One-loop renormalization and asymptotic behaviour of a higher-derivative scalar theory in curved spacetime

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Abstract

A higher-derivative, interacting, scalar field theory in curved spacetime with the most general action of sigma-model type is studied. The one-loop counterterms of the general theory are found. The renormalization group equations corresponding to two different, multiplicatively renormalizable variants of the same are derived. The analysis of their asymptotic solutions shows that, depending on the sign of one of the coupling constants, we can construct an asymptotically free theory which is also asymptotically conformal invariant at strong (or small) curvature. The connection that can be established between one of the multiplicatively renormalizable variants of the theory and the effective theory of the conformal factor, aiming at the description of quantum gravity at large distances, is investigated.

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1. Introduction. There are several motivations for the study of quantum field theories involving higher-derivative terms. To begin, it is well known in string theory (see [1], for a review) that if one wants to study the massive higher-spin modes, one has to modify the standard σ -model action, by adding to it an infinite number of terms containing all possible derivatives. In particular, the terms with two derivatives in that σ -model describe the massless modes of the string, the terms with four derivatives describe the massive higher-spin modes at first level, and so on.

Moreover, also the effective action of (super)string theory can be presented under the form of a certain derivative expansion [1] involving higher-derivative terms of any order. Such models —as well as some other higher-derivative gravitational theories— often admit singularity-free solutions (for a recent discussion together with a list of references, see [2, 3]), whose behaviour is, as a rule, in agreement with the violation of the energy conditions, which is in accordance with the singularity theorems [4].

As a second motivation, higher-derivative theories —forgetting about the standard unitarity problem which, most probably, will be solved only in a non-perturbative approach—often exhibit better renormalization properties. A nice example on this line is quantum R^2 -gravity (for a review and a list of references, see [5]), which (unlike Einstein's gravity) is multiplicatively renormalizable [6] and even asymptotically free.

Third, a scalar theory with higher-order derivatives, in the infrared stable fixed point, has been used recently by Antoniadis and Mottola [7] (see also [8, 9]) in order to describe a truncated version of quantum gravity at large distances. This theory, which was obtained by integration over the conformal anomaly [10], was found by these authors to be superrenormalizable.

In the present letter we start a systematic study of higher-derivative scalar theory in curved spacetime. An action of the σ -model type, which is renormalizable in the generalized sense, will be considered (the scalar field will be chosen to be dimensionless). The one-loop divergences of the effective action will be found. For two different variants of this theory (both of them multiplicatively renormalizable), the corresponding renormalization (RG) equations to one loop order are constructed and the asymptotic behaviour of their solutions is investigated in detail. It is shown that, depending on the sign of the initial values of the coupling constants, the theory under discussion can be asymptotically free. Moreover, at strong curvature such asymptotically free theory becomes also asymptotically conformally invariant.

2. One-loop divergences. As starting point, let us impose the condition that the scalar field φ be dimensionless in four-dimensional curved spacetime, namely that $[\varphi] = 0$. Also, there is just one dimensional constant, which has dimensions of mass squared, i.e. $[m^2] = 2$. Then, dimensional considerations lead us to the following action (of sigma-model type and which is renormalizable in the generalized sense)

$$S = \int d^{4}x \sqrt{-g} \left\{ b_{1}(\varphi) \left(\Box \varphi \right)^{2} + b_{2}(\varphi) \left(\nabla_{\mu} \varphi \right) \left(\nabla^{\mu} \varphi \right) \Box \varphi + b_{3}(\varphi) \left[\left(\nabla_{\mu} \varphi \right) \left(\nabla^{\mu} \varphi \right) \right]^{2} \right.$$

$$\left. + b_{4}(\varphi) \left(\nabla_{\mu} \varphi \right) \left(\nabla^{\mu} \varphi \right) + b_{5}(\varphi) + c_{1}(\varphi) R \left(\nabla_{\mu} \varphi \right) \left(\nabla^{\mu} \varphi \right) + c_{2}(\varphi) R^{\mu\nu} \left(\nabla_{\mu} \varphi \right) \left(\nabla_{\nu} \varphi \right) \right.$$

$$\left. + c_{3}(\varphi) R \Box \varphi + a_{1}(\varphi) R_{\mu\nu\alpha\beta}^{2} + a_{2}(\varphi) R_{\mu\nu}^{2} + a_{3}(\varphi) R^{2} + a_{4}(\varphi) R \right\}. \tag{1}$$

