The Schwinger Action Principle and the Feynman Path Integral for Quantum Mechanics in Curved Space

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Abstract

The Feynman path integral approach to quantum mechanics is examined in the case where the configuration space is curved. It is shown how the ambiguity that is present in the choice of path integral measure may be resolved if, in addition to general covariance, the path integral is also required to be consistent with the Schwinger action principle. On this basis it is argued that in addition to the natural volume element associated with the curved space, there should be a factor of the Van Vleck-Morette determinant present. This agrees with the conclusion of an approach based on the link between the path integral and stochastic differential equations.

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I. INTRODUCTION

There is a large body of work on the Feynman path integral [1] approach to quantum mechanics on a curved background, and the Schrödinger equation which results from the path integral. (See some of the references described below.) It is widely known that, depending upon what is chosen for the measure in the configuration space path integral, there is a multiple of the scalar curvature to be added to the canonical Hamiltonian in the resulting Schrödinger equation. This is demonstrated explicitly by Parker [2] who considers including an arbitrary power of the Van Vleck-Morette [3, 4] determinant $\Delta^p(x, x')$, where p is an arbitrary real number, in the measure in addition to the natural curved space volume element. The choice p = 0, which corresponds to choosing the natural volume element for the path integral measure, and p = 1/2, which is motivated by the WKB approximation, were examined originally by DeWitt [5]. The choice p = 1 was shown by Parker [2] to lead to no additional curvature modification of the canonical Hamiltonian. For any value of p the path integral measure is invariant under an arbitrary change of coordinates for the curved space. General covariance alone of the configuration space path integral does not prescribe a unique measure.

It might be thought that this lack of uniqueness in the path integral approach to quantum mechanics in curved space disappears if a phase space path integral is used instead of one in configuration space. In this case there is a unique choice of measure; namely, the Liouville measure. However, Kuchař [6] has shown in a very lucid paper that the lack of uniqueness associated with the configuration space path integral measure reappears in phase space as the lack of uniqueness of the skeletonization of the action. (As discussed by DeWitt [5] Hamilton-Jacobi theory leads to a covariant skeletonization in configuration space.) Lack of uniqueness in the quantum theory is therefore inherent in the curved space Feynman path integral, and cannot be eliminated solely on the grounds of general covariance.

If this lack of uniqueness is to be eliminated from the Feynman path integral in curved space, the only possibility is to impose some further property in addition to general covariance under a change of coordinates in the curved space. One example of an extra property done within the minisuperspace approach to quantum cosmology is to demand conformal invariance [7, 8]. However, this seems rather specific to quantum cosmology, and not very compelling as a general principle, since there is no reason to believe that conformal invari-

ance is a fundamental symmetry of nature. Another more general principle, based on the relationship between the Feynman path integral and stochastic differential equations, was given in Ref. [9]. This corresponds to the choice p = 1, and hence gives no curvature modification to the canonical Schrödinger equation. As we will see, the approach adopted in the present paper supports the conclusion of Ref. [9].

We will examine what happens if the Feynman path integral is required to be equivalent to the Schwinger action principle [10, 11]. It was known almost from its inception, that for flat spaces the Schwinger action principle is completely equivalent to the path integral [12, 13]. This equivalence holds equally well for quantum field theory, which is usually regarded as involving a flat configuration space. (This is how DeWitt [14], for example, arrives at the path integral.) In this paper, I wish to show how the Schwinger action principle may be used to fix the measure in the Feynman path integral for quantum mechanics on a curved space. If the notation is interpreted in the spirit of DeWitt's [14] condensed notation, then the result holds equally well for quantum field theory with a curved configuration space. (We will comment briefly on this in Sec. IV.) It will be shown that equivalence between the Schwinger action principle and the Feynman path integral is only achieved if there is a single factor of the Van Vleck-Morette determinant in the measure, (i.e. p = 1 above.) This is completely consistent with the result of Ref. [9] which was based on totally different reasoning.

