Analogue Gravity

Carlos Barceló

Instituto de Astrofísica de Andalucía (IAA-CSIC)
Glorieta de la Astronomía,
18008 Granada, Spain
email: carlos@iaa.es
http://www.iaa.csic.es/

Stefano Liberati

SISSA
International School for Advanced Studies
Via Bonomea 265, I-34136 Trieste, Italy,
and
INFN, Sezione di Trieste, Trieste, Italy
email: liberati@sissa.it
http://www.sissa.it/~liberati

Matt Visser

School of Mathematics, Statistics, and Operations Research
Victoria University of Wellington; PO Box 600
Wellington 6140, New Zealand
email: matt.visser@msor.vuw.ac.nz
http://www.msor.victoria.ac.nz/~visser

Abstract

Analogue gravity is a research programme which investigates analogues of general relativistic gravitational fields within other physical systems, typically but not exclusively condensed matter systems, with the aim of gaining new insights into their corresponding problems. Analogue models of (and for) gravity have a long and distinguished history dating back to the earliest years of general relativity. In this review article we will discuss the history, aims, results, and future prospects for the various analogue models. We start the discussion by presenting a particularly simple example of an analogue model, before exploring the rich history and complex tapestry of models discussed in the literature. The last decade in particular has seen a remarkable and sustained development of analogue gravity ideas, leading to some hundreds of published articles, a workshop, two books, and this review article. Future prospects for the analogue gravity programme also look promising, both on the experimental front (where technology is rapidly advancing) and on the theoretical front (where variants of analogue models can be used as a springboard for radical attacks on the problem of quantum gravity).

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11 May 2011: Completely revised and updated previous version. Significantly extended Sections 2.4, 3, 4.1, 4.2, 5, and 7. Introduced new Section 6. Eight new figures have been added. The number of references increased from 434 to 702.

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1 Introduction

> And I cherish more than anything else the Analogies, my most trustworthy masters.
> They know all the secrets of Nature, and they ought to be least neglected in Geometry.
> – Johannes Kepler
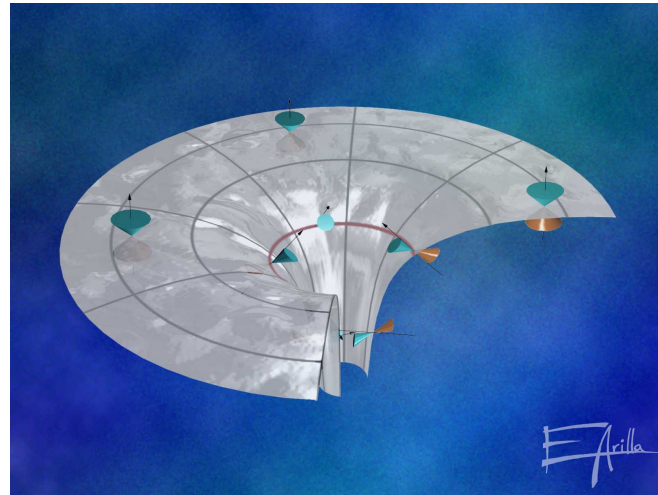
Figure 1: Artistic impression of cascading sound cones (in the geometrical acoustics limit) forming an acoustic black hole when supersonic flow tips the sound cones past the vertical.
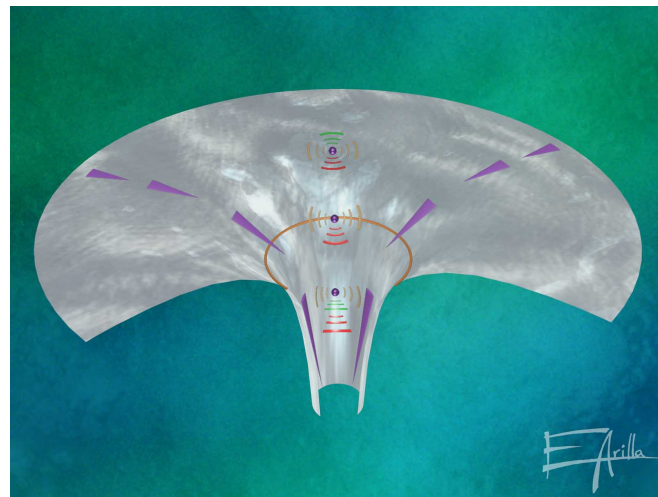
Figure 2: Artistic impression of trapped waves (in the physical acoustics limit) forming an acoustic black hole when supersonic flow forces the waves to move downstream.

Analogies have played a very important role in physics and mathematics – they provide new ways of looking at problems that permit cross-fertilization of ideas among different branches of science. A carefully chosen analogy can be extremely useful in focusing attention on a specific problem, and in suggesting unexpected routes to a possible solution. In this review article we will focus on "analogue gravity", the development of analogies (typically but not always based on

condensed matter physics) to probe aspects of the physics of curved spacetime – and in particular to probe aspects of curved space quantum field theory, and to obtain lessons of potential relevance on the road towards a theory of quantum gravity.

The most well-known of these analogies is the use of sound waves in a moving fluid as an analogue for light waves in a curved spacetime. Supersonic fluid flow can then generate a "dumb hole", the acoustic analogue of a "black hole", and the analogy can be extended all the way to mathematically demonstrating the presence of phononic Hawking radiation from the acoustic horizon. This particular example provides (at least in principle) a concrete laboratory model for curved-space quantum field theory in a realm that is technologically accessible to experiment.

There are many other "analogue models" that may be useful for this or other reasons – some of the analogue models are interesting for experimental reasons, others are useful for the way they provide new light on perplexing theoretical questions. The information flow is, in principle, bi-directional and sometimes insights developed within the context of general relativity can be used to understand aspects of the analogue model.

Of course, analogy is not identity, and we are in no way claiming that the analogue models we consider are completely equivalent to general relativity – merely that the analogue model (in order to be interesting) should capture and accurately reflect a sufficient number of important features of general relativity (or sometimes special relativity). The list of analogue models is extensive, and in this review we will seek to do justice both to the key models, and to the key features of those models.

1.1   Overview

In the following sections we shall:

- Discuss the flowing fluid analogy in some detail.

- Summarise the history and motivation for various analogue models.

- Discuss the many physics issues various researchers have addressed.

- Provide a representative catalogue of extant models.

- Discuss the main physics results obtained to date, both on the theoretical and experimental sides.

- Outline some of the many possible directions for future research.

- Summarise the current state of affairs.

By that stage the interested reader will have had a quite thorough introduction to the ideas, techniques, and hopes of the analogue gravity programme.

1.2   Motivations

The motivation for these investigations (both historical and current) is rather mixed. In modern language, the reasons to investigate analogue models are:

- Partly to use condensed matter to gain insight into classical general relativity.

- Partly to use condensed matter to gain insight into curved-space quantum field theory.

- Partly to develop an observational window on curved-space quantum field theory.

- Partly to use classical general relativity to gain insight into condensed matter physics.

- Partly to gain insight into new and radically-different ways of dealing with "quantum/emergent gravity".

1.3 Going further

Apart from this present review article, and the references contained herein, there are several key items that stand out as starting points for any deeper investigation:

- The book "Artificial Black Holes", edited by Novello, Visser, and Volovik [470].

- The archival website for the "Analogue models" workshop:

    - http://www.msor.victoria.ac.nz/~visser/Analog/

- The book "The Universe in a Helium Droplet", by Volovik [660].

- The *Physics Reports* article, "Superfluid analogies of cosmological phenomena", by Volovik [655].

- The review article by Balbinot, Fabbri, Fagnocchi, and Parentani [21] (focussing largely on backreaction and short-distance effects).

- The *Lecture Notes in Physics* volume on "Quantum Analogies" edited by Unruh and Schütz-hold [614].

2   The Simplest Example of an Analogue Spacetime

Acoustics in a moving fluid is the simplest and cleanest example of an analogue model [607, 622, 626, 624]. The basic physics is simple, the conceptual framework is simple, and specific computations are often simple (whenever, that is, they are not impossibly hard).[1]

2.1   Background

The basic physics is this: A moving fluid will drag sound waves along with it, and if the speed of the fluid ever becomes supersonic, then in the supersonic region sound waves will never be able to fight their way back upstream [607, 622, 626, 624]. This implies the existence of a "dumb hole", a region from which sound can not escape.[2] Of course this sounds very similar, at the level of a non-mathematical *verbal* analogy, to the notion of a "black hole" in general relativity. The real question is whether this verbal analogy can be turned into a precise *mathematical* and *physical* statement – it is only after we have a precise mathematical and physical connection between (in this example) the physics of acoustics in a fluid flow and at least some significant features of general relativity that we can claim to have an "analogue model of (some aspects of) gravity".
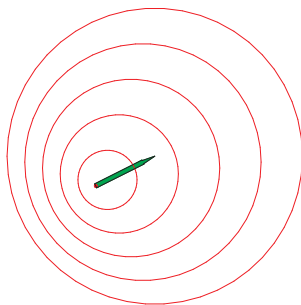
Figure 3: A moving fluid will drag sound pulses along with it.

Now the features of general relativity that one typically captures in an "analogue model" are the *kinematic* features that have to do with how fields (classical or quantum) are defined on curved spacetime, and the *sine qua non* of any analogue model is the existence of some "effective metric" that captures the notion of the curved spacetimes that arise in general relativity. (At the very least, one might wish to capture the notion of the Minkowski geometry of special relativity.) Indeed, the verbal description above (and its generalizations in other physical frameworks) *can* be converted into a precise mathematical and physical statement, which ultimately is the reason that analogue models are of physical interest. The analogy works at two levels:

- Geometrical acoustics.

- Physical acoustics.

The advantage of geometrical acoustics is that the derivation of the precise mathematical form of the analogy is so simple as to be almost trivial, and that the derivation is extremely general. The

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[1] The need for a certain degree of caution regarding the allegedly straightforward physics of simple fluids might be inferred from the fact that the Clay Mathematics Institute is currently offering a US$ 1,000,000 Millennium Prize for significant progress on the question of existence and uniqueness of solutions to the Navier–Stokes equation. See http://www.claymath.org/millennium/ for details.

[2] In correct English, the word "dumb" means "mute", as in "unable to speak". The word "dumb" does *not* mean "stupid", though even many native English speakers get this wrong.

disadvantage is that in the geometrical acoustics limit one can deduce only the causal structure of the spacetime, and does not obtain a unique effective metric. The advantage of physical acoustics is that, while the derivation of the analogy holds in a more restricted regime, the analogy can do more for you in that it can now uniquely determine a specific effective metric and accommodate a wave equation for the sound waves.

2.2 Geometrical acoustics

At the level of geometrical acoustics we need only assume that:

- The speed of sound c, relative to the fluid, is well defined.

- The velocity of the fluid $\mathbf{v}$, relative to the laboratory, is well defined.

Then, relative to the laboratory, the velocity of a sound ray propagating, with respect to the fluid, along the direction defined by the unit vector $\mathbf{n}$, is

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = c\,\mathbf{n} + \mathbf{v}. \tag{1}$$

This defines a sound cone in spacetime given by the condition $\mathbf{n}^2 = 1$, i.e.,

$$-c^2\,\mathrm{d}t^2 + (\mathrm{d}\mathbf{x} - \mathbf{v}\,\mathrm{d}t)^2 = 0\,. \tag{2}$$

That is

$$-[c^2 - v^2]\,\mathrm{d}t^2 - 2\mathbf{v}\cdot\mathrm{d}\mathbf{x}\,\mathrm{d}t + \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = 0\,. \tag{3}$$
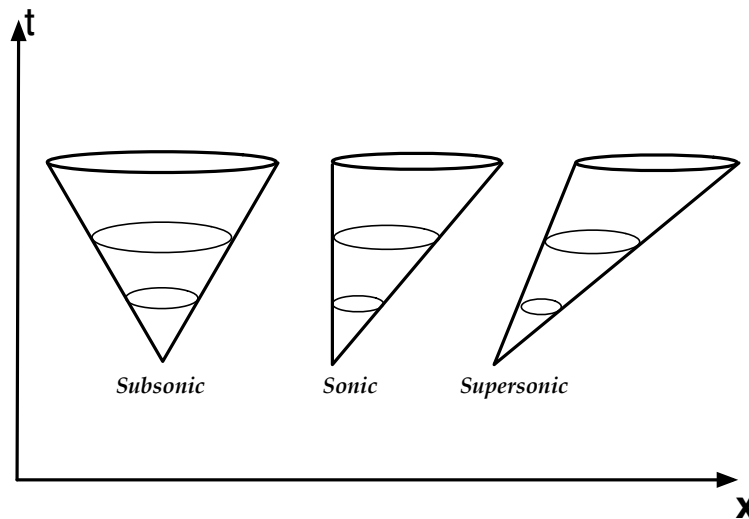
Figure 4: A moving fluid will tip the "sound cones" as it moves. Supersonic flow will tip the sound cones past the vertical.

Solving this quadratic equation for $\mathrm{d}\mathbf{x}$ as a function of $\mathrm{d}t$ provides a double cone associated with each point in space and time. This is associated with a conformal class of Lorentzian metrics [607, 622, 626, 624, 470]

$$g = \Omega^2 \left[\begin{array}{c|c} -(c^2 - v^2) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I} \end{array} \right]\,, \tag{4}$$

where Ω is an unspecified but non-vanishing function.

The virtues of the geometric approach are its extreme simplicity and the fact that the basic structure is dimension-independent. Moreover, this logic rapidly (and relatively easily) generalises to more complicated physical situations.[3]

2.3 Physical acoustics

It is well known that for a static homogeneous inviscid fluid the propagation of sound waves is governed by the simple wave equation [370, 372, 443, 577]

$$\partial_t^2 \phi = c^2 \, \nabla^2 \phi. \tag{5}$$

Generalizing this result to a fluid that is non-homogeneous, or to a fluid that is in motion, possibly even in non-steady motion, is more subtle than it at first would appear. To derive a wave equation in this more general situation we shall start by adopting a few simplifying assumptions to allow us to derive the following theorem.

Theorem. *If a fluid is barotropic and inviscid, and the flow is irrotational (though possibly time dependent) then the equation of motion for the velocity potential describing an acoustic disturbance is identical to the d'Alembertian equation of motion for a minimally-coupled massless scalar field propagating in a (3+1)-dimensional Lorentzian geometry*

$$\Delta \phi \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \, g^{\mu\nu} \, \partial_\nu \phi \right) = 0. \tag{6}$$

Under these conditions, the propagation of sound is governed by an acoustic metric – $g_{\mu\nu}(t, \mathbf{x})$. *This acoustic metric describes a (3+1)-dimensional Lorentzian (pseudo–Riemannian) geometry. The metric depends algebraically on the density, velocity of flow, and local speed of sound in the fluid. Specifically*

$$g_{\mu\nu}(t, \mathbf{x}) \equiv \frac{\rho}{c} \begin{bmatrix} -(c^2 - v^2) & \vdots & -\mathbf{v}^T \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ -\mathbf{v} & \vdots & \mathbf{I} \end{bmatrix}. \tag{7}$$

(Here $\mathbf{I}$ is the 3×3 identity matrix.) In general, when the fluid is non-homogeneous and flowing, the acoustic Riemann tensor *associated with this Lorentzian metric will be nonzero.*

Comment. It is quite remarkable that even though the underlying fluid dynamics is Newtonian, nonrelativistic, and takes place in flat space-plus-time, the fluctuations (sound waves) are governed by a curved (3+1)-dimensional Lorentzian (pseudo-Riemannian) spacetime geometry. For practitioners of general relativity this observation describes a very simple and concrete physical model for certain classes of Lorentzian spacetimes, including (as we shall later see) black holes. On the other hand, this discussion is also potentially of interest to practitioners of continuum mechanics and fluid dynamics in that it provides a simple concrete introduction to Lorentzian differential geometric techniques.

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[3] For instance, whenever one has a system of PDEs that can be written in first-order quasi-linear symmetric hyperbolic form, then it is an exact non-perturbative result that the matrix of coefficients for the first-derivative terms can be used to construct a conformal class of metrics that encodes the causal structure of the system of PDEs. For barotropic hydrodynamics this is briefly discussed in [138]. This analysis is related to the behaviour of characteristics of the PDEs, and ultimately can be linked back to the Fresnel equation that appears in the eikonal limit.

Proof. The fundamental equations of fluid dynamics [370, 372, 443, 577] are the equation of continuity

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \, \mathbf{v}) = 0, \tag{8}$$

and Euler's equation (equivalent to $\mathbf{F} = m\mathbf{a}$ applied to small lumps of fluid)

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \equiv \rho \left[\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} \right] = \mathbf{f}. \tag{9}$$

Start the analysis by assuming the fluid to be inviscid (zero viscosity), with the only forces present being those due to pressure.[4] Then, for the force density, we have

$$\mathbf{f} = -\boldsymbol{\nabla}p. \tag{10}$$

Via standard manipulations the Euler equation can be rewritten as

$$\partial_t \mathbf{v} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) - \frac{1}{\rho} \, \boldsymbol{\nabla}p - \boldsymbol{\nabla} \left(\frac{1}{2} \, v^2 \right). \tag{11}$$

Now take the flow to be *vorticity free*, that is, *locally irrotational*. Introduce the velocity potential ϕ such that $\mathbf{v} = -\boldsymbol{\nabla}\phi$, at least locally. If one further takes the fluid to be *barotropic* (this means that ρ is a function of p only), it becomes possible to define

$$h(p) = \int_0^p \frac{\mathrm{d}p'}{\rho(p')}; \qquad \text{so that} \qquad \boldsymbol{\nabla}h = \frac{1}{\rho} \, \boldsymbol{\nabla}p. \tag{12}$$

Thus, the specific enthalpy, $h(p)$, is a function of p only. Euler's equation now reduces to

$$-\,\partial_t\phi + h + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 = 0. \tag{13}$$

This is a version of Bernoulli's equation.

Now linearise these equations of motion around some assumed background (ρ_0, p_0, ϕ_0). Set

$$\rho = \rho_0 + \epsilon\rho_1 + O(\epsilon^2), \tag{14}$$
$$p = p_0 + \epsilon p_1 + O(\epsilon^2), \tag{15}$$
$$\phi = \phi_0 + \epsilon\phi_1 + O(\epsilon^2). \tag{16}$$

Sound is *defined* to be these linearised fluctuations in the dynamical quantities. Note that this is the *standard definition* of (linear) sound and more generally of acoustical disturbances. In principle, of course, a fluid mechanic might really be interested in solving the complete equations of motion for the fluid variables (ρ, p, ϕ). In practice, it is both traditional and extremely useful to separate the exact motion, described by the exact variables, (ρ, p, ϕ), into some average bulk motion, (ρ_0, p_0, ϕ_0), plus low amplitude acoustic disturbances, $(\epsilon\rho_1, \epsilon p_1, \epsilon\phi_1)$. See, for example, [370, 372, 443, 577].

Since this is a subtle issue that we have seen cause considerable confusion in the past, let us be even more explicit by asking the rhetorical question: *"How can we tell the difference between a wind gust and a sound wave?"* The answer is that the difference is to some extent a matter of convention – sufficiently low-frequency long-wavelength disturbances (wind gusts) are conventionally lumped in with the average bulk motion. Higher-frequency, shorter-wavelength disturbances are conventionally described as acoustic disturbances. If you wish to be hyper-technical, we can introduce a high-pass filter function to define the bulk motion by suitably averaging the exact

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[4] It is straightforward to add external forces, at least conservative body forces such as Newtonian gravity.

fluid motion. There are no deep physical principles at stake here – merely an issue of convention. The place where we are making a specific physical assumption that restricts the validity of our analysis is in the requirement that the amplitude of the high-frequency short-wavelength disturbances be small. This is the assumption underlying the linearization programme, and this is why sufficiently high-amplitude sound waves must be treated by direct solution of the full equations of fluid dynamics.

Linearizing the continuity equation results in the pair of equations

$$\partial_t \rho_0 + \boldsymbol{\nabla} \cdot (\rho_0 \, \mathbf{v}_0) = 0, \tag{17}$$

$$\partial_t \rho_1 + \boldsymbol{\nabla} \cdot (\rho_1 \, \mathbf{v}_0 + \rho_0 \, \mathbf{v}_1) = 0. \tag{18}$$

Now, the barotropic condition implies

$$h(p) = h\left(p_0 + \epsilon p_1 + O(\epsilon^2)\right) = h_0 + \epsilon \, \frac{p_1}{\rho_0} + O(\epsilon^2). \tag{19}$$

Use this result in linearizing the Euler equation. We obtain the pair

$$-\partial_t \phi_0 + h_0 + \frac{1}{2}(\boldsymbol{\nabla}\phi_0)^2 = 0. \tag{20}$$

$$-\partial_t \phi_1 + \frac{p_1}{\rho_0} - \mathbf{v}_0 \cdot \boldsymbol{\nabla}\phi_1 = 0. \tag{21}$$

This last equation may be rearranged to yield

$$p_1 = \rho_0 \, \left(\partial_t \phi_1 + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\phi_1\right). \tag{22}$$

Use the barotropic assumption to relate

$$\rho_1 = \frac{\partial \rho}{\partial p} \, p_1 = \frac{\partial \rho}{\partial p} \, \rho_0 \, (\partial_t \phi_1 + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\phi_1). \tag{23}$$

Now substitute this consequence of the linearised Euler equation into the linearised equation of continuity. We finally obtain, up to an overall sign, the wave equation:

$$-\partial_t \left(\frac{\partial \rho}{\partial p}\rho_0 \, (\partial_t \phi_1 + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\phi_1) \right) + \boldsymbol{\nabla} \cdot \left(\rho_0 \boldsymbol{\nabla}\phi_1 - \frac{\partial \rho}{\partial p}\rho_0 \mathbf{v}_0 \, (\partial_t \phi_1 + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\phi_1) \right) = 0. \tag{24}$$

This wave equation describes the propagation of the linearised scalar potential ϕ_1. Once ϕ_1 is determined, Equation (22) determines p_1, and Equation (23) then determines ρ_1. Thus this wave equation completely determines the propagation of acoustic disturbances. The background fields p_0, ρ_0 and $\mathbf{v}_0 = -\boldsymbol{\nabla}\phi_0$, which appear as time-dependent and position-dependent coefficients in this wave equation, are constrained to solve the equations of fluid motion for a barotropic, inviscid, and irrotational flow. Apart from these constraints, they are otherwise permitted to have *arbitrary* temporal and spatial dependencies.

Now, written in this form, the physical import of this wave equation is somewhat less than pellucid. To simplify things algebraically, observe that the local speed of sound is defined by

$$c^{-2} \equiv \frac{\partial \rho}{\partial p}. \tag{25}$$

Now construct the symmetric 4×4 matrix

$$f^{\mu\nu}(t, \mathbf{x}) \equiv \frac{\rho_0}{c^2} \left[\begin{array}{c:c} -1 & -v_0^j \\ \hdashline -v_0^i & (c^2 \, \delta^{ij} - v_0^i v_0^j) \end{array} \right]. \tag{26}$$

(Greek indices run from $0-3$, while Roman indices run from $1-3$.) Then, introducing (3+1)-dimensional space-time coordinates, which we write as $x^\mu \equiv (t; x^i)$, the above wave Equation (24) is easily rewritten as

$$\partial_\mu(f^{\mu\nu}\, \partial_\nu \phi_1) = 0. \tag{27}$$

This remarkably compact formulation is completely equivalent to Equation (24) and is a much more promising stepping-stone for further manipulations. The remaining steps are a straightforward application of the techniques of curved space (3+1)-dimensional Lorentzian geometry.

Now in any Lorentzian (i.e., pseudo–Riemannian) manifold the curved space scalar d'Alembertian is given in terms of the metric $g_{\mu\nu}(t, \mathbf{x})$ by

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\; g^{\mu\nu}\, \partial_\nu \phi\right). \tag{28}$$

(See, for example, [213, 446, 597, 445, 275, 670].) The inverse metric, $g^{\mu\nu}(t, \mathbf{x})$, is pointwise the matrix inverse of $g_{\mu\nu}(t, \mathbf{x})$, while $g \equiv \det(g_{\mu\nu})$. Thus one can rewrite the physically derived wave Equation (24) in terms of the d'Alembertian provided one identifies

$$\sqrt{-g}\; g^{\mu\nu} = f^{\mu\nu}. \tag{29}$$

This implies, on the one hand,

$$\det(f^{\mu\nu}) = (\sqrt{-g})^4\; g^{-1} = g. \tag{30}$$

On the other hand, from the explicit expression (26), expanding the determinant in minors yields

$$\det(f^{\mu\nu}) = \left(\frac{\rho_0}{c^2}\right)^4 \cdot \left[(-1)\cdot(c^2 - v_0^2) - (-v_0)^2\right] \cdot \left[c^2\right] \cdot \left[c^2\right] = -\frac{\rho_0^4}{c^2}. \tag{31}$$

Thus,

$$g = -\frac{\rho_0^4}{c^2}; \qquad \sqrt{-g} = \frac{\rho_0^2}{c}. \tag{32}$$

Therefore, we can pick off the coefficients of the *inverse* (contravariant) acoustic metric

$$g^{\mu\nu}(t, \mathbf{x}) \equiv \frac{1}{\rho_0 c} \begin{bmatrix} -1 & \vdots & -v_0^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots \\ -v_0^i & \vdots & (c^2\, \delta^{ij} - v_0^i\, v_0^j) \end{bmatrix}. \tag{33}$$

We could now determine the metric itself simply by inverting this 4×4 matrix (and if the reader is not a general relativist, proceeding in this direct manner is definitely the preferred option). On the other hand, for general relativists it is even easier to recognise that one has in front of one a specific example of the Arnowitt–Deser–Misner split of a (3+1)-dimensional Lorentzian spacetime metric into space + time, more commonly used in discussing initial value data in general relativity. (See, for example, [445, pp. 505–508].) The (covariant) acoustic metric is then read off by inspection

$$g_{\mu\nu} \equiv \frac{\rho_0}{c} \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdots\cdots & \\ -v_0^i & \vdots & \delta_{ij} \end{bmatrix}. \tag{34}$$

Equivalently, the acoustic interval (acoustic line-element) can be expressed as

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu = \frac{\rho_0}{c} \left[-c^2\, dt^2 + (dx^i - v_0^i\, dt)\, \delta_{ij}\, (dx^j - v_0^j\, dt)\right]. \tag{35}$$

This completes the proof of the theorem. $\square$

We have presented the theorem and proof, which closely follows the discussion in [624], in considerable detail because it is a standard template that can be readily generalised in many ways. This discussion can then be used as a starting point to initiate the analysis of numerous and diverse physical models.

2.4 General features of the acoustic metric

A few brief comments should be made before proceeding further:

- Observe that the signature of this effective metric is indeed $(-, +, +, +)$, as it should be to be regarded as Lorentzian.

- Observe that in physical acoustics it is the inverse metric density,

$$f^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} \tag{36}$$

  that is of more fundamental significance for deriving the wave equation than is the metric $g_{\mu\nu}$ itself. (This observation continues to hold in more general situations where it is often significantly easier to calculate the tensor density $f^{\mu\nu}$ than it is to calculate the effective metric $g_{\mu\nu}$.)

- It should be emphasised that there are two distinct metrics relevant to the current discussion:

  - The *physical spacetime metric* is, in this case, just the usual flat metric of Minkowski space:

    $$\eta_{\mu\nu} \equiv (\mathrm{diag}[-c_{\mathrm{light}}^2, 1, 1, 1])_{\mu\nu}. \tag{37}$$

    (Here c_{light} is the speed of light in vacuum.) The fluid particles couple only to the physical metric $\eta_{\mu\nu}$. In fact the fluid motion is completely non-relativistic, so that $||v_0|| \ll c_{\mathrm{light}}$, and it is quite sufficient to consider Galilean relativity for the underlying fluid mechanics.

  - Sound waves on the other hand, do not "see" the physical metric at all. Acoustic perturbations couple only to the effective *acoustic metric* $g_{\mu\nu}$.

- It is quite remarkable that (to the best of our knowledge) a version of this acoustic metric was first derived and used in Moncrief's studies of the relativistic hydrodynamics of accretion flows surrounding black holes [448]. Indeed, Moncrief was working in the more general case of a curved background "physical" metric, in addition to a curved "effective" metric. We shall come back to this work later on, in our historical section. (See also Section 4.1.2.)

- However, the geometry determined by the acoustic metric does inherit some key properties from the existence of the underlying flat physical metric. For instance, the *topology* of the manifold does not depend on the particular metric considered. The acoustic geometry inherits the underlying topology of the physical metric – ordinary $\Re^4$ – with possibly a few regions excised (due to whatever hard-wall boundary conditions one might wish to impose on the fluid). In systems constrained to have effectively less than 3 space-like dimensions one can reproduce more complicated topologies (consider, for example, an effectively one-dimensional flow in a tubular ring).

- Furthermore, the acoustic geometry automatically inherits from the underlying Newtonian time parameter, the important property of "stable causality" [275, 670]. Note that

$$g^{\mu\nu}\,(\nabla_\mu t)\,(\nabla_\nu t) = -\frac{1}{\rho_0\, c} < 0. \tag{38}$$

This precludes some of the more entertaining causality-related pathologies that sometimes arise in general relativity. (For a general discussion of causal pathologies in general relativity, see, for example, [275, 272, 273, 125, 274, 630]).

- Other concepts that translate immediately are those of "ergo-region", "trapped surface", "apparent horizon", and "event horizon". These notions will be developed more fully in the following subsection.

- The properly normalised four-velocity of the fluid is

$$V^\mu = \frac{(1;\ v_0^i)}{\sqrt{\rho_0\ c}},\tag{39}$$

so that

$$g_{\mu\nu}\, V^\mu\, V^\nu = g(V,V) = -1.\tag{40}$$

This four-velocity is related to the gradient of the natural time parameter by

$$\nabla_\mu t = (1,0,0,0);\qquad\qquad \nabla^\mu t = -\frac{(1;\ v_0^i)}{\rho_0\ c} = -\frac{V^\mu}{\sqrt{\rho_0\ c}}.\tag{41}$$

Thus the integral curves of the fluid velocity field are orthogonal (in the Lorentzian metric) to the constant time surfaces. The acoustic proper time along the fluid flow lines (streamlines) is

$$\tau = \int \sqrt{\rho_0\ c}\ \mathrm{d}t,\tag{42}$$

and the integral curves are geodesics of the acoustic metric if and only if $\rho_0\, c$ is position independent.

- Observe that in a completely general (3+1)-dimensional Lorentzian geometry the metric has 6 degrees of freedom per point in spacetime. (4×4 symmetric matrix $\Rightarrow$ 10 independent components; then subtract 4 coordinate conditions).

  In contrast, the acoustic metric is more constrained. Being specified completely by the three scalars $\phi_0(t, \mathbf{x})$, $\rho_0(t, \mathbf{x})$, and $c(t, \mathbf{x})$, the acoustic metric has, at most, 3 degrees of freedom per point in spacetime. The equation of continuity actually reduces this to 2 degrees of freedom, which can be taken to be $\phi_0(t, \mathbf{x})$ and $c(t, \mathbf{x})$.

  Thus, the simple acoustic metric of this section can, at best, reproduce some subset of the generic metrics of interest in general relativity.

- A point of notation: Where the general relativist uses the word "stationary" the fluid dynamicist uses the phrase "steady flow". The general-relativistic word "static" translates to a rather messy constraint on the fluid flow (to be discussed more fully below).

- Finally, we should emphasise that in Einstein gravity the spacetime metric is related to the distribution of matter by the nonlinear Einstein–Hilbert differential equations. In contrast, in the present context, the acoustic metric is related to the distribution of matter in a simple algebraic fashion.

2.4.1   Horizons and ergo-regions

In the next two subsections we shall undertake to more fully explain some of the technical details underlying the acoustic analogy. Concepts and quantities such as horizons, ergo-regions and "surface gravity" are important features of standard general relativity, and analogies are useful only insofar as they adequately preserve these notions.

Let us start with the notion of an ergo-region: Consider integral curves of the vector

$$K^\mu \equiv (\partial/\partial t)^\mu = (1,0,0,0)^\mu. \tag{43}$$

If the flow is steady, then this is the time translation Killing vector. Even if the flow is not steady the background Minkowski metric provides us with a natural definition of "at rest". Then[5]

$$g_{\mu\nu} \, (\partial/\partial t)^\mu \, (\partial/\partial t)^\nu = g_{tt} = -[c^2 - v^2]. \tag{44}$$

This quantity changes sign when $||\mathbf{v}|| > c$. Thus, any region of supersonic flow is an ergo-region. (And the boundary of the ergo-region may be deemed to be the ergo-surface.) The analogue of this behaviour in general relativity is the ergosphere surrounding any spinning black hole – it is a region where space "moves" with superluminal velocity relative to the fixed stars [445, 275, 670].

A trapped surface in acoustics is defined as follows: Take any closed two-surface. If the fluid velocity is everywhere inward-pointing and the normal component of the fluid velocity is everywhere greater than the local speed of sound, then no matter what direction a sound wave propagates, it will be swept inward by the fluid flow and be trapped inside the surface. The surface is then said to be outer-trapped. (For comparison with the usual situation in general relativity see [275, pp. 319–323] or [670, pp. 310–311].) Inner-trapped surfaces (anti-trapped surfaces) can be defined by demanding that the fluid flow is everywhere outward-pointing with supersonic normal component. It is only because of the fact that the background Minkowski metric provides a natural definition of "at rest" that we can adopt such a simple and straightforward definition. In ordinary general relativity we need to develop considerable additional technical machinery, such as the notion of the "expansion" of bundles of ingoing and outgoing null geodesics, before defining trapped surfaces. That the above definition for acoustic geometries is a specialization of the usual one can be seen from the discussion in [275, pp. 262–263]. The acoustic trapped region is now defined as the region containing outer trapped surfaces, and the acoustic (future) apparent horizon as the boundary of the trapped region. That is, the acoustic apparent horizon is the two-surface for which the normal component of the fluid velocity is everywhere equal to the local speed of sound. (We can also define anti-trapped regions and past apparent horizons but these notions are of limited utility in general relativity.)[6]

The fact that the apparent horizon seems to be fixed in a foliation-independent manner is only an illusion due to the way in which the analogies work. A particular fluid flow reproduces a specific "metric", (a matrix of coefficients in a specific coordinate system, not a "geometry"), and, in particular, a specific foliation of spacetime. (Only "internal" observers see a "geometry"; see the discussion in Section 7.4). The same "geometry" written in different coordinates would give rise to a different fluid flow (if at all possible, as not all coordinate representations of a fixed geometry give rise to acoustic metrics) and, therefore, to a different apparent horizon.

The event horizon (absolute horizon) is defined, as in general relativity, by demanding that it be the boundary of the region from which null geodesics (phonons) cannot escape. This is actually the future event horizon. A past event horizon can be defined in terms of the boundary of the region

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[5] Henceforth, in the interests of notational simplicity, we shall drop the explicit subscript 0 on background field quantities unless there is specific risk of confusion.

[6] This discussion naturally leads us to what is perhaps the central question of analogue models – just how much of the standard "laws of black hole mechanics" [51, 671] carry over into these analogue models? Quite a lot but not everything – that is our main topic for the rest of the review.
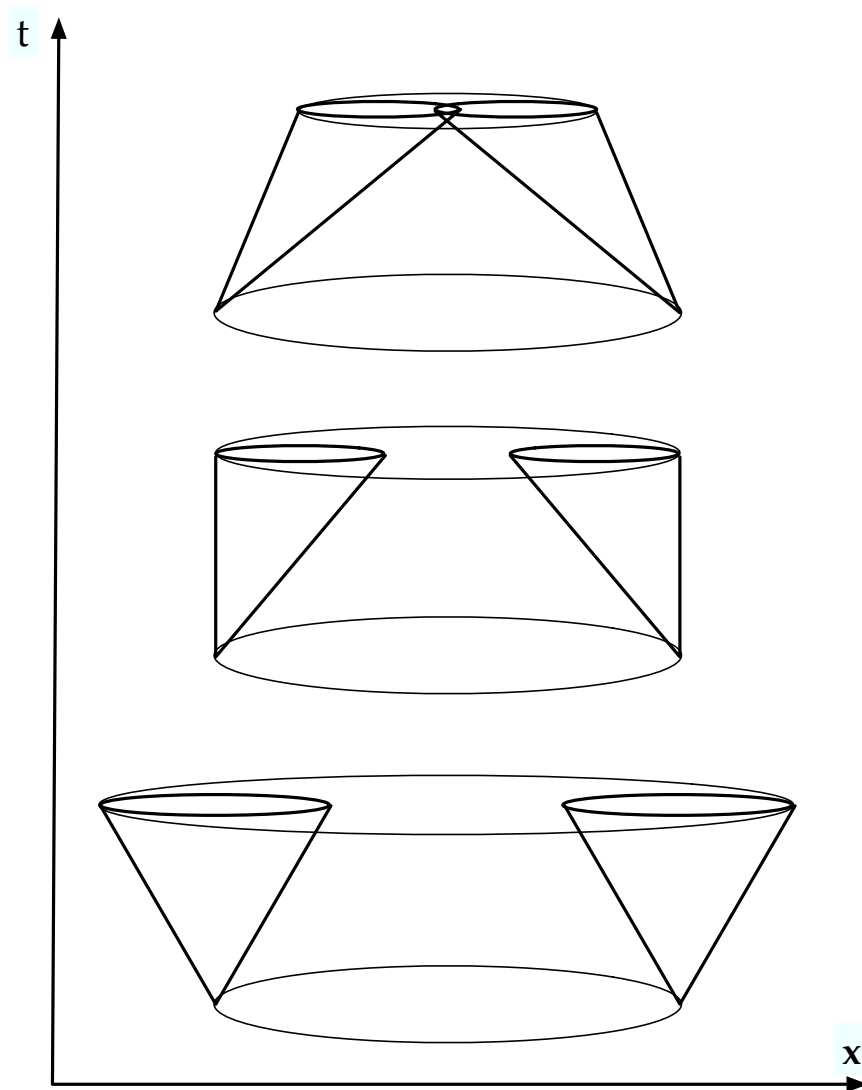
Figure 5: A moving fluid can form "trapped surfaces" when supersonic flow tips the sound cones past the vertical.
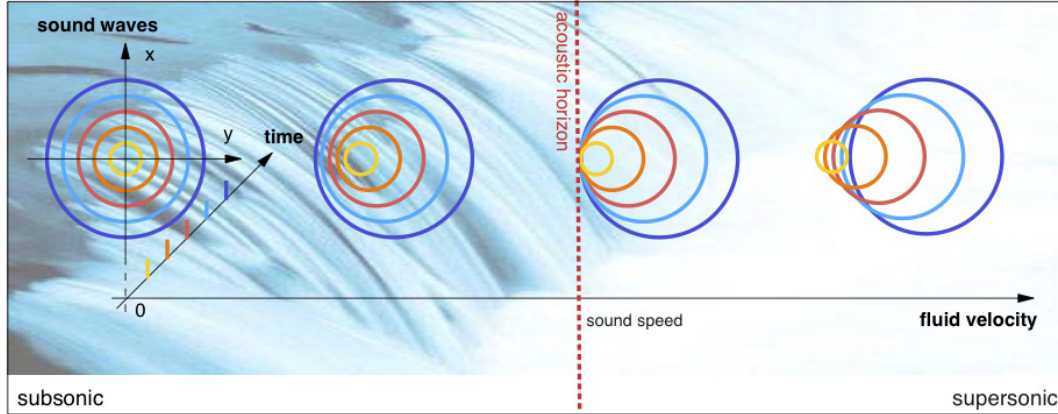
Figure 6: A moving fluid can form an "acoustic horizon" when supersonic flow prevents upstream motion of sound waves.

that cannot be reached by incoming phonons – strictly speaking this requires us to define notions of past and future null infinities, but we will simply take all relevant incantations as understood. In particular, the event horizon is a null surface, the generators of which are null geodesics.

