The History and Present Status of Quantum Field Theory in Curved Spacetime

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Abstract

Quantum field theory in curved spacetime is a theory wherein matter is treated fully in accord with the principles of quantum field theory, but gravity is treated classically in accord with general relativity. It is not expected to be an exact theory of nature, but it should provide a good approximate description in circumstances where the quantum effects of gravity itself do not play a dominant role. Some of the earliest applications of the theory were to study particle creation effects in an expanding universe. A major impetus to the theory was provided by Hawking's calculation of particle creation by black holes, showing that black holes radiate as perfect black bodies. During the past 30 years, considerable progress has been made in giving a mathematically rigorous formulation of quantum field theory in curved spacetime. Major issues of principle with regard to the formulation of the theory arise from the lack of Poincare symmetry, the absence of a preferred vacuum state, and, in general, the absence of asymptotic regions in which particle states can be defined. By the mid-1980's, it was understood how all of these difficulties could be overcome for free (i.e., non-self-interacting) quantum fields by formulating the theory via the algebraic approach and focusing attention on the local field observables rather than a notion of "particles". However, these ideas, by themselves, were not adequate for the formulation of interacting quantum field theory, even at a perturbative level, since standard renormalization prescriptions in Minkowski spacetime rely heavily on Poincare invariance and the existence of a Poincare invariant vacuum state. However, during the past decade, great progress has been made, mainly due to the importation into the theory of the methods of "microlocal analysis". This article will describe the historical development of the subject and describe some of the recent progress.

1 Introduction

Quantum field theory in curved spacetime is the theory of quantum fields propagating in a classical curved spacetime. Here the spacetime is described, in accord with general relativity, by a manifold, M, on which is defined a Lorentz metric, g_{ab} . In order to assure that classical dynamics is well defined on (M, g_{ab}) , we restrict attention to the case where (M, g_{ab}) is globally hyperbolic (see, e.g., [1]). In the framework of quantum field theory in curved spacetime, back-reaction of the quantum fields on the spacetime geometry can be taken into account by imposing the semi-classical Einstein equation $G_{ab} = 8\pi \langle T_{ab} \rangle$. However, I will not consider issues associated with back-reaction here, so in the following, (M, g_{ab}) may be taken to be an arbitrary, fixed globally hyperbolic spacetime.

This article will focus primarily on issues concerning the formulation of quantum field theory in curved spacetime. I will describe some aspects of the historical evolution of the subject through the mid-1970's, at which point it had become clear that a proper formulation of the theory could not be based upon a notion of "particles". I will then describe how the major conceptual obstacles to formulating the theory in the case of free quantum fields were overcome by adopting the algebraic approach. Finally, I will describe some of the progress that has been made during the past decade toward the formulation of interacting quantum field theory in curved spacetime.

Much of the quantum theory of a free field follows directly from the analysis of an ordinary quantum mechanical harmonic oscillator, described by the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2. \tag{1}$$

By introducing the "lowering" (or "annihilation") operator

$$a \equiv \sqrt{\frac{\omega}{2}} \ q + i\sqrt{\frac{1}{2\omega}} \ p \tag{2}$$

we can rewrite H as

$$H = \omega(a^{\dagger}a + \frac{1}{2}I) \tag{3}$$

where a^{\dagger} is referred to as the "raising" (or "creation") operator, and we have the commutation relations

$$[a, a^{\dagger}] = I, \ [H, a] = -\omega a.$$
 (4)

It then follows that in the Heisenberg representation, the position operator, q_H , is given by

$$q_H = \sqrt{\frac{1}{2\omega}} \left(e^{-i\omega t} a + e^{i\omega t} a^{\dagger} \right) \tag{5}$$

so that a is seen to be the positive frequency part of the Heisenberg position operator. The ground state, $|0\rangle$, of the harmonic oscillator is determined by

$$a|0\rangle = 0. (6)$$

All other states of the harmonic oscillator are obtained by successive applications of a^{\dagger} to $|0\rangle$.

Consider, now, a free Klein-Gordon scalar field, ϕ , in Minkowski spacetime. Classically, ϕ satisfies the wave equation

$$\partial^a \partial_a \phi - m^2 \phi = 0. (7)$$

To avoid technical awkwardness, it is convenient to imagine that the scalar field resides in a cubic box of side L with periodic boundary conditions. In that case, $\phi(t, \vec{x})$ can be decomposed in terms of a Fourier series in \vec{x} . In terms of the Fourier coefficients

$$\phi_{\vec{k}} \equiv L^{-3/2} \int e^{-i\vec{k}\cdot\vec{x}} \phi(t,\vec{x}) \ d^3x \tag{8}$$

where

$$\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3) \tag{9}$$

we have

$$H = \sum_{\vec{k}} \frac{1}{2} \left(|\dot{\phi}_{\vec{k}}|^2 + \omega_{\vec{k}}^2 |\phi_{\vec{k}}|^2 \right)$$
 (10)

