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Improving the Effective Potential

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ABSTRACT

A general procedure is presented how to improve the effective potential by using the renormalization group equation (RGE) in $\overline{\mathrm{MS}}$ scheme. If one knows the L-loop effective potential and the RGE coefficient functions up to $(L+1)$-loop level, this procedure gives an improved potential which satisfies the RGE and contains all of the leading, next-to-leading, $\cdots$, and L-th-to-leading log terms.

Since the work of Georgi, Quinn and Weinberg,[1] it is a standard procedure to use renormalization group equation (RGE) to discuss the low-energy physics, in a system which is supposedly described by a certain unified theory at a very large energy scale. In recent active investigations in the minimal supersymmetric standard model, people also discuss the effective potential for the Higgs fields in which the coupling constants and masses are running parameters which depend on the renormalization point μ. It is quite legitimate to use such renormalized parameters with renormalization point μ chosen to be a value of the order of the energy scale at which we discuss the physics.

It was, however, found that the tree effective potential with such running parameters inserted is too sensitive to the choice of the renormalization point μ: for instance, the vacuum expectation value (VEV) of the Higgs field rapidly varies against a small change of the renormalization point, so that no reliable prediction can be made. To save this situation it was proposed to use the effective potential at the 1-loop level instead.[2] This indeed improved the situation and the μ-dependence, *e.g.*, of the Higgs VEV, became much milder than in the tree case. But, in some cases,[3] the stability against μ achieved by this is not enough and the Higgs VEV still shows a rapid μ-dependence for μ one-order apart from the supersymmetry breaking scale.

This is of course a problem which arises from the fact that the used effective potential itself is not satisfying the RGE: the RGE differential operator is just a total derivative $d/d\ln\mu$, so if the effective potential satisfies the RGE, it *is* μ-independent and the VEV realized as its minimum should show only a very mild (logarithmic) μ-dependence of the wave function renormalization. The purpose of this letter is to present a simple procedure to improve effective potential so as to satisfy the RGE.

It is a bit surprising why such a procedure has not been known for a general system. Coleman and Weinberg[4] are probably the first to improve the effective potential using RGE. Their method, however, relied on a special definition of the

$\lambda\phi^4$ coupling constant as a fourth derivative of the potential, and was restricted to massless systems. Very recently Kastening wrote a remarkable paper,[5] in which he presented two method to obtain RGE improved effective potential in massive $\lambda\phi^4$ theory: one method uses a detailed form of the effective potential expressed as a power series in several variables and determines the coefficients of the series by inserting that form in the RGE. This is probably too complicated to extend it to more realistic systems. Another method is a smarter one which may have a possibility of generalization. However this method as it stands also has problems: it still expands the effective potential in a power series in a certain variable and solves the RGE order by order, which is in fact a complicate procedure for a realistic system. Another problem is that he had to make a peculiar ansatz for the form of the vacuum energy (*i.e.*, ϕ-independent) term of the effective potential. [It is peculiar since it diverges when λ goes to zero.]

Our present work is in a sense a re-organization of his second method. The above mentioned second problem has a simple solution: if we properly take the renormalization of the vacuum energy term into account, we do not need any special ansatz for it. [In any case it is very interesting that such vacuum energy term becomes relevant to the ϕ-dependent terms of the effective potential.] As for the first problem we do not solve the RGE for effective potential order by order but uses the well-known full order solution itself. This is the main point of our method, which greatly simplifies the procedure and makes it possible to apply to more general systems. Indeed we shall show that this procedure applies to any system which has essentially a unique mass scale. [Actually, with a suitable modification, it is extendible also to completely general system possessing many mass scales, as will be shown in a separate paper.[6]]

To explain the essence of our procedure, we consider the simplest case of $\lambda\phi^4$ model of a real scalar field. The Lagrangian of this system is given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 - hm^4 \ . \tag{1}$$

