fixed points of the group are connected by an infinite set of classical trajectories, characterized by the numbers n_j. The action for a given trajectory is

$$A_v(\hat{g}, t) = (2t)^{-1} \Lambda(\varphi + 2\pi v)^2 = (2t)^{-1} \sum_{j,k=1}^{r} S_{jk}(\psi_j + 2\pi n_j)(\psi_k + 2\pi n_k) \tag{3.12}$$

where $S_{jk} = \Lambda(\tilde{\gamma}^j \tilde{\gamma}^k)$, $\varphi = \sum \psi_j \tilde{\gamma}^j$, and ψ_j are normalized coordinates on the maximal torus, $-\pi \leqq \psi_j < \pi$ (see Appendix C).

In view of Eqs. (3.7) and (3.12) one should emphasize a fact important for the following. In operators and functions defined in two different spaces, the Lie group and the Lie algebra, we use as a rule the same notation for the argument — the group element $\hat{g}$. Of course, one should have in mind that the functions on the algebra depend on the element $\hat{x} = \xi^a \hat{X}_a$, i.e. on the coordinates ξ^a, that have unbounded variations. The group element is $\hat{g} = \exp \hat{x}$. So if the argument for the algebra is written as $\hat{g} = \hat{g}_1 \hat{g}_2$, we mean the element of the algebra $\hat{x}(\exp \hat{x} = \exp \hat{x}_1 \cdot \exp \hat{x}_2)$ defined by the Baker-Campbell-Hausdorff series. The coordinates are then functions of $\xi_1{}^a$ and $\xi_2{}^a$; the first terms of the power expansion of this functions are given in (B.3).

4. The Quantal Dynamics

The canonical quantization on a Riemannian space may be started from construction of the momentum operator

$$\hat{p}_a = -i\hbar \left(\partial_a + \frac{1}{2} \Gamma_a\right) \tag{4.1}$$

where $\Gamma_a \equiv \Gamma^c_{ac} = \partial_a(\ln \sqrt{g})$, $g = \det g_{ab}$. This approach was discussed by De Witt [32] (see also a more recent note by Chand and Casanova [33]). The operator (4.1) is Hermitian if the scalar product of wave functions is defined by integration with the invariant measure $\sqrt{g} \prod_a d\xi^a$. In case of the group space the Christoffel symbol is given by (B.11), and

$$\Gamma_a = -M_b{}^c \partial_a L_c{}^b = -M_a{}^c \partial_b L_c{}^b \tag{4.2}$$

(the last identity is a consequence of (B.1)). The operators of generalized momenta and energy are also Hermitian and are of the form

$$\begin{aligned} \hat{P}_a &= \frac{1}{2} (L_a{}^b \hat{p}_b + \hat{p}_b L_a{}^b) = -i\hbar L_a{}^b \partial_b \\ H &= \frac{1}{2} G^{ab} \hat{P}_a \hat{P}_b = -\frac{1}{2} \hbar^2 \Delta \end{aligned} \tag{4.3}$$

where Δ is the Laplace operator (B.16).

The operator $\hat{P}_a$ is proportional to the left shift operator (B.1) and the Poisson brackets (3.9) become the commutator

$$[\hat{P}_a, \hat{P}_b] = -i\hbar C^c_{ab} \hat{P}_c. \tag{4.4}$$

The solutions of the Schroedinger equations are the matrix elements of the group representations (cf. (B.20)), and the energy levels are proportional to the Casimir operator eigen-values:

$$H\psi_l(\hat{g}) = E_l\psi_l(\hat{g})$$
$$\psi_l(\hat{g}) = d_l^{1/2}D^l(\hat{g}); \qquad E_l = \frac{1}{2}\,\hbar^2\lambda(l) \tag{4.5}$$

where $\boldsymbol{l}$ is the highest weight and λ is given in (A.11). The ψ functions are orthonormal (cf. (A.32)).

The Green's function, just as the classical action, depends only on the radial coordinates $\boldsymbol{\varphi}$ and is expanded over the representation characters:

$$\begin{aligned}\mathcal{K}(\hat{g}_t, \hat{g}_0; t) = K(\hat{g}; t) &= \sum_l d_l \,\mathrm{Sp}\left[D^l(\hat{g}_t)\,\overset{+}{D}{}^l(\hat{g}_0)\right] \exp\left(-\frac{i}{\hbar}\,E_l t\right)\\ &= \sum_l d_l\chi_l(\boldsymbol{\varphi}) \exp\left(-\frac{i}{2}\,\hbar\lambda(l)\,t\right)\end{aligned} \tag{4.6}$$

where $\hat{g} = \hat{g}_t\hat{g}_0^{-1} = \hat{v}^{-1}\hat{h}(\boldsymbol{\varphi})\,\hat{v}$. Using the invariance of the measure, Eq. (1.2) may be also written as follows

$$K(\hat{g}; t_2 + t_1) = \int K(\hat{g}_1; t_1)\,K(\hat{g}_1^{-1}\hat{g}; t_2)\,d\hat{g}_1. \tag{4.7}$$

It follows also from (4.6) that

$$\int K(\hat{g}; t)\,d\hat{g} = 1. \tag{4.8}$$

Now we present some explicit formulas for the $SU(2)$ group. Write the group element by means of the Pauli matrices[1]): $\hat{g} = \exp i\sigma_a\xi^a$, where $a = 1, 2, 3$, $\boldsymbol{\xi} = \varphi\boldsymbol{\omega}$, $\boldsymbol{\omega}$ is a unit vector, $0 \leqq \varphi < \pi$. For the rotation group $SO(3)$ the parameter 2φ is the rotation angle, $\boldsymbol{\omega}$ is the axis, and $0 \leqq \varphi < \pi/2$.

Using Eqs. (B.5) one gets

$$M_b{}^a(\xi) = \frac{\sin 2\varphi}{2\varphi}\,(\delta_{ab} - \omega_a\omega_b) + \omega_a\omega_b + \frac{1-\cos\varphi}{2\varphi}\,\varepsilon_{abc}\omega_c$$
$$g_{ab}(\xi) = 8\left[\frac{\sin^2\varphi}{\varphi^2}\,(\delta_{ab} - \omega_a\omega_b) + \omega_a\omega_b\right] \tag{4.9}$$

and

$$d\hat{g} = \sin^2\varphi\,d\varphi\,\frac{d^2\omega}{2\pi^2}. \tag{4.10}$$

Green's function is a sum over all angular momenta

$$K(\hat{g}; t) = \sum_j (2j+1)\,\chi^j(\varphi)\exp\left(-\frac{i}{4}\,\hbar j(j+1)\,t\right)$$
$$\chi^j(\varphi) = \frac{\sin(2j+1)\,\varphi}{\sin\varphi}. \tag{4.11}$$

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[1]) We use here such a normalization that $\alpha = 2$, but not $\sqrt{2}$ as in Eq. (A.41), so $\varrho = 1$ and $\Lambda = 8$.

The energy operator is

$$H = \frac{1}{16}\hat{\boldsymbol{P}}^2 = -\frac{\hbar^2}{16}\left[\frac{1}{\sin^2\varphi}\frac{\partial}{\partial\varphi}\sin^2\varphi\frac{\partial}{\partial\varphi} + \frac{\Delta_\omega}{\sin^2\varphi}\right] \tag{4.12}$$

where Δ_ω is the Laplace operator on the sphere S^2. The operator H is the Hamiltonian for symmetrical top (see e.g. the textbook by Landau and Lifshitz [*34*], § 103). However, here it is expressed in terms of the variables φ, $\boldsymbol{\omega}$, and not the Euler angles. The eigen-functions for which the variables $(\varphi, \boldsymbol{\omega})$ are separated, are obtained in [*35*]. On the other hand, one easily recognizes in (4.12) the Laplace operator for the sphere S^3 the four-dimensional points of which are $(\sin\varphi\cdot\boldsymbol{\omega}, \cos\varphi)$, cf. Section 7. The "radial part" of the Laplace operator may be given in form (B.17)

$$\mathring{\Delta} = \frac{1}{\sin\varphi}\frac{\partial^2}{\partial\varphi^2}\sin\varphi + 1. \tag{4.13}$$

The constant $+1$ here provides the ground state with the zero energy.

5. Semiclassical Approximation

In this Section we present the semiclassical expressions for the Green's function on the Lie algebra and the Lie group and prove that they are the exact solutions.

For the Lie algebra the semiclassical approximation is given by Eq. (1.6) with action (3.7). The coordinates are the parameters ξ^a, the integration measure is the Riemannian one and given in Eq. (B.12), so that $\varrho(\xi) = C_R\sqrt{g}$. In fact, the Lie algebra may be treated as a Riemannian space with the trivial topology; an element of the algebra is in one-to-one correspondence with the coordinates ξ^a, that are unbounded. Therefore one may deduce Eq. (1.6) for the Lie algebra, following De Witt [3].

Calculate the pre-exponential factor. Using the metric tensor (B.10) as well as the definition (B.1) one gets

$$\begin{aligned}\det\left[-\frac{\partial^2\mathcal{A}}{\partial\xi_t^{\,b}\,\partial\xi_0^{\,a}}\right][\det g_{ab}(\xi_0)\det g_{ab}(\xi_t)]^{-1/2} \\ = [\det G_{ab}]^{-1}\det[-\nabla_a(\hat{g}_t)\,\nabla_b(\hat{g}_0)\,\mathcal{A}(\hat{g}_t, \hat{g}_0; t)].\end{aligned} \tag{5.1}$$

For the free motion, the action depends on $\hat{g} = \hat{g}_t\hat{g}_0^{-1}$ only, and using Eqs. (B.13) and (B.14) one gets with the function defined in (3.6)

$$\det[-\nabla_a(\hat{g}_t)\,\nabla_b(\hat{g}_0)\,\mathcal{A}(\hat{g}_t, \hat{g}_0; t)] = \det[\nabla_a(\hat{g})\,\nabla_b(\hat{g})\,A(\hat{g}, t)]. \tag{5.2}$$

Using now the explicit form for the action (3.7) as well as Eqs. (B.7) and (B.8) one obtains for the semiclassical Green's function on the Lie algebra $\mathfrak{A}$:

$$K_{\mathfrak{A}}^{\rm cl}(\hat{g}; t) = C_R^{-1}(2\pi i\hbar t)^{-n/2}\,B(\hat{g})\exp\left[i\Lambda\frac{\varphi^2}{2\hbar t}\right] \tag{5.3a}$$

where

$$B(\hat{g}) = (\det L_a{}^b(\xi))^{1/2} = \left(\frac{\det G_{ab}}{\det g_{ab}(\xi)}\right)^{1/4} = \prod_{\alpha>0}\frac{\alpha\varphi}{2\sin\dfrac{\alpha\varphi}{2}}. \tag{5.3b}$$

The $t\to 0$ limit of $K_{\mathfrak{A}}^{\rm cl}$ is the δ-function on the Lie algebra. This is a result of the fact that at $\hbar\to 0$ the semiclassical expression satisfies (1.2) up to some terms of higher order in $\hbar$. One may use this property to find the pre-exponential factor, calculating the integral in Eq. (1.2) by means of the stationary phase method.

Prove now that the exact Green's function equals to $K_{\mathfrak{A}}{}^{\mathrm{cl}}$ times a trivial factor

$$K_{\mathfrak{A}}(\hat{g}\,;t) = K_{\mathfrak{A}}{}^{\mathrm{cl}}(\hat{g}\,;t)\exp\left(i\hbar\,\frac{nt}{48}\right). \tag{5.4}$$

At $t \to 0$ the initial condition is correct,

$$K_{\mathfrak{A}}(\hat{g}\,;t) \to C_R{}^{-1}\delta^{(n)}(\xi)\,,$$

so one has to check that (5.4) satisfies the Schroedinger equation $i\hbar\,\partial K/\partial t = HK$. With the Hamiltonian $H = -1/2\,\hbar^2 G^{ab}\nabla_a\nabla_b$ we establish that the equation is satisfied if the following relations are valid:

$$\frac{1}{2}\,G^{ab}(\nabla_a A)\,(\nabla_b A) = -\frac{\partial A}{\partial t} \tag{5.5}$$

$$\Delta A + 2G^{ab}\nabla_a A B^{-1}\nabla_b B = n \tag{5.6}$$

$$B^{-1}\Delta B = n/24. \tag{5.7}$$

The first one is the Hamilton-Jacobi equation (3.10). To check (5.6), note that $B^{-1}\nabla_b B = -1/2\,\nabla_b \ln g = -1/2\,L_b{}^a\Gamma_a$. Using now the Δ operator in form of Eq. (B.16), we get for l.h.s. of Eq. (5.6)

$$\partial_a(g^{ab}\,\partial_b A) = \partial_a(L_p{}^a G^{pq} L_q{}^b\,\partial_b A)\,.$$

By means of Eq. (B.8) this expression is reduced to $\partial_a\xi^a = n$ in agreement with r.h.s. To check (5.7) use the explicit form of the "radial part" of the Laplace operator (B.17) and reduce Eq. (5.7) with account of (B.18) to the identity

$$\frac{\partial^2}{\partial\boldsymbol{\varphi}^2}\prod_{\alpha>0}(\alpha\boldsymbol{\varphi}) = 0. \tag{5.8}$$

This is true because l.h.s. is homogeneous in $\boldsymbol{\varphi}$, while its correctness at $\boldsymbol{\varphi} \to 0$ is a consequence of Eq. (B.19). So $K_{\mathfrak{A}}$ in (5.4) is indeed the exact Green's function for the Lie algebra.

