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Unconventional large-N limit of the Gaussian effective potential and the phase transition in $\lambda\phi^4$ theory

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Abstract. In the conventional large-N limit the coupling constant λ is required to scale as $1/N$. While the Gaussian effective potential (GEP) is known to contain the exact result in this limit, it shows a phase transition only when $\lambda \gg 1/N$ (in units of the renormalized mass in the symmetric vacuum). Here we determine the asymptotic behaviour, as $N \to \infty$, of λ and other quantities at the phase transition of the GEP. We find λ_{crit} to be finite in $0+1$ dimensions; of order $1/\ln N$ in $1+1$ dimensions; $1/N^{\frac{1}{3}}$ in $2+1$ dimensions; and $1/\sqrt{N}$ in $3+1$ dimensions. The GEP's first-order phase transition is shown to become asymptotically second-order in $1+1$ dimensions and below. We also discuss non-integer dimensions and the approach to the non-trivial "autonomous" theory in $3+1$ dimensions.

1 Introduction

The phase transition in $O(N)$-symmetric $\lambda\phi^4$ theory has long been an important paradigm in both statistical mechanics and elementary particle physics. The Higgs mechanism that generates W, Z masses is traditionally viewed as arising from spontaneous symmetry breaking in the $O(4)$ $\lambda\phi^4$ theory that constitutes the Higgs sector of the Standard Model. However, this picture has been challenged by the claims that the $3+1$ dimensional $\lambda\phi^4$ theory is "trivial" [1]. This situation makes it important to try to better understand the phase transition in $\lambda\phi^4$ theory, through nonperturbative methods.

The Gaussian effective potential (GEP) is a simple, nonperturbative method founded upon intuitive ideas familiar in ordinary quantum mechanics [2, 3]. It is known to contain the one-loop and leading-order $1/N$ effective potential results in the appropriate limiting cases [2–4]. Intriguingly, the GEP indicates that a non-trivial $3+1$ dimensional theory exists and can have spontaneously broken symmetry [5, 6]. This form of the theory has been called "autonomous" [4, 7] because it is quite separate from the perturbative form of the theory. The "autonomous" theory is not seen in the $1/N$ expansion [8], but this, we argue, has to do with the limitations of the 1/N method. To elucidate this point we consider here the behaviour of the GEP, at and near its phase transition, when N becomes infinitely large.

The GEP is a variational approximation to the effective potential which uses a trial wave-functional of Gaussian form. The Gaussian wavefunctional describes a free field theory vacuum state and contains two parameters, Ω and ω, which are the masses associated with the radial and transverse fields, respectively (where the "radial" direction in the $O(N)$ space is that picked out by the classical field). The expectation value of the Hamiltonian density:

$$\mathscr{H} = \tfrac{1}{2}(\pi^a\pi^a + \partial_i\phi^a\partial_i\phi^a + m_B^2\phi^a\phi^a) + \lambda_B(\phi^a\phi^a)^2. \tag{1.1}$$

(where $a = 1, \ldots, N$) in the Gaussian trial state is [4, 9]:

$$\begin{aligned}\langle\mathscr{H}\rangle = {} & J(\Omega) + (N-1)J(\omega) \\ & + \tfrac{1}{2}m_B^2(I_0(\Omega) + \phi_c^2 + (N-1)I_0(\omega)) \\ & + \lambda_B[3(I_0(\Omega)+\phi_c^2)^2 + 2(N-1)I_0(\omega)(I_0(\Omega)+\phi_c^2) \\ & + (N^2-1)I_0(\omega)^2 - 2\phi_c^4],\end{aligned} \tag{1.2}$$

with (in $\nu+1$ dimensions)*

$$I_0(m) = \tfrac{1}{2}\int \frac{d^\nu p}{(2\pi)^\nu}\frac{1}{\sqrt{p^2+m^2}}$$

and

$$J(m) = \tfrac{1}{4}\int \frac{d^\nu p}{(2\pi)^\nu}\frac{2p^2+m^2}{\sqrt{p^2+m^2}}. \tag{1.3}$$

The GEP itself, $\bar{V}_G(\phi_c)$, is obtained by minimizing $\langle\mathscr{H}\rangle$ with respect to the variational parameters Ω and ω. This leads to optimization equations for Ω and ω:

$$\Omega^2 = m_B^2 + 4\lambda_B[(N-1)I_0(\omega) + 3I_0(\Omega) + 3\phi_c^2], \tag{1.4}$$

$$\omega^2 = m_B^2 + 4\lambda_B[(N+1)I_0(\omega) + I_0(\Omega) + \phi_c^2]. \tag{1.5}$$

* In terms of the notation of [3, 4], $J(m) = I_1(m) - \frac{1}{2}m^2 I_0(m)$

We define the renormalized mass m_R to be the physical particle mass in the symmetric vacuum at $\phi_c=0$. This is given, in the Gaussian approximation, by the value of the mass parameter Ω (or equivalently ω) at the origin. (m_R^2 is also equal to the second derivative of the GEP at the origin.) The mass renormalization is accomplished by using the relationship:

$$m_B^2=m_R^2-4(N+2)\lambda_B I_0(m_R) \tag{1.6}$$

to eliminate the bare mass m_B in favour of the renormalized mass m_R. Having made this substitution, and subtracted the vacuum-energy constant, so as to have $\bar{V}_G=0$ at the origin, one obtains equations that have exactly the same form as (1.2), (1.4), and (1.5), except that the divergent integrals $I_0(m)$ are replaced by $\Delta(m)$ and $\Delta J(m)$, defined by:

$$\Delta(m)=I_0(m)-I_0(m_R) \quad \text{and} \quad \Delta J(m)=J(m)-J(m_R). \tag{1.7}$$

In less than $3+1$ dimensions these functions are finite, and are given explicitly by:

$$\Delta(m)=\begin{cases} (1/m-1/m_R)/2 & \nu=0, \\ -\ln(m^2/m_R^2)/(4\pi) & \nu=1, \\ -(m-m_R)/(4\pi) & \nu=2, \end{cases} \tag{1.8}$$

$$\Delta J(m)=\begin{cases} (m-m_R)/4 & \nu=0, \\ (m^2-m_R^2)/(8\pi) & \nu=1, \\ (m^3-m_R^3)/(24\pi) & \nu=2. \end{cases} \tag{1.9}$$

(Note that $\Delta(m)$ is always negative for $m>m_R$, whereas $\Delta J(m)$ is positive.)

