Gaussian and $1/N$ approximations in semiclassical cosmology

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We study the $\lambda\phi^4$ theory and the interacting O(N) model in a curved background using the Gaussian approximation for the former and the large-N approximation for the latter. We obtain the renormalized version of the semiclassical Einstein equations having in mind a future application of these models to investigate the physics of the very early Universe. We show that, while the Gaussian approximation has two different phases, in the large-N limit only one is present. The different features of the two phases are analyzed at the level of the effective field equations. We discuss the initial-value problem and find the initial conditions that make the theory renormalizable. As an example, we study the de Sitter self-consistent solutions of the semiclassical Einstein equations. Finally, for an identically zero mean value of the field we find the evolution equations for the classical field $\Omega(x)=(\lambda\langle\phi^2\rangle)^{1/2}$ and the spacetime metric. They are very similar to the ones obtained by replacing the classical potential by the one-loop effective potential in the classical equations but do not have the drawbacks of the one-loop approximation.

I. INTRODUCTION

Inflationary models[1] have been extensively studied in recent years because of their success in solving old cosmological problems of the standard big-bang scenario. The basic equations that are used in order to build up these models are the evolution equations for the matter fields (interacting scalar fields in a typical simple model) coupled with the Einstein equations. These equations have been analyzed in several approximations: on one hand, some quantum effects have been incorporated replacing the classical potential of the scalar field by the one-loop effective potential. In these models, inflation is produced by an effective cosmological constant that dominates the energy-momentum tensor if the system is in the false-vacuum state or is slow rolling down to the true vacuum. On the other hand, another successful way of producing an inflationary period in the history of the Universe is through the effect of the vacuum-polarization terms in the energy-momentum tensor. In fact, without the necessity of interacting fields, quantum free fields can be used to build up an inflationary scenario as was done by Starobinski.[2] In this case it is necessary to include, in the gravitational part of the action, terms that are quadratic in the curvature (see Ref. 3 for a review of the so-called R^2 cosmologies).

There exists an important difference between the two approaches mentioned above: in the first case the source of the Einstein equations is usually taken as being the energy-momentum tensor of the matter fields (with the effective potential instead of the classical one) evaluated in the mean value of the fields while in the second approach the source is taken to be the expectation value of the energy-momentum-tensor operator.[4]

In this paper we are interested in studying interacting quantum fields in curved spacetime, the metric of the latter being determined self-consistently through the semiclassical Einstein equations (SEE). In this framework, we derived in a previous work[5] the equations for the metric and the mean value of the field in the one-loop approximation. These equations include vacuum-polarization effects, cosmological particle production, and the corrections due to the use of the complete effective action instead of the effective potential. They encompass the two above-mentioned approaches and are the ones that should be used to describe the evolution of the Universe within this approximation (some recent related works on the subject are listed in Ref. 6).

However, the one-loop approximation is not the end of the story and one can analyze the problem using other methods such as the Gaussian approximation (GA) and the $1/N$ approximation. The GA is based on a variational principle (see, for instance, Refs. 7 and 8) and has been recently used to compute the effective potential for several field theories[9] and to study time-dependent systems both in quantum mechanics[10] and field theory.[11] Some work has also been done in order to study two-dimensional theories with solitons.[9] On the other hand, the $1/N$ approximation[12] is usually invoked in quantum field theory in curved spaced in order to neglect the graviton contribution to the SEE (Ref. 13). However, as far as we know, the SEE have not been studied in the large-N limit for interacting fields (although the mean-value equation has been recently studied in Ref. 14 in Robertson-Walker metrics).

In a recent work,[15] we wrote down the Gaussian equations for the mean value of an interacting field in an arbitrary background gravitational field, showing its renormalizability as well as the appearance of the so-called "precarious" phase[8] found in flat space.

The aim of this work is to show the renormalizability of the SEE in both Gaussian and $1/N$ approximations, and to discuss their relevance in cosmological models. Before doing this we will reobtain some of the results of Ref. 15 (the paper will thus be self-contained), showing in addition that the GA also admits in curved space the so-called "autonomous" phase.[16,17] The comparison with related works will be much more detailed than in Ref. 15 and of course we will also include the analysis of the mean-value equation in the large-N approximation. We think that these results are interesting not only for cosmological applications but have their own theoretical relevance.

The paper is organized as follows. In Sec. II we will describe the main features of the GA and will obtain the renormalized evolution equation for the mean value of a single self-interacting scalar field. We will show in an easy way how two nontrivial phases appear: the abovementioned precarious and autonomous phases.

In Sec. III we will study a theory with N interacting scalar fields [with an O(N) symmetry] in the large-N approximation. We will compare the results with those obtained using the GA, since the equations obtained in this section will be very similar to the ones described in Sec. II. However, there will be some differences that will cause, for instance, the nonappearance of the autonomous phase.

In Sec. IV we will deal with the SEE. We will prove the renormalizability of these equations for the theories described in Secs. II and III. We will also comment on the restrictions that the renormalizability of the theory imposes on the possible initial data of the problem, showing its relation with well-known restrictions that appear even for free fields in curved spacetime.[4,18] Finally, as an example, we will discuss the existence of the de Sitter-type solutions of the SEE, showing the new features introduced by the "beyond-one-loop" approximations.

In Sec. V we will present an interesting result which can be deduced from our formalism. Setting the mean value of the field equal to zero, we will obtain the effective equations satisfied by a "classical field" which is related to $\langle \phi^2 \rangle^{1/2}$ and by the spacetime metric. The results are, in an adiabatic approximation, surprisingly analogous to the ones obtained in the one-loop approximation. Nevertheless, as we will comment in this section, the analogy is only apparent.

In Sec. VI we will present a summary and the conclusions of this work.

Throughout the paper we will deal with a theory defined by the action

$$S = S_{\text{grav}} + S_{\text{matt}} . \tag{1.1}$$

The gravitational part is given by

$$S_{\text{grav}} = \int (-g)^{1/2} d^n x [\kappa^{-1}(R - 2\Lambda) - \epsilon_1 R^2 - \epsilon_2 R^{ab} R_{ab} - \epsilon_3 R^{abcd} R_{abcd}]/2 , \tag{1.2}$$

where R_{abcd} are the components of the Riemann tensor, $R_{ab} = g^{cd} R_{cadb}$ and $R = R^a{}_a$. $g(x)$ denotes the determinant of the metric tensor $g_{ab}(x)$, and the signature is $- + \cdots + +$. The indexes $a,b,c,d, \ldots$ will run from 0 to $n-1$ and n will be the dimension of the spacetime. The covariant derivative will be denoted with a comma. All the parameters appearing in (1.2) should be understood as bare coupling constants. $\kappa = 8\pi G$, where G is the Newton constant, ϵ_i ($i = 1,2,3$) are dimensionless parameters, and Λ is the cosmological constant. The metric will be considered as classical while the matter fields will be quantized. The matter action S_{matt} is going to be defined in Secs. II and III.

II. THE GAUSSIAN APPROXIMATION

Let us consider a single self-interacting scalar field with the action

$$S_{\text{matt}} = -\int (-g)^{1/2} d^n x (g^{ab} \phi_{,a} \phi_{,b} + m^2 \phi^2 + \xi R \phi^2 + \lambda \phi^4/12)/2 . \tag{2.1}$$

The classical field equation is thus

$$(-\Box + m^2 + \xi R + \lambda \phi^2/6)\phi = 0 . \tag{2.2}$$

When the theory is quantized, $\phi(x)$ is considered as an operator acting on a Hilbert space of states. We will denote with $\phi_0(x)$ the mean value of the field and with $\hat{\phi}(x) = \phi(x) - \phi_0(x)$ the fluctuation field. As in Ref. 15, we will implement the GA considering "Gaussian states" for the fluctuation field. These states are such that

$$\langle \hat{\phi}^3 \rangle = 0 ,$$
$$\langle \{ \hat{\phi}^3(x), \hat{\phi}(x') \} \rangle = 3 G_1(x,x') G_1(x,x)/2 ,$$

where $G_1(x,x') = \langle \{ \hat{\phi}(x), \hat{\phi}(x') \} \rangle$. From Eq. (2.2) it is easy to find the evolution equations for $\phi_0(x)$ and $G_1(x,x')$. They are[15]

$$-\Box\phi_0 + m^2\phi_0 + \xi R \phi_0 + \lambda \phi_0^3/6 + \lambda \phi_0 [G_1]/4 = 0 , \tag{2.3}$$

$$(-\Box + m^2 + \xi R + \lambda \phi_0^2/2 + \lambda [G_1]/4) G_1(x,x') = 0 , \tag{2.4}$$

where we are using the notation $[G_1] = \lim_{x \to x'} G_1(x,x')$.