To construct Green's function on a group one has to have in mind that in mapping the algebra on the group the sets of points $\boldsymbol{\varphi} + 2\pi\boldsymbol{\nu}$ with various $\boldsymbol{\nu}$ (see Eq. (3.11)) are identified, so the function for the group is to be invariant under the translation $\boldsymbol{\varphi} \to \boldsymbol{\varphi} + 2\pi\boldsymbol{\nu}$. A natural way to obtain such a function is to sum $K_{\mathfrak{A}}$ over all possible values of $\boldsymbol{\nu}$. Note, however, that the sector $\boldsymbol{\nu} = 0$ for the algebra is defined in the Weyl chamber (A.24) while the translation, in general, leads the point out of the chamber. Using the invariance of $K_{\mathfrak{A}}$ with respect to the Weyl group W, one is able to extend the support of $K_{\mathfrak{A}}$ on the whole Euclidean r-dimensional space of vectors $\boldsymbol{\varphi}$. We name the space of the algebra parameters obtained in this manner, the direct product $\mathfrak{A} \otimes W$. Extending the region of the parameters one has to insert a factor $N_W{}^{-1}$ (N_W is the order of the Weyl group) into Green's function, so that the normalization (1.3) be conserved. (This factor arises from the fact that the $\boldsymbol{\varphi}$-vector space is a superposition of N_W Weyl chambers. The Weyl reflections are linear transformations of the root system. At any reflection $\boldsymbol{\varphi}$ is transferred in a copy of the Weyl chamber).

Summing up the resulting Green's function over the lattice formed by allowed values of $\boldsymbol{v}$, we get Green's function on the Lie group

$$K(\hat{g}; t) = \sum_{\boldsymbol{v}} \frac{B_{\boldsymbol{v}}(\hat{g})}{C_R N_W (2\pi i \hbar t)^{n/2}} \exp\left[\frac{i}{\hbar} A_{\boldsymbol{v}}(\hat{g}; t) + i\hbar \frac{nt}{48}\right] \tag{5.9a}$$

where

$$B_{\boldsymbol{v}}(\hat{g}) = \prod_{\alpha>0} \frac{(\alpha, \boldsymbol{\varphi} + 2\pi \boldsymbol{v})}{2 \sin \frac{1}{2} (\alpha, \boldsymbol{\varphi} + 2\pi \boldsymbol{v})}. \tag{5.9b}$$

In this formula the vector $\boldsymbol{\varphi}$ is the coordinate on the maximal torus of the group (it changes inside the elementary cell of the lattice). The constant C_R may be determined from comparison of (5.9) with the spectral expansion (4.6). The result is

$$C_R^{-1} = (2\pi)^{p+r} N_W \Lambda^{n/2} \sqrt{\Delta_\gamma} \prod_{\alpha>0} (\alpha\varrho)^{-1} \tag{5.10}$$

where $\alpha > 0$ are the positive roots, p is their number, $\Delta_\gamma = \det \tilde{\gamma}^j \tilde{\gamma}^k$, $\tilde{\gamma}^j = 2\gamma^j/(\gamma^j)^2$, γ^j is a simple root ($j, k = 1, \ldots, r$, see Appendix C), $\varrho = 1/2 \sum_{\alpha>0} \alpha$. The expression (5.10) will be obtained in the next Section where an alternative deduction of the semiclassical series (5.9) at small t is presented.

The representation (5.9) is true because it satisfies the Schroedinger equation by construction, has the correct periodical property on the group, at its limit at $t \to 0$ is the δ-function on the group. The latter is checked by comparison with the spectral expansion (see the next Section).

6. The Path Integral

To construct the path integral one needs Green's function at small times $\hbar\delta t \equiv \tau = t\hbar/N$, $N \to \infty$.

In case of the Lie algebra, effective region of ξ^a (and φ) in (5.3) is small at small t. The expansion of $B(\hat{g})$ at $\varphi \to 0$ is

$$B(\hat{g}) = 1 + \frac{1}{24} \sum_{\alpha>0} (\alpha\boldsymbol{\varphi})^2 + \cdots = 1 + \frac{1}{48} G_{ab}\xi^a\xi^b + \cdots \tag{6.1}$$

where we have used Eqs. (A.5), (A.6) and (A.23). For the path integral only linear in τ terms in Green's function are necessary. Within this accuracy, taking into account the exponent, one may substitute $G_{ab}\xi^a\xi^b \to i\tau G_{ab}G^{ab}$, $G_{ab}G^{ab} = n$ (see the works [3, 4]). The resulting approximate expression is

$$K_{\mathfrak{A}}(\hat{g}, t) = C_R^{-1} (2\pi i \tau)^{-n/2} \exp\left[\frac{i}{\hbar} A(\hat{g}; t) + \frac{i\tau n}{24}\right]. \tag{6.2}$$

Green's function at finite $t = N\tau/\hbar$ is obtained by integration of product of N functions given by (6.2). The result is the path integral representation at $N \to \infty$.

$$K_{\mathfrak{A}}(\hat{g}; t) = \int \mathcal{D}_{\mathfrak{A}}\hat{g} \exp \frac{i}{\hbar} \left[\int_0^t \mathcal{L}\, d\tau + \hbar^2 \frac{nt}{24}\right] \tag{6.3}$$

where

$$\mathscr{D}_{\mathfrak{A}}\hat{g} = C_R^{-1} \lim_{N\to\infty} (2\pi i\tau)^{-n/2} \prod_{j=1}^{N-1} \left[\sqrt{g(\xi_j)}\, \frac{d^n\xi_j}{(2\pi i\tau)^{n/2}}\right]. \tag{6.4}$$

The integration is over the whole space of ξ_j, from $-\infty$ up to ∞; $\hat{g} = \hat{g}_t\hat{g}_0^{-1}$,

$$\xi_{j=0} = \xi_0, \qquad \xi_{j=N} = \xi_t.$$

Note that in Eq. (6.3) a correction to the classical Lagrangian arises, $\hbar^2 n/24 = \hbar^2 R/6$, cf. (B.11). For the general case of the Riemannian space this correction was found by De Witt [3].

It is clear from the construction that the integral $\int \mathscr{L}\, d\tau$ in Eq. (6.3) is approximated by the sum $\sum\limits_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \mathscr{L}\, d\tau$ where $\hbar(t_j - t_{j-1}) = \tau \to 0$, while on any small time interval the integral is calculated along the classical trajectory (geodesic) and equals to the action (3.7). So the paths in Eq. (6.3) are chains composed of segments of geodesics. In the conventional Feynman path integral the paths are composed of straight linear segments. The difference of $\int\limits_{t_{j-1}}^{t_j} \mathscr{L}\, d\tau$ along the straight line connecting ξ_j and ξ_{j-1} from $\mathscr{A}(\hat{g}_j, \hat{g}_{j-1}; \tau)$ is of order τ. Therefore, if one prefers to understand Eq. (6.3) as an integral over piecewise linear paths, one has to add more corrections to $\mathscr{L}$, that may be interpreted as an effective potential $U_{\text{eff}} \sim \hbar^2$ (see the work [4]). We mean that at small times the action calculated along a classical trajectory, may be represented as follows

$$\frac{1}{2}\, g_{ab}(\bar{\xi})\, \frac{\Delta\xi^a \Delta\xi^b}{\delta t} - \delta t\, U_{\text{eff}}(\xi_j) \tag{6.5}$$

where $\Delta\xi = \xi_j - \xi_{j-1}$. Correspondingly, the action may be considered as an integral along a straight line with $\mathscr{L}_{\text{eff}} = \mathscr{L} - U_{\text{eff}}$. The shape of the function $U_{\text{eff}}(\xi)$ is determined by the choice of the average point $\bar{\xi}$. For instance, in the work [4] $\bar{\xi} = 1/2(\xi_j + \xi_{j-1})$, and for an arbitrary Riemannian space

$$U_{\text{eff}} = \frac{\hbar^2}{8}\,(g^{ab}\,\partial_b\Gamma^c_{ac} + g^{ab}\Gamma^c_{ad}\Gamma^d_{bc}) + \frac{\hbar^2}{24}\,R. \tag{6.6a}$$

(This result is obtained from the corresponding formulas in [4] by means of a series of transformations).

If $\bar{\xi} = \xi_j$ (as in [3])or if $\bar{\xi}$ is the average point on the geodesic, connecting ξ_j and ξ_{j-1}, then U_{eff} is of an another form. For example, in the latter case

$$U_{\text{eff}} = \frac{\hbar^2}{8}\,(g^{ab}\partial_b\Gamma^c_{ac} - g^{ab}\Gamma^c_{ad}\Gamma^d_{bc} - g^{bc}\Gamma^a_{bc}\Gamma_a) + \frac{\hbar^2}{24}\,R. \tag{6.6b}$$

For the Lie algebra in both cases U_{eff} depends on the "radial" variables φ, but it is a rather complicated function and we do not present it here.

To construct the path integral for the Lie group there are two possibilities. First one may take the expression (6.3) for the Lie algebra and to sum up over the lattice (see discussion before Eq. (5.9)). The result is

$$K(\hat{g}_t\hat{g}_0^{-1}; t) = \sum_{\nu} \int \mathscr{D}\hat{g}\, \exp \frac{i}{\hbar}\left[\int\limits_0^t \mathscr{L}\, d\tau + \hbar^2\, \frac{nt}{24}\right] \tag{6.7}$$

where $\mathscr{D}\hat{g} = N_W^{-1}\mathscr{D}_{\mathfrak{A}}\hat{g}$. The measure $\mathscr{D}_{\mathfrak{A}}\hat{g}$ is defined in Eq. (6.3) and

$$\hat{g}_0 = \exp \xi_0{}^a \hat{X}_a$$

$$\hat{g}_t = \exp \xi_t{}^a \hat{X}_a = \hat{v}^{-1}(\omega_t) \exp [\boldsymbol{\varphi}_t \hat{\boldsymbol{H}}]\, \hat{v}(\omega_t); \qquad \xi_t{}^a = A_j{}^a(\omega_t)\, \varphi_t{}^j.$$

See (A.22) and (A.27). Similarly, $\xi_0{}^a = A_j{}^a(\omega_0)\, \varphi_0{}^j$. The integration over $\xi_1{}^a, \ldots, \xi_{N-1}^a$ is on the infinite domain. The coordinates of the end points of the paths in Eq. (6.7) are

$$\begin{aligned} \xi_N{}^a &= \xi_t{}^a + 2\pi A_k{}^a(\omega_t)\, \nu^k = A_k{}^a(\omega_t)\, (\varphi_t{}^k + 2\pi\nu^k) \\ \xi_0{}^a &= A_k{}^a(\omega_0)\, \varphi_0{}^k. \end{aligned} \tag{6.8}$$

The vectors $\boldsymbol{\varphi}_0$ and $\boldsymbol{\varphi}_t$ are the coordinates on the maximal torus, the vector $\boldsymbol{\nu}$ is given by Eq. (3.11), the sum in Eq. (6.7) is fulfilled over the group lattice. The final point of the path on the algebra is shifted in (6.8), one may shift the initial point as well. Recall that in case of the free motion Green's function depends only on $\boldsymbol{\varphi}$ where $\hat{g}_t\hat{g}_0{}^{-1} = \hat{v}^{-1} \exp \boldsymbol{\varphi}\hat{\boldsymbol{H}}\hat{v}$. In the following we mention the representation (6.7) as the first-kind path integral or PI-1.