In all stationary geometries the apparent and event horizons coincide, and the distinction is immaterial. In time-dependent geometries the distinction is often important. When computing the surface gravity, we shall restrict attention to stationary geometries (steady flow). In fluid flows of high symmetry (spherical symmetry, plane symmetry), the ergosphere may coincide with the acoustic apparent horizon, or even the acoustic event horizon. This is the analogue of the result in general relativity that for static (as opposed to stationary) black holes the ergosphere and event horizon coincide. For many more details, including appropriate null coordinates and Carter–Penrose diagrams, both in stationary and time-dependent situations, see [37].

2.4.2   Surface gravity

Because of the definition of event horizon in terms of phonons (null geodesics) that cannot escape the acoustic black hole, the event horizon is automatically a null surface, and the generators of the event horizon are automatically null geodesics. In the case of acoustics, there is one particular parameterization of these null geodesics that is "most natural", which is the parameterization in terms of the Newtonian time coordinate of the underlying physical metric. This allows us to unambiguously define a "surface gravity" even for non-stationary (time-dependent) acoustic event horizons, by calculating the extent to which this natural time parameter fails to be an affine parameter for the null generators of the horizon. (This part of the construction fails in general relativity where there is no universal natural time-coordinate unless there is a time-like Killing vector – this is why extending the notion of surface gravity to non-stationary geometries in general relativity is so difficult.)

When it comes to explicitly calculating the surface gravity in terms of suitable gradients of the fluid flow, it is nevertheless very useful to limit attention to situations of steady flow (so that the acoustic metric is stationary). This has the added bonus that for stationary geometries the notion of "acoustic surface gravity" in acoustics is unambiguously equivalent to the general relativity definition. It is also useful to take cognizance of the fact that the situation simplifies considerably for static (as opposed to merely stationary) acoustic metrics.

To set up the appropriate framework, write the general stationary acoustic metric in the form

$$\mathrm{d}s^2 = \frac{\rho}{c} \left[-c^2 \, \mathrm{d}t^2 + (\mathrm{d}\mathbf{x} - \mathbf{v} \, \mathrm{d}t)^2 \right]. \tag{45}$$

The time translation Killing vector is simply $K^\mu = (1; \vec{0})$, with

$$K^2 \equiv g_{\mu\nu} K^\mu K^\nu \equiv -||\mathbf{K}||^2 = -\frac{\rho}{c}[c^2 - v^2]. \tag{46}$$

The metric can also be written as

$$\mathrm{d}s^2 = \frac{\rho}{c} \left[-(c^2 - v^2) \, \mathrm{d}t^2 - 2\mathbf{v} \cdot \mathrm{d}\mathbf{x} \, \mathrm{d}t + (\mathrm{d}\mathbf{x})^2 \right]. \tag{47}$$

Static acoustic spacetimes: Now suppose that the vector $\mathbf{v}/(c^2 - v^2)$ is integrable, (the gradient of some scalar), then we can define a new time coordinate by

$$\mathrm{d}\tau = \mathrm{d}t + \frac{\mathbf{v} \cdot \mathrm{d}\mathbf{x}}{c^2 - v^2}. \tag{48}$$

Substituting this back into the acoustic line element gives

$$\mathrm{d}s^2 = \frac{\rho}{c} \left[-(c^2 - v^2) \, \mathrm{d}\tau^2 + \left\{ \delta_{ij} + \frac{v^i v^j}{c^2 - v^2} \right\} \mathrm{d}x^i \, \mathrm{d}x^j \right]. \tag{49}$$

In this coordinate system the absence of the time-space cross-terms makes manifest that the acoustic geometry is in fact static (there exists a family of space-like hypersurfaces orthogonal to the time-like Killing vector). The condition that an acoustic geometry be static, rather than merely stationary, is thus seen to be

$$\boldsymbol{\nabla} \times \left\{ \frac{\mathbf{v}}{(c^2 - v^2)} \right\} = 0. \tag{50}$$

That is, (since in deriving the existence of the effective metric we have already assumed the fluid to be irrotational),

$$\mathbf{v} \times \boldsymbol{\nabla}(c^2 - v^2) = 0. \tag{51}$$

This requires the fluid flow to be parallel to another vector that is not quite the acceleration but is closely related to it. (Note that, because of the vorticity free assumption, $\frac{1}{2}\boldsymbol{\nabla}v^2$ is just the three-acceleration of the fluid, it is the occurrence of a possibly position dependent speed of sound that complicates the above.) Note that because of the barotropic assumption we have

$$\boldsymbol{\nabla}c^2 = \frac{\partial c^2}{\partial \rho} \frac{\partial \rho}{\partial p} \boldsymbol{\nabla}p = \frac{\partial^2 p}{\partial \rho^2} \frac{\partial \rho}{\partial p} \rho \, \mathbf{a}. \tag{52}$$

That is

$$\boldsymbol{\nabla}(c^2 - v^2) = \left(\frac{\partial^2 p}{\partial \rho^2} \frac{\partial \rho}{\partial p} \rho - 2 \right) \mathbf{a}. \tag{53}$$

So (given that the geometry is already stationary) the condition for a static acoustic geometry reduces to

$$\left(\frac{\partial^2 p}{\partial \rho^2} \frac{\partial \rho}{\partial p} \rho - 2 \right) \mathbf{v} \times \mathbf{a} = 0. \tag{54}$$

This condition can be satisfied in two ways, either by having $\mathbf{v} \parallel \mathbf{a}$, or by having the very specific (and not particularly realistic) equation of state

$$p = \frac{1}{3}k\rho^3 + C; \qquad c^2 = k\rho^2. \tag{55}$$

Note that for this particular barotropic equation of state the conformal factor drops out.

Once we have a static geometry, we can of course directly apply all of the standard tricks [601] for calculating the surface gravity developed in general relativity. We set up a system of fiducial observers (FIDOS) by properly normalizing the time-translation Killing vector

$$\mathbf{V}_{\text{FIDO}} \equiv \frac{\mathbf{K}}{||\mathbf{K}||} = \frac{\mathbf{K}}{\sqrt{(\rho/c)\,[c^2 - v^2]}}. \tag{56}$$

The four-acceleration of the FIDOS is defined as

$$\mathbf{A}_{\text{FIDO}} \equiv (\mathbf{V}_{\text{FIDO}} \cdot \boldsymbol{\nabla})\mathbf{V}_{\text{FIDO}}, \tag{57}$$

and using the fact that $\mathbf{K}$ is a Killing vector, it may be computed in the standard manner

$$\mathbf{A}_{\text{FIDO}} = +\frac{1}{2}\frac{\boldsymbol{\nabla}||\mathbf{K}||^2}{||\mathbf{K}||^2}. \tag{58}$$

That is

$$\mathbf{A}_{\text{FIDO}} = \frac{1}{2}\left[\frac{\boldsymbol{\nabla}(c^2 - v^2)}{(c^2 - v^2)} + \frac{\boldsymbol{\nabla}(\rho/c)}{(\rho/c)}\right]. \tag{59}$$

The surface gravity is now defined by taking the norm $||\mathbf{A}_{\text{FIDO}}||$, multiplying by the lapse function, $||\mathbf{K}|| = \sqrt{(\rho/c)\,[c^2 - v^2]}$, and taking the limit as one approaches the horizon: $|v| \to c$ (remember that we are currently dealing with the static case). The net result is

$$||\mathbf{A}_{\text{FIDO}}||\,||\mathbf{K}|| = \frac{1}{2}\mathbf{n} \cdot \boldsymbol{\nabla}(c^2 - v^2) + O(c^2 - v^2), \tag{60}$$

so that the surface gravity is given in terms of a normal derivative by[7]

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v^2)}{\partial n}\bigg|_H = c_H\,\frac{\partial|c - v|}{\partial n}\bigg|_H. \tag{61}$$

This is not quite Unruh's result [607, 608] since he implicitly took the speed of sound to be a position-independent constant. (This is of course a completely appropriate approximation for water, which was the working fluid he was considering.) The fact that prefactor ρ/c drops out of the final result for the surface gravity can be justified by appeal to the known conformal invariance of the surface gravity [315]. Though derived in a totally different manner, this result is also compatible with the expression for "surface-gravity" obtained in the solid-state black holes of Reznik [523], wherein a position dependent (and singular) refractive index plays a role analogous to the acoustic metric. As a further consistency check, one can go to the spherically symmetric case and check that this reproduces the results for "dirty black holes" enunciated in [621]. Finally, note that we can also write the expression for surface gravity as

$$g_H = \left|1 - \frac{1}{2}\frac{\partial^2 p}{\partial \rho^2}\,\frac{\partial \rho}{\partial p}\,\rho\right|\,||\mathbf{a}||, \tag{62}$$

demonstrating that (in a static acoustic spacetime) the surface gravity is (up to a dimensionless factor depending on the equation of state) directly related to the acceleration of the fluid as it crosses the horizon. For water $p = k\rho + C$, $c^2 = k$, and $g_H = ||\mathbf{a}||$; for a Bose–Einstein condensate (BEC) we shall later on see that $p = \frac{1}{2}k\rho^2 + C$, implying $c^2 = k\rho$, which then leads to the simple result $g_H = \frac{1}{2}||\mathbf{a}||$.

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[7] Because of the background Minkowski metric there can be no possible confusion as to the definition of this normal derivative.

Since this is a static geometry, the relationship between the Hawking temperature and surface gravity may be verified in the usual fast-track manner – using the Wick rotation trick to analytically continue to Euclidean space [245]. If you don't like Euclidean signature techniques (which are in any case only applicable to equilibrium situations) you should go back to the original Hawking derivations [270, 271].[8]

We should emphasize that the formula for the Hawking temperature contains both the surface gravity g_H and the speed of sound c_H at the horizon [624]. Specifically

$$kT_H = \frac{\hbar g_H}{2\pi c_H}. \tag{63}$$

In view of the explicit formula for g_H above, this can also be written as

$$kT_H = \frac{\hbar}{2\pi} \left. \frac{\partial |c - v|}{\partial n} \right|_H, \tag{64}$$

which is closer to the original form provided by Unruh [607] (which corresponds to c being constant). Purely on dimensional grounds it is a spatial derivative of velocity (which has the same engineering dimension as frequency) that is the determining factor in specifying the physically-normalised Hawking temperature. (Since there is a strong tendency in classical general relativity to adopt units such that $c \to 1$, and even in these analogue models it is common to adopt units such that $c_H \to 1$, this has the potential to lead to some confusion. If you choose units to measure the surface gravity as a physical acceleration, then it is the quantity g_H/c_H, which has the dimensions of frequency that governs the Hawking flux [624].)

One final comment to wrap up this section: The coordinate transform we used to put the acoustic metric into the explicitly static form is perfectly good mathematics, and from the general relativity point of view is even a simplification. However, from the point of view of the underlying Newtonian physics of the fluid, this is a rather bizarre way of deliberately de-synchronizing your clocks to take a perfectly reasonable region – the boundary of the region of supersonic flow – and push it out to "time" plus infinity. From the fluid dynamics point of view this coordinate transformation is correct but perverse, and it is easier to keep a good grasp on the physics by staying with the original Newtonian time coordinate.

Stationary (non-static) acoustic spacetimes: If the fluid flow does not satisfy the integrability condition, which allows us to introduce an explicitly static coordinate system, then defining the surface gravity is a little trickier.

Recall that by construction the acoustic apparent horizon is, in general, defined to be a two-surface for which the normal component of the fluid velocity is everywhere equal to the local speed of sound, whereas the acoustic event horizon (absolute horizon) is characterised by the boundary of those null geodesics (phonons) that do not escape to infinity. In the stationary case these notions coincide, and it is still true that the horizon is a null surface, and that the horizon can be ruled by an appropriate set of null curves. Suppose we have somehow isolated the location of the acoustic horizon, then, in the vicinity of the horizon, we can split up the fluid flow into normal and tangential components

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel; \qquad \text{where} \qquad \mathbf{v}_\perp = v_\perp \, \hat{\mathbf{n}}. \tag{65}$$

Here (and for the rest of this particular section) it is essential that we use the natural Newtonian time coordinate inherited from the background Newtonian physics of the fluid. In addition $\hat{\mathbf{n}}$ is

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[8] There are a few potential subtleties in the derivation of the existence of Hawking radiation, which we are, for the time being, glossing over; see Section 5.1 for details.

a unit vector field that, at the horizon, is perpendicular to it, and away from the horizon is some suitable smooth extension. (For example, take the geodesic distance to the horizon and consider its gradient.) We only need this decomposition to hold in some open set encompassing the horizon and do not need to have a global decomposition of this type available. Furthermore, by definition we know that $v_\perp = c$ at the horizon. Now consider the vector field

$$L^\mu = (\, 1;\ v_\parallel^i\,). \tag{66}$$

Since the spatial components of this vector field are by definition tangent to the horizon, the integral curves of this vector field will be generators for the horizon. Furthermore, the norm of this vector (in the acoustic metric) is

$$||\mathbf{L}||^2 = -\frac{\rho}{c}\left[\,-(c^2-v^2)-2\mathbf{v}_\parallel\cdot\mathbf{v}+\mathbf{v}_\parallel\cdot\mathbf{v}_\parallel\right] = \frac{\rho}{c}\,(c^2-v_\perp^2). \tag{67}$$

In particular, on the acoustic horizon L^μ defines a null vector field, the integral curves of which are generators for the acoustic horizon. We shall now verify that these generators are geodesics, though the vector field $\mathbf{L}$ is not normalised with an affine parameter, and in this way shall calculate the surface gravity.

Consider the quantity $(\mathbf{L}\cdot\boldsymbol{\nabla})\mathbf{L}$ and calculate

$$L^\alpha\nabla_\alpha L^\mu = L^\alpha(\nabla_\alpha L_\beta - \nabla_\beta L_\alpha)g^{\beta\mu} + \frac{1}{2}\nabla_\beta(L^2)g^{\beta\mu}. \tag{68}$$

To calculate the first term note that

$$L_\mu = \frac{\rho}{c}\,(-[c^2-v_\perp^2];\mathbf{v}_\perp). \tag{69}$$

Thus,

$$L_{[\alpha,\beta]} = -\begin{bmatrix} 0 & \vdots & -\nabla_i\left[\frac{\rho}{c}(c^2-v_\perp^2)\right] \\ \cdots\cdots\cdots & \cdot & \cdots\cdots \\ +\nabla_j\left[\frac{\rho}{c}(c^2-v_\perp^2)\right] & \vdots & \left(\frac{\rho}{c}\,v^\perp\right)_{[i,j]} \end{bmatrix}. \tag{70}$$

And so:

$$L^\alpha L_{[\beta,\alpha]} = \left(\mathbf{v}_\parallel\cdot\boldsymbol{\nabla}\left[\frac{\rho}{c}(c^2-v_\perp^2)\right];\nabla_j\left[\frac{\rho}{c}(c^2-v_\perp^2)\right] + v_\parallel^i\left(\frac{\rho}{c}\,|v^\perp|\hat{n}\right)_{[j,i]}\right). \tag{71}$$

On the horizon, where $c = v_\perp$, and additionally assuming $\mathbf{v}_\parallel\cdot\boldsymbol{\nabla}\rho = 0$ so that the density is constant over the horizon, this simplifies tremendously

$$(L^\alpha L_{[\beta,\alpha]})|_{\text{horizon}} = -\frac{\rho}{c}\,\left(0;\nabla_j(c^2-v_\perp^2)\right) = -\frac{\rho}{c}\,\frac{\partial(c^2-v_\perp^2)}{\partial n}\,(0;\hat{n}_j). \tag{72}$$

Similarly, for the second term we have

$$\nabla_\beta(L^2) = \left(0;\nabla_j\left[\frac{\rho}{c}(c^2-v_\perp^2)\right]\right). \tag{73}$$

On the horizon this again simplifies

$$\nabla_\beta(L^2)|_{\text{horizon}} = +\frac{\rho}{c}\,\left(0;\nabla_j(c^2-v_\perp^2)\right) = +\frac{\rho}{c}\,\frac{\partial(c^2-v_\perp^2)}{\partial n}\,(0;\hat{n}_j). \tag{74}$$

There is partial cancellation between the two terms, and so

$$(L^\alpha\nabla_\alpha L_\mu)_{\text{horizon}} = +\frac{1}{2}\,\frac{\rho}{c}\,\frac{\partial(c^2-v_\perp^2)}{\partial n}\,(0;\hat{n}_j), \tag{75}$$

while

$$(L_\mu)_{\text{horizon}} = \frac{\rho}{c} \; (0; c \; \hat{n}_j) \,. \tag{76}$$

Comparing this with the standard definition of surface gravity [670][9]

$$(L^\alpha \nabla_\alpha L_\mu)_{\text{horizon}} = +\frac{g_H}{c} \; (L_\mu)_{\text{horizon}} \,, \tag{77}$$

we finally have

$$g_H = \frac{1}{2} \frac{\partial(c^2 - v_\perp^2)}{\partial n} = c \; \frac{\partial(c - v_\perp)}{\partial n} \,. \tag{78}$$

This is in agreement with the previous calculation for static acoustic black holes, and insofar as there is overlap, is also consistent with results of Unruh [607, 608], Reznik [523], and the results for "dirty black holes" [621]. From the construction it is clear that the surface gravity is a measure of the extent to which the Newtonian time parameter inherited from the underlying fluid dynamics fails to be an affine parameter for the null geodesics on the horizon.[10]

Again, the justification for going into so much detail on this specific model is that this style of argument can be viewed as a template – it will (with suitable modifications) easily generalise to more complicated analogue models.

2.4.3   Example: vortex geometry

As an example of a fluid flow where the distinction between ergosphere and acoustic event horizon is critical, consider the "draining bathtub" fluid flow. We shall model a draining bathtub by a (3+1) dimensional flow with a linear sink along the z-axis. Let us start with the simplifying assumption that the background density ρ is a position-independent constant throughout the flow (which automatically implies that the background pressure p and speed of sound c are also constant throughout the fluid flow). The equation of continuity then implies that for the radial component of the fluid velocity we must have

$$v^{\hat{r}} \propto \frac{1}{r} \,. \tag{79}$$

In the tangential direction, the requirement that the flow be vorticity free (apart from a possible delta-function contribution at the vortex core) implies, via Stokes' theorem, that

$$v^{\hat{t}} \propto \frac{1}{r} \,. \tag{80}$$

(If these flow velocities are nonzero, then following the discussion of [641] there must be some external force present to set up and maintain the background flow. Fortunately it is easy to see that this external force affects only the background flow and does not influence the linearised fluctuations we are interested in.)

For the background velocity potential we must then have

$$\phi(r, \theta) = -A \; \ln(r/a) - B \; \theta. \tag{81}$$

Note that, as we have previously hinted, the velocity potential is not a true function (because it has a discontinuity on going through 2π radians). The velocity potential must be interpreted as

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[9] There is an issue of normalization here. On the one hand we want to be as close as possible to general relativistic conventions. On the other hand, we would like the surface gravity to really have the dimensions of an acceleration. The convention adopted here, with one explicit factor of c, is the best compromise we have come up with. (Note that in an acoustic setting, where the speed of sound is not necessarily a constant, we cannot simply set $c \to 1$ by a choice of units.)

[10] There are situations in which this surface gravity is a lot larger than one might naively expect [398].

being defined patch-wise on overlapping regions surrounding the vortex core at $r = 0$. The velocity of the fluid flow is

$$\mathbf{v} = -\boldsymbol{\nabla}\phi = \frac{(A\,\hat{r} + B\,\hat{\theta})}{r}. \tag{82}$$

Dropping a position-independent prefactor, the acoustic metric for a draining bathtub is explicitly given by

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \left(\mathrm{d}r - \frac{A}{r}\mathrm{d}t\right)^2 + \left(r\,\mathrm{d}\theta - \frac{B}{r}\mathrm{d}t\right)^2 + \mathrm{d}z^2. \tag{83}$$

Equivalently

$$\mathrm{d}s^2 = -\left(c^2 - \frac{A^2 + B^2}{r^2}\right)\mathrm{d}t^2 - 2\frac{A}{r}\,\mathrm{d}r\,\mathrm{d}t - 2B\,\mathrm{d}\theta\,\mathrm{d}t + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + \mathrm{d}z^2. \tag{84}$$

A similar metric, restricted to $A = 0$ (no radial flow), and generalised to an anisotropic speed of sound, has been exhibited by Volovik [646], that metric being a model for the acoustic geometry surrounding physical vortices in superfluid ^{3}He. (For a survey of the many analogies and similarities between the physics of superfluid ^{3}He, see Section 4.2.2 and references therein. For issues specifically connected to the Standard Electroweak Model see [667].) Note that the metric given above is *not* identical to the metric of a spinning cosmic string, which would instead take the form [623]

$$\mathrm{d}s^2 = -c^2(\mathrm{d}t - \tilde{A}\,\mathrm{d}\theta)^2 + \mathrm{d}r^2 + (1 - \tilde{B})r^2\mathrm{d}\theta^2 + \mathrm{d}z^2. \tag{85}$$
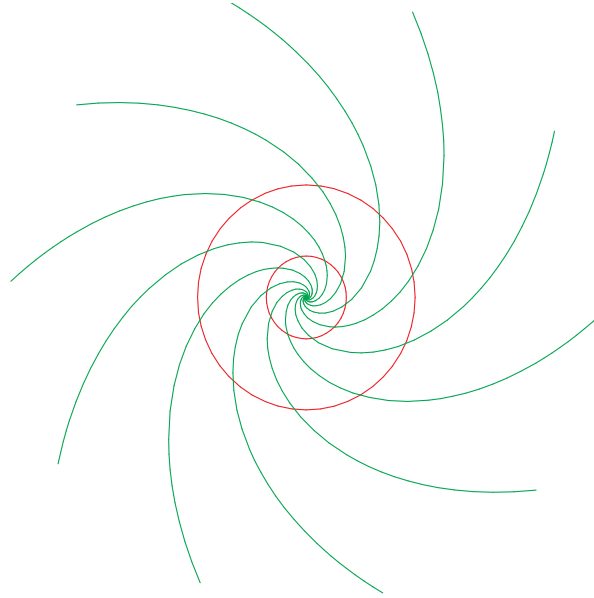
Figure 7: A collapsing vortex geometry (draining bathtub): The green spirals denote streamlines of the fluid flow. The outer circle represents the ergo-surface (ergo-circle) while the inner circle represents the [outer] event horizon.

In conformity with previous comments, the vortex fluid flow is seen to possess an acoustic metric that is stably causal and which does not involve closed time-like curves. At large distances it is possible to *approximate* the vortex geometry by a spinning cosmic string [646], but this approximation becomes progressively worse as the core is approached. Trying to force the existence

of closed time-like curves leads to the existence of evanescent waves in what would be the achronal region, and therefore to the breakdown of the analogue model description [8].

The ergosphere (ergocircle) forms at

$$r_{\text{ergosphere}} = \frac{\sqrt{A^2 + B^2}}{c}. \tag{86}$$

Note that the sign of A is irrelevant in defining the ergosphere and ergo-region: It does not matter if the vortex core is a source or a sink.

The acoustic event horizon forms once the radial component of the fluid velocity exceeds the speed of sound, that is, at

$$r_{\text{horizon}} = \frac{|A|}{c}. \tag{87}$$

The sign of A now makes a difference. For $A < 0$ we are dealing with a future acoustic horizon (acoustic black hole), while for $A > 0$ we are dealing with a past event horizon (acoustic white hole).

2.4.4   Example: slab geometry

A popular model for the investigation of event horizons in the acoustic analogy is the one-dimensional slab geometry where the velocity is always along the z direction and the velocity profile depends only on z. The continuity equation then implies that $\rho(z)\, v(z)$ is a constant, and the acoustic metric becomes

$$\mathrm{d}s^2 \propto \frac{1}{v(z)\, c(z)} \left[-c(z)^2\, \mathrm{d}t^2 + \{\mathrm{d}z - v(z)\, \mathrm{d}t\}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 \right]. \tag{88}$$

That is

$$\mathrm{d}s^2 \propto \frac{1}{v(z)\, c(z)} \left[-\left\{ c(z)^2 - v(z)^2 \right\} \mathrm{d}t^2 - 2v(z)\, \mathrm{d}z\, \mathrm{d}t + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right]. \tag{89}$$

If we set $c = 1$ and ignore the x, y coordinates and conformal factor, we have the toy model acoustic geometry discussed in many papers. (See for instance the early papers by Unruh [608, p. 2828, Equation (8)], Jacobson [310, p. 7085, Equation (4)], Corley and Jacobson [148], and Corley [145].) Depending on the velocity profile one can simulate black holes, white holes or black hole-white hole pairs, interesting for analyzing the black hole laser effect [150] or aspects of the physics of warpdrives [197].

In this situation one must again invoke an external force to set up and maintain the fluid flow. Since the conformal factor is regular at the event horizon, we know that the surface gravity and Hawking temperature are independent of this conformal factor [315]. In the general case it is important to realise that the flow can go supersonic for either of two reasons: The fluid could speed up, or the speed of sound could decrease. When it comes to calculating the "surface gravity" both of these effects will have to be taken into account.

2.4.5   Example: Painlevé–Gullstrand geometry

To see how close the acoustic metric can get to reproducing the Schwarzschild geometry it is first useful to introduce one of the more exotic representations of the Schwarzschild geometry: the Painlevé–Gullstrand line element, which is simply an unusual choice of coordinates on the Schwarzschild spacetime.[11] In modern notation the Schwarzschild geometry in outgoing (+) and

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[11] The Painlevé–Gullstrand line element is sometimes called the Lemaître line element.

ingoing (−) Painlevé–Gullstrand coordinates may be written as:

$$ds^2 = -dt^2 + \left(dr \pm \sqrt{\frac{2GM}{r}}\, dt \right)^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right). \tag{90}$$

Equivalently

$$ds^2 = -\left(1 - \frac{2GM}{r} \right) dt^2 \pm \sqrt{\frac{2GM}{r}}\, dr \, dt + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right). \tag{91}$$

This representation of the Schwarzschild geometry was not (until the advent of the analogue models) particularly well-known, and it has been independently rediscovered several times during the 20th century. See, for instance, Painlevé [484], Gullstrand [264], Lemaître [379], the related discussion by Israel [305], and more recently, the paper by Kraus and Wilczek [364]. The Painlevé–Gullstrand coordinates are related to the more usual Schwarzschild coordinates by

$$t_{\mathrm{PG}} = t_{\mathrm{S}} \pm \left[4M \ \mathrm{arctanh}\left(\sqrt{\frac{2GM}{r}} \right) - 2\sqrt{2GMr} \right]. \tag{92}$$

Or equivalently

$$dt_{\mathrm{PG}} = dt_{\mathrm{S}} \pm \frac{\sqrt{2GM/r}}{1 - 2GM/r}\, dr. \tag{93}$$

With these explicit forms in hand, it becomes an easy exercise to check the equivalence between the Painlevé–Gullstrand line element and the more usual Schwarzschild form of the line element. It should be noted that the + sign corresponds to a coordinate patch that covers the usual asymptotic region plus the region containing the future singularity of the maximally-extended Schwarzschild spacetime. Thus, it covers the future horizon and the black hole singularity. On the other hand the − sign corresponds to a coordinate patch that covers the usual asymptotic region plus the region containing the past singularity. Thus it covers the past horizon and the white hole singularity.

As emphasised by Kraus and Wilczek, the Painlevé–Gullstrand line element exhibits a number of features of pedagogical interest. In particular the constant-time spatial slices are completely flat. That is, the curvature of space is zero, and all the spacetime curvature of the Schwarzschild geometry has been pushed into the time–time and time–space components of the metric.

Given the Painlevé–Gullstrand line element, it might seem trivial to force the acoustic metric into this form: Simply take ρ and c to be constants, and set $v = \sqrt{2GM/r}$. While this certainly forces the acoustic metric into the Painlevé–Gullstrand form, the problem with this is that this assignment is incompatible with the continuity equation $\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) \neq 0$ that was used in deriving the acoustic equations.

The best we can actually do is this: Pick the speed of sound c to be a position-independent constant, which we normalise to unity ($c = 1$). Now set $v = \sqrt{2GM/r}$, and use the continuity equation $\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0$ plus spherical symmetry to deduce $\rho|\mathbf{v}| \propto 1/r^2$ so that $\rho \propto r^{-3/2}$. Since the speed of sound is taken to be constant, we can integrate the relation $c^2 = dp/d\rho$ to deduce that the equation of state must be $p = p_\infty + c^2\rho$ and that the background pressure satisfies $p - p_\infty \propto c^2 r^{-3/2}$. Overall, the acoustic metric is now

$$ds^2 \propto r^{-3/2} \left[-dt^2 + \left(dr \pm \sqrt{\frac{2GM}{r}}\, dt \right)^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]. \tag{94}$$

So we see that the net result is conformal to the Painlevé–Gullstrand form of the Schwarzschild geometry but not identical to it. For many purposes this is good enough. We have an event

horizon; we can define surface gravity; we can analyse Hawking radiation.[12] Since surface gravity and Hawking temperature are conformal invariants [315] this is sufficient for analysing basic features of the Hawking radiation process. The only way in which the conformal factor can influence the Hawking radiation is through backscattering off the acoustic metric. (The phonons are minimally-coupled scalars, not conformally-coupled scalars, so there will in general be effects on the frequency-dependent greybody factors.) If we focus attention on the region near the event horizon, the conformal factor can simply be taken to be a constant, and we can ignore all these complications.

2.4.6 Causal structure

We can now turn to another aspect of acoustic black holes, i.e., their global causal structure, which we shall illustrate making use of the *Carter–Penrose* conformal diagrams [445, 275]. A systematic study in this sense was performed in [37] for 1+1 geometries (viewed either as a dimensional reduction of a physical 3+1 system, or directly as geometrical acoustic metrics). The basic idea underlying the conformal diagram of any non-compact 1+1 manifold is that its metric can always be conformally mapped to the metric of a compact geometry, with a boundary added to represent events at infinity. Since compact spacetimes are in some sense "finite", they can then properly be drawn on a sheet of paper, something that is sometimes very useful in capturing the essential features of the geometry at hand.

The basic steps in the acoustic case are the same as in standard general relativity: Starting from the coordinates (t, x) as in Equation (35), one has to introduce appropriate null coordinates (analogous to the Eddington–Finkelstein coordinates) (u, v), then, by exponentiation, null Kruskal-like coordinates (U, W), and finally compactify by means of a new coordinate pair $(\mathcal{U}, \mathcal{W})$ involving a suitable function mapping an infinite range to a finite one (typically the arctan function). We shall explicitly present only the conformal diagrams for an acoustic black hole, and a black hole-white hole pair, as these particular spacetimes will be of some relevance in what follows. We redirect the reader to [37] for other geometries and technical details.

Acoustic black hole: For the case of a single isolated black-hole horizon we find the Carter–Penrose diagram of Figure 8. (In the figure we have introduced an aspect ratio different from unity for the coordinates $\mathcal{U}$ and $\mathcal{W}$, in order to make the various regions of interest graphically more clear.) As we have already commented, in the acoustic spacetimes, with no periodic identifications, there are two clearly-differentiated notions of asymptotia, "right" and "left". In all our figures we have used subscripts "right" and "left" to label the different null and spacelike infinities. In addition, we have denoted the different sonic-point boundaries with $\Im^{\pm}_{\text{right}}$ or $\Im^{\pm}_{\text{left}}$ depending on whether they are the starting point (– sign) or the ending point (+ sign) of the null geodesics in the right or left parts of the diagram.

In contradistinction to the Carter–Penrose diagram for the Schwarzschild black hole (which in the current context would have to be an eternal black hole, not one formed via astrophysical stellar collapse) there is no singularity. On reflection, this feature of the conformal diagram should be obvious, since the fluid flow underlying the acoustic geometry is nowhere singular.

Note that the event horizon $\mathcal{H}$ is the boundary of the causal past of future right null infinity; that is, $\mathcal{H} = \dot{J}^-(\Im^+_{\text{right}})$, with standard notations [445].

Acoustic black-hole–white-hole pair: The acoustic geometry for a black-hole–white-hole combination again has no singularities in the fluid flow, and no singularities in the spacetime

---

[12] Similar constructions work for the Reissner–Nordström geometry [398], as long as one does not get too close to the singularity. (With $c = 1$ one needs $r > Q^2/(2m)$ to avoid an imaginary fluid velocity.) Likewise, certain aspects of the Kerr geometry can be emulated in this way [641]. (One needs $r > 0$ in the Doran coordinates [176, 633] to avoid closed timelike curves.) As a final remark, let us note that de Sitter space corresponds to $v \propto r$ and $\rho \propto 1/r^3$. For further details see Section 2.5.
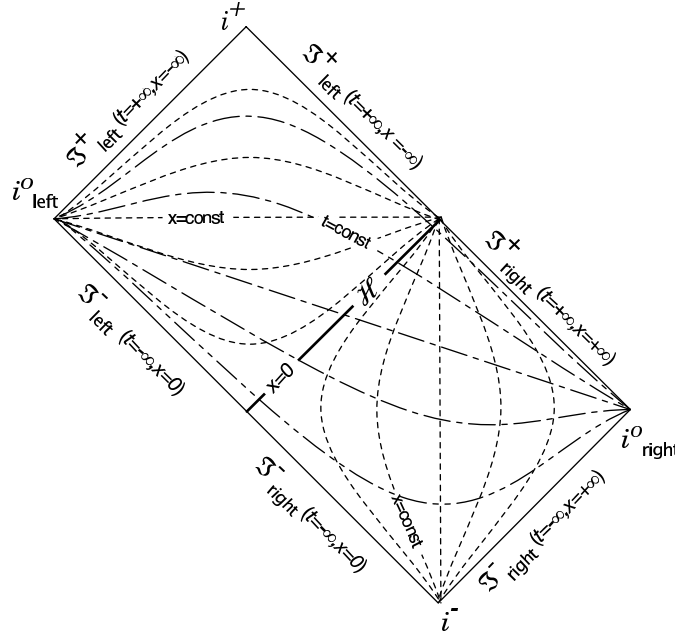
Figure 8: Conformal diagram of an acoustic black hole.

curvature. In particular, from Figure 9, we note the complete absence of singularities.

2.5   Cosmological metrics

In a cosmological framework the key items of interest are the Friedmann–Robertson–Walker (FRW) geometries, more properly called the Friedmann–Lemaître–Robertson–Walker (FLRW) geometries. The simulation of such geometries has been considered in various works such as [46, 47, 105, 106, 194, 195, 196, 328, 403, 674, 675, 676, 683, 677] with a specific view to enhancing our understanding of "cosmological particle production" driven by the expansion of the universe.

Essentially there are two ways to use the acoustic metric, written as

$$ \mathrm{d}s^2 = \frac{\rho}{c_\mathrm{s}} \left[-(c_\mathrm{s}^2 - v^2)\, \mathrm{d}t^2 - 2\mathbf{v} \cdot \mathbf{dx}\, \mathrm{d}t + \mathbf{dx}^2 \right], \tag{95} $$

to reproduce cosmological spacetimes. One is based on physical explosion, the other on rapid variations in the "effective speed of light".

2.5.1   Explosion

We can either let the explosion take place more or less spherically symmetrically, or through a pancake-like configuration, or through a cigar-like configuration.

Three-dimensional explosion: Following the cosmological ideas of [46, 47, 105, 106, 195, 674, 675, 676], and the BEC technologies described in [126, 336, 337], one can take a homogeneous system $\rho(t)$, $c_\mathrm{s}(t)$ and a radial profile for the velocity $\mathbf{v} = (\dot{b}/b)\mathbf{r}$, with b a scale factor depending only on t. (This is actually very similar to the situation in models for Newtonian cosmology, where
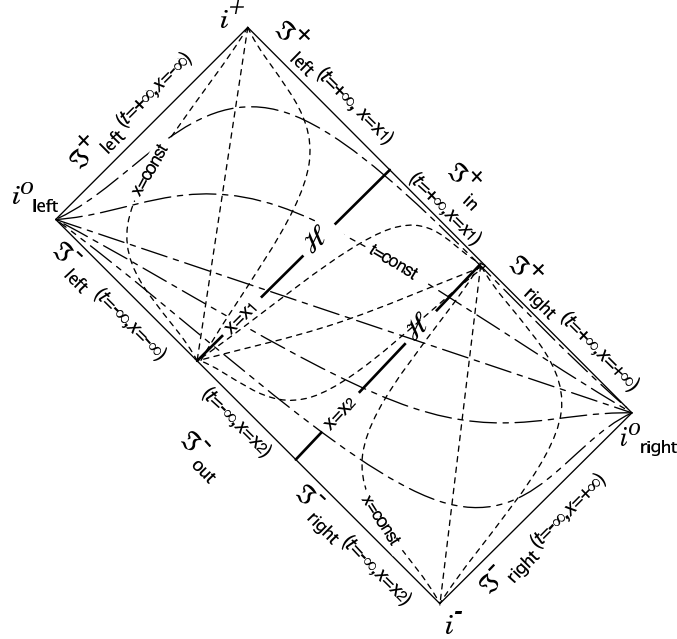
Figure 9: Conformal diagram of an acoustic black-hole–white-hole pair. Note the complete absence of singularities.

position is simply related to velocity via "time of flight".) Then, defining a new radial coordinate as $r_b = r/b$ the metric can be expressed as

$$\mathrm{d}s^2 = \frac{\rho}{c_\mathrm{s}} \left[-c_\mathrm{s}^2 \, \mathrm{d}t^2 + b^2 (\mathrm{d}r_b^2 + r_b^2 \, \mathrm{d}\Omega_2^2) \right]. \tag{96}$$

Introducing a Hubble-like parameter,

$$H_b(t) = \frac{\dot{b}(t)}{b(t)}, \tag{97}$$

the equation of continuity can be written as

$$\dot{\rho} + 3H_b(t)\,\rho = 0; \qquad \Rightarrow \qquad \rho(t) = \frac{\rho_0}{b^3(t)}, \tag{98}$$

with ρ_0 constant. Finally, we arrive at the metric of a spatially-flat FLRW geometry

$$\mathrm{d}s^2 = -T^2(t)\,\mathrm{d}t^2 + a_s^2(t)\,(\mathrm{d}r_b^2 + r_b^2\,\mathrm{d}\Omega_2^2), \tag{99}$$

with

$$T(t) \equiv \sqrt{\rho\,c_\mathrm{s}}; \qquad a_s(t) \equiv \sqrt{\frac{\rho}{c_\mathrm{s}}}\,b. \tag{100}$$

The proper Friedmann time, τ, is related to the laboratory time, t, by

$$\tau = \int T(t)\,\mathrm{d}t. \tag{101}$$

Then,

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a_s^2(\tau)\,(\mathrm{d}r_b^2 + r_b^2\,\mathrm{d}\Omega_2^2). \tag{102}$$

The "physical" Hubble parameter is

$$H = \frac{1}{a_s}\,\frac{da_s}{d\tau}. \tag{103}$$

If one now wishes to specifically mimic de Sitter expansion, then we would make

$$a_s(\tau) = a_0\,\exp(H_0\tau). \tag{104}$$

Whether or not this can be arranged (in this explosive model with comoving coordinates) depends on the specific equation of state (which is implicitly hidden in $c_s(t)$) and the dynamics of the explosion (encoded in $b(t)$).

Two-dimensional explosion: By holding the trap constant in the z direction, and allowing the BEC to expand in a pancake in the x and y directions (now best relabeled as r and ϕ) one can in principle arrange

$$\mathrm{d}s^2 = \frac{\rho(x,t)}{c_s(x,t)}\left[-\{c_s^2(x,t) - v(r,t)^2\}\,\mathrm{d}t^2 - 2v(r,t)\mathrm{d}r\,\mathrm{d}t + \mathrm{d}r^2 + r^2\mathrm{d}\phi^2 + \mathrm{d}z^2\right]. \tag{105}$$

One-dimensional explosion: An alternative "explosive" route to FLRW cosmology is to take a long thin cigar-shaped BEC and let it expand along its axis, while keeping it trapped in the transverse directions [196, 194]. The relevant acoustic metric is now

$$\mathrm{d}s^2 = \frac{\rho(x,t)}{c_s(x,t)}\left[-\{c_s^2(x,t) - v(x,t)^2\}\,\mathrm{d}t^2 - 2v(x,t)\mathrm{d}x\,\mathrm{d}t + \mathrm{d}x^2 + \{\mathrm{d}y^2 + \mathrm{d}z^2\}\right]. \tag{106}$$

The virtue of this situation is that one is keeping the condensate under much better control and has a simpler dimensionally-reduced problem to analyze. (Note that the true physics is 3+1 dimensional, albeit squeezed along two directions, so the conformal factor multiplying the acoustic metric is that appropriate to 3+1 dimensions. See also Section 2.7.2.)

