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Nontriviality of spontaneously broken $\lambda\phi^4$ theories

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By using nonperturbative functional methods it is argued that $\lambda\phi^4$ theories, undergoing spontaneous symmetry breaking, are asymptotically free. Comparison with Hamiltonian methods and with results previously obtained by different authors is presented.

When dealing with phenomena such as spontaneous symmetry breaking, gluon condensation, etc., it is usually assumed that the details of the ground state are not relevant at short distances. This point of view, based on the assumption that the ultraviolet divergences of the true theory are the same as in perturbation theory, implies that the various renormalization-group (RG) properties, such as the scaling behavior of the ground-state energy, may be deduced, in a weak-coupling regime, from the perturbative evaluation of the relevant quantities (β function, anomalous dimensions of the operators). This procedure amounts to the truncation of the power-series expansion about an unstable vacuum and may conceivably produce wrong results if, for example, there are nonanalytic effects in the coupling constant. The aim of this Brief Report is to discuss the RG properties of the effective potential for scalar self-interacting theories and provide strong evidence that the usually accepted point of view concerning their "triviality" has to be reconsidered in the presence of spontaneous symmetry breaking. Before entering the details of the problem a few general considerations are in order. Two basically different approaches to the calculation of the effective potential exist. The first one, based on a semiclassical approximation, the loop expansion,[1] can be applied in the presence of classically stable configurations and in a situation where quantum fluctuations are "small." This approach, essentially of perturbative nature, does not enjoy any stability property. As shown in Ref. 2 for $\lambda\phi^4$ theory and in Ref. 3 for Yang-Mills theories the one-loop effective potential is obtained by minimizing the expectation value of the shifted, linearized Hamiltonian in a Gaussian state. Now, the linearized Hamiltonian may exhibit well-known pathologies, such as unboundedness from below, which show up into the appearance of unphysical imaginary parts,[4] and there is no guarantee, in principle, that, by increasing the accuracy in $\hbar$, one also gets a better estimate of the ground-state energy.

The second approach, based on the variational method, is clearly advantageous and necessary in those cases where quantum fluctuations can sizably change the naive expectations based on the classical potential. As shown in Ref. 5, in the Gaussian approximation, spontaneous symmetry breaking is discovered as a sensible phenomenon in a pure massless $\lambda\phi^4$ theory, differently from the usual one-loop approximation where this is true only in the presence of gauge bosons, i.e., in scalar electrodynamics.[1] Moreover, in Ref. 5, it was implicitly assumed that the cutoff regulated $\lambda\phi^4$ theory was not allowing the "continuum limit," i.e., cutoff to infinity, in a consistent way. However, in a weak-coupling regime, the cutoff was exponentially decoupled from the spontaneously broken phase, thus implying a meaningful physical framework in a low-energy region. An important progress was obtained in Ref. 6 by Stevenson and Tarrach. These authors, by applying mass renormalization conditions close to the ones of Ref. 5, suggested that the effective potential of Ref. 5 could be renormalized by allowing for an infinite (in the infinite cutoff limit) wave-function renormalization of the scalar field. This feature, well known in Yang-Mills theories in the background gauge due to asymptotic freedom, seems to be unavoidable in $\lambda\phi^4$ theories in four space-time dimensions as discussed in Ref. 7 by Frohlich. This point of view has been pursued in Ref. 8 where the constraints on the true β function of the theory obtained by variational arguments, lead to the conclusion that indeed scalar self-interacting theories, in the presence of spontaneous symmetry breaking, are asymptotically free. In view of the importance of this result for quantum field theories and for its implications on our present understanding of electroweak interactions, it is worthwhile to support Ref. 8 by providing different arguments.

We shall employ, rather than the purely Hamiltonian formalism, the covariant effective potential for composite operators introduced in Ref. 9. At the same time, to avoid any criticism related to the use of an ("old fashioned") ultraviolet cutoff in momentum space we shall adopt, here, the modified minimal subtraction ($\overline{\rm MS}$) prescription of dimensional regularization. Finally, the connection with the results of Ref. 6 will be clarified.

