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Dimensional continuation and the two $(\lambda\phi^4)_4$ theories

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Abstract. The Gaussian effective potential for $(\lambda\phi^4)_4$ theory indicates that two distinct, non-trivial, theories may exist. In the context of dimensional regularization these can be understood as the $d\to 4$ limits of two separate theories inhabiting $d\geqq 4$ and $d\leqq 4$ dimensions, respectively.

1 Introduction

Recent studies of the Gaussian effective potential (GEP) for $(\lambda\phi^4)_4$ suggest that there may be *two* distinct such theories [1–4]. One version [1] is related to perturbation theory and to the $1/N$ expansion results [5, 6]. This theory has negative λ_B, and has not been addressed in the Monte Carlo studies [7] or in the celebrated "triviality" analyses [8]. However, there are now several very different approaches which indicate that a negative-λ_B theory is meaningful [9–14]. While the GEP method has no pretensions to the rigor of these other approaches, it does shed some light on the crucial question of stability. It seems that although the negative-λ_B theory is unstable when regularized on the lattice (or with any sort of ultraviolet cutoff), it can become stable, or, rather, infinitely metastable, as the ultraviolet cutoff is removed [1]. Hence, we refer to this version of the theory as "precarious ϕ^4."

The GEP also indicates another possibility [2, 3], christened "autonomous ϕ^4," because it appears to be quite separate from perturbation theory and inaccessible to the $1/N$ expansion [4]. This theory has positive λ_B and requires a wavefunction (field-strength) renormalization. It can display spontaneous symmetry breaking, but in its symmetric phase the particles are massless. This unexpected feature could explain how this theory has eluded the Monte Carlo studies [7]. It is also possible that this theory would fall through one of the known gaps in the incomplete "triviality" proofs [8, 15], but this suggestion requires more study.

The purpose of this paper is to re-examine the GEP results from the perspective of dimensional continuation [16]*. That is, instead of regarding the divergent integrals

$$I_N(\Omega)=\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{2\omega_{\underline{k}}}[\omega_{\underline{k}}^2]^N, \tag{1}$$

where

$$\omega_{\underline{k}}\equiv(\mathbf{k}^2+\Omega^2)^{1/2},$$

as regulated by an (implicit) ultraviolet cutoff as in [1–4], we shall treat them by analytic continuation in the spacetime dimension, d. The integrals are thereby assigned the values

$$I_{-1}(\Omega)=\frac{A}{\varepsilon}\Omega^{-\varepsilon},\qquad I_0(\Omega)=\frac{-A}{\varepsilon(2-\varepsilon)}\Omega^{2-\varepsilon},$$
$$I_1=\frac{-A}{\varepsilon(2-\varepsilon)(4-\varepsilon)}\Omega^{4-\varepsilon}, \tag{2}$$

where

$$A\equiv\Gamma(1+\varepsilon/2)(4\pi)^{\varepsilon/2}/(4\pi^2)$$

and $d=4-\varepsilon$. For the purposes of this paper, we propose to regard this procedure as having some kind of "physical" meaning. While we would not necessarily advocate this view in general, it deserves to be taken seriously, especially in view of the popularity of Kaluza-Klein ideas. We do not mean to imply that there is anything wrong with an ultraviolet-cutoff or lattice-regularization approach: one would expect the physical results to be independent of the regularization used. Indeed, this is basically what we find here.

* This augments an earlier such study, pre-dating "autonomous ϕ^4," by Bollini and Giambiagi [17]

However, dimensional regularization provides an interesting new angle on the existing GEP results, and allows us to make connections between results in different dimensions.

Our main points are that (i) "precarious ϕ^4" arises as the limit $d \to 4_+$ of a theory that "exists" and is stable only in *more* than 4 dimensions, while (ii) "autonomous ϕ^4" can be obtained as a limit $d \to 4_-$ of the GEP results in *less* than 4 dimensions. The latter is important, since there is no serious reason to doubt that the GEP results for the 2 and 3 dimensional theories have something to do with real physics.*

We begin by recalling that the GEP is a variational approximation to the effective potential obtained using Gaussian trial wavefunctionals. It is easily computed by writing $\phi = \phi_0 + \hat{\phi}(\Omega)$, where $\hat{\phi}(\Omega)$ is a free quantum field of mass Ω, and calculating ${}_\Omega\langle 0|\mathcal{H}|0\rangle_\Omega \equiv V_G(\phi_0, \Omega)$ in the free-field vacuum $|0\rangle_\Omega$,

$$\begin{aligned} V_G(\phi_0, \Omega) = {} & I_1(\Omega) + \tfrac{1}{2}(m_B^2 - \Omega^2)\, I_0(\Omega) \\ & + \tfrac{1}{2} m_B^2 \phi_0^2 + \lambda_B \phi_0^4 \\ & + 6\lambda_B I_0(\Omega)\, \phi_0^2 + 3\lambda_B I_0^2(\Omega), \end{aligned} \tag{3}$$