where

$$\omega_{\vec{k}}^2 = |\vec{k}|^2 + m^2 \,. \tag{11}$$

Thus, a free Klein-Gordon field, ϕ , is seen to be nothing more than an infinite collection of decoupled harmonic oscillators. The quantum field theory associated to ϕ can therefore be obtained by quantizing each of these oscillators. It follows immediately that the Heisenberg field operator $\phi(t, \vec{x})$ should be given by the formula

$$\phi(t, \vec{x}) = L^{-3/2} \sum_{\vec{k}} \frac{1}{2\omega_{\vec{k}}} \left(e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t} a_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}}t} a_{\vec{k}}^{\dagger} \right). \tag{12}$$

However, the sum on the right side of this equation does not converge in any sense that would allow one to define the operator ϕ at the point (t, \vec{x}) . Roughly speaking, the infinite number of arbitrarily high frequency oscillators fluctuate too much to allow $\phi(t, \vec{x})$ to be defined. However, this difficulty can be overcome by "smearing" ϕ with an arbitrary "test function", f (i.e., f is a smooth function of compact support), so as to define

$$\phi(f) = \int f(t, \vec{x})\phi(t, \vec{x})d^4x \tag{13}$$

rather than $\phi(t, \vec{x})$. The resulting formula for $\phi(f)$ can be shown to make rigorous mathematical sense, thus defining ϕ as an "operator-valued distribution".

The ground state, $|0\rangle$, of ϕ is simply simultaneous ground state of all of the harmonic oscillators that comprise ϕ , i.e., it is the state satisfying $a_{\vec{k}}|0\rangle = 0$ for all \vec{k} . In quantum field theory, this state is interpreted as representing the "vacuum". A state of the form $(a^{\dagger})^n|0\rangle$ is interpreted as a state where a total of n particles are present. In an interacting theory, the state of the field may be such that the field behaves like a free field at early

and late times. In that case, we would have a particle interpretation of the states of the field at early and late times. The relationship between the early and late time particle descriptions of a state—given by the S-matrix—contains a great deal of the dynamical information about the interacting theory, and, indeed, contains all of the information relevant to laboratory scattering experiments.

The particle interpretation/description of quantum field theory in flat spacetime has been remarkably successful—to the extent that one might easily get the impression from the way the theory is normally described that, at a fundamental level, quantum field theory is really a theory of particles. However, the definition of particles relies on the decomposition of ϕ into annihilation and creation operators in eq.(12). This decomposition, in turn, relies heavily on the time translation symmetry of Minkowski spacetime, since the "annihilation part" of ϕ is its positive frequency part with respect to time translations. In a curved spacetime that does not possess a time translation symmetry, it is far from obvious how a notion of "particles" should be defined.

2 The Development of Quantum Field Theory in Curved Spacetime from the mid-1960's Through the mid-1970's

Beginning in the mid-1960's, Parker investigated effects of particle creation in an expanding universe [2, 3]. Consider a spatially flat Friedmann-Lemaitre-Robertson-Walker spacetime, with metric

$$ds^{2} = -dt^{2} + a^{2}(t)[dx^{2} + dy^{2} + dz^{2}].$$
(14)

Consider, first the (highly artificial) case where a(t) is constant for $t < t_0$ and is again constant for $t > t_1$, but goes through a time-dependent phase at intermediate times, $t_0 \leq t \leq t_1$. In the "in" region $t < t_0$, spacetime is locally indistinguishable from a corresponding portion of Minkowski spacetime, so a given state of a free quantum field will have a particle interpretation in that region, i.e., it can be characterized by its "particle content". Similarly, in the "out" region, $t > t_1$, the same state will also have a particle interpretation. However, on account of the time dependence of the metric in the intermediate region, a classical solution of the Klein-Gordon equation that corresponds to a purely positive frequency solution in the "in" region will not correspond to a purely positive frequency solution in the "out region". This means that the "in" and "out" annihilation and creation operators of the quantum field (corresponding to the decomposition of eq.(12) for the "in" and "out" regions) will be different. This, in turn, implies that the particle content in the "in" and "out" regions will be different. In other words, the expansion of the universe will result in spontaneous particle creation. Quite generally, the relationship between the "in" and "out" annihilation and creation operators is given by a Bogoliubov transformation, whose coefficients are determined by the classical scattering. It is not difficult to derive a general formula for the resulting S-matrix in terms of these coefficients and, in particular, an expression for the spontaneous particle creation from the vacuum (see, e.g., [4] for a general derivation).