The last term hm^4 is the vacuum energy term which is usually omitted. But surprisingly, in the mass-independent renormalization scheme, it becomes relevant to us in the calculation of effective potential as we shall see below. Renormalization of the vacuum energy term is performed in the $\overline{\rm MS}$ scheme simply by omitting the 'divergent' term proportional to positive powers of $1/\bar{\varepsilon} \equiv 2/(4-n)-\gamma+\ln 4\pi$ in the calculated vacuum energy $V(\phi=0)$. The corresponding counter-terms are supplied by the renormalization of the bare vacuum-energy parameter $h_0 = Z_h \mu^{4-n} h$, from which the renormalized vacuum-energy parameter h becomes dependent on the renormalization point μ.⋆

The effective potential at the 1-loop level is calculated in $\overline{\rm MS}$ scheme as

$$
\begin{aligned}
V_1 &= V^{(0)} + V^{(1)} \ , \\
V^{(0)} &= \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 + h m^4 \ , \\
V^{(1)} &= \frac{1}{4 \cdot 16\pi^2} M_\phi^4 \Big(\ln \frac{M_\phi^2}{\mu^2} - \frac{3}{2} \Big) \ ,
\end{aligned}
\tag{2}
$$

where

$$
M_\phi^2 \equiv \frac{1}{2} \lambda \phi^2 + m^2 \tag{3}
$$

is the scalar mass in the presence of scalar background ϕ.

Renormalization theory tells us that effective potential satisfies RGE:

$$
\mathcal{D} V(\phi, m^2, \lambda, h;\, \mu) = 0 \ , \tag{4}
$$

with

$$
\mathcal{D} = \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} - \gamma_m m^2 \frac{\partial}{\partial m^2} - \gamma_\phi \phi \frac{\partial}{\partial \phi} + \beta_h \frac{\partial}{\partial h} \ . \tag{5}
$$

⋆ In the orthodox mass-independent renormalization[7] with cutoff regularization, one should impose the following three renormalization conditions to renormalize the vacuum energy term (0-point function) $\Gamma^{(0)}$ as a function of m^2:

$$
\Gamma^{(0)}(m^2)\Big|_{m^2=0} = \frac{\partial}{\partial m^2} \Gamma^{(0)}(m^2)\Big|_{m^2=0} = 0 \ , \quad \Big(\frac{\partial}{\partial m^2}\Big)^2 \Gamma^{(0)}(m^2)\Big|_{m^2=\mu^2} = -h \ .
$$

These are realized by counter-terms of the form $A + Bm^2 + Cm^4$.

The well-known solution is given in the form:

$$V(\phi, m^2, \lambda, h;\, \mu^2) = V\big(\bar\phi(t), \bar m^2(t), \bar\lambda(t), \bar h(t);\, e^{2t}\mu^2\big)\ , \tag{6}$$

where $\bar\lambda$, $\bar m^2$, $\bar\phi$ and $\bar h$ are running parameters whose t-dependence is determined by

$$\begin{aligned} \frac{d\bar\lambda(t)}{dt} &= \beta\big(\bar\lambda(t)\big)\ , \\ \frac{d\bar X(t)}{dt} &= -\gamma_X\big(\bar\lambda(t)\big)\,\bar X(t) \qquad \text{for} \quad X = m^2, \phi, \\ \frac{d\bar h(t)}{dt} &= \beta_h\big(\bar\lambda(t), \bar h(t)\big)\ , \end{aligned} \tag{7}$$

with the boundary condition that they reduce to the unbarred parameters at $t = 0$. Note that the vacuum-energy parameter h can affect only the evolution of itself.

The general solution (6) gives full information of RGE: it says that, as a result of the fact that RGE is a first order differential equation, the effective potential is determined once its function form is known at a certain value of t. Namely, RGE reduces the number of variables on which the effective potential depends by one. That is all. So, to derive useful information from RGE, we need to know anyway the function of effective potential at a certain value of t, a 'boundary' or 'initial' function.