and then minimizing with respect to the variational parameter Ω, which gives

$$\Omega^2 = m_B^2 + 12\lambda_B (I_0(\Omega) + \phi_0^2), \tag{4}$$

though in certain cases the global minimum occurs at the endpoint $\Omega = 0$. (Ω is positive definite, because the trial wavefunctional must be normalizable). In the Gaussian approximation, the physical particle mass in the $\phi_0 = 0$ vacuum is given by Ω at $\phi_0 = 0$, so we define the renormalized mass parameter by

$$m_R^2 = m_B^2 + 12\lambda_B I_0(m_R). \tag{5}$$

2 Precarious ϕ^4

In this section we shall write $\delta = -\varepsilon$ and consider $\delta > 0$, together with a coupling constant of the form

$$\lambda_B = \frac{-1}{6 I_{-1}(\Lambda)} = \frac{\delta}{6A} \Lambda^{-\delta}, \tag{6}$$

where Λ is a finite mass scale. The mass renormalization (5) becomes

$$m_B^2 = m_R^2 \left[1 - \frac{2}{2+\delta} \left(\frac{m_R}{\Lambda} \right)^{\delta} \right], \tag{7}$$

and we define a parameter

$$\kappa = (2/\delta)\, [(m_R/\Lambda)^\delta - 1] \tag{8}$$

* Some quantitative support is provided by numerical studies of bound states in ":$\phi^6 - \phi^4$:" theories, which agree well with GEP predictions [18]

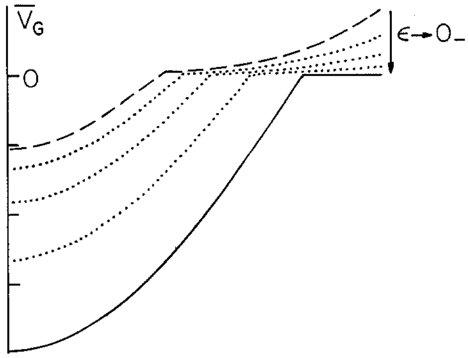

Fig. 1. The GEP for the "precarious" version (see (6), (7)) of $\lambda\phi^4$ in $d = 4 - \varepsilon$ dimensions. The 5-dimensional theory ($\varepsilon = -1$) is shown by the dashed line, with the dotted curves representing $\varepsilon = -0.75$, -0.5, -0.25. The solid line shows the $\varepsilon \to 0$ limit. [Special case $\kappa = 1$ (i.e., $m_B^2 = 0$). Units $m_R = 1$]

which reduces to $\kappa = \ln(m_R^2/\Lambda^2)$ as $\delta \to 0$, as in [1], and serves as a reciprocal renormalized coupling constant. For the Ω equation's solution to represent a minimum of V_G one requires $I_{-1}(\Omega)(1 + 6\lambda_B I_{-1}(\Omega)) > 0$, which implies $\Omega > \Lambda$ always, and hence $\kappa > 0$.

The Ω equation becomes

$$\begin{aligned} & x[(2+\delta) - (2+\kappa\delta)\, x^{\delta/2}]/\delta \\ & \quad = 1 - \kappa + (2+\delta)(2+\kappa\delta)\, \hat{\phi}_0^2/A, \end{aligned} \tag{9}$$

with

$$x \equiv \Omega^2/m_R^2, \qquad \hat{\phi}_0^2 = \phi_0^2/m_R^{2+\delta},$$

and $\overline{V}_G(\phi_0)$ can be written as

$$\begin{aligned} \overline{V}_G(\phi_0) = {} & \frac{A\, m_R^{4+\delta}\, x^{2+\delta/2}}{\delta(2+\delta)(4+\delta)} \\ & \cdot \left[1 - \frac{(4+\delta)(2+\kappa\delta)}{4(2+\delta)} x^{\delta/2} \right] + V_{\rm cl}(\phi_0), \end{aligned} \tag{10}$$

with

$$\begin{aligned} V_{\rm cl}(\phi_0) & \equiv \tfrac{1}{2} m_B^2 \phi_0^2 + \lambda_B \phi_0^4 \\ & = \frac{1}{2} \delta\, m_R^{4+\delta} \left[\frac{(1-\kappa)}{(2+\delta)} \hat{\phi}_0^2 + \frac{(2+\kappa\delta)}{6A} \hat{\phi}_0^4 \right]. \end{aligned} \tag{11}$$

Equation (9) ceases to have a solution once $\hat{\phi}_0$ gets too big, but before this happens the $\Omega = 0$ endpoint takes over as the global minimum of V_G, giving $\overline{V}_G = V_{\rm cl}$ in the large-ϕ_0 region.

