Five-loop vacuum energy β function in ϕ^4 theory with O(N)-symmetric and cubic interactions

Boris Kastening
Albert-Ludwigs-Universität Freiburg, Fakultät für Physik, Hermann-Herder-Straße 3, D-79104 Freiburg, Germany

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The beta function of the vacuum energy density is analytically computed at the five-loop level in O(N)-symmetric ϕ^4 theory, using dimensional regularization in conjunction with the $\overline{\text{MS}}$ scheme. The result for the case of cubic anisotropy is also given. It is pointed out how to also obtain the beta function of the coupling and the gamma function of the mass from vacuum graphs. This method may be easier than traditional approaches. [S0556-2821(98)05506-4]

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I. INTRODUCTION

In this article, we extend earlier work [1], where the beta function β_v of the vacuum energy density in O(N)-symmetric ϕ^4 theory was computed at the four-loop level, to five loops. Integrals are dimensionally regulated and divergences removed by modified minimal subtraction ($\overline{\text{MS}}$). For motivations for this work, see [1]. We employ two different methods to arrive at our result for the O(N)-symmetric case. In Sec. II, we use the scheme from [1] to determine the five-loop contributions to the vacuum energy renormalization constant Z_v and to β_v. All necessary subtractions result from consistency requirements while renormalizing the vacuum energy. As explained in [1], this yields as a by-product γ_m through four loops and β_g through three loops. In Sec. III, we check the relevant recursion relations for the contributions to Z_v. In Sec. IV, we give β_v through five loops for the O(N)-symmetric case. In Sec. V, we use a more traditional approach to check our result for β_v. In Sec. VI, we extend it for the case of an additional cubic interaction. Our work brings the evaluation of β_v on a par with that of the other beta and gamma functions in ϕ^4 theory (see [2,3] and references therein).

For definitions and conventions, the reader is referred to the detailed article in [1]. The only exception is β_v itself, which we define here as

$$\beta_{v,\epsilon}(g,\epsilon)=\frac{\mu^{2+\epsilon}}{m^4}\left[\frac{\partial}{\partial\mu^2}\left(\frac{m^4h}{\mu^\epsilon g}\right)\right]_B, \tag{1}$$

where the subscript B indicates that bare quantities are kept fixed. I.e. β_v and $\beta_{v,\epsilon}$ in this article are just β_v and $\beta_{v,\epsilon}$ from [1] divided by the renormalized coupling g. The connection to the constant $Z_v=1+h^{-1}\sum_{k=1}^{\infty}Z_{v,k}(g)/\epsilon^k$, which renormalizes the vacuum energy, is now

$$\beta_{v,\epsilon}(g,\epsilon)=\beta_v(g)=\frac{1}{2}Z'_{v,1}. \tag{2}$$

Our revised definition is more natural since even in the noninteracting theory, i.e. for $g=0$, the vacuum energy density acquires a divergent part from the zero-point fluctuations of the uncoupled harmonic oscillators. It has to be renormalized and therefore runs with μ, such that

$$\beta_v=\frac{N}{4} \tag{3}$$

for the free theory. E.g., the renormalization group equation for the effective potential in $d=4-\epsilon$ dimensions for $h=0$ reads now [1]

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta_{g,\epsilon}\frac{\partial}{\partial g}+\gamma_{m,\epsilon}m^2\frac{\partial}{\partial m^2}-\gamma_{\phi,\epsilon}\phi^2\frac{\partial}{\partial\phi^2}\right]V_\epsilon(g,m^2,\phi^2,h=0,\mu^2)=-\frac{\beta_{v,\epsilon}m^4}{(4\pi)^2\mu^\epsilon}. \tag{4}$$

For the free theory, we have $\beta_{g,\epsilon}=\gamma_{m,\epsilon}=\gamma_{\phi,\epsilon}=0$ and, going to four dimensions, the $\overline{\text{MS}}$ vacuum energy

$$V(g=0,m^2,\phi^2=0,h=0,\mu^2)=\frac{Nm^4}{4(4\pi)^2}\left(\ln\frac{m^2}{\overline{\mu^2}}-\frac{3}{2}\right) \tag{5}$$

with the $\overline{\text{MS}}$ renormalization scale

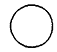
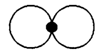
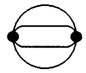
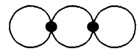
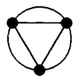
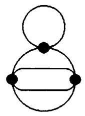
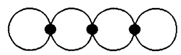
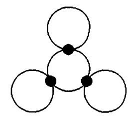
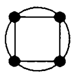
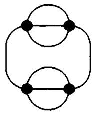
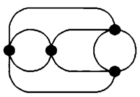
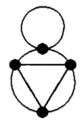
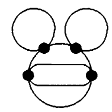
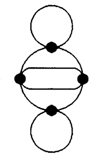
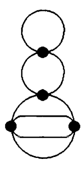
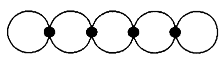
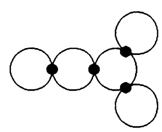
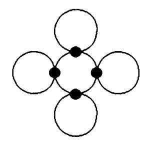
$$\overline{\mu^2}\equiv 4\pi\mu^2e^{-\gamma_E} \tag{6}$$

is immediately seen to satisfy Eq. (4) with β_v given by Eq. (3) with no spurious appearances of a coupling.

II. FIVE-LOOP CONTRIBUTION TO Z_v AND β_v

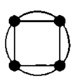
In this section, we determine the five-loop contributions to Z_v and to β_v in the O(N)-symmetric model. We proceed in the same way as in [1]: We employ modified Feynman rules, where the one-loop mass correction is absorbed into a modified mass in the propagator. This reduces the full set of diagrams in Table I to the reduced set in Table II, where we have also included the six-loop diagrams, whose evaluation we save for a later day. We arrange that the appearances of the wave function renormalization constant Z_ϕ in the propa-

TABLE I. Vacuum diagrams through five loops and their symmetry factors. In the equations in the text the symmetry factor is considered part of each respective diagram.

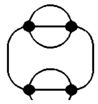
number of loops	order in g	diagrams and symmetry factors
0	g^{-1}	1 $\bullet$
1	g^{0}	$\frac{1}{2}$ 
2	g^{1}	$\frac{1}{8}$ 
3	g^{2}	$\frac{1}{48}$  $\frac{1}{16}$ 
4	g^{3}	$\frac{1}{48}$  $\frac{1}{24}$  $\frac{1}{32}$  $\frac{1}{48}$ 
5	g^{4}	$\frac{1}{128}$  $\frac{1}{144}$  $\frac{1}{32}$  $\frac{1}{16}$  $\frac{1}{48}$  $\frac{1}{32}$  $\frac{1}{48}$  $\frac{1}{64}$  $\frac{1}{32}$  $\frac{1}{128}$ 

gator and coupling cancel from the outset. There are no extra counterterm rules, since all counterterms are already contained in the Feynman rules for vacuum energy, propagator and coupling; i.e. we use bare values for vacuum energy, mass and coupling in the Feynman rules and expand our results for diagrams to the necessary order in the renormalized coupling g. For this purpose, we have to formally construct the renormalization constants Z_m, $Z_{\bar{m}}$ and Z_g, for mass, modified mass and coupling, through four, four and three loops, respectively, from γ_m and β_g [1].