Let us first see the logarithm structure of the effective potential. We note that the L-loop ($L \geq 1$) level contribution to the effective potential has the following form:

$$V^{(L)} = \lambda^{L-1} M_\phi^4 \times \big[\text{polynomial in} \quad \ln\frac{M_\phi^2}{\mu^2} \ \text{and}\ \frac{\lambda\phi^2}{M_\phi^2}\big]\ . \tag{8}$$

This can be understood most easily in the following way: To compute the effective potential $V(\phi)$, we first rewrite the quantum Lagrangian in the form

$$\mathcal{L} = \frac{1}{\lambda}\left[\frac{1}{2}(\partial(\sqrt{\lambda}\Phi))^2 - \frac{1}{2}m^2(\sqrt{\lambda}\Phi)^2 - \frac{1}{4!}(\sqrt{\lambda}\Phi)^4 - \lambda h m^4\right]\ , \tag{9}$$

and then make the field shift $\Phi \to \Phi + \phi$ and regard $\sqrt{\lambda}\Phi$ as our basic quantum field. In this form the parameters characterizing the theory are only the scalar mass

$M_\phi^2 = \frac{1}{2}\lambda\phi^2 + m^2$, the cubic coupling $\sqrt{\lambda}\phi$ and λ (aside from the vacuum-energy term). Moreover the last parameter λ is no longer the quartic coupling constant but an overall factor in front of the action just like Planck constant $\hbar$. Then the above form of Eq.(8) is self-evident. Now let us introduce the following variables:

$$\begin{aligned} s &\equiv \lambda \ln \frac{M_\phi^2}{\mu^2} \ , \\ x &\equiv \frac{\frac{\lambda}{2}\phi^2}{M_\phi^2} \ , \\ z &\equiv \lambda h \frac{m^4}{M_\phi^4} \ . \end{aligned} \tag{10}$$

In terms of these variables together with λ, the 1-loop potential (2), for example, can be expressed as

$$V_1 = \frac{M_\phi^4}{\lambda}\Big[x(1-x) + \frac{1}{3!}x^2 + z + \frac{1}{64\pi^2}\big(s - \frac{3}{2}\lambda\big)\Big] \ . \tag{11}$$

Since we know that the logarithms appear only up to L-th power at the L loop level, the L-loop contribution (8) takes the form

$$V^{(L)} = \frac{M_\phi^4}{\lambda} \times \big[v_0^{(L)}(x)s^L + \lambda v_1^{(L)}(x)s^{L-1} + \lambda^2 v_2^{(L)}(x)s^{L-2} + \cdots + \lambda^L v_L^{(L)}(x)\big] \ , \tag{12}$$

so that the full effective potential has the form:

$$V = \frac{M_\phi^4}{\lambda}\sum_{\ell=0}^{\infty}\lambda^\ell\big[f_\ell(s,x) + z\delta_{\ell,0}\big] \tag{13}$$

$$\equiv M_\phi^4 \widetilde{V}(s,x,z,\lambda) \ , \tag{14}$$

$$f_\ell(s,x) = \sum_{L=\ell}^{\infty} v_\ell^{(L)}(x)s^{L-\ell} \ . \tag{15}$$

This form of expansion (13) in powers of λ, which was first derived by Kastening,[5] just gives a *leading-log series expansion*: namely, the functions $f_0, f_1, \cdots$ correspond to the leading, next-to-leading, $\cdots$ log terms, respectively. So the explicit λ factors which appear when the expression is written in terms of variables s, x, z and λ show the order in this leading-log series expansion. We refer to the term proportional to $\lambda^{\ell-1}$ in V as ℓ-th-to-leading log term. Unlike Kastening, we do not write the RGE for the functions $f_\ell(s,x)$ separately since we do already know the solution (6) to the RGE for the full effective potential. This greatly simplifies the procedure for the practical use.

In this form, the above solution (6) of RGE is rewritten into

$$V = M_\phi^4 \widetilde{V}(s,x,z,\lambda) = \overline{M}_\phi^4(t)\, \widetilde{V}\big(\bar{s}(t), \bar{x}(t), \bar{z}(t), \bar{\lambda}(t)\big) \ , \tag{16}$$

where the barred quantities $\overline{M}_\phi^2(t)$, $\bar{s}(t)$, $\bar{x}(t)$ and $\bar{z}(t)$ are the variables M_ϕ^2, s, x and z at 'time' t: *e.g.*,

$$\begin{aligned} \overline{M}_\phi^2(t) \equiv& \frac{1}{2}\bar{\lambda}(t)\bar{\phi}^2(t) + \bar{m}^2(t) \ , \\ \bar{s}(t) \equiv& \bar{\lambda}(t) \ln \frac{\overline{M}_\phi^2(t)}{e^{2t}\mu^2} \ . \end{aligned} \tag{17}$$

Since this expression (16) says that it is t-independent, we can put any t and we should look for such t at which we can calculate the function form of effective potential.