In the conventional large-N limit one re-scales the field and the coupling constant:

$$\lambda_B=\tilde{\lambda}/N, \qquad \phi_c^2=N\tilde{\phi}^2, \tag{1.10}$$

and takes the limit $N\to\infty$ with $\tilde{\lambda}$ and $\tilde{\phi}$ kept finite. In this limit, where $\bar{V}_G$ becomes of order N and all terms involving Ω drop out, it is easily shown [4] (see also [10]) that $\bar{V}_G$ coincides with the leading-order $1/N$ expansion result: the parameter ω corresponds to the auxiliary field χ introduced in the $1/N$ approach [8].

However, from numerical studies of the GEP we find that the parameter Ω plays a crucial role at the phase transition, even when N is large. The contributions from the single radial field turn out to be comparable to those from the $(N-1)$ transverse fields because the radial mass Ω is so much bigger than the transverse mass ω at the non-trivial minimum of the GEP. This fact can be viewed as the approximate realization of the Goldstone theorem in the Gaussian approximation [4, 9].

Numerical results [4] also reveal that the value of λ_B at which the GEP shows a phase transition, although decreasing with N, does not fall as fast as $1/N$. Therefore, the conventional large-N limit does not allow one to reach the phase transition. (Above $1+1$ dimensions, one can get to the other side of the phase transition by using a different mass renormalization with $-m_R^2$ in place of m_R^2, but one still cannot get to the phase transition region itself.) This is particularly unsatisfactory if we wish to consider the $3+1$ dimensional case, where coupling-constant renormalization is necessary, since one is doomed to find "triviality" unless λ_B is allowed to be sufficiently close to its critical value. Indeed, the whole point of coupling constant renormalization is in a sense to keep λ_B sufficiently close to its phase-transition value.

In this paper we ask how λ_B needs to scale with N, when $N\to\infty$, if we are to find the phase transition of the GEP. We also determine the asymptotic N-dependence of Ω_v and ω_v; the mass parameters at the non-trivial minimum, $\phi_c=\phi_v$ (the v subscript indicating "vacuum value"); and the shape of the GEP at the critical λ_B. Remarkably, it turns out to be possible to find relatively simple analytic solutions which give some valuable insight into the behaviour of the ϕ^4-model in its large-N limit.

The plan of the paper is as follows. In Sect. 2 we set up the analysis in general. Then in Sects. 3–5 we consider in turn $2+1$, $1+1$, and $0+1$ dimensions. (The analysis is easiest in $2+1$ dimensions, and so we begin with that case.) In Sect. 6, we consider non-integer dimensions, which interpolate between these results. Finally, in Sect. 7, we consider the limit as the spatial dimension ν tends to 3, which links up with earlier results on "autonomous" $\lambda\phi^4$ theory [7]. Our conclusions are summarized in Sect. 8.

2 General analysis

Our goal is to find, for $N\to\infty$, the critical λ_B at which the GEP shows a phase transition. This is determined by four equations. Firstly, there are the Ω and ω equations (in mass-renormalized form). Next we have the equation $\mathrm{d}\bar{V}_G/\mathrm{d}\phi_c=0$, which determines ϕ_v, the location of the non-trivial minimum of the GEP. Upon using the Ω equation, this simplifies to:

$$\Omega_v^2=8\lambda_B\phi_v^2. \tag{2.1}$$

Finally, the critical λ_B of the phase transition is determined by requiring the non-trivial vacuum to be degenerate with the symmetric vacuum at the origin (see Fig. 1); i.e.,

$$\bar{V}_G(\phi_c=\phi_v, \lambda_B=\lambda_{\text{crit}})=0. \tag{2.2}$$

In this section we put these equations into convenient form. We make a crude distinction between "larger" and "smaller" terms based on the expectation that when N is large ϕ_v^2 will be large and Ω_v^2 will be much greater than either m_R^2 or ω_v^2. The actual situation is quite complicated, involving a whole hierarchy of powers and/or logarithms of N, depending upon spatial dimension ν. Nevertheless, this crude partitioning into "larger" and "smaller" terms is very useful as a starting point. Except in the last paragraph, though, we shall not actually discard any terms in this section.

First, by subtracting the ω equation from the Ω equation, we obtain:

$$\Omega^2-8\lambda_B\phi_c^2=\omega^2+8\lambda_B(\Delta(\Omega)-\Delta(\omega)). \tag{2.3}$$

The left-hand side consists of two "larger" terms which almost cancel, leaving the "smaller" remainder on the right-hand side. (This will mean that for large N we may approximate Ω^2 by $8\lambda_B\phi_c^2$ at essentially all ϕ_c.) At the

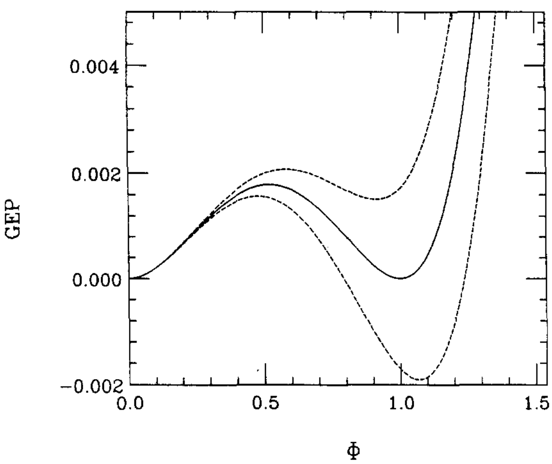

Fig. 1. The GEP, $\bar{V}_G(\Phi)$, at λ_{crit} in 2+1 dimensions as given in (3.16) (solid line), together with curves showing the GEP slightly below and above the critical coupling (dashed lines)

non-trivial minimum, where (2.1) holds exactly, we have:

$$\omega_v^2 + 8\lambda_B(\Delta(\Omega_v) - \Delta(\omega_v)) = 0. \tag{2.4}$$

Next, by re-arranging the ω equation, we can write:

$$\phi_c^2 + N\Delta(\omega) = \frac{1}{4\lambda_B}[\omega^2 - m_R^2 - 4\lambda_B(\Delta(\Omega) + \Delta(\omega))] \equiv \xi, \tag{2.5}$$

in which the left-hand side again consists of two "larger" terms which almost cancel. Next, we use this equation to eliminate ϕ_c^2 in $\bar{V}_G$. The crucial point is that $\langle\mathscr{H}\rangle$ in (1.2) contains three "super-large" terms; $\lambda_B\phi_c^4$, $\lambda_B N^2\Delta(\omega)^2$, and $2\lambda_B N\Delta(\omega)\phi_c^2$, which almost cancel. By substituting for ϕ_c^2 we accomplish this cancellation and, after simplification, we obtain:

$$\bar{V}_G(\phi_c) = [\Delta J(\Omega) + N\Delta J(\omega) - 2N\lambda_B\Delta(\omega)(2\Delta(\Omega) - \Delta(\omega))] + \eta, \tag{2.6}$$

where the "smaller" terms, denoted collectively by η, are given by:

$$\eta \equiv -\Delta J(\omega) + \tfrac{1}{2}m_R^2(\Delta(\Omega) - \Delta(\omega) + \xi) + \lambda_B[3\Delta(\Omega)^2 - 2\Delta(\Omega)\Delta(\omega) - \Delta(\omega)^2 + 2\xi(3\Delta(\Omega) - \Delta(\omega)) + \xi^2]. \tag{2.7}$$

If we use (2.1) to eliminate ϕ_v^2, we are left with three coupled equations to solve for Ω_v, ω_v, and λ_{crit}; namely, (2.4), (2.5), and (2.6) evaluated at the vacuum and equated to zero. In the following sections we proceed to find the asymptotic solution of these three equations.

Before turning to this, we first make a few general observations, valid in all dimensions, about the shape of the GEP at large N when λ_B is at or near λ_{crit}. At the origin, both Ω and ω are equal to m_R, and $\bar{V}_G$ starts out growing as $\frac{1}{2}m_R^2\phi_c^2$. Then Ω grows rapidly with ϕ_c and, once away from the origin, is given approximately by:

$$\Omega^2 \approx 8\lambda_B\phi_c^2. \tag{2.8}$$

The transverse mass ω also grows with ϕ_c, but much more slowly, governed approximately by:

$$N\Delta(\omega) \approx -\phi_c^2. \tag{2.9}$$

The GEP is given approximately by:

$$\bar{V}_G(\phi_c) \approx \Delta J(\Omega) + N\Delta J(\omega) + 4\lambda_B\Delta(\Omega)\phi_c^2 + 2\lambda_B\phi_c^4/N, \tag{2.10}$$

in which only the third term is negative (because $\Delta(\Omega)$ is negative): It is this term, working against the other terms, which brings the GEP down to create the non-trivial minimum.

3 2+1 dimensions

Upon substituting the 2+1 dimensional forms of $\Delta(\Omega)$ and $\Delta(\omega)$ into (2.4) we obtain:

$$\omega_v^2 - \left(\frac{2\lambda_B}{\pi}\right)(\Omega_v - \omega_v) = 0. \tag{3.1}$$

Using (2.1) in (2.5) and discarding the ξ term gives us:

$$\Omega_v^2 = \left(\frac{2\lambda_B}{\pi}\right)N(\omega_v - m_R). \tag{3.2}$$

Substituting the 2+1 dimensional forms of $\Delta J(\Omega)$ and $\Delta J(\omega)$ into (2.6), evaluating at the vacuum and equating to zero, neglecting η, gives the equation determining λ_{crit}:

$$(\Omega_v^3 - m_R^3) + N(\omega_v^3 - m_R^3) - 3N\frac{\lambda_{\text{crit}}}{\pi}(\omega_v - m_R)(2\Omega_v - \omega_v - m_R) = 0. \tag{3.3}$$

We are expecting Ω_v to be much greater than m_R, whereas we anticipate ω_v and m_R to be roughly comparable. The last equation suggests a solution in which Ω_v^3 is of order N. Equation (3.2) would then imply that λ_{crit} is of order $1/N^{1/3}$. We therefore look for an asymptotic solution of the form:

$$\frac{\omega_v}{m_R} = g, \quad \frac{\Omega_v}{m_R} = GN^{1/3}, \quad \frac{\lambda_{\text{crit}}}{m_R} = \frac{A}{N^{1/3}}, \tag{3.4}$$

where g, G, and A are order-one numbers to be determined.

From (3.1), up to terms suppressed by $1/N^{1/3}$, we immediately get:

$$A = \frac{\pi}{2}\frac{g^2}{G}. \tag{3.5}$$

Substitution into (3.2) yields:

$$G^3 = g^2(g-1). \tag{3.6}$$

Equation (3.3) then becomes:

$$G^3 + (g^3 - 1) - 3g^2(g-1) = 0, \tag{3.7}$$

which factorizes to give:

$$(g-1)(g^2 - g - 1) = 0. \tag{3.8}$$

Clearly, $g=1$ is not an acceptable solution, so g turns out to be the "golden mean":

$$g = \frac{1+\sqrt{5}}{2}. \tag{3.9}$$

Note that $G = g^{1/3}$ in consequence.

Summarizing, the critical coupling has the asymptotic form:

$$\frac{\lambda_{\text{crit}}}{m_R}=\frac{\pi}{2}g^{5/3}\frac{1}{N^{1/3}}, \tag{3.10}$$

as $N\to\infty$, with the mass parameters being:

$$\frac{\omega_v}{m_R}=g, \quad \frac{\Omega_v}{m_R}=g^{1/3}N^{1/3}, \tag{3.11}$$

and

$$\frac{\phi_v^2}{m_R}=\frac{N}{4\pi g}. \tag{3.12}$$

There are corrections of relative order $1/N^{1/3}$, which can easily be obtained if desired. The "smaller" terms ξ and η, neglected in this Section, are suppressed by *two* powers of $1/N^{1/3}$, relative to the terms retained.