It is interesting to notice that non-minimal scalar-gravitational interactions of a new type (with couplings c_1 , c_2 and c_3) appear in action (1). Also, in Eq. (1) all generalized coupling

constants are dimensionless, except for b_4 , b_5 and a_4 , for which we have: $[b_4(\varphi)] = 2$, $[b_5(\varphi)] = 4$, $[a_4(\varphi)] = 2$. All the possible structures which can appear at dimension 4 can be obtained from the terms of Eq. (1) with the only help of integration by parts. We must also take into account that total derivative terms (such as $\Box^2 a(\varphi)$, $\Box R$, etc.) have been dropped from Eq. (1) and will not be considered below. As a further observation, we note that the action (1) gives a way for obtaining a gravitationally induced scalar mass and also (if the variant of (1) under discussion is symmetric under $\varphi \to -\varphi$) curvature induced spontaneous symmetry breaking at tree level. This generalizes the corresponding phenomenon which was found in ref. [14] for the first time in the case of a theory with a scalar potential of the form

$$\lambda \varphi^4 + \xi R \varphi^2. \tag{2}$$

Now we start the discussion of the renormalization structure of the action (1). For the sake of simplicity, in the present paper we will restrict ourselves to the case where $b_1(\varphi) \equiv b_1$, $b_2(\varphi) \equiv b_2$ and $b_3(\varphi) \equiv b_3$ are simply constants (they have no dependence on φ). In order to obtain the one-loop effective action divergences, Γ_{div} , extremely lengthy and tedious calculations are necessary. They rely on the use of the background field method (for a review see [5]) and of Schwinger-De Witt techniques for the computation of the one-loop effective action divergences (see [11] for an introduction and [12] for the generalization to the higher derivative case). Details of this very involved calculation will be given elsewhere. The result we have obtained is

$$\Gamma_{div} = -\frac{2}{\epsilon} \int d^4x \sqrt{-g} \left\{ \frac{5b_2^2}{4b_1^2} (\Box \varphi)^2 + \frac{5b_2b_3}{b_1^2} (\nabla_{\mu}\varphi) (\nabla^{\mu}\varphi) \Box \varphi + \frac{5b_3^2}{b_1^2} [(\nabla_{\mu}\varphi) (\nabla^{\mu}\varphi)]^2 \right. \\
+ \left[\frac{6b_3b_4(\varphi) - 3b_2b_4'(\varphi)}{2b_1^2} - \frac{b_4''(\varphi)}{2b_1} \right] (\nabla_{\mu}\varphi) (\nabla^{\mu}\varphi) + \frac{b_4^2(\varphi) - b_1b_5''(\varphi)}{2b_1^2} \\
+ \left[\frac{4b_3 (-9c_3'(\varphi) + 9c_1(\varphi) + 2c_2(\varphi)) - 8b_1b_3 - b_2c_2'(\varphi)}{12b_1^2} - \frac{c_1''(\varphi)}{2b_1} \right] R (\nabla_{\mu}\varphi) (\nabla^{\mu}\varphi) \\
+ \left[\frac{2b_3c_2(\varphi) - b_2^2 + b_2c_2'(\varphi) + 4b_1b_3}{6b_1^2} - \frac{c_2''(\varphi)}{2b_1} \right] R^{\mu\nu} (\nabla_{\mu}\varphi) (\nabla_{\nu}\varphi) \\
+ \left[\frac{b_2 (-9c_3'(\varphi) + 9c_1(\varphi) + 2c_2(\varphi)) - 2b_1b_2}{6b_1^2} - \frac{c_3''(\varphi)}{2b_1} \right] R \Box \varphi \\
- \frac{a_1''(\varphi)}{2b_1} R_{\mu\nu\alpha\beta}^2 + \left[\frac{c_2^2(\varphi)}{24b_1^2} + \frac{c_2(\varphi)}{6b_1} + \frac{1}{30} - \frac{a_2''(\varphi)}{2b_1} \right] R_{\mu\nu}^2 \\
+ \left[\frac{3 (4c_3'(\varphi) - 4c_1(\varphi) - c_2(\varphi))^2 - c_2^2(\varphi)}{96b_1^2} + \frac{2c_3'(\varphi) - 2c_1(\varphi) - c_2(\varphi)}{12b_1} + \frac{1}{60} \right] \\
- \frac{a_3''(\varphi)}{2b_1} R^2 + \left[\frac{b_4 (-4c_3'(\varphi) + 4c_1(\varphi) + c_2(\varphi))}{4b_1^2} - \frac{b_4(\varphi)}{6b_1} - \frac{a_4''(\varphi)}{2b_1} \right] R \right\}. \tag{3}$$

Here $\epsilon = (4\pi)^2(n-4)$, φ is the background scalar field, and no surface terms have been kept. Notice that, as one can see, the terms describing non-minimal interaction of gravity with the scalar field (i.e., those involving c_1 , c_2 and c_3) are induced even for constant couplings. To check our results, we have considered the case of a flat background and there we have made use of standard Feynman graph analysis (which is also quite involved, due to the presence of derivatives at the vertices).

