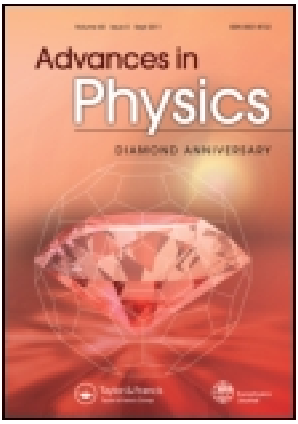

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Quantum field theory, horizons and thermodynamics

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Quantum field theory, horizons and thermodynamics†

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Abstract

The aim of the article is to obtain an intuitive understanding of the recently explored deep connections between thermal physics, quantum field theory and general relativity. A special case in which a detector moves with constant acceleration through a quantum vacuum is examined to clarify the fact that such a detector becomes thermally excited, with a temperature proportional to its acceleration. An elementary physical explanation of this fundamental result is provided. The uniformly accelerated observer finds his space–time manifold bounded by an event horizon and so realizes a 'model' black hole. Real black holes also have thermal properties when quantum effects are taken into account; these are described and the correspondences with the accelerated case are pointed out. In particular, an elementary account is given of the thermal Hawking radiation emitted by the black holes formed by collapsed stars.

CONTENTS

	PAGE
§1. Introduction.	328
§2. Accelerated detectors in Minkowski space–time.	329
2.1. Why a uniformly accelerated detector sees a truly thermal spectrum.	335
2.2. The quantum–classical correspondence.	336
§3. Alternative vacua and their thermal properties.	342
3.1. Black-hole thermodynamics.	347
3.2. Field theory on black-hole space–times.	349
3.3. The rate of area decrease.	356
3.4. De Sitter space–time.	357
Appendix A. Analytic properties of the manifold: the rotation to complex time.	360
Appendix B. On Bekenstein's heuristic derivation of the temperature of a black hole.	361
Appendix C. Thermal relationship between the Minkowski and Fulling vacua.	362
Appendix D. On the radiation reaction suffered by a uniformly accelerated charge.	364
References.	365

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§ 1. Introduction

There have been many recent demonstrations that natural vacuum states possess thermal properties and a number of striking and fundamental phenomena have been predicted. A deep connection between thermal physics, quantum theory and general relativity lies at the base of this. In this paper our purpose is to develop an intuitive understanding of the physical effects involved by viewing them in terms of more familiar physical processes.

We begin, not with the famous result of Hawking [1] that black holes emit thermal radiation, but with a calculation by Unruh [2] of the consequences of acceleration for a model particle detector. The model detector is essentially a single atom, initially in its ground state and weakly coupled to the quantum field under consideration. An atom may usually be said to have detected a particle of appropriate energy whenever it is excited to a higher energy state but this interpretation becomes problematic for accelerated motion. It turns out that the response of a detector undergoing uniform acceleration is the same as it would be if it were immersed in a heat bath at a temperature $\hbar\alpha/2\pi ck$, where k is Boltzmann's constant, c the velocity of light and $\hbar$ Planck's constant. The implicit connection between relativity, quantum theory and thermal physics is evident.

In § 2 we seek the origin of the effect in the zero-point fluctuations of the quantum field which are present even *in vacuo*. The stochastic properties of these fluctuations are reminiscent of those of a gas of real particles in equilibrium. For a detector at rest, the excitations caused by these zero-point fluctuations are precisely cancelled by its spontaneous emission rate. When it is in *uniform* motion, the Lorentz invariance of the vacuum state ensures that there is still no net excitation. But when the detector accelerates, the correlations in the zero-point fluctuations of the field *along the detector's world-line* no longer precisely balance its own zero-point fluctuations. The detector is consequently excited. *It has not, however, detected a particle* in the external quantum field, which by hypothesis starts in its vacuum state.

To explain the precisely thermal character of these excess excitations we appeal to the stability of the vacuum state with respect to the switching-on of a small self-interaction of the field, together with the staticity of the regime internal to a uniformly accelerated detector. These two properties together are sufficient [3] to ensure that an ensemble of such detectors would come into equilibrium with the vacuum at a non-zero temperature. A closely related argument is based on another presumed property of the vacuum state, namely its role as the state of lowest energy. No cyclic process in the accelerated detector could extract work from the quantum field. We then discuss in more detail the similarities between quantum fields and classical stochastic fields.

We change, in § 3, from a local point of view to a completely global one, to compare the thermal properties which arise in the following cases: Minkowski space–time from the point of view of the accelerated observer, black-hole space–times and De Sitter space–time.

The stress-energy tensor $T_{\mu\nu}$ is an invariant 'measure of vacuum activity'. Taking the case of electrodynamics as an example, it is well known [5] that the naïve vacuum expectation value of $\hat{T}_{\mu\nu}$ is divergent because each electromagnetic field mode of angular frequency ω contributes a zero-point energy $\frac{1}{2}\hbar\omega$ to the vacuum. It is the *renormalized* expectation value $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{Ren}}$, from which the zero-point divergences have been removed in an invariant way, which is directly measurable. It determines, for example, the back-reaction of matter fields on the geometry. We quote the results of calculations of $\langle \hat{T}_{\mu\nu} \rangle_{\mathrm{Ren}}$ for massless fields of spin 0, $\frac{1}{2}$ and 1, in various vacuum states. The spectrum is always thermal, though a density-of-states factor differing from that in Minkowski space–time gives it a non-Planckian form for non-zero spin. We stress the fact that each of the inequivalent field quantizations treated represents a different physical situation, and there is no question of one of the vacuum states being the correct or preferred one.

The thermal properties of black holes had first been predicted by Bekenstein [6] on the basis of heuristic arguments and thought experiments, and the black-hole analogues of the four laws of thermodynamics were also tentatively proposed [7] before Hawking's calculation; we review these laws and explain their significance.

We make a tentative step towards an attack on the back-reaction problem for radiating black holes by proving from classical general relativity that the rate of change of the black-hole mass is the negative of its luminosity. This is of course no more than a matter of dynamical self-consistency, but we regard the result as a reassuring connection between theory and thought experiment.

In Appendix A we discuss 'rotation to complex time' as a mathematical device for facilitating various calculations for static analytic manifolds. Appendix B contains a refutation of one of Bekenstein's thought experiment arguments (which we replace by a different argument). In Appendix C we give a formal proof that the Minkowski vacuum has all the properties of a thermal state when considered from the point of view of accelerated observers.

§2. Accelerated detectors in Minkowski space–time

The response of an idealized 'particle detector' is an interesting problem with a long history. In 1900, Planck [8] studied the rate at which a charged harmonic oscillator of frequency ω absorbs energy from a stochastic radiation field. He found that this rate is essentially determined by the noise power in the radiation field. Suppose for simplicity that a scalar field ϕ is being detected, then this result may be expressed by saying that the energy absorption rate of the detector is essentially determined by the quantity

$$\Pi(\omega) = \int_{-\infty}^{\infty} dt \exp(-i\omega t) \langle\langle \phi(t)\phi(0) \rangle\rangle,$$

where $\langle\langle \ldots \rangle\rangle$ denotes an ensemble average. By the Wiener–Khinchin theorem $N(\omega)\Pi(\omega)$ is the *power spectrum of the fluctuating field*, where $N(\omega)$ is the density of normal modes per unit frequency interval. The corresponding quantum problem has also been the object of considerable study not least in quantum optics [9]. The result for the quantum case is entirely analogous:

$$\Pi(\omega) = \int_{-\infty}^{\infty} dt \exp(-i\omega t) \langle \hat{\phi}(t)\hat{\phi}(0) \rangle,$$

where $\hat{\phi}$ denotes a field operator and $\langle \ldots \rangle$ denotes a quantum expectation value.

There is an aspect of this process which requires attention and that is the role of the *motion* of the detector. If the detector as a whole is unaccelerated, so that its centre of mass moves on a timelike geodesic, then the effect of its motion is easily allowed for. Indeed we know that we can discuss the measurement process in the rest frame of the detector, which would be an inertial frame. The time coordinate t which we have already referred to is the ordinary inertial time coordinate and the frequency variable is the one associated with this time coordinate. More generally, we would have as a frequency variable for a moving inertial detector the quantity $u^{\mu}k_{\mu}$, where u^{μ} is the four-velocity of the detector and k_{μ} is the propagation vector of the field being measured. The Doppler shift factor would then be automatically included. In particular a positive frequency would be transformed into a positive frequency under a Lorentz (velocity) transformation.

For our present purpose we are more interested in a detector whose centre of mass is being accelerated by the action on it of an external force. We may distinguish two possible influences of this force on the characteristics of the detector: (*a*) a structure-sensitive effect and (*b*) a universal kinematic effect. We would expect effect (*a*) to be present; to take an extreme case, the measuring capacity of the detector might be completely destroyed by the force acting on it. In a less extreme case we would have to apply a correction factor to allow for the effect of the force. This factor would depend in detail on the structure of the detector and is of no interest in the present context. To avoid having to consider it, we shall restrict ourselves to forces so small that the correction factor differs negligibly from unity. For convenience we shall refer to the detector as robust. Where necessary we shall also consider it to be small in some relevant sense.

It is not quite so obvious how to handle the universal kinematic effect arising from the fact that the world-line of the detector's centre of mass is not a geodesic. We are already familiar with a universal effect arising from motion; the frequency measured by a uniformly moving detector, for example, is multiplied by a Doppler shift factor. What universal factor must we introduce to allow for the acceleration of the detector?

There is no self-evident answer to this question; indeed a physical hypothesis is required. In relativity one assumes that the time measured by an arbitrarily moving small robust detector is *the proper time along its world-line*. It is important to emphasize that this is a physical hypothesis, and should not simply be taken for granted, as it often is. In fact in the early discussions of the gravitational red shift in light coming from the sun this assumption was itself under question. The reason is that the atoms on the sun's surface are on average at rest in the sun's gravitational field only because they are constantly being supported by atomic impacts from below. Why should these impacts not have a kinematical effect on the time-keeping and so on the wavelength of the light which the atoms emit? However, the standard Einstein red shift is obtained only if one assumes that the accelerated atoms on the sun's surface continue to measure proper time.

Point was added to this controversy by Weyl's original unified field theory (1918) according to which the time measured by a detector does depend on its world-line (the extra effect being due to the electromagnetic field). Einstein objected to this ingenious suggestion on the grounds that (apart from the gravitational red shift itself and Doppler shifts) spectral lines from atoms on the sun have essentially the same wavelengths as the corresponding lines produced on the Earth, although the world-lines involved are very different. In any case, the question is clearly an empirical one. There is now strong observational evidence in favour of the standard relativistic assumption. For example, measurements of the Mössbauer effect on crystals of ^{57}Fe show that any specific influence of the accelerations occurring in the thermal motion of the Fe atoms on the emitted frequency is less than one part in 10^{13}. Yet these accelerations are large, of order $10^{16}\,g$ [10]. We may add that the ^{57}Fe crystal is also a robust detector, since the distortion suffered by a ^{57}Fe nucleus due to its acceleration corresponds to a maximum relative displacement of its constituent neutrons and protons of only 10^{-13} of its nuclear diameter [10]†.

One consequence of the change in the meaning of frequency when a detector accelerates is that a non-radiative (zero-frequency) field for an inertial observer can become a radiative (non-zero-frequency) field for an accelerating one. This has been shown for a Coulomb field by Mould [11] who examined how an explicitly constructed detector of electromagnetic fields would behave under acceleration. This verified in detail the point made by Rohrlich [12] that the criterion for a field to be non-radiative is Lorentz invariant, but not invariant under an acceleration transformation.

We wish to compare the responses of two detectors, one of which moves inertially while the other is uniformly accelerated. Consider the autocorrelation function (the Wightman function) $\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle$ for the zero-point fluctuations of a real scalar field $\hat{\phi}(x)$ relative to the usual Poincaré invariant vacuum state. Standard analysis reveals that

$$\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle=\frac{1}{4\pi^2}\frac{1}{[-(t-t'-i\varepsilon)^2+(\mathbf{x}-\mathbf{x}')^2]}, \tag{1}$$

where $x=(t,x,y,z)$, $\mathbf{x}=(x,y,z)$ and we have introduced an infinitesimal ε to specify correctly the singularities of the function.

If the detector moves inertially then we may refer the calculation to the rest frame of the detector, whence

$$\Pi_{\text{inertial}}(\omega)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\frac{dt\exp(-i\omega t)}{(t-i\varepsilon)^2}$$
$$=\frac{-\omega}{2\pi}\theta(-\omega),$$

where θ denotes the step function and the last line follows on evaluating the integral by residues.

Since Π_{inertial} contains only negative frequencies we have recovered the obvious result that if the detector is prepared in its ground state then it will never be found in an excited state. If, however, it is prepared in an excited state then at a later time we may find it in a lower state, the decay being induced by the fluctuations of the field.

Let us now examine the response of a uniformly accelerated detector—uniform acceleration being defined as a motion such that the acceleration as measured in the instantaneous rest frame of the body is a constant. This is also the motion produced by the application of a constant force.

A useful parametrization [13] of the space–time path of the detector is obtained by writing

$$\left.\begin{aligned}x&=\alpha^{-1}\cosh\tau,\\ t&=\alpha^{-1}\sinh\tau.\end{aligned}\right\} \tag{2}$$

Then α is the constant proper acceleration of the detector and $\alpha^{-1}\tau$ is its proper time.

† A similar remark applies to muons moving in circular orbits with an acceleration $\sim 10^{18}$ g. Their half-life undergoes the usual velocity-dependent time-dilation but is otherwise independent of the acceleration to one part in a thousand (Bailey, J., *et al.*, 1977, *Nature, Lond.*, **268**, 301).

If we substitute the parametrization (2) into the expression (1) for the autocorrelation function we find

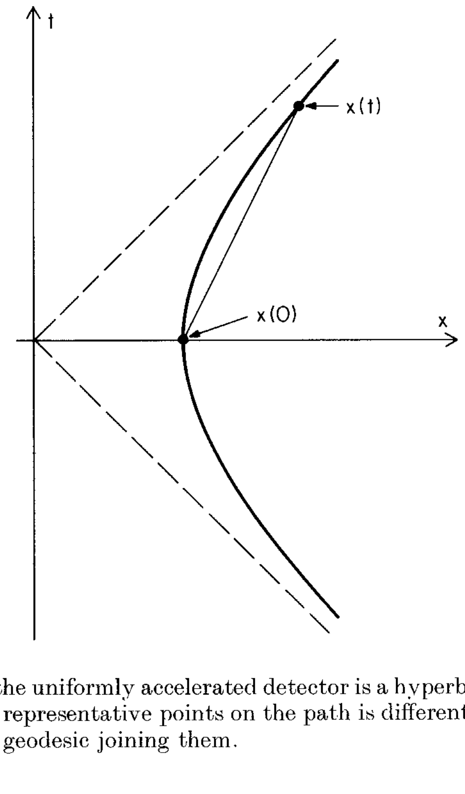
$$\langle 0|\hat{\phi}(\tau)\hat{\phi}(0)|0\rangle = -\frac{\alpha^2}{16\pi^2 \sinh^2\left(\dfrac{\tau}{2} - i\varepsilon\right)},$$

where $\hat{\phi}(\tau)$ denotes $\hat{\phi}(x)$ evaluated at the point $x(\tau) = (\tau, \xi, y, z)$. Thus the autocorrelation function assumes a different functional form, when expressed in terms of the detector's proper time, from that appropriate to inertial motion. This is simply the observation that the interval between two points on the hyperbolic world-line is different if measured along the world-line or along the geodesic joining them (see fig. 1). Note also that we are still evaluating the autocorrelation function relative to the Poincaré invariant vacuum since we wish to see how an accelerated observer perceives this familiar state.

