Effective Lagrangian and the back-reaction problem in a self-interacting $O(N)$ scalar theory in curved spacetime

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A derivation of the one-loop effective Lagrangian in the self-interacting $O(N)$ scalar theory, in slowly varying gravitational fields, is presented (using ζ-function regularization and heat-kernel techniques). The result is given in terms of the expansion in powers of the curvature tensors (up to quadratic terms) and their derivatives, as well as in derivatives of the background scalar field (up to second derivatives). The renormalization group improved effective Lagrangian is studied, which gives the leading-log approach of the whole perturbation theory. An analysis of the effective equations (back-reaction problem) on the static hyperbolic spacetime $\mathbb{R}^2 \times H^2/\Gamma$ is carried out for the simplest version of the theory: $m^2 = 0$ and $N = 1$. The possibility of the existence of the solution $\mathbb{R}^2 \times H^2/\Gamma$, induced by purely quantum effects, is discussed.

I. INTRODUCTION

There is little to say about the importance of the effective action formalism in modern quantum-field theory and quantum cosmology. If we had at our disposal a closed form for the action of the theory of quantum gravity, which still does not exist as a consistent theory, we would be able to construct a reasonable picture of the early Universe, which, presumably, would contain the inflationary stage (for a review see [1–3]). Unfortunately, even in frames of the rather simple approximations to quantum gravity based on quantum-field theory in curved space-time (for a review see [4–6]), it is impossible to find the effective action in a closed form.

First of all, one has to work in perturbation theory and, as a rule, in the simplest one-loop approximation. However, even in this case there appear some limitations, connected with the fact that it is impossible to calculate the one-loop effective action in general curved space-time.

For that reason, there are a lot of explicit calculations of one-loop effective actions, mainly for some simple models (such as the $\lambda\varphi^4$ theory and scalar electrodynamics) in specific backgrounds, paying special attention to the role

of constant curvature [7–11], topology [12–22], a combination of both [23–25], and, finally, to the anisotropy in different Bianchi-type universes [26–33].

If we still want to obtain information connected with general properties of curved space-time, a very useful method is the quasilocal approximation scheme for the effective action in curved space-time [6] and [34–37]. This will be the main method applied in the present work, where we continue the study of the effective Lagrangian in curved space-time. In particular, using heat-kernel techniques, we will derive the one-loop effective Lagrangian in a scalar self-interacting $O(N)$ theory, thus generalizing existing results for the $\lambda\varphi^4$ theory in curved space-time. [For the $O(N)$ σ model in two dimensions see [38].]

Some technical problems, which appear in such calculations, are discussed in Secs. II and III, where the computation of the one-loop effective action and its renormalization is performed. Section IV is devoted to an attempt to discuss higher-loop effects. For that reason we use the renormalization group (RG) to improve the effective Lagrangian and to obtain it in the leading-log approximation of the whole perturbation theory. In Sec. V we analyze the back-reaction problem, following the early attempts of [32] and [39–42] (for a review see [5]), using the static hyperbolic space $\mathbb{R}^2 \times H^2/\Gamma$ with a varying radius as a background, and using two different approximations for the effective action: the exact one-loop effective action [25] and the RG improved effective action. The existence of a quantum solution induced by matter is discussed. Section VI summarizes the main results. Finally, Appendix A presents the technique of

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diagonalization of the matrix describing the potential of the O(N) field, which we need for the calculation of the one-loop effective action.

II. THE ONE-LOOP EFFECTIVE ACTION

The aim of this section is to derive a quasilocal approximation for the effective action in a self-interacting O(N)-symmetric model described by the action

$$S[\tilde{\phi}, g_{\mu\nu}] = -\int_{\mathcal{M}} d^n x |g(x)|^{1/2} [\tfrac{1}{2} \tilde{\phi} \Delta \tilde{\phi} - \tilde{U}(\tilde{\phi})] . \quad (2.1)$$

Here $\mathcal{M}$ is a smooth n-dimensional manifold with Lorentzian metric $g_{\mu\nu} = \text{diag}(-, +, \ldots, +)$, $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_N)$ is an N-component scalar field, and $\tilde{U}(\tilde{\phi})$ is a potential describing the self-interaction of the scalar fields, which contains, furthermore, local expressions of dimension n, involving curvature tensors and non-quadratic terms in the field, independent up to a total divergence. The latter terms have to be included to ensure renormalizability of the theory.

Expanding the action (2.1) around its classical minimum $\hat{\phi}$ in powers of the fluctuations $\phi = \tilde{\phi} - \hat{\phi}$ leads to

$$S[\tilde{\phi}, g_{\mu\nu}] = S^{(0)} + S^{(1)} + S^{(2)} + \cdots .$$

Here $S^{(0)}$ describes the contribution of the classical background field $\hat{\phi}$, $S^{(0)} = S[\hat{\phi}, g_{\mu\nu}]$, $S^{(1)}$ is linear in ϕ, and

$$S^{(2)} = \tfrac{1}{2} \int_{\mathcal{M}} d^n x |g(x)|^{1/2}$$
$$\times \int_{\mathcal{M}} d^n y |g(y)|^{1/2} \phi_i(x) \mathcal{M}_{ij} \phi_j(y)$$

contains the relevant quadratic contributions of the fluctuations, the only relevant ones in the one-loop calculation we are going to perform. Obviously,

$$\mathcal{M}_{ij} = \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \bigg|_{\phi_i = \hat{\phi}_i} ,$$

leading to the fluctuation operator

$$D = -\Delta + U(\hat{\phi}) , \quad (2.2)$$

with

$$U(\hat{\phi})_{ij} = \tilde{U}''(\hat{\phi}) e_i e_j + \frac{\tilde{U}'(\hat{\phi})}{\hat{\phi}} (\delta_{ij} - e_i e_j) , \quad (2.3)$$

where the normalized background field $e_i = \hat{\phi}_i / \sqrt{\hat{\phi}^2}$ has been used.

The effective action of the theory is then expanded in powers of $\hbar$ as

$$\Gamma[\hat{\phi}] = S[\hat{\phi}, g_{\mu\nu}] + \Gamma^{(1)} + \Gamma' , \quad (2.4)$$

with the one-loop contribution $\Gamma^{(1)}$ to the action being

$$\Gamma^{(1)} = \frac{i}{2} \ln \det \frac{D}{\mu^2} \quad (2.5)$$

and higher-loop quantum corrections Γ'. The introduction of the arbitrary mass parameter μ is necessary in order to keep the action dimensionless.