Having now the explicit form of the one-loop divergences of the theory under discussion, we are able to see the structure of its one-loop renormalization. For doing this we shall consider some specific variants of action (1). The standard conditions for the one-loop renormalizability of the theory are obtained from the constraint that the functional form of the one-loop counterterms must repeat the form of the corresponding generalized couplings. These conditions look here as follows (for b_1 , b_2 and b_3 they are trivially satisfied, owing to the fact that these couplings are constants)

$$d_{1}b_{4}(\varphi) = \frac{6b_{3}b_{4}(\varphi) - 3b_{2}b'_{4}(\varphi)}{2b_{1}^{2}} - \frac{b''_{4}(\varphi)}{2b_{1}},$$

$$d_{2}b_{5}(\varphi) = \frac{b''_{4}(\varphi) - b_{1}b''_{5}(\varphi)}{2b_{1}}, \dots$$
(4)

where d_1 and d_2 are arbitrary constants, and similarly for the other couplings. One must solve Eqs. (4) for all the coupling constants in order to obtain the whole class of one-loop renormalizable theories. However, in this paper we will focus our interest in multiplicatively renormalizable theories only, which form a subclass of the whole class of one-loop renormalizable scalar theories (1).

The analysis of Eqs. (4) and the structure of the divergence index of the theory, as well as dimensional considerations, show that two variants of the theory given by the action (1) are multiplictively renormalizable, namely the following ones.

(1) The theory where all the generalized coupling constants, except c_3 , are constants, that is

$$b_4(\varphi) = b_4, \quad b_5(\varphi) = b_5, \quad c_1(\varphi) = c_1, \quad c_2(\varphi) = c_2, \quad c_3(\varphi) = c_{31}\varphi + c_{32},$$

 $a_1(\varphi) = a_1, \quad a_2(\varphi) = a_2, \quad a_3(\varphi) = a_3, \quad a_4(\varphi) = a_4.$ (5)

This particular choice of the constants permits us to have the following conformally invariant 4th order action, which is particular case of conformal version of the theory (1) [13]

$$\int d^4x \sqrt{-g}\varphi \left[\Box^2 + 2R^{\mu\nu}\nabla_{\mu}\nabla_{\nu} - \frac{2}{3}R\Box + \frac{1}{3}\left(\nabla^{\mu}R\right)\nabla_{\mu}\right]\varphi,\tag{6}$$

which corresponds to the following choice of constants

$$b_1 = 1, c_1 = \frac{2}{3}, c_2 = -2 (7)$$

with all the other couplings set equal to zero. Notice that we could also put $c_{31} = 0$ in (5), and we would get in this way another variant of the above theory. (2) The second multiplicatively renormalizable case of the theory (1) is the one given by the following couplings:

$$b_4(\varphi) = b_4 e^{\alpha \varphi}, \quad b_5(\varphi) = b_5 e^{2\alpha \varphi}, \quad a_4(\varphi) = a_4 e^{\alpha \varphi},$$
 (8)

while the remaining generalized coupling constants are chosen as before, in Eq. (5). Notice that this is the theory which, on a flat background, has been recently used in refs. [7, 9] in order to describe a truncated theory of quantum gravity at large distances.

3. Renormalization and RG equations. We will here obtain the RG equations corresponding to the renormalizable variant (5) of the theory under discussion. In this section we set $b_1 = 1$, since the corresponding term plays the role of the higher-derivative kinetic term.

We must now consider the combination of (1) and (3),

$$S - \Gamma_{div},$$
 (9)

for the choice of the coupling constants, as in (5) and (8). Expression (8) defines the renormalization of all the coupling constants and scalar field φ .