It is interesting to obtain the form of the GEP at the critical λ_B. We first define a dimensionless, rescaled field:

$$\Phi^2=\frac{\phi_c^2}{\phi_v^2}=\frac{4\pi g}{m_R}\frac{\phi_c^2}{N}, \tag{3.13}$$

such that $\Phi=1$ at the non-trivial minimum. From (2.8) we have

$$\frac{\Omega}{m_R}\approx g^{1/3}N^{1/3}|\Phi|. \tag{3.14}$$

and from (2.9) we have:

$$\frac{\omega}{m_R}\approx\left(1+\frac{1}{g}\Phi^2\right). \tag{3.15}$$

Upon substituting into $\bar{V}_G$, (2.10), we obtain, after discarding subleading terms:

$$\frac{\bar{V}_{G,\,\text{crit}}}{N}=\frac{m_R^3}{24\pi g^3}\left[3g^2\Phi^2-2g^4|\Phi|^3+3g\Phi^4+\Phi^6\right]. \tag{3.16}$$

This form of $\bar{V}_G(\Phi)$ at λ_{crit} is shown in Fig. 1, together with curves showing the GEP slightly above and below the critical coupling.

Note that $\bar{V}_G$ scales as N, so that the energy density per degree of freedom is finite. Thus, the barrier height remains finite (though small – only $0.002\ m_R$), and the phase transition remains first order. This contrasts with the behaviour we shall find in the $1+1$-dimensional case discussed in the next section.

4 1+1 dimensions

Using the $1+1$ dimensional forms of Δ and ΔJ, and neglecting the "smaller" terms, (2.4)–(2.6) become:

$$\omega_v^2-\left(\frac{2\lambda_B}{\pi}\right)\left(\ln\frac{\Omega_v^2}{m_R^2}-\ln\frac{\omega_v^2}{m_R^2}\right)=0, \tag{4.1}$$

$$\Omega_v^2=\left(\frac{2\lambda_B}{\pi}\right)N\ln\frac{\omega_v^2}{m_R^2}, \tag{4.2}$$

$$(\Omega_v^2-m_R^2)+N(\omega_v^2-m_R^2)$$
$$-N\frac{\lambda_{\text{crit}}}{\pi}\ln\frac{\omega_v^2}{m_R^2}\left(2\ln\frac{\Omega_v^2}{m_R^2}-\ln\frac{\omega_v^2}{m_R^2}\right)=0. \tag{4.3}$$

An asymptotic solution to these equations can be found, after a certain amount of trial and error. The key realization is that $\ln(\omega_v^2/m_R^2)$ must be small. One then finds the solution:

$$\frac{\lambda_{\text{crit}}}{m_R^2}=\frac{\pi}{2}\frac{1}{\ln N}\left[1+\frac{(2\ln\ln N+2-\ln 2)}{\ln N}+\dots\right], \tag{4.4}$$

$$\frac{\Omega_v^2}{m_R^2}=\frac{2N}{(\ln N)^2}+\dots, \quad \frac{\omega_v^2}{m_R^2}=1+\frac{2}{\ln N}+\dots, \tag{4.5}$$

$$\phi_v^2=\frac{1}{2\pi}\frac{N}{\ln N}. \tag{4.6}$$

Convergence towards the asymptotic solution is very slow because of $\ln\ln N/\ln N$ type corrections. We have included sub-leading terms in λ_{crit} and ω^2 because these play an important role. A solution containing all orders in $1/\ln N$ can be obtained, implicitly, from (4.1)–(4.3) above. That is so because the ξ and η terms neglected are suppressed by a whole power of N, as is easily checked. Thus, in $1+1$ dimensions, the "larger" terms are of order N, and the 'smaller" terms are of order unity, up to logarithms.

We can now obtain the form of $\bar{V}_G$ at the critical coupling in terms of the re-scaled field Φ, where:

$$\Phi^2\equiv\frac{\phi_c^2}{\phi_v^2}=\frac{2\pi\ln N}{N}\phi_c^2. \tag{4.7}$$

From (2.8) and (2.9) we have:

$$\frac{\Omega}{m_R}\approx\frac{\sqrt{2N}}{\ln N}|\Phi|, \tag{4.8}$$

$$\frac{\omega}{m_R}\approx 1+\frac{\Phi^2}{\ln N}+\frac{1}{2}\frac{\Phi}{(\ln N)^2}+\dots, \tag{4.9}$$

and substituting in (2.10), we observe a cancellation of the $N/\ln N$ terms – which makes it necessary to include the sub-leading terms in λ_{crit} – leaving:

$$\frac{\bar{V}_{G,\,\text{crit}}}{N}\approx\frac{1}{(\ln N)^2}\frac{m_R^2}{4\pi}\left[-\Phi^2(\ln\Phi^2+1)+\Phi^4\right]. \tag{4.10}$$

This can be viewed as an "infinitesimally weak" first-order phase transition, in that the barrier height per degree of freedom tends to zero like $1/(\ln N)^2$. Moreover, if we make the more usual scaling $\tilde{\phi}^2=\phi_c^2/N$, then the vacuum value of $\tilde{\phi}^2$ would vanish like $1/\ln N$, and only the $\tilde{\phi}^4$ term from (4.10) would survive.

Actually, on a scale where $\tilde{\phi}^2$ is considered finite, one must use $\omega^2/m_R^2=\exp(4\pi\tilde{\phi}^2)$ in place of (4.9). On this scale one finds that the first and last terms of $\bar{V}_G$ in (2.10) are suppressed by $1/\ln N$, so that one obtains:

$$\frac{\bar{V}_G}{N}=\frac{m_R^2}{8\pi}\left[(\exp(4\pi\tilde{\phi}^2)-1)-\frac{\lambda_B}{\lambda_{\text{crit}}}4\pi\tilde{\phi}^2\right]+\mathcal{O}\left(\frac{\ln\ln N}{\ln N}\right), \tag{4.11}$$

for λ_B at or near to λ_{crit}. In this form, one sees a second-order phase transition: the $\tilde{\phi}^2$ term, in the expansion about the origin, goes through zero when $\lambda_B=\lambda_{\text{crit}}$. However, as we have seen in (4.10), the potential actually has a "fine structure" when $\tilde{\phi}^2$ is small, of order $1/\ln N$. (Furthermore, on a still finer scale, $\tilde{\phi}^2$ of order $1/N$, the apparent $\Phi^2 \ln \Phi^2$ non-analyticity is smoothed out so that the second derivative right at the origin is always m_R^2.)

