Quantum simulation of quantum field theory using scalable architecture

Kevin Marshall,[1] Raphael Pooser,[2,3] George Siopsis,[3,*] and Christian Weedbrook[4]

[1] *Department of Physics, University of Toronto, Toronto, M5S 1A7, Canada*
[2] *Quantum Information Science Group, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, U.S.A*
[3] *Department of Physics and Astronomy, The University of Tennessee, Knoxville, Tennessee 37996-1200, U.S.A.*
[4] *QKD Corp, 60 St. George St., Toronto, M5S 1A7, Canada*

(Dated: April 13, 2015)

The year 1982 is often credited as the year that theoretical quantum computing was started with a 'key note speech' by Richard Feynman who proposed a universal quantum simulator. The idea being that if you had such a machine you could in principle "imitate any quantum system, including the physical world". With that in mind, we present a new algorithm for a scalable continuous-variable quantum computing architecture which gives an exponential speedup over the best-known classical methods. Specifically, this relates to efficiently calculating the scattering amplitudes in scalar bosonic quantum field theory, a problem that is believed to be hard using a classical computer. Building on this, we give an experimental implementation based on continuous-variable cluster states that is feasible with today's technology.

PACS numbers: 42.50.Ex, 03.70.+k, 42.50.Dv, 03.67.Lx

Introduction - Quantum field theory (QFT) [1] unites the discipline of quantum mechanics with special relativity to provide us with our best understanding of the world around us and what it is made of; notwithstanding that it has yet to be reconciled with general relativity. Typically, the best-known algorithms for calculations in field theories are very difficult on classical computers. One method is lattice field theory [2] which discretizes space into a finite set of points. Unfortunately, classical computations on the lattice increase exponentially with the number of sites, making it unfeasible. Quantum algorithms [3] have been proposed to accomplish a variety of fundamental tasks more quickly than any known classical counterpart, most famously Shor's factoring algorithm [4] and Grover's searching algorithm [5]. When Feynman first proposed the notion of quantum computing [6], he had a different idea in mind, namely the ability of one quantum system to simulate another [7].

In this paper, we keep true to the spirit of Feynman's vision by presenting a method of calculating scattering amplitudes in a scalar bosonic QFT with a quartic self-interaction on a quantum computing substrate that has shown to be scalable, i.e., a continuous-variable (CV) quantum computer. In fact, we show one can obtain an exponential speedup over the best known classical algorithms. A discrete version of this algorithm was originally shown in Refs. [8, 9] for a quantum computer based on qubits. Further work extended this result to fermionic QFTs [10], as well as using wavelets for multi-scale simulations [11].

The field of quantum computing [12] using CVs [13, 14] has also progressed significantly in the last few years. From its original conception in 1999 [13], progress began to accelerate after a cluster state [15] version was established in 2006 [16, 17], leading to something significantly more tangible for experimentalists. This resulted in numerous proof-of-principle demonstrations [18–21], currently culminating in an extremely large 10,000 node cluster [22] created 'on-the-go' along with a 60 node cluster created simultaneously [23]. From a theoretical perspective, much progress has been made [24–32], including recently, an important fault tolerant architecture [33], achieved by leveraging the Gottesman-Kitaev-Preskill (GKP) encoding [34]. However, one area that is significantly underdeveloped is that of algorithms for a CV quantum computer. Thus far there only exists CV versions of quantum searching [35] and the Deutsch-Jozsa algorithm [36–39].

Typically, q and p are the CVs spreading across all real numbers. To encode them in qubits, one needs a whole register of qubits at each point in space. However, with CVs, there is a 1-to-1 mapping to qumodes (the CV equivalent of a qubit). In fact it is arguable that a CV quantum computer is the natural choice for such a QFT problem given that the fields are continuous variables. Thus, the value of the field at a given point in space can be mapped onto a qumode naturally. If qubits are used, instead, the qumode needs to be replaced by a register of M qubits which only allows the field to take on 2^M discrete values. Brennen *et al.* describe both possibilities in Ref. [11], although they do not explain how to implement the quartic phase gate with CVs, which we do here. Furthermore, the quartic vertex in wavelets becomes very complicated. Implementing it would require gates acting on more than two modes (resulting in logarithmic overhead in complexity).