The most interesting quantity for us in this section will be the derivative expansion of the effective action (2.4): that is,

$$\Gamma[\hat{\phi}, g_{\mu\nu}] = \int_{\mathcal{M}} d^n x |g(x)|^{1/2}$$
$$\times [V(\hat{\phi}) + \tfrac{1}{2} \mathcal{Z}_1(\hat{\phi})(\hat{\phi} \Delta \hat{\phi}) + \tfrac{1}{2} \mathcal{Z}_2(\hat{\phi}) \Delta \hat{\phi}^2$$
$$+ \tfrac{1}{2} \mathcal{Z}_3(\hat{\phi}) \frac{\Delta \hat{\phi}^4}{\hat{\phi}^2} + \cdots] , \quad (2.6)$$

where as many as possible of the derivative terms are assumed to be written in a form depending only on O(N)-symmetric quantities.

Our first goal will be to compute the quantities $V(\hat{\phi})$ and $\mathcal{Z}_i(\hat{\phi})$ in Eq. (2.6) up to quadratic powers in the curvature tensors and up to second derivatives of the curvature terms. The tool we shall use to achieve this result will be ζ-function regularization in combination with heat-kernel techniques. In the ζ-function regularization scheme, the functional determinant in Eq. (2.5) is defined by [43,44]

$$\Gamma^{(1)}[\hat{\phi}] = -\frac{i}{2}[\zeta_D'(0) - \zeta_D(0) \ln \mu^2] , \quad (2.7)$$

where $\zeta_D(s)$ is the ζ function associated with the operator D, Eq. (2.2), and the prime denotes differentiation with respect to s. This means that, λ_j being the eigenvalues of D, the ζ function $\zeta_D(s)$ is defined by

$$\zeta_D(s) = \sum_j \lambda_j^{-s} = \frac{i^s}{\Gamma(s)} \sum_j \int_0^\infty dt \, t^{s-1} e^{-i\lambda_j t}$$
$$= \frac{i^s}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{tr} K(x, x, t) , \quad (2.8)$$

where the kernel $K(x, x', t)$ satisfies the equations

$$i \frac{\partial}{\partial t} K(x, x', t) = D K(x, x', t) ,$$
$$\lim_{t \to 0} K(x, x', t) = |g|^{-1/2} \delta(x, x') . \quad (2.9)$$

In order to obtain the derivative expansion, Eq. (2.6), of the effective action, the following ansatz by Parker and Toms is suggested [45,46]:

$$K(x, x', t) = -i \frac{\Delta_{\text{VM}}(x, x')}{(4\pi i t)^{n/2}} \Omega(x, x', t) \exp \left\{ i \left[\frac{\sigma^2(x, x')}{4t} - t[U(\hat{\phi}) - \tfrac{1}{6} R] \right] \right\} , \quad (2.10)$$

where $\sigma(x, x')$ is the proper arc length along the geodesic x' to x and $\Delta_{\rm VM}(x, x')$ is the Van Vleck–Morette determinant. For $t \to 0$ the function $\Omega(x, x', t)$ may be expanded in an asymptotic series

$$\Omega(x, x', t) = \sum_{l=0}^{\infty} a_l(x, x')(it)^l , \tag{2.11}$$

where the coefficients a_l have to satisfy some recurrence relations. Using the ansatz (2.10), it has been shown in Ref. [47] that the dependence of a_l, $l = 1, \ldots, \infty$, on the field $\hat{\phi}$ is only through derivatives of the field.

Up to the order to which we are going to calculate the effective action, we need only include contributions up to a_3. Then, for the ζ function Eq. (2.8) one finds

$$\zeta_D(s) = -\frac{ii^s}{\Gamma(s)(4\pi i)^{n/2}} \int_0^\infty dt\, t^{s-1-n/2}$$
$$\times \text{tr}\big(\exp\{-it[U(\hat{\phi}) - \tfrac{1}{6}R]\}[a_0 - a_2 t^2 - ia_3 t^3 + \cdots]\big) , \tag{2.12}$$

where $a_1 = 0$ has been used. In order to calculate the trace in Eq. (2.12), one has to diagonalize the matrix $U(\hat{\phi}) - \tfrac{1}{6}R$. This is done in Appendix A.

For the calculation of the effective action let us restrict ourselves to the most interesting case $\tilde{U}(\hat{\phi}) = \tfrac{1}{2}(m^2 + \xi R)\hat{\phi}^2 + (\lambda/4!)\hat{\phi}^4$ in $n = 4$ dimensions. Then, as is seen by diagonalizing the matrix $U(\hat{\phi})$, two different masses,

$$m_1^2 = m^2 + (\xi - \tfrac{1}{6})R + \frac{\lambda}{2}\hat{\phi}^2 , \quad m_2^2 = m^2 + (\xi - \tfrac{1}{6})R + \frac{\lambda}{6}\hat{\phi}^2 ,$$

arise. By denoting $\Gamma_i^{(1)}$ the part of the effective action that results from the coefficient a_i and by using

$$b_2 = \tfrac{1}{180}(\Delta R + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu})$$

for the purely geometrical part of the a_2 coefficient, we find

$$\Gamma_0^{(1)} = \frac{1}{32\pi^2} \int_{\mathcal{M}} d^4x |g(x)|^{1/2} \left\{ \frac{m_1^4}{2}\left[\ln\left(\frac{m_1^2}{\mu^2}\right) - \frac{3}{2}\right] + (N-1)\frac{m_2^4}{2}\left[\ln\left(\frac{m_2^2}{\mu^2}\right) - \frac{3}{2}\right] \right\} , \tag{2.13}$$

$$\Gamma_2^{(1)} = \frac{1}{32\pi^2} \int_{\mathcal{M}} d^4x |g(x)|^{1/2}$$
$$\times \left\{ \frac{\lambda}{18}\ln\left(\frac{m_1^2}{m_2^2}\right)\left[-2\hat{\phi}\Delta\hat{\phi} + \frac{1}{2}\Delta\hat{\phi}^2 - \frac{1}{4\hat{\phi}^2}\Delta\hat{\phi}^4\right] + \ln\left(\frac{m_1^2}{m_2^2}\right)[b_2 - \tfrac{1}{6}\Delta m_2^2] \right.$$
$$\left. + \ln\left(\frac{m_2^2}{\mu^2}\right)[Nb_2 - \tfrac{1}{6}(N-1)\Delta m_2^2 - \tfrac{1}{6}\Delta m_1^2] \right\} , \tag{2.14}$$

$$\Gamma_3^{(1)} = \frac{1}{32\pi^2} \int_{\mathcal{M}} d^4x |g(x)|^{1/2} \left\{ -\frac{\lambda^2}{108m_1^2}[-\hat{\phi}^2(\hat{\phi}\Delta\hat{\phi}) - \tfrac{1}{2}\hat{\phi}^2\Delta\hat{\phi}^2 + \tfrac{1}{2}\Delta\hat{\phi}^4] \right.$$
$$\left. - \frac{\lambda^2}{108m_2^2}\left[-\hat{\phi}^2(\hat{\phi}\Delta\hat{\phi}) - \frac{N-4}{4}\hat{\phi}^2\Delta\hat{\phi}^2 + \frac{N-2}{8}\Delta\hat{\phi}^4\right] \right\} . \tag{2.15}$$

The complete one-loop effective action is then given as the sum

$$\Gamma^{(1)} = \Gamma_0^{(1)} + \Gamma_2^{(1)} + \Gamma_3^{(1)} \tag{2.16}$$

of the contributions in Eqs. (2.13)–(2.15).