2.5.2 Varying the effective speed of light

The other avenue starts from a fluid at rest $v = 0$ with respect to the laboratory at all times:

$$\mathrm{d}s^2 = -\rho\,c_s\,\mathrm{d}t^2 + \frac{\rho}{c_s}\,\mathbf{dx}^2. \tag{107}$$

Now it is not difficult to imagine a situation in which ρ remains spatially and temporally constant, in a sufficiently large region of space, while the speed of sound decreases with time (e.g., we shall see that this can be made in analogue models based on Bose–Einstein condensates by changing with time the value of the scattering length [46, 47, 328, 676, 683, 677]). This again reproduces an expanding spatially-flat FLRW Universe. The proper Friedmann time, τ, is again related to the laboratory time, t, by

$$\tau = \int \sqrt{\rho\,c_s}\,\mathrm{d}t. \tag{108}$$

Then, defining $a_s(t) \equiv \sqrt{\rho/c_s}$, we have

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a_s^2(\tau)\,(\mathrm{d}r_b^2 + r_b^2\,\mathrm{d}\Omega_2^2). \tag{109}$$

If one now specifically wishes to specifically mimic de Sitter expansion then we would wish

$$a_s(\tau) = a_0 \, \exp(H_0 \tau). \tag{110}$$

Whether or not this can be arranged (now in this non-explosive model with tunable speed of sound) depends on the specific manner in which one tunes the speed of sound as a function of laboratory time.

2.6 Regaining geometric acoustics

Up to now, we have been developing general machinery to force acoustics into Lorentzian form. This can be justified either with a view to using fluid mechanics to teach us more about general relativity, or to using the techniques of Lorentzian geometry to teach us more about fluid mechanics.

For example, given the machinery developed so far, taking the short wavelength/high frequency limit to obtain geometrical acoustics is now easy. Sound rays (phonons) follow the *null geodesics* of the acoustic metric. Compare this to general relativity, where in the geometrical optics approximation, light rays (photons) follow *null geodesics* of the physical spacetime metric. Since null geodesics are insensitive to any overall conformal factor in the metric [445, 275, 670], one might as well simplify life by considering a modified conformally-related metric

$$h_{\mu\nu} \equiv \begin{bmatrix} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots \\ -v_0^i & \vdots & \delta^{ij} \end{bmatrix}. \tag{111}$$

This immediately implies that, in the geometric acoustics limit, sound propagation is insensitive to the density of the fluid. In this limit, acoustic propagation depends only on the local speed of sound and the velocity of the fluid. It is only for specifically wave-related properties that the density of the medium becomes important.

We can rephrase this in a language more familiar to the acoustics community by invoking the eikonal approximation. Express the linearised velocity potential, ϕ_1, in terms of an amplitude, a, and phase, φ, by $\phi_1 \sim a e^{i\varphi}$. Then, neglecting variations in the amplitude a, the wave equation reduces to the *eikonal equation*

$$h^{\mu\nu} \, \partial_\mu \varphi \, \partial_\nu \varphi = 0. \tag{112}$$

This eikonal equation is blatantly insensitive to any overall multiplicative prefactor (conformal factor). As a sanity check on the formalism, it is useful to re-derive some standard results. For example, let the null geodesic be parameterised by $x^\mu(t) \equiv (t; \mathbf{x}(t))$. Then the null condition implies

$$
\begin{aligned}
& h_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t} = 0 \\
\Longleftrightarrow \; & -(c^2 - v_0^2) - 2v_0^i \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^i}{\mathrm{d}t} = 0 \\
\Longleftrightarrow \; & \left\| \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{v}_0 \right\| = c.
\end{aligned}
\tag{113}
$$

Here the norm is taken in the flat physical metric. This has the obvious interpretation that the ray travels at the speed of sound, c, relative to the moving medium.

Furthermore, if the geometry is stationary (steady flow) one can do slightly better. Let $x^\mu(s) \equiv (t(s); \mathbf{x}(s))$ be some null path from $\mathbf{x}_1$ to $\mathbf{x}_2$, parameterised in terms of physical arc length (i.e.,

$||d\mathbf{x}/ds|| \equiv 1$). Then the tangent vector to the path is

$$\frac{dx^\mu}{ds} = \left(\frac{dt}{ds}; \frac{dx^i}{ds}\right). \tag{114}$$

The condition for the path to be null (though not yet necessarily a null geodesic) is

$$g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \tag{115}$$

Using the explicit algebraic form for the metric, this can be expanded to show

$$-(c^2 - v_0^2)\left(\frac{dt}{ds}\right)^2 - 2v_0^i\left(\frac{dx^i}{ds}\right)\left(\frac{dt}{ds}\right) + 1 = 0. \tag{116}$$

Solving this quadratic

$$\left(\frac{dt}{ds}\right) = \frac{-v_0^i\left(\frac{dx^i}{ds}\right) + \sqrt{c^2 - v_0^2 + \left(v_0^i\,\frac{dx^i}{ds}\right)^2}}{c^2 - v_0^2}. \tag{117}$$

Therefore, the total time taken to traverse the path is

$$T[\gamma] = \int_{\mathbf{x}_1}^{\mathbf{x}_2}(dt/ds)\,ds = \int_\gamma \frac{1}{c^2 - v_0^2}\left\{\sqrt{(c^2 - v_0^2)ds^2 + (v_0^i\,dx^i)^2} - v_0^i\,dx^i\right\}. \tag{118}$$

If we now recall that extremising the total time taken is Fermat's principle for sound rays, we see that we have checked the formalism for stationary geometries (steady flow) by reproducing the discussion of Landau and Lifshitz [372, p. 262].[13]

2.7 Generalizing the physical model

There are a large number of ways in which the present particularly-simple analogue model can be generalised. Obvious issues within the current physical framework are:

- Adding external forces.

- Working in truly (1+1) or (2+1) dimensional systems.

- Adding vorticity, to go beyond the irrotational constraint.

Beyond these immediate questions, we could also seek similar effects in other physical or mathematical frameworks.

2.7.1 External forces

Adding external forces is (relatively) easy, an early discussion can be found in [624] and more details are available in [641]. The key point is that with an external force one can to some extent shape the background flow (see for example the discussion in [249]). However, upon linearization, the fluctuations are insensitive to any external force.

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[13] Mathematically, one can view the time taken to traverse such a path as a particular instance of Finsler distance – it is, in fact, the distance function associated with a Randers metric. See [246], and brief discussion in Section 3.2.10.

2.7.2 The role of dimension

The role of spacetime dimension in these acoustic geometries is sometimes a bit surprising and potentially confusing. This is important because there is a real physical distinction, for instance, between truly (2+1)-dimensional systems and effectively (2+1)-dimensional systems in the form of (3+1)-dimensional systems with cylindrical (pancake-like) symmetry. Similarly, there is a real physical distinction between a truly (1+1)-dimensional system and a (3+1)-dimensional system with transverse (cigar-like) symmetry. We emphasise that in Cartesian coordinates the wave equation

$$\frac{\partial}{\partial x^\mu}\left(f^{\mu\nu}\frac{\partial}{\partial x^\nu}\,\phi\right) = 0, \tag{119}$$

where

$$f^{\mu\nu} = \left[\begin{array}{c|c} -\rho/c^2 & -\rho\,v^j/c^2 \\ \hline -\rho\,v^i/c^2 & \rho\,\{\delta^{ij} - v^i v^j/c^2\} \end{array}\right], \tag{120}$$

holds *independent* of the dimensionality of spacetime. It depends only on the Euler equation, the continuity equation, a barotropic equation of state, and the assumption of irrotational flow [607, 622, 626, 624].

Introducing the inverse acoustic metric $g^{\mu\nu}$, defined by

$$f^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}; \qquad g = \frac{1}{\det(g^{\mu\nu})}, \tag{121}$$

the wave Equation (119) corresponds to the d'Alembertian wave equation in a curved space-time with contravariant metric tensor:

$$g^{\mu\nu} = \left(\frac{\rho}{c}\right)^{-2/(d-1)}\left[\begin{array}{c|c} -1/c^2 & -\mathbf{v}^T/c^2 \\ \hline -\mathbf{v}/c^2 & \mathbf{I}_{d\times d} - \mathbf{v}\otimes\mathbf{v}^T/c^2 \end{array}\right], \tag{122}$$

where d is the dimension of *space* (not spacetime). The covariant acoustic metric is then

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right)^{2/(d-1)}\left[\begin{array}{c|c} -\left(c^2 - v^2\right) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I}_{d\times d} \end{array}\right]. \tag{123}$$

d = 3: The acoustic line element for three space and one time dimension reads

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right)\left[\begin{array}{c|c} -\left(c^2 - v^2\right) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I}_{3\times 3} \end{array}\right]. \tag{124}$$

d = 2: The acoustic line element for two space and one time dimension reads

$$g_{\mu\nu} = \left(\frac{\rho}{c}\right)^2\left[\begin{array}{c|c} -\left(c^2 - v^2\right) & -\mathbf{v}^T \\ \hline -\mathbf{v} & \mathbf{I}_{2\times 2} \end{array}\right]. \tag{125}$$

This situation would be appropriate, for instance, when dealing with surface waves or excitations confined to a particular substrate.

d = 1: The naive form of the acoustic metric in (1+1) dimensions is ill-defined, because the conformal factor is raised to a formally infinite power. This is a side effect of the well-known conformal invariance of the Laplacian in 2 dimensions. The wave equation in terms of the densitised inverse metric $f^{\mu\nu}$ continues to make good sense; it is only the step from $f^{\mu\nu}$ to the effective metric that breaks down. Acoustics in intrinsically (1+1) dimensional systems does not reproduce the conformally-invariant wave equation in (1+1) dimensions.

Note that this issue only presents a difficulty for physical systems that are *intrinsically* one-dimensional. A three-dimensional system with plane symmetry, or a two-dimensional system with line symmetry, provides a perfectly well-behaved model for (1+1) dimensions, as in the cases $d = 3$ and $d = 2$ above.

2.7.3   Adding vorticity

For the preceding analysis to hold, it is necessary and sufficient that the flow locally be vorticity free, $\nabla \times \mathbf{v} = 0$, so that velocity potentials exist on an atlas of open patches. Note that the irrotational condition is automatically satisfied for the superfluid component of physical superfluids. (This point has been emphasised by Comer [143], who has also pointed out that in superfluids there will be multiple acoustic metrics – and multiple acoustic horizons – corresponding to first and second sound.) Even for normal fluids, vorticity-free flows are common, especially in situations of high symmetry. Furthermore, the previous condition enables us to handle vortex filaments, where the vorticity is concentrated into a thin vortex core, provided we do not attempt to probe the vortex core itself. It is not necessary for the velocity potential ϕ to be globally defined.

Though physically important, dealing with situations of distributed vorticity is much more difficult, and the relevant wave equation is more complicated in that the velocity scalar is now insufficient to completely characterise the fluid flow.[14] An approach similar to the spirit of the present discussion, but in terms of Clebsch potentials, can be found in [502]. The eikonal approximation (geometrical acoustics) leads to the same conformal class of metrics previously discussed, but in the realm of physical acoustics the wave equation is considerably more complicated than a simple d'Alembertian. (Roughly speaking, the vorticity becomes a source for the d'Alembertian, while the vorticity evolves in response to gradients in a generalised scalar potential. This seems to take us outside the realm of models of direct interest to the general relativity community.)[15]

2.8   Simple Lagrangian meta-model

As a first (and rather broad) example of the very abstract ways in which the notion of an acoustic metric can be generalised, we start from the simple observation that irrotational barotropic fluid mechanics can be described by a Lagrangian, and ask if we can extend the notion of an acoustic metric to all (or at least some wide class of) Lagrangian systems?

Indeed, suppose we have a single scalar field ϕ whose dynamics is governed by some generic Lagrangian $\mathcal{L}(\partial_\mu \phi, \phi)$, which is some arbitrary function of the field and its first derivatives (here we will follow the notation and ideas of [44]). In the general analysis that follows the previous irrotational and inviscid fluid system is included as a particular case; the dynamics of the scalar field ϕ is now much more general. We want to consider linearised fluctuations around some background solution $\phi_0(t, \mathbf{x})$ of the equations of motion, and to this end we write

$$\phi(t, \mathbf{x}) = \phi_0(t, \mathbf{x}) + \epsilon \phi_1(t, \mathbf{x}) + \frac{\epsilon^2}{2} \phi_2(t, \mathbf{x}) + O(\epsilon^3). \tag{126}$$

---

[14] Vorticity is automatically generated, for instance, whenever the background fluid is non-barotropic, and, in particular, when $\nabla \rho \times \nabla p \neq 0$. Furthermore, it has been argued in [559] that quantum backreaction can also act as a source for vorticity.

[15] In [233, 234, 235, 236, 237, 238, 239, 240] the author has attempted to argue that vorticity can be related to the concept of torsion in a general affine connexion. We disagree. Although deriving a wave equation in the presence of vorticity very definitely moves one beyond the realm of a simple Riemannian spacetime, adding torsion to the connexion is not sufficient to capture the relevant physics.

Now use this to expand the Lagrangian around the classical solution $\phi_0(t, \mathbf{x})$:

$$
\begin{aligned}
\mathcal{L}(\partial_\mu \phi, \phi) = {} & \mathcal{L}(\partial_\mu \phi_0, \phi_0) + \epsilon \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \, \partial_\mu \phi_1 + \frac{\partial \mathcal{L}}{\partial \phi} \, \phi_1 \right] \\
& + \frac{\epsilon^2}{2} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \, \partial_\mu \phi_2 + \frac{\partial \mathcal{L}}{\partial \phi} \, \phi_2 \right] \\
& + \frac{\epsilon^2}{2} \left[\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial(\partial_\nu \phi)} \, \partial_\mu \phi_1 \, \partial_\nu \phi_1 + 2 \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial \phi} \, \partial_\mu \phi_1 \, \phi_1 + \frac{\partial^2 \mathcal{L}}{\partial \phi \, \partial \phi} \, \phi_1 \, \phi_1 \right] + O(\epsilon^3).
\end{aligned}
\tag{127}
$$

It is particularly useful to consider the action

$$
S[\phi] = \int \mathrm{d}^{d+1}x \, \mathcal{L}(\partial_\mu \phi, \phi),
\tag{128}
$$

since doing so allows us to integrate by parts. (Note that the Lagrangian $\mathcal{L}$ is taken to be a scalar density, not a true scalar.) We can now use the Euler–Lagrange equations for the background field

$$
\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0,
\tag{129}
$$

to discard the linear terms (remember we are linearizing around a solution of the equations of motion) and so we get

$$
\begin{aligned}
S[\phi] = S[\phi_0] + \frac{\epsilon^2}{2} \int \mathrm{d}^{d+1}x \Bigg[& \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial(\partial_\nu \phi)} \right\} \, \partial_\mu \phi_1 \, \partial_\nu \phi_1 \\
& + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi \, \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial \phi} \right\} \right) \, \phi_1 \, \phi_1 \Bigg] + O(\epsilon^3).
\end{aligned}
\tag{130}
$$

Having set things up this way, the equation of motion for the linearised fluctuation is now easily read off as

$$
\partial_\mu \left(\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial(\partial_\nu \phi)} \right\} \partial_\nu \phi_1 \right) - \left(\frac{\partial^2 \mathcal{L}}{\partial \phi \, \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial \phi} \right\} \right) \phi_1 = 0.
\tag{131}
$$

This is a second-order differential equation with position-dependent coefficients (these coefficients all being implicit functions of the background field ϕ_0).

This can be given a nice clean geometrical interpretation in terms of a d'Alembertian wave equation – provided we *define* the effective spacetime metric by

$$
\sqrt{-g} \, g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial(\partial_\nu \phi)} \right\} \Bigg|_{\phi_0}.
\tag{132}
$$

Note that this is another example of a situation in which calculating the inverse metric density is easier than calculating the metric itself.

Suppressing the ϕ_0 except when necessary for clarity, this implies [in (d+1) dimensions, d space dimensions plus 1 time dimension]

$$
(-g)^{(d-1)/2} = - \det \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi) \, \partial(\partial_\nu \phi)} \right\}.
\tag{133}
$$

Therefore,

$$g^{\mu\nu}(\phi_0) = \left(-\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right)^{-1/(d-1)}\Bigg|_{\phi_0}\;\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\Bigg|_{\phi_0}. \tag{134}$$

And, taking the inverse,

$$g_{\mu\nu}(\phi_0) = \left(-\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right)^{1/(d-1)}\Bigg|_{\phi_0}\;\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}^{-1}\Bigg|_{\phi_0}. \tag{135}$$

We can now write the equation of motion for the linearised fluctuations in the geometrical form

$$[\Delta(g(\phi_0)) - V(\phi_0)]\,\phi_1 = 0, \tag{136}$$

where Δ is the d'Alembertian operator associated with the effective metric $g(\phi_0)$, and $V(\phi_0)$ is the background-field-dependent (and so, in general, position-dependent) "mass term":

$$V(\phi_0) = \frac{1}{\sqrt{-g}}\;\left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\right\}\right) \tag{137}$$

That is,

$$V(\phi_0) = \left(-\det\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial(\partial_\nu\phi)}\right\}\right)^{-1/(d-1)} \times \left(\frac{\partial^2\mathcal{L}}{\partial\phi\,\partial\phi} - \partial_\mu\left\{\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\,\partial\phi}\right\}\right). \tag{138}$$

Thus, $V(\phi_0)$ is a true scalar (not a density). Note that the differential Equation (136) is automatically formally self-adjoint (with respect to the measure $\sqrt{-g}\,\mathrm{d}^{d+1}x$).

It is important to realise just how general the result is (and where the limitations are): It works for *any* Lagrangian depending only on a single scalar field and its first derivatives. The linearised PDE will be *hyperbolic* (and so the linearised equations will have wave-like solutions) if and only if the effective metric $g_{\mu\nu}$ has Lorentzian signature $\pm[-, +^d]$. Observe that if the Lagrangian contains nontrivial second derivatives you should not be too surprised to see terms beyond the d'Alembertian showing up in the linearised equations of motion.

As a specific example of the appearance of effective metrics due to Lagrangian dynamics we reiterate the fact that inviscid irrotational barotropic hydrodynamics naturally falls into this scheme (which is why, with hindsight, the derivation of the acoustic metric presented earlier in this review was so relatively straightforward). In inviscid irrotational barotropic hydrodynamics the lack of viscosity (dissipation) guarantees the existence of a Lagrangian; which a priori could depend on several fields. Since the flow is irrotational $\mathbf{v} = -\boldsymbol{\nabla}\phi$ is a function only of the velocity potential, and the Lagrangian is a function only of this potential and the density. Finally, the equation of state can be used to eliminate the density leading to a Lagrangian that is a function only of the single field ϕ and its derivatives [44].

2.9   Going further

The class of analogue models based on fluid mechanics is now quite large and the literature is extensive. Most of the relevant detailed discussion will be deferred until subsequent sections, so for the time being we shall just mention reasonably immediate generalizations such as:

- Working with specific fluids.
    - Superfluids.
    - Bose–Einstein condensates.
- Abstract generalizations.
    - Normal modes in generic systems.
    - Multiple signal speeds.

We next turn to a brief historical discussion, seeking to place the work of the last two decades into its proper historical perspective.

3 History

From the point of view of the general relativity community the history of analogue models can reasonably neatly (but superficially) be divided into an "historical" period (essentially pre-1981) and a "modern" period (essentially post-1981). We shall in the next Section 3.1 focus more precisely on the early history of analogue models, and specifically those that seem to us to have had a direct historical connection with the sustained burst of work carried out in the last 20 years.

3.1 Historical period

Of course, the division into pre-1981 and post-1981 articles is at a deeper level somewhat deceptive. There have been several analogue models investigated over the years, with different aims, different levels of sophistication, and ultimately different levels of development. Armed with a good library and some hindsight it is possible to find interesting analogues in a number of places.[16]

3.1.1 Optics – the Gordon metric

Perhaps the first paper to seriously discuss analogue models and effective metric techniques was that of Gordon (yes, he of the Klein–Gordon equation) [258]. Note that Gordon seemed largely interested in trying to describe dielectric media by an "effective metric". That is: Gordon wanted to use a gravitational field to mimic a dielectric medium. What is now often referred to as the Gordon metric is the expression

$$[g_{\text{effective}}]_{\mu\nu} = \eta_{\mu\nu} + \left[1 - n^{-2}\right] V_\mu V_\nu, \tag{139}$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric, n is the position-dependent refractive index, and V_μ is the (constant) 4-velocity of the medium.

After that, there was sporadic interest in effective metric techniques. An historically-important contribution was one of the *problems* in the well-known book "The Classical Theory of Fields" by Landau and Lifshitz [373]. See the end of Chapter 10, Paragraph 90, and the problem immediately thereafter: "Equations of electrodynamics in the presence of a gravitational field". Note that in contrast to Gordon, here the interest is in using dielectric media to mimic a gravitational field.

In France the idea was taken up by Pham Mau Quan [503], who showed that (under certain conditions) Maxwell's equations can be expressed directly in terms of the effective metric specified by the coefficients

$$[g_{\text{effective}}]_{\mu\nu} = g_{\mu\nu} + \left[1 - \frac{1}{\epsilon\mu}\right] V_\mu V_\nu, \tag{140}$$

where $g_{\mu\nu}$ is the ordinary spacetime metric, ϵ and μ are the permeability and permittivity, and V_μ is the 4-velocity of the medium. The trajectories of the electromagnetic rays are interpreted in this case as geodesics of null length of this new effective metric.

Three articles that directly used the dielectric analogy to analyse specific physics problems are those of Skrotskii [576], Balazs [18], and Winterberg [689]. The general formalism was more fully developed in articles such as those by Plebański [511, 510], and a good summary of this classical period can be found in the article by de Felice [168]. In summary and with the benefit of hindsight:

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[16] Indeed, historically, though not of direct relevance to general relativity, analogue models played a key role in the development of electromagnetism – Maxwell's derivation of his equations for the electromagnetic field was guided by a rather complicated "analogue model" in terms of spinning vortices of aether. Of course, once you have the equations in hand you can treat them in their own right and forget the model that guided you – which is exactly what happened in this particular case.

An arbitrary gravitational field can always be represented as an equivalent optical medium, but subject to the somewhat unphysical restriction that

$$[\text{magnetic permitivity}] \propto [\text{electric permeability}]. \tag{141}$$

If an optical medium does not satisfy this constraint (with a position independent proportionality constant) then it is not completely equivalent to a gravitational field. For a position dependent proportionality constant complete equivalence can be established in the geometric optics limit, but for wave optics the equivalence is not complete.

Subsequently, Anderson and Spiegel [7] extended and modified the notion of the Gordon metric to allow the medium to be a flowing fluid – so the 4-vector V^a is no longer assumed to be a constant. In hindsight, using modern notation and working in full generality, the Gordon metric can be justified as follows: Consider an arbitrary curved spacetime with physical metric g_{ab} containing a fluid of 4-velocity V^a and refractive index n. Pick a point in the manifold and adopt Gaussian normal coordinates around that point so that $g_{ab} \to \eta_{ab}$. Now perform a Lorentz transformation to go to to a local inertial frame co-moving with the fluid, so that $V^a \to (1; \vec{0})$ and one is now in Gaussian normal coordinates co-moving with the fluid. In this coordinate patch, the light rays, by definition of the refractive index, locally propagate along the light cones

$$-\frac{1}{n^2} \, \mathrm{d}t^2 + ||\mathrm{d}\vec{x}||^2 = 0, \tag{142}$$

implying in these special coordinates the existence of an optical metric

$$\mathcal{G}_{\mu\nu} \propto \begin{bmatrix} -1/n^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{143}$$

That is, transforming back to arbitrary coordinates:

$$\mathcal{G}_{\mu\nu} \propto -\frac{1}{n^2} V_\mu V_\nu + \{g_{\mu\nu} + V_\mu V_\nu\} \propto g_{\mu\nu} + \left\{1 - \frac{1}{n^2}\right\} V_\mu V_\nu. \tag{144}$$

Note that in the ray optics limit, because one only has the light cones to work with, one can neither derive nor is it even meaningful to specify the overall conformal factor.

3.1.2 Acoustics

After the pioneering hydrodynamical paper by White in 1973 [687], which studied acoustic ray tracing in non-relativistic moving fluids, there were several papers in the 1980s using an acoustic analogy to investigate the propagation of shockwaves in astrophysical situations, most notably those of Moncrief [448] and Matarrese [433, 434, 432]. In particular, in Moncrief's work [448] the linear perturbations of a relativistic perfect fluid on a Schwarzschild background were studied, and it was shown that the wave equation for such perturbations can be expressed as a relativistic wave equation on some effective (acoustic) metric (which can admit acoustic horizons). In this sense [448] can be seen as a precursor to the later works on acoustic geometries and acoustic horizons. Indeed, because they additionally permit a general relativistic Schwarzschild background, the results of Moncrief [448] are, in some sense, more general than those considered in the mainstream acoustic gravity papers that followed.

In spite of these impressive results, we consider these papers to be part of the "historical period" for the main reason that such works are philosophically orthogonal to modern developments in analogue gravity. Indeed the main motivation for such works was the study of perfect fluid

dynamics in accretion flows around black holes, or in cosmological expansion, and in this context the description via an acoustic effective background was just a tool in order to derive results concerning conservation laws and stability. This is probably why, even if temporally, [448] pre-dates Unruh's 1981 paper by one year, and while [433, 434, 432] post-date Unruh's 1981 paper by a few years, there seems to have not been any cross-connection.

3.1.3 Surface waves

Somewhat ironically, 1983 marked the appearance of some purely experimental results on surface waves in water obtained by Badulin et al. [17]. At the time these results passed unremarked by the relativity community, but they are now of increasing interest, and are seen to be precursors of the theoretical work reported in [560, 531] and the modern experimental work reported in [532, 682].

3.2 Modern period

3.2.1 The years 1981–1999

The key event in the "modern" period (though largely unrecognised at the time) was the 1981 publication of Unruh's paper "Experimental black hole evaporation" [607], which implemented an analogue model based on fluid flow, and then used the power of that analogy to probe fundamental issues regarding Hawking radiation from "real" general-relativistic black holes.

We believe that Unruh's 1981 article represents the first observation of the now widely established fact that Hawking radiation has nothing to do with general relativity *per se*, but that Hawking radiation is instead a fundamental curved-space quantum field theory phenomenon that occurs whenever a horizon is present in an effective geometry.[17] Though Unruh's 1981 paper was seminal in this regard, it lay largely unnoticed for many years.

Some 10 years later Jacobson's article "Black-hole evaporation and ultra-short distances" [307] used Unruh's analogy to build a physical model for the "trans-Planckian modes" believed to be relevant to the Hawking radiation process. Progress then sped up with the relatively rapid appearance of [308] and [608]. (This period also saw the independent rediscovery of the fluid analogue model by one of the present authors [622], and the first explicit consideration of superfluids in this regard [143].)

The later 1990s then saw continued work by Jacobson and his group [309, 310, 148, 146, 150, 322], with new and rather different contributions coming in the form of the solid-state models considered by Reznik [523, 522]. [285] is an attempt at connecting Hawking evaporation with the physics of collapsing bubbles. This was part of a more general programme aimed at connecting black-hole thermodynamics with perfect-fluid thermodynamics [286]. This period also saw the introduction of the more general class of superfluid models considered by Volovik and his collaborators [644, 645, 359, 183, 649, 647, 648, 326, 651, 652], more precise formulations of the notions of horizon, ergosphere, and surface gravity in analogue models [624, 626], and discussions of the implications of analogue models regarding Bekenstein–Hawking entropy [625, 626]. Optical models were considered in [389]. Finally, analogue spacetimes based on *special* relativistic acoustics were considered in [72].

By the year 2000, articles on one or another aspect of analogue gravity were appearing at the rate of over 20 per year, and it becomes impractical to summarise more than a selection of them.

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[17] We emphasise: To get Hawking radiation you need an effective geometry, a horizon, and a suitable quantum field theory on that geometry.

3.2.2 The year 2000

Key developments in 2000 were the introduction, by Garay and collaborators, of the use of Bose–Einstein condensates as a working fluid [231, 232], and the extension of those ideas by the present authors [43]. Further afield, the trans-Planckian problem also reared its head in the context of cosmological inflation, and analogue model ideas previously applied to Hawking radiation were reused in that context [341, 457].

That year also marked the appearance of a review article on superfluid analogues [655], more work on "near-horizon" physics [210], and the transference of the idea of analogue-inspired "multiple metric" theories into cosmology, where they can be used as the basis for a precise definition of what is meant by a VSL ("variable speed of light") cosmology [58]. Models based on nonlinear electrodynamics were investigated in [26], ^{3}He-A based models were reconsidered in [316, 653], and "slow light" models in quantum dielectrics were considered in [390, 391, 382]. The most radical proposal to appear in 2000 was that of Laughlin et al. [131]. Based on taking a superfluid analogy rather literally, they mooted an actual physical breakdown of general relativity at the horizon of a black hole [131].

Additionally, the workshop on "Analogue models of general relativity", held at CBPF (Rio de Janeiro) gathered some 20 international participants and greatly stimulated the field, leading ultimately to the publication of a book [470] in 2002.

3.2.3 The year 2001

This year saw more applications of analogue-inspired ideas to cosmological inflation [179, 441, 440, 343, 459], to neutron star cores [116], and to the cosmological constant [656, 658].

Closer to the heart of the analogue programme were the development of a "normal mode" analysis in [44, 45, 637], the development of dielectric analogues in [557], speculations regarding the possibly emergent nature of Einstein gravity [50, 637], and further developments regarding the use of ^{3}He-A [178] as an analogue for electromagnetism. Experimental proposals were considered in [48, 637, 539]. Vorticity was discussed in [502], and the use of BECs as a model for the breakdown of Lorentz invariance in [636]. Analogue models based on nonlinear electrodynamics were discussed in [169]. Acoustics in an irrotational vortex were investigated in [207].

The excitation spectrum in superfluids, specifically the fermion zero modes, were investigated in [654, 303], while the relationship between rotational friction in superfluids and super-radiance in rotating spacetimes was discussed in [104]. More work on "slow light" appeared in [91]. The possible role of Lorentz violations at ultra-high energy was emphasised in [312].

3.2.4 The year 2002

"What did we learn from studying acoustic black holes?" was the title and theme of Parentani's article in 2002 [492], while Schützhold and Unruh developed a rather different fluid-based analogy based on gravity waves in shallow water [560, 560]. Super-radiance was investigated in [57], while the propagation of phonons and quasiparticles was discussed in [209, 208]. More work on "slow light" appeared in [211, 509]. Applications to inflationary cosmology were developed in [460], while analogue spacetimes relevant to braneworld cosmologies were considered in [28].

The stability of an acoustic white hole was investigated in [386], while further developments regarding analogue models based on nonlinear electrodynamics were presented by Novello and collaborators in [170, 171, 468, 464, 214]. Though analogue models lead naturally to the idea of high-energy violations of Lorentz invariance, it must be stressed that definite observational evidence for violations of Lorentz invariance is lacking – in fact, there are rather strong constraints on how strong any possible Lorentz violating effect might be [318, 317].

3.2.5 The year 2003

This year saw further discussion of analogue-inspired models for black-hole entropy and the cosmological constant [661, 668], and the development of analogue models for FRW geometries [195, 194, 46, 177, 403]. There were several further developments regarding the foundations of BEC-based models in [47, 196], while analogue spacetimes in superfluid neutron stars were further investigated in [117].

Effective geometry was the theme in [466], while applications of nonlinear electrodynamics (and its effective metric) to cosmology were presented in [467]. Super-radiance was further investigated in [56, 54], while the limitations of the "slow light" analogue were explained in [612]. Vachaspati argued for an analogy between phase boundaries and acoustic horizons in [615]. Emergent relativity was again addressed in [378]. The review article by Burgess [98] emphasised the role of general relativity as an effective field theory – the *sine qua non* for any attempt at interpreting general relativity as an emergent theory. The lecture notes by Jacobson [313] give a nice introduction to Hawking radiation and its connection to analogue spacetimes.

3.2.6 The year 2004

The year 2004 saw the appearance of some 30 articles on (or closely related to) analogue models. Effective geometries in astrophysics were discussed by Perez Bergliaffa [501], while the physical realizability of acoustic Hawking radiation was addressed in [159, 616]. More cosmological issues were raised in [616, 675], while a specifically astrophysical use of the acoustic analogy was invoked in [160, 161, 162]. BEC-based horizons were again considered in [249, 247], while backreaction effects were the focus of attention in [25, 23, 344]. More issues relating to the simulation of FRW cosmologies were raised in [204, 206].

Unruh and Schützhold discussed the universality of the Hawking effect [613], and a new proposal for possibly detecting Hawking radiation in an electromagnetic wave guide [562]. The causal structure of analogue spacetimes was considered in [37], while quasinormal modes attracted attention in [69, 392, 112, 452]. Two dimensional analogue models were considered in [101].

There were attempts at modeling the Kerr geometry [641], and generic "rotating" spacetimes [132], a proposal for using analogue models to generate massive phonon modes in BECs [640, 642], and an extension of the usual formalism for representing weak-field gravitational lensing in terms of an analogue refractive index [81]. Finally, we mention the development of yet more strong observational bounds on possible ultra-high-energy Lorentz violation [319, 320].

3.2.7 The year 2005

The year 2005 saw continued and vigorous activity on the analogue model front. More studies of the super-resonance phenomenon appeared [55, 193, 347, 578], and a mini-survey was presented in [111]. Quasinormal modes again received attention in [135], while the Magnus force was reanalysed in terms of the acoustic geometry in [699]. Singularities in the acoustic geometry are considered in [102], while backreaction has received more attention in [559].

The original version of this Living Review appeared in May of 2005 [49], and since then activity has, if anything, increased. Work on BEC-related models included [401, 402, 678, 642, 679, 605, 205], while additional work on super-radiance [136], the background fluid flow [103], and quasinormal modes [537] also appeared. Dynamical phase transitions were considered in [551], and astrophysical applications to accretion flow in [2, 165]. The connection between white hole horizon and the classical notion of a "hydraulic jump" was explored in [662] and in [572]. A "spacetime condensate" point of view was advocated in [299], and analogue applications to "quantum teleportation" were considered in [243]. A nice survey of analogue ideas and backreaction effects was presented in [21] (and related articles [23, 25]). Finally, we mention the appearance in 2005 of

another Living Review, one that summarises and systematises the very stringent bounds that have been developed on possible ultra-high-energy Lorentz violation [435].

3.2.8   The year 2006

A key article, which appeared in 2006, involved the "inverse" use of the acoustic metric to help understand hydrodynamic fluid flow in quark-gluon plasma [123]. The relationship between modified dispersion relations and Finsler spacetimes was discussed in [251]. Backreaction effects were again considered in [20].

Using analogue ideas as backdrop, Markopoulou developed a pre-geometric model for quantum gravity in [421]. Analogue implications vis-a-vis entanglement entropy were discussed in [226, 225]. A microscopic analysis of the micro-theory underlying acoustic Hawking radiation in a "piston" geometry appeared in [248]. Volovik extended and explained his views on quantum hydrodynamics as a model for quantum gravity in [664]. Applications to the cosmological constant were considered in [543]. More BEC-related developments appeared in [680, 38, 39, 29, 24, 188, 189]. An analogue model based on a suspended "shoestring" was explored in [282]. Super-resonance was again discussed in [352]. More analogue-inspired work on black-hole accretion appeared in [164], while "ripplons" (quantised surface capillary waves) were discussed in [663]. Modified dispersion relations again attracted attention [250], and analogue inspired ideas concerning constrained systems were explored in [357]. Quasinormal frequencies were considered in [134]. Finally, we emphasise particularly the realization that the occurrence of Hawking-like radiation does not require the presence of an event horizon or even a trapped region [38, 39].

3.2.9   The year 2007

Emergent geometry [16, 14, 15, 618] was an important theme in 2007, as were efforts at moving beyond the semiclassical description [491]. BEC-based analogue models were adapted to investigating "signature change events" in [684]. Acoustic cross-sections were considered in [156]. "Rimfall" was discussed in [609].

Analogue-inspired ideas were adapted to the study of gravitational collapse in [40], while the importance of nonlocal correlations in the Hawking flux was emphasized in [22]. Quantum field theoretic anomalies were considered in [345], while entanglement entropy was investigated in [324]. The specific shape of the de Laval nozzle needed to acoustically reproduce linearised perturbations of the Schwarzschild geometry was discussed in [1], and quasinormal frequencies were investigated in [692]. Superradiance and disclinations were considered in [167]. Theoretical aspects of the circular hydraulic jump were investigated in [519].

The use of analogue spacetimes as "toy models" for quantum gravity was emphasized in [643, 632]. Within the optics community Philbin, Leonhardt, and co-workers initiated the study of "fibre-optic black holes" [505, 504]. Within the fluid dynamics community, wave-tank experiments were initiated [532] by Rousseaux and co-workers, who demonstrated the presence of negative phase-velocity waves. Dissipation-induced breakdown of Lorentz invariance was considered by Parentani in [494, 495], and BEC-based models continued to attract attention [676, 30, 31, 333]. While analyzing quark matter, acoustic metrics were found to be useful in [419] (see also [418]).

Analogues based on ion traps were considered by Schützhold in [558], while a toy model for backreaction was explored in [414]. Further afield, analogue models were used to motivate a "Abraham–Lorentz" interpretation of relativity in terms of a physically-real aether and physically-real Lorentz–FitzGerald contraction [36]. In a similar vein analogue models were used to motivate a counter-factual counter-historical approach to the Bohm versus Copenhagen interpretations of quantum physics [463]. Analogue-inspired ideas regarding the possible "localization" of the origin of the Hawking flux were investigated by Unruh in [610]. Additionally, an analogue inspired analysis

of accretion appeared in [163], while astrophysical constraints on modified dispersion relations were improved and extended in [409, 394].

3.2.10   The year 2008

This year saw the introduction of "quantum graphity" [358, 356], an analogue-inspired model for quantum gravity. The BEC theme continued to generate attention [368], in particular regarding cosmological particle production [677, 683], and Hawking radiation [119]. A mini-review appeared in [553]. The theme of "emergence" was also represented in articles such as [255, 252, 329]. Localization of the Hawking radiation was again addressed in [563], while sensitivity of the Hawking flux to the presence of superluminal dispersion was considered in [33]. Astrophysical constraints on modified dispersion relations were again considered in [408, 410], while applications to quark matter were investigated in [415, 417]. Possible applications to high-temperature superconductivity were reported in [444]. Quantum field theoretic anomalies in an acoustic geometry were considered in [60, 691], while (2+1) acoustic black-hole thermodynamics were investigated in [346]. Gravitational collapse was again discussed in [40].

Gibbons and co-workers used analogue-based ideas in their analysis of general stationary spacetimes, demonstrating that the spatial slices of stationary spacetimes are best thought of as a special class of Finsler spaces, in particular, Randers spaces [246]. Attempts at developing a generally useful notion of Finsler *spacetime* were discussed in [573, 574], with Finslerian applications to the Higgs mechanism being investigated in [569]. Signature change, now not in a BEC context, was again addressed in [686]. Black-hole lasers were considered in [388], and the fluid-gravity correspondence in [5]. Backreaction of the Hawking flux was again considered in [556], while the analogue physics of a "photon fluid" was considered in [420]. In [73] analogue ideas were applied to polytrope models of Newtonian stars, while super-radiance was considered in [598].