Another benefit to our approach is in the development of the initial cluster state. Here we show how to create the initial CV cluster state as well as suggesting an experimental implementation based on standard linear optics. Furthermore, we also note that in the preparation of the initial state we see a slight improvement over the original

qubit approach of Ref. [8]. There they require $O(N^{2.376})$ gates to engineer the ground cluster state; whereas in our scheme, we require slightly less than that, specifically, $O(N^2)$ gates.

Discretization in one-dimension - We consider a relativistic scalar field ϕ in one spatial dimension including a quartic self-interaction. We shall outline the discretization specifically in the one-dimensional case so as not to clutter the notation unnecessarily, but generalization to higher dimensions is straightforward and is discussed in the supplementary material. We note that the field ϕ is a function of x and t (time), $\phi(x,t)$. All three parameters are continuous. In our approach, we discretize x, but not ϕ or t. In the case of qubits, one would discretize x and ϕ, but not t. In classical lattice calculations, one discretizes all three ϕ, x, and t.

In the continuum, the one-dimensional free scalar QFT is given by the Hamiltonian

$$H_0 = \frac{1}{2}\int_0^L dx \left[\pi^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + m^2\phi^2\right] \tag{1}$$

where ϕ is the scalar field and π the conjugate momentum field. They obey commutation relations $[\phi(x), \pi(x')] = i\delta(x-x')$ where we choose units in which $\hbar = 1$.

We discretize space by letting $x = na$, $n = 0, 1, \ldots, N-1$, where a is the lattice spacing and $L = Na$ is the finite length of the spatial dimension ($L \gg a$). We choose units in which $a = 1$, for simplicity, and denote $Q_n = \phi(x)$, $P_n = \pi(x)$. The discretized variables obey standard commutator relations, $[Q_n, P_m] = i\delta_{nm}$. The Hamiltonian becomes

$$H_0 = \sum_{n=0}^{N-1} \frac{P_n^2 + m^2 Q_n^2}{2} + \frac{1}{2}\sum_{n=0}^{N-1}(Q_n - Q_{n+1})^2 \tag{2}$$

where we employed periodic boundary conditions and defined $Q_N \equiv Q_0$.

It is useful to define creation and annihilation operators, $A_n^\dagger$ and A_n, respectively, by $A_n = (Q_n + iP_n)/\sqrt{2}$. They obey the commutation relations $[A_n, A_m^\dagger] = \delta_{nm}$ and the Hamiltonian can then be written as

$$H_0 = \frac{1}{2}\mathbf{P}^T\mathbf{P} + \frac{1}{2}\mathbf{Q}^T\mathbf{V}\mathbf{Q} \tag{3}$$

where $\mathbf{P} \equiv [P_0, P_1, \ldots, P_{N-1}]^T$ and $\mathbf{Q} \equiv [Q_0, Q_1, \ldots, Q_{N-1}]^T$. The eigenvalues of the matrix $\mathbf{V}$ and the components of the corresponding normalized eigenvectors $\mathbf{e}^n$ are, respectively, $\omega_n^2 = m^2 + 4\sin^2\frac{n\pi}{N}$, and $\mathbf{e}_k^n = \frac{1}{\sqrt{N}}e^{2\pi ikn/N}$, $k = 0, \ldots, N-1$. Notice that the massless case is special because it contains a zero mode (for $m = 0$, $\omega_0 = 0$), so the matrix $\mathbf{V}$ is not invertible. To avoid the problems that arise, we can shift the mass by a small amount $\sim 1/N$, which vanishes in the continuum limit ($N \to \infty$).

We also wish to add a quartic interaction, $H_{int} = \frac{\lambda}{4!}\int_0^L dx\phi^4 \to \frac{\lambda}{4!}\sum_n Q_n^4$ which necessitates the addition of a mass counter term $H_{c.t.} = \frac{\delta_m}{2}\int_0^L dx\phi^2 \to \frac{\delta_m}{2}\sum_n Q_n^2$ due to renormalization, as explained in the supplementary material. We find that for weak coupling, the physically interesting case is stable for $\lambda > 0$.

