On the low-energy spectrum of spontaneously broken Φ^4 theories

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Abstract

The low-energy spectrum of a one-component, spontaneously broken Φ^4 theory is generally believed to have the same simple massive form $\sqrt{\mathbf{p}^2 + m_h^2}$ as in the symmetric phase where $\langle\Phi\rangle = 0$. However, in lattice simulations of the 4D Ising limit of the theory, the two-point connected correlator and the connected scalar propagator show deviations from a standard massive behaviour that do not exist in the symmetric phase. As a support for this observed discrepancy, I present a variational, analytic calculation of the energy spectrum $E_1(\mathbf{p})$ in the broken phase. This analytic result, while providing the trend $E_1(\mathbf{p}) \sim \sqrt{\mathbf{p}^2 + m_h^2}$ at large $|\mathbf{p}|$, gives an energy gap $E_1(0) < m_h$, even when approaching the infinite-cutoff limit $\Lambda \to \infty$ with that infinitesimal coupling $\lambda \sim 1/\ln\Lambda$ suggested by the standard interpretation of "triviality" within leading-order perturbation theory. I also compare with other approaches and discuss the more general implications of the result.

1. Introduction

In the case of a one-component, spontaneously broken Φ^4 theory, one usually assumes a form of single-particle energy spectrum, say $E_1(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_h^2}$, as in a simple massive theory with no qualitative difference from the symmetric phase where $\langle\Phi\rangle = 0$.

One can objectively test [1] this expectation with lattice simulations, performed in the 4D Ising limit of the theory, and study the exponential decay of the connected two-point correlator $C_1(\mathbf{p}, t) \sim e^{-E_1(\mathbf{p})t}$ and the connected scalar propagator $G(p)$. Differently from the symmetric phase, where the simple massive picture works to very high accuracy, the results of the low-temperature phase show unexpected deviations. Namely, when the 3-momentum $\mathbf{p} \to 0$, the fitted $E_1(\mathbf{p})$ deviates from (the lattice version of) the standard massive form $\sqrt{\mathbf{p}^2 + \text{const.}}$ and, when the 4-momentum $p_\mu \equiv (\mathbf{p}, p_4) \to 0$, the measured $G(p)$ deviates from (the lattice version of) the form $1/(p^2 + \text{const.})$.

After the first indications of Ref.[1], Stevenson [2] checked independently the existence of this discrepancy in the lattice data of other authors. To this end, he started from the lattice data of Ref.[3] for the time slices of $C_1(\mathbf{p} = 0, t)$ and used the Fourier-transform relation to generate equivalent data for the connected scalar propagator $G(p)$. The resulting behaviour of $G(p)$ is in complete agreement with the analogous plots obtained from Ref.[1] (compare Figs.6c, 7, 8 and 9 of Ref.[2]).

The whole issue was later re-considered in Ref.[4]. According to these authors, at the present, after taking into account various theoretical uncertainties, the deviations are not so statistically compelling. In their opinion, the conventional scenario of a simple, weakly coupled, massive theory, "unfortunately can only be nailed down by analytic proofs".

The aim of this Letter is to present, in Sects.2 and 3, a possible analytic proof under the form of a variational calculation of the energy spectrum in the broken-symmetry phase. This analytic result, while indeed providing a behaviour $E_1(\mathbf{p}) \sim \sqrt{\mathbf{p}^2 + m_h^2}$ at larger $|\mathbf{p}|$, gives theoretical support for deviations in the $\mathbf{p} \to 0$ limit. In particular, the energy gap $E_1(0)$ is definitely smaller than the m_h parameter that enters the asymptotic form of the spectrum. I emphasize that the estimate, being of variational nature, constrains from above the ratio $\frac{E_1(0)}{m_h}$ whose value, by enlarging the variational subspace, can only decrease. In addition, the result persists when taking the infinite cutoff limit $\Lambda \to \infty$ with the typical trend of the coupling constant $\lambda \sim 1/\ln\Lambda$ that is expected in the standard interpretation of "triviality" [5] within leading-order perturbation theory. Finally, in Sect.4, I will also compare with other approaches and discuss the more general implications of the result.

