PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

Point Transformations in Quantum Mechanics

BRYCE SELIGMAN DEWITT*
Ecole d'Eté de Physique Théorique de l'Université de Grenoble, Les Houches, Haute Savoie, France

(Received September 14, 1951)

An isomorphism is shown to exist between the group of point transformations in classical mechanics and a certain subgroup of the group of all unitary transformations in quantum mechanics. This isomorphism is used to indicate that the quantum analogs of physically significant classical expressions can be constructed uniquely in any coordinate system. There is no ambiguity in the ordering of noncommuting quantum operators, and the method of constructing the quantum analogs is covariant under general coordinate transformations. The method is actually only applicable to systems having Lagrangians which are at most quadratic in the velocities, but this includes all systems which are presently of interest in physics. The method is applied to two intrinsically nonlinear examples, one of which is the gravitational field. The correct Hamiltonian operator for a quantized version of Einstein's gravitational theory is constructed.

1. INTRODUCTION

PRESENT day methods of formulating quantum mechanics are based more or less completely on analogy with classical mechanics. There are certain well-known rules for passing from the classical theory to the quantum theory. One replaces ordinary numbers by operators and Poisson brackets by commutator brackets. In principle, however, an ambiguity always presents itself when one is faced with the task of constructing the quantum analog of a classical expression which involves the product of two factors whose poisson bracket does not vanish. One does not know, *a priori*, how the corresponding quantum factors should be ordered.

Fortunately, the systems occurring in nature are for the most part simple enough in their mathematical description so that one has no trouble in deciding what the correct order should be. Nevertheless the aforementioned ambiguity represents a real deficiency in the present theory, because (1) the simplicity of natural systems is only apparent and is due to the fact that for such systems there usually exist what may be called "natural" coordinates in which the dynamical equations take particularly simple forms, and (2) the transformation theory of dynamics, which plays such an important role in the quantum theory, owes its validity to the invariance of classical Hamiltonian systems under a much wider group of transformations than one has heretofore been able to incorporate sensibly into the quantum scheme, owing to said ambiguity.

It is known that a true correspondence between the classical and quantum theories exists with respect to a certain subgroup of the group of all canonical transformations, namely the subgroup of all linear inhomogeneous canonical transformations. If one restricts oneself to this subgroup, then an isomorphism can be set up between classical quantities and their quantum analogs, when these quantities are at most quadratic in the canonical variables. A similar isomorphism does not exist, however, for other classical quantities, even under this restricted subgroup.[1] The question therefore arises: Is it possible, for a given dynamical system, to choose the canonical variables in such a way that the important physical quantities, energy, momentum, etc., become quadratic in these variables? Unfortunately, the answer to this question is no in many cases of importance, e.g., interacting fields.

Even in the case of interacting systems, however, no ambiguity in formulating the quantum theory has arisen in practice, because one has always supposed that a clear distinction could be made between the various systems in interaction, and one has usually imagined that it makes sense to talk about "free systems" and to treat the interactions as perturbations. For the "free systems" the answer to the above question is in the affirmative and a set of "natural" dynamical variables does exist. But, as we have already remarked, the existence of "natural" variables is more apparent than real, and may be more a reflection of the way our minds work than of the way nature works.

More pertinent to the present discussion is the fact that the linear inhomogeneous subgroup of canonical transformations is never used, as such, in practice. Indeed, the restriction to this subgroup is highly artificial. A type of canonical transformation which has much more physical content and which is much more frequently used in solving actual problems is a general transformation of the coordinate variables, i.e., a so-called point transformation.

In using point transformations in quantum theory, one usually first "quantizes" a given system in a set of "natural" coordinates (e.g., rectilinear coordinates) and then carries out the coordinate changes afterwards. However, if we adopt seriously the philosophy of general relativity, then we should say that one coordinate system is as good as another, and we need not have felt obliged to carry out the quantization in a "natural" coordinate system. Our rules of quantization, as well as our quantum-mechanical equations, should be

* Now a Fulbright grantee at the Tata Institute of Fundamental Research, Bombay, India.

[1] For a full discussion of this point, see L. Van Hove, "Sur le problème des Relations entre les Transformations Unitaires de la Mécanique Quantique et les Transformations Canoniques de la Mécanique Classique." (To be published.)

covariant under general point transformations. These requirements become all the more necessary when it is pointed out that there do exist systems in nature for which there are no "natural" coordinates; e.g., a particle constrained to move on a curved surface; the gravitational field in the general theory of relativity (see Sec. 6 below); any intrinsically nonlinear system.

It is the purpose of this paper to show (1) that it is possible to formulate a set of well-defined rules for passing from the classical to the quantum theory in a variety of cases which appears to be wide enough to include all systems occuring in nature, (2) that these rules are covariant under the canonical subgroup of all point transformations of a certain very general type, (3) that there exists an isomorphism between this subgroup and a corresponding subgroup of the group of all unitary transformations in Hilbert space, and, finally, (4) that there exists a unique correspondence between the classical expressions for the important physically observable quantities, and their quantum analogs, which persists under these subgroups of transformations. In connection with this last assertion it must be pointed out that the quantum analog of a classical expression may contain, in addition to a term which has the same appearance as the classical expression, also a term which vanishes as $\hbar \to 0$. All that one must require is that the form of this nonclassical term as well as that of the classical term, remain invariant under point transformations.

2. REPRESENTATION THEORY IN GENERAL COORDINATES

We begin by considering an arbitrary set of general coordinates x^i $(i=1\cdots n)$ and conjugate momenta p_i which may be used to describe the dynamical behavior of some system. In the quantum theory these coordinates and momenta are Hermitian operators. We shall work in the Schrödinger representation in which the algebra generated by the x's and p's refers to a fixed instant of time, while the actual temporal behavior of the system is described by a time dependent state vector $|\ \rangle_t$. We shall suppose that each of the operators x^i possesses a continuum of eigenvalues ranging from $-\infty$ to ∞. We shall further suppose that the x's are coordinates in an n-dimensional Riemannian manifold possessing a metric tensor g_{ij}. We shall see later that the metric tensor, which is a function of the x's alone, is determined in each case by the dynamical system itself.