The starting point for our analysis is Eq. (2.9a) of Ref. 9, where the effective action $\Gamma[\phi,G]$ ($\hbar=1$) up to a constant

$$\Gamma[\phi,G]=I[\phi]+\frac{i}{2}\mathrm{Tr}\ln DG^{-1}+\frac{i}{2}\mathrm{Tr}\mathcal{D}^{-1}(\phi)G+\Gamma_2[\phi,G] \quad (1)$$

is explicitly given. For our simple case, described by the Lagrangian ($\lambda_B>0$)

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\Phi)^2-\frac{\lambda_B}{4!}\Phi^4 , \quad (2)$$

we obtain (we restrict to the case of constant ϕ)

$$iD^{-1}(x,y)=-\Box_x\delta^4(x-y) , \quad (3)$$

$$i\mathcal{D}^{-1}=iD^{-1}(x,y)-\frac{\lambda_B}{2}\phi^2\delta^4(x-y) , \quad (4)$$

$$I[\phi]=-\int d^4x\frac{\lambda_B}{4!}\phi^4 , \quad (5)$$

and $G(x,y)$ admits the general, space-time translationally invariant, form

$$G(x,y)=i\int\frac{d^3\mathbf{k}}{(2\pi)^3}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\int\frac{d\omega}{2\pi}\frac{e^{-i\omega(x_0-y_0)}}{\omega^2-\omega^2(k)+i\epsilon} \quad (6)$$

in terms of the unknown function $\omega(k)$.

Finally $\Gamma_2[\phi,G]$ is obtained by the sum of all two-particle-irreducible vacuum-vacuum graphs of the shifted theory with propagators set equal to G. As shown in Ref. 9 the usual effective action $\Gamma[\phi]$ is obtained by minimizing $\Gamma[\phi,G]$ with respect to G, i.e.,

$$\Gamma[\phi]=\Gamma[\phi,G_0(\phi)] , \quad (7)$$

where $G_0(\phi)$ is defined as the solution of

$$\left.\frac{\delta\Gamma}{\delta G}\right|_{G=G_0(\phi)}=0 . \quad (8)$$

The stability of the underlying theory can be investigated through the relation

$$\Gamma[\phi,G]_{\mathrm{static}}=-\int dt\, E[\phi,G] , \quad (9)$$

$\Gamma[\phi,G]_{\mathrm{static}}$ being defined in Ref. 12, and ($\langle\psi|\psi\rangle=1$)

$$E[\phi,G]=\min_{\{\psi\}}\langle\psi|\hat{H}|\psi\rangle , \quad (10)$$

with the conditions

$$\langle\psi|\hat{\Phi}(\mathbf{x})|\psi\rangle=\phi , \quad (11)$$

$$\langle\psi|\hat{\Phi}(\mathbf{x})\hat{\Phi}(\mathbf{y})|\psi\rangle=\phi^2+G(\mathbf{x},\mathbf{y}) , \quad (12)$$

where

$$G(\mathbf{x},\mathbf{y})=G(x,y)\Big|_{x_0=y_0} . \quad (13)$$

We stress that the above equivalence is only true for the exact quantity $\Gamma[\phi,G]$ and that the problem of finding approximations to $E[\phi,G]$, which enjoy stability properties, by truncating the infinite series in $\Gamma_2[\phi,G]$, is highly nontrivial. The only known case is the Hartree-Fock contribution since it corresponds to a systematic variational procedure within the class of the Gaussian state functionals.[10]

For constant fields the effective potential $V(\phi)$ is obtained by minimizing $E[\phi,G]$ with respect to G, and the ground state energy density $\mathcal{W}^0$ corresponds to the absolute minimum of the effective potential: $V(\phi_0)$. Correspondingly, in the Gaussian subspace, we have $E_g[\phi,G]$ and the approximated values $\bar{\phi}$, $\bar{G}$, $V_g(\bar{\phi})\equiv\overline{\mathcal{W}}_g$, instead of the exact minimum configuration ϕ_0, $G_0, V(\phi_0)$.

