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QUANTUM MECHANICS AND PATH INTEGRALS IN SPACES WITH CURVATURE AND TORSION*

H. KLEINERT
Institut für Theorie der Elementarteilchen, Freie Universität Berlin, Arnimallee 14, 1000 Berlin 33, Germany

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We point out that there is a natural geometric procedure for constructing the quantum theory of a particle in a general metric-affine space with curvature and torsion. Quantization rules are presented and expressed in the form of a simple path integral formula which specifies compactly a new combined equivalence and correspondence principle. The associated Schrödinger equation has no extra curvature nor torsion terms that have plagued earlier attempts. Several well-known physical systems are invoked to suggest the correctness of the proposed theory.

The quantum theory of particles in spaces with curvature is beset with ambiguities. It was emphasized by De Witt[1] that, depending on the place at which one imposes the corresponding principle,[a] one obtains for a non-relativistic particle of unit mass a Schrödinger equation

$$\left(i\hbar\partial_t + \frac{1}{2}\hbar^2 D^*D - \lambda\hbar^2\bar{R}\right)\psi(q,t) = 0 \tag{1}$$

with $\lambda = 0$, 1/6 or 1/12, where D^*D is the Laplace Beltrami operator

$$D^*D = \frac{1}{\sqrt{g}}\,\partial_\mu\sqrt{g}\,g^{\mu\nu}\partial_\nu, \tag{2}$$

[a] Canonical quantization fails, mainly due to ordering problems. The case $\lambda = 0$ in Eq. (1) has been advocated by L. D. Landau and F. M. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Reading, 1965, Ch. 9, and since 1980, by C. DeWitt-Morette on the basis of stochastic differential equations in curved spaces; see here 1989 Erice Lectures in *Quantum Mechanics in Curved Spacetime*, ed. V. De Sabbata, Plenum Press, New York, 1990 or the original article, C. DeWitt-Morette, K. D. Elworthy, B. Nelson, G. S. Sammelmann, Ann. Inst. Henry Poincarè **32** (1980) 327.

A simple time-sliced propagator composed à la Dirac and Feynman from a product of short-time pieces, each evaluated along a short, classical trajectories, gives $\lambda = 1/6$, see K. S. Cheng, *J. Math. Phys.* **13** (1972) 1723.

If in each piece, the semi-classical limit is used as proposed by C. Morette, *Phys. Rev.* **81** (1951) 848, the Van Vleck determinant gives an additional $\Delta\lambda = -1/12$.

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t is the time and $\bar{R}$ the scalar curvature of space formed from the netric tensor $g_{\mu\nu}$ ($\partial_\mu = \partial/\partial q^\mu$). There are even proposals which lead to extra non-covariant term. DeWitt[2] does not state any preference for λ.

The same ambiguity arises in spacetime q^μ with t playing the role of a proper time parameter. Then the Fourier transform $\int dt\, e^{im^2 t}$ of Eq. (1) describes a Klein Gordon particle of mass m. For $m = 0$, Penrose prefers $\lambda = 1/6$ to have conformal invariance.[3] Texts on quantum gravity usually leave λ a free parameter to be determined in the future.[4]

The situation becomes even more dramatic if the space also carries torsion.[5,6] Then already the correspondence principle which prescribes how to find the classical motion in curved space from that in flat space is ambiguous. Imposing the principle at the level of the action one finds the classical orbits by transforming the flat-space action to curvilinear coordinates

$$\mathcal{A} \equiv \frac{1}{2} \int dt\, g_{\mu\nu}(q(t))\, \dot{q}^\mu(t) \dot{q}^\nu(t) \tag{3}$$

and extremizing it. Then the orbits coincide with the geodesics

$$\ddot{q}^\lambda + \bar{\Gamma}_{\mu\nu}{}^\lambda \dot{q}^\mu \dot{q}^\nu \tag{4}$$

where $\bar{\Gamma}_{\mu\nu}{}^\lambda \equiv \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$ denotes the usual Christoffel symbols. On purely geometric grounds, however, it would be just as acceptable to have particles move along the so-called straightest lines[7]

$$\ddot{q}^\lambda + \Gamma_{\mu\nu}{}^\lambda \dot{q}^\mu \dot{q}^\mu = 0 \tag{5}$$

with the full affine connection $\Gamma_{\mu\nu}{}^\lambda$, which are related to $\bar{\Gamma}_{\mu\nu}{}^\lambda$ by

$$\Gamma_{\mu\nu}{}^\lambda = \bar{\Gamma}_{\mu\nu}{}^\lambda + K_{\mu\nu}{}^\lambda, \tag{6}$$

where $K_{\mu\nu\lambda} \equiv S_{\mu\nu\lambda} - S_{\nu\lambda\mu} + S_{\lambda\mu\nu}$ is the contortion tensor, a combination of torsion tensors $S_{\mu\nu}{}^\lambda \equiv [\Gamma_{\mu\nu}{}^\lambda - (\mu \leftrightarrow \nu)]/2$. The straightest lines arise by applying the correspondence principle to the free particle equation of motion, transforming it to curvilinear coordinates rather than the action.