The state vector $|\ \rangle_t$ may be expanded in terms of the eigenvectors $|x'\rangle$ of the x's in the following fashion,

$$|\ \rangle_t = \int |x'\rangle\langle x'|\ \rangle_t d\omega'. \tag{2.1}$$

$d\omega'$ denotes the volume element of the manifold, and the integration is to be carried out over the entire range of coordinate values. The quantity $\langle x'|\ \rangle_t$ is the wave function of the system, and may be denoted in more customary form by

$$\psi(x',t) \equiv \langle x'|\ \rangle_t. \tag{2.2}$$

$|\psi(x',t)|^2 d\omega'$ is the probability that the coordinates of the system will be found in the volume $d\omega'$ in the neighborhood of the point x' at the time t. This interpretation of $\psi(x',t)$ as a "probability amplitude" depends upon a normalization condition for the eigenvectors $|x'\rangle$ which may be expressed in the form

$$\langle x'|x''\rangle = \delta(x',x''). \tag{2.3}$$

$\delta(x',x'')$ is the "generalized delta-function" for the manifold and is defined by the conditions

$$\delta(x',x'')=0 \quad \text{for } x'\neq x'', \tag{2.4}$$

and

$$\int f(x')\delta(x',x'')d\omega' = f(x'') \tag{2.5}$$

for all "reasonably well-behaved" functions $f(x')$.

The representation of the invariant volume element $d\omega'$ in the form $d\omega' = g^{\frac{1}{2}}(x')dx'$, where $g(x')$ is the determinant of the metric tensor at the point x', enables one to put the generalized delta-function into the following analytic form involving the ordinary delta-function

$$\delta(x',x'') = g^{-\frac{1}{2}}(x')\delta(x'-x'') = g^{-\frac{1}{2}}(x'')\delta(x'-x''). \tag{2.6}$$

Using this form, one may readily derive the following two important identities

$$(x'^i - x''^i)\frac{\partial}{\partial x'^j}\delta(x',x'') = -\delta_j{}^i\delta(x',x''), \tag{2.7}$$

$$\frac{\partial}{\partial x'^i}\delta(x',x'') = -\frac{\partial}{\partial x''^i}\delta(x',x'') - \left\{ {j \atop j\ i} \right\}\delta(x',x''). \tag{2.8}$$

The contracted Christoffel symbol appearing in (2.8) may be evaluated at either x' or x''.

The adoption of the above so-called "coordinate representation," in which the eigenvectors of the x's are chosen as basis vectors, depends on the existence of the commutation relations

$$[x^i, x^j] = 0 \quad \text{for all } i,j. \tag{2.9}$$

The matrix elements of the operators x^i in the coordinate representation are given by

$$\langle x'|x^i|x''\rangle = x'^i\delta(x',x'') = x''^i\delta(x',x''). \tag{2.10}$$

The corresponding matrix elements of the operators p_i are obtained from the commutation relations

$$[x^i, p_j] = i\hbar\delta_j{}^i, \tag{2.11}$$

$$[p_i, p_j] = 0. \tag{2.12}$$

In matrix form, (2.11) becomes

$$\begin{aligned} i\hbar\delta_j{}^i\delta(x',x'') &= \int \{\langle x'|x^i|x'''\rangle\langle x'''|p_j|x''\rangle \\ &\qquad - \langle x'|p_j|x'''\rangle\langle x'''|x^i|x''\rangle\} d\omega''' \qquad (2.13) \\ &= (x'^i - x''^i)\langle x'|p_j|x''\rangle, \end{aligned}$$

of which the most general solution is (see (2.7))

$$\langle x'|p_j|x''\rangle = -i\hbar\frac{\partial}{\partial x'^j}\delta(x', x'')+F_j\delta(x', x''), \quad (2.14)$$

where F_j is an arbitrary function of the coordinates which may be evaluated at either x' or x''.

The solution (2.14) must also satisfy relations (2.12). This imposes certain conditions on the functions F_j. Introducing the functions

$$\bar{F}_i = F_i + i\hbar\left\{\begin{matrix} j \\ j\ i\end{matrix}\right\},$$

we have

$$\begin{aligned} 0 &= \int \{\langle x'|p_i|x'''\rangle\langle x'''|p_j|x''\rangle \\ &\qquad - \langle x'|p_j|x'''\rangle\langle x'''|p_i|x''\rangle\}d\omega''' \\ &= g^{-\frac{1}{2}}(x')\int\left\{\left[-i\hbar\frac{\partial}{\partial x'^i}\delta(x'-x''')+\bar{F}_i(x')\delta(x'-x''')\right]\right. \\ &\qquad\times\left[-i\hbar\frac{\partial}{\partial x'''^j}\delta(x'''-x'')+\bar{F}_j(x'')\delta(x'''-x'')\right] \\ &\qquad-\left[-i\hbar\frac{\partial}{\partial x'^j}\delta(x'-x''')+\bar{F}_j(x')\delta(x'-x''')\right] \\ &\qquad\left.\times\left[-i\hbar\frac{\partial}{\partial x'''^i}\delta(x'''-x'')+\bar{F}_i(x'')\delta(x'''-x'')\right]\right\}dx''' \\ &= i\hbar\left[\frac{\partial}{\partial x'^j}\bar{F}_i(x')-\frac{\partial}{\partial x'^i}\bar{F}_j(x')\right]\delta(x', x''), \end{aligned} \quad (2.15)$$

which implies that the $\bar{F}_i$ are of the form

$$\bar{F}_i(x') \equiv \frac{\partial}{\partial x'^i}\bar{F}(x') \quad (2.16)$$

where $\bar{F}(x')$ is some function independent of the index i. Since the Christoffel symbol $\left\{\begin{matrix} j \\ j\ i\end{matrix}\right\}$ is also of this form, being in fact equal to $\frac{1}{2}\partial(\log g)/\partial x'^i$, we may write

$$\langle x'|p_i|x''\rangle = -i\hbar\frac{\partial}{\partial x'^i}\delta(x', x'')+\frac{\partial F(x')}{\partial x'^i}\delta(x', x'') \quad (2.17)$$

where $F=\bar{F}-i\hbar\log g^{\frac{1}{4}}$.

Owing to the fact that the p's are Hermitian operators, the function F is not entirely arbitrary. The Hermitian condition may be expressed in the form

$$\langle x'|p_i|x''\rangle = \langle x''|p_i|x'\rangle^*, \quad (2.18)$$

which leads, in virtue of the identity (2.8), to the condition

$$F^* = F + i\hbar\log g^{\frac{1}{2}}. \quad (2.19)$$

F must therefore have the form

$$F = R - \tfrac{1}{2}i\hbar\log g^{\frac{1}{2}}, \quad (2.20)$$

where R is a real function of the coordinates.