Now we come to the point. The form (15) tells us that at $s=0$ the ℓ-th-to-leading log function f_ℓ is given solely in terms of ℓ-loop level potential: $f_\ell(s=0,x) = v_\ell^{(L=\ell)}(x) = \widetilde{V}^{(\ell)}(s=0,x,\lambda)/\lambda^{\ell-1}$. So, if we calculate the effective potential up to L-loop level, $V_L = V^{(0)} + V^{(1)} + \cdots + V^{(L)}$, then at $s=0$ it already gives the function *exact* up to L-th-to-leading log order:

$$V = M_\phi^4 \widetilde{V}(s=0,x,z,\lambda) = M_\phi^4 \widetilde{V}_L(s=0,x,z,\lambda) + O(\lambda^L) \ . \tag{18}$$

That is, we can use the function $V_L\big|_{s=0}$ as a 'boundary' function required in the RHS of the solution (6) or (16) of RGE. Therefore, with the L-loop potential V_L

at hand, the effective potential satisfying the RGE can be given by

$$
\begin{aligned}
M_\phi^4 \widetilde{V}(s,x,z,\lambda) &= \overline{M}_\phi^4(t)\, \widetilde{V}_L\big(\bar{s}(t)=0, \bar{x}(t), \bar{z}(t), \bar{\lambda}(t)\big) \\
&= \overline{M}_\phi^4(t) \sum_{\ell=0}^{L} \bar{\lambda}^{\ell-1}(t) \Big[v_\ell^{(\ell)}\big(\bar{x}(t)\big) + \bar{z}(t)\delta_{\ell,0} \Big]_{\bar{s}(t)=0} \ ,
\end{aligned}
\tag{19}
$$

or, equivalently,

$$
V(\phi, m^2, \lambda, h;\, \mu^2) = V_L\big(\bar{\phi}(t), \bar{m}^2(t), \bar{\lambda}(t), \bar{h}(t);\, e^{2t}\mu^2\big)\Big|_{\bar{s}(t)=0} \ . \tag{20}
$$

The barred quantities in these expressions should of course be evaluated at t satisfying $\bar{s}(t) = 0$.

The process of solving $\bar{s}(t) = 0$ with respect to t, if one wishes, may be bypassed as follows. Running of this variable $\bar{s} \equiv \bar{s}(t)$ is determined by the differential equation:

$$
\begin{aligned}
&\frac{d\bar{s}}{dt} = \beta_s(\bar{s}, \bar{m}^2, \bar{\lambda}, \bar{\phi}) \equiv \bar{\beta}_s \ , \\
&\beta_s = \lambda\Big[\frac{\beta}{\lambda^2}s - 2\Big] + \lambda^2\Big[\Big(\frac{\beta}{\lambda^2} - 2\frac{\gamma}{\lambda}\Big)x - \frac{\gamma_m}{\lambda}(1-x)\Big] \ .
\end{aligned}
\tag{21}
$$

We can switch to use the variable $\bar{s}$ itself in place of the 'time' variable t, and then we regard the running quantities $\bar{\lambda}, \bar{m}^2, \bar{\phi}$ and $\bar{h}$ as functions of $\bar{s}$ (and of initial parameters λ, m^2, ϕ and h, of course). Their runnings with respect to $\bar{s}$ are of course determined by equations

$$
\frac{d\bar{X}}{d\bar{s}} = \frac{1}{\bar{\beta}_s}\frac{d\bar{X}}{dt} = \frac{\bar{\beta}_X}{\bar{\beta}_s} \ . \tag{22}
$$

Then the quantities $\bar{\phi}(t), \bar{m}^2(t), \bar{\lambda}(t)$ and $\bar{h}(t)$ in the RHS of the solution (20) are simply obtained by setting their argument $\bar{s}$ equal to zero.