Of course, we do not believe that the universe began or will end in a phase where a(t) is constant. How do we analyze and describe particle creation in a more realistic context? If the universe is expanding sufficiently slowly, an approximate notion of an "adiabatic vacuum state" can be defined [3], and a notion of "particles" relative to this adiabatic vacuum can be introduced. Such a notion of particles is completely adequate in the present universe to describe quantum field phenomena on scales small compared with the Hubble radius, i.e., only when we consider modes of the field whose oscillation period is comparable to or larger than the Hubble time (i.e., 10^{10} years in the present universe) does the notion of "particles" become genuinely ambiguous. However, at or very near the "big bang" singularity, the notion of "particles" is highly ambiguous.

The next major steps in the development of quantum field theory in curved spacetime came from the application of the theory to black holes. By definition, a black hole in an asymptotically flat spacetime is a region of spacetime from which nothing can escape to infinity. A black holes is believed to be the endpoint of the complete gravitational collapse of a body. The "time reverse" of a black hole—i.e., a region of spacetime that is impossible to enter if one starts from infinity—is called a white hole. It is believed that white holes cannot occur in nature. (The asymmetry between the expected occurrence of black holes and the expected non-occurrance of white holes in nature is undoubtedly closely related to the second law of thermodynamics.) However, black holes are expected to "settle down" to a stationary final state, and if one extends the idealized stationary final state metric of a black hole backward in time preserving the stationary symmetry, one obtains a spacetime containing a white hole region. Thus, although white holes are not expected to occur in nature, they do occur in the mathematically idealized solutions used to describe the final stationary states of black holes.

Since, by definition, nothing can escape from a black hole, it would seem that black holes would be one of the least promising places to seek any observable effects of particle creation. However, the study of effects of particle creation by black holes arose quite naturally for reasons that I shall now explain. Outside of a rotating black hole is a region, called the ergosphere, where the Killing field that describes time translations at infinity becomes spacelike. This means that an observer in the ergosphere cannot "stand still" relative to a stationary observer at infinity. In fact, an observer in the ergosphere must rotate relative to infinity in the same direction as the rotation of the black hole; this is an extreme example of the "dragging of inertial frames" effect in general relativity. The prime importance of the ergosphere is that, since the time translation Killing field is spacelike there, it is possible to have classical particles whose total energy (including rest mass energy) relative to infinity is negative. Consequently, as Penrose realized in 1969, one can extract energy from a black hole by sending a body into the ergosphere and having it break up into two fragments, one of which has negative total energy. The negative energy fragment then falls into the black hole (thereby reducing its mass), but it can be arranged that the positive energy fragment emerges to infinity, carrying greater total energy than the original body.

Not long after Penrose's discovery, Misner (unpublished) and Zel'dovich and Starobinski [5, 6] realized that there is a wave analog of the Penrose energy extraction process. Instead of sending in a classical particle and having it break up into two fragments, one can simply have a classical wave impinge upon a rotating black hole. Part of this wave will be absorbed by the black hole and part will return to infinity. However, if the frequency and angular dependence of the wave are chosen to lie in the appropriate range, then the part of the wave that is absorbed by the black hole will carry negative energy relative to infinity. The portion of the wave which returns to infinity will thereby have greater energy and amplitude than the initial wave. This phenomenon is known as *superradiant scattering*.

Thus, when a wave of superradiant frequency and angular dependence impinges upon a rotating black hole, the black hole amplifies the wave just like a laser. Superradiant scattering thus appears to be a direct analog of stimulated emission. However, in quantum theory, it is well known that in circumstances where stimulated emission occurs, spontaneous emission will also occur. This suggested that for a rotating black hole, "spontaneous emission"—i.e., spontaneous particle creation from the vacuum—should occur. This was noted by Starobinski [6] and confirmed by Unruh [7].

The fact that spontaneous particle creation occurs near rotating black holes did not cause much surprise or excitement. The effect is negligibly small for macroscopic black holes such as those that would be produced by the collapse of rotating stars, so unless tiny black holes were produced in the early universe, the effect is not of astrophysical importance. While it is an interesting phenomenon as a matter of principle, it was not surprising or unexpected in view of the ability to extract energy from a rotating black hole by classical processes. However, it led directly to a further development that caused a genuine revolution.

The calculation of particle creation by a rotating black hole was done in the idealized spacetime representing the stationary final state of the black hole. As explained above, this spacetime also contains a white hole. Consequently, in the particle creation calculation one has to impose initial conditions on the white hole horizon that express the condition that no particles are emerging from the white hole. In the calculation of Unruh [7], a seemingly natural choice of "in" vacuum state on the white hole horizon was made. But it was not obvious that this choice was physically correct.

In 1974, Hawking [8] realized that this difficulty could be overcome by considering the more physically relevant case of a spacetime describing gravitational collapse to a black hole rather than the idealized spacetime describing a stationary black hole (and white hole). When he carried through the calculation, he found that the results were significantly altered from the results obtained for the idealized stationary black hole using the seemingly natural choice of vacuum state on the white hole horizon. Remarkably, Hawking found that even for a non-rotating black hole, particle creation occurs at late times and produces a steady, non-zero flux of particles to infinity. Even more remarkably, he found that, for a non-rotating black hole, the spectrum of particles emitted to infinity at late times is precisely thermal in character, at a temperature $T = \kappa/2\pi$, where κ denotes the surface gravity of the black hole.