The variational content of the Gaussian approximation allows us to deduce the fundamental inequality

$$\mathcal{W}^0\le\overline{\mathcal{W}}_g , \quad (14)$$

which is hardly to be recovered by employing different truncations of $\Gamma_2(\phi,G)$.

By evaluating $E[\phi,G]$, whose dependence on the unknown function $\omega(k)$ is well known to be extremized by the form

$$\omega^2(k)=\mathbf{k}^2+\Omega^2 , \quad (15)$$

and by using dimensional regularization, we obtain ($\mathcal{V}$=three dimensional volume)

$$\frac{1}{\mathcal{V}}E_g(\phi_B,\Omega)=\frac{\lambda_B}{4!}\phi_B^4-\frac{\Omega^4}{64\pi^2}(x+\tfrac{1}{2})+\frac{\lambda_B\Omega^2\phi_B^2}{64\pi^2}x+\frac{\lambda_B\Omega^4}{2048\pi^4}x^2 , \quad (16)$$

where ($\epsilon=n-4$)

$$x=\frac{2}{\epsilon}+\ln\Omega^2+\gamma+\ln\pi=1 ,$$

and we have for the sake of clarity added a subscript B to the field to indicate that we are dealing with bare quantities before renormalization. Equation (16) is equivalent to the result obtained in Ref. 5 by introducing a bare mass counterterm and by regulating the divergences in three-momentum space by means of an ultraviolet cutoff.

As discussed in detail in Ref. 5 one gets the extremum conditions

$$\frac{1}{\mathcal{V}}\frac{\partial E_g}{\partial\phi_B}=\lambda_B\phi_B\left[\frac{\phi_B^2}{6}+\frac{\Omega^2}{32\pi^2}x\right]=0 , \quad (17)$$

$$\frac{1}{\mathcal{V}}\frac{\partial E_g}{\partial\Omega}=\frac{\Omega}{16\pi^2}(x+1)\left[\frac{\lambda_B}{2}\phi_B^2-\Omega^2+\frac{\lambda_B\Omega^2}{32\pi^2}x\right]=0 , \quad (18)$$

whose solutions, $\pm\bar{\phi}_B$ and $\bar{\Omega}$, given by

$$\bar{\Omega}^2=\frac{\lambda_B}{3}\bar{\phi}_B^2 , \quad (19)$$

$$\lambda_B\bar{x}=-16\pi^2 , \quad (20)$$

produce the variational estimate for the energy density

$$\overline{\mathcal{W}}_g(\epsilon,\lambda_B)=\frac{\bar{\Omega}^4}{128\pi^2} . \quad (21)$$

Note that, from Eq. (20) and from the positivity of λ_B, it follows that the continuum limit $n \to 4$ should be taken from below, i.e., $\epsilon \to 0^-$, thus avoiding the "triviality bound" of Ref. 7 which holds for $\epsilon \to 0^+$.

In order to eliminate any polar term from the minimum of the Gaussian effective potential, which is the only point that may, possibly, have a physical meaning, we introduce a $\overline{\text{MS}}$ running coupling constant $\lambda(\mu)$ through the relation

$$\lambda(\mu)=\frac{\lambda_B}{1+\dfrac{\lambda_B}{16\pi^2}\left[\dfrac{2}{\epsilon}+\ln\mu^2+\gamma+\ln\pi\right]}, \tag{22}$$

so that Eq. (20) becomes

$$\ln\frac{\mu^2}{\overline{\Omega}^2}=\frac{16\pi^2}{\lambda(\mu)}-1 . \tag{23}$$

From Eq. (22), it follows immediately that, by defining the function $\beta_g(\lambda)$,

$$\beta_g(\lambda)=-\frac{\lambda^2}{8\pi^2}+O(\lambda^3) , \tag{24}$$

$\overline{\mathcal{W}}_g$ satisfies the equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta_g(\lambda)\frac{\partial}{\partial\lambda}\right]\overline{\mathcal{W}}_g(\mu,\lambda)=0 , \tag{25}$$

and $\overline{\mathcal{W}}_g(\mu,\lambda)$ is the same quantity of Eq. (21) after replacing λ_B in terms of $\lambda(\mu)$.