3.2.11   The year 2009

This year saw intriguing and unexpected relations develop between analogue spacetimes and Hořava gravity [584, 585, 634, 666, 570]. These connections seem primarily related to the way Hořava's projectability condition interleaves with the ADM decomposition of the metric, and to the manner in which Hořava's distinguished spacetime foliation interleaves with the preferred use of Painlevé–Gullstrand-like coordinates.

The theme of emergent gravity continues to play a role [300, 395, 571], as does the theme of nontrivial dispersion relations [413, 412, 411, 396, 437]. A variant of quantum graphity was further developed in [268], and a matrix model implementation of analogue spacetime was developed in [220]. Quasinormal modes were considered in [174, 700], with a survey appearing in [70]. Acoustic scattering was considered in [173]. Applications to quark matter were again investigated in [416]. In [340], the universe was interpreted as a "soap film". Nonlinear electrodynamics was again considered in [260]. A model based on liquid crystals appeared in [500]. Attempts at including backreaction in a cosmological fluid context were investigated in [451, 450]. Possible experimental implementations of acoustic black holes using circulating ion rings are discussed in [290], while ultra-short laser pulses are considered in [187]. Signature change events were again considered in [685], while analogue-inspired lessons regarding the fundamental nature of time were investigated in [330] and [254]. Non-canonical quantum fields were considered in [304].

In [217, 218] analogue ideas are implemented in an unusual direction: fluid dynamics is used to model aspects of quantum field theory. That the trans-Planckian and information loss problems are linked is argued in [397]. The BEC paradigm for acoustic geometry is again discussed in [331] and [520], while universal aspects of super-radiant scattering are considered in [524]. Most remarkably, a BEC-based black hole analogue was experimentally realised in [369]. (See Sections 6.2 and 7.13.)

3.2.12 The year 2010

This current year has already (September) seen some 50 articles appear that can legitimately claim to have either direct or tangential relationships to the analogue spacetime programme. Being necessarily very selective, we first mention work related to "entropic" attempts at understanding the "emergence" of general relativity and the spacetime "degrees of freedom" from the quantum regime [483, 482, 130, 590, 355]. The use of correlations as a potential experimental probe has been theoretically investigated in [19, 118, 190, 496, 512, 564, 604], while an analysis of optimality conditions for the detection of Hawking–Unruh radiation appeared in [13]. Entanglement issues were explored in [427]. More work on "emergent horizons" has appeared in [552, 554, 555]. Relativistic fluids have been revisited in [639], with specific applications to relativistic BECs being reported in [191]. Possible measurement protocols for Hawking radiation in ionic systems were discussed in [291]. In [34] analogue spacetimes were used to carefully separate the notion of "emergent manifold" from that of "emergent curvature". Quantum graphity was again considered in [110].

Step-function discontinuities in BECs were considered in [186, 438]. (Signature change can be viewed as an extreme case of step-function discontinuity [684, 685, 686].) Black holes induced by dielectric effects, and their associated Hawking radiation, were considered in [63]. The acoustic geometry of polytrope rotating Newtonian stars was considered in [74]. Random fluids were investigated in [365]. Finsler spacetime geometries were again considered in [575], while the relationship between analogue spacetimes and foundational mathematical relativity was discussed in [139]. Further afield, analogue spacetimes were used as an aid to understanding "warp drive" spacetimes [32].

Quantum sound in BECs was again investigated in [35], BEC-based particle creation in [367], and BEC-based black hole lasers in [199]. Optical effective geometries in Kerr media were discussed in [100]. 2+1 dimensional draining-bathtub geometries were probed in [477]. Theoretical and historical analyses of surface waves in a wave tank were presented in [531]. Finally, we mention the stunning experimental verification by Weinfurtner et al. of the existence of classical stimulated Hawking radiation in a wave flume [682], and the experimental detection of photons associated with a phase velocity horizon by Belgiorno et al. [66].

3.2.13 The future?

Interest in analogue models, analogue spacetimes, and analogue gravity is intense and shows no signs of abating. Interest in these ideas now extends far beyond the general relativity community, and there is significant promise for direct laboratory-based experimental input. We particularly wish to encourage the reader to keep an eye out for future developments regarding the possible experimental verification of the existence of Hawking radiation or the closely related Unruh radiation.

3.3 Going further

To further complicate the history, there is large body of work for which analogue spacetime ideas provide part of the background gestalt, even if the specific connection may sometimes be somewhat tenuous. Among such articles, we mention:

- Analogue-based "geometrical" interpretations of pseudo-momentum, Iordanskii forces, Magnus forces, and the acoustic Aharanov–Bohm effect [227, 592, 593, 594, 595, 650].

- An analogue-inspired interpretation of the Kerr spacetime [267].

- The use of analogies to clarify the Newtonian limit of general relativity [602], to provide heuristics for motivating interest in specific spacetimes [525, 631], and to discuss a simple interpretation of the notion of a horizon [473].

- Discrete [583, 281, 436] and non-commutative [140] spacetimes partially influenced and flavoured by analogue ideas.

- Analogue-based hints on how to implement "doubly special relativity" (DSR) [361, 362, 363, 599], and a cautionary analysis of why this might be difficult [561].

- Possible black-hole phase transitions placed in an analogue context [591].

- Attempts at deriving inertia and passive gravity, (though not active gravity), from analogue ideas [442].

- Applications of analogue ideas in braneworld [241] and Kaluza–Klein [242] settings.

- Analogue inspired views on the "mathematical universe" [332].

- Cosmological structure formation viewed as noise amplification [568].

- Modified inflationary scenarios [140, 142].

- Discussions of unusual topology, "acoustic wormholes", and unusual temporal structure [453, 455, 514, 581, 582, 701, 702].

- Magnon contributions to BEC physics [97, 665].

- Analogue models based on plasmon physics [581, 582].

- Abstract quantum field theoretic considerations of the Unruh effect [695].

- The interpretation of black-hole entropy in terms of universal "near horizon" behaviour of quantum fields living on spacetime [114, 115].

- Numerous suggestions regarding possible trans-Planckian physics [12, 59, 121, 128, 129, 284, 440, 528, 600].

- Numerous suggestions regarding a minimum length in quantum gravity [71, 78, 93, 144, 230, 293, 294, 295, 361, 362, 363, 405, 407, 406, 579].

- Standard quantum field theory physics reformulated in the light of analogue models [9, 10, 203, 215, 393, 398, 399, 400, 422, 471, 472, 486, 485, 487, 493, 515, 695].

- Standard general relativity supplemented with analogue viewpoints and insights [354, 376, 422, 456, 325, 475].

- The discussion of, and argument for, a possible reassessment of fundamental features of quantum physics and general relativity [11, 259, 342, 377, 400, 488, 535, 544].

- Non-standard viewpoints on quantum physics and general relativity [157, 292, 479, 529, 527, 545, 546, 547, 548].

- Soliton physics [497], defect physics [172], and the Fizeau effect [454], presented with an analogue flavour.

- Analogue-inspired models of black-hole accretion [516, 517, 518].

- Cosmological horizons from an analogue spacetime perspective [244].

- Analogue-inspired insights into renormalization group flow [107].

- An analysis of "wave catastrophes" inspired by analogue models [348].

- Improved numerical techniques for handling wave equations [688], and analytic techniques for handling wave tails [80], partially based on analogue ideas.

From the above the reader can easily appreciate the broad interest in, and wide applicability of, analogue spacetime models.

There is not much more that we can usefully say here. We have doubtless missed some articles of historical importance, but with a good library or a fast Internet connection the reader will be in as good a position as we are to find any additional historical articles.

4   A Catalogue of Models

In this section, we will attempt to categorise the very many analogue models researchers have
investigated. Perhaps the most basic subdivision is into classical models and quantum models, but
even then many other levels of refinement are possible. Consider for instance the following list:

- Classical models:

    - Classical sound.
    - Sound in relativistic hydrodynamics.
    - Water waves (gravity waves).
    - Classical refractive index.
    - Normal modes.

- Quantum models:

    - Bose–Einstein condensates (BECs).
    - The heliocentric universe.
      (Helium as an exemplar for just about anything.)
    - Slow light.

We will now provide a few words on each of these topics.

4.1   Classical models

4.1.1   Classical sound

Sound in a non-relativistic moving fluid has already been extensively discussed in Section 2, and
we will not repeat such discussion here. In contrast, sound in a solid exhibits its own distinct and
interesting features, notably in the existence of a generalization of the normal notion of birefringence
– longitudinal modes travel at a different speed (typically faster) than do transverse modes. This
may be viewed as an example of an analogue model which breaks the "light cone" into two at
the classical level; as such this model is not particularly useful if one is trying to simulate special
relativistic kinematics with its universal speed of light, though it may be used to gain insight into
yet another way of "breaking" Lorentz invariance (and the equivalence principle).

4.1.2   Sound in relativistic hydrodynamics

When dealing with relativistic sound, key historical papers are those of Moncrief [448] and Bilic [72],
with astrophysical applications being more fully explored in [162, 161, 160], and with a more recent
pedagogical follow-up in [639]. It is convenient to first quickly *motivate* the result by working in
the limit of relativistic ray acoustics where we can safely ignore the wave properties of sound. In
this limit we are interested only in the "sound cones". Let us pick a curved manifold with physical
spacetime metric $g_{\mu\nu}$, and a point in spacetime where the background fluid 4-velocity is V^μ while
the speed of sound is c_s. Now (in complete direct conformity with our discussion of the generalised
optical Gordon metric) adopt Gaussian normal coordinates so that $g_{\mu\nu} \to \eta_{\mu\nu}$, and go to the local
rest frame of the fluid, so that $V^\mu \to (1; \vec{0})$ and

$$
g_{\mu\nu} \to \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \qquad h_{\mu\nu} = g_{\mu\nu} + V_\mu V_\nu \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{145}
$$

In the rest frame of the fluid the sound cones are (locally) given by

$$- c_s^2 \, \mathrm{d}t^2 + ||\mathrm{d}\vec{x}||^2 = 0, \tag{146}$$

implying in these special coordinates the existence of an acoustic metric

$$\mathcal{G}_{\mu\nu} \propto \begin{bmatrix} -c_s^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{147}$$

That is, transforming back to arbitrary coordinates:

$$\mathcal{G}_{\mu\nu} \propto -c_s^2 V_\mu V_\nu + \{g_{\mu\nu} + V_\mu V_\nu\} \propto g_{\mu\nu} + \{1 - c_s^2\} V_\mu V_\nu. \tag{148}$$

Note again that in the ray acoustics limit, because one only has the sound cones to work with, one can neither derive (nor is it even meaningful to specify) the overall conformal factor. When going beyond the ray acoustics limit, seeking to obtain a relativistic wave equation suitable for describing physical acoustics, all the "fuss" is simply over how to determine the overall conformal factor (and to verify that one truly does obtain a d'Alembertian equation using the conformally-fixed acoustic metric).

One proceeds by combining the relativistic Euler equation, the relativistic energy equation, an assumed barotropic equation of state, and assuming a relativistic irrotational flow of the form [639]

$$V^\mu = \frac{g^{\mu\nu} \, \nabla_\nu \Theta}{||\nabla\Theta||}. \tag{149}$$

In this situation the relativistic Bernoulli equation can be shown to be

$$\ln ||\nabla\Theta|| = + \int_0^p \frac{\mathrm{d}p}{\varrho(p) + p}, \tag{150}$$

where we emphasize that ϱ is now the energy density (not the mass density), and the total particle number density can be shown to be

$$n(p) = n_{(p=0)} \, \exp\left[\int_{\varrho(p=0)}^{\varrho(p)} \frac{\mathrm{d}\varrho}{\varrho + p(\varrho)} \right]. \tag{151}$$

After linearization around some suitable background [448, 72, 639], the perturbations in the scalar velocity potential Θ can be shown to satisfy a dimension-independent d'Alembertian equation

$$\nabla_\mu \left\{ \frac{n_0^2}{\varrho_0 + p_0} \, \left[-\frac{1}{c_s^2} V_0^\mu V_0^\nu + h^{\mu\nu} \right] \nabla_\nu \Theta_1 \right\} = 0, \tag{152}$$

which leads to the identification of the relativistic acoustic metric as

$$\sqrt{-\mathcal{G}} \, \mathcal{G}^{\mu\nu} = \frac{n_0^2}{\varrho_0 + p_0} \, \left[-\frac{1}{c_s^2} V_0^\mu V_0^\nu + h^{\mu\nu} \right]. \tag{153}$$

The dimension-dependence now comes from solving this equation for $\mathcal{G}^{\mu\nu}$. Therefore, we finally have the (contravariant) acoustic metric

$$\mathcal{G}^{\mu\nu} = \left(\frac{n_0^2 \, c_s^{-1}}{\varrho_0 + p_0} \right)^{-2/(d-1)} \left\{ -\frac{1}{c_s^2} V_0^\mu V_0^\nu + h^{\mu\nu} \right\}, \tag{154}$$

and (covariant) acoustic metric

$$\mathcal{G}_{\mu\nu} = \left(\frac{n_0^2 \, c_s^{-1}}{\varrho_0 + p_0} \right)^{2/(d-1)} \left\{ -c_s^2 \, [V_0]_\mu [V_0]_\nu + h_{\mu\nu} \right\}. \tag{155}$$

In the non-relativistic limit $p_0 \ll \varrho_0$ and $\varrho_0 \approx \bar{m} \, n_0$, where $\bar{m}$ is the average mass of the particles making up the fluid (which by the barotropic assumption is a time-independent and position-independent constant). So in the non-relativistic limit we recover the standard result for the conformal factor [639]

$$\frac{n_0^2 \, c_s^{-1}}{\varrho_0 + p_0} \to \frac{n_0}{\bar{m} c_s} = \frac{1}{\bar{m}^2} \frac{\rho_0}{c_s} \propto \frac{\rho_0}{c_s}. \tag{156}$$

Under what conditions is the fully general relativistic discussion of this section necessary? (The non-relativistic analysis is, after all, the basis of the bulk of the work in "analogue spacetimes", and is perfectly adequate for many purposes.) The current analysis will be needed in three separate situations:

- when working in a nontrivial curved general relativistic background;

- whenever the fluid is flowing at relativistic speeds;

- less obviously, when the internal degrees of freedom of the fluid are relativistic, even if the overall fluid flow is non-relativistic. (That is, in situations where it is necessary to distinguish the energy density ϱ from the mass density ρ; this typically happens in situations where the fluid is strongly self-coupled – for example in neutron star cores or in relativistic BECs [191]. See Section 4.2.)

4.1.3 Shallow water waves (gravity waves)

A wonderful example of the occurrence of an effective metric in nature is that provided by gravity waves in a shallow basin filled with liquid [560]. (See Figure 10.)[18] If one neglects the viscosity and considers an irrotational flow, $\mathbf{v} = \boldsymbol{\nabla}\phi$, one can write Bernoulli's equation in the presence of Earth's gravity as

$$\partial_t \phi + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 = -\frac{p}{\rho} - gz - V_\parallel. \tag{157}$$

Here ρ is the density of the fluid, p its pressure, g the gravitational acceleration and $V_\parallel$ a potential associated with some external force necessary to establish an horizontal flow in the fluid. We denote that flow by $\mathbf{v}_{\mathrm{B}}^\parallel$. We must also impose the boundary conditions that the pressure at the surface, and the vertical velocity at the bottom, both vanish. That is, $p(z = h_{\mathrm{B}}) = 0$, and $v_\perp(z = 0) = 0$.

Once a horizontal background flow is established, one can see that the perturbations of the velocity potential satisfy

$$\partial_t \, \delta\phi + \mathbf{v}_{\mathrm{B}}^\parallel \cdot \boldsymbol{\nabla}_\parallel \delta\phi = -\frac{\delta p}{\rho}. \tag{158}$$

If we now expand this perturbation potential in a Taylor series

$$\delta\phi = \sum_{n=0}^{\infty} \frac{z^n}{n!} \, \delta\phi_n(x, y), \tag{159}$$

---

[18] Of course, we now mean "gravity wave" in the traditional fluid mechanics sense of a water wave whose restoring force is given by ordinary Newtonian gravity. Waves in the fabric of spacetime are more properly called "gravitational waves", though this usage seems to be in decline within the general relativity community. Be very careful in any situation where there is even a possibility of confusing the two concepts.

---

it is not difficult to prove [560] that surface waves with long wavelengths (long compared with the depth of the basin, $\lambda \gg h_{\mathrm{B}}$), can be described to a good approximation by $\delta\phi_0(x,y)$ and that this field "sees" an effective metric of the form

$$\mathrm{d}s^2 = \frac{1}{c^2}\left[-(c^2 - v_{\mathrm{B}}^{\|2})\,\mathrm{d}t^2 - 2\mathbf{v}_{\mathrm{B}}^{\|}\cdot\mathbf{dx}\,\mathrm{d}t + \mathbf{dx}\cdot\mathbf{dx}\right], \tag{160}$$

where $c \equiv \sqrt{gh_{\mathrm{B}}}$. The link between small variations of the potential field and small variations of the position of the surface is provided by the following equation

$$\delta v_{\perp} = -h_{\mathrm{B}}\,\nabla_{\|}^2\delta\phi_0 = \partial_t\,\delta h + \mathbf{v}_{\mathrm{B}}^{\|}\cdot\boldsymbol{\nabla}_{\|}\delta h = \frac{\mathrm{d}}{\mathrm{d}t}\,\delta h. \tag{161}$$

The entire previous analysis can be generalised to the case in which the bottom of the basin is not flat, and the background flow not purely horizontal [560]. Therefore, one can create different effective metrics for gravity waves in a shallow fluid basin by changing (from point to point) the background flow velocity and the depth, $h_{\mathrm{B}}(x,y)$.
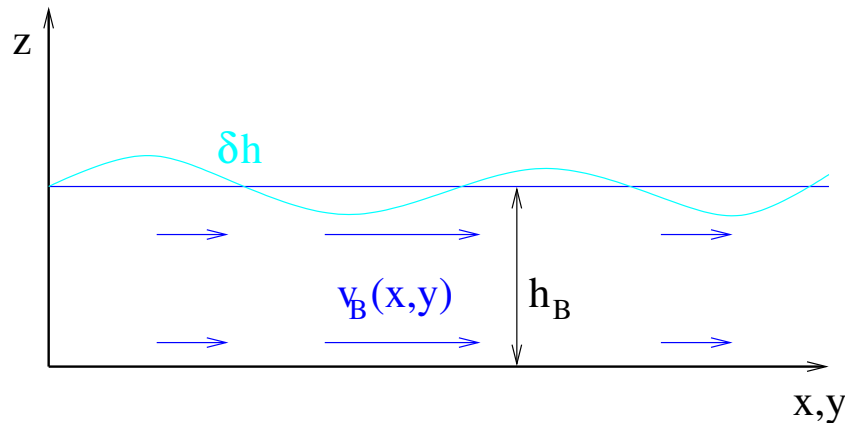
Figure 10: Gravity waves in a shallow fluid basin with a background horizontal flow.

The main advantage of this model is that the velocity of the surface waves can very easily be modified by changing the depth of the basin. This velocity can be made very slow, and consequently, the creation of ergoregions should be relatively easier than in other models. As described here, this model is completely classical given that the analogy requires long wavelengths and slow propagation speeds for the gravity waves. Although the latter feature is convenient for the practical realization of analogue horizons, it is a disadvantage in trying to detect analogue Hawking radiation as the relative temperature will necessarily be very low. (This is why, in order to have a possibility of experimentally observing (spontaneous) Hawking evaporation and other quantum phenomena, one would need to use ultra-cold quantum fluids.) However, the gravity wave analogue can certainly serve to investigate the classical phenomena of mode mixing that underlies the quantum processes.

It is this particular analogue model (and its extensions to finite depth and surface tension) that underlies the experimental [532] and theoretical [531] work of Rousseaux et al., the historically-important experimental work of Badulin et al. [17], and the very recent experimental verification by Weinfurtner et al. of the existence of classical stimulated Hawking radiation [682].

4.1.4 More general water waves

If one moves beyond shallow-water surface waves the physics becomes more complicated. In the shallow-water regime (wavelength λ much greater than water depth d) the co-moving dispersion

relation is a simple linear one $\omega = c_s k$, where the speed of sound can depend on both position and time. Once one moves to finite-depth ($\lambda \sim d$) or deep ($\lambda \ll d$) water, it is a standard result that the co-moving dispersion relation becomes

$$\omega = \sqrt{gk \tanh(kd)} = c_s k \sqrt{\frac{\tanh(kd)}{kd}}. \tag{162}$$

See, for instance, Lamb [370] §228, p. 354, Equation (5). A more modern discussion in an analogue spacetime context is available in [643]. Adding surface tension requires a brief computation based on Lamb [370] §267 p. 459, details can be found in [643]. The net result is

$$\omega = c_s k \sqrt{1 + k^2/K^2} \sqrt{\frac{\tanh(kd)}{kd}}. \tag{163}$$

Here $K^2 = g\rho/\sigma$ is a constant depending on the acceleration due to gravity, the density, and the surface tension [643]. Once one adds the effects of fluid motion, one obtains

$$\omega = \mathbf{v} \cdot \mathbf{k} + c_s k \sqrt{1 + k^2/K^2} \sqrt{\frac{\tanh(kd)}{kd}}. \tag{164}$$

All of these features, fluid motion, finite depth, and surface tension (capillarity), seem to be present in the 1983 experimental investigations by Badulin et al. [17]. All of these features should be kept in mind when interpreting the experimental [532] and theoretical [531] work of Rousseaux et al., and the very recent experimental work by Weinfurtner et al. [682].

A feature that is sometimes not remarked on is that the careful derivation we have previously presented of the acoustic metric, or in this particular situation the derivation of the shallow-water-wave effective metric [560], makes technical assumptions tantamount to asserting that one is in the regime where the co-moving dispersion relation takes the linear form $\omega \approx c_s k$. Once the co-moving dispersion relation becomes nonlinear, the situation is more subtle, and based on a geometric acoustics approximation to the propagation of signal waves one can introduce *several* notions of conformal "rainbow" metric (momentum-dependent metric). Consider

$$g_{ab}(k^2) \propto \left[\begin{array}{c|c} - \left\{ c^2(k^2) - \delta_{ij}\, v^i\, v^j \right\} & +v_j \\ \hline + v_i & +h_{ij} \end{array} \right], \tag{165}$$

and the inverse

$$g^{ab}(k^2) \propto \left[\begin{array}{c|c} - 1 & +v^j \\ \hline + v^i & c^2(k^2)\, h^{ij} - v^i\, v^j \end{array} \right]. \tag{166}$$

At a minimum we could think of using the following notions of propagation speed

$$c(k^2) \to \begin{cases} c_{\text{phase}}(k^2); \\ c_{\text{group}}(k^2); \\ c_{\text{sound}} = \lim_{k \to 0} c_{\text{phase}}(k^2) \text{ provided this equals } \lim_{k \to 0} c_{\text{group}}(k^2); \\ c_{\text{signal}} = \lim_{k \to \infty} c_{\text{phase}}(k^2). \end{cases} \tag{167}$$

Brillouin, in his classic paper [92], identified at least six useful notions of propagation speed, and many would argue that the list can be further refined. Each one of these choices for the rainbow metric encodes different physics, and is useful for different purposes. It is still somewhat unclear as to which of these rainbow metrics is "best" for interpreting the experimental results reported in [17, 532, 682].

4.1.5   Classical refractive index

The macroscopic Maxwell equations inside a dielectric take the well-known form

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{E} + \partial_t \mathbf{B} = 0, \tag{168}$$

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0, \quad \boldsymbol{\nabla} \times \mathbf{H} - \partial_t \mathbf{D} = 0, \tag{169}$$

with the constitutive relations $\mathbf{H} = \boldsymbol{\mu}^{-1} \cdot \mathbf{B}$ and $\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E}$. Here, $\boldsymbol{\epsilon}$ is the 3×3 permittivity tensor and $\boldsymbol{\mu}$ the 3×3 permeability tensor of the medium. These equations can be written in a condensed way as

$$\partial_\alpha \left(Z^{\mu\alpha\nu\beta} \, F_{\nu\beta} \right) = 0 \tag{170}$$

where $F_{\nu\beta} = A_{[\nu,\beta]}$ is the electromagnetic tensor,

$$F_{0i} = -F_{i0} = -E_i, \quad F_{ij} = \varepsilon_{ijk} B^k, \tag{171}$$

and (assuming the medium is at rest) the non-vanishing components of the 4th-rank tensor Z are given by

$$Z^{0i0j} = -Z^{0ij0} = Z^{i0j0} = -Z^{i00j} = -\frac{1}{2} \epsilon^{ij}; \tag{172}$$

$$Z^{ijkl} = \frac{1}{2} \varepsilon^{ijm} \, \varepsilon^{kln} \, \mu_{mn}^{-1}; \tag{173}$$

supplemented by the conditions that Z is antisymmetric on its first pair of indices and antisymmetric on its second pair of indices. Without significant loss of generality we can ask that Z also be symmetric under pairwise interchange of the first pair of indices with the second pair – thus Z exhibits most of the algebraic symmetries of the Riemann tensor, though this appears to merely be accidental, and not fundamental in any way.

If we compare this to the Lagrangian for electromagnetism in curved spacetime

$$\mathcal{L} = \sqrt{-g} \, g^{\mu\alpha} \, g^{\nu\beta} \, F_{\mu\nu} \, F_{\alpha\beta} \tag{174}$$

we see that in curved spacetime we can also write the electromagnetic equations of motion in the form (170) where now (for some constant K):

$$Z^{\mu\nu\alpha\beta} = K \, \sqrt{-g} \, \left\{ g^{\mu\alpha} \, g^{\nu\beta} - g^{\mu\beta} \, g^{\nu\alpha} \right\}. \tag{175}$$

If we consider a static gravitational field we can always re-write it as a conformal factor multiplying an ultra-static metric

$$g_{\mu\nu} = \Omega^2 \, \left\{ -1 \oplus g_{ij} \right\} \tag{176}$$

then

$$Z^{0i0j} = -Z^{0ij0} = Z^{i0j0} = -Z^{i00j} = -K \, \sqrt{-g} \, g^{ij}; \tag{177}$$

$$Z^{ijkl} = K \, \sqrt{-g} \left\{ g^{ik} \, g^{jl} - g^{il} \, g^{jk} \right\}. \tag{178}$$

The fact that Z is independent of the conformal factor Ω is simply the reflection of the well-known fact that the Maxwell equations are conformally invariant in (3+1) dimensions. Thus, if we wish to have the analogy (between a static gravitational field and a dielectric medium at rest) hold *at the level of the wave equation* (physical optics) we must satisfy the two stringent constraints

$$K \sqrt{-g} \, g^{ij} = \frac{1}{2} \, \epsilon^{ij}; \tag{179}$$

$$K \sqrt{-g} \left\{ g^{ik} \, g^{jl} - g^{il} \, g^{jk} \right\} = \frac{1}{2} \, \varepsilon^{ijm} \, \varepsilon^{kln} \, \mu_{mn}^{-1}. \tag{180}$$

The second of these constraints can be written as

$$K \sqrt{-g} \, \varepsilon_{ijm} \, \varepsilon_{kln} \left\{ g^{ik} \, g^{jl} \right\} = \mu_{mn}^{-1}. \tag{181}$$

In view of the standard formula for 3×3 determinants

$$\varepsilon_{ijm} \, \varepsilon_{kln} \left\{ X^{ik} \, X^{jl} \right\} = 2 \, \det X \, X_{mn}^{-1}, \tag{182}$$

this now implies

$$2K \frac{g_{ij}}{\sqrt{-g}} = \mu_{ij}^{-1}, \tag{183}$$

whence

$$\frac{1}{2K} \sqrt{-g} \, g^{ij} = \mu^{ij}. \tag{184}$$

Comparing this with

$$2K \sqrt{-g} \, g^{ij} = \epsilon^{ij}, \tag{185}$$

we now have:

$$\epsilon^{ij} = 4 \, K^2 \, \mu^{ij}; \tag{186}$$

$$g^{ij} = \frac{4 \, K^2}{\det \epsilon} \, \epsilon^{ij} = \frac{1}{4 \, K^2 \, \det \boldsymbol{\mu}} \, \mu^{ij}. \tag{187}$$

To rearrange this, introduce the matrix square root $[\boldsymbol{\mu}^{1/2}]^{ij}$, which always exists because $\boldsymbol{\mu}$ is real positive definite and symmetric. Then

$$g^{ij} = \left[\left\{ \frac{\boldsymbol{\mu}^{1/2} \, \boldsymbol{\epsilon} \, \boldsymbol{\mu}^{1/2}}{\det(\boldsymbol{\mu} \, \boldsymbol{\epsilon})} \right\}^{1/2} \right]^{ij}. \tag{188}$$

Note that if you are given the static gravitational field (in the form Ω, g_{ij}) you can always solve it to find an equivalent analogue in terms of permittivity/permeability (albeit an analogue that satisfies the mildly unphysical constraint $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$).[19] On the other hand, if you are given permeability and permittivity tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$, then it is *only* for that subclass of media that satisfy $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ that one can perfectly mimic *all* of the electromagnetic effects by an equivalent gravitational field. Of course, this can be done provided one only considers wavelengths that are sufficiently long for the macroscopic description of the medium to be valid. In this respect it is interesting to note that the behaviour of the refractive medium at high frequencies has been used to introduce an effective cutoff for the modes involved in Hawking radiation [523]. We shall encounter this model (which is also known in the literature as a solid state analogue model) later on when we consider the trans-Planckian problem for Hawking radiation. Let us stress that if one were able to directly probe the quantum effective photons over a dielectric medium, then one would be dealing with a quantum analogue model instead of a classical one.

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[19] The existence of this constraint has been independently re-derived several times in the literature. In contrast, other segments of the literature seem blithely unaware of this important restriction on just when permittivity and permeability are truly equivalent to an effective metric.

Eikonal approximation: With a bit more work this discussion can be extended to a medium in motion, leading to an extension of the Gordon metric. Alternatively, one can agree to ask more limited questions by working at the level of geometrical optics (adopting the eikonal approximation), in which case there is no longer any restriction on the permeability and permittivity tensors. To see this, construct the matrix

$$C^{\mu\nu} = Z^{\mu\alpha\nu\beta} \, k_\alpha \, k_\beta. \tag{189}$$

The dispersion relations for the propagation of photons (and therefore the sought for geometrical properties) can be obtained from the reduced determinant of C (notice that the [full] determinant of C is identically zero as $C^{\mu\nu}k_\nu = 0$; the reduced determinant is that associated with the three directions orthogonal to $k_\nu = 0$). By choosing the gauge $A_0 = 0$ one can see that this reduced determinant can be obtained from the determinant of the 3×3 sub-matrix C^{ij}. This determinant is

$$\det(C^{ij}) = \frac{1}{8} \det \left(-\omega^2 \epsilon^{ij} + \varepsilon^{ikm} \, \varepsilon^{jln} \, \mu_{mn}^{-1} k_k k_l \right) \tag{190}$$

or, after making some manipulations,

$$\det(C^{ij}) = \frac{1}{8} \det \left[-\omega^2 \epsilon^{ij} + (\det \boldsymbol{\mu})^{-1} (\mu^{ij}\mu^{kl} k_k k_l - \mu^{im} k_m \mu^{jl} k_l) \right]. \tag{191}$$

To simplify this, again introduce the matrix square roots $[\boldsymbol{\mu}^{1/2}]^{ij}$ and $[\boldsymbol{\mu}^{-1/2}]_{ij}$, which always exist because the relevant matrices are real positive definite and symmetric. Then define

$$\tilde{k}^i = [\boldsymbol{\mu}^{1/2}]^{ij} \, k_j \tag{192}$$

and

$$[\tilde{\boldsymbol{\epsilon}}]^{ij} = \det(\boldsymbol{\mu}) \, [\boldsymbol{\mu}^{-1/2} \, \boldsymbol{\epsilon} \, \boldsymbol{\mu}^{-1/2}]_{ij} \tag{193}$$

so that

$$\det(C^{ij}) \propto \det \left\{ -\omega^2 \, [\tilde{\boldsymbol{\epsilon}}]^{ij} + \delta^{ij} \, [\delta_{mn} \, \tilde{k}^m \, \tilde{k}^n] - \tilde{k}^i \, \tilde{k}^j \right\}. \tag{194}$$

The behaviour of this dispersion relation now depends critically on the way that the eigenvalues of $\tilde{\boldsymbol{\epsilon}}$ are distributed.

3 degenerate eigenvalues: If all eigenvalues are degenerate then $\tilde{\boldsymbol{\epsilon}} = \tilde{\epsilon} \, \mathbf{I}$, implying $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ but now with the possibility of a position-dependent proportionality factor (in the case of physical optics the proportionality factor was constrained to be a position-independent constant). In this case we now easily evaluate

$$\boldsymbol{\epsilon} = \frac{\operatorname{tr}(\boldsymbol{\epsilon})}{\operatorname{tr}(\boldsymbol{\mu})} \, \boldsymbol{\mu} \qquad \text{and} \qquad \tilde{\epsilon} = \det \boldsymbol{\mu} \, \frac{\operatorname{tr}(\boldsymbol{\epsilon})}{\operatorname{tr}(\boldsymbol{\mu})}, \tag{195}$$

while

$$\det(C^{ij}) \propto \omega^2 \, \left\{ \omega^2 - [\tilde{\epsilon}^{-1} \, \delta_{mn} \, \tilde{k}^m \, \tilde{k}^n] \right\}^2. \tag{196}$$

That is

$$\det(C^{ij}) \propto \omega^2 \left\{ \omega^2 - [g^{ij} \, k_i \, k_j] \right\}^2, \tag{197}$$

with

$$g^{ij} = \frac{1}{\tilde{\epsilon}} \, [\boldsymbol{\mu}]^{ij} = \frac{\operatorname{tr}(\boldsymbol{\mu}) \, [\boldsymbol{\mu}]^{ij}}{\operatorname{tr}(\boldsymbol{\epsilon}) \, \det \boldsymbol{\mu}} = \frac{\operatorname{tr}(\boldsymbol{\epsilon}) \, [\boldsymbol{\epsilon}]^{ij}}{\operatorname{tr}(\boldsymbol{\mu}) \, \det \boldsymbol{\epsilon}}. \tag{198}$$

This last result is compatible with but more general than the result obtained under the more restrictive conditions of physical optics. In the situation where both permittivity and permeability are isotropic, ($\epsilon^{ij} = \epsilon\,\delta^{ij}$ and $\mu^{ij} = \mu\,\delta^{ij}$) this reduces to the perhaps more expected result

$$g^{ij} = \frac{\delta^{ij}}{\epsilon\,\mu}. \tag{199}$$

2 distinct eigenvalues: If $\tilde{\epsilon}$ has two distinct eigenvalues then the determinant $\det(C^{ij})$ factorises into a trivial factor of ω^2 and two quadratics. Each quadratic corresponds to a distinct effective metric. This is the physical situation encountered in uni-axial crystals, where the *ordinary* and *extraordinary* rays each obey distinct quadratic dispersion relations [82]. From the point of view of analogue models this corresponds to a two-metric theory.

3 distinct eigenvalues: If $\tilde{\epsilon}$ has three distinct eigenvalues then the determinant $\det(C^{ij})$ is the product of a trivial factor of ω^2 and a *non-factorizable quartic*. This is the physical situation encountered in bi-axial crystals [82, 638], and it seems that no meaningful notion of the effective Riemannian metric can be assigned to this case. (The use of Finsler geometries in this situation is an avenue that may be worth pursuing [306]. But note some of the negative results obtained in [573, 574, 575].)

Abstract linear electrodynamics: Hehl and co-workers have championed the idea of using the linear constitutive relations of electrodynamics as the primary quantities, and then treating the spacetime metric (even for flat space) as a *derived* concept. See [474, 276, 371, 277].

Nonlinear electrodynamics: In general, the permittivity and permeability tensors can be modified by applying strong electromagnetic fields (this produces an effectively nonlinear electrodynamics). The entire previous discussion still applies if one considers the photon as the linear perturbation of the electromagnetic field over a background configuration

$$F_{\mu\nu} = F^{\mathrm{bg}}_{\mu\nu} + f^{\mathrm{ph}}_{\mu\nu}. \tag{200}$$

The background field $F^{\mathrm{bg}}_{\mu\nu}$ sets the value of $\epsilon^{ij}(F^{\mathrm{bg}})$, and $\mu^{ij}(F^{\mathrm{bg}})$. Equation (170) then becomes an equation for $f^{\mathrm{ph}}_{\mu\nu}$. This approach has been extensively investigated by Novello and co-workers [465, 469, 170, 468, 466, 467, 464, 214].

Summary: The propagation of photons in a dielectric medium characterised by 3×3 permeability and permittivity tensors constrained by $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ is equivalent (at the level of geometric optics) to the propagation of photons in a curved spacetime manifold characterised by the ultra-static metric (198), provided one only considers wavelengths that are sufficiently long for the macroscopic description of the medium to be valid. If, in addition, one takes a fluid dielectric, by controlling its flow one can generalise the Gordon metric and again reproduce metrics of the Painlevé–Gullstrand type, and therefore geometries with ergo-regions. If the proportionality constant relating $\boldsymbol{\epsilon} \propto \boldsymbol{\mu}$ is position independent, one can make the stronger statement (187) which holds true at the level of physical optics. Recently this topic has been revitalised by the increasing interest in (classical) meta-materials.

4.1.6   Normal mode meta-models

We have already seen how linearizing the Euler–Lagrange equations for a single scalar field naturally leads to the notion of an effective spacetime metric. If more than one field is involved the situation

becomes more complicated, in a manner similar to that of geometrical optics in uni-axial and bi-axial crystals. (This should, with hindsight, not be too surprising since electromagnetism, even in the presence of a medium, is definitely a Lagrangian system and definitely involves more than one single scalar field.) A normal mode analysis based on a general Lagrangian (many fields but still first order in derivatives of those fields) leads to a concept of *refringence*, or more specifically *multi-refringence*, a generalization of the birefringence of geometrical optics. To see how this comes about, consider a straightforward generalization of the one-field case.

We want to consider linearised fluctuations around some background solution of the equations of motion. As in the single-field case we write (here we will follow the notation and conventions of [45])

$$\phi^A(t, \vec{x}) = \phi_0^A(t, \vec{x}) + \epsilon\, \phi_1^A(t, \vec{x}) + \frac{\epsilon^2}{2}\, \phi_2^A(t, \vec{x}) + O(\epsilon^3). \tag{201}$$

Now use this to expand the Lagrangian

$$\begin{aligned}
\mathcal{L}(\partial_\mu \phi^A, \phi^A) = {} & \mathcal{L}(\partial_\mu \phi_0^A, \phi_0^A) + \epsilon \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\, \partial_\mu \phi_1^A + \frac{\partial \mathcal{L}}{\partial \phi^A}\, \phi_1^A \right] \\
& + \frac{\epsilon^2}{2} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)}\, \partial_\mu \phi_2^A + \frac{\partial \mathcal{L}}{\partial \phi^A}\, \phi_2^A \right] \\
& + \frac{\epsilon^2}{2} \left[\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)}\, \partial_\mu \phi_1^A\, \partial_\nu \phi_1^B \right. \\
& \left. + 2 \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial \phi^B}\, \partial_\mu \phi_1^A\, \phi_1^B + \frac{\partial^2 \mathcal{L}}{\partial \phi^A\, \partial \phi^B}\, \phi_1^A\, \phi_1^B \right] \\
& + O(\epsilon^3).
\end{aligned} \tag{202}$$

Consider the action

$$S[\phi^A] = \int \mathrm{d}^{d+1}x\; \mathcal{L}(\partial_\mu \phi^A, \phi^A). \tag{203}$$

Doing so allows us to integrate by parts. As in the single-field case we can use the Euler–Lagrange equations to discard the linear terms (since we are linearizing around a solution of the equations of motion) and so get

$$\begin{aligned}
S[\phi^A] = {} & S[\phi_0^A] \\
& + \frac{\epsilon^2}{2} \int \mathrm{d}^{d+1}x \left[\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right\}\, \partial_\mu \phi_1^A\, \partial_\nu \phi_1^B \right. \\
& \left. + 2 \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial \phi^B} \right\}\, \partial_\mu \phi_1^A\, \phi_1^B + \left\{ \frac{\partial^2 \mathcal{L}}{\partial \phi^A\, \partial \phi^B} \right\}\, \phi_1^A\, \phi_1^B \right] \\
& + O(\epsilon^3).
\end{aligned} \tag{204}$$

Because the fields now carry indices (AB) we cannot cast the action into quite as simple a form as was possible in the single-field case. The equation of motion for the linearised fluctuations are now read off as

$$\begin{aligned}
& \partial_\mu \left(\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right\}\, \partial_\nu \phi_1^B \right) + \partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial \phi^B}\, \phi_1^B \right) \\
& \quad - \partial_\mu \phi_1^B\, \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial \phi^A} - \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^A\, \partial \phi^B} \right) \phi_1^B = 0.
\end{aligned} \tag{205}$$

This is a linear second-order *system* of partial differential equations with position-dependent coefficients. This system of PDEs is automatically self-adjoint (with respect to the trivial "flat" measure $d^{d+1}x$).

