THE RETURN OF $\lambda\phi^4$

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A re-examination of the gaussian effective potential – allowing for an infinite wavefunction renormalization – indicates the existence of a stable, non-trivial form of $(\lambda\phi^4)_{3+1}$ with positive, infinitesimal λ_B. The theory exhibits a massless symmetric phase and a massive broken-symmetry phase.

According to many experts, the $(\lambda\phi^4)_{3+1}$ theory is probably "trivial" – either non-interacting or else inconsistent [1,2]. In more than four dimensions there are rigorous proofs of triviality, but in exactly four dimensions the proofs are incomplete [2]. A very physical, albeit non-rigorous, approach to this question is provided by the gaussian effective potential (GEP) [‡1] [3,4], which is basically a variational approximation to the effective potential.

Previous studies of the GEP for $(\lambda\phi^4)_{3+1}$ [3–5] seemed consistent with the "triviality" scenario for positive λ. An apparently viable, though "precarious", theory with *negative*, and infinitesimal, λ_B was found. Fascinating as this theory is, we forbear to discuss it further here. The purpose of this letter is to point out that the GEP allows another *nontrivial* solution with *positive* (infinitesimal) λ_B, and which (unlike "precarious" ϕ^4) can exhibit spontaneous symmetry breaking (SSB). This solution was not seen in the previous analyses [3–5] because these studies did not allow for a wavefunction renormalization. Our results turn out to be closely connected to earlier work by Consoli and Ciancitto [6], as we discuss later on.

In the absence of SSB the new version of ϕ^4 is a massless theory. The physics of this phase may well be very subtle, and we suspect that it might thereby fall through one of the known gaps in the incomplete triviality proofs.

We begin by recalling the bare form of the GEP. It is obtained from the hamiltonian density

$$H = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m_B^2\phi^2 + \lambda_B\phi^4, \tag{1}$$

by writing $\phi = \phi_0 + \hat{\phi}$, where $\hat{\phi}$ is a free quantum field of mass Ω, and computing the expectation value ${}_{\Omega}\langle 0 | H | 0\rangle_{\Omega}$ in the free-field vacuum state $|0\rangle_{\Omega}$. A straightforward calculation gives

$$\begin{aligned} V_G(\phi_0, \Omega) = {} & I_1(\Omega) + \tfrac{1}{2}(m_B^2 - \Omega^2) I_0(\Omega) \\ & + \tfrac{1}{2}m_B^2\phi_0^2 + \lambda_B\phi_0^4 \\ & + 6\lambda_B I_0(\Omega)\phi_0^2 + 3\lambda_B I_0^2(\Omega), \end{aligned} \tag{2}$$

where

$$I_n(\Omega) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} (\omega_k^2)^n, \quad \omega_k^2 = \boldsymbol{k}^2 + \Omega^2. \tag{3}$$

An ultraviolet cutoff, Λ, is to be understood in all integrals (which can be manipulated conveniently using formulas in ref. [4]). The GEP itself, $\bar{V}_G(\phi_0)$,

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[‡1] References to earlier work on this method can be found in refs. [3,4].

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is obtained by minimizing with respect to the variational parameter Ω. It can easily be seen from (2) that the minimum occurs either at

$$\Omega^2 = m_B^2 + 12\lambda_B(I_0 + \phi_0^2), \tag{4}$$

or at the lower endpoint of its allowed range, $\Omega = 0$. It can furthermore be shown that at any minimum of $\bar{V}_G(\phi_0)$, Ω corresponds to the particle mass in that vacuum [4].

For the present, we assume that (4) is always true (i.e. $\Omega = 0$ never applies). Then the expressions for the first and second derivatives of $\bar{V}_G(\phi_0)$ can be obtained directly from (2) and (4),

$$\mathrm{d}\bar{V}_G/\mathrm{d}\phi_0 = \phi_0(\Omega^2 - 8\lambda_B\phi_0^2), \tag{5}$$

$$\frac{\mathrm{d}^2\bar{V}_G}{\mathrm{d}\phi_0^2} = \Omega^2 - \frac{I_{-1}(\Omega)(12\lambda_B)^2\phi_0^2}{1 + 6\lambda_B I_{-1}(\Omega)}. \tag{6}$$

To make physical sense of the theory we must ascribe to the bare parameters m_B^2, λ_B some particular dependence on the UV-cutoff, Λ, so that the regulator can be removed eventually. In the previous analyses [3–5] it seemed that the only non-trivial possibility was $\lambda_B \sim -1/6I_{-1}$. We show here that, if one also allows a re-scaling of the field ϕ_0, there is another, more attractive, alternative. Before presenting the result, we briefly discuss how we were led to it.