To diagonalize the Hamiltonian, we introduce new creation and annihilation operators, $a_k^\dagger$ and a_k, respectively, defined by $a_k = \sqrt{\frac{\omega_k}{2}}(\mathbf{e}^\dagger\mathbf{Q})_k + \frac{i}{\sqrt{2\omega_k}}(\mathbf{e}^\dagger\mathbf{P})_k$ where $\mathbf{e}$ is the matrix of the eigenvectors. Notice that $\mathbf{e}$ is unitary, $\mathbf{e}^\dagger\mathbf{e} = \mathbf{I}$. These operators obey standard commutation relations, $[a_k, a_l^\dagger] = \delta_{kl}$ and the free Hamiltonian reads

$$H_0 = \sum_{k=0}^{N-1}\omega_k\left(a_k^\dagger a_k + \frac{1}{2}\right). \tag{4}$$

In this form, it is straightforward to construct the states in the Hilbert space.

Initial cluster state preparation - For the initial cluster state, in Refs. [8, 11] the excited state was created *after* creating the ground state. This is difficult because it involves manipulating a large number of qubits. In our approach, we create a single photon state in a single mode *before* creating the cluster state. This is more accessible, as it involves creating the state $|1\rangle$ for a single mode. It can be done in a variety of ways, via a heralded single photon source, for instance. At the end of the computation, the field modes are all measured and the distribution of single photons across them determines the result.

To begin with, we build the system with N oscillators representing the variables (Q_n, P_n), $n = 0, 1, 2, \ldots$. The nth oscillator has a Hilbert space constructed by successive application of the creation operator $A_n^\dagger$ on the vacuum $|0\rangle_n$, which is annihilated by A_n. Here $|0\rangle_n$ is shorthand for a product state of vacuum fields

$$|0\rangle = |0\rangle_0 \otimes |0\rangle_1 \otimes \cdots \otimes |0\rangle_{N-1}\ , \tag{5}$$

with $A_n|0\rangle = 0$. For a scattering process, we are given an initial state typically consisting of a fixed number of particles, usually two, which undergoes evolution and then a measurement is performed (detection of particles) on the final state. Both initial and final states asymptote to eigenstates of the free Hamiltonian H_0. Thus quantum computation starts with preparation of an eigenstate of H_0.

First, we consider the ground state of H_0. It is the cluster state $|\Omega\rangle$ annihilated by all a_k, i.e., $a_k|\Omega\rangle = 0$ for $k = 0, 1, \ldots, N-1$. It can be constructed from the vacuum state (5) by acting with the Gaussian unitary $U^\dagger$, where $a_n = U^\dagger A_n U$. Noticing the relationship between the operators a_k and A_k we can use the Bloch-Messiah reduction [40] to determine $U = VSW^\dagger$ as a decomposition involving a multiport interferometer (V) followed by single mode squeezing (S) followed by a final multiport

interferometer (W). These unitary operators can be realized with $O(N^2)$ gates [41]. This is in contrast to the qubit version [8] where they require $O(N^{2.376})$ gates.

To implement U we first perform the rotation

$$\begin{aligned} A_0 \to A_0' &= \sum_{k=0}^{N-1} A_k \\ A_n \to A_n' &= \sum_{k=0}^{N-1} \cos\frac{2\pi nk}{N} A_k \\ A_{N-n} \to A_{N-n}' &= \sum_{k=0}^{N-1} \sin\frac{2\pi nk}{N} A_k \end{aligned} \tag{6}$$