The power spectrum of the fluctuations along the accelerated world-line is therefore

$$\prod\nolimits_{\text{accelerated}}(\omega) = \frac{-1}{4\pi^2}\int_{-\infty}^{\infty} d\tau \frac{\alpha \exp(-i\omega\tau/\alpha)}{4\sinh^2\left(\dfrac{\tau}{2} - i\varepsilon\right)}.$$

Fig. 1



The space–time path of the uniformly accelerated detector is a hyperbolic curve. The elapsed time between two representative points on the path is different if measured along the path or along the geodesic joining them.

It is again convenient to evaluate this integral by residues. To this end consider the result of integrating the function

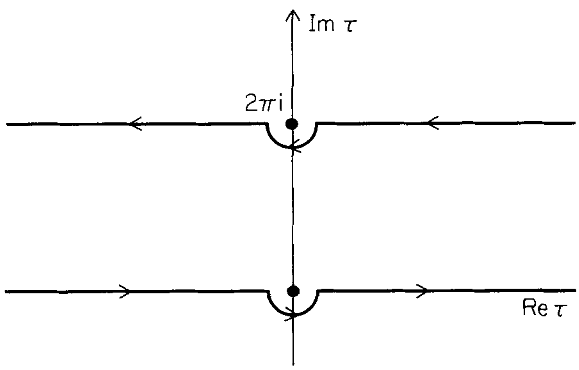
$$f(\tau) = -\frac{\alpha}{16\pi^2}\frac{\exp(-i\omega\tau/\alpha)}{\sinh^2(\tau/2)}$$

around the contour C as shown in fig. 2. The integral over the lower part of the contour yields $\Pi(\omega)$ while that over the upper part yields $-\exp(2\pi\omega/\alpha)\Pi(\omega)$. The sum of these contributions is related to the residue of $f(\tau)$ at $\tau=0$. Thus we find

$$\Pi_{\text{accelerated}}(\omega) = \frac{1}{\alpha\pi}\frac{\omega}{(\exp(2\pi\omega/\alpha)-1)}.$$

Hence (restoring the dimensions) the uniformly accelerated detector will be excited to a temperature $T = \hbar\alpha/2\pi kc$ as first pointed out by Unruh [2] (see also [14] for a related result). Moreover, the uniformly accelerated detector considers itself to be in a heat bath at temperature T not simply in terms of the mean noise power or second-order correlation function which it experiences. *All the stochastic properties of this noise power are the same as in thermal equilibrium at temperature T.* This follows essentially because the detector is moving through a Gaussian random process and therefore still sees a Gaussian random process. This process is completely determined by the second-order correlation function which, as we have seen, is that of a thermal distribution at temperature T.

Fig. 2



The contour of integration appropriate to the evaluation of $\Pi(\omega)$.

Since an understanding of the readings of accelerated detectors is central to our entire discussion it may be helpful to express our results in a slightly different way which refers more directly to the internal degrees of freedom of the detector.

We may regard the state of the detector as being determined by the balance between two competing processes, namely spontaneous emission by the fluctuations of the detector's own dipole moment and absorption (or stimulated emission) arising from the vacuum fluctuations of the ambient field. The magnitude of these separate effects can be varied by means of transformations of variables using the field equations and commutation relations for the coupled system of detector plus vacuum fluctuations [15]. The balance between these effects cannot, of course, be varied in the sense that a definite result can be obtained for any observable effect. It turns out that we can choose the variables in such a way that if the detector is in its excited state, then the quantum fluctuations of its own dipole moment contribute *equally* with emission stimulated by the quantum fluctuations of the ϕ field to determine the de-excitation rate [16]. When the detector is in its ground state there is no stimulated emission, and the spontaneous emission due to the zero-point fluctuations of the detector is now precisely balanced by absorption from the zero-point fluctuations of the ϕ field (note the analogy to detailed balancing in thermal equilibrium). This precise balance implies that a detector in its ground state remains in its ground state. This is true at all times since there are no positive frequency components in the autocorrelation function of the vacuum (the ground state energy is an eigenstate of the full Hamiltonian). We may summarize this situation by saying that the quantum noise in the detector completely masks the signal.

When the detector accelerates this balance is upset. Because the detector is robust, the autocorrelation function of the minimum detectable signal is the same as before, in terms of the proper time *along* its (curved) world-line. On the other hand, the relevant autocorrelation function of the ϕ field involves the proper time along *geodesics* between pairs of points, one of which is fixed and the other runs along the world-line of the detector.

Our central result is that, for a uniform acceleration, the 'signal' now exceeds the noise in the detector, that this excess has a thermal spectrum and that the fluctuations in the signal also correspond to those in thermal equilibrium at temperature T.

This result is of particular interest because the stochastic properties of any equilibrium state actually realized in nature would be expected to be only approximately Gaussian. For instance, radiation in a box would be Gaussian, if the law of large numbers could be applied to the atomic emitters of the radiation in the walls of the box. If these emitters radiate independently of one another then this procedure is valid, but in any practical case there might be phase relations between the emitters. Moreover, the absence of such phase relations raises deep questions about the origin of the second law of thermodynamics and ultimately of the presumed absence of correlations at the origin of the Universe. By contrast, the Gaussian character of the thermal radiation observed by a detector accelerating uniformly through the Minkowski vacuum is guaranteed by the Gaussian character of the quantum stochastic properties of a single harmonic oscillator. In this sense the quantum-promoted thermal radiation is more 'truly thermal' than is ordinary thermal radiation.

It must be recognized, however, that in other respects the thermal state we have been describing has some unusual aspects. For example, the temperature depends on the acceleration of the observer but is independent of his velocity. Thus two neighbouring observers with the same acceleration, but with velocities differing by nearly the velocity of light would observe the *same* temperature—there would be no Doppler transformation from one temperature to the other.

The point that deserves emphasis since it is often the source of confusion is that, while it is sometimes convenient to describe the readings of the detector by saying that the detector perceives itself to be immersed in black-body radiation, this is in part just a form of words and need not be interpreted as meaning that the detected 'particles' are 'real'. As we have stressed the detector does not so much detect particles as the spectrum of vacuum fluctuations. In particular the vacuum expectation value of the renormalized stress-energy tensor is zero in the Minkowski vacuum and since this quantity is a tensor it must be zero to every observer including an accelerated one. We shall examine further in § 3 the circumstances in which our model detector can be said to detect 'real' particles.

This discussion leaves open one final question. Granted that a uniformly accelerating detector observes excess radiation, what is the essential reason why this excess has a thermal spectrum? We shall now see that this is not just a coincidence, but has a deep meaning which is closely related to the Gaussian character of the process.

2.1. *Why a uniformly accelerated detector sees a truly thermal spectrum*

It is important at the outset to distinguish between two aspects of a radiation field, its frequency spectrum and its stochastic properties. As we have seen, its frequency spectrum is the Fourier transform of its second-order correlation function. Its stochastic properties concern the structure of the nth-order correlation functions. These aspects are independent of one another. Thus a radiation field could be Gaussian without having a Planck spectrum, and it could have a Planck spectrum without being Gaussian. Only when both properties hold is the radiation field truly thermal.

In order to see why a vacuum state looks truly thermal to a uniformly accelerating observer we need to discover adequate analogies between a vacuum state and a thermal state. One important analogy, which we shall shortly discuss, is that both states have a Gaussian probability structure. We now aim to explain why this analogy is deeper than may appear at first sight, and that this extra depth provides just what we need for understanding the thermal effects observed by a uniformly accelerating detector.

Let us consider first the thermal state. It is well known that this can be calculated for a harmonic oscillator from the requirement that the entropy be a maximum for a given mean energy. It is also well known that an oscillator in weak contact with a thermal reservoir will eventually achieve this thermal state [17]. The idea, which stems originally from Einstein [18], is that the oscillator undergoes a Brownian motion as a result of its interaction with the reservoir. It is subject to a fluctuating force which has a Gaussian probability distribution (by the central limit theorem) and a damping whose rate is proportional to the strength of the coupling between oscillator and reservoir and to the density of states in the reservoir in the vicinity of the oscillator's natural frequency. The technical problem involved is to average over the variables of the reservoir to produce a contracted description in which only the damping coefficient and the temperature of the reservoir survive. This is a well studied problem in the literature.

One can also carry out the quantum analogue of this calculation either at finite temperatures or even for a reservoir at the absolute zero. This was first done by Senitzky [19] (see also [20–22]). Two new points arise here. The first is that the driven oscillator is Gaussian by the quantum form of the central limit theorem [23, 24]. Since the quantum fluctuations of the undriven oscillator are also Gaussian, the stochastic properties of the oscillator when averaged over several of its cycles are unchanged by switching on a weak coupling with a reservoir and waiting for a long time. Secondly, the commutation relations satisfied by the *operator* fluctuating force exerted by the reservoir guarantee that the coordinate and momentum of the driven oscillator satisfy the correct commutation relations at all times. This is essential for consistency, since the initial values of both these variables are damped out after the coupling is switched on. Thus *a weakly coupled vacuum, in which each mode is driven by* *all the others, is indistinguishable from an uncoupled vacuum.* Of course this result is also a consequence of the uniqueness of the uncoupled vacuum. This deep consistency property is just what we want. It means that the uncoupled vacuum state is 'already equilibrated'. Put more formally we can say that the uncoupled vacuum is stable to the switching on of weak coupling between its modes. This is also a characteristic property of the thermal state. We can see now why a uniformly accelerating observer sees a thermal state. For he observes a time-independent distribution of modes which is stable to the switching on of weak coupling between the modes. Moreover, since a linear combination of Gaussian probability distributions is Gaussian, he will find Gaussian fluctuations in the state which he observes. Thus the quantum Gaussian fluctuations are promoted to thermal Gaussian fluctuations, and he observes a truly thermal state.

2.2. *The quantum–classical correspondence*

The remainder of this section is devoted to a fuller discussion of the detection process emphasizing the close parallels between the quantum and classical regimes. Since it is intended as a supplement to our previous treatment rather than as preparation for the discussion of alternative vacua the reader may proceed directly to the next section if he so wishes.

2.2.1. *The classical detection process*

We are free to make a variety of assumptions about our model classical detector, but in order to maintain a close analogue with the quantum case we shall assume that it is *sluggish*, that is its time constant is assumed to be long compared to the time-scale of fluctuations in the signal. In other words, we are assuming that the detector cannot follow the high-frequency components of the signal. This is the classical counterpart of the usual quantum assumption that the linewidths of the excited states of the detector are small compared with the bandwidth of the signal. By the uncertainty principle this is equivalent to assuming that the excitation rate and decay time of these excited states are long compared with the time-scale of the signal. If this assumption is not correct, as might be the case for highly coherent light from a laser, the detection process would be substantially altered [25].

An elegant theory of sluggish classical detectors has been devised by Gabor [26]. Its starting point is the remark that such a detector, in the presence of a high-frequency monochromatic disturbance $2A \cos \omega t$, would register only the average intensity A^2. To avoid actually performing an average over time, Gabor represented this intensity in the form

$$V^*(t)\, V(t),$$

where

$$V(t) = A \exp(-i\omega t),$$

and the * represents the complex conjugate.

Since

$$2 \cos \omega t = \exp(-i\omega t) + \exp(i\omega t),$$

this procedure amounts to throwing away the negative frequency component in the exponential representation of the signal. However, since the original disturbance is real, the function we throw away is the complex conjugate of the one we keep, and so no information is lost: the negative frequency component can always be retrieved.

Gabor generalized this procedure for signals containing a range of high frequencies by first obtaining the Fourier transform $V(\omega)$ of the real signal $\phi(t)$:

$$V(\omega)=\int_{-\infty}^{\infty}\exp(i\omega t)\phi(t)\,dt.$$

He then defined the complex signal $V(t)$ in terms of the positive frequency components of $V(\omega)$ by

$$V(t)=\frac{1}{2\pi}\int_{0}^{\infty}\exp(-i\omega t)V(\omega)\,d\omega. \tag{3}$$

Again no information is lost since

$$V(-\omega)=V^*(\omega).$$

In fact the original signal can easily be recovered since

$$\phi(t)=2\,\mathrm{Re}\,V(t).$$

As we shall see shortly, $V(t)$ has an important analyticity property if the defining equation (3) is regarded as holding for complex values of t. For this reason it is called the *complex analytic signal* associated with the original real disturbance $\phi(t)$. It has a particularly clear physical meaning if the range of frequencies present in $\phi(t)$ is small compared with the mean frequency $\bar{\omega}$. This can be seen by writing

$$V(t)=A(t)\exp[i(\Phi(t)-\bar{\omega}t)],$$

where $A(\geqslant 0)$ and $\Phi(t)$ are real. We would then have

$$\left.\begin{aligned} A(t)\exp[i\Phi(t)]&=\frac{1}{2\pi}\int_{0}^{\infty}\exp[-i(\omega-\bar{\omega})t]V(\omega)\,d\omega,\\ &=\frac{1}{2\pi}\int_{-\bar{\omega}}^{\infty}g(\mu)\exp(-i\mu t)\,d\mu,\end{aligned}\right\} \tag{4}$$

where

$$g(\mu)=2V(\bar{\omega}+\mu).$$

If the amplitude of the signal is small unless $\omega\sim\bar{\omega}$, $|g(\mu)|$ would be appreciable only near $\mu=0$. Accordingly the integral (4) would represent a superposition of harmonic components of low frequencies, and both $A(t)$ and $\Phi(t)$ would vary slowly in comparison with $\cos\bar{\omega}t$. The real and imaginary parts of V are given by

$$V^{(\mathrm{r})}(t)=A(t)\cos[\Phi(t)-\bar{\omega}t],$$

$$V^{(\mathrm{i})}(t)=A(t)\sin[\Phi(t)-\bar{\omega}t].$$

Thus $V^{(\mathrm{r})}$ and $V^{(\mathrm{i})}$ are represented as modulated signals of carrier frequency $\bar{\omega}$, the slowly varying function $A(t)$ giving the envelope of the real signal. We now see that *the complex analytic signal is a measure of the amplitude of this envelope* since

$$V^*(t)\,V(t)=A^2(t).$$

We might expect that a variety of simple sluggish detectors would measure $V^*(t)\,V(t)$, and detailed calculations show that this is so (for example, for a receiver coupled to a square-law detector followed by a smoothing circuit of time constant $\gg 1/\bar{\omega}$). If the detector is a resonator at frequency ω it would not measure the instantaneous signal $V^*(t)\,V(t)$, but rather the component of this signal with frequency ω. If the signal is a stationary stochastic one then the average quantity that would be measured is

$$\int_{-\infty}^{\infty} \exp(-i\omega t)\,\langle V^*(t)\,V(0)\rangle\,dt,$$

where the brackets represent an ensemble average. By the Wiener–Khinchin theorem [27] this integral gives the noise power at frequency ω present in the complex analytic signal.