2. Stability analysis of Φ^4 theory

The preliminary starting point, for any variational calculation in the broken-symmetry phase of a one-component Φ^4 theory, is the basic Hamiltonian operator ($\lambda > 0$)

$$\hat{H} = \ :\int d^3x \left[\frac{1}{2}\left(\Pi^2 + (\nabla\Phi)^2 + \Omega_o^2\Phi^2\right) + \frac{\lambda}{4!}\Phi^4\right]: \tag{1}$$

where ($\omega_{\mathbf{k}}(\Omega) = \sqrt{\mathbf{k}^2 + \Omega^2}$)

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{\sqrt{2\omega_{\mathbf{k}}(\Omega_o)(2\pi)^3}} \left(a_{\mathbf{k}} \exp i\mathbf{k}\cdot\mathbf{x} + a^\dagger_{\mathbf{k}} \exp -i\mathbf{k}\cdot\mathbf{x}\right) \tag{2}$$

and

$$\Pi(\mathbf{x}) = i\int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\omega_{\mathbf{k}}(\Omega_o)}{2}} \left(a^\dagger_{\mathbf{k}} \exp -i\mathbf{k}\cdot\mathbf{x} - a_{\mathbf{k}} \exp i\mathbf{k}\cdot\mathbf{x}\right) \tag{3}$$

In Eq.(1) normal ordering is defined with respect to a reference state $|0\rangle$ which is the vacuum of the creation and annihilation operators ($a_{\mathbf{k}}|0\rangle = \langle 0|a^\dagger_{\mathbf{k}} = 0$) with commutation relations $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$. The standard stability analysis for the above Hamiltonian is performed in the class of the normalized gaussian ground states $|\Psi^{(0)}\rangle \equiv |\Psi^{(0)}(\Omega,\varphi)\rangle$ with [6, 7]

$$\langle\Psi^{(0)}|\Phi|\Psi^{(0)}\rangle = \varphi \tag{4}$$

and

$$\langle\Psi^{(0)}|\Phi(\mathbf{x})\Phi(\mathbf{y})|\Psi^{(0)}\rangle = \varphi^2 + G(\mathbf{x},\mathbf{y}) \tag{5}$$

where

$$G(\mathbf{x},\mathbf{y}) = \int \frac{d^3k}{2\omega_{\mathbf{k}}(\Omega)(2\pi)^3} \exp i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) \tag{6}$$

is the equal-time propagator of the shifted fluctuation field

$$h(\mathbf{x}) = \Phi(\mathbf{x}) - \varphi \tag{7}$$

with

$$h(\mathbf{x}) = \int \frac{d^3k}{\sqrt{2\omega_{\mathbf{k}}(\Omega)(2\pi)^3}} \left(b_{\mathbf{k}} \exp i\mathbf{k}\cdot\mathbf{x} + b^\dagger_{\mathbf{k}} \exp -i\mathbf{k}\cdot\mathbf{x}\right) \tag{8}$$

Thus, the relation with the reference vacuum state is $|0\rangle \equiv |\Psi^{(0)}(\Omega_o, \varphi = 0)\rangle$ at which $b_{\mathbf{k}} \equiv a_{\mathbf{k}}$. Equivalently, one could switch to a functional formalism where the gaussian ground states are described by the class of functionals [8]

$$\Psi^{(0)}[\Phi] = (\mathrm{Det}\ G)^{-1/4} \exp -\frac{1}{4}\int d^3x \int d^3y (\Phi(\mathbf{x}) - \varphi)G^{-1}(\mathbf{x},\mathbf{y})(\Phi(\mathbf{y}) - \varphi) \tag{9}$$

In this equivalent approach, the field operator $\Phi(\mathbf{x})$ acts on $\Psi^{(0)}[\Phi]$ multiplicatively while the momentum operator acts by functional differentiation

$$\Pi(\mathbf{x})\Psi^{(0)}[\Phi] = \frac{1}{i}\frac{\delta}{\delta\Phi(\mathbf{x})}\Psi^{(0)}[\Phi] \tag{10}$$

In the following, I shall maintain the standard second-quantized representation (1)-(8) for its more intuitive character.

As shown in Ref.[8], the states $|\Psi^{(0)}(\Omega,\varphi)\rangle$ can be represented as coherent states built up with the original $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ operators. In this sense, they represent forms of condensed vacua and the old operators are related to the new "quasiparticle" $b_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ operators (whose vacuum is $|\Psi^{(0)}(\Omega,\varphi)\rangle$) by a Bogolubov transformation that includes a shift of the zero-momentum mode.