First of all, we obtain the one-loop renormalization of the scalar field in the following form

$$\varphi_0 = Z_{\varphi}^{1/2} \varphi, \qquad Z = 1 + \frac{5b_2^2}{2\epsilon}.$$
(10)

Using expression (10) in order to find the one-loop renormalization of all the coupling constants (5), we are able to construct the system of one-loop RG equations for the theory with couplings given by (5):

$$\frac{db_2(t)}{dt} = -10b_2(t)b_3(t) + \frac{15}{4}b_2^3(t), \qquad b_2(0) = b_2,
\frac{db_3(t)}{dt} = -10b_3^2(t) + 5b_2^2(t)b_3(t), \qquad b_3(0) = b_3,
\frac{db_4(t)}{dt} = -6b_3(t)b_4(t) + \frac{5}{2}b_2^2(t)b_4(t), \qquad b_4(0) = b_4,
\frac{db_5(t)}{dt} = -b_4^2(t), \qquad b_5(0) = b_5.$$
(11)

This system (11) has to be solved in the first place, since the corresponding running couplings do not change when we come to flat space. The RG equations which give the behaviour of the scalar-gravitational couplings are the following

$$\frac{dc_1(t)}{dt} = \frac{2}{3}b_3(t)\left[9c_{31}(t) - 9c_1(t) - 2c_2(t) + 2\right] + \frac{5}{2}b_2^2(t)c_1(t), \qquad c_1(0) = c_1,
\frac{dc_2(t)}{dt} = -\frac{1}{3}\left[2c_2(t)b_3(t) - b_2^2(t) + 4b_3(t)\right] + \frac{5}{2}b_2^2(t)c_2(t), \qquad c_2(0) = c_2,
\frac{dc_{31}(t)}{dt} = \frac{5}{2}b_2^2(t)c_{31}(t), \qquad c_{31}(0) = c_{31},
\frac{dc_{32}(t)}{dt} = \frac{1}{3}b_2(t)\left[9c_{31}(t) - 9c_1(t) - 2c_2(t) + 2\right] + \frac{5}{4}b_2^2(t)c_{32}(t), \qquad c_{32}(0) = c_{32}. \tag{12}$$

Finally, the RG equations for the effective vacuum couplings have the form

$$\frac{da_1(t)}{dt} = 0, \quad a_1(0) = a_1, \qquad \frac{da_2(t)}{dt} = -\frac{1}{12}c_2^2(t) - 2c_2(t) - \frac{1}{15}, \quad a_2(0) = a_2,
\frac{da_3(t)}{dt} = \frac{-3[4c_1(t) + c_2(t) - 4c_{31}(t)]^2 + c_2^2(t)}{48} - \frac{2c_{31}(t) - 2c_1(t) - c_2(t)}{6} - \frac{1}{30}, \quad a_3(0) = a_3,
\frac{da_4(t)}{dt} = \frac{1}{2}b_4(t)\left[4c_{31}(t) - 4c_1(t) - c_2(t) + \frac{2}{3}\right], \quad a_4(0) = a_4.$$
(13)

When comparing with the ordinary versions of the RG equations, notice that in Eqs. (11)-(13) the following change of RG parameter has to be made (with respect to the standard one, t): $t \longrightarrow (4\pi)^{-2}t$. Notice also that, as is usual in curved space, we adopt the curved spacetime version of the RG equations (for an introduction, see [5]). In particular, the UF

limit $t \to \infty$ in flat space, corresponds in curved spacetime to the strong curvature —or small distances— limit (instead of the momentum rescaling $p \to e^t p$, in curved space the rescaling $g_{\mu\nu} \to e^{-2t} g_{\mu\nu}$ is used). Notice also that we are not taking into account classical dimensions in the RG equations for the dimensional couplings.

4. Asymptotic solutions. It is remarkable that the RG equations above can be solved in an exact way. Its asymptotic analysis can be carried out completely and we will obtain all possible asymptotic behaviours of the solutions compatible with the equations. At the same time, well behaved, approximate solutions will be found, starting from some initial conditions $b_i(0)$, $c_i(0)$ and $a_i(0)$.

Eqs. (11)-(13) can be solved iteratively. In fact, we can start from the two first Eqs. (11), proceed then with the third (after having substituted the solutions of the first two), then with the fourth, and so on. In particular, these two first equations constitute an exact differential equation once the integrating factor $\mu = b_2^9 b_3^8$ is introduced. The equation for the orbit turns out to be a kind of higer-order hyperbola $12b_2^{10}b_3^{10} - 5b_2^{12}b_3^9 = c$, which is, however, unstable (and, therefore, not relevant for the asymptotic analysis that follows). All the possible asymptotic behaviours of the solutions turns out to be rather simple to find. In fact, as $t \to +\infty$ only the three exclusive cases: (i) $b_2^2(t) << b_3(t)$, (ii) $b_2^2(t) >> b_3(t)$ and (iii) $b_2^2(t) \simeq k \, b_3(t)$, with k finite and $k \neq 0$, can occur. It is easy to see that the second case is incompatible (when substituted back into the differential equations). For the first one, under the only conditions that $|b_2|, |b_3| < 1$ and that $b_3 > 0$ (in order to prevent singularities from occuring) and after some work, we obtain

$$b_2(t) \simeq \frac{b_2}{1 + 10b_3t}, \qquad b_3(t) \simeq \frac{b_3}{1 + 10b_3t}.$$
 (14)

