Dynamical symmetry breaking of $\lambda\phi^4$ theory in the two loop effective potential

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The two loop effective potential of massless $\lambda\phi^4$ theory is presented in several regularization and renormalization prescriptions and the dynamical symmetry breaking solution is obtained in the strong-coupling situation in several prescriptions except the Coleman-Weinberg prescription. The beta function in the broken phase becomes negative and the UV fixed point turns out to be a strong-coupling one, and its numeric value varies with the renormalization prescriptions, a detail which is different from the asymptotic-free solution in the one loop case. The symmetry-breaking phase is shown to be an entirely strong-coupling phase. The reason for the relevance of the renormalization prescriptions is shown to be due to the nonperturbative nature of the effective potential. We also reanalyze the two loop effective potential by adopting a differential equation approach based on the understanding that all the quantum field theories are ill-defined formulations of the "low-energy" effective theories of a complete underlying theory. The relevance of the prescriptions of fixing the local ambiguities to physical properties such as symmetry breaking is further emphasized. We also tentatively propose a rescaling insensitivity argument for fixing the quadratic ambiguities. Some detailed properties of the strongly coupled broken phase and related issues are discussed.

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I. INTRODUCTION

The standard model (SM) has now been firmly established with most of its predictions experimentally confirmed. New physics beyond the SM are being intensively explored from theoretical perspective, but no concrete experimental evidence has yet been found. A major motivation to go beyond the SM has been to get rid of those theoretically unsatisfactory aspects of the SM such as the hierarchy or naturalness problem [1] and the triviality [2] of the Higgs sector, and that there are too many parameters to be explained. Thus most particle theorists believe that the SM is only an effective theory of a fundamental theory. The currently prevailing direction to go beyond the SM has been string theory [3] and/or supersymmetric field theories [4]. These theories modify the SM profoundly. As a matter of fact, the most demanding task in and beyond SM physics is to find the true mechanism of symmetry breaking to replace the Higgs sector that suffers the above-mentioned defects and is held as phenomenological. In this connection, there has been another important theoretical direction that does not modify the SM so profoundly: the technicolor model and its descendants [5]. All the above theoretical constructions share a common feature: the elementary Higgs scalar fields are excluded and the solution to the hierarchy and triviality problem must be in a nonperturbative regime [4].

However, more than a decade ago, there were some efforts to revive the $\lambda\phi^4$ interaction from the perturbative triviality by showing that the one loop effective potential of the massless $\lambda\phi^4$ permitted a nontrivial nonperturbative renormalization [6], i.e., $\beta(\lambda)<0$, in contrast with the perturbative renormalization, where $\beta(\lambda)>0$ (leading to triviality). On the other hand, it has been recently proposed that color confinement is closely related to flavor symmetry breaking [7] and even that color symmetry be realized via the Higgs mechanism [8]. In a sense, the Higgs model or $\lambda\phi^4$ interaction is still useful and should be explored further to search for nontrivial solutions of the model. If symmetry breaking can be dynamically realized together with asymptotic freedom or nontriviality, then it will shed new light on the confinement of color and symmetry breaking of the standard model. Thus it is worthwhile to see if the interesting nontrivial one loop solution can still exist after including higher loop corrections or how it "evolves" in the presence of higher-order quantum corrections.

In this paper, we provide a detailed report of our recent investigation of the existence and new features (if any) of the nontrivial dynamical symmetry-breaking solution of the quartic interaction by studying the two loop effective potential [9]. For convenience, we will consider the simplest scalar model—the massless $\lambda\phi^4$ model with Z_2 symmetry—with which the first example of dynamical symmetry breaking was demonstrated [10]. There is also a technical concern in choosing the massless scalar theory: there is a nonconvexity in the tree interactions that affects the Higgs model and often complicates the use of effective potential methods [11], while in massless models the tachyon mass term does not exist and the configuration of the expectation value of the scalar field can be naturally interpreted as the homogeneous argument of the effective potential.

In the meantime, we need to consider the regularization and renormalization problems in the nonperturbative regime. Since the effective potential is nonperturbative in nature, its regularization and renormalization might become more subtle. There have long been standard procedures to carry out perturbative renormalization, but in nonperturbative contexts the renormalization often needs to be dealt with case by case, example by example. Moreover, the nonperturbative context sometimes allows for an alternative renormalization solution, for example the nontrivial or asymptotic free solution for the one loop potential of $\lambda\phi^4$ mentioned above [6,12]. Other examples of the subtleties associated with regularization and renormalization can be found in the recent applications of the effective-field-theory method [13] to nucleon interactions [14–17], where the framework in use is necessarily nonperturbative. We hope our experiences here might be useful in carrying out renormalization within nonperturbative contexts.

The paper is organized in the following way. The two loop effective potential will be given in dimensional and cutoff regularization, respectively, in Sec. II. The bare and renormalized effective potentials obtained in different schemes will also be listed there. Then in Sec. III we investigate the existence and the properties of the dynamical symmetry breaking solution(s) via the effective potentials obtained with various intermediate renormalization prescriptions. The prescription dependence of the solution is exhibited and explained. Sec. IV will be devoted to a new approach for evaluating the loop diagrams, and the relevance of the intermediate renormalization is highlighted. Some properties and features of the symmetry-breaking solution in the two effective potentials are also presented. Some discussion and a summary will be given in the final section.

II. REGULARIZATION AND RENORMALIZATION

As is stated in the Introduction, we will consider the massless $\lambda\phi^4$ model with Z_2 symmetry: invariance under the transformation of $\phi\rightarrow-\phi$. The algorithm for the two loop effective potential is well known according to Jackiw [18],

$$L=\frac{1}{2}(\partial\phi)^2-\lambda\phi^4, \tag{1}$$

$$V_{(2l)}\equiv\lambda\phi^4+\frac{1}{2}I_0(\Omega)+3\lambda I_1^2(\Omega)-48\lambda^2\phi^2I_2(\Omega), \tag{2}$$

$$\Omega\equiv\sqrt{12\lambda\phi^2}, \tag{3}$$

$$I_0(\Omega)=\int\frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\Omega^2}{k^2}\right), \tag{4}$$

$$I_1(\Omega)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\Omega^2}, \tag{5}$$

$$I_2(\Omega)=\int\frac{d^4kd^4l}{(2\pi)^8}\times\frac{1}{(k^2+\Omega^2)(l^2+\Omega^2)[(k+l)^2+\Omega^2]}. \tag{6}$$

Here we have Wick-rotated all the loop integrals into Euclidean space. Let us calculate the three integrals in two regularization schemes: dimensional and cutoff. As these integrals have already been calculated in the literature both in dimensional regularization and in cutoff schemes, we will only need to list the results here.

A. Dimensional and cutoff regularizations

In dimensional regularization, these integrals have been calculated in the literature, see [19]. Here we list the two loop diagram (the sunset diagram) for example; the other integrals will be delegated to Appendix A:

$$\begin{aligned}\mu^{4\epsilon}I_2^{(D)}(\Omega)&=\int\frac{\mu^{4\epsilon}d^Dkd^Dl}{(2\pi)^{2D}}\frac{1}{(k^2+\Omega^2)(l^2+\Omega^2)[(k+l)^2+\Omega^2]}\\&=-\frac{3\Omega^2}{2(4\pi)^4\epsilon^2}\{1+(3-2\overline{L})\epsilon+[2\overline{L}^2-6\overline{L}+7+6S-\tfrac{5}{3}\zeta(2)]\epsilon^2\}\end{aligned} \tag{7}$$

with $S=\Sigma_{n=0}^{\infty}[1/(2+3n)^2]$, $\overline{L}=L+\gamma-\ln 4\pi$, and $L=\ln(\Omega^2/\mu^2)$.

Similarly, in cutoff regularization, we find, from Ref. [18],

$$\begin{aligned}I_2^{(\Lambda)}(\Omega)&=\int_\Lambda\frac{d^4kd^4l}{(2\pi)^8}\frac{1}{(k^2+\Omega^2)(l^2+\Omega^2)[(k+l)^2+\Omega^2]}\\&=\frac{1}{(4\pi)^4}\left\{2\Lambda^2-\frac{3\Omega^2}{2}\ln^2\frac{\Omega^2}{\Lambda^2}+3\Omega^2\ln\frac{\Omega^2}{\Lambda^2}+o(\Lambda^{-2})\right\}.\end{aligned} \tag{8}$$

Note that the $\sim\Lambda^2$ term in the two loop integral is not explicitly given in [18].