5 0+1 dimensions

Of course in $0+1$ dimensions there is properly speaking no phase transition. Nevertheless, there is a rather marked change from "single-well" behaviour to "double-well" behaviour as we pass to larger values of the dimensionless ratio λ_B/m_R^3 (See [3]). This provides a direct analog of the phase transition in higher dimensions, and for convenience we continue to use the same terminology.

The main difference in the $0+1$ dimensional case is that now $\Delta(\Omega)$ does not become large and negative when Ω becomes large, but tends to a constant $-1/(2m_R)$. This means that the energy-decreasing term in (2.10) becomes essentially $-(2\lambda_B/m_R)\phi_c^2$. Because this term does not grow faster than ϕ_c^2, as was the case in higher dimensions, we basically expect a *second order* phase transition to occur when this term overwhelms the (renormalized) mass term $\frac{1}{2}m_R^2\phi_c^2$; i.e., when $\lambda_B/m_R^3=\frac{1}{4}$. This heuristic argument is not quite right, but it aids in finding the actual asymptotic solution. The three equations (2.4)–(2.6), neglecting "smaller" terms, become in $0+1$ dimensions:

$$\omega_v^2+4\lambda_B\left(\frac{1}{\Omega_v}-\frac{1}{\omega_v}\right)=0, \tag{5.1}$$

$$\Omega_v^2=-4\lambda_B N\left(\frac{1}{\omega_v}-\frac{1}{m_R}\right), \tag{5.2}$$

$$(\Omega_v-m_R)+N(\omega_v-m_R)$$
$$-4N\lambda_{\text{crit}}\left(\frac{1}{\omega_v}-\frac{1}{m_R}\right)\left(\frac{1}{\Omega_v}-\frac{1}{2\omega_v}-\frac{1}{2m_R}\right)=0. \tag{5.3}$$

The asymptotic solution to these equations, outlined below, does indeed show that λ_B/m_R^3 tends towards $\frac{1}{4}$. It also shows that Ω_v becomes large, of order $N^{1/3}$, while ω_v remains close to m_R, so that $\Delta(\omega)$ is small, of order $1/N^{1/3}$. To find the asymptotic solution it is actually necessary to keep both leading and sub-leading terms in an expansion in powers of $1/N^{1/3}$. Making the ansatz:

$$\frac{\lambda_B}{m_R^3}=\frac{1}{4}\left(1+\frac{J}{N^{1/3}}+\dots\right), \tag{5.4}$$

$$\phi_v^2 m_R=\frac{K^2}{2}N^{2/3}\left(1+\frac{L}{N^{1/3}}+\dots\right), \tag{5.5}$$

where J, K, L are numerical coefficients to be determined, we obtain from $\Omega_v^2=8\lambda_B\phi_v^2$:

$$\frac{\Omega_v}{m_R}=KN^{1/3}\left(1+\frac{\frac{1}{2}(J+L)}{N^{1/3}}+\dots\right). \tag{5.6}$$

Substituting in (5.2) gives:

$$\frac{\omega_v}{m_R}=1+\frac{K^2}{N^{1/3}}+\frac{K^2(K^2+L)}{N^{2/3}}+\dots. \tag{5.7}$$

Substituting in (5.1), the leading terms cancel, and the sub-leading terms cancel if:

$$J=3K^2+1/K. \tag{5.8}$$

Finally, substituting in (5.3), the leading terms again cancel, and the subleading terms cancel if:

$$J=\tfrac{3}{2}K^2+2/K, \tag{5.9}$$

which gives

$$J=3(3/2)^{1/3}, \quad K=(2/3)^{1/3}. \tag{5.10}$$

The coefficient L, although it enters among the sub-leading terms, actually cancels out, and so is not determined by this analysis. It can easily be checked that the "smaller" terms, ξ and η are of order unity, and hence are sub-sub-leading, so that their neglect was in fact justified.

As before, we can find the form of the GEP at the critical λ_B in terms of the re-scaled field:

$$\Phi^2\equiv\frac{\phi_c^2}{\phi_v^2}=\frac{3^{2/3}2^{1/3}m_R}{N^{2/3}}\phi_c^2. \tag{5.11}$$

From (2.8) and (2.9) we find:

$$\frac{\Omega}{m_R}\approx KN^{1/3}|\Phi|, \tag{5.12}$$

$$\frac{\omega}{m_R}\approx 1+\frac{K^2}{N^{1/3}}\Phi^2+\frac{K^4}{N^{2/3}}\Phi^4+\dots, \tag{5.13}$$

with $K=(2/3)^{1/3}$. Substituting in (2.10), we observe a cancellation of leading-order terms which leaves:

$$\frac{\bar{V}_{G,\text{crit}}}{N}\approx\frac{m_R}{3^{1/3}2^{5/3}}\frac{1}{N^{2/3}}(2|\Phi|-3\Phi^2+\Phi^4). \tag{5.14}$$

Again the "phase transition", although first order, becomes infinitesimally weak, in that the barrier height per degree of freedom is vanishingly small, $O(1/N^{2/3})$.

As in the $1+1$ dimensional case, the effectively second-order nature of the transition becomes evident if we consider $\bar{V}_G$ on a scale where $\tilde{\phi}^2\equiv\phi^2/N$ is considered finite. On this scale, the $\Delta J(\Omega)$ term, proportional to $|\tilde{\phi}|$, is only of order $\sqrt{N}$, whereas the other terms in (2.10) are of order N, so that:

$$\frac{\bar{V}_G}{N}=\frac{m_R}{4}\left[(1-2m_R\tilde{\phi}^2)^{-1}-1\right]$$
$$-\frac{2\lambda_B}{m_R}\tilde{\phi}^2+2\lambda_B\tilde{\phi}^4+\mathcal{O}\left(\frac{1}{\sqrt{N}}\right). \tag{5.15}$$

If this is expanded in powers of $\tilde{\phi}^2$, one sees that the $\tilde{\phi}^2$ term goes through zero at $\lambda_B/m_R^3=1/4$, producing a second-order "phase transition". At very small $\tilde{\phi}^2$, of order $1/N^{1/3}$, there is, however, a "fine structure" to the potential, described by (5.14)

Finally, we have to note that (5.15) is only valid for $\tilde{\phi}^2<1/(2m_R)$, since the pole in ω, and hence in $\bar{V}_G$, must be spurious. In fact, if $\tilde{\phi}^2$ approaches within $O(1/\sqrt{N})$ of $1/(2m_R)$, then ω^2 grows to becomes of order N, so that the $\omega^2/(4\lambda_B)$ term in ξ cannot be neglected in (2.5).