Although the solution (20) is 'exact' only up to L-th-to-leading log order, it satisfies the RGE *exactly* if the runnings of the barred quantities are solved exactly (,which is independent of the choice of the 'boundary' function). If the runnings of the parameters $\bar{\lambda}/\lambda, \bar{\phi}/\phi, \bar{m}^2/m^2$ and $\bar{h}/h$ are solved correctly only up to L-*th power in* λ in the sense of leading log series expansion, our solution (20) satisfies the RGE up to L-th-to-leading log order and is 'exact' in that order. This can be understood from the second expression in (19); if $\bar{\lambda}/\lambda, \bar{\phi}/\phi, \bar{x}/x$ and $\bar{z}/z$ are determined correctly up to the L-th power in λ, their errors are of the order $O(\lambda^{L+1})$ and can change our effective potential in (19) at most by an $O(\lambda^L)$ quantity.⋆

To achieve this 'exactness' up to L-th-to-leading log order, it is sufficient to know the $(L+1)$-*loop RGE coefficient functions* β, γ and so on. This is because we need those coefficient functions $\beta/\lambda,\ \gamma,\ \gamma_m$ and β_h/h correct up to $O(\lambda^{L+1})$ (but not $O(\lambda^L)$), since $\bar{\beta}_s$ in the RG running eq.(22) with respect to $\bar{s}$ is of $O(\lambda^1)$. Thus, *with L-loop effective potential and* $(L+1)$*-loop RGE coefficient functions, we can obtain an RGE improved effective potential which is exact up to L-th-to-leading log order.*

That is all of our procedure: the Eq.(20) gives the final answer of our improved effective potential.

Let us now demonstrate these processes by explicit computations to the leading log order (*i.e.*, $L = 0$). The coefficient functions of RGE are calculated at the 1-loop level as

$$\begin{aligned}
\beta &= \frac{1}{16\pi^2} 3\lambda^2 \equiv \beta_1 \lambda^2 \ , \\
\gamma_m &= -\frac{1}{16\pi^2} \lambda \equiv \gamma_{m1} \lambda \ , \\
\gamma &= \frac{1}{16\pi^2} \times 0 \equiv \gamma_1 \lambda \ , \\
\beta_h &= +2h\gamma_m + \frac{1}{16\pi^2}\frac{1}{2} \equiv 2h\gamma_m + \beta_{h1} \ .
\end{aligned} \tag{23}$$

⋆ It may be of help to give here a 'quick table' showing which quantities are of which order in λ (as the expansion parameter of the leading-log series expansion):

$$\bar{\lambda}(t) \ \sim \ O(\lambda^1), \quad \bar{\phi}(t) \ \sim \ O(\lambda^{-\frac{1}{2}}), \quad \bar{m}^2(t) \ \sim \ O(1), \quad \bar{h}(t) \ \sim \ O(\lambda^{-1}) \ .$$

This follows from the RG running equation and the fact that $m^2, \lambda\phi^2$ and λh are of $O(1)$ in the leading-log series expansion.

Noting that $\bar{\beta}_s = (\beta_1 \bar{s} - 2)\bar{\lambda} + O(\lambda^2)$, the RGE for $\bar{\lambda}$

$$\bar{\beta}_s \frac{d\bar{\lambda}}{d\bar{s}} = \bar{\beta} \tag{24}$$

gives in the lowest order in λ:

$$(\beta_1 \bar{s} - 2)\bar{\lambda} \frac{d\bar{\lambda}}{d\bar{s}} = \beta_1 \bar{\lambda}^2 \ . \tag{25}$$

This is integrated as

$$\int_{\lambda}^{\bar{\lambda}} \frac{d\bar{\lambda}}{\bar{\lambda}} = \int_{s}^{\bar{s}} \frac{d\bar{s}}{\bar{s} - \frac{2}{\beta_1}} \ , \tag{26}$$

and gives

$$\bar{\lambda} = \lambda \frac{1 - \frac{\beta_1}{2}\bar{s}}{1 - \frac{\beta_1}{2}s} \ . \tag{27}$$