From (2.4) we can see that the fluctuation field $\hat{\phi}(x)$ can be considered as a free field with a variable *self-consistent* mass given by $M^2 = m^2 + \lambda \phi_0^2/2 + \lambda [G_1]/4$. This point is very important because in this case it is possible to use the (modified) Schwinger-DeWitt expansion (SDWE) in order to obtain an asymptotic representation for the two-point function G_1. There are many versions of this expansion[19] and we will use the one that is most appropriate for solving a problem with a variable mass. The asymptotic expansion for the solution of the equation $(-\Box + M^2 + \xi R) G_1 = 0$ is given by (see Ref. 5)

$$[G_1] = (8\pi)^{-1} (M^2/\mu^2)^{(n-4)/2} \times \sum_{j \geq 0} [W_j] (M^2)^{1-j} \Gamma(1 + j - n/2) , \tag{2.5}$$

where n is the spacetime dimension, μ is an arbitrary parameter with mass dimension, and the functions W_j are defined as the solutions of a coupled system of linear equations. As an example, and for future use, we will give the following explicit expressions:

$$[W_0] = 1 , \tag{2.6a}$$

$$[W_1] = (\tfrac{1}{6} - \xi) R , \tag{2.6b}$$

$$[W_{1,ab}] = M^2{}_{,ab}/6 + (\tfrac{3}{20} - \xi) R_{,ab}/3 + \Box R_{ab}/60 - R_{ac} R^c{}_b/45 + (R_{acde} R_b{}^{cde} + R_{acbd} R^{cd})/90 , \tag{2.6c}$$

$$[W_2]=-\Box M^2/6+(\tfrac{1}{5}-\xi)\Box R/6+(\tfrac{1}{6}-\xi)^2R^2/2 +(R_{abcd}R^{abcd}-R_{ab}R^{ab})/180\ , \tag{2.6d}$$

$$[W_3]=M^{2,a}M^{2,a}/12 +(\xi-\tfrac{3}{10})R_{,a}M^{2,a}-\Box^2M^2/60 -R_{ab}M^{2,a}M^{2,b}+\mathrm{GT}\ , \tag{2.6e}$$

where GT means "geometric terms" (which can be found, for example, in Ref. 20). Expanding Eq. (2.5) around $n=4$, we get

$$[G_1]=\frac{1}{8\pi^2}\left[\frac{2}{n-4}+\ln\frac{M^2}{\mu^2}\right]\left[\frac{2M^2}{n-2}+(\xi-\tfrac{1}{6})R\right] +\frac{1}{8\pi^2}\sum_{j\geq 0}\frac{j![W_{j+2}]}{M^{2(j+1)}}\ . \tag{2.7}$$

Now if we want to obtain an effective equation for ϕ_0 we can proceed (as we did in Ref. 15) in the following way: using Eq. (2.3) the equation for G_1 becomes

$$[-\Box+(\Box\phi_0/\phi_0+\lambda\phi_0^2/3)]G_1=0\ . \tag{2.8}$$

The next step is the use of (2.7) for the coincidence limit of the propagator which can be computed choosing $M^2(x)=(\Box\phi_0/\phi_0+\lambda\phi_0^2/3)$ [and setting $\xi=0$ in formulas (2.6)]. After doing that, we can replace the expression in Eq. (2.3). In that way we obtained Eq. (3.5) of Ref. 15. Here we will proceed in a different way that will allow us to compare more directly our results with those obtained by people working with the GA at the level of the effective potential. We can directly use Eq. (2.7) taking $M^2=m^2+\lambda\phi_0^2/2+\lambda[G_1]/4$. In this way, we obtain the following equation for $[G_1]$ (that has to be solved self-consistently):

$$8\pi^2[G_1]H=\left[\frac{m^2+\lambda\phi_0^2/2}{\frac{n}{2}-1}+\frac{\lambda}{4}[G_1]\right]\ln\left[\frac{m^2+\lambda\phi_0^2/2+\lambda[G_1]/4}{\mu^2}\right]+\frac{2}{n-4}\left[\frac{m^2+\lambda\phi_0^2/2}{\frac{n}{2}-1}+(\xi-\tfrac{1}{6})R\right] +\sum_{j\geq 2}\frac{j![W_j]}{(m^2+\lambda\phi_0^2/2+\lambda[G_1]/4)^{j-1}}\ , \tag{2.9}$$

where $H=[1-\lambda/8\pi^2(n-4)(n-2)]$. Multiplying by $\lambda/4$ and defining $\Omega^2(x)=\lambda[G_1]/4$, this equation can be thought as an equation for Ω^2 as a function of ϕ_0 (and the geometry). It reads

$$32\pi^2\Omega^2\frac{H}{\lambda}=\left[\frac{m^2+\lambda\phi_0^2/2}{\frac{n}{2}-1}+\Omega^2\right]\ln\left[\frac{m^2+\lambda\phi_0^2/2+\Omega^2}{\mu^2}\right]+\frac{2}{n-4}\left[\frac{m^2+\lambda\phi_0^2/2}{\frac{n}{2}-1}+(\xi-\tfrac{1}{6})R\right] +\sum_{j\geq 2}\frac{j![W_j]}{(m^2+\lambda\phi_0^2/2+\Omega^2)^{j-1}}\ . \tag{2.10}$$

At this point we can compare our approach with the one that is used by people who computed the GEP using a variational definition of the GA (such as Stevenson, Tarrach, and others[8,16,17]). In order to compute GEP, Eq. (2.9) must be evaluated in a $\phi_0=$const configuration, assuming also that Ω^2 is constant (of course this is not true in curved spacetime). In this way we obtain an equation for $\Omega(\phi_0)$. This equation (in the flat spacetime case) is nothing but the so-called Ω *equation* that is obtained in the variational approach.[8] So, our Eq. (2.10) can be interpreted as the generalization of the Ω equation to nonstatic problems and to curved spacetimes.

There is an obvious problem with Eqs. (2.9) and (2.10): they are ill defined in $n=4$ (four spacetime dimensions). Thus we have to face the problem of *renormalization,* that is to say, we must find a *reparametrization* of the bare parameters in terms of the renormalized ones such that the equations become well defined. It is worth noting that it is possible that several reparametrizations exist. Each of them gives a different theory. Those theories are usually known as different "phases" of the original theory. In particular, we will show how two *nontrivial* phases appear in the $\lambda\phi^4$ theory. They are named the *precarious phase* and the *autonomous phase.*

A. Precarious phase

In Ref. 15 we studied this phase using the abovementioned effective equation for ϕ_0. The precarious phase is defined by the reparametrization

$$\lambda=16\pi^2(n-4)+B(n-4)^2+\cdots\ , \tag{2.11a}$$

$$m^2=C(n-4)+\cdots\ , \tag{2.11b}$$

$$\xi=\tfrac{1}{6}+D(n-4)+\cdots\ , \tag{2.11c}$$

where B, C, and D are arbitrary finite constants. Replacing (2.11) in (2.10), the Ω equation in this phase can be easily written. The kinetic terms (that come from the coefficients $[W_j]$) are important. Taking into account that in the coefficients $[W_2]$ and $[W_3]$ appear the factors $(-\Box\Omega^2/6)$ and $(\Omega^2{}_{,a}\Omega^{2,a}/12)$, respectively, we find

$$-\Box\Omega+6(C+DR+16\pi^2\phi_0^2)\Omega-3\Omega^3\left[\tfrac{1}{2}-B-\ln\frac{\Omega^2}{\mu^2}\right]+3\left[\frac{a_2}{\Omega^2}+\frac{a_3}{\Omega^4}+\sum_{j\geq 4}\frac{(j-2)![W_j]}{\Omega^{2(j-1)}}\right]=0\ , \tag{2.12}$$

where $a_2=[W_2]+\Box\Omega^2/6$ and $a_3=[W_3]-\Omega^2{}_{,a}\Omega^{2,a}/12$.