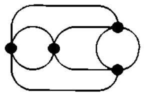
The calculation through four loops is detailed in [1]. Continuing to five loops, we keep all divergent terms in zero through five loops through order g^4. The relevant five-loop diagrams are



$$= \frac{N(N+2)(N^2+6N+20)}{10368}(4\pi)^8 g^4 Z_g^4 Z_{\bar{m}}^{2-\frac{5}{2}\epsilon} I_{5a}\,, \tag{7}$$



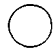
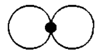
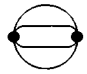
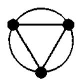
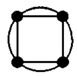
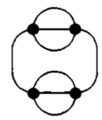
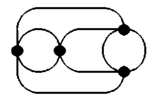
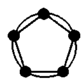
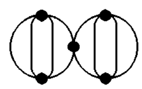
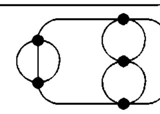
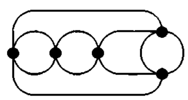
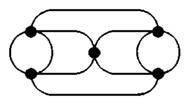
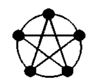
$$= \frac{N(N+2)^2}{1296}(4\pi)^8 g^4 Z_g^4 Z_{\bar{m}}^{2-\frac{5}{2}\epsilon} I_{5b}\,, \tag{8}$$



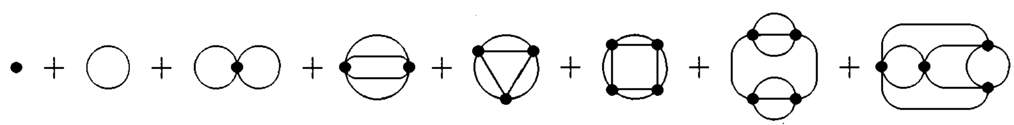
$$= \frac{N(N+2)(5N+22)}{2592}(4\pi)^8 g^4 Z_g^4 Z_{\bar{m}}^{2-\frac{5}{2}\epsilon} I_{5c}\,, \tag{9}$$

where I_{5a}, I_{5b} and I_{5c} are defined, their evaluation sketched and their results given in Appendix A. Using these, the formally

TABLE II. Remaining diagrams through six-loop order after absorption of the one-loop mass correction into a modified mass in the propagator, i.e. after a careful resummation of the quadratic part of the Lagrange density.

number of loops	order in g	remaining diagrams and revised symmetry factors
0	g^{-1}	1 •
1	g^0	$\frac{1}{2}$ 
2	g^1	$-\frac{1}{8}$ 
3	g^2	$\frac{1}{48}$ 
4	g^3	$\frac{1}{48}$ 
5	g^4	$\frac{1}{128}$  $\frac{1}{144}$  $\frac{1}{32}$ 
6	g^5	$\frac{1}{320}$  $\frac{1}{288}$  $\frac{1}{48}$  $\frac{1}{32}$  $\frac{1}{16}$  $\frac{1}{120}$ 

reconstructed Z_m, $Z_{\bar{m}}$ and Z_g, as well as the results of [1], one gets



$$
\begin{aligned}
&= -\frac{m^4}{(4\pi)^2 g}\Big\{h \\
&\quad +\Big[Z^v_{15} + \frac{5N(N+2)}{144}\Big(\beta_3 - \frac{33N^2+922N+2960+96(5N+22)\zeta(3)}{432}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)^2 \\
&\quad +\frac{N}{4}\Big(\alpha_4 - \frac{(N+2)[N^2-7578N-31060-48(3N^2+10N+68)\zeta(3)-288(5N+22)\zeta(4)]}{15552}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big) \\
&\qquad +\frac{N(N+2)}{622080}\Big(627N^2-51124N-224192+4320\beta_3 \\
&\qquad\qquad +4320\Big(\beta_3-\frac{33N^2+922N+2960+96(5N+22)\zeta(3)}{432}\Big)\zeta(2) \\
&\qquad\qquad -(144N^2-13536N-60480)\zeta(3)-(1152N^2+11232N+42048)\zeta(4) \\
&\qquad\qquad +(15360N+67584)\zeta(5)\Big)\Big]\frac{g^5}{\epsilon} \\
&\quad +\Big[Z^v_{25} - \frac{N(N+2)}{36}\Big(\beta_3 - \frac{33N^2+922N+2960+96(5N+22)\zeta(3)}{432}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big) \\
&\quad -\frac{N}{2}\Big(\alpha_4 + \frac{(N+2)[-501N^2+226N+6404+96(4N^2+39N+146)\zeta(3)+288(5N+22)\zeta(4)]}{77760}\Big)\Big]\frac{g^5}{\epsilon^2} \\
&\quad +\Big[Z^v_{35} - \frac{N(N+2)}{18}\Big(\beta_3 + \frac{873N^2+12314N+35296+96(5N+22)\zeta(3)}{2160}\Big)\Big]\frac{g^5}{\epsilon^3} \\
&\quad +\Big[Z^v_{45} + \frac{N(N+2)(293N^2+2624N+5840)}{4860}\Big]\frac{g^5}{\epsilon^4} \\
&\quad +\Big[Z^v_{55} - \frac{N(N+2)(N+4)(N+5)(5N+28)}{810}\Big]\frac{g^5}{\epsilon^5} + \mathcal{O}(g^6,\epsilon^0)\Big\}.
\end{aligned} \tag{10}
$$

Demanding the cancellation of logarithmic terms gives the three-loop coefficient of β_g and the four-loop coefficient of γ_m,

$$\beta_3=\frac{33N^2+922N+2960+96(5N+22)\zeta(3)}{432}, \tag{11}$$

$$\alpha_4=\frac{(N+2)[N^2-7578N-31060-48(3N^2+10N+68)\zeta(3)-288(5N+22)\zeta(4)]}{15552}, \tag{12}$$

which coincide with known results (see, e.g., [2,3]). Demanding subsequently Eq. (10) to be finite as $\epsilon\to 0$ gives

$$Z_{15}^v=\frac{N(N+2)[-319N^2+13968N+64864+16(3N^2-382N-1700)\zeta(3)+96(4N^2+39N+146)\zeta(4)-1024(5N+22)\zeta(5)]}{207360}, \tag{13}$$

$$Z_{25}^v=-\frac{N(N+2)[31N^2+2354N+9306+3(7N^2-28N+48)\zeta(3)+72(5N+22)\zeta(4)]}{9720}, \tag{14}$$

$$Z_{35}^v=\frac{N(N+2)[519N^2+8462N+25048+288(5N+22)\zeta(3)]}{19440}, \tag{15}$$

$$Z_{45}^v=-\frac{N(N+2)(293N^2+2624N+5840)}{4860}, \tag{16}$$

$$Z_{55}^v=\frac{N(N+2)(N+4)(N+5)(5N+28)}{810}. \tag{17}$$

The relation (2) then gives us the five-loop coefficient of $\beta_v=\Sigma_{k=1}^{\infty}\delta_k g^{k-1}$ through $\delta_5=5Z_{15}^v/2$.