III. RENORMALIZATION

The quantum correction $\Gamma^{(1)}[\hat{\phi}]$ depends on the arbitrary renormalization scale μ. This dependence will be now removed by the renormalization procedure, which we are going to describe. As is well known by now, one is always forced to take into consideration the most general quadratic gravitational Lagrangian [48], and so one is led to consider the classical Lagrangian

$$L_{\rm cl} = \eta \,\Box\, \hat{\phi}^2 - \tfrac{1}{2}\hat{\phi}\,\Box\,\hat{\phi} + \Lambda + \tfrac{1}{24}\lambda\hat{\phi}^4 + \tfrac{1}{2}m^2\hat{\phi}^2 + \tfrac{1}{2}\xi R\hat{\phi}^2$$
$$+ \varepsilon_0 R + \tfrac{1}{2}\varepsilon_1 R^2 + \varepsilon_2 C + \varepsilon_3 G + \varepsilon_4\,\Box\, R , \tag{3.1}$$

with the corresponding counterterm contributions

$$\delta L_{\rm cl} = \delta\eta\,\Box\,\hat{\phi}^2 + \delta\Lambda + \tfrac{1}{24}\delta\lambda\hat{\phi}^4 + \tfrac{1}{2}\delta m^2\hat{\phi}^2 + \tfrac{1}{2}\delta\xi R\hat{\phi}^2$$
$$+ \delta\varepsilon_0 R + \tfrac{1}{2}\delta\varepsilon_1 R^2 + \delta\varepsilon_2 C + \delta\varepsilon_3 G + \delta\varepsilon_4\,\Box\, R , \tag{3.2}$$

which are necessary in order to renormalize all coupling constants. With C and G we indicate, respectively, the square of the Weyl tensor and the Gauss-Bonnet density. They read

$$C = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2 , \qquad (3.3)$$

$$G = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 . \qquad (3.4)$$

The renormalization conditions are given by [35]

$$\Lambda = L|_{\hat{\phi}=\varphi_0, R=0} ,$$

$$\lambda = \frac{\partial^4 L}{\partial \hat{\phi}^4}\bigg|_{\hat{\phi}=\varphi_1, R=0} ,$$

$$m^2 = \frac{\partial^2 L}{\partial \hat{\phi}^2}\bigg|_{\hat{\phi}=0, R=0} ,$$

$$\xi = \frac{\partial^3 L}{\partial R \partial \hat{\phi}^2}\bigg|_{\hat{\phi}=\varphi_3, R=R_3} ,$$

$$\qquad (3.5)$$

$$\varepsilon_0 = \frac{\partial L}{\partial R}\bigg|_{\hat{\phi}=0, R=0} ,$$

$$\varepsilon_1 = \frac{\partial^2 L}{\partial R^2}\bigg|_{\hat{\phi}=0, R=R_5} ,$$

$$\varepsilon_2 = \frac{\partial L}{\partial C}\bigg|_{\hat{\phi}=0, R=R_6} ,$$

$$\varepsilon_3 = \frac{\partial L}{\partial G}\bigg|_{\hat{\phi}=0, R=R_7} ,$$

$$\varepsilon_4 = \frac{\partial L}{\partial \Box R}\bigg|_{\hat{\phi}=0, R=R_8} ,$$

$$\eta = \frac{\partial L}{\partial \Box \hat{\phi}^2}\bigg|_{\hat{\phi}=\varphi_9, R=0} .$$

Conditions (3.5) determine the counterterms to be

$$64\pi^2\delta\Lambda = -64\pi^2\left(\frac{m^2\varphi_0^2}{2} + \frac{\lambda\varphi_0^4}{24}\right)$$
$$+ z_0(M_{10}, M_{11}, \lambda)$$
$$+ (N-1)z_0(M_{20}, M_{21}, \lambda/3) ,$$

$$64\pi^2\delta\lambda = z_1(M_{11}, \lambda) + (N-1)z_1(M_{21}, \lambda/3) ,$$

$$64\pi^2\delta m^2 = z(\lambda) + (N-1)z(\lambda/3) ,$$

$$64\pi^2\delta\xi = z_3(M_{13}, \lambda) + (N-1)z_3(M_{23}, \lambda/3) ,$$

$$64\pi^2\delta\varepsilon_0 = 2Nm^2(\xi - \tfrac{1}{6})\left(1 - \ln\frac{m^2}{\mu^2}\right) ,$$

$$64\pi^2\delta\varepsilon_1 = -2N(\xi - \tfrac{1}{6})^2 \ln\frac{M_{15}^2}{\mu^2} ,$$

$$64\pi^2\delta\varepsilon_2 = -\frac{1}{60}\ln\frac{M_{16}^2}{\mu^2} - \frac{1}{60}(N-1)\ln\frac{M_{26}^2}{\mu^2} ,$$

$$64\pi^2\delta\varepsilon_3 = \frac{1}{180}\ln\frac{M_{17}^2}{\mu^2} + \frac{1}{180}(N-1)\ln\frac{M_{27}^2}{\mu^2} ,$$

$$64\pi^2\delta\varepsilon_4 = \frac{1}{3}N(\xi - \tfrac{1}{5})\ln\frac{M_{18}^2}{\mu^2} ,$$

$$64\pi^2\delta\eta = \frac{\lambda}{6}\ln\frac{M_{19}^2}{\mu^2} + \frac{\lambda}{18}(N-1)\ln\frac{M_{29}^2}{\mu^2}$$
$$- \frac{\lambda^2\varphi_9^2}{108M_{19}^2} - \frac{\lambda^2\varphi_9^2(N-4)}{216M_{29}^2} , \qquad (3.6)$$

where we have introduced

$$M_{1j}^2 = m^2 + (\xi - \tfrac{1}{6})R_j + \frac{\lambda}{2}\varphi_j^2 ,$$

$$M_{2j}^2 = m^2 + (\xi - \tfrac{1}{6})R_j + \frac{\lambda}{6}\varphi_j^2 ,$$

and the functions

$$z_0(M_0, M_1, \lambda) = -\lambda m^2\varphi_0^2 + \lambda m^2\varphi_0^2 \ln\frac{m^2}{\mu^2} + \frac{\lambda^2\varphi_0^4}{4}\ln\frac{M_1^2}{\mu^2}$$
$$- M_0^4 \ln\frac{M_0^2}{\mu^2} + \frac{3M_0^4}{2}$$
$$- \frac{\lambda^4\varphi_0^4\varphi_1^4}{12M_1^4} + \frac{\lambda^3\varphi_0^4\varphi_1^2}{2M_1^2} ,$$

$$z_1(M_1, \lambda) = -6\lambda^2 \ln\frac{M_1^2}{\mu^2} + \frac{2\lambda^4\varphi_1^4}{M_1^4} - \frac{12\varphi_1^2\lambda^3}{M_1^2} ,$$

$$z(\lambda) = 2\lambda m^2\left(1 - \ln\frac{m^2}{\mu^2}\right) ,$$

$$z_3(M_3, \lambda) = -2\lambda(\xi - \tfrac{1}{6})\ln\frac{M_3^2}{\mu^2} - \frac{2\lambda^2(\xi - \tfrac{1}{6})\varphi_3^2}{M_3^2} ,$$

which describe essentially the counterterms of a single scalar field.