The function R may always be removed from the scene by a trivial unitary transformation. One introduces a new set of basic eigenvectors $|x'\rangle^\dagger$, which are connected with the old eigenvectors by the relation

$$|x'\rangle^\dagger = \exp[-iR(x')/\hbar]|x'\rangle. \quad (2.21)$$

That is, the two bases differ only by a phase transformation. We shall suppose this transformation already to have been carried out. The matrix elements of the p_i then reduce to

$$\langle x'|p_i|x''\rangle = -i\hbar\frac{\partial}{\partial x'^i}\delta(x', x'')-\tfrac{1}{2}i\hbar\left\{\begin{matrix} j \\ j\ i\end{matrix}\right\}\delta(x', x''). \quad (2.22)$$

If we confine ourselves to coordinate space representations, the operators x^i, p_i may be regarded as being able to act directly on the wave function $\psi(x', t)$ itself, by means of the definitions

$$\langle x'|x^i|\ \rangle_t = x^i\psi(x', t), \quad \langle x'|p_i|\ \rangle_t = p_i\psi(x', t). \quad (2.23)$$

The x's and p's then have the following coordinate space representations,[2]

$$x^i = x'^i, \quad p_i = -i\hbar\frac{\partial}{\partial x'^i}-\tfrac{1}{2}i\hbar\left\{\begin{matrix} j \\ j\ i\end{matrix}\right\}(x'). \quad (2.24)$$

In subsequent sections we shall drop the primes used to designate particular eigenvalues of the coordinate operators. The use of primes will be reserved to designate general point transformations.

3. POINT TRANSFORMATIONS

In classical mechanics, for every transformation of the generalized coordinates of the form[3]

$$x'^i = x'^i(x), \quad i=1, \cdots n, \quad (3.1)$$

there exists a corresponding transformation of the conjugate momenta, of the form

$$p_i' = (\partial x^j/\partial x'^i)p_j, \quad (3.2)$$

which preserves the canonical nature of the x's and p's. Equations (3.1) and (3.2) together define what is called a point transformation of the x's and p's.

Point transformations may also be defined in quantum mechanics, in an unambiguous manner. Since the coordinates x^i all commute with one another there is clearly no ambiguity in writing down the quantum analog of Eq. (3.1). There is likewise actually no ambiguity in

[2] The fact that the representation of the momentum operators in curvilinear coordinates differs from that in rectilinear coordinates in flat space, by the presence of the contracted Christoffel symbol, has already been noted. See, for example, W. Pauli, *Wellenmechanik*, Handbuch der Physik, Band 24, I, p. 120 (1933).

[3] The Jacobian of the transformation is assumed to be nonvanishing.

formulating the quantum analog of Eq. (3.2). For the only problem here is that of correctly symmetrizing the right-hand side of (3.2) so as to make it Hermitian. One may easily convince oneself that all methods of symmetrization lead to the same result, namely,[4]

$$p_i'=\tfrac{1}{2}[\partial x^j/\partial x'^i,\, p_j]_+. \tag{3.3}$$

That Eq. (3.3) gives the correct quantum transformation law for the momentum operators may be shown by making explicit use of expressions (2.24). We have

$$p_i'=\frac{\partial x^j}{\partial x'^i}p_j+\frac{1}{2}\left[p_j,\frac{\partial x^j}{\partial x'^i}\right]=-i\hbar\frac{\partial}{\partial x'^i}-\frac{1}{2}i\hbar\begin{Bmatrix} j \\ j\; i\end{Bmatrix}', \tag{3.4}$$

where

$$\begin{Bmatrix} i \\ j\; i\end{Bmatrix}'=\frac{\partial x^i}{\partial x'^i}\begin{Bmatrix} k \\ k\; j\end{Bmatrix}+\frac{\partial}{\partial x^j}\frac{\partial x^j}{\partial x'^i}. \tag{3.5}$$

Equation (3.5) is, however, just the usual transformation law for the contracted Christoffel symbol. Expressions (2.24) are therefore covariant under point transformations.

That there exists an isomorphism between the group of point transformations in classical mechanics and a corresponding subgroup of the group of all unitary transformations in quantum mechanics is thus quite evident. The group property ensures that each point transformation has an inverse. It is instructive to display explicitly the inverse of Eq. (3.3). We write

$$\begin{aligned}
&\tfrac{1}{2}[\partial x'^j/\partial x^i,\, p_j']_+\\
&=\tfrac{1}{4}[\partial x'^j/\partial x^i,\,[\partial x^k/\partial x'^j,\, p_k]_+]_+\\
&=\tfrac{1}{4}[[\partial x'^j/\partial x^i,\, \partial x^k/\partial x'^j]_+,\, p_k]_+\\
&\qquad+\tfrac{1}{4}[\partial x^k/\partial x'^j,\,[p_k,\,\partial x'^j/\partial x^i]]\\
&=\tfrac{1}{2}[\delta_i{}^k,\, p_k]_+-\tfrac{1}{4}i\hbar[\partial x^k/\partial x'^j,\,\partial^2 x'^j/\partial x^k\partial x^i]=p_i,
\end{aligned} \tag{3.6}$$

which shows that the inverse transformation has the same form as (3.3).

The unitary representations of the point-transformation group may be obtained by determining the infinitesimal generators of the group. An infinitesimal point transformation may be expressed in the form

$$x'^i=x^i+\epsilon\Lambda^i(x), \tag{3.7}$$

$$p_i'=p_i-\tfrac{1}{2}\epsilon[(\partial/\partial x^i)\Lambda^j(x),\, p_j]_+, \tag{3.8}$$

where ϵ is an infinitesimal constant and Λ^i is a function of the x's. More generally, every function f of the x's and p's transforms under (3.7, 8) according to

$$f'=f+(\epsilon/i\hbar)[f,S] \tag{3.9}$$

where

$$S=\tfrac{1}{2}[\Lambda^i(x),\, p_i]_+. \tag{3.10}$$

S is the generator of the infinitesimal point transformation.

The subgroup of unitary transformations in quantum mechanics which corresponds isomorphically to the group of all point transformations in classical mechanics is given by the set of all unitary operators $\exp(\tau S/i\hbar)$, where S has the form (3.10) and where τ is an arbitrary parameter. Each set of functions Λ^i defines a one-parameter subgroup of the point-transformation group.

4. DYNAMICAL SYSTEMS IN GENERAL COORDINATES

In this section we shall consider the set of all dynamical systems which, in the classical theory, have a Lagrangian function of the form

$$L=\tfrac{1}{2}G_{ij}\dot{x}^i\dot{x}^j+A_i\dot{x}^i-V, \tag{4.1}$$

where G_{ij}, A_i, and V are functions of the x's and where the matrix $\|G_{ij}\|$ is symmetric and nonsingular. We assert that this set includes all systems in nature which satisfy Bose-Einstein statistics, i.e., for which Poisson brackets, involving coordinates and momenta singly as well as multiply, correspond to commutator brackets in the quantum theory. The case of Fermi-Dirac systems will be discussed briefly in Sec. 7.