If one attempts to quantize the orbits in the presence of torsion by generalizing any of the various existing path integral procedures,[2] the Schrödinger equation picks up not only a $\lambda\hbar^2\bar{R}$ term but also different possible torsion terms so that there are even more ambiguities. In particular, it is unclear whether $\bar{R}$ in (1) should be replaced by R formed from the covariant curl of $\Gamma_{\mu\nu}{}^\lambda$ or by some mixture of $\bar{R}$ and R. Obviously, before we possess an observable physical system with space dependent curvature and torsion and known classical motion as well as quantum spectrum, we shall be unable to eliminate these ambiguities.

The purpose of this note is to point out that there exists such a system which, moreover, is one of the best known quantum systems of all, theoretically as well as experimentally. The quantization rules required to explain its properties will serve us to set up a unique quantization scheme.

The system with these virtues is, surprisingly, the non-relativistic H-atom in three dimensions with the action (in units charge $e^2 = 1/2$ and mass $M = 1$).

$$\mathcal{A} = \int dt \left(\frac{\dot{\mathbf{x}}^2}{2} + \frac{1}{2r} \right) \tag{7}$$

($r \equiv |\mathbf{x}|$). The equation of motion is

$$\ddot{x}_i - \partial_i \left(\frac{1}{2r} \right) = 0, \quad i = 1, 2, 3, \tag{8}$$

and the Schrödinger equation at fixed energy $E = -\omega^2/2$

$$\left(\hbar^2 \partial_i^2 - \omega^2 + \frac{1}{r} \right) \psi(\mathbf{x}) = 0. \tag{9}$$

It is well known that the simplest way of solving the classical[8] as well as the quantum mechanics[9,10] of this system is to imagine the atom to live in a fictitious four-dimensional space x^a, $a = 1, 2, 3, 4$ (maintaining $r =$ length of three-vector), and eliminate the extra degree of freedom later, by a supplementary condition $\ddot{x}^4 = 0$. Quantum mechanically, this implies that in Eq. (9), ∂_i^2 replaced by ∂_a^2. The elimination of x^4 done at the end by considering the subset of solutions with classical momentum $p^4 = 0$, or states with $\partial_{x^4} \psi(x) = 0$.

The point is that now in this extended space, we can perform a non-linear transformation introduced in the context of celestial mechanics by Krustaanheimo and Stiefel (see Ref. 10 for details)

$$\dot{x}^a = e^a_\mu(q) \dot{q}^\mu \tag{10}$$

with the basis tetrad

$$e^a{}_\mu(q) = \begin{vmatrix} q^3 & q^4 & q^1 & q^2 \\ q^4 & -q^3 & -q^2 & q^1 \\ q^1 & q^2 & -q^3 & -q^4 \\ q^2 & -q^1 & q^4 & -q^3 \end{vmatrix}. \tag{11}$$

The first three equations can be integrated to $x^a = e^a{}_\mu q^\mu/2 (a = 1, 2, 3)$. Together they give $(x^i)^2 = r^2 = q^4$. The fourth equation, however, is not integrable. In fact,

$O \equiv 1/2\, e^4{}_\mu\, q^\mu$. Yet, the fourth equation in (10) ensures that each orbit $x^a(t)$ ($a = 1, 2, 3, 4$) winds up at a unique final place in q^μ space.

