arXiv:hep-ph/9207276v1 31 Jul 1992

Gaussian Effective Potential for the

U(1) Higgs Model

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Abstract:

In order to investigate the Higgs mechanism nonperturbatively, we compute the Gaussian effective potential (GEP) of the U(1) Higgs model ("scalar electrodynamics"). We show that the same simple result is obtained in three different formalisms. A general covariant gauge is used, with Landau gauge proving to be optimal. The renormalization generalizes the "autonomous" renormalization for $\lambda \phi^4$ theory and requires a particular relationship between the bare gauge coupling e_B and the bare scalar self-coupling λ_B. When both couplings are small, then λ is proportional to e^4 and the scalar/vector mass-squared ratio is of order e^2, as in the classic 1-loop analysis of Coleman and Weinberg. However, as λ increases, e reaches a maximum value and then decreases, and in this "nonperturbative" regime the Higgs scalar can be much heavier than the vector boson. We compare our results to the autonomously renormalized 1-loop effective potential, finding many similarities. The main phenomenological implication is a Higgs mass of about 2 TeV.

1 Introduction

The Higgs mechanism [1] is a vital, but problematic, aspect of the Standard Model. At the classical level it is clear that spontaneous symmetry breaking (SSB) in the $\lambda\phi^4$ scalar sector, through its coupling to the gauge sector, generates gauge-boson mass terms. The issue of how – or whether – this works in the full quantum theory can be addressed using the effective potential [2], and traditionally the 1-loop approximation [3, 4] has been used. However – at least as it is conventionally renormalized – the 1-loop effective potential (1LEP) is closely tied to perturbation theory. The possibility that a perturbative approach is totally misleading must be raised by the claims that the $(\lambda\phi^4)_4$ theory is actually "trivial" [5], and by the failure of lattice Monte-Carlo calculations to find a non-trivial, interacting theory [6]. Thus, it is very important to study $(\lambda\phi^4)_4$ theory and the Higgs mechanism with nonperturbative methods.

A simple, nonperturbative method, founded upon intuitive ideas familiar in ordinary quantum mechanics, is the Gaussian effective potential (GEP) [7, 8]. In the appropriate limiting cases it contains the one-loop and leading-order $1/N$ effective potential results [7, 8, 9, 10]. The GEP for O(N)-symmetric $(\lambda\phi^4)_4$ theory can be renormalized in *two* different ways [11]: the "precarious" renormalization, with a *negative* infinitesimal λ_B [12, 9, 10], yields essentially the leading-order $1/N$ result [13]. The resulting effective potential does not, however, display SSB. The other, "autonomous", renormalization [14, 15, 10], which can have SSB, is characterized by a *positive* infinitesimal λ_B and an infinite re-scaling of the classical field. The resulting theory is asymptotically free [16], which can explain why the "triviality" proofs [5] do not apply. Particle masses turn out to be proportional to $\langle\phi\rangle$, so in the unbroken-symmetry phase the particles must be massless. This can perhaps explain the negative findings of most lattice calculations [6]. (See Ref. [17] for an interesting comparison of recent lattice results [18] with the Gaussian approximation.) The "autonomous" theory cannot be obtained in the $1/N$ expansion because λ_B must behave as $1/\sqrt{N}$, not as $1/N$, when $N \to \infty$ [10, 19].

We used to believe that the "autonomous" theory could only be seen with the Gaussian (or some still-better) approximation. However, it has been shown recently by Consoli and collaborators [20] that the unrenormalized 1LEP can also be renormalized in an "autonomous"-like way. This result generalizes to more complicated theories [21]. Applied to the SU(2)×U(1) electroweak theory it predicts a Higgs mass of about 2 TeV [20, 21].

In this paper we calculate the GEP for the U(1)-Higgs model, which is O(2) $\lambda\phi^4$ theory coupled to a U(1) gauge field. We show that it can be renormalized in an "autonomous"-like fashion, and that the vector boson aquires a mass proportional to $\langle\phi\rangle$, just as in the traditional description of the Higgs mechanism. The bare gauge coupling constant e_B^2 and the bare scalar self-coupling λ_B, both infinitesimal, are related such that for a given e_B^2 (below some maximum value) there are *two* allowed values of λ_B. One of these lies in a "perturbative" regime in which $\lambda \sim e^4$, where the results agree with the classic 1-loop analysis of Coleman and Weinberg [3]. The other lies in a "nonperturbative" regime, where it is possible to have a Higgs particle which

is arbitrarily heavy compared to the vector boson. (See Figs. 1 and 2.) Our results have much in common with the "autonomously renormalized" 1LEP [20, 21], and thus tend to support the expectation of a 2 TeV Higgs mass.

The layout of the paper is as follows: After some preliminaries in Sect. 2, we outline three separate calculations of the GEP for the U(1) Higgs model using three different formalisms: Sect. 3 describes a canonical, Hamiltonian-based calculation, as in [8, 9]; Sect. 4 gives a covariant "δ expansion" calculation, as in [22] (see also [23, 24, 25]); and Sect. 5 outlines a covariant variational calculation, as in [26]. We think it is very instructive, as well as reassuring, to see how the same result emerges from these very different approaches. Some comments on the unrenormalized result are given in Sect. 6, where we show that the optimal gauge parameter is $\xi = 0$ (Landau gauge). The renormalization of the GEP is carried out in Sect. 7. To conclude, we discuss the comparison to the 1LEP results and the implications for the Higgs mass in Sect. 8.

2 Preliminaries

We first recall the integrals which play a central rôle in the GEP. The expectation value of ϕ^2 for a single scalar field yields the quadratically divergent integral

$$I_0(\Omega) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\omega_{\underline{p}}}, \qquad \omega_{\underline{p}} \equiv \sqrt{\vec{p}^2 + \Omega^2}, \tag{2.1}$$

which is equivalent to the contracted Euclidean propagator $G(x,x)$ ("tadpole diagram") integral

$$I_0(\Omega) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 + \Omega^2} \tag{2.2}$$

that arises in the manifestly covariant formalism. The vacuum energy for a free scalar field of mass Ω is given by the quartically divergent integral

$$I_1(\Omega) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2} \omega_{\underline{p}}, \tag{2.3}$$

which is the sum of the zero-point energies for each momentum mode. This integral is familiar from the 1LEP and in the covariant formalism it arises in the form

$$\frac{1}{2}\mathrm{Tr}\,\ln\left[G^{-1}(x,y)\right]/\mathcal{V} = I_1(\Omega) = \frac{1}{2}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \ln\left(p^2 + \Omega^2\right), \tag{2.4}$$

where $\mathcal{V}$ is the spacetime volume. (Actually, this form of I_1 is only equivalent to the canonical form (2.3) up to an infinite constant [9].) The GEP also naturally involves the combination

$$J(\Omega) \equiv I_1(\Omega) - \frac{1}{2}\Omega^2 I_0(\Omega), \tag{2.5}$$

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which arises from the expectation value of a *massless* scalar-theory Hamiltonian (i.e., kinetic terms only), evaluated in the vacuum of a free field theory with mass Ω.

The GEP is essentially a variational calculation: one first obtains a function V_G of the classical field φ_c and of the mass parameters, and then one has to minimize with respect to the mass parameters. This leads to coupled "optimization equations" for the optimal mass parameters (denoted by overbars). In carrying out the minimization one needs the formal result:

$$\frac{\mathrm{d}I_1(\Omega)}{\mathrm{d}\Omega^2} = \frac{1}{2}I_0(\Omega). \tag{2.6}$$

Further discussion of these divergent integrals is postponed until Section 7.

The quantization of gauge theories in a covariant gauge always involves Faddeev-Popov ghosts. However, in the U(1) case the ghosts are free. Since they do not couple to the other fields, they have no effect, *except* for their contribution to the vacuum energy [27]. Because the ghosts correspond to two free, massless, anticommuting degrees of freedom, their contribution is easily seen to be $-2I_1(0)$. (In the covariant formalism this term would come from performing the functional integral over the ghost fields.) Since this contribution is φ_c independent, it will drop out when the infinite vacuum-energy constant is subtracted off. This happens automatically in dimensional regularization, which effectively sets $I_1(0) = 0$. We shall therefore ignore ghosts in the following calculations.