6 Non-integer dimensions

Analytic continuation in the spatial dimension ν, familiar from dimensional regularization and the ε expansion, provides some useful insight into the previous results. The

Δ and ΔJ functions are given by:

$$\Delta(m)=\gamma(m^{\nu-1}-m_R^{\nu-1}), \tag{6.1}$$

$$\Delta J(m)=\frac{\gamma}{2}\left(\frac{1-\nu}{1+\nu}\right)(m^{\nu+1}-m_R^{\nu+1}), \tag{6.2}$$

where

$$\gamma\equiv\frac{\Gamma\left(\frac{1-\nu}{2}\right)}{(4\pi)^{(1+\nu)/2}}. \tag{6.3}$$

The three equations determining λ_{crit} are then:

$$\omega_v^2+8\lambda_B\gamma(\Omega_v^{\nu-1}-\omega_v^{\nu-1})=0, \tag{6.4}$$

$$\Omega_v^2=-8N\lambda_B\gamma(\omega_v^{\nu-1}-m_R^{\nu-1}), \tag{6.5}$$

$$(\Omega_v^{\nu+1}-m_R^{\nu+1})+N(\omega_v^{\nu+1}-m_R^{\nu+1})+4N\gamma\left(\frac{\nu+1}{\nu-1}\right)$$
$$\cdot\lambda_{\mathrm{crit}}(\omega_v^{\nu-1}-m_R^{\nu-1})(2\Omega_v^{\nu-1}-\omega_v^{\nu-1}-m_R^{\nu-1})=0. \tag{6.6}$$

In less than $1+1$ dimensions the large-N behaviour is basically similar to $\nu=0$. Effectively, the phase transition becomes second order and the critical coupling approaches a constant value. In terms of $\tilde{\phi}^2\equiv\phi_c^2/N$, the leading contributions to the effective potential are:

$$\frac{\bar{V}_G}{N}\approx\frac{\gamma}{2}\left(\frac{1-\nu}{1+\nu}\right)m_R^{1+\nu}\left[\left(1-\frac{\tilde{\phi}^2 m_R^{1-\nu}}{\gamma}\right)^{-\left(\frac{1+\nu}{1-\nu}\right)}-1\right]$$
$$-4\gamma\frac{\lambda_B\tilde{\phi}^2}{m_R^{1-\nu}}+2\lambda_B\tilde{\phi}^4. \tag{6.7}$$

If this is expanded in powers of $\tilde{\phi}^2$, one finds that the coefficient of $\tilde{\phi}^2$ goes through zero at:

$$\lambda_{\mathrm{crit}}=\frac{m_R^{3-\nu}}{8\gamma}. \tag{6.8}$$

The value of the critical coupling starts at $\frac{1}{4}$ in $0+1$ dimensions. As ν is increased, λ_{crit} initially increases slightly, then decreases, tending to zero as $\nu\to1$ on account of the pole of the gamma function at $\nu=1$. This is in accord with the result of Sect. 4, where we found that λ_{crit} goes to zero like $1/\ln N$ for $\nu=1$.

One can give a more detailed analysis akin to that of Sect. 5, showing that Ω_v is of order $N^{1/(3-\nu)}$ and ϕ_v^2 is of order $N^{2/(3-\nu)}$, with sub-leading terms suppressed by factors of $1/N$ to the power $(1-\nu)/(3-\nu)$ (which is a suppression factor provided $\nu<1$). In terms of $\Phi^2\equiv\phi_c^2/\phi_v^2$, one would find a "fine structure" to the potential, involving $|\Phi|^{1+\nu}$, Φ^2, and Φ^4 terms.

In more than $1+1$ dimensions, one can straightforwardly generalize the $2+1$ dimensional analysis of Sect. 3. by making the ansatz:

$$\frac{\omega_v}{m_R}=g,\quad \frac{\Omega_v}{m_R}=GN^{\frac{1}{\nu+1}},\quad \frac{\lambda_{\mathrm{crit}}}{m_R^{3-\nu}}=\frac{A}{N^{\frac{\nu-1}{\nu+1}}}, \tag{6.9}$$

where g, G, and A are order-one numbers to be determined. From (6.4), we find:

$$A=\frac{1}{8|\gamma|}\frac{g^2}{G^{\nu-1}}. \tag{6.10}$$

Note that the assumption $\nu>1$ is crucial here, since we have neglected a term suppressed by $1/N$ to the power $(\nu-1)/(\nu+1)$. (Note also that γ is negative for $1<\nu<3$: it equals $-1/(4\pi)$ for $\nu=2$.) Next, substitution into (6.5) yields:

$$G^{\nu+1}=g^2(g^{\nu-1}-1). \tag{6.11}$$

Equation (6.6) then becomes:

$$G^{\nu+1}+(g^{\nu+1}-1)-\left(\frac{\nu+1}{\nu-1}\right)g^2(g^{\nu-1}-1)=0, \tag{6.12}$$

which simplifies to:

$$g^{\nu+1}(\nu-3)+2g^2-(\nu-1)=0. \tag{6.13}$$

This equation has a trivial root $g=1$, but the relevant root is the non-trivial one with $g>1$. As ν increases from 1 to 3, g starts from 1 and increases, going to infinity as $\nu\to3$. The coefficient A in the critical coupling starts from 0 at $\nu=1$ and increases to about 4 around $\nu=2.3$, then falls to zero as $\nu\to3$. We shall consider the $\nu\to3$ limit in the next section.