For the sake of generality, we chose different values φ_i, R_i for the definitions of the physical coupling constants. This is due to the fact that, in general, they are measured at different scales, the behavior with respect to a change of scale being determined by the renormalization-group equations we shall derive below.

After some calculation one finds the renormalized effective action. According to Eq. (2.16), we split it into three parts:

$$\Gamma_{\text{ren}}^{(1)} = \Gamma_{0,\text{ren}}^{(1)} + \Gamma_{2,\text{ren}}^{(1)} + \Gamma_{3,\text{ren}}^{(1)} . \qquad (3.7)$$

As we have seen in the renormalization, it is useful to introduce functions describing the contributions coming from the different masses. Thus, we define

$$\gamma_0(m_1, M_{1i}, \lambda) = m^4 \ln \frac{m_1^2}{M_{10}^2} + \lambda m^2 \varphi_0^2 \left(\ln \frac{m^2}{M_{10}^2} + \frac{1}{2} \right)$$

$$+ 2m^2 (\xi - \tfrac{1}{6}) R \left(\ln \frac{m_1^2}{m^2} - \frac{1}{2} \right) + (\xi - \tfrac{1}{6})^2 R^2 \left(\ln \frac{m_1^2}{M_{15}^2} - \frac{3}{2} \right)$$

$$- \frac{\lambda^2 \varphi_0^4}{4} \left[\ln \frac{M_{10}^2}{M_{11}^2} - \frac{3}{2} - \frac{4(M_{11}^2 - m^2)(2M_{11}^2 + m^2)}{3 M_{11}^4} \right]$$

$$+ \left\{ (\xi - \tfrac{1}{6}) R \left[\ln \frac{m_1^2}{M_{13}^2} - \frac{3}{2} - \frac{\lambda \varphi_3^2}{M_{13}^2} \right] + m^2 \left[\ln \frac{m_1^2}{m^2} - \frac{1}{2} \right] \right\} \lambda \hat{\phi}^2$$

$$+ \left\{ \left(\ln \frac{m_1^2}{M_{11}^2} - \frac{25}{6} \right) + \frac{4 m^2 (m^2 + M_{11}^2)}{3 M_{11}^4} \right\} \frac{\lambda^2 \hat{\phi}^4}{4} \tag{3.8}$$

and

$$\gamma_2(m_1^2, M_{1j}^2) = \frac{C}{60} \ln \left(\frac{m_1^2}{M_{16}^2} \right) - \frac{G}{180} \ln \left(\frac{m_1^2}{M_{17}^2} \right) - \tfrac{1}{3}(\xi - \tfrac{1}{5}) \Delta R \ln \left(\frac{m_1^2}{M_{18}^2} \right) . \tag{3.9}$$

Using this, the renormalized versions of (2.13) and (2.14) read, respectively,

$$64\pi^2 L_{0,\mathrm{ren}}^{(1)} = -32\pi^2 m^2 \varphi_0^2 - \frac{8\pi^2 \lambda \varphi_0^4}{3} + \gamma_0(m_1^2, M_{1j}^2, \lambda) + (N-1)\gamma_0(m_2^2, M_{2j}^2, \lambda/3) ,$$

$$64\pi^2 L_{2,\mathrm{ren}}^{(1)} = \frac{\lambda}{9} \ln \left(\frac{m_1^2}{m_2^2} \right) \left[-2\hat{\phi}\Delta\hat{\phi} + \frac{1}{2}\Delta\hat{\phi}^2 - \frac{1}{4\hat{\phi}^2}\Delta\hat{\phi}^4 \right]$$

$$- \frac{\lambda}{18}(N-1)\Delta\hat{\phi}^2 \ln \left(\frac{m_2^2}{M_{29}^2} \right) - \frac{\lambda}{6}\Delta\hat{\phi}^2 \ln \left(\frac{m_2^2}{M_{19}^2} \right)$$

$$- \frac{\lambda}{18}\Delta\hat{\phi}^2 \ln \left(\frac{m_1^2}{m_2^2} \right) + \gamma_2(m_1^2, M_{1j}^2) + (N-1)\gamma_2(m_2^2, M_{2j}^2) . \tag{3.10}$$

The part $L_{3,\mathrm{ren}}$ is directly given by Eq. (2.15) because here no renormalization of this term is necessary.

When considering the case of a constant background field, $\hat{\phi}$ =const, the term $L_{2,\mathrm{ren}}^{(1)}$ reduces to the contributions of γ_2. Furthermore, $L_{3,\mathrm{ren}}^{(1)}$ is equal to zero by construction.

Summarizing, we have obtained here the one-loop renormalized effective Lagrangian in the $O(N)$ model in curved space-time. The next section will be devoted to an attempt (using RG) to include higher-loop effects by improving the effective Lagrangian.

IV. RG IMPROVED EFFECTIVE LAGRANGIAN FOR THE $O(N)$ THEORY

We discuss in this section the RG improvement of the effective Lagrangian. As is well known, the RG improved effective Lagrangian sums all leading logarithms of perturbation theory. We start again from the multiplicatively renormalizable Lagrangian (3.1) in curved space-time, where curvature invariants are necessary in order to make the theory multiplicatively renormalizable. Multiplicative renormalizability results in the standard RG equation for effective actions (and, here, also for the ef-

fective Lagrangian):

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_i \frac{\partial}{\partial \lambda_i} - \gamma \hat{\phi} \frac{\partial}{\partial \hat{\phi}} \right) L_{\mathrm{eff}}(\mu, \lambda_i, \hat{\phi}) = 0 , \tag{4.1}$$

where $\lambda_i = \{\lambda, m^2, \xi, \eta, \Lambda, \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ are all the coupling constants of the theory in the matter sector as well as in the external gravitational field sector, β_i are the corresponding β functions, and γ is the anomalous dimension of the scalar field.