To simplify the notation we introduce a number of definitions. First

$$f^{\mu\nu}{}_{AB} \equiv \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\, \partial(\partial_\mu \phi^B)} \right). \tag{206}$$

This quantity is independently symmetric under interchange of μ, ν and A, B. We will want to interpret this as a generalization of the "densitised metric", $f^{\mu\nu}$, but the interpretation is not as straightforward as for the single-field case. Next, define

$$\Gamma^\mu{}_{AB} \equiv +\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial\phi^B} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial\phi^A}$$
$$+\frac{1}{2}\partial_\nu\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\, \partial(\partial_\mu \phi^B)} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right). \tag{207}$$

This quantity is anti-symmetric in A, B. One might want to interpret this as some sort of "spin connection", or possibly as some generalization of the notion of "Dirac matrices". Finally, define

$$K_{AB} = -\frac{\partial^2 \mathcal{L}}{\partial\phi^A\, \partial\phi^B} + \frac{1}{2}\partial_\mu\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial\phi^B} \right) + \frac{1}{2}\partial_\mu\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial\phi^A} \right). \tag{208}$$

This quantity is by construction symmetric in (AB). We will want to interpret this as some sort of "potential" or "mass matrix". Then the crucial point for the following discussion is to realise that Equation (205) can be written in the compact form

$$\partial_\mu\left(f^{\mu\nu}{}_{AB}\, \partial_\nu \phi_1^B \right) + \frac{1}{2}\left[\Gamma^\mu{}_{AB}\, \partial_\mu \phi_1^B + \partial_\mu(\Gamma^\mu{}_{AB}\, \phi_1^B) \right] + K_{AB}\, \phi_1^B = 0. \tag{209}$$

Now it is more transparent that this is a formally self-adjoint second-order linear *system* of PDEs. Similar considerations can be applied to the linearization of any hyperbolic system of second-order PDEs.

Consider an eikonal approximation for an arbitrary direction in field space; that is, take

$$\phi^A(x) = \epsilon^A(x)\, \exp[-i\varphi(x)], \tag{210}$$

with $\epsilon^A(x)$ a slowly varying amplitude, and $\varphi(x)$ a rapidly varying phase. In this eikonal approximation (where we neglect gradients in the amplitude, and gradients in the coefficients of the PDEs, retaining only the gradients of the phase) the linearised system of PDEs (209) becomes

$$\left\{ f^{\mu\nu}{}_{AB}\, \partial_\mu \varphi(x)\, \partial_\nu \varphi(x) + \Gamma^\mu{}_{AB}\, \partial_\mu \varphi(x) + K_{AB} \right\}\, \epsilon^B(x) = 0. \tag{211}$$

This has a nontrivial solution if and only if $\epsilon^A(x)$ is a null eigenvector of the matrix

$$f^{\mu\nu}{}_{AB}\, k_\mu\, k_\nu + \Gamma^\mu{}_{AB}\, k_\mu + K_{AB}, \tag{212}$$

where $k_\mu = \partial_\mu \varphi(x)$. Now, the condition for such a null eigenvector to exist is that

$$F(x,k) \equiv \det\left\{ f^{\mu\nu}{}_{AB}\, k_\mu\, k_\nu + \Gamma^\mu{}_{AB}\, k_\mu + K_{AB} \right\} = 0, \tag{213}$$

with the determinant to be taken on the field space indices AB. This is the natural generalization to the current situation of the Fresnel equation of birefringent optics [82, 375]. Following the analogy with the situation in electrodynamics (either nonlinear electrodynamics, or more prosaically

propagation in a birefringent crystal), the null eigenvector $\epsilon^A(x)$ would correspond to a specific "polarization". The Fresnel equation then describes how different polarizations can propagate at different velocities (or in more geometrical language, can see different metric structures). In the language of particle physics, this determinant condition $F(x, k) = 0$ is the natural generalization of the "mass shell" constraint. Indeed, it is useful to define the mass shell as a subset of the cotangent space by

$$\mathcal{F}(x) \equiv \left\{ k_\mu \,\middle|\, F(x, k) = 0 \right\}. \tag{214}$$

In more mathematical language we are looking at the null space of the determinant of the "symbol" of the system of PDEs. By investigating $F(x, k)$ one can recover part (not all) of the information encoded in the matrices $f^{\mu\nu}{}_{AB}$, $\Gamma^\mu{}_{AB}$, and K_{AB}, or equivalently in the "generalised Fresnel equation" (213). (Note that for the determinant equation to be useful it should be non-vacuous; in particular one should carefully eliminate all gauge and spurious degrees of freedom before constructing this "generalised Fresnel equation", since otherwise the determinant will be identically zero.) We now want to make this analogy with optics more precise, by carefully considering the notion of characteristics and characteristic surfaces. We will see how to extract from the the high-frequency high-momentum regime described by the eikonal approximation all the information concerning the causal structure of the theory.

One of the key structures that a Lorentzian spacetime metric provides is the notion of causal relationships. This suggests that it may be profitable to try to work backwards from the causal structure to determine a Lorentzian metric. Now the causal structure implicit in the system of second-order PDEs given in Equation (209) is described in terms of the characteristic surfaces, and it is for this reason that we now focus on characteristics as a way of encoding causal structure, and as a surrogate for some notion of a Lorentzian metric. Note that, via the Hadamard theory of surfaces of discontinuity, the characteristics can be identified with the infinite-momentum limit of the eikonal approximation [265]. That is, when extracting the characteristic surfaces we neglect subdominant terms in the generalised Fresnel equation and focus only on the leading term in the symbol ($f^{\mu\nu}{}_{AB}$). In the language of particle physics, going to the infinite-momentum limit puts us on the light cone instead of the mass shell; and it is the light cone that is more useful in determining causal structure. The "normal cone" at some specified point x, consisting of the locus of normals to the characteristic surfaces, is defined by

$$\mathcal{N}(x) \equiv \left\{ k_\mu \,\middle|\, \det\left(f^{\mu\nu}{}_{AB}\; k_\mu\, k_\mu\right) = 0 \right\}. \tag{215}$$

As was the case for the Fresnel Equation (213), the determinant is to be taken on the field indices AB. (Remember to eliminate spurious and gauge degrees of freedom so that this determinant is not identically zero.) We emphasise that the algebraic equation defining the normal cone is the leading term in the Fresnel equation encountered in discussing the eikonal approximation. If there are N fields in total then this "normal cone" will generally consist of N nested sheets each with the topology (not necessarily the geometry) of a cone. Often several of these cones will coincide, which is not particularly troublesome, but unfortunately it is also common for some of these cones to be degenerate, which is more problematic.

It is convenient to define a function $Q(x, k)$ on the co-tangent bundle

$$Q(x, k) \equiv \det\left(f^{\mu\nu}{}_{AB}(x)\; k_\mu\, k_\mu\right). \tag{216}$$

The function $Q(x, k)$ defines a completely-symmetric spacetime tensor (actually, a tensor density) with $2N$ indices

$$Q(x, k) = Q^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_N\nu_N}(x)\; k_{\mu_1}\, k_{\nu_1}\, k_{\mu_2}\, k_{\nu_2} \cdots k_{\mu_N}\, k_{\nu_N}. \tag{217}$$

(Remember that $f^{\mu\nu}{}_{AB}$ is symmetric in both $\mu\nu$ and AB independently.) Explicitly, using the expansion of the determinant in terms of completely antisymmetric field-space Levi–Civita tensors

$$Q^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_N\nu_N} = \frac{1}{N!}\, \epsilon^{A_1 A_2\cdots A_N}\, \epsilon^{B_1 B_2\cdots B_N}\, f^{\mu_1\nu_1}{}_{A_1 B_1}\, f^{\mu_2\nu_2}{}_{A_2 B_2}\, \cdots f^{\mu_N\nu_N}{}_{A_N B_N}. \tag{218}$$

In terms of this $Q(x,k)$ function, the normal cone is

$$\mathcal{N}(x) \equiv \left\{ k_\mu \ \middle|\ Q(x,k) = 0 \right\}. \tag{219}$$

In contrast, the "Monge cone" (*aka* "ray cone", *aka* "characteristic cone", *aka* "null cone") is the envelope of the set of characteristic surfaces through the point x. Thus the "Monge cone" is dual to the "normal cone", its explicit construction is given by (Courant and Hilbert [154, vol. 2, p. 583]):

$$\mathcal{M}(x) = \left\{ t^\mu = \frac{\partial Q(x,k)}{\partial k_\mu} \ \middle|\ k_\mu \in \mathcal{N}(x) \right\}. \tag{220}$$

The structure of the normal and Monge cones encode all the information related with the causal propagation of signals associated with the system of PDEs. We will now see how to relate this causal structure with the existence of effective spacetime metrics, from the experimentally favoured single-metric theory compatible with the Einstein equivalence principle to the most complicated case of pseudo-Finsler geometries [306].

- Suppose that $f^{\mu\nu}{}_{AB}$ factorises

$$f^{\mu\nu}{}_{AB} = h_{AB}\, f^{\mu\nu}. \tag{221}$$

  Then

$$Q(x,k) = \det(h_{AB})\, [f^{\mu\nu}\, k_\mu\, k_\nu]^N. \tag{222}$$

  The Monge cones and normal cones are then true geometrical cones (with the N sheets lying directly on top of one another). The normal modes all see the same spacetime metric, defined up to an unspecified conformal factor by $g^{\mu\nu} \propto f^{\mu\nu}$. This situation is the most interesting from the point of view of general relativity. Physically, it corresponds to a single-metric theory, and mathematically it corresponds to a strict algebraic condition on the $f^{\mu\nu}{}_{AB}$.

- The next most useful situation corresponds to the commutativity condition:

$$f^{\mu\nu}{}_{AB}\, f^{\alpha\beta}{}_{BC} = f^{\alpha\beta}{}_{AB}\, f^{\mu\nu}{}_{BC}; \qquad \text{that is} \qquad [\, \boldsymbol{f}^{\mu\nu}, \boldsymbol{f}^{\alpha\beta}\,] = 0. \tag{223}$$

  If this algebraic condition is satisfied, then for all spacetime indices $\mu\nu$ and $\alpha\beta$ the $f^{\mu\nu}{}_{AB}$ can be simultaneously diagonalised in field space leading to

$$\bar{f}^{\mu\nu}{}_{AB} = \mathrm{diag}\{\bar{f}_1^{\mu\nu}, \bar{f}_2^{\mu\nu}, \bar{f}_3^{\mu\nu}, \ldots, \bar{f}_N^{\mu\nu}\} \tag{224}$$

  and then

$$Q(x,k) = \prod_{A=1}^{N} [\bar{f}_A^{\mu\nu}\, k_\mu\, k_\nu]. \tag{225}$$

  This case corresponds to an N-metric theory, where up to an unspecified conformal factor $g_A^{\mu\nu} \propto \bar{f}_A^{\mu\nu}$. This is the natural generalization of the two-metric situation in bi-axial crystals.

- If $f^{\mu\nu}{}_{AB}$ is completely general, satisfying no special algebraic condition, then $Q(x,k)$ does not factorise and is, in general, a polynomial of degree $2N$ in the wave vector k_μ. This is the natural generalization of the situation in bi-axial crystals. (And for any deeper analysis of this situation one will almost certainly need to adopt pseudo-Finsler techniques [306]. But note some of the negative results obtained in [573, 574, 575].)

The message to be extracted from this rather formal discussion is that effective metrics are rather general and mathematically robust objects that can arise in quite abstract settings – in the abstract setting discussed here it is the algebraic properties of the object $f^{\mu\nu}{}_{AB}$ that eventually leads to mono-metricity, multi-metricity, or worse. The current abstract discussion also serves to illustrate, yet again,

1. that there is a significant difference between the levels of physical normal modes (wave equations), and geometrical normal modes (dispersion relations), and

2. that the densitised inverse metric is in many ways more fundamental than the metric itself.

4.2 Quantum models

4.2.1 Bose–Einstein condensates

We have seen that one of the main aims of research in analogue models of gravity is the possibility of simulating semiclassical gravity phenomena, such as the Hawking radiation effect or cosmological particle production. In this sense systems characterised by a high degree of quantum coherence, very cold temperatures, and low speeds of sound offer the best test field. One could reasonably hope to manipulate these systems to have Hawking temperatures on the order of the environment temperature (~ 100 nK) [48]. Hence it is not surprising that in recent years Bose–Einstein condensates (BECs) have become the subject of extensive study as possible analogue models of general relativity [231, 232, 45, 48, 47, 195, 194].

Let us start by very briefly reviewing the derivation of the acoustic metric for a BEC system, and show that the equations for the phonons of the condensate closely mimic the dynamics of a scalar field in a curved spacetime. In the dilute gas approximation, one can describe a Bose gas by a quantum field $\widehat{\Psi}$ satisfying

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \widehat{\Psi} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + \kappa(a)\,\widehat{\Psi}^\dagger \widehat{\Psi} \right) \widehat{\Psi}. \tag{226}$$

Here κ parameterises the strength of the interactions between the different bosons in the gas. It can be re-expressed in terms of the scattering length a as

$$\kappa(a) = \frac{4\pi a \hbar^2}{m}. \tag{227}$$

As usual, the quantum field can be separated into a macroscopic (classical) condensate and a fluctuation: $\widehat{\Psi} = \psi + \widehat{\varphi}$, with $\langle \widehat{\Psi} \rangle = \psi$. Then, by adopting the self-consistent mean-field approximation (see, for example, [261])

$$\widehat{\varphi}^\dagger \widehat{\varphi} \widehat{\varphi} \simeq 2 \langle \widehat{\varphi}^\dagger \widehat{\varphi} \rangle\, \widehat{\varphi} + \langle \widehat{\varphi} \widehat{\varphi} \rangle\, \widehat{\varphi}^\dagger, \tag{228}$$

one can arrive at the set of coupled equations:

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(t,\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + \kappa\, n_c \right) \psi(t,\mathbf{x})$$
$$+\kappa \left\{ 2\tilde{n}\psi(t,\mathbf{x}) + \tilde{m}\psi^*(t,\mathbf{x}) \right\}; \tag{229}$$

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \widehat{\varphi}(t,\mathbf{x}) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{x}) + \kappa\, 2n_T \right) \widehat{\varphi}(t,\mathbf{x})$$
$$+\kappa\, m_T\, \widehat{\varphi}^\dagger(t,\mathbf{x}). \tag{230}$$

Here

$$n_c \equiv |\psi(t,\mathbf{x})|^2\,; \quad m_c \equiv \psi^2(t,\mathbf{x}); \tag{231}$$

$$\tilde{n} \equiv \langle \widehat{\varphi}^\dagger\, \widehat{\varphi} \rangle; \qquad \tilde{m} \equiv \langle \widehat{\varphi}\, \widehat{\varphi} \rangle; \tag{232}$$

$$n_T = n_c + \tilde{n}; \qquad m_T = m_c + \tilde{m}. \tag{233}$$

The equation for the classical wave function of the condensate is closed only when the backreaction effect due to the fluctuations is neglected. (This backreaction is hiding in the parameters $\tilde{m}$ and $\tilde{n}$.) This is the approximation contemplated by the Gross–Pitaevskii equation. In general, one will have to solve both equations simultaneously. Adopting the Madelung representation for the wave function of the condensate

$$\psi(t,\mathbf{x}) = \sqrt{n_c(t,\mathbf{x})}\ \exp[-\mathrm{i}\theta(t,\mathbf{x})/\hbar], \tag{234}$$

and defining an irrotational "velocity field" by $\mathbf{v} \equiv \boldsymbol{\nabla}\theta/m$, the Gross–Pitaevskii equation can be rewritten as a continuity equation plus an Euler equation:

$$\frac{\partial}{\partial t}n_c + \boldsymbol{\nabla}\cdot(n_c\mathbf{v}) = 0, \tag{235}$$

$$m\frac{\partial}{\partial t}\mathbf{v} + \boldsymbol{\nabla}\left(\frac{mv^2}{2} + V_{\mathrm{ext}}(t,\mathbf{x}) + \kappa n_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right) = 0. \tag{236}$$

These equations are completely equivalent to those of an irrotational and inviscid fluid apart from the existence of the *quantum potential*

$$V_{\mathrm{quantum}} = -\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}, \tag{237}$$

which has the dimensions of an energy. Note that

$$n_c\,\nabla_i V_{\mathrm{quantum}} \equiv n_c\,\nabla_i\left[-\frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right] = \nabla_j\left[-\frac{\hbar^2}{4m}\,n_c\,\nabla_i\nabla_j\ln n_c\right], \tag{238}$$

which justifies the introduction of the *quantum stress tensor*

$$\sigma_{ij}^{\mathrm{quantum}} = -\frac{\hbar^2}{4m}\,n_c\,\nabla_i\nabla_j\ln n_c. \tag{239}$$

This tensor has the dimensions of pressure, and may be viewed as an intrinsically quantum anisotropic pressure contributing to the Euler equation. If we write the mass density of the Madelung fluid as $\rho = m\,n_c$, and use the fact that the flow is irrotational, then the Euler equation takes the form

$$\rho\left[\frac{\partial}{\partial t}\mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right] + \rho\,\boldsymbol{\nabla}\left[\frac{V_{\mathrm{ext}}(t,\mathbf{x})}{m}\right] + \boldsymbol{\nabla}\left[\frac{\kappa\rho^2}{2m^2}\right] + \boldsymbol{\nabla}\cdot\sigma^{\mathrm{quantum}} = 0. \tag{240}$$

Note that the term V_{ext}/m has the dimensions of specific enthalpy, while $\kappa\rho^2/(2m)$ represents a bulk pressure. When the gradients in the density of the condensate are small one can neglect the quantum stress term leading to the standard hydrodynamic approximation. Because the flow is irrotational, the Euler equation is often more conveniently written in Hamilton–Jacobi form:

$$m\frac{\partial}{\partial t}\theta + \left(\frac{[\boldsymbol{\nabla}\theta]^2}{2m} + V_{\mathrm{ext}}(t,\mathbf{x}) + \kappa n_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right) = 0. \tag{241}$$

Apart from the wave function of the condensate itself, we also have to account for the (typically small) quantum perturbations of the system (230). These quantum perturbations can be described in several different ways, here we are interested in the "quantum acoustic representation"

$$\widehat{\varphi}(t, \mathbf{x}) = e^{-i\theta/\hbar} \left(\frac{1}{2\sqrt{n_c}} \, \widehat{n}_1 - i \, \frac{\sqrt{n_c}}{\hbar} \, \widehat{\theta}_1 \right), \tag{242}$$

where $\widehat{n}_1, \widehat{\theta}_1$ are real quantum fields. By using this representation, Equation (230) can be rewritten as

$$\partial_t \widehat{n}_1 + \frac{1}{m} \boldsymbol{\nabla} \cdot \left(n_1 \, \boldsymbol{\nabla}\theta + n_c \, \boldsymbol{\nabla}\widehat{\theta}_1 \right) = 0, \tag{243}$$

$$\partial_t \widehat{\theta}_1 + \frac{1}{m} \boldsymbol{\nabla}\theta \cdot \boldsymbol{\nabla}\widehat{\theta}_1 + \kappa(a) \, n_1 - \frac{\hbar^2}{2m} \, D_2 \widehat{n}_1 = 0. \tag{244}$$

Here D_2 represents a second-order differential operator obtained by linearizing the quantum potential. Explicitly:

$$D_2 \, \widehat{n}_1 \equiv -\frac{1}{2} n_c^{-3/2} \, [\nabla^2(n_c^{+1/2})] \, \widehat{n}_1 + \frac{1}{2} n_c^{-1/2} \, \nabla^2(n_c^{-1/2} \, \widehat{n}_1). \tag{245}$$

The equations we have just written can be obtained easily by linearizing the Gross–Pitaevskii equation around a classical solution: $n_c \to n_c + \widehat{n}_1$, $\phi \to \phi + \widehat{\phi}_1$. It is important to realise that in those equations the backreaction of the quantum fluctuations on the background solution has been assumed negligible. We also see in Equations (243) and (244) that time variations of V_{ext} and time variations of the scattering length a appear to act in very different ways. Whereas the external potential only influences the background Equation (241) (and hence the acoustic metric in the analogue description), the scattering length directly influences both the perturbation and background equations. From the previous equations for the linearised perturbations it is possible to derive a wave equation for $\widehat{\theta}_1$ (or alternatively, for $\widehat{n}_1$). All we need is to substitute in Equation (243) the $\widehat{n}_1$ obtained from Equation (244). This leads to a PDE that is second-order in time derivatives but infinite-order in space derivatives – to simplify things we can construct the symmetric 4×4 matrix

$$f^{\mu\nu}(t, \mathbf{x}) \equiv \begin{bmatrix} f^{00} & \vdots & f^{0j} \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \\ f^{i0} & \vdots & f^{ij} \end{bmatrix}. \tag{246}$$

(Greek indices run from 0–3, while Roman indices run from 1–3.) Then, introducing $(3+1)$-dimensional space-time coordinates,

$$x^\mu \equiv (t; x^i) \tag{247}$$

the wave equation for θ_1 is easily rewritten as

$$\partial_\mu(f^{\mu\nu} \, \partial_\nu \widehat{\theta}_1) = 0. \tag{248}$$

Where the $f^{\mu\nu}$ are *differential operators* acting on space only:

$$f^{00} = -\left[\kappa(a) - \frac{\hbar^2}{2m} \, D_2 \right]^{-1} \tag{249}$$

$$f^{0j} = -\left[\kappa(a) - \frac{\hbar^2}{2m} \, D_2 \right]^{-1} \frac{\nabla^j \theta_0}{m} \tag{250}$$

$$f^{i0} = -\frac{\nabla^i \theta_0}{m} \left[\kappa(a) - \frac{\hbar^2}{2m} \, D_2 \right]^{-1} \tag{251}$$

$$f^{ij} = \frac{n_c \, \delta^{ij}}{m} - \frac{\nabla^i \theta_0}{m} \left[\kappa(a) - \frac{\hbar^2}{2m} \, D_2 \right]^{-1} \frac{\nabla^j \theta_0}{m}. \tag{252}$$

Now, if we make a spectral decomposition of the field $\widehat{\theta}_1$ we can see that for wavelengths larger than $\hbar/mc_s$ (this corresponds to the "healing length", as we will explain below), the terms coming from the linearization of the quantum potential (the D_2) can be neglected in the previous expressions, in which case the $f^{\mu\nu}$ can be approximated by (momentum independent) numbers, instead of differential operators. (This is the heart of the acoustic approximation.) Then, by identifying

$$\sqrt{-g}\, g^{\mu\nu} = f^{\mu\nu}, \tag{253}$$

the equation for the field $\widehat{\theta}_1$ becomes that of a (massless minimally coupled) quantum scalar field over a curved background

$$\Delta\theta_1 \equiv \frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu}\, \partial_\nu \right) \widehat{\theta}_1 = 0, \tag{254}$$

with an effective metric of the form

$$g_{\mu\nu}(t,\mathbf{x}) \equiv \frac{n_c}{m\, c_s(a,n_c)} \left[\begin{array}{c:c} -\{c_s(a,n_c)^2 - v^2\} & -v_j \\ \hdotsfor{1} & \hdotsfor{1} \\ -v_i & \delta_{ij} \end{array} \right]. \tag{255}$$

Here, the magnitude $c_s(n_c, a)$ represents the speed of the phonons in the medium:

$$c_s(a, n_c)^2 = \frac{\kappa(a)\, n_c}{m}. \tag{256}$$

With this effective metric now in hand, the analogy is fully established, and one is now in a position to start asking more specific physics questions.

Lorentz breaking in BEC models – the eikonal approximation: It is interesting to consider the case in which the above "hydrodynamical" approximation for BECs does not hold. In order to explore a regime where the contribution of the quantum potential cannot be neglected we can use the *eikonal* approximation, a high-momentum approximation where the phase fluctuation $\widehat{\theta}_1$ is itself treated as a slowly-varying amplitude times a rapidly varying phase. This phase will be taken to be the same for both $\widehat{n}_1$ and $\widehat{\theta}_1$ fluctuations. In fact, if one discards the unphysical possibility that the respective phases differ by a time-varying quantity, any time-constant difference can be safely reabsorbed in the definition of the (complex) amplitudes. Specifically, we shall write

$$\widehat{\theta}_1(t,\mathbf{x}) = \mathrm{Re}\left\{ \mathcal{A}_\theta\, \exp(-i\phi) \right\}, \tag{257}$$

$$\widehat{n}_1(t,\mathbf{x}) = \mathrm{Re}\left\{ \mathcal{A}_\rho\, \exp(-i\phi) \right\}. \tag{258}$$

As a consequence of our starting assumptions, gradients of the amplitude, and gradients of the background fields, are systematically ignored relative to gradients of ϕ. (Warning: What we are doing here is not quite a "standard" eikonal approximation, in the sense that it is not applied directly on the fluctuations of the field $\psi(t,\mathbf{x})$ but separately on their amplitudes and phases ρ_1 and ϕ_1.) We adopt the notation

$$\omega = \frac{\partial\phi}{\partial t}; \qquad k_i = \nabla_i\phi. \tag{259}$$

Then the operator D_2 can be approximated as

$$D_2\, \widehat{n}_1 \equiv -\frac{1}{2} n_c^{-3/2} \left[\Delta(n_c^{+1/2}) \right] \widehat{n}_1 + \frac{1}{2} n_c^{-1/2}\, \Delta(n_c^{-1/2}\widehat{n}_1) \tag{260}$$

$$\approx +\frac{1}{2} n_c^{-1}\, [\Delta\widehat{n}_1] \tag{261}$$

$$= -\frac{1}{2} n_c^{-1}\, k^2\, \widehat{n}_1. \tag{262}$$

A similar result holds for D_2 acting on $\widehat{\theta}_1$. That is, under the eikonal approximation we effectively replace the *operator* D_2 by the *function*

$$D_2 \rightarrow -\frac{1}{2} n_c^{-1} k^2. \tag{263}$$

For the matrix $f^{\mu\nu}$ this effectively results in the (explicitly momentum dependent) replacement

$$f^{00} \rightarrow -\left[\kappa(a) + \frac{\hbar^2 \, k^2}{4m \, n_c}\right]^{-1} \tag{264}$$

$$f^{0j} \rightarrow -\left[\kappa(a) + \frac{\hbar^2 \, k^2}{4m \, n_c}\right]^{-1} \frac{\nabla^j \theta_0}{m} \tag{265}$$

$$f^{i0} \rightarrow -\frac{\nabla^i \theta_0}{m} \left[\kappa(a) + \frac{\hbar^2 \, k^2}{4m \, n_c}\right]^{-1} \tag{266}$$

$$f^{ij} \rightarrow \frac{n_c \, \delta^{ij}}{m} - \frac{\nabla^i \theta_0}{m} \left[\kappa(a) + \frac{\hbar^2 \, k^2}{4m \, n_c}\right]^{-1} \frac{\nabla^j \theta_0}{m}. \tag{267}$$

As desired, this has the net effect of making $f^{\mu\nu}$ a matrix of (explicitly momentum dependent) numbers, not operators. The physical wave equation (248) now becomes a nonlinear dispersion relation

$$f^{00} \, \omega^2 + (f^{0i} + f^{i0}) \, \omega \, k_i + f^{ij} \, k_i \, k_j = 0. \tag{268}$$

After substituting the approximate D_2 into this dispersion relation and rearranging, we see (remember: $k^2 = ||k||^2 = \delta^{ij} \, k_i \, k_j$)

$$-\omega^2 + 2 \, v_0^i \, \omega k_i + \frac{n_c k^2}{m} \left[\kappa(a) + \frac{\hbar^2}{4mn_c} k^2\right] - (v_0^i \, k_i)^2 = 0. \tag{269}$$

That is:

$$\left(\omega - v_0^i \, k_i\right)^2 = \frac{n_c k^2}{m} \left[\kappa(a) + \frac{\hbar^2}{4mn_c} k^2\right]. \tag{270}$$

Introducing the speed of sound $c_{\rm s}$, this takes the form:

$$\omega = v_0^i \, k_i \pm \sqrt{c_{\rm s}^2 k^2 + \left(\frac{\hbar}{2m} \, k^2\right)^2}. \tag{271}$$

At this stage some observations are in order:

1. It is interesting to recognize that the dispersion relation (271) is exactly in agreement with that found in 1947 by Bogoliubov [79] (reprinted in [508]; see also [374]) for the collective excitations of a homogeneous Bose gas in the limit $T \rightarrow 0$ (almost complete condensation). In his derivation Bogoliubov applied a diagonalization procedure for the Hamiltonian describing the system of bosons.

2. Coincidentally this is the same dispersion relation that one encounters for shallow-water surface waves in the presence of surface tension. See Section 4.1.4.

3. Because of the explicit momentum dependence of the co-moving phase velocity and co-moving group velocity, once one goes to high momentum the associated effective metric should be thought of as one of many possible "rainbow metrics" as in Section 4.1.4. See also [643]. (At low momentum one, of course, recovers the hydrodynamic limit with its uniquely specified standard metric.)

4. It is easy to see that Equation (271) actually interpolates between two different regimes depending on the value of the wavelength $\lambda = 2\pi/||k||$ with respect to the "acoustic Compton wavelength" $\lambda_c = h/(mc_{\rm s})$. (Remember that $c_{\rm s}$ is the speed of sound; this is not a standard particle physics Compton wavelength.) In particular, if we assume $v_0 = 0$ (no background velocity), then, for large wavelengths $\lambda \gg \lambda_c$, one gets a standard phonon dispersion relation $\omega \approx c||k||$. For wavelengths $\lambda \ll \lambda_c$ the quasi-particle energy tends to the kinetic energy of an individual gas particle and, in fact, $\omega \approx \hbar^2 k^2/(2m)$.

We would also like to highlight that in relative terms, the approximation by which one neglects the quartic terms in the dispersion relation gets worse as one moves closer to a horizon where $v_0 = -c_{\rm s}$. The non-dimensional parameter that provides this information is defined by

$$\delta \equiv \frac{\sqrt{1 + \frac{\lambda_c^2}{4\lambda^2}} - 1}{(1 - v_0/c_{\rm s})} \simeq \frac{1}{(1 - v_0/c_{\rm s})} \frac{\lambda_c^2}{8\lambda^2}. \tag{272}$$

As we will discuss in Section 5.2, this is the reason why sonic horizons in a BEC can exhibit different features from those in standard general relativity.

5. The dispersion relation (271) exhibits a contribution due to the background flow $v_0^i \, k_i$, plus a quartic dispersion at high momenta. The group velocity is

$$v_g^i = \frac{\partial \omega}{\partial k_i} = v_0^i \pm \frac{\left(c^2 + \frac{\hbar^2}{2m^2} k^2\right)}{\sqrt{c^2 k^2 + \left(\frac{\hbar}{2m} \, k^2\right)^2}} \, k^i \, . \tag{273}$$

Indeed, with hindsight, the fact that the group velocity goes to infinity for large k was pre-ordained: After all, we started from the generalised nonlinear Schrödinger equation, and we know what its characteristic curves are. Like the diffusion equation the characteristic curves of the Schrödinger equation (linear or nonlinear) move at infinite speed. If we then approximate this generalised nonlinear Schrödinger equation in any manner, for instance by linearization, we cannot change the characteristic curves: For any well-behaved approximation technique, at high frequency and momentum we should recover the characteristic curves of the system we started with. However, what we certainly do see in this analysis is a suitably large region of momentum space for which the concept of the effective metric both makes sense, and leads to finite propagation speed for medium-frequency oscillations.

Relativistic BEC extension: Bose–Einstein condensation can occur not only for non-relativistic bosons but for relativistic ones as well. The main differences between the thermodynamical properties of these condensates at finite temperature are due both to the different energy spectra and also to the presence, for relativistic bosons, of anti-bosons. These differences result in different conditions for the occurrence of Bose–Einstein condensation, which is possible, e.g., in two spatial dimensions for a homogeneous relativistic Bose gas, but not for its non-relativistic counterpart – and also, more importantly for our purposes, in the different structure of their excitation spectra.

In [191] an analogue model based on a relativistic BEC was studied. We summarise here the main results. The Lagrangian density for an interacting relativistic scalar Bose field $\hat{\phi}(\mathbf{x}, t)$ may be written as

$$\hat{\mathcal{L}} = \frac{1}{c^2} \frac{\partial \hat{\phi}^\dagger}{\partial t} \frac{\partial \hat{\phi}}{\partial t} - \nabla \hat{\phi}^\dagger \cdot \nabla \hat{\phi} - \left(\frac{m^2 c^2}{\hbar^2} + V(t, \mathbf{x})\right) \hat{\phi}^\dagger \hat{\phi} - U(\hat{\phi}^\dagger \hat{\phi}; \lambda_i) \, , \tag{274}$$

where $V(t, \mathbf{x})$ is an external potential depending both on time t and position $\mathbf{x}$, m is the mass of the bosons and c is the light velocity. U is an interaction term and the coupling constant $\lambda_i(t, \mathbf{x})$

can depend on time and position too (this is possible, for example, by changing the scattering length via a Feshbach resonance [151, 175]). U can be expanded as

$$U(\hat{\phi}^\dagger \hat{\phi}; \lambda_i) = \frac{\lambda_2}{2}\hat{\rho}^2 + \frac{\lambda_3}{6}\hat{\rho}^3 + \cdots \tag{275}$$

where $\hat{\rho} = \hat{\phi}^\dagger \hat{\phi}$. The usual two-particle $\lambda_2 \hat{\phi}^4$-interaction corresponds to the first term $(\lambda_2/2)\hat{\rho}^2$, while the second term represents the three-particle interaction and so on.

The field $\hat{\phi}$ can be written as a classical field (the condensate) plus perturbation:

$$\hat{\phi} = \phi(1 + \hat{\psi}) \,. \tag{276}$$

It is worth noticing now that the expansion in Equation (276) can be linked straightforwardly to the previously discussed expansion in phase and density perturbations $\hat{\theta}_1$, $\hat{\rho}_1$, by noting that

$$\frac{\hat{\rho}_1}{\rho} = \frac{\hat{\psi} + \hat{\psi}^\dagger}{2}, \quad \hat{\theta}_1 = \frac{\hat{\psi} - \hat{\psi}^\dagger}{2i} \,.$$

Setting $\psi \propto \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ one then gets from the equation of motion [191]

$$\left(-\frac{\hbar}{m}\mathbf{q} \cdot \mathbf{k} + \frac{u^0}{c}\omega - \frac{\hbar}{2mc^2}\omega^2 + \frac{\hbar}{2m}k^2\right)\left(\frac{\hbar}{m}\mathbf{q} \cdot \mathbf{k} - \frac{u^0}{c}\omega - \frac{\hbar}{2mc^2}\omega^2 + \frac{\hbar}{2m}k^2\right)$$
$$-\left(\frac{c_0}{c}\right)^2 \omega^2 + c_0^2 k^2 = 0 \,, \tag{277}$$

where, for convenience, we have defined the following quantities as

$$u^\mu \equiv \frac{\hbar}{m}\eta^{\mu\nu}\partial_\nu \theta \,, \tag{278}$$

$$c_0^2 \equiv \frac{\hbar^2}{2m^2}U''(\rho; \lambda_i)\rho \,, \tag{279}$$

$$\mathbf{q} \equiv m\mathbf{u}/\hbar \,. \tag{280}$$

Here $\mathbf{q}$ is the speed of the condensate flow and c is the speed of light. For a condensate at rest ($\mathbf{q} = 0$) one then obtains the following dispersion relation

$$\omega_\pm^2 = c^2\left\{k^2 + 2\left(\frac{mu^0}{\hbar}\right)^2\left[1 + \left(\frac{c_0}{u^0}\right)^2\right] \pm 2\left(\frac{mu^0}{\hbar}\right)\sqrt{k^2 + \left(\frac{mu^0}{\hbar}\right)^2\left[1 + \left(\frac{c_0}{u^0}\right)^2\right]^2}\right\} \,. \tag{281}$$

The dispersion relation (281) is sufficiently complicated to prevent any obvious understanding of the regimes allowed for the excitation of the system. It is much richer than the non-relativistic case. For example, it allows for both a massless/gapless (phononic) and massive/gapped mode, respectively for the ω_- and ω_+ branches of (281). Nonetheless, it should be evident that different regimes are determined by the relative strength of the the first two terms on the right-hand side of Equation (281) (note that the same terms enter in the square root). This can be summarised, in low and high momentum limits respectively, for k much less or much greater than

$$\frac{mu^0}{\hbar}\left[1 + \left(\frac{c_0}{u^0}\right)^2\right] \equiv \frac{mu^0}{\hbar}(1 + b), \tag{282}$$

where b encodes the relativistic nature of the condensate (the larger b the more the condensate is relativistic).

Table 1: Dispersion relation of gapless and gapped modes in different regimes. Note that we have $c_s^2 = c^2 b/(1+b)$, and $c_{s,\mathrm{gap}}^2 = c^2(2+b)/(1+b)$, while $m_{\mathrm{eff}} = 2(\mu/c^2)(1+b)^{3/2}/(2+b)$.

		Gapless		Gapped
		$b \ll 1$	$b \gg 1$	
$\|k\| \ll \frac{mu^0(1+b)}{\hbar}$	$\|k\| \ll \frac{2mc^0}{\hbar}$	$\omega^2 = c_s^2 k^2$	$\omega^2 = c_s^2 k^2$	$\omega^2 = \frac{m_{\mathrm{eff}}^2 c_{s,\mathrm{gap}}^4}{\hbar^2} + c_{s,\mathrm{gap}}^2 k^2$
	$\frac{2mc^0}{\hbar} \ll \|k\| \ll \frac{mu^0}{\hbar}$	$\hbar\omega = \frac{(\hbar ck)^2}{2\mu}$		
$\|k\| \gg \frac{mu^0(1+b)}{\hbar}$		$\omega^2 = c^2 k^2$		

A detailed discussion of the different regimes would be inappropriately long for this review; it can be found in [191]. The results are summarised in Table 1. Note that $\mu \equiv mcu^0$ plays the role of the chemical potential for the relativistic BEC. One of the most remarkable features of this model is that it is a condensed matter system that interpolates between two different Lorentz symmetries, one at low energy and a different Lorentz symmetry at high energy.

Finally, it is also possible to recover an acoustic metric for the massless (phononic) perturbations of the condensate in the low momentum limit ($k \ll mu^0(1+b)/\hbar$):

$$g_{\mu\nu} = \frac{\rho}{\sqrt{1 - u_\sigma u^\sigma/c_o^2}} \left[\eta_{\mu\nu} \left(1 - \frac{u_\sigma u^\sigma}{c_0^2} \right) + \frac{u_\mu u_\nu}{c_0^2} \right] . \tag{283}$$

As should be expected, it is just a version of the acoustic geometry for a relativistic fluid previously discussed, and in fact can be cast in the form of Equation (155) by suitable variable redefinitions [191].

4.2.2 The heliocentric universe

Helium is one of the most fascinating elements provided by nature. Its structural richness confers on helium a paradigmatic character regarding the emergence of many and varied macroscopic properties from the microscopic world (see [660] and references therein). Here, we are interested in the emergence of effective geometries in helium, and their potential use in testing aspects of semiclassical gravity.