Then $d\overline{\Omega}/d\mu=0$, and Eq. (19) can be expressed as

$$\overline{\Omega}^2=\frac{\lambda(\mu)}{3}\overline{\phi}^2(\mu) , \tag{26}$$

where $\overline{\phi}(\mu)$ is automatically defined by Eqs. (19), (22), and (26). Now let us consider the exact ground-state energy. In this case, from the inequality (14), for the unrenormalized quantities, i.e.,

$$\mathcal{W}^0(\epsilon,\lambda_B)\le\overline{\mathcal{W}}_g(\epsilon,\lambda_B) ,$$

a suitable definition of the running coupling constant

$$\lambda^T(\mu)=\frac{\lambda_B}{1+\dfrac{\lambda_B}{16\pi^2 c}\left[\dfrac{2}{\epsilon}+\ln\mu^2+\gamma+\ln\pi\right]} \tag{27}$$

(the superscript $T=$ "true") may be introduced, in terms of an unknown constant c, to renormalize $\mathcal{W}^0$, at least for small value of λ_B. It should be clear that Eq. (27) is the only alternative left out by our variational upper bound since, otherwise, the resulting unboundness from below (for $n \to 4$) of the exact theory, for positive λ_B, would be very difficult to understand. As a consequence, the inequality (14) can be read as

$$\mathcal{W}^0(\mu,\lambda^T(\mu))\le\overline{\mathcal{W}}_g(\mu,\lambda(\mu)) ,$$

and the RG equation

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda}\right]\mathcal{W}^0(\mu,\lambda)=0 \tag{28}$$

defines the "true" β function of the theory $\beta(\lambda)$:

$$\beta(\lambda)=-\frac{\lambda^2}{8\pi^2 c}+O(\lambda^3) . \tag{29}$$

Alternatively one can start from Eq. (28), and the general form (29) to obtain Eq. (27).

Note that in Eqs. (25) and (28) μ and λ are independent variables, as in a partial differential equation, but only along integral curves $\lambda=\lambda(\mu)$ and $\lambda=\lambda^T(\mu)$ with the same boundary conditions we are allowed to compare $\overline{\mathcal{W}}_g$ and $\mathcal{W}^0$.

We note that the possibility to renormalize $\mathcal{W}^0$, as in (27), is equivalent to solving Eq. (28) for arbitrarily large values of μ only if c is positive as in the Gaussian approximation ($c=1$). In this case we can consider the same region in the (μ,λ) plane (large μ and small λ) as the integral curves of both the exact theory and its Gaussian approximation are driven toward the ultraviolet fixed point at $\lambda=0$. On the other hand, if one would follow the perturbative indications ($c=-\frac{2}{3}$) the exact theory and its Gaussian approximation decouple in the continuum limit. In this case, surprising as it may be, the Gaussian approximation would be, nevertheless, very appealing for consistent quantum-field-theoretical models of weak and electromagnetic interactions as it exhibits both spontaneous symmetry breaking and asymptotic freedom. Henceforth we shall adopt the more realistic point of view that the Gaussian approximation, at least for weak coupling, well reproduces the properties of the exact theory thus providing the correct sign of c. By assuming the "Lipschitz continuity" of both β and β_g, the existence and uniqueness theorem of Ref. 11 states that only one integral curve (of β or β_g) can pass through any nonsingular point in our plane. By considering the two distinct characteristics, one of β and one of β_g, crossing a given (μ_0,λ_0) point, thus corresponding to the same λ_B, we can find a relation between the scale parameter M associated with the true ground-state energy $\mathcal{W}^0$ and our variational quantity $\overline{\Omega}$.