II. THE SCHWINGER ACTION PRINCIPLE

Let q^i denote a set of local coordinates on some manifold M. The classical motion of a particle moving on M is given by $q^i(t)$ where $q^i(t)$ is a solution to the Euler-Lagrange equations. We will adopt a configuration space rather than a phase space approach. In quantum mechanics we are interested in computing the transition amplitude $\langle q_2, t_2 | q_1, t_1 \rangle$ where $|q_1, t_1\rangle$ represents the quantum state at time t_1 , and $|q_2, t_2\rangle$ represents the state at time $t_2 \geq t_1$. These states are chosen to be eigenstates of the position operator \hat{q}^i :

$$\hat{q}^{i}|q_{\alpha},t_{\alpha}\rangle = q^{i}(t_{\alpha})|q_{\alpha},t_{\alpha}\rangle \quad (\alpha=1,2) .$$
 (2.1)

The Schwinger action principle [10, 11] states that

$$\delta\langle q_2, t_2 | q_1, t_1 \rangle = \frac{i}{\hbar} \langle q_2, t_2 | \delta S | q_1, t_1 \rangle , \qquad (2.2)$$

where S represents the action obtained by the replacement of q^i in the action for the classical theory with \hat{q}^i , along with an operator ordering which leads to S being self-adjoint. δ in Eq. (2.2) represents any possible variation, including variations with respect to the times t_1, t_2 , the dynamical variables q^i , or the structure of the Lagrangian. The variations of the dynamical variables δq^i will be chosen to be c-numbers, appropriate to bosonic theories. This choice was also made in flat space by Schwinger [10], and in curved space by Kawai [15, 16]. The case of fermionic variables will not be considered here.

Suppose that we add a source term to the action. Normally this is done so that differentiation with respect to the source generates the n-point functions of the theory. If the space is flat, this may be accomplished by simply taking

$$S_J[q] = S[q] + \int_{t_1}^{t_2} dt \ J_i(t)q^i(t) , \qquad (2.3)$$

where S[q] is the original action, and $J_i(t)$ is an external source which is turned on at time t_1 and off at time t_2 . However, on a curved space where q^i are coordinates rather than vectors, the addition of the source term in Eq. (2.3) does not result in a covariant expression. This is also the case in a flat space if the coordinates are chosen to be curvilinear rather than Cartesian.

The covariant generalization of Eq. (2.3) is obtained as follows. First, it may be noted that if the field space is flat, the same classical theory is obtained from

$$S_J[q, q_*] = S[q] + \int_{t_1}^{t_2} dt \ J_i(t) \left(q^i(t) - q_*^i(t) \right) , \qquad (2.4)$$

as from as from Eq. (2.3), where the coordinates are assumed to be Cartesian, and where q_*^i is regarded as a fixed point in the configuration space M. (q_*^i plays the role that the background field [14] does in quantum field theory.) The coordinate difference ($q^i - q_*^i$) then represents a vector which connects the fixed reference point q_*^i to the point q_*^i . Equivalently, ($q^i - q_*^i$) represents the tangent vector to the geodesic connecting q_*^i to q_*^i , which for M flat is just a straight line segment. This indicates that the natural replacement for the coordinate difference ($q^i - q_*^i$) in a general space M is just the tangent vector at q_* to the geodesic connecting q_*^i to q_*^i . One way of introducing this tangent vector is by means of the geodetic interval $\sigma(q_*;q)$. (See Refs. [14, 17, 18].) By definition,

$$\sigma^{i}(q_{*};q) = \frac{1}{2}\ell^{2}(q_{*};q) , \qquad (2.5)$$

where $\ell(q_*;q)$ is the length of the geodesic connecting q_* to q. The tangent vector to the geodesic at q_* is

$$\sigma^{i}(q_{*};q) = g^{ij}(q_{*}) \frac{\partial}{\partial q_{*}^{j}} \sigma(q_{*};q) , \qquad (2.6)$$

where M is assumed to have a metric tensor g_{ij} . If M is flat, and q^i are Cartesian coordinates (so that $g_{ij} = \delta_{ij}$), then

$$\sigma^{i}(q_{*};q) = -(q^{i} - q_{*}^{i}). {2.7}$$

The natural replacement for $(q^i - q_*^i)$ in Eq. (2.4) is therefore $-\sigma^i(q_*;q)$ resulting in

$$S_J[q, q_*] = S[q] - \int_{t_1}^{t_2} dt \ J_i(t)\sigma^i(q_*; q) \ . \tag{2.8}$$

 $\sigma^i(q_*;q)$ transforms like a vector under a change of coordinates q_* , and as a scalar under a change of coordinates q. If the source $J_i(t)$ is required to transform like a covariant vector at q_* , and be independent of q, then Eq. (2.8) is a completely covariant definition. It is important that $J_i(t)$ be independent of q if it is to fulfill its role as an external source. If M is flat, but q^i are not Cartesian coordinates, it is possible to derive Eq. (2.8) from Eq. (2.4) using the approach of Ref. [19].