The second possibility to construct the path integral is to find Green's function at small times $\delta t = \hbar^{-1}\tau$, and then to iterate Eq. (1.2). Start from the spectral expansion (4.6) at small δt. Write Green's function in form of the Fourier series on the torus (cf. Eq. (C.4)):

$$\sum_l d_l \chi_l(\boldsymbol{\varphi}) \exp(-iE_l\tau) = \sum_{\mu} f(\boldsymbol{\mu}, \tau)\, e^{i\boldsymbol{\mu}\boldsymbol{\varphi}} \tag{6.9}$$

where $\boldsymbol{\mu} = \sum_{j=1}^{r} m^j \boldsymbol{\beta}_j$, the vectors $\boldsymbol{\beta}_j$ form the basis of the inverse lattice, and $\boldsymbol{\mu}\boldsymbol{\varphi} = \sum_{j=1}^{r} m^j \psi_j$. The sum in (6.9) is over all integer m_j. Note a consequence of invariance of the characters with respect to the Weyl group: $f(\sigma\boldsymbol{\mu}, \tau) = f(\boldsymbol{\mu}, \tau)$. Using the orthonormal property of the characters for the Weyl measure one gets a set of relations for the function $f(\boldsymbol{\mu}, \tau)$:

$$d_l e^{-iE_l\tau} = \sum_{\mu} f(\boldsymbol{\mu}, \tau) \int e^{i\boldsymbol{\mu}\boldsymbol{\varphi}} \chi_l{}^*(\boldsymbol{\varphi})\, d\hat{g}. \tag{6.10}$$

Using Eqs. (A.29) and (A.35) as well as the symmetry of $f(\boldsymbol{\mu}, \tau)$ these relations are reduced to

$$d_l \exp(-iE_l\tau) = \sum_{\sigma} \varepsilon_\sigma f(\boldsymbol{L} - \sigma\boldsymbol{\varrho}, \tau). \tag{6.11}$$

Suppose that at small τ the function is of the form

$$f(\boldsymbol{\mu}, \tau) \simeq f_0(\tau)\, [\exp(-ic\boldsymbol{\mu}^2\tau) + 0(\tau^2)] \tag{6.12}$$

where c is a constant. Substituting this form into Eq. (6.11) and using the explicit expression (4.5) for E_l as well as Eqs. (A.12) and (A.29) we obtain $c = \hbar/2A$ and:

$$f_0(\tau) = \left(\frac{A}{i\tau}\right)^p \prod_{\alpha>0} (\boldsymbol{\alpha}\boldsymbol{\varrho})^{-1} \left[\exp\left(i\frac{\tau\varrho^2}{A}\right) + O(\tau^2)\right]. \tag{6.13}$$

Recall that $\varrho^2/A = n/24$. Thus at small τ Green's function is reduced to the Θ function (C.5)

$$K(\hat{g}, \tau) = f_0(\tau) \sum_{\mu} \exp\left(-i\frac{\boldsymbol{\mu}^2}{2A}\tau + i\boldsymbol{\mu}\boldsymbol{\varphi}\right). \tag{6.14}$$

Transforming it to the form of Eq. (C.7) and using (6.13), one gets the result

$$K(\hat{g}, \tau) = C(2\pi i\tau)^{-n/2} \sum_{\nu} \exp \frac{i}{\hbar} \left[A_\nu(\hat{g}, \tau) + \hbar \frac{n\tau}{24} \right] \tag{6.15}$$

where

$$C = (2\pi)^{p+r} \Lambda^{n/2} \sqrt{\Delta_\gamma} \prod_{\alpha>0} (\alpha\varrho)^{-1}. \tag{6.16}$$

Comparing this expression with (5.9a) at $\tau \to 0$, $\varphi \to 0$, we find that $C_R = C^{-1}N_W^{-1}$, so that Eq. (5.10) for C_R arises. Trying to get Eq. (6.15) directly from Eq. (5.9) at small t one meets some mathematical questions. The problem is that expanding the preexponential factor one has to take into account that Green's function is periodical in $\boldsymbol{\varphi}$. One might also doubt whether it is possible to substitute $\xi^2 \to \tau$ in expressions like (6.1) just in the same manner as in Gaussian integrals over an infinite space. However, in fact these actions are loyal because one can obtain (6.15) from (5.9) in an independent manner. One must have in mind that at small τ all the integrals with the function (5.9) may be calculated by means of the stationary phase method and with account of two main terms of the asymptotics, the function (5.9) being approximated by the Gaussian contributions near the saddle points $\boldsymbol{\varphi} + 2\pi\boldsymbol{\nu}$. We shall use this procedure in the next section considering the spheres for which no other way is at hand.

The Green's function at a finite time $\hbar t = N\tau$ is constructed, as usual, integrating the product of N small-time functions (6.15) for consecutive intervals $\tau = (t_{j+1} - t_j)\,\hbar$. The integral is with the Weyl measure and the domain of ξ^a is compact. The resulting representation at $N \to \infty$ is the "path integral" comprising at any time interval not only the integration over the group, but also the sum over $\boldsymbol{\nu}$ that takes into account the infinite set of classical trajectories connecting any pair of consecutive configurations. We call this representation the second-kind path integral, *PI*-2. Just this variant was considered for unitary groups in our previous work [*21*].

In case of $U(1)$, discussed in Section 2, the equivalence of *PI*-1 and *PI*-2 was shown manifestly (see the paragraph after Eq. (2.6)). The direct proof of the equivalence for an arbitrary group is a very cumbersome task.

7. Path Integral for Sphere of Arbitrary Dimensionality

Consider dynamics on the n-dimensional sphere S^n. The coordinates are components of a unit $(n + 1)$-dimensional vector $\boldsymbol{\omega}$. The free Lagrangian is

$$L = \frac{1}{2} \dot{\boldsymbol{\omega}}^2. \tag{7.1}$$

The classical trajectories are the great circles. The action for the trajectory connecting $\boldsymbol{\omega}_0$ and $\boldsymbol{\omega}_t$ and evolving k times around the circle is

$$\mathcal{A}_k(\boldsymbol{\omega}_t, \boldsymbol{\omega}_0, t) = A(\theta_k; t) = \frac{\theta_k^2}{2t} \tag{7.2}$$

where $\theta_k = \theta + 2\pi k$, $\cos\theta = \boldsymbol{\omega}_0\boldsymbol{\omega}_t$ and k is an arbitrary integer.

The vector $\boldsymbol{\omega}$ may be parametrized by explicit discrimination of its $(n+1)$-th component

$$\boldsymbol{\omega} = (\boldsymbol{\varkappa} \sin\theta, \cos\theta) \equiv \left(\boldsymbol{q}, \pm\sqrt{1 - \boldsymbol{q}^2}\right) \tag{7.3}$$

where $\varkappa$ and $\boldsymbol{q}$ are n-dimensional vectors, $\varkappa^2 = 1$, $\boldsymbol{q}^2 \leqq 1$. The Riemannian measure on S^n and the Laplace operator on S^n are given by

$$d\omega = V_n^{-1} \sin^{n-1} \theta \, d\varkappa = V_n^{-1} \frac{d\boldsymbol{q}}{\sqrt{1 - \boldsymbol{q}^2}} \tag{7.4}$$

$$\Delta_\omega = \frac{\partial^2}{\partial \theta^2} + (n-1) \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \cdot \Delta_\varkappa \tag{7.5}$$

$$V_n = \frac{2\pi^{\nu+1}}{\Gamma(\nu+1)}, \qquad \nu = \frac{n-1}{2}. \tag{7.6}$$

Here $\Delta_\varkappa$ is the Laplace operator for S^{n-1}, V_n is the volume of S^n, $\int d\omega = 1$.
The quantal Hamiltonian is $H = -1/2\, \hbar^2 \Delta_\omega$. The energy spectrum and the eigen-functions are

$$E_l = \frac{\hbar^2}{2} l(l + n - 1), \qquad \psi_{lM}(\omega) \sim t^l_{0M}\big(\hat{g}(\omega)\big) \tag{7.7}$$

where $l = 0, 1, 2, \ldots$, t^l_{0M} is a matrix element of the zero column of a representation of the group $SO(n+1)$; the element $\hat{g}(\omega)$ of this group is a rotation of a fixed unit vector $\boldsymbol{e}$ (say, with $\theta = 0$ in (7.3)) to the position of $\boldsymbol{\omega}$. The symbol M is a set of $(n-1)$ indices numerating the component of the ψ-function, the set 0 consists of zeroes (see the book by Vilenkin [36]). The spectral expansion of Green's function is based on functions

$$t^l_{00}(\hat{g}_{t0}) = \sum_M t^l_{0M}\big(\hat{g}(\omega_t)\big) \overset{*}{t}{}^l_{0M}\big(\hat{g}(\omega_0)\big)$$

where $\hat{g}_{t0}$ is a rotation of ω_0 into ω_t. The element t_{00} depends only on the angle θ and is proportional to the Gegenbauer polynomial. The resulting expansion is

$$\mathcal{K}(\omega_t, \omega_0; t) = K(\theta; t) = \sum_{l=0}^{\infty} \frac{l+\nu}{\nu} C_l^\nu(\cos\theta) \exp\left[-\frac{i\hbar t}{2} l(l+2\nu)\right] \tag{7.8}$$

where $C_l^\nu(\cos \theta)$ is the Gegenbauer polynomial, $\nu = (n-1)/2$. Green's function is normalized for normalized measure (7.4)

$$\int \mathcal{K}(\omega_t, \omega_0; t)\, d\omega_0 = 1. \tag{7.9}$$

To construct the path integral, find $K(\theta; t)$ at small $t \to \delta t$ by means of the semiclassical approximation (1.6). Using the q-parameters (7.3) one may show that

$$\det\left[-\frac{\partial^2 A}{\partial q_0^j\, \partial q_t^k}\right] = t^{-n} \left[(1 - q_0^2)(1 - q_t^2)\right]^{-1/2} \left(\frac{\theta}{\sin\theta}\right)^{n-1}. \tag{7.10}$$

So the semiclassical approximation is

$$K^{\mathrm{cl}}(\theta, \delta t) \equiv \sum_{k=-\infty}^{\infty} F(\theta_k) = \frac{1}{(2\pi i \tau)^{n/2}} \sum_{k=-\infty}^{\infty} \left(\frac{\theta_k}{\sin \theta_k}\right)^{(n-1)/2} \exp \frac{i}{\hbar} A(\theta_k, \delta t) \tag{7.11}$$

where $A(\theta_k, \delta t)$ is given by Eq. (7.2) and $\tau = \hbar \delta t$. Substituting this form into the Schroedinger equation one gets

$$\left(i\hbar \frac{\partial}{\partial t} - H\right) F(\theta) = -\hbar^2 \left[\frac{n(n-1)}{12} + \chi(\theta)\right] F(\theta) \tag{7.12}$$

$$\chi(\theta) = \frac{(n-1)(n-3)}{8}\left[\theta^{-2} - \cot^2\theta - \frac{2}{3}\right]. \tag{7.12}$$

Thus K^{cl} is not a solution of the equation for $n \neq 1$. However, the function $K^{\mathrm{cl}}(\theta; \delta t) \times \exp\big(i\tau\, n(n-1)/12\big)$ is the solution with the desired accuracy. The fact is at $\theta^2 \leqq \tau$ the term with $\chi(\theta)$ is inessential and leads to corrections $\sim\tau^2$ in Green's function.