III. RECURSION RELATIONS FOR THE Z_{kl}^v

The relevant recursion relations among the renormalization constants are [1]

$$Z_{25}^v=\frac{1}{5}[(2Z_{11}^m+3Z_{11}^g)Z_{14}^v+2(2Z_{12}^m+2Z_{12}^g)Z_{13}^v+3(2Z_{13}^m+Z_{13}^g)Z_{12}^v+4(2Z_{14}^m)Z_{11}^v], \tag{18}$$

$$Z_{35}^v=\frac{1}{5}[(2Z_{11}^m+3Z_{11}^g)Z_{24}^v+2(2Z_{12}^m+2Z_{12}^g)Z_{23}^v+3(2Z_{13}^m+Z_{13}^g)Z_{22}^v], \tag{19}$$

$$Z_{45}^v=\frac{1}{5}[(2Z_{11}^m+3Z_{11}^g)Z_{34}^v+2(2Z_{12}^m+2Z_{12}^g)Z_{33}^v], \tag{20}$$

$$Z_{55}^v=\frac{1}{5}(2Z_{11}^m+3Z_{11}^g)Z_{44}^v. \tag{21}$$

Using the results from [1], it is straightforward to check that all of the above relations hold.

IV. β_v THROUGH FIVE LOOPS

Combining our result for δ_5 with the four-loop result from [1], we arrive at the five-loop result for the O(N)-symmetric case,

$$\begin{aligned}\beta_v(g)=&\frac{N}{4}+\frac{N(N+2)}{96}g^2+\frac{N(N+2)(N+8)[12\zeta(3)-25]}{1296}g^3\\&+\frac{N(N+2)[-319N^2+13968N+64864+16(3N^2-382N-1700)\zeta(3)+96(4N^2+39N+146)\zeta(4)-1024(5N+22)\zeta(5)]}{82944}g^4\\&+\mathcal{O}(g^5).\end{aligned} \tag{22}$$

We note that among the beta and gamma functions, β_v is the easiest to compute. Therefore, it would be the prime candidate for the first complete non-trivial six-loop calculation, since only the last six diagrams in Table II have to be computed, of which the first three are essentially trivial. After converting the quartically divergent integrals into logarithmically divergent ones by twice differentiating with respect to m^2 (see Appendix A), a subset of the diagrams necessary for the six-loop renormalization of the coupling has to be evaluated. Since this subset consists of the diagrams where the four external lines are attached to only two vertices, the most difficult topologies are absent and the computation should be considerably more easy than the full coupling renormalization at this level. A similar statement is true for the comparison with mass and wave function renormalization.

We further note that the k-loop coefficients α_k of γ_m we get as by-products here and in [1] come from $\epsilon^{-1}\ \ln(m^2/\overline{\mu}^2)$ terms—and therefore originally from ϵ^{-2} terms—of $(k+1)$-loop vacuum diagrams. Similarly, the k-loop coefficients β_k of β_g we get come from $\epsilon^{-1}\ \ln^2(m^2/\overline{\mu}^2)$ terms—and therefore originally from ϵ^{-3} terms—of $(k+2)$-loop vacuum diagrams. These may be easier to compute than the ϵ^{-1} terms of generic k-loop diagrams contributing to β_k and α_k. In particular, and in contrast to [9], we do not have to consider the topology of eight-loop vacuum diagrams to obtain the five-loop contribution to β_g, but only to compute the $1/\epsilon^3$ terms of seven-loop vacuum diagrams. To obtain the five-loop contribution to γ_m, we have to to compute the $1/\epsilon^2$ terms of six-loop vacuum diagrams.

However, in the current approach of evaluating loop integrals (to be described in Appendix A), the divergences related to the two-point function at one lower loop (and therefore contributing to γ_m at one lower loop) and to the four-point function at two lower loops (and therefore contributing to β_g at two lower loops) than the considered vacuum graph integrals need to be evaluated as subdivergences of these vacuum graphs anyway. Therefore, when evaluating the integrals as described in Appendix A, no advantage has been gained for computing γ_m and β_g, at least as far as the necessary computational techniques are concerned.

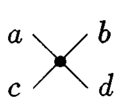
V. β_v AND Z_v FROM THE STANDARD METHOD

Along more traditional lines, we have

$$Z_v = 1 + \frac{g}{hm^4}\ \mathcal{K}\overline{R}\sum \quad \text{vacuum graphs}, \tag{23}$$

where $\overline{R}$ removes subdivergences of graphs and $\mathcal{K}$ isolates the negative powers in ϵ (see [10] for a clear definition of $\mathcal{K}$ and $\overline{R}$). The sum in Eq. (23) goes over the graphs in Table I and the Feynman rules to be used are

$$a \;\text{——}\; b \;=\; \frac{\delta_{ab}}{p^2 + m^2}\,, \tag{24}$$



$$= -[\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}]\,\frac{(4\pi)^2 g}{3} \tag{25}$$

with renormalized quantities m^2 and g, since all subdivergences are removed by the $\overline{R}$ operation. We have carried out this program from one through five loops in order to also check the results in [1]. Using the results of Appendix B, it is easy to see that we get the same Z_v and therefore also β_v as in the approach above.

If a diagram D separates into n diagrams D_k with independent integrations, then the standard definition of $\mathcal{K}\overline{R}$ yields

$$\mathcal{K}\overline{R}D = \mathcal{K}\overline{R}\prod_{k=1}^{n} D_k = (-1)^{n+1}\prod_{k=1}^{n}\ \mathcal{K}\overline{R}D_k\,. \tag{26}$$

Since $\mathcal{K}$ picks out only the pole terms in ϵ, the $1/\epsilon$ terms of Z_v and therefore also β_v receive only contributions from diagrams whose integrations are not separable. Thus, for the calculation of β_v, it is again sufficient to consider the diagrams of Table II. The two-loop diagram is no longer needed, though. However, since, within this program, separable diagrams can be simply put algebraically together from lower-loop diagrams, including them in the calculation provides together with the recursion relations between the Z_x [1] a convenient cross-check in the determination of Z_v and β_v.