The implications of Hawking's result were enormous. It established that black holes are perfect black bodies in the thermodynamic sense at a non-zero temperature. This tied in beautifully with the mathematical analogy that had previously been discovered between certain laws of black hole physics and the ordinary laws of thermodynamics, giving clear

evidence that the similarity of these laws is much more than a mere mathematical analogy. The identification of these laws led to the identification of A/4 as representing the physical entropy of a black hole, where A denotes the area of the event horizon. These and other ramifications of Hawking's results have provided us with some of the deepest insights we presently have regarding the nature of quantum gravity.

However, the Hawking calculation also had other major ramifications for the development of quantum field theory in curved spacetime, and these are the ones that I wish to emphasize here. Although Hawking's results were too beautiful to be disbelieved, there was a very disturbing feature of the calculation: Using a seemingly natural notion of "particles" near the event horizon of the black hole, there appeared to be a divergent density of ultra-high-frequency particles present there. What do these "particles" mean? Does their presence destroy the black hole?

To gain insight into this issue, Unruh [9] proceeded by taking a purely operational viewpoint regarding the notion of "particles": A "particle" is a state of the field that makes a particle detector register. Unruh then showed that in Minkowski spacetime, when a quantum field is in it's ordinary vacuum state, a particle detector carried by an accelerating observer will get excited. Indeed, he showed that a uniformly accelerating observer "sees" an exactly thermal spectrum of particles, at a temperature $T = a/2\pi$, where a denotes the acceleration of the observer. This result provided an explanation of the meaning of the divergent density of ultra-high-frequency particles present near the event horizon of a black hole. Such particles would be "seen" by a stationary observer just outside the black hole. Indeed, such an observer would have to undergo an enormous acceleration in order to remain stationary, and what this observer sees corresponds exactly to the Unruh effect in Minkowski spacetime. However, an observer who freely falls into the black hole would not "see" these particles, just as an inertial observer in Minkowski spacetime does not see the particles associated with the accelerating observer. Furthermore, there are no significant stress-energy effects associated with the quantum field near the horizon of the black hole, so the presence of these "particles" as seen by the stationary observer outside the black hole does not have a significant back-reaction effect and, in particular, does not destroy the black hole.

The clear lesson from Unruh's work is that one cannot view the notion of "particles" as fundamental in quantum field theory. As its name suggests, quantum field theory is truly the quantum theory of *fields*, not particles. If one views the local fields as the fundamental objects in the theory, the Unruh effect is seen to be a simple consequence of how these fields interact with other quantum mechanical systems (i.e., "particle detectors"). If one attempts to view "particles" as the fundamental entities in the theory, the Unruh effect becomes incomprehensible.

Furthermore, with the exception of stationary spacetimes (and certain other spacetimes with very special properties), there is no preferred notion of a "vacuum state" in quantum field theory in curved spacetime and, correspondingly, there is no preferred notion of "particles". The difficulty is not that there is no notion of a vacuum state but rather that there are many, and, in a general spacetime, none can be uniquely singled out as having distinguished properties. Thus, for this reason alone, it clearly would be preferable to have a formulation of quantum field theory in curved spacetime that does not require one to specify a vacuum state or a notion of "particles" at the outset.

The usual way of constructing the theory of a free quantum field would be to choose a vacuum state and then take the Hilbert space of states to be the Fock space based upon this choice of vacuum state. The field operator can then be defined (as an operator-valued-distribution) by the analog of eq.(12). If different choices of vacuum state corresponded to a mere relabeling of the states in terms of their particle content, then it would make sense to construct the theory in this manner, despite the lack of a preferred vacuum state. However, in general, it turns out that different choices of vacuum state will give rise to unitarily inequivalent theories, so the choice does matter. Since there does not appear to be a preferred construction, how does one formulate quantum field theory in a general curved spacetime?

By the mid-1980's, it was well understood—via the efforts of Ashtekar [10], Sewell [11], Kay [12], and others—that the theory of a free quantum field in curved spacetime could be formulated in an entirely satisfactory manner via the algebraic approach. I shall now describe this formulation.