Similarly, we have

$$\begin{aligned} (\beta_1 \bar{s} - 2)\bar{\lambda} \frac{d\bar{\phi}}{d\bar{s}} &= -\gamma_1 \bar{\lambda} \bar{\phi} \ , \\ (\beta_1 \bar{s} - 2)\bar{\lambda} \frac{d\bar{m}^2}{d\bar{s}} &= -\gamma_{m1} \bar{\lambda} \bar{m}^2 \ , \end{aligned} \tag{28}$$

so that we get

$$\begin{aligned} \bar{\phi} &= \phi \left(\frac{1 - \frac{\beta_1}{2}\bar{s}}{1 - \frac{\beta_1}{2}s} \right)^{-\frac{\gamma_1}{\beta_1}} \ , \\ \bar{m}^2 &= m^2 \left(\frac{1 - \frac{\beta_1}{2}\bar{s}}{1 - \frac{\beta_1}{2}s} \right)^{-\frac{\gamma_{m1}}{\beta_1}} \ . \end{aligned} \tag{29}$$

Finally, instead of $\bar{h}$, we write the RGE for $\bar{h} \times \bar{m}^4$ which reads

$$\begin{aligned} (\beta_1 \bar{s} - 2)\bar{\lambda} \frac{d(\bar{h}\bar{m}^4)}{d\bar{s}} &= \bar{m}^4 \left[\bar{\beta}_h - 2\bar{\gamma}_m h \right]_{\text{lowest order in } \lambda} \\ &= \beta_{h1} \bar{m}^4 \ . \end{aligned} \tag{30}$$

Note that the $2h\gamma_m$ part in β_h is cancelled. We obtain

$$\begin{aligned}\int\limits_{hm^4}^{\bar{h}\bar{m}^4} d(\bar{h}\bar{m}^4) &= \beta_{h1}\frac{m^4}{\lambda}\big(1-\frac{\beta_1}{2}s\big)^{1+\frac{2\gamma_{m1}}{\beta_1}}(-\frac{1}{2})\int\limits_{s}^{\bar{s}} d\bar{s}\,\big(1-\frac{\beta_1}{2}\bar{s}\big)^{-2(1+\frac{\gamma_{m1}}{\beta_1})} \\ &= \frac{m^4}{\lambda}\frac{\beta_{h1}}{\beta_1+2\gamma_{m1}}\left[1-\big(\frac{1-\frac{\beta_1}{2}s}{1-\frac{\beta_1}{2}\bar{s}}\big)^{1+\frac{2\gamma_{m1}}{\beta_1}}\right]\end{aligned} \tag{31}$$

so that

$$\bar{h}\bar{m}^4 = hm^4 + \frac{m^4}{\lambda}\frac{\beta_{h1}}{\beta_1+2\gamma_{m1}}\left[1-\big(\frac{1-\frac{\beta_1}{2}s}{1-\frac{\beta_1}{2}\bar{s}}\big)^{1+\frac{2\gamma_{m1}}{\beta_1}}\right]\ . \tag{32}$$

Now that we have determined the RG running of all the relevant parameters, we can write down the improved effective potential in the leading-log order by inserting them into the 'boundary' function. The 'boundary' function to the leading-log order is given by the tree potential $V^{(0)}$. But we use here the 1-loop potential $V_1 = V^{(0)} + V^{(1)}$ with s set equal to zero, since it gives in any case a better approximation in the region in which $\ln(M_\phi^2/\mu^2)$ is not so large (and to keep it is harmless in the sense of leading log expansion). Then the potential at $s=0$ is given simply by setting $\mu^2 = M_\phi^2$ *directly* in $V = V^{(0)} + V^{(1)}$ and then replace all the parameters there by the above obtained barred ones with $\bar{s}=0$ substituted. Thus the leading-log order effective potential is found to be:

$$V = \frac{1}{2}\bar{m}^2\bar{\phi}^2 + \frac{1}{4!}\bar{\lambda}\bar{\phi}^4 + \bar{h}\bar{m}^4 - \frac{3}{2}\frac{1}{64\pi^2}\big(\frac{1}{2}\bar{\lambda}\bar{\phi}^2+\bar{m}^2\big)^2 \tag{33}$$

with

$$\begin{aligned}\bar{\phi} &= \phi\ , \\ \bar{\lambda} &= \lambda\left(1-\frac{3\lambda}{32\pi^2}\ln\frac{\frac{1}{2}\lambda\phi^2+m^2}{\mu^2}\right)^{-1}\ , \\ \bar{m}^2 &= m^2\left(1-\frac{3\lambda}{32\pi^2}\ln\frac{\frac{1}{2}\lambda\phi^2+m^2}{\mu^2}\right)^{-\frac{1}{3}}\ , \\ \bar{h}\bar{m}^4 &= hm^4 + \frac{1}{2}\frac{m^4}{\lambda}\left[1-\left(1-\frac{3\lambda}{32\pi^2}\ln\frac{\frac{1}{2}\lambda\phi^2+m^2}{\mu^2}\right)^{\frac{1}{3}}\right]\ .\end{aligned} \tag{34}$$