Integrating Green's function $\mathcal{K}(\omega_t, \omega_0; t)$ with a smooth function on the sphere $f(\omega_0)$, it is useful to put the origin into the point ω_t. With the measure (7.4) note that at small t dominating contribution is from the saddle points in θ. The expansion of the preexponential factor near the saddle point $\theta_k = 0$ is

$$\left(\frac{\theta_k}{\sin\theta_k}\right)^{(n-1)/2} \simeq 1 + \frac{n-1}{12}\,\theta_k^2. \tag{7.13}$$

With the assumed accuracy $\theta_k^2 \simeq i\tau n$, so that the asymptotic expansion is

$$K(\theta, \delta t) \simeq V_n(2\pi i\tau)^{-n/2}\, e^{i\tau n(n-1)/6} \sum_{k=-\infty}^{\infty} \exp\frac{i}{\hbar}\, A(\theta_k, \delta t). \tag{7.14}$$

Integrating the product of N such functions one gets

$$\mathcal{K}(\omega_t, \omega_0; t) = \lim_{N\to\infty} \mathcal{K}^N(\omega_t, \omega_0; t)$$

$$\mathcal{K}^N(\omega_t, \omega_0; t) = V_n(2\pi i\tau)^{-n/2} \sum_{k_1\cdots k_N=-\infty}^{\infty} \int \prod_{j=1}^{N-1} \frac{d^n\omega_j}{(2\pi i\tau)^{n/2}}\,. \tag{7.15}$$

$$\exp\frac{i}{\hbar}\sum_{j=1}^{N}\left[\mathcal{A}_{k_j}(\omega_j, \omega_{j-1}; \delta t) + \frac{1}{6}\,\hbar R\tau\right]$$

where $\omega_N = \omega_t$, $\tau = \hbar\delta t = \hbar t/N$, $\mathcal{A}_k$ is the classical action (7.2), $R = n(n-1)$ is the Riemannian curvature of the sphere S^n. This representation is of the type PI-2 discussed at the end of Section 6. Remind that it is much more complicated than the usual path integral for the n-dimensional Euclidean space because all the intermediate integrals are finite, while the action is a rather nontrivial function of the end points of the trajectories. In two particular cases $n = 1$ and $n = 3$ the representation (7.15) is just PI-2 for the groups $U(1)$ and $SU(2)$, respectively. Note that the manifold of $SU(2)$ is the 3-dimensional sphere with the radius $\sqrt{8}$ (cf. the metrics in Eq. (4.9)). Therefore the energy spectrum as well as the curvature for S^3 and $SU(2)$ are equal up to a factor of 8. With this in view, the expansion (4.11) and (7.8) coincide at $l = 2j$.

An another variant of the path integral may be constructed if we consider the motion on S^n as a motion on the $(n+1)$-dimensional Euclidean space with the constraint $\boldsymbol{x}^2 = 1$ where $\boldsymbol{x}$ is the $(n+1)$-dimensional vector. To deduce this variant, start from the spectral expansion and write it at small $t = \delta t$ for the effective region of angles $(1 - \cos\theta) \sim \hbar\delta t \to 0$. The series (7.8) is approximated by the expression

$$K \simeq V_n(2\pi i\tau)^{-n/2}\left(1 + \frac{i\tau}{8}\, n(n-2)\right). \tag{7.16}$$

$$\left[1 + \frac{i\tau}{24}\, n(n-4) - \frac{1}{6}\,(n-1)(1-\cos\theta) - \frac{i}{6\tau}\,(1-\cos\theta)^2\right]\exp\frac{i}{\tau}\,(1-\cos\theta).$$

Two main terms are retained here in region $(1 - \cos\theta) \lesssim \tau$. In order to obtain (7.16) one may consider the known expansion of the exponent

$$\exp\left(-\frac{i}{\tau}\cos\theta\right) = 2\tau^\nu \Gamma(\nu+1) \sum_{l=0}^{\infty} \frac{l+\nu}{\nu} (-1)^l J_{l+\nu}\left(\frac{1}{\tau}\right) C_l^\nu(\cos\theta) \tag{7.17}$$

and to use the asymptotical expansion of the Bessel function $J_{l+\nu}(1/\tau)$ at large values of its argument and index $(l+\nu) \sim \theta^{-1} \sim \tau^{-1/2} \gg 1$; see the book [37], Section 8.453. As usual, we assume that at small τ Green's function is defined as a limit from the lower half-plane of the complex τ, i.e. $\mathrm{Im}\,\tau < 0$. Comparing the obtained series with (7.8) we establish that

$$\begin{gathered}\exp\left(-\frac{i}{\tau}\cos\theta\right) \simeq V_n^{-1}(2\pi i\tau)^{n/2}\left(1 - \frac{i\tau}{8}n(n-2)\right)\cdot(1+\hat{O})\,K(\theta;\delta t) \\ \hat{O} = \frac{i}{2}\tau^2\frac{\partial}{\partial\tau} + \frac{i}{6}\tau^3\frac{\partial^2}{\partial\tau^2} \sim \tau.\end{gathered} \tag{7.18}$$

Applying the operator $(1 - \hat{O})$ to this idendity and calculating the derivatives we obtain (7.16).

Remark that Eq. (7.16) may be used for an independent deduction of Eq. (7.14). Actually, expanding (7.16) near the saddle points one gets

$$\begin{aligned}K(\theta;\delta t) \simeq{}& V_n(2\pi i\tau)^{-n/2}\left(1 + \frac{i\tau}{8}n(n-2)\right) \\ &\times \sum_{k=-\infty}^{\infty}\left(1 + \frac{i\tau}{24}n(n-4) - \frac{1}{12}(n-1)\,\theta_k^2 - \frac{i}{12\tau}\theta_k^4\right)\exp\frac{i}{2\tau}\theta_k^2\end{aligned} \tag{7.19}$$

where $\theta_k = \theta + 2\pi k$. With account of the exponent and of the measure (7.4), one may put within the prescribed accuracy

$$\theta_k^2 = i\tau n, \qquad \theta_k^4 = (i\tau)^2 n(n+2) \tag{7.20}$$

and one gets Eq. (7.14) once more.

To embed S^n into the Euclidean space, rewrite the integral with the measure (7.4) as follows

$$\int V_n\,d\omega = \int d^{n+1}x\,d\lambda\,\exp\frac{i}{2}\lambda(\boldsymbol{x}^2 - 1). \tag{7.21}$$

Now note that $(1 - \cos\theta) = 1/2\,(\boldsymbol{x}_t - \boldsymbol{x}_0)^2$ and then Eq. (7.16) leads to

$$\begin{aligned}K(\theta;\delta t) \simeq{}& V_n(2\pi i\tau)^{-n/2}\left[1 + \frac{i\tau}{12}n(2n-5) - \frac{1}{12}(n-1)(\Delta\boldsymbol{x})^2 - \frac{i}{24\tau}(\Delta\boldsymbol{x})^4\right] \\ &\times \exp\frac{i}{2\tau}(\Delta\boldsymbol{x})^2\end{aligned} \tag{7.22}$$

where $\Delta\boldsymbol{x} = \boldsymbol{x}_t - \boldsymbol{x}_0$, $\boldsymbol{x}_t^2 = \boldsymbol{x}_0^2 = 1$. With account of the exponent and of the Euclidean measure, one can substitute up to higher terms in τ

$$(\Delta\boldsymbol{x})^2 = i\tau(n+1); \qquad (\Delta\boldsymbol{x})^4 = (i\tau)^2(n+1)(n+3). \tag{7.23}$$

The resulting asymptotic expansion is

$$K(\theta;\delta t) \simeq \frac{V_n}{(2\pi i\tau)^{n/2}}\left[1+\frac{i\tau}{8}(n-1)^2+\frac{i\tau}{12}\right]\exp\frac{i}{2\tau}(\boldsymbol{x}_t-\boldsymbol{x}_0)^2 . \tag{7.24}$$

As usual, Green's function at finite times is constructed integrating a product of N small-time functions. The intermediate integrals are now calculated with the measure (7.21). The result is (cf. Eq. (2.11)):

$$\mathcal{K}(\omega_t,\omega_0;t) = \int \mathcal{D}x\tilde{\mathcal{D}}\lambda \exp\frac{i}{\hbar}\int_0^t \mathcal{L}_{\text{eff}}\,d\tau \tag{7.25}$$

where

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\dot{\boldsymbol{x}}^2+\frac{1}{2}\lambda(\boldsymbol{x}^2-1)+\hbar^2\left[\frac{(n-1)^2}{8}+\frac{1}{12}\right]$$

$$\mathcal{D}\boldsymbol{x} = V_n(2\pi i\tau)^{-(n+1)/2}\prod_{j=1}^{N-1}\left[(2\pi i\tau)^{-(n+1)/2}\,d^{n+1}\boldsymbol{x}_j\right] \tag{7.26}$$

$$\tilde{\mathcal{D}}\lambda = (2\pi i\tau)^{1/2}\prod_{j=1}^{N-1}(2\pi i\tau)^{1/2}\,d\lambda_j .$$

The integrals in $\boldsymbol{x}$ and λ are for infinite space, the paths are piecewise linear, as in the conventional Feymann integral. The second term in L_{eff} is interpreted as the Lagrange constraint, λ the indefinite multiplier, and the third term is a "quantal potential" compensating the contribution of fluctuations deviating the point from the sphere.
Green's function for S^n may also be constructed using the coset space representation: $S^n = SO(n+1)/SO(n)$. According to Eq. (1.8), the result is obtained integrating Green's function for $SO(n+1)$ over the subgroup $SO(n)$. For the case S^2 one has to integrate over $U(1)$ because $S^2 = SO(3)/SO(2) \sim SU(2)/U(1)$. A detailed discussion of this approach is outside the framework of the present paper.

8. Conclusion

We have obtained various representations of Green's functions for compact groups and for spheres in form of the path integrals. The main results are given in Eqs. (2.7) and (2.11) for the $U(1)$ group, Eq. (6.3) for the simple Lie algebra, Eq. (6.7) for the compact Lie group, Eqs. (7.15) and (7.25) for the sphere of arbitrary dimensionality. These examples show that the general method by De Witt [3] developed for noncompact Riemannian spaces, is quite appropriate for compact manifolds. Still one has to supplement the integration over the intermediate points of the paths with summing up the sets of classical trajectories connecting the consecutive points.
Another path-integral representation is also valid for the groups. The Green's function is written as a sum of the Feynman-De Witt path integrals for an infinite space (the Lie algebra) over the equivalent for the group initial (or final) points of the paths. There is another way to write the path integral for spheres also. It results from embedding S^n into the $(n+1)$-dimensional Euclidean space where the motion is mastered by the Lagrange constraint.
In all the considered cases quantal corrections arise in the path integrals in addition to the classical Lagrangian.

Classical and quantal dynamics is also discussed in some detail for groups and spheres. For the free motion on a group, as in case of the Euclidean spaces, the semiclassical approximation provides with an exact description of the quantal dynamics (cf. (1.6) and (5.9)). In general, this is not true for spheres.

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Appendix

A. Some Results of the Theory of Lie Algebras and Lie Groups

A.1. The Lie algebras

The multiplication law (commutator) in a Lie algebra $\mathfrak{A}$ is denoted by the bracket $[\hat{x}, \hat{y}] = -[\hat{y}, \hat{x}]$, where $\hat{x}$ and $\hat{y}$ are elements of $\mathfrak{A}$. Any element may be decomposed over a set of basis elements $\hat{X}_a$, $a = 1, \ldots, n$:

$$\hat{x} = \xi^a \hat{X}_a, \qquad [\hat{X}_a, \hat{X}_b] = C^c_{ab} \hat{X}_c. \tag{A.1}$$

Here n is the dimensionality of the algebra $\mathfrak{A}$, C^c_{ab} are the structure constants, ξ^a are real or complex numbers. The Cartan criterion states that the Lie algebra is semisimple if the Killing matrix

$$G_{ab} = -C^c_{ad} C^d_{bc} \tag{A.2}$$

is not singular. The symmetrical Killing tensor G_{ab} and its inverse G^{ab}, $G_{ac}G^{bc} = \delta_a{}^b$, enable one to set up the Euclidean space and to consider cogradient and contragradient tensor components. The Jacobi identity

$$\big[\hat{x}[\hat{y}, \hat{z}]\big] + \big[\hat{z}[\hat{x}, \hat{y}]\big] + \big[\hat{y}[\hat{z}, \hat{x}]\big] = 0$$

for any elements $\hat{x}$, $\hat{y}$, $\hat{z}$ results in the total antisymmetry of the structure tensor with 3 lower indeces

$$C^d_{ab} G_{cd} \equiv C_{abc} = -C_{acb} = -C_{bac}. \tag{A.3}$$

In a semisimple Lie algebra a *canonical basis* $\hat{X}_a \to (\hat{H}_j, \hat{E}_\alpha)$ may be chosen in such a way that the set of the commutators for the basis elements acquires the most simple form

$$\begin{aligned} [\hat{H}_j, \hat{H}_k] &= 0; \qquad [\hat{H}_j, \hat{E}_\alpha] = i\alpha_j \hat{E}_\alpha \\ [\hat{E}_\alpha, \hat{E}_{-\alpha}] &= i\tilde{\alpha}^j \hat{H}_j; \qquad [\hat{E}_\alpha, \hat{E}_\beta] = N(\alpha, \beta)\hat{E}_{\alpha+\beta}. \end{aligned} \tag{A.4}$$

Here $1 \leqq j, k \leqq r$, where r is the rank of the algebra, i.e. the number of the commuting basis elements. The commuting elements of the Lie algebra form a subalgebra, named the Cartan subalgebra. At linear transformations of the subalgebra the numbers α_j behave as the components of a vector. This r-dimensional vector $\boldsymbol{\alpha}$ is named *the root* of the algebra. In view of the Jacobi identity, the numbers $\tilde{\alpha}^j$ are related to α_j (see further Eq. (A.6)). The number function $N(\alpha, \beta)$ does not vanish only if α, β, and $\alpha + \beta$ are roots of the algebra; the explicit expression may be found in the book by Gilmore [26], p. 280.