The resulting GEP is illustrated in Fig. 1 for $\kappa = 1$, which is fairly typical (and corresponds to having m_B exactly zero). The vacuum always corresponds to $\phi_0 = 0$, and as one approaches 4 dimensions the large-ϕ_0 part of the curve, governed by the classical potential terms, becomes flatter and flatter. In dimensions lower than 4 the potential would curve downwards at infinity, and so this version of the theory would become unstable. The reason is obvious: the λ_B of (6) is then negative.

We note that in dimensional regularization there appears to be a non-trivial, stable, 5-dimensional theory. However, we stress that it probably "exists" only in the context of dimensional regularization, where the usual power-counting rules for divergences do not apply. In other regularizations the usual diseases of non-renormalizability would show up. It is therefore doubtful whether the 5 dimensional results relate to a physically acceptable theory. There is no such problem for the 4-dimensional "precarious ϕ^4" itself, because those results are regularization independent. Nevertheless, it is noteworthy that precarious ϕ^4 – the form of $(\lambda\phi^4)_4$ to which the familiar perturbative and $1/N$-expansion results refer – is obtained by approaching 4 dimensions from *above*, (cf. [14]).

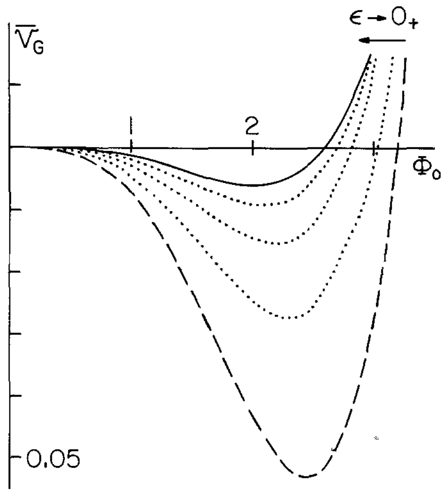

Fig. 2. The GEP for the "autonomous" version (see (12)–(14)) of $\lambda\phi^4$ in $d=4-\varepsilon$ dimensions. The 3-dimensional theory ($\varepsilon=1$) is shown by the dashed line, with the dotted curves representing $\varepsilon=0.75$, 0.5, 0.25. The solid line shows the $\varepsilon\to 0_+$ limit. [Special case $m^2{}_0=0$ (i.e., $m_B^2=0$). Units $\Lambda=1$]

3 Autonomous ϕ^4

In 2 and 3 dimensions there is nothing very mysterious about $\lambda\phi^4$. The mass renormalization (5) is sufficient to render the GEP manifestly finite, and one obtains the same results from the dimensional regularization formulas (2) as from the usual cutoff procedure [1]. Dimensional continuation allows us to extrapolate from these reasonably well-understood theories towards 4 dimensions. We now show that, by approaching the 4-dimensional limit in a very particular way, one obtains a non-trivial GEP, which is the "autonomous ϕ^4" of [2–4].

We shall need (i) λ_B vanishing as ε, times a specific coefficient, (ii) the physical mass squared m_R^2 also vanishing as ε (so, in the absence of spontaneous symmetry breaking, the particles will be massless), and (iii) a re-scaling of ϕ_0; i.e., we adjust the scale on the ϕ_0 axis so that the interesting part of the picture remains in view. Specifically, we define

$$\lambda_B=\frac{1}{12 I_{-1}(\Lambda)}=\frac{\varepsilon}{12A}\Lambda^{\varepsilon}, \tag{12}$$

$$m_0^2=\frac{2A}{3\varepsilon}m_B^2=\frac{2A}{3\varepsilon}m_R^2\left[1+\frac{1}{(2-\varepsilon)}\left(\frac{\Lambda}{m_R}\right)^{\varepsilon}\right], \tag{13}$$

$$\Phi_0^2=\phi_0^2(\varepsilon/A), \tag{14}$$

and take the limit $\varepsilon\to 0_+$ with the parameters Λ, m_0, and the re-scaled field Φ_0 remaining finite. Note that in, and around, 3 dimensions these equations are finite, and represent a mere re-parametrization of the theory, leaving the physics entirely unaffected. However, they serve to define a *path* by which we may approach a non-trivial 4-dimensional limit.