The fact that $V(t)$ contains no negative frequency components gives rise to the analyticity property to which we have referred. If one extends the definition of $V(t)$ to a complex argument $t+i\tau$ by

$$V(t+i\tau)=\int_{0}^{\infty} \exp[-i\omega(t+i\tau)]\,V(\omega)\,d\omega,$$

then this defines a function which is analytic and regular in the lower half of the complex t plane, where $\tau<0$ (essentially because the exponential does not blow up as $\tau\to-\infty$ since $\omega>0$). It then follows [28] that the real and imaginary parts of $V(t)$ are related to one another by Hilbert transformations, that is one has

$$\operatorname{Im} V(t)=\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\operatorname{Re} V(t')}{t'-t}\,dt',$$

$$\operatorname{Re} V(t)=-\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{\operatorname{Im} V(t')}{t'-t}\,dt',$$

where P denotes that the principal value of the integral is to be taken at the singular point $t=t'$.

These equations are reminiscent of dispersion relations for the complex response of a linear system to an external disturbance [29]. The underlying mathematical reasons are similar (the response is zero before a disturbance is applied and so vanishes for negative t) but in that case there is a physical constraint (causality) which is responsible, while the complex analytic signal vanishes for negative frequencies by construction.

There is no doubt that this construction is elegant and gains from its direct contact with the powerful theory of functions of a complex variable. Nevertheless for us its main importance is the light it sheds on the detection theory of quantum fields. We shall see that there, too, positive frequency complex functions play a special role. In fact the complex analytic signal V will turn out to be an eigenvalue of the operator used to represent the annihilation of a particle which occurs when a quantum field $\hat{\phi}$ is measured.

2.2.2. *The stochastic properties of a stationary random classical radiation field*

We saw in the last section that a simple sluggish detector measures the (Fourier transform of the) autocorrelation function of the complex analytic signal. More complicated detectors would be expected to measure higher moments of this random signal of the form

$$\langle V^*(t_1)\,V^*(t_2)\ldots V^*(t_n)\,V(t_{n+1})\ldots V(t_{2n})\rangle$$

This collection of moments for arbitrary n determines the measurable stochastic properties of the given stationary random field. Of particular interest for us is a Gaussian process, since a radiation field in thermal equilibrium is just such a process. It can be characterized by the property that the nth-order correlation functions are determined by the second-order function according to the scheme

$$\langle V^*(t_1)\dots V(t_{2n})\rangle = \langle V^*(t_1)\, V(t_{n+1})\rangle \dots \langle V^*(t_n)\, V(t_{2n})\rangle + \dots, \tag{5}$$

where the sum is taken over all the ways the $2n$ values of t can be permuted.

This basic result is well known, but we give a proof here since it uses ideas which are also important in the quantum treatment. The main idea [30] is to use a generating function for the nth-order correlations, what in probability theory is called the characteristic function. This function involves the Fourier amplitudes a_ω given by

$$a_\omega = \frac{1}{T}\int_0^T \exp(i\omega t)\, V(t)\, dt.$$

The characteristic function is defined as

$$\langle \exp(i\xi a_\omega - i\xi^* a_\omega^*)\rangle, \tag{6}$$

where ξ is a complex constant. The coefficient of $\xi^m\xi^{*n}$ in the series expansion of the exponential gives the moment $\langle a^m a^{*n}\rangle$. These moments are directly related to the correlation functions. For example,

$$\langle a_\omega^* a_\omega\rangle = \frac{1}{T^2}\int_0^T\int_0^T dt_1\, dt_2 \exp(-i\omega t_1)\exp(i\omega t_2)\langle V^*(t_1)\, V(t_2)\rangle$$

and similarly for the higher moments.

Since the averaging process involved in the characteristic function is taken with respect to the probability distribution for the amplitudes a_ω, we see that the characteristic function is simply the Fourier transform of this distribution. In the present case this distribution is a Gaussian, and since its Fourier transform is also a Gaussian, we have

$$\langle \exp(i\xi a_\omega - i\xi^* a_\omega^*)\rangle = \exp(-\xi\xi^*/\xi_0^2).$$

Expanding each side as a power series in ξ and ξ^* and equating coefficients we obtain

$$\langle a_\omega^n a_\omega^{*n}\rangle = 1\times 3\times 5\times \dots \times (2n-1)\langle a_\omega a_\omega^*\rangle. \tag{7}$$

Thus the nth-order moments are determined by the second moment. By the symmetry in the time variables, the corresponding relation for the correlation functions is given by (5).

So far in our discussion we have considered the stochastic signal to be a function only of a time variable t. In the application we have in mind this signal is really a *field* distributed in space as well as in time. Since we are concerned with measurements of the field by a small detector we can retain our simple notation if we regard t_1, t_2, etc. as referring to *events on the world-line of the detector*. If the detector is inertial this world-line would be a geodesic, but if it is accelerated it would be a more general time like line. Thus the essential point is, as we have emphasized, that the noise power (and the higher moments) measured by the detector are Fourier transforms of correlation functions evaluated along the world-line of the detector.

2.2.3. The quantum detection process

This process has been discussed by Glauber [24] with applications to quantum optics in mind. He takes as the detector a single atom in its ground state Ψ_i which is shielded from the radiation field ϕ it is detecting except during the time interval from t_0 to t. Using standard first-order perturbation theory and the dipole approximation, one finds, for the probability that the atom is in an excited state Ψ_f at time t, an expression involving the squared modulus of the quantity

$$\frac{ie}{\hbar}\int_{t_0}^{t} \exp(i\omega_{fi}t')M\Psi_f^*\hat{\phi}(\mathbf{x},t')\Psi_i\,dt', \tag{8}$$

where

$$\hbar\omega_{fi}=E_f-E_i>0$$

and M is an atomic matrix element. The argument now proceeds by analogy with the classical case. We distinguish between the positive and negative frequency components of $\hat{\phi}$ and note that the negative frequency components lead to rapidly oscillating integrands of the form $\exp[i(\omega_{fi}+\omega)t]$, which when integrated have a large denominator. On the other hand, the positive frequency components lead to integrands of the form $\exp[i(\omega_{fi}-\omega)t]$ which make an important contribution when $\omega\sim\omega_{fi}$ (corresponding to the approximate conservation of energy, when one neglects the linewidth of the excited state), If the time $t-t_0$ during which the atom is coupled to the radiation field is very much greater than $1/\omega_{fi}$ then energy is very accurately conserved and the contribution of the negative frequency components of $\hat{\phi}$ is negligible. This argument is analogous to the classical one which also leads to the singling out of the positive frequency components of the field and the introduction of the complex analytic signal. In the quantum case we can also use the language of particle (photon) states for the field $\hat{\phi}$, and say that the atom detects the field by absorbing a photon, this absorption being represented by the action of an annihilation operator for photons which involves the positive frequency components of the field.

When one takes the squared modulus of (8) one obtains for the detection probability a product of the atomic factor $|M|^2$ and the quantity

$$\int_{t_0}^{t} dt \int_{t_0}^{t} dt' \exp[i\omega_{fi}(t-t')]\langle\hat{\phi}(t)\hat{\phi}(t')\rangle.$$

Thus as in the classical case, the excitation rate depends on the noise power, this noise power involving the positive frequency component of the Fourier transform of the autocorrelation function of the field evaluated along the world-line of the detector. It involves a purely quantum average as well as a classical-type ensemble average if the field is in a mixed case.

2.2.4. The stochastic properties of a quantum field

As in the classical case we would expect a more complicated detector to be able to measure the positive frequency component of nth-order correlation or Wightman functions $\langle\hat{\phi}(t_1)\ldots\hat{\phi}(t_n)\rangle$. Since $\hat{\phi}$ is now an operator, the evaluation of these expectation values is often more complicated than in the classical case. Nevertheless there is a close correspondence between the two cases which is summed up in Sudarshan's optical equivalence theorem [23]. This correspondence is best expressed in terms of a basis for the quantum states which makes them most resemble classical states, namely the so-called coherent states [23, 24]. This resemblance arises from the property (which can be taken to define them) that the joint uncertainty in pairs of canonical variables achieves the minimum prescribed by the Heisenberg uncertainty relation. It can then be shown that these states are eigenstates of the positive frequency (annihilation) component of the operator $\hat{\phi}$, whose eigenvalues are complex analytic signals. The quantum correlation functions for operators can then be evaluated as though they involved only the apparently classical analytic signals. This is the essential content of the optical equivalence theorem. It should be stressed that no approximation of a semi-classical nature is being made here. The averages obtained are precisely the quantum mechanical ones. This quantum character is revealed by the fact that the weight function used in the 'classical' averaging (which is the quantum density operator expressed in a coherent state basis) is not necessarily positive-definite, and so cannot in general be regarded as a classical probability density. However, in the application of greatest interest to us, namely to a thermal equilibrium state for $\hat{\phi}$, this weight function is positive-definite, and thermal averages can be taken as though the state were classical (so long as the quantum (Planck) form of the energy spectrum is maintained).

The only place where care is needed is in handling the vacuum state. This state can be defined in terms of the *absence* of positive frequency components of the correlation functions, and so from the classical point of view would correspond to no activity at all. Of course this is not so in the quantum case, and this is of direct interest to us. Indeed this whole article is concerned with the close relation between the vacuum state and a thermal state when a horizon is present in the space–time. A clue to this relation may be found in the well-known fact that *the quantum mechanical probability distribution for the vacuum, like the thermal probability distribution for an equilibrium state, has a Gaussian character.*

This is a deep result, whose significance we shall attempt to elucidate later. For the moment we shall be content to express the result in as compact a form as possible. Since the radiative part of the quantum field may be regarded as made up of a set of harmonic oscillators, we may confine ourselves to the stochastic properties of a quantum oscillator in thermal equilibrium at temperature T (which for the vacuum fluctuations alone would be set equal to zero). These properties are best summed up using a quantum form of the characteristic function. We obtain this by replacing the classical variables a_ω, a^*_ω of (6) with operators $\hat{a}_\omega$, $\hat{a}^\dagger_\omega$ which possess the usual quantum commutation relations. We are therefore interested in $\langle\langle \exp[i(\xi\hat{a}_\omega + \xi^*\hat{a}^\dagger_\omega)]\rangle\rangle$, where the brackets refer to an average of both a quantum and a thermal kind.

This problem was solved by Bloch [31], and his result is now usually called Bloch's theorem. He found that

$$\langle\langle \exp[i(\xi\hat{a}_\omega + \xi^*\hat{a}^\dagger_\omega)]\rangle\rangle = \exp[-(\pi\xi\xi^*\bar{\varepsilon}/\omega^2)], \tag{9}$$

where

$$\bar{\varepsilon} = \tfrac{1}{2}\hbar\omega + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}.$$

We see that the dispersion is determined by $\bar{\varepsilon}$ which is the mean energy of a harmonic oscillator at temperature T, *including the zero-point energy*. Thus we are dealing with a Gaussian random process at the absolute zero as well as at finite temperatures. In particular, if we expand both sides of (9) we obtain in quantum operator form a relation between the nth-order correlation functions and the second-order one. This relation is the same as the classical one (5) if the appropriate operator ordering is maintained.

§3. Alternative vacua and their thermal properties

In this section we shall examine and elucidate the nature of the quantum states that have been proposed as suitable candidates for the physical vacuum in black-hole and cosmological space–times.

We begin with a discussion of an accelerated state in flat space–time which by contrast to our previous discussion concerning the readings of *accelerated detectors* in the Poincaré invariant (inertial) vacuum refers to a vacuum which is itself accelerated.

Historically Fulling [32] noticed that although the formalism of Lagrangian field theory has the appearance of being covariant, the choice of a time coordinate defines what is meant by positive and negative frequency thereby defining what is meant by the vacuum state. It was later pointed out by Unruh that the field theory that is 'natural' in Rindler coordinates is not unitarily equivalent to the usual one: a fact that is intimately linked with the existence of a horizon.

In order to examine this state of affairs let us introduce Rindler coordinates (ξ, τ) in the x–t plane by a transformation regular in the region $x > |t|$

$$x = \xi \cosh \tau,$$

$$t = \xi \sinh \tau.$$

The Minkowski space line element becomes

$$ds^2 = -\xi^2 d\tau^2 + d\xi^2 + dy^2 + dz^2.$$

We recall from the previous section that the curves $\xi = \text{constant}$, $\mathbf{x} = \text{constant}$ are curves of constant proper acceleration ξ^{-1}, $\xi\tau$ being the proper time along such a curve.

The curves $\xi = \text{constant}$ are all asymptotic to the planes $x = \pm t$. It is clear from fig. 3 that these planes are horizons for uniformly accelerated observers since an observer following one of the curves $\xi = \text{constant}$ cannot communicate with any space–time point in regions II or P and cannot receive any communication from regions II or F. These interesting causal properties associated with uniformly accelerated observers in Minkowski space–time have long been known and have received a certain amount of attention over the years, much of it related to the question whether a uniformly accelerated charge radiates.

Following Fulling [32] let us consider the problem of quantizing, say, a massless scalar field ϕ using Rindler coordinates instead of the more usual Minkowski set. The wave equation

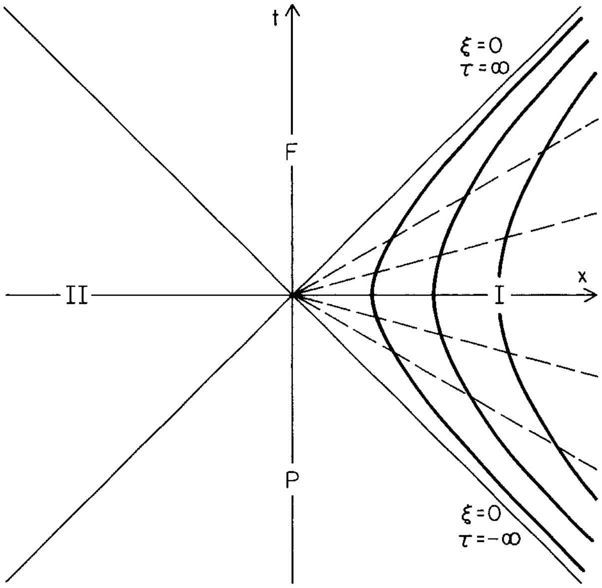
$$\Box \phi = 0$$

can be separated in Rindler coordinates and possesses normal mode solutions that are of positive frequency with respect to Rindler time

$$u(x|v, \mathbf{k}) = \frac{\sqrt{(\sinh v\pi)}}{2\pi^2} \exp(-iv\tau) K_{iv}(k\xi) \exp(i\mathbf{k} \cdot \mathbf{x}), \tag{10}$$

where $\mathbf{x}$ denotes the two-vector (y, z). The normalization in (10) has been chosen such that $u(x|v, \mathbf{k})$ is of unit norm with respect to a Klein Gordon inner product taken over

Fig. 3



Rindler coordinates cover the region $x > |t|$. The curves of constant ξ are hyperbolae while those of constant τ are straight lines through the origin.

any of the surfaces $\tau = \text{constant}$ (and hence over any spacelike surface originating at the origin in fig. 3.