The introduction of the canonical basis is, in fact, a reduction of the system of $\underset{\sim}{C}_j$ $n \times n$-matrices ($j = 1, \ldots, r$) with elements equal to the structure constants C^a_{jb}, to a diagonal form. If there is a basis for which all the structure constants are real (in this case the Lie

algebra is named real), the matrices C_j are skew-Hermitean, the eigen-values are pure imaginary, and the root vectors are real. If α is a root, $-\alpha$ is also a root, so that the total number of roots is even, $n - r = 2p$. The canonical basis of a real algebra is reduced to a real form by means of the linear transformation $\hat{E}_{\pm\alpha} = \hat{J}_\alpha \mp i\hat{K}_\alpha$.
For the canonical basis the Killing tensor is separated into two sectors

$$G_{jk} = \sum_\alpha \alpha_j\alpha_k; \qquad G_{\alpha\beta} = G_\alpha\delta(\alpha + \beta); \qquad G_{\alpha j} = G_{j\alpha} = 0$$
$$G_\alpha = 2(\alpha_j\tilde{\alpha}^j) + \sum_{\alpha'} N(\alpha, \alpha')\, N(\alpha + \alpha', -\alpha) \tag{A.5}$$

while $\tilde{\alpha}^j = G_\alpha G^{jk}\alpha_k$. The canonical commutators are covariant under linear transformations of the Cartan subalgebra, as well as under gauge transformations of the elements $\hat{E}_\alpha$: $\hat{E}_\alpha \to \lambda_\alpha\hat{E}_\alpha$ where λ_α is a number. Using this freedom, one may reduce the Killing tensor to isotropic form

$$G_{jk} = \Lambda\delta_{jk}, \qquad G_\alpha = \Lambda, \qquad \tilde{\alpha}^j = \alpha_j \tag{A.6}$$

where $\Lambda = r^{-1}\sum \alpha^2$ is a number, not fixed yet and determined by a common scale of the root vectors, i.e. in essence by the general normalization of the basis elements of the Lie algebra. The root space has an Euclidean metrics given by Eq. (A.6).
For any root α there is reflection in the orthogonal hyperplane

$$\sigma_\alpha\alpha = -\alpha; \qquad \sigma_\alpha \equiv \sigma_{-\alpha}; \qquad \sigma_\alpha{}^2 = 1. \tag{A.7}$$

It is proven (see e.g. the lecture notes by Racah [27]), that if β is a root then $\sigma_\alpha\beta$ is also a root, so that σ_α transforms the root system into itself. All the reflections and all their products form a finite group W, the *Weyl group*. The elements of W are permutations in the root system, so the order of the Weyl group is a divisor of the number $(n-r)!$. The orders of the Weyl group for all simple groups are calculated in a paper by Cartan [22]. All the nonzero vectors in the root space may be divided between two groups: "positive" and "negative"; say, in accordance with the sign of the first nonvanishing component in a fixed basis. If a root α is positive its partner $-\alpha$ is negative. The highest root is that root α^1, for which $\alpha^1 - \alpha^\mu > 0$ for any $\mu = 2, \ldots, 2p$.

A.2. Matrix representations

Elements of an abstract Lie algebra are represented by matrices, $\hat{x} \to X$, for which the multiplication in the algebra is just the commutator. If the structure constants are real, there are equivalent commutators of Hermitean conjugated basis matrices:

$$[X_a{}^+, X_b{}^+] = -C^c_{ab}X_c{}^+; \qquad -X_a{}^+ = \Gamma X_a\Gamma^{-1} \tag{A.8}$$

where Γ is a nonsingular matrix. In particular, there are real representations, for which X_a are skew-Hermitean and $\Gamma = 1$.
Any matrix representation is determined, up to an isomorphism, by the eigenvalues of those representing the Cartan basis H_j. If they all are in diagonal form, one may write

$$(H_j)_{\mu\nu} = iw_j{}^{(\mu)}\delta_{\mu\nu}; \qquad \mu, \nu = 1, \ldots d \tag{A.9}$$

where d is the dimensionality of the representation, and $w_j{}^{(\mu)}$ are components of the *weight vectors* $\boldsymbol{w}^{(\mu)}$. The weight vectors belong to the same r-dimensional Euclidean space, as the roots, while any difference $\boldsymbol{w}^{(\mu)} - \boldsymbol{w}^{(\nu)}$ is a linear combination of the roots with integer coefficients. Just as the root system, the weight system for any representation has the symmetry with respect to the Weyl group. If $\boldsymbol{w}$ is a weigth, $\sigma_\alpha \boldsymbol{w} = \boldsymbol{w} - 2(\boldsymbol{w}\boldsymbol{\alpha})\,\boldsymbol{\alpha}/\alpha^2$ is also a weight in the same representation for any $\boldsymbol{\alpha}$. Therefore the number $2(\boldsymbol{w}\boldsymbol{\alpha})/\alpha^2$ is integer for any $\boldsymbol{\alpha}$ and $\boldsymbol{w}$. An ordering in the root space generates the ordering in the weight system; usually one assumes that $\mu < \nu$ if $\boldsymbol{w}^{(\mu)} - \boldsymbol{w}^{(\nu)} > 0$. A representation is totally determined by its *highest weight* $\boldsymbol{l} = \boldsymbol{w}^{(1)}$. Other weights are obtained from $\boldsymbol{l}$ by subtracting some combinations of positive roots. The number $2(\boldsymbol{l}\boldsymbol{\alpha})/\alpha^2$ is a non-negative integer for any positive root $\boldsymbol{\alpha} > 0$.

Let $\psi^{(\boldsymbol{w})}$ be a normalized eigen-vector of the matrices H_j, corresponding to the eigenvalues iw_j. A consequence of the commutators in Eqs. (A.4) is

$$E_\alpha \psi^{(\boldsymbol{w})} = \varepsilon \psi^{(\boldsymbol{w}+\boldsymbol{\alpha})} \tag{A.10}$$

where ε is a number function. If the vector $\boldsymbol{w} + \boldsymbol{\alpha}$ is not a weight in the system for the representation in view, then $\varepsilon(\boldsymbol{w}, \boldsymbol{\alpha}) = 0$. By definition of the highest weight, $E_\alpha \psi^{(\boldsymbol{l})} = 0$ for any $\boldsymbol{\alpha} > 0$. Any d-dimensional vector $\psi^{(\boldsymbol{w})}$ is obtained from $\psi^{(\boldsymbol{l})}$ by the action of a product of matrices $E_{-\alpha}$ with $\boldsymbol{\alpha} > 0$.

The invariant operators for the algebra are constructed by means of a convolution of a product of basis elements with the metric tensor (A.2). The invariant operators are represented by matrices, that commute with any elements of the representation; so for an irreducible representation any such matrix is unity multiplied by a factor (the eigenvalue). The simplest and the most interesting is the second-order *Casimir operator*

$$C_2 = -G^{ab} X_a X_b = -\Big(\sum_j H_j H_j + \sum_\alpha E_\alpha E_{-\alpha}\Big)\,\Lambda^{-1}. \tag{A.11}$$

Its eigen-value λ for a given representation, $C_2 = \lambda \cdot I$, as well as the dimensionality of the presentation is written in terms of the highest weight $\boldsymbol{l}$:

$$\begin{aligned} \Lambda\,\lambda(\boldsymbol{l}) &= l^2 + \sum_{\alpha>0} (\boldsymbol{\alpha}\boldsymbol{l}) = L^2 - \varrho^2 \\ d_l &= \prod_{\alpha>0} \frac{(\boldsymbol{\alpha}\boldsymbol{L})}{(\boldsymbol{\alpha}\boldsymbol{\varrho})} \end{aligned} \tag{A.12}$$

where $\boldsymbol{\varrho} = 1/2 \sum_{\alpha>0} \boldsymbol{\alpha}$, $\boldsymbol{L} = \boldsymbol{l} + \boldsymbol{\varrho}$. This expression for $\lambda(\boldsymbol{l})$ may be rather easily obtained in a pure algebraic way (see e.g. RACAH [27]). The result for the dimensionality d_l was obtained by Weyl from the formula for the representation character. The vector $\boldsymbol{\varrho}$ is involved also in some other formulas of the representation theory. A direct calculation shows, by the way, that for any simple Lie algebra $\varrho^2/\Lambda = n/24$. An immediate consequence of (A.11) is that for the regular representation (see the next Section) $\lambda = 1$. Note that for any irreducible matrix representation

$$\operatorname{Sp} X_a = 0, \qquad \operatorname{Sp} X_a X_b = -G_{ab}\,\frac{\lambda d}{n}. \tag{A.13}$$

A.3. The regular representation

Consider n matrices with elements equal to the structure constants

$$(\underset{\sim}{C}_a)_b{}^c \equiv C^c_{ab}\,. \tag{A.14}$$

In view of the Jacobi identity, they satisfy the commutation relations (A.1)

$$[\underset{\sim}{C}_a, \underset{\sim}{C}_b] = C^c_{ab}\underset{\sim}{C}_c. \tag{A.15}$$

Thus the structure constants provide with a matrix representation $\hat{X}_a \to \underset{\sim}{C}_a$, that is named the *regular representation*. Its dimensionality equals to that of the algebra. If the structure constants are real, the matrices $\underset{\sim}{C}_a$ are skew-Hermitean in the sense of the Killing metrics:

$$(\underset{\sim}{C}_a{}^+)_b{}^c = -G^{bd}(\underset{\sim}{C}_a)_d{}^e G_{ce}. \tag{A.16}$$

The set of the weight vectors for the regular representation is quite clear in the canonical basis. The index μ has n values: $1, \ldots, r, \alpha^{(1)} \ldots \alpha^{(k)}, \ldots, \alpha^{(p)}, -\alpha^{(p)} \ldots -\alpha^{(k)} \ldots -\alpha^{(1)}$ while $\boldsymbol{w}^{(j)} = 0$ for the Cartan subspace, and $\boldsymbol{w}^{(\alpha)} = \alpha$ for any α. The highest weight is the highest root. The structure constants are imaginary in the canonical basis; it is easily seen that

$$\underset{\sim}{C}_j{}^+ = -C_j; \qquad \underset{\sim}{C}_\alpha{}^+ = -\underset{\sim}{C}_{-\alpha}; \qquad (\underset{\sim}{C}_j)_k{}^l = 0; \qquad (\underset{\sim}{C}_j)_\alpha{}^\alpha = i\alpha_j. \tag{A.17}$$

Other matrix elements of $\underset{\sim}{C}_j$ are zeros.
The structure constants behave as components of a rank-3 tensor at arbitrary linear transformations of the Lie algebra. Among the transformations there are those conserving the structure constants. Such transformations (though not all of them) may be found by means of the regular representation of the algebra. Define a $n \times n$-matrix

$$A(\eta) = \exp\,(\eta^a \underset{\sim}{C}_a) \tag{A.18}$$

where η^a are numbers. Then

$$A^{-1}(\eta)\, \underset{\sim}{C}_a\, A(\eta) = A_k{}^b(\eta)\, \underset{\sim}{C}_b \tag{A.19}$$

where $A_a{}^b(\eta)$ are matrix elements of $A(\eta)$. To prove this relation, it is sufficient to expand l.h.s. as a series over multiple commutators and to use then Eq. (A.15). Eq. (A.19) may also be written in the following form

$$A_a{}^p(\eta)\, A_b{}^q(\eta)\, C^r_{pq} A_r{}^c(-\eta) = C^c_{ab}. \tag{A.20}$$