VI. CUBIC ANISOTROPY

Here we give β_v for the case of a cubic anisotropy [3,9]. We rename $g \to g_1$ and introduce a second coupling g_2 through

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_{Bi}\partial_\mu\phi_{Bi} + \frac{1}{2}m_B^2\phi_{Bi}\phi_{Bi} + \frac{(4\pi)^2}{4!}(g_{1_B}T^{(1)}_{ijkl} + g_{2_B}T^{(2)}_{ijkl})\phi_i\phi_j\phi_k\phi_l + \frac{m_B^4 h_B}{(4\pi)^2 g_{1_B}}, \tag{27}$$

where repeated indices are summed over ($\mu = 1,\ldots,d$ and $i,j,k,l = 1,\ldots,N$), the subscript B refers to bare quantities and where

$$T^{(1)}_{ijkl} = \frac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{28}$$

$$T^{(2)}_{ijkl} = \delta_{ijkl} \equiv \begin{cases} 1, & i=j=k=l, \\ 0, & \text{else}. \end{cases} \tag{29}$$

Using standard methods [4], one arrives at

$$\beta_v = \frac{1}{2g_1}\left(g_1\frac{\partial Z_{v,1}}{\partial g_1} + g_2\frac{\partial Z_{v,1}}{\partial g_2}\right). \tag{30}$$

The seemingly asymmetric treatment of g_1 and g_2 is entirely due to our definition of the constant term in Eq. (27), since the definitions (28) and (29) are not used for the derivation of Eq. (30). However, it is easy to see that e.g. replacing the constant term in Eq. (27) by $m_B^4 h_B/[(4\pi)^2 g_{2_B}]$ and redefining Z_v accordingly exchanges g_1 and g_2 in Eq. (30), but leaves β_v invariant.

Through five loops, we get

$$\beta_\nu(g_1,g_2)=\frac{N}{4}+\frac{N(N+2)}{96}g_1^2+\frac{N}{16}g_1g_2+\frac{N}{32}g_2^2+\frac{N(N+2)(N+8)[12\zeta(3)-25]}{1296}g_1^3+\frac{N(N+8)[12\zeta(3)-25]}{144}g_1^2g_2+\frac{N[12\zeta(3)-25]}{16}g_1g_2^2$$
$$+\frac{N[12\zeta(3)-25]}{48}g_2^3$$
$$+\frac{N(N+2)[-319N^2+13968N+64864+16(3N^2-382N-1700)\zeta(3)+96(4N^2+39N+146)\zeta(4)-1024(5N+22)\zeta(5)]}{82944}g_1^4$$
$$+\frac{N[-319N^2+13968N+64864+16(3N^2-382N-1700)\zeta(3)+96(4N^2+39N+146)\zeta(4)-1024(5N+22)\zeta(5)]}{6912}g_1^3g_2$$
$$+\frac{N[(6431N+229108)-48(71N+2008)\zeta(3)+864(5N+58)\zeta(4)-3072(N+26)\zeta(5)]}{13824}g_1^2g_2^2$$
$$+\frac{N[26171-11088\zeta(3)+6048\zeta(4)-9216\zeta(5)]}{2304}g_1g_2^3+\frac{N[26171-11088\zeta(3)+6048\zeta(4)-9216\zeta(5)]}{9216}g_2^4, \tag{31}$$

which reduces to Eq. (22) for $(g_1,g_2)\rightarrow(g,0)$.

Note Added. Recently, the result (22) has been confirmed in an independent calculation by S.A. Larin, M. Mönnigmann, M. Strösser and V. Dohm, cond-mat/9711069, where also its application to three-dimensional systems via the ϵ expansion is considered.

ACKNOWLEDGMENTS

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APPENDIX A: FIVE-LOOP INTEGRALS

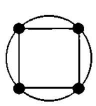
Our strategy for computing the five-loop integrals I_{5a},I_{5b},I_{5c} is different from the way of computing integrals in [1]. Here we take two derivatives of each integral with respect to m^2 to convert it into a sum of logarithmically divergent integrals. Then we subtract subdivergences using the $\mathcal{K}\overline{R}$ operation, so that we can set most masses in propagators in the resulting expression to zero without changing its divergent part. This allows us to do the relevant integrals using the methods of infrared rearrangement [5], (our own modified version of) the R^* operation [6] and the results of [7], being in turn partially based also on the integration-by-parts algorithm [8]. For an introduction to these techniques and the multi-loop renormalization of ϕ^4 theory in general, see [9]. At last, we evaluate the terms containing subdivergences that we have subtracted above and add them again. These terms have less than five loops and can be computed by either recursively continuing this procedure or by using the results from [1].

One might argue that this method of computing integrals is not independent of the standard method referred to in Sec. V. Nevertheless, it provides us with many more cross-checks as described in Secs. II and III.

All diagrams in this section only refer to momentum space integrals without symmetry and group factors. To get the notation in line with Appendix B, we rename the integrals I_2^{cc} and I_3^{cc} from [1] I_{3a} and I_{4a}, respectively, so that their finite parts defined in [1] are now $I_{3a,f}=I_{2,f}^{cc}$ and $I_{4a,f}=I_{3,f}^{cc}$.

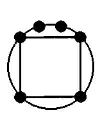
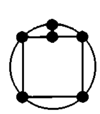
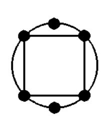
1. I_{5a}

Define I_{5a} by

$I_{5a}=$  $=\int_{kpqrs}\frac{1}{(p^2+m^2)[(k+p)^2+m^2](q^2+m^2)[(k+q)^2+m^2](r^2+m^2)[(k+r)^2+m^2](s^2+m^2)[(k+s)^2+m^2]}.$ (A1)

Since $I_{5a}\propto(m^2)^{5/2\,d-8}=(m^2)^{2-5/2\,\epsilon}$ we can write

$$I_{5a}=\frac{m^4}{(2-\frac{5}{2}\epsilon)(1-\frac{5}{2}\epsilon)}\left(\frac{\partial}{\partial m^2}\right)^2 I_{5a}$$

$=\frac{8m^4}{(2-\frac{5}{2}\epsilon)(1-\frac{5}{2}\epsilon)}\Big[2$  $+$  $+6$  $\Big].$ (A2)

