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Mode-dependent field renormalization and triviality in $\lambda\Phi^4$ theory

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Abstract

A continuum limit of the $(\lambda\Phi^4)_4$ theory exists which is "trivial" (all scattering amplitudes vanish), but which does not correspond to the zero-renormalized-coupling limit of conventional perturbation theory. This allows Spontaneous Symmetry Breaking and a very heavy, but still weakly interacting, Higgs boson. We answer some criticisms of this work, emphasizing that results from renormalization-group-improved perturbation theory are not valid. The phase transition encountered as the bare mass-squared is made more and more negative is *first order*, contrary to the assumption of the conventional perturbative "RG-improvement" program.

1. Conventionally, it is assumed that the "triviality" of $\lambda\Phi^4$ theory [1] implies that one may do renormalization-group-improved perturbation theory (RGIPT) in terms of a renormalized coupling λ_R, provided that λ_R is then sent to zero as $1/\ln(\text{cutoff})$. The Higgs mass, m_h, which is proportional to $(\lambda_R v^2)^{1/2}$ in RGIPT, would then go to zero in the continuum limit, if the vacuum expectation value v (phenomenologically believed to be 246 GeV) is taken to be finite. In this picture, the only way to have a viable Higgs mass is to keep the cutoff finite and not too large.

In contrast, we have argued [2] that there is a non-perturbative continuum limit in which all scattering amplitudes vanish ("triviality"), but which has, by construction, spontaneous symmetry breaking (SSB) and a *finite* m_h. The underlying physical picture is very simple and has important phenomenological consequences as we shall briefly recapitulate in the following.

Firstly, one should understand that "triviality" is a simple but technical statement concerning the Gaussian nature of the continuum limit in $(\lambda\Phi^4)_4$ theories. It does not preclude a non-vanishing vacuum expectation value $\langle\Phi\rangle$ for the scalar field. However, it means that the 3- and higher-point *connected* Green's functions of the *shifted field* $h(x) = \Phi(x) - \langle\Phi\rangle$ must vanish in the continuum limit. Consistency with the Källen-Lehmann decomposition then requires the h-field propagator to exhibit a trivial residue $Z_h \to 1$ at its pole $s = m_h^2$, thus ensuring that the spectral function becomes consistently $\rho(s) \to \delta(s - m_h^2)$, as expected for a non-interacting field [3].

The crucial point is that, even though there are no physical scattering processes (with $p_\mu \neq 0$) and the S-matrix reduces to the trivial identity, one can nevertheless obtain a change in the vacuum structure. Statistical mechanics furnishes many such examples. In superconductivity, for example, an *arbitrarily small* 2-

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body interaction produces macroscopic effects if there are sufficiently many states at the Fermi surface. Ordinary perturbation theory fails completely in predicting the basic features of the superconducting ground state.

A statistical-mechanics approach to $(\lambda\Phi^4)_4$ theory is enlightening.[1] Consider a large box containing a gas of "atoms" (quanta of the symmetric phase) that interact via a repulsive $\delta^{(3)}(\boldsymbol{r})$ potential plus an attractive, long-range interaction. The latter arises from exchange of particle pairs, and behaves as $-1/r^3$ in the limit of massless atoms. Even with an infinitesimal strength, such an interaction can cause an instability of the symmetric vacuum: i.e., an "empty box" may have higher energy than one containing an infinite number of spontaneously created atoms, Bose condensed in the zero-momentum mode. The quantum excitations of this condensate ("phonons") correspond to the Higgs particles. Although phonon-phonon scattering vanishes, the phonons are non-trivially related to the atoms. Heating the system above a critical temperature (*finite* in units of the phonon mass m_h) destroys the Bose condensate, giving symmetry restoration. These features point out the deep difference between a "trivial" theory and a free-field theory.