This is an evolution equation for $\Omega=(\lambda\langle\hat{\phi}^2\rangle/2)^{1/2}$. The coefficients a_2, a_3, and $[W_j]$ depend on geometric quantities (a_3 and $[W_j]$ depend also on Ω derivatives). Note that, as we have written everything in terms of renormalized constants, Ω is a finite quantity. In flat spacetime, the last equation is much more simple: all in the $[W_j]$ disappears excepting those terms that depend on Ω derivatives.

We will show now how to relate the finite constants B, C, and D with the physical coupling constants. The reasonable definition of the derivative of the GEP is [see Eq. (2.3)]

$$\frac{dV_{\rm GEP}}{d\phi_0}=m^2\phi_0+\xi R\phi_0+\lambda\phi_0^3/6+\phi_0\Omega^2 . \qquad (2.13a)$$

So, after the reparametrization (renormalization) it turns out to be

$$\frac{dV_{\rm GEP}}{d\phi_0}=R\phi_0/6+\phi_0\Omega^2 . \qquad (2.13b)$$

We will define the "physical coupling constants" as

$$M^2_{\rm ren}=\left.\frac{d^2V_{\rm GEP}}{d\phi_0^2}\right|_{\phi_0=0,\ R_{abcd}=0} , \qquad (2.14a)$$

$$\lambda_{\rm ren}=\left.\frac{d^4V_{\rm GEP}}{d\phi_0^4}\right|_{\phi_0=0,\ R_{abcd}=0} , \qquad (2.14b)$$

$$\xi_{\rm ren}=\left.\frac{d^3V_{\rm GEP}}{d\phi_0^2dR}\right|_{\phi_0=0,\ R_{abcd}=0} . \qquad (2.14c)$$

In order to perform the derivatives involved in these equations, we can evaluate the Ω equation in a $\phi_0=$const configuration and write it in the following way:

$$0=\Omega^2[\tfrac{1}{2}-B-\ln(\Omega^2/\mu^2)/2]-(C+DR+16\pi^2\phi_0^2)+F(\Omega^2,R_{abcd}) , \qquad (2.15)$$

where the function $F(\Omega^2,R_{abcd})$ is such that $F(\Omega,R_{abcd}=0)=0$ and

$$dF(\Omega^2,R_{abcd})/d\Omega^2|_{R_{abcd}=0}=0 .$$

Taking into account the last equations, the physical coupling constants can be related to the parameters B, C, and D through the relations

$$M^2_{\rm ren}=C/(\tfrac{1}{2}-B) , \qquad (2.16a)$$

$$\lambda_{\rm ren}=-48\pi^2/B , \qquad (2.16b)$$

$$\xi_{\rm ren}=\tfrac{1}{6}-[(C-\tfrac{1}{36})+\alpha/24]/B , \qquad (2.16c)$$

where $\alpha=dF/dR|_{R_{abcd}=0}$.

In conclusion, this precarious phase is a nontrivial (i.e., interacting) theory that can be completely described by the Ω equation (2.12) coupled with the ϕ_0 equation (2.3). Ω^2 is a finite quantity that can be interpreted (if ϕ_0 is constant and the spacetime is flat) as the self-consistent mass of the fluctuation quanta. Finally we would like to mention the only difference between our approach and the variational one. When computing the GEP, we have to solve the Ω equation. This equation could have more than one solution $\Omega(\phi_0)$ (in fact, this is the case in flat spacetime as noted in Ref. 8). Within the variational approach we have a criterion that can be used to disregard one of the solutions (there is only one that corresponds to a minimum of $\langle H\rangle$). As we did not make use of this variational approach we have not such a criterion at hand from the beginning. However, as we know that the equations obtained are equivalent, we can appeal to the "variational selection rule" at the end of the calculations.

B. Autonomous phase

Besides the precarious phase, there exist another nontrivial phase of the $\lambda\phi^4$ theory. This phase was first discussed in Ref. 16. It involves a "wave-function" renormalization (or a ϕ_0 rescaling). In Ref. 17 it was found the mean-value evolution equation through a computation of the complete effective action of the theory using the GA (see also Ref. 21). Here we will generalize this equation (that was obtained in flat spacetime) to an arbitrary background metric. Defining $\tilde{\phi}_0=Z^{-1/2}\phi_0$ and using Eqs. (2.3), (2.4), and (2.5) we get the evolution equation

$$-\Box\tilde{\phi}_0+m^2H^{-1}\tilde{\phi}_0+\left[\xi-\frac{\lambda}{96\pi^2(n-4)}\right]H^{-1}R\tilde{\phi}_0+\frac{\lambda}{6}Z\tilde{\phi}_0^3H^{-1}+\frac{\lambda^2Z\tilde{\phi}_0^3H^{-1}}{48\pi^2(n-2)}\left[\frac{2}{n-4}+\ln\left[\frac{\Box\tilde{\phi}_0/\tilde{\phi}_0+\lambda Z\tilde{\phi}_0^2/3}{\mu^2}\right]\right]+\cdots=0 . \qquad (2.17)$$

Now we have to find an appropriate reparametrization. If we assume that $\lambda Z\to\infty$ for $n\to 4$, in order to have an interacting (nontrivial) theory we must require that

$$\lambda^2Z=a=\text{finite const} , \qquad (2.18a)$$

$$\lambda Z/6+\lambda^2Z[2/(n-4)+\ln(\lambda Z)]/96\pi^2=b=\text{finite const} . \qquad (2.18b)$$

From these equations we can obtain λ and Z as functions of n (and the new finite parameters a and b)

$$Z^{-1}=64\pi^2(n-4)^2/a+\cdots , \qquad (2.19a)$$

$$\lambda=-8\pi^2(n-4)\{1-2(n-4)\ln[-a(n-4)/8\pi^2]+\cdots\} . \qquad (2.19b)$$

Replacing in Eq. (2.17) we get

$$-\Box\tilde{\phi}_0+2m^2\tilde{\phi}_0/3+(2\xi/3+1/18)R\tilde{\phi}_0+2b\tilde{\phi}_0^3/3+a\tilde{\phi}_0^3\ln(\tilde{\phi}_0^2/\mu^2)/144\pi^2=0 . \qquad (2.20)$$

Note that all the extra terms that appeared in (2.17) (denoted by the ellipsis) vanished because they were proportional to $\lambda(\lambda Z)^{-k}$ $(k>0)$. So in this phase, no new kinetic terms appear in the mean-value equation. When computing the effective action for this phase of the theory only the classical kinetic term is found.[17] This means that within this approximation we cannot study some interesting things such as the dependence of the four-point proper vertices on the momenta. In Sec. IV we will show that this phase has other problems: the renormalized version of the SEE will not contain important physical information such as vacuum-polarization terms, particle creation, etc. These properties make the autonomous phase less interesting than the precarious one.

There are several differences between both phases. One of them can be seen by examining the Ω equation. In this case this equation can be obtained from (2.10) replacing ϕ_0 by $Z^{1/2}\tilde{\phi}_0$. If we make this replacement and also use Eqs. (2.19), it is easy to show that, unlike in the precarious phase, Ω is divergent when $n \to 4$. So in this case Ω has no direct physical meaning. Another difference between the two phases is that they cannot coexist in the same number of spacetime dimensions. In fact, the autonomous phase can only exist if $n<4$ because of the logarithm in (2.19) while the precarious phase can only exist if $n>4$ because of the $\ln[(n-4)\phi_0^2/2]$ that appears[15] in the effective equation for ϕ_0 (in the case $\phi_0=$const, for example). This fact has been noted in Ref. 22.

III. O(N) MODEL

Let us now consider the following action for the matter fields:

$$S_{\text{matt}}=-\int d^n x\,(-g)^{1/2}[g^{ab}\phi^\alpha{}_{,a}\phi^\alpha{}_{,b}+(m^2+\xi R)\phi^\alpha\phi^\alpha +\lambda(\phi^\alpha\phi^\alpha)^2/4N]/2\ , \tag{3.1}$$

where the internal index α runs from 1 to N. This is a O(N)-symmetric generalization of the $\lambda\phi^4$ theory described in the previous section. In what follows we will obtain the renormalized equations for the mean values of the N scalar fields $\langle\phi^\alpha\rangle=\phi_0^\alpha$ using the large-N approximation. Note that a factor N^{-1} has been explicitly included in the quartic coupling constant λ, in this way the energy is of order $O(N)$ for large values of N (of course the N dependence of the coupling constants must always be adjusted in order to obtain a well-behaved theory for $N=\infty$, see, for example, the study of QCD in this limit[23]).

