Stochastic Quantization of Autonomous Φ^4

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Abstract: The non-perturbative autonomous renormalization of the scalar Φ^4-model is applied in the framework of stochastic quantization. I show that this requires a selective, momentum-dependent renormalization of the Onsager coefficient λ, a direct consequence of the characteristic wavefunction renormalization applied. As a result, I obtain a Langevin equation for the renormalized constant mode of the field, which is solved numerically. It is demonstrated for temperature zero that, starting from specified initial conditions, the system relaxes to its equilibrium state, the symmetry-breaking vacuum of the "static" Φ^4-theory.

Recently there has been much interest in the autonomous renormalization (AR) of the Higgs field [1]. The AR emerges very naturally as an alternative solution to the renormalization group equation [2, 3], where the major difference to the conventional perturbative procedure is an infinite wavefunction renormalization [4, 5]. Concerning the number of free parameters the AR turns out to be very economical; there is only one for the scale invariant theory . As a consequence, it is possible to make a concrete prediction for the mass of the Higgs boson, which comes out surprisingly heavy, at about 2 TeV [2, 6, 7]. For more technical and phenomenological details concerning the AR, I refer to the literature cited above.

The question posed in this Letter is whether the AR can also be applied in the context of stochastic quantization. The basic idea of this method is to assign a purely relaxational dynamics in a *fictitious* time coordinate, which I will simply call time in the following, to the system [8]. Starting from some non-equilibrium initial state, on account of ergodicity and detailed balance, the system relaxes to its equilibrium state, the ground state of the euclidean quantum field theory. The latter will be called the *static* theory below. The dynamics is described by a Langevin equation and has many formal analogues in statistical physics. There, however, a real time-dependent process, like the dynamics of a ferromagnet close to its Curie point, is described by the equations. Reasons for introducing stochastic quantization have been advantages in gauge theories [8] and numerical simulations [9].

Consider now the scale invariant euclidean Φ^4 theory in a finite periodic hypercube with volume $\Omega = L^d$, described by the action

$$S[\phi_B] = \int_\Omega \mathrm{d}^d x \left[\frac{1}{2}(\nabla\phi_B)^2 + \frac{g}{4!}\,\phi_B^4 \right] \; . \tag{1}$$

From the static calculation one knows that the AR requires a *selective* wavefunction renormalization, *i.e.*, one has to distinguish between constant and finite-momentum (FM) modes [4]. While the FM modes are essentially not renormalized, the (bare) constant mode has to be rescaled by a logarithmically divergent Z-factor. In a previous work on the static theory [3], it was demonstrated that the most natural starting point for the AR is a finite system. Although one is not really interested in finite-size effects, many features of the model, like the convexity of the effective potential, the emergence of spontaneous symmetry breaking, and finite-temperature effects, can best be understood by starting from a finite system and then taking carefully the infinite-volume limit. Much the same strategy will be pursued in this Letter for the stochastically quantized model. For the sake of simplicity, the calculations will be restricted to the zero-temperature case. To treat the more general situation, one dimension of the volume Ω has to be kept finite [3].

Following Consoli and Stevenson [4], I denote (in slightly adapted form) the relation between

bare and renormalized Fourier amplitudes of the field by

$$\tilde{\phi}_B(\mathbf{p}, t) = Z(\mathbf{p})^{\frac{1}{2}} \tilde{\phi}(\mathbf{p}, t) = \left[\sqrt{3} \, \epsilon^{-\frac{1}{2}} \delta_{\mathbf{p},\mathbf{0}} + (1 - \delta_{\mathbf{p},\mathbf{0}}) \right] \tilde{\phi}(\mathbf{p}, t) . \tag{2}$$

Since I will use dimensional regularization in the following, logarithmic divergences are expressed by powers of $1/\epsilon$ (with $\epsilon = 4 - d$) from the start [10]. The vector $\mathbf{p}$ in (2) has d components with discrete values $p_i = 2\pi k_i/L$, where $k_i \in \mathbb{Z}$. The factor $\sqrt{3}$ in (2) is a result of the static calculation [2]. It comes from a finite rescaling of the constant mode (called z_0 in [2]), which guarantees the proper normalization of the (static) propagator.