Note that the leading "low-energy" content of the sunset diagram (the double-log term) obtained in dimensional regularization differs from that obtained in the cutoff scheme. However, this does not matter because, after subtracting the subdivergences in such diagrams [19], the "nonlocal" term will be the same.[1]

B. Bare and renormalized effective potential

With the preceding preparations, we can write down the bare effective potential obtained, respectively, in dimensional and cutoff regularizations:

$$V_{(2l)}^{(D)}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{-\frac{1}{\epsilon}+\bar{L}-\frac{3}{2}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}\left[\left(-\frac{1}{\epsilon}+\bar{L}-1\right)^2+(\bar{L}-1)^2+2\left(-\frac{1}{\epsilon}+\bar{L}-\frac{3}{2}\right)^2+2\left(\bar{L}-\frac{3}{2}\right)^2+7+6S-\frac{5}{3}\zeta(2)\right]\right\}, \tag{9}$$

$$V_{(2l)}^{(\Lambda)}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{L_\Lambda^\Omega-\frac{1}{2}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}[(L_\Lambda^\Omega)^2-2+2(L_\Lambda^\Omega-1)^2]\right\}+\Omega^2\left\{\frac{2\Lambda^2}{(8\pi)^2}+\frac{3\lambda\Lambda^2L_\Lambda^\Omega}{(4\pi)^4}-\frac{8\lambda\Lambda^2}{(4\pi)^4}\right\}, \tag{10}$$

where $L_\Lambda^\Omega=\ln(\Omega^2/\Lambda^2)$. Here we have omitted all the field-independent terms. In the remaining part of this section, we will focus on the renormalization of $V_{(2l)}^{(D)}(\Omega)$ and $V_{(2l)}^{(\Lambda)}(\Omega)$.

The renormalization will be done in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme for $V_{(2l)}^{(D)}(\Omega)$ (cf. [19]), while for $V_{(2l)}^{(\Lambda)}(\Omega)$ the renormalization will be done in three prescriptions: the one defined by Jackiw [18], the one adopted by Coleman and Weinberg [10], and a new prescription, μ_Λ^2 (a simulation of $\overline{\mathrm{MS}}$, see Appendix B). The results read

$$V_{(2l)}^{(\overline{\mathrm{MS}})}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{\bar{L}-\frac{3}{2}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}\times[3\bar{L}^2-10\bar{L}+11+12S-\tfrac{8}{9}\pi^2]\right\}, \tag{11}$$

$$V_{(2l)}^{(\mu_\Lambda^2)}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{\tilde{L}-\frac{1}{2}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}[3\tilde{L}^2-4\tilde{L}]\right\}, \tag{12}$$

$$V_{(2l)}^{(\mathrm{Jackiw})}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{\check{L}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}[3\check{L}^2-\check{L}]\right\}, \tag{13}$$

$$V_{(2l)}^{(\mathrm{CW})}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{\check{L}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}\times\left[3\check{L}^2-\check{L}+\frac{205}{12}\right]\right\} \tag{14}$$

with the notations defined as $\tilde{L}=\ln(\Omega^2/\mu_\Lambda^2)$, $\check{L}=\ln(\Omega^2/12\lambda\mu_{\mathrm{Jackiw}}^2)$, and $\check{L}=\ln(\Omega^2/12\lambda\mu_{\mathrm{CW}}^2)-\frac{25}{6}$. In all the above formulas the scheme dependence of field strength and coupling constant are understood. Note that the μ_Λ^2, Jackiw, and Coleman-Weinberg prescriptions were applied to the same bare effective potential, i.e., that calculated in the cutoff scheme.

C. Prescription dependence

Upon appropriate rescaling of the subtraction scales, all versions of the effective potential take the following form (we will drop all the dressing symbols):

$$V_{(2l)}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{L-1/2}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}[L^2+2(L-1)^2+\alpha]\right\} \tag{15}$$

with $L\equiv\ln(\Omega^2/\mu^2)$. Now we see the explicit dependence of

[1]In a diagrammatic or perturbative framework, such independence of regularization schemes is beyond doubt. But in a nonperturbative framework, such independence is controversial [14–17,20]. Since our calculation is a systematic summation of infinite diagrams (hence nonperturbative) where only a few diagrams are UV ill-defined, this subtle point does not concern us here. There are also some references [21] where related issues are discussed.

TABLE I. Values of α in various schemes.

Scheme	α
$\overline{\mathrm{MS}}$	-2.6878
μ_Λ^2	-2
Jackiw	$-\frac{5}{4}$
Coleman-Weinberg	$16\frac{1}{3}$

the effective potential upon the intermediate renormalization prescriptions expressed by α, which varies across schemes as exhibited in Table I.

Here the scheme dependence (regularization and/or renormalization) of the effective potential as a nonperturbative quantity (summing over infinitely many one and two loop one-particle irreducible diagrams) differs from that of the perturbative framework [22] that arises from the truncation of perturbation series (a sum of a finite number of connected diagrams).[2] The difference in α could not be removed through redefinition of the coupling constant (and perhaps of field strength) without changing the functional dependence upon the field expectation value, ϕ. This is a crucial difference. The main obstacles here are (i) the presence of the double log dependence on ϕ (in $[\ln(12\lambda\phi^2/\mu^2)]^2$) and (ii) the nonperturbative feature of the effective potential, i.e., the sum over infinitely many diagrams.

If one redefines the coupling constant and expands the new coupling constant in terms of the old one like in the perturbative case ($\lambda'=\lambda+a\lambda^2+b\lambda^3+\cdots$), one could at best arrive at the other schemes' results *plus* extra higher-order terms that take the form $\sim\lambda^n\phi^4\ln(\phi^2/\mu^2), n\geqslant 3$. The same is true for the redefinition of ϕ or Ω. Since the effective potential is nonperturbative in terms of λ and ϕ in nature, one should not discard such higher-order terms due to consistency due to their nontrivial dependence upon ϕ, which will affect the symmetry-breaking status, unlike in the perturbative case. Otherwise, as will be clear shortly, even if one puts the consistency aside and discards such terms, the symmetry breaking behavior *will be changed* due to such redefinition and approximation. Thus even with the intermediate renormalization done in the standard way, the nonperturbative results depend on the prescriptions quite nontrivially. To the best of our knowledge, this new feature in the nonperturbative framework has not been explicitly and particularly pointed out.

If there are no double log terms present in the effective potential except the single log terms, then the constant terms can be easily redefined away or absorbed into the single log terms without leading to a new extra functional dependence upon ϕ that can affect the symmetry breaking. In gauge theories, there are only single log terms in the sum of one-particle irreducible diagrams. While here we encounter the essential presence of double log terms in the sum of one-particle irreducible diagrams at the two loop level (recall that the effective potential is the generating functional of the one-particle irreducible diagrams), it is not difficult to see that a still higher power of log terms can generally show up in higher loop one-particle irreducible diagrams.

[2]Rigorously speaking, this property has been established only in mass-independent subtraction schemes or in massless theories or in the high-energy region where mass effects are negligible. The nontrivial influence of renormalization prescriptions in defining masses has been recently emphasized [23] in theories with unstable elementary particles (like $W^\pm, Z^0$ bosons in electroweak theory).

III. EFFECTIVE POTENTIAL AND THE SYMMETRY-BREAKING SOLUTION

Now let us start to determine the minima of the two loop effective potentials that are renormalized in the prescriptions specified in the preceding section. We will work with the general parametrization form of Eq. (15). Our goal is to solve the first-order equation

$$\frac{dV_{(2l)}(\sqrt{12\lambda\phi^2})}{d\phi}=0, \tag{16}$$

which becomes the following equation upon substituting Eq. (15) into it:

$$24\lambda\phi\Omega^2\left(\frac{2V_{(2l)}(\Omega^2)}{\Omega^4}+\frac{1}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}(6L-4)\right)=0. \tag{17}$$

An obvious solution is $\phi=0$, which is the symmetric solution in the perturbative (weak-coupling) regime, while the existence of the nonzero expectation value solution is determined by the existence of a real number solution of L to the following algebraic equation:

$$3L^2+\left(\frac{4\pi^2}{3\lambda}-1\right)L+\alpha+\frac{16\pi^4}{27\lambda^2}=0. \tag{18}$$

Here it is obvious that the existence of real number solutions depends on both α and λ. Since α is renormalization-prescription-dependent, it is natural to expect that the solution and its existence are also prescription-dependent. Since symmetry breaking is a physical phenomenon, one usually anticipates that the occurrence of symmetry breaking should be independent of a manipulation of infinities, that is, independent of renormalization schemes. Here we see a counterexample. In this connection, we would like to mention other nonperturbative examples discussed in Ref. [24], where the physical predictions depend on the renormalization (and regularization) prescription in use. The reason is basically the same as was given in the preceding subsection.