The shape of $\bar{V}_G$ at the critical λ_B can now be obtained. The re-scaled field is:

$$\Phi^2\equiv\frac{\phi_c^2}{\phi_v^2}=\frac{1}{|\gamma|(g^{\nu-1}-1)m_R^{\nu-1}}\frac{\phi_c^2}{N}. \tag{6.14}$$

Note that above $1+1$ dimensions ϕ_v^2 is proportional to N, so that this field re-scaling is the 'usual' one, as far as N dependence is concerned. From (2.8) and (2.9) we have:

$$\frac{\Omega}{m_R}\approx GN^{\frac{1}{\nu+1}}|\Phi|, \tag{6.15}$$

$$\frac{\omega}{m_R}\approx[1+(g^{\nu-1}-1)\Phi^2]^{\frac{1}{\nu-1}}. \tag{6.16}$$

Substituting into (2.10) we observe that the last term in that equation is suppressed by a factor of $1/N$ to the power $(\nu-1)/(\nu+1)$, relative to the others. Furthermore, the first and third terms are both proportional to $|\Phi|^{\nu+1}$, and they combine to give a net negative contribution. We obtain then:

$$\frac{\bar{V}_{G,\,\mathrm{crit}}}{N}=m_R^{\nu+1}\frac{|\gamma|}{2(\nu+1)}\left\{-2G^{\nu+1}|\Phi|^{\nu+1}\right.$$
$$\left.+(\nu-1)\left[\left(1+\frac{G^{\nu+1}}{g^2}\Phi^2\right)^{\frac{\nu+1}{\nu-1}}-1\right]\right\}. \tag{6.17}$$

7 Towards 3+1 dimensions

We now consider the results of the last section in the limit $\nu\to3$. Defining $\nu=3-\varepsilon$ and considering $\varepsilon\to0$, we can express the solution to the g equation, (6.13), as:

$$g^2=\frac{2}{\varepsilon}\left[1-\frac{\varepsilon}{2}\left(\ln\frac{\varepsilon}{2}+1\right)+\ldots\right]. \tag{7.1}$$

Hence, from (6.11):

$$G^{4-\varepsilon}=\frac{4}{\varepsilon^2}\left[1-\frac{\varepsilon}{2}\left(\ln\frac{\varepsilon}{2}+3\right)+\ldots\right]. \tag{7.2}$$

Recalling the definitions of g and G in (6.9), one sees that:

$$\frac{\omega_v^2}{m_R^2}\sim\frac{2}{\varepsilon},\quad \frac{\Omega_v^2}{m_R^2}\sim\frac{2}{\varepsilon}\sqrt{N}. \tag{7.3}$$

Thus, we can obtain a sensible solution provided that the renormalization is arranged so that m_R^2 is of order ε. This means that in $3+1$ dimensions the particles in the symmetric vacuum at $\phi_c=0$ must be *massless*. This is a crucial feature of the "autonomous" $\lambda\phi^4$ theory [4–7, 11].

From (6.9) and (6.10), noting that $|\gamma|\to 1/(8\pi^2\varepsilon)$, we find the critical bare coupling constant to be:

$$\lambda_{B,\,\mathrm{crit}}\sim\frac{\pi^2\varepsilon}{\sqrt{N}}(1+O(\varepsilon)). \tag{7.4}$$

This agrees precisely with the "autonomous" solution: In [4] λ_B is given in the form:

$$\lambda_B=\alpha/I_{-1}(\mu), \tag{7.5}$$

where $I_{-1}(\mu)$ is a logarithmically divergent integral which in dimensional regularization takes the form:

$$I_{-1}(\mu)=\frac{\mu^{-\varepsilon}}{4\pi^2\varepsilon}\Gamma\left(1+\frac{\varepsilon}{2}\right)(4\pi)^{\varepsilon/2}=2|\gamma|\mu^{-\varepsilon}\left(1-\frac{\varepsilon}{2}\right), \tag{7.6}$$

μ being a dimensional-transmutation scale parameter. The numerical coefficient α is [4, 9]:

$$\alpha=\frac{1}{4(\sqrt{N+3}+1)} \tag{7.7}$$

which tends to $1/(4\sqrt{N})$ for large N. Note that the $1/\sqrt{N}$ dependence of λ_B puts it out of reach of the $1/N$ expansion.

From $\phi_v^2=\Omega_v^2/(8\lambda_B)$ we obtain, at the phase transition:

$$\phi_v^2\sim\frac{1}{4\pi^2}\frac{m_R^2}{\varepsilon^2}N. \tag{7.8}$$

With m_R^2 of order ε, this still leaves ϕ_v^2 of order $1/\varepsilon$. Thus, a wavefunction-renormalization factor is required so that the renormalized classical field is finite. This corresponds to the re-scaling $\phi_B^2=I_{-1}\phi_R^2$ of [4, 6, 7].

From (6.17) we can find the form of the GEP at the phase transition in terms of the dimensionless field $\Phi^2\equiv\phi_c^2/\phi_v^2$ (which naturally takes care of both the N and the ε re-scalings). It is important to carefully take account of all subleading terms, since there is a cancellation between the most singular terms. To begin, we note that:

$$\frac{G^{\nu+1}}{g^2}\Phi^2=\frac{2}{\varepsilon}\Phi^2[1-\varepsilon+\ldots], \tag{7.9}$$

so that:

$$\left(1+\frac{G^{\nu+1}}{g^2}\Phi^2\right)^{\frac{\nu+1}{\nu-1}}=\frac{4}{\varepsilon^2}\Phi^4\left[1+\frac{\varepsilon}{2}\left(\ln\Phi^2-\ln\frac{\varepsilon}{2}-4+\frac{2}{\Phi^2}\right)+\ldots\right]. \tag{7.10}$$

Subtracting 1 from this affects only the "$+\ldots$" terms, so by multiplying by $\nu-1=2-\varepsilon$ we obtain the second term in the curly brackets in (6.17). Next, we evaluate the first term, obtaining:

$$-2G^{\nu+1}|\Phi|^{\nu+1}=-\frac{8}{\varepsilon^2}\Phi^4\left[1-\frac{\varepsilon}{2}\left(\ln\Phi^2+\ln\frac{\varepsilon}{2}+3\right)+\ldots\right]. \tag{7.11}$$

Thus, in (6.17) the Φ^4/ε^2 terms cancel. There is an overall $1/\varepsilon$ factor coming from $|\gamma|$, and so we obtain:

$$\frac{\bar V_{G,\,\mathrm{crit}}}{N}\sim\frac{m_R^{4-\varepsilon}}{8\pi^2}\frac{1}{\varepsilon^2}[\Phi^4(\ln\Phi^2-1)+\Phi^2]. \tag{7.12}$$

This is finite if we take m_R^2 to be of order ε. The form of the potential is exactly that of the "autonomous" theory [4, 6] (in the special case that we are precisely at the phase transition of the theory). It is amusing to note that if we had not been so careful with the subleading terms, and had just set $\nu=3$ (and $G\sim g$) in (6.17), then we would have found "triviality" – an effective potential just proportional to Φ^2.