Suppose that an SSB vacuum exists at $\phi_0 = \pm\phi_v$. From eq. (5) the particle mass obeys $\Omega_v^2 = 8\lambda_B\phi_v^2$. Now suppose ϕ_v becomes infinitely large when $\Lambda \to \infty$, $= Z^{1/2}v$, where v is finite and Z is an infinite wavefunction renormalization constant [‡2]. We demand, though, that the physics in this vacuum be finite. To begin with, we shall need $\lambda_B \to 0$ like Z^{-1} so that Ω_v stays finite. Also, from eq. (6) we see that

$$\bar{V}_G''|_{\phi_0=\phi_v} = \Omega_v^2 \frac{1 - 12\lambda_B I_{-1}(\Omega_v)}{1 + 6\lambda_B I_{-1}(\Omega_v)}, \tag{7}$$

which in general is finite. This is not acceptable because the rescaling of ϕ_0 by $Z^{-1/2}$ will rescale $\bar{V}_G''$ by an infinite factor Z. To have the rescaled result finite requires that (7) be vanishingly small, which requires $\lambda_B = 1/12I_{-1}(\mu) = 1/12I_{-1}(\Omega_v) + O(1/I_{-1}^2)$, where μ is a finite parameter with dimensions of mass. Hence we need $Z = \zeta I_{-1}(\mu)$, where ζ is finite and can, without loss of generality, be set equal to unity.

We next observe that, by use of eq. (4) at $\phi_0 = \phi_v$,

$$\begin{aligned} m_B^2 &+ 12\lambda_B I_0(0) \\ &= \Omega_v^2 + 12\lambda_B[I_0(0) - I_0(\Omega_v)] - 12\lambda_B\phi_v^2 \\ &= \Omega_v^2 + \frac{1}{I_{-1}(\mu)}\left[\tfrac{1}{2}\Omega_v^2 I_{-1}(\Omega_v) + \Omega_v^2/16\pi^2\right] - \tfrac{3}{2}\Omega_v^2 \\ &= O(1/I_{-1}). \end{aligned} \tag{8}$$

This means, if one considers eq. (4) at $\phi_0 = 0$, that Ω at the origin is vanishingly small.

The above observations motivate us to consider a renormalization of the theory in which

$$\lambda_B = 1/12I_{-1}(\mu), \tag{9}$$

$$m_B^2 + 12\lambda_B I_0(0) = \tfrac{3}{2}m_0^2/I_{-1}(\mu), \tag{10}$$

and the field rescaled by

$$\phi_0^2 = I_{-1}(\mu)\Phi_0^2. \tag{11}$$

We now show that this "works": it leads to a GEP which, as $\Lambda \to \infty$, remains manifestly finite in terms of Φ_0, m_0 and μ (apart from a divergent vacuum-energy constant). Note that although our motivational discussion assumed SSB, the analysis itself, which starts from (9)–(11), does not make this assumption.

It is easiest to proceed via the second derivative of $\bar{V}_G$, at a general point, ϕ_0. Substituting for ϕ_0 and λ_B in (6) one obtains

$$\mathrm{d}^2\bar{V}_G/\mathrm{d}\Phi_0^2 = I_{-1}(\mu)\left(\Omega^2 - \tfrac{2}{3}\Phi_0^2\left\{1 - (1/12\pi^2)\ln(\Omega^2/\mu^2)[I_{-1}(\mu)]^{-1}\right\}\right) \tag{12}$$

up to terms vanishing as $1/I_{-1}$. The logarithmic term arises from $I_{-1}(\Omega) - I_{-1}(\mu)$. The Ω equation, (5), can be rewritten as

[‡2] Recall that the effective potential is related to the zero-momentum Green's functions by

$$V_{\mathrm{eff}}(\phi_0) = -\sum[(2n)!]^{-1}(\phi_0^2)^n\Gamma^{(2n)}(0,\ldots,0).$$

Thus, a re-scaling of the classical field ϕ_0 is equivalent to a multiplicative wavefunction renormalization of the Green's functions.

$$\Omega^2 = \tfrac{2}{3}\Phi_0^2 + [I_{-1}(\mu)]^{-1} \times \{m_0^2 + (\Omega^2/24\pi^2)[\ln(\Omega^2/M^2) - 1]\}. \tag{13}$$

[One can check that indeed the Ω equation is applicable, except when $m_0^2 < 0$ and $\Phi_0^2 \leq -\frac{3}{2}m_0^2/I_{-1}(\mu)$, in which case $\Omega = 0$.] Substituting (13) into (12) one sees that the divergent terms cancel. In the remaining terms it is sufficient to replace Ω^2 by $\frac{2}{3}\Phi_0^2$, since we now take $\Lambda \to \infty$ so that $O(1/I_{-1})$ terms can be dropped. One thus obtains

$$\mathrm{d}^2\overline{V}_G/\mathrm{d}\Phi_0^2 = m_0^2 + (\Phi_0^2/36\pi^2)[3\ln(\Phi_0^2/\tfrac{3}{2}\mu^2) - 1]. \tag{14}$$

