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Massless ϕ^4 theory is not asymptotically free

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An ambiguity in the renormalization group β-function of massless ϕ^4 theory is resolved, leaving the known one-loop expression intact. Contrary to recent claims, this theory is therefore not asymptotically free.

The one-loop perturbative renormalization group (RG) β-function for massless $\lambda\phi^4$ theory is known to be

$$\beta_{\text{1-loop}} = \frac{3h\lambda^2}{16\pi^2} \,. \tag{1}$$

The positivity of the β-function at the origin ensures that $\lambda\phi^4$ is not asymptotically free. Although this conclusion is based on the perturbative β-function, it may be challenged if the exact β-function is known to contain a non-perturbative part which is non-analytic at $\lambda=0$ and in which the negative contribution dominates over the leading perturbative term as $h\lambda\to 0$. However, an analytic non-perturbative expression for the β-function, defined at $\lambda=0$, cannot differ from the perturbative β-function as $h\lambda\to 0$. Any such discrepancy would lead to conflicting physical consequences, such as the asymptotic behaviour of Green functions in momentum space [1] or the asymptotic behaviour of the effective potential in field space [2].

It has, however, been claimed [3] recently that such a non-perturbative β-function, defined and analytic at $\lambda=0$ and in contradiction with eq. (1), does in fact exist for $\lambda\phi^4$. This function is given as

$$\beta = \frac{-3h\lambda^2}{16\pi^2} + O(\lambda^3) \,. \tag{2}$$

The derivation of this result is based on the renormalization group equation (RGE) of the effective potential $V(\phi)$:

$$\left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} - \gamma\phi\frac{\partial}{\partial\phi} \right) V(\phi,\lambda,M) = 0 \,, \tag{3}$$

and the one-loop expression $V_{\text{1-loop}}(\phi)$ for $V(\phi)$.

Assuming that $\lambda\phi^4$ theory undergoes spontaneous symmetry breaking (SSB), the value $V(\bar\phi,\lambda,M)$ of the effective potential at its absolute minimum $\phi=\bar\phi$ satisfies

$$\left(M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} \right) V(\bar\phi,\lambda,M) = 0 \,. \tag{4}$$

This equation determines the β-function when $V(\bar\phi, \lambda, M)$ is known. The expression for $V_{\text{1-loop}}(\phi)$ [4] is

$$V_{\text{1-loop}}(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\lambda^2\phi^4}{256\pi^2}\left(\log\frac{\lambda\phi^2}{2M^2} - \frac{1}{2}\right) . \tag{5}$$

From this one finds,

$$\bar\phi^2_{\text{1-loop}} = \frac{2M^2}{\lambda}\exp\left(\frac{-32\pi^2}{3\lambda}\right) , \tag{6}$$

so that

$$V_{\text{1-loop}}(\bar\phi) = -\frac{\lambda^2\bar\phi^4}{512\pi^2} \,. \tag{7}$$

Substitution from this equation into eq. (4) yields the result in eq. (2) for β. It is then claimed that one has obtained asymptotic freedom for $\lambda\phi^4$ as a consequence of its vacuum instability i.e. its property of SSB as demonstrated by $V_{\text{1-loop}}(\phi)$.

Now, although it is true that eq. (4) determines β

non-perturbatively, it is necessary to use a non-perturbative expression for $V(\bar{\phi}, \lambda, M)$ before one can claim [3,5] independence of perturbation. It is clear that when $V_{1\text{-loop}}(\phi)$ is used for $V(\phi)$, eqs. (5), (6) and (7) are all perturbative and there is no way for β of eq. (2) to be non-perturbative.

In fact the minimum of $V_{1\text{-loop}}(\phi)$ at $\phi=\bar{\phi}$ has been known, since the original paper of Coleman and Weinberg [4], to be false, i.e. it is known not to approximate a true minimum of the exact effective potential $V(\phi)$ of massless $\lambda\phi^4$ theory. The use of eq. (7) in eq. (4) is therefore unjustified and should not be expected to give even a correct perturbative result for β.