The expectation value of the Hamiltonian in the class of the gaussian ground states gives the gaussian energy density $W_G(\varphi,\Omega)$

$$\langle\Psi^{(0)}|\hat{H}|\Psi^{(0)}\rangle = \int d^3x\ (W_G(\varphi,\Omega) - W_G(0,\Omega_o)) \tag{11}$$

where $(I_o(\Omega) = G(\mathbf{x},\mathbf{x}),\ I_1(\Omega) = \frac{1}{8}G^{-1}(\mathbf{x},\mathbf{x})\ )$

$$W_G(\varphi,\Omega) = I_1(\Omega) + \frac{1}{2}m_B^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\left(m_B^2 + \frac{\lambda}{2}\varphi^2 - \Omega^2 + \frac{\lambda}{4}I_o(\Omega)\right) I_o(\Omega) \tag{12}$$

and, just for simplicity of notation, the quantity

$$m_B^2 \equiv \Omega_o^2 - \frac{\lambda}{2}I_o(\Omega_o) \tag{13}$$

has been introduced. It plays the role of a 'bare mass' for the quantum theory but his origin depends on the normal ordering prescription adopted for the Hamiltonian Eq.(1).

Now, the existence of the Φ^4 critical point [9] implies that, for sufficiently large and negative values of m_B^2, the cutoff theory will exhibit spontaneous symmetry breaking. In this regime, one can explore the conditions for non-trivial minima with $\varphi \neq 0$. Minimization of W_G with respect to φ gives

$$\frac{\partial W_G(\varphi,\Omega)}{\partial\varphi} = \varphi\left(m_B^2 + \frac{\lambda}{6}\varphi^2 + \frac{\lambda}{2}I_o(\Omega)\right) = 0 \tag{14}$$

while minimization with respect to Ω yields

$$\Omega^2(\varphi) = m_B^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{2}I_o(\Omega) \tag{15}$$

Finally, the replacement $\Omega = \Omega(\varphi)$ in $W_G(\varphi, \Omega)$ provides the gaussian effective potential (GEP)

$$V_G(\varphi) = W_G(\varphi, \Omega(\varphi)) - W_G(0, \Omega_o) \tag{16}$$

By combining Eqs.(14) and (15), non-trivial extrema $\varphi \neq 0$ can only occur at those values $\varphi = \pm v$ where

$$\Omega^2(v) = \frac{\lambda}{3} v^2 \equiv m_h^2 \tag{17}$$

The standard identification of m_h with the energy-gap of the broken phase derives from the following argument. At the absolute minima $\varphi = \pm v$, the same Hamiltonian in Eq.(1) becomes also normal ordered in the creation and annihilation operators $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ [8], namely one finds

$$\hat{H} = E_0 + \hat{H}_2 + \hat{H}_{\rm int} \tag{18}$$

Here

$$E_o = \int d^3x \; V_G(v) < 0 \tag{19}$$

is the gaussian ground-state energy. The quadratic operator

$$\hat{H}_2 = \int d^3p \; \omega_{\mathbf{p}}(m_h) \; b^\dagger_{\mathbf{p}} b_{\mathbf{p}} \tag{20}$$

describes free-field quanta with energies $\omega_{\mathbf{p}}(m_h) = \sqrt{\mathbf{p}^2 + m_h^2}$ and finally

$$\hat{H}_{\rm int} = \int d^3x : \left(\frac{\lambda v}{3!} h^3(\mathbf{x}) + \frac{\lambda}{4!} h^4(\mathbf{x}) \right) : \tag{21}$$

takes into account the residual self-interactions that have not been reabsorbed into the vacuum structure and in the mass parameter m_h. In the above relation, normal ordering of the $b^\dagger_{\mathbf{p}}$ and $b_{\mathbf{p}}$ operators is now defined with respect to one of the two equivalent absolute minima of the GEP for $\varphi = \pm v$. In this way, by introducing the one-quasiparticle states (see Eq.(6.4) of Ref.[7])

$$|1, \mathbf{p}\rangle = b^\dagger_{\mathbf{p}} |\Psi^{(0)}\rangle \sqrt{2\omega_{\mathbf{p}} (2\pi)^3} \tag{22}$$

one finds

$$\frac{\langle 1, \mathbf{p}|(\hat{H} - E_0)|1, \mathbf{p}\rangle}{\langle 1, \mathbf{p}|1, \mathbf{p}\rangle} = \sqrt{\mathbf{p}^2 + m_h^2} \tag{23}$$

and it becomes natural to identify m_h with the energy-gap of the broken phase. In the following section, I will check this expectation with a variational calculation.