where $1 \le n \le N/2$, which can be expressed in terms of rotations each involving only a couple of oscillators. Notice that if N is even, $A_{N/2}$ does not have a partner; we obtain $A_{N/2} \to \sum_k (-)^k A_k$. Next, we squeeze each mode as $A_n' \to A_n'' = \cosh r_n A_n' + \sinh r_n A_n'^\dagger$ where $e^{2r_n} = \omega_n$ for $n \le N/2$, and $e^{-2r_n} = \omega_n$, for $n > N/2$. Finally, we untangle the pairs by rotating them, $A_k'' \to a_k$ where $a_0 = A_0''$, $a_n = (A_n'' + iA_{N-n}'')/\sqrt{2}$, and $a_{N-n} = (iA_n'' + A_{N-n}'')/\sqrt{2}$. Excited states can be constructed with the same number of gates, e.g., the single-particle state $|k\rangle \equiv a_k^\dagger|\Omega\rangle$ can be constructed by acting upon the vacuum with $A_k^\dagger$. This turns the initial state of the kth mode into a one-photon state, $A_k^\dagger|0\rangle_k$, which can be accomplished in a variety of ways; see supplementary material. Having engineered $A_k^\dagger|0\rangle_k$, we then apply the Gaussian unitary $U^\dagger$, to obtain the one-particle state

$$a_k{}^\dagger|\Omega\rangle = U^\dagger A_k^\dagger|0\rangle \tag{7}$$

Extending the above to the engineering of multi-particle states, $|k_1, k_2, \dots\rangle \propto a_{k_1}{}^\dagger a_{k_2}{}^\dagger \cdots |0\rangle$, is straightforward.

Quantum computation - We wish to calculate a general scattering amplitude, which can be written as

$$\mathcal{A} = \langle out|T\exp\left\{i\int_{-T}^{T} dt(H_{int}(t) + H_{c.t.}(t))\right\}|in\rangle \tag{8}$$

in the limit $T \to \infty$, where time evolution is defined with respect to the non-interacting Hamiltonian.

We start by preparing the initial state $|in\rangle$ as in the previous section and define initial time as $t = -T$. Then we act successively with evolution operators of the form

$$U(t) = \exp\{i\delta t(H_{int}(t) + H_{c.t.}(t))\} \tag{9}$$

Time dependence is obtained via the free Hamiltonian,

$$Q_i(t) = e^{itH_0}Q_i(0)e^{-itH_0} \tag{10}$$

Therefore, the evolution (9) can be implemented as

$$U(t) = e^{itH_0}e^{i\delta t(H_{int}+H_{c.t.})}e^{-itH_0} \tag{11}$$

We deduce

$$\mathcal{A} = \langle out|\left[e^{i\delta tH_0}e^{i\delta t(H_{int}+H_{c.t.})}\right]^N|in\rangle \tag{12}$$

where we divided the time interval into $N = \frac{2T}{\delta t}$ segments.

The coupling constants in (9) are turned on and off adiabatically. This is achieved by splitting the time interval $[-T, T]$ into three segments, $[-T, -T_1]$, $[-T_1, T_1]$, and $[T_1, T]$. For $t \in [-T, -T_1]$, we turn the coupling constants on by replacing $\lambda \to \lambda(t)$, $\delta m \to \delta m(t)$, so that $\lambda(-T) = \delta m(-T) = 0$, and $\lambda(-T_1) = \lambda$, $\delta m(-T_1) = \delta m$. Then for $t \in [-T_1, T_1]$ the coupling constants are held fixed. Finally, for $t \in [T_1, T]$, they are turned off adiabatically by reversing the process in the first time interval. In the case of small λ, the time dependence of the coupling constants can be chosen efficiently by making use of perturbative renormalization. Renormalization informs the choice (see supplementary material) $\lambda(t) = \frac{T+t}{T-T_1}\lambda$, $\delta m(t) = \frac{\lambda(t)}{8\pi}\log\frac{64}{m^2}$, for $-T \le t \le -T_1$.

The unitary operators $e^{i\delta tH_0}$ and $e^{i\delta tH_{c.t.}}$ are Gaussian and can be implemented with second order nonlinear optical interactions and linear optics beam splitter networks. The interaction is implemented through a *quartic* phase gate for each mode,

$$e^{i\delta tH_{int}} = \prod_n e^{i\gamma Q_n^4} \quad , \quad \gamma = \delta t\frac{\lambda}{4!} \tag{13}$$

The quartic phase gate may be implemented in a similar manner to the cubic phase gate previously proposed [24].