The assumption that $\phi=\bar{\phi}$ is an approximate minimum of $V(\phi)$ was in fact shown in ref. [6] to be inconsistent with the RGE, where it was pointed out that, if a minimum at $\phi=\bar{\phi}$ exists, then the RGE requires the small-λ asymptotic form of $\bar{\phi}$ to be given by $\bar{\phi}^2 \sim \exp(32\pi^2/3\lambda)$ whereas $V_{1\text{-loop}}(\phi)$ gives $\bar{\phi}^2_{1\text{-loop}} \sim \exp(-32\pi^2/3\lambda)$. Our interpretation of the result of ref. [3] is that it demonstrates the same contradiction with the RGE. Ref. [3] makes the assumption of the validity of the minimum at $\phi=\bar{\phi}$ and obtains a β-function different from the known expression of the β-function for $\lambda\phi^4$ theory in a region where both are assumed valid i.e. for small λ. This demonstration of the inconsistency of SSB in $V_{1\text{-loop}}(\phi)$ and the RGE for $\lambda\phi^4$ is of the same type as that of ref. [6], i.e. a difference in sign that leads to $\bar{\phi}_{1\text{-loop}} \to 0$ as $\lambda \to 0$ in one case and to $\bar{\lambda}_{1\text{-loop}}(t) \to 0$ as $t \to \infty$ in the other case, contradicting the RGE in both cases.

The same remarks apply to the gaussian approximation for $\bar{\phi}$, i.e. $\bar{\phi}^2_{\text{Gauss}} \sim \exp(-16\pi^2/\lambda)$ is inconsistent with the RGE for $\bar{\phi}$ and it cannot therefore approximate the true $\bar{\phi}$ of $\lambda\phi^4$ theory, if it exists. In fact the assumption that $\bar{\phi}_{\text{Gauss}}$ satisfies the RGE for $\bar{\phi}$ leads [3] to

$$\beta_{\text{Gauss}} = \frac{-\lambda^2}{8\pi^2}, \tag{8}$$

different from both previous expressions.

We now show that if one recognizes the perturbative nature of the loop-expansion, for both the effective potential and the RG-functions as explained in ref. [7], one does not obtain eq. (2) from $V_{1\text{-loop}}(\phi)$ For, in this case, one has from eq. (3)

$$M\frac{\partial V_1}{\partial M} + \left(\beta_1 \frac{\partial}{\partial\lambda} - \gamma_1 \phi \frac{\partial}{\partial\phi}\right) V_0 = 0, \tag{9}$$

where

$$V = V_0 + hV_1 + \dots, \quad \beta = h\beta_1 + \dots, \quad \gamma = h\gamma_1 + \dots,$$

$$V_0 = \frac{\lambda}{4!}\phi^4, \quad V_1 = \frac{\lambda^2\phi^4}{256\pi^2}\left(\log\frac{\lambda\phi^2}{2M^2} - \frac{1}{2}\right). \tag{10}$$

Eq. (9) is then an identity in ϕ, which for all $\phi \neq 0$ leads to

$$\beta_1 - 4\lambda\gamma_1 = \frac{3}{16\pi^2}\lambda^2. \tag{11}$$

No more conditions on β and γ may be obtained from a proper application of the RGE to $V_{1\text{-loop}}(\phi)$. Since eq. (11) does not enable a simultaneous determination of both β_1 and γ_1, one must feed information from renormalization of Green functions, which gives $\gamma_1 = 0$ so that eq. (11) yields eq. (1) for β_1.

Finally we remark that the result (2) has nothing to do with SSB. It can be obtained on the basis of the assumption that $V_{1\text{-loop}}(\phi)$ is an exact solution to the RGE for $V(\phi)$. For, this assumption leads to an identity in ϕ that yields the two equations

$$\beta - 4\lambda\gamma = \frac{3h}{16\pi^2}\lambda^2, \quad \beta - 2\lambda\gamma = 0, \tag{12}$$

from the independent terms proportional to ϕ^4 and $\phi^4\log(\lambda\phi^2/2M^2)$ respectively. The solutions to eqs. (12) are β of eq. (2) and

$$\gamma = -\frac{3h\lambda}{32\pi^2}, \tag{13}$$

which is also reported in ref. [3]. The assumption that $V_{1\text{-loop}}(\phi)$ exactly solves the RGE is, however, obviously inadmissible. Eqs. (2) and (13) are equivalent to it.

Thus, in conclusion, the use of a perturbative $V_{\text{eff}}(\phi)$ in the RGE will not lead to expressions for the RG functions that are different from the conventional ones. In particular there is no perturbative or non-perturbative alternative to the one-loop β-function in eq. (1) for small λ. If non-trivial, massless ϕ^4 is therefore not asymptotically free whether or not SSB occurs.

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