It proves convenient to adopt condensed notation [14] at this stage, and to write Eq. (2.8) as

$$S_J[q, q_*] = S[q] - J_i \sigma^i(q_*; q)$$
 (2.9)

where the index i is now understood to include the time label, and a repeated index includes integration over time. Instead of regarding $S_J[q, q_*]$ as a functional of q, q_* , it is convenient to regard it instead as a functional $\tilde{S}_J[q_*; \sigma^i(q_*; q)]$ which is defined using the covariant Taylor expansion [17]

$$S[q] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_{;(i_1,\dots i_n)}[q_*] \sigma_1^i(q_*;q) \cdots \sigma_n^i(q_*;q)$$
 (2.10)

in Eq. (2.9). (The semicolon denotes the usual covariant derivative using the Christoffel connection constructed from the metric G_{ij} on M.)

Let $\langle q_2, t_2 | q_1, t_1 \rangle [J]$ be the transition amplitude for the theory with action $S_J[q; q_*]$ in Eq. (2.9). The Schwinger action principle gives

$$\delta\langle q_2, t_2 | q_1, t_1 \rangle [J] = \frac{i}{\hbar} \langle q_2, t_2 | \delta S_J | q_1, t_1 \rangle [J] . \qquad (2.11)$$

If the variation is taken to be one that is with respect to the dynamical variables q^i leaving the values fixed at times t_1 and t_2 , then the amplitude will not change under the variation, and we have

$$0 = \langle q_2, t_2 | \delta S_J | q_1, t_1 \rangle [J] , \qquad (2.12)$$

from which the equation of motion may be inferred:

$$\frac{\delta \tilde{S}_J}{\delta \sigma^i} - J_i = 0 \ . \tag{2.13}$$

Since we may regard $\langle q_2, t_2 | q_1, t_1 \rangle [J]$ a functional of the source J_i , it may be expanded in a Taylor series about $J_i = 0$:

$$\langle q_2, t_2 | q_1, t_1 \rangle [J] = \sum_{n=0}^{\infty} \frac{1}{n!} J_{i_1} \cdots J_{i_n} \frac{\delta^n \langle q_2, t_2 | q_1, t_1 \rangle [J]}{\delta J_{i_1} \cdots \delta J_{i_n}} \bigg|_{J=0}$$
 (2.14)

We will now evaluate the n^{th} derivative of the amplitude which occurs in Eq. (2.14) using the Schwinger action principle. The method is just that used originally by Schwinger [10].

Suppose that the variation in Eq. (2.11) is one with respect to the external source J_i . Since the dependence of the action on the source is given in Eq. (2.9), we have

$$\frac{\delta\langle q_2, t_2|q_1, t_1\rangle[J]}{\delta J_i} = -\frac{i}{\hbar}\langle q_2, t_2|\sigma^i(q_*; q)|q_1, t_1\rangle[J] . \qquad (2.15)$$

If we now perform a further variation of Eq. (2.15) with respect to the source, we have

$$\delta \frac{\delta \langle q_2, t_2 | q_1, t_1 \rangle [J]}{\delta J_i} = -\frac{i}{\hbar} \delta \langle q_2, t_2 | \sigma^i(q_*; q) | q_1, t_1 \rangle [J] . \tag{2.16}$$