A. Determinants of the symmetry-breaking solution

Now let us examine the symmetry-breaking solution in more detail. Since we must start from a stable micropotential, the coupling λ must be a positive real number. Now let us closely examine Eq. (18). For Eq. (18) to possess a finite real number solution, we must impose the following criterion in terms of α and λ:

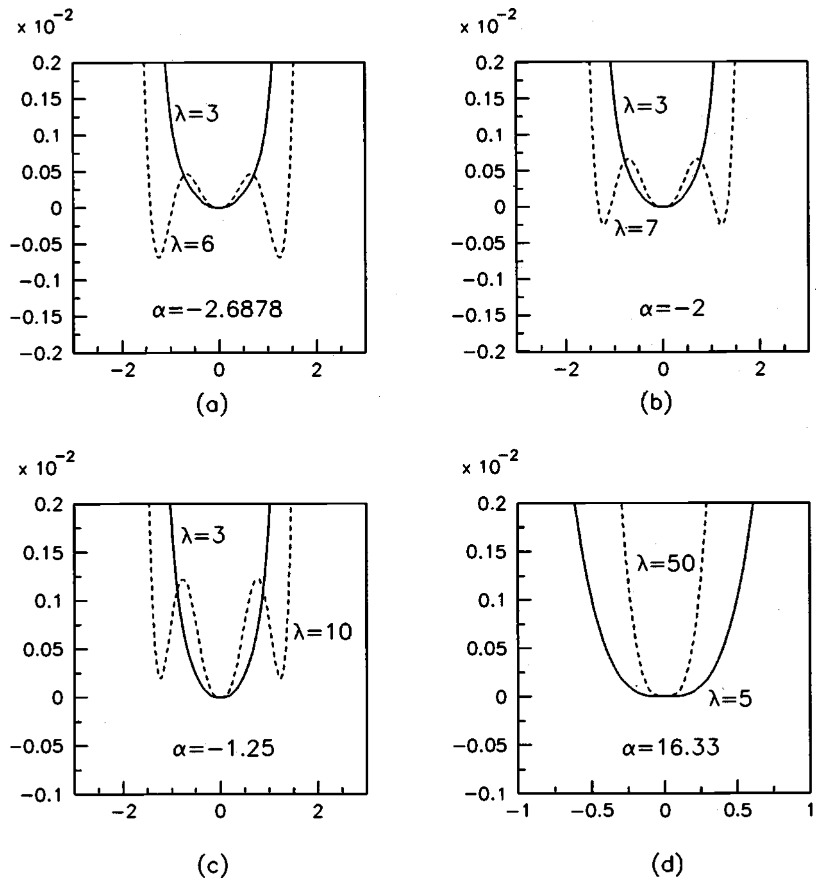

FIG. 1. The two loop effective potential in various renormalization prescriptions with different values of coupling constant. In all the four prescriptions [(a)–(d)], the horizontal axis represents the quantity $\sqrt{12\lambda}\phi/\mu$ while the vertical axis represents $V_{(2l)}/\mu^4$.

$$\Delta \equiv \left(\frac{4\pi^2}{3\lambda}-1\right)^2 - 12\left(\alpha + \frac{16\pi^4}{27\lambda^2}\right)$$

$$= \frac{1}{3}\left[4-36\alpha - \left(1+\frac{4\pi^2}{\lambda}\right)^2\right] \geqslant 0. \tag{19}$$

This inequality is only valid for certain ranges of α and λ:

$$\alpha < \frac{1}{12}, \tag{20}$$

$$\lambda \geqslant \lambda_{cr} \equiv \frac{4\pi^2}{\sqrt{4-36\alpha}-1}. \tag{21}$$

Then the solutions to Eq. (18) can be found provided the above two requirements are satisfied in certain schemes,

$$L_{\pm}(\lambda) = \frac{1}{6}\left[1 - \frac{4\pi^2}{3\lambda} \pm \sqrt{\Delta}\right]. \tag{22}$$

From this and the definitions $L \equiv \ln(\Omega^2/\mu^2), \Omega \equiv \sqrt{12\lambda\phi^2}$, we can find the nonzero solutions of ϕ, which read

$$\phi_{\pm}^2(\lambda;[\mu,\alpha]) = \frac{\mu^2}{12\lambda}\exp\left\{\frac{1}{6}\left[1-\frac{4\pi^2}{3\lambda} \pm \sqrt{\Delta}\right]\right\}. \tag{23}$$

But the solutions corresponding to $L_-(\lambda)$ are local maxima (tachyonic); only the $L_+(\lambda)$ solutions are local minima. This can be seen from the second-order derivative of the effective potential at $\Omega_{\pm}^2$ (which is exactly the effective mass),

$$m_{\mathrm{eff};\pm}(\lambda) \equiv \frac{\partial^2 V_{(2l)}}{(\partial\phi)^2}\Bigg\|_{\phi^2=\phi_{\pm}} = \pm \frac{18\lambda^2\Omega_{\pm}^2}{(2\pi)^4}\sqrt{\Delta}. \tag{24}$$

Because of the presence of the local maxima $(\pm\sqrt{(\mu^2/12\lambda)}\exp\{\frac{1}{12}[1-(4\pi^2/3\lambda)-\sqrt{\Delta}]\})$ between the local minima $\phi=0$ and $\pm\sqrt{(\mu^2/12\lambda)}\exp\{\frac{1}{12}[1-(4\pi^2/3\lambda)+\sqrt{\Delta}]\}$, the symmetry breaking must be a first-order phase transition when it happens, in accordance with the recent results [25] obtained through other approaches. This is also clear from Fig. 1, in which the shape of the effective potential is depicted in several renormalization prescriptions (α) for different values of the coupling constant.

The inequality (20) tells us that the renormalization prescriptions do affect physical contents in the nonperturbative framework: Not all prescriptions could be compatible with symmetry breaking as far as the two loop effective potential is concerned (the stability of such solutions will be discussed shortly). From Table I in Sec. II we see the following: For the two loop effective potential, the Coleman-Weinberg scheme failed to predict dynamical symmetry breaking as the critical inequality (20) is badly violated there, $\alpha_{\mathrm{CW}}=16\frac{1}{3}=\frac{196}{12}\gg\frac{1}{12}$, while the other three schemes do allow for symmetry-breaking solutions. The situation is not affected by the rescaling of the subtraction points. One can check that even in the original form [cf. Eq. (3.17) in Ref. [18]], the inequality corresponding to Eq. (19) could not be satisfied (see Appendix C), in fact the corresponding Δ is strictly negative for non-negative values of the renormalized coupling λ. Figure 1 also exhibits such a prescription dependence.

Now we find a strong dependence of ''physical'' properties upon renormalization prescriptions, though it is demonstrated within a model that is not quite realistic. This is not totally unexpected if one recalls that the effective potential is a nonperturbative object, as was noted in the preceding section. The only unexpected point is that the pioneering prediction of dynamical symmetry breaking has been made in the Coleman-Weinberg scheme used in the *one loop* effective potential, while this scheme becomes incompatible with symmetry breaking after the two loop contributions were included. In fact, the freedom of renormalization prescription choices will be further restricted after imposing the stability condition for the solutions, which be made clear in the next subsection.

B. Stability of symmetry breaking and the criterion for the coupling constant

From the above discussions, it is not clear yet whether the symmetry-breaking solutions are stable or not, i.e., we have not confronted our intermediately renormalized effective potential with physical conditions or requirements, which corresponds to solving the renormalized quantities in terms of physical quantities. To this end, let us calculate the vacuum energy density of the symmetry-breaking phase. Using Eq. (17), we have

$$E_{+}(\lambda,\mu)\equiv V_{(2l)}(\sqrt{12\lambda\phi^{2}})|_{\phi^{2}=\phi_{+}^{2}} = -\frac{(12\lambda\phi_{+}^{2})^{2}}{2(8\pi)^{2}}\left[1+\frac{3\lambda}{2\pi^{2}}(3L_{+}-2)\right] \tag{25}$$

with the symbols defined in the previous subsections. Since the weak-coupling vacuum state ($\phi=0$) energy is zero, for the symmetry-breaking states to be stable we must require that

$$E_{+}(\lambda,\mu)\leqslant 0, \tag{26}$$

that is,

$$L_{+}\geqslant\frac{2}{3}-\frac{2\pi^{2}}{9\lambda}. \tag{27}$$

TABLE II. Critical values of the coupling constant in various schemes.