Thus the structure constants are invariant under the linear transformations of the basis

$$\hat{X}_a \to A_a{}^b(\eta)\, \hat{X}_b. \tag{A.21}$$

The decomposition of the Lie algebra elements over the real basis, $\hat{x} = \xi^a \hat{X}_a$, $-\infty < \xi^a < \infty$, introduces a Cartesian-type coordinate system. The algebra is considered as a linear space, where a group of linear transformations, $\xi^a \to A_b{}^a(\eta)\, \xi^b$, acts, called the *adjoint group*. The antisymmetric "triple products" $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\} = \xi_1{}^a \xi_2{}^b \xi_3{}^c\, C_{abc}$, are invariants of the ajoint group. The Lie group is also the Euclidean space, the binary scalar product is defined by means of the Killing tensor, $(\hat{x}_1, \hat{x}_2) = \xi_1{}^a \xi_2{}^b G_{ab}$.
Instead of the Cartesian coordinates one may also define a generalization of the polar coordinates. To introduce the *polar coordinates*, the matrix of the regular representation is reduced to a diagonal form

$$\xi^a \underset{\sim}{C}_a = A^{-1}(\omega)\, (\varphi^j H_j)\, A(\omega). \tag{A.22a}$$

In other terms, the variables ξ^a and (φ, ω) are related by the equation

$$\xi^a = A_j{}^a(\omega)\, \varphi^j. \tag{A.22b}$$

The radial coordinates $\boldsymbol{\varphi}$ determine the length of the vector $\hat{x}$, as

$$(\hat{x}, \hat{x}) = G_{ab} \xi^a \xi^b = \Lambda \boldsymbol{\varphi}^2. \tag{A.23}$$

The angular parameters ω are defined by Eq. (A.22) with an ambiguity: the result is the same if one substitutes $A(\omega) \to A(\omega_1) A(\omega)$, where the matrix $A(\omega_1)$ commutes with $\boldsymbol{\varphi H}$. To fix the choice, one may assume for instance, that $A(\omega) = \exp\left(\sum_\alpha \omega_\alpha E_\alpha\right)$, $\omega_\alpha^* = \omega_{-\alpha}$. The Weyl reflections are elements of the ajoint group and they permute the diagonal elements of $\boldsymbol{\varphi H}$. To avoid this sort of ambiguity, some additional restrictions are usually applied to the parameters $\boldsymbol{\varphi}$:

$$(\boldsymbol{\alpha\varphi}) \geqq 0, \qquad \boldsymbol{\alpha} > 0. \tag{A.24}$$

This domain in the space of radial coordinates $\boldsymbol{\varphi}$ is named the *Weyl chamber*. The Euclidean space of $\boldsymbol{\varphi}$ is divided by the hyperplanes $(\boldsymbol{\alpha\varphi}) = 0$ into N_W regions congruent to the Weyl chamber (N_W is the order of the Weyl group). The Weyl transformations permute these regions

A.4. Compact groups

Elements of a continuous group G, corresponding to the Lie algebra $\mathfrak{A}$ are constructed by means of the formal exponential, that acquires the literal meaning for a matrix representation

$$\hat{g}(\xi) = \exp(\xi^a \hat{X}_a) \to D(\hat{g}) = \exp(\xi^a X_a). \tag{A.25}$$

Here $\hat{g}$ is an element of the abstract group G, ξ^a are the group parameters, X_a is the matrix representation of the basis element of the algebra $\mathfrak{A}$, and $D(\hat{g})$ is the matrix representation of $\hat{g}$. We assume that the Lie algebra is real, then the Lie group is real, i.e. ξ^a are real numbers at $X_a^+ = -X_a$, while $D(\hat{g})$ is unitary. It is assumed also that the Killing tensor is positive definite. In this case the exp-operation is a mapping of the Lie algebra on the *compact group*; i.e. the group manifold is compact.
The transformation $\hat{g} \to \hat{g}_1^{-1} \hat{g} \hat{g}_1$, where $\hat{g}_1$ is a fixed element, is an automorphism of the group. A linear transformation from the ajoint group acts in the space of the group parameters

$$\hat{g}^{-1}(\eta) \exp(\xi^a \hat{X}_a)\, \hat{g}(\eta) = \exp\left(\xi^b A_b{}^a(\eta)\, \hat{X}_a\right). \tag{A.26}$$

Applying such a transformation, any element of the group may be represented as follows

$$\hat{g} = \hat{v}^{-1} \hat{h}(\boldsymbol{\varphi})\, \hat{v}, \qquad \hat{h}(\boldsymbol{\varphi}) = \exp \varphi^j H_j. \tag{A.27}$$

(Cf. the analogous Eq. (A.22) for the Lie algebra, here $\hat{v} = \hat{g}(\omega)$). At fixed $\hat{v}$, the elements (A.27) form an Abelian subgroup. Evidently, the group elements depend on $\boldsymbol{\varphi}$ periodically. Actually, for any representation $2(\boldsymbol{\alpha w})/\alpha^2$ are integers, so $\exp(4\pi \boldsymbol{\alpha H}/\alpha^2)$ is the unit element. Thus

$$\hat{h}(\boldsymbol{\varphi} + 2\pi\tilde{\boldsymbol{\alpha}}) = \hat{h}(\boldsymbol{\varphi}), \qquad \tilde{\boldsymbol{\alpha}} = 2\boldsymbol{\alpha}/\alpha^2 \tag{A.28}$$

and the manifold of parameters of the Abelian subgroup is manifestly compact. All such subgroups are isomorphic, and the structure is named the *maximal torus* of the Lie group. The vectors $\tilde{\boldsymbol{\alpha}}$ are periods on the torus. More details are given in Appendix C.
The matrix representations of a group are identified, just as those for the Lie algebra, by means of the highest weight vector $\boldsymbol{l}$. For any positive root $\boldsymbol{\alpha}$ number $2(\boldsymbol{l\alpha})/\alpha^2$ is a non-negative integer. A group representation is determined by a set of r non-negative integers $l^j = 2(\boldsymbol{l}\gamma^j)/(\gamma^j)^2$, where γ^j is a simple root (see Appendix C). Any such set $(l^1, l^2, \ldots l^r)$ corresponds to a unitary representation.

A.5. Integration on a group

Integration on a group is introduced by means of an invariant measure: $d\hat{g} = d(\hat{g}_1\hat{g})$, where $\hat{g}_1$ is an arbitrary group element. The invariance determines the dependence of the measure on the radial parameters (φ in Eq. (A.27)). This result is due to WEYL (see [24] and [25])

$$d\hat{g} = C_W\,|J(\boldsymbol{\varphi})|^2\, d^r\boldsymbol{\varphi}\; d\hat{v}$$

$$J(\boldsymbol{\varphi}) = \sum_{\sigma} \varepsilon_\sigma \exp i(\sigma\varrho, \boldsymbol{\varphi}) = \prod_{\alpha>0} 2i \sin(\alpha\boldsymbol{\varphi}/2). \tag{A.29}$$

Here C_W is a constant, $d^r\boldsymbol{\varphi} = \prod_{j=1}^{r} d\varphi^j$, $d\hat{v}$ is an element of the $(n - r)$-dimensional volume in space of the angular variables. The sum in Eq. (A.29) is over the elements of the Weyl group, $\varepsilon_\sigma = \det\sigma = \pm 1$. The constant C_W is fixed by the requirement that the volume of the group manifold equals to 1. Asuming that the integral over the angular variables is also normalized to 1, one finds C_W:

$$\int_G d\hat{g} = 1 \to C_W^{-1} = \int |J(\boldsymbol{\varphi})|^2\, d^r\boldsymbol{\varphi} = N_W V_0. \tag{A.30}$$

Here N_W is the order of the Weyl group, V_0 is the volume of the maximal torus. The domain of the $\boldsymbol{\varphi}$ variables and, respectively, V_0 is discussed in Appendix C.
The invariant integral may be also defined, if one considers the group manifold as the Riemannian space (see Appendix B). The result obtained in that way coincides with (A.29).
Any smooth function on the group may be expanded over the unitary representations, and a sort of the Fourier series arises:

$$F(\hat{g}) = \sum_{l} d_l\, \mathrm{Sp}\,[F^{(l)} D^{(l)}(\hat{g})], \qquad F^{(l)} = \int F(\hat{g})\, D^{(l)\dagger}(\hat{g})\, d\hat{g}. \tag{A.31}$$

Here d_l is the dimensionality of the representation with the highest weight $\boldsymbol{l}$, $F^{(l)}$ is a d_l-matrix. The representation matrices form an orthonormal system

$$\int D_\nu^{(l)\mu}(\hat{g})\, \overset{*}{D}{}_{\nu'}^{(l')\mu'}(\hat{g})\, d\hat{g} = \delta(\nu, \nu')\, \delta(\mu, \mu')\, \delta(\boldsymbol{l}, \boldsymbol{l}')\, d_l^{-1} \tag{A.32}$$

the indices $\mu, \nu = 1, \ldots, d_l$ numerate the rows and the columns of the matrix.
Functions, constant on classes of equivalent elements, $F(\hat{g}_1^{-1}\hat{g}\hat{g}_1) = F(\hat{g})$ for any $\hat{g}_1$, are of special interest. Such functions are totally determined by their values on the maximal torus. In this case any coefficient matrix $F^{(l)}$ is proportional to the unit matrix, and the series (A.31) is reduced to the following

$$\begin{gathered} F(\hat{g}) \equiv f(\boldsymbol{\varphi}) = \sum_{l} f_l \chi^{(l)}(\boldsymbol{\varphi}); \qquad f_l = \int f(\boldsymbol{\varphi})\, \overset{*}{\chi}{}^{l}(\boldsymbol{\varphi})\, |J(\boldsymbol{\varphi})|^2\, d^r\boldsymbol{\varphi} \\ \chi^l(\boldsymbol{\varphi}) = \mathrm{Sp}\, D^{(l)}(\hat{g}) = \sum_{\mu} \exp i\boldsymbol{w}^{(\mu)}\boldsymbol{\varphi}. \end{gathered} \tag{A.33}$$

The functions $\chi^{(l)}(\boldsymbol{\varphi})$, the *representation characters*, form a system on the torus, orthonormal with respect to the Weyl measure:

$$\begin{gathered} C_W \int \chi^{(l)}(\boldsymbol{\varphi})\, \overset{*}{\chi}{}^{(l')}(\boldsymbol{\varphi})\, |J(\boldsymbol{\varphi})|^2\, d^r\boldsymbol{\varphi} = \delta(\boldsymbol{l}, \boldsymbol{l}') \\ C_W \sum_{l} \chi^{(l)}(\boldsymbol{\varphi}_1)\, \overset{*}{\chi}{}^{(l)}(\boldsymbol{\varphi}_2) = |J(\boldsymbol{\varphi}_1)|^{-2}\, \delta(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2) \end{gathered} \tag{A.34}$$

where $\delta(\varphi_1, \varphi_2)$ is the usual δ-function on the torus. The orthogonality of the characters is quite clear with the remarkable Weyl formula for the character [24]:

$$\chi^{(l)}(\varphi) = [J(\varphi)]^{-1} \sum_\sigma \varepsilon_\sigma \exp [i(\sigma L, \varphi)]$$
$$L = l + \varrho. \tag{A.35}$$

A direct consequence of the definition of the character is the expression for the representation dimensionality, $d_l = \lim\limits_{\varphi \to 0} \chi^{(l)}(\varphi)$. The characters are eigen-functions of a differential operator on the torus, cf. Eq. (B.20).