3   Canonical GEP calculation

The Lagrangian for the U(1) Higgs model (ignoring ghosts) is:

$$\mathcal{L} = \mathcal{L}_{\text{Gauge}} + \mathcal{L}_{\text{Scalar}}, \tag{3.1}$$

where $\mathcal{L}_{\text{Gauge}}$ is discussed below, and $\mathcal{L}_{\text{Scalar}}$ is the Lagrangian for a complex scalar field, ϕ, with the derivative replaced by the covariant derivative: [28]

$$\mathcal{L}_{\text{Scalar}} = (D_\mu\phi)^* (D_\mu\phi) - m_B^2\phi^*\phi - 4\lambda_B \left(\phi^*\phi\right)^2, \tag{3.2}$$

with

$$D_\mu = \partial_\mu + ie_B A_\mu, \tag{3.3}$$

where e_B is the bare gauge coupling constant. Replacing the complex field by two real fields:

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right), \tag{3.4}$$

we find the O(2)-symmetric $\lambda\phi^4$-theory Lagrangian plus coupling terms to the gauge field:

$$\mathcal{L}_{\text{Scalar}} = \frac{1}{2}(\partial_\mu\phi_1 - e_B A_\mu\phi_2)^2 + \frac{1}{2}(\partial_\mu\phi_2 + e_B A_\mu\phi_1)^2 - \frac{1}{2}m_B^2(\phi_1^2 + \phi_2^2) - \lambda_B(\phi_1^2 + \phi_2^2)^2. \tag{3.5}$$

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Forming the Hamiltonian density

$$\mathcal{H}_{\text{Scalar}} \equiv \dot{\phi}_1 \Pi_1 + \dot{\phi}_2 \Pi_2 - \mathcal{L}_{\text{Scalar}}, \tag{3.6}$$

with $\Pi_i \equiv \delta \mathcal{L}/\delta \dot{\phi}_i$, we obtain:

$$\mathcal{H}_{\text{Scalar}} = \mathcal{H}_{O(2)} - e_B \vec{A}.(\phi_1 \vec{\nabla} \phi_2 - \phi_2 \vec{\nabla} \phi_1) - \frac{1}{2} e_B^2 A_\mu A^\mu (\phi_1^2 + \phi_2^2), \tag{3.7}$$

where $\mathcal{H}_{O(2)}$ is the Hamiltonian density for O(2)-symmetric $\lambda \phi^4$ theory.

Without loss of generality, we can choose the classical field φ_c to lie in the ϕ_1 direction. Our trial vacuum $|0>_G$ is a direct product of the free-field vacua for the $\hat{\phi}_1$ "radial" field (with $\hat{\phi}_1 \equiv \phi_1 - \varphi_c$), with mass Ω; for the ϕ_2 "transverse" field, with mass ω; and for the gauge fields (to be discussed below). The middle term in (3.7) therefore gives no contribution when we take the expectation value of $\mathcal{H}_{\text{Scalar}}$ in the trial state $|0>_G$. Hence, we find:

$$< \mathcal{H}_{\text{Scalar}} > = V_G^{O(2)} - \frac{1}{2} e_B^2 <A_\mu A^\mu> (\varphi_c^2 + I_0(\Omega) + I_0(\omega)), \tag{3.8}$$

where the first term is the O(2) $\lambda \phi^4$-theory result [29, 10]:

$$\begin{aligned} V_G^{O(2)} &= J(\Omega) + J(\omega) + \frac{1}{2} m_B^2 (\varphi_c^2 + I_0(\Omega) + I_0(\omega)) \\ &\quad + \lambda_B \left[3(I_0(\Omega) + \varphi_c^2)^2 + 2I_0(\omega)(I_0(\Omega) + \varphi_c^2) + 3I_0(\omega)^2 - 2\varphi_c^4 \right]. \end{aligned} \tag{3.9}$$

The gauge-field Lagrangian, including gauge-fixing terms, can be written as:

$$\mathcal{L}_{\text{Gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu B) A^\mu + \frac{1}{2} \xi B^2, \tag{3.10}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The last two terms, involving the Nakanishi-Lautrup [30] auxiliary field B, are equivalent to the usual covariant gauge-fixing term $-\frac{1}{2\xi}(\partial \cdot A)^2$, where ξ is the gauge parameter. [To see this one integrates by parts to get $-B(\partial \cdot A) + \frac{1}{2} \xi B^2$, and then eliminates B by its equation of motion $B = (\partial \cdot A)/\xi$.]

By itself $\mathcal{L}_{\text{Gauge}}$ would just describe a set of free massless fields. We want to consider a generalization of this Lagrangian that includes a mass term:

$$\mathcal{L}_{\text{Trial}} = \mathcal{L}_{\text{Gauge}} + \frac{1}{2} \Delta^2 A_\mu A^\mu. \tag{3.11}$$

The ground state of this "trial theory" will provide us with our trial vacuum state, with the mass Δ playing the rôle of a variational parameter. To construct the GEP we shall then need to take the expectation value of $\mathcal{H}_{\text{Gauge}}$ (which we can obtain from $\mathcal{H}_{\text{Trial}}$ by setting $\Delta = 0$) in the vacuum state of $\mathcal{H}_{\text{Trial}}$.

The content of the "trial theory" is made plain by defining

$$\mathcal{A}_\mu \equiv A_\mu + \frac{1}{\Delta^2} \partial_\mu B, \tag{3.12}$$

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which de-couples $\mathcal{L}_{\text{Trial}}$ into separate $\mathcal{A}_\mu$ and B sectors:

$$\mathcal{L}_{\text{Trial}} = \left(-\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\Delta^2 \mathcal{A}_\mu \mathcal{A}^\mu\right) - \frac{1}{\Delta^2}\left(\frac{1}{2}\partial_\mu B \partial^\mu B - \frac{1}{2}\xi\Delta^2 B^2\right). \tag{3.13}$$

The $\mathcal{A}_\mu$ field is thus a free, massive vector field, and its equation of motion $\partial_\mu \mathcal{F}^{\mu\nu} + \nabla^2 \mathcal{A}^\nu = 0$ yields both $(\partial^2 + \Delta^2)\mathcal{A}^\nu = 0$ and $\partial\cdot\mathcal{A} = 0$. The B field is a normal scalar field, mass $\sqrt{\xi}\Delta$, except that its Lagrangian has the "wrong sign" and has an overall factor $1/\Delta^2$. It is now a relatively straightforward exercise to obtain the Hamiltonian and canonically quantize the theory, and we just list some of the key steps below.

The plane-wave expansion for the $\mathcal{A}_\mu$ field is:

$$\mathcal{A}_\mu = \sum_\lambda \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\omega_{\underline{k}}(\Delta)}\left[a(\vec{k},\lambda)\epsilon_\mu(\vec{k},\lambda)\mathrm{e}^{-ik.x} + \text{h.c.}\right], \tag{3.14}$$

in which $k^0 = \omega_{\underline{k}}(\Delta) \equiv \sqrt{\vec{k}^2 + \Delta^2}$, and the three polarization vectors $\epsilon_\mu(\vec{k},\lambda)$, with $\lambda = -1, 0, 1$ being the helicity label, satisfy the usual completeness relation:

$$\sum_\lambda \epsilon_\mu^*(\vec{k},\lambda)\epsilon_\nu(\vec{k},\lambda) = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{\Delta^2}\right). \tag{3.15}$$

The creation-annihilation operators obey

$$[a(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = 2\omega_{\underline{k}}(\Delta)(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}')\delta_{\lambda\lambda'}. \tag{3.16}$$

The plane-wave expansion for B is

$$B = \Delta \int \frac{\mathrm{d}^3 k}{(2\pi)^3\, 2\omega_{\underline{k}}(\sqrt{\xi}\Delta)}\left[a(\vec{k},B)\mathrm{e}^{-ik.x} + \text{h.c.}\right], \tag{3.17}$$

in which $k^0 = \omega_{\underline{k}}(\sqrt{\xi}\Delta) \equiv \sqrt{\vec{k}^2 + \xi\Delta^2}$, and the operators obey a "wrong-sign" commutation relation:

$$[a(\vec{k},B), a^\dagger(\vec{k}',B)] = -2\omega_{\underline{k}}(\sqrt{\xi}\Delta)(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'). \tag{3.18}$$

Our trial vacuum state $|0>_G$ is, by definition, annihilated by the operators $a(\vec{k},\lambda)$ ($\lambda = -1, 0, 1$) and $a(\vec{k},B)$. To construct the GEP we need to substitute the above plane-wave expansions into $\mathcal{H}_{\text{Gauge}}$ (conveniently obtained from $\mathcal{H}_{\text{Trial}}$ by dropping all terms involving Δ) and then sandwich the result between $_G<0|$ and $|0>_G$. From the $\mathcal{A}_\mu$ fields one obtains

$$<\mathcal{H}_\mathcal{A}> = 3J(\Delta), \tag{3.19}$$

which is three times (because of the three polarization states) the usual GEP result for a massless-scalar Hamiltonian evaluated in a free-field vacuum state of mass Δ. The B fields give

$$<\mathcal{H}_B> = J(\sqrt{\xi}\Delta), \tag{3.20}$$

which is positive because the "wrong sign" of the Hamiltonian is compensated by the "wrong sign" of the commutator. Therefore, in total we have

$$<\mathcal{H}_{\text{Gauge}}> = 3J(\Delta) + J(\sqrt{\xi}\Delta). \tag{3.21}$$

[As a check, note that if we were considering the gauge sector by itself, minimization of (3.21) would yield $\bar{\Delta} = 0$ and the result would reduce to $4J(0) = 4I_1(0)$. Recalling that the ghosts contribute $-2I_1(0)$, the total is $2I_1(0)$, which is the vacuum energy associated with two massless, bosonic degrees of freedom. These correspond to the two transverse polarizations of the massless vector field. Thus, we see explicitly, for any value of the gauge parameter ξ, how the ghosts act to cancel out the vacuum-energy contributions from the unphysical components of the gauge field [27].]