3. Variational calculation of the energy gap in the broken phase

The variational procedure is of the same type considered by Di Leo and Darewych [10] and by Siringo [11] when discussing the bound-state problem in the Higgs sector, namely

$$|\Psi_1\rangle = A(\mathbf{q}) b^\dagger_{\mathbf{q}} |\Psi^{(0)}\rangle + \int d^3k \; B(\mathbf{k},\mathbf{q}) b^\dagger_{\mathbf{k}+\mathbf{q}} b^\dagger_{-\mathbf{k}} |\Psi^{(0)}\rangle \tag{24}$$

with $B(\mathbf{k},\mathbf{q}) = B(-\mathbf{k}-\mathbf{q},\mathbf{q})$.

The two complex functions $A(\mathbf{q})$ and $B(\mathbf{p},\mathbf{q})$ have to be determined in order to solve the eigenvalue problem for the Hamiltonian $\hat{H}$ Eq.(18) in the chosen subspace. By denoting with $E_1 = E_1(\mathbf{q})$ the corresponding eigenvalue, one gets coupled equations (everywhere $\omega_{\mathbf{p}} = \omega_{\mathbf{p}}(m_h)$)

$$\frac{\delta\langle\Psi_1|(\hat{H} - E_o - E_1)|\Psi_1\rangle}{\delta A^*(\mathbf{q})} = A(\mathbf{q})(\omega_{\mathbf{q}} - E_1) + f(\mathbf{q}) = 0 \tag{25}$$

and

$$\frac{\delta\langle\Psi_1|(\hat{H} - E_o - E_1)|\Psi_1\rangle}{\delta B^*(\mathbf{k},\mathbf{q})} = 2B(\mathbf{k},\mathbf{q})[\omega_{\mathbf{k}} + \omega_{\mathbf{k}+\mathbf{q}} - E_1] + g(\mathbf{k},\mathbf{q}) = 0 \tag{26}$$

In Eqs.(25) and (26) $f(\mathbf{q})$ and $g(\mathbf{k},\mathbf{q})$ are defined as

$$f(\mathbf{q}) = \frac{\lambda v}{8\pi^{3/2}\sqrt{\omega_{\mathbf{q}}}} \int d^3k \; \frac{B(\mathbf{k},\mathbf{q})}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}+\mathbf{q}}}} \tag{27}$$

and

$$g(\mathbf{k},\mathbf{q}) = \frac{\lambda v}{8\pi^{3/2}\sqrt{\omega_{\mathbf{q}}}} \frac{A(\mathbf{q})}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}+\mathbf{q}}}} + \frac{\lambda}{32\pi^3\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}+\mathbf{q}}}} \int d^3p \; \frac{B(\mathbf{p},\mathbf{q})}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}+\mathbf{q}}}} \tag{28}$$

The two functions $f(\mathbf{q})$ and $g(\mathbf{k},\mathbf{q})$ contain the same integral up to numerical factors. This allows to eliminate *exactly* $B(\mathbf{k},\mathbf{q})$ in favour of $A(\mathbf{q})$ as

$$B(\mathbf{k},\mathbf{q}) = \frac{A(\mathbf{q})}{8v\;\pi^{3/2}} \sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{k}}\omega_{\mathbf{k}+\mathbf{q}}}} \left(\frac{\omega_{\mathbf{q}} - E_1 - \frac{3m_h^2}{2\omega_{\mathbf{q}}}}{\omega_{\mathbf{k}} + \omega_{\mathbf{k}+\mathbf{q}} - E_1} \right) \tag{29}$$

after using the relation (17) $m_h^2 = \frac{\lambda v^2}{3}$. By replacing in Eq.(25), one obtains

$$A(\mathbf{q})(\omega_{\mathbf{q}} - E_1) + A(\mathbf{q}) \left(\omega_{\mathbf{q}} - E_1 - \frac{3m_h^2}{2\omega_{\mathbf{q}}} \right) \frac{\lambda}{16\pi^2} J(\mathbf{q}) = 0 \tag{30}$$

where

$$J(\mathbf{q}) = \frac{1}{4\pi} \int \frac{d^3p}{\omega_{\mathbf{p}}\omega_{\mathbf{p}+\mathbf{q}}[\omega_{\mathbf{p}} + \omega_{\mathbf{p}+\mathbf{q}} - E_1(\mathbf{q})]} \tag{31}$$