After evolution, we must project onto the state $|out\rangle$. This is similar to the state $|in\rangle$, and its construction depends on the number of desired particles. The latter are excitations created with $a_n^\dagger$, so in general,

$$|out\rangle = a_{n_1}^\dagger a_{n_2}^\dagger \cdots |\Omega\rangle = U^\dagger A_{n_1}^\dagger A_{n_2}^\dagger \cdots |0\rangle \tag{14}$$

It follows that the next step is to *uncompute* by applying the Gaussian unitary U (which is the inverse operation to the preparation of the initial state), and then measure the number of photons in each mode. The final uncompute step projects the set of output modes onto the Fock basis. Thus, the scattering amplitude calculation is a mapping from one set of field modes on the input to a separate set of field modes on the output, as expected. That is, for each click on the photodetector for mode n, there is an operator $a_n^\dagger$ present in the final state (14). If the QFT calculation involved an initial input state with two excitations spread across 100 field modes, say, then the entire calculation would involve two photons, for instance. We note that the calculation has made use of a quartic phase gate up to this point, and thus technically speaking a non-Gaussian operation would not be necessary during this measurement step in order to achieve an exponential speedup over the classical QFT algorithm.

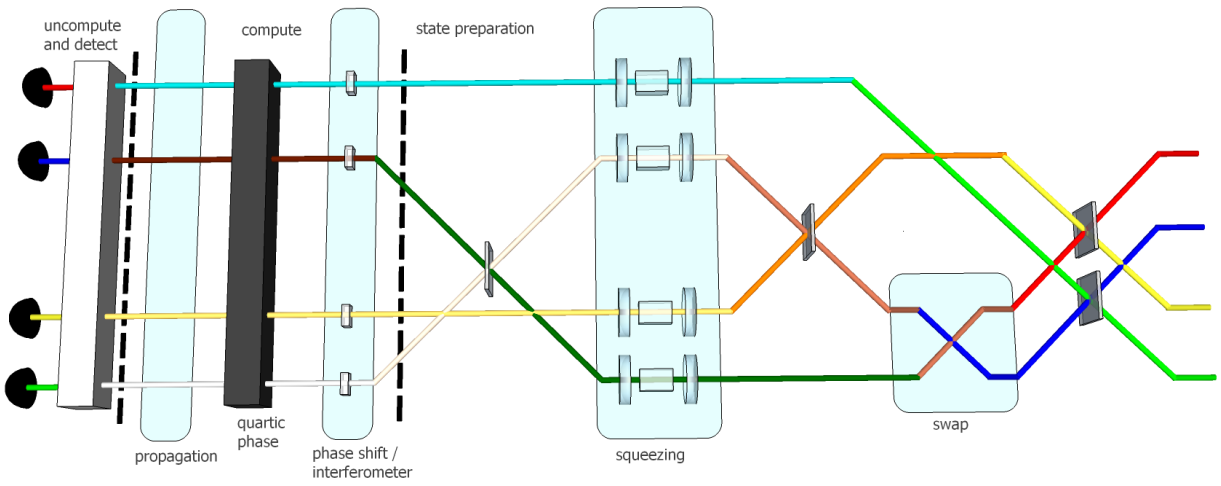

FIG. 1: (Color Online) Sketch of an experimental setup for electromagnetic field modes used as qudits in a QFT calculation involving four field modes. The modes are encoded into electric field modes (colored red, blue, yellow, green), which are then prepared via beam splitters, swap gates, and squeezers for the compute stage. The compute stage consists of an interferometer, a quartic phase gate (black box, see Ref. [24]), and free propagation. An uncompute stage, which is the inverse of the preparation stage, and a detection stage in the Fock basis, yield the scattering amplitudes into the four QFT field modes.

However, in order to achieve high accuracy in the final result, photon number resolving detectors with high efficiency [42] would be desirable for the measurment phase.