To obtain the total GEP we combine $<\mathcal{H}_{\text{Gauge}}>$ from (3.21) with $<\mathcal{H}_{\text{Scalar}}>$ from (3.8). A short calculation gives:

$$
\begin{aligned}
<A_\mu A^\mu> &= <\mathcal{A}_\mu \mathcal{A}^\mu> + \frac{1}{\Delta^4} <\partial_\mu B \partial^\mu B> \\
&= -3I_0(\Delta) - \xi I_0(\sqrt{\xi}\Delta),
\end{aligned}
\tag{3.22}
$$

so we obtain finally:

$$
\begin{aligned}
V_G(\varphi_c; \Omega, \omega, \Delta) &= V_G^{O(2)} + 3J(\Delta) + J(\sqrt{\xi}\Delta) \\
&\quad + \frac{1}{2}e_B^2(3I_0(\Delta) + \xi I_0(\sqrt{\xi}\Delta))(\varphi_c^2 + I_0(\Omega) + I_0(\omega)).
\end{aligned}
\tag{3.23}
$$

Minimization with respect to the mass parameters Δ, Ω, and ω leads to:

$$\bar{\Delta}^2 = e_B^2[\varphi_c^2 + I_0(\bar{\Omega}) + I_0(\bar{\omega})], \tag{3.24}$$

$$\bar{\Omega}^2 = m_B^2 + 4\lambda_B[3I_0(\bar{\Omega}) + I_0(\bar{\omega}) + 3\varphi_c^2] + e_B^2[3I_0(\bar{\Delta}) + \xi I_0(\sqrt{\xi}\bar{\Delta})], \tag{3.25}$$

$$\bar{\omega}^2 = m_B^2 + 4\lambda_B[I_0(\bar{\Omega}) + 3I_0(\bar{\omega}) + \varphi_c^2] + e_B^2[3I_0(\bar{\Delta}) + \xi I_0(\sqrt{\xi}\bar{\Delta})]. \tag{3.26}$$

4   Covariant δ-expansion Calculation

In this section we perform the calculation in the Euclidean functional-integral formalism in the manner of Ref. [22]. Note that in passing to the Euclidean formalism the Minkowski scalar product $a^\mu b_\mu$ goes to $-a_\mu b_\mu$; thus terms with just one pair of contracted indices change sign relative to other terms. The Euclidean action reads:

$$
\begin{aligned}
S &= \int d^4x \left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2\xi}(\partial_\mu A_\mu)^2 \right. \\
&\quad \left. + (D_\mu \phi)^* (D_\mu \phi) + m_B^2 \phi^* \phi + 4\lambda_B (\phi^*\phi)^2 \right].
\end{aligned}
\tag{4.1}
$$

Rewritten in terms of the real scalar fields ϕ_1 and ϕ_2, the action becomes

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \frac{1}{2} \phi_1 \left(-\partial^2 + m_B^2 \right) \phi_1 \right.$$
$$+ \frac{1}{2} \phi_2 \left(-\partial^2 + m_B^2 \right) \phi_2 + \lambda_B \left(\phi_1^2 + \phi_2^2 \right)^2$$
$$\left. + \frac{1}{2} e_B^2 A_\mu A_\mu \left(\phi_1^2 + \phi_2^2 \right) + e_B A_\mu \left(\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1 \right) \right], \tag{4.2}$$

where $\partial^2 \equiv \partial_\mu \partial_\mu$. Introducing a source for the ϕ_1 field, the generating functional is given by:

$$Z[j] = \int D\left[\phi_1, \phi_2, A_\mu\right] \exp\left(- S + \int d^4x \, j(x)\phi_1(x) \right), \tag{4.3}$$

and the effective action is obtained by the Legendre transformation

$$\Gamma[\varphi_c] = \ln Z[j] - \int d^4x \, j(x)\varphi_c(x), \tag{4.4}$$

where the classical field, $\varphi_c(x)$, is the vacuum expectation value of the field $\phi_1(x)$ in the presence of the source $j(x)$. The effective potential itself, $V_{\text{eff}}(\varphi_c)$, is obtained from $\Gamma[\varphi_c]$ by setting $\varphi_c(x)$ to be a constant, φ_c, and dividing out a minus sign and a spacetime volume factor.

Generalizing the procedure of Ref. [22] (see also [23, 24, 25]), we can calculate the GEP from a first-order expansion in a nonstandard kind of perturbation theory. First we introduce the shifted fields:

$$\hat{\phi}_1(x) = \phi_1(x) - \varphi_c, \qquad \hat{\phi}_2(x) = \phi_2(x). \tag{4.5}$$

(Notice that we have taken the shift parameter to be exactly φ_c, the vacuum expectation value of $\phi(x)$: Although not obligatory [22], this simplifies the calculation.)

We then split the (Euclidean) Lagrangian into two parts:

$$\mathcal{L} = \left(\mathcal{L}_0 + \mathcal{L}_{int} \right)_{\delta=1}, \tag{4.6}$$

where $\mathcal{L}_0$ is a sum of three free-field Lagrangians: one for the vector field A_μ, of mass Δ; one for the radial scalar field, $\hat{\phi}_1$, with mass Ω; and one for the transverse scalar field, $\hat{\phi}_2$, with mass ω:

$$\mathcal{L}_0 = \frac{1}{2} A_\mu(x) \left[\left(-\partial^2 + \Delta^2 \right) \delta_{\mu\nu} + \left(1 - \frac{1}{\xi} \right) \partial_\mu \partial_\nu \right] A_\nu(x)$$
$$+ \frac{1}{2} \hat{\phi}_1(x) \left(-\partial^2 + \Omega^2 \right) \hat{\phi}_1(x) + \frac{1}{2} \hat{\phi}_2(x) \left(-\partial^2 + \omega^2 \right) \hat{\phi}_2(x). \tag{4.7}$$

The interaction Lagrangian is then:

$$\mathcal{L}_{int} = \delta \left[v_0 + v_1 \hat{\phi}_1 + v_2 \hat{\phi}_1^2 + v_3 \hat{\phi}_1^3 + \lambda_B \hat{\phi}_1^4 + v_2' \hat{\phi}_2^2 + \lambda_B \hat{\phi}_2^4 \right.$$
$$+ 4\lambda_B \varphi_c \, \hat{\phi}_1 \hat{\phi}_2^2 + 2\lambda_B \, \hat{\phi}_1^2 \hat{\phi}_2^2 + \frac{1}{2} \left(e_B^2 \varphi_c^2 - \Delta^2 \right) A_\mu A_\mu$$
$$+ e_B \varphi_c \, A_\mu \, \partial_\mu \hat{\phi}_2 + e_B A_\mu \left(\hat{\phi}_1 \partial_\mu \hat{\phi}_2 - \hat{\phi}_2 \partial_\mu \hat{\phi}_1 \right)$$
$$\left. + e_B^2 \varphi_c \, \hat{\phi}_1 A_\mu A_\mu + \frac{1}{2} e_B^2 A_\mu A_\mu \left(\hat{\phi}_1^2 + \hat{\phi}_2^2 \right) \right]. \tag{4.8}$$

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The "coupling constants" v_0, v_1, v_2, v_2' and v_3, which are φ_c dependent, are the same as in the $\lambda\phi^4$ case [22]:

$$
\begin{aligned}
v_0 &= \frac{1}{2}m_B^2\varphi_c^2 + \lambda_B\varphi_c^4, \\
v_1 &= \left(m_B^2 + 4\lambda_B\varphi_c^2\right)\varphi_c, \\
v_2 &= \frac{1}{2}\left(m_B^2 - \Omega^2\right) + 6\lambda_B\varphi_c^2, \\
v_2' &= \frac{1}{2}\left(m_B^2 - \omega^2\right) + 2\lambda_B\varphi_c^2, \\
v_3 &= 4\lambda_B\varphi_c.
\end{aligned}
\tag{4.9}
$$

The artificial expansion parameter δ has been introduced in $\mathcal{L}_{int}$ in order to keep track of the order of approximation, which consists in obtaining a (truncated) Taylor series in δ, about $\delta = 0$, which is then used to extrapolate to $\delta = 1$.