Helium four, a bosonic system, becomes superfluid at low temperatures (2.17 K at vapour pressure). This superfluid behaviour is associated with condensation in the vacuum state of a macroscopically large number of atoms. A superfluid is automatically an irrotational and inviscid fluid, so, in particular, one can apply to it the ideas worked out in Section 2. The propagation of classical acoustic waves (scalar waves) over a background fluid flow can be described in terms of an effective Lorentzian geometry: the acoustic geometry. However, in this system one can naturally go considerably further, into the quantum domain. For long wavelengths, the quasiparticles in this system are quantum phonons. One can separate the classical behaviour of a background flow (the effective geometry) from the behaviour of the quantum phonons over this background. In this way one can reproduce, in laboratory settings, different aspects of quantum field theory over curved backgrounds. The speed of sound in the superfluid phase is typically on the order of cm/sec. Therefore, at least in principle, it should not be too difficult to establish configurations with supersonic flows and their associated ergoregions.

Helium three, the fermionic isotope of helium, in contrast, becomes superfluid at much lower temperatures (below 2.5 milli-K). The reason behind this rather different behaviour is the pairing

of fermions to form effective bosons (Cooper pairing), which are then able to condense. In the ^{3}He-A phase, the structure of the fermionic vacuum is such that it possesses two Fermi points, instead of the more typical Fermi surface. In an equilibrium configuration one can choose the two Fermi points to be located at $\{p_x = 0, p_y = 0, p_z = \pm p_F\}$ (in this way, the z-axis signals the direction of the angular momentum of the pairs). Close to either Fermi point the spectrum of quasiparticles becomes equivalent to that of Weyl fermions. From the point of view of the laboratory, the system is not isotropic, it is axisymmetric. There is a speed for the propagation of quasiparticles along the z-axis, $c_\parallel \simeq$ cm/sec, and a different speed, $c_\perp \simeq 10^{-5} c_\parallel$, for propagation perpendicular to the symmetry axis. However, from an internal observer's point of view this anisotropy is not "real", but can be made to disappear by an appropriate rescaling of the coordinates. Therefore, in the equilibrium case, we are reproducing the behaviour of Weyl fermions over Minkowski spacetime. Additionally, the vacuum can suffer collective excitations. These collective excitations will be experienced by the Weyl quasiparticles as the introduction of an effective electromagnetic field and a curved Lorentzian geometry. The control of the form of this geometry provides the sought for gravitational analogy.

Apart from the standard way to provide a curved geometry based on producing nontrivial flows, there is also the possibility of creating topologically nontrivial configurations with a built-in nontrivial geometry. For example, it is possible to create a domain-wall configuration [327, 326] (the wall contains the z-axis) such that the transverse velocity $c_\perp$ acquires a profile in the perpendicular direction (say along the x-axis) with $c_\perp$ passing through zero at the wall (see Figure 11). This particular arrangement could be used to reproduce a black-hole–white-hole configuration only if the soliton is set up to move with a certain velocity along the x-axis. This configuration has the advantage that it is dynamically stable, for topological reasons, even when some supersonic regions are created.
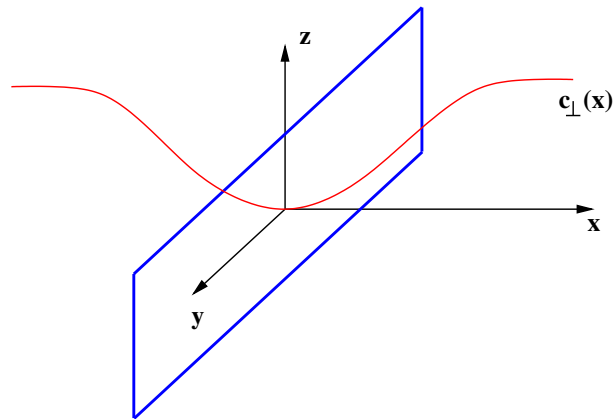
Figure 11: Domain wall configuration in ^{3}He.

A third way in which superfluid helium can be used to create analogues of gravitational configurations is the study of surface waves (or ripplons) on the interface between two different phases of ^{3}He [657, 659]. In particular, if we have a thin layer of ^{3}He-A in contact with another thin layer of ^{3}He-B, the oscillations of the contact surface "see" an effective metric of the form [657, 659]

$$ds^2 = \frac{1}{(1 - \alpha_A \alpha_B U^2)} \left[- \left(1 - W^2 - \alpha_A \alpha_B U^2\right) \, dt^2 - 2\mathbf{W} \cdot d\mathbf{x} \, dt + d\mathbf{x} \cdot d\mathbf{x} \right], \tag{284}$$
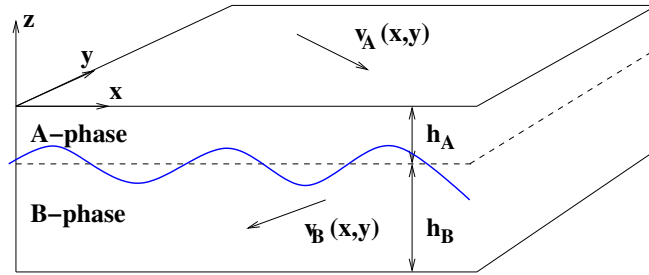
where

$$\mathbf{W} \equiv \alpha_A \mathbf{v_A} + \alpha_B \mathbf{v_B}, \qquad \mathbf{U} \equiv \mathbf{v_A} - \mathbf{v_B}, \tag{285}$$

and

$$\alpha_A \equiv \frac{h_B \rho_A}{h_A \rho_B + h_B \rho_A}; \qquad \alpha_B \equiv \frac{h_A \rho_B}{h_A \rho_B + h_B \rho_A}. \tag{286}$$

(All of this provided that we are looking at wavelengths larger than the layer thickness, $k\, h_A \ll 1$ and $k\, h_B \ll 1$.)

Figure 12: Ripplons in the interface between two sliding superfluids.

The advantage of using surface waves instead of bulk waves in superfluids is that one could create horizons without reaching supersonic speeds in the bulk fluid. This could alleviate the appearance of dynamical instabilities in the system, that in this case are controlled by the strength of the interaction of the ripplons with bulk degrees of freedom [657, 659].

4.2.3   Slow light in fluids

The geometrical interpretation of the motion of light in dielectric media leads naturally to conjecture that the use of flowing dielectrics might be useful for simulating general relativity metrics with ergoregions and black holes. Unfortunately, these types of geometry require flow speeds comparable to the group velocity of the light. Since typical refractive indexes in non-dispersive media are quite close to unity, it is then clear that it is practically impossible to use them to simulate such general relativistic phenomena. However recent technological advances have radically changed this state of affairs. In particular the achievement of controlled slowdown of light, down to velocities of a few meters per second (or even down to complete rest) [617, 338, 96, 353, 506, 603, 565], has opened a whole new set of possibilities regarding the simulation of curved-space metrics via flowing dielectrics.

But how can light be slowed down to these "snail-like" velocities? The key effect used to achieve this takes the name of Electromagnetically Induced Transparency (EIT). A laser beam is coupled to the excited levels of some atom and used to strongly modify its optical properties. In particular one generally chooses an atom with two long-lived metastable (or stable) states, plus a higher energy state that has some decay channels into these two lower states. The coupling of the excited states induced by the laser light can affect the transition from a lower energy state to the higher one, and hence the capability of the atom to absorb light with the required transition energy. The system can then be driven into a state where the transitions between each of the lower energy states and the higher energy state exactly cancel out, due to quantum interference, at some specific resonant frequency. In this way the higher-energy level has null averaged occupation number. This state is hence called a "dark state". EIT is characterised by a transparency window, centered around the resonance frequency, where the medium is *both* almost transparent *and* extremely dispersive (strong dependence on frequency of the refractive index). This in turn implies that the group velocity of any light probe would be characterised by very low real group velocities (with almost vanishing imaginary part) in proximity to the resonant frequency.

Let us review the most common setup envisaged for this kind of analogue model. A more detailed analysis can be found in [383]. One can start by considering a medium in which an EIT window is opened via some control laser beam which is oriented perpendicular to the direction of the flow. One then illuminates this medium, now along the flow direction, with some probe light (which is hence perpendicular to the control beam). This probe beam is usually chosen to be weak with respect to the control beam, so that it does not modify the optical properties of the medium. In the case in which the optical properties of the medium do not vary significantly over several wavelengths of the probe light, one can neglect the polarization and can hence describe the propagation of the latter with a simple scalar dispersion relation [390, 211]

$$k^2 = \frac{\omega^2}{c^2} \left[1 + \chi(\omega) \right], \tag{287}$$

where χ is the susceptibility of the medium, related to the refractive index n via the simple relation $n = \sqrt{1 + \chi}$.

It is easy to see that in this case the group and phase velocities differ

$$v_{\mathrm{g}} = \frac{\partial \omega}{\partial k} = \frac{c}{\sqrt{1+\chi} + \frac{\omega}{2n}\frac{\partial \chi}{\partial \omega}}; \qquad v_{\mathrm{ph}} = \frac{\omega}{k} = \frac{c}{\sqrt{1+\chi}}. \tag{288}$$

So even for small refractive indexes one can get very low group velocities, due to the large dispersion in the transparency window, and in spite of the fact that the phase velocity remains very near to c. (The phase velocity is exactly c at the resonance frequency ω_0). In an ideal EIT regime the probe light experiences a vanishing susceptibility χ near the the critical frequency ω_0, this allows us to express the susceptibility near the critical frequency via the expansion

$$\chi(\omega) = \frac{2\alpha}{\omega_0} \left(\omega - \omega_0 \right) + O\left[(\omega - \omega_0)^3 \right], \tag{289}$$

where α is sometimes called the "group refractive index". The parameter α depends on the dipole moments for the transition from the metastable states to the high energy one, and most importantly depends on the ratio between the probe-light energy per photon, $\hbar\omega_0$, and the control-light energy per atom [383]. This might appear paradoxical because it seems to suggest that for a dimmer control light the probe light would be further slowed down. However this is just an artificial feature due to the extension of the EIT regime beyond its range of applicability. In particular in order to be effective the EIT requires the control beam energy to dominate all processes and hence it cannot be dimmed at will.

At resonance we have

$$v_{\mathrm{g}} = \frac{\partial \omega}{\partial k} \to \frac{c}{1+\alpha} \approx \frac{c}{\alpha}; \qquad v_{\mathrm{ph}} = \frac{\omega}{k} \to c. \tag{290}$$

We can now generalise the above discussion to the case in which our highly dispersive medium flows with a characteristic velocity profile $\mathbf{u}(\mathbf{x}, t)$. In order to find the dispersion relation of the probe light in this case we just need to transform the dispersion relation (287) from the comoving frame of the medium to the laboratory frame. Let us consider for simplicity a monochromatic probe light (more realistically a pulse with a very narrow range of frequencies ω near ω_0). The motion of the dielectric medium creates a local Doppler shift of the frequency

$$\omega \to \gamma \left(\omega_0 - \mathbf{u} \cdot \mathbf{k} \right), \tag{291}$$

where γ is the usual relativistic factor. Given that $k^2 - \omega^2/c^2$ is a Lorentz invariant, it is then easy to see that this Doppler detuning affects the dispersion relation (287) only via the susceptibility

dependent term. Given further that in any realistic case one would deal with non-relativistic fluid velocities $\mathbf{u} \ll c$ we can then perform an expansion of the dispersion relation up to second order in u/c. Expressing the susceptibility via Equation (289) we can then rewrite the dispersion relation in the form [390]

$$g^{\mu\nu}k_\mu k_\nu = 0, \tag{292}$$

where

$$k_\nu = \left(\frac{\omega_0}{c}, -\mathbf{k}\right), \tag{293}$$

and

$$g^{\mu\nu} = \left[\begin{array}{c|c} -(1+\alpha u^2/c^2) & -\alpha\mathbf{u}^T/c^2 \\ \hline -\alpha\mathbf{u}/c^2 & \mathbf{I}_{3\times 3} - 4\alpha\mathbf{u}\otimes\mathbf{u}^T/c^2 \end{array}\right]. \tag{294}$$

(Note that most of the original articles on this topic adopt the opposite signature $(+---)$.) The inverse of this tensor will be the covariant effective metric experienced by the probe light, whose rays would then be null geodesics of the line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$. In this sense the probe light will propagate as in a curved background. Explicitly one finds the covariant metric to be

$$g_{\mu\nu} = \left[\begin{array}{c|c} -A & -B\mathbf{u}^T \\ \hline -B\mathbf{u} & \mathbf{I}_{3\times 3} - C\mathbf{u}\otimes\mathbf{u}^T \end{array}\right], \tag{295}$$

where

$$A = \frac{1 - 4\alpha u^2/c^2}{1 + (\alpha^2 - 3\alpha)u^2/c^2 - 4\alpha^2 u^4/c^4}; \tag{296}$$

$$B = \frac{1}{1 + (\alpha^2 - 3\alpha)u^2/c^2 - 4\alpha^2 u^4/c^4}; \tag{297}$$

$$C = \frac{1 - (4/\alpha + 4u^2/c^2)}{1 + (\alpha^2 - 3\alpha)u^2/c^2 - 4\alpha^2 u^4/c^4}. \tag{298}$$

Several comments are in order concerning the metric (295). First of all, it is clear that, although more complicated than an acoustic metric, it will still be possible to cast it into the Arnowitt–Deser–Misner-like form [627]

$$g_{\mu\nu} = \left[\begin{array}{c|c} -[c_{\mathrm{eff}}^2 - g_{ab}u_{\mathrm{eff}}^a u_{\mathrm{eff}}^b] & [u_{\mathrm{eff}}]_i \\ \hline [u_{\mathrm{eff}}]_j & [g_{\mathrm{eff}}]_{ij} \end{array}\right], \tag{299}$$

where the effective speed $\mathbf{u}_{\mathrm{eff}}$ is proportional to the fluid flow speed $\mathbf{u}$ and the three-space effective metric g_{eff} is (rather differently from the acoustic case) nontrivial.

In any case, the existence of this ADM form already tells us that an ergoregion will always appear once the norm of the effective velocity exceeds the effective speed of light (which for slow light is approximately c/α, where α can be extremely large due to the huge dispersion in the transparency window around the resonance frequency ω_0). However, a trapped surface (and hence an optical black hole) will form only if the *inward normal component* of the effective flow velocity exceeds the group velocity of light. In the slow light setup so far considered such a velocity turns out to be $u = c/(2\sqrt{\alpha})$.

The realization that ergoregions and event horizons can be simulated via slow light may lead one to the (erroneous) conclusion that this is an optimal system for simulating particle creation by gravitational fields. However, as pointed out by Unruh in [470, 612], such a conclusion would turn out to be over-enthusiastic. In order to obtain particle creation through "mode mixing", (mixing between the positive and negative norm modes of the incoming and outgoing states), an

inescapable requirement is that there must be regions where the frequency of the quanta as seen by a local comoving observer becomes negative.

In a flowing medium this can, in principle, occur thanks to the tilting of the dispersion relation due to the Doppler effect caused by the velocity of the flow Equation (291); but this also tells us that the negative norm mode must satisfy the condition $\omega_0 - \mathbf{u} \cdot \mathbf{k} < 0$, but this can be satisfied only if the velocity of the medium exceeds $|\omega_0/k|$, which is the *phase* velocity of the probe light, not its group velocity. This observation suggests that the existence of a "phase velocity horizon" is an essential ingredient (but not the only essential ingredient) in obtaining Hawking radiation. A similar argument indicates the necessity for a specific form of "group velocity horizon", one that lies on the negative norm branch. Since the phase velocity in the slow light setup we are considering is very close to c, the physical speed of light in vacuum, not very much hope is left for realizing analogue particle creation in this particular laboratory setting.

However, it was also noticed by Unruh and Schützhold [612] that a different setup for slow light might deal with this and other issues (see [612] for a detailed summary). In the setup suggested by these authors there are two strong-background counter-propagating control beams illuminating the atoms. The field describing the beat fluctuations of this electromagnetic background can be shown to satisfy, once the dielectric medium is in motion, the same wave equation as that on a curved background. In this particular situation the phase velocity and the group velocity are approximately the same, and both can be made small, so that the previously discussed obstruction to mode mixing is removed. So in this new setup it *is* concretely possible to simulate classical particle creation such as, e.g., super-radiance in the presence of ergoregions.

Nonetheless, the same authors showed that this does not open the possibility for a simulation of quantum particle production (e.g., Hawking radiation). This is because that effect also requires the commutation relations of the field to generate the appropriate zero-point energy fluctuations (the vacuum structure) according to the Heisenberg uncertainty principle. This is not the case for the effective field describing the beat fluctuations of the system we have just described, which is equivalent to saying that it does not have a proper vacuum state (i.e., analogue to any physical field). Hence, one has to conclude that any simulation of quantum particle production is precluded.

4.2.4 Slow light in fibre optics

In addition to the studies of slow light in fluids, there has now been a lot of work done on slow light in a fibre-optics setting [505, 504, 64, 63], culminating in recent experimental detection of photons apparently associated with a phase-velocity horizon [66]. The key issue here is that the Kerr effect of nonlinear optics can be used to change the refractive index of an optical fibre, so that a "carrier" pulse of light traveling down the fibre carries with it a region of high refractive index, which acts as a barrier to "probe" photons (typically at a different frequency). If the relative velocities of the "carrier" pulse and "probe" are suitably arranged then the arrangement can be made to mimic a black-hole–white-hole pair. This system is described more fully in Section 6.4.

4.2.5 Lattice models

The quantum analogue models described above all have an underlying discrete structure: namely the atoms they are made of. In abstract terms one can also build an analogue model by considering a quantum field on specific lattice structures representing different spacetimes. In [310, 149, 322] a falling-lattice black-hole analogue was put forward, with a view to analyzing the origin of Hawking particles in black-hole evaporation. The positions of the lattice points in this model change with time as they follow freely falling trajectories. This causes the lattice spacing at the horizon to grow approximately linearly with time. By definition, if there were no horizons, then for long wavelengths compared with the lattice spacing one would recover a relativistic quantum field theory over a classical background. However, the presence of horizons makes it impossible to analyze the

field theory only in the continuum limit, it becomes necessary to recall the fundamental lattice nature of the model.

4.2.6 Graphene

A very interesting addition to the catalogue of analogue systems is the graphene (see, for example, these reviews [127, 339]). Although graphene and some of its peculiar electronic properties have been known since the 1940s [672], only recently has it been specifically proposed as a system with which to probe gravitational physics [153, 152]. Graphene (or mono-layer graphite) is a two-dimensional lattice of carbon atoms forming a hexagonal structure (see Figure 13). From the perspective of this review, its most important property is that its Fermi surface has two independent Fermi points (see Section 4.2.2 on helium). The low-energy excitations around these points can be described as massless Dirac fields in which the light speed is substituted by a "sound" speed c_s about 300 times smaller:

$$\partial_t \psi_j = c_s \sum_{k=1,2} \sigma^k \partial_{x_k} \psi_j \,. \tag{300}$$

Here ψ_j with $j = 1, 2$ represent two types of massless spinors (one for each Fermi point), the σ^k are the Pauli sigma matrices, and $c_s = 3ta/2$, with $a = 1.4$ Å being the interatomic distance, and $t = 2.8$ eV the hopping energy for an electron between two nearest-neighboring atoms.
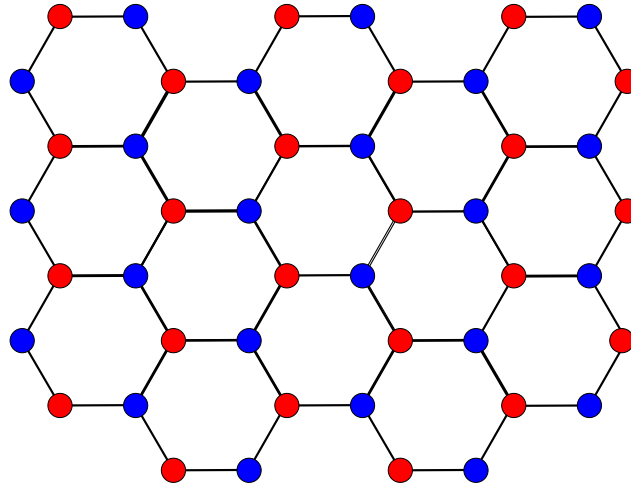
Figure 13: The graphene hexagonal lattice is made of two inter-penetrating triangular lattices. Each is associated with one Fermi point.

From this perspective graphene can be used to investigate ultra-relativistic phenomena such as the Klein paradox [339]. On the other hand, graphene sheets can also acquire curvature. A nonzero curvature can be produced by adding strain fields to the sheet, imposing a curved substrate, or by introducing topological defects (e.g., some pentagons within the hexagonal structure) [669]. It has been suggested that, regarding the electronic properties of graphene, the sheet curvature promotes the Dirac equation to its curved space counterpart, at least on the average [153, 152]. If this proves to be experimentally correct, it will make graphene a good analogue model for a diverse set of spacetimes. This set, however, does not include black-hole spacetimes, as the curvatures mentioned above are purely spatial and do not affect the temporal components of the metric.

4.3   Going further

We feel that the catalogue we have just presented is reasonably complete and covers the key items. For additional background on many of these topics, we would suggest sources such as the books "Artificial Black Holes" [470] and "The Universe in a Helium Droplet" [660]. For more specific detail, check this review's bibliography, and use SPIRES (or the beta version of INSPIRE) to check for recent developments.

5 Phenomenology of Analogue Models

Of course, the entire motivation for looking at analogue models is to be able to learn more physics. One could start studying analogue models with the idea of seeing whether it is possible (either theoretically or in practice) to reproduce in the laboratory various gravitational phenomena whose real existence in nature cannot be currently checked. These are phenomena that surpass our present (and foreseeable) observational capabilities, but yet, we believe in their existence because it follows from seemingly strong theoretical arguments within the standard frameworks of general relativity and field theory in curved space. However, the interest of this approach is not merely to reproduce these gravitational phenomena in some formal analogue model, but to see which departures from the ideal case show up in real analogue models, and to analyse whether similar deviations are likely to appear in real gravitational systems.

When one thinks about emergent gravitational features in condensed-matter systems, one immediately realises that these features only appear in the low-energy regime of the analogue systems. When these systems are probed at high energies (short length scales) the effective geometrical description of the analogue models break down, as one starts to be aware that the systems are actually composed of discrete pieces (atoms and molecules). This scenario is quite similar to what one expects to happen with our geometrical description of the Universe, when explored with microscopic detail at the Planck scale.

That is, the study of analogue models of general relativity is giving us insights as to how the standard theoretical picture of different gravitational phenomena could change when taking into account additional physical knowledge coming from the existence of an underlying microphysical structure. Quite robustly, these studies are telling us already that the first deviations from the standard general relativity picture can be encoded in the form of high-energy violations of Lorentz invariance in particle dispersion relations. Beyond these first deviations, the analogue models of general relativity provide well-understood examples (the underlying physics is well known) in which a description in terms of fields in curved spacetimes shows up as a low-energy-regime emergent phenomena.

The analogue models are being used to shed light on these general questions through a number of specific routes. Let us now turn to discussing several specific physics issues that are being analysed from this perspective.

5.1 Hawking radiation

5.1.1 Basics

As is well known, in 1974 Stephen Hawking announced that quantum mechanically black holes should emit radiation with a spectrum approximately that of a black body [270, 271]. We shall not re-derive the existence of Hawking radiation from scratch, but will instead assume a certain familiarity with at least the basics of the phenomenon. (See, for instance, [95] or [309].) The collapse of a distribution of matter will end up forming an evaporating black hole emitting particles from its horizon toward future null infinity. Hawking radiation is a quantum-field-in-curved-space effect: The existence of radiation emission is a kinematic effect that does not rely on Einstein's equations. Therefore, one can aim to reproduce it in a condensed-matter system. Within standard field theory, a minimal requirement for having Hawking radiation is the existence in the background configuration of an apparent horizon [629]. So, in principle, to be able to reproduce Hawking radiation in a laboratory one would have to fulfill at least two requirements:

1. To choose an adequate analogue system; it has to be a quantum analogue model (see Section 4) such that its description could be separated into a classical effective background spacetime plus some standard relativistic quantum fields living on it (it can happen that

the quantum fields do not satisfy the appropriate commutation or anti-commutation relations [612]).

2. To configure the analogue geometry such that it includes some sort of horizon. That is, within an appropriate quantum analogue model, the formation of an apparent horizon for the propagation of the quantum fields should excite the fields such as to result in the emission of a thermal distribution of field particles. [20]

This is a straightforward and quite naive translation of the standard Hawking effect derivation to the condensed matter realm. However, in reality, this translation process has to take into account additional issues that, as we are trying to convey, instead of problems, are where the interesting physics lies.

1. The effective description of the quantum analogue systems as fields over a background geometry breaks down when probed at sufficiently short length scales. This could badly influence the main features of Hawking radiation. In fact, immediately after the inception of the idea that black holes radiate, it was realised that there was a potential problem with the calculation [606]. It strongly relies on the validity of quantum field theory on curved backgrounds up to arbitrary high energies. Following a wave packet with a certain frequency at future infinity backwards in time, we can see that it had to contain arbitrarily large frequency components with respect to a local free fall observer (well beyond the Planck scale) when it was close to the horizon. In principle, any unknown physics at the Planck scale could strongly influence the Hawking process so that one should view it with caution. This is the *trans-Planckian* problem of Hawking radiation. To create an analogue model exhibiting Hawking radiation will be, therefore, equivalent to giving a solution to the trans-Planckian problem.

2. In order to clearly observe Hawking radiation, one should first be sure that there is no other source of instabilities in the system that could mask the effect. In analogue models such as liquid helium or BECs the interaction of a radial flow (with speed on the order of the critical Landau speed, which in these cases coincides with the sound speed [359]) with the surface of the container (an electromagnetic potential in the BECs case) might cause the production of rotons and quantised vortices, respectively. Thus, in order to produce an analogue model of Hawking radiation, one has to be somewhat ingenious. For example, in the liquid helium case, instead of taking acoustic waves in a supersonic flow stream as the analogue model, it is preferable to use as analogue model ripplons in the interface between two different phases, A and B phases, of liquid helium three [657]. Another option is to start from a moving domain wall configuration. Then, the topological stability of the configuration prevents its destruction when creating a horizon [326, 327]. In the case of BECs, a way to suppress the formation of quantised vortices is to take effectively one-dimensional configurations. If the transverse dimension of the flow is smaller than the healing length, then there is no space for the existence of a vortex [48]. In either liquid helium or BECs, there is also the possibility of creating an apparent horizon by rapidly approaching a critical velocity profile (see Figure 14), but without actually crossing into the supersonic regime [37], softening in this way the appearance of dynamical instabilities.

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[20] One could also imagine systems in which the effective metric fails to exist on one side of the horizon (or even more radically, on both sides). The existence of particle production in this kind of system will then depend on the specific interactions between the sub-systems characterizing each side of the horizon. For example, in stationary configurations it will be necessary that these interactions allow negative energy modes to disappear beyond the horizon, propagating forward in time (as happens in an ergoregion). Whether these systems will provide adequate analogue models of Hawking radiation or not is an interesting question that deserves future analysis. Certainly systems of this type lie well outside the class of usual analogue models.
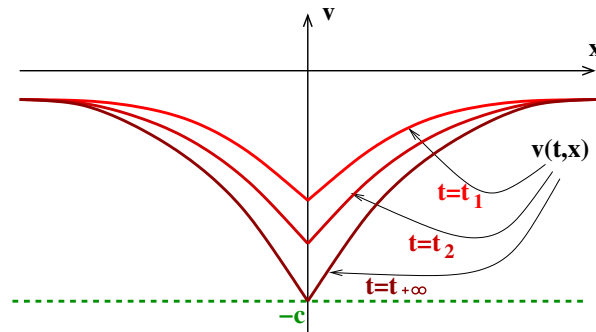
Figure 14: Velocity profile for a left going flow; the profile is dynamically modified with time so that it reaches the profile with a sonic point at the asymptotic future.

3. Real analogue models cannot, strictly speaking, reproduce eternal black-hole configurations. An analogue model of a black hole has always to be created at some finite laboratory time. Therefore, one is forced to carefully analyse the creation process, as it can greatly influence the Hawking effect. Depending on the procedure of creation, one could end up in quite different quantum states for the field and only some of them might exhibit Hawking radiation. This becomes more important when considering that the analogue models incorporate modified dispersion relations. An inappropriate preparation, together with modified dispersion relation effects, could completely eliminate Hawking radiation [613, 35].

4. Another important issue is the need to characterise "how quantum" a specific analogue model is. Even though, strictly speaking, one could say that any system undergoes quantum fluctuations, the point is how important they are in its description. In trying to build an analogue model of Hawking's quantum effect, the relative value of Hawking temperature with respect to the environment is going to tell us whether the system can be really thought of as a quantum analogue model or as effectively classical. For example, in our standard cosmological scenario, for a black hole to radiate at temperatures higher than that of the Cosmic Microwave Background, ≈ 3 K, the black hole should have a diameter on the order of micrometers or less. We would have to say that such black holes are no longer classical, but semiclassical. The black holes for which we have some observational evidence are of much higher mass and size, so their behaviour can be thought of as completely classical. Estimates of the Hawking temperature reachable in BECs yield $T \sim 100$ nK [48]. This has the same order of magnitude of the temperature as the BECs themselves. This is telling us that, regarding the Hawking process, BECs can be considered to be highly-quantum analogue models.

5. There is also the very real question of whether one should trust semiclassical calculations *at all* when it comes to dealing with backreaction in the Hawking effect. See, for instance, the arguments presented by Helfer ([278, 279, 280], and references therein).

Because of its importance, let us now review what we know about the effects of high-energy dispersion relations on the Hawking process.

5.1.2   UV robustness

We saw in the introduction to this section that the trans-Planckian problem of Hawking radiation was one of the strongest motivations for the modern research into analogue models of gravity. In

fact, it was soon realised that such models could provide a physical framework within which a viable solution of the problem could be found. Let us explain why and how.

As we have said, the requirement of a reservoir of ultra-high frequency modes near the horizon seems to indicate a possible (and worrisome) sensitivity of the black-hole radiation to the microscopic structure of spacetime. Actually, by assuming exact Lorentz invariance, one could, in principle, always *locally* transform the problematic ultra-high frequency modes to low energy ones via some appropriate Lorentz transformation [307]. However there are (at least) two problems in doing so:

1. One has to rely on the physics of reference frames moving ultra fast with respect to us, as the reference frame needed would move arbitrarily close to the speed of light. Hence, we would have to apply Lorentz invariance in a regime of arbitrarily large boosts that is as yet untested, and in principle never completely testable given the non-compactness of the boost subgroup. The assumption of an exact boost symmetry is linked to the scale-free nature of spacetime given that unbounded boosts expose ultra-short distances. Hence, the assumption of *exact* Lorentz invariance needs, in the end, to rely on some ideas regarding the nature of spacetime at ultra-short distances.

2. Worse, even given these assumptions, "one cannot boost away an *s*-wave". That is, given the expected isotropy of Hawking radiation, a boost in any given direction could, at most, tame the trans-Planckian problem only in that specific direction. Indeed, the problem is then not ameliorated in directions orthogonal to the boost, and would become even worse on the opposite side of the black hole.

It was this type of reasoning that led in the nineties to a careful reconsideration of the crucial ingredients required for the derivation of Hawking radiation [307, 308, 608]. In particular investigators explored the possibility that spacetime microphysics could provide a short-distance, Lorentz-breaking cutoff, but at the same time leave Hawking's results unaffected at energy scales well below that set by the cutoff.

Of course, ideas about a possible cutoff imposed by the discreteness of spacetime at the Planck scale had already been discussed in the literature well before Unruh's seminal paper [607]. However, such ideas were running into serious difficulties given that a naive short-distance cutoff posed on the available modes of a free field theory results in a complete removal of the evaporation process (see, e.g., Jacobson's article [307] and references therein, and the comments in [278, 279, 280]). Indeed there are alternative ways through which the effect of the short-scale physics could be taken into account, and analogue models provide a physical framework where these ideas could be put to the test. In fact, analogue models provide explicit examples of emergent spacetime symmetries; they can be used to simulate black-hole backgrounds; they may be endowed with quantizable perturbations and, in most of the cases, they have a well-known microscopic structure. Given that Hawking radiation can be, at least in principle, simulated in such systems, one might ask how and if the trans-Planckian problem is resolved in these cases.

Modified dispersion relations: The general feature that most of the work on this subject has focused on is that in analogue models the quasi-particles propagating on the effective geometry are actually collective excitations of atoms. This generically implies that their dispersion relation will be a relativistic one only at low energies (large scales),[21] and in each case there will be some short length scale (e.g., intermolecular distance for a fluid, coherence length for a superfluid, healing

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[21] Actually, even relativistic behaviour at low energy can be non-generic, but we assume in this discussion that an analogue model by definition is a system for which all the linearised perturbations do propagate on the same Lorentzian background at low energies.

length for a BEC) beyond which deviations will be non-negligible. In general, such microphysics-induced corrections to the dispersion relation take the form

$$E^2 = c^2 \left(m^2 c^2 + k^2 + \Delta(k, K) \right), \tag{301}$$

where K is the scale that describes the transition to the full microscopic system (what we might call – within this section – the "analogue Planck scale").

In general, the best one can do is to expand $\Delta(k, K)$ around $k = 0$, obtaining an infinite power series (of which it will be safe to retain only the lowest-order terms), although in some special models (like BEC) the series is automatically finite due to intrinsic properties of the system. (In any case, one can see that most of the analogue models so far considered lead to modifications of the form $\pm k^3/K$ or $\pm k^4/K^2$.) Depending on the sign in front of the modification, the group velocity at high energy can be larger (+) or smaller (−) than the low energy speed of light c. These cases are usually referred to in the literature as "superluminal" and "subluminal" dispersion relations.

Most of the work on the trans-Planckian problem in the 1990s focused on studying the effect on Hawking radiation due to such modifications of the dispersion relations at high energies in the case of acoustic analogues [307, 308, 608, 148], and the question of whether such phenomenology could be applied to the case of real black holes (see e.g., [94, 310, 148, 490]).[22] In all the aforementioned works, Hawking radiation can be recovered under some suitable assumptions, as long as neither the black-hole temperature nor the frequency at which the spectrum is considered are too close to the scale of microphysics K. However, the applicability of these assumptions to the real case of black hole evaporation is an open question. It is also important to stress that the mechanism by which the Hawking radiation is recovered is conceptually rather different depending on the type of dispersion relation considered. We concisely summarise here the main results (but see, e.g., [613] for further details).

Subluminal dispersion relations: This was the case originally considered by Unruh [608],

$$\omega = K \left(\tanh(k/K)^n \right)^{1/n}, \tag{302}$$

and by Corley and Jacobson [148]

$$\omega^2 = k^2 - k^4/K^2, \tag{303}$$

where both dispersion relations are given in the co-moving frame.

The key feature is that in the presence of a subluminal modification the group velocity of the modes increases with k only up to some turning point (which is equivalent to saying that the group velocity does not asymptote to c, which could be the speed of sound, but instead is upper bounded). For values of k beyond the turning point the group velocity decreases to zero (for Equation (302)) or becomes imaginary (for Equation (303)). See Figure 15 for a schematic representation of this effect (for Equation (302)). In the latter case, this can be interpreted as signifying the breakdown of the regime where the dispersion relation (303) can be trusted. The picture that emerged from these analyses concerning the origin of the outgoing Hawking modes at infinity is quite surprising. In fact, if one traces back in time an outgoing mode, as it approaches the horizon, it decreases its group velocity below the speed of sound. At some point before reaching the horizon, the outgoing mode will appear as a combination of ingoing modes dragged into the black hole by the flow. Stepping further back in time it is seen that such modes are located at larger and larger distances from the horizon, and tend to have very high wave numbers far away

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[22] However, see also [522, 528] for a radically different alternative approach based on the idea of "superoscillations" where ultra-high frequency modes near the horizon can be mimicked (to arbitrary accuracy) by the exponential tail of an exponentially-large amplitude mostly hidden behind the horizon.
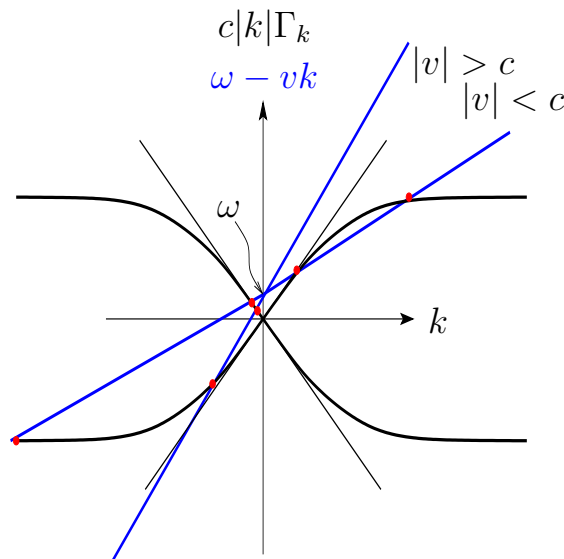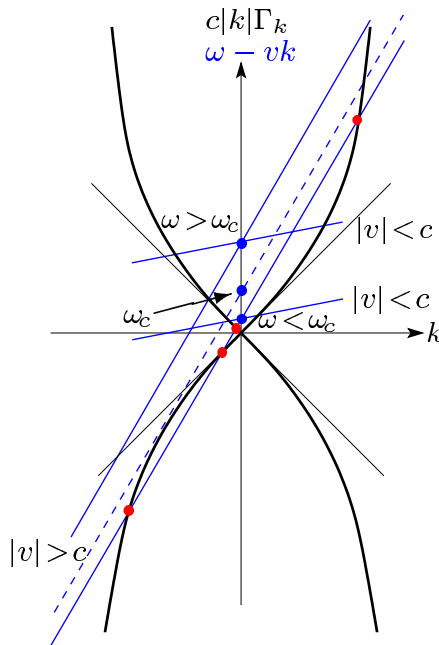
Figure 15: The picture shows a subsonic dispersion relation as a relation $\omega - vk = \pm c|k|\Gamma_k$. In particular we plot a dispersion of the type $\Gamma_k = K\sqrt{\tanh(k^2/K^2)}$ originally employed by Unruh. For supersonic velocities the dispersion relation has two real roots. For subsonic velocities and ω greater than a critical frequency ω_c, the dispersion relation has four real roots.

at early times. The important point is that one of the modes has "negative norm". (That is, the norm is negative in the appropriate Klein–Gordon-like inner product [35].) In this way one finds what might be called a "mode conversion", where the origin of the outgoing Hawking quanta seems to originate from ingoing modes, which have "bounced off" the horizon without reaching trans-Planckian frequencies in its vicinities. Several detailed analytical and numerical calculations have shown that such a conversion indeed happens [608, 94, 148, 147, 283, 538, 613, 413, 412] and that the Hawking result, specifically the presence of a Planckian spectrum of particles at a temperature $T \simeq \kappa/(2\pi)$ moving outwards toward the asymptotic region, can be recovered for $K \gg \kappa$ where κ is the black-hole surface gravity. In fact, the condition to recover a Planckian spectrum is a bit more subtle. In the mode conversion process every frequency experiences the horizon as located at a different position, and so, having a different surface gravity (a rainbow geometry). To have an approximately Planckian spectrum one would need this frequency-dependent surface gravity to stay almost constant for frequencies below and around κ (see, for example, the analysis in [413, 412, 199]). This condition seems to be easily realizable. In addition, we would like to point out another subtlety of these analogue black-hole configurations: It is necessary that the constant flow velocity reached at the asymptotic region is different from zero. For the particular dispersion relation in Equation (303), $|v| \geq \kappa/K$ so that the Planckian form is not cut down at frequencies around κ.

Let us mention here (more details in Section 6.1) that the classical counterpart of the above-described mode-conversion mechanism has recently been observed for the first time in a wave tank experiment [682]. It is remarkable that the exponential factor associated with a black body spectrum is clearly observed even with the inherent noise of the experiment. We will have to wait for the repetition of the same sort of experiment in a more explicitly quantum system to observe the spontaneous production of particles with a Planckian spectrum.