Therefore, for $A(\mathbf{q}) \neq 0$, one obtains the final relation for the eigenvalue

$$E_1(\mathbf{q}) = \omega_{\mathbf{q}} \left(1 - \frac{3m_h^2}{2\omega_{\mathbf{q}}^2} F(\mathbf{q})\right) \tag{32}$$

where

$$F(\mathbf{q}) = \frac{\frac{\lambda}{16\pi^2} J(\mathbf{q})}{1 + \frac{\lambda}{16\pi^2} J(\mathbf{q})} \tag{33}$$

Now, the integral in Eq.(31) diverges logarithmically

$$J \sim \int_0^\Lambda \frac{p^2 dp}{2(p^2 + m_h^2)^{3/2}} \sim \frac{1}{2} \ln \frac{\Lambda}{m_h} \tag{34}$$

so that any conclusion on the energy spectrum depends on the possible behaviours of the coupling constant λ when the ultraviolet cutoff $\Lambda \to \infty$. A straightforward $\Lambda \to \infty$ limit for λ =fixed would yield $F(\mathbf{q}) \to 1$ and a negative $E_1(0)$. However, a more meaningful continuum limit could be obtained, for instance, by interpreting λ as the value of a running coupling $\lambda(\mu)$ at some scale μ and then requiring $\lambda(\mu) \sim 1/\ln(\Lambda/\mu)$ as suggested by the standard interpretation of "triviality" within leading-order perturbation theory.

For a self-consistent derivation of this trend within our Hamiltonian formalism, let us return to equation (15) and use relation (13) to replace the bare mass. For simplicity, I shall first consider the case $\Omega_o = 0$, i.e.

$$m_B^2 = -\frac{\lambda}{2} I_o(0) \tag{35}$$

By using the identity of Ref.[7]

$$I_o(\Omega) - I_o(0) = -\frac{\Omega^2}{8\pi^2} \left(\ln \frac{\Lambda}{\Omega} + \frac{1}{2}\right) \tag{36}$$

equation (15) for $\varphi = \pm v$, where Ω is given in Eq.(17), reduces to the relation

$$1 = \frac{\lambda}{8\pi^2} \left(\ln \frac{\Lambda}{m_h} + \frac{1}{2}\right) \tag{37}$$

One can give different interpretations to this equation. On the one hand, if Φ^4 theory were just considered a cutoff theory, it might simply express m_h in terms of the two basic, fixed parameters λ and Λ. On the other hand, in a Renormalization Group (RG) perspective, it could also be used to determine a suitable flow of the coupling constant $\lambda = \lambda(\Lambda)$, in the two-parameter (λ, Λ) space, that corresponds to the same value of m_h. As anticipated, from this latter RG point of view and within leading-order perturbation theory, the resulting trend $\lambda \sim 1/\ln \frac{\Lambda}{m_h}$ would be similar to the $\Lambda-$dependence of the "renormalized" coupling λ_R, usually identified with the value of a running coupling $\lambda(\mu)$ at a typical finite scale $\mu \sim m_h$. However, in principle, λ might also be considered a "bare" coupling λ_B, and thus identified with a running coupling $\lambda(\mu)$ at an asymptotic ultraviolet scale $\mu \sim \Lambda$. As discussed in Ref.[12], this latter point of view cannot be ruled out. In fact, the trend $\lambda_B \sim 1/\ln\Lambda$ represents a completely consistent solution that yields "triviality" (i.e. $\lambda_R = 0$) to any finite order in perturbation theory by avoiding the problems posed by the 1-loop, 3-loop, 5-loop,.. Landau poles and by the 2-loop, 4-loop,... spurious ultraviolet fixed points at finite coupling that arise in the conventional interpretation. In the more general context of the $\epsilon-$expansion, these two distinct points of view might also reflect the existence of two separate Φ^4 theories inhabiting in $d = 4 + \epsilon$ and $d = 4 - \epsilon$ space-time dimensions [13].

In any case, regardless of these interpretative aspects, the consistency of the whole calculation requires to adopt Eq.(37) to fix the (λ, Λ, m_h) interdependence. In this way, one can control the ultraviolet divergence in $J(\mathbf{q})$ and obtain a finite value for $F(\mathbf{q})$. Notice however that, independently of the given finite value of $F(\mathbf{q})$, one gets

$$E_1(\mathbf{q}) \sim \sqrt{\mathbf{q}^2 + m_h^2} \tag{38}$$

at large $|\mathbf{q}|$ and

$$E_1(0) < m_h \tag{39}$$

consistently with $J(0)$ and $F(0)$ being positive-definite quantities for any $E_1(0) < 2m_h$.