The expansion in $\mathcal{L}_{int}$ (or equivalently in δ) is now quite straightforward, following standard perturbation theory procedures. To first order in δ it yields:

$$
\begin{aligned}
\Gamma[\varphi_c] &= \Gamma^{O(2)}[\varphi_c] - \frac{1}{2}\,\mathrm{Tr}\,\ln\left[G_{\mu\nu}^{-1}(x,y)\right] \\
&\quad - \delta\int\mathrm{d}^4x\,\left\{\frac{1}{2}\left(e_B^2\varphi_c^2 - \Delta^2\right)G_{\mu\mu}(x,x)\right. \\
&\qquad\qquad \left. + \frac{1}{2}e_B^2\left[I_0(\Omega) + I_0(\omega)\right]G_{\mu\mu}(x,x)\right\} + O\left(\delta^2\right),
\end{aligned}
\tag{4.10}
$$

where $\Gamma^{O(2)}[\varphi_c]$ is the first-order action for the scalar sector, and

$$
G_{\mu\nu}(x,y) = \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{1}{p^2+\Delta^2}\left[\delta_{\mu\nu} + (\xi-1)\frac{p_\mu p_\nu}{p^2+\xi\Delta^2}\right]e^{-ip\cdot(x-y)}
\tag{4.11}
$$

is the gauge field propagator, $<A_\mu(x)A_\nu(y)>$. The Trace-log term gives

$$
\mathrm{Tr}\,\ln\left[G_{\mu\nu}^{-1}(x,y)\right] = 2\mathcal{V}\left[3I_1(\Delta) + I_1(\sqrt{\xi}\,\Delta)\right],
\tag{4.12}
$$

where $\mathcal{V}$ is the infinite spacetime volume, and the contracted propagator gives

$$
G_{\mu\mu}(x,x) = 3I_0(\Delta) + \xi I_0(\sqrt{\xi}\,\Delta).
\tag{4.13}
$$

To obtain the GEP we discard the $O(\delta^2)$ terms and set $\delta = 1$ and divide through $-\mathcal{V}$ to obtain

$$
\begin{aligned}
V_G &= V_G^{O(2)} + 3I_1(\Delta) + I_1(\sqrt{\xi}\Delta) + \frac{1}{2}\left(e_B^2\varphi_c^2 - \Delta^2\right)\left(3I_0(\Delta) + \xi I_0(\sqrt{\xi}\,\Delta)\right) \\
&\quad + \frac{1}{2}e_B^2\left[I_0(\Omega) + I_0(\omega)\right]\left(3I_0(\Delta) + \xi I_0(\sqrt{\xi}\,\Delta)\right).
\end{aligned}
\tag{4.14}
$$

Recalling that $J(\Delta) \equiv I_1(\Delta) - \frac{1}{2}\Delta^2 I_0(\Delta)$, one sees that this result coincides with the result obtained from the canonical calculation, Eq. (3.23).

5 Covariant Variational Calculation

In this section we use the method developed in [26] based on Feynman's variational principle [31] applied to the Euclidean action. This in turn follows from Jensen's inequality for expectation values of convex functions; in particular, exponential functions:

$$\int d\mu(\phi) \exp g(\phi) \geq \exp\left(\int d\mu(\phi) g(\phi)\right), \tag{5.1}$$

for a normalized integration measure $d\mu(\phi)$. The inequality applies only to commuting fields, but happily in the U(1) case the anticommuting ghost fields can be integrated out exactly. The remaining action can be written in the following form (using the shifted fields $\hat{\phi}_1 = \phi_1 - \varphi_c$, $\hat{\phi}_2 = \phi_2$, as in the last section):

$$S[A_\mu, \hat{\phi}_1, \hat{\phi}_2] = S_A[A_\mu] + S_{A,\phi}[A_\mu, \hat{\phi}_1, \hat{\phi}_2] + S_\phi[\hat{\phi}_1, \hat{\phi}_2], \tag{5.2}$$

where

$$S_A = \frac{1}{2}\int d^4x A_\mu(x)\left[-\partial^2\delta_{\mu\nu} + (1-\frac{1}{\xi})\partial_\mu\partial_\nu\right]A_\nu(x), \tag{5.3}$$

$$\begin{aligned} S_{A,\phi} = \int d^4x \Big\{ &\frac{1}{2}e_B^2(\varphi_c^2 + \hat{\phi}_1^2 + \hat{\phi}_2^2)A_\mu(x)A_\mu(x) \\ &+ e_B A_\mu\left[(\hat{\phi}_1 + \varphi_c)\partial_\mu\hat{\phi}_2 + \hat{\phi}_2\partial_\mu(\hat{\phi}_1 + \varphi_c)\right] + e_B^2\varphi_c\hat{\phi}_1 A_\mu A_\mu\Big\}, \end{aligned} \tag{5.4}$$

and S_ϕ is given by the usual O(2) $\lambda\phi^4$ action.

Following Ref. [26], we now apply the Feynman-Jensen inequality to $Z[j]$, Eq (4.3), with $d\mu(\phi) = \mathcal{N}^{-1}DA_\mu D\hat{\phi}_1 D\hat{\phi}_2 e^{-S_G}$ and $g(\phi) = S_G - S + \int j\phi$, where

$$\mathcal{N} = \int DA_\mu D\hat{\phi}_1 D\hat{\phi}_2\, e^{-S_G}, \tag{5.5}$$

and where S_G is a quadratic "trial action":

$$S_G = \frac{1}{2}\int d^4x \left\{A_\mu G_{\mu\nu}^{-1}A_\nu + \hat{\phi}_1 G_1^{-1}\hat{\phi}_1 + \hat{\phi}_2 G_2^{-1}\hat{\phi}_2\right\}, \tag{5.6}$$

involving adjustable kernels G^{-1}. Taking the Legendre transform, (4.4), we obtain the "Gaussian effective action" [26]:

$$\bar{\Gamma}^{GEA}[\varphi_c] = \max_G \left\{\log(\mathcal{N}) + \mathcal{N}^{-1}\int DA_\mu D\hat{\phi}_1 D\hat{\phi}_2\, e^{-S_G}\,(S_G - S)\right\}, \tag{5.7}$$

as a lower bound on the exact effective action (which will hence yield an upper bound on the effective potential). Since the kernels involve differential operators, it is convenient to go to momentum space, using Fourier transforms (indicated by tildes) and the convenient notation $\int_p = \int d^4p/(2\pi)^4$, $\bar{\delta}(p) = (2\pi)^4\delta(p)$. We can then write the trial action as

$$S_G = \frac{1}{2}\int_p\int_q \left\{\tilde{A}_\mu(p)\tilde{G}_{\mu\nu}^{-1}(p,q)\tilde{A}_\nu(q) + \tilde{\phi}_1(p)\tilde{G}_1^{-1}(p,q)\tilde{\phi}_1(q) + \tilde{\phi}_2(p)\tilde{G}_2^{-1}(p,q)\tilde{\phi}_2(q)\right\}, \tag{5.8}$$

in which the $\tilde{G}^{-1}$'s are the inverses of the momentum-space propagators.

Evaluation of the Gaussian functional integrals involved in (5.7) is straightforward, and yields

$$\bar{\Gamma}^{GEA} = \max_{\tilde{G}} \left\{ \Gamma^{O(2)}[\tilde{\varphi}_c, \tilde{G}_1, \tilde{G}_2] - \frac{1}{2}\text{Tr } \ln[\tilde{G}_{\mu\nu}^{-1}(p,q)] \right.$$

$$-\frac{1}{2}\int_p \left[p^2\delta_{\mu\nu} + e_B^2\int_r (\tilde{G}_1(r,-r) + \tilde{G}_2(r,-r))\delta_{\mu\nu} - (1-\frac{1}{\xi})p_\mu p_\nu \right] \tilde{G}_{\mu\nu}(p,-p)$$

$$\left. -\frac{1}{2}e_B^2\int_{pqrs} \tilde{\varphi}_c(r)\tilde{\varphi}_c(s)\tilde{G}_{\mu\mu}(p,q)\bar{\delta}(p+q+r+s) \right\}. \tag{5.9}$$

Maximization yields optimization equations determining the optimal $\tilde{G}$ propagators, denoted by $\bar{G}(p,q)$:

$$\bar{G}_{\mu\nu}^{-1}(p,q) = \left[p^2\delta_{\mu\nu} + e_B^2\int_r (\bar{G}_1(r,-r) + \bar{G}_2(r,-r))\delta_{\mu\nu} - (1-\frac{1}{\xi})p_\mu p_\nu \right] \bar{\delta}(p+q)$$

$$+ e_B^2\int_{rs} \tilde{\varphi}_c(r)\tilde{\varphi}_c(s)\bar{\delta}(p+q+r+s)\delta_{\mu\nu}, \tag{5.10}$$

$$\bar{G}_1^{-1}(p,q) = \left[p^2 + m_B^2 + e_B^2\int_r \bar{G}_{\mu\mu}(r,-r) \right] \bar{\delta}(p+q)$$

$$+ 4\lambda_B\int_{rs} [3\bar{G}_1(r,s) + \bar{G}_2(r,s) + 3\tilde{\varphi}_c(r)\tilde{\varphi}_c(s)]\bar{\delta}(p+q+r+s), \tag{5.11}$$

$$\bar{G}_2^{-1}(p,q) = \left[p^2 + m_B^2 + e_B^2\int_r \bar{G}_{\mu\mu}(r,-r) \right] \bar{\delta}(p+q)$$

$$+ 4\lambda_B\int_{rs} [\bar{G}_1(r,s) + 3\bar{G}_2(r,s) + \tilde{\varphi}_c(r)\tilde{\varphi}_c(s)]\bar{\delta}(p+q+r+s). \tag{5.12}$$