We may now expand $\hat{\phi}$ in the form

$$\hat{\phi}(x) = \int_0^\infty d\nu \int d^2\mathbf{k} \frac{\sqrt{(\sinh \nu\pi)}}{2\pi^2} K_{i\nu}(k\xi)$$

$$\times \{\exp(-i\nu\tau + i\mathbf{k}\cdot\mathbf{x})\hat{a}(\nu, k) + \exp(i\nu\tau - i\mathbf{k}\cdot\mathbf{x})\hat{a}\dagger(\nu, k)\},$$

where '†' denotes Hermitian conjugation.

The canonical commutation relations then require that

$$[a(\nu, \mathbf{k}), a\dagger(\nu', \mathbf{k}')] = \delta(\nu - \nu')\delta(\mathbf{k} - \mathbf{k}')$$

with all commutators independent of this one vanishing. Let us now define a 'vacuum state' $|\mathrm{F}\rangle$ by the requirement that it be annihilated by the annihilation operators, i.e.

$$\hat{a}(\nu, \mathbf{k})|\mathrm{F}\rangle = 0, \quad \text{for all } \nu, \mathbf{k}.$$

Unruh's observation was that the theory that is thereby constructed is not unitarily equivalent to the usual free field theory on Minkowski space. Of even greater surprise was the subsequently discovered fact that the usual Poincaré invariant vacuum state appropriate to Minkowski space, which we shall henceforth denote by $|\mathrm{M}\rangle$, contains a thermal distribution of quanta with respect to the Fulling Fock space.

Indeed $|\mathrm{M}\rangle$ has properties identical with a thermal Fulling *mixed case* of local temperature $(2\pi\xi)^{-1}$. If the field is in the Minkowski vacuum state, then the *quantum expectation value* of any observable on the Fulling Fock space is the same as its *statistical ensemble average* in thermal equilibrium at a temperature $(2\pi\xi)^{-1}$. (Two bodies can be in thermal equilibrium at different points in a gravitational field if and only if the ratio of their temperatures is equal to the gravitational red shift that light would suffer in travelling from one to the other. Thus the dependence of the temperature on position should cause no surprise, being just that which is required to maintain equilibrium.)

What, more precisely, is meant by this is as follows. We have said that for any observable $\hat{O}[\hat{\phi}(x)]$ confined to the Rindler wedge, that is when $\hat{O}$ depends only on the field $\hat{\phi}(x)$ at points for which $x>|t|$,

$$\langle \mathrm{M}|\hat{O}[\hat{\phi}(x)]|\mathrm{M}\rangle = \mathrm{Tr}\,\hat{\rho}_{2\pi\xi}\hat{O}, \tag{11}$$

where $\hat{\rho}_{2\pi\xi}$ is the density operator

$$\left\{\sum_{|m,n\rangle} \exp(-2\pi E_m\xi)\right\}^{-1} \sum_{|m,n\rangle} \exp(-2\pi E_m\xi)\,|m,n\rangle\langle m,n| \tag{12}$$

of a thermal Fulling state of temperature $(2\pi\xi)^{-1}$. $|m,n\rangle$ is the nth Fulling state of energy E_m. The substantiation of (11) and (12) may be found in Appendix C.

We shall now seek to gain a physical understanding for the Fulling vacuum $|\mathrm{F}\rangle$. Rindler coordinates are suggestive of acceleration so let us, with a certain prescience, examine the effect of accelerating the vacuum by taking an infinite plane conductor and accelerating it normal to itself with acceleration a. If we choose coordinates appropriately, then the path swept out by the mirror can be taken to be the line $\xi=a^{-1}$ in fig. 3. This situation is moreover relevant to gravity since instead of regarding it as an accelerated mirror which drives the vacuum before it we might alternatively care to think of it as the vacuum in the gravitational field of an infinite flat earth. The vacuum would, in some sense, like to fall through the floor but it is prevented from doing so by the fact that the floor is perfectly reflecting. The vacuum adjusts to this situation by acquiring a non-vanishing vacuum expectation value for $\hat{T}_{\mu\nu}$. Thus the ether takes on the appearance of an equilibrium atmosphere loaded under the action of a gravitational field. The relevant field theory calculations can be performed exactly [33] and we shall briefly review the results of these calculations. We have used the language of electromagnetism but it is equally possible to consider analogous problems for fields of different spin.

Far above the mirror (i.e. as $\xi a\to\infty$) we find

$$\langle \hat{T}^{\nu}_{\mu}\rangle \underset{\xi\to\infty}{\sim} -\frac{h(s)}{2\pi^2\xi^4}\int_0^\infty \frac{d\nu\,\nu(\nu^2+s^2)}{\exp(2\pi\nu)-(-1)^{2s}}\,\mathrm{diag}\left(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right), \tag{13}$$

where $h(s)$ is the number of helicity states of a massless field of spin s. Thus far above the mirror we find that $\langle \hat{T}^{\nu}_{\mu}\rangle$ is reduced below zero by an amount corresponding to black-body radiation at a 'temperature' $(2\pi\xi)^{-1}$, the dependence of the 'temperature' on position being just that which would be required to maintain thermal equilibrium in a gas of massless scalar particles in the gravitational field of an 'infinite flat earth'. This asymptotic expression for $\langle \hat{T}^{\nu}_{\mu}\rangle$ is independent of the acceleration of the barrier in the sense that it depends only on the acceleration of the local Killing trajectory. It may also be shown that the relation (13) is independent of the precise nature of the boundary conditions applied to the field at the mirror.

The form of the stress tensor seems rather remarkable at first, though with the benefit of hindsight we can see that some of the features of $\langle \hat{T}^{\nu}_{\mu}\rangle_{\mathrm{Ren}}$ might have been anticipated had we had the confidence to assert that the precise nature of the boundary condition becomes unimportant far from the barrier. In virtue of the symmetries of the problem the expectation value of the stress-energy tensor would be expected to assume the form

$$\langle \hat{T}^{\nu}_{\mu} \rangle_{\mathrm{Ren}} = \mathrm{diag}\,(\alpha, \beta, \gamma, \gamma)$$

with α, β, γ functions of ξ only. In order that $\langle \hat{T}^{\nu}_{\mu} \rangle_{\mathrm{Ren}}$ be trace-free and conserved these functions are required to satisfy

$$\text{and} \qquad \left.\begin{aligned} \alpha + \beta + 2\gamma &= 0 \\ \alpha &= \frac{d}{d\xi}(\beta\xi). \end{aligned}\right\} \tag{14}$$

If the components of the stress tensor are to be 'local', i.e. asymptotically independent of a, then on dimensional grounds they must vary as ξ^{-4}. Equations (14) then determine the ratios $\alpha : \beta : \gamma$ so that up to a constant of proportionality $\langle \hat{T}^{\nu}_{\mu} \rangle_{\mathrm{Ren}}$ becomes

$$\xi^{-4}\,\mathrm{diag}\left(-1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{15}$$

Now the Minkowski vacuum is that state which minimizes the energy subject to the constraint of invariance under the full Poincaré group. The ground state for the present problem is that state which (aside from boundary effects) minimizes the energy subject to the constraint of invariance under the homogeneous Lorentz transformations of the x–t plane and the Euclidean transformations of the y–z plane. Since this latter symmetry group is a subgroup of the Poincaré group the energy is to be minimized subject to a less onerous constraint, resulting in the attainment of a lower minimum. We might therefore expect the energy density far from the barrier to be depressed *below* that of the Minkowski vacuum and the constant that multiplies (15) to be negative.

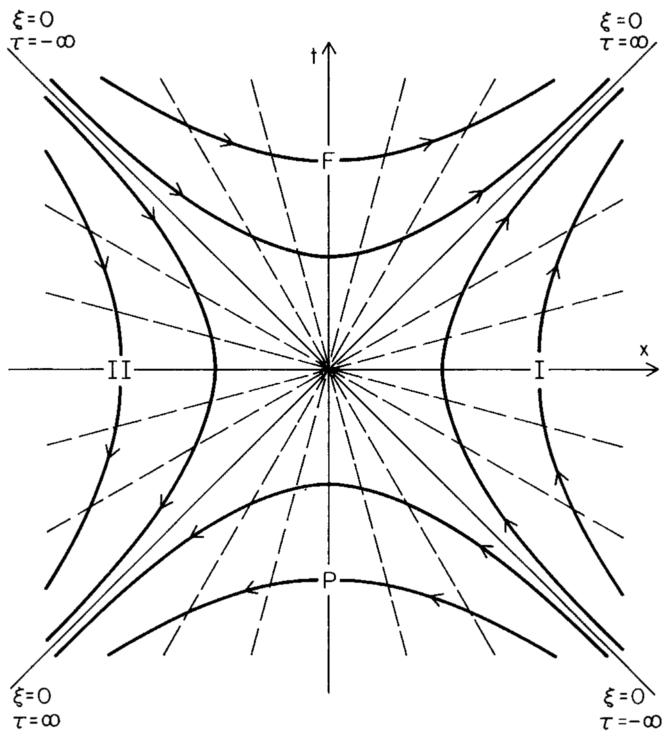
We see from (13) that the energy density has a thermal distribution. The factor $(\nu^2 + s^2)$ that occurs in the integrand is a density-of-states factor. The reason that one would normally expect this factor to be ν^2 is that usually the density-of-states factor is computed under the assumption that the wavelengths involved are very much smaller than the size of the container. Now when note is taken of the fact that the proper time measured along a world-line is $\xi\tau$ rather than τ then it is readily seen from (13) that the relevant 'wavelengths' are of order ξ and hence are of the same order as the distance to the mirror. The surprise then, if any, is that the deviation of the spectrum from the more usual Planckian form is so simple. The reason for this is that the density-of-states factor is a property which the Rindler manifold shares with a related manifold which has the metric

$$d\bar{s}^2 = -d\tau^2 + \frac{d\xi^2 + dy^2 + dz^2}{\xi^2}. \tag{16}$$

It is a somewhat remarkable fact that the spatial metric in (16) is the metric on the three-hyperboloid. It is the simplicity of the geometry of this related space that is responsible for the concise form that the density-of-states factor assumes.

We have not discussed the form of $\langle \hat{T}^{\nu}_{\mu} \rangle_{\mathrm{Ren}}$ when ξ is comparable to a^{-1}. In this regime the effects and the precise nature of the boundary conditions become

Fig. 4



The extended Rindler coordinates. The arrows denote the direction in which τ increases.

important. The point that we wish to stress is the fact that for the region $\xi a \gg 1$ the vacuum state above an accelerated mirror is a very good approximation to the Fulling vacuum. The agreement between these states becomes better the greater the acceleration of the mirror, and becomes exact in the limit that the acceleration of the mirror becomes infinite. In this limit the motion of the mirror is as in fig. 4 with the mirror initially travelling in the negative x direction at the speed of light until it reaches the origin and then instantaneously reversing its motion. In this limit the expression (13) for $\langle \hat{T}^{\nu}_{\mu} \rangle$ is exact and is valid for all ξ. The components of $\langle \hat{T}^{\nu}_{\mu} \rangle$ are seen to be unbounded as $\xi \to 0$ but this is hardly surprising since the vacuum is being made to accelerate infinitely hard there.

We wish to stress that the Fulling vacuum and the corresponding field theory are not to be regarded as some sort of wrong quantization but rather a quantum field theory adapted to a particular physical situation. It is thus hardly surprising that this field theory should not be unitarily equivalent to the more usual Poincaré invariant one.

Both the Poincaré invariant vacuum and the Fulling vacuum have close analogues in black-hole space–times and, just as is the case in flat space–time, these different vacua are most easily discussed in terms of different coordinate systems. In order that we may shortly draw attention to these similarities we are led to ask if there is any 'natural' way to extend the Rindler coordinate system in such a way as to cover the entire space–time. If we wish to do this in such a way that $\partial/\partial\tau$ remains a generator of boosts then we are led to the following sets of transformations:

$$\left.\begin{aligned} x &= \xi \cosh \tau \\ t &= \xi \sinh \tau \end{aligned}\right\} \quad \text{region I,}$$

$$\left.\begin{aligned} x &= \xi \sinh \tau \\ t &= \xi \cosh \tau \end{aligned}\right\} \quad \text{region F}$$

$$\left.\begin{aligned} x &= -\xi \cosh \tau \\ t &= -\xi \sinh \tau \end{aligned}\right\} \quad \text{region II,}$$

$$\left.\begin{aligned} x &= -\xi \sinh \tau \\ t &= -\xi \cosh \tau \end{aligned}\right\} \quad \text{region P.}$$

The line element induced by these transformations is

$$ds^2 = \mp \xi^2 d\tau^2 \pm d\xi^2 + dy^2 + dz^2 \tag{17}$$

with the upper signs holding in I and II and the lower signs holding in F and P. We note that τ is a time coordinate in I and a space coordinate in F and P. In region II, τ is again a time coordinate but runs backwards. We shall also meet this phenomenon in the curved space–times with horizons that we shall consider. We would also remark that even the above set does not quite cover the space–time since the transformations are not regular on the planes $x = \pm t$. We shall return to this point when we come to discuss the rotation to imaginary time.

3.1. *Black-hole thermodynamics*

The first hint that black holes are endowed with thermodynamic properties was the result of Penrose and Floyd [34] and its generalization by Hawking [35] to the theorem that the total area A of the horizons in black-hole spaces must increase in all reasonable physical processes. It must be stressed that these results used only classical general relativity together with certain reasonable assumptions about the behaviour of matter.

The analogy between Hawking's area theorem and the second law of thermodynamics was elaborated and generalized by Bekenstein [6] and by Bardeen *et al.* [7]. The latter authors produced a striking summary of various results of the classical theory of black holes in the form of the 'four laws of black-hole mechanics'. The first law is just the law of conservation of energy. This holds as well for systems of interacting black holes (provided we take proper account of gravitational waves and restrict our attention to asymptotically flat space–times) as for any other physical system. If δM is the infinitesimal change of mass of a rotating black hole of angular velocity Ω and angular momentum J then

$$\delta M = \frac{1}{8\pi}\kappa \delta A + \Omega \delta J,$$

where A is the area of the horizon and

$$\kappa=\frac{(M^4-J^2)^{1/2}}{2M[M^2+(M^4-J^2)^{1/2}]}$$

is the 'surface gravity' of the black hole. This is interpreted as showing that the change in the internal energy of the black hole equals the work done on it plus the 'heat', $\kappa\delta A$, 'flowing into it', where κ is proportional to a 'temperature' and A to an 'entropy'.