The numerical estimate of the energy gap can be obtained from the relation

$$E_1(0) = m_h \left(1 - \frac{3}{2} \frac{\frac{\lambda}{16\pi^2} J(0)}{1 + \frac{\lambda}{16\pi^2} J(0)}\right) \tag{40}$$

with

$$J(0) = \int_0^\Lambda \frac{p^2 dp}{(p^2 + m_h^2)[2\sqrt{p^2 + m_h^2} - E_1(0)]} \tag{41}$$

Thus, by defining $p = m_h \sinh t$ and introducing $\epsilon_1 \equiv E_1(0)/m_h$, one obtains

$$J(0) = \int_0^{t_{\max}} \frac{\sinh^2 t \; dt}{\cosh t[2\cosh t - \epsilon_1]} \tag{42}$$

or

$$J(0) = \frac{t_{\max}}{2} + \frac{1}{\epsilon_1}\left[\frac{\pi}{2} - \sqrt{1 - \frac{\epsilon_1^2}{4}}\left(\arcsin(\epsilon_1/2) + \frac{\pi}{2}\right)\right] \tag{43}$$

where $t_{\max} = \ln(2\Lambda/m_h)$. In this way, in a double limit $t_{\max} \to \infty$ and $\lambda \to 0$, such that $\lambda t_{\max}$ is finite, ϵ_1 is definitely smaller than unity. With the trend in Eq.(37), one finds

$\frac{\lambda}{16\pi^2}J(0) = 1/4 + \mathcal{O}(\frac{1}{t_{\max}})$ or $F(0) = 1/5 + \mathcal{O}(\frac{1}{t_{\max}})$ so that

$$\frac{E_1(0)}{m_h} = \epsilon_1 = 0.7 \left(1 + \mathcal{O}(\frac{1}{t_{\max}})\right) \tag{44}$$

In the same approximation, where also $F(\mathbf{q}) - F(0)$ represents a non-leading $\mathcal{O}(\frac{1}{t_{\max}})$ effect, the form of the spectrum becomes very simple and one finds

$$E_1(\mathbf{q}) \sim \omega_{\mathbf{q}} - \frac{3}{10}\frac{m_h^2}{\omega_{\mathbf{q}}} \tag{45}$$

I emphasize that the result in Eq.(44) is of variational nature. Therefore, by maintaining the same relation Eq.(37) for the coupling constant, and by enlarging the variational subspace for the Hamiltonian Eq.(18) to include higher-order components $|b^\dagger b^\dagger b^\dagger\rangle$, $|b^\dagger b^\dagger b^\dagger b^\dagger\rangle$,..., the ratio $\frac{E_1(0)}{m_h}$ can only decrease.

Exactly the same procedure can be repeated in the more general case where the Ω_o mass parameter of the symmetric phase is non vanishing. As one can check, by requiring the broken phase to represent anyway the absolute minimum of the gaussian effective potential, Eq.(37) can only be modified up to non-leading $\mathcal{O}(\lambda)$ terms. As a consequence, Eq.(44) is also modified up to non-leading $\mathcal{O}(\frac{1}{t_{\max}})$ terms and the basic result remains unaffected.

Before concluding this section, I have to explain the considerable differences between the conclusions of the present Letter and those of Ref.[11]. There, the analysis was performed directly in the broken-symmetry phase without considering the overall stability of the basic Φ^4 Hamiltonian (1) in the class of the gaussian ground states. For this reason, there was no obvious guiding principle to relate λ to the ultraviolet cutoff Λ and to m_h as in Eq.(37). Thus, differently from the approach followed in the present Letter, one could try to take the $\Lambda \to \infty$ limit at $\lambda =$ fixed in such a way that $\lambda J(0) \to \infty$ and

$$F(0) = \frac{\frac{\lambda}{16\pi^2}J(0)}{1 + \frac{\lambda}{16\pi^2}J(0)} \to 1 \tag{46}$$

In this framework, it was adopted a particular mass renormalization condition (see Eqs.(20), (21), (27) and (30) of Ref.[11])

$$\delta m^2 = -\lambda v^2 F(0) \tag{47}$$

in order to get, in the broken-symmetry phase, an exactly free massive spectrum up to terms that vanish in the $\Lambda \to \infty$ limit. Now, it would be very hard to understand the choice of such a vacuum-dependent mass counterterm in the context of the basic Hamiltonian (1). Therefore, it should not come as a surprise that, by changing the renormalization conditions, the same type of variational structure can lead to different physical conclusions.