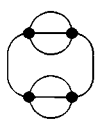
The three integrals on the right hand side can now be evaluated as described above. The result is

$$I_{5a}=\frac{m^4}{(4\pi)^{10}}\Bigg\{\frac{192}{5\epsilon^5}+\frac{1}{\epsilon^4}\Bigg[-96\ln\frac{m^2}{\bar{\mu}^2}+\frac{848}{5}\Bigg]+\frac{1}{\epsilon^3}\Bigg[120\ln^2\frac{m^2}{\bar{\mu}^2}-424\ln\frac{m^2}{\bar{\mu}^2}+\Bigg(\frac{2004}{5}+24\zeta(2)\Bigg)\Bigg]+\frac{1}{\epsilon^2}\Bigg[116\ln^3\frac{m^2}{\bar{\mu}^2}$$
$$-298\ln^2\frac{m^2}{\bar{\mu}^2}+[258+156\zeta(2)]\ln\frac{m^2}{\bar{\mu}^2}+\Bigg(-\frac{317}{5}-170\zeta(2)+\frac{88}{5}\zeta(3)\Bigg)\Bigg]+\frac{1}{\epsilon}\Bigg[\frac{31}{2}\ln^4\frac{m^2}{\bar{\mu}^2}-45\ln^3\frac{m^2}{\bar{\mu}^2}$$
$$+\Bigg(\frac{107}{2}+45\zeta(2)\Bigg)\ln^2\frac{m^2}{\bar{\mu}^2}-\Bigg(\frac{19}{2}+71\zeta(2)+12\zeta(3)\Bigg)\ln\frac{m^2}{\bar{\mu}^2}+\Bigg(-\frac{609}{20}+\frac{59}{2}\zeta(2)+\frac{82}{5}\zeta(3)$$
$$+\frac{141}{5}\zeta(4)+\frac{21}{2}\zeta(2)^2\Bigg)\Bigg]\Bigg\}+\frac{24}{(4\pi)^4\epsilon^2}I_{3a,f}+\frac{8}{(4\pi)^2\epsilon}I_{4a,f}+I_{5a,f}, \qquad \text{(A3)}$$

where $I_{5a,f}=\mathcal{O}(\epsilon^0)$.

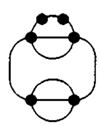
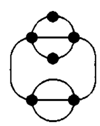
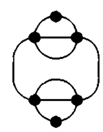
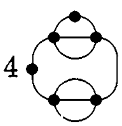
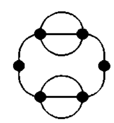
2. I_{5b}

Define I_{5b} by

$$I_{5b}=$$  $$=\int_{kpqrs}\frac{1}{(k^2+m^2)^2[(k+p+q)^2+m^2](p^2+m^2)(q^2+m^2)[(k+r+s)^2+m^2](r^2+m^2)(s^2+m^2)}. \qquad \text{(A4)}$$

We can write

$$I_{5b}=\frac{m^4}{(2-\frac{5}{2}\epsilon)(1-\frac{5}{2}\epsilon)}\left(\frac{\partial}{\partial m^2}\right)^2 I_{5b}$$

$$=\frac{6m^4}{(2-\frac{5}{2}\epsilon)(1-\frac{5}{2}\epsilon)}\Bigg[2$$  $$+2$$  $$+3$$  $$+4$$  $$+$$  $$\Bigg] \qquad \text{(A5)}$$

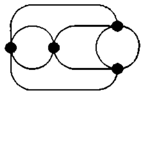
and evaluate this to give

$$I_{5b}=\frac{m^4}{(4\pi)^{10}}\Bigg\{\frac{192}{5\epsilon^5}+\frac{1}{\epsilon^4}\Bigg[-96\ln\frac{m^2}{\bar{\mu}^2}+\frac{452}{5}\Bigg]+\frac{1}{\epsilon^3}\Bigg[120\ln^2\frac{m^2}{\bar{\mu}^2}-226\ln\frac{m^2}{\bar{\mu}^2}+\Bigg(\frac{521}{5}+24\zeta(2)\Bigg)\Bigg]+\frac{1}{\epsilon^2}\Bigg[-46\ln^3\frac{m^2}{\bar{\mu}^2}$$
$$+\frac{151}{2}\ln^2\frac{m^2}{\bar{\mu}^2}+\Bigg(\frac{109}{2}-6\zeta(2)\Bigg)\ln\frac{m^2}{\bar{\mu}^2}+\Bigg(-\frac{2463}{20}-\frac{25}{2}\zeta(2)+4\zeta(3)\Bigg)\Bigg]+\frac{1}{\epsilon}\Bigg[-\frac{47}{4}\ln^4\frac{m^2}{\bar{\mu}^2}+\frac{1225}{12}\ln^3\frac{m^2}{\bar{\mu}^2}$$
$$-\Bigg(\frac{2633}{8}+\frac{39}{2}\zeta(2)\Bigg)\ln^2\frac{m^2}{\bar{\mu}^2}+\Bigg(\frac{931}{2}+\frac{233}{4}\zeta(2)-10\zeta(3)\Bigg)\ln\frac{m^2}{\bar{\mu}^2}+\Bigg(-\frac{1214}{5}-\frac{385}{8}\zeta(2)+\frac{61}{6}\zeta(3)$$
$$-\frac{3}{2}\zeta(4)+\frac{3}{4}\zeta(2)^2\Bigg)\Bigg]\Bigg\}+\frac{6}{(4\pi)^4}\Bigg[\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\Bigg(-\ln\frac{m^2}{\bar{\mu}^2}+\frac{1}{2}\Bigg)\Bigg]I_{3a,f}+I_{5b,f}, \qquad \text{(A6)}$$

where $I_{5b,f}=\mathcal{O}(\epsilon^0)$.

3. I_{5c}

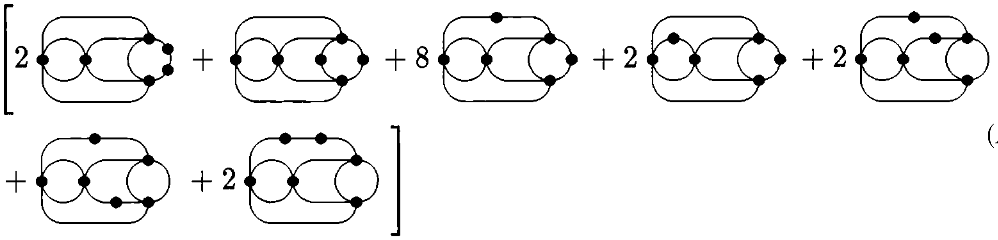
Define I_{5c} by

$$I_{5c}\equiv$$  $$=\int_{kpqrs}\frac{1}{(p^2+m^2)[(p+k)^2+m^2](q^2+m^2)[(q+k)^2+m^2](r^2+m^2)[(r+p-q)^2+m^2](s^2+m^2)[(s+k)^2+m^2]}. \qquad \text{(A7)}$$