Scheme	λ_{cr}	$\hat{\lambda}_{\mathrm{cr}}$
$\overline{\mathrm{MS}}$	4.368	5.2024
μ_{Λ}^{2}	5.1152	6.5797
Jackiw	6.5797	10.698

This criterion turns out to be a requirement of the renormalized coupling constant, i.e.,

$$\lambda\geqslant\hat{\lambda}_{\mathrm{cr}}\equiv\frac{4\pi^{2}}{\sqrt{4-36\alpha-27}-1}\times\left(>\lambda_{\mathrm{cr}}=\frac{4\pi^{2}}{\sqrt{4-36\alpha}-1}\right). \tag{28}$$

In all the schemes with symmetry breaking, the two critical values of the coupling constant are greater than 1. We can conclude that symmetry breaking could not happen in the weak-coupling regime. The critical couplings in various prescriptions are exhibited in Table II.

Now we see that dynamical symmetry breaking does happen in certain renormalization schemes in the strong-coupling regime. On the other hand, the stability requirement also imposes a further constraint on the prescription choices in order to predict symmetry breaking. In this connection, note that the stable condition (27) amounts to the following mathematical requirement:

$$\left(1+\frac{4\pi^{2}}{\lambda}\right)^{2}\leqslant -23-36\alpha. \tag{29}$$

Since the left-hand side of this inequality could not be less than 1^{+}, we obtain the following criterion for α, or for scheme choices:

$$\alpha\leqslant-\tfrac{2}{3}, \tag{30}$$

which is a more stringent requirement than $\alpha<\frac{1}{12}$.

C. RG invariance of vacuum energy and beta function

Since the vacuum energy is a physical entity, it must be renormalization-group-invariant, i.e., insensitive to the choice of subtraction point within a scheme,

$$\mu\frac{dE_{+}(\lambda,\mu)}{d\mu}=0. \tag{31}$$

We must stress that this condition in fact defines a fundamental physical scale as input in this broken phase that should be obtained from some kind of experimental measurements, corresponding to the important and necessary step after renormalization is done, i.e., to confront the renormalized amplitudes with experiments or other physical inputs or conditions where the physical scales come from [26]. Consequently, a fundamental physical scale is introduced into the effective potential.

From this equation, we can determine the beta function of λ as was done in Ref. [6]. First, let us rewrite the vacuum energy density as

$$E_+ = -\frac{\mu^4}{2(8\pi)^2}\varepsilon_+(\lambda)e^{2L_+(\lambda)}, \tag{32}$$

with $\varepsilon_+(\lambda)\equiv 1+(3\lambda/2\pi^2)[3L_+(\lambda)-2] = 3\lambda/4\pi^2(\sqrt{\Delta}-3)$. Then we find from Eq. (31) that

$$4\varepsilon_+(\lambda)e^{2L_+(\lambda)} + \{\varepsilon_+(\lambda)e^{2L_+(\lambda)}\}'\beta(\lambda) = 0, \tag{33}$$

or equivalently,

$$\beta(\lambda) \equiv \mu\frac{d\lambda}{d\mu} = -4\frac{\varepsilon_+(\lambda)e^{2L_+(\lambda)}}{\{\varepsilon_+(\lambda)e^{2L_+(\lambda)}\}'}. \tag{34}$$

Since $\varepsilon_+(\lambda)$ is positive definite provided the symmetry-breaking solution is stable,

$$\{\varepsilon_+(\lambda)e^{2L_+(\lambda)}\}' = \left\{\frac{\varepsilon_+(\lambda)}{\lambda}\left(1+\frac{4\pi^2}{9\lambda}\right) + \frac{(1+4\pi^2/\lambda)}{3\lambda}\right\}e^{2L_+(\lambda)} > 0 \tag{35}$$

and hence the beta function is negative definite as long as the broken phase is stable,

$$\beta(\lambda) = -\frac{12\lambda\varepsilon_+(\lambda)}{\left\{\varepsilon_+(\lambda)\left(3+\frac{4\pi^2}{3\lambda}\right)+1+4\pi^2/\lambda\right\}} < 0. \tag{36}$$

This is true for all three schemes allowing for the symmetry-breaking solution. When the coupling becomes infinitely strong, i.e., $\lambda\to\infty$, the beta function approaches a straight line:

$$\beta(\lambda)|_{\lambda\to\infty} \to -4\lambda, \tag{37}$$

while when the coupling approaches the critical value $\hat{\lambda}_{cr}$, the beta function also approaches a straight line with the same ratio:

$$\beta(\lambda)|_{\lambda\to\hat{\lambda}_{cr}^+} \sim -4(\lambda-\hat{\lambda}_{cr}). \tag{38}$$

The wonderful thing that enhances our faith in the two loop effective potential is that all schemes (except the Coleman-Weinberg scheme) predict the same kind of running behavior of the coupling (the same kind of beta function),[3] and we could roughly imitate the true beta function with the following qualitative approximation:

[3]This is true in fact for all the prescriptions as long as the criterion (30) is satisfied, since the beta function is basically the same except the UV fixed point varies with prescription.

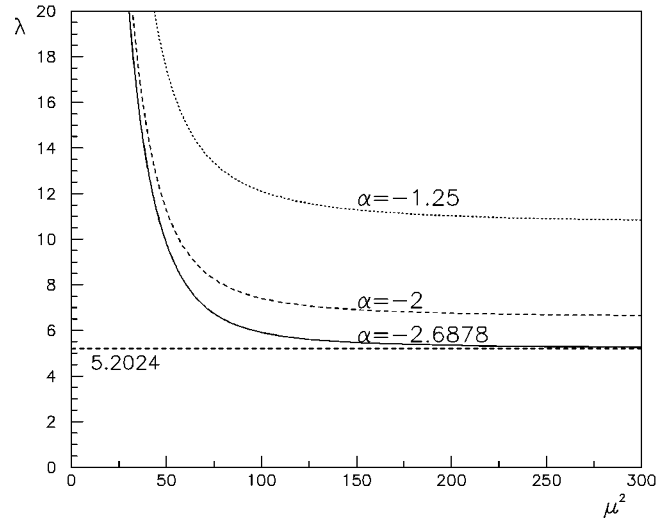

FIG. 2. The running behavior of the coupling constant in various prescriptions. The UV fixed points can be found as the asymptotic lines. We have exhibited the UV fixed point for the $\overline{\mathrm{MS}}$ case.

$$\beta_{\mathrm{appr}}(\lambda) = -4(\lambda-\hat{\lambda}_{cr}), \quad \lambda\in(\hat{\lambda}_{cr},\infty), \tag{39}$$

with the obvious solution

$$\lambda-\hat{\lambda}_{cr} = \frac{\mu_0^4}{\mu^4}, \qquad \mu\in(0,\infty) \tag{40}$$

which could also be obtained as a crude approximation of Eq. (32). The RG-invariant scale μ_0^2 should be a function of the vacuum energy density as the fundamental physical scale for this theory. Moreover, the running is relatively milder in the UV region, which means that the coupling constant does not become very large at energies that are not too low. The true running behavior defined by Eq. (32) has been plotted in Fig. 2.

Now it is clear that we obtained a *nontrivial* theory with a nonzero UV fixed point, $\hat{\lambda}_{cr}$, i.e., a strong coupling, as is clear from Table II, in contrast to the one loop case. From Eq. (40) we can identify an IR pole in terms of μ^2, unlike the IR Landau pole in QCD, thus it is new at least in a theoretical sense. No matter what kind of phenomenon it defines, it is clear that within the two loop effective potential, the dynamical symmetry-breaking phase is nontrivial without asymptotic freedom, which means this phase is a totally strong-coupling phase. Since this property is true in a number of renormalization prescriptions that satisfy the criterion (30), we feel that it is at least an interesting phenomenon that deserves further examination. We emphasize that our derivation here has not employed any unconventional or special assumptions or approximations. All the techniques and arguments are well known and well established. From now on, we denote this solution as SCRDSB for the strong coupling regime beta dynamical symmetry breaking.

D. Further details about the SCRDSB

Before speculating on this SCRDSB solution, let us examine the scale dependence patterns of the main quantities of interests.