This agrees with the result by Kastening[5] aside from the next-to-leading log terms and the ϕ-independent constant terms. [Note that the singularity at $\lambda = 0$ is automatically absent here contrary to Kastening.] An important point in this calculation is to demonstrate explicitly that we need not even to rewrite the effective potential in terms of the chosen variables s, x, z. We have used those variables just to see the correctness of our effective potential to a certain order in the leading-log series expansion. In practice we can simply substitute the running barred parameters directly in the potential without rewriting it in terms of s, x, z.

We add a remark on the numerical applications. In cases in which there are many coupling constants and masses (masses should be of the same order), it is necessary to carry out the calculation using computer. In such cases of numerical work, the above process of changing the variable from t to s in solving the RG running is quite extraneous. (Doing so introduces unnecessary complications.) The solution in the form (20) before doing that already gives the final answer: We first solve the RG running of $\bar{\lambda}(t)$, $\bar{\phi}(t)/\phi$, $\bar{m}^2(t)$ *etc.*, for a given set of initial coupling constants and mass parameters at μ (in which we do not yet need to specify ϕ). At the same time as we vary the parameter t in solving these differential equations, we can find the corresponding ϕ by $\bar{s}(t) = 0$; *i.e.*,

$$\phi^2 = \frac{2}{\bar{\lambda}(t)} \Big(\frac{\bar{\phi}(t)}{\phi} \Big)^{-2} \big(e^{2t}\mu^2 - \bar{m}^2(t) \big) \ . \tag{35}$$

[Note that the RHS does not depend on ϕ.] Then putting this value of ϕ and the running parameters into the RHS of (20), we obtain the effective potential at that point ϕ. Namely the effective potential is obtained simultaneously as we solve the RG running of the parameters. Moreover, since we make no further approximation in solving the RG running of the barred quantities in this process, the obtained effective potential satisfies exactly the RGE with given $(L+1)$-loop coefficient functions.

Our procedure described in this paper is applicable to any complicated system *provided that the relevant mass scales are essentially unique.* To explain this, let us now consider the general situation. Generically, if a system consists of several particles labeled by j ($j = 1, 2, \cdots, n$), the effective potential contains logarithm factors of the form:

$$\lambda_j \ln \frac{M_j^2(\phi)}{\mu^2} \ , \tag{36}$$

where $M_j(\phi)$ is a mass of the j-th particle on the background in which scalar fields take VEV's $\phi = (\phi_1, \cdots \phi_n)$, and takes the form

$$M_j^2(\phi) = \sum_i \lambda_{ji} \phi_i^2 + m_j^2 \ . \tag{37}$$

Here λ_j and λ_{ji} are (certain linear combinations of) coupling constants and m_j is the mass of j-th particle in the absence of scalar field background. First problem we encounter here is which log-factor among these we should choose as the s variable with which we sum up the leading log, next-to-leading log, $\cdots$ terms. The best choice would be to take a particle whose coupling constant λ_j is the largest; namely, calling that particle by label $j = 0$, we take the corresponding log-factor as the s variable: $s \equiv \lambda_0 \ln(M_0^2(\phi)/\mu^2)$. Then all the other log-factors are rewritten in the form

$$\lambda_j \ln \frac{M_j^2(\phi)}{\mu^2} = \frac{\lambda_j}{\lambda_0} s + u_j, \tag{38}$$