3 The Algebraic Formulation of Free Quantum Field Theory in Curved Spacetime

In the algebraic formulation of quantum field theory in an arbitrary globally hyperbolic, curved spacetime, (M, g_{ab}) , one begins by specifying an algebra of field observables. For a free Klein-Gordon field, a suitable algebra can be defined as follows. Start with the free *-algebra, \mathcal{A}_0 , generated by a unit element I and expressions of the form " $\phi(f)$ ", where f is a test function on M. In other words, \mathcal{A}_0 consists of all formal finite linear combinations of finite products of ϕ 's and ϕ *'s, e.g., an example of an element of \mathcal{A}_0 is $c_1\phi(f_1)\phi(f_2)+c_2\phi^*(f_3)\phi(f_4)\phi^*(f_5)$. Now impose the following relations on \mathcal{A}_0 : (i) linearity of $\phi(f)$ in f; (ii) reality of ϕ : $\phi^*(f)=\phi(\bar{f})$ where \bar{f} denotes the complex conjugate of f; (iii) the Klein-Gordon equation: $\phi([\nabla^a\nabla_a-m^2]f)=0$; (iv) the canonical commutation relations:

$$[\phi(f), \phi(g)] = -i\Delta(f, g)I, \qquad (15)$$

where Δ denotes the advanced minus retarded Green's function. The desired *-algebra, \mathcal{A} , is simply \mathcal{A}_0 factored by these relations. Note that the observables in \mathcal{A} correspond to the correlation functions of the quantum field ϕ .

In the algebraic approach, a *state*, ω , is simply a linear map $\omega : \mathcal{A} \to \mathbb{C}$ that satisfies the positivity condition $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. The quantity $\omega(A)$ is interpreted as the expectation value of the observable A in the state ω .

States in the usual Hilbert space sense give rise to algebraic states as follows: Suppose \mathcal{H} is a Hilbert space which carries a representation, π , of \mathcal{A} , i.e., for each $A \in \mathcal{A}$, the quantity $\pi(A)$ is an operator on \mathcal{H} , and the association $A \to \pi(A)$ preserves the algebraic relations of \mathcal{A} . Let $\Psi \in \mathcal{H}$ be such that it lies in the common domain of all operators $\pi(A)$. Then the map $\omega : \mathcal{A} \to \mathbf{C}$ given by

$$\omega(A) = \langle \Psi | \pi(A) | \Psi \rangle \tag{16}$$

defines a state on \mathcal{A} .

Conversely, given a state, ω , on \mathcal{A} , we can use it to define a (pre-)inner-product on \mathcal{A} by

$$(A_1, A_2) = \omega(A_1^* A_2). \tag{17}$$

This may fail to define an inner product on \mathcal{A} because, although $(A, A) \geq 0$ for all $A \in \mathcal{A}$, there may exist nonzero elements for which (A, A) = 0. However, if this happens, we may factor the space by such zero-norm vectors. We may then complete the resulting space to get a Hilbert space \mathcal{H} which carries a natural representation, π , of \mathcal{A} . The vector $\Psi \in \mathcal{H}$ corresponding to $I \in \mathcal{A}$ then satisfies $\omega(A) = \langle \Psi | \pi(A) | \Psi \rangle$ for all $A \in \mathcal{A}$. The usual quantum mechanical probability rules for determining values of the operator $\pi(A)$ in the state Ψ can then be taken over to define probability rules for the observable A in the state ω . We thereby obtain a complete specification of the quantum field of a Klein-Gordon field on an arbitrary globally hyperbolic curved spacetime insofar as the local field observables appearing in \mathcal{A} are concerned, i.e., in any state we can provide the probabilities for measuring the possible values of all observables in \mathcal{A} . No preferred notion of "vacuum state" or "particles" need be introduced, although, of course, one is free to introduce such notions in particular spacetimes if one wishes.

As has just been seen, every state in the algebraic sense corresponds to a state in the usual Hilbert space sense. What, then is the advantage of formulating the theory via the algebraic approach? The main advantage is that one is not forced to make a particular choice of representation at the outset, i.e., one may simultaneously consider all states arising in all Hilbert space constructions of the theory. As a result, one may define the theory without first having to make a choice of "vacuum state" or introduce any other problematical notions. In addition, it is worth noting that the algebraic notion of states dispenses with the unphysical states in the Hilbert space that do not lie in the domain of the observables of the theory; vectors in a Hilbert space representation of the theory that do not lie in the domain of all $\pi(A)$ do not define states in the algebraic sense.

The above provides a completely satisfactory construction of a free Klein-Gordon field in curved spacetime insofar as observables in \mathcal{A} are concerned. Similar constructions can be done for all other free (i.e., non-self-interacting) quantum fields. However, the overall situation is still quite incomplete and unsatisfactory for at least the following two reasons: First, even if we were only interested in the theory of a free Klein-Gordon field, there are many observables of interest that are not represented in A. Indeed, the observables in Aare merely the n-point functions of the linear field ϕ ; they do not even include polynomial functions of ϕ and its derivatives ("Wick polynomials"). A prime example of an observable of great physical importance that is not represented in \mathcal{A} is the stress-energy tensor, T_{ab} , which would be needed to estimate back-reaction effects of the quantum field on the spacetime metric. Therefore, we would like to enlarge the algebra \mathcal{A} so that it includes at least the Wick polynomials of ϕ . Second, we do not believe that the quantum fields occurring in nature are described by free fields, so we would like to extend the theory to nonlinear fields. Even in Minkowski spacetime, one understands how to do this only at a perturbative level, but one would like to at least generalize these perturbative rules to curved spacetime. These perturbative rules require that one be able to define Wick polynomials of the free field as well as time-ordered-products of polynomial expressions

in the field. Again, we need to enlarge the algebra \mathcal{A} so that it includes such quantities.