In order to evaluate the right hand side of this expression, insert unity in the form $1 = \int dv'|q',t'\rangle\langle q',t'|$ where $t_1 < t' < t_2$, and $dv' = d^nq'g^{1/2}(q')$ is the invariant volume element on M. If the time included in the index i of $\sigma^i(q_*;q)$ lies to the past of t', then we will change the source only to the future of t' and the past of t_2 . Assuming causal boundary conditions, $\delta\langle q_2,t_2|\sigma^i(q_*;q)|q_1,t_1\rangle[J]$ cannot be affected by such a change in the source. Thus,

$$\delta\langle q_2, t_2 | \sigma^i | q_1, t_1 \rangle [J] = \int dv' \delta\langle q_2, t_2 | q', t' \rangle [J] \langle q', t' | \sigma^i | q_1, t_1 \rangle [J]$$

$$= -\frac{i}{\hbar} \int dv' \delta J_j \langle q_2, t_2 | \sigma^j | q', t' \rangle [J] \langle q', t' | \sigma^i | q_1, t_1 \rangle [J]$$

$$= -\frac{i}{\hbar} \delta J_j \langle q_2, t_2 | \sigma^j \sigma^i | q_1, t_1 \rangle [J] , \qquad (2.17)$$

where we have dropped the argument $(q_*;q)$ on σ^i and σ^j for brevity. Note that the time corresponding to the condensed index j lies to the future of that corresponding to i.

Conversely, if i lies to the future of t', but to the past of t_2 , a similar argument shows that

$$\delta\langle q_2, t_2 | \sigma^i | q_1, t_1 \rangle [J] = -\frac{i}{\hbar} \delta J_j \langle q_2, t_2 | \sigma^i \sigma^j | q_1, t_1 \rangle [J] . \qquad (2.18)$$

Both situations in Eqs. (2.17,2.18) may be summarized compactly by

$$\delta\langle q_2, t_2 | \sigma^i | q_1, t_1 \rangle [J] = -\frac{i}{\hbar} \delta J_j \langle q_2, t_2 | T(\sigma^i \sigma^j) | q_1, t_1 \rangle [J] , \qquad (2.19)$$

where T is the chronological, or time ordering, symbol. It then follows that

$$\frac{\delta^2 \langle q_2, t_2 | q_1, t_1 \rangle [J]}{\delta J_j \delta J_i} = \left(-\frac{i}{\hbar} \right)^2 \langle q_2, t_2 | T(\sigma^i \sigma^j) | q_1, t_1 \rangle [J] . \tag{2.20}$$

It is easily established by induction that

$$\frac{\delta^n \langle q_2, t_2 | q_1, t_1 \rangle [J]}{\delta J_{i_1} \cdots \delta J_{i_n}} = \left(-\frac{i}{\hbar}\right)^n \langle q_2, t_2 | T(\sigma^{i_1} \cdots \sigma^{i_n}) | q_1, t_1 \rangle [J] . \tag{2.21}$$

Use of Eq. (2.21) in Eq. (2.14) leads to

$$\langle q_{2}, t_{2} | q_{1}, t_{1} \rangle [J] = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar} \right)^{n} J_{i_{1}} \cdots J_{i_{n}} \langle q_{2}, t_{2} | T(\sigma^{i_{1}} \cdots \sigma^{i_{n}}) | q_{1}, t_{1} \rangle [J=0]$$

$$= \langle q_{2}, t_{2} | T\left\{ \exp\left(-\frac{i}{\hbar} J_{i} \sigma^{i} \right) \right\} | q_{1}, t_{1} \rangle [J=0] . \tag{2.22}$$

(The exponential in the last line is understood to be defined in terms of its Taylor series as in the preceding line.)

Define

$$E_i \left[q_*; \sigma^i(q_*; q) \right] = \frac{\delta \tilde{S}}{\delta \sigma^i} \,, \tag{2.23}$$

so that the operator equation of motion Eq. (2.13) becomes

$$E_i\left[q_*;\sigma^i(q_*;q)\right] = J_i. \tag{2.24}$$

We can view $E_i\left[q_*;\sigma^i(q_*;q)\right]$ as defined in terms of the Taylor series obtained by differentiating Eq. (2.10). Now consider $E_i\left[q_*;-\frac{\hbar}{i}\frac{\delta}{\delta J_i}\right]$ where σ^i in the Taylor series for $E_i\left[q_*;\sigma^i(q_*;q)\right]$ is replaced by $-\frac{\hbar}{i}\frac{\delta}{\delta J_i}$. Using Eq. (2.22), it is clear that