The theorem [36] that the surface gravity $\kappa(M, J)$ is constant over the horizon of a stationary black hole is analogous to the zeroth law of thermodynamics. We can interpret it as saying that equilibrium between two points is possible only if they have the same temperature.

The analogue of the third law must now be that no physical process can lower the 'temperature' κ to absolute zero, which occurs when $J=M$. Although black-hole space–times with all real values of κ exist as solutions of the Einstein field equations, their causal structure changes discontinuously when κ descends through zero. Arbitrarily small perturbations in the geometry of the $\kappa=0$ solution produce naked singularities. It has been a long-standing conjecture that such pathologies cannot develop from regular initial data. This is known as the 'cosmic censorship' hypothesis. The third law of black-hole mechanics is a special case of the hypothesis. The irreversibility implied by the Hawking theorem was taken even more seriously by Bekenstein [37] who suggested that A be identified (up to a constant of proportionality) as the black-hole entropy S_{BH}, and that the second law of thermodynamics be generalized to the requirement that the sum of S_{BH} and the 'ordinary' entropy S be a non-decreasing function of time

$$\frac{\partial}{\partial t}(S_{\mathrm{BH}}+S)\geqslant 0$$

with the equality holding only for reversible processes. Bekenstein appealed to heuristic physical arguments, of the kind we are seeking in this paper, estimating the effect of quantum theory on physical processes in the neighbourhood of a black hole. At the time, no actual calculation of a black hole's thermodynamic properties had been made, and when they were, Bekenstein's prediction, together with his implicit assumption that these properties would be quantum field theoretic in origin, were shown to be correct.

Central to Bekenstein's heuristic arguments was a thought experiment designed to show that quantum effects reduce the efficiency of heat engine processes which extract work from a black hole, and thereby to predict the black hole's temperature T_{BH}. In Appendix B we describe the thought experiment and show why it is fallacious, even though the *result* and its generalization to charged, rotating black holes has been vindicated by subsequent exact calculations [1]. This example illustrates the lack of, and the need for, an intuitive derivation of the Hawking process and related effects, which we shall give later in this section.

Because of the work of many authors, reviewed in this paper, the thermodynamic properties of black holes are now generally accepted. The four laws of black-hole mechanics really are special cases of the ordinary laws of thermodynamics, formulated with κ as the black-hole temperature and A as its entropy.

3.2. *Field theory on black-hole space–times*

We wish now to discuss quantum field theory in a black-hole space–time; for simplicity we shall take the black hole to be non-rotating. We begin by briefly reviewing the geometry of the extended Schwarzschild manifold. The close analogies between the structure of Schwarzschild space–time and the properties of accelerated observers in Minkowski space–time will become apparent as we proceed.

The Schwarzschild metric describes the gravitational influence of a spherically symmetric body of mass M. The metric was first presented in the form

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

It was not until 1960 [38] that the nature of the coordinate singularity at $r=2M$ was properly elucidated. Schwarzschild coordinates are singular at $r=2M$ but the curvature of the manifold is not. It is possible to introduce non-singular coordinates which may be used to analytically continue the manifold from the domain of its original definition $r>2M$ to encompass the points for which $r<2M$. It is usual to do this in terms of Kruskal coordinates in which the metric takes the form

$$ds^2 = \frac{32M^3}{r}\exp(-r/2M)(-dv^2+du^2) + r^2(d\theta^2+\sin^2\theta d\phi^2)$$

in which r is to be understood as a function of u and v given implicitly by

$$\left(\frac{r}{2M}-1\right)\exp(r/2M) = u^2 - v^2.$$

This metric is singular only at the *curvature* singularities where $r=0$, and with the coordinate ranges

$$-\infty < v < \infty, \quad -\infty < u < \infty, \quad u^2 - v^2 > -1$$

represents the maximal analytic extension of the Schwarzschild manifold. In addition to the Kruskal coordinates we shall have occasion to refer to the null coordinates U and V defined by

$$U = v - u,$$

$$V = v + u.$$

Figure 5 is a Kruskal diagram which exhibits the structure of the u–v plane. The transformation between the Kruskal coordinates and Schwarzschild coordinates take the following explicit forms:

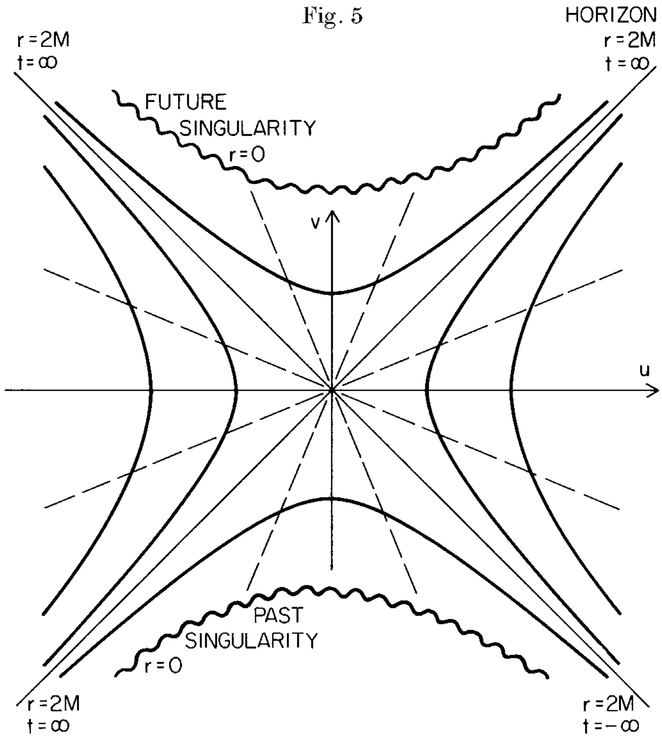
$$\left.\begin{aligned} u&=\left(\frac{r}{2M}-1\right)^{1/2}\exp\left(r/4M\right)\cosh\left(t/4M\right)\\ v&=\left(\frac{r}{2M}-1\right)^{1/2}\exp\left(r/4M\right)\sinh\left(t/4M\right)\end{aligned}\right\}\ \text{region I,}$$

$$\left.\begin{aligned} u&=\left(1-\frac{r}{2M}\right)^{1/2}\exp\left(r/4M\right)\sinh\left(t/4M\right)\\ v&=\left(1-\frac{r}{2M}\right)^{1/2}\exp\left(r/4M\right)\cosh\left(t/4M\right)\end{aligned}\right\}\ \text{region F,}$$

$$\left.\begin{aligned} u&=-\left(\frac{r}{2M}-1\right)^{1/2}\exp\left(r/4M\right)\cosh\left(t/4M\right)\\ v&=-\left(\frac{r}{2M}-1\right)^{1/2}\exp\left(r/4M\right)\sinh\left(t/4M\right)\end{aligned}\right\}\ \text{region II,}$$

$$\left.\begin{aligned} u&=-\left(1-\frac{r}{2M}\right)^{1/2}\exp\left(r/4M\right)\sinh\left(t/4M\right)\\ v&=-\left(1-\frac{r}{2M}\right)^{1/2}\exp\left(r/4M\right)\cosh\left(t/4M\right)\end{aligned}\right\}\ \text{region P.}$$

Fig. 5

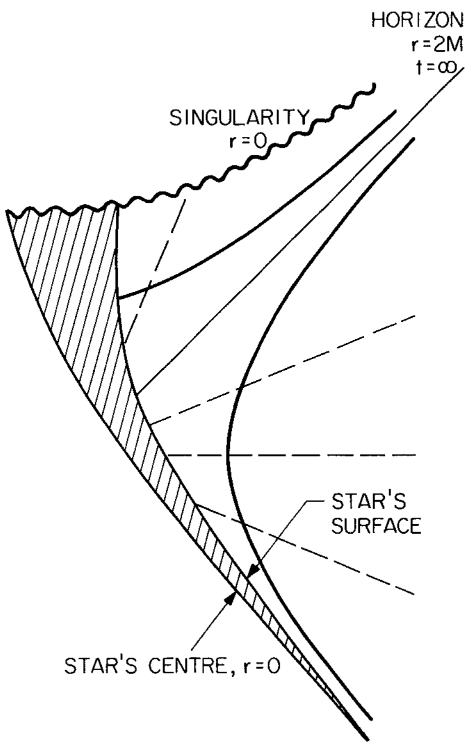


The analytically extended Schwarzschild manifold. The hyperbolae represent curves of constant r while the straight lines represent those of constant t.

The complete analogy between the relationship that Kruskal coordinates bear to Schwarzschild coordinates and the relationship that Minkowski coordinates bear to Rindler coordinates is clear from the above. The fact that the maximally extended Schwarzschild manifold contains two singularities and two asymptotically flat regions is perhaps surprising on first acquaintance. It should, however, be kept in mind that if a black hole is formed by the collapse of an extended object then it is only a portion of this manifold that is relevant as the solution exterior to the extended body and this must be joined to an interior solution. In fig. 6 we present a diagram of the space–time appropriate to a star that collapses to form a black hole.

Because it is easier to handle the equations on the Kruskal manifold rather than to work on the non-analytic manifold which is the one that in fact corresponds to the way that a black hole would actually form, many authors have considered the problem how to 'formulate correctly' a quantum field theory on the Kruskal manifold. It is to this problem that we now turn. Our conclusion will be that the problem consists not so much of how to formulate correctly such a field theory but rather of recognizing that different theories correspond to different physical situations.

Fig. 6



The manifold appropriate to an extended body that collapses to form a black hole. The Schwarzschild metric describes the geometry exterior to the body. When this is smoothly joined to the geometry interior to the body the unphysical regions II and P are eliminated.

Let us consider a scalar field theory. In the exterior region of Schwarzschild space–time a complete set of normalized basis functions that satisfy the equation

$$\Box u = 0$$

is [5]

$$\vec{u}_{\omega lm}(x) = (4\pi\omega)^{-1/2} \exp(-i\omega t)\vec{R}_l(\omega|r)\, Y_{lm}(\theta, \phi),$$

$$\overleftarrow{u}_{\omega lm}(x) = (4\pi\omega)^{-1/2} \exp(-i\omega t)\overleftarrow{R}_l(\omega|r)\, Y_{lm}(\theta, \phi).$$

The Y_{lm} are the spherical harmonics and the radial functions have the asymptotic forms

$$\vec{R}_l(\omega|r) \sim \begin{cases} \dfrac{1}{r}\exp(i\omega r_*) + \dfrac{1}{r}\vec{A}_l(\omega)\exp(-i\omega r_*), & r \to 2M, \\[2ex] \dfrac{1}{r} B_l(\omega)\exp(i\omega r_*), & r \to \infty, \end{cases}$$

$$\overleftarrow{R}_l(\omega|r) \sim \begin{cases} \dfrac{1}{r} B_l(\omega)\exp(-i\omega r_*), & r \to 2M, \\[2ex] \dfrac{1}{r}\exp(-i\omega r_*) + \dfrac{1}{r}\overleftarrow{A}_l(\omega)\exp(i\omega r_*), & r \to \infty, \end{cases}$$

in which

$$r_* = r + 2M \log\left(\frac{r}{2M} - 1\right)$$

denotes the Regge–Wheeler coordinate. The physical interpretation of these modes is illustrated in fig. 7. The $\vec{u}$ modes emerge from the past horizon and the $\overleftarrow{u}$ modes come in from infinity.

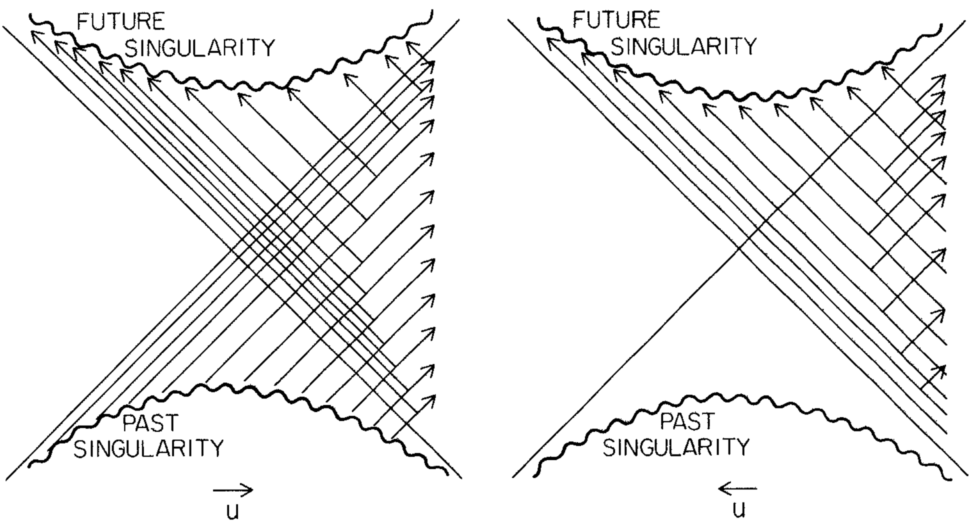
Three states have been proposed as suitable candidates for the vacuum of a black-hole space–time: these are as follows.

(i) The Boulware [39] vacuum $|\mathrm{B}\rangle$, defined by requiring normal modes to be positive frequency with respect to the Killing vector $\partial/\partial t$ with respect to which the exterior region is static.

(ii) The Unruh [2] vacuum $|\mathrm{U}\rangle$, defined by taking modes that are incoming from infinity to be positive frequency with respect to $\partial/\partial t$, while those that emanate from the past horizon are taken to be positive frequency with respect to U, the null coordinate on the past horizon.

(iii) The Hartle–Hawking [40] vacuum $|\mathrm{H}\rangle$ defined by taking incoming modes to be positive frequency with respect to V, the null coordinate on the future horizon, and outgoing modes to be positive frequency with respect to U. A detailed statement of what constitutes being of positive frequency with respect to U and V may be found in Appendix C.

Fig. 7



A schematic representation of the behaviour of the outgoing and incoming modes. The outgoing modes $\vec{u}$ represent waves that are outgoing across the past horizon. The space–time curvature deflects part of the wave across the future horizon. The incoming modes $\overleftarrow{u}$ represent waves that are incoming from infinity. The geometry deflects part of the wave while the remainder is transmitted across the future horizon.

It is possible to examine these three states by calculating the values of various physical observables. Of the various possible observables we shall choose to examine the vacuum expectation values of the stress-energy operator and the response of an Unruh box since, as we shall see, there is a sense in which these are complementary measures of vacuum activity. The results of the calculations for the response of an Unruh box are summarized in the table. From these results we find the following consistent interpretations.

The asymptotic values of the function

$$\Pi(\omega|r) = \int_{-\infty}^{\infty} dt \exp(-i\omega t) \langle 0|\hat{\phi}(t)\hat{\phi}(0)|0\rangle$$

which determines the response of an Unruh box. θ denotes the step function.