The dynamics assigned to the system is described by the Langevin equation

$$\partial_t \tilde{\phi}_B(\mathbf{p}, t) + \lambda_B(\mathbf{p}) \frac{\delta S[\phi_B]}{\delta \tilde{\phi}_B(-\mathbf{p}, t)} = \tilde{\xi}_B(\mathbf{p}, t) , \tag{3}$$

where $\tilde{\xi}_B(\mathbf{p}, t)$ is a Gaussian random "force" with mean zero and variance

$$\langle \tilde{\xi}(\mathbf{p}, t) \, \tilde{\xi}(B\mathbf{q}, t') \rangle = 2 \lambda_B(\mathbf{p}) \, \delta_{\mathbf{p},-\mathbf{q}} \, \delta(t - t') . \tag{4}$$

From work on classical statistical dynamics [11], it is well known that the renormalization of the Onsager coefficient λ is closely related to the wavefunction renormalization. Hence, from this point of view, it is natural to infer a relation of the form

$$\lambda_B(\mathbf{p}) = [z_\lambda \, \epsilon^\sigma \, \delta_{\mathbf{p},\mathbf{0}} + (1 - \delta_{\mathbf{p},\mathbf{0}})] \, \lambda , \tag{5}$$

where the finite normalization constant z_λ and the exponent σ are to be determined below.

Like in the static calculation [3], the first step towards a solution of (3) is the approximative integration of the FM-modes in a one-loop type procedure. Similar methods have been applied in the context of critical dynamics in finite-size systems [12, 13]. Here I am following closely the method suggested by Goldschmidt [13]. After integrating out the FM modes, one obtains the Langevin equation for the constant mode:

$$\partial_t \phi_0(t) + \epsilon^\sigma z_\lambda \lambda \frac{g}{6} \left(3 \, C(t) + \phi_0(t)^2 \right) \phi_0(t) = \xi_0(t) , \tag{6}$$

where I have defined

$$\phi_0(t) \equiv \Omega^{-\frac{1}{2}} \tilde{\phi}_B(\mathbf{0}, t) \qquad \text{such that} \qquad \phi_0(t) = \frac{1}{\Omega} \int \mathrm{d}^d x \, \phi_B(\mathbf{x}, t) ,$$

and $\xi_0(t)$ is related to $\tilde{\xi}(\mathbf{0}, t)$ analogously.

In (6), the term $C(t)$ represents the effect of the FM modes. It is the time-dependent analogue of the tad-pole graph of the static one-loop calculation [3] and may be represented in the form

$$C(t) = 2 \lambda L^{-d} \int_0^t \mathrm{d}s \sum_{\mathbf{p} \neq \mathbf{0}} \exp\left(-2\lambda \int_{t-s}^t \mathrm{d}s' \left[\mathbf{p}^2 + \frac{g}{2} \phi_0^2(s') \right] \right) . \tag{7}$$

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Suppose that for large time $\phi_0(t)$ approaches a constant equilibrium value. Then (7) reduces to the well-known static result

$$C(t = \infty) = L^{-d} \sum_{\mathbf{p} \neq \mathbf{0}} \frac{1}{\mathbf{p}^2 + g\,\phi_0(t = \infty)^2/2} \ . \tag{8}$$

For the further evaluation, (7) may be rewritten as

$$C(t) = 2\lambda\, L^{-d} \int_0^t \mathrm{d}s\, \mathcal{B}\left(\frac{8\pi^2 \lambda s}{L^2}\right)\, \exp\left(-\lambda\, g \int_0^s \mathrm{d}s'\, \phi_0(t - s')^2\right) \tag{9}$$

with

$$\mathcal{B}(x) = \mathcal{A}(x)^d - 1 \qquad \text{and} \qquad \mathcal{A}(x) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-k^2 x} \ .$$

The fact that the sum in (7) does not extend over zero momentum has no influence on the UV-behavior. Consequently, $C(t)$ contains the typical $1/\epsilon$ pole. In (9) it is caused by the divergence of the integrand for small argument, where $\mathcal{B}(x) \sim (\pi/x)^{d/2}$.

At this point a brief comment on the *initial conditions* is appropriate: For the evolution of the field expectation, one initial condition will be to assign some specified value to the constant mode at $t = 0$. Moreover, from (7) and (9), it becomes obvious that another "initial condition" is $C(t = 0) = 0$, *i.e.*, fluctuations of the FM modes are suppressed at $t = 0$, which, in a sense, emerges here as the natural initial state of the relaxational process.