A.6. Unitary groups

All skew-Hermitean matrices of a given rank N with zero trace form a Lie algebra with the commutation as the binary operation. This is an algebra of generators of the SU(N) group, the group of unitary $N \times N$ matrices with unit determinant. The dimensionality is $n = N^2 - 1$, the rank $r = N - 1$, the number of positive roots $p = N(N-1)/2$. We illustrate some general formulae for this class of the simple Lie algebras.
Consider a total orthonornal system of N-dimensional Euclidean vectors

$$\{\boldsymbol{v}^{(\mu)}\}, \qquad \mu = 1, 2, \dots N; \; (\boldsymbol{v}^{(\mu)}\boldsymbol{v}^{(\nu)}) = \delta_{\mu\nu}.$$

Take the following basis of the Cartan subalgebra

$$(H_j)_{\mu\nu} = i v_\mu{}^{(j)} \delta_{\mu\nu}; \qquad j = 1, 2 \dots N-1; \qquad \nu, \mu = 1, 2, \dots N. \tag{A.36}$$

Let $\boldsymbol{v}^{(N)}$ be a vector with identical components, $v_\mu{}^{(N)} = N^{-1/2}$, then Sp $H_j = (\boldsymbol{v}^{(j)}\boldsymbol{v}^{(N)}) = 0$. Note that due to the completeness

$$\sum_{j=1}^{N-1} v_\mu{}^{(j)} v_\nu{}^{(j)} = \delta_{\mu\nu} - v_\mu{}^{(N)} v_\nu{}^{(N)} = \delta_{\mu\nu} - 1/N. \tag{A.37}$$

Let any matrix E_α has only one nonzero element off the diagonal $(E_\alpha)_{\mu\nu} = i$, $\mu \neq \nu$. Then the roots are numerated by the pairs of the numbers μ, ν:

$$\alpha^{(\mu\nu)} = -\alpha^{(\nu\mu)}, \qquad E_{-\alpha} = -(E_\alpha)^+, \qquad \alpha^{(\mu\nu)} > 0$$

if $\mu < \nu$. Substituting the matrices into the commutators (A.4), one obtains the roots

$$\alpha_j{}^{(\mu\nu)} = v_\mu{}^{(j)} - v_\nu{}^{(j)}, \qquad \alpha^2 = 2. \tag{A.38}$$

The Weyl reflection σ_α (A.7) for a root $\alpha^{(\mu\nu)}$ is an exchange of $v_\mu{}^{(j)}$ and $v_\nu{}^{(j)}$ for any j. Therefore the Weyl group consists of all the simultaneous permutations of the components of the vectors $\boldsymbol{v}^{(j)}$ and is of the order $N_W = N!$
The described N-dimensional representation of the Lie algebra is named the *fundamental representation*. The weights are just the components of $\boldsymbol{v}^{(j)}$: $\hat{w}_j{}^{(\mu)} = v_\mu{}^{(j)}$. The eigen-value of the second-order Casimir operator is $\lambda = (1 - N^{-2})/2$. By means of (A.37), it is easily seen that

$$\varrho_j = \frac{1}{2} \sum_{\mu=1}^{N} (N - 2\mu + 1)\, v_\mu{}^{(j)}, \qquad \Lambda = 2N, \qquad \varrho^2 = \frac{N(N^2-1)}{12}. \tag{A.39}$$

For the ajoint representation the highest weight is $\boldsymbol{l} = \boldsymbol{\alpha}^{(1N)}$, $l^2 = 2$, and $\lambda = 1$. For an arbitrary representation of $SU(N)$ the weights may be decomposed over the system of N weights of the fundamental representation $\mathring{\boldsymbol{w}}$: $\boldsymbol{w} = \sum_{\mu=1}^{N} \omega_\mu \mathring{\boldsymbol{v}}^{(\mu)}$, while any number $2(\boldsymbol{w}\boldsymbol{\alpha})/\alpha^2 = \omega_\mu - \omega_\nu$ is an integer. (Note that the weights are $(N-1)$-dimensional, and the system is overcomplete). For the highest weight $\boldsymbol{l}$ these integers are non-negative at $\mu < \nu$, and $\boldsymbol{l}$ is determined by $N - 1$ numbers

$$m_k = \omega_k - \omega_N, \qquad k = 1, 2, \dots N-1, \qquad m_1 \geqq m_2 \geqq \cdots \geqq m_{N-1} \geqq 0$$

For the fundamental representation $\{m\} = (1, 0, \dots, 0)$, for the ajoint one $\{m\} = (2, 1, \dots, 1)$. Calculating with Eq. (A.12) one obtains

$$\begin{gathered} 2N\lambda(l) = \sum_{k=1}^{N-1} [m_k{}^2 + (N - 2k + 1)\, m_k] - N^{-1}[\textstyle\sum m_k]^2 \\ d_l = \prod_{\mu<\nu}^{N} \frac{\omega_\mu - \omega_\nu + \nu - \mu}{\nu - \mu}. \end{gathered} \tag{A.40}$$

Constructing a representation by means of irreducible tensors in the complex N-dimensional space, one identifies m_k as a number of cells in the k-th line of the corresponding Young diagram. The representation may be also described with components of the highest weight ω_μ with a supplementary condition $\sum \omega_\mu = 0$.

Consider now two most frequently mentioned cases: the Lie algebras of $SU(2)$ and $SU(3)$.

i) $SU(2)$. The root space is a line, the roots $\alpha^{(12)} = -\alpha^{(21)} = \sqrt{2}$, $\varrho = 1/\sqrt{2}$, $v^{(1,2)} = 1/\sqrt{2}$ $(1, \mp 1)$. Let $\boldsymbol{j}$ is the usual angular momentum vector, then

$$H = i\sqrt{2}\, j_3, \qquad E_{\pm\alpha} = i(j_1 \pm j_2), \qquad l = j\sqrt{2}, \qquad \lambda = \frac{1}{2} j(j+1). \tag{A.41}$$

The representation dimensionality is $2j + 1$, the weights $w^{(\mu)} = \mu\sqrt{2}$, where $\mu = j$, $j - 1, \dots, -j$. For the fundamental representation $j = 1/2$, for the regular one $j = 1$.

ii) $SU(3)$. The root space is a plane; the positive roots $\boldsymbol{\alpha}^{(13)} = (\sqrt{2}, 0)$, $\boldsymbol{\alpha}^{(12)} = (1/\sqrt{2}, \sqrt{3/2})$, $\boldsymbol{\alpha}^{(23)} = (1/\sqrt{2}, -\sqrt{3/2})$; $\boldsymbol{\varrho} = \boldsymbol{\alpha}^{(13)}$. The Jacobi system may be choosen for the vectors $\boldsymbol{v}^{(\mu)}$: $\boldsymbol{v}^{(1)} = 1/\sqrt{2}\,(1, -1, 0)$, $\boldsymbol{v}^{(2)} = 1/\sqrt{6}(1, 1, -2)$, $\boldsymbol{v}^{(3)} = 1/\sqrt{3}\,(1, 1, 1)$. The highest weight has the components $\boldsymbol{l} = \big((m_1 - m_2)/\sqrt{2}, (m_1 + m_2)/\sqrt{6}\big)$. The representation may also be identified by a pair of non-negative integers (p, q), while $m_1 = p + q$, $m_2 = q$.

Another description of the $SU(N)$ group is frequently used in the literature. The basis of the Lie algebra is a set of N^2 generators $\hat{X}_{\mu\nu}$ with a supplementary condition $\sum_\mu \hat{X}_{\mu\mu} = 0$ (see e.g. the paper of Racah [27]). In this variant the commutators are

$$[\hat{X}_{\mu\nu}, \hat{X}_{\varrho\sigma}] = \hat{X}_{\mu\sigma}\delta_{\nu\varrho} - \hat{X}_{\varrho\nu}\delta_{\mu\sigma}. \tag{A.42}$$

The fundamental representation is given by

$$(X_{\mu\nu})_{\sigma\varrho} = \delta_{\mu\sigma}\delta_{\nu\varrho}, \qquad (\nu \neq \mu); \qquad (X_{\nu\nu})_{\varrho\sigma} = \left(e_\nu{}^{(\sigma)} - \frac{1}{N}\right)\delta_{\varrho\sigma} \tag{A.43}$$

where $e^{(1)}, e^{(2)}, \dots, e^{(N)}$ is an orthonormal basis in the N-dimensional space, $e_\nu{}^{(\sigma)} = \delta_{\nu\sigma}$. The roots are

$$\bar{\alpha}_\sigma{}^{(\mu\nu)} = -\bar{\alpha}_\sigma{}^{(\nu\mu)} = e_\sigma{}^{(\mu)} - e_\sigma{}^{(\nu)} \tag{A.44}$$

and they are normal to the vector $(e^{(1)} + e^{(2)} + \cdots + e^{(N)})$. By means of a rotation in the N-dimensional space, one may change the basis, so that the last component of any root vanishes

$$(\bar{\alpha}_j^{(\mu\nu)}, \bar{\alpha}_N^{(\mu\nu)}) \to (\alpha_j^{(\mu\nu)}, \alpha_N^{(\mu\nu)} = 0).$$

Eq. (A.38) presents the root components just in such a basis. In this transformation there is still a freedom to rotate without changing the N-th component. This freedom is the ambiguity in the choice of the basis $\boldsymbol{v}^{(\mu)}$ in Eq. (A.36), i.e. the freedom to choose the Cartan basis $\hat{H}_j$.

B. Group as a Riemannian Space

One may realize elements of a Lie algebra as first-order differential operators on a homogeneous space. The group itself is such a homogeneous space, while the operators are

$$\hat{X}_a \to \nabla_a(\hat{g}) = L_a{}^b(\xi)\,\partial_b; \qquad L_a{}^p\,\partial_p L_b{}^c - L_b{}^p\,\partial_p L_a{}^c = C^p_{ab}\,L_p{}^c \tag{B.1}$$

where $\partial_a = \partial/\partial\xi^a$. The operators $\nabla_a(\hat{g})$ generate left shifts on the group if

$$\exp\,[\eta^a\nabla_a]\,\hat{g}(\xi) = \hat{g}(\zeta) = \hat{g}(\eta)\,\hat{g}(\xi), \qquad L_a{}^b(\xi) = \left.\frac{\partial\zeta^b}{\partial\eta^a}\right|_{\eta\to 0} \tag{B.2}$$

where $\zeta^a = Z^a(\eta, \xi)$ is the multiplication law in terms of the coordinates resulting from the product of the exponents (A.18). The function $Z(\eta, \xi)$ is given explicitly by means of the Baker-Campbell-Hausdorff

$$Z^a = \eta^a + \xi^a + \frac{1}{2}\,C^a_{bc}\eta^b\xi^c + \frac{1}{12}\,C^p_{bc}C^a_{pd}\eta^b\xi^c(\xi^d - \eta^d) + \cdots \tag{B.3}$$

Calculate the derivative of $\hat{g}(\zeta)$ in (B.2), using the known integral representation for the matrix exponent

$$\left[\frac{\partial\hat{g}(\zeta)}{\partial\eta^a}\right]\hat{g}^{-1}(\zeta) = \int_0^1 \hat{g}(\tau\zeta)\,\hat{X}_b\,\frac{\partial\zeta^b}{\partial\eta^a}\,\hat{g}^{-1}(\tau\zeta)\,d\tau. \tag{B.4}$$

Hence at $\eta \to 0$, $\zeta \to \xi$, one gets

$$\hat{X}_a = L_a{}^b(\xi)\int_0^1 \hat{g}(\tau\xi)\,\hat{X}_b\hat{g}^{-1}(\tau\xi)\,d\tau$$

and using Eq. (A.26) one obtains the matrix field M, inverse to L,

$$M_a{}^b(\xi) = \int_0^1 A_a{}^b(\tau\xi)\,d\tau; \qquad M_a{}^cL_c{}^b = \delta_a{}^b. \tag{B.5}$$

Introduce the polar coordinates (A.27) and use explicitly the form of the structure constants in the canonical basis, Eq. (A.17). The result is

$$M_a{}^b(\xi) = A_a{}^p(-\omega)\,M_p{}^q(\boldsymbol{\varphi})\,A_q{}^b(\omega)$$

$$M_a{}^b(\boldsymbol{\varphi}) = \int_0^1 A_a{}^b(\tau\boldsymbol{\varphi})\,d\tau$$

$$M_k{}^l(\varphi) = \delta_k{}^l, \qquad M_\alpha{}^\beta(\varphi) = -i\delta(\alpha, \beta)\,\frac{\exp i(\alpha\varphi) - 1}{\alpha\varphi}. \tag{B.6}$$

It is seen, in particular, that

$$\det\left(L_a{}^b(\xi)\right) = [\det M_a{}^b(\varphi)]^{-1} = \prod_{\alpha>0}\left[\frac{\frac{1}{2}\,\alpha\varphi}{\sin\frac{1}{2}\,\alpha\varphi}\right]^2. \tag{B.7}$$

Note also some useful properties of the matrix field $L(\xi)$, resulting from Eqs. (B.6), (A.16), and (A.22a):

$$L_a{}^b(\xi)\,\xi^a = \xi^b, \qquad L_a{}^b(\xi)\,G_{bc}\xi^c = G_{ac}\xi^c \qquad L_a{}^b(-\xi) = A_a{}^c(\xi)\,L_c{}^b(\xi). \tag{B.8}$$

The field $M(\xi)$ determines essentially the geometry of the Lie algebra and of the Lie group. To elucidate its geometrical meaning, consider an infinitesimal shift in (B.2)

$$\eta^a \equiv \delta\xi^a \to 0, \qquad \zeta^a = \xi^a + d\xi^a, \qquad d\xi^a = L_b{}^a\delta\xi^b.$$

The differentials $\delta\xi$ form a tangent linear space at the point ξ of the group manifold. The projection of $d\xi$ on this space is given by the matrix M

$$\delta\xi^a = M_b{}^a(\xi)\,d\xi^b, \qquad \delta\xi^a\nabla_a = d\xi^a\,\partial_a.$$

In view of Eqs. (B.1) the matrix field $M(\xi)$ satisfies a set of differential equations

$$\partial_b M_a{}^c - \partial_a M_b{}^c = M_a{}^p M_b{}^q C^c_{pq}. \tag{B.9}$$

Metrics on the group manifold is induced by the Killing metrics (A.2) on the tangent space

$$\begin{aligned} ds^2 &= G_{ab}\delta\xi^a\delta\xi^b = g_{ab}(\xi)\,d\xi^a\,d\xi^b \\ g_{ab}(\xi) &= M_a{}^p(\xi)\,M_b{}^q(\xi)\,G_{pq}. \end{aligned} \tag{B.10}$$

Thus the group is provided with the properties of the Riemannian space. The field $M(\xi)$ is just the analog to the vierbein field, used in general relativity. The group manifold has a specific property that the field satisfies the equations (B.9).