By integrating twice one obtains $\overline{V}_G(\Phi_0)$ up to a constant term $D \equiv \overline{V}_G(\Phi_0 = 0)$,

$$\overline{V}_G(\Phi_0) - D = \tfrac{1}{2}m_0^2\Phi_0^2 + (\Phi_0^4/144\Pi^2)\{\ln(\Phi_0^2/\tfrac{3}{2}\mu^2) - \tfrac{3}{2}\}. \tag{15}$$

This is the GEP. The expression is valid for all values of Φ_0 [for $m_0^2 < 0$ the region where (13) does not apply shrinks, as $\Lambda \to \infty$, to just $\Phi_0 = 0$, which for $m_0^2 < 0$ is anyway a maximum, not a minimum]. Eq. (15) allows SSB when m_0^2 is negative or not too large. It is then convenient to rewrite it in terms of the vacuum value of Φ_0, denoted v, which is given by

$$\ln(v^2/\tfrac{3}{2}\mu^2) = 1 - 36\pi^2 m_0^2/v^2. \tag{16}$$

Eliminating μ in favour of v gives

$$\overline{V}_G(\Phi_0) - D = \tfrac{1}{2}m_0^2\Phi_0^2(1 - \tfrac{1}{2}\Phi_0^2/v^2) + (\Phi_0^4/144\pi^2)[\ln(\Phi_0^2/v^2) - \tfrac{1}{2}]. \tag{17}$$

This is strangely similar to the result of Coleman and Weinberg (CW) [7]. In fact, for $m_0 = 0$ our potential is identical (up to a finite rescaling of the field) to CW's one-loop result for massless ϕ^4. [One first needs to rewrite their eq. (3.10) in terms of $v \equiv \phi_0|_{\text{vacuum}}$; cf. eqs. (4.6)–(4.9). However, this is more or less a coincidence. It does *not* arise because the renormalization (9)–(11) makes negligible those terms by which the GEP differs from the one-loop effective potential. Moreover, our result with non-zero m_0, being non-analytic at the origin, is quite unlike the massive one-loop result (CW eq. (B5)). The "coincidence" of the massless results is probably just due to approximate scale invariance, which constrains the potential to be proportional to Φ_0^4 up to logarithms. Adding the assumption that only a single logarithm is present leads uniquely to the form of (17) with $m_0 = 0$.

Even though, for $m_0 = 0$, the result coincides in form with CW's, there is a big difference in interpretation. CW point out that in the vicinity of the minimum their one-loop approximation is not trustworthy, and that by changing the renormalization scheme ("renormalization-group improvement") the minimum can be made to disappear. Thus, the interpretation is unclear. For the GEP there is no such difficulty, since the result is renormalization-group invariant up to rescaling of Φ_0. The non-trivial minimum is unambiguously present.

Another important difference is that the one-loop calculation has $Z = 1$, whereas for us an infinite wavefunction renormalization is crucial. Indeed, since $Z \propto 1/\lambda_B$, the derivation above would seem to be intrinsically nonperturbative. Apart from the $m_0 = 0$ "coincidence", there seems to be no point of contact with perturbation theory.

Returning to consider (15) one finds that SSB occurs for m_0^2 negative, zero, or even for sufficiently small positive values. However, when $m_0^2/\mu^2 > \sqrt{e}/48\pi^2$ the SSB vacuum becomes a false vacuum, and for $m_0^2/\mu^2 > 1/24\pi^2$ [when eq. (16) has no solution for v] it ceases to exist. In these cases the vacuum corresponds to $\Phi_0 = 0$. Since Ω vanishes at the origin we expect this phase to contain massless particles. Beyond this we can say little about this phase. We should not look for more detailed information from our approximation, since its variational parameter has here been driven to the endpoint of its acceptable range. [We note only that the singular fourth derivative at the origin [‡3], which naively indicates

[‡3] We caution that the usual formula for $\mathrm{d}^4\overline{V}_G/\mathrm{d}\phi_0^4|_0$ [3,4] is not valid when Ω vanishes as $\phi_0 \to 0$.

infinite coupling, is not necessarily a physical pathology, since infrared effects render $2 \to 2$ scattering an unobservable process. Infrared subtleties may also take care of the seeming contradiction between the physical mass, Ω, being zero, while the inverse propagator at zero momentum ($\mathrm{d}^2\bar{V}_G/\mathrm{d}\Phi_0^2|_0$) is not.] None of these difficulties arise for the SSB phase, of course.