We can write

$$I_{5c} = \frac{m^4}{(2-\frac{5}{2}\epsilon)(1-\frac{5}{2}\epsilon)}\left(\frac{\partial}{\partial m^2}\right)^2 I_{5c}$$

$$= \frac{4m^4}{(2-\frac{5}{2}\epsilon)(1-\frac{5}{2}\epsilon)}\Big[$$



$$\Big] \qquad \text{(A8)}$$

and evaluate this to give

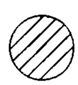
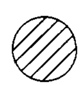
$$I_{5c}=\frac{m^4}{(4\pi)^{10}}\left\{\frac{96}{5\epsilon^5}+\frac{1}{\epsilon^4}\left[-48\ \ln\frac{m^2}{\overline{\mu}^2}+\frac{456}{5}\right]+\frac{1}{\epsilon^3}\left[60\ \ln^2\frac{m^2}{\overline{\mu}^2}-228\ \ln\frac{m^2}{\overline{\mu}^2}+\frac{3542}{15}+12\zeta(2)+\frac{32}{5}\zeta(3)\right]+\frac{1}{\epsilon^2}\left[58\ \ln^3\frac{m^2}{\overline{\mu}^2}\right.\right.$$
$$-129\ \ln^2\frac{m^2}{\overline{\mu}^2}+\left(\frac{119}{3}+78\zeta(2)-16\zeta(3)\right)\ln\frac{m^2}{\overline{\mu}^2}+\left(\frac{457}{6}-81\zeta(2)+\frac{84}{5}\zeta(3)+\frac{24}{5}\zeta(4)\right)\Bigg]+\frac{1}{\epsilon}\left[\frac{31}{4}\ln^4\frac{m^2}{\overline{\mu}^2}-\frac{19}{6}\ln^3\frac{m^2}{\overline{\mu}^2}\right.$$
$$+\left(\frac{5}{12}+\frac{45}{2}\zeta(2)+20\zeta(3)\right)\ln^2\frac{m^2}{\overline{\mu}^2}+\left(-\frac{773}{12}-\frac{19}{2}\zeta(2)-26\zeta(3)-12\zeta(4)\right)\ln\frac{m^2}{\overline{\mu}^2}+\left(\frac{2309}{24}-\frac{107}{12}\zeta(2)+\frac{5}{3}\zeta(3)\right.$$
$$\left.\left.\left.+\frac{201}{10}\zeta(4)-\frac{64}{5}\zeta(5)+\frac{21}{4}\zeta(2)^2+4\zeta(2)\zeta(3)\right)\right]\right\}+\frac{1}{(4\pi)^4}\left(\frac{12}{\epsilon^2}+\frac{4}{\epsilon}\right)I_{3a,f}+\frac{4}{(4\pi)^2\epsilon}I_{4a,f}+I_{5c,f}, \qquad \text{(A9)}$$

where $I_{5c,f}=\mathcal{O}(\epsilon^0)$.

APPENDIX B: $\mathcal{K}\overline{R}$ ON QUARTICALLY DIVERGENT DIAGRAMS

Using standard methods, we have evaluated the quartically divergent diagrams from Table I with subdivergences removed, as needed for the determination of Z_v in Sec. V. As in the last section, the diagrams in the following refer to momentum space integrals, while symmetry factors S_x and group factors $G_x(g_1,g_2)$ are written out separately. As indicated, we define the group factors to contain not only the contributions from the contraction of the δ_{ij} and δ_{ijkl} tensors, but also their accompanying factors $-g_1/3$ and $-g_2$, respectively. For diagrams with separable integrations, we only give the O(N)-symmetric result $G_x(g,0)$, since these diagrams are exclusively used for cross-checks, which we have performed only for the O(N)-symmetric case. For diagrams whose integrations do not separate, we give $G_x(g_1,g_2)$.

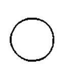
Our strategy to compute the quartically divergent integrals with subdivergences removed is to convert quartically divergent integrals by twice differentiating with respect to m^2 to logarithmically divergent ones and subsequently evaluate these by the methods of [5–8]. We exploit the fact that the operator $\partial/\partial m^2$ commutes with $\mathcal{K}\overline{R}$ (in the same way as differentiation with respect to an external momentum commutes with $\mathcal{K}\overline{R}$, see [10]) and that the result of $\mathcal{K}\overline{R}$ acting on a quartically divergent diagram is proportional to m^4,

$$\mathcal{K}\bar{R}$$  $$=\frac{1}{2}m^4\mathcal{K}\bar{R}\left(\frac{\partial}{\partial m^2}\right)^2$$  $$. \qquad \text{(B1)}$$

We have evaluated all logarithmically divergent diagrams encountered *en route* and found agreement with the respective results in [9]. Therefore, we do not display the results for the logarithmically divergent diagrams here. In order to also check our results in [1], we have carried out this program from one through five loops and give the results for all diagrams of Table I below.

The notation of $\overline{I_x}$ refers to the integral I_x with subdivergences removed by $\mathcal{K}\overline{R}$. The integrals I_{3a} and I_{4a} below are identical to I_2^{cc} and I_3^{cc} in [1], respectively.

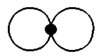
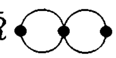
1. One loop

$$\bar{I}_{1a}\equiv\mathcal{K}\bar{R}$$  $$=\frac{1}{2}m^4\mathcal{K}\bar{R}$$  $$=\frac{m^4}{(4\pi)^2\epsilon}, \qquad \text{(B2)}$$

$$S_{1a}=\frac{1}{2}, \qquad \text{(B3)}$$

$$G_{1a}(g_1,g_2)=N. \tag{B4}$$

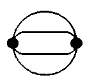
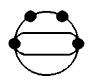
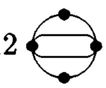
2. Two loops

$$\bar{I}_{2a} \equiv \mathcal{K}\bar{R}$$  $$= m^4\mathcal{K}\bar{R}$$  $$= -\frac{4m^4}{(4\pi)^4\epsilon^2}, \tag{B5}$$

$$S_{2a}=\frac{1}{8}, \tag{B6}$$

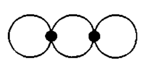
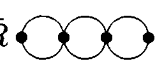
$$G_{2a}(g,0)=-\frac{1}{3}N(N+2)g. \tag{B7}$$