The classical equations of motion derived from (3.1) are

$$(-\Box+m^2+\xi R+\lambda\phi^2/2N)\phi^\alpha=0\ , \tag{3.2}$$

where ϕ^2 denotes $\phi^\alpha\phi^\alpha$. In terms of the two-point function

$$G_1^{\alpha\beta}(x,x')=\langle\{\phi^\alpha(x),\phi^\beta(x')\}\rangle-2\phi_0^\alpha(x)\phi_0^\beta(x') \tag{3.3}$$

the large-N approximation reads[12,14]

$$G_1^{\alpha\beta}(x,x')\simeq\delta^{\alpha\beta}G_1(x,x')\ , \tag{3.4a}$$

$$\langle\phi^2(x)\phi^\alpha(x')\rangle\simeq\phi_0^\alpha(x')\langle\phi^2(x)\rangle\ , \tag{3.4b}$$

$$\langle\{\phi^\alpha(x'),\phi^\beta(x)\phi^2(x)\}\rangle\simeq\delta^{\alpha\beta}NG_1(x,x)G_1(x,x')/2\ . \tag{3.4c}$$

In consequence, after a rescaling of the mean value of the field $\phi_0^\beta\to N^{1/2}\phi_0^\beta$, the evolution equations in this approximation become

$$(-\Box+M^2+\xi R)\phi_0^\beta=0\ , \tag{3.5a}$$

$$(-\Box+M^2+\xi R)G_1(x,x')=0\ , \tag{3.5b}$$

where $M^2=m^2+\lambda(\phi_0^2+[G_1]/2)/2$. Equations (3.5) are our starting point. It is worth noting that the structure of these equations is similar to the ones derived in the GA [cf. Eqs. (2.3) and (2.4)]. However, there is a difference that will be important in what follows: unlike in the equations of the previous section, the self-consistent mass M^2 appears both in the mean value and G_1 equations (we will turn to this point below).

To proceed, we follow the same steps given in Sec. II. After the use of Eq. (3.5a), Eq. (3.5b) becomes

$$(-\Box+\Box\phi_0/\phi_0)G_1(x,x')=0\ , \tag{3.6}$$

where we dropped the index α in $\Box\phi_0/\phi_0$ since it is in fact α independent. Using the modified SDWE with $M^2=\Box\phi_0/\phi_0$ and introducing the result into Eq. (3.5a) we obtain

$$-\Box\phi_0^\alpha+m^2H^{-1}\phi_0^\alpha+H^{-1}\left[\xi-\frac{\lambda}{96\pi^2(n-4)}\right]R\,\phi_0^\alpha+\frac{\lambda H^{-1}}{2}\phi_0^2\phi_0^\alpha+\frac{\lambda}{32\pi^2}H^{-1}\phi_0^\alpha\left[\left[\frac{\Box\phi_0}{\phi_0}-\frac{R}{6}\right]\ln\left[\frac{\Box\phi_0}{\phi_0\mu^2}\right]+\cdots\right]=0\ . \tag{3.7}$$

Like in the GA, the mean-value equation becomes finite choosing the bare parameters as in Eqs. (2.11). However, this is the only allowed phase: if one performs a wave-function renormalization $\phi_0^\alpha\to Z^{-1/2}\phi_0^\alpha$, the equation for the renormalized field becomes

$$-\Box\tilde{\phi}_0^\alpha+m^2H^{-1}\tilde{\phi}_0^\alpha+H^{-1}\left[\xi-\frac{\lambda}{96\pi^2(n-4)}\right]R\,\tilde{\phi}_0^\alpha+\frac{\lambda Z}{2}H^{-1}\tilde{\phi}_0^2\tilde{\phi}_0^\alpha+\frac{\lambda}{32\pi^2}H^{-1}\tilde{\phi}_0^\alpha\left[\left[\frac{\Box\tilde{\phi}_0}{\tilde{\phi}_0}-\frac{R}{6}\right]\ln\left[\frac{\Box\tilde{\phi}_0}{\tilde{\phi}_0\mu^2}\right]+\cdots\right]=0\ . \tag{3.8}$$

From (3.8) we see that λH and λHZ must be finite so λ must be given again by Eq. (2.11a). In consequence, Z must be a finite constant and only a trivial finite wave-function renormalization is allowed, so a proper new phase does not exist.

All this treatment is of course valid only if $\tilde{\phi}_0^\alpha \neq 0$, since only in this case Eq. (3.6) follows from Eqs. (3.5). As in the previous section one can develop a formalism valid in the general case, which leads to the Ω equation [cf. Eq. (2.12)]. Equation (3.5b) for the two-point function in the O(N) model coincides with the one in the GA so the Ω equation for the former coincides with the one correspondent to the latter.

To end this section we would like to note that our condition to obtain a finite theory in the large-N limit (which reduces to impose the finiteness of λH^{-1}) is the dimensionally regulated version of the condition imposed by other authors using an adiabatic regularization in particular examples. This is[11,14] $\lambda/(1+\lambda\delta\lambda)=$finite where $\delta\lambda$ is the cutoff regulated integral:

$$\delta\lambda=(64\pi^2)^{-1}\int d^3k\,(k^2+M^2)^{-3/2} .$$

Calculating this integral in n dimensions, it is easy to see that $\delta\lambda=-[16\pi^2(n-4)]^{-1}$, so the above-mentioned condition is equivalent to $\lambda H^{-1}=$finite.

IV. THE SEMICLASSICAL EINSTEIN EQUATIONS

Up to now we considered only the equations for the mean value of the scalar fields, in what follows we will analyze the Einstein equations. The classical equations can be derived from Eqs. (1.1) and (1.2) and read

$$\kappa^{-1}(R_{ab}-Rg_{ab}/2+\Lambda g_{ab}+\epsilon_1 H_{ab}^{(1)}+\epsilon_2 H_{ab}^{(2)}+\epsilon_3 H_{ab} \equiv \kappa^{-1}\mathbb{G}_{ab}=T_{ab}(\phi) , \quad (4.1)$$

where

$$H_{ab}^{(1)}=2R_{;ab}-2g_{ab}\Box R+g_{ab}R^2/2-2RR_{ab} , \quad (4.2a)$$

$$H_{ab}^{(2)}=R_{;ab}-g_{ab}\Box R/2-\Box R_{ab}-g_{ab}R_{cd}R^{cd}/2 -2R^{cd}R_{cadb} , \quad (4.2b)$$

$$H_{ab}=2R_{;ab}+4R_{ac}R^c{}_b-4R^{cd}R_{cadb}-2R_{acde}R_b{}^{cde} -4\Box R_{ab}+g_{ab}R_{cdef}R^{cdef}/2 , \quad (4.2c)$$

and

$$T_{ab}=2(-g)^{-1/2}\delta S_{\text{matt}}/\delta g^{ab} .$$

For the O(N) model the classical energy-momentum tensor is

$$T_{ab}=(1-2\xi)\phi^\alpha{}_{,a}\phi^\alpha{}_{,b}+(2\xi-\tfrac{1}{2})g_{ab}\phi^\alpha{}_{,c}\phi^\alpha{}_{,c} -2\xi\phi^\alpha\phi^\alpha{}_{,ab}+2\xi g_{ab}\phi^\alpha\Box\phi^\alpha +[\xi(R_{ab}-Rg_{ab}/2)-g_{ab}(m^2+\lambda\phi^2/12)/2]\phi^2 . \quad (4.3)$$

In the quantum theory the computation of the expectation value of the operator T_{ab} can be done by using the point-splitting method.[24] In fact, $\langle T_{ab}\rangle$ can be expressed in terms of the coincidence limit of the two-point function $G_1(x,x')$ and its derivatives (see Refs. 5 and 25). In the large-N approximation it is given by

$$\langle T_{ab}\rangle=N\sum_{k=1}^{N}T_{ab}(\phi_0^\alpha)+N\langle\tilde{T}_{ab}\rangle+\lambda Ng_{ab}[G_1]^2/32 +O(N^0) , \quad (4.4)$$

where

$$\langle\tilde{T}_{ab}\rangle=-[G_{1,ab}]/2+(\tfrac{1}{2}-\xi)[G_1]_{,ab}/2 +(\xi-\tfrac{1}{4})g_{ab}\Box[G_1]/2+\xi R_{ab}[G_1]/4 \quad (4.5)$$

and we have redefined ϕ_0^α as in Eqs. (3.5).