The geometric characteristics of the group, the Christoffel symbol and the Riemann tensor, are calculated using Eqs. (B.9), while the total antisymmetry of C_{abc}, given in Eq. (A.3), is extremely essential. The result is

$$\begin{aligned} &\Gamma^c_{ab} = \frac{1}{2}\,L_p{}^c(\partial_a M_b{}^p + \partial_b M_a{}^p), \qquad R^a_{bcd} = \frac{1}{4}\,K^a_{bp}K^p_{cd} \\ &K^a_{bc} \equiv L_p{}^a C^p_{qr} M_b{}^q M_c{}^r, \qquad R_{ab} = R^c_{acb} = \frac{1}{4}\,g_{ab} \\ &R = g^{ab}R_{ab} = \frac{n}{4}. \end{aligned} \tag{B.11}$$

A remarkable fact is that the Riemann curvature of the group is a constant determined only by the group dimensionality.

The invariant measure for a group is defined as for any Riemannian space, as follows

$$d\hat{g} = C_R \sqrt{g} \prod_a d\xi^a$$
$$\sqrt{g} = (\det g_{ab})^{1/2} = (\det G)^{1/2} \det M_a{}^b \tag{B.12}$$

where $\det G = \Lambda^n$, C_R is a constant. It may be shown that the Jacobian of the transformation from Cartesian coordinates ξ^a to polar coordinates (φ, ω), Eq. (A.27), is of the form $\prod_{\alpha>0} (\alpha\varphi)^2 f(\omega)$, where $f(\omega)$ is a function of the angular variables. With this in view, taking (B.7) and adjusting the constant C_R in such a way that the group volume be equal to 1, we see that the measure (B.12) coincides with the Weyl measure (A.29). The explicit form of C_R is given in Eq. (5.10).

To be definite we considered the left shifts on the group. It is for the left shift operator that Eq. (B.3) is valid. The right shifts may be considered in the same manner. Writting the right shift as $\hat{g}\hat{g}_1 = (\hat{g}\hat{g}_1\hat{g}^{-1})\,\hat{g}$ and using the property of the ajoint representation, Eq. (A.26), we easily relate the right-shift operator to $\nabla_a(\hat{g})$:

$$\nabla_a{}^{\text{right}}(\hat{g}) = A_a{}^b(-\xi)\,\nabla_b(\hat{g}). \tag{B.13a}$$

The left-shift operators are invariant under right shifts (and vice versa):

$$\nabla(\hat{g}\hat{g}_1) = \nabla_a(\hat{g}), \qquad \nabla_a{}^{\text{right}}(\hat{g}_1\hat{g}) = \nabla_a{}^{\text{right}}(\hat{g}). \tag{B.13b}$$

This important fact is a consequence of that the multiplication law is associative $Z\big(\eta, Z(\xi, \xi_1)\big) = Z\big(Z(\eta, \xi), \xi_1\big)$ Differentiating this identity over η, one gets Eq. (B.13b). Another useful relation is obtained from Eq. (B.8)

$$\nabla_a(\hat{g}^{-1}) = -A_a{}^b(-\xi)\,\nabla_b(\hat{g}). \tag{B.14}$$

One may state that the field $L(\xi)$ transfers the derivative operation from the vicinity of the unit group element ($\xi = 0$) to the element with any ξ. In view of Eq. (B.13b), the action of the left-shift operator on the representation matrix is evident:

$$\nabla_a D(\hat{g}) = X_a D(\hat{g}). \tag{B.15}$$

With this result and using the Fourier expansion on the group, one may obtain basic properties of the ∇_a operator.

The Laplace operator for a group, as for any Riemannian space, is defined as the invariant second-order differential operator. By means of Eqs. (B.10) and (B.11) it is easy to see that it is just the differential realization of the second-order Casimir operator

$$\Delta = G^{ab}\nabla_a\nabla_b = g^{-1/2}\,\partial_a(g^{1/2}g^{ab}\,\partial_b) = \partial_a g^{ab}\,\partial_b + \Gamma_a g^{ab}\,\partial_b \tag{B.16}$$

where

$$\Gamma_a \equiv \Gamma^b_{ab} = \partial_a \ln \sqrt{g}, \qquad g^{ab} = L_p{}^a L_q{}^b G^{pq}.$$

For functions, depending in the radial coordinates only, Eq. (A.33), the Laplace operator is of the form

$$\Delta F(\hat{g}) = \mathring{\Delta} f(\varphi) = \Lambda^{-1}\left[J^{-1}\frac{\partial^2(Jf)}{\partial\varphi^2} + \varrho^2 f\right] \tag{B.17}$$

where the function $J(\varphi)$ is defined in Eq. (A.29). The operator $\mathring{\Delta}$ is the radial part of the Laplace operator, it was analysed by BEREZIN [*30*]. Note once more that for any simple group

$$\varrho^2/A = n/24 = R/6. \tag{B.18}$$

This term in $\mathring{\Delta}$ appears as a consequence of the requirement $\mathring{\Delta} \cdot 1 = 0$. It is evident if one puts $J(\varphi)$, given by the sum over the Weyl group, into Eq. (B.17). On the other hand, using the second form of $J(\varphi)$ as product of the sines, one gets a remarkable identity

$$\sum_{\alpha,\beta>0} (\alpha,\beta) \frac{\cos\frac{1}{2}(\alpha-\beta,\varphi)}{\sin\frac{1}{2}\alpha\varphi \sin\frac{1}{2}\beta\varphi} = \sum_{\alpha>0} \frac{\alpha^2}{\sin^2\frac{1}{2}\alpha\varphi}. \tag{B.19}$$

Matrix elements of the representations, as well as the characters, are the eigen-functions of the Laplace operator

$$\begin{aligned} \Delta D^{(l)}(\hat{g}) &= -\lambda(l)\, D^{(l)}(\hat{g}) \\ \mathring{\Delta}\chi^{(l)}(\varphi) &= -\lambda(l)\, \chi^{(l)}(\varphi). \end{aligned} \tag{B.20}$$

This is a direct consequence of Eq. (B.15). The eigen-values $\lambda(l)$ are given in Eq. (A.12).

C. Maximal Torus

In the root space one may use an affine basis formed by r *simple roots* $\gamma^{(j)}$, $j = 1, 2, \ldots, r$. (See the article [*28*]; the root is named simple if it is positive and can not be represented by a sum of other positive roots). Any positive root may be decomposed over this basis, and the coefficients are non-negative integers.

Functions on the maximal torus are periodical, cf. Eq. (A.28),

$$f(\varphi + 2\pi\boldsymbol{v}) = f(\varphi) \tag{C.1}$$

and any period $\boldsymbol{v}$ may be written as follows

$$\boldsymbol{v} = \sum_{j=1}^{r} n_j \tilde{\gamma}^{(j)}; \qquad \tilde{\gamma}^{(j)} = 2\gamma^{(j)}/(\gamma^{(j)})^2 \tag{C.2}$$

while n_j are integers. The vectors $\tilde{\gamma}^{(j)}$ form a basis of the r-dimensional lattice Γ consisting of all the period vectors $\boldsymbol{v}$. The vector φ is represented by its components in this basis, $\varphi = \sum_j \psi_j \tilde{\gamma}^{(j)}$. Taking Eq. (C.1) into account, one may assume that $-\pi < \psi_j < \pi$ for any j. Name this domain in the φ space the torus T_1. Its volume in the Euclidean measure is easily calculated

$$\begin{aligned} V_1 &= \int_{T_1} d^r\varphi = (2\pi)^r \det(\tilde{\gamma}_k^{(j)}) = (2\pi)^r \Delta_\gamma^{1/2} \\ \Delta_\gamma &= \det(\tilde{\gamma}^{(j)}\tilde{\gamma}^{(k)}) = \det(\gamma^{(j)}\gamma^{(k)}) \prod_j \left(\frac{1}{2}\gamma^{(j)2}\right)^{-2}. \end{aligned} \tag{C.3}$$

Example: The $SU(N)$ group. The simple roots are $\gamma^{(j)} = \alpha^{(j,j+1)}$ (see (A.38)), $j = 1, \ldots, N-1$. Now

$$(\gamma^{(j)}\gamma^{(k)}) = 2\delta_{jk} - \delta_{j,k-1} - \delta_{j,k+1}; \qquad \gamma^2 = 2, \qquad \Delta_\gamma = N.$$

Beside the periodical property, functions on the Lie group have a symmetry under the Weyl transformations (A.7). So the functions are determined by their values on the Weyl chamber (A.24), and the maximal torus is a part of the Weyl chamber, lying inside the torus T_1. (Recall that we deal with simply connected groups, i.e. the universal covering groups of the ajoint groups, having no center). Considering a function, invariant under the Weyl group, one may extend it to the whole region T_1. This enables one to integrate always over T_1, where the boundaries of the φ domain are simpler.
Functions on the torus are represented by the Fourier series

$$
\begin{aligned}
f(\varphi) &= \sum_{l^1,\dots l^2} c(l) \exp\,(il^{(j)}\varphi_j) \equiv \sum c(\lambda) \exp i\lambda\varphi \\
c(\lambda) &= V_1^{-1} \int_{T_1} f(\varphi) \exp\,(-i\lambda\varphi)\, d^r\varphi\,.
\end{aligned}
\tag{C.4}
$$

Here the sum is over all integers l^j, and the vector λ may be written in the biorthogonal basis

$$
\lambda = \sum_{j=1}^{r} l^j \beta_{(j)}\,, \qquad (\beta_{(j)}\tilde{\gamma}^{(k)}) = \delta_{jk}\,.
$$

A Θ-function on the torus T_1 is defined by means of the Fourier series

$$
\Theta(\varphi, A) = \sum_{\lambda} \exp i \left(\lambda\varphi - \frac{1}{2}\,(\lambda, A\lambda)\right)
\tag{C.5}
$$

where A is a linear operator

$$
(\lambda, A\lambda) \equiv \sum_{j,k} l^j l^k A_{jk}\,, \qquad A_{jk} = (\beta_{(j)}, A\beta_{(k)})\,.
$$

The matrix A is invariant under the Weyl reflections, $\sigma A \sigma = A$, if we deal with the Θ-function on the torus T_1. In case the matrix $(\beta_{(j)}, A\beta_{(k)})$ is diagonal, the function $\Theta(\varphi, A)$ is a product of usual elliptic functions (definition see e.g. in [37]):

$$
\Theta(\psi, \tau) \equiv \Theta_3\left(\frac{\psi}{2\pi}, \frac{i\tau}{2\pi}\right) = \sum_{l=-\infty}^{\infty} \exp i \left(l\psi - \frac{1}{2}\, l^2\tau\right).
\tag{C.6}
$$

The one-dimensional Θ-function may be also represented by another series

$$
\Theta(\psi, \tau) = \left(-\frac{2\pi i}{\tau}\right)^{1/2} \sum_{n=-\infty}^{\infty} \exp \frac{i}{2\tau}\,(\psi + 2\pi n)^2.
\tag{C.7}
$$

An analogous transformation is applicable also to the multi-dimensional case, if the matrix A is not singular

$$
\theta(\varphi, A) = (-2\pi i)^{r/2} \Delta_\gamma^{1/2} (\det A)^{-1/2} \sum_{\nu} \exp \frac{i}{2}\,[(\varphi + 2\pi\nu),\; A^{-1}(\varphi + 2\pi\nu)].
\tag{C.8}
$$

Note the limit of $(2\pi)^{-r}\, \Theta(\varphi, A)$ at $A \to 0$ is the δ-function on the torus T_1.

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