First let us look at the order parameter of the symmetry-breaking, i.e., the square vacuum expectation value of the scalar field ϕ_+^2, or, equivalently, Ω_+^2. Inverting the dimensionless function of coupling, we can express the running of the coupling in the following form by taking the vacuum energy density as fundamental in Eq. (32), i.e.,

$$\mu^2 = 8\pi e^{-L_+(\lambda)}\sqrt{\frac{-2E_+}{\varepsilon_+(\lambda)}}. \qquad (41)$$

Combining this relation with the definition of L_+, we find the dependence of Ω^2 upon the running scale,

$$\Omega_+^2(\lambda) = \mu^2 e^{L_+(\lambda)} = 8\pi\sqrt{-2E_+/\varepsilon_+(\lambda)}, \qquad (42)$$

or, equivalently,

$$\phi_+^2(\lambda) = \frac{2\pi\sqrt{-2E_+/\varepsilon_+(\lambda)}}{3\lambda}. \qquad (43)$$

Since the coupling runs, the order parameter also runs from its dependence upon $\varepsilon_+(\lambda)$, therefore we need to study the running behavior of $\varepsilon_+(\lambda)$. Bearing in mind the running behavior described in Eq. (40), we have

$$\varepsilon_+(\lambda)\|_{\lambda\to\infty} \to \frac{3\lambda}{4\pi^2}\sqrt{(1-12\alpha)},$$

$$\varepsilon_+(\lambda)\|_{\lambda\to\hat{\lambda}_{cr}^+} \to \frac{\delta(\delta-1)}{12\pi^2}(\lambda-\hat{\lambda}_{cr}^+), \qquad (44)$$

with $\delta \equiv \sqrt{4-36\alpha-27}$ being positive definite in all the three schemes compatible with symmetry-breaking. Using Eq. (40), we find that in both IR and UV regions,

$$\varepsilon_+(\mu) \propto \frac{1}{\mu^4}. \qquad (45)$$

Then we obtain the asymptotic behaviors of the order parameter Ω_+^2 in both IR and UV regions,

$$\Omega_+^2(\mu) \propto \mu^2. \qquad (46)$$

But the asymptotic behavior of ϕ_+^2 is somewhat different,

$$\phi_+^2(\mu)\|_{\mu\to\infty} \propto \mu^2, \quad \phi_+^2(\mu)\|_{\mu\to 0} \propto \mu^6, \qquad (47)$$

which means that the square vacuum expectation value of the field vanishes more rapidly than Ω_+^2. We note that due to the extra term of L_+ in the vacuum energy density, the parameter Ω^2 is no longer RG-invariant [6], in contrast to the one loop effective potential case.

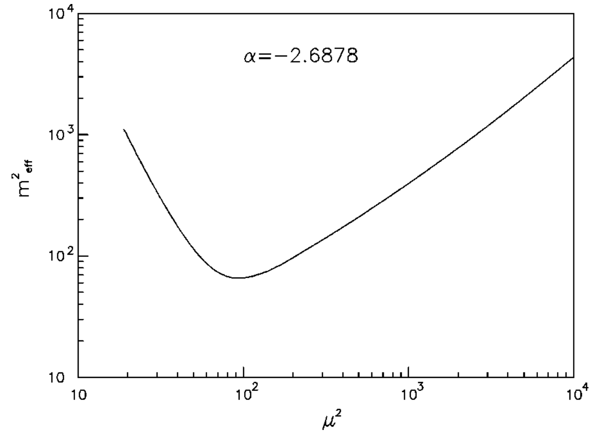

FIG. 3. The running behavior of the effective mass in the $\overline{\text{MS}}$ scheme with $-2E_+$ set to 1.

Similarly, we can obtain the asymptotic behavior of the effective mass defined in Eq. (24). Using Eq. (42), we have

$$m_{\text{eff}}^2(\lambda) = \frac{18\lambda^2}{(2\pi)^2}\sqrt{\frac{-2E_+}{\varepsilon_+(\lambda)}}. \qquad (48)$$

With the above preparations, we find that

$$m_{\text{eff}}^2(\lambda)\|_{\lambda\sim\infty} \sim \lambda^{3/2}, \quad m_{\text{eff}}^2(\lambda)\|_{\lambda\sim\hat{\lambda}_{cr}^+} \sim (\lambda-\hat{\lambda}_{cr})^{-1/2}, \qquad (49)$$

or in terms of the running scale,

$$m_{\text{eff}}^2(\mu^2)\|_{\mu\sim 0} \sim \frac{1}{\mu^6}, \quad m_{\text{eff}}^2(\mu^2)\|_{\mu\sim\infty} \sim \mu^2. \qquad (50)$$

Here we found new asymptotic behaviors that differ from both the asymptotic freedom and the triviality solutions. The effective mass (self-energy at zero momentum) becomes singular at both IR and UV ends. Only in the moderate energy region characterized by the typical energy scale—the vacuum energy density—can we have finite effective mass. (Of course we must be aware that since the dynamics of SCRDSB exists entirely in a strong-coupling regime, the uncalculated higher-order loop corrections will probably change the situation obtained here and make it even more complicated.) The running behavior of the effective mass is plotted in Fig. 3.

At this stage, one would naturally ask about the asymptotic behaviors of the effective coupling, defined as $\lambda_{\text{eff}}(\lambda) \equiv \partial^4 V_{(2l)}/(\partial\phi)^4\|_{\phi^2=\phi_+^2}$. The dependence of this four-point vertex function upon the renormalized coupling λ reads

$$\lambda_{\text{eff}}(\lambda) = \frac{3\lambda}{2\pi^4}\{16\pi^4 + 132\pi^2\lambda + 9(61+3\alpha)\lambda^2 + 9\lambda(4\pi^2+63\lambda)L_+ + 81\lambda^2 L_+^2\}. \qquad (51)$$

After some calculations, we have

$$\lambda_{\rm eff}(\lambda)\|_{\lambda\sim\infty}\sim 10^1\lambda^3, \quad \lambda_{\rm eff}(\lambda)\|_{\lambda\sim\hat{\lambda}_{\rm cr}}\sim 10^2\hat{\lambda}_{\rm cr}, \tag{52}$$

or equivalently

$$\lambda_{\rm eff}(\mu)\|_{\mu\sim 0}\sim\frac{1}{\mu^{12}}, \quad \lambda_{\rm eff}(\mu)\|_{\mu\sim\infty}\sim 10^3. \tag{53}$$

Note that the effective coupling becomes more singular than the effective mass does in the IR limit.

Since both the effective mass and the effective coupling become extremely singular in the IR limit, it is not difficult to see that in the low-energy region, the kinetic energy of the scalar field is negligible and the static potential energy dominates, thus it seems impossible to find free scalar field quanta as asymptotic states. In this sense the elementary scalar field seems to be ''confined'' somehow even in the high-energy ranges. We might detect some kind of bound states of such a scalar field, with the new bound states also being scalar states. So, even though we found a scalar particle, there may be another problem with regard to whether the detected particles are elementary ones or bound states of the elementary ones. In addition, the coupling is still strong in the high-energy region, though not infinitely strong. The situation encountered here seems to indicate that the Higgs model can allow for another scenario and symmetry-breaking mechanism provided one explores it nonperturbativey. The Higgs scalar quanta seem to be hidden ''heros'' that did not like to be ''shown off'' in the asymptotic states.

IV. A DIFFERENTIAL EQUATION APPROACH ANALYSIS

Now we employ a new approach without explicit regulators or deformations to calculate the loop diagrams. This approach is based on the standard point of view that all the known quantium field theories (QFT's) are effective theories for a completely well-defined quantum theory containing ''correct'' high-energy details [27]. We should make it clear that the UV structures of our present QFT's are inevitably incorrect or inaccurate and should be replaced by the ''correct'' underlying ones that are unknown to us yet, hence certain diagrams cannot be directly computed within the present formulation of QFT's. (In conventional methods, one introduces artificial regularizations to imitate the underlying UV structures.)

Fortunately, since differentiating a loop diagram with respect to its ''low-energy'' parameters [momenta and mass(es) that characterize the ''effective'' QFT's] amounts to inserting ''low-energy'' vertices to this diagram (this is valid in both the underlying theory and the effective theories), which in turn reduces the divergence degree of the diagram in terms of the effective QFT's, we can compute a potentially divergent loop diagram after differentiating them with respect to the (external) momenta and/or mass(es) for appropriate times. In other words, we can calculate the ill-defined diagrams by solving certain well-defined differential equations [28]. In this approach, the solutions would naturally contain unknown constants parametrizing the ill-definedness or incompleteness (to be fixed by physical ''boundary conditions'') of the effective theories. It is obvious that this approach *needs neither artificial regularizations nor complicated procedures*.