There exist, to be sure, Bose-Einstein systems which have Lagrangians of the form (4.1) but for which the matrix $\|G_{ij}\|$ is singular. The singularity of the matrix, however, simply implies that the momenta are not all independent, and the lagrangian for such a system can always be replaced by a Lagrangian for which $\|G_{ij}\|$ is nonsingular, together with a set of supplementary conditions expressing the relations between the momenta. The existence of such supplementary conditions does not alter the discussion which follows.

Under general coordinate transformations the quantities V, A_i, and G_{ij} transform like a scalar, a covariant vector, and a covariant tensor respectively. V and A_i have respectively the nature of a scalar and a vector potential. G_{ij} can likewise be regarded as a tensor potential. However, it is a true potential only if it cannot be "transformed away," i.e., if there exists no coordinate system in which it is everywhere constant. We shall tentatively identify G_{ij} with the metric tensor of the manifold of the x's—or rather with some constant multiple of it,

$$G_{ij}=\mu g_{ij}. \tag{4.2}$$

We shall subsequently discuss in fuller detail the reasons for this identification.

The Hamiltonian function corresponding to the Lagrangian (4.1) has the form

$$H=(1/2\mu)g^{ij}(p_i-A_i)(p_j-A_j)+V, \tag{4.3}$$

where g^{ij} is the contravariant inverse of the metric tensor and the momenta are given by

$$p_i=\mu g_{ij}\dot{x}^j+A_i. \tag{4.4}$$

[4] For example, one might expand $\partial x^j/\partial x'^i$ in a power series in the x's. The operator p could then be inserted between the x's in any symmetrical fashion in each term of the series. The result of commuting p symmetrically to the left and to the right through the x's is to produce two terms of order $\hbar$ which cancel each other, leaving simply the expression (3.3).

The question we now have to answer is how to put expression (4.3) into quantum form in a covariant and unambiguous manner. Actually there is no ambiguity in the symmetrization of the terms linear in the p's, so the question reduces simply to that of finding the quantum analog of $g^{ij}p_ip_j$.

Let us first investigate the transformation law of the wave function $\psi(x, t)$, because, when all is said and done, it is the Schrödinger equation

$$i\hbar\partial\psi/\partial t=H\psi, \tag{4.5}$$

which must transform covariantly. If the quantized system has no nonclassical degrees of freedom, such as spin, then the interpretation of $|\psi|^2$ as a probability per unit volume of the x-manifold implies that ψ is a simple scalar quantity. If, however, ψ is composed of a multiplet of components describing spin angular momentum, then the spin transformations of ψ are not independent of coordinate transformations and ψ is no longer a scalar.[5] We shall here consider only the scalar case, as the generalization to the other cases will be obvious at the end of the discussion.[6]

The quantum analog of the expression $g^{ij}p_ip_j$ is now readily determined by considering the well-known special case in which the x-manifold is flat and the coordinates are rectilinear and normalized. The expression then takes the form p_ip_i and the corresponding quantum operator is simply $-\hbar^2\partial^2/\partial x^i\partial x^i$. Using the convenient shorthand notation in which indices following a comma indicate ordinary differentiation with respect to the corresponding coordinates, we may write

$$p_ip_i\psi\equiv-\hbar^2\psi_{,ii}. \tag{4.6}$$

The generalization of expression (4.6) to the case of curvilinear coordinates is

$$-\hbar^2g^{ij}\psi_{\cdot ij}, \tag{4.7}$$

where the dot $\cdot$ denotes covariant differentiation.

The simplest method of symmetrizing the expression $g^{ij}p_ip_j$ in the quantum theory is to write it in the form $p_ig^{ij}p_j$. We are thus led to investigate the quantity $p_ig^{ij}p_j\psi+\hbar^2g^{ij}\psi_{\cdot ij}$. Using the explicit expression (2.24) for the momentum operators, and remembering that ψ is a scalar in calculating its covariant derivatives, we find

$$-\hbar^2g^{ij}\psi_{\cdot ij}-p_ig^{ij}p_j\psi=2\mu\hbar^2Q\psi \tag{4.8}$$

where

$$Q=\frac{1}{4\mu}g^{ij}\left[\left\{\begin{matrix}k\\ k\;i\end{matrix}\right\}_{,j}-\left\{\begin{matrix}k\\ i\;j\end{matrix}\right\}\left\{\begin{matrix}l\\ l\;k\end{matrix}\right\}-\frac{1}{2}\left\{\begin{matrix}k\\ k\;i\end{matrix}\right\}\left\{\begin{matrix}l\\ l\;j\end{matrix}\right\}\right]. \tag{4.9}$$

[5] In the case of isotopic spin, on the other hand, the components of ψ are each scalar quantities, for isotopic spin rotations bear no connection with coordinate transformations.

[6] The extension to the nonzero integral-spin cases is well known in the theory of tensors. For the theory of spinors in general coordinates see W. Pauli, Ann. Physik (to be published).

If we now write the quantum analog of the Hamiltonian function in the form

$$H=(1/2\mu)(p_i-A_i)g^{ij}(p_j-A_j)+\hbar^2Q+V \tag{4.10}$$

the Schrödinger equation takes the covariant form

$$\begin{aligned}i\hbar\partial\psi/\partial t&=H\psi\\ &=-(\hbar^2/2\mu)g^{ij}\psi_{\cdot ij}+(i\hbar/\mu)(A^i\psi_{\cdot i}+\tfrac{1}{2}A^i{}_{\cdot i}\psi)\\ &\quad+[(1/2\mu)A^iA_i+V]\psi.\end{aligned} \tag{4.11}$$

The quantity $\hbar^2Q$ may be regarded as a kind of quantum-mechanical potential which goes to zero as $\hbar\rightarrow0$. It is the quantity which must be added to the covariant classical Hamiltonian in order to produce the covariant quantum Hamiltonian. From the covariance of Eq. (4.11) it is evident that the form of expression (4.10) remains invariant under all point transformations (3.1). Furthermore, we have been led to this quantum expression in an unambiguous manner.[7]

5. THE HEISENBERG REPRESENTATION AND THE EQUATIONS OF MOTION

The passage to the time-independent state vector of the Heisenberg representation is effected by the unitary transformation

$$|\;\rangle_0=\exp(iHt/\hbar)|\;\rangle_t, \tag{5.1}$$

together with a corresponding transformation of all operators. The new Hamiltonian operator will have the same form as (4.10), except that all operators will be written in bold face type to indicate that we are working in the Heisenberg representation. The time dependence of the Heisenberg operators is given by the canonical equation

$$d\mathbf{f}/dt=(1/i\hbar)[\mathbf{f},\mathbf{H}]. \tag{5.2}$$

In particular we have

$$d\mathbf{x}^i/dt=(1/2\mu)[\mathbf{g}^{ij},\mathbf{p}_j]_+-(1/\mu)\mathbf{A}^i, \tag{5.3}$$

which may be solved for the $\mathbf{p}$'s to give

$$\mathbf{p}_i=\tfrac{1}{2}\mu[\mathbf{g}_{ij},d\mathbf{x}^j/dt]_++\mathbf{A}_i. \tag{5.4}$$