If a quantum field were well defined at a (sharp) spacetime event p, it would be straightforward to define polynomial quantities in ϕ as well as time-ordered-products. However, we have already noted below eq.(12) that a quantum field makes sense only as a distribution on spacetime. Consequently, a priori, a naive attempt to define, say, $[\phi(p)]^2$ is not likely to make any more sense than an attempt to define the square of a Dirac delta-function. In particular, it would be natural to attempt to define the smeared Wick power $\phi^2(f)$ by a formula like

$$\phi^{2}(f) = \lim_{n \to \infty} \int \phi(x)\phi(y)f(x)F_{n}(x,y)d^{4}xd^{4}y$$
(18)

where $F_n(x, y)$ is a sequence of smooth functions that approaches the Dirac delta-function $\delta(x, y)$. However, the right hand side diverges in the limit, so some sort of "regularization" of this expression must first be done in order to make the limit well behaved.

Once the Wick powers $\phi^k(f)$ have been defined, it would be an easy matter to define the time-ordered-product $T(\phi^{k_1}(f_1)\dots\phi^{k_n}(f_n))$ by a straightforward "time ordering" of the factors in the case where the supports of f_1, \dots, f_n have suitable causal properties so that they can be put in a well defined time order. Indeed, using induction in the number, n, of factors, it is straightforward to define $T(\phi^{k_1}(f_1)\dots\phi^{k_n}(f_n))$ whenever the intersection of the supports of f_1, \dots, f_n vanishes. However, it is not straightforward to extend this distribution to the "total diagonal", i.e., to the case where the supports of f_1, \dots, f_n have nonvanishing mutual intersection.

From the way I have described the regularization issues above, it might seem that the most difficult problem would be to define Wick polynomials and that the definition of time-ordered-products would be a minor addendum to this problem. In fact, however, in Minkowski spacetime Wick polynomials are easily defined by a "normal ordering" prescription, which can be interpreted as subtracting off the vacuum expectation value of the field quantities before taking the kind of limit appearing on the right side of eq.(18). On the other hand, the problem of extending time-ordered-products to the total diagonal corresponds to the problem of renormalizing all Feynman diagrams—an extremely difficult and complex problem.

There are major issues of principle that must be overcome in order to extend the Minkowski spacetime regularization and renormalization prescriptions to curved spacetime. The normal ordering prescription for defining Wick polynomials in Minkowski spacetime relies on the existence of a preferred vacuum state, with respect to which the "normal ordering" is carried out. However, we have already seen that in a general curved spacetime, there does not appear to exist any notion of a preferred vacuum state. Furthermore, the renormalization prescriptions used to define time-ordered-products in Minkowski spacetime make use of "momentum space methods" (i.e., global Fourier transforms of quantities) and/or "Euclidean methods" (i.e., analytic continuation of expressions defined on Euclidean space rather than Minkowski spacetime). These methods, in turn, require Poincare symmetry, the existence of a preferred, Poincare invariant vacuum state, and/or the ability to "Euclideanize" Minkowski spacetime by the transformation $t \to it$. All of these features are absent in a general, curved spacetime.

It was already understood by the late 1970's that it should be possible to define the stress-energy tensor, T_{ab} , of a quantum field ϕ only on a restricted class of states, namely the so-called Hadamard states, ω_H , whose two-point distribution $\omega_H(\phi(x), \phi(y))$ has a short distance singularity structure as $y \to x$ of a particular form (see, e.g., [4]). For Hadamard states, a prescription for defining the expectation value, $\omega_H(T_{ab})$, can be given that involves the subtraction from $\omega_H(\phi(x), \phi(y))$ of a locally and covariantly constructed Hadamard parametrix rather than a vacuum expectation value [4]. The resulting prescription defines $\omega_H(T_{ab})$ in an entirely satisfactory manner that does not require a choice of vacuum state. Indeed, this prescription is local and covariant in the sense that the value of $\omega_H(T_{ab})$ at a point p depends only on the spacetime geometry and the behavior of ω_H in an arbitrarily small neighborhood of p. It is not difficult to show that, even if one could make a unique choice of vacuum state in all spacetimes, normal ordering would not provide a local and covariant definition of $\omega_H(T_{ab})$.