Superluminal dispersion relations: The case of a superluminal dispersion relation is quite different and, as we have seen, has some experimental interest, given that this is the kind of dispersion relation that arises in some promising analogue models (e.g., BECs). In this situation, the outgoing modes are actually seen as originating from behind the horizon (see Figure 16). This implies that these modes somehow originate from the singularity (which can be a region of high turbulence in acoustic black-hole analogues), and hence it would seem that not much can be said in this case. However, it is possible to show that if one imposes vacuum boundary conditions on these modes near the singularity, then it is still possible to recover the Hawking result, i.e., thermal radiation outside the black hole [147].

Figure 16:   The picture shows a supersonic dispersion relation as a relation $\omega - vk = \pm c|k|\Gamma_k$. In particular, we plot a dispersion of the type $\Gamma_k = \sqrt{1 + k^2/K^2}$. For supersonic velocities the dispersion relation has two real roots. For supersonic velocities and ω less than a critical frequency ω_c, the dispersion relation has four real roots.

The understanding of the physics behind the presence of Hawking radiation in superluminal dispersive theories has greatly improve recently through the detailed analysis of 1+1 stationary configurations possessing one black or white horizon connecting two asymptotic regions [413] (for previous work dealing with mode conversion through a horizon, see [386]). One of the asymptotic regions corresponds to the asymptotic region of a black-hole spacetime and is subsonic; the other asymptotic region is supersonic and replaces the internal singularity. Once the appropriate acoustic geometry is defined, this analysis considers a Klein–Gordon field equation in this geometry, modified by a ∂_x^4 term that gives rise to the quartic dispersion relation $\omega^2 = k^2 + k^4/K^2$. For this setup it has been shown that the relevant Bogoliubov coefficients have the form $\beta_{\omega',i;\omega,j} = \delta(\omega - \omega')\beta_{\omega;ij}$, with $\beta_{\omega;ij}$ a 3×3 matrix. To recover from these coefficients a Planckian spectrum of particles at the external asymptotic region, the relevant condition happens to be not $\kappa \ll K$ but $\kappa \ll \omega_c$ where $\omega_c = K f(v_{\text{int}-\text{asym}})$ and with $f(\cdot)$ a specific function of $v_{\text{int}-\text{asym}}$, the value of the flow velocity at the internal (or supersonic) asymptotic region [413]. In all cases, for $\omega > \omega_c$ the Bogoliubov coefficients are exactly zero. When $v_{\text{int}-\text{asym}}$ is just above $c = 1$, that is, when the flow is only

"slightly supersonic", the function $f(\cdot)$ can become very small $f(v_{\text{int}-\text{asym}}) \propto (v_{\text{int}-\text{asym}} - 1)^{3/2}$, and thus also the critical frequency ω_c at which particle production is cut off. If this happens at frequencies comparable with κ the whole Planckian spectrum will be truncated and distorted. Therefore, to recover a Planckian spectrum of particles at the external asymptotic region one needs to have a noticeable supersonic region.

As with subsonic dispersion, the existence of a notion of rainbow geometry makes modes of different frequency experience different surface gravities. One can think of the distortion of the Planckian spectrum as having a running $\kappa(\omega)$ that, for these configurations, interpolates between its low energy, or geometric value, $\kappa(\omega = 0) = \kappa_0$ and a value of zero for $\omega \to \omega_c$. This transition is remarkably sudden for the smooth profiles analyzed. If $\kappa_0 \ll \omega_c$, then $\kappa(\omega)$ will stay constant and equal to κ_0 throughout the relevant part of the spectrum reproducing Hawking's result. However, in general terms one can say that the spectrum takes into account the characteristic of the profile deep inside the supersonic region (the analogue of the black hole interior). The existence of superluminal modes makes it possible to obtain information from inside the (low energy) horizon.

In these analyses based on stationary configurations, the quantum field was always assumed to be in the *in*-vacuum state. Is this vacuum state, which has thermal properties, in terms of *out*-observers? However, it is interesting to realise that the dispersive character of the theory allows one to select for these systems a different, perfectly regular state, which is empty of both incoming and outgoing particles in the external asymptotic region [35]. This state can be interpreted as the regular generalization to a dispersive theory of the Boulware state for a field in a stationary black hole (let us recall that this state is not regular at the horizons). This implies that, in principle, one could set up a semi-classically stable acoustic black hole geometry. Another important result of the analysis in [413] is that stationary white holes do also Hawking radiate and in a very similar way to black holes.

These analyses have been repeated for the specific case of the fluctuations of a BEC [412] with identical qualitative results. The reasons for this is that, although the Bogoliubov–de Gennes system of equations is different from the modified scalar field equation analyzed in previous papers, they share the same quartic dispersion relation. Apart from the more formal treatments of BECs, Carusotto et al. reported [119, 118] the numerical observation of the Hawking effect in simulations in which a black-hole horizon has been created dynamically from an initially homogeneous flow. The observation of the effect has been through the calculation of two-point correlation functions (see Section 5.1.6 below). These simulations strongly suggest that any non-quasi-static formation of a horizon would give place to Hawking radiation.

It is particularly interesting to note that this recovery of the standard result is not always guaranteed in the presence of superluminal dispersion relations. Corley and Jacobson [150] in fact discovered a very peculiar type of instability due to such superluminal dispersion in the presence of black holes with inner horizons. The net result of the investigation carried out in [150] is that the compact ergo-region characterizing such configurations is unstable to self-amplifying Hawking radiation. The presence of such an instability was also identified in the dynamical analysis carried on in [231, 232, 29] where Bose–Einstein condensate analogue black holes were considered. As we have already mentioned, the spectrum associated with the formation of a black hole horizon in a superluminal dispersive theory depends, in a more or less obvious fashion, on the entire form of the internal velocity profile. In the case in which the internal region contains an additional white horizon, the resulting spectrum is completely changed. These configurations, *in addition to* a steady Hawking flux, produce a self-amplified particle emission; from this feature arises their name "black-hole lasers". In the recent analyses in [155, 199] it has been shown that the complete set of modes to be taken into account in these configurations is composed of a continuous sector with real frequencies, plus a discrete sector with complex frequencies of positive imaginary part. These discrete frequencies encode the unstable behaviour of these configurations, and are generated as resonant modes inside the supersonic cavity encompassed between the two horizons.

5.1.3 General conditions for Hawking radiation

Is it possible to reduce the rather complex phenomenology just described to a few basic assumptions that must be satisfied in order to recover Hawking radiation in the presence of Lorentz-violating dispersion relations? A tentative answer is given in [613], where the robustness of the Hawking result is considered for general modified (subluminal as well as superluminal) dispersion relations. The authors of [613] assume that the geometrical optics approximation breaks down only in the proximity of the event horizon (which is equivalent to saying that the particle production happens only in such a region). Here, the would-be trans-Planckian modes are converted into sub-Planckian ones. Then, they try to identify the minimal set of assumptions that guarantees that such "converted modes" are generated in their ground states (with respect to a freely falling observer), as this is a well-known condition in order to recover Hawking's result. They end up identifying three basic assumptions that guarantee such emergence of modes in the ground state at the horizon.

- First, the preferred frame selected by the breakdown of Lorentz invariance must be the freely falling one instead of the rest frame of the static observer at infinity (which coincides in this limit with the laboratory observer).

- Second, the Planckian excitations must start off in the ground state with respect to freely falling observers.

- Finally, they must evolve in an adiabatic way (i.e., the Planck dynamics must be much faster than the external sub-Planckian dynamics).

Of course, although several systems can be found in which such conditions hold, it is also possible to show [613] that realistic situations in which at least one of these assumptions is violated can be imagined. Hence, it is still an open question whether real black hole physics does indeed satisfy such conditions, and whether it is therefore robust against modifications induced by the violation of Lorentz invariance.

5.1.4 Source of the Hawking quanta

There is a point of view (not universally shared within the community) that asserts that the trans-Planckian problem also makes it clear that the ray-optics limit cannot be the whole story behind Hawking radiation. Indeed, it is precisely the ray optics approximation that leads to the trans-Planckian problem. Presumably, once one goes beyond ray optics, to the wave optics limit, it will be the region within a wavelength or so of the horizon (possibly the region between the horizon and the unstable circular photon orbit) that proves to be quantum-mechanically unstable and will ultimately be the "origin" of the Hawking photons. If this picture is correct, then the black-hole particle production is a low-frequency and low-wavenumber process. See, for instance, [563, 610, 611]. Work along these lines is continuing.

5.1.5 Which surface gravity?

One issue that has become increasingly important, particularly in view of recent experimental advances, is the question of exactly which particular definition of surface gravity is the appropriate one for controlling the temperature of the Hawking radiation. In standard general relativity with Killing horizons there is no ambiguity, but there is already considerable maneuvering room once one goes to evolving horizons in general relativity, and even more ambiguities once one adopts modified dispersion relations (as is very common in analogue spacetimes).

   Already at the level of time-dependent systems in standard general relativity there are two reasonably natural definitions of surface gravity, one in terms of the inaffinity of null geodesics skimming along the event horizon, and another in terms of the peeling properties of those null

geodesics that escape the black hole to reach future null infinity. It is this latter definition that is relevant for Hawking-like fluxes from non-stationary systems (e.g., evaporating black holes) and in such systems it never coincides with the inaffinity-based definition of the surface gravity except possibly at asymptotic future-timelike infinity i^+. Early comments along these lines can be found in [94, 95]; more recently this point was highlighted in [42, 41].

Regarding analogue models of gravity, the conclusions do not change when working in the hydrodynamic regime (where there is a strict analogy with GR). This point was implicitly made in [37, 41] and clearly stressed in [412]. If we now add modified dispersion relations, there are additional levels of complication coming from the distinction between "group velocity horizons", and "phase velocity horizons", and the fact that null geodesics have to be replaced by modified characteristic curves. The presence of dispersion also makes explicit that the crucial notion underlying Hawking emission is the "peeling" properties of null ray characteristics. For instance, the relevant "peeling" surface gravity for determining Hawking fluxes has to be determined locally, in the vicinity of the Killing horizon, and over a finite frequency range. (See for instance [612, 505, 504, 532, 66, 682, 412] for some discussion of this and related issues.) This "surface gravity" is actually an emergent quantity coming from averaging the naive surface gravity (the slope of the c–v profile) on a finite region around the would-be Killing horizon associated with the acoustic geometry [200]. Work on these important issues is ongoing.

5.1.6 How to detect Hawking radiation: Correlations

While the robustness of Hawking radiation against UV violations of the acoustic Lorentz invariance seems a well-established feature by now (at least in static or stationary geometries), its strength is indubitably a main concern for a future detection of this effect in a laboratory. As we have seen, the Hawking temperature in acoustic systems is simply related to the gradient of the flow velocity at the horizon (see Equation (61)). This gradient cannot be made arbitrarily large and, for the hydrodynamic approximation to hold, one actually needs it to be at least a few times the typical coherence length (e.g., the healing length for a BEC) of the superfluid used for the experiment. This implies that in a cold system, with low speed of sound, like a BEC, the expected power loss due to the Hawking emission could be estimated to be on the order of $P \approx 10^{-48}$ W (see, e.g., [48]): arguably too faint to be detectable above the thermal phonon background due to the finite temperature of the condensate (alternatively, one can see that the Hawking temperature is generally below the typical temperature of the BEC, albeit they are comparable and both in the nano-Kelvin range). Despite this, it is still possible that a detection of the spontaneous quantum particle creation can be obtained via some other feature rather than the spectrum of the Hawking flux. A remarkable possibility in this sense is offered by the fact that vacuum particle creation leads generically to a spectrum, which is (almost) Planckian but not thermal (in the sense that all the higher order field correlators are trivial combinations of the two-point one).[23]

Indeed, particles created by the mode mixing (Bogoliubov) mechanism are generically in a squeezed state (in the sense that the *in* vacuum appears as a squeezed state when expressed in terms of the *out* vacuum) [313] and such a state can be distinguished from a real thermal one exactly by the nontrivial structure of its correlators. This discrimination mechanism was suggested a decade ago in the context of dynamical Casimir effect explanations of Sonoluminescence [65], and later envisaged for analogue black holes in [48], but was finally investigated and fully exploited only recently in a stream of papers focussed on the BEC set up [22, 119, 412, 520, 19, 118, 190, 496, 512, 564, 604]. The outcome of such investigations (carried out taking into account the full Bogoliubov spectrum) is quite remarkable as it implies that indeed, while the Hawking flux is

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[23] Note however, that for a real black hole and Lorentz-invariant physics, the spectrum observed at infinity is indistinguishable from thermal given that no correlation measurement is allowed, since the Hawking partners are space-like separated across the horizon. This fact is indeed the root of the *information paradox.*

generically out-powered by the condensate intrinsic thermal bath, it is, in principle, possible to have a clear cut signature of the Hawking effect by looking at the density-density correlator for phonons on both sides of the acoustic horizon. In fact, the latter will show a definite structure totally absent when the flow is always subsonic or always supersonic. Even more remarkably, it was shown, both via numerical simulations as well as via a detailed analytical investigation, that a realistic finite temperature background does not spoil the long distance correlations which are intrinsic to the Hawking effect (and, indeed, for non-excessively large condensate temperatures the correlations can be amplified).

This seems to suggest that for the foreseeable future the correlation pattern will offer the most amenable route for obtaining a clean signature of the (spontaneous) Hawking effect in acoustic analogues. (For stimulated Hawking emission, see [682].) Finally, it is interesting to add that the correlator analysis can be applied to a wider class of analogue systems, in particular it has been applied to analogue black holes based on cold atoms in ion rings [290], or extended to the study of the particle creation in time-varying external fields (dynamical Casimir effect) in Bose–Einstein condensates (where the time-varying quantity is the scattering length via a Feshbach resonance) [118]. As we shall see, such studies are very interesting for their possible application as cosmological particle production simulations (possibly including Lorentz-violations effects). (See Section 5.4, and [512].)

5.1.7   Open issues

In spite of the remarkable insight given by the models discussed above (based on modified dispersion relations) it is not possible to consider them fully satisfactory in addressing the trans-Planckian problem. In particular, it was soon recognised [149, 311] that in this framework it is not possible to explain the origin of the short wavelength incoming modes, which are "progenitors" of the outgoing modes after bouncing off in the proximity of the horizon. For example, in the Unruh model (302), one can see that if one keeps tracking a "progenitor" incoming mode back in time, then its group velocity (in the co-moving frame) drops to zero as its frequency becomes more and more blue shifted (up to arbitrarily large values), just the situation one was trying to avoid. This is tantamount to saying that the trans-Planckian problem has been moved from the region near the horizon out to the region near infinity. In the Corley–Jacobson model (303) this unphysical behaviour is removed thanks to the presence of the physical cutoff K. However, it is still true that in tracking the incoming modes back in time one finally sees a wave packet so blue shifted that $|k| = K$. At this point one can no longer trust the dispersion relation (303) (which in realistic analogue models is emergent and not fundamental anyway), and hence the model has no predictive power regarding the ultimate origin of the relevant incoming modes.

These conclusions regarding the impossibility of clearly predicting the origin at early times of the modes ultimately to be converted into Hawking radiation are not specific to the particular dispersion relations (302) or (303) one is using. In fact, the Killing frequency is conserved on a static background; thus, the incoming modes must have the same frequency as the outgoing ones. Hence, in the case of strictly Lorentz invariant dispersion relations there can be no mode-mixing and particle creation. This is why one actually has to assume that the WKB approximation fails in the proximity of the horizon and that the modes are there in the vacuum state for the co-moving observer. In this sense, the need for these assumptions can be interpreted as evidence that these models are not yet fully capable of solving the trans-Planckian problem. Ultimately, these issues underpin the analysis by Schützhold and Unruh regarding the spatial "origin" of the Hawking quanta [563, 610, 611].

5.1.8   Solid state and lattice models

It was to overcome this type of issue that alternative ways of introducing an ultra-violet cutoff due to the microphysics were considered [522, 523, 149]. In particular, in [523] the transparency of the refractive medium at high frequencies has been used to introduce an effective cutoff for the modes involved in Hawking radiation in a classical refractive index analogue model (see Section 4.1.5). In this model an event horizon for the electromagnetic field modes can be simulated by a surface of singular electric and magnetic permeabilities. This would be enough to recover Hawking radiation but it would imply the unphysical assumption of a refractive index, which is valid at any frequency. However, it was shown in [523] that the Hawking result can be recovered even in the case of a dispersive medium, which becomes transparent above some fixed frequency K (which we can imagine as the plasma frequency of the medium); the only (crucial) assumption being again that the "trans-Planckian" modes with $k > K$ are in their ground state near the horizon.

An alternative avenue was considered in [149]. There a lattice description of the background was used for imposing a cutoff in a more physical way with respect to the continuum dispersive models previously considered. In such a discretised spacetime, the field takes values only at the lattice points, and wavevectors are identified modulo $2\pi/\ell$ where ℓ is the lattice characteristic spacing; correspondingly one obtains a sinusoidal dispersion relation for the propagating modes. Hence, the problem of recovering a smooth evolution of incoming modes to outgoing ones is resolved by the intrinsically-regularised behaviour of the wave vectors field. In [149] the authors explicitly considered the Hawking process for a discretised version of a scalar field, where the lattice is associated with the free-fall coordinate system (taken as the preferred system). With such a choice, it is possible to preserve a discrete lattice spacing. Furthermore, the requirement of a fixed short-distance cutoff leads to the choice of a lattice spacing constant at infinity, and that the lattice points are at rest at infinity and fall freely into the black hole.[24] In this case, the lattice spacing grows in time and the lattice points spread in space as they fall toward the horizon. However, this time dependence of the lattice points is found to be of order $1/\kappa$, and hence unnoticeable to long-wavelength modes and relevant only for those with wavelengths on the order of the lattice spacing. The net result is that, on such a lattice, long wavelength outgoing modes are seen to originate from short wavelength incoming modes via a process analogous to the Bloch oscillations of accelerated electrons in crystals [149, 311].

5.1.9   Analogue spacetimes as background gestalt

In addition, among the many papers using analogue spacetimes as part of their background mindset when addressing these issues we mention:

- "Top down" calculations of Hawking radiation starting from some idealised model of quantum gravity [4, 335, 461, 462].

- "Bottom up" calculations of Hawking radiation starting from curved space quantum field theory [52, 53, 108, 109, 124, 184, 185, 216, 428, 430, 429, 431, 489, 526].

- Trans-Hawking versions of Hawking radiation, either as reformulations of the physics, or as alternative scenarios [62, 120, 222, 223, 257, 266, 278, 279, 280, 384, 478, 480, 530, 536, 541, 542, 549, 550, 567, 591].

- Black hole entropy viewed in the light of analogue spacetimes [158].

- Hawking radiation interpreted as a statement about particles traveling along complex space-time trajectories [481, 567, 586].

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[24] [149] also considered the case of a lattice with proper distance spacing constant in time but this has the problem that the proper spacing of the lattice goes to zero at spatial infinity, and hence there is no fixed short-distance cutoff.

5.2   Dynamical stability of horizons

Although the two issues are very closely related, as we will soon see, we have to carefully distinguish between the stability analysis of the modes of a linear field theory (with or without modified dispersion relations – MDR) over a fixed background, and the stability analysis of the background itself.

5.2.1   Classical stability of the background (no MDR)

Let us consider a three-dimensional irrotational and inviscid fluid system with a stationary sink-type of flow – a "draining bathtub" flow – (see Figures 1 and 2). The details of the configuration are not important for the following discussion, only the fact that there is a spherically-symmetric fluid flow accelerating towards a central sink, that sink being surrounded by a sphere acting as a sonic horizon. Then, as we have discussed in Section 2, linearizing the Euler and continuity equations leads to a massless scalar field theory over a black-hole–like spacetime. (We are assuming that the hydrodynamic regime remains valid up to arbitrarily-short length scales; for instance, we are neglecting the existence of MDR.) To be specific, let us choose the geometry of the canonical acoustic black-hole spacetime described in [624]:

$$ds^2 = -c^2 \left(1 - \frac{r_0^4}{r^4}\right) d\tau^2 + \left(1 - \frac{r_0^4}{r^4}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right). \tag{304}$$

In this expression we have used the Schwarzschild time coordinate τ instead of the lab time t; c is constant. If we expand the field in spherical harmonics,

$$\phi_{lm}(\tau, r, \theta, \varphi) \equiv e^{-i\omega\tau} \, \frac{\chi_{lm}(r)}{r} \, Y_{lm}(\theta, \varphi), \tag{305}$$

we obtain the following equation for the radial part of the field:

$$\frac{\omega^2}{c^2} \, \chi = \left(-\frac{d^2}{dr^{*2}} + V_l(r)\right) \chi; \tag{306}$$

where

$$V_l(r) = \left(1 - \frac{r_0^4}{r^4}\right) \left[\frac{l(l+1)}{r^2} + \frac{4r_0^4}{r^6}\right]. \tag{307}$$

Here

$$r^* \equiv r - \frac{r_0}{4} \, \left\{\ln\left[\frac{(r+r_0)}{(r-r_0)}\right] + 2\arctan\left(\frac{r}{r_0}\right)\right\} \tag{308}$$

is a "tortoise" coordinate.

In a normal mode analysis one requires boundary conditions such that the field is regular everywhere, even at infinity. However, if one is analysing the solutions of the linear field theory as a way of probing the stability of the background configuration, one can consider less restrictive boundary conditions. For instance, one can consider the typical boundary conditions that lead to quasinormal modes: These modes are defined to be purely out-going at infinity and purely in-going at the horizon; but one does not require, for example, the modes to be normalizable. The quasinormal modes associated with this sink configuration have been analysed in [69]. The results found are qualitatively similar to those in the classical linear stability analysis of the Schwarzschild black hole in general relativity [619, 620, 521, 698, 447]. Of course, the gravitational field in general relativity has two dynamical degrees of freedom – those associated with gravitational waves – that have to be analysed separately; these are the "axial" and "polar" perturbations. In contrast, in

the present situation we only have scalar perturbations. Nevertheless, the potentials associated with "axial" and "polar" perturbations of Schwarzschild spacetime, and that associated with scalar perturbations of the canonical acoustic black hole, produce qualitatively the same behaviour: There is a series of *damped* quasinormal modes – proving the linear stability of the system – with higher and higher damping rates.

An important point we have to highlight here is that, although in the linear regime the dynamical behaviour of the acoustic system is similar to general relativity, this is no longer true once one enters the nonlinear regime. The underlying nonlinear equations in the two cases are very different. The differences are so profound, that in the general case of acoustic geometries constructed from compressible fluids, there exist sets of perturbations that, independent of how small they are initially, can lead to the development of shocks, a situation completely absent in vacuum general relativity.

5.2.2 Semiclassical stability of the background (no MDR)

Now, given an approximately stationary, and at the very least metastable, classical black-hole-like configuration, a standard quantum mode analysis leads to the existence of Hawking radiation in the form of phonon emission. This shows, among other things, that quantum corrections to the classical behaviour of the system must make the configuration with a sonic horizon dynamically unstable against Hawking emission. As a consequence, in any system (analogue or general relativistic) with quantum fluctuations that maintain strict adherence to the equivalence principle (no MDR), it must then be impossible to create an isolated *truly stationary* horizon by merely setting up external initial conditions and letting the system evolve by itself. However, in an analogue system a truly stationary horizon can be set up by providing an external power source to stabilise it against Hawking emission. Once one compensates, by manipulating external forces, for the backreaction effects that in a physical general relativity scenario cause the horizon to shrink or evaporate, one would be able to produce, in principle, an analogue system exhibiting precisely a stationary horizon and a stationary Hawking flux.

Let us describe what happens when one takes into account the existence of MDR. Once again, a wonderful physical system that has MDR explicitly incorporated in its description is the Bose–Einstein condensate. The macroscopic wave function of the BEC behaves as a classical irrotational fluid but with some deviations when short length scales become involved. (For length scales on the order of, or shorter than, the healing length.) What are the effects of the MDR on the dynamical stability of a black-hole-like configuration in a BEC? The stability of a sink configuration in a BEC has been analysed in [231, 232] but taking the flow to be effectively one-dimensional. What these authors found is that these configurations are dynamically unstable: There are modes satisfying the appropriate boundary conditions such that the imaginary parts of their associated frequencies are positive. These instabilities are associated basically with the bound states inside the black hole. The dynamical tendency of the system to evolve is suggestively similar to that in the standard evaporation process of a black hole in semiclassical general relativity.

5.2.3 Classical stability of the background (MDR in BECs)

Before continuing with the discussion of the instability of configurations with horizons, and in order not to cause confusion between the different wording used when talking about the physics of BECs and the emergent gravitational notions on them, let us write down a quite loose but useful translation dictionary:

- The "classical" or macroscopic wave function of the BEC represents the classical spacetime of GR, but only when probed at long-enough wavelengths such that it behaves as pure hydrodynamics.

- The "classical" long-wavelength perturbations to a background solution of the Gross–Pitaevskii equation correspond to classical gravitational waves in GR. Of course, this analogy does not imply that these are spin 2 waves; it only points out that the perturbations are made from the same "substance" as the background configuration itself.

- The macroscopic wave function of the BEC, without the restriction of being probed only at long wavelengths, corresponds to some sort of semiclassical vacuum gravity. Its "classical" behaviour (in the sense that does not involve any probability notion) is already taking into account, in the form of MDR, its underlying quantum origin.

- The Bogoliubov quantum quasiparticles over the "classical" wave function correspond to a further step away from semiclassical gravity in that they are analogous to the existence of quantum gravitons over a (semiclassical) background spacetime.

At this point we would like to remark, once again, that the analysis based on the evolution of a BEC has to be used with care. For example, they cannot directly serve to shed light on what happens in the final stages of the evaporation of a black hole, as the BEC does not fulfil, at any regime, the Einstein equations. Summarizing:

- If the perturbations to the BEC background configuration have "classical seeds" (that is, are describable by the linearised Gross–Pitaevskii equation alone), then, one will have "classical" instabilities.

- If the perturbations have "quantum seeds" (that is, are described by the Bogoliubov equations), then, one will have "quantum" instabilities.

5.2.4 Black holes, white holes, and rings

In the light of the acoustic analogies it is natural to ask whether there are other geometric configurations with horizons of interest, besides the sink type of configurations (these are the most similar to the standard description of black holes in general relativity, but probably not the simplest in terms of realizability in a real laboratory; for an entire catalogue of them see [37]). Here, let us mention four effectively one-dimensional configurations: a black hole with two asymptotic regions, a white hole with two asymptotic regions, a black-hole–white-hole in a straight line and the same in a ring (see Figures 17, 18, 19 and 20, respectively).
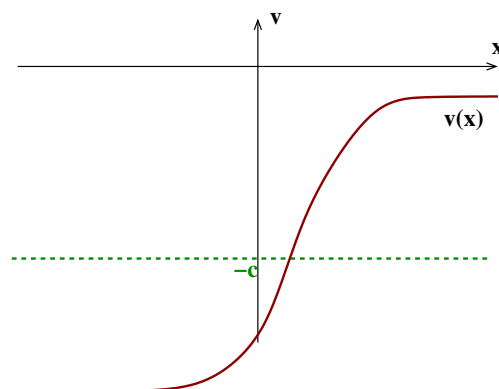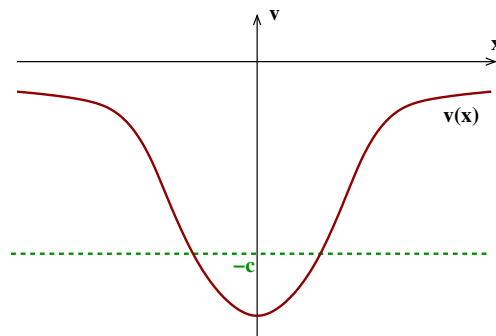

Figure 17: One-dimensional velocity profile with a black-hole horizon.
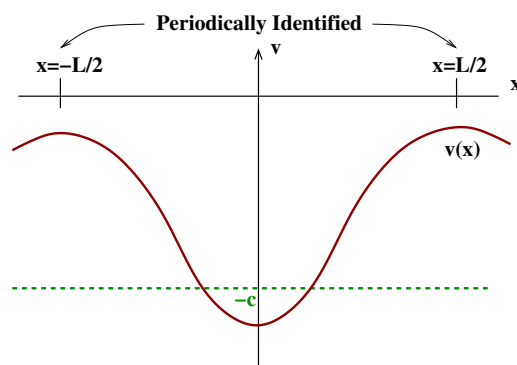
There are several classical instability analyses of these types of configurations in the literature [231, 232, 386, 29, 155, 199]. In these analyses one looks for the presence or absence of
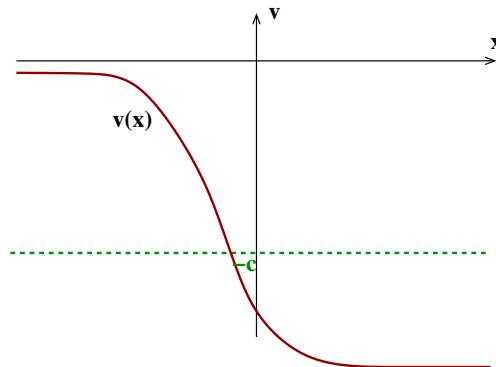
Figure 18: One-dimensional velocity profile with a white-hole horizon.

Figure 19: One-dimensional velocity profile with a black-hole horizon and a white-hole horizon.

Figure 20: One-dimensional velocity profile in a ring; the fluid flow exhibits two sonic horizons, one of black hole type and the other of white-hole type.

modes with a positive-imaginary-part eigenfrequency, under certain appropriate boundary conditions. The boundary condition in each asymptotic region can be described as outgoing, as in quasi-normal modes, or as convergent, meaning that at a particular instant of time the mode is exponentially damped towards the asymptotic region. Let us mention that in Lorentz invariant theory these two types of conditions are not independent: any unstable mode is at the same time both convergent and outgoing. However, in general, in dispersive theories, once the frequency is extended to the complex plane, these two types of conditions become, at least in principle, independent.

Under outgoing and convergent boundary conditions in both asymptotic regions, in [29] it was concluded that there are no instabilities in any of the straight line (non-ring) configurations. If one relaxed the convergence condition in the downstream asymptotic region, (the region that substitutes the unknown internal region, and so the region that might require a different treatment for more realistic black hole configurations), then the black hole is still stable, while the white hole acquires a continuous region of instability, and the black-hole–white-hole configuration shows up as a discrete set of unstable modes. The white-hole instability was previously identified in [386]. Let us mention here that the stable black-hole configuration has been also analyzed in terms of stable or quasinormal modes in [30]. It was found that, although the particular configurations analyzed (containing idealised step-like discontinuities in the flow) did not posses quasinormal modes in the acoustic approximation, the introduction of dispersion produced a continuous set of quasinormal modes at trans-Planckian frequencies.

Continuing with the analysis of instabilities, in contrast to [29], the more recent analysis in [155, 199] consider only convergent boundary conditions in both asymptotic regions. They argue that the ingoing contributions that these modes sometimes have always correspond to waves that do not carry energy, so that they have to be kept in the analysis, as their ingoing character should not be interpreted as an externally-provoked instability[25]. If this is confirmed, then the appropriate boundary condition for instability analysis under dispersion would be just the convergent condition, as in non-dispersive theories.

Under these convergent conditions, the authors of [155, 199] show that the previously-considered black-hole and white-hole configurations in BECs are stable. (Let us remark that this does not mean that configurations with a more complicated internal region need be stable.) However, black-hole–white-hole configurations do show a discrete spectrum of instabilities. In these papers, one can find a detailed analysis of the strength of these instabilities, depending on the form and size of the intermediate supersonic region. For instance, it is necessary that the supersonic region acquire a minimum size so that the first unstable mode appears. (This feature was also observed in [29].) When the previous mode analysis is used in the context of a quantum field theory, as we mention in Section 5.1, one is led to the conclusion that black-hole–white-hole configurations emit particles in a self-amplified (or runaway) manner [150, 155, 199]. Although related to Hawking's process, this phenomenon has a quite different nature. For example, there is no temperature associated with it.

When the black-hole–white-hole configuration is compactified in a ring, it is found that there are regions of stability and instability, depending on the parameters characterizing the configuration [231, 232]. We suspect that the stability regions appear because of specific periodic arrangements of the modes around the ring. Among other reasons, these arrangements are interesting because they could be easier to create in the laboratory with current technology, and their instabilities easier to detect than Hawking radiation itself.

To conclude this subsection, we would like to highlight that there is still much to be learned by studying the different levels of description of an analogue system, and how they influence the stability or instability of configurations with horizons.

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[25] Personal communication by R. Parentani

5.3 Super-radiance

Another phenomenon that has been (and is being) analysed from the analogue gravity perspective is super-radiance. The rotational kinetic energy accumulated in a rotating black hole can be extracted from it by scattering into it waves of sufficiently low frequency and high angular momentum. In general, in order that the wave can extract rotational energy from the system, it has to satisfy the condition

$$\omega < m\,\Omega, \tag{309}$$

where Ω is the angular speed of the black hole at the event horizon and m is the harmonic azimuthal number of the wave. This is a purely classical process first considered by Penrose [499]. When dealing with quantum fields, as opposed to classical fields, this process can proceed spontaneously. Quantum mechanically, a rotating black hole will tend to radiate away all of its angular momentum, eventually approaching a non-rotating Schwarzschild black hole [696, 697]. This process is known as super-radiance. (The term super-radiance was already used in condensed matter to describe processes in which there was some coherent emission of radiation from an otherwise disordered system.)

Again, these processes have a purely kinematical origin, so they are perfectly suitable for being reproduced in an analogue model. Regarding these processes, the simplest geometry that one can reproduce, thinking of analogue models based on fluid flows, is that of the draining bathtub of Section 2. Of course, this metric does not exactly correspond to Kerr geometry, nor even to a section of it [641, 633]. However, it is *qualitatively* similar. It can be used to simulate both Penrose's classical process and quantum super-radiance, as these effects do not depend on the specific multipole decomposition of Kerr's geometry, but only on its rotating character. A specific experimental setup has been put forward by Schützhold and Unruh using gravity waves in a shallow basin mimicking an ideal draining bathtub [560]. Equivalent to what happens with Kerr black holes, this configuration is classically stable in vacuum (in the linear regime) [69]. A word of caution is in order here: Interactions of the gravity surface waves with bulk waves (neglected in the analysis) could cause the system to become unstable [657]. This instability has no counterpart in standard general relativity (though it might have one in braneworld theories). Super-resonant scattering of waves in this rotating sink configuration, or in a simple purely rotating vortex, could in principle be observed in this and other analogue models. There are already several articles dealing with this problem [55, 57, 56, 112, 193, 392]. Most recently, see [524], where necessary and sufficient conditions for super-radiance were investigated.

A related phenomenon one can consider is the black-hole bomb mechanism [513]. One would only have to surround the rotating configuration by a mirror for it to become grossly unstable. What causes the instability is that those in-going waves that are amplified when reflected in the ergosphere would then in turn be reflected back toward the ergoregion, due to the exterior mirror, thus being amplified again, and so on.

An interesting phenomenon that appears in many condensed matter systems is the existence of quantised vortices. The angular momentum of these vortices comes in multiples of some fundamental unit (typically $\hbar$ or something proportional to $\hbar$). The extraction of rotational energy by a Penrose process in these cases could only proceed via finite-energy transitions. This would supply an additional specific signature to the process. In such a highly quantum configuration, it is also important to look for the effect of having high-energy dispersion relations. For example, in BECs, the radius of the ergoregion of a single quantised vortex is on the order of the healing length, so one cannot directly associate an effective Lorentzian geometry with this portion of the configuration. Any analysis that neglects the high-energy terms is not going to give any sensible result in these cases.

5.4 Cosmological particle production

Analogue model techniques have also been applied to cosmology. Models considered to date focus on variants of the BEC-inspired analogues:

- Fedichev and Fischer [195, 194] have investigated WKB estimates of the cosmological particle production rate and (1+1) dimensional cosmologies, both in expanding BECs.

- Lidsey [403], and Fedichev and Fischer [196] have focussed on the behaviour of cigar-like condensates in grossly-asymmetric traps.

- Barceló et al. [46, 47] have focussed on BECs and tried to mimic FLRW behaviour as closely as possible, both via free expansion, and via external control of the scattering length using a Feshbach resonance.

- Fischer and Schützhold [206] propose the use of two-component BECs to simulate cosmic inflation.

- Weinfurtner [674, 675] has concentrated on the approximate simulation of de Sitter space-times.

- Weinfurtner, Jain, et al. have undertaken both numerical [328] and general theoretical [677, 683] analyses of cosmological particle production in a BEC-based FLRW universe.

In all of these models the general expectations of the relativity community have been borne out – the theory definitely predicts particle production, and the very interesting question is the extent to which the formal predictions are going to be modified when working with real systems experimentally [47]. We expect that these analogue models provide us with new insights as to how their inherent modified-dispersion relations affect cosmological processes such as the generation of a primordial spectrum of perturbations (see, for example, [85, 84, 86, 87, 88, 89, 90, 122, 179, 269, 296, 297, 343, 380, 381, 406, 426, 423, 424, 425, 458, 459, 460, 487, 566, 587, 588, 589, 600] where analogue-like ideas are applied to cosmological inflation).

An interesting side-effect of the original investigation, is that birefringence can now be used to model "variable speed of light" (VSL) geometries [58, 181]. Since analogue models quite often lead to two or more "excitation cones", rather than one, it is quite easy to obtain a bimetric or multi-metric model. If one of these metrics is interpreted as the "gravitational" metric and the other as the "photon" metric, then VSL cosmologies can be given a mathematically well-defined and precise meaning [58, 181].

5.5 Bose novae: an example of the reverse flow of information?

As we have seen in the previous Sections ?? and 5.4, analogue models have in the past been very useful in providing new, condensed-matter-physics–inspired ideas about how to solve longstanding problems of semiclassical gravity. In closing this section, it is interesting to briefly discuss what perhaps represents, so far, the only attempt to use analogue models in the reverse direction; that is to import well-known concepts of semiclassical gravity into condensed matter frameworks.

The phenomenon we are referring to is the "Bose nova" [175]. This is an experiment dealing with a gas of a few million ^{85}Rb atoms at a temperature of about 3 nK. The condensate is rendered unstable by exploiting the possibility of tuning the interaction (more precisely the scattering length) between the atoms via a magnetic field. Reversing the sign of the interaction, making it attractive, destabilises the condensate. After a brief waiting time (generally called t_{collapse}), the condensate implodes and loses a sizable fraction of its atoms in the form of a "nova burst". If left to evolve undisturbed, the number of atoms in the burst stabilises and a remnant condensate is left. However,

if the condensate interaction is again made repulsive after some time t_{evolve}, before the condensate has sufficient time to stabilise, then the formation of "jets" of atoms is observed, these jets being characterised by lower kinetic energy and a distinct shape with respect to the burst emission.

Interestingly, an elegant explanation of such a phenomenology was proposed in [105, 106], based on the well-known semiclassical gravity analysis of particle creation in an expanding universe. In fact, the dynamics of quantum excitations over the collapsing BEC were shown to closely mimic that for quantum excitations in a time-reversed (collapsing instead of expanding) scenario for cosmological particle creation. This is not so surprising as the quantum excitations above the BEC ground state feel a time-varying background during the collapse, and, as a consequence, one then expects squeezing of the vacuum state and mode mixing, which are characteristic of quantum field theory in variable external fields.

However, the analogy is even deeper than this. In fact, in [105, 106] a key role in explaining the observed burst and jets is played by the concepts of "frozen" versus "oscillating" modes – borrowed from cosmology – (although with a reverse dynamics with respect to the standard (expanding) cosmological case). In the case of Bose novae, the modes, which are amplified, are those for which the physical frequency is smaller than the collapse rate, while modes with higher frequencies remain basically unaffected and their amplitudes obey a harmonic oscillator equation. As the collapse rate decreases, more and more modes stop growing and start oscillating, which is equivalent to a creation of particles from the quantum vacuum. In the case of a sudden stop of the collapse by a new reversal of the sign of the interaction, all of the previously growing modes are suddenly converted into particles, explaining in this way the generation of jets and their lower energy (they correspond to modes with lower frequencies with respect to those generating the bursts).