Due to the non-integrability of the transformation, the q^μ space carries curvature and torsion. From the metric $g_{\mu\nu} = e^a{}_\mu\, e^a{}_\nu = q^2 \delta_{\mu\nu} = r\delta_{\mu\nu}$ (the metric in x^a space is $\eta_{ab} = \delta_{ab}$) and the connection $\Gamma_{\mu\nu}{}^\lambda = e^{a\lambda}\, \partial_\mu e^a{}_\nu$. We find that although the scalar curvature formed from the covariant curl of $\Gamma_{\mu\nu}{}^\lambda$ vanishes, the Riemannian scalar curvature formed from $\bar{\Gamma}_{\mu\nu}{}^\lambda$ does not, $\bar{R} = -18/q^4$. The contracted torsion tensor $S_{\mu\nu}{}^\lambda$ is obtained right-away observing that $\bar{\Gamma}^{\mu\nu}_{\mu} = -\partial_\mu (\sqrt{g}\, g^{\mu\nu})/\sqrt{g} = -\, 2q^\mu/q^4$ while $\Gamma^{\mu\nu}_{\mu} = -e^{a\mu}\, \partial_\mu\, e^{a\nu} = \partial_\mu\, (u^2\, e^{a\mu})\, e^{a\nu}/u^2 = 0$ where $e^{a\nu} = (u^2)^{-1}\, e^a{}_\nu$ is the reciprocal basis tetrad, i.e. $e^a{}_\nu\, e^{a\mu} = \delta_\nu{}^\mu$, which satisfies $\partial_\mu (u^2 e^{a\mu}) = \partial_\mu e^a{}_\mu = 0$. Hence $K^{\mu\nu}_\mu = -2S^\nu{}_\lambda{}^\lambda = 2q^\nu/q^4$. The other non-zero components are (writing $\tilde{q}^\mu$ for q^μ/q^2)

$$S_{\substack{12\\34}}{}^1 = -\tilde{q}^2, \quad S_{\substack{12\\34}}{}^2 = \tilde{q}^1, \quad S_{\substack{12\\34}}{}^3 = -\tilde{q}^4, \quad S_{\substack{12\\34}}{}^4 = \tilde{q}^3. \tag{12}$$

Such non-integrable transformations which locally carry flat into non-flat spaces are familiar in the theory of plasticity where they introduce defects into a perfect crystal (dislocations $\triangleq$ torsion and disclinations $\triangleq$ curvature).[11] The mapping (10) may be viewed as a "plastic deformation" of the x^a space.

What are the orbits of the H-atom in the metric-affine q^μ space? They are obtained by a direct transformation of (8) via (10). In the absence of a potential they satisfy $\ddot{x}^a = (e^a{}_\mu\, \dot{q}^\mu) = 0$, i.e. they are the straightest lines (5) rather than the geodesics (4).

How about the quantum theory? Dividing (9) by r and going over to q^μ space we find directly

$$\left[\hbar^2 g^{\mu\nu}(q)\partial_\mu\partial_\nu - \omega^2 + \frac{1}{q^2}\right]\psi(q) = 0. \tag{13a}$$

Upon multiplying by q^2 this becomes the Schrödinger equation of the four dimensional harmonic oscillator

$$(\hbar^2 \partial_\mu{}^2 - \omega^2 q^2 + 1)\psi(q) = 0. \tag{13b}$$

This is known to give the correct bound state spectrum and wave functions of the Schrödinger equation (after the projection $\partial_{x^4}\psi = e_4{}^\mu \partial_\mu \psi(q) = 0$) as well as the correct continuum after analytic continuation.[10]

What does this certainly correct Schrödinger equation in a space with curvature and torsion imply for the various options in Eq. (1)? None of them is correct! Equation (13) has the general form

$$\left[i\hbar\partial_t + \frac{1}{2}\hbar^2 D^\mu D_\mu - \frac{1}{2}\omega^2 + \frac{1}{2q^2}\right]\psi(q, t) = 0 \tag{14}$$

where $D^\mu D_\mu$ is not the Laplace-Beltrami operator but the square of the covariant derivative D_μ

$$D^\mu D_\mu = g^{\mu\nu}\,\partial_\mu\partial_\nu - \Gamma_\mu{}^{\mu\nu}\,\partial_\mu \tag{15}$$

formed with the connection $\Gamma_{\mu\nu}{}^\lambda$ rather than $\bar{\Gamma}_{\mu\nu}{}^\lambda$. The torsion enforces that $\Gamma_\mu{}^{\mu\nu} = \bar{\Gamma}_\mu{}^{\mu\nu} + K_\mu{}^{\mu\nu} = 0$ which is why (13b) is so simple. Moreover, the Schrödinger equation contains no curvature term of the $\bar{R}$ type.