Application of the method of characteristics to (4.1) immediately yields the solution

$$L_{\mathrm{eff}}(\mu, \lambda_i, \hat{\phi}) = L_{\mathrm{eff}}(\mu(t), \lambda_i(t), \hat{\phi}(t)) , \tag{4.2}$$

where

$$\frac{d\lambda_i(t)}{dt} = \beta_i(\lambda_i(t)) , \quad \lambda_i(0) = \lambda_i , \tag{4.3}$$

$$\mu(t) = \mu e^t , \quad \hat{\phi}(t) = \hat{\phi} \exp \left(- \int_0^t dt' \, \gamma(t') \right) .$$

The one-loop β functions of the theory can be easily found, from the calculations in the preceding section, in the form

$$\beta_\lambda = \frac{(N+8)\lambda^2}{3(4\pi)^2} \ , \quad \gamma = 0 \ ,$$

$$\beta_{m^2} = \frac{N+2}{3(4\pi)^2}\lambda m^2 \ ,$$

$$\beta_\xi = \frac{N+2}{3(4\pi)^2}\lambda(\xi - \tfrac{1}{6}) \ ,$$

$$\beta_\eta = -\frac{\lambda(N+2)}{36(4\pi)^2} \ ,$$

$$\beta_\Lambda = \frac{Nm^4}{2(4\pi)^2} \ , \tag{4.4}$$

$$\beta_{\epsilon_0} = \frac{Nm^2(\xi - \tfrac{1}{6})}{(4\pi)^2} \ ,$$

$$\beta_{\epsilon_1} = \frac{2N(\xi - \tfrac{1}{6})^2}{(4\pi)^2} \ ,$$

$$\beta_{\epsilon_2} = \frac{N}{120(4\pi)^2} \ ,$$

$$\beta_{\epsilon_3} = \frac{N}{360(4\pi)^2} \ .$$

The solution of the RG equations (4.3) is easily found to be

$$\lambda(t) = \lambda\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{-1} \ ,$$

$$m^2(t) = m^2\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{-(N+2)/(N+8)} \ ,$$

$$\xi(t) = \tfrac{1}{6} + (\xi - \tfrac{1}{6})\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{-(N+2)/(N+8)} \ ,$$

$$\eta(t) = \eta + \frac{N+2}{12(N+8)}\ln\left[1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right] \ ,$$

$$\Lambda(t) = \Lambda - \frac{3Nm^4}{2\lambda(4-N)}$$
$$\times \left[\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{(4-N)/(N+8)} - 1\right] \ ,$$

$$\epsilon_0(t) = \epsilon_0 - \frac{3Nm^2(\xi - \tfrac{1}{6})}{\lambda(4-N)}$$
$$\times \left[\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{(4-N)/(N+8)} - 1\right] \ ,$$

$$\epsilon_1(t) = \epsilon_1 - \frac{3N(\xi - \tfrac{1}{6})^2}{\lambda(4-N)}$$
$$\times \left[\left(1 - \frac{(N+8)\lambda t}{3(4\pi)^2}\right)^{(4-N)/(N+8)} - 1\right] \ ,$$

$$\epsilon_2(t) = \epsilon_2 + \frac{Nt}{120(4\pi)^2} \ ,$$

$$\epsilon_3(t) = \epsilon_3 - \frac{Nt}{360(4\pi)^2} \ .$$

Note that, as it follows from those expressions, in the IR limit $(t \to -\infty)$, where the theory is asymptotically free, the matter-sector coupling constants tend to approach their conformally invariant values.

Now, using the classical Lagrangian (3.1) as boundary function, we get the RG improved effective Lagrangian in the leading-log approximation:

$$L_{\text{eff}} = \tfrac{1}{2}g^{\mu\nu}\partial_\mu\hat\phi\partial_\nu\hat\phi + \eta(t)\,\Box\,\hat\phi^2 + \lambda(t)\frac{\hat\phi^4}{4!}$$
$$+\tfrac{1}{2}[m^2(t) + \xi(t)R]\hat\phi^2 + \Lambda(t)$$
$$+\epsilon_0(t)R + \tfrac{1}{2}\epsilon_1(t)R^2$$
$$+\epsilon_2(t)C + \epsilon_3(t)G + \epsilon_4(t)\,\Box\,R \ . \tag{4.5}$$

Here, the question of the choice of t appears. As we see from the results of the one-loop analysis of the preceding section, there are two effective masses:

$$m_1^2 = m^2 + \frac{\lambda\hat\phi^2}{2} + (\xi - \tfrac{1}{6})R \ ,$$

$$m_2^2 = m^2 + \frac{\lambda\hat\phi^2}{6} + (\xi - \tfrac{1}{6})R \ .$$

Evidently, there is no choice of t, which eliminates all logarithms of the perturbation theory. However, we may, of course, consider separately the different regions for the parameters of the theory.

In particular, for large $\hat\phi^2$ we have the natural choice $t = \tfrac{1}{2}\ln(\hat\phi^2/\mu^2)$. Then Eq. (4.5) controls the large $\hat\phi^2$ behavior of the effective potential. Similarly, for large values of the mass, $m^2 > \hat\phi^2$, $m^2 > |R|$, we have another choice $t = \tfrac{1}{2}\ln(m^2/\mu^2)$. Eventually, we reach the conclusion that we do not know how to solve this problem of writing the RG improved effective potential for the theory under discussion in a *unique* form valid for all regions of the parameters of the theory.

For the massless theory in flat space, the potential (4.5) adopts the Coleman-Weinberg form [49] (for a discussion of RG improved effective potentials, see also [50–53]). As has been recognized recently [54–56], the RG improved effective potential for massive theories in flat space includes some term corresponding to the running of the vacuum energy (the potential at zero $\hat\phi^2$). In the above approach this term appears as the effective cosmological constant $\Lambda(t)$, which provides a very natural interpretation of it.

Let us restrict ourselves to the case when $N = 1$. Then the RG improved effective potential is given again by (4.5) (see also [57]), with the natural choice

$$t = \frac{1}{2}\ln\frac{m^2 + \lambda\hat\phi^2/2 + (\xi - \tfrac{1}{6})R}{\mu^2} \ . \tag{4.6}$$

So, we are able to construct a RG improved potential, which has the same form for the whole parameter region in the case $N = 1$.

Another case of particular interest is the massless one. Limiting ourselves here to linear curvature terms, we get the following RG improved effective Lagrangian for the $O(N)$ theory:

$$L_{\text{eff}} = \tfrac{1}{2}g^{\mu\nu}\partial_\mu\hat\phi\partial_\nu\hat\phi + \eta(t)\,\Box\,\hat\phi^2 + \tfrac{1}{2}\xi(t)R\hat\phi^2$$
$$+\lambda(t)\frac{\hat\phi^4}{4!} + \Lambda(t) + \epsilon_0(t)R \ , \tag{4.7}$$

where $t = \frac{1}{2}\ln(\hat{\phi}^2/\mu^2)$. Of course, curvature should be chosen to be slowly varying. In a similar way, taking into account the fact that the choice of t is actually unique, one can get the RG improved effective Lagrangian for a large variety of massless theories in curved spacetime [58].