There are two points that we want to stress about Eq. (4.4), on one hand, the ϕ_0^α-dependent part of $\langle T_{ab}\rangle$ is $O(N)$ since each term in the summation is of order $1/N$ due to the field redefinition. On the other hand, $\langle\tilde{T}_{ab}\rangle$ can be interpreted as the energy-momentum tensor of a free field with mass $M^2=m^2+\lambda(\phi_0^2+[G_1]/2)/2$.

All this calculation can also be done for the single scalar field theory defined by the action Eq. (2.1). The result is analogous to the one of the O(N) model, one must simply set $N=1$ in Eq. (4.4). To understand this result, note that in the GA one has

$$\lambda\langle\hat{\phi}^4\rangle=3\lambda[G_1]^2/4 ,$$

while, in the large-N limit,

$$\lambda\langle(\hat{\phi}^\alpha\hat{\phi}^\alpha)^2\rangle=N[G_1]^2/4 ,$$

as in the action for both models we defined the quartic coupling with a factor $3/N$ of difference [compare Eqs. (2.1) and (3.1)], then this factor cancels, yielding the same result for both energy-momentum tensors. For this reason the G_1 equations are also the same in both models.

Introducing the SDWE for the propagator into Eq. (4.5) we get

$$\langle\tilde{T}_{ab}\rangle=\frac{1}{16\pi^2}\left[\frac{M^2}{\mu^2}\right]^{n-4/2}\sum_{j\geq 0}\left[\Gamma\left[j-\frac{n}{2}\right]M^{2(2-j)}[W_j]g_{ab} +\Gamma\left[1+j-\frac{n}{2}\right]M^{2(1-j)}\{(\xi-\tfrac{1}{6})[W_j]R_{ab}-[W_{j,ab}]+(\tfrac{1}{2}-\xi)[W_j]_{,ab} +(\xi-\tfrac{1}{4})g_{ab}\Box[W_j]\}\right.$$

$$+\Gamma\left[2+j-\frac{n}{2}\right]M^{-2j}\left[\frac{M^2_{,a}}{2}[W_{j,b}]-\frac{M^2_{,b}}{2}[W_{j,a}]+[W_j][\xi M^2_{,ab}-(\xi-\tfrac{1}{4})g_{ab}\Box M^2]\right.$$
$$\left.+(\xi-\tfrac{1}{2})(M^2_{,a}[W_j]_{,b}-M^2_{,b}[W_j]_{,a}+2(\xi-\tfrac{1}{4})g_{ab}[W_j]_{,c}M^{2,c}\right]$$
$$+\Gamma\left[3+j-\frac{n}{2}\right]M^{-2(j+1)}[W_j](\xi-\tfrac{1}{4})(-M^2_{,a}M^2_{,b}+g_{ab}M^2_{,c}M^{2,c})\Bigg] . \tag{4.6}$$

In this expression it is easy to recognize the (possible) infinite terms in the limit $n\to 4$, we will compute these terms and show that after the use of the reparametrizations given in Secs. II and III the SEE become explicitly finite (of course, a reparametrization of the gravitational constants will also be needed). It will be useful to write the term quadratic in $[G_1]$ that appears in (4.4) as

$$\lambda g_{ab}[G_1]^2/32=g_{ab}(M^2-m^2-\lambda\phi_0^2/2)^2/2 . \tag{4.7}$$

A. Renormalizability of the SEE

Let us first consider the precarious phase. As the bare parameters m^2, λ, and $(\xi-\frac{1}{6})$ are in this case infinitesimal, the only divergences appearing in $\langle T_{ab}\rangle+\lambda g_{ab}[G_1]^2/32$ must be purely geometric, since $T_{ab}(\phi_0)$ is finite. From Eqs. (4.6) and (4.7) one has

$$\langle \tilde{T}_{ab}\rangle+\frac{\lambda g_{ab}}{32}[G_1]^2=\frac{1}{16\pi^2}\left[\frac{M^2}{\mu^2}\right]^{(n-4)/2}\left[\Gamma\left[-\frac{n}{2}\right]M^2\frac{g_{ab}}{2}+\Gamma\left[1-\frac{n}{2}\right]M^2\left[[W_1]\frac{g_{ab}}{2}+(\xi-\tfrac{1}{6})R_{ab}\right]\right.$$
$$+\Gamma\left[2-\frac{n}{2}\right]\left[[W_2]\frac{g_{ab}}{2}+(\xi-\tfrac{1}{6})R_{ab}[W_1]-[W_1]_{,ab}(\xi-\tfrac{1}{2})-[W_{1,ab}]\right.$$
$$\left.\left.+(\xi-\tfrac{1}{6})g_{ab}\Box[W_1]+\xi M^2_{,ab}+(\tfrac{1}{4}-\xi)g_{ab}\Box M^2\right]\right]$$
$$+\frac{g_{ab}}{2\lambda}\left[M^2-m^2-\frac{\lambda\phi_0^2}{2}\right]^2+\text{finite terms} . \tag{4.8}$$

Using Eqs. (2.6) and (2.11) one obtains

$$\langle \tilde{T}_{ab}\rangle+\frac{\lambda g_{ab}}{32}[G_1]^2=\frac{\Gamma\left[2-\frac{n}{2}\right]}{2880\pi^2}(H_{ab}-H^{(2)}_{ab})+\frac{D}{8\pi^2}(\Omega^2G_{ab}-\Omega^2_{,ab}+g_{ab}\Box\Omega^2)-\frac{g_{ab}\Omega^2}{16\pi^2}(C+8\pi^2\phi_0^2)$$
$$+(\tfrac{3}{4}-B)\frac{\Omega^4g_{ab}}{32\pi^2}+\frac{1}{64\pi^2}\ln\frac{\Omega^2}{\mu^2}[-g_{ab}\Omega^4+\tfrac{1}{45}(H^{(2)}_{ab}-H_{ab})]+\text{finite terms} . \tag{4.9}$$

The infinite terms containing derivatives of M^2 multiplied by $\Gamma(2-n/2)$ in Eq. (4.8) have canceled out due to the relation $\xi-\frac{1}{6}=O(n-4)$. On the other hand, the divergence of the first term $\Gamma(-n/2)M^4(32\pi^2)^{-1}$ is canceled by the one of $\lambda g_{ab}[G_1]^2/32$ due to the fact that $m^2=O(n-4)$ and $\lambda=16\pi^2(n-4)$. Thus we see that in order to arrive at a final result with only geometric divergences [as is the case in Eq. (4.9)] the choice of the bare parameters is very important.

The gravitational bare constants that multiply $H^{(2)}_{ab}$ and H_{ab} (i.e., ϵ_2 and ϵ_3) must be reparametrized as

$$\epsilon_2=-\Gamma(2-n/2)/2880\pi^2+\epsilon_{2r} , \tag{4.10a}$$

$$\epsilon_3=\Gamma(2-n/2)/2880\pi^2+\epsilon_{3r} , \tag{4.10b}$$

where ϵ_{2r} and ϵ_{3r} are arbitrary finite constants, to be fixed by the renormalization procedure. It is worth noting that, unlike in the one-loop approximation,[5] the other gravitational bare constants are finite. This is so because the infinites in κ^{-1}, $\Lambda\kappa^{-1}$, and ϵ_1 in the one-loop approximation are proportional to $\xi-\frac{1}{6}$ and m^2, as in the precarious phase these parameters are infinitesimal then the infinites do not appear. On the other hand, when computing a general $\langle T_{ab}\rangle_{\text{ren}}$, i.e.,

$$\langle T_{ab}\rangle_{\text{ren}}=\langle T_{ab}\rangle-\langle T_{ab}\rangle_{\text{SDWE(4)}}$$

and not the one computed using the SDWE, the geometric divergence of Eq. (4.9) will give rise to the correct trace-anomaly term; an explicit example on this will be shown in sec. IV B.