Although this simple model cannot explain all the details of the Bose novae phenomenology, we think it is remarkable how far it can go in explaining several observed features by exploiting the language and techniques so familiar to quantum cosmology. In this sense, the analysis presented in [105, 106] primarily shows a possible new application of analogue models, where they could be used to lend ideas and techniques developed in the context of gravitational physics to the explanation of condensed matter phenomena.

5.6   Romulan cloaking devices

A wonderful application of analogue gravity techniques is the design of *cloaking devices* [385, 498, 387]. How to achieve invisibility, or more properly, low observability has been a matter of extensive study for decades. With the appearance of a technology capable of producing and controlling meta-materials and plasmonic structures [3], cloaking is becoming a real possibility.

To achieve cloaking, one needs to ensure that light rays (beyond geometric optics it is impossible to produce perfect invisibility [449, 690]) effectively behave as if they were propagating in Minkowski spacetime, although in reality they are bending around the invisible compact region. One way of producing this is, precisely, to make rays propagate in Minkowski spacetime but using non-Cartesian coordinates. Take the Minkowski metric in some Cartesian coordinates x', $\eta_{\mu\nu}$, and apply a diffeomorphism, which is different from the identity only inside the compact region. One then obtains a different representation $g_{\mu\nu}(x)$ of the flat geometry. Now take the x to be the Cartesian coordinates of the real laboratory spacetime, and build this metric with the meta-material. By construction, the scattering process with the compact region will not change the directions of the rays, making everything within the compact region invisible. Recent implementations of these ideas have investigated the concept of a "spacetime cloak" or "history editor" that cloaks a particular event, not a particular region [439].

To end this brief account we would like to highlight the broad scope of application of these ideas: Essentially the cloaking techniques can be applied to any sort of wave, from acoustic cloaking [133] to earthquake damage prevention [192]. More radically, and with enough civil engineering, one

might adapt the ideas of Berry [67, 68] to anti-tsunami cloaking.

5.7 Going further

For more details on the trans-Planckian problem, some of the key papers are the relatively early papers of Unruh [608] and Jacobson [307, 308]. For super-radiance and cosmological issues (especially particle production) there seems to be considerable ongoing interest, and one should carefully check SPIRES (or the beta version of INSPIRE) for the most recent articles.

6 Experimental efforts

In recent years several efforts towards the detection of analogue Hawking radiation have been carried out with different physical systems. Here we report a brief list of the ongoing efforts. There is already definite evidence for mode conversion and negative-norm modes (group velocity opposite to phase velocity) [532], and more recently for stimulated Hawking emission [682]. There are very recent claims of photon detection from a phase velocity horizon in an optical fibre [66], and for white-hole–like behaviour in hydraulic jumps [334]. It seems that reliable and reproducible experimental probes of spontaneous quantum Hawking radiation might be just a few years in the future.

6.1 Wave tank experiments

As we have seen, waves in shallow water can be considered to be a particularly simple analogue gravity system. (See Section 4.1.3.) Experimentally, water basins are relatively cheap and easy to construct and handle. In particular, shallow water basins (more precisely, wave tanks or wave flumes) have acquired a prominent role in recent years. In particular, such technology underlay the 1983 work of Badulin et al. [17], and such wave tanks are currently used by groups in Nice, France [532] and Vancouver, Canada [682]. The Nice experiments have been carried out using a large wave-tank 30 m long, 1.8 m wide and 1.8 m deep. The simplest set-up with such a device is to send water waves (e.g., produced by a piston) against a fluid flow produced by a pump. To generate a water-wave horizon, a ramp is placed in the water, with positive and negative slopes separated by a flat section. When a train of waves is sent against the reverse fluid flow there will be a place where the flow speed equals the group velocity of the waves – there a group velocity horizon will be created. (Remember that in the shallow-water regime the low momentum co-moving dispersion relation for surface water waves is $\omega^2 = gk \tanh(kh)$ where g denotes the gravitational acceleration of the Earth at the water surface and h is the height of the channel.) For incident waves moving against the flow it would be impossible to cross such a horizon, and in the sense that the system is the analogue of a white-hole horizon, the time reversal of a black-hole horizon. (In the engineering and fluid mechanics literature this effect is typically referred to as "wave blocking".)

Remarkably, in 2007 Rousseaux et al. [532] reported the first direct observation of negative-frequency waves, converted from positive-frequency waves in a moving medium, albeit the degree of mode conversion appears to be significantly higher than that expected from theory. The same group has now set up a more compact experiment based on the hydraulic jump, wherein measurements of the "Froude cones" convincingly demonstrate the presence of a surface-wave white hole [334], (as described in Sections 4.1.3 and 4.1.4 and possibly implicit in the results of Badulin et al. [17]).

A related experiment, with a smaller water basin, performed under the auspices of the gravitation theory group (Physics) and the fluid mechanics group (Civil Engineering) at the University of British Columbia, has recently (August 2010) reported the detection of stimulated Hawking emission [682]. This stimulated Hawking emission is a classical effect that (via the usual discussion in terms of Einstein A and B coefficients) is rather closely related to spontaneous quantum Hawking emission. A central result of the Weinfurtner et al. group is that the relevant Bogoliubov coefficients have been experimentally measured and are observed to satisfy the expected Boltzmann relation

$$\frac{|\beta|^2}{|\alpha|^2} = \exp\left\{-\frac{2\pi\omega}{g_H/c_H}\right\}. \tag{310}$$

More details of the experimental setup can be found in [682], and additional details will soon be available in planned follow-up articles.

6.2   Bose–Einstein condensate experiments

We have already extensively discussed the theoretical aspects of Bose–Einstein-condensate–based analogue models. Regarding the actual experimental possibility of generating BEC-based acoustic black holes, several options have been envisaged in the literature. In particular, commonly proposed settings are long thin condensates in a linear or circular trap [231, 232] as well as Laval-nozzle–shaped traps [48, 539, 228, 476]. However, it is only very recently that an experiment aimed at the formation of a sonic black hole in a BEC has been set up, and the creation of a sonic horizon has been convincingly argued for [369]. In this case, a sonic horizon was achieved by a counterintuitive effect of "density inversion", in which a deep potential minimum creates a region of low density, as would a potential maximum. This low density region corresponds to a slower-than-normal speed of sound, and hence to the possibility for the flow speed to exceed the speed of sound and generate sonic horizons at the crossing points (where the speed of the flow and that of sound coincide). The density inversion is achieved by overlapping a low-frequency (broad) harmonic potential and a high-frequency (narrow) Gaussian potential generated via an elongated laser. In this manner a sonic black hole was generated, and kept stable for about 8 ms. The Hawking radiation predicted for the system as realised has a temperature of about $0.20 - 0.35$ nK; unfortunately, one order of magnitude smaller than the lowest temperature allowed by the size of the system. (Lower temperatures for the condensate permit longer-wavelength characteristic Hawking quanta, which must still fit into the condensate. So there is a trade-off between Hawking temperature and physical size of the condensate.) However, higher densities could allow one to increase the Hawking temperature, and $T_H \approx$ a few nK seems within experimental reach. Given that 8 ms would correspond to one cycle of 6 nK Hawking radiation, it appears that increases in T_H together with amelioration of the lifetime of the sonic black hole might put the detection of the analogue spontaneous quantum Hawking effect within experimental reach (via correlation experiments) in the near future.

6.3   Differentially-rotating flows in superfluid helium

At the end of Section 4.2.2 we briefly described an analogue model based on the ripplons in the surface separating two differentially-moving superfluids, in particular an AB-interphase in ^{3}He. These interphases are being produced in Helsinki's Low Temperature Lab [76, 77, 201, 202]. The AB-interphase is prepared in a small quartz cylinder (3 mm radius times 11 cm long) inside a rotating cryostat. The ^{3}He-A is rotating with the cryostat while the ^{3}He-B remains at rest with respect to the lab. Among other things, in this setting the critical values at which instabilities appear as functions of the temperature, and the nature of these instabilities, are being investigated. These instabilities are related to the appearance of an ergoregion in the analogue metric for ripplons and to the Kelvin–Helmholtz instability [657]. In particular, this has been the first time that the Kelvin–Helmholtz instability has been observed in superfluids [76]. The nature of the instability in these experiments is controlled by the difference in velocities between the normal and superfluid components of the flow. It still remains to further lower the ambient temperature so as to probe the nature of the instabilities in the absence of any normal fluid component.

6.4   Fibre-optic models

A recent implementation of analogue models based on electrodynamics is that based on fibre-optic engineering [504, 505]. The basic idea in this case is to use long dispersive light pulses (solitons), generated with a suitable laser, to create a propagating front at which the refractive index of the fibre changes suddenly (albeit by a small amount). Basically, the refractive index of the fibre, n_0, acquires a time- and position-dependent correction δn, which is proportional to the instantaneous pulse intensity I at a give space-time position, $\delta n \propto I(t, x)$. The wavefront at which this change in the refractive index occurs will move naturally at a speed close to the speed of light (and fibre

optic engineering allows one to control this feature). If one now sends a continuous wave of light, what we might call a probe, along the fibre in such a manner that the probe group velocity in the fibre is arranged to be slightly larger than the pulse group velocity, then it will be possible to obtain horizon-like effects. In fact, as the probe wave reaches the back of the pulse, the increase in the refractive index will slow it down, until the probe group velocity will match the pulse one. Effectively, the rear end of the pulse will act as a white hole for the probe wave. Similarly, there will be a point on the front side of the pulse where the two group velocities will match. This will be the equivalent of a black-hole horizon for the probe wave. In [504, 505] the behaviour of the probe waves at the pulse was investigated, and it was shown for the white hole case that the expected classical behaviour is theoretically reproduced. Since this behaviour lies at the core of the mechanism responsible for the mode conversion underlying the Hawking effect, it is then expected that the quantum counterpart should also be reproducible in this manner. Indeed, very recently Belgiorno et al. have reported experimental detection of photons from a black-hole–white-hole configuration possessing a "phase velocity horizon" [66]. The underlying theory behind their specific experiment is considered in [63, 64].

6.5 Going further

There seems to be considerable ongoing interest in experimental probes of analogue spacetimes, and quantum effects in analogue spacetimes, and one should carefully check SPIRES (or the beta version of INSPIRE) for the most recent articles.

7 Towards a Theory of Quantum Gravity?

The key question one should ask at this stage is this: "Where can we go from here?" Apart from continuing with the analysis of the issues described in the previous two sections, it is natural to explore whether the analogue gravity programme could be extended to the point of yielding a theory of "Quantum Gravity". In particular, the following topics come to mind as steps on a path towards a possible theory of Quantum Gravity:

- Backreaction.

- Equivalence principle.

- Diffeomorphism invariance.

- Effective spin-two excitations.

- Weinberg–Witten theorem.

- Emergent gravity.

- The cosmological constant problem.

- Quantum gravity phenomenology.

- Quantum gravity.

Some work has already been done dealing with these topics in the context of analogue gravity. Let us now expand on these issues a little.

7.1 Backreaction

There are important phenomena in gravitational physics whose understanding needs analysis well beyond classical general relativity and field theory on (fixed) curved background spacetimes. The black-hole evaporation process can be considered as paradigmatic among these phenomena. Here, we confine our discussion to this case. Since we are currently unable to analyse the entire process of black-hole evaporation within a complete quantum theory of gravity, a way of proceeding is to analyse the simpler (but still extremely difficult) problem of semiclassical backreaction (see, for example, [166, 137, 75, 224, 95, 431]). One takes a background black-hole spacetime, calculates the expectation value of the quantum energy-momentum tensor of matter fields in the appropriate quantum state (the Unruh vacuum state for a radiating black hole), and then takes this expectation value as a source for the perturbed Einstein equations. This calculation gives us information about the tendency of spacetime to evolve under vacuum polarization effects.

A nice feature of analogue models of general relativity is that, although the underlying classical equations of motion have nothing to do with Einstein equations, the tendency of the analogue geometry to evolve due to quantum effects is formally equivalent (approximately, of course) to that in semiclassical general relativity. Therefore, the onset of the backreaction effects (if not their precise details) can be simulated within the class of analogue models. An example of the type of backreaction calculations one can perform are those in [23, 25]. These authors started from an effectively one-dimensional acoustic analogue model, configured to have an acoustic horizon by using a Laval nozzle to control the flow's speed. They then considered the effect of quantizing the acoustic waves over the background flow. To calculate the appropriate backreaction terms they took advantage of the classical conformal invariance of the (1+1)-dimensional reduction of the system. In this case, we know explicitly the form of the expectation value of the energy-momentum tensor trace (via the trace anomaly). The other two independent components of the

energy-momentum tensor were approximated by the Polyakov stress tensor. In this way, what they found is that the tendency of a left-moving flow with one horizon is for it to evolve in such a manner as to push the horizon down-stream at the same time that its surface gravity is decreased. This is a behaviour similar to what is found for near-extremal Reissner–Nordström black holes. (However, we should not conclude that acoustic black holes are, in general, closely related to near-extremal Reissner–Nordström black holes, rather than to Schwarzschild black holes. This result is quite specific to the particular one-dimensional configuration analysed.)

Can we expect to learn something new about gravitational physics by analysing the problem of backreaction in different analogue models? As we have repeatedly commented, the analyses based on analogue models force us to consider the effects of modified high-energy dispersion relations. For example, in BECs, they affect the "classical" behaviour of the background geometry as much as the behaviour of the quantum fields living on the background. In seeking a semiclassical description for the evolution of the geometry, one would have to compare the effects caused by the modified dispersion relations to those caused by pure semiclassical backreaction (which incorporates deviations from standard general relativity as well). In other words, one would have to understand the differences between the standard backreaction scheme in general relativity, and that based on Equations (229) and (230).

To end this subsection, we would like to comment that one can go beyond the semiclassical backreaction scheme by using the *stochastic semiclassical gravity programme* [298, 301, 302]. This programme aims to pave the way from semiclassical gravity toward a complete quantum-gravitational description of gravitational phenomena. This stochastic gravity approach not only considers the expectation value of the energy-momentum tensor but also its fluctuations, encoded in the semiclassical Einstein–Langevin equation. In a very interesting paper [490], Parentani showed that the effects of the fluctuations of the metric (due to the in-going flux of energy at the horizon) on the out-going radiation led to a description of Hawking radiation similar to that obtained with analogue models. It would be interesting to develop the equivalent formalism for quantum analogue models and to investigate the different emerging approximate regimes.

7.2 Equivalence principle

Analogue models are of particular interest in that the analogue spacetimes that emerge often violate, to some extent, the Einstein equivalence principle [45, 638]. This is the heart and soul of any metric theory of gravity and is basically the requirement of the universality of free fall, plus local Lorentz invariance and local position invariance of non-gravitational experiments.

As such, the Einstein equivalence principle is a "principle of universality" for the geometrical structure of spacetime. Whatever the spacetime geometrical structure is, if all excitations "see" the same geometry, one is well on the way to satisfying the observational and experimental constraints. In a metric theory, this amounts to the demand of mono-metricity: A *single* universal metric must govern the propagation of all excitations.

Now it is this feature that is relatively difficult to arrange in analogue models. If one is dealing with a single degree of freedom, then mono-metricity is no great constraint. But with multiple degrees of freedom, analogue spacetimes generally lead to refringence – that is the occurrence of Fresnel equations that often imply multiple propagation speeds for distinct normal modes. To even obtain a bi-metric model (or, more generally, a multi-metric model), requires an algebraic constraint on the Fresnel equation that it completely factorises into a product of quadratics in frequency and wavenumber. Only if this algebraic constraint is satisfied can one assign a "metric" to each of the quadratic factors. If one further wishes to impose mono-metricity, then the Fresnel equation must be some integer power of some single quadratic expression, an even stronger algebraic statement [45, 638]. Faced with this situation, there are two ways in which the analogue gravity community might proceed:

1. Try to find a broad class of analogue models (either physically based or mathematically idealised) that naturally lead to mono-metricity. Little work along these lines has yet been done; at least partially because it is not clear what features such a model should have in order to be "clean" and "compelling".

2. Accept refringence as a common feature of the analogue models and attempt to use refringence to ones benefit in one or more ways:

   - There are real physical phenomena in non-gravitational settings that definitely do exhibit refringence and sometimes multi-metricity. Though situations of this type are not directly relevant to the gravity community, there is significant hope that the mathematical and geometrical tools used by the general relativity community might in these situations, shed light on other branches of physics.

   - Use the refringence that occurs in many analogue models as a way of "breaking" the Einstein equivalence principle, and indeed as a way of "breaking" even more fundamental symmetries and features of standard general relativity, with a view to exploring possible extensions of general relativity. While the analogue models are not themselves primary physics, they can nevertheless be used as a way of providing *hints* as to how more fundamental physics might work.

7.3 Nontrivial dispersion as Einstein-aether theory

There is a certain precise sense in which nontrivial dispersion relations can effectively be viewed as implicitly introducing an "aether field", in the sense of providing a kinematic (but not dynamic) implementation of Einstein-aether theory [323, 180, 219, 314]. The point is that to define nontrivial dispersion one needs to pick a rest frame V^a, and then assert $\omega^2 = f(k^2)$ in this rest frame. But one can then re-write this dispersion relation (in the eikonal approximation) as

$$\left[-(V^a \partial_a)^2 + f([g^{ab} + V^a V^b]\partial_a \partial_b) \right] \Psi(x) = 0. \tag{311}$$

That is, using $f(w) = j(w) + w$,

$$\left[g^{ab}\partial_a\partial_b + j([g^{ab} + V^a V^b]\partial_a \partial_b) \right] \Psi(x) = 0. \tag{312}$$

As long as the background is slowly varying, this can be re-written as:

$$\left[\Delta_{d+1} + j(\Delta_d) \right] \Psi(x) = 0, \tag{313}$$

with $\Delta_{d+1} = g^{ab}\nabla_a\nabla_b$ and with the aether field V^a hiding in the definition of the spatial Laplacian $\Delta_d = [g^{ab} + V^a V^b]\nabla_a\nabla_b$. This procedure allows us to take a quantity that is manifestly not Lorentz invariant, the dispersion relation $\omega^2 = f(k^2)$, and nevertheless "covariantise" it via the introduction of new structure — a locally specified preferred frame defined by the (possibly position- and time-dependent) aether 4-velocity V^a.

Of course, in standard analogue models such an aether field does not come with its own dynamics: It is a background structure which breaks the physically-relevant content of what is usually called diffeomorphism invariance (see next Section 7.4). However, in a gravitation theory context one might still want to require background independence taking it as a fundamental property of any gravity theory, even a Lorentz breaking one. In this case one has to provide the aether field with a suitable dynamics; we can then rephrase much of the analogue gravity discussion in the presence of nontrivial dispersion relations in terms of a variant of the Einstein-aether models [323, 180, 219, 314].

7.4   Diffeomorphism invariance

When looking at the analogue metrics one problem immediately comes to mind. The laboratory in which the condensed-matter system is set up provides a privileged coordinate system. Thus, one is not really reproducing a geometrical configuration but only a specific metrical representation of it. This naturally raises the question of whether or not diffeomorphism invariance is lost in the analogue spacetime construction. Indeed, if all the degrees of freedom contained in the metric had a physical role, as opposed to what happens in a general relativistic context in which only the geometrical degrees of freedom (metric modulo diffeomorphism gauge) are physical, then diffeomorphism invariance would be violated. Here we are thinking of "active" diffeomorphisms, not "passive" diffeomorphisms (coordinate changes). As is well known, *any* theory can be made invariant under passive diffeomorphisms (coordinate changes) by adding a sufficient number of external/background/non-dynamical fields (prior structure). See, for instance, [256]. Invariance under active diffeomorphisms is equivalent to the assertion that there is no "prior geometry" (or that the prior geometry is undetectable). Many readers may prefer to re-phrase the current discussion in terms of the undetectability of prior structure.

The answer to this question is that active diffeomorphism invariance is maintained but only for (low-energy) internal observers, i.e., those observers who can only perform (low-energy) experiments involving the propagation of the relativistic collective fields. By revisiting classic Lorentz–FitzGerald ideas on length contraction, and analyzing the Michelson–Morley experiment in this context, it has been explicitly shown in [36] that (low-energy) Lorentz invariance is not broken, i.e., that an internal observer cannot detect his absolute state of motion. (For earlier suggestions along these lines, see, for example, [400] and [660].)

The argument is the same for curved spacetimes; the internal observer would have no way to detect the "absolute" or fixed background. So the apparent background dependence provided by the (non-relativistic) condensed-matter system will not violate active diffeomorphism invariance, at least not for these internal inhabitants. These internal observers will then have no way to collect any metric information beyond what is coded into the intrinsic geometry (i.e., they only get metric information up to a gauge or diffeomorphism equivalence factor). Internal observers would be able to write down diffeomorphism invariant Lagrangians for relativistic matter fields in a curved geometry. However, the dynamics of this geometry is a different issue. It is a well-known issue that the expected relativistic dynamics, i.e., the Einstein equations, have to date not been reproduced in any known condensed-matter system.

7.5   Effective spin-two particles

Related to the previous point is the possibility of having quantum systems with no pre-geometric notions whatsoever (i.e., a condensed-matter–like system) that still exhibit in their low-energy spectrum effective massless spin-two excitations. This precise question has been investigated in [262, 263, 693, 694] for abstract quantum systems based on the underlying notion of qubits. Although not fully conclusive, these works indicate the possible existence of systems exhibiting purely helicity ± 2 excitations. One crucial ingredient in these constructions is the existence of a specific vacuum state with the characteristics of a string-net condensate.

These authors also show that it is not easy to have just helicity ± 2 excitations – typically one would also generate helicity ± 1 and massless scalar excitations. This is what would happen, for instance, in the emergent gravity scenario inspired in the phenomenology of ^{3}He, which will be described below.

7.6   Weinberg–Witten theorem

The Weinberg–Witten theorem [673] has often been interpreted as an insurmountable obstacle for obtaining massless spin-two excitations as effective degrees of freedom emerging from any reasonable underlying quantum field theory. However, the status of the Weinberg–Witten theorem [673] insofar as it applies to analogue models is rather subtle. First, note that whenever one's main concern is in developing an analogue spacetime at the purely kinematic level of an effective metric, then the Weinberg–Witten theorem has nothing to say. (This includes, for instance, all analogue experiments probing the Hawking effect or cosmological particle production; these are purely kinematic experiments that do not probe the dynamics of the effective spacetime.) When one turns to the dynamics of the effective spacetime, desiring, for instance, to investigate quantum fluctuations of the effective geometry (gravitons), then one should bear in mind that the Weinberg–Witten theorem is derived under specific technical assumptions (strict Lorentz invariance in flat spacetime) that are not applicable in the current context. Furthermore, even if the specific technical assumptions are satisfied, then those authors state [673]:

> Of course, there are acceptable theories that have massless charged particles with spin $j > 1/2$ (such as the massless version of the original Yang–Mills theory), and also theories that have massless particles with spin $j > 1$ (such as supersymmetry theories or general relativity). Our theorem does not apply to these theories because they do not have Lorentz-covariant conserved currents or energy-momentum tensors, respectively.

Furthermore, when it comes to Sakharov-style induced gravity those authors explicitly state [673]:

> However, the theorem dearly does not apply to theories in which the gravitational field is a basic degree of freedom but the Einstein action is induced by quantum effects.

That is: The Weinberg–Witten theorem has no direct application to analogue spacetimes – at the kinematic level it has nothing to say, at the dynamic level its applicability is rather limited by the stringent technical assumptions invoked – specifically exact Lorentz invariance at all scales – and the fact that these technical assumptions are not applicable in the current context. For careful discussions of the technical assumptions see [596, 366, 212, 404]. Note particularly the comment by Kubo [366]

> ... the powerful second part of the theorem becomes empty in the presence of gravity ...

Finally we mention that, though motivated by quite different concerns, the review article [61] gives a good overview of the Weinberg–Witten theorem, and the ways in which it may be evaded.

7.7   Emergent gravity

One of the more fascinating approaches to "quantum gravity" is the suggestion, typically attributed to Sakharov [540, 628], that gravity itself may not be "fundamental physics". Indeed it is now a relatively common opinion, maybe not mainstream but definitely a strong minority opinion, that gravity (and in particular the whole notion of spacetime and spacetime geometry) might be no more "fundamental" than is fluid dynamics. The word "fundamental" is here used in a rather technical sense – fluid mechanics is not fundamental because there is a known underlying microphysics, that of molecular dynamics, of which fluid mechanics is only the low-energy low-momentum limit. Indeed, the very concepts of density and velocity field, which are so central to the Euler and continuity equations, make no sense at the microphysical level and emerge only as one averages over time-scales and distance-scales larger than the mean free time and mean free path.

In the same way, it is plausible (even though no specific and compelling model of the relevant microphysics has yet emerged) that the spacetime manifold and spacetime metric might arise only once one averages over suitable microphysical degrees of freedom.

7.8 One specific route to the Einstein equations?

In fact, there is a specific route to reproduce Einstein equations within a Fermi-liquid-like system advocated by G. Volovik [660]. In a Fermi liquid like ^{3}He-A phase, there exist two important energy scales. One is the energy scale E_B at which bosonization in the system start to develop. This scale marks the onset of the superfluid behaviour of helium three. At energies below E_B, the different bosons appearing in the systems condense so that they can exhibit collective behaviour. The other energy scale, E_L, is the Lorentz scale at which the quasiparticles of the system start to behave relativistically (as Weyl spinors). As we mention in Section 4.2.2 discussing the "heliocentric universe", this occurs in the ^{3}He-A phase because the vacuum has Fermi points. It is in the immediate surroundings of these Fermi points where the relativistic behaviour shows up. There are additional relevant scales in these systems, but in this section we are going to talk exclusively of these two.

When one is below both energy scales, one can describe the system as a set of Weyl spinors coupled to background electromagnetic and gravitational fields. For a particular Fermi point, the electromagnetic and gravitational fields encode, respectively, its position and its "light-cone" structure through space and time. Both electromagnetic and gravitational fields are built from bosonic degrees of freedom, which have condensed. Apart from any predetermined dynamics, these bosonic fields will acquire additional dynamical properties through the Sakharov-induced gravity mechanism. Integrating out the effect of quantum fluctuations in the Fermionic fields *à la* Sakharov, one obtains a one-loop effective action for the geometric field, to be added to the tree-level contribution (if any). This integration cannot be extended beyond E_B, as at that energy scale the geometrical picture based on the bosonic condensate disappears. Thus, E_B will be the cut-off of the integration.

Now, in order that the geometrical degrees of freedom follows an Einstein dynamics, we need three conditions (which we shall see immediately are really just two):

1. $E_L > E_B$: For the induction mechanics to give rise to an Einstein–Hilbert term, $\sqrt{-g}R$, in the effective Lagrangian we have to be sure that the fluctuating Fermionic field "feels" the geometry (fulfilling a locally-Lorentz-invariant equation) at all scales up to the cut-off. The term $\sqrt{-g}R$ will appear multiplied by a constant proportional to E_B^2. That is why from now on we can called E_B alternatively the Planck energy scale $E_P \equiv E_B$.

2. *Special relativity dominance or $E_L \gg E_P$*: The E_P^2 dependence of the gravitational coupling constant tells us that the fluctuations that are more relevant in producing the Einstein–Hilbert term are those with energies close to the cut-off, that is, around the Planck scale. Therefore, to assure the induction of an Einstein–Hilbert term one needs the Fermionic fluctuation with energies close to the Planck scale to be perfectly Lorentzian to a high degree. This can only be assured if $E_L \gg E_B$.

3. *Sakharov one-loop dominance*: Finally, one also needs the induced dynamical term to dominate over the pre-existing tree-level contribution (if any).

Unfortunately, what we have called special-relativity dominance is not implemented in helium three, nor in any known condensed-matter system. In helium three the opposite happens: $E_B \gg E_L$. Therefore, the dynamics of the gravitational degrees of freedom is non-relativistic but of fluid-mechanical type. That is, the dynamics of the gravitational degrees of freedom is not Einstein, but of fluid-mechanical type. The possible emergence of gravitational dynamics in the context of a condensed-matter system has also been investigated for BECs [252, 571]. It has been shown that, for the simple gravitational dynamics in these systems, one obtains a modified Poisson equation, and so it is completely non-relativistic, giving place to a short range interaction (on the order of the healing length). However, starting from abstract systems of PDFs with *a priori* no geometrical

information, the emergence of Nordström spin-0 gravity has been shown to be possible [253]; this is relativistic though not Einstein.

In counterpoint, in Hořava gravity the graviton appears to be fundamental, and need not be emergent [287, 288, 289]. Additionally, the Lorentz breaking scale and the Planck scale are in this class of models distinct and unconnected, with the possibility of driving the Lorentz breaking scale arbitrarily high [585, 584, 635, 681]. In this sense the Hořava models are a useful antidote to the usual feeling that Lorentz violation is typically Planck-scale.

7.9   The cosmological constant problem

The condensed matter analogies offer us an important lesson concerning the cosmological constant problem [660]. Sakharov's induced gravity not only can give rise to an Einstein–Hilbert term under certain conditions, it also gives rise to a cosmological term. This contribution would depend on the cut-off as E_P^4, so if there were not additional contributions counterbalancing this term, emergent gravity in condensed-matter-like systems will always give place to an enormous cosmological constant inducing a strongly-repulsive force between quasiparticles.

However, we know that at low temperatures, depending on the microscopic characteristics of the system we can have quite different situations. Remarkably, liquid systems (as opposed to gases) can remain stable on their own, without requiring any external pressure. Their total internal pressure at equilibrium is (modulo finite-size effects) always zero. This implies that, at zero temperature, if gravity emerges from a liquid-like system, the total vacuum energy $\Lambda \propto \rho_V = -p_V$ will be automatically forced to be (relatively) small, and not a large number. The E_P^4 contribution coming from quasiparticle fluctuations will be exactly balanced by contributions from the microphysics or "trans-Planckian" contributions.

If the temperature is not zero there will be a pressure p_M associated with the thermal distribution of quasiparticles, which constitute the matter field of the system. Then, at equilibrium one will have $p_M + p_V = 0$, so that there will be a small vacuum energy $\Lambda \propto p_M$. This value is not expected to match exactly the preferred value of Λ obtained in the standard cosmological model (ΛCDM) as we are certain to be out of thermodynamic equilibrium. However, it is remarkable that it matches its order of magnitude, (at least for the current epoch), albeit any dynamical model implementing this idea will probably have to do so only at late times to avoid possible tension with the observational data. Guided by these lessons, there are already a number of heuristic investigations about how a cosmological term could dynamically adapt to the evolution of the matter content, and which implications it could have for the evolution of the universe [27, 350, 349, 351].

As final cautionary remark let us add that consideration of an explicit toy model for emergent gravity [252] shows that the quantity that actually gravitates cannot be so easily predicted without an explicit derivation of the analogue gravitational equations. In particular in [198] it was shown that the relevant quantity entering the analogue of the cosmological constant is a contribution coming only from the excitations above the condensate.

7.10   Other pieces of the puzzle

Sakharov had in mind a specific model in which gravity could be viewed as an "elasticity" of the spacetime medium, and where gravity was "induced" via one-loop physics in the matter sector [540, 628]. In this way, Sakharov had hoped to relate the observed value of Newton's constant (and the cosmological constant) to the spectrum of particle masses.

More generally, the phrase "emergent gravity" is now used to describe the whole class of theories in which the spacetime metric arises as a low-energy approximation, and in which the microphysical degrees of freedom might be radically different. Analogue models, and in particular analogue models based on fluid mechanics or the fluid dynamic approximation to BECs, are specific

examples of "emergent physics" in which the microphysics is well understood. As such, they are useful for providing *hints* as to how such a procedure might work in a more fundamental theory of quantum gravity.

7.11   Quantum gravity – phenomenology

Over the last few years a widespread consensus has emerged that observational tests of quantum gravity are for the foreseeable future likely to be limited to precision tests of dispersion relations and their possible deviations from Lorentz invariance [435, 321]. The key point is that at low energies (well below the Planck energy) one expects the locally-Minkowskian structure of the spacetime manifold to guarantee that one sees only special relativistic effects; general relativistic effects are negligible at short distances. However, as ultra-high energies are approached (although still below Planck-scale energies) several quantum-gravity models seem to predict that the locally Euclidean geometry of the spacetime manifold will break down. There are several scenarios for the origin of this breakdown ranging from string theory [360, 182] to brane worlds [99] and loop quantum gravity [229]. Common to all such scenarios is that the microscopic structure of spacetime is likely to show up in the form of a violation of Lorentz invariance leading to modified dispersion relations for elementary particles. Such dispersion relations are characterised by extra terms (with respect to the standard relativistic form), which are generally expected to be suppressed by powers of the Planck energy. Remarkably, the last years have seen a large wealth of work in testing the effects of such dispersion relations and in particular strong constraints have been cast by making use of high energy astrophysics observations (see, for example, [6, 141, 318, 317, 319, 320, 321, 435, 579, 396] and references therein).

Several of the analogue models are known to exhibit similar behaviour, with a low-momentum effective Lorentz invariance eventually breaking down at high momentum once the microphysics is explored.[26] Thus, some of the analogue models provide controlled theoretical laboratories in which at least some forms of subtle high-momentum breakdown of Lorentz invariance can be explored. As such, the analogue models provide us with *hints* as to what sort of modified dispersion relation might be natural to expect given some general characteristics of the microscopic physics. Hopefully, an investigation of appropriate analogue models might be able to illuminate possible mechanisms leading to this kind of quantum gravity phenomenology, and so might be able to provide us with new ideas about other effects of physical quantum gravity that might be observable at sub-Planckian energies.

7.12   Quantum gravity – fundamental models

When it comes to dealing with "fundamental" theories of quantum gravity, the analogue models play an interesting role which is complementary to the more standard approaches. The search for a quantum theory of gravity is fundamentally a search for an appropriate mathematical structure in which to simultaneously phrase both quantum questions and gravitational questions. More precisely, one is searching for a mathematical framework in which to develop an abstract quantum theory which then itself encompasses classical Einstein gravity (the general relativity), and reduces to it in an appropriate limit [113, 580, 221].

The three main approaches to quantum gravity currently in vogue, "string models" (also known as "M-models"), "loop space" (and the related "spin foams"), and "lattice models" (Euclidean or Lorentzian) all share one feature: They attempt to develop a "pre-geometry" as a replacement for classical differential geometry (which is the natural and very successful mathematical language used to describe Einstein gravity) [113, 580, 221, 534, 533, 83, 507]. The basic idea is that the mooted replacement for differential geometry would be relevant at extremely small distances (where the quantum aspects of the theory would be expected to dominate), while at larger distances (where the classical aspects dominate) one would hope to recover both ordinary differential geometry and specifically Einstein gravity or possibly some generalization of it. The "string", "loop", and

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[26] However, it is important to keep in mind that not all the above-cited quantum gravity models violate the Lorentz symmetry in the same manner. The discreteness of spacetime at short scales is not the only way of breaking Lorentz invariance.

"lattice" approaches to quantum gravity differ in detail in that they emphasise different features of the long-distance model, and so obtain rather different short-distance replacements for classical differential geometry. Because the relevant mathematics is extremely difficult, and by and large not particularly well understood, it is far from clear which, if any, of these three standard approaches will be preferable in the long run [580].

A recent (Jan. 2009) development is the appearance of Hořava gravity [287, 288, 289]. This model is partially motivated by condensed matter notions such as (deeply non-perturbative) anomalous scaling and the existence of a "Lifshitz point", and additionally shares with most of the analogue spacetimes the presence of modified dispersion relations and high-energy deviations from Lorentz invariance [287, 288, 289, 634, 635, 585, 584, 681]. Though Hořava gravity is not directly an analogue model *per se*, there are deep connections – with some steps toward an explicit connection being presented in [694].

We feel it likely that analogue models can shed new light on this very confusing field by providing a concrete specific situation in which the transition from the short-distance "discrete" or "quantum" theory to the long-distance "continuum" theory is both well understood and non-controversial. Here we are specifically referring to fluid mechanics, where, at short distances, the system must be treated using discrete atoms or molecules as the basic building blocks, while, at large distances, there is a well-defined continuum limit that leads to the Euler and continuity equations. Furthermore, once one is in the continuum limit, there is a well-defined manner in which a notion of "Lorentzian differential geometry", and in particular a "Lorentzian effective spacetime metric" can be assigned to any particular fluid flow [607, 624, 470]. Indeed, the "analogue gravity programme" is extremely successful in this regard, providing a specific and explicit example of a "discrete" → "continuum" → "differential geometry" chain of development. What the "analogue gravity programme" does not seem to do as easily is to provide a natural direct route to the Einstein equations of general relativity, but that merely indicates that current analogies have their limits and therefore, one should not take them too literally [624, 470]. Fluid mechanics is a guide to the mathematical possibilities, not an end in itself. The parts of the analogy that do work well are precisely the steps where the standard approaches to quantum gravity have the most difficulty, and so it would seem useful to develop an abstract mathematical theory of the "discrete" → "continuum" → "differential geometry" chain using this fluid mechanical analogy (and related analogies) as inspiration.

7.13 Going further

Beyond the various theoretical issues we have discussed above there is the important question of "experimental analogue gravity" – to what extent can all these ideas be tested in direct laboratory experiments? Currently several experimental groups are investigating analogue models – surface wave experiments [17, 532, 682, 334], a BEC-based experiment [369], and the fibre-optic experiment [66]. Broadly speaking, for any experimental group interested in analogue spacetimes the two key issues to address are:

- Identify a particular analogue model easily amenable to laboratory investigation, and double check the extent to which the model provides a theoretically robust and clean analogue to general relativistic curved spacetime.

- Identify the technical issues involved in actually setting up a laboratory experiment.

While the consensus in the theoretical community is that Bose–Einstein condensates are likely to provide the best working model for analogue gravity, it is possible that we might still be surprised by experimental developments. We leave this as an open challenge to the experimental community.

8 Conclusions

In this review article we have seen the interplay between standard general relativity and various analogies that can be used to capture aspects of its behaviour. These analogies have ranged from rather general but very physical analogue models based on fluid-acoustics, geometrical optics, and wave optics, to highly specific models based on BECs, liquid helium, slow light, etc. Additionally, we have seen several rather abstract mathematical toy models that bring us to such exotic structures and ideas as birefringence, bimetricity, Finsler spaces, and Sakharov's induced gravity.

The primary reason that these analogies were developed within the general relativity community was to help in the understanding of general relativity by providing very down-to-earth models of otherwise subtle behaviour in general relativity. Secondary reasons include the rather speculative suggestion that there may be more going on than just analogy – it is conceivable (though perhaps unlikely) that one or more of these analogue models could suggest a relatively simple and useful way of quantizing gravity that side-steps much of the technical machinery currently employed in such efforts. A tertiary concern (at least as far as the general relativity community is concerned) is the use of relativity and differential geometric techniques to improve understanding of various aspects of condensed matter physics.

The authors expect interest in analogue models to continue unabated, and suspect that there are several key but unexpected issues whose resolution would be greatly aided by the analysis of appropriate analogue models.

8.1 Going further

Though every practicing scientist already knows this, for the sake of any student reading this we mention the following resources:

- http://www.slac.stanford.edu/spires/ – the SPIRES bibliographic database for keeping track of (almost all of) the general relativity and particle physics aspects of the relevant literature.

- http://inspirebeta.net/ – the new INSPIRE bibliographic database (currently beta version) for keeping track of (almost all of) the general relativity and particle physics aspects of the relevant literature.

- http://ads.harvard.edu/ – the SAO/NASA ADS bibliographic database for keeping track of (almost all of) the astrophysical aspects of the relevant literature.

- http://www.arXiv.org – the electronic-preprint (e-print) database for accessing the text of (almost all, post 1992) relevant articles.

- http://relativity.livingreviews.org/ – the *Living Reviews in Relativity* journal.

Those five access points should allow you to keep abreast of what is going on in the field.

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