However, although the above prescription provides a satisfactory definition of the expectation value of the stress-energy tensor in Hadamard states and can be generalized to define the expectation value of higher Wick powers, it does not define T_{ab} or other Wick powers as an element of an enlarged algebra. Indeed, in a Hilbert space representation of the theory, the above prescription would merely define T_{ab} as a quadratic form on Hadamard states rather than as an operator-valued-distribution, so no probability rules for measuring the possible values of T_{ab} would be available. Furthermore, it is worth mentioning that the characterization of "Hadamard states" in terms of their short-distance singularity structure is extremely cumbersome to work with. Finally, by the mid-1990's it was still very far from clear how to perform the much more difficult and complex renormalizations that would be needed to define time-ordered-products in curved spacetime.

4 Progress Since the Mid-1990's

During the past decade, the algebra of observables for a free quantum field has been extended to include all Wick polynomials and time-ordered-products, so that, in particular, the perturbative renormalization of interacting quantum fields in curved spacetime is now rigorously well defined. Much of this progress has resulted from the importation of methods of "microlocal analysis" into the theory. In essence, microlocal analysis provides a refined characterization of the singularities of a distribution. If one has a distribution, α , defined in a neighborhood of point p on a manifold M, one can multiply α by a smooth function f with support in an arbitrarily small neighborhood of p, such that $f(p) \neq 0$. One can then examine the decay properties of the Fourier transform of $f\alpha$ (where the Fourier transform can be defined by choosing an arbitrary embedding of a neighborhood of the support of $f\alpha$ into Euclidean space). If α were smooth in a neighborhood of p, then, for p with support in this neighborhood, the Fourier transform of p would decay rapidly in all directions in Fourier transform space, p, as p, as p. Thus, the failure of the Fourier transform of p to decay rapidly characterizes the singular behavior of p at p. If, for all choices of p, the Fourier transform of p does not decay rapidly in a neighborhood

of the direction k, then one says that the pair (p,k) lies in the wavefront set, WF(α), of α . One can naturally identify WF(α) with a subset of the cotangent bundle of the manifold, M. The wavefront set thereby provides a characterization of not only the points in M at which α is singular, but also the directions (in the cotangent space) at which it is singular. This refined characterization of the singularities of distributions can enable one to define operations that normally are ill defined. For example, if α and β are distributions, then it normally will not make mathematical sense to take their product. However, if it is the case that whenever $(p,k) \in \mathrm{WF}(\alpha)$, we have that $(p,-k) \notin \mathrm{WF}(\beta)$, then the product $\alpha\beta$ can be defined in a natural way via the Fourier convolution formula.

By providing rules for, e.g., when products of distributions are well defined as distributions, microlocal analysis provides an extremely useful calculus for determining whether proposed regularization/renormalization schemes are well defined. Since the analysis is completely local in nature, it provides an ideal tool for analyzing the behavior of local field observables.

The first significant application of microlocal analysis to quantum field theory in curved spacetime occurred in the Ph.D. thesis of Radzikowski [13], a student of Wightman. Radzikowski was concerned with proving a conjecture, due to Bernard Kay, that stated that if a quantum state had a two-point function whose short-distance singularities are of the Hadamard form, then it could not have any additional singularities at large spacelike separations ("local Hadamard form implies global Hadamard form"). Radzikowski employed the tools of microlocal analysis to prove this conjecture. In particular, in the course of his analysis, he proved that the (quite cumbersome) characterization of Hadamard states in terms of the detailed local singularity structure of $\omega_H(\phi(x), \phi(y))$ is equivalent to a very simple condition on the wavefront set of this distribution, namely WF[$\omega_H(\phi(x), \phi(y))$] is the subset of the cotangent bundle of $M \times M$ consisting of all points (x, y; k, l) such that x and y are connected by a null geodesic γ with future-directed tangent $k^a = g^{ab}k_b$ at x and with l_a being minus the parallel transport along γ of k_a to y.

It is worth mentioning that there was an interesting historical interplay between microlocal analysis and quantum field theory in curved spacetime. In the late 1960's Hormander visited the Institute for Advanced Study in Princeton and interacted with Wightman. Wightman explained to Hormander what the "Feynman propagator" is in Minkowski spacetime, and a characterization of "Feynman parametrices" in a general curved spacetime in terms of wavefront set properties can be found in the classic paper of Duistermaat and Hormander [14]. Conversely, Wightman realized that the methods of microlocal analysis could potentially be useful in the formulation of quantum field theory in curved spacetime. For example, in de Sitter spacetime, there is no globally timelike Killing field and therefore no global notion of energy that is positive. Therefore, it does not appear that one could impose a global spectral condition on a quantum field analogous to the requirement of positivity of energy in the Minkowski case. However, one might be able to impose a "microlocal spectral condition" on the local quantum field observables. Shortly after his interactions with Hormander, Wightman had a student, S. Fulling, who was interested in quantum field theory in curved spacetime, and he suggested to Fulling that he investigate the possible application of microlocal analysis to quantum field theory