	Boulware vacuum	Unruh vacuum	Hartle–Hawking vacuum
$r \to 2M$	$\dfrac{-\omega\theta(-\omega)}{2\pi\left(1-\dfrac{2M}{r}\right)}$	$\dfrac{\omega}{2\pi\left(1-\dfrac{2M}{r}\right)[\exp(2\pi\omega/\kappa)-1]}$	$\dfrac{\omega}{2\pi\left(1-\dfrac{2M}{r}\right)[\exp(2\pi\omega/\kappa)-1]}$
$r \to \infty$	$-\dfrac{\omega}{2\pi}\theta(-\omega)$	$\dfrac{1}{r^2}\dfrac{\sum(2l+1)\lvert B_l(\omega)\rvert^2}{8\pi\omega[\exp(2\pi\omega/\kappa)-1]} - \dfrac{\omega}{2\pi}\theta(-\omega)$	$\dfrac{\omega}{2\pi(\exp(2\pi\omega/\kappa)-1)}$

(i) The Boulware vacuum corresponds to our familiar concept of an empty state at large radii, but is pathological at the horizon in the sense that the renormalized expectation value of the stress tensor, in a freely falling frame, diverges as $r \to 2M$. For a massless conformally invariant field of spin $s(=0, \frac{1}{2}, 1)$ the leading behaviour of $\langle \mathrm{B}|\hat{T}^{\nu}_{\mu}|\mathrm{B}\rangle_{\mathrm{Ren}}$ near the horizon is given, in Schwarzschild coordinates, by

$$\langle \mathrm{B}|\hat{T}^{\nu}_{\mu}|\mathrm{B}\rangle_{\mathrm{Ren}} \underset{r\to 2M}{\sim} -\frac{h(s)}{2\pi^2\left(1-\dfrac{2M}{r}\right)^2}\int_0^{\infty}\frac{d\omega\,\omega(\omega^2+s^2\kappa^2)}{\exp(2\pi\omega/\kappa)-(-1)^{2s}}\,\mathrm{diag}\left(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),$$

which corresponds to the absence from the vacuum of black-body radiation at the black-hole temperature $(8\pi M)^{-1}$. This result is in precise analogy to the result for $\langle \hat{T}^{\nu}_{\mu}\rangle_{\mathrm{Ren}}$ above the infinitely accelerated plane conductor that we have previously considered.

(ii) In the Unruh vacuum we find that $\langle \hat{T}^{\nu}_{\mu}\rangle_{\mathrm{Ren}}$ is regular, in a freely falling frame, on the future horizon but not on the past horizon. At infinity this vacuum corresponds to an outgoing flux of black-body radiation at the black-hole temperature.

(iii) In the Hartle–Hawking 'vacuum' $\langle \hat{T}^{\nu}_{\mu}\rangle_{\mathrm{Ren}}$ is well behaved, in a freely falling frame, on both the future and past horizons. The price paid for this is that the state is not empty at infinity but corresponds instead to a thermal distribution of quanta at the black-hole temperature. That is, the Hartle–Hawking 'vacuum' corresponds to a black hole in equilibrium with an infinite sea of black-body radiation.

This equilibrium is actually unstable for a black hole in contact with an infinite radiation bath since the temperature of the hole varies inversely with its mass. Thus if, in virtue of a fluctuation, the black hole were to absorb more radiation than it emitted, its mass would increase and hence its temperature would fall. It would therefore absorb even more radiation, cool further and so on. If, on the other hand, the black hole were initially to emit more radiation than it absorbed then its temperature would rise and it would radiate even more rapidly. Thus in either case the system would tend to evolve away from equilibrium. The stability of the equilibrium can be restored, however, by enclosing the black hole in a suitably small box [41].

We conclude from this that it is the Unruh vacuum that best approximates the vacuum relevant to the gravitational collapse of a massive body. It is also apparent what situation the Boulware vacuum is appropriate to: it is relevant to the region exterior to a massive body that is only just outside its Schwarzschild radius. One could, for ease of visualization, consider a spherical body with a perfectly conducting surface just outside the body's Schwarzschild radius. The vacuum, in some sense, would like to fall through the surface but is prevented from doing so by the boundary condition. It therefore comes to equilibrium loaded under the action of the gravitational field, and its response to this loading is to set up in itself certain energy densities and stresses. If the body is now slowly shrunk down towards its Schwarzschild radius then the stresses that are required to support the vacuum become larger and ultimately infinite in the limit that the radius of the body actually equals its Schwarzschild radius. Thus the ether is no more able to support itself under such circumstances than is a more conventional atmosphere.

Note also the manner in which $\langle B|\hat{T}_{\mu\nu}|B\rangle_{\text{Ren}}$ becomes infinite on the horizon; it corresponds in some sense to the absence of black-body radiation and becomes infinite on the horizon because the local temperature contains a gravitational red shift factor. We might care to think of $\langle B|\hat{T}_{\mu\nu}|B\rangle_{\text{Ren}}$ as the 'pure vacuum polarization part' of $\langle H|T_{\mu\nu}|H\rangle_{\text{Ren}}$ and thereby gain further understanding of the necessity of Hawking radiation. The 'radiation part' of the stress tensor also becomes infinite on the horizon but the sum of the 'pure vacuum polarization part' and the 'radiation part' remains finite.

The globally defined state $|H\rangle$ bears essentially the same relationship to Boulware states as the Minkowski vacuum $|M\rangle$ does to Fulling states. For observables $\hat{O}[\hat{\phi}(x)]$ confined to region I, we have analogously to (11),

$$\langle H|\hat{O}|H\rangle = \operatorname{Tr} \hat{\rho}_{2\pi/\kappa}\hat{O}, \tag{18}$$

where

$$\hat{\rho}_{2\pi/\kappa} = \left\{ \sum_{|m,n\rangle} \exp(-2\pi E_m/\kappa) \right\}^{-1} \sum_{|m,n\rangle} \exp(-2\pi E_m/\kappa)\, |m,n\rangle\langle m,n|, \tag{19}$$

where $|m,n\rangle$ is now the nth Boulware state of energy E_m. $\hat{\rho}_{2\pi/\kappa}$ is a density matrix representing thermal equilibrium at a temperature $2\pi/\kappa$.

One way of obtaining the results (18) and (19) is to regard $\hat{\rho}_{2\pi/\kappa}$ as a partial trace of $|H\rangle\langle H|$ over a complete set of states *outside* region I [42]. $|H\rangle\langle H|$ is thereby projected to a mixed case because it contains global correlations not confined to region I. So much is physically obvious, but it is not, perhaps, so obvious why the mixed case should be precisely a thermal one. Part of the reason is as follows. The state $|H\rangle$ possesses all the symmetries of Minkowski space–time. In particular, it is locally invariant under the action of $\partial/\partial t$, the Killing vector field with respect to which region I is static. Therefore the restriction of $|H\rangle\langle H|$ to region I must be a *static* case with respect to $\partial/\partial t$, (even though the full state $|H\rangle$, of course, is not because $\partial/\partial t$ is not everywhere timelike). We are almost home because it is intuitively obvious (and rigorously proved [3]) that every static, mixed case which is also stable under all sufficiently small perturbations of the Hamiltonian must be in thermal equilibrium at some non-zero temperature. $|H\rangle$, being the global state of least energy, ought to have this stability, and more: there should exist no physical process, operating in a cycle, which could extract work from it. Consider a box at rest at a fixed distance from a black hole in the state $|H\rangle$. The walls of the box are constructed so that we can, at will, either isolate the interior of the box or put it into thermal contact with the outside world. The states of a quantum field inside the box, when it is isolated, are well approximated (ignoring boundary effects, which are irrelevant to the present argument) by the states of the Boulware Fock space, since the box is at rest with respect to the Schwarzschild time t. When we allow thermal contact between the interior of the box and the outside the combined interior + exterior near the box must settle down to the case $|H\rangle\langle H|$, after any transient Hartle–Hawking particles have been radiated away. When, after a sufficiently long time, the box is sealed again, the interior must be represented by the restriction to it of $|H\rangle\langle H|$, i.e.

$$\hat{\rho}_{2\pi/\kappa} = \operatorname{Tr}_{\text{outside}} |H\rangle\langle H|,$$

when the trace is over a complete set of states outside the box. *If this is not a thermal case, then work can be extracted from it*, and the process can be repeated in a cycle, contrary to the assumption that $|H\rangle$ is a ground state.

3.3. *The rate of area decrease*

One of the main motivations in studying the expectation value of the stress tensor has been to see whether the semi-classical Einstein equations

$$G_{\mu\nu}=\langle\hat{T}_{\mu\nu}\rangle$$

make sense. Clearly this equation would cause some embarrassment if, say, $\langle\hat{T}_{\mu\nu}\rangle$ were to prove to be infinite along the future horizon of an extended body that collapsed to form a black hole. It is of interest to note that it is possible to solve the linearized Einstein equations to show that the rate at which the area of the black hole decreases is just that which is to be expected from a knowledge of the flux at infinity. This result is, of course, completely obvious on physical grounds but the fact that it may be derived by explicit solution of an Einstein equation may be regarded as a tentative step towards the solution of the back-reaction problem.

The rate of area decrease is determined by the Newman–Penrose equation [43]

$$\frac{d\rho}{dv}=\kappa\rho+\rho^2+\langle\hat{\sigma}\hat{\sigma}\dagger\rangle+4\pi\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{Ren}}\,l^\mu l^\nu, \tag{20}$$

where ρ is the convergence and σ the shear of the null congruence l^μ which generates the horizon. In (20) ρ^2 and $d\rho/d\sigma$ are second-order quantities. If we neglect also the back-reaction due to the radiation of gravitons (we shall return to this point) we may omit the $\sigma\sigma\dagger$ term. The lowest-order solution to (20) is then

$$\rho=-\frac{4\pi}{\kappa}\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{Ren}}\,l^\mu l^\nu,$$

where

$$l^\mu=\left(\frac{1}{2},\frac{1}{2},0,0\right)$$

in (t,r_*,θ,ϕ) coordinates.

Thus

$$\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{Ren}}\,l^\mu l^\nu=\frac{1}{4}\langle\hat{T}_{tt}\rangle_{\mathrm{Ren}}+\langle\hat{T}_{r_*r_*}\rangle_{\mathrm{Ren}}+2\langle\hat{T}_{tr_*}\rangle_{\mathrm{Ren}}$$

$$\rightarrow\langle\hat{T}_{tr_*}\rangle \quad \text{as} \quad r\rightarrow 2M,$$

where the last line follows from the fact that we must have

$$\langle\hat{T}_{tt}\rangle_{\mathrm{Ren}}+\langle\hat{T}_{r_*r_*}\rangle_{\mathrm{Ren}}-2\langle\hat{T}_{tr_*}\rangle=O(r-2M)^2 \quad \text{as} \quad r\rightarrow 2M$$

in order for $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{Ren}}$ to be regular in a freely falling frame on the future horizon.

The divergence condition for the flux components implies that

$$\langle\hat{T}_{tr_*}\rangle=\frac{L}{4\pi r^2}$$

with L the luminosity of the black hole at infinity.

Now the rate of area decrease is given by

$$\frac{dA}{dv}=-2\int\rho\,dA, \tag{21}$$

from which it follows that

$$\frac{dA}{dv} = -\frac{2AL}{M},$$

which is (of course) just the relation that was to be expected, since with $A = 16\pi M^2$ this equation becomes

$$\frac{dM}{dv} = -L.$$

It is a curious fact that one may deduce the value of $\langle \hat{T}_{\mu\nu} \rangle_{\text{Ren}} l^\mu l^\nu$ on the horizon without actually doing any renormalization; simply the knowledge that the renormalized stress tensor is finite there suffices.

From the way that $\sigma\sigma\dagger$ enters into the Newman–Penrose equation (20) it would seem that we must also have

$$\langle \hat{\sigma}\hat{\sigma}\dagger \rangle_{\text{Ren}} \to -\frac{1}{4M^2} L_{\text{gravitons}},$$

as $r \to 2M$ and a little analysis reveals this is indeed the case [44, 45].

3.4. *De Sitter space–time*

In our discussion of uniformly accelerated observers in Minkowski space we saw that they perceived the vacuum as a bath of black-body radiation. Thay they should perceive the vacuum as non-empty is perhaps not too surprising if we recall that it is only supposed that the Minkowski vacuum is invariant under Poincaré transformations and not under acceleration transformations. To put the matter more graphically we might say that it is hardly surprising that detectors make strange readings when accelerated through the vacuum since the Minkowski vacuum is really tailored to the motion of inertial observers. With this in mind we now wish to discuss field theory on De Sitter space–time since this theory manifests the curious property that even inertially moving detectors register black-body radiation in the vacuum.

Before proceeding we shall briefly review the geometry of the manifold.

De Sitter space is a maximally symmetric space of constant (positive) curvature. It may be realized as the hyperboloid of revolution

$$\frac{1}{K} = -\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2$$

in a five-dimensional Minowski space with metric

$$ds^2 = -d\xi_0^2 + d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\xi_4^2.$$

The Riemann tensor in De Sitter space can be simply expressed in terms of the metric as

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}).$$

We may choose coordinates, defined for $Kr^2<1$, according to the following relations:

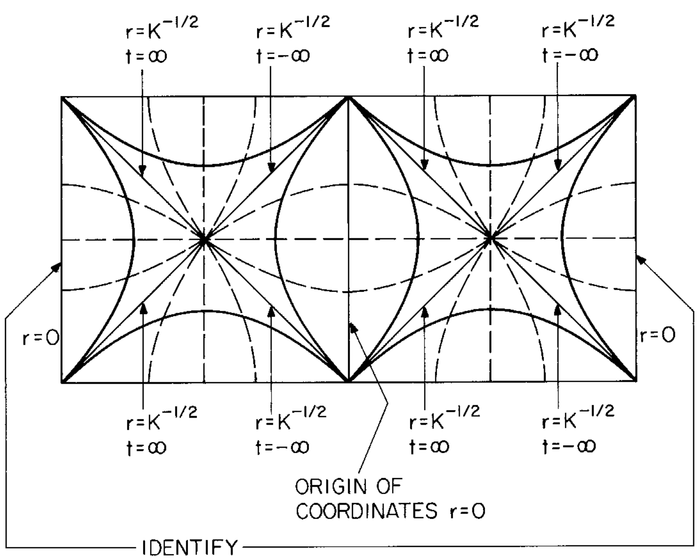
$$\left.\begin{aligned}
\xi^0 &= K^{-1/2}(1-Kr^2)^{1/2}\sinh(K^{1/2}t),\\
\xi^1 &= K^{-1/2}(1-Kr^2)^{1/2}\cosh(K^{1/2}t),\\
\xi^2 &= r\sin\theta\cos\phi,\\
\xi^3 &= r\sin\theta\sin\phi,\\
\xi^4 &= r\cos\theta.
\end{aligned}\right\} \tag{22}$$

In terms of these coordinates the line element becomes

$$ds^2 = -(1-Kr^2)\,dt^2 + \frac{dr^2}{(1-Kr^2)} + r^2(d\theta^2+\sin^2\theta d\phi^2).$$

In this form the metric is static but has a coordinate singularity where $Kr^2=1$ corresponding to cosmological event horizons. These are event horizons for observers who follow trajectories of the Killing vector $\partial/\partial t$ and on these surfaces $\partial/\partial t$ becomes null. The coordinates (22) may be related to global coordinates in much the same way as Rindler coordinates are related to Minkowski coordinates, or for that matter as Schwarzschild coordinates are related to Kruskal coordinates. Just as with these other manifolds it is possible to introduce four coordinate patches similar to (22) in order to cover the entire space–time (with the exception of the Killing horizons themselves). Figure 8 is the Penrose diagram for De Sitter space in which the dotted lines represent the trajectories of the Killing vector $\partial/\partial t$. We see, in a manner

Fig. 8



A conformal diagram of De Sitter space. For ease of representation De Sitter space, which can be thought of as a hyperboloid of one sheet, has been conformally deformed into a cylinder and the cylinder has been unrolled. Lines at 45° represent null rays and two dimensions have been surpassed so that each point, except those for which $r=0$, represents a two-sphere.

reminiscent of our previous examples, that $\partial/\partial t$ becomes spacelike in regions F and P and is timelike but with reversed orientation in region II. In fact (22) may be regarded as the restriction to the De Sitter hyperboloid of a Rindler coordinate system in the five-dimensional embedding space.