V. THE BACK-REACTION EFFECT IN HYPERBOLIC SPACE-TIME

As a further application of the results of the previous sections, we would like here to consider the back-reaction effect. In order to obtain some concrete information on the back reaction, we have to restrict ourselves to some space-time with a higher symmetry. Note that for time-dependent backgrounds the back-reaction problem is very difficult to study, even in the case of free fields ([5] and [40–42]), due to the fact that the effective equations include higher derivative terms as is clearly seen also from our effective Lagrangian (3.10). That is why, unfortunately, we were not able to apply our quite general formalism of the first sections to the back-reaction problem on time-varying backgrounds (that demands very involved numerical calculations). Note, however, that recently an interesting approach to the back-reaction has been developed [59], which shows the way to reduce the problem to the low-derivatives case. Instead, we limit ourselves to the space of constant curvature. We choose the space-time $\mathbb{R}^2 \times H^2/\Gamma$, where Γ is a cocompact discrete group in $PSL(2,\mathbb{R})$ containing only hyperbolic elements [60]. This choice is interesting for the following reasons. First of all, it has been shown that nontrivial topology might have considerably influenced the early universe and is, in principle, observable ([61] and [62]). From another point of view, for that space-time exact results for the one-loop effective potential are at hand [25], and one is able to see in which specific range the effective potential obtained in this paper is a good approximation to the exact result, and where it fails to be so.

Let us first start our considerations from the RG improved effective potential (4.5), choosing for simplicity $N = 1$ and $m^2 = 0$. We are aware of the fact that in this case our approximation is not quite good, since we are loosing the part of the whole information connected with the nontrivial topology. However, we hope to get some hint from the results obtained because we have included some higher-loop corrections in this approach and, also,

below we will present a similar analysis using the outcome of the mentioned explicit one-loop calculation of the effective action on $\mathbb{R}^2 \times H^2/\Gamma$, thus showing the influence of the topology in detail. This will also justify our qualitative result for the purely quantum solution [see (5.8)].

Taking into account the properties of the space under consideration, namely,

$$R = -\frac{2}{\rho^2} , \quad \int d^4x\sqrt{g} = V(F)\rho^2 , \tag{5.1}$$

we obtain

$$\frac{S_{\text{eff}}}{V(F)} = \lambda(t)\frac{\hat{\phi}^4\rho^2}{4!} - \xi(t)\hat{\phi}^2$$
$$+\Lambda(t)\rho^2 - 2\epsilon_0(t) + \frac{2\epsilon_1(t)}{\rho^2} , \tag{5.2}$$

where

$$t = \frac{1}{2}\ln\frac{\lambda\hat{\phi}^2/2 - 2(\xi - \frac{1}{6})\rho^{-2}}{\mu^2} ,$$

$$\lambda(t) = \lambda\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1} ,$$

$$\xi(t) = \frac{1}{6} + (\xi - \frac{1}{6})\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-1/3} ,$$

$$\epsilon_1(t) = \epsilon_1 - \frac{1}{\lambda}(\xi - \frac{1}{6})^2\left[\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{1/3} - 1\right] .$$

Notice that here the only nontrivial parameter of the metric is ρ. Hence, the effective equations are given by

$$\frac{\partial S_{\text{eff}}}{\partial\hat{\phi}} = 0 , \quad \frac{\partial S_{\text{eff}}}{\partial\rho} = 0 , \quad \epsilon_0(t) = \epsilon_0 , \quad \Lambda(t) = \Lambda . \tag{5.3}$$

Then, using S_{eff} (5.2), we get

$$\hat{\phi}\left[\frac{1}{6}\lambda(t)\rho^2\hat{\phi}^2 - 2\xi(t) + B(t,\xi,\rho\hat{\phi})\frac{\lambda}{2}\right] = 0 , \tag{5.4}$$

$$\frac{1}{12}\lambda(t)\rho^4\hat{\phi}^4 + 2\rho^4\Lambda - 4\epsilon_1(t) + 2(\xi - \frac{1}{6})B(t,\xi,\rho\hat{\phi}) = 0 ,$$

being

$$B(t,\xi,\rho\hat{\phi}) \equiv \left[\frac{\lambda\rho^4\hat{\phi}^4}{4!}\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-2} - \frac{1}{3}(\xi - \frac{1}{6})\rho^2\hat{\phi}^2\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-4/3}\right.$$
$$\left. + \frac{2}{3\lambda}(\xi - \frac{1}{6})^2\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-2/3}\right]\frac{3\lambda}{(4\pi)^2}\left[\frac{\lambda\rho^2\hat{\phi}^2}{2} - 2(\xi - \frac{1}{6})\right]^{-1} . \tag{5.5}$$

Let us now consider some simple, different cases.

(1) Case $\hat{\phi} = 0$. Here we have two possibilities.

(1a) Subcase $\Lambda \neq 0$. Then we obtain a perturbative solution of the form

$$\rho = \left\{ \frac{2\epsilon_1}{\Lambda} - \frac{2}{\lambda\Lambda}\left(\xi - \frac{1}{6}\right)^2\left[\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{1/3} - 1\right] + \frac{1}{(4\pi)^2\Lambda}\left(\xi - \frac{1}{6}\right)^2\left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{-2/3}\right\}^{1/4} \simeq \left(\frac{2\epsilon_1}{\Lambda}\right)^{1/4} + O(\hbar) .$$
(5.6)

(1b) Subcase $\Lambda = \epsilon_1 = 0$. In this special case we obtain a purely quantum solution, by solving

$$1 - \frac{3\lambda t}{(4\pi)^2} - \left(1 - \frac{3\lambda t}{(4\pi)^2}\right)^{2/3} - \frac{\lambda}{2(4\pi)^2} = 0 ,$$
(5.7)

with $t = \frac{1}{2}\ln[-2(\xi - 1/6)(\rho\mu)^{-2}]$. Assuming that λ is small, we get (with very good approximation): $t \simeq -2$, that is

$$(\rho\mu)^{-2} \simeq -\frac{1}{2e^2(\xi - 1/6)} ,$$
(5.8)

which has sense only for $\xi < 1/6$. Although, as we mentioned, one may not trust the approximation, we will see that also the one-loop effective potential leads to the existence of a purely quantum solution. That we will show below using the one-loop effective potential calculated with the help of the Selberg trace formula on $\mathbb{R}^2 \times H^2/\Gamma$ in [25].