4. Summary and outlook

In Sects.2 and 3, I have illustrated an analytic, variational calculation of the energy spectrum for the broken-symmetry phase of the basic Φ^4 Hamiltonian (1). In a continuum limit where the ultraviolet cutoff $\Lambda \to \infty$ and the coupling constant $\lambda \to 0$, such that $\lambda \ln \Lambda$ is finite, the variationally determined spectrum $E_1(\mathbf{p})$ approaches the free-field form at large $|\mathbf{p}|$, namely

$$E_1(\mathbf{p}) \to \sqrt{\mathbf{p}^2 + m_h^2}\left(1 + \mathcal{O}(\frac{m_h^2}{\mathbf{p}^2})\right) \tag{48}$$

However, in the same continuum limit, the energy-gap $E_1(0)$ remains definitely smaller than the m_h parameter that controls the asymptotic shape of the spectrum. With the trend for the coupling coupling constant in Eq.(37), which is self-consistently determined by the overall minimization of the effective potential, one finds

$$\frac{E_1(0)}{m_h} = 0.7\left(1 + \mathcal{O}(\frac{1}{\ln \Lambda})\right) \tag{49}$$

and the simple leading behaviour Eq.(45). The variational nature of the result implies that, by enlarging the subspace to include higher-order $|b^\dagger b^\dagger b^\dagger\rangle$, $|b^\dagger b^\dagger b^\dagger b^\dagger\rangle$,...contributions in the Fock space, the ratio $\frac{E_1(0)}{m_h}$ can only decrease.

A possible objection might concern the simplest form Eq.(1) adopted for the Hamiltonian operator. Would the variational result persist by employing for the contact interaction more sophisticated de-singularized operators as, for instance, the generalized normal-ordering prescriptions of Ref.[14] ? There is no obvious answer to this question. By replacing the Hamiltonian operator $\hat{H}$ of Eq.(1) with a new operator, say $\hat{H}'$, one should first repeat the whole stability analysis within the class of the gaussian ground states and later check the consistency of $\hat{H}'$ with the variational calculation in the $|b^\dagger\rangle$ and $|b^\dagger b^\dagger\rangle$ sectors. In our case, by using Eq.(1) (or equivalently the 'bare mass' in Eq.(13)), one obtains finite results at all stages, once Eq.(37) is used self-consistently to determine the cutoff dependence of the coupling constant. For this reason, the operator $\Delta\hat{H} = \hat{H}' - \hat{H}$ should only introduce non-leading divergent terms in the calculations, at least if the trend $\lambda \sim 1/\ln \Lambda$ has to be maintained.

Therefore, one is naturally driven to interpret the peculiar infrared behaviour of the broken phase as a true physical effect due to the existence of a non-trivial vacuum condensate associated with the typical scale m_h. When the momentum increases, the differences with the trivial empty vacuum become unimportant and the energy spectrum approaches a standard massive form with $m_h^2 \sim \lambda v^2$. However, when $\mathbf{p} \to 0$, the presence of the condensate cannot be reabsorbed into the mass term alone due to the strong attraction among the bare massive states which is induced by the cubic interaction proportional to λv. For this reason, it becomes important to understand how fast $E_1(0)$ decreases by improving on the variational procedure and, in particular, whether it remains non vanishing in the $\Lambda \to \infty$ limit.

This aspect is closely related to the comparison with the lattice data mentioned in the Introduction and deserves additional comments. In general, one can express the inverse connected propagator as

$$G^{-1}(p) = p^2 + M^2(p^2) \tag{50}$$

If the single-particle spectrum approximates the form $\sqrt{\mathbf{p}^2 + m_h^2}$ at large $|\mathbf{p}|$ and tends to $E_1(0) < m_h$ when $\mathbf{p} \to 0$, one can imagine various interpolating forms for $M(p^2)$ in Euclidean space but, in any case, one expects the propagator to deviate from the simple form $1/(p^2+m_h^2)$ by approaching the $p_\mu \to 0$ limit. These deviations can be parameterized by using Stevenson's sensitive variable [2]

$$\zeta(p, m) \equiv (p^2 + m^2)G(p) \tag{51}$$

In terms of this variable, by introducing the mass value $m \sim m_h$ that well describes the high-momentum propagator data, one gets from Ref.[1] a zero-momentum value