A. Recalculating the loop diagrams

Now we demonstrate this method with the sunset diagram, the two loop integral $I_\theta(\Omega)$.

(i) First, we differentiate it twice with respect to mass square (Ω^2) to remove all overall ill-definedness (divergence),

$$\frac{\partial^2}{\partial_{(\Omega)^2}} I_\theta(\Omega)\equiv 6I_{\theta:(3;1;1)}(\Omega)+3I_{\theta:(2;2;1)}(\Omega)+3I_{\theta:(2;1;2)}(\Omega) \tag{54}$$

with

$$I_{\theta:(\alpha;\beta;\gamma)}(\Omega)\equiv\int\frac{d^4k d^4l}{(2\pi)^8(k^2+\Omega^2)^\alpha[(k+l)^2+\Omega^2]^\beta(l^2+\Omega^2)^\gamma}. \tag{55}$$

The result is a sum of new diagrams without any overall divergence. Among these diagrams, $I_{\theta:(3;1;1)}(\Omega)$ still contains a subdivergence in the l integration,

$$I_{(1;1)}(\Omega,k^2)\equiv\int\frac{d^4l}{(2\pi)^4[(k+l)^2+\Omega^2](l^2+\Omega^2)}.$$

(ii) Second, we treat this divergent subdiagram in the same way to arrive at the following inhomogeneous differential equation:

$$\partial_{\Omega^2} I_{(1;1)}(\Omega,k^2)=\frac{-1}{(4\pi)^2}\int_0^1\frac{dx}{\Omega^2+(x-x^2)k^2} \tag{56}$$

and its solution

$$I_{(1;1)}(\Omega,k^2)=\frac{-1}{(4\pi)^2}\int_0^1 dx\left\{\ln\frac{\Omega^2+(x-x^2)k^2}{\mu^2_{\rm PDE}}+c_1\right\}, \tag{57}$$

with c_1 being the integration constants to be fixed through physical ''boundary conditions.''

(iii) Now we can compute the right-hand side of Eq. (54) and obtain again an inhomogeneous differential equation as below,

$$\frac{\partial^2}{\partial_{(\Omega)^2}} I_\theta(\Omega) = -\frac{3\left(\ln\frac{\Omega^2}{\mu_{PDE}^2}+c_1-1\right)}{(4\pi)^4\Omega^2}, \tag{58}$$

and the solution to it reads

$$I_2(\Omega) = -\frac{3}{2(4\pi)^4}\left\{\Omega^2\left[\left(\ln\frac{\Omega^2}{\mu_{PDE}^2}+c_1-2\right)^2 - (c_1-1)^2+1+2c_1^\theta\right]+2c_2^\theta\right\} \tag{59}$$

with μ_{PDE}^2, c_1^θ, and c_2^θ being the constants (independent of masses, coupling, and momenta) to be fixed by ''boundary conditions.'' The single loop integrals can be done in the same way and are listed in Appendix A.

It is not difficult to see that, before fixing the constants, this differential equation approach provides the most general parametrization of the ill-defined loop diagrams. Any consistent regularization and/or renormalization *should be* a special solution to these differential equations provided the counterterms are local functions of the momenta and mass(es). One might feel that this approach is merely another form of the powerful Bogolobov-Parasiuk-Hepp-Zimmermann (BPHZ) program [29]. To respond, we note the following. First, one must employ a regularization method in BPHZ. Second, the local terms in BPHZ are prefixed through the Taylor expansion of the amplitudes, a crucial technical point, while in the differential equation approach the local terms are *to be fixed physically*. Third, the BPHZ ends up with the introduction of infinite bare quantities while there is no room in principle for such infinite quantities at all if one adopts the underlying theory standpoint. Fourth, the application of BPHZ (and other conventional programs) in nonperturbative circumstances is rather involved, which might preclude any useful (or trustworthy) predictions [24], while the differential equation approach makes the calculation easier and the physical predictions more accessible [24].

In fact, one often relies on a good regularization method to make the subtraction simpler, e.g., dimensional regularization for gauge theories. Recently, it has been shown that in dimensional regularization some subtraction is done implicitly without introducing counterterms [20]. That is, we rely heavily upon the regularization method, which could discard divergences ''invisibly.'' If one disregards the underlying theory point of view, in which there is no divergence but there are ambiguities, then there seems to be no good reason to prefer the regularization methods that *simply discard some of the divergences without subtraction*. For example, the modified minimal subtraction in dimensional regularization does not lead to useful predictions in the nonperturbative applications of the effective-field-theory method [13] to nucleon interactions [15], which is followed by the works that employ unconventional renormalization methods [14,17,30]. Applying the underlying-theory-based differential equation approach, will make the problem easier to resolve [31].

B. Relevance of the fixing of the local ambiguities

Now we arrive at the following general form of the effective potential with unknown constants to be fixed:

$$V_{(2l)}^{(\mathrm{PDE})}(\Omega) = \Omega^4\left\{\frac{1}{144\lambda}+\frac{\hat{L}-\frac{3}{2}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}[(\hat{L}-1)^2+2(\hat{L}-2)^2-2(c_1-1)^2+2+4c_1^\theta]\right\} + \Omega^2\left\{\frac{c_2}{2(4\pi)^2}+\frac{6\lambda(c_2[\hat{L}-1]+2c_2^\theta)}{(4\pi)^4}\right\}, \tag{60}$$

where $\hat{L}=\ln(\Omega^2/\mu_{\mathrm{PDE}}^2)+c_1$ and all the ϕ-independent constant terms are discarded as they are irrelevant to our discussions here. Naive dimensional analysis tells us that we have three-dimensional constants, μ_{PDE}^2, c_2, and c_2^θ, and two dimensionless constants, c_1 and c_1^θ. In all the conventional prescriptions, the terms quadratic in Ω are discarded somehow: In dimensional regularization, it is done due to the vanishing (the ''invisible'' subtraction) of power divergences, while in cutoff regularization it is just subtracted away by counterterms. Here we must fix it via sound physical arguments.

We may expect that there should be at least a fundamental scale to characterize the *quantum* fluctuations of the scalar field. As we are mainly concerned with the symmetry-breaking solution, we temporarily take the vacuum energy density to play the role of the fundamental scale. Generally speaking, all the three-dimensional constants should be of the same order of magnitude were they not zero. Then the signs and magnitudes of c_2, c_2^θ, c_1, and c_1^θ will be crucial to the existence of symmetry-breaking solutions. c_1 and c_1^θ can be put into one constant α as this will not change the problem. Then Eq. (60) becomes

$$V_{(2l)}^{(\mathrm{PDE})}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{\hat{L}-\frac{3}{2}}{(8\pi)^2}+\frac{3\lambda}{(4\pi)^4}\right.$$
$$\left.\times[(\hat{L}-1)^2+2(\hat{L}-2)^2+\alpha]\right\}$$
$$+\Omega^2\left\{\frac{c_2}{2(4\pi)^2}+\frac{6\lambda(c_2[\hat{L}-1]+2c_2^{\theta})}{(4\pi)^4}\right\}. \tag{61}$$

With the presence of c_2 and c_2^{θ}, we will find that no matter what number we assign to α, there might be symmetry-breaking in this effective potential provided the c_2 and c_2^{θ} are appropriately chosen, say, $c_2>0, c_2^{\theta}=0$. This is because when Ω becomes vanishingly small, the potential reduces to

$$V_{(2l)}^{(\mathrm{PDE})}(\Omega)\sim c_2\frac{6\lambda\Omega^2\hat{L}}{(4\pi)^4}, \tag{62}$$

where $\phi=0$ is a local maximum (clear evidence of symmetry-breaking), which is true even if c_2^{θ} is not zero as long as it is not too large compared to c_2. Of course, if we set both c_2 and c_2^{θ} equal to zero, then α will determine the existence of symmetry-breaking solutions.