with introducing new variables

$$u_j \equiv \lambda_j \ln \frac{M_j^2(\phi)}{M_0^2(\phi)} \ . \tag{39}$$

Assume now that all the masses $M_j^2(\phi)$ here in the presence of scalar field background, are of the same order *independently* of the background ϕ. This is the situation which we meant in the above by "relevant mass scales are essentially unique". [This happens, for instance, if we are considering the effective potential as a function of a single scalar field ϕ with other VEV's set equal to zero, and the 'bare' masses m_j (and the coupling constants λ_j) are all of the same order; indeed in such a case, $M_j^2(\phi)$'s take the form $\lambda_j\phi^2+m_j^2$ and are of the same order as $\lambda_0\phi^2+m_0^2$ independently of the value of ϕ.] Then, these variables u_j in (39) remain always of the order $O(\lambda_j) \lesssim O(\lambda_0)$ at most, and therefore all the log-factors (36) can be treated essentially as s, (or $(\lambda_j/\lambda_0)s$ more precisely,) since the differences u_j in Eq.(38) are higher order (in the leading-log series expansion) than the first s term.

The L-th power terms in s and u_j's come from L-(or higher-)loop contributions to the effective potential. But the variable s among them can be set equal to zero when we obtain the boundary function. The other variables u_j remain as they are. However, since u_j are of $O(\lambda_0)$ under the above constraints, the $(L+1)$-loop or higher-loop contributions to the effective potential after setting $s=0$, $\sum_{\ell=L+1}^{\infty} V^{(\ell)}\big|_{s=0}$, can be of the order λ_0^L at most. Therefore the previous order-counting argument of the leading-log series expansion remains unchanged; namely, the L-th-to-leading-log exact boundary function can be obtained by the L-loop potential V_L simply by setting $\mu^2 = M_0^2(\phi)$ (*i.e.*, $s=0$). The runnings of the barred quantities, $\bar{\lambda}_j, \bar{m}_j^2$ *etc.*, substituted there can of course be computed correctly by using the $(L+1)$-loop RGE coefficient functions.

This argument also clarifies the problem which occurs when the above constraints are not met. In such a case the variables u_j in (39) no longer remain small depending on the region of ϕ_j's. For instance, even when we are discussing the effective potential of a single scalar field, $\phi_j=\phi$, if a particle is massless, $m_j=0$, then the corresponding u_j is given by $\lambda_j \ln\left(\lambda_j\phi^2/(\lambda_0\phi^2+m_0^2)\right)$. This is of $O(\lambda_j) \lesssim O(\lambda_0)$ for large ϕ, $\lambda_0\phi^2 \geq m_0^2$, but becomes very large for small ϕ in the region $\lambda_0\phi^2 \ll m_0^2$. So we have to keep any higher powers of such u_j in our leading-log series expansion.⋆ This implies that we cannot find a good boundary function by the present procedure: if the boundary function is calculated by L-loop potential V_L with s set equal to zero, then it is correct only up to L-th power in

⋆ This difficulty due to the presence of multi-scales has been noticed by several authors.[8,9]

the u_j variables, and so it becomes completely unreliable in the small ϕ region $\lambda_0\phi^2 \ll m_0^2$.

This problem, of course, stems from our careless treatment of different mass scales by a single scale parameter μ. It turns out to be overcome by a proper use of decoupling theorem or the so-called effective field theory. Renormalization group equation in fact contains this notion of effective field theory in a very natural form. Using this we can still have a simple procedure of improving effective potential for completely general system. This will be given in a separate paper.[6]

REFERENCES

1. H. Georgi, H. Quinn and S. Weinberg, *Phys. Rev. Lett.* **33** (1974) 451.
2. G. Gamberini, G. Ridolfi and F. Zwirner, *Nucl. Phys.* **B331** (1990) 331.
3. M. Bando, T. Kugo, N. Maekawa and H. Nakano, Preprint KUNS 1129 (1992).
4. S. Coleman and S. Weinberg, *Phys. Rev.* **D7** (1973) 1888.
5. B. Kastening, *Phys. Lett.* **B283** (1992) 287.
6. M. Bando, T. Kugo, N. Maekawa and H. Nakano, Preprint KUNS 1162 (1992).
7. T. Kugo, *Soryusiron Kenkyu* **53** (1976) 1, see also *Prog. Theor. Phys.* **57** (1977) 593.
8. M. Sher, *Phys. Rep.* **179** (1989) 273.
9. M. B. Einhorn and D. R. T. Jones, *Nucl. Phys.* **B230[FS10]** (1990) 261.