Experimental implementation - An example of the experimental implementation is given in Fig. 1. For brevity the setup for calculating four space time points is given. For the electromagnetic field, the initial unitary rotation involves weighted beam splitters with the appropriate splitting to achieve the desired sums over the field operators (see supplementary material). A swap gate is involved in the input state preparation stage. We note that a swap gate contains essentially the CV version of the CNOT operator along with parity operators [43], but in some cases the gate can be simplified to a beam splitter interaction [44] such as for the electromagnetic field. Here we use a mode label swap operator, which is possible in systems with movable qubits, such as CV optical fields. Next, $H_{c.t.}$ is quadratic in position quadrature operators, which can be implemented with a series of phase shifts [44]. The non-Gaussian piece of the computation is then the quartic phase gate contained in H_{int}, which can be implemented via repeated application of the photon number-dependent phase gate [24]. Lastly, the free propagation H_0 can be implemented by a calibrated free propagation before the uncompute stage. We note that the QFT field modes are encoded into the qudits which are themselves electromagnetic field modes, meaning that the free propagation contained in H_0 is not arbitrary. It must conform to the calculated QFT free propagation distance, and phase stability must be maintained throughout.

In conclusion, we developed a new algorithm for a continuous-variable quantum computer which gave an exponential speedup over the best known classical algorithms. This algorithm was the calculation of the scattering amplitudes in scalar bosonic quantum field theory, and as previously mentioned, arguably a natural choice for a continuous variable quantum computer to solve. At weak coupling, analytic calculations are possible, however, at strong coupling no such calculations are generally available, and one has to rely on numerical techniques. A widely used framework is lattice field theory which is based on the discretization of space into a finite set of points. The complexity of classical computations on a lattice increases exponentially with the number of lattice sites [2].

Quantum computations offer a distinct advantage (first shown in Ref. [9] for qubits, and here for qumodes), since complexity only grows polynomially. Using continuous variables we also see an advantage over the original qubit proposal; specifically, in the preparation of the initial cluster state. There they required $O(N^{2.376})$ gates to synthesize the ground state. However, in our scheme, we required slightly less, $O(N^2)$ gates. Finally, we also gave an example of an experimental implementation on a continuous-variable cluster state quantum computer that calculated four space time points. We noted that such a scheme is feasible with current linear optical technology and consisted of a set of Gaussian operations along with the non-Gaussian quartic phase gate.

We are grateful to Peter Rohde for feedback. K. M. acknowledges support from NSERC. R. C. P. performed

portions of this work at Oak Ridge National Laboratory, operated by UT-Battelle for the US Department of Energy under Contract No. DE-AC05-00OR22725.

* siopsis@tennessee.edu

[1] M. E. Peskin, and D.V. Schroeder, An introduction to quantum field theory. (Westview, 1995).

[2] M. Creutz, Quarks, gluons and lattices. Vol. 8. (Cambridge University Press, 1985).

[3] M. A. Nielsen, and I.L. Chuang, Quantum computation and quantum information. (Cambridge university press, 2010).

[4] P. Shor, SIAM J. Comput. **26**, 1484 (1997).

[5] L. K. Grover, Phys. Rev. Lett. **79**, 325 (1997).

[6] R. Feynman, Int. J. Theor. Phys. **21**(6), 467 (1982).

[7] I. M. Georgescu, S. Ashhab, and F. Nori, Rev. Mod. Phys. **86**, 153 (2014).

[8] S. P. Jordan, K. S. M. Lee, and J. Preskill, Science **336**, 1130-1133 (2012).

[9] S. P. Jordan, K. S. M. Lee, and J. Preskill, Quantum Information and Computation **14**, 1014-1080 (2014).

[10] S. P. Jordan, K. S. M. Lee, and J. Preskill, arXiv:1404.7115 (2014).

[11] G. K. Brennen, P. Rohde, B. C. Sanders, and S. Singh, arXiv:1412.0750 (2014).

[12] T. D. Ladd et al., Nature 464, 45 (2010).