It is clear that, for any t > 0, Eqs. (14) are self-consistent approximate solutions of the first two Eqs. (10), since always $|b_2(t)|, |b_3(t)| < 1$. Moreover, they are in fact asymptotic solutions, because $|b_2(t)|, |b_3(t)| \to 0$, as $t \to +\infty$. Thus, as we see, in the ultraviolet limit $(t \to +\infty)$ the theory is asymptotically free, and this property is obtained by just choosing the initial value $b_3 > 0$. For $b_3 < 0$ we would have the usual zero-charge problem in the UF limit.

In the same way, we obtain

$$b_4(t) \simeq \frac{b_4}{(1+10b_3t)^{3/5}}, \quad b_5(t) \simeq b_5 + \frac{b_4^2}{2b_3} \left[\frac{1}{(1+10b_3t)^{1/5}} - 1 \right],$$
 (15)

with the only additional hypothesis that $|b_4| < 1$ (no restriction on b_5 needs to be imposed). That is, (14) and (15) are the asymptotic solutions $(t \to +\infty)$ of Eqs. (10), provided $|b_2|, |b_4| < 1$ and $0 < b_3 < 1$. Moreover, they are also approximate solutions for any t > 0. Notice that we have found an asymptotical solution of a specific form, which is self-consistent (when substituted back into the full set of Eqs. (11)) and asymptotically free.

As for the case (iii) of the above analysis, we may just consider it asymptotically or either we can impose it rigorously (special solution of the RG equations)

$$b_3(t) = kb_2^2(t), \quad k \neq 0,$$
 (16)

and then obtain an exact solution of the whole system. A detailed study yields, with the constrains k = 1/4 and $b_3 = b_2^2/4$ for the initial values, the solution:

$$b_{2}(t) = b_{2} \left(1 - \frac{5}{2} b_{2}^{2} t \right)^{-1/2}, \qquad b_{3}(t) = \frac{b_{2}^{2}}{4} \left(1 - \frac{5}{2} b_{2}^{2} t \right)^{-1},$$

$$b_{4}(t) = b_{2} \left(1 - \frac{5}{2} b_{2}^{2} t \right)^{-2/5}, \qquad b_{5}(t) = b_{5} - \frac{2b_{4}^{2}}{b_{2}^{2}} \left[\left(1 \frac{5}{2} b_{2}^{2} t \right)^{1/5} - 1 \right]. \tag{17}$$

We again have an asymptotically free type solution, but in the IR limit only. In the case that (16) is only satisfied asymptotically, then Eqs. (17) just reflect the behaviour of the $b_i(t)$ for large t, with the difference that instead of b_2 we must put a constant, c, and that now the above constraint on the initial values b_2 and b_3 need not be satisfied.

The analysis of Eqs. (11) and (12) is carried out in the same way. For the case (i), the result reads as follows:

$$c_{1}(t) \simeq \frac{2(c_{2}+2)}{(1+10b_{3}t)^{1/15}} - \alpha_{1}, \qquad \alpha_{1} \equiv -c_{31} \exp\left(\frac{b_{2}^{2}}{4b_{3}}\right) - \frac{2}{3},$$

$$c_{2}(t) \simeq \frac{c_{2}+2}{(1+10b_{3}t)^{1/15}} - 2,$$

$$c_{31}(t) \simeq c_{31} \exp\left[\frac{b_{2}^{2}}{4b_{3}}\left(1 - \frac{1}{1+10b_{3}t}\right)\right],$$

$$c_{32}(t) \simeq c_{32} + \frac{b_{2}(c_{2}+2)}{b_{3}} \left[\frac{1}{(1+10b_{3}t)^{1/15}} - 1\right],$$
(18)