For the further evaluation of (9) it is convenient to isolate the pole by rewriting the whole expression in the form

$$C(t) = C_{\mathrm{div}}(t) + \Delta C(t) + \mathcal{O}\left(L^{-d}\right). \tag{10}$$

The contributions $\mathcal{O}\left(L^{-d}\right)$ do not influence the dynamics in the infinite-volume limit.

The UV-divergent part C_{div} is given by

$$C_{\mathrm{div}}(t) = \frac{(2\lambda)^{1-d/2}}{(4\pi)^{d/2}} \int_0^t \mathrm{d}s\, s^{-d/2} \exp\left(-\lambda\, g\, s\, \phi_0(t)^2\right) \ . \tag{11}$$

The integral can be calculated exactly, and, after analytic continuation to $4 - \epsilon$ dimensions, the result may be expanded in a power series in ϵ:

$$C_{\mathrm{div}}(t) = -\frac{g\phi_0(t)^2}{(4\pi)^2 \epsilon} + C_{\mathrm{div}}^{(0)}(t) + \mathcal{O}(\epsilon), \tag{12}$$

where the finite contribution is given by

$$C_{\mathrm{div}}^{(0)} = -\frac{\mathrm{e}^{-\Theta}}{(4\pi)^2\, 2\lambda t} + \frac{g\, \phi_0^2}{2(4\pi)^2}\left[\log\left(\frac{g\, \phi_0^2}{16\pi^2\, \Phi_v^2}\right) - \log(\Theta)\,\mathrm{e}^{-\Theta} - \left.\frac{\mathrm{d}\,\gamma(x, \Theta)}{\mathrm{d}x}\right|_{x=1} - C_E\right] \tag{13}$$

with the abbreviation $\Theta = \Theta(t) = \lambda t\, g\phi_0^2(t)$. In (13), $\gamma(x, y)$ denotes the incomplete Gamma function and C_E Euler's constant. Φ_v emerges in this stage of the calculation as an arbitrary

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momentum scale generated by dimensional transmutation, but its normalization is chosen such that it later on also turns out as the vacuum expectation value of the field.

The second term on the rhs of (10) is a memory term given by

$$\Delta C(t) = \frac{(2\lambda)^{1-d/2}}{(4\pi)^{d/2}} \int_0^t \mathrm{d}s\, s^{-d/2} \left[\exp\left(-\lambda g \int_{t-s}^t \mathrm{d}s'\, \phi_0(s')^2 \right) - \exp\left(-\lambda s\, g\phi_0(t)^2 \right) \right]. \tag{14}$$

Like in the static calculation [3], the pole in (11) is cancelled by the nonlinear term in (6) by assigning the UV-flows

$$\phi_0(t) = \sqrt{3}\, \epsilon^{-1/2} \Phi(t) \qquad \text{and} \qquad g = \frac{(4\pi)^2 \epsilon}{3} \tag{15}$$

to the constant mode (*cf.* Eqn. (2)) and the bare coupling, respectively. After this cancellation, the remainder $C_{\mathrm{div}}^{(0)}(t) + \Delta C(t)$ on the lhs of (6) is of order ϵ^0. Thus, together with the factor g, the deterministic "force" would be of order ϵ. A finite contribution may be obtained by choosing $\sigma = -1$ in (5). Moreover, the normalization z_λ can be fixed by demanding that the deterministic part is equal to the derivative of the *static* effective potential for $t \to \infty$. In this limit, the memory term $\Delta C(t)$ tends to zero, and only the first term in square brackets in (13) survives. As a result, one derives the relation

$$\frac{\partial V(\Phi)}{\partial \Phi} = \frac{4\pi^2 z_\lambda}{3} \Phi^3 \log\left(\frac{\Phi^2}{\Phi_v^2} \right), \tag{16}$$

from which it is also obvious that $\Phi = \Phi_v$ (where the logarithm has its zero) corresponds to the asymmetric ground state. Now the second derivative of V with respect to the renormalized field must be equal to the squared propagator mass, called M_H in the following. The latter can be read from (8), and inserting the UV-flows (15) one finds

$$M_H^2 = 8\pi^2\, \Phi_v^2\,.$$

Eventually, the second derivative of V, calculated from (16), is equal to M_H^2 if $z_\lambda = 3$.