1. The rescaling insensitivity requirement and fine tuning

The most important point is that if one adopts a fixing prescription so that the quadratic terms are present, then we can in no way remove them by redefinition of the coupling constant (and perhaps ϕ) without altering the symmetry-breaking status. That means the fixing schemes with quadratic terms are at least inequivalent to those without. As the underlying theory is still unknown, we have to resort to experimental or other physical means to fix them. Of course for such an unrealistic model, experimental data are unavailable, thus we need to search for physical arguments. In the absence of obvious good clues to use, a tentative argument might be that, due to the presence of the dimensional constants c_2 and c_2^{θ} as the coefficient of the quadratic terms, the effective potential would be rather sensitive to the rescaling of these dimensional constants, in contrast to the relatively milder rescaling behavior described by the logarithmic dependence upon the dimensional constant μ_{PDE}^2. Then for the ''low-energy'' effective potential to be less sensitive to the rescaling of the underlying details, we must fix the dimensional constants c_2 and c_2^{θ} to be zero.

One might argue that this is just the unnatural fine tuning. If the differential equation approach is taken as another way to ''renormalize'' QFT's, this is true. However, if we adopt the underlying theory point of view, we feel that this is a very natural argument. This is because in the underlying-theory-based differential equation there are no divergences to be subtracted but only ambiguities to be fixed (a big improvement). Then setting these dimensional constants equal to zero based on the insensitivity argument is just like what we usually do in solving the Laplace equation or the Schrödinger equation, namely imposing sound boundary conditions. Thus the underlying theory and differential equation approach offer a new way to understand the vanishing of the quadratic divergence: not from the symmetry argument but from the insensitivity of the effective theories' quantities to the rescaling of the underlying structures (represented by the arbitrary constants).

2. Relevance of dimensionless constant(s)

Of course there might be other possibilities. We will no longer investigate this topic here. Now let us temporarily adopt the rescaling insensitivity requirement and focus on the other constants in the effective potential, i.e., μ_{PDE}^2 and α in the following form of the effective potential:

$$V_{(2l)}^{(\mathrm{PDE})}(\Omega)=\Omega^4\left\{\frac{1}{144\lambda}+\frac{\tilde{L}-1/2}{(8\pi)^2}\right.$$
$$\left.+\frac{3\lambda}{(4\pi)^4}[\tilde{L}^2+2(\tilde{L}-1)^2+\alpha]\right\} \tag{63}$$

where $\tilde{L}=\ln(\Omega^2/\mu_{\mathrm{PDE}}^2)-1$. Now since α is dimensionless and μ_{PDE}^2 only appears in the logarithmic functions, the rescaling insensitivity requirement is basically satisfied (which is just the variant form of renormalization-group invariance). However, this requirement does not automatically avoid the additional ''sensitivity'' to the definition of the dimensionless constant α (or c_1 and c_1^{θ}). The reason for this has already been given in Sec. II.

3. Nonexistence of asymptotic freedom

As a by-product we can determine whether the UV fixed point could be zero within the two loop effective potential. Here is the reasoning. In order to get the asymptotic free solution, i.e., $\hat{\lambda}_{\mathrm{cr}}=0$, it is clear from Eq. (28) that we must require the constant α to be infinitely large,

$$\frac{4\pi^2}{\sqrt{4-36\alpha-27}-1}=0\rightarrow\alpha=-\infty. \tag{64}$$

This is in fact a divergent constant. No sensible renormalization prescription could allow for such a divergent number. If one adopts the underlying theory point of view, it is also an unacceptable choice of definition. Otherwise, it might imply that the underlying structures do not decouple with the ''low-energy'' effective theories. Therefore, we conclude that the UV fixed point at the two loop level cannot be zero, i.e., the solution cannot be an asymptotic one, if we accept the parametrization Eq. (63) or Eq. (15). Generally the magnitude of α should be of order not much bigger than 10^2; then the magnitude of the UV fixed point value of λ should be around $4\pi^2/60\sim 0.6$, that is, roughly of the order 1, which means that the broken phase cannot be a weak coupling one even in the high-energy region.

V. DISCUSSIONS AND SUMMARY

To recapitulate, in Secs. II and III, we made use of the well-known two loop calculations to search for the symmetry-breaking solutions. Our results here are new in two respects. (i) First, the striking prescription dependence of the nonperturbative framework differs from that of the standard perturbative framework [22], in other words, the perturbative scheme dependence pattern is no longer valid in nonperturbative contexts. Thus we can understand the relevance of the prescription found in nonperturbative applications as in Refs. [14–17] and [24]; (ii) Second, we found (in a number of renormalization prescriptions) that the massless $\lambda\phi^4$ model could also allow for a totally (nonperturbative) strong coupling dynamics regime with a negative beta function (SCRDSB), and therefore could be nontrivial, at least in the two loop effective potential.

Although this phenomenon (SCRDSB) is only discovered in the two loop effective potential, we found that there is at least one thing in common with the one loop case, namely the existence of a nontrivial phase of dynamics with broken symmetry that is strongly coupled at least in the IR region. Considering the new kind of diagrams beginning to appear from the two loop level (the sunset diagram, etc.), such ‘‘consensus’’ is conspicuous. We think the nontrivial solution might persist after including still higher-order contributions, with the running behaviors being more complicated, perhaps with more stringent constraints on the scheme choices.

As far as the two loop effective potential is concerned, it is very difficult to define asymptotic final states, thus the scalar field theories with quartic interaction is rather different from the gauge theories: it may have a broken phase that exists entirely in the strong-coupling regime. Thus such scalar field theories with quartic interactions might not permit the elementary scalar fields to appear in the final asymptotic states. This scenario might be of certain reference value to Higgs physics.

Another main task that has been performed is that we reanalyzed the loop diagrams from the underlying theory point of view, which takes all the presently known QFT's that suffer UV ill-definedness to be ill-defined formulations of the effective ‘‘low-energy’’ sectors. Then we showed clearly that the prescriptions or choices for fixing the local ambiguities *are* relevant to physical properties encoded in the nonperturbative effective potential, especially for the quadratic terms. In contrast to the conventional regularization and renormalization programs where power divergences are present and must be subtracted carefully (fine tuning), in the underlying theory understanding, we can fix them to be zero under the insensitivity requirement. This is a natural procedure that is usually done in electrodynamics and quantum mechanics, i.e., imposing appropriate boundary conditions on the solutions obtained from the Laplace or Schrödinger equation. In this way, we arrive at a new understanding of the naturalness problem.

Furthermore, we also showed that there could not be reasonable prescriptions that would allow for asymptotic freedom in the broken phase as long as the two loop effective potential is considered. In the underlying theory understanding, this is also true.

Since more efforts in the realistic model are needed, we will refrain here from making further comments. Our only aim here is to call attention to the reexamination of our triviality conviction about the $\lambda\phi^4$ model and to the investigation of its new nonperturbative properties (the perturbative regime is unavoidably trivial).

In summary, we reconsidered the massless $\lambda\phi^4$ model with Z_2 symmetry and found that at two loop level the nonperturbative effective potential's predictability of symmetry breaking depends upon the renormalization prescriptions in use. The prescription used by Coleman and Weinberg in their pioneering work [10] was shown to be incompatible with symmetry-breaking in the two loop effective potential, while the modified minimal subtraction in dimensional regularization, Jackiw's prescription, and others were shown to be able to accommodate the symmetry-breaking solution in the two loop effective potential. The reason for the relevance of the prescriptions in nonperturbative contexts was given. The potential was also recalculated and reanalyzed in a differential equation approach based on the standard point of view that a complete theory underlies all the QFT's that suffer UV divergences. The relevance of the prescriptions for fixing the local ambiguities was stressed and the rationality of this approach was highlighted.