Finally, the SEE in this phase can be written as

$$\kappa^{-1}\left[R_{ab}-R\frac{g_{ab}}{2}+\Lambda g_{ab}\right]+\left[\alpha_1-\frac{1}{2880\pi^2}\ln\frac{\Omega^2}{\mu^2}\right]H^{(1)}_{ab}+\left[\alpha_2+\frac{1}{960\pi^2}\ln\frac{\Omega^2}{\mu^2}\right]H^{(2)}_{ab}$$
$$=\sum_{\beta=1}^{N}T_{ab}(\phi_0^\beta)+\frac{D}{8\pi^2}(\Omega^2G_{ab}-\Omega^2_{,ab}+g_{ab}\Box\Omega^2)-g_{ab}\frac{\Omega^2}{16\pi^2}(C+8\pi^2\phi_0^2)+\frac{g_{ab}\Omega^4}{32\pi^2}\left[\tfrac{3}{4}-B-\tfrac{1}{2}\ln\frac{\Omega^2}{\mu^2}\right]+\text{finite terms}\ , \tag{4.11}$$

where $T_{ab}(\phi_0^\beta)$ is the energy-momentum tensor of a free, massless, and conformally coupled field and we used the linear dependence between $H^{(1)}_{ab}$, $H^{(2)}_{ab}$, and H_{ab} in four dimensions (the three finite constants ϵ_1, ϵ_{2R}, and ϵ_{3R} combine then into two independent constants α_1 and α_2). We also absorbed the factor N that was multiplying the right-hand side in the gravitational constants; as we mentioned in Sec. III, this must be done in order to obtain a sensible theory in the large-N limit. The finite terms in Eq. (4.11) can be read from Eq. (4.6) setting $n=4$.

It is worth noting that, if one would include the graviton contribution to the SEE (as must be done in order to have a consistent quantum theory[26]), then it would disappear in the large-N limit since the contribution is of order $1/N$. Of course one should include this contribution in the GA to the single field theory.

Now we consider the autonomous phase of the theory described in Sec. II B. The calculations to be done in order to show the renormalizability are analogous to the ones we did above, using now the reparametrization given in Eqs. (2.19). However, there is a difference. The wave-function renormalization introduces a factor Z in the term $T_{ab}(\phi_0^\beta)$ which appears on the right-hand side of the SEE so, in order to obtain finite equations, it is also reasonable to redefine the gravitational constants in such a way that the generalized Einstein tensor $\mathbb{G}_{ab}$ acquires the same factor. In fact, it can be easily shown that if one does not perform this redefinition then there is no way to absorb the divergences.

The original equations are

$$\kappa^{-1}\mathbb{G}_{ab}=T_{ab}(\phi_0)+\langle\tilde{T}_{ab}(\tilde{\phi})\rangle+\lambda g_{ab}[G_1]^2/32\ , \tag{4.12}$$

where the effective mass of the free field energy-momentum tensor $\langle\tilde{T}_{ab}\rangle$ is $M^2_{\text{eff}}=\Box\phi_0/\phi_0+\lambda\phi_0^3/3$. Performing the wave-function renormalization and defining $\kappa_r=\kappa Z$ Eq. (4.12) becomes

$$\kappa_r^{-1}\mathbb{G}_{ab}=T_{ab}(\tilde{\phi}_0)+\langle\tilde{T}_{ab}(\hat{\phi})\rangle/Z+\lambda g_{ab}[G_1]^2/32Z\ , \tag{4.13}$$

where now the effective mass is $M^2_{\text{eff}}=\Box\tilde{\phi}_0/\tilde{\phi}_0+\lambda Z\tilde{\phi}_0^2/3$. After a long calculation (similar to the one presented for the precarious phase) we find

$$\kappa_r^{-1}\mathbb{G}_{ab}=T^{\text{eff}}_{ab}/(\phi_0)\ , \tag{4.14}$$

where $T^{\text{eff}}_{ab}(\phi_0)$ is the energy-momentum tensor of a classical field with the potential

$$V_{\text{eff}}(\tilde{\phi}_0)=m^2\tilde{\phi}_0^2/3+(\xi+\tfrac{1}{12})R\tilde{\phi}_0^2/3 +\tilde{\phi}_0^4[b-a/192\pi^2+a\ln(\tilde{\phi}_0^2/3\mu^2)/96\pi^2]\ , \tag{4.15}$$

i.e., $T^{\text{eff}}_{ab}(\phi_0)$ is the classical energy-momentum tensor of a field that satisfies Eq. (2.20). This remarkably simple result is thus coherent with the one obtained in Sec. II B: the autonomous phase does not include the effects of particle production, trace anomaly, etc.; all the new kinetic terms are erased by the wave-function renormalization.

To end the general discussion of the renormalizability, we will stress an important point. The analysis that we have done both with the mean value and SEE was based on the use of the SDWE for the propagator $G_1(x,x')$ in n dimensions. This was crucial when proving the renormalizability of the theory.

When solving a concrete problem, the propagator is fixed imposing some boundary conditions (for example, given its Cauchy data on a spacelike hypersurface). In order to assure the renormalizability, the boundary conditions must be such that the singular structure of the propagator is the same that the one of the SDW propagator (i.e., the renormalizability implies strong restrictions on the allowed Cauchy data, note that these restrictions appear even in flat space). This problem is well known in the context of quantum field theory in curved spaces. In fact, when studying quantum *free* fields on a given gravitational background, the quantity $\langle T_{ab}\rangle$ is renormalized substrating $\langle T_{ab}\rangle_{\text{SDW4}}$, which is the energy-momentum tensor computed through Eq. (4.6) (with $\lambda=0$ of course) and using the first terms of the SDWE (Ref. 4). Thus one sees that, even in this simple case, not all the two-point functions are adequate since $\langle T_{ab}\rangle-\langle T_{ab}\rangle_{\text{SDW4}}$ must be finite. A similar situation arises for interacting fields in the one-loop approximation, we studied it in detail in Ref. 5 and the restrictions are of the same type. So, what we will find here is nothing else but the generalization of all these well-known results. In flat space we have the same phenomenon but, as the initial-value problem is not a usual problem in field theory, it has been noticed in the literature only very recently[11,27] and may appear as surprising.

Let us now be a little more specific. In the calculations done in Sec. II, we used only the first two terms of the SDWE, that is to say,

$$[G_1]_{\text{SDW2}}=(8\pi^2)^{-1}(M^2/\mu^2)^{(n-4)/2} \times\{\Gamma(1-n/2)M^2+\Gamma(2-n/2)[W_1]\}\ . \tag{4.16}$$

Equation (4.16) is the second-order adiabatic expansion for the propagator and thus the allowed solutions of the G_1 equation are those which satisfy

$$[G_1]=[G_1]_{\text{SDW2}}+\text{finite terms}\ . \tag{4.17}$$

The dimensional regularization may not be very useful in some specific calculations so it is interesting to write the restriction (4.17) in a different way. For instance, if the regularization is performed through the point-splitting method, Eq. (4.17) becomes

$$G_1(x,t,x',t)=2/\sigma+[M^2+(\xi-\tfrac{1}{6})R]\ln\sigma+O(\sigma\ln\sigma)\ , \tag{4.18}$$

i.e., the propagator must have Hadamard singularities[18] [the function M^2 in Eqs. (4.16) and (4.18) should be determined self-consistently, as we mentioned in Sec. II].

Another possibility is to use the adiabatic regularization method, that is particularly useful when solving some concrete problems, for example, when the metric and the mean values depend only on time. It is well known, as was proved in the context of QFT in curved spacetime, that the adiabatic propagator is of the Hadamard type (see Ref. 18, and references therein). This result allows the use of the method, as was done in Refs. 11, 14, and 27 when studying the propagator equation in flat (or Robertson-Walker) space. Here we showed which are the restrictions imposed by the renormalizability when the metric of the spacetime and the mean value of the field are arbitrary.

The conditions (4.17) and (4.18) are sufficient to guarantee the renormalizability of the mean-value equation in a fixed (but, arbitrary) gravitational background. If we also consider the SEE, then the restrictions that arise are stronger: the allowed solutions are those which reproduce the SDWE up to the fourth adiabatic order (in the point-splitting method the term $\sigma\ln\sigma$ must also be reproduced[4,18]).

B. de Sitter self-consistent solutions

The task of finding solutions of the SEE is a rather difficult one. The main problem is that these equations are strongly coupled with the mean-value and propagator equations. Some exact solutions to this system are well known for free fields. In fact, in this case, the computation of the renormalized energy-momentum tensor can be done in several background metrics. After doing that, one can verify if the SEE are satisfied and, if one is lucky, some exact solutions of the system can be found. The most famous solution is the de Sitter spacetime.