in curved spacetime. However, after spending some effort in studying microlocal analysis, Fulling decided that his efforts would be better spent on other projects. Among the other projects that Fulling then investigated in his thesis was the inequivalence of different quantization schemes. In particular, he showed that quantization in the "Rindler wedge" of Minkowski spacetime using a Lorentz boost Killing field to define a notion of "time translations" gave rise to a different notion of "vacuum state" than the restriction of the usual Minkowski vacuum to this region. This work provided the mathematical basis for Unruh's subsequent analysis discussed above [9]. However, Wightman had to wait another twenty years before he had another student interested in quantum field theory in curved spacetime. When Radzikowski began to apply the methods of microlocal analysis to analyze Kay's conjecture, Wightman was well prepared to provide plenty of encouragement.

After Radzikowski's work, it became clear to Fredenhagen and collaborators that microlocal analysis should provide the needed tools for analyzing the divergences occurring in quantum field theory in curved spacetime. Brunetti, Fredenhagen, and Kohler [15] showed that if one considers a Fock representation associated with an arbitrary Hadamard vacuum state ω_0 , then normal ordering can be used to define Wick polynomials as operator-valued-distributions on this Hilbert space. Indeed, a larger algebra, \mathcal{W} , of field observables—large enough to include all time-ordered-products—can be defined in this manner. Brunetti and Fredenhagen [16] also formulated a microlocal spectral condition that should be imposed on time-ordered-products. However, as previously mentioned, a normal ordering prescription cannot yield a local and covariant definition of Wick polynomials. Furthermore, the construction of \mathcal{W} given in [15] invokes an arbitrary choice of Hadamard vacuum state ω_0 . Nevertheless, it can be shown that, as an abstract algebra, \mathcal{W} does not depend on the choice of ω_0 , so it is a legitimate candidate for the desired enlarged algebra of observables. Thus, the key remaining issue was to determine which elements of \mathcal{W} properly represent the "true" Wick polynomials and time-ordered-products.

A key condition to be imposed on the definition of Wick polynomials and time-ordered-products is that they be local and covariant fields. As mentioned in the previous section, this condition had been imposed on the definition of the expectation value of the stress-energy. However, the formulation of this notion given in [4] was not adequate for the present purpose, and a more general formulation had to be given [17, 18].

With these key ideas and constructions in place, it was possible to prove the following results [17], [19]-[21]: (1) There exists a well defined prescription for defining all Wick polynomials that is local and covariant and satisfies a list of additional reasonable properties, including appropriate scaling behavior and continuous/analytic variation under continuous/analytic changes in the metric [17]. This prescription is unique up to certain "local curvature ambiguities". For example, for a Klein-Gordon field, ϕ , the prescription for ϕ^2 is unique up to

$$\phi^2 \to \phi^2 + (c_1 R + c_2 m^2) I$$
 (19)

where c_1, c_2 are arbitrary constants, R denotes the scalar curvature, and I denotes the identity element of W. For a massless field in Minkowski spacetime, all of the ambiguities disappear, and the prescription agrees with normal ordering with respect to the usual Minkowski vacuum state. However, on a general curved spacetime, the prescription for

defining ϕ^2 and other Wick polynomials does not agree with normal ordering with respect to any choice of vacuum state. (2) There exists a prescription for defining all time-ordered-products that is local and covariant, that satisfies the microlocal spectral conditions [16], and that satisfies a list of additional reasonable properties [19]. This prescription is unique up to "renormalization ambiguities" of the type expected from Minkowski spacetime analyses, but with additional local curvature ambiguities. (3) Theories that are renormalizable in Minkowski spacetime remain renormalizable in curved spacetime. For renormalizable theories, renormalization group flow can be defined in terms of the behavior of the quantum field theory under scaling of the spacetime metric, $g_{ab} \rightarrow \lambda^2 g_{ab}$, [20]. (4) Additional renormalization conditions on time-ordered-products can be imposed so that, order by order in perturbation theory, for an arbitrary (not necessarily renormalizable) interaction, (i) the interacting field satisfies the classical interacting equation of motion and (ii) the stress-energy tensor of the interacting field is conserved [21]. All of the above results have been obtained without any appeal to a notion of "vacuum" or "particles".

These and other results of the past decade have demonstrated that quantum field theory in curved spacetime has a mathematical structure that is comparable in depth to such theories as classical general relativity. In particular, it is highly nontrivial that quantum field theory in curved spacetime appears to be mathematically consistent. Although quantum field theory in curved spacetime cannot be a fundamental description of nature since gravity itself is treated classically, it seems hard to believe that it is not capturing some fundamental properties of nature.

The above results suffice to define interacting quantum field theory in curved spacetime at a perturbative level. However, it remains very much an open issue as to how to provide a non-perturbative formulation of interacting quantum field theory in curved spacetime. It is my hope that significant progress will be made on this issue in the coming years.

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