For a spatially constant classical field we have $\tilde{\varphi}_c(p) = \varphi_c\bar{\delta}(p)$, and the above equations then dictate that the propagators all become proportional to $\bar{\delta}(p+q)$, so we may write them in the form

$$\bar{G}_{\mu\nu}^{-1}(p,q) = \left[(p^2 + \bar{\Delta}^2)\delta_{\mu\nu} - (1-\frac{1}{\xi})p_\mu p_\nu \right] \bar{\delta}(p+q), \tag{5.13}$$

$$\bar{G}_1^{-1}(p,q) = (p^2 + \bar{\Omega}^2)\bar{\delta}(p+q), \tag{5.14}$$

$$\bar{G}_2^{-1}(p,q) = (p^2 + \bar{\omega}^2)\bar{\delta}(p+q), \tag{5.15}$$

where the optimal mass parameters $\bar{\Delta}$, $\bar{\Omega}$, and $\bar{\omega}$ are given by

$$\bar{\Delta}^2 = e_B^2[\varphi_c^2 + I_0(\bar{\Omega}) + I_0(\bar{\omega})], \tag{5.16}$$

$$\bar{\Omega}^2 = m_B^2 + 4\lambda_B[3I_0(\bar{\Omega}) + I_0(\bar{\omega}) + 3\varphi_c^2] + e_B^2\mathcal{V}^{-1}\int_p \bar{G}_{\mu\mu}(p,-p), \tag{5.17}$$

$$\bar{\omega}^2 = m_B^2 + 4\lambda_B[I_0(\bar{\Omega}) + 3I_0(\bar{\omega}) + \varphi_c^2] + e_B^2\mathcal{V}^{-1}\int_p \bar{G}_{\mu\mu}(p,-p). \tag{5.18}$$

As usual, factors of "$\bar{\delta}(0)$" have been interpreted as spacetime volume factors $\mathcal{V}$. The integral $\int_p \bar{G}_{\mu\mu}(p,-p)$, where

$$\bar{G}_{\mu\nu}(p,-p) = \frac{\mathcal{V}}{p^2 + \bar{\Delta}^2}\left[\delta_{\mu\nu} + (\xi-1)\frac{p_\mu p_\nu}{p^2 + \xi\bar{\Delta}^2} \right], \tag{5.19}$$

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can be evaluated in terms of I_0 integrals:

$$\int_p \bar{G}_{\mu\mu}(p, -p) = \mathcal{V}[3I_0(\bar{\Delta}) + \xi I_0(\sqrt{\xi}\bar{\Delta})]. \tag{5.20}$$

The Trace-log term can be taken from Eq. (4.12), so that we obtain finally the same result as in Eq. (3.23).

6   Comments on the Unrenormalized Result

The Gaussian-approximation result shows a dependence upon the gauge parameter ξ. This means that our Gaussian approximation does not fully respect gauge invariance. However, we argue that this is inevitable and not fatal. It is inevitable because, for the O(2) scalars, we have to use "Cartesian-coordinate" fields $\hat{\phi}_1$, $\hat{\phi}_2$ rather than "polar-coordinate" fields, so that, when $\varphi_c \neq 0$, the O(2) symmetry is not being fully respected. In pure $\lambda\phi^4$ theory this produces an apparent conflict with Goldstone's theorem, in that the transverse mass parameter, ω, is non-zero [10, 29]. However, the point is that the transverse field $\hat{\phi}_2$ is not the true "polar-angle", Goldstone field. In the U(1) Higgs model, in a covariant gauge, the Goldstone field becomes an unphysical degree of freedom [1], but the problem remains in the form of gauge-parameter dependence. This just means, though, that we have a "non-invariant" approximation – which is where the exact result is known to be independent of some parameter, but the approximate result has a dependence on that parameter. This is actually quite a common occurrence, and can be dealt with by "optimizing" the unphysical parameter; requiring the approximate result to be stationary, or more generally "minimally sensitive," to the unphysical parameter [32]. One is in a still better position when the approximation has a variational character, because then the optimal choice for the unphysical parameter is unquestionably determined by minimization.

Our calculation here does indeed have a variational character. [One might well have been unsure, with the canonical calculation alone, whether or not the variational inequality is valid in the presence of a "wrong-sign" field (and hence negative-norm states). However, this doubt is allayed by the covariant variational calculation: the Jensen inequality just depends on the the the convexity of the exponential function and is equally valid for $\exp(-g(\phi))$ and $\exp(+g(\phi))$.] By differentiating $\bar{V}_G$ one finds that the optimal gauge is the Landau gauge, $\xi = 0$. This can easily be seen by noting that, by virtue of the $\bar{\Delta}$ equation, the ξ-dependence in $\bar{V}_G$ comes only from an $I_1(\sqrt{\xi}\bar{\Delta})$ term. Since I_1 is (formally) an increasing function of its argument, the energy is minimized when $\xi = 0$.

With $\xi = 0$, and discarding a vacuum-energy contribution $I_1(0)$, the GEP and its optimization equations simplify to:

$$V_G(\varphi_c; \Omega, \omega, \Delta) = V_G^{O(2)} + 3J(\Delta) + \frac{3}{2}e_B^2 I_0(\Delta)(\varphi_c^2 + I_0(\Omega) + I_0(\omega)), \tag{6.1}$$

with $V_G^{O(2)}$ given by (3.9), and

$$\bar{\Delta}^2 = e_B^2[\varphi_c^2 + I_0(\bar{\Omega}) + I_0(\bar{\omega})], \tag{6.2}$$

$$\bar{\Omega}^2 = m_B^2 + 4\lambda_B[3I_0(\bar{\Omega}) + I_0(\bar{\omega}) + 3\varphi_c^2] + 3e_B^2I_0(\bar{\Delta}), \tag{6.3}$$

$$\bar{\omega}^2 = m_B^2 + 4\lambda_B[I_0(\bar{\Omega}) + 3I_0(\bar{\omega}) + \varphi_c^2] + 3e_B^2I_0(\bar{\Delta}). \tag{6.4}$$

Note that if the $\bar{\Delta}$ equation, (6.2), is substituted back into V_G, (6.1), then we can write the GEP as

$$\bar{V}_G(\varphi_c) = V_G^{O(2)} + 3I_1(\bar{\Delta}), \tag{6.5}$$

with separate contributions from the scalar and gauge sectors. This observation applies to the Gaussian effective action, too, since Eq. (5.10) substituted back into (5.9) yields

$$\bar{\Gamma}^{GEA} = \Gamma^{O(2)} - \frac{1}{2}\text{Tr } \ln[\bar{G}_{\mu\nu}^{-1}(p,q)]. \tag{6.6}$$

Note, however, that the optimization equations for $\bar{\Delta}$, $\bar{\Omega}$, and $\bar{\omega}$, (6.2–6.4), remain coupled.

We may also remark that the generalization of the result to $\nu + 1$ dimensions is trivial: the integrals need to be re-defined in an obvious way, and the factors of 3 associated with the Δ integrals need to be replaced by ν, since these factors correspond to the number of polarization states of a massive vector field.

Finally, we briefly comment upon some previous work relating to the GEP and the U(1) Higgs model. (i) Allès and Tarrach [33] used a somewhat naive canonical approach which we believe is valid in Feynman gauge ($\xi=1$) only. Their treatment of the scalar sector effectively sets $\omega \equiv \Omega$, which is sub-optimal. In renormalizing their result, Allès and Tarrach used a generalization of the "precarious" $\lambda\phi^4$ theory, which does not have SSB. (ii) Cea [34] describes a temporal-gauge GEP calculation, but contents himself with demonstrating that the 1-loop terms are recovered correctly. (iii) The papers of Ref. [35] make a comprehensive study of the Schrödinger wavefunctional formalism, and try hard to maintain gauge invariance and compliance with the Goldstone theorem. Our view, as discussed above, is less puritanical. (iv) Kovner and Rosenstein [36] use yet another formulation of the Gaussian approximation, based on truncating the Dyson-Schwinger equations. Their renormalization is quite different from ours: it appears to be related to the "precarious" $\lambda\phi^4$ renormalization, but it somehow transfers the negative sign from λ_B to wavefunction renormalization factors.

The "precarious" renormalization of the U(1)-Higgs-model GEP is a topic which we do not pursue here, but it could be of theoretical interest: We would expect the results to be similar to the $1/N$-expansion analysis of Kang [37].