In quantum field theoretical language, the intuitive concept of an infinitely large atom density in the zero-momentum mode is reflected in a rescaling factor Z_ϕ for the vacuum field that is quite distinct from the trivial $Z_h = 1$ rescaling of the fluctuation field. Although novel, such special treatment of the field's zero-4-momentum mode violates no sacred principles. It can be easily implemented in momentum space by introducing suitable projections that select or remove the zero four-momentum state [2]. In x-space this corresponds to writing the bare field as $\Phi_B(x) = \phi_B + h_B(x)$ (where $\int d^4x\, h_B(x) = 0$), and defining the renormalized field as $\Phi_R(x) = \phi_R + h_R(x)$ with $\phi_R = Z_\phi^{-1/2}\phi_B$ and $h_R(x) = Z_h^{-1/2}h_B(x)$. Notice that although "triviality" and the Källen-Lehmann representation require $Z_h = 1$ in the continuum theory, they place no constraint whatsoever on Z_ϕ. This crucial quantity is fixed by the relation between the physical Higgs mass and the second derivative of the effective potential at its minimum $\phi_B = v_B$. As we shall see below, one finds that $Z_\phi \to \infty$. Then, although the ratio m_h^2/v_B^2 goes to zero, the Higgs mass m_h is finite in units of the physical vacuum field $v = Z_\phi^{-1/2}v_B$ defined through

$$\left.\frac{d^2V_{\text{eff}}}{d\phi_R^2}\right|_{\phi_R=\pm v} = m_h^2. \tag{1}$$

Also, the vacuum energy $W = V_{\text{eff}}(\pm v_B)$, and hence the symmetry-restoration temperature, is finitely related to v^4 and *not* to v_B^4. The physical weak scale, 246 GeV (related to the Fermi constant) is thus to be identified with v and *not* with v_B. Although $m_h^2/v_B^2 \to 0$ in the continuum limit, we have m_h^2/v^2= cutoff-independent, and so the ultraviolet cutoff can be removed, leaving m_h finite in GeV units. The resulting Higgs phenomenology will not conform to conventional ideas, because now m_h^2/v^2 is not proportional to the strength of any *observable* interaction process: large Higgs mass does not imply strong scalar-sector interactions.

2. In describing our picture concretely, the key quantity is the effective potential, V_{eff}. In terms of it, the deep difference between a "trivial" theory and a free-field theory is this: In a "trivial" theory V_{eff} must arise solely from the classical potential plus the zero-point energies of the free h-field fluctuations — however, it need not be the quadratic function found in free-field theory.

A brief outline of our picture was provided in Ref [4]. A more sophisticated version of the calculation has been provided by Ritschel [5], who explicitly uses a finite-volume formalism (in which separating out the zero-momentum mode is completely well defined). Following Brézin [6], one first integrates over the h modes, leaving the integration over the ϕ mode for the last step. The h integration can be performed if one neglects the h^3, h^4 interactions arising from expanding $(\phi + h(x))^4$. The subtlety is that this is not a crude approximation, but is *effectively an exact ansatz*.

From a conventional, perturbative viewpoint, this is an extraordinary claim. However, it just corresponds to assuming "triviality" as an input. If all scattering amplitudes must vanish (for which there is overwhelming evidence [1]), that means that the h modes are effectively governed by a quadratic Lagrangian. (However, it says nothing about the ϕ mode, for which it is meaningless to speak of "scattering".) Discarding the h^3, h^4 terms is just the simplest of a *class* of

[1] We shall describe this approach in more detail in a future paper.

"triviality-compatible" approximations in which the h modes are effectively governed by some quadratic Lagrangian, with the h propagator determined by solving exactly a nonperturbative gap equation. In any such approximation – though intermediate details may well be different – the final, physical, renormalized results are always the same. The Gaussian approximation provides an explicit example of this [2]. (See also Ritschel's post-Gaussian calculation [7].) In this sense, the "triviality" of $(\lambda\Phi^4)_4$ theory implies that the bare interaction terms h^3, h^4 are effectively *irrelevant* operators, in that all their effects can be re-absorbed in the renormalization process.

If we proceed with the *ansatz* of dropping the h^3, h^4 terms we obtain a free h field with a mass $\frac{1}{2}\lambda_B\phi_B^2$ and an effective potential that is just the classical potential plus the zero-point energy of the h-field fluctuations. This is formally the same as the unrenormalized one-loop effective potential. However, it is very important to stress that we are *not* saying that the one-loop result is good because the loop expansion is super-rapidly convergent. On the contrary, the loop expansion is very badly behaved (it basically behaves as $1-1+1-1+\ldots$). What we are saying is that, in a "trivial" theory, the effective potential must be interpretable as the classical potential plus free-field zero-point energies – any other form of $V_{\rm eff}$ would imply non-zero $h-h$ interactions. Therefore, the one-loop potential is just the *prototype* of all approximations that respect the non-interacting nature of the field $h(x)$.