If $\mathbf{f}$ is a function of the $\mathbf{x}$'s alone, then Eq. (5.3) tells us that

$$[\mathbf{f},d\mathbf{x}^i/dt]=(i\hbar/\mu)\mathbf{g}^{ij}\mathbf{f}_{,j}. \tag{5.5}$$

Using Eqs. (5.2), (5.4), and (5.5) one may readily show that the time derivative of $\mathbf{f}$ is given by

$$d\mathbf{f}/dt=\tfrac{1}{2}[\mathbf{f}_{,i},d\mathbf{x}^i/dt]_+. \tag{5.6}$$

[7] There does remain a slight ambiguity in the fact that we could add to expression (4.10) a term of the form $\alpha(\hbar^2/\mu)R$, where R is the contracted curvature tensor of the x-manifold and α is an arbitrary number. However, α would then be an extra dimensionless "constant of nature" for which there seems to be no present need. More interesting, perhaps, is the possibility of introducing constants having the dimensions of length by adding invariant terms proportional to R^2, $R^{ij}R_{ij}$, etc.

The time dependence of the momentum operators is a little more complicated. One finds

$$d\mathbf{p}_i/dt=\tfrac{1}{2}\mu(d\mathbf{x}^j/dt)\mathbf{g}_{jk,\,i}d\mathbf{x}^k/dt+\tfrac{1}{2}[\mathbf{A}_{j,\,i},\,d\mathbf{x}^j/dt]_+ -(\hbar^2\mathbf{Q}+\mathbf{V})_{,\,i}+\hbar^2\mathbf{W}_i \quad (5.7)$$

where

$$\mathbf{W}_i=(1/4\mu)[\mathbf{g}^{km}(\mathbf{g}^{jl}{}_{,\,l}\mathbf{g}_{jk,\,i})_{,\,m}+\tfrac{1}{2}\mathbf{g}^{jl}{}_{,\,l}\mathbf{g}^{km}{}_{,\,m}\mathbf{g}_{jk,\,i}]. \quad (5.8)$$

Finally, taking the time derivative of Eq. (5.3), using Eqs. (5.6) and (5.7), and juggling factors a bit, one obtains the following equations of motion for the $\mathbf{x}$'s:

$$\frac{d^2\mathbf{x}^i}{dt^2}+\frac{d\mathbf{x}^j}{dt}\left\{\begin{matrix}\mathbf{i}\\ \mathbf{j}\;\mathbf{k}\end{matrix}\right\}\frac{d\mathbf{x}^k}{dt}-\frac{1}{2\mu}\left[\mathbf{F}^i{}_j,\frac{d\mathbf{x}^j}{dt}\right]_+ +\frac{1}{\mu}\mathbf{g}^{ij}(\hbar^2\mathbf{Q}+\mathbf{V})_{,\,j}-\frac{\hbar^2}{\mu}\mathbf{Z}^i=0, \quad (5.9)$$

where

$$\mathbf{F}_{ij}=\mathbf{A}_{j,\,i}-\mathbf{A}_{i,\,j} \quad (5.10)$$

and

$$\mathbf{Z}^i=\mathbf{g}^{ij}\mathbf{W}_j-(1/4\mu)\mathbf{g}^{ln}[(\mathbf{g}^{km}\mathbf{g}^{ij}{}_{,\,km}\mathbf{g}_{jl})_{,\,n} +\mathbf{g}^{ij}{}_{,\,l}(\mathbf{g}^{km}\mathbf{g}_{jk,\,m})_{,\,n}+(\mathbf{g}^{km}\mathbf{g}^{ij}{}_{,\,m}\mathbf{g}_{kl,\,j})_{,\,n}]. \quad (5.11)$$

6. EXAMPLES OF INTRINSICALLY NONLINEAR SYSTEMS

Perhaps the simplest example of a system for which there exist no "natural" coordinates and for which the methods of the preceding sections are therefore mandatory if a quantum theory of the system is desired, is that of a particle which is constrained to move on the surface of a sphere but which is otherwise free.

It will immediately be objected that this system does not satisfy one of the conditions which we imposed at the outset, namely that it be describable in terms of coordinates whose values range from $-\infty$ to ∞. However, we may suppose that the particle, instead of being constrained to move on the surface of a sphere, is constrained to move on a surface which has the topological properties of an infinite plane but which, in a certain region, is shaped like a portion of a sphere. It is this portion that we will be interested in. (Alternatively, we may say that the particle is indeed constrained to move on the spherical surface, but that it is forbidden to approach a certain point on that surface.) This means that if we construct a wave packet to represent the particle, the theory of the preceding sections—in particular, the equations of motion (5.9)—will describe the behavior of the wave packet only so long as its dimensions remain small compared to those of the sphere. When its dimensions are no longer small, the Schrödinger equation (4.11) will still remain valid, but it will then be necessary to impose cyclic conditions on the wave function ψ.

We shall use customary spherical coordinates, in which the element of length is given by $r^2(d\theta^2+\sin^2\theta d\varphi^2)$, r being the radius of the sphere. The momentum operators conjugate to the coordinates θ and φ have the forms

$$p_\theta=-i\hbar(\partial/\partial\theta+\tfrac{1}{2}\cot\theta), \quad (6.1)$$

$$p_\varphi=-i\hbar\partial/\partial\varphi. \quad (6.2)$$

The "quantum-mechanical potential" Q is given by

$$Q=-(8\mu r^2)^{-1}(\csc^2\theta+1), \quad (6.3)$$

where μ may now be identified with the mass of the particle. The operator equations of motion become

$$\frac{d^2\boldsymbol{\theta}}{dt^2}-\frac{1}{2}\frac{d\boldsymbol{\phi}}{dt}(\sin 2\boldsymbol{\theta})\frac{d\boldsymbol{\phi}}{dt}+\frac{\hbar^2}{4\mu^2r^4}\cot\boldsymbol{\theta}\csc\boldsymbol{\theta}=0, \quad (6.4)$$

$$\frac{d^2\boldsymbol{\phi}}{dt^2}+\frac{d\boldsymbol{\phi}}{dt}(\cot\boldsymbol{\theta})\frac{d\boldsymbol{\theta}}{dt}+\frac{d\boldsymbol{\theta}}{dt}(\cot\boldsymbol{\theta})\frac{d\boldsymbol{\phi}}{dt}+\frac{3\hbar^2}{\mu^2r^4}\cot\boldsymbol{\theta}\csc^2\boldsymbol{\theta}=0. \quad (6.5)$$

As $\hbar\to 0$, or as $\mu\to\infty$, these equations are seen to reduce to the classical forms.

