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Effective Potential Calculation for a Nonlocal Scalar Field Theory

by

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ABSTRACT

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In the Standard Model, particles acquire mass due to interactions with a spontaneous vacuum condensate of scalar particles. The formation of the condensate can be understood as being due to a balance between the fundamental repulsive $\lambda\phi^4$ interaction and an induced long-range attractive interaction. In order to study the role of this induced long-range interaction, we consider a model theory in which it is included, along with the standard $\lambda\phi^4$ term, in the starting Lagrangian. We calculate the field-theoretic effective potential of this theory, working to second order in both of the interaction coupling strengths. We then investigate the renormalization of this result, in both the perturbative and the "autonomous" frameworks.

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1 Introduction: The Higgs Mechanism

Scalar field theory is a vital part of the Standard Model, as the scalar Higgs field imparts mass to the other fundamental particles (quarks, leptons, and weak vector bosons). Local weak isospin invariance of the Standard Model Lagrangian prevents the inclusion of a mass term for any particle that carries weak isospin. The textbook solution to this problem[1] is to include instead an interaction term with a new spin-0 field, the scalar Higgs field. The Higgs field is thought to have a non-zero field expectation value in the ground state (so-called spontaneous symmetry breaking). That is, the ground state of the theory (the lowest-energy configuration of the system) does not correspond to a vacuum empty of particles, but rather a vacuum containing a "spontaneous condensate" of spin-0 particles. Various particles acquire their observed masses through their interaction strengths with the background condensate of scalar particles. The key question we will attempt to address is whether or not symmetry breaking occurs in a particular nonlocal model theory, after the lowest-order quantum effects have been taken into account.

To see how spontaneous symmetry breaking works, consider a (Euclideanized[2]) Lagrangian density for a single scalar field:

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)}_{\text{kinetic}} + \underbrace{\tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4!}\lambda\phi^4}_{\text{potential } \equiv V_{\text{cl}}}. \tag{1.1}$$

In this Lagrangian, ϕ represents a field of spin-0 particles. It is a scalar in the sense that it is invariant under Lorentz transformations (as opposed to a vector or a spinor, whose components would change under rotations and/or boosts). Classically, ϕ would be a function of the space-time coordinates (the 4-vector position $x = (t, \vec{x})$). Quantum mechanically, it is an operator that acts on some state in a Hilbert space to return a field value (in units of energy), and is also a function of x. In Quantum Field Theory, the Hilbert space of states contains not just single particle states but states with multiple particles, and the vacuum (which may or may not be empty).

The parameters m and λ that weight the terms in the Lagrangian are constant in x, and are not operators. The parameter m has dimensions of energy, and corresponds to the bare particle

[1]A good source for background on standard Quantum Field Theory topics is reference [1].

[2]For our purposes, Euclideanizing the metric just flips the sign of the potential terms in the Lagrangian. Equivalently, we could work in normal Minkowski space and Wick-rotate all the integrals later on, but we have avoided this technical complication by working in Euclidean space from the outset.

mass[3]. The dimensionless parameter λ is the bare coupling strength for the scalar self-interaction term.

While the Lagrangian (1.1) is symmetric under $\phi \to -\phi$, the minimum of the potential is not necessarily at $\phi = 0$. A scalar field is allowed both quadratic and quartic field-strength terms while still retaining renormalizability and if the coupling λ is positive while m^2 is negative there will be two global minima of the classical potential $V_{\rm cl}$ at $\phi = \pm v$ (Figure 1). Classically, the ground state is at one of these, say $\phi = v$. Defining a new field h in terms of deviations from this vacuum value, $h = \phi - v$, the manifest $\phi \to -\phi$ symmetry of the original Lagrangian is hidden, or "broken". An interaction term between the scalar field and another field generates a mass term; for example a Yukawa interaction with a fermion field becomes

$$g_1\phi\overline{\psi}\psi = m_\psi\overline{\psi}\psi + g_1 h\overline{\psi}\psi; \qquad m_\psi \equiv g_1 v. \tag{1.2}$$

The Standard Model actually requires four scalar fields arranged in a complex SU(2) doublet, and the asymmetric vacuum is one of a continuum of choices. At the classical level the situation is easy to visualize in the case of two scalar fields with O(2) symmetry (Figure 2). The asymmetric vacuum is at any point on the circle where the classical potential is minimized. Choosing a particular vacuum state and defining a new set of fields corresponding to the radial and angular degrees of freedom, it is easy to see that at the classical level the radial field has a physical mass while the transverse fields are massless since the potential is constant along those directions. Goldstone's theorem proves that quantum modifications of the effective potential respect the angular symmetry, leaving the potential constant along the transverse directions. (In the Standard Model, the massless would-be Goldstone bosons are "eaten" by the weak vector bosons, endowing them with longitudinal, as well as transverse, polarizations.)