7 "Autonomous" Renormalization of the GEP

7.1 The divergent integrals

The GEP involves the quartically and quadratically divergent integrals I_1 and I_0. Another related integral:

$$I_{-1}(\Omega) \equiv -2\frac{\mathrm{d}I_0}{\mathrm{d}\Omega^2} = \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{1}{2(\omega_{\underline{p}})^3} = 2\int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{1}{(p^2 + \Omega^2)^2}, \tag{7.1}$$

which is logarithmically divergent, will play a crucial role. Ref. [9] derives useful formulas for these integrals by Taylor-expanding the integrands about $\Omega^2 = m^2$, and then re-summing the terms that give convergent integrals. From these we can obtain the still more convenient formulas:

$$I_1(\Omega) = I_1(0) + \frac{\Omega^2}{2}I_0(0) - \frac{\Omega^4}{8}I_{-1}(\mu) + f(\Omega^2), \tag{7.2}$$

$$I_0(\Omega) = I_0(0) - \frac{\Omega^2}{2}I_{-1}(\mu) + 2f'(\Omega^2), \tag{7.3}$$

$$I_{-1}(\Omega) = I_{-1}(\mu) - \frac{1}{8\pi^2}\ln\frac{\Omega^2}{\mu^2}, \tag{7.4}$$

where

$$f(\Omega^2) = \frac{\Omega^4}{64\pi^2}\left[\ln\frac{\Omega^2}{\mu^2} - \frac{3}{2}\right], \tag{7.5}$$

and $f'(\Omega^2)$ is its derivative with respect to Ω^2. These formulas are valid in any regularization scheme that preserves the property $\mathrm{d}I_n/\mathrm{d}\Omega^2 = (n - \frac{1}{2})I_{n-1}$. This allows one, at least in the $\lambda\phi^4$ case, to discuss the renormalization procedure in a completely regularization-independent manner. However, in gauge theories, most cutoff-based renormalizations – because they interfere with gauge invariance – have problems with quadratic divergences in the vector self-energy. Here these problems would manifest themselves as quadratic divergences in the vector-mass parameter $\bar{\Delta}^2$, Eq. (6.2) [35]. (Unlike the scalar case, these cannot be simply absorbed into a bare-mass.) It is well known in other contexts that, with sufficient technical virtuosity, these problems can be shown to be spurious [38]. However, it is much simpler to appeal to dimensional regularization, or some such scheme, in which one can justify setting the scale-less integrals, $I_0(0)$ and $I_1(0)$, equal to zero. This automatically elimates any problem with quadratic divergences. All the remaining divergences can be written in terms of I_{-1}, which has a $1/\epsilon$ pole in dimensional regularization: Explicitly:

$$I_{-1}(\Omega) = \frac{A}{\epsilon}\Omega^{-\epsilon}, \qquad A \equiv \frac{1}{4\pi^2}\Gamma(1 + \epsilon/2)(4\pi)^{\epsilon/2}. \tag{7.6}$$

7.2 Renormalization: Part I

To renormalize $\bar{V}_G$ we use an "autonomous" renormalization (Cf. [15, 10]), characterized by an infinite re-scaling of the classical field and infinitesimal bare coupling constants:

$$\varphi_c^2 = Z_\phi \Phi_c^2 = z_0 I_{-1}(\mu)\Phi_c^2, \tag{7.7}$$

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$$\lambda_B = \frac{\eta}{I_{-1}(\mu)}\,, \qquad e_B^2 = \frac{\gamma}{I_{-1}(\mu)}\,, \tag{7.8}$$

where z_0, η and γ are finite, and μ is a finite mass scale. For the present, we assume that all the I_{-1} factors have the same argument, μ. We shall also take $m_B^2 = 0$. These simplifying assumptions will be removed later in subsection 7.4. We shall also postpone the determination of the finite wavefunction-renormalization factor z_0 to that subsection.

First, we substitute the renormalization equations into the optimization equations (6.2 – 6.4) and use the key formula for I_0, (7.3), setting $I_0(0) = 0$. Keeping only the finite terms, for the present, we obtain:

$$\begin{aligned}
\bar{\Delta}^2 &= \gamma(z_0\Phi_c^2 - \tfrac{1}{2}\bar{\Omega}^2 - \tfrac{1}{2}\bar{\omega}^2) + \epsilon_\Delta, \\
\bar{\Omega}^2 &= 4\eta(-\tfrac{3}{2}\bar{\Omega}^2 - \tfrac{1}{2}\bar{\omega}^2 + 3z_0\Phi_c^2) - \tfrac{3}{2}\gamma\bar{\Delta}^2 + \epsilon_\Omega, \\
\bar{\omega}^2 &= 4\eta(-\tfrac{1}{2}\bar{\Omega}^2 - \tfrac{3}{2}\bar{\omega}^2 + z_0\Phi_c^2) - \tfrac{3}{2}\gamma\bar{\Delta}^2 + \epsilon_\omega,
\end{aligned} \tag{7.9}$$

where the ϵ_Δ, ϵ_Ω, ϵ_ω terms are infinitesimal, $\mathcal{O}(1/I_{-1})$, terms. Ignoring the ϵ terms the equations are linear and homogeneous, so that each mass parameter is proportional to Φ_c. The equations can be straightforwardly solved to yield:

$$\begin{aligned}
\bar{\Delta}^2 &= \frac{2\gamma}{(2 + 16\eta - 3\gamma^2)}z_0\Phi_c^2 + \mathcal{O}(1/I_{-1}), \\
\bar{\Omega}^2 &= \frac{[8\eta(3 + 16\eta) - 3\gamma^2(1 + 8\eta)]}{(1 + 4\eta)(2 + 16\eta - 3\gamma^2)}z_0\Phi_c^2 + \mathcal{O}(1/I_{-1}), \\
\bar{\omega}^2 &= \frac{(8\eta - 3\gamma^2)}{(1 + 4\eta)(2 + 16\eta - 3\gamma^2)}z_0\Phi_c^2 + \mathcal{O}(1/I_{-1}).
\end{aligned} \tag{7.10}$$

Since $\partial V_G/\partial\Omega = 0$, etc., by virtue of the gap equations, the total derivative of $\bar{V}_G$ with respect to φ_c is equal to its partial derivative, and so can be calculated very easily:

$$\begin{aligned}
\frac{\mathrm{d}\bar{V}_G}{\mathrm{d}\varphi_c} = \frac{\partial V_G}{\partial\varphi_c} &= \varphi_c[m_B^2 + 4\lambda_B(3I_0(\bar{\Omega}) + I_0(\bar{\omega}) + \varphi_c^2) + 3e_B^2 I_0(\bar{\Delta})] \\
&= \varphi_c(\bar{\Omega}^2 - 8\lambda_B\varphi_c^2).
\end{aligned} \tag{7.11}$$

The last equality follows from the optimization equation for $\bar{\Omega}$, (6.3), and yields the same expression as in pure $\lambda\phi^4$ theory [10]. In order for $\bar{V}_G$ to be finite in terms of the re-scaled field Φ_c, we must have a cancellation between the finite part of $\bar{\Omega}^2$ and $8\lambda_B\varphi_c^2 = 8\eta z_0\Phi_c^2$. This condition implies a constraint on the coefficients η and γ of the λ_B and e_B^2 coupling constants. This can be expressed as:

$$\gamma^2 = \frac{8\eta}{3}\frac{(1 - 8\eta - 64\eta^2)}{(1 - 32\eta^2)}, \tag{7.12}$$

and will be discussed further in the next subsection.

Using the constraint one can simplify the expressions for the optimal mass parameters to:

$$\begin{aligned}
\bar{\Delta}^2 &= \gamma \frac{1 - 32\eta^2}{1 + 4\eta} z_0 \Phi_c^2 + \mathcal{O}(1/I_{-1}), \\
\bar{\Omega}^2 &= 8\eta z_0 \Phi_c^2 + \mathcal{O}(1/I_{-1}), \\
\bar{\omega}^2 &= \frac{32\eta^2}{1 + 4\eta} z_0 \Phi_c^2 + \mathcal{O}(1/I_{-1}).
\end{aligned} \tag{7.13}$$

The renormalized GEP is most easily obtained from the expression for its first derivative, (7.11). The leading terms cancel, so one needs to obtain the infinitesimal, $\mathcal{O}(1/I_{-1})$, part of $\bar{\Omega}^2$. The calculation is straightforward, if tedious. One needs to obtain the explicit form of $\epsilon_\Delta, \epsilon_\Omega, \epsilon_\omega$, in Eqs. (7.9) by going back to Eqs. (6.2 – 6.4). One can then solve for the $\mathcal{O}(1/I_{-1})$ correction to $\bar{\Omega}$ in Eq. (7.13). After some algebra, one finds that the coefficients of the three f' terms match those in (7.13) above, so that one can write:

$$\frac{d\bar{V}_G}{d\Phi_c} = 2\Phi_c \left[3 \left(\frac{d\bar{\Delta}^2}{d\Phi_c^2} \right) f'(\bar{\Delta}^2) + \left(\frac{d\bar{\Omega}^2}{d\Phi_c^2} \right) f'(\bar{\Omega}^2) + \left(\frac{d\bar{\omega}^2}{d\Phi_c^2} \right) f'(\bar{\omega}^2) \right]. \tag{7.14}$$