We should now like to discuss the connection between our results and the work of Consoli and Ciancitto [6]. These authors described a version of $\lambda\phi^4$ with SSB, positive λ_B, and which had $\Omega = 0$ at the origin. This corresponds to our eqs. (9), (10) in the case $m_0 = 0$. [Note that $\Omega_v^2 = \frac{2}{3}v^2$, and hence eq. (16) for $m_0 = 0$ gives $\ln(\Omega_v^2/\mu^2) = 1$, so that $I_{-1}(\Omega_v) = I_{-1}(\mu) - 1/8\pi^2$. The equivalence between our eq. (9) and their eq. (3.37) follows immediately.] They found only the $m_0 = 0$ case because they assumed $\Omega|_0$ to be *strictly* zero, whereas in general it is only vanishingly small.

Unfortunately, ref. [6] left it unclear whether or not the ultraviolet cutoff could be removed entirely. Comments in the subsequent literature [4,8] implied that this form of the theory was cutoff-sensitive, albeit only logarithmically, and so could, at best, be viewed as an "effective", approximate theory. The virtue of the present analysis is to show clearly that this is a bonafide renormalizable theory: the cutoff can – and should – be sent to infinity, with all the physics remaining finite. The key ingredient is to allow an infinite rescaling of the field.

One might ask whether the inclusion of wave-function renormalization opens up still other possibilities for renormalizing the GEP of $(\lambda\phi^4)_{3+1}$. In fact, not. The removal of quadratic divergences requires the bare mass to satisfy

$$m_B^2 + 12\lambda_B I_0(0) = f(\lambda_B, I_{-1}), \tag{18}$$

where the right-hand side contains only logarithmic divergences. Rescaling the field in (6) gives

$$\frac{\mathrm{d}^2\bar{V}_G}{\mathrm{d}\Phi_0^2} = Z\left(\Omega^2 - \frac{(12\lambda_B)^2 I_{-1}(\Omega) Z\Phi_0^2}{1 + 6\lambda_B I_{-1}(\Omega)}\right), \tag{19}$$

and this expression must be finite. Also, Ω must be finite. [Possibilities with $\Omega =$ (infinitesimal) everywhere may be discounted since, with the variational parameter always driven to the end of its range, the gaussian ansatz would not be credible.] Now, if $Z \to 0$ then either (19) will vanish everywhere, or it will be negative everywhere (recall that $1 + 6\lambda_B I_{-1}(\Omega) > 0$ if Ω is a *minimum* of V_G [4]). Such possibilities are uninteresting or unacceptable. If, instead, $Z \to \infty$ then there must be a cancellation between the two terms of (19), so that $\Omega^2 \propto \Phi_0^2$, up to infinitesimal corrections. Consistency with the Ω equation then requires $\lambda_B I_{-1} \to$ (finite constant), which moreover must be precisely 1/12. This leads uniquely to the theory described above.

Thus, the GEP indicates that there are two (and only two) distinct, non-trivial versions of $(\lambda\phi^4)_{3+1}$. One is the "precarious" theory with $\lambda_B = -1/6I_{-1}(\mu)$, which has some relation to perturbation theory, although not entirely straightforwardly [4]. The other is the version of $\lambda\phi^4$ described here, which has $\lambda_B = 1/12I_{-1}(\mu)$, and seems to be intrinsically nonperturbative. Depending on the ratio m_0^2/μ^2, the theory may or may not show SSB. In fact, a plot of eq. (15) for various m_0^2/μ^2 looks qualitatively very similar to the results for lower-dimensional $\lambda\phi^4$ theories [4]. We may also observe that the approximate proportionality of Ω^2 to ϕ_0^2 at large ϕ_0, as here, is also characteristic of the lower-dimensional theories. These similarities give us confidence in the physical relevance of our results. Furthermore, the $\Omega = 0$ endpoint plays essentially no role here (unlike the "precarious" case), so that there is nothing that indicates that the gaussian approximation is inadequate.

We believe that the results here, and their precursors in ref. [6], represent the "first sighting" of the true, physically relevant, interacting $(\lambda\phi^4)_{3+1}$ theory, which has been missing for so long. As we can testify ourselves [4,5], it is easy to miss. In its unbroken-symmetry phase the new theory seems to be very subtle, since it has massless particles and a singular coupling. This may explain how it has escaped the Monte Carlo studies [1] and the rigorous analyses [2]. The physics of the SSB phase appears to be more straightforward. We hope that the results here will encourage and guide new searches for $(\lambda\phi^4)_{3+1}$ by other techniques.

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