The crucial question that motivates this thesis is whether or not symmetry breaking occurs *after* quantum effects have been included and renormalization has been performed. To address this question one must look not at the classical potential $V_{\rm cl}$, but rather at the "effective potential" (defined in the next section). The essential issues can be studied in the relatively simple theory

[3]This is apparent if one sets $\lambda = 0$ in the Minkowski-space version of the Lagrangian (1.1), then employs the Euler-Lagrange equations to obtain a wave equation for ϕ. One thus obtains the Klein-Gordon equation, which describes a free spin-0 field of mass m. The latter correspondence follows from the usual canonical quantization of the relativistic mass-energy relation.

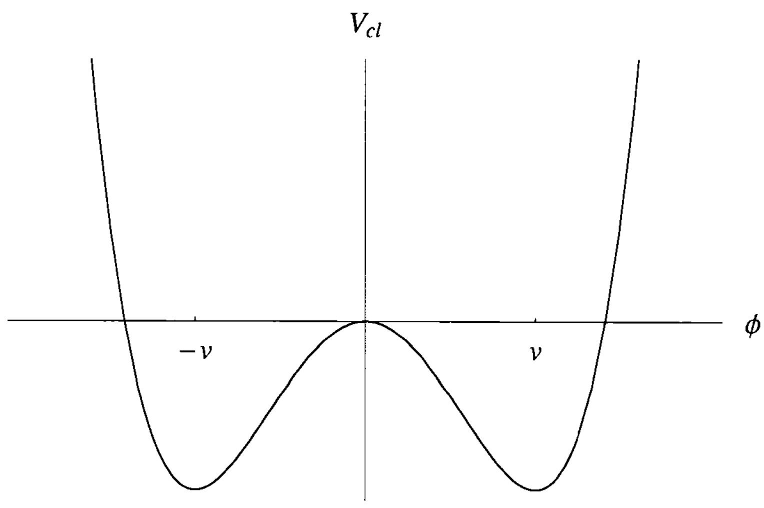


Figure 1: Classical potential for one scalar field, $V_{\rm cl} \equiv \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4;\ m^2 < 0,\ \lambda > 0$

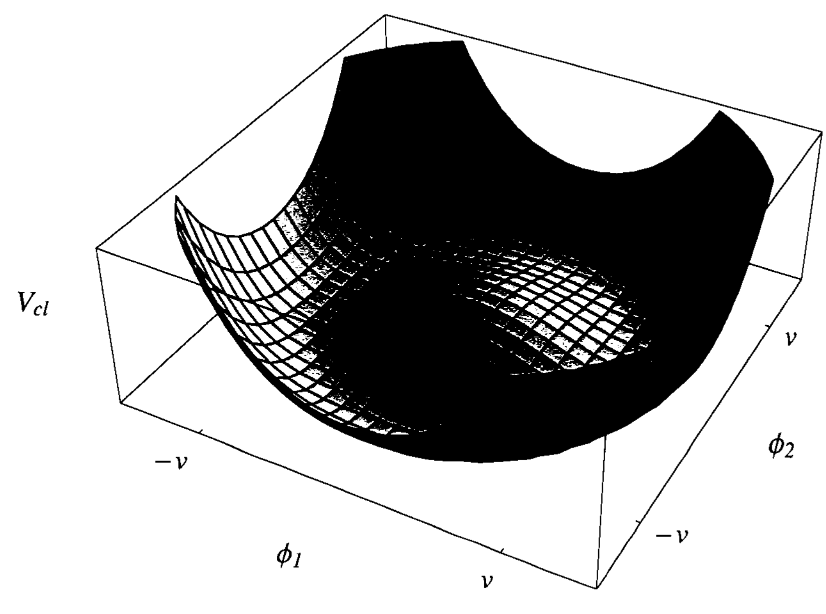


Figure 2: Classical potential for two scalar fields, $V_{\rm cl} \equiv \frac{1}{2}m^2\left(\phi_1^2 + \phi_2^2\right) + \frac{1}{4!}\lambda\left(\phi_1^2 + \phi_2^2\right)^2$

with a single scalar field, since the generalization to multiple scalar fields should be straighforward.