Thus, by integrating with respect to Φ_c, one obtains the renormalized GEP as just:

$$\bar{V}_G = 3f(\bar{\Delta}^2) + f(\bar{\Omega}^2) + f(\bar{\omega}^2), \tag{7.15}$$

where f is the function defined in Eq. (7.5). The GEP is thus a sum of $\Phi_c^4 \ln \Phi_c^2$ and Φ_c^4 terms. If we swap the parameter μ for the vacuum value Φ_v (defined as the position of the minimum of $\bar{V}_G$), we can write the GEP simply as

$$\bar{V}_G = K z_0^2 \Phi_c^4 \left(\ln \left(\frac{\Phi_c^2}{\Phi_v^2} \right) - \frac{1}{2} \right), \tag{7.16}$$

where

$$K = \frac{\eta}{8\pi^2} \left[\frac{(1 + 8\eta)(1 - 8\eta + 32\eta^2 + 256\eta^3)}{(1 + 4\eta)^2} \right]. \tag{7.17}$$

7.3 Discussion

The constraint (7.12) arises from the requirement that the divergent I_{-1} terms in $\bar{V}_G$ cancel. The equivalent constraint in pure O(2) $\lambda\phi^4$ analysis [10, 29] would fix the coefficient η to be the positive root of the numerator factor, $(1 - 8\eta - 64\eta^2)$, which is

$$\eta_0 \equiv \frac{1}{4(1 + \sqrt{5})} = 0.0773. \tag{7.18}$$

Here, however, one has instead a relationship between the two coupling coefficients, which is shown in Fig. 1. It is easily established that only the region between $\eta = 0$ and $\eta = \eta_0$ is physically relevant. This is because (i) γ, being proportional to e_B^2, must be positive; and (ii) the vector mass-squared $\bar{\Delta}^2$ must be positive, which precludes η^2 from being larger than 1/32

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(see Eq. (7.13)). From the figure we see that there is a "perturbative region" in which both η and γ are small, with $\gamma^2 \approx (8/3)\eta$. This corresponds to $e^4 \sim \mathcal{O}(\lambda)$, as in Coleman and Weinberg (CW) [3]. However, as η increases, γ^2 reaches a maximum and then starts to decrease, going to zero at $\eta = \eta_0$. This extreme case corresponds to a free vector theory completely decoupled from a self-interacting $\lambda\phi^4$ theory.

The vector-boson and Higgs masses come directly from Eq. (7.13), evaluated at $\Phi_c = \Phi_v$. Their ratio is given by:

$$\frac{M_H^2}{M_V^2} = \frac{\bar{\Omega}_v^2}{\bar{\Delta}_v^2} = \frac{8\eta(1 + 4\eta)}{\gamma(1 - 32\eta^2)}, \tag{7.19}$$

which is just a function of η, since γ is determined by the constraint (7.12). The mass-squared ratio is plotted in Fig. 2. In the "perturbative regime" the Higgs is much lighter than the vector boson, by a factor of 3γ, which is $\mathcal{O}(e^2)$ as in CW. However, for most of the range of η the Higgs has a mass comparable to the vector. When η becomes close to η_0 the Higgs can be much heavier than the vector.

The other mass parameter, $\bar{\omega}^2$, does not have a direct physical meaning. It corresponds to the mass of the transverse scalar field, which is, approximately, the Goldstone field. In the covariant-gauge Higgs mechanism [1] the Goldstone field is an unphysical degree of freedom. As discussed in Sect. 6, the fact that $\bar{\omega}^2$ is non-zero is due to our approximation being unable to fully respect the O(2) symmetry. We can therefore be pleased by the fact that ω^2 is small (dashed line in Fig. 2).

7.4 Renormalization: Part II

We initially assumed that the mass-scale in the I_{-1} denominator of e_B^2 was the same as the mass-scale μ in λ_B (see Eq. (7.8)). If this is not so then, using (7.4), we can re-write e_B^2 as:

$$e_B^2 = \frac{\gamma}{I_{-1}(\mu)} + \frac{\gamma_2}{(I_{-1}(\mu))^2} + \dots, \tag{7.20}$$

where γ_2 is a coefficient proportional to the logarithm of the ratio of the two mass-scales. The subleading $\gamma_2/(I_{-1}(\mu))^2$ term leads to an extra contribution, proportional to Φ_c^2, in the infinitesimal part of $\bar{\Omega}^2$. Thus, when $\bar{V}_G$ is obtained by integrating (7.11), we obtain an extra finite contribution proportional to Φ_c^4 in Eq. (7.15). However, if we then re-parametrize the GEP in terms of the vacuum value Φ_v, we obtain Eq. (7.16) *unchanged*: all the differences are absorbed into the relationship of Φ_v to μ and γ_2. Exactly the same argument applies if the scale in the I_{-1} factor of Z_ϕ is different from that in λ_B [15]. [The argument also applies if one wants to insist upon replacing the factors of 3 in the GEP, representing the number of polarization states of a massive vector field, by $3 - \epsilon$ in dimensional regularization.]

Note that, for $m_B^2 = 0$, the bare Lagrangian is characterized by just two bare parameters; λ_B and e_B. Thus, we expect the renormalized GEP to be characterized by two parameters. This is indeed the case, and in the final form, (7.16), these are η and Φ_v. (We shall shortly see

that z_0 is fixed in terms of η by Eq. (7.24) below.) The Φ_v parameter has dimensions of mass, and its appearance constitutes the "dimensional transmutation" phenomenon [3]. Originally, the "autonomous" renormalization conditions (7.7) and (7.8) introduced a superfluity of parameters; η, γ, z_0, and the scale arguments of the I_{-1} factors. As just discussed, it does not matter if all these mass-scales are different, since they are eventually subsumed in a single scale, Φ_v. We saw earlier that γ was fixed in terms of η by the constraint (7.12), required for the I_{-1} divergences to cancel. It remains to show how z_0 is determined, and we turn to this topic next.

The "autonomous" renormalization involves a wavefunction renormalization constant $Z_\phi = z_0 I_{-1}(\mu)$. The $\lambda\phi^4$ analysis in Refs. [15, 10] set $z_0 = 1$ arbitrarily (although the possibility of further finite re-scalings of the field was considered). However, as Ref. [20] has pointed out, z_0 is actually fixed uniquely by the following argument. The bare and renormalized two-point functions are related by

$$\Gamma_B^{(2)} = Z_\phi^{-1}\Gamma_R^{(2)}. \tag{7.21}$$

Let us consider this relation at zero momentum in the vacuum $\varphi_c = \varphi_v$. $\Gamma_B^{(2)}$ is then given by the second derivative of the effective potential, with respect to the bare field, at $\varphi_c = \varphi_v$. This is easily calculated from (7.16):

$$\left.\frac{d^2\bar{V}_G}{d\varphi_c^2}\right|_{\varphi_c=\varphi_v} = \frac{1}{Z_\phi}\left.\frac{d^2\bar{V}_G}{d\Phi_c^2}\right|_{\Phi_c=\Phi_v} = \frac{1}{Z_\phi}8Kz_0^2\Phi_v^2. \tag{7.22}$$

The renormalized (Euclidean) two-point function (i.e., inverse propagator), $\Gamma_R^{(2)}$, is just $p^2 + \bar{\Omega}^2$ in the Gaussian approximation. At zero momentum and at $\varphi_c = \varphi_v$ it therefore becomes the physical Higgs mass squared $M_H^2 = \bar{\Omega}_v^2 = 8\eta z_0\Phi_v^2$. Hence, Eq. (7.21) gives

$$\frac{1}{Z_\phi}8Kz_0^2\Phi_v^2 = \frac{1}{Z_\phi}8\eta z_0\Phi_v^2, \tag{7.23}$$

which implies

$$z_0(m_B{=}0) = \frac{\eta}{K} = 8\pi^2\left[\frac{(1+4\eta)^2}{(1+8\eta)(1-8\eta+32\eta^2+256\eta^3)}\right]. \tag{7.24}$$

The factor in square brackets varies between 1 and 1.536 for η between 0 and η_0. (See Fig. 3.)

Finally, we remove our initial simplifying assumption that the bare mass vanishes identically. A finite bare mass would spoil the cancellation of I_{-1} divergences, but an infinitesimal bare mass,

$$m_B^2 = m_0^2/I_{-1}(\mu), \tag{7.25}$$

is allowed (Cf. Ref. [15] with $I_0(0) = 0$). This produces an extra, Φ_c-independent, contribution to the $1/I_{-1}(\mu)$ part of $\bar{\Omega}^2$. Thus, when we integrate (7.11), we obtain an extra finite contribution, proportional to Φ_c^2, in the GEP, Eq. (7.15). The result, conveniently re-parametrized by Φ_v and a new parameter m^2 (trivially related to m_0^2), takes the form:

$$\bar{V}_G = Kz_0^2\Phi_c^4\left(\ln\left(\frac{\Phi_c^2}{\Phi_v^2}\right) - \frac{1}{2}\right) + \frac{1}{2}m^2z_0\Phi_c^2\left(1 - \frac{1}{2}\frac{\Phi_c^2}{\Phi_v^2}\right). \tag{7.26}$$

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As before, K is given by (7.17) and Φ_v corresponds to the position of the minimum of $\bar{V}_G$ [39].