and

$$a_{1}(t) = a_{1}, a_{2}(t) \simeq a_{2} + \frac{18}{5}t,$$

$$a_{3}(t) \simeq a_{3} - \alpha_{2}t, \alpha_{2} \equiv \frac{1}{4} \left[2c_{31} \exp\left(\frac{b_{2}^{2}}{4b_{3}}\right) + 2\alpha_{1} + 1 \right]^{2} + \frac{1}{6} \left[c_{31} \exp\left(\frac{b_{2}^{2}}{4b_{3}}\right) + \alpha_{1} + 1 \right] - \frac{1}{20},$$

$$a_{4}(t) \simeq a_{4} + \alpha_{3} \left[1 - (1 + 10b_{3}t)^{2/5} \right], \alpha_{3} \equiv \frac{b_{4}}{2b_{3}} \left[-c_{31} \exp\left(\frac{b_{2}^{2}}{4b_{3}}\right) - \alpha_{1} - \frac{2}{3} \right]. (19)$$

Eqs. (14)–(19) are the asymptotic solutions of Eqs. (10)–(12), for $t \to +\infty$. The only requeriments for self-consistency are the ones stablished before: $|b_2|, |b_4| < 1$ and $0 < b_3 < 1$. Furthermore, they are approximate solutions for any t > 0. Note also that in the ultraviolet UF limit, the non-minimal coupling constants (18) tend to their asymptotically conformal invariant values (7), independently of their initial values. Notice that this phenomenon (asymptotical conformal invariance) has been already found to occur in several asymptotically free GUTs [5], but it is somewhat surprising to observe that it can persist in higher-derivative theories. It means, for example, that particle creation of higher-derivative scalar bosons (coming from our model) in a Friedmann-Robertson-Walker universe is asymptotically suppressed at strong curvature, as it happens for scalar particles in the asymptotically conformal invariant GUTs [5]. It would be interesting to understand if this phenomenon holds for the generalized theory (1) as well (in terms of the generalized RG). It is also interesting to observe that the induced running gravitational constant $a_4(t)$ has similar behaviour as in the case of GUTs in curved spacetime [5].

The analysis of the infrared limit is pretty much the same as that for the ultraviolet. In fact, the only change in the asymptotic behaviour concerns the sign of b_3 , which should be now reversed in order to avoid the singularity (zero-charge problem). Thus, the final expressions are exactly the same as (14)–(18) being now the requeriments for self-consistency: $|b_2|, |b_4| < 1$ and $-1 < b_3 < 0$. Again, with this proviso expressions (14)–(18) are approximate solutions for any t < 0 big enough in absolute value, and with $-1 < b_3 < 0$ the theory is asymptotically free in the IR region (weak curvature limit). Moreover, the non-minimal scalar graviton couplings, $c_1(t)$, $c_2(t)$ and $c_3(t)$, tend to their conformal invariant values (7) at weak curvature. Thus, our theory is asymptotically conformal invariant at weak curvature.

It is intesting to note that asymptotical conformal invariance may also occur in general theory (1). The values corresponding to the conformal version of (1) are [13] $b_1(\varphi) = f(\varphi), b_2(\varphi) = f'(\varphi), b_3(\varphi) = b_4(\varphi) = b_5(\varphi) = c_3(\varphi) = a_4(\varphi) = 0, c_1 = \frac{2}{3}f(\varphi), c_2 = -2f(\varphi), a_1 = q(\varphi), a_2 = -2q(\varphi), a_3 = \frac{1}{3}q(\varphi)$ where $f(\varphi)$ and $q(\varphi)$ are arbitrary functions. It is natural to suppose that this relations will hold for the one-loop divergences.

5. RG equations for the conformal sector of quantum gravity. Let us now consider the multiplicatively renormalizable theory which is obtained by choosing the generalized couplings in the form (7). In flat space $(g_{\mu\nu} = \eta_{\mu\nu})$, if we make the following identification

$$b_1 = -\frac{\theta^2}{(4\pi)^2}, \quad b_2 = -2\zeta\alpha, \quad b_3 = -\zeta\alpha^2, \quad b_4(\varphi) = \gamma \exp(2\alpha\varphi), \quad b_5(\varphi) = -\frac{\lambda}{\alpha^2} \exp(4\alpha\varphi),$$
(20)

we find that our theory corresponds to the one which was used in Ref. [7] in the infrared stable fixed point $\zeta = 0$ to describe the truncated theory of quantum gravity at large distances. In (20) we have used the same notations as in Ref. [7]. It is interesting to note that the theory (20) admits the interesting vacuum (for simplicity we set $\alpha = 1$) $e^{2\varphi} = \gamma/(2\lambda)$, already at the tree level. The situation becomes more complicated at one-loop order [9]. It would be worth, perhaps, to develop some kind of gaussian effective potential approach (see, for example, [15]) in order to try understand the non-perturbative structure of this theory.