(2) Case $\hat{\phi} \neq 0$, $\xi = 1/6$. Here, assuming again λ small, from Eq. (5.4) the following expressions are easily obtained:

$$\lambda\rho^2\hat{\phi}^2 \simeq 2 , \quad \lambda\rho^4\hat{\phi}^4 + 24\Lambda\rho^4 \simeq 48\epsilon_1 ,$$
(5.9)

and combining them, we get

$$\rho \simeq \left(\frac{2\epsilon_1 - 1/(6\lambda)}{\Lambda}\right)^{1/4} , \quad \hat{\phi}^2 \simeq \left(\frac{2\Lambda}{\lambda}\right)^{1/2}\frac{1}{\sqrt{\epsilon_1\lambda - 1/12}} ,$$
(5.10)

which has sense only in the case $\Lambda < 0$, $\epsilon_1 < 1/(12\lambda)$, or else $\Lambda > 0$, but then $\epsilon_1 > 1/(12\lambda)$. And those are the only possibilities that come out of the simple cases considered here. All the other cases are difficult to handle analytically.

To get an idea of the reliability of the approximation, we must first state the exact result for the $\lambda\varphi^4$ theory derived in [25] and then perform a similar analysis. Without going into the details of the calculation (the interested reader may consult [25]) the effective potential reads

$$S_{\text{eff}} = -\xi\hat{\phi}^2 + \frac{2\epsilon_1}{\rho^2} + \Lambda\rho^2 + \frac{\lambda}{4!}\rho^2\hat{\phi}^4 - 2\epsilon_0$$

$$- \frac{\hbar}{128\pi^2}\left\{\frac{25}{12}\lambda^2\hat{\phi}^4\rho^2 + 2\lambda\hat{\phi}^2[\frac{9}{8} - 7\xi] + \frac{4}{\rho^2}[3\xi^2 - \frac{11}{12}\xi + \frac{79}{960}] - \frac{8X}{\rho^2} + \frac{16\pi H}{V(\mathcal{F})\rho^2}\right.$$

$$\left. - \left[\frac{1}{2}\lambda^2\hat{\phi}^4\rho^2 - 4\lambda\hat{\phi}^2(\xi - \frac{1}{6}) + \frac{8}{\rho^2}[(\xi - \frac{1}{6})^2 + \frac{1}{180}]\right]\left[\ln\left(\frac{\hat{\phi}^2}{M^2}\right) + \ln\left(1 + \frac{2\epsilon}{\lambda\hat{\phi}^2\rho^2}\right)\right]\right\} ,$$
(5.11)

where

$$X = \int_0^\infty dr\, r(r^2 + \delta^2)\ln\left(1 + \frac{r^2}{\delta^2}\right)(1 - \tanh\pi r) ,$$

$$H = \int_0^\infty dy(y^2 + 2y\delta)\frac{\mathcal{Z}'}{\mathcal{Z}}(y + \delta + \frac{1}{2}) .$$
(5.12)

$\mathcal{Z}$ is a Selberg-type ζ function and, furthermore, $\delta^2 = [\xi R + (\lambda/2)\hat{\phi}^2]\rho^2$ and $\epsilon = \xi - \frac{1}{8}$. It may be shown that in the limit of small curvature the resulting equation (5.11) reduces to the adiabatic approximation (3.10), apart from finite renormalization terms. From (5.11) one easily finds the effective equation (5.3) determining the back reaction:

$$V^{-1}(\mathcal{F})\frac{\partial S_{\text{eff}}}{\partial\hat{\phi}} = -2\xi\hat{\phi} + \frac{\lambda}{6}\rho^2\hat{\phi}^3$$

$$- \frac{\hbar}{128\pi^2}\left\{\frac{25}{3}\lambda^2\rho^2\hat{\phi}^3 + 4\lambda\hat{\phi}[\frac{9}{8} - 7\xi] - \frac{8}{\rho^2}\left(\frac{\partial X}{\partial\hat{\phi}}\right) + \frac{16\pi}{V(\mathcal{F})\rho^2}\left(\frac{\partial H}{\partial\hat{\phi}}\right)\right.$$

$$- [2\lambda^2\rho^2\hat{\phi}^3 - 8\lambda\hat{\phi}(\xi - \frac{1}{6})]\left[\ln\left(\frac{\hat{\phi}^2}{M^2}\right) + \ln\left(1 + \frac{2\epsilon}{\lambda\hat{\phi}^2\rho^2}\right)\right]$$

$$\left. - \left[\frac{1}{2}\lambda^2\hat{\phi}^4\rho^2 - 4\lambda\hat{\phi}^2(\xi - \frac{1}{6}) + \frac{8}{\rho^2}[(\xi - \frac{1}{6})^2 + \frac{1}{180}]\right]\frac{2\lambda\rho^2\hat{\phi}}{\lambda\hat{\phi}^2\rho^2 + 2\epsilon}\right\} = 0 ,$$
(5.13)

$$V^{-1}(\mathcal{F})\frac{\partial S_{\text{eff}}}{\partial \rho} = -\frac{4\epsilon_1}{\rho^3} - 2\rho\Lambda + \frac{\lambda}{12}\rho\hat{\phi}^4$$

$$-\frac{\hbar}{128\pi^2}\left\{\tfrac{25}{6}\lambda^2\rho\hat{\phi}^4 + -\frac{8}{\rho^3}[3\xi^2 - \tfrac{11}{12}\xi + \tfrac{79}{960}] + \frac{16X}{\rho^3}\right.$$

$$-\frac{32\pi H}{V(\mathcal{F})\rho^3} - \frac{8}{\rho^2}\left(\frac{\partial X}{\partial \rho}\right) + \frac{16\pi}{V(\mathcal{F})\rho^2}\left(\frac{\partial H}{\partial \rho}\right)$$

$$-\left[\lambda^2\rho\hat{\phi}^4 - \frac{16}{\rho^3}[(\xi - \tfrac{1}{6})^2 + \tfrac{1}{180}]\right]\left[\ln\left(\frac{\hat{\phi}^2}{M^2}\right) + \ln\left(1 + \frac{2\epsilon}{\lambda\hat{\phi}^2\rho^2}\right)\right]$$

$$\left.+\left[\frac{1}{2}\lambda^2\hat{\phi}^4\rho^2 - 4\lambda\hat{\phi}^2(\xi - \tfrac{1}{6}) + \frac{8}{\rho^2}[(\xi - \tfrac{1}{6})^2 + \tfrac{1}{180}]\right]\frac{4\epsilon}{\rho}\frac{1}{\lambda\hat{\phi}^2\rho^2 + 2\epsilon}\right\} = 0 . \tag{5.14}$$