$$\zeta(0, m_h) = m_h^2 G(p = 0) > 1 \tag{52}$$

The data also indicate that, by approaching the continuum limit of the lattice theory, $\zeta(0, m_h)$ becomes larger and larger while the deviations from $\zeta \sim 1$ are also confined to a smaller and smaller region of momenta near $p_\mu = 0$ (compare Figs. 3, 4 and 5 of Ref.[1]). Thus, in the continuum limit, both the "zero-momentum mass" $\sqrt{G^{-1}(p = 0)}$ and the peculiar infrared region $|p| \lesssim \delta$ where the propagator deviates from the simple massive form, might vanish in units of the higher-momentum parameter m_h. In this scenario there would be a hierarchy of scales $\delta \ll m_h \ll \Lambda$ such that $\frac{\delta}{m_h} \to 0$ when $\frac{m_h}{\Lambda} \to 0$ (as for instance with the relation $\delta \sim m_h^2/\Lambda$). However, if $\frac{G^{-1}(p=0)}{m_h^2} \to 0$, also $E_1(0)/m_h \to 0$ and thus both the range $|\mathbf{p}| \lesssim \delta$ and the corresponding portion of the energy spectrum $E_1(\mathbf{p})$ would shrink to the zero-measure set $p_\mu = 0$. In this picture of the continuum limit, where m_h can be taken to define the unit mass scale, the energy spectrum becomes discontinuous, namely $E_1(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m_h^2}$ for $\mathbf{p} \neq 0$ and $E_1(\mathbf{p}) = 0$ for $\mathbf{p} = 0$. Notice that this would represent a Lorentz-invariant decomposition because the value $(E_1 = 0, \mathbf{p} = 0)$ or $p_\mu = 0$ forms a Lorentz-invariant subset.

This point of view agrees well with Stevenson's recent analysis [15] of the propagator in the broken-symmetry phase. In his view, a more faithful representation of the continuum limit can be obtained by starting from the non-local action

$$\int d^4x \int d^4y \ \Phi^2(x)U(x-y)\Phi^2(y) \tag{53}$$

The kernel $U(x-y)$ contains, besides the repulsive contact $\delta-$function term, say $U_{\text{core}}(x-y)$, also an effective long-range attraction for $x \neq y$, say $U_{\text{tail}}(x-y)$. The latter, which is essential for a physical description of spontaneous symmetry breaking as a true condensation process [16], originates from *ultraviolet-finite* parts of higher-order Feynman graphs and has never been considered in the perturbative RG$-$approach. Instead, by taking into account both U_{core} and U_{tail} (and avoiding double counting), one can define a modified RG$-$expansion [15], as in a theory with two coupling constants. In the end, by taking the $\Lambda \to \infty$ limit, the resulting connected Euclidean propagator $G(p)$ has the standard massive form $G^{-1}(p) = (p^2 + m_h^2)$ except for a discontinuity at $p_\mu = 0$ where $G^{-1}(p = 0) = 0$. This type of structure, implying the existence of a branch of the spectrum whose energy $E_1(\mathbf{p}) \to 0$ in the $\mathbf{p} \to 0$ limit, would indeed support the previous idea that, at least for $\Lambda \to \infty$, the exact result is $E_1(0) = 0$.

Finally, this discontinuous nature of $G^{-1}(p = 0)$ would also be in agreement with the analogous indication of Ref.[17] that, in the broken-symmetry phase and quite independently of the Goldstone phenomenon, the zero-momentum connected propagator of the shifted fluctuation field is a two-valued function that, in addition to the standard value $G_a^{-1}(p = 0) = m_h^2$, includes the solution $G_b^{-1}(p = 0) = 0$ as in a massless theory. It is conceivable that such a subtle, nearly point-like, effect around $p_\mu = 0$ might have been missed in most conventional approximation schemes. At the same time, the idea of an infrared sector which is richer than expected might have far reaching phenomenological implications. For instance, by using the general properties of the Fourier transform, any $G^{-1}(p)$ that smoothly interpolates in an infinitesimal momentum region $|p| \sim \delta \ll m_h$, between $G_b^{-1}(p = 0) = 0$ and $G_a^{-1}(p) \sim (p^2 + m_h^2)$, would yield a long-range $1/r$ potential of infinitesimal strength δ^2/m_h^2 [18].

In conclusion, for the conceptual relevance of the problem and its potential phenomenological implications, it seems worth to sharpen our understanding of the low-momentum region of spontaneously broken Φ^4 theories both analytically and with a new generation of numerical simulations on those very large 4D lattices (e.g. 100^4) that should now be available with the present computer technology.

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