Now, we will study the de Sitter-type solutions of the SEE that appear when the quantum fields are self-interacting. We will do it as an application of the results obtained previously. The first thing that we have to note is that both in the $1/N$ and GA the de Sitter solution is possible only if $\phi_0=0$. We can show this in several ways, the most direct one is to examine the equations for the fluctuation propagator. In both cases, if ϕ_0 is constant and nonvanishing, they reduce to

$$\Box G_1(x,x')=0 \tag{4.19}$$

after the use of the mean-value equation. Thus, the fluctuation field is effectively a free field with vanishing mass and minimal coupling. It is well known that, in that case, a de Sitter-invariant vacuum state does not exist (see, for example, Ref. 28). This means that a state in which $[G_1]$ is constant does not exist and, if this is so, the SEE cannot be satisfied for a de Sitter metric.

Taking this into account, we are going to study the $\phi_0=0$ case and to do this we will make use of the Ω equation. The quantum state of the fluctuation field will be chosen as the Bunch-Davies vacuum. In this state the coincidence limit of the propagator of a free field with mass M is given by the equation[4]

$$\begin{aligned}[G_1]=\ &2H^2\Gamma(\nu_n-\tfrac{1}{2}+n/2)\Gamma(-\nu_n-\tfrac{1}{2}+n/2)\\ &\times\Gamma(1-n/2)(4\pi H^2)^{-n/2}\Gamma^{-1}(\nu_n+\tfrac{1}{2})\\ &\times\Gamma^{-1}(-\nu_n+\tfrac{1}{2})\ ,\end{aligned} \tag{4.20}$$

where $R=12H^2$, and $\nu_n=(n-1)^2/4-M^2H^2-\xi n(n-1)$. Setting $M^2=m^2+\Omega^2$, expanding around $n=4$ and using the reparametrization (2.11) Eq. (4.20) gives

$$\begin{aligned}\Omega^2[\tfrac{1}{2}-B-\ln(\Omega^2/\mu^2)/2]-(C+DR)+R/36&\\ -\Omega^2[\psi_++\psi_-+\ln(R/12\Omega^2)]/2=0\ ,&\end{aligned} \tag{4.21}$$

where $\psi_\pm=\psi(\tfrac{3}{2}\pm\nu_4)$, $\psi(z)=[\ln\Gamma(z)]'$.

This is the Ω equation in de Sitter space. Comparing it with (2.15) we can read which is, in this case, the function $F(\Omega^2,R_{abcd})$ that we defined there. Using this result we find

$$\xi_{\rm ren}=\tfrac{1}{6}-D/B\ . \tag{4.22}$$

Now we can replace the change of variables defined by (2.16a) and (2.16b) and by (4.22) in the Ω equation (4.21) obtaining

$$\begin{aligned}\Omega^2=\ &M^2_{\rm ren}+(\xi_{\rm ren}-\tfrac{1}{6})R\\ &-(\xi_{\rm ren}-\tfrac{1}{9})R/(1+96\pi^2/\lambda_{\rm ren})\\ &+\Omega^2[\psi_++\psi_-+\ln(R/12M^2_{\rm ren})]/(1+96\pi^2/\lambda_{\rm ren})\ .\end{aligned} \tag{4.23}$$

In order to write the trace of the SEE and to renormalize it, we can proceed in a similar way. The expectation value of the trace of the energy-momentum tensor can be written as

$$\langle T^a_a\rangle=-M^2[G_1]/2+n(M^2-m^2)^2/2\lambda\ . \tag{4.24}$$

In this equation, we can replace the "closed" expression of $[G_1]$ that we computed before. As it should, the $[G_1]$ obtained from (4.20) has the same residue in $n=4$ as the Schwinger-DeWitt one. Moreover, it contains (at least) the first three terms of the SDWE (2.5). In this case (de Sitter spacetime), the third term of the sum does not contribute to the divergences appearing in $\langle T_{ab}\rangle$. However, in general it does [it gives a purely geometric divergent term that is proportional to $(n-4)^{-1}(H_{ab}-H^{(1)}_{ab})$]. As we want to apply a renormalization recipe valid for an arbitrary spacetime, we have to remember this and absorb this term in the bare parameters appearing on the left-hand side of the SEE. With this in mind, we can write the SEE after some calculations as

$$R-4\Lambda=\kappa N(16\pi^2)^{-1}[M^2_{\rm ren}(1+96\pi^2/\lambda_{\rm ren})\Omega^2-\Omega^4/2+R^2/2160] \tag{4.25}$$

[for one scalar field in the GA we must set $N=1$ in (4.25)]. This equation is obviously coupled with the Ω equation (4.23). Both together determine the aspect of the de Sitter solution to the back-reaction problem. There is something peculiar about (4.25): in the weak-coupling limit ($\lambda_{\rm ren}\to 0$), a huge cosmological constant appears [note that in this limit from Eq. (4.23) one finds that $\Omega^2=M^2_{\rm ren}+(\xi_{\rm ren}-\frac{1}{6})R$]. This is a typical nonperturbative result that was noted first in Ref. 8 when computing the GEP and evaluating $V_{\rm GEP}(\phi_0=0)$.

V. EFFECTIVE EQUATIONS FOR THE CLASSICAL FIELD

In order to study phase transitions in the early Universe, some authors argued that, due to the $\phi\to-\phi$ symmetry, the mean value of the field remains zero for all time. In this case, the order parameter of the phase transition turns out to be the classical field $\phi_{\rm cl}=\langle\phi^2\rangle^{1/2}$. In fact, Hawking and Moss have shown[29] in particular examples that $\phi_{\rm cl}$ satisfies a classical evolution equation with the classical potential replaced by the one-loop effective potential. Of course the quantity $\langle\phi^2\rangle$ is divergent and must be regulated.

In the formalism we presented in the previous sections, there is a natural way to address this issue. In fact, one can set $\phi_0=0$ and study the effective equations satisfied by the *finite* quantity Ω, which is our natural "classical field," and the spacetime metric. We will show these equations in what follows.

The analysis can be done simultaneously for the single field theory in the GA and the O(N) model in the large-N approximation since, as we mentioned before, the G_1 equation and the SEE are formally the same in both cases. We will restrict ourselves to the precarious phase since only in this case Ω is a finite function. In Sec. II we proved that, after the use of the adiabatic approximation,

$$-\Box\Omega+6(C+DR)\Omega+6\Omega^3[B-\tfrac{1}{2}+\ln(\Omega^2/\mu^2)/2]+\cdots=0\ . \tag{5.1}$$

This is a very interesting result. Starting with the GA and neglecting the terms that contain more than two derivatives of Ω, one ends with an equation for the classical field which is similar to the one used in the one-loop approximation. Ω is a field that satisfies a classical equation with "effective potential"

$$V_{\rm eff}(\Omega)=3(C+DR)\Omega^2+\tfrac{3}{2}\Omega^4\left[\tfrac{1}{2}\ln\frac{\Omega^2}{\mu^2}+B-\tfrac{3}{4}\right]+\cdots\ . \tag{5.2}$$

Thus, although the "real effective potential" of the theory (which must be read off the mean-value equation) does not have a closed form, we obtained an expression for the effective potential of the classical field Ω. The ellipsis in Eq. (5.2) denote terms that are nonzero only in curved spacetime. In flat space, where these additional terms vanish, the form of the effective potential resembles the Coleman-Weinberg one.[30] However, we stress that in fact it has nothing to do with the one-loop approximation; we obtained it in a very different context. Of course all the (kinetic) terms neglected in Eq. (5.1) can be systematically included in order to improve the approximation.