As we have already remarked the response of an Unruh box is essentially determined by the Fourier transform of the autocorrelation function of the field. Thus for a scalar field the quantity that is measured by the box is

$$\Pi(\omega) = \int_{-\infty}^{\infty} dt \exp(-i\omega t) \langle 0|\hat{\phi}(t)\hat{\phi}(0)|0\rangle$$

in which $\hat{\phi}(t)$ denotes $\hat{\phi}(x)$ evaluated at

$$x(t) = (t, 0, \theta, \phi)$$

and where $|0\rangle$ denotes the unique De Sitter invariant vacuum [46–48]. It is a simple matter to deduce the value of the autocorrelation function from the known form of the Feynman propagator on the De Sitter manifold. We find

$$\langle 0|\hat{\phi}(t)\hat{\phi}(0)|0\rangle = \frac{-K}{16\pi^2 \sinh^2\left(\dfrac{t}{2} - i\varepsilon\right)}$$

and hence

$$\Pi(\omega) = \frac{1}{2\pi} \frac{\omega}{(\exp(2\pi K^{-1/2}\omega) - 1)}. \tag{23}$$

We may deduce from this that Unruh's box responds as if it were immersed in black-body radiation at a temperature $(2\pi K^{-1/2})^{-1}$.

Now it is clear from the symmetry of the problem that the vacuum expectation value of the stress tensor must be maximally symmetric, i.e. it must be a multiple of the metric tensor. In fact for our massless scalar field we have

$$\langle 0|\hat{T}_\mu^\nu|0\rangle_{\text{Ren}} = -\frac{K^2}{960\pi^2} g_\mu^\nu. \tag{24}$$

Thus $\langle \hat{T}_{\mu\nu}\rangle_{\text{Ren}}$ looks nothing like black-body radiation. By now this should not occasion the reader any surprise since, as we have already emphasized, the response of Unruh's box and the vacuum expectation values of local observables are simply different and largely independent measures of vacuum activity. The lesson that should be reinforced by the results (23) and (24) is that it is not in general possible to divide $\langle \hat{T}_{\mu\nu}\rangle_{\text{Ren}}$ into a 'real particle part' and a 'vacuum polarization part' in an unambiguous way.

It is, of course, possible to define an 'observer-dependent vacuum' for De Sitter space–time which has the property that an Unruh box moving inertially through the space–time will not register. This is accomplished by choosing a coordinate system (22) such that the path of the detector coincides with the line $r=0$. Positive and negative frequencies are then defined with respect to the Killing vector $\partial/\partial t$. This vacuum is pathological, however, since it is defined so as to favour those particular observers that follow the trajectories of $\partial/\partial t$. To underscore this fact we quote the expectation value of the stress tensor in this state [49]. We have

$$\langle 0|\hat{T}^{\nu}_{\mu}|0\rangle_{\mathrm{Ren}} = -\frac{K^2}{960\pi^2}g^{\nu}_{\mu} - \frac{1}{480\pi^2}\frac{K^2}{(1-Kr^2)^2}\mathrm{diag}\left(-1,\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),$$

which exhibits the by now familiar divergence on the Killing horizons where $\partial/\partial t$ becomes null. A physical realization of this state would involve accelerated mirrors whose motion followed the curves of constant r, with r taking a value only slightly less than $K^{-1/2}$. This may seem a somewhat extragavant state of affairs; however, there is room in the theory for such an arrangement.

APPENDIX A

Analytic properties of the manifold: the rotation to complex time

Consider the result of setting $\tau = -i\eta$ in the (ξ, τ) part of the line element (17), we find

$$ds^2 = \xi^2 d\eta^2 + d\xi^2 + dy^2 + dz^2.$$

If we take η to be real then we see that the (ξ, η) part of the metric describes a flat two space in polar coordinates. The metric possesses a coordinate singularity at the origin of these polar coordinates where $\xi = 0$. Potentially the point $\xi = 0$ is a conical point of this manifold but it is a regular point if η is a periodic coordinate with period 2π. We may also note that if we continue the Minkowski time coordinate t to imaginary values according to the replacement $t = -i\lambda$ and simultaneously write $\tau = -i\eta$ we find that

$$\lambda = \xi \sin\eta, \quad x = \xi\cos\eta$$

and we again find that ξ and η are to be interpreted as polar coordinates.

Consider now the problem of determining the Feynman propagator for, say, a massless scalar field appropriate to the entire Minkowski manifold, in other words we are seeking the usual propagator of free field theory. If we work in Rindler coordinates then we are faced with the problem that the coordinate patch covers only part of the space–time and it is thus not immediately apparent how to apply the usual boundary conditions that the propagator should contain only positive frequencies in the remote future and only negative frequencies in the remote past.

We may proceed however by writing $\tau = -i\eta$ in the wave equation for the propagator, which then becomes

$$\left\{\frac{1}{\xi^2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{\xi}\frac{\xi}{\partial\xi}\left(\frac{\xi\partial}{\partial\xi}\right) + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right\}G(x,x') = -\frac{i}{\xi}\delta(\eta-\eta')\delta(\xi-\xi')\delta(\mathbf{x}-\mathbf{x}'). \quad (\mathrm{A}\,1)$$

Now the horizons in the indefinite metric are located at the points for which $\xi = 0$ which are, of course, perfectly regular points when regarded as points of the Minkowski manifold. If we wish the propagator to reflect this regularity then we should demand that the propagator should be regular at the points for which $\xi = 0$ in the analytically continued space; this requires that G should be a periodic function of the coordinate η with period 2π. Equivalently G must be periodic *qua* function of τ with period $2\pi i$. If we solve (A 1) incorporating this periodicity requirement then we recover the familiar Feynman propagator for Minkowski space. The fact that the

Minkowski propagator is periodic in Rindler time with period $2\pi i$ is intimately related to the fact that the Minkowski vacuum contains a thermal distribution of quanta with respect to the Fulling Fock space.

APPENDIX B

On Bekenstein's heuristic derivation of the temperature of a black hole

Consider the process of transferring heat to a Schwarzschild black hole of mass M from a black-body photon reservoir at temperature T, at infinity. Photons are lowered quasi-statically towards the black hole, and their gravitational potential energy is converted into mechanical energy stored, say, in a flywheel. If the box could be lowered to a proper distance d from the horizon then, owing to the gravitational red shift, the frequency of the photon as viewed from infinity would be reduced by a factor

$$\frac{\nu_\infty}{\nu_{\text{local}}} \sim \frac{d}{2M}$$

(assuming $d \ll M$) and if it is then dropped into the black hole, its energy has been converted into work with an efficiency

$$\varepsilon = 1 - \frac{d}{2M}.$$

If d could be made arbitrarily small, then ε could approach unity.

But (Bekenstein's argument runs) we must have

$$d \gtrsim T^{-1},$$

since 'the box must be large enough for the wavelengths characteristic of radiation of temperature T to fit into it'. Therefore it appears that the maximum efficiency is

$$1 - \frac{\alpha}{MT}$$

for some α of order unity, and that the black hole acts as a heat sink of temperature $\sim M^{-1}$. It is true that when the energy transferred to a cavity (the box) is small compared with the energy E_0 of the lowest excitation of the cavity, the thermodynamic approximation breaks down for the behaviour of a single cavity. However, when discussing the thermodynamics of heat engines, one is concerned with the behaviour of *ensembles* rather than of individual systems. An ensemble of cavities has no difficulty in acquiring an average energy $\langle E \rangle \ll E_0$: most of the copies of the cavity are not excited at all, and a few are excited to energy E_0. When calculating the efficiency of a heat engine one must use this ensemble average, which has a meaningful value even when $d \ll T^{-1}$. Therefore it is indeed possible with the help of a black hole to convert heat into work with an efficiency arbitrarily close to unity. Bekenstein's black hole is at *zero* temperature. Moreover, with the benefit of hindsight this is to be expected, for in Bekenstein's thought experiment it is assumed that there is no radiation in the environment of the black hole, which means that it must be in the 'Boulware vacuum state' appropriate to the exterior region $r > 2M$. It is the *Hartle–Hawking* vacuum state, appropriate to the Kruskal manifold which possesses thermal properties. If one calculates the efficiency of the Bekenstein process, taking into account the presence of radiation at a temperature T_{BH} (unspecified) in equilibrium with the black hole, one finds that the efficiency is indeed $1 - (T_{\text{BH}}/T)$, but there is no restriction on the value of T_{BH}.

APPENDIX C

Thermal relationship between the Minkowski and Fulling vacua

A complete set of solutions of the wave equation in the 'Rindler wedge' (region I in figs. 3 and 4) is

$$u_{\nu\mathbf{k}}(x)=N_{\nu\mathbf{k}}\exp(-i\nu\tau+i\mathbf{k}\cdot\mathbf{y})K_{i\nu}(k\xi), \tag{C 1}$$

where the normalization constant $N_{\nu\mathbf{k}}$ is chosen so that $u_{\nu\mathbf{k}}$ satisfies the canonical Klein Gordon orthonormality condition

$$i\int_{\Sigma^{(+)}(\tau)} u^*_{\nu\mathbf{k}}(x)\overleftrightarrow{\partial}_{\alpha}u_{\nu'\mathbf{k}'}(x)n^{\alpha}\,d\Sigma=\delta(\nu-\nu')\,\delta(\mathbf{k}-\mathbf{k}'). \tag{C 2}$$

n^{α} is the unit future-pointing normal to the surface $\Sigma^{(+)}(\tau)$, which is the hypersurface $\tau=$constant in region I. The Fulling quantization of the scalar field in the Rindler wedge is a canonical quantization based on the modes I, with the direction $\partial/\partial\tau$ defining 'positive' or 'negative' frequency. The $u_{\nu\mathbf{k}}(x)$ are positive frequency Fulling modes in region I, and are singular on the horizon. It is our task to relate this Fulling quantization to the more usual 'Minkowski' one, in which the modes are analytic everywhere and positive and negative frequency is determined by $\partial/\partial t$. To this end we now present, following Unruh, Fulling, and Israel (42) a derivation of the thermal properties of the Minkowski vacuum state with respect to the Fulling quantization.

The Fulling modes in region II have the same functional form as (C 1) but because of the opposite sense of $\partial/\partial\tau$ in II it is the $u^*_{\nu\mathbf{k}}(x)$ which have a positive frequency there. Now let $u^{(+)}_{\nu\mathbf{k}}(x)$ be that solution of the wave equation which vanishes in II but agrees with $u_{\nu\mathbf{k}}(x)$ in I, and let $u^{(-)}_{\nu\mathbf{k}}(x)$ have the corresponding definition with I and II interchanged. The Fulling quantization of the scalar field in I *and* II is thus based on the expression

$$\hat{\phi}(x)=\int_0^{\infty}d\nu\int\frac{d^2\mathbf{k}}{(2\pi)^2}\{\hat{a}^{(+)}_{\nu\mathbf{k}}u^{(+)}_{\nu\mathbf{k}}(x)+\hat{a}^{(-)}_{\nu\mathbf{k}}u^{(-)}_{\nu\mathbf{k}}(x)+\hat{a}^{(+)*}_{\nu\mathbf{k}}u^{(+)*}_{\nu\mathbf{k}}(x)+\hat{a}^{(-)*}_{\nu\mathbf{k}}u^{(-)*}_{\nu\mathbf{k}}(x)\} \tag{C 3}$$

with the canonical commutation relation

$$[\hat{a}^{(\varepsilon)}_{\nu\mathbf{k}},\hat{a}^{(\varepsilon')*}_{\nu'\mathbf{k}'}]=\delta_{\varepsilon\varepsilon'}\delta(\nu-\nu')\delta(\mathbf{k}-\mathbf{k}'). \tag{C 4}$$

These two regions, taken together, contain a global Cauchy surface. The quantization in regions F and P is therefore also determined. All commutators independent of (C 4) vanish and ε ranges over '$\pm$'.

Linear combinations of the $u^{(\pm)}_{\nu\mathbf{k}}$ are clearly analytic everywhere except possibly on the horizons

$$\left.\begin{aligned}&\mathscr{H}^+(\tau\to\infty,\quad ln\,\xi\to-\infty,\quad \tau+ln\,\xi\text{ finite})\\ &\mathscr{H}^-(\tau\to-\infty,\quad ln\,\xi\to-\infty,\quad \tau-ln\,\xi\text{ finite}).\end{aligned}\right\} \tag{C 5}$$

and

Now near $\mathscr{H}^+$

$$u_{\nu\mathbf{k}}(x)\sim\exp[-i\nu(\tau+ln\,\xi)]+(\text{rapidly oscillating term}),$$

and the rapidly oscillating term does not contribute to wave packets of frequency near ν. Thus effectively $u^{(+)}_{\nu\mathbf{k}}\sim|t+x|^{-i\nu}$, which is not analytic. However, the linear combination

$$u^{(+)}_{\nu\mathbf{k}}(x)+\exp(-\nu\pi)u^{(-)}_{\nu\mathbf{k}}(x)\sim(t+x-i\delta)^{-i\nu} \tag{C 6}$$

is analytic on $\mathscr{H}^+$ Moreover, for complex values of

$$\tau + ln\, \xi = t + x, \tag{C 7}$$

(C 6) is analytic in the lower half $t+x$ plane, and hence in the lower half t plane, and is therefore a positive *Minkowski* frequency mode function. Similarly one verifies that the functions

$$v^{(\varepsilon)}_{\nu\mathbf{k}} = (1-\exp(-2\pi\nu))^{-1/2}\,[u^{(\varepsilon)}_{\nu\mathbf{k}}(x) + \exp(-\pi\nu)u^{(-\varepsilon)}_{\nu\mathbf{k}}(x)] \tag{C 8}$$

are analytic on $\mathscr{H}^+$ and $\mathscr{H}^-$, that $v^{(-)}_{\nu\mathbf{k}}$ is a negative Minowski frequency function and that the $v^{(\varepsilon)}_{\nu\mathbf{k}}$ are conventionally normalized over the Cauchy surface $\Sigma^{(+)}(\tau)\cup\Sigma^{(-)}(\tau)$.