Nothing else is affected except the determination of z_0. The second derivative of the GEP at the vacuum is now given by:

$$\left. \frac{d^2 \bar{V}_G}{d\varphi_c^2} \right|_{\varphi_c = \varphi_v} = \frac{1}{Z_\phi} \left. \frac{d^2 \bar{V}_G}{d\Phi_c^2} \right|_{\Phi_c = \Phi_v} = \frac{1}{Z_\phi} \left(8K z_0^2 \Phi_v^2 - 2m^2 z_0 \right), \tag{7.27}$$

which replaces the left-hand side of Eq. (7.23), so that we obtain

$$z_0 = \frac{1}{K} \left(\eta + \frac{1}{4} \frac{m^2}{\Phi_v^2} \right). \tag{7.28}$$

Note that m is not a particle mass. In the symmetric vacuum all the particles would be massless, for any m^2. In the SSB vacuum the particle masses are affected by m^2 only through its effect on z_0.

8 Summary, Comparison to 1LEP, and Implications for the Higgs Mass

We have calculated, with three different formalisms, the GEP of the U(1) Higgs model. The unrenormalized result, in a general covariant gauge, is given at the end of Sect. 3. In the optimal gauge, $\xi = 0$, the result is given in Sect. 6.

To renormalize the GEP we postulated the infinitesimal forms $\lambda_B = \eta/I_{-1}$, $e_B^2 = \gamma/I_{-1}$, $m_B^2 = m_0^2/I_{-1}$ for the bare parameters, and an infinite re-scaling of the classical field, $\varphi_c^2 = z_0 I_{-1} \Phi_c^2$, where I_{-1} is a log-divergent integral. The cancellation of I_{-1} divergences in the GEP gave the constraint

$$\gamma^2 = \frac{8\eta}{3} \frac{(1 - 8\eta - 64\eta^2)}{(1 - 32\eta^2)}. \tag{8.1}$$

(See Fig. 1.) The vector and Higgs masses were found to be given by

$$M_V^2 = \gamma \frac{1 - 32\eta^2}{1 + 4\eta} z_0 \Phi_v^2, \tag{8.2}$$

$$M_H^2 = 8\eta z_0 \Phi_v^2. \tag{8.3}$$

(See Fig. 2.) The z_0 factor in the φ_c^2 re-scaling was obtained in Eq. (7.28): in the $m_B = 0$ case it varies between $8\pi^2$ and $(8\pi^2) \times (1.536)$ (see (7.24) and Fig. 3). The renormalized GEP, Eq. (7.26) is a sum of $\Phi_c^4 \ln \Phi_c^2$, Φ_c^4, and Φ_c^2 terms.

At the unrenormalized level, we can recover the 1LEP simply by discarding all the I_0^2 terms in Eq. (6.1), since each I_0 and I_1 is really accompanied by an $\hbar$ factor. Consequently, the optimization equations, (6.2 – 6.4), would be reduced to the classical expressions, $\Delta_c^2 = e_B^2 \varphi_c^2$,

$\Omega_c^2 = m_B^2 + 12\lambda_B\varphi_c^2$, $\quad \omega_c^2 = m_B^2 + 4\lambda_B\varphi_c^2$, and Eq. (6.5) would reduce to the familiar (unrenormalized) 1-loop result [3, 4]:

$$V_{1l} = \frac{1}{2}m_B^2\varphi_c^2 + \lambda_B\varphi_c^4 + 3I_1(\Delta_c) + I_1(\Omega_c) + I_1(\omega_c). \tag{8.4}$$

Conventionally, the 1LEP is renormalized in a perturbative fashion, with $\lambda_R = \lambda_B(1 + \mathcal{O}(\lambda_B\hbar I_{-1}) + \ldots)$, etc.. However, it has been realized recently [20, 21] that the 1LEP can also be renormalized in an "autonomous" fashion. The analysis exactly parallels the GEP case, and can be made even simpler by directly using (7.2) for I_1 [21]. In the 1-loop case the constraint needed to cancel the I_{-1} divergences is:

$$\tilde{\gamma}^2 = \frac{8}{3}\tilde{\eta}(1 - 20\tilde{\eta}), \tag{8.5}$$

with tilde's distinguishing the 1-loop quantities from their GEP counterparts. The vector and Higgs masses are given by

$$M_V^2 = \tilde{\gamma}\tilde{z}_0\Phi_c^2, \tag{8.6}$$
$$M_H^2 = 12\tilde{\eta}\tilde{z}_0\Phi_c^2. \tag{8.7}$$

The $\tilde{z}_0$ factor in the massless case is $12\pi^2$ (so one can regard $12\pi^2\tilde{\gamma}$ as the renormalized e^2). The renormalized 1LEP emerges (modulo the qualifications mentioned below) as:

$$V_{1l} = 3f(\Delta_c^2) + f(\Omega_c^2) + f(\omega_c^2), \tag{8.8}$$

in terms of the function f defined in Eq. (7.5), and so is a mixture of $\Phi_c^4 \ln \Phi_c^2$ and Φ_c^4 terms. [Actually, this result assumes $m_B = 0$, and that all the I_{-1} factors have the same scale μ. These assumptions are easily removed, as discussed in "Part II" of the GEP analysis (Sect. 7.4): one simply gets additional Φ_c^2 and Φ_c^4 terms with free-parameter coefficients. The final result can again be parametrized in the form (7.26).]

Clearly, the autonomously renormalized 1LEP and GEP results have much in common. The 1LEP constraint equation and M_H^2/M_V^2 ratio are plotted in Figs. 4 and 5. Qualitatively, these closely resemble the GEP results in Figs. 1 and 2. In the 1-loop case the maximum $\tilde{\eta}$ is $1/20 = 0.05$, rather than $\eta_0 = 0.077$, but a rescaling of the η and γ axes almost entirely absorbs the differences between the 1LEP and GEP results. The form of the renormalized potentials is also remarkably similar, both when we compare (8.8) with (7.15), and when we note that they share the same final form (7.26). In the 1-loop case the coefficient $\tilde{K}$ would be given by $\tilde{\eta}/(8\pi^2)$ instead of by Eq. (7.17). These differ by the same factor that occurs in the GEP's z_0; a factor that lies between 1 and 1.536.

To see the implications for phenomenology, we can consider $\Phi_v = (\sqrt{2}G_F)^{-1/2} = 246$ GeV, and $M_V \sim 90$ GeV. This implies a very small γ, and hence we must either be in the perturbative regime where both η and γ are small, or near to the maximum allowed η. The former case gives

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a light Higgs, as in CW [3]: The latter case gives a Higgs that is much heavier than the vector boson. In fact, the Higgs mass would be almost exactly that of a pure O(2) $\lambda\phi^4$ theory whose Φ_v was 246 GeV. For definiteness let us assume the attractive possibility that the bare mass is zero [3]. From the 1-loop result (8.7), with $\tilde{\eta} = 1/20$, $\tilde{z}_0 = 12\pi^2$, we would obtain $M_H = 2.07$ TeV. From the GEP result (8.3), with $\eta = \eta_0 = 0.0773$, $z_0 = (8\pi^2) \times (1.533)$, we would obtain $M_H = 2.13$ TeV. These results agree remarkably well.

Of course, these results are for the U(1) Higgs model, not the actual SU(2)×U(1) theory. However, the 1-loop analysis is easily extended to that theory [21], and yields $M_H = 1.89$ TeV. The GEP calculation for SU(2)×U(1) is a more difficult matter. However, it is clear that the phenomenological result would be essentially governed by the scalar sector, which is an O(4), rather than an O(2), $\lambda\phi^4$ theory. The GEP results for the O(N) case can be obtained from Ref. [10], supplemented by a quick calculation of the proper z_0 factor, as explained in Sect. 7.4. For zero bare mass, this gives:

$$z_0[\mathrm{O}(N)\,\lambda\phi^4] = \frac{\pi^2}{2\eta_0} \frac{(1 + 4\eta_0)}{(1 - 4\eta_0)}, \tag{8.9}$$

where η_0 in the O(N) case is

$$\eta_0 = \frac{1}{4(1 + \sqrt{N + 3})}. \tag{8.10}$$

The Higgs mass is again given by the form (8.3). [If the bare mass is non-zero, then the result is affected only by an $\mathcal{O}(m^2/\Phi_v^2)$ correction to z_0.] The O(4) case gives us a GEP prediction for the Standard-Model Higgs mass:

$$M_H = 2.05\,\mathrm{TeV}.$$

Consoli *et al.* [20] argue that the nonperturbative renormalization used here implies a vanishing renormalized scalar self-coupling (see also [18]). That would drastically suppress the Higgs-to-longitudinal-W, Z couplings, leaving the Higgs with a relatively narrow width. The phenomenology of such a Higgs deserves urgent attention.

Acknowledgements

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