Let us now consider the SEE. From Eq. (4.11) we see that the source of the Einstein equations (for $\phi_0=0$) is

$$T^{\rm eff}_{ab}=D(\Omega^2G_{ab}-\Omega^2{}_{,ab}+g_{ab}\Box\Omega^2)/8\pi^2-g_{ab}\Omega^2C/16\pi^2+g_{ab}\Omega^4[\tfrac{3}{4}-B-(\ln\Omega^2/\mu^2)/2]/32\pi^2+\Sigma_{ab}\ . \tag{5.3}$$

Retaining again no more than two derivatives of Ω in the summation one finds

$$\Sigma_{ab}=\Omega^{-2}(g_{ab}[W_3]-2[W_2]_{,ab}+\tfrac{1}{6}\Omega^2{}_{,a}\Omega^2{}_{,b}-\tfrac{1}{6}g_{ab}\Omega^2{}_{,c}\Omega^{2,c})/32\pi^2\ . \tag{5.4}$$

In this approximation $[W_3]=\Omega^2{}_{,c}\Omega^{2,c}/12+\cdots$ and $[W_{2,ab}]=\cdots$ where the dots denote terms which do not contain derivatives of Ω. Replacing Eq. (5.4) into (5.3) we get

$$T^{\rm eff}_{ab}(\Omega)=\frac{1}{48\pi^2}\left[\Omega_{,a}\Omega_{,b}-\frac{g_{ab}}{2}\Omega_{,c}\Omega^{,c}+6D(\Omega^2G_{ab}-\Omega^2_{,ab}+g_{ab}\Box\Omega^2)-3\Omega^2Cg_{ab}-\tfrac{3}{2}g_{ab}\Omega^4\left[\tfrac{1}{2}\ln\frac{\Omega^2}{\mu^2}+B-\tfrac{3}{4}\right]\right], \tag{5.5}$$

i.e., $T^{\rm eff}_{ab}(\Omega)$ is the energy-momentum tensor of a classical field which satisfies Eq. (5.1). The SEE are then

$$\kappa^{-1}(G_{ab}+\Lambda g_{ab})+[\alpha_1-(\ln\Omega^2/\mu^2)/2880\pi^2]H^{(1)}_{ab}+[\alpha_2+(\ln\Omega^2/\mu^2)/960\pi^2]H_{ab}=T^{\rm eff}_{ab}(\Omega)\ . \tag{5.6}$$

We have thus shown that the effective equations for the classical field Ω in the precarious phase of the Gaussian or large-N approximation resemble the one-loop "approximate" approach, in which the quantum corrections are taken into account only through the use of the effective potential. In addition, the expressions obtained here for the effective potential and the energy-momentum tensor are valid for all values of the field Ω, unlike in the one-loop approximation where the region of validity is restricted.[30]

VI. CONCLUSIONS

In this work we studied the $\lambda\phi^4$ theory in an arbitrary curved spacetime using the GA and the interacting O(N) model using the large-N approximation. We showed, through the analysis of the mean-value equation, that the single scalar theory admits two very different phases in the GA, while using the $1/N$ approximation only one phase of the O(N) model can be studied. The two mentioned phases have been previously found in flat space; we obtain here (using a different technique more adapted to curved-space problems) their generalization to nonflat spacetimes. In the precarious phase the bare parameters are finite or infinitesimal and a closed form for the effective action (or the mean-value equation) does not exist. The quantity $(\lambda[G_1]/4)^{1/2}=\Omega(g_{ab},\phi_0)$ is finite and can be interpreted as a classical field, being $\Omega(\eta_{ab},0)=M^2$ the renormalized mass of the field. On the other hand, in the autonomous phase, an infinite wave-function renormalization is performed and Ω diverges. It is easy to find in this phase a closed form for the effective action and for the mean-value equation; nevertheless this phase shows some unusual properties as the final result is simply

$$L_{\text{eff}}=(-g)^{-1/2}[g_{ab}\tilde{\phi}_0^{;a}\tilde{\phi}_0^{;b}/2-V_{\text{eff}}(\tilde{\phi}_0)]$$

then some important physical information is absent.

We also analyzed the SEE for the model and phases above mentioned. We showed that the SEE are finite once the bare parameters are chosen in such a way that the mean-value equation is finite. Of course there are some geometric divergences that can be absorbed in the bare gravitational constants. In the autonomous phase, we found necessary to renormalize the gravitational constants with the factor Z which defines the wave-function renormalization; like for the mean-value equation, this renormalization again erases terms of physical interest (as, for instance, the trace anomaly and particle production). It should also be stressed that the only way to find this phase is to renormalize the mean value of the field instead of the complete field operator. This point deserves further study.

An important point is that, besides a clever choice of the bare parameters, it is also necessary to restrict the initial conditions in order to ensure the renormalizability of the theory. In fact, the initial values that fix the two-point function G_1 must be such that the propagator has the SDWE singularities. This situation generalizes a well-known result that arises even for free fields in curved space: the two-point function must copy the SDWE singularities (in other words it must have Hadamard singularity structure) in order to obtain finite SEE.

As an example of application we briefly analyzed the self-consistent de Sitter solutions for both models showing that there are no self-consistent solutions if the mean value of the field is different from zero. For an identically zero mean value, we wrote down the equations for the order parameter (assumed constant). In the free field limit, one surprisingly finds a large cosmological constant which is absent if one treats the problem setting $\lambda=0$ from the beginning.

Our main motivation in studying interacting scalar theories in curved backgrounds using different approximations and including the SEE is their application to the analysis of the quantum effects in the very early Universe. In this paper we presented a formalism that can be used to address this issue. In addition, we have shown that the standard approximate calculations used in order to build up inflationary models can be understood using the precarious phase. In fact, setting $\phi_0=0$ we found the effective equations that satisfy the (finite) classical field $\Omega(x)$; in the quasistatic approximation these equations are very similar (in form) to the usually considered ones. Nevertheless, they do not have the drawbacks of the one-loop equations and new corrections can be included through a derivative expansion.

[1] A. Linde, Rep. Prog. Phys. **47**, 925 (1984); R. Brandenberger, Rev. Mod. Phys. **51**, 1 (1985).
[2] A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[3] A. Starobinsky, Pis'ma Astron. Zh. **10**, 323 (1984) [Sov. Astron. Lett. **10**, 135 (1984)]; M. Mijic, M. Morris, and W. Suen, Phys. Rev. D **34**, 2934 (1986).
[4] N. Birrell and P. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).
[5] J. Paz and F. Mazzitelli, Phys. Rev. D **37**, 2170 (1988).
[6] B. Hu and J. O'Connor, Phys. Rev. D **30**, 743 (1984); A. Ringwald, Ann. Phys. (N.Y.) **177**, 129 (1987); J. Guven, Phys. Rev. D **35**, 2378 (1987); B. Hu and Y. Zhang, *ibid.* **37**, 2151 (1988).
[7] R. Jackiw and A. Kerman, Phys. Lett. **71A**, 158 (1979).
[8] P. Stevenson, Phys. Rev. D **32**, 1389 (1985).
[9] M. Altenbokum, U. Kaulfuss, and J. Verbaarschot, Phys. Rev. D **34**, 1840 (1986); M. Altenbokum and U. Kaulfuss, *ibid.* **35**, 609 (1987).
[10] F. Cooper, S. Pi, and P. Stancioff, Phys. Rev. D **34**, 3831 (1986).
[11] S. Pi and M. Samiullah, Phys. Rev. D **36**, 3128 (1987).
[12] J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974); G. Mazenko, Phys. Rev. Lett. **54**, 2163 (1985); Phys. Rev. D **34**, 2223 (1986).
[13] J. Hartle, in *Quantum Gravity,* proceedings of the Second Seminar, Moscow, USSR, 1981, edited by M. A. Markov and P. C. West (Plenum, New York, 1984).
[14] F. Cooper and E. Mottola, Mod. Phys. Lett. **A2**, 635 (1987).
[15] F. Mazzitelli and J. Paz, Phys. Rev. D **37**, 3533 (1988).
[16] P. Stevenson and R. Tarrach, Phys. Lett. B **176**, 436 (1986); P. Stevenson, B. Alles, and R. Tarrach, Phys. Rev. **D 35**, 2407 (1987).
[17] S. Paban and R. Tarrach, Phys. Lett. B **197**, 383 (1987).
[18] F. Mazzitelli, J. Paz, and M. Castagnino, Phys. Rev. D **36**, 2994 (1987).
[19] I. Jack and L. Parker, Phys. Rev. D **31**, 2439 (1985).
[20] A. Barbinsky and G. Vilkovisky, Phys. Rep. **119**, 1 (1985).
[21] J. Soto, Phys. Lett. B **188**, 340 (1987).
[22] P. Stevenson, Rice University report (unpublished).
[23] E. Witten, Nucl. Phys. **B160**, 57 (1979).
[24] S. Christensen, Phys. Rev. D **14**, 2490 (1976).
[25] M. Castagnino, D. Harari, and C. Nunez, J. Math. Phys. **28**, 184 (1987).
[26] M. Duff, in *Quantum Gravity 2,* proceedings of the Oxford Conference, Oxford, England, 1980, edited by C. Isham, R. Penrose, and D. Sciama (Clarendon, Oxford, 1981), p. 81.
[27] F. Cooper and E. Mottola, Phys. Rev. D **36**, 3114 (1987).
[28] B. Allen and A. Follacci, Phys. Rev. D **35**, 3771 (1987).
[29] S. Hawking and I. Moss, Nucl. Phys. **B224**, 180 (1983).
[30] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1889 (1973).