Explicitly (taking, for simplicity, the "classically scale-invariant" (CSI) case where the mass renormalization is required to give exactly zero mass in the symmetric phase), we have (as in Ref. [8], Eq. (3.4))

$$V_{\rm eff} = \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda_B^2\phi_B^4}{256\pi^2}\left(\ln\frac{\frac{1}{2}\lambda_B\phi_B^2}{\Lambda^2} - \frac{1}{2}\right), \tag{2}$$

where Λ is an ultraviolet cutoff. Accepting "triviality," a consistent renormalization procedure should not mention any "λ_R," but should simply require the physical mass m_h and the effective potential $V_{\rm eff}$ to be finite. Now, the $V_{\rm eff}$ above is just a sum of $\phi_B^4\ln\phi_B^2$ and ϕ_B^4 terms. It has a pair of minima at $\phi_B = \pm v_B$, and using v_B as a parameter it may be re-written as:

$$V_{\rm eff} = \frac{\lambda_B^2\phi_B^4}{256\pi^2}\left(\ln\frac{\phi_B^2}{v_B^2} - \frac{1}{2}\right). \tag{3}$$

Equating these two forms yields v_B in terms of Λ. Hence, the mass-squared of the $h(x)$ fluctuation field, $\frac{1}{2}\lambda_B\phi_B^2$, when evaluated in the SSB vacuum, is

$$m_h^2 = \tfrac{1}{2}\lambda_B v_B^2 = \Lambda^2\exp\left(-\frac{32\pi^2}{3\lambda_B}\right). \tag{4}$$

The vacuum energy density is

$$W = V_{\rm eff}(\pm v_B) = -\frac{\lambda_B^2 v_B^4}{512\pi^2} = -\frac{m_h^4}{128\pi^2}. \tag{5}$$

Demanding that the particle mass m_h, and hence the vacuum energy W, be finite requires λ_B to be infinitesimal:

$$\lambda_B = \frac{32\pi^2}{3}\frac{1}{\ln(\Lambda^2/m_h^2)} \to 0_+. \tag{6}$$

The effective potential can then be made manifestly finite just by re-scaling ϕ_B. We define ϕ_R as $Z_\phi^{-1/2}\phi_B$, with $Z_\phi \propto 1/\lambda_B \to \infty$, so that the combination $\xi \equiv \frac{1}{2}\lambda_B Z_\phi$ remains finite. The physical mass is then finitely proportional to $v = Z_\phi^{-1/2}v_B$; i.e., $m_h^2 = \xi v^2$. The requirement in Eq. (1) fixes ξ to be $8\pi^2$. [These results can be generalized straightforwardly to the general, non-CSI case [2]. In general, $V_{\rm eff}$ is a sum of $\phi^4\ln\phi$, ϕ^4 and ϕ^2 terms.]

Because we obtain an infinitesimal λ_B, the scattering amplitudes produced by the h^3, h^4 terms are vanishingly small. This is true to all orders in the interaction, as one can show by a simple argument [2,4], based on the fact that each loop in a diagram generates at most one extra logarithm of the cutoff. This means that our ansatz self-consistently gives "triviality" as an output.

Lattice calculations confirm our predicted form of $V_{\rm eff}$ to high precision [9].

3. A recent Letter [10] has criticized our work. In responding, we hope to clarify some key aspects of our picture. In outline, our points are: (i) the renormalization procedure considered in Ref. [10] is not the same as ours; (ii) Ref. [10]'s discussion assumes results from perturbation theory that are not valid; and (iii) there is nothing physically pathological about the zero-momentum limit of our effective action.