3. Three loops

$$\bar{I}_{3a} \equiv \mathcal{K}\bar{R}$$  $$= \tfrac{1}{2}m^4\mathcal{K}\bar{R}\Big[8$$  $$+ 12$$  $$\Big] = \frac{m^4}{(4\pi)^6}\left(\frac{16}{\epsilon^3}-\frac{40}{3\epsilon^2}+\frac{1}{\epsilon}\right), \tag{B8}$$

$$S_{3a}=\frac{1}{48}, \tag{B9}$$

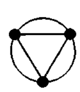
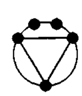
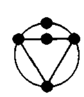
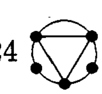
$$G_{3a}(g_1,g_2)=\frac{1}{3}N(N+2)g_1^2+2Ng_1g_2+Ng_2^2. \tag{B10}$$

$$\bar{I}_{3b} \equiv \mathcal{K}\bar{R}$$  $$= m^4\mathcal{K}\bar{R}$$  $$= \frac{8m^4}{(4\pi)^6\epsilon^3}, \tag{B11}$$

$$S_{3b}=\frac{1}{16}, \tag{B12}$$

$$G_{3b}(g,0)=\frac{1}{9}N(N+2)^2g^2. \tag{B13}$$

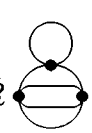
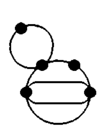
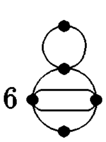
4. Four loops

$$\bar{I}_{4a} \equiv \mathcal{K}\bar{R}$$  $$= \tfrac{1}{2}m^4\mathcal{K}\bar{R}\Big[12$$  $$+ 6$$  $$+ 24$$  $$\Big]$$

$$= \frac{m^4}{(4\pi)^8}\left[-\frac{24}{\epsilon^4}+\frac{44}{\epsilon^3}-\frac{42}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{25}{2}-6\zeta(3)\right)\right], \tag{B14}$$

$$S_{4a}=\frac{1}{48}, \tag{B15}$$

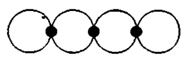
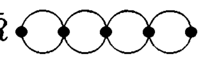
$$G_{4a}(g_1,g_2)=-\frac{1}{27}N(N+2)(N+8)g_1^3-\frac{1}{3}N(N+8)g_1^2g_2-3Ng_1g_2^2-Ng_2^3. \tag{B16}$$

$$\bar{I}_{4b} \equiv \mathcal{K}\bar{R}$$  $$= \tfrac{1}{2}m^4\mathcal{K}\bar{R}\Big[4$$  $$+ 6$$  $$\Big] = \frac{m^4}{(4\pi)^8}\left(-\frac{16}{\epsilon^4}+\frac{40}{3\epsilon^3}-\frac{1}{\epsilon^2}\right), \tag{B17}$$

$$S_{4b}=\frac{1}{24}, \tag{B18}$$

$$G_{4b}(g,0)=-\frac{1}{9}N(N+2)^2g^3. \tag{B19}$$

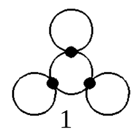
(B20)

$\bar{I}_{4c} \equiv \mathcal{K}\bar{R}$  $= m^4\mathcal{K}\bar{R}$  $= -\frac{16m^4}{(4\pi)^8\epsilon^4}$,

$$S_{4c}=\frac{1}{32}, \tag{B21}$$

$$G_{4c}(g,0)=-\frac{1}{27}N(N+2)^3g^3. \tag{B22}$$

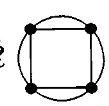
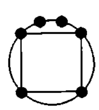
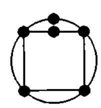
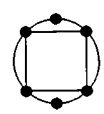
(B23)

$\bar{I}_{4d} \equiv \mathcal{K}\bar{R}$  $= 0$,

$$S_{4d}=\frac{1}{48} \tag{B24}$$

$$G_{4d}(g,0)=-\frac{1}{27}N(N+2)^3g^3,. \tag{B25}$$

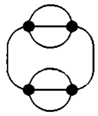
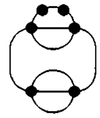
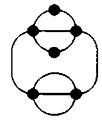
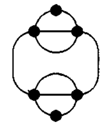
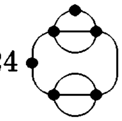
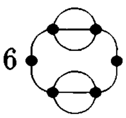
5. Five loops

$\bar{I}_{5a} \equiv \mathcal{K}\bar{R}$  $= \frac{1}{2}m^4\mathcal{K}\bar{R}\Big[16$  $+ 8$  $+ 48$  $\Big]$

$$= \frac{m^4}{(4\pi)^{10}}\left[\frac{192}{5\epsilon^5} - \frac{352}{5\epsilon^4} + \frac{144}{5\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{168}{5} - \frac{272}{5}\zeta(3)\right) + \frac{1}{\epsilon}\left(-\frac{319}{20} + \frac{12}{5}\zeta(3) + \frac{96}{5}\zeta(4)\right)\right] , \tag{B26}$$

$$S_{5a}=\frac{1}{128}, \tag{B27}$$

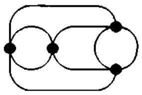
$$G_{5a}(g_1,g_2)=\frac{1}{81}N(N+2)(N^2+6N+20)g_1^4+\frac{4}{27}N(N^2+6N+20)g_1^3g_2+\frac{2}{3}N(N+8)g_1^2g_2^2+4Ng_1g_2^3+Ng_2^4. \tag{B28}$$

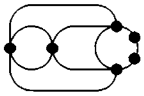
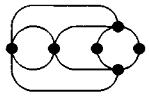
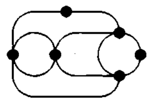
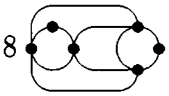
$\bar{I}_{5b} \equiv \mathcal{K}\bar{R}$  $= \frac{1}{2}m^4\mathcal{K}\bar{R}\Big[12$  $+ 12$  $+ 18$  $+ 24$  $+ 6$  $\Big]$

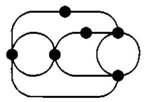
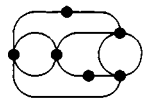
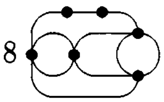
$$= \frac{m^4}{(4\pi)^{10}}\left(\frac{192}{5\epsilon^5} - \frac{208}{5\epsilon^4} - \frac{74}{5\epsilon^3} + \frac{138}{5\epsilon^2} - \frac{71}{20\epsilon}\right) , \tag{B29}$$

$$S_{5b}=\frac{1}{144}, \tag{B30}$$

$$G_{5b}(g_1,g_2)=\frac{1}{9}N(N+2)^2g_1^4+\frac{4}{3}N(N+2)g_1^3g_2+\frac{2}{3}N(N+8)g_1^2g_2^2+4Ng_1g_2^3+Ng_2^4. \tag{B31}$$

$$\bar{I}_{5c} \equiv \mathcal{K}\bar{R}$$ 

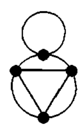
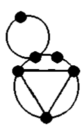
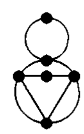
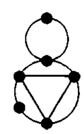
$$= \tfrac{1}{2}m^4\mathcal{K}\bar{R}\Big[8$$  $$+4$$  $$+32$$  $$+8$$ 