When considering the specific theory which is obtained with the choice of coupling constants (8) —and having in mind the conformal anomaly-induced theory of Ref. [7] as a particular case of our general theory—we come to the conclusion that the coupling constants b_2 and b_3 may be connected (in particular, for $\alpha = 1$ they are simply proportional). As one can see from (3), its one-loop renormalization may still respect this connection—while it will be destroyed already at one-loop level if we perform the one-loop renormalization of the scalar field, as in (8), for arbitrary b_1 . That is why we prefer here not to do the renormalization of the scalar field, and to consider instead all coupling constants, including b_1 (σ -model type renormalization). In this case, the one-loop RG equations can be easily derived from (3):

$$\dot{b}_1 = -\frac{5b_2^2}{2b_1^2}, \ \dot{b}_2 = -\frac{10b_2b_3}{b_1^2}, \ \dot{b}_3 = -\frac{10b_3^2}{b_1^2}, \ \dot{b}_4 = \frac{-6b_3b_4 + 3\alpha b_2b_4}{b_1^2} + \frac{\alpha^2b_4}{b_1}, \ \dot{b}_5 = -\frac{b_4^2}{b_1^2} + \frac{4\alpha^2b_5}{b_1}, \ \dot{b}_{10} = -\frac{b_1^2}{b_1^2} + \frac{4\alpha^2b_5}{b_1}, \ \dot{b}_{11} = -\frac{b_1^2}{b_1^2} + \frac{4\alpha^2b_5}{b_1^2}, \ \dot{b}_{12} = -\frac{b_1^2}{b_1^2} + \frac{4\alpha^2b_5}{b_1^2}, \ \dot{b}_{13} = -\frac{b_1^2}{b_1^2} + \frac{4\alpha^2b_5}{b_1^2}, \ \dot{b}_{13} = -\frac{b_1^2}{b_1^2} + \frac{b_1^2}{b_1^2} + \frac{b_$$

and

$$\dot{c}_1 = \frac{2b_3(9c_{31} - 9c_1 - 2c_2)}{3b_1^2} + \frac{4b_3}{3b_1}, \ \dot{c}_2 = \frac{b_2^2 - 2b_3c_2}{3b_1^2} - \frac{4b_3}{3b_1}, \ \dot{c}_3 = \frac{b_2(9c_{31} - 9c_1 - c_2)}{3b_1^2} + \frac{2b_2}{3b_1}. \ (22)$$

To simplify the notation, we have not written explicitly the t-dependence in Eqs. (21) and (22). Also, α and c_{31} are constants, and $b_i(0) = b_i$, $c_i(0) = c_i$.

The vacuum RG equations are

$$\dot{a}_1 = 0, \qquad \dot{a}_2 = -\frac{c_2^2}{12b_1^2} - \frac{c_2}{3b_1} - \frac{1}{15},
\dot{a}_3 = -\frac{3(4c_{31} - 4c_1 - c_2)^2 + c_2^2}{48} - \frac{2c_{31} - 2c_1 - c_2}{6b_1} - \frac{1}{30},
\dot{a}_4 = \frac{b_4(4c_{31} - 4c_1 - c_2)}{2b_1^2} + \frac{b_4}{3b_1} + \frac{\alpha^2 a_4}{2b_1}.$$
(23)

For simplicity, and having in mind possible applications to the conformal sector of quantum gravity [7], we will present here the solutions corresponding to the matter sector only. Now, the t-dependences will be considered again. The first three equations (22) are equivalent to the following:

$$b_3(t) = \frac{b_3}{b_2}b_2(t), \ b_2(t) = \frac{4b_3}{b_2}b_1(t) + b_2 - \frac{4b_1b_3}{b_2}, \ \frac{b_1^2(t)\dot{b}_1(t)}{\left[b_1(t) - b_1 + b_2^2/(4b_3)\right]^2} = -\frac{40b_3^2}{b_2^2}, \tag{24}$$

where the t-dependence has been written explicitly (remember that the constants are the initial values at t = 0). As before, even without being able to solve the sistem of differential equations exactly, the analysis of all posible asymptotic behaviours of the solutions can be carried out completely. What distinguishes now the different possibilities is the behaviour of $b_1(t)$ as $t \to \infty$. It can tend (i) to zero, (ii) to a non-zero constant, k, or (iii) to infinity. The case (i) turns out to be incompatible. The case (ii) is the most interesting. The constant k is constrained to be