In general, it is impossible to solve Eqs. (5.13) and (5.14), explicitly. However, it is easily seen that $\hat{\phi} = 0$ is an exact solution. In this case equation (5.14) reduces to

$$0 = \frac{4\epsilon_1}{\rho^4} - 2\Lambda$$

$$+\frac{\hbar}{128\pi^2\rho^4}\left\{-8(3\xi - \tfrac{11}{12}\xi + \tfrac{79}{960}) + 16X - \frac{32\pi H}{V(\mathcal{F})}\right.$$

$$\left.+16\left[(\xi - \tfrac{1}{6})^2 + \tfrac{1}{180}\right][\ln(2\epsilon) - \ln(\lambda\rho^2 M^2)] + 16[(\xi - \tfrac{1}{6})^2 + \tfrac{1}{180}]\right\} . \tag{5.15}$$

This can be written as

$$\frac{1}{\rho^4}[a + b\ln(\rho^2 M^2)] - c = 0 , \tag{5.16}$$

where we have introduced

$$a = -4\epsilon_1 - \frac{\hbar}{128\pi^2}\left\{-8[3\xi - \tfrac{11}{12}\xi + \tfrac{79}{960}] + 16X - \frac{32\pi H}{V(\mathcal{F})} + 16[(\xi - \tfrac{1}{6})^2 + \tfrac{1}{180}]\left[1 + \ln\left(\frac{2\epsilon}{\lambda}\right)\right]\right\} ,$$

$$b = \frac{\hbar}{8\pi^2}[(\xi - \tfrac{1}{6})^2 + \tfrac{1}{180}] ,$$

$$c = 2\Lambda .$$

VI. CONCLUSIONS

For $\lambda \neq 0$, using the ansatz $\rho = \exp[-a/(2b)]\tilde{\rho}$ (one always has $b > 0$), Eq. (5.16) reads

$$ce^{-2a/b}\tilde{\rho}^4 = \ln(\tilde{\rho}M) . \tag{5.17}$$

Depending on the values of a, b, and c, the number of solutions may be two, one, or none.

If $\Lambda = 0$, $\epsilon_1 = 0$, then one finds the purely quantum solution $(\rho M)^2 = \exp(-a/b)$, where one-loop topological effects are included exactly and are seen to be of importance. However, it is of course difficult to find the explicit values of a, since H is very hard to calculate. Notice that, in principle, in order to be consistent one should include into the discussion quantum gravity (for example, in frames of Einstein gravity). We expect that such an inclusion will not qualitatively change the solutions of the effective equations. Note, finally, that the analysis of the case $\hat{\phi} \neq 0$ is extremely complicated, due once more to the structure of H, and will not be presented here.

In new inflationary models the effective cosmological constant is obtained from an effective potential, which includes quantum corrections to the classical potential of a scalar field [49]. For that reason an intensive effort has been dedicated to the analysis of the one-loop effective action of a self-interacting scalar field in curved space-time. In this paper we have continued this research by considering a self-interacting $O(N)$ scalar theory in curved space-time. First of all we have obtained the derivative expansion of the effective action of the theory (Secs. II and III). It is seen from the analysis, that two different effective masses arise, the expansion being consistent if both effective masses are large as compared to a typical magnitude of the curvature and to the variation of the background field. Based on the one-loop analysis we have then discussed the RG improvement in Sec. IV. As an application of this study we have considered the back-reaction problem in hyperbolic space-times of the type

$\mathbb{R}^2 \times H^2/\Gamma$. The specific way the hyperbolic space is induced as a result of quantum effects is clearly seen for some values of the parameters. When comparing the RG improved result with the exact result available in that case [25], we have seen that topology plays in fact an important role in this kind of space-times. This, however (as is well known), may not be seen by using the local adiabatic approximation scheme.

Let us also say that one may arrive, qualitatively, to the same conclusions when considering the space-time $\mathbb{R} \times H^3/\Gamma$, using the results in [24]. Finally, it would be very interesting (albeit not very easy) to extend the above results to dynamical space-times.

ACKNOWLEDGMENTS

APPENDIX: DIAGONALIZATION OF THE MATRIX

The aim of this appendix is the diagonalization of the matrix $U(\hat{\phi}) - \frac{1}{6}R$, which is needed for solving Eq. (2.12). As given in Eq. (2.3), we have

$$[U(\hat{\phi}) - \tfrac{1}{6}R]_{ij} = d\delta_{ij} + fe_ie_j , \tag{A1}$$

with

$$d = \frac{U'(\hat{\phi})}{\hat{\phi}} - \tfrac{1}{6}R , \quad f = U''(\hat{\phi}) - \frac{U'(\hat{\phi})}{\hat{\phi}} . \tag{A2}$$

In order to diagonalize the matrix in Eq. (A1), we have to solve the eigenvalue equation

$$U(\hat{\phi})\mathbf{v}^{(j)} = \lambda_j\mathbf{v}^{(j)} , \quad j = 1,\ldots,N . \tag{A3}$$

By doing this for $N = 2,3$, it is possible to guess the result for arbitrary N. We have found

$$\lambda_1 = d + f; \quad \lambda_2 = \cdots = \lambda_N = d , \tag{A4}$$

with the corresponding eigenfunctions

$$v_l^{(1)} = e_l ;$$

$$v_1^{(l)} = e_l , \quad v_j^{(j)} = -e_1 , \quad v_k^{(j)} = 0 , \quad k \neq 1 ,j . \tag{A5}$$

This determines the matrix S, which diagonalizes the matrix $U(\hat{\phi})$ by $U_{\text{diag}}(\hat{\phi}) = S^{-1}U(\hat{\phi})S$, to be

$$S = \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & e_N \\ e_2 & -e_1 & 0 & \cdots & 0 \\ e_3 & 0 & -e_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_N & 0 & 0 & \cdots & -e_1 \end{pmatrix} , \tag{A6}$$

and its inverse

$$S^{-1} = \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & e_N \\ e_2 & -\frac{1-e_2^2}{e_1} & \frac{e_3}{e_1}e_2 & \cdots & \frac{e_N}{e_1}e_3 \\ e_3 & \frac{e_2}{e_1}e_3 & -\frac{1-e_3^2}{e_1} & \cdots & \frac{e_N}{e_1}e_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_N & \frac{e_2}{e_1}e_N & \frac{e_3}{e_1}e_N & \cdots & -\frac{1-e_N^2}{e_1} \end{pmatrix} . \tag{A7}$$

With these results at hand, Eqs. (2.13)–(2.15) are found from

$$\text{tr}\left(\exp\{-it[U(\hat{\phi}) - \tfrac{1}{6}R]\}a_i \right) = \text{tr}(S^{-1}\exp\{-it[U(\hat{\phi}) - \tfrac{1}{6}R]\}SS^{-1}a_iS)$$

$$= \text{tr}\left\{ \begin{pmatrix} \exp(-it[d+f]) & 0 & \cdots & 0 \\ 0 & \exp(-itd) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp(-itd) \end{pmatrix} S^{-1}a_iS \right\} . \tag{A8}$$

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