The above analysis has, of course, been neglecting the "smaller" terms ξ and η. One can check that these are relatively suppressed by $1/\sqrt{N}$. It is true that the η terms in $\bar V_G$, (2.6), contain m_R^4/ε^3 terms, which are more singular as $\varepsilon\to 0$ than our result in (7.12) above. However, these terms will cancel provided we use the exact $1/(\sqrt{N+3}+1)$ dependence of λ_B (see (7.7)) in place of the asymptotic $1/\sqrt{N}$ behaviour in (7.4).

One can check that our results agree in every detail with those of [4] for N large. However, the parameters m_0 and μ used there have to be assigned a particular scaling with N. From their respective definitions, one can show that m_0^2 corresponds to $-4\sqrt{N}I_{-1}m_R^2$, for large N. Also, the scale parameter μ must have an $N^{1/8}$ factor, as one can derive from comparing the sub-leading terms in λ_B: From our (6.9) we see that the N-dependence of λ_{crit} is $1/N$ to the power $\frac{1}{2}-\frac{\varepsilon}{8}+\ldots$ and the sub-leading $N^{\varepsilon/8}$ factor must be absorbed by the μ^ε in [4]'s form (see (7.5), (7.6) above). Provided that these subtleties are taken into account, there seems to be no problem with taking the $N\to\infty$ and $\varepsilon\to 0$ limits in either order. This is not so for the conventional large-N limit, which accounts for the inability of the $1/N$-expansion method to find "autonomous" $\lambda\phi^4$ theory [4, 11].

8 Summary and conclusions

The phase transition of $O(N)$-symmetric $\lambda\phi^4$ theory in the large-N limit has been investigated, using the Gaussian effective potential (GEP). We have found analytic solutions to the GEP-equations near the phase transition in the large-N limit. Our main point is that the critical coupling constant is much larger than $O(1/N)$, so that the phase transition lies outside the reach of the $1/N$-expansion method. In units of the renormalized mass in the symmetric vacuum λ_{crit} is $\frac{1}{4}$ in $0+1$ dimensions; $\frac{\pi}{2}/\ln N$ in $1+1$ dimensions; $\frac{\pi}{2}g^{5/3}/N^{1/3}$ with $g=(\sqrt{5}+1)/2$ in $2+1$ dimensions; and $\pi^2\varepsilon/\sqrt{N}$ in $(3-\varepsilon)+1$ dimensions (for $\varepsilon\ll 1$) – this last case corresponding to the positive, infinitesimal bare coupling constant of "autonomous" ϕ^4 theory [4–7].

One important result concerns the order of the phase transition. Since its second derivative at the origin is always positive ($=m_R^2$), the GEP always shows a *first-order* phase transition, whereas it is known rigorously that in $1+1$ dimensions the actual transition is *second order* [12]. However, in $1+1$ dimensions (and below) the GEP's first-order phase transition becomes infinitesimally weak as $N \to \infty$, i.e., both the order parameter (the expectation value of the conventionally re-scaled field $\tilde{\phi}$), and the potential barrier between the symmetric and asymmetric minima vanish asymptotically. Hence, the phase transition becomes effectively second order, asymptotically. On the other hand, above $1+1$ dimensions we clearly find a first-order transition persisting in the $N \to \infty$ limit.

There is an interesting resemblance between our results and those obtained for models, such as the RP^{N-1} or CP^{N-1} models. Whereas expansions in space dimension and the conventional $1/N$ expansion find continuous phase transitions for these systems [13], mean-field theory and Monte-Carlo calculations indicate first-order transitions in more than two dimensions [14].

All of our analysis is based on the Gaussian approximation. The degree of validity of this approximation in quantum field theory has not, of course, been rigorously established. Nevertheless, it is known that in the well-understood $0+1$ dimensional case the GEP gives a reasonably good, often very good, description for all values of the parameters [3]. Furthermore, the GEP is known to become exact in the conventional large-N limit, where λ is of order $1/N$ [4, 10]. We believe that the GEP remains a good approximation at large N, even for the much larger λ of our unconventional limit. Indeed, it might even be true that the GEP is *exact* in this limit. This speculation is motivated by two observations: (i) the "Goldstone ratio", ω^2/Ω^2, which is a measure of the Gaussian approximation's violation of the Goldstone theorem, tends to zero in our unconventional large-N limit, and (ii) the phase transition in $1+1$ dimensions (and below) becomes second order asymptotically, in accordance with rigorous results [12]. If this conjecture is correct then it should be possible to show that the corrections to the Gaussian approximation are suppressed as $N \to \infty$, even with our unconventional scaling of λ_B. This problem could be addressed in the two complementary approaches that have recently been developed to go beyond the Gaussian approximation. These approaches are the post-Gaussian expansion [15], also known as the "linear δ-expansion" [10, 16], and the variational method employing non-linear canonical transformations [17].

Finally, we emphasize that, contrary to the "triviality" scenario [1], we find, as in previous GEP studies [4–7, 11], a *non-trivial* $3+1$ dimensional theory. Our analysis, like [7] for the $N=1$ case, implies that this "autonomous" theory can be seen as a very natural, smooth, but slightly subtle extrapolation from the lower-dimensional $\lambda\phi^4$ theories. The fact that λ_B must scale as $1/\sqrt{N}$ not $1/N$ explains why the "autonomous" theory is not seen by the $1/N$ expansion method. The delicate cancellations involved, and the masslessness of the symmetric phase probably explain how it has eluded other studies. The important point is that to find a non-trivial theory the bare coupling constant must be sufficiently close to the critical phase-transition value. This is a delicate matter, since, in the context of dimensional regularization, $\lambda_{B,\,\mathrm{crit}}$ is itself infinitesimal, of order ε, while "sufficiently close" means within $O(\varepsilon^2)$.

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