$$k = b_1 - \frac{b_2^2}{4b_3} \equiv b_{1\infty},\tag{25}$$

and we obtain

$$b_2(t) \simeq \frac{b_{1\infty}^2 b_2}{b_{1\infty}^2 + 10b_3 t}, \qquad b_3(t) \simeq \frac{b_{1\infty}^2 b_3}{b_{1\infty}^2 + 10b_3 t},$$

$$b_4(t) \simeq b_4 e^{b_{1\infty} t}, \qquad b_5(t) \simeq -\frac{b_4^2}{2b_{1\infty}^3} e^{2b_{1\infty} t} + \left(b_5 + \frac{b_4^2}{2b_{1\infty}^3}\right) e^{4b_{1\infty} t}. \tag{26}$$

As one can see we found asymptotically free type solutions. Notice that when $b_{1\infty} < 0$ (that is, $b_1 < b_2^2/(4b_3)$) both $b_4(t)$ and $b_5(t)$ decay exponentially. The last term of $b_5(t)$ is then subleading, but we have included it here in order to show how the initial value b_5 is just washed out for large t. Finally, in the case (iii) the following asymptotic behaviour is easily inferred

$$b_1(t) = -\frac{40b_3^2}{b_2^2}t + \left(\frac{b_2^2}{2b_3} - 2b_1\right)\ln\left(1 - \frac{160b_3^3}{b_2^2}t\right) + b_1 + \mathcal{O}(t^{-1}),\tag{27}$$

valid both for $t \to +\infty$ (with $b_3 > 0$) and for $t \to -\infty$ (with $b_3 < 0$), and similar behaviours for b_2 and b_3 (as is clear from (24)). On the other hand, for the remaining b-functions we have

$$b_4(t) \simeq b_4 \left(1 - \frac{40b_3^2}{b_1 b_2^2} t\right)^{(\alpha_2 b_2^2)/(40b_3^2)},$$
 (28)

and

$$b_5(t) \simeq b_5 \left(1 - \frac{40b_3^2}{b_1 b_2^2} t \right)^{(-2\alpha_1^2 b_2^2)/(40b_3^2)}$$

$$(b_4(t) \to 0),$$

$$b_5(t) \simeq b_5 + \frac{b_4^2}{b_1 (2\alpha_2 - 40b_3^2/b_2^2)} \left(1 - \frac{40b_3^2}{b_1 b_2^2} t \right)^{(\alpha_2 b_2^2)/(20b_3^2) - 1}$$

$$(b_4(t) \to \infty), \quad (29)$$

where

$$\alpha_1 \equiv \frac{b_2^2}{4b_3} - b_1, \quad \alpha_2 \equiv \left(\frac{6b_3}{b_2} - 3\alpha_1\right) \frac{4b_3}{b_2} - \alpha_1^2.$$
 (30)

The different asymptotic behaviours for $b_5(t)$ come from the two possible (in principle) behaviours of $b_4(t)$ (depending on the sign of the exponent in (28)). As one can see, in this case (iii) the coupling constants do not display an asymptotically free behaviour. Notice that using the same prescription for the renormalization of the scalar theory as in Sect. 3, we would have obtained the same behaviour for the effective couplings $b_2(t)$ and $b_3(t)$ as in Sect. 4, while the behaviours of $b_4(t)$ and $b_5(t)$ would be quite different.

6. Conclusions. We have discussed in this work the renormalization of a higher derivative interacting scalar field theory in curved spacetime. Some variants of this theory have been shown to be multiplicatively renormalizable and asymptotically free. The RG structure of the effective theory of the conformal factor has been also investigated.

There are many interesting directions in which the theory that we have here introduced deserves further study. One possibility is to find the one-loop counterterms for the general model (1) (with $b_1(t)$, $b_2(t)$ and $b_3(t)$ being arbitrary functions of the scalar field) and to construct the generalized RG equations and investigate their flows. Such approach may be of interest for some cosmological applications also, if one would hope to describe our early universe using a kind of higher-derivative Brans-Dicke theory (for a recent discussion and a list of references on scalar-tensor gravity theories, see [16]). In this case, to complete the study it would be necessary, as a final step, to discuss the renormalization structure of the theory (1) with the scalar and the gravitational fields being both quantized. This is a quite complicated program on which we expect to report elsewhere.

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