Another system of interest for which there exist no "natural" coordinates is the gravitational field. In this case, the "coordinates" are the gravitational potentials, which are customarily chosen to be the components $g_{\mu\nu}$ of the metric tensor of space-time. To say that there exist no "natural" coordinates for the gravitational field is to say that there exists no representation in which the gravitational field equations become linear. For example, we might use the contravariant density $(-g)^{\frac{1}{2}}g^{\mu\nu}$ instead of $g_{\mu\nu}$. This would be a "coordinate" or "point" transformation, but in terms of the new "coordinates" the field equations would still be nonlinear.

Recently Pirani and Schild,[8] using some methods developed by Dirac,[9] have constructed an explicit Hamiltonian for the gravitational field. Their ultimate purpose in doing this was to obtain an eventual quantization of the gravitational field. However, one of the terms in their Hamiltonian is quadratic in the momenta conjugate to the $g_{\mu\nu}$, and thus they are faced with the factor-ordering ambiguity. One could attempt to obtain a quantum Hamiltonian from their classical Hamiltonian by using the simplest possible symmetrization procedure, but then one would not know whether or not an equivalent quantum theory would have been obtained if another set of gravitational "coordinates," such as $(-g)^{\frac{1}{2}}g^{\mu\nu}$, had been used instead, with a similar symmetrization procedure. What is necessary, in order to remove this uncertainty, is to calculate and include the "quantum-mechanical potential" $\hbar^2 Q$ for the problem.

We shall need for this purpose only the quadratic term of the Pirani-Schild Hamiltonian. This has the

[8] F. A. E. Pirani and A. Schild, Phys. Rev. 79, 986 (1950).
[9] P. A. Dirac, Can. J. Math. 2, 129 (1950); see also Mimeographed Notes, Canadian Mathematical Congress, Vancouver, B. C., (Summer, 1949).

form

$$H_{\text{quad}}=\frac{1}{2}\int \dot{x}^\sigma l_\sigma l^{-2}(-g)^{-\frac{1}{2}}(g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\beta}g_{\gamma\delta})\pi^{\alpha\beta}\pi^{\gamma\delta}du. \quad (6.6)$$

The integral on the right is a triple integral, with respect to three parameters u^1, u^2, u^3. Without entering into too much detail, we may say that the points x of space-time are labeled by these parameters, together with a fourth parameter t, which plays the same role as the time in conventional Hamiltonian theory. A dot denotes differentiation with respect to t. l_σ is a vector normal to the space-like surface $t=$constant, and is defined by $l_\sigma \equiv t_{,\sigma}\partial(x)/\partial(u,t)$. Also, $l^2=g^{\alpha\beta}l_\alpha l_\beta$. The coordinates x^σ on any space-like surface are regarded, in the present scheme, as dynamical variables on an equal footing with the $g_{\mu\nu}$. Since the parameters u^i, t are completely arbitrary, the "velocities" $\dot{x}^\sigma$ are arbitrary. For each such arbitrary "velocity variable" there exists what Dirac[9] calls a "first class ϕ." Only a part of one of the first class ϕ's occuring in gravitational theory appears in expression (6.6).

In the quantized theory the momenta $\pi^{\alpha\beta}$ will satisfy the commutation relation

$$[g_{\alpha\beta}(u),\, \pi^{\gamma\delta}(u')]=\tfrac{1}{2}i\hbar(\delta_\alpha{}^\gamma\delta_\beta{}^\delta+\delta_\alpha{}^\delta\delta_\beta{}^\gamma)\delta(u-u'). \quad (6.7)$$

The π's do not correspond to the momenta of systems of a finite number of degrees of freedom. We must replace them by p's given by $p^{\alpha\beta}=[\delta(0)]^{-1}\pi^{\alpha\beta}$ in order that we have commutation relations of the form

$$[g_{\alpha\beta}(u),\, p^{\gamma\delta}(u)]=\tfrac{1}{2}i\hbar(\delta_\alpha{}^\gamma\delta_\beta{}^\delta+\delta_\alpha{}^\delta\delta_\beta{}^\gamma) \quad (6.8)$$

for each u. The Hamiltonian (6.6) then appears as the Hamiltonian of a nondenumerably infinite set of identical systems, each having 10-degrees of freedom and a Hamiltonian of the form

$$\mathcal{H}_{\text{quad}}=\tfrac{1}{2}\dot{x}^\sigma l_\sigma[\delta(0)]^2(du)l^{-2}(-g)^{-\frac{1}{2}}\times(g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\beta}g_{\gamma\delta})p^{\alpha\beta}p^{\gamma\delta}. \quad (6.9)$$

If the "mass constant" μ of each system is taken to be

$$1/\mu=2\dot{x}^\sigma l_\sigma[\delta(0)]^2du, \quad (6.10)$$

then the "contravariant" metric tensor of the $g_{\mu\nu}$-manifold is

$$G^{(\alpha\beta)(\gamma\delta)}=\tfrac{1}{2}l^{-2}(-g)^{-\frac{1}{2}}(g_{\alpha\gamma}g_{\beta\delta}+g_{\alpha\delta}g_{\beta\gamma}-g_{\alpha\beta}g_{\gamma\delta}). \quad (6.11)$$

The "covariant" metric tensor is

$$G_{(\alpha\beta)(\gamma\delta)}=\tfrac{1}{2}l^2(-g)^{\frac{1}{2}}(g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\delta}g^{\beta\gamma}-g^{\alpha\beta}g^{\gamma\delta}), \quad (6.12)$$

satisfying

$$G_{(\alpha\beta)(\epsilon\zeta)}G^{(\epsilon\zeta)(\gamma\delta)}=\tfrac{1}{2}(\delta_\gamma{}^\alpha\delta_\delta{}^\beta+\delta_\delta{}^\alpha\delta_\gamma{}^\beta). \quad (6.13)$$