Another theoretical puzzle here is the issue of so-called "triviality" [2, 3, 4, 5]. Strong arguments have been made that in scalar $\lambda\phi^4$ theory the renormalized coupling λ_R and hence all observable scattering amplitudes must vanish in the limit that the ultraviolet cutoff Λ is taken to infinity. There is some controversy as to how this triviality property affects the issue of spontaneous symmetry breaking. Conventional Renormalization-Group Improved perturbation theory (RGIPT) methods indicate that spontaneous symmetry breaking exists, but only for $m^2 < 0$ as in the classical picture shown above. Some authors [6, 7, 8] argue that the RGIPT methods are misleading and that an alternate approach is necessary. In this so called "autonomous" aproach SSB is possible even with positive m^2, if m is sufficiently small. This approach also implies a much larger Higgs mass, ≈ 2 TeV, than the conventional approach, which expects a mass in the 100 to 400 GeV range. We will present both of these possible approaches to renormalization of the effective potential for a model nonlocal scalar field theory.

2 Scalar $\lambda\phi^4$ Theory, Renormalization, and Triviality

2.1 The Effective Potential

The effective potential is essentially the classical potential plus quantum corrections, and can be defined in a number of different but equivalent ways. One intuitive definition due to Symanzik [9] is the minimum energy density subject to the constraint that the field has a particular expectation value,

$$V_{\text{eff}}(\phi_c) \equiv \min_{\psi}\langle\psi|\mathcal{H}|\psi\rangle; \qquad \langle\psi|\phi|\psi\rangle = \phi_c. \tag{2.1}$$

This definition provides a natural starting point for variational methods. The gaussian effective potential approximation [10, 11, 12], for example, is obtained by assuming that ψ is a gaussian wavefunctional centered at v with some width that is determined by the minimization condition.

Another definition proceeds in analogy with statistical mechanics [1] and employs the partition functional for the field in the presence of an external source $J(x)$. One starts with the Euclideanized action,

$$\mathcal{S}[\phi] \equiv \int d^4x\, \mathcal{L}[\phi, \partial_\mu\phi], \tag{2.2}$$

then adds on a term representing the interaction with the external source, $J(x)$, and functionally integrates the exponential of the modified action over all possible field configurations to obtain the partition functional,

$$\mathcal{Z}[J] \equiv \int [D\phi] e^{-\left(\mathcal{S}[\phi] + \int d^4x\, J(x)\phi(x)\right)}. \tag{2.3}$$

Then, in analogy with the Helmholtz free energy, one takes the logarithm:

$$\mathcal{W}[J] \equiv -\text{Log}\ \mathcal{Z}[J]. \tag{2.4}$$

Note that the variation of $\mathcal{W}$ with respect to J is

$$\frac{\delta\mathcal{W}[J]}{\delta J(x)} = -\frac{1}{\mathcal{Z}[J]}\frac{\delta\mathcal{Z}[J]}{\delta J(x)} = \langle\phi(x)\rangle_J \equiv \phi_c(x) \tag{2.5}$$

where $\phi_c(x) = \phi_{\text{classical}}(x)$ is the expectation value of $\phi(x)$ in the presence of the external source $J(x)$. In this sense $\phi_c(x)$ is conjugate to $J(x)$. In analogy with the Gibbs free energy, one Legendre transforms to a functional of $\phi_c(x)$:

$$\Gamma[\phi_c] \equiv \mathcal{W}[J] - \int d^4x\, J(x)\phi_c(x). \tag{2.6}$$

This functional is called the effective action. By construction, the Legendre transform produces a functional that satisfies an identity analogous to (2.5):

$$\begin{aligned}\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} &= \frac{\delta\mathcal{W}[J]}{\delta\phi_c(x)} - \int d^4y\,\frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y) - J(x)\\ &= \int d^4y\,\frac{\delta\mathcal{W}[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi_c(x)} - \int d^4y\,\frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y) - J(x)\\ &= -J(x).\end{aligned} \tag{2.7}$$

As a result, when the external source is removed ($J(x) = 0$) the ground state of the theory corresponds to a minimum of Γ. Now if we assume a translationally invariant vacuum, $\phi_c(x) = \phi_c$ = constant, then as in thermodynamics Γ is extensive, so we divide out an infinite 4-volume factor $\mathcal{V}$ and define the result as the effective potential:

$$V_{\text{eff}}(\phi_c) \equiv \frac{1}{\mathcal{V}}\Gamma[\phi_c(x)]\bigg|_{\phi_c(x)=\phi_c}. \tag{2.8}$$

The equation

$$\frac{dV_{\text{eff}}(\phi_c)}{d\phi_c} = 0 \tag{2.9}$$

then provides the vacuum expectation value of the field taking into account an average over quantum configurations. This is exactly what we need to determine if spontaneous symmetry breaking occurs in a particular theory.