$$E_{i}\left[q_{*}; -\frac{\hbar}{i}\frac{\delta}{\delta J_{i}}\right] \langle q_{2}, t_{2}|q_{1}, t_{1}\rangle[J] = \langle q_{2}, t_{2}|T\left\{E_{i}\left[q_{*}; \sigma^{i}\right] \exp\left(-\frac{i}{\hbar}J_{i}\sigma^{i}\right\}\right) |q_{1}, t_{1}\rangle[J=0]$$

$$= J_{i}\langle q_{2}, t_{2}|q_{1}, t_{1}\rangle[J], \qquad (2.25)$$

noting Eqs. (2.24,2.22). This last result provides a functional-differential equation for the amplitude $\langle q_2, t_2 | q_1, t_1 \rangle [J]$ which has followed from the Schwinger action principle. Integration of Eq. (2.25) will provide the link between the Schwinger action principle and the Feynman path integral.

III. THE FEYNMAN PATH INTEGRAL

In order to solve Eq. (2.25), let

$$\langle q_2, t_2 | q_1, t_1 \rangle [J] = \int \left(\prod_i d\sigma^i(q_*; q) \right) F\left[q_*; \sigma^i(q_*; q) \right] \exp\left(-\frac{i}{\hbar} J_i \sigma^i \right) , \qquad (3.1)$$

for some function F. The integration is assumed to extend over all $\sigma^i(q_*;q)$ (or equivalently over all q^i , as will be seen below) for which $\sigma^i(q_*;q) = \sigma^i(q_*;q_1)$ at time $t = t_1$, and $\sigma^i(q_*;q) = \sigma^i(q_*;q_2)$ at time $t = t_2$. If Eq. (3.1) is to solve Eq. (2.25), we must have

$$0 = \int \left(\prod_{i} d\sigma^{i}\right) \left\{ E_{i}[q_{*}; \sigma^{i}(q_{*}; q)] - J_{i} \right\} F\left[q_{*}; \sigma^{i}(q_{*}; q)\right] \exp\left(-\frac{i}{\hbar} J_{i} \sigma^{i}\right)$$
$$= \int \left(\prod_{i} d\sigma^{i}\right) \left\{ E_{i}[q_{*}; \sigma^{i}] F\left[q_{*}; \sigma^{i}\right] + \frac{\hbar}{i} F\left[q_{*}; \sigma^{i}\right] \frac{\delta}{\delta \sigma^{i}} \right\} \exp\left(-\frac{i}{\hbar} J_{i} \sigma^{i}\right) .$$

If we integrate the second term in the last line by parts, then

$$0 = \int \left(\prod_{i} d\sigma^{i} \right) \left\{ E_{i}[q_{*}; \sigma^{i}] F\left[q_{*}; \sigma^{i}\right] - \frac{\hbar}{i} \frac{\delta F\left[q_{*}; \sigma^{i}\right]}{\delta \sigma^{i}} \right\} \exp\left(-\frac{i}{\hbar} J_{i} \sigma^{i}\right) + \frac{\hbar}{i} F\left[q_{*}; \sigma^{i}(q_{*}; q)\right] \exp\left(-\frac{i}{\hbar} J_{i} \sigma^{i}\right) \Big|_{q_{1}}^{q_{2}}.$$
(3.2)

Because $E_i = \delta \tilde{S}/\delta \sigma^i$, if we assume that the surface term in Eq. (3.2) vanishes, then the solution to Eq. (3.2) is

$$F\left[q_*; \sigma^i(q_*; q)\right] = f(q_*) \exp\left(\frac{i}{\hbar} \tilde{S}(q_*; \sigma^i)\right) , \qquad (3.3)$$

for arbitrary $f(q_*)$. The condition for the surface term in Eq. (3.2) to vanish is then that $S[q = q_1] = S[q = q_2]$ (assuming that the source J_i is only non-zero for $t_1 < t < t_2$.) This condition is often met in field theory by assuming that q^i is in the vacuum state at $t = t_1$ and at $t = t_2$.