Using the relations

$$(\partial/\partial g_{\mu\nu})(-g)^{\frac{1}{2}}=\tfrac{1}{2}(-g)^{\frac{1}{2}}g^{\mu\nu}, \quad (6.14)$$

$$(\partial/\partial g_{\mu\nu})g^{\alpha\beta}=-\tfrac{1}{2}(g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}), \quad (6.15)$$

$$(\partial/\partial g_{\mu\nu})l^2=-l^\mu l^\nu, \quad (6.16)$$

we may readily construct the Christoffel symbols for the 10-dimensional $g_{\mu\nu}$-manifold. We have

$$\begin{aligned}
&[(\alpha\beta)(\gamma\delta),\,(\epsilon\zeta)]\\
&=\tfrac{1}{2}[G_{(\alpha\beta)(\epsilon\zeta),\,(\gamma\delta)}+G_{(\gamma\delta)(\epsilon\zeta),\,(\alpha\beta)}-G_{(\alpha\beta)(\gamma\delta),\,(\epsilon\zeta)}]\\
&=\tfrac{1}{2}l^{-2}[-l^\gamma l^\delta G_{(\alpha\beta)(\epsilon\zeta)}-l^\alpha l^\beta G_{(\gamma\delta)(\epsilon\zeta)}+l^\epsilon l^\zeta G_{(\alpha\beta)(\gamma\delta)}]\\
&\quad+\tfrac{1}{8}l^2(-g)^{\frac{1}{2}}[g^{\gamma\delta}(g^{\alpha\epsilon}g^{\beta\zeta}+g^{\alpha\zeta}g^{\beta\epsilon}-g^{\alpha\beta}g^{\epsilon\zeta})\\
&\quad-g^{\beta\zeta}(g^{\alpha\gamma}g^{\epsilon\delta}+g^{\alpha\delta}g^{\epsilon\gamma})-g^{\alpha\epsilon}(g^{\beta\gamma}g^{\zeta\delta}+g^{\beta\delta}g^{\zeta\gamma})\\
&\quad-(g^{\alpha\gamma}g^{\zeta\delta}+g^{\alpha\delta}g^{\zeta\gamma})g^{\beta\epsilon}-g^{\alpha\zeta}(g^{\beta\gamma}g^{\epsilon\delta}+g^{\beta\delta}g^{\epsilon\gamma})\\
&\quad+(g^{\alpha\gamma}g^{\beta\delta}+g^{\alpha\delta}g^{\beta\gamma})g^{\epsilon\zeta}+g^{\alpha\beta}(g^{\epsilon\gamma}g^{\zeta\delta}+g^{\epsilon\delta}g^{\zeta\gamma})]
\end{aligned} \quad (6.17)$$

which gives

$$\left\{\begin{matrix}(\alpha\beta)\\(\alpha\beta)(\gamma\delta)\end{matrix}\right\}=G^{(\alpha\beta)(\epsilon\zeta)}[(\alpha\beta)(\gamma\delta),\,(\epsilon\zeta)]=-5l^{-2}l^\gamma l^\delta, \quad (6.18)$$

$$G^{(\alpha\beta)(\gamma\delta)}\left\{\begin{matrix}(\epsilon\zeta)\\(\alpha\beta)(\gamma\delta)\end{matrix}\right\}=2l^{-2}(-g)^{-\frac{1}{2}}(2l^{-2}l_\epsilon l_\zeta-g_{\epsilon\zeta}), \quad (6.19)$$

$$G^{(\gamma\delta)(\epsilon\zeta)}\left\{\begin{matrix}(\alpha\beta)\\(\alpha\beta)(\gamma\delta)\end{matrix}\right\}_{,(\epsilon\zeta)}=(35/2)l^{-2}(-g)^{-\frac{1}{2}}, \quad (6.20)$$

and finally, the "quantum-mechanical potential" for each system,

$$\hbar^2Q=\hbar^2(85/8)\dot{x}^\sigma l_\sigma l^{-2}(-g)^{-\frac{1}{2}}[\delta(0)]^2du. \quad (6.21)$$

The correct quantum-mechanical Hamiltonian is therefore written in the form

$$H_{\text{quad}}=\int \dot{x}^\sigma l_\sigma[\pi^{\alpha\beta}G^{(\alpha\beta)(\gamma\delta)}\pi^{\gamma\delta}+\hbar^2(85/8)l^{-2}(-g)^{-\frac{1}{2}}[\delta(0)]^2]du. \quad (6.22)$$

The "quantum-mechanical potential" is seen to be a strongly divergent quantity in the present case. It must nevertheless be retained in all quantum expressions in order that the Hamiltonian operator be invariant under changes in the variables used to describe the gravitational field.

Those terms in the total Hamiltonian which do not appear in (6.22) are (at most) linear in the π's, and their symmetrization does not give us any trouble.

7. CONCLUSION

The formalism of the preceding sections is unsymmetrical in the coordinates and momenta. Such an asymmetry does not exist in the transformation theory of classical mechanics, where one deals only with Poisson brackets in which coordinates and momenta are on a completely equal and complementary footing.

Actually, a kind of symmetry can be retained in the formalism of Sec. 2. In any given coordinate system there do exist eigenvectors $|p'\rangle$ of the momentum operators, satisfying

$$p_i|p'\rangle = p_i'|p'\rangle. \tag{7.1}$$

The coordinate-space representation of these eigenvectors may be taken in the form

$$\langle x'|p'\rangle = (2\pi\hbar)^{-\frac{1}{2}n} f^{-\frac{1}{4}}(p') g^{-\frac{1}{4}}(x') \exp(ip_i'x'^i/\hbar), \tag{7.2}$$

and any coordinate-space wave function can then be transformed into a momentum-space wave function by the transformation

$$\begin{aligned}\langle p'|\ \rangle &= \int \langle p'|x'\rangle\langle x'|\ \rangle d\omega' \\ &= (2\pi\hbar)^{-\frac{1}{2}n} f^{-\frac{1}{4}}(p') \int g^{\frac{1}{4}}(x') \\ &\qquad \times \exp(-ip_i'x'^i/\hbar)\langle x'|\ \rangle dx'.\end{aligned} \tag{7.3}$$

The normalization condition

$$\langle p'|p''\rangle = \int \langle p'|x'\rangle\langle x'|p''\rangle d\omega' = f^{-\frac{1}{2}}(p')\delta(p'-p''), \tag{7.4}$$

and the representation of the operators x^i in momentum-space in the form

$$\begin{aligned}\langle p'|x^i|p''\rangle &= \int \langle p'|x'\rangle x'^i \langle x'|p''\rangle d\omega' \\ &= i\hbar \frac{\partial}{\partial p_i'}[f^{-\frac{1}{2}}(p')\delta(p'-p'')] \\ &\quad + \frac{i\hbar}{4}\frac{\partial}{\partial p_i'}[\log f(p')]f^{-\frac{1}{2}}(p')\delta(p'-p''),\end{aligned} \tag{7.5}$$

allow one to identify $f^{\frac{1}{2}}(p')dp'$ as the volume element of momentum-space. The function $f(p')$ is completely arbitrary.