We are then led to generalize these observations and postulate the following new quantum rules for particles moving in a general metric-affine space: At any point in the q^μ space, find a non-integrable coordinate transformation $dx^a = e^a{}_\mu(q)\,dq^\mu$ which maps an entire neighborhood of q^μ into a patch of euclidean space. In the neighborhood of that point, the differential equation is Schrödinger's $[i\hbar\,\partial_t + (\hbar^2/2)\partial_a{}^2 - V]\psi = 0$. Then return to the general space by the inverse of the same coordinate transformation, substituting $\partial_a \rightarrow e_a{}^\mu\partial_\mu$, and obtain

$$\left[\hbar i\partial_t + \left(\frac{\hbar^2}{2}\right)D_\mu D^\mu - V\right]\psi(q, t) = 0 \tag{16}$$

where $D_\mu D^\mu$ can also be written as $D^*_\mu D^\mu - 2S^\lambda\partial_\lambda$ (with $S^\mu \equiv S^\mu{}_\lambda{}^\lambda$).

We now specify the new path integral that leads to this quantum theory. We slice the time axis into $N+1$ small pieces t and describe the fluctuating particle path by a vector $q_n{}^\mu$ ($n = 0, \ldots, N+1$). In each time slice we perform a non-integrable transformation of the coordinate differences $\Delta q^\mu_n \equiv q^\mu_n - q^\mu_{n-1}$ to intervals $\Delta x^a{}_n \equiv x^a{}_n - x^a{}_{n-1}$ in a locally flat reference space via the following expansion $\Delta x^a = x^a(q) - x^a(q - \Delta q) = e^a_\mu(q)\,\Delta q^\mu - 1/2\ e^a{}_{\mu,\nu}(q)\ \Delta q^\mu\Delta q^\nu + 1/6 \times e^a{}_{\mu,\nu\lambda}(q)\ \Delta q^\mu\Delta q^\nu\Delta q^\nu$ (omitting the subscripts n). In the flat reference space we postulate the measure of functional integration

$$\int d\mu^{\text{new}} \equiv \prod_{n=2}^{N+1}\left[\int \frac{d\Delta x^a_n}{\sqrt{2\pi i\varepsilon\hbar}}\right]\exp\left(\frac{i}{\hbar}\sum_{n=1}^{N+1}\frac{(\Delta x^a_n)^2}{2\varepsilon}\right). \tag{17}$$

In a flat space with euclidean coordinates, this coincides with Feynman's (and Wiener's) original measure

$$\int d\mu^{\text{old}} \equiv \prod_{n=1}^{N} \left[\int \frac{dx_n^a}{\sqrt{2\pi i \varepsilon \hbar}} \right] \exp\left(\frac{i}{\hbar} \sum_{n=1}^{N+1} \frac{(\Delta x_n^a)^2}{2\varepsilon} \right). \tag{18}$$

The same thing holds for a flat space parametrized with curvilinear coordinates. In a general metric-affine space, however, the new measure leads to a new quantum physics.

Consider the path integral composed of a large product of many short-time amplitudes. Displaying explicitly the last integral in the product we may write

$$\psi(q, t) = \sqrt{g(q)} \int \frac{d\Delta q^\mu}{\sqrt{2\pi i \varepsilon \hbar}} \exp\left\{ \frac{i}{\hbar} (\mathcal{A}^\varepsilon + \mathcal{A}^J) \right\} \psi(q - \Delta q, t - \varepsilon) \tag{19}$$

where ψ is the product of all factors up to the time t (i.e. the wave function that develops from a localized state). The right-hand ψ is to be expanded in powers of Δq^μ around the final point q^μ. In the exponent, $\mathcal{A}^\varepsilon$ is the short-time action in a post-point expansion,

$$\begin{aligned} \varepsilon \mathcal{A}^\varepsilon = \; & \frac{1}{2} (\Delta x^a)^2 = \frac{1}{2} g_{\mu\nu} \Delta q^\mu \Delta q^\nu - \frac{1}{2} e^a{}_\mu e^s{}_{\nu,\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda \\ & + \left(\frac{1}{6} e^a{}_\mu e^a{}_{\nu,\lambda,\varkappa} + \frac{1}{8} e^a{}_{\mu,\nu} e^a{}_{\lambda,\varkappa} \right) \Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\varkappa + \dots \end{aligned} \tag{20}$$

and $(i/\hbar) \mathcal{A}^J$ is the logarithm of the Jacobian.