(i) Ref. [10] also considers a re-scaling of the zero-momentum mode of the field (and hence of its vacuum value v) that is distinct from the wavefunction renormalization of the finite-momentum modes. That appears to grant our key point that it is an allowable procedure. However, Ref. [10]'s actual re-scalings are not the same as ours. This fact is somewhat obscured by different terminology and notation, but can be seen as follows: In our work the key requirement is that the combination $\lambda_B v_B^2$, governing the physical mass, should be finite. In our notation λ_B is the bare coupling constant, which tends to zero like $1/\ln\Lambda$, and the finite, physical v is related to the bare field by

$$v_B = Z_\phi^{1/2} v \tag{7}$$

with $Z_\phi \sim \ln\Lambda$, so that $1/Z_\phi$ *scales like* λ_B. In Ref. [10] the corresponding equation is in the last line of Eq. (20):

$$\text{“}v_R = Z_A^{-1/2} A\text{”}, \tag{8}$$

where "v_R" is essentially our v_B (it is "$Z_R^{-1/2} v_B$" with "Z_R" ~ 1) and "A" is the finite quantity (our v). Thus, "Z_A" is our $1/Z_\phi$. However, in Ref. [10]'s continuum limit ("$\tau \to 0$"), based on RGIPT, Z_A scales as $|\ln\tau|^{-1/2}$ (Eq. (21)) while the coupling scales as $|\ln\tau|^{-1}$ (Eq. (19)). Thus, Ref. [10]'s renormalization cannot be reduced to ours. This difference accounts for the fact that no mass term survives in Ref. [10]'s renormalized effective action (Eq. (23)), in contrast to ours.

Although, for reasons to be explained below, we do not accept Ref. [10]'s initial premise, Eq. (17), it might be instructive to point out that a more accurate caricature of our picture could have been produced by replacing the postulated Eq. (18) with

$$a \sim \tau^{1/2}\,|\ln\tau|^{-1/6}\,L. \tag{9}$$

This would yield our re-scaling for v and also an "m_R" that is finite in physical units. Superficially, it leads to an effective potential that is of order $\ln\Lambda$, but in our picture, as originally in Ref. [11], this is remedied by a cancellation. This cancellation is simply the fact that a function made up of a log-divergent ϕ^4 term and a finite $\phi^4\ln\phi^2$ term can always be re-written as $\phi^4(\ln\phi^2/v^2 - \frac{1}{2})$, with the divergence absorbed into the vacuum value v.

(ii) Ref. [10]'s starting point, Eq. (17), relies on results from RGIPT. It is claimed that these results are "very solidly founded, because RGIPT is, at low energies, and *because of triviality* [our italics], very reliable." As explained at the beginning of this Letter, this is a common misconception. It falsely assumes that a small (or vanishingly small) renormalized coupling is a sufficient condition for RGIPT to work. In the case of $(\lambda\Phi^4)_4$ theory, RGIPT and "triviality" are inherently contradictory about the continuum limit; the former begins by postulating a finite, *non-zero* renormalized coupling constant, and "triviality" says that there can be no such thing.

[In [12] we discuss exactly what goes wrong with RGIPT: Its re-summation of leading logs tries to re-sum a geometric series that is inevitably *divergent* when one tries to take the continuum limit. Our not-entirely-trivial continuum limit arises precisely where the leading-log series becomes $1-1+1-\ldots$, which RGIPT assumes will re-sum to $1/(1+1) = 1/2$. There are instances in physics where such an illegal re-summation happens to give the right answer — but this is not one of them.]

Ref. [10]'s Eq. (17) assumes, based on perturbation theory, that SSB in lattice $(\lambda\Phi^4)_4$ theory corresponds to a *second-order* phase transition. We claim, with strong support from recent lattice data [9], that this is not true. Consider varying the bare-mass-squared parameter $r \equiv m_B^2$ while holding the bare coupling constant fixed. *A priori*, one can define two distinct critical values of r; one, $r_{\rm PhT}$, is where the SSB phase transition actually occurs; the other, $r_{\rm CSI}$, is where the mass gap of the symmetric phase becomes exactly zero (the "classically scale-invariant" (CSI) case). If these two values *exactly* coincide then the transition is second order. If that were so, then a continuum limit could be obtained for *any* λ_B by taking the limit $\tau \to 0$, where $\tau = |1 - \frac{r}{r_{\rm CSI}}|$), since the physical correlation length would then diverge in units of the lattice spacing.