What, then, happens to the noise in view of finite and infinite renormalizations of the field? Carrying out the rescalings in (6) gives rise to a renormalized Langevin equation of the form $\partial_t \Phi(t) + \text{finite} = 3^{-1/2}\epsilon^{1/2}\xi(t)$. Thus, the most natural UV-flow for the *constant* mode of the noise is

$$\xi(t) = \sqrt{3}\, \epsilon^{-1/2} \Xi(t)\,. \tag{17}$$

The renormalized noise $\Xi(t)$ has mean zero, and with (4) and (5) one obtains for the variance:

$$\langle \Xi(t)\Xi(t') \rangle = 2\lambda\, L^{-d}\, \delta(t-t')\,. \tag{18}$$

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The average magnitude of the noise is $\mathcal{O}(L^{-d/2})$, and, therefore, in the infinite-volume limit the dynamics becomes essentially deterministic. This is completely analogous to the static case, where a saddle-point approximation for the path integral over the constant mode becomes exact in the infinite-volume limit [3].

With (13), (14), and (15) in (6) and after introducing dimensionless variables $\bar{t} = 2\lambda t M_H^2$ and $\bar{\Phi} = \Phi/\Phi_v$, the equation of motion for the constant mode reads

$$
\begin{aligned}
\partial_{\bar{t}}\bar{\Phi} \;=\; & \frac{1}{4}\,\bar{\Phi}\left\{\bar{\Phi}^2\left[\log(\bar{t}\,\bar{\Phi}^2)\exp(-\bar{t}\,\bar{\Phi}^2) + \left.\frac{\mathrm{d}\,\gamma\,(x,\bar{t}\,\bar{\Phi}^2)}{\mathrm{d}x}\right|_{x=1} + C_E - \log(\bar{\Phi}^2)\right]\right. \\
& \left. + \frac{\exp(-\bar{t}\,\bar{\Phi}^2)}{\bar{t}} - \int_0^{\bar{t}}\mathrm{d}s\,s^{-2}\left[\exp\left(-\int_{\bar{t}-s}^{\bar{t}}\mathrm{d}s'\,\bar{\Phi}(s')^2\right) - \exp\left(-s\,\bar{\Phi}^2\right)\right]\right\}, \quad (19)
\end{aligned}
$$

where the argument of $\bar{\Phi}$ has been suppressed when it is $\bar{t}$.

There is one special feature, which one encounters on the way to a numerical solution of (19). The first term on the rhs of (13) and, in turn, the first term in the second line of (19) are singular for $\bar{t} \to 0$. This is due to the initial state discussed above, and the singularity could be avoided by choosing different, though more complicated, initial conditions. Similar so-called short-time singularities are well known from statistical physics, where they give rise to anomalous behavior of relaxational processes in their early stages [14]. In stochastic quantization, where one is mainly interested in the equilibrium state, the short-time behavior seems to be merely a technical problem. A non-singular equation may be obtained with the ansatz $\bar{\Phi}(\bar{t}) = \bar{t}^{1/4}\Psi(\bar{t})$ with finite initial value $\Psi(0)$.

A number of numerical solutions is depicted in the Figure. $\Psi(0)$ serves as a parameter. As a consequence of the short-time singularity, all trajectories of $\bar{\Phi}$ have to start from zero. Then the expectation value increases and overshoots the equilibrium value, but asymptotically, for large time, all solutions approach $\bar{\Phi} = 1$ from above. There are no solutions that cross the origin.

In summary, I have applied the autonomous renormalization to the stochastically quantized Φ^4-theory. The most important difference compared with conventional perturbation theory is a selective renormalization of the Onsager coefficient λ as formulated in Eqn. (5). A finite equation of motion, which becomes purely deterministic in the infinite-volume limit, has been derived and solved numerically. Asymptotically, all solutions approach the equilibrium state, the asymmetric ground state of the static Φ^4-model.

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Figure captions

Fig. Numerical solutions of Eqn. (19) with the ansatz $\bar{\Phi}(\bar{t}) = \bar{t}^{1/4}\Psi(\bar{t})$, where $\Psi(0)$ serves as the parameter.

This figure "fig1-1.png" is available in "png" format from:

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