$$\frac{\partial(\Delta x^a)}{\partial(\Delta q^\mu)} = \sqrt{g} \det \left(\delta^\lambda{}_\mu - e^{a\lambda} e^a{}_{\mu,\nu} \Delta q^\nu + \frac{1}{2} e^{a\lambda} e^a{}_{\mu,\nu\lambda} \Delta q^\nu \Delta q^\lambda + \dots \right). \tag{21}$$

The hook underneath the indices denotes symmetrization. In terms of the connection, the action becomes

$$\begin{aligned} \varepsilon \mathcal{A}^\varepsilon = \; & \frac{1}{2} g_{\mu\nu} \Delta q^\mu \Delta q^\nu - \frac{1}{2} \Gamma_{\mu\nu\lambda} \Delta q^\mu \Delta q^\nu \Delta q^\lambda + \left[\frac{1}{6} g_{\mu\tau} (\partial_\varkappa \Gamma_{\lambda\nu}{}^\tau + \Gamma_{\lambda\nu}{}^\delta \Gamma_{\varkappa\delta}{}^\tau) \right. \\ & \left. + \frac{1}{8} \Gamma_{\mu\lambda\sigma} \Gamma_{\nu\varkappa}{}^\sigma \right] \Delta q^\mu \Delta q^\nu \Delta q^\lambda \Delta q^\varkappa \end{aligned} \tag{22}$$

while the Jacobian contribution reads

$$\frac{i}{\hbar} \mathcal{A}^J = - \Gamma_{\nu\mu}{}^\mu \Delta q^\nu + \frac{1}{2} [\partial_\varkappa \Gamma_{\lambda\nu}{}^\nu + \Gamma_{\lambda\nu}{}^\delta \Gamma_{\varkappa\delta}{}^\nu - \Gamma^\mu{}_{\lambda\sigma} \Gamma_{\mu\varkappa}{}^\sigma] \Delta q^\lambda \Delta q^\varkappa. \tag{23}$$

We have omitted all higher order terms that will be irrelevant for small ε, by virtue of the Trotter formula. The kernel in (19), the short-time amplitude

$$K^{\varepsilon}(\Delta q) = \exp\left\{\frac{i}{\hbar}(\mathcal{A}^{\varepsilon} + \mathcal{A}^{J})\right\}, \tag{24}$$

has the property that for small ε

$$\sqrt{g(q)} \int \frac{d\Delta q^{\mu}}{\sqrt{2\pi i\varepsilon\hbar}} K^{\varepsilon}(\Delta q) \left\{\begin{matrix} 1 \\ \Delta q^{\mu} \\ \Delta q^{\mu}\,\Delta q^{\nu} \end{matrix}\right\} = i\varepsilon\hbar \left\{\begin{matrix} 1 \\ \Gamma_{\nu}{}^{\nu\mu}/2 \\ g^{\mu\nu} \end{matrix}\right\} + \mathcal{O}(\varepsilon^{2}). \tag{25}$$

This is precisely what is needed to make the path integral (19) develop in time according to the Schrödinger equation (16) (with (15)). Its wave packets propagate along the straightest orbits (5).

Note that we can replace a complicated kernel effectively by either of the two expressions

$$\begin{aligned} K^{\varepsilon}_{\text{eff}}(\Delta q) &= \left(1 + \frac{1}{2}\Gamma_{\nu}{}^{\nu}{}_{\mu}(q)\Delta q^{\mu}\right) \exp\left\{\frac{i}{\hbar}\frac{1}{2\varepsilon} g_{\mu\nu}(q)\Delta q^{\mu}\Delta q^{\nu}\right\} \\ &= \exp\left\{\frac{i}{\hbar}\frac{1}{2\varepsilon} g_{\mu\nu}(q)\Delta q^{\mu}\Delta q^{\nu}\left(1 - \frac{i}{D+2}\Gamma_{\nu}{}^{\nu}{}_{\mu}(q)\Delta q^{\mu}(\Delta q)^{2}\right)\right\} \end{aligned} \tag{26}$$

where $\Gamma_{\nu}{}^{\nu}{}_{\mu}$ and $g_{\mu\nu}$ are taken at the post-point q^{μ} and D = space dimension.

In the special case of a conformally flat space with a metric $g_{\mu\nu} = \rho(q)\delta_{\mu\nu}$, the present construction can be short-circuited by looking at the Green function at fixed energy and performing a Duru-Kleinert time transformation[9,10] to an auxiliary pseudo-time

$$ds = dt\,\rho(q). \tag{27}$$

The associated pseudo-Hamiltonian

$$\mathcal{H} = \rho(H - E) \tag{28}$$

has then right-away the desired flat-space kinetic term $p^{2}/2$ with unique quantization. Indeed, the operator in (13b) is precisely of this type with $\rho = q^{2}$.