[13] S. Lloyd and S. L. Braunstein, Phys. Rev. Lett. **82**, 1784 (1999).

[14] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

[15] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[16] J. Zhang and S. L. Braunstein, Phys. Rev. A **73**, 032318 (2006).

[17] N. C. Menicucci et al., Phys. Rev. Lett. **97**, 110501 (2006).

[18] S. Yokoyama, R. Ukai, S. C. Armstrong, J.-i. Yoshikawa, P. van Loock, and A. Furusawa, arXiv:1410.0472 (2014).

[19] K. Miyata, H. Ogawa, P. Marek, R. Filip, H. Yonezawa, J.-i. Yoshikawa, and A. Furusawa, Phys. Rev. A **90**, 060302(R) (2014).

[20] M. Pysher, Y. Miwa, R. Shahrokhshahi, R. Bloomer, and O. Pfister, Phys. Rev. Lett. **107**, 030505 (2011).

[21] S. Takeda, T. Mizuta, M. Fuwa, J.-i. Yoshikawa, H. Yonezawa, and A. Furusawa, Phys. Rev. A **87**, 043803 (2013).

[22] S. Yokoyama, R. Ukai, S. C. Armstrong, C. Sornphiphatphong, T. Kaji, S. Suzuki, J.-i. Yoshikawa, H. Yonezawa, N. C. Menicucci, and A. Furusawa, Nat. Phot. **7**, 982 (2013).

[23] M. Chen, N. C. Menicucci, and O. Pfister, Phys. Rev. Lett. **112**, 120505 (2014).

[24] K. Marshall, R. Pooser, G. Siopsis, and C. Weedbrook, Phys. Rev. A **91**, 032321 (2015).

[25] H.-K. Lau and C. Weedbrook, Phys. Rev. A **88**, 042313 (2013).

[26] P. van Loock, C. Weedbrook, and M. Gu, Phys. Rev. A **76**, 032321 (2007).

[27] M. Gu, C. Weedbrook, N. Menicucci, T. Ralph, and P. van Loock, Phys. Rev. A **79**, 062318 (2009).

[28] R. N. Alexander, S. C. Armstrong, R. Ukai, N. C. Menicucci, Phys. Rev. A **90**, 062324 (2014).

[29] T. F. Demarie, T. Linjordet, N. C. Menicucci, and G. K. Brennen, New J. Phys. **16**, 085011 (2014).

[30] N. C. Menicucci, T. F. Demarie, G. K. Brennen, arXiv:quant-ph/1503.00717 (2015).

[31] P. Wang, M. Chen, N. C. Menicucci, and O. Pfister, Phys. Rev. A **90**, 032325 (2014).

[32] N. C. Menicucci, Phys. Rev. A **83**, 062314 (2011).

[33] N. C. Menicucci, Phys. Rev. Lett. **112**, 120504 (2014).

[34] D. Gottesman, A. Kitaev, and J. Preskill, Phys. Rev. A **64**, 012310 (2001).

[35] A. K. Pati, S. L. Braunstein, and S. Lloyd, arXiv:quant-ph/0002082v2 (2000).

[36] A. K. Pati, and S. L. Braunstein, in Quantum information with continuous variables, edited by S. L. Braunstein, and A. K. Pati, (Kluwer Academic, Dordrecht), 31-36 (2003).

[37] M. R. A. Adcock, P. Høyer, and B. C. Sanders, New J. Phys. **11**, 103035 (2009).

[38] M. Zwierz, C. A. Pérez-Delgado, and P. Kok, Phys. Rev. A **82**, 042320 (2010).

[39] M. R. A. Adcock, P. Høyer, and B. C. Sanders, Quantum Information Processing,**12**, 1759-1779 (2013).

[40] S. L. Braunstein, Phys. Rev. A **71**, 055801 (2005).

[41] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

[42] A. E. Lita, A. J. Miller, and S. Nam, Optics Express, **16**, 3032 (2008).

[43] X. Wang, J. Phys. A: Math. Gen. **34**, 9577 (2001).

[44] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).