Re-written in terms of the new parameters the Ω equation becomes

$$\Omega^2\chi(\Omega)=\frac{2}{3}\Lambda^{\varepsilon}\Phi_0^2+\varepsilon m_0^2/A, \tag{15}$$

where

$$\chi(\Omega)\equiv\frac{2}{3}\left[1+\frac{1}{2-\varepsilon}\left(\frac{\Lambda}{\Omega}\right)^{\varepsilon}\right], \tag{16}$$

which tends to $1-\varepsilon(\ln\Omega^2/\Lambda^2-1)/6$ as $\varepsilon\to 0$. The GEP itself (discarding the vacuum-energy constant) can be expressed as

$$\bar{V}_G(\Phi_0)=\frac{1}{2}m_0^2\Phi_0^2+\frac{A\Lambda^{-\varepsilon}}{8\varepsilon}\frac{\Omega^4}{(4-\varepsilon)}\cdot[2\varepsilon+6(2-\varepsilon)\chi-3(4-\varepsilon)\chi^2]. \tag{17}$$

In the limit $\varepsilon\to 0$, $\Omega^2\to\frac{2}{3}\Phi_0^2+0(\varepsilon)$ and a $\Phi_0^4\ln\Phi_0^2$ term appears in $\bar{V}_G$, and we reproduce the results of [3].

The evolution of the GEP from 3 to 4 dimensions is illustrated in Fig. 2 for the special case $m_0=0$. The limit is quite smooth. The relation $\Omega^2=\frac{2}{3}\Phi_0^2$ is always true at the non-trivial minimum of the GEP, and the region of its approximate validity spreads out until it holds everywhere in the limit $\varepsilon\to 0$.

It is important to realize that one cannot continue to ε negative ($d>4$), because the field re-scaling (14) would make Φ_0 imaginary. It is also interesting that the dimensional-continuation approach allows only

$m_0^2 \geqq 0$ (because m_R^2, being Ω^2 at $\phi_0 = 0$, cannot be negative). In the cutoff-based approach [3, 4] there seems to be no restriction on the sign of m_0^2.

The fact that "autonomous ϕ^2" emerges naturally as a smooth limit of the fairly well-understood lower-dimensional theories seems to us very important. It is good evidence that "autonomous ϕ^4" is not some fantastic mirage, but a glimpse of some real, and very interesting physics.

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References

1. P.M. Stevenson: Z. Phys. C – Particles and Fields 24 (1984), 87; Phys. Rev. D32 (1985), 1389
2. M. Consoli, A. Ciancitto: Nucl. Phys. B254 (1985), 653
3. P.M. Stevenson, R. Tarrach: Phys. Lett. 176B (1986) 436
4. P.M. Stevenson, B. Alles, R. Tarrach: Phys. Rev. D35 (1987), 2407
5. L.F. Abbott, J.S. Kang, H.J. Schnitzer: Phys. Rev. D13 (1976), 2212, and references therein
6. W.A. Bardeen, M. Moshe: Phys. Rev. D28 (1983), 1372
7. K.G. Wilson, J. Kogut: Phys. Rep. 12C (1974), 75; G.A. Baker, J.M. Kincaid: Phys. Rev. Lett. 42 (1979), 1431; B. Freedman, P. Smolensky, D. Weingarten: Phys. Lett. 113B (1982), 481; D.J.E. Callaway, R. Petronzio: Nucl. Phys. B240 (1984), 577; I.A. Fox, I.G. Halliday: Phys. Lett. 159B (1985), 148; C.B. Lang: Nucl. Phys. B265 (1986), 630
8. M. Aizenman: Phys. Rev. Lett. 47 (1981), 1; Commun. Math. Phys. 86 (1982), 1; J. Fröhlich: Nucl. Phys. B200 (1982), 281, and references therein. A. Sokal: Ann. Inst. H. Poincare, 37 (1982), 317
9. K. Symanzik: Lett. Nuovo Cimento 6 (1973) 77
10. F. Cooper, G.S. Guralnik, S.H. Kasdan: Phys. Rev. D14 (1976) 1607
11. G. 't Hooft: Phys. Lett. 119B (1982), 369; Commun. Math. Phys. 88 (1983) 1; ibid. 86 (1982), 449; V. Rivasseau: Commun. Math. Phys. 95 (1984), 445
12. S. Albeverio et al.: Phys. Lett. 104B (1984), 396
13. K. Gawedzki, A. Kupiainen: Commun. Math. Phys. 99 (1985), 197; Nucl. Phys. B257 (1985), 474
14. G. Felder: Commun. Math. Phys. 102 (1985), 139
15. G. Gallavotti, V. Rivasseau: Ann. Inst. H. Poincaré 40 (1984), 185
16. C.G. Bollini, J.J. Giambiagi: Nuovo Cimento 12B (1972), 20; G. 't Hooft, M. Veltman: Nucl. Phys. B44 (1972), 189
17. C.G. Bollini, J.J. Giambiagi: Nuovo Cimento A93 (1986), 113
18. J.W. Darewych, M. Horbatsch, R. Koniuk: Phys. Rev. Lett. 54 (1985), 2188; Phys. Rev. D33 (1986), 2316