It is easy to verify that our quantization rules allow for a consistent transformation of the time sliced path integral of the $D = 3$ dimensional H-atom into the time sliced harmonic oscillator in the same way as done in Ref. 13 for $D = 2$. None of the earlier rules would do. Similarly, path integrals with centrifugal $1/r^2$ or angular $1/\sin^2\theta$ barriers can be transformed to integrable time-sliced expressions without singularities.[13]

Our rules are also in accordance with recent experience in string theory. The quantum theory associated with an action $\int d^2\xi\sqrt{g}$ is possible only when going to an orthonormal coordinate system in which it becomes $\int d^2\xi\, 1/2\, (\partial_\mu x)^2$. In Polyakov's quantization scheme,[14] on the other hand, the action $1/2 \times \int d^2\xi\sqrt{g}\, g^{\mu\nu}\, \partial_\mu x \partial_\nu x$ makes proper sense with respect to x integrations only in the conformal gauge where it becomes $1/2 \times \int d^2\xi(\partial_\mu x)^2$.

We expect that also in quantum gravity where the metric tensor itself moves through a space of metrics (with hypermetric ${}_m g^{\mu\nu,\sigma\tau} = 1/2 \times \sqrt{g}\,(g^{\mu\sigma} g^{\nu\tau} + g^{\mu\tau} g^{\nu\sigma} + \lambda\, g^{\mu\nu} g^{\sigma\tau})$). The quantization, which presently is a matter of controversy,[15] can be made unique by following an analogous procedure. In particular, the ordering problem in the kinetic term of the Wheeler-DeWitt equation[16] that governs the wave function of the universe[17] has a unique form. It is given by the Laplace-Beltrami operator in the space of 3-metrics with $\lambda = -1$ and $\sqrt{{}_m g} \propto g^{(D-4)(D+1)/8}$ in D space dimensions and no extra curvature term (which in this space would be given by ${}_m\bar{R} = (4D)^{-1}(D+2)(D-1)[(1+4\lambda)D^2 + 9D - 4]\, Dg^{-1/2}/8(2+\lambda D)$ at $\lambda = -1$ (see Ref. 15)). The minisuperspace model of the universe in Ref. 15 (Sec. V) has then, after a time reparametrization (25) with $\rho = a$, a well-defined Hamiltonian operator $\mathscr{H}$ à la Eq. (26).

Also the two-dimensional quantum gravity emerging from Polyakov's action after the x integration complies with our rules. In the conformal gauge, a transformation $\varphi = \log \rho$ brings the gradient terms to the flat-space form $1/2 \times \int d^2\xi(\partial\varphi)^2$ and this is why it becomes a bona fide quantum theory.

It goes without saying that, certainly, any earlier difficulties in quantizing systems with curvilinear coordinates in flat space (for instance with radial coordinates) are absent in the new path integral.

Faced with all these evidences in favor of the present path integral, the sceptic reader may wonder what happened to the sacrosaint action principle, according to which classical motion should run along the orbits of minimal action (3), i.e. along the geodesics. This principle can be salvaged if we remember that the only source of torsion in the universe is spinning matter. Therefore $S_{\mu\nu}{}^\lambda$ always carries a factor of $\hbar$ relative to the Christoffel symbol and in the limit $\hbar \to 0$, the straightest orbit reduces to the shortest ones after all. This one should write the decomposition (6) more clearly as $\Gamma_{\mu\nu}{}^\lambda = \bar{\Gamma}_{\mu\nu}{}^\lambda + \hbar K_{\mu\nu}{}^\lambda$.

Let us point out that at present, there exists no consistent purely classical theory of gravity with curvature and torsion. The actions and field equations that have been proposed for this purpose[6] must be considered as effective actions (i.e. as Legendre transforms of the generating functional of the full quantum theory to be extremized in metric and torsion fields). As such, however, they are only an incomplete collection of all $\hbar^2$ correction terms in the energy: At the same order of

$\hbar$ there will exist also loop corrections from metric fluctuations, which nobody knows how to regularize (except in somewhat artificial string models which replace the cutoff mass at the Planck scale by a tension parameter).

Finally, let us note that in the kernels (26) an auxiliary momentum integration can be introduced to obtain a canonical expressions for the path integral. In it, there are no problems in performing canonical transformations.

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