Therefore a Minkowski decomposition of $\hat{\phi}(x)$ is

$$\hat{\phi}(x) = \int_0^\infty d\nu \int \frac{d^2\mathbf{k}}{(2\pi)^2}\left\{\hat{b}^{(+)}_{\nu\mathbf{k}}v^{(+)}_{\nu\mathbf{k}}(x) + \hat{b}^{(-)}_{\nu\mathbf{k}}v^{(-)*}_{\nu\mathbf{k}}(x) + \hat{b}^{(+)*}_{\nu\mathbf{k}}v^{(+)*}_{\nu\mathbf{k}} + \hat{b}^{(-)*}_{\nu\mathbf{k}}v^{(-)}_{\nu\mathbf{k}}\right\}. \tag{C 9}$$

The $\hat{b}^{(\varepsilon)}_{\nu\mathbf{k}}$ also obey the canonical commutation relations

$$[\hat{b}^{(\varepsilon)}_{\nu\mathbf{k}}, b^{(\varepsilon')*}_{\nu'\mathbf{k}'}] = \delta_{\varepsilon\varepsilon'}\delta(\nu-\nu')\delta(\mathbf{k}-\mathbf{k}'). \tag{C 10}$$

From (C 3) and (C 9) we have

$$\hat{b}^{(\varepsilon)}_{\nu\mathbf{k}} = [1-\exp(-2\pi\nu)]^{-1/2}(\hat{a}^{(\varepsilon)}_{\nu\mathbf{k}} - \exp(-\pi\nu)\hat{a}^{(-\varepsilon)*}_{\nu\mathbf{k}}). \tag{C 11}$$

The Minkowski vacuum state $|\mathrm{M}\rangle$ is of course defined by

$$\hat{b}^{(\varepsilon)}_{\nu\mathbf{k}}|\mathrm{M}\rangle = 0 \tag{C 12}$$

and the Fulling vacuum state $|\mathrm{F}\rangle^{(\varepsilon)}$ by

$$\hat{a}^{(\varepsilon)}_{\nu\mathbf{k}}|\mathrm{F}\rangle^{(\varepsilon)} = 0. \tag{C 13}$$

Now

$$|\mathrm{M}\rangle \propto \prod_{\nu\mathbf{k}} \sum_{n_{\nu\mathbf{k}}=0}^{\infty} \exp(-\pi n_{\nu\mathbf{k}}\nu)\,|n_{\nu\mathbf{k}}\rangle^{(+)}\,|n_{\nu\mathbf{k}}\rangle^{(-)}, \tag{C 14}$$

as can easily be verified by direct application of (C 12).

With a slight abuse of notation we have denoted the state with $n_{\nu\mathbf{k}}$ and $m_{\nu\mathbf{k}}$ particles in the Fulling modes $u^{(+)}_{\nu\mathbf{k}}(x)$ and $u^{(-)}_{\nu\mathbf{k}}(x)$ respectively by

$$\prod_{\nu\mathbf{k}} |n_{\nu\mathbf{k}}\rangle^{(+)}\,|m_{\nu\mathbf{k}}\rangle^{(-)}. \tag{C 15}$$

Thus if $\hat{O}^{(+)}[\hat{\phi}(x)]$ is any functional of $\hat{\phi}(x)$ depending only on its restriction to region I, we have

$$\langle\mathrm{M}|\hat{O}^{(+)}[\hat{\phi}(x)]|\mathrm{M}\rangle = \left\{\mathrm{Tr}\left[\prod_{\nu\mathbf{k}}\sum_{n_{\nu\mathbf{k}}}|n_{\nu\mathbf{k}}\rangle^{(+)}\langle n_{\nu\mathbf{k}}|^{(+)}\exp(-2\pi n_{\nu\mathbf{k}}\nu)\right]\right\}^{-1}$$

$$\times\,\mathrm{Tr}\left\{\hat{O}[\hat{\phi}(x)]\prod_{\nu\mathbf{k}}\sum_{n_{\nu\mathbf{k}}}|n_{\nu\mathbf{k}}\rangle\langle n_{\nu\mathbf{k}}|\exp(-2\pi n_{\nu\mathbf{k}}\nu)\right\}, \tag{C 16}$$

which is the desired result (11) and (12).

APPENDIX D

On the radiation reaction suffered by a uniformly accelerated charge

Armed with the understanding that we have gained of the relationship between the Minkowski and Fulling vacua we shall, in this appendix, re-examine the classic issue of the radiation reaction suffered by a uniformly accelerated charge.

The nature of the problem

In order to avoid boundary effects that would otherwise obscure the issue, we shall consider only motions of the charge such that the agency producing the acceleration has finite duration. A realization of such a motion is: the charge initially moves freely; at time $t=0$ it enters a region where there is a uniform electric field: it remains in the field until $t=t_1$ at which time it leaves the region. Owing to the phenomenon of pre-acceleration the acceleration of the charge is non-zero before $t=0$ and is non-uniform before $t=t_1$. However, if the time t_1 is long compared with the characteristic time

$$\tau_0=\frac{2}{3}\frac{e^2}{mc^3}$$

(the time that light takes to traverse the classical electron radius), then the motion of the electron is in effect inertial for $t<0$, uniformly accelerated for $0<t<t_1$ and inertial again for $t>t_1$. The feature of this motion that has attracted so much attention over the years is the seemingly paradoxical relation between the radiation rate and the radiation reaction force. Following the detailed work of Bradbury [50] we note that the following facts are pertinent:

(i) By direct computation from the Lienard–Wiechart potentials it is straightforward to show that at each instant of retarded time the charge radiates energy at a rate P given by the Larmor formula

$$P=\frac{2}{3}\frac{e^2a^2}{c^3}$$

with a the magnitude of the proper acceleration of the charge. In particular it radiates at a uniform rate whenever a is constant.

(ii) The classical radiation reaction force is given, in the instantaneous rest frame of the charge, by

$$\mathbf{F}_{\mathrm{rad}}=\frac{2}{3}\frac{e^2}{c^3}\frac{d^2}{dt^2}\mathbf{v},$$

which depends on the second derivarive of the charge's velocity and hence vanishes when a is constant.

(iii) The magnetic field of the charge is zero everywhere in the 'accelerated frame of the charge while it undergoes uniform acceleration. More precesely: if the charge follows the world-line $\xi=a^{-1}$ in the Rindler coordinate frame of §3 then the magnetic components of the electromagnetic field tensor $F_{\xi y}$, $F_{\xi z}$ and F_{yz} are zero on the Rindler manifold.

The apparent paradox is the seeming contradication between (i) and (ii) whereby during the period of uniform acceleration there is a uniform rate of radiation of energy yet no radiation reaction force. That there is in fact no blatant violation of the principle of the conservation of energy follows from the following fact.

(iv) The radiation reaction force acts, during the initial and final periods of *non-uniform* acceleration, in just such a way as to ensure that the total work done by the agency accelerating the charge is equal to the sum of the change in the charge's kinetic energy and the total amount of energy radiated to infinity.

This last statement amounts to the assertion that the time integral of the rate at which work is done against the radiation reaction force is equal to the total amount of energy radiated and is assured, mathematically, by an integration by parts. The integrated term vanishes provided that the motion is inertial at sufficiently early and sufficiently late times.

Although there is in reality no difficulty posed by overall conservation of energy, the fact that the force of radiation reaction vanishes during the period of uniform acceleration seems counter-intuitive—especially in view of the observation first made by Callen and Welton [51] of the intimate relation between radiation reaction and the zero-point fluctuations of the electromagnetic field. The nature of this relation was elucidated by Senitzky [52] and by Milonni *et al.* [53] (see also [15]) who showed that, in the limit of weak coupling, the energy loss suffered by an accelerating electron can be equivalently viewed as either the action of the self-field of the electron on itself or as the effect of fluctuations in the electromagnetic vacuum.

It is instructive to consider the absence of a radiative reaction force on a uniformly accelerated charge with regard to this duality. On the one hand, the self-force vanishes because the second derivative of the velocity does. On the other hand, we might say that the charge perceives the vacuum fluctuations as co-moving and comprising a thermal bath. Thus if the charge is constrained to move with constant acceleration *there can be no net transfer of energy or momentum* between the charge and the vacuum as seen in the accelerated frame. This explains (ii) as well as drawing a close parallel with (iii) which is the statement that the field of the charge is non-radiative as seen in the accelerated frame.

It is also worth noting that the Unruh heat bath observed in the constantly accelerating frame is subject to the Gaussian fluctuations of energy density which a conventional heat bath would possess. If the charge were acted on by a constant force, the pressure fluctuations associated with these energy fluctuations would confer on the charge an irregular motion. This motion would represent a non-constant acceleration and so would also lead to a systematic radiation damping force acting on the charge. We would expect this combination of forces to lead to a situation the importance of which was so often emphasized by Einstein, namely that in which an irregular activating force and a systematic damping one lead to a steady state in which the system's momentum distribution is that given by Maxwell for thermal equilibrium. In the present case this would mean that a charge subject to a constant external force would come to have the momentum distribution appropriate to the Unruh temperature of the ambient quantum vacuum. If, on the other hand, the charge were constrained to have exactly constant acceleration, then the external force would have to fluctuate to compensate for the pressure fluctuations in the Unruh heat bath.

References

[1] Hawking, S. W., 1974, *Nature, Lond.*, **248,** 30; 1975, *Commun. Math. Phys.*, **43,** 199; 1975, *Quantum Gravity: an Oxford Symposium*, edited by C. J. Isham, R. Penrose and D. W. Sciama (Oxford Press: Clarendon), pp. 219–267.

[2] Unruh, W. G., 1976, *Phys. Rev.* D, **14,** 870.

[3] Haag, R., Kastler, D., and Trich-Pohlmeyer, E. B., 1974, *Commun. Math. Phys.*, **38**, 173.
[4] Carter, B., 1969, *J. Math. Phys.*, **10**, 70.
[5] DeWitt, B. S., 1975, *Physics Rep.* C, **19**, 6, 295.
[6] Bekenstein, J. D., 1973, *Phys. Rev.* D, **7**, 2333; 1974, *Ibid.*, **9**, 3292.
[7] Bardeen, J. M., Carter, B., and Hawking, S. W., 1973, *Commun. Math. Phys.*, **31**, 161.
[8] Planck, M., 1900, *Annln Phys.*, **1**, 69.
[9] Glauber, R. J., 1963, *Phys. Rev.*, **131**, 2766.
[10] Sherwin, C. W., 1960, *Phys. Rev.*, **120**, 17.
[11] Mould, R. A., 1964, *Ann. Phys.*, **27**, 1.
[12] Rohrlich, F., 1963, *Ann. Phys.*, **22**, 169.
[13] Rindler, W., 1969, *Essential Relativity: Special, General and Cosmological* (New York: Van Nostrand).
[14] Davies, P. C. W., 1975, *J. Phys.* A, **8**, 609.
[15] Milonni, P. W., 1976, *Physics Rep.*, **25**, 1.
[16] Fain, V. M., and Khanin, Ya. M., 1969, *Quantum Electronics*, Vol. 1, translated by H. S. H. Massey (Oxford: Pergamon Press).
[17] Ford, G. W., Kac, M., and Mazur, P., 1965, *J. Math. Phys.*, **6**, 504.
[18] Einstein, A., 1905, *Annln Phys.*, **17**, 549.
[19] Senitzky, I. R., 1959, *Phys. Rev.*, **115**, 227; 1960, *Ibid.*, **119**, 670; 1961, *Ibid.*, **124**, 642.
[20] George, C., 1960, *Physica*, **26**, 453.
[21] Schwinger, J., 1961, *J. Math. Phys.*, **2**, 407.
[22] Lax, M., 1966, *Phys. Rev.*, **145**, 110.
[23] Klauder, J. R., and Sudarshan, E. C. G., 1968, *Fundamentals of Quantum Optics* (New York: Benjamin), p. 63.
[24] Glauber, R. J., 1963, *Phys. Rev.*, **131**, 2766.
[25] Smith, R. A., 1978, *Proc. R. Soc.* A, **362**, 1.
[26] Gabor, D., 1946, *J. Inst. elect. electron. Engrs*, **93**, 429.
[27] Wang, M. C., and Uhlenbeck, G. E., 1945, *Rev. mod. Phys.*, **17**, 323.
[28] Titchmarsh, E. D., 1949, *The Theory of Functions* (Oxford University Press), p. 128.
[29] Landau, L. D., and Lifshitz, E. M., 1958, *Statistical Physics* (Oxford: Pergamon Press).
[30] Siegert, A. J. F., 1900, cited in Wang and Uhlenbeck [27].
[31] Bloch, F., 1932, *Z. Phys.*, **74**, 295.
[32] Fulling, S. A., 1973, *Phys. Rev.* D, **7**, 2850.
[33] Candelas, P., and Deutsch, D., 1977, *Proc. R. Soc.* A, **354**, 79; 1978, *Ibid.*, **362**, 251.
[34] Penrose, R., and Floyd, R. M., 1971, *Nature, Lond.*, **229**, 177.
[35] Hawking, S. W., 1971, *Phys. Rev. Lett.*, **26**, 1344.
[36] Carter, B., 1973, *Black Holes*, edited by B. DeWitt and C. DeWitt (New York: Gordon & Breach).
[37] Bekenstein, J. D., 1974, *Phys. Rev.* D, **9**, 3292.
[38] Kruskal, M. D., 1960, *Phys. Rev.*, **119**, 1743.
[39] Boulware, D. G., 1975, *Phys. Rev.* D, **11**, 1404.
[40] Hartle, J. B., and Hawking, S. W., 1976, *Phys. Rev.* D, **13**, 2188.
[41] Gibbons, G. W., and Perry, M. J., 1978, *Proc. R. Soc.* A, **358**, 467.
[42] Israel, W., 1976, *Physics Lett.* A, **57**, 107.
[43] See, for example, Hawking, S. W., 1973, *Black Holes*, edited by B. DeWitt and C. DeWitt (New York: Gordon & Breach).
[44] Palmer, T. M., 1978, D. Phil. Thesis, Oxford University.
[45] Candelas, P., Chrzanowski, P., and Howard, K., 1981, University of Texas preprint.
[46] Tagirov, E. A., 1973, *Ann. Phys.*, **76**, 561.
[47] Candelas, P., and Raine, D. J., 1975, *Phys. Rev.* D, **12**, 965.
[48] Dowker, J. S., and Critchley, R., 1976, *Phys. Rev.* D, **13**, 3224.
[49] Candelas, P., and Dowker, J. S., 1979, *Phys. Rev.* D, **19**, 2902.
[50] Bradbury, T. C., 1962, *Ann. Phys.*, **19**, 323.
[51] Callen, H. B., and Welton, T. A., 1951, *Phys. Rev.*, **83**, 34.
[52] Senitzky, I. R., 1973, *Phys. Rev. Lett.*, **31**, 955.
[53] Milonni, P. W., Ackerhalt, J. R., and Smith, W. A., 1973, *Phys. Rev. Lett.*, **31**, 958.