$$+8$$  $$+4$$  $$+8$$  $$\Big]$$

$$= \frac{m^4}{(4\pi)^{10}}\left[\frac{96}{5\epsilon^5} - \frac{304}{5\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{1592}{15} + \frac{192}{5}\zeta(3)\right) + \frac{1}{\epsilon^2}\left(-\frac{340}{3} + \frac{24}{5}\zeta(3) - \frac{96}{5}\zeta(4)\right) + \frac{1}{\epsilon}\left(\frac{329}{8} - 16\zeta(3) + \frac{18}{5}\zeta(4) - \frac{64}{5}\zeta(5)\right)\right], \tag{B32}$$

$$S_{5c} = \frac{1}{32}, \tag{B33}$$

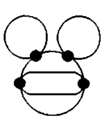
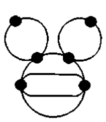
$$G_{5c}(g_1,g_2) = \frac{1}{81}N(N+2)(5N+22)g_1^4 + \frac{4}{27}N(5N+22)g_1^3 g_2 + \frac{2}{9}N(N+26)g_1^2 g_2^2 + 4Ng_1 g_2^3 + Ng_2^4. \tag{B34}$$

$$\bar{I}_{5d} \equiv \mathcal{K}\bar{R}$$  $$= \tfrac{1}{2}m^4\mathcal{K}\bar{R}\Big[4$$  $$+2$$  $$+8$$  $$\Big]$$

$$= \frac{m^4}{(4\pi)^{10}}\left[\frac{16}{\epsilon^5} - \frac{88}{3\epsilon^4} + \frac{28}{\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{25}{3} + 4\zeta(3)\right)\right], \tag{B35}$$

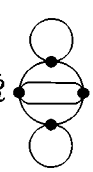
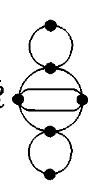
$$S_{5d} = \frac{1}{16}, \tag{B36}$$

$$G_{5d}(g,0) = \frac{1}{81}N(N+2)^2(N+8)g^4. \tag{B37}$$

$$\bar{I}_{5e} \equiv \mathcal{K}\bar{R}$$  $$= m^4\mathcal{K}\bar{R}$$  $$= \frac{m^4}{(4\pi)^{10}}\left(\frac{8}{3\epsilon^4} - \frac{3}{\epsilon^3}\right), \tag{B38}$$

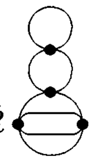
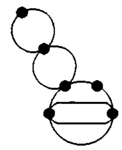
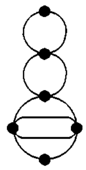
$$S_{5e} = \frac{1}{48}, \tag{B39}$$

$$G_{5e}(g,0) = \frac{1}{27}N(N+2)^3 g^4. \tag{B40}$$

$$\bar{I}_{5f} \equiv \mathcal{K}\bar{R}$$  $$= m^4\mathcal{K}\bar{R}$$  $$= \frac{m^4}{(4\pi)^{10}}\left(\frac{32}{3\epsilon^5} - \frac{32}{3\epsilon^4} + \frac{8}{3\epsilon^3}\right), \tag{B41}$$

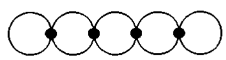
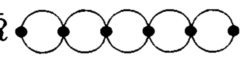
$$S_{5f} = \frac{1}{32}, \tag{B42}$$

$$G_{5f}(g,0)=\frac{1}{27}N(N+2)^3g^4. \tag{B43}$$

$\bar{I}_{5g} \equiv \mathcal{K}\bar{R}$  $= \frac{1}{2}m^4\mathcal{K}\bar{R}\Bigg[4$  $+6$  $\Bigg] = \frac{m^4}{(4\pi)^{10}}\left(\frac{32}{\epsilon^5}-\frac{80}{3\epsilon^4}+\frac{2}{\epsilon^3}\right),$ (B44)

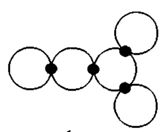
$$S_{5g}=\frac{1}{48}, \tag{B45}$$

$$G_{5g}(g,0)=\frac{1}{27}N(N+2)^3g^4. \tag{B46}$$

$\bar{I}_{5h} \equiv \mathcal{K}\bar{R}$  $= m^4\mathcal{K}\bar{R}$  $= \frac{32m^4}{(4\pi)^{10}\epsilon^5},$ (B47)

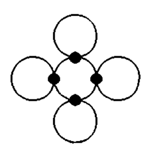
$$S_{5h}=\frac{1}{64}, \tag{B48}$$

$$G_{5h}(g,0)=\frac{1}{81}N(N+2)^4g^4. \tag{B49}$$

$\bar{I}_{5i} \equiv \mathcal{K}\bar{R}$  $=0,$ (B50)

$$S_{5i}=\frac{1}{32}, \tag{B51}$$

$$G_{5i}(g,0)=\frac{1}{81}N(N+2)^4g^4. \tag{B52}$$

$\bar{I}_{5j} \equiv \mathcal{K}\bar{R}$  $=0,$ (B53)

$$S_{5j}=\frac{1}{128}, \tag{B54}$$

$$G_{5j}(g,0)=\frac{1}{81}N(N+2)^4g^4. \tag{B55}$$

[1] B. Kastening, Phys. Rev. D **54**, 3965 (1996).

[2] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin and S. A. Larin, Phys. Lett. B **272**, 39 (1991); **319**, 545(E) (1993).

[3] H. Kleinert and V. Schulte-Frohlinde, Phys. Lett. B **342**, 284 (1995).

[4] J. C. Collins and A. J. Macfarlane, Phys. Rev. D **10**, 1201 (1974).

[5] A. A. Vladimirov, Teor. Mat. Fiz. **43**, 210 (1980) [Theor. Math. Phys. **43**, 417 (1980)].

[6] K. G. Chetyrkin and F. V. Tkachov, Phys. Lett. **114B**, 340 (1982); K. G. Chetyrkin and V. A. Smirnov, *ibid.* **114B**, 419 (1984).

[7] D. J. Broadhurst, ''Massless Scalar Feynman Diagrams: Five Loops and Beyond,'' Open University Report No. OUT-4102-18, 1985.

[8] F. V. Tkachov, Phys. Lett. **100B**, 65 (1981); K. G. Chetyrkin and F. V. Tkachov, Nucl. Phys. **B192**, 159 (1981).

[9] V. Schulte-Frohlinde, Ph.D. thesis, Freie Universität Berlin, 1996.

[10] W. E. Caswell and A. D. Kennedy, Phys. Rev. D **25**, 392 (1982).