The symmetry between coordinates and momenta which is obtained via Eq. (7.2) is, however, quite artificial, for the concept of "plane wave," as expressed by this equation is not a covariant one. If a state vector is an eigenvector of the momentum operator in one coordinate system, it will not, in general, be a momentum eigenvector in another coordinate system.

When the formalism is applied to actual physical systems then, of course, the symmetry between coordinates and momenta disappears completely, for the Hamiltonian will, in general, be quite unsymmetric in the two sets of conjugate variables. This type of asymmetry exists in both the classical and quantum theories. For this reason, it is quite natural in both theories to speak of a state in which the position (i.e., configuration) of a system has a fixed definite value regardless of what momentum it may have, while it may be quite unnatural to speak of a state in which the momentum of the system has a fixed numerical value which is the same regardless of what its configuration is.

Only for certain special systems is it possible to bring about a complete symmetry between coordinate and momentum operators, namely systems having Hamiltonians which are at most quadratic in all the canonical variables taken together. These systems we have called "free" systems in this paper, following the nomenclature of quantum field theory, in which, for example, an unperturbed harmonic oscillator is regarded as a "free" system.

For "free" systems, the subgroup of linear inhomogeneous canonical transformations have some meaning. Coordinates and momenta can be mixed together or interchanged, sometimes to advantage. A consideration of these systems alone, however, provides a very inadequate description of nature, for the really important aspect of nature is change, and change involves interactions which destroy linearity and nullify the existence of "natural" coordinates.

We have asserted that all Bose-Einstein systems in nature, even those in interaction, may nevertheless still be characterized by the fact that their Hamiltonians are at most quadratic in the momenta. This is not true of Fermi-Dirac systems, whose Hamiltonians may involve the momenta to powers higher than the second. Fermi-Dirac Hamiltonians, however, are always characterized as being expressible as functions of the n^2 products p_ix^j:

$$H = H(p_ix^j, f_\alpha). \tag{7.6}$$

The f_α may be functions of the dynamical variables of other systems which are in interaction with the Fermi-Dirac system in question, but they are independent of the p's and x's of the system itself. The p's and x's satisfy anticommutation relations among themselves,

$$[x^i, x^j]_+ = 0, \tag{7.7}$$

$$[p_i, p_j]_+ = 0, \tag{7.8}$$

$$[x^i, p_j]_+ = i\hbar\delta_j{}^i, \tag{7.9}$$

but they commute with the dynamical variables of the other interacting systems. In constructing the quantum Hamiltonian (or operators corresponding to other physical observables) for such systems, the p's must always be placed to the left of the x's. The anticommutation relation (7.9) requires that the x's and p's must be non-Hermitian, or complex quantities.[10]

[10] In practice, the Hermitian adjoints of the coordinates will be related to the momenta by means of supplementary conditions of the form $p_i = A_{ij}x^{j*}$, where the A_{ij} are coefficients which may involve the dynamical variables of other interacting systems.

There is nothing analogous to point transformations for Fermi-Dirac systems, and, in fact, the dynamical variables are themselves not physically observable. Only products of p's and x's correspond to physical quantities. Fermi-Dirac systems depend for their existence on the fact that there are for them what amounts to "natural" coordinates, because their Hamiltonians must always be functions of the bilinear forms $p_i x^j$. The existence of these "natural" coordinates is, in this case, independent of interactions with other systems.

As we have already remarked, point transformations can be introduced in a very natural way into the quantum theory of systems having Hamiltonians quadratic in the momenta, and there is no ambiguity in the proper ordering of noncommuting factors. If there were systems in nature having Hamiltonians of a more complicated type, then the situation would be quite different. To be sure, in the classical theory, one can make a canonical transformation of a completely arbitrary type, thus destroying the simplicity of a given Hamiltonian. The question that imposes itself is therefore the following: Given a Hamiltonian function, What conditions must it satisfy in order that it be transformable by a canonical transformation into a Hamiltonian which is quadratic in one of the sets of canonical variables?

The answer to the preceding question is not easy. However, supposing we do arrive at a Hamiltonian of the prescribed type, we must then decide what the metric tensor of its coordinate manifold is. The choice of a metric tensor is intimately related with the process of measurement. In simple cases, such as that of a particle constrained to move on a surface, the choice is obvious. In more complicated examples, such as the gravitational Hamiltonian discussed above, there may appear to be some uncertainty.

Suppose we have arbitrarily chosen the metric tensor, and that it corresponds to a flat manifold. In order that there be no factor-ordering ambiguity, the tensor G_{ij} of (4.1) must, in a set of rectilinear coordinates, satisfy the equation

$$G_{ij,j}=0. \tag{7.10}$$

The generalization of this condition to curved manifolds is

$$g^{jk}G_{ij\cdot k}=0. \tag{7.11}$$

For a given metric there are an infinity of tensor G_{ij} satisfying (7.11). Conversely, for a given tensor G_{ij}, there are an infinity of metrics which will cause (7.11) to be satisfied. Even if we impose the stronger condition

$$G_{ij\cdot k}=0 \tag{7.12}$$

the metric will still be undetermined. It is only by adopting the point of view that everything about a given system should be described by its Hamiltonian, that we are led to the identification (4.2).[11] It is to be noted that the choice of the "mass constant" μ does not affect the "quantum-mechanical potential" $\hbar^2 Q$.

The author is indebted to Professor Pauli for reading the manuscript and making helpful suggestions prior to publication.

[11] A very special uncertainty exists in the case of Hamiltonians reducible to the form

$$H=(\tfrac{1}{2}A^{ij}p_ip_j+B^ip_i+C)(\tfrac{1}{2}D_{kl}x^kx^l+E_kx^k+F),$$

where $\|A^{ij}\|$ and $\|D_{kl}\|$ are nonsingular matrices and the expressions inside the parentheses are positive-definite inhomogeneous quadratic forms. In this case the Hamiltonian is quadratic in both the coordinates and the momenta taken separately, and there is no way of deciding, *a priori*, whether the coordinates, or the momenta, are the variables which should be allowed to suffer general point transformations. Once the decision is made, however, such point transformations will quickly destroy the special symmetry which the Hamiltonian above possesses. The quantum Hamiltonian operator will be different depending on which transformation group is selected. The "quantum-mechanical potential" for x-transformations has the form

$$\hbar^2Q=\tfrac{1}{32}\hbar^2n^2(\tfrac{1}{2}D_{mn}x^mx^n+E_mx^m+F)^{-1}A^{ij}(D_{ik}x^k+E_i)\times(D_{jl}x^l+E_j)-\tfrac{1}{8}nA^{ij}D_{ij},$$

where n is the number of degrees of freedom of the system. The quantum-mechanical potential for p-transformations is given by a similar expression involving the p's.