However, to find out whether $r_{\rm CSI}$ and $r_{\rm PhT}$ coincide, one must explore the effective potential of the theory. In any approximation consistent with "triviality," the effective potential in the CSI case has the form $\phi^4(\ln\phi^2/v^2 - \frac{1}{2})$. This is confirmed to great accuracy by lattice simulations [9]. Since this has SSB, one sees that the CSI case lies within the broken phase. Thus, $r_{\rm CSI}$ is distinct from (and more negative than) $r_{\rm PhT}$.

Since $r_{\rm CSI}$ and $r_{\rm PhT}$ differ, the phase transition is first-order. In order to obtain a continuum limit, one needs the physical correlation length ξ_h of the *broken* phase to be infinite in units of the lattice spacing a. In other words, the mass $m_h \sim 1/\xi_h$ of the fluctuations about the SSB vacuum must be much, much less than the cutoff. As discussed above, this requires λ_B to tend to zero like $1/\ln\Lambda$ [2].

With such an infinitesimal value of λ_B, the difference between $r_{\rm CSI}$ and $r_{\rm PhT}$ is only infinitesimal, even in physical units: each one is negative and huge, of order $-\Lambda^2$, while their difference is infinitesimal, of order $1/\ln\Lambda$. [For this reason, the phase-transition is so weakly first-order that, in a lattice calculation, the correlation length $\xi_h \sim a\exp(1/\lambda_B)$, although finite for finite λ_B, is larger than any presently attainable lattice size.]

Superficially, since $r_{\rm CSI} \to r_{\rm PhT}$ and $\xi_h/a \to \infty$, the phase transition becomes an *infinitesimally weak* first-order transition, which is practically indistinguishable from a second-order transition. However, the difference between $r_{\rm CSI}$ and $r_{\rm PhT}$, although infinitesimal, remains crucial: all the interesting physics occurs when r is varied over an infinitesimal range around $r_{\rm PhT}$. This is because such tiny variations in r cause *finite* changes (i) in the particle mass of the broken vacuum, (ii) in the energy-density difference between the two phases, and (iii) in the barrier between them. The problem with the conventional approach is that it looks at the phase transition on too coarse a scale — making finite variations in r. Viewed on that scale the transition appears indistinguishable from a second-order transition and the not-entirely-trivial physics is not seen.

(iii) Ref. [10] complains that our effective action is "infrared divergent." It is not clear what is meant by this. There is, of course, the usual infinite-volume factor between the effective action and the effective potential: In a derivative expansion of the effective action the term with no derivatives is $-\int d^4x\, V_{\rm eff}(\Phi(x))$, so that if $\Phi(x) = \phi$ = constant one gets $-(\int d^4x)\, V_{\rm eff}(\phi)$ (see, eg. [8,13]). This is the origin of terms proportional to $\delta^4(0)$ in momentum space. Physically, this is natural — the energy diverges with the volume if the energy density is finite — but, for mathematical respectability, some regularization is called for. Ref. [10] objects to having a *constant* source, and hence a constant ϕ, $\neq v$, insisting that all sources should fall off to zero at infinity. However, that is only one way of regularizing. More conveniently, as explicitly done in Ritschel's calculation [5], the theory can be formulated in finite volume with periodic boundary conditions; there is then no problem with considering a source that is constant over that volume. In any case the treatment of this nicety has nothing to do with the ultraviolet renormalization.

It is true that our renormalized effective action is discontinuous at zero momentum, in that the renormalized proper n-point functions ($n \geq 3$) are zero at finite momentum, but are non-zero at zero momentum [2]. [Our renormalized 2-point function, however, has *no* discontinuity at zero momentum; our field renormalization is precisely what is needed to ensure this.] However, this discontinuity could never be directly revealed experimentally, because scattering experiments with exactly zero-momentum particles are inherently impossible. Moreover, S-matrix elements are more directly related, not to the proper Green's functions generated by the effective action, but to the *full* Green's functions. The latter are inherently singular at $p = 0$, whenever there is SSB, because they contain disconnected pieces proportional to $\delta^{(4)}(p)$. Therefore, smoothness at $p \to 0$ is not to be expected. Again, this reflects the underlying physics, namely Bose condensation.

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