We have therefore found that

$$\langle q_2, t_2 | q_1, t_1 \rangle [J] = f(q_*) \int \left(\prod_i d\sigma^i \right) \exp \left\{ \frac{i}{\hbar} (\tilde{S} - J_i \sigma^i) \right\} .$$
 (3.4)

The integration in Eq. (3.4) may be changed to one over the more conventional variable q^i as follows. The usual rule for a change of variable gives

$$\left(\prod_{i} d\sigma^{i}(q_{*}; q)\right) = \left|\det \frac{\delta}{\delta q^{j}} \sigma^{i}(q_{*}; q)\right| \left(\prod_{i} dq^{i}\right) . \tag{3.5}$$

Noting from Eq. (2.6) that $\sigma^i(q_*;q) = g^{ik}(q_*)\delta\sigma(q_*;q)/\delta q_*^k$, and that the Van Vleck-Morette determinant [3, 4] is defined by

$$\Delta(q_*; q) = |g(q_*)|^{-1/2} |g(q)|^{-1/2} \det\left(-\frac{\delta^2 \sigma(q_*; q)}{\delta q^i \delta q_*^j}\right) , \qquad (3.6)$$

Eq. (3.5) becomes

$$\left(\prod_{i} d\sigma^{i}(q_{*}; q)\right) = \left(\prod_{i} dq^{i}\right) |g(q)|^{1/2} |\Delta(q_{*}; q)| |g(q_{*})|^{-1/2}.$$
(3.7)

Here g(q) denotes det $g_{ij}(q)$, and the factors of g(q), $g(q_*)$ have been chosen to make $\Delta(q_*;q)$ a scalar in each argument.

With the change of variable described above, the expression for the transition amplitude becomes

$$\langle q_2, t_2 | q_1, t_1 \rangle [J] = |g(q_*)|^{-1/2} f(q_*) \int \left(\prod_i dq^i \right) |g(q)|^{1/2} |\Delta(q_*; q)| \exp \left\{ \frac{i}{\hbar} (\tilde{S} - J_i \sigma^i) \right\}.$$
(3.8)

The amplitude must be invariant under the change of coordinates $q_*^i \to q_*'^i$. This is seen to constrain $|g(q_*)|^{-1/2}f(q_*)$ to transform like a scalar. This scalar is irrelevant since we typically only compare one amplitude with another. In fact, if we require the expression to reduce to that of Feynman when the space is flat, then $|g(q_*)|^{-1/2}f(q_*)$ must be a constant. In any case, because $|g(q_*)|^{-1/2}f(q_*)$ has no dependence on the dynamical variables q^i , we may simply take

$$\langle q_2, t_2 | q_1, t_1 \rangle [J] = \int \left(\prod_i dq^i \right) |g(q)|^{1/2} |\Delta(q_*; q)| \exp \left\{ \frac{i}{\hbar} (\tilde{S} - J_i \sigma^i) \right\} , \qquad (3.9)$$

as the path integral representation for the amplitude. As promised in the introduction, in addition to the natural volume element $\left(\prod_i dq^i\right) |g(q)|^{1/2}$, the additional factor of $\Delta(q_*;q)$ is seen to be present. As we have already mentioned, this agrees with the conclusion of Ref. [9] and as shown by Parker [2] results in a Schrödinger equation with no explicit dependence on the scalar curvature.

IV. DISCUSSION AND CONCLUSIONS

In addition to the natural volume element in the path integral measure, we have shown that there is an additional term which involves the Van Vleck-Morette determinant. The origin of this term can be traced to consistency between the Schwinger action principle and the Feynman path integral. In the special case of a flat space, $\Delta(q_*;q) = 1$, so that this additional term disappears even if curvilinear coordinates are used. As mentioned in the introduction, the existence of the term $|\Delta(q_*;q)|$ in the measure leads to the normal Schrödinger equation without any additional modifications due to the curvature [2].

There are of course other ways to derive the factor of $\Delta(q_*;q)$ in the measure. One is the previously mentioned method of DeWitt-Morette et al. [9]. Another approach, which is independent of the Schwinger action principle, is to postulate that the amplitude satisfy the equation of motion Eq. (2.25). This is what would be done following Symanzik [13] for example. The steps leading up to the end result of Eq. (3.9) are identical. We chose instead to postulate the more general action principle of Schwinger, and to derive Eq. (2.25) as one of its many consequences.

It is of interest to explore the consequences of the measure found in this paper in the case of quantum field theory. A covariant approach to quantum field theory has been advocated by Vilkovisky [20]. It would be of interest to study the implications for gauge theories, particularly in relation to the geometrical analysis of the measure presented in Refs. [21, 22].

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