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APPENDIX A

In this appendix, we write down all the needed one loop integrals calculated, respectively, in the dimensional, cutoff, and differential equation approach:

$$\mu^{2\epsilon}I_0^{(D)}(\Omega)=\int\frac{\mu^{2\epsilon}d^Dk}{(2\pi)^D}\ln\left(1+\frac{\Omega^2}{k^2}\right)=-\frac{\Omega^4\Gamma(1+\epsilon)}{(4\pi)^2\epsilon(1-\epsilon)(2-\epsilon)}\left(\frac{4\pi\mu^2}{\Omega^2}\right)^{\epsilon}, \tag{A1}$$

$$\mu^{2\epsilon}I_1^{(D)}(\Omega)=\int\frac{\mu^{2\epsilon}d^Dk}{(2\pi)^D}\frac{1}{k^2+\Omega^2}$$
$$=-\frac{\Omega^2\Gamma(1+\epsilon)}{(4\pi)^2\epsilon(1-\epsilon)}\left(\frac{4\pi\mu^2}{\Omega^2}\right)^{\epsilon}, \quad \text{(A2)}$$

$$I_0^{(\Lambda)}(\Omega)=\int_\Lambda\frac{d^4k}{(2\pi)^4}\ln\left(1+\frac{\Omega^2}{k^2}\right)$$
$$=\frac{1}{2(4\pi)^2}\left\{\Lambda^4\ln\frac{\Lambda^2+\Omega^2}{\Lambda^2}-\Omega^4\ln\frac{\Lambda^2+\Omega^2}{\Omega^2}+\Lambda^2\Omega^2\right\}, \quad \text{(A3)}$$

$$I_1^{(\Lambda)}(\Omega)=\int_\Lambda\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\Omega^2}$$
$$=\frac{1}{(4\pi)^2}\left\{-\Omega^2\ln\frac{\Lambda^2+\Omega^2}{\Omega^2}+\Lambda^2\right\}, \quad \text{(A4)}$$

$$I_0(\Omega)=\frac{1}{(4\pi)^2}\left\{\frac{\Omega^4}{2}\left[\ln\frac{\Omega^2}{\mu^2_{\rm PDE}}+c_1-3/2\right]+c_2\Omega^2+c_3\right\}, \quad \text{(A5)}$$

$$I_1(\Omega)=\frac{1}{(4\pi)^2}\left\{\Omega^2\left[\ln\frac{\Omega^2}{\mu^2_{\rm PDE}}+c_1-1\right]+c_2\right\}. \quad \text{(A6)}$$

Note that in the differential equation approach there appear unknown constants in the integrals parametrizing the ill definedness. The constants should be fixed by "boundary conditions" as discussed above.

APPENDIX B

In this section, we describe the μ_Λ^2 scheme that mimics the $\overline{\rm MS}$ scheme in dimensional regularization, i.e., we merely subtract the cutoff-containing parts. Let us demonstrate it with the sunset diagram.

First let us list all the relevant integrals or diagrams:

$$I_1^{(\Lambda)}(\Omega)=\frac{1}{(4\pi)^2}\left\{-\Omega^2\ln\frac{\Lambda^2+\Omega^2}{\Omega^2}+\Lambda^2\right\}$$
$$=\frac{1}{(4\pi)^2}\left\{\Lambda^2-\Omega^2\ln\frac{\Lambda^2}{\Omega^2}+o(\Lambda^{-2})\right\}, \quad \text{(B1)}$$

$$I^\Lambda_{(1;1)}(\Omega,k^2)=\frac{1}{(4\pi)^2}\left(\int_0^1 dx\,\ln\frac{\Lambda^2}{\Omega^2+x(1-x)k^2}-1\right)+o(\Lambda^{-2}), \quad \text{(B2)}$$

$$I_2^{(\Lambda)}(\Omega)=\frac{1}{(4\pi)^4}\left\{2\Lambda^2-\frac{3\Omega^2}{2}\ln^2\frac{\Omega^2}{\Lambda^2}+3\Omega^2\ln\frac{\Omega^2}{\Lambda^2}+o(\Lambda^{-2})\right\}. \quad \text{(B3)}$$

The counterterm for subdivergence in $I_2^{(\Lambda)}(\Omega)$ comes from the log in $I^\Lambda_{(1;1)}(\Omega,k^2)$ containing Λ as an argument, i.e., from $\ln(\Lambda^2/\mu^2)$ together with factors from graph topology and angular integration. Thus the counterterm for the sunset diagram reads

$$\text{c.t.}(1)=\frac{(12\lambda)^2}{(4\pi)^2}\phi^2\left(\ln\frac{\Lambda^2}{\mu^2}\right)\times I_1^{(\Lambda)}(\Omega)$$
$$=\frac{12\lambda\Omega^2}{(4\pi)^4}\left\{\Omega^2\ln\frac{\Lambda^2}{\mu^2}\ln\frac{\Omega^2}{\Lambda^2}+\Lambda^2\ln\frac{\Lambda^2}{\mu^2}\right\}. \quad \text{(B4)}$$

Here we selected an arbitrary scale to balance the dimension in the argument of log. After removing the subdivergence, we get for the sunset diagram

$$-48\lambda^2\phi^2I_2^{(\Lambda)}(\Omega)+\text{c.t.}(1)=\frac{6\lambda\Omega^4}{(4\pi)^4}\left\{\left(\ln\frac{\Omega^2}{\Lambda^2}-1\right)^2-1\right\}-\frac{8\lambda\Lambda^2\Omega^2}{(4\pi)^4}+\frac{12\lambda\Omega^2}{(4\pi)^4}\left\{\Omega^2\ln\frac{\Lambda^2}{\mu^2}\ln\frac{\Omega^2}{\Lambda^2}+\Lambda^2\ln\frac{\Lambda^2}{\mu^2}\right\}$$
$$=\frac{6\lambda\Omega^4}{(4\pi)^4}\left\{\left(\ln\frac{\Omega^2}{\mu^2}\right)^2-2\ln\frac{\Omega^2}{\mu^2}\right\}+\frac{6\lambda\Omega^4\ln\frac{\Lambda^2}{\mu^2}}{(4\pi)^4}\left\{2-\left(\ln\frac{\Lambda^2}{\mu^2}\right)\right\}+\frac{6\lambda\Lambda^2\Omega^2}{(4\pi)^4}\ln\frac{\Lambda^2}{\mu^2}-\frac{8\lambda\Lambda^2\Omega^2}{(4\pi)^4}. \quad \text{(B5)}$$

Now we see all remaining divergences are purely ''local'' and can be removed through introducing a second counterterm c.t.(2),

$$\mathrm{c.t.}(2) = -\frac{6\lambda\Omega^4 \ln\frac{\Lambda^2}{\mu^2}}{(4\pi)^4}\left\{2-\left(\ln\frac{\Lambda^2}{\mu^2}\right)\right\} - \frac{6\lambda\Lambda^2\Omega^2}{(4\pi)^4}\ln\frac{\Lambda^2}{\mu^2} + \frac{8\lambda\Lambda^2\Omega^2}{(4\pi)^4}, \tag{B6}$$

which contains no finite part, and the renormalized sunset diagram now takes the following form:

$$\begin{aligned}[-48\lambda^2\phi^2 I_2^{(\Lambda)}(\Omega)]^{(\mu_\Lambda^2)} &\equiv -48\lambda^2\phi^2 I_2^{(\Lambda)}(\Omega) + \mathrm{c.t.}(1) + \mathrm{c.t.}(2) \\ &= \frac{6\lambda\Omega^4}{(4\pi)^4}\left\{\left(\ln\frac{\Omega^2}{\mu^2}\right)^2 - 2\ln\frac{\Omega^2}{\mu^2}\right\}.\end{aligned} \tag{B7}$$

APPENDIX C

In this appendix, we verify that even in the original parametrization the Coleman-Weinberg scheme is still incompatible with the DSB solution. The two loop effective potential in Ref. [18] reads

$$\tilde{V}_{(2l)}^{(CW)} = \frac{\lambda\phi^4}{4!}\left(1 + a\lambda l + a^2\lambda^2 l^2 + b\lambda^2 l + a^2\lambda^2\frac{205}{36}\right),$$

$$a = \frac{3}{32\pi^2}, \quad b = \frac{-3}{4(4\pi)^4}, \quad l = \ln\frac{\phi^2}{M^2} - \frac{25}{6}. \tag{C1}$$

From the first-order condition, we find the following equation:

$$2\left(1 + a\lambda l + a^2\lambda^2 l^2 + b\lambda^2 l + a^2\lambda^2\frac{205}{36}\right) + a\lambda + 2a^2\lambda^2 l + b\lambda^2 = 0, \tag{C2}$$

that is,

$$2a^2\lambda^2 l^2 + (2a\lambda + 2b\lambda^2 + 2a^2\lambda^2)l + 2 + a\lambda + b\lambda^2 + a^2\lambda^2\frac{205}{18} = 0. \tag{C3}$$

The corresponding delta reads

$$\begin{aligned}\tilde{\Delta} &\equiv (2a\lambda + 2b\lambda^2 + 2a^2\lambda^2)^2 - 8a^2\lambda^2\left(2 + a\lambda + b\lambda^2 + a^2\lambda^2\frac{205}{18}\right) \\ &= -\left(\frac{3\lambda}{16\pi^2}\right)^2\left\{3 + \frac{\lambda}{16\pi^2} + \frac{195\lambda^2}{4(4\pi)^4}\right\} < 0.\end{aligned} \tag{C4}$$

This inequality implies the incompatibility of the Coleman-Weinberg scheme with symmetry-breaking at two loop level.

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