From the general solution of Eq. (28),

$$\mathcal{W}^0=-\mu_0^4\exp\left[-4\int^{\lambda_0}\frac{dx}{\beta(x)}\right] , \tag{30}$$

and from Eq. (29) one gets

$$\mathcal{W}^0=-\frac{M^4}{128\pi^2} , \tag{31}$$

where

$$M=\mu_0\exp\left[-\frac{8\pi^2 c}{\lambda_0}[1+O(\lambda_0)]\right]$$

is to be compared with the Gaussian approximation result, Eq. (22):

$$\overline{\Omega}=\mu_0\exp\left[-\frac{8\pi^2}{\lambda_0}[1+O(\lambda_0)]\right] . \tag{32}$$

In the weak-coupling limit, and by using the fundamental inequality (14), we deduce $0<c\le 1$ [or $\beta(\lambda)\le\beta_g(\lambda)$].

It should be clear that asymptotic freedom is, in general, a nonperturbative statement concerning Eq. (28).

Spontaneous symmetry breaking, signaling the "essential instability" of massless self-interacting scalar theories, prevents the possibility of any consistent perturbation theory in the unbroken phase.

We note, incidentally, that from our results it follows automatically the "triviality" of the $\lambda\phi^4$ theory in the symmetric phase. Indeed by using the definition of the leading-logarithmic perturbative coupling constant $\lambda^P(\mu)$ one gets the chain inequalities

$$\lambda(\mu) > \lambda_B > \lambda^P(\mu) .$$

Since (when $\epsilon \to 0^-$) $\lambda_B \to 0^+$, $\lambda^P(\mu)$ identically vanishes at any scale.

Connection with the results of Ref. 6 can be obtained by the explicit expression for the renormalized Gaussian effective potential around one of its absolute minima ($\pm\bar{\phi}$),

$$V_g(\phi) = \frac{\lambda^2\phi^4}{576\pi^2}\left[\ln\frac{\phi^2}{\bar{\phi}^2} - \frac{1}{2}\right] , \tag{33}$$

from which the wave-function renormalization Z can be derived through the general expression

$$\left.\frac{d^2V(\phi)}{d\phi^2}\right|_{\phi=\bar{\phi}} = \frac{m_R^2}{Z} . \tag{34}$$

In our case, m_R^2 is $\bar{\Omega}^2$.

Therefore we obtain, in the Gaussian approximation,

$$Z_g = \frac{24\pi^2}{\lambda(\mu)} , \tag{35}$$

or, by introducing explicitly the mass $\bar{\Omega}^2$,

$$Z_g\left[\frac{\mu^2}{\bar{\Omega}^2}\right] = \tfrac{3}{2}\ln\left[\frac{\mu^2}{\bar{\Omega}^2}\right] , \tag{36}$$

the infinite rescaling of Ref. 6 in the limit $\mu \to \infty$, and $\bar{\Omega}^2$ fixed. Equation (35) may be checked by means of an explicit calculation in the shifted theory.[12]

As stressed in Ref. 6, the formal identity (up to a finite rescaling of the field) of Eq. (33) with the one-loop result of Ref. 1 is accidental. Equation (33) is nonperturbative and has *no* leading-logarithmic corrections.

Finally the generalization of our approach to the continuous symmetry case $O(N)$ is straightforward following Ref. 13. In that paper, by introducing *two* variational masses, corresponding to the "radial" (physical Higgs boson) and "angular" excitations (Goldstone modes), it is found that the Goldstone theorem is recovered exactly in the limit $N \to \infty$. This result, a property of the continuum theory, is obtained by keeping fixed the physical Higgs-boson mass, to set up the scale of the theory, in the infinite cutoff limit, as in the discrete symmetry case analyzed in this paper. As a consequence, if one accepts that spontaneous symmetry breaking is discovered as sensible result in pure $\lambda\phi^4$ theories, the existence of an ultraviolet fixed point is unavoidable.

Because of the nonuniformity of the two limits $N \to \infty, \mu \to \infty$, the conclusion of Ref. 14, that an $O(N)$-invariant self-interacting scalar theory is trivial, in the large-N limit, should be limited to the symmetric phase.

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