It is worth noting that thus far the expression given for V_{eff} is exact but not of practical use, as calculation of the functional integrals is difficult if not impossible. In order to proceed, it is necessary to make some sort of approximation. The conventional approximation method is the "loop expansion", which in our scalar theory is equivalent to a perturbative expansion in powers of λ. Define the split[4] $\phi(x) = \phi_c + h(x)$ and Taylor expand the argument of the exponential in $\mathcal{Z}[J]$ about ϕ_c:

$$\begin{aligned}\mathcal{Z}[J] = \int Dh(x)\exp\Bigg[&-\int d^4x\,\left(V_{\text{cl}}(\phi_c) + J(x)\phi_c\right) - \int d^4x\, h(x)\left(\frac{\delta \mathcal{S}}{\delta\phi}\bigg|_{\phi=\phi_c} + J\right)\\ &-\tfrac{1}{2}\int\int d^4x\, d^4y\, h(x)\left(\frac{\delta^2\mathcal{S}}{\delta\phi(x)\delta\phi(y)}\bigg|_{\phi=\phi_c}\right)h(y) - \cdots\Bigg].\end{aligned} \tag{2.10}$$

The terms explicitly shown correspond to a free field, while the terms not shown correspond to interactions that generate a series of Feynman diagrams. At the "one-loop" level, discussed here, we may neglect these terms. (The term linear in $h(x)$ vanishes by construction in the limit $J \to 0$ so we will drop it.)

[4]There is a subtlety here. One may separate out the zero-momentum mode by setting $\phi(x) = \phi_c + h(x)$ with $\int d^4x\; h(x) = 0$ so that ϕ_c is the DC offset in a classical sense. However $D\phi_c \neq 0$, that is ϕ_c is spatially constant but not variationally constant. We could define a different decomposition $\phi(x) = \phi_{\text{constant}} + h'(x)$ so that the zero-point fluctuations of ϕ_c are absorbed into $h'(x)$ leaving ϕ_{constant} as a true constant both spatially and variationally, with the cost that now $\int d^4x\; h'(x) \neq 0$. We will gloss over this ambiguity since the contribution of the zero-point fluctuations of the single zero-momentum mode is suppressed by an infinite volume factor.

The $\hbar^2$ term is in gaussian form and can be integrated exactly, yielding

$$\mathcal{Z}[J] = \left(\text{Det}\left[\left.\frac{\delta^2 \mathcal{S}}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_c}\right]\right)^{-\frac{1}{2}} e^{-\left(\int d^4x\ (V_{\text{cl}}(\phi_c) + J(x)\phi_c)\right)}. \tag{2.11}$$

Hence, using the definitions (2.4) and (2.6) one obtains

$$\begin{aligned}\Gamma[\phi_c] &= -\log \mathcal{Z}[J] - \int d^4x\, J(x)\phi_c \\ &= \int d^4x\, V_{\text{cl}}(\phi_c) + \tfrac{1}{2}\text{Log Det}\left[\left.\frac{\delta^2 \mathcal{S}}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_c}\right] \\ &= \mathcal{V}V_{\text{cl}}(\phi_c) + \tfrac{1}{2}\text{Tr Log}\left[G(x,y)^{-1}\right],\end{aligned} \tag{2.12}$$

using the functional operator identity Log Det $A =$ Tr Log A and defining the inverse propagator

$$G(x,y)^{-1} \equiv \left.\frac{\delta^2 \mathcal{S}}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_c}. \tag{2.13}$$

For the Lagrangian (1.1) it is

$$G(x,y)^{-1} = (-\delta^2 + M^2)\delta^{(4)}(x-y); \qquad M^2 \equiv m^2 + \tfrac{1}{2}\lambda\phi_c^2. \tag{2.14}$$

which may be diagonalized by transforming to momentum space,

$$G(x,y) = \int \frac{d^4p}{(2\pi)^4} G(p) e^{-ip\cdot(x-y)}, \tag{2.15}$$

so that one obtains

$$G(p) = \frac{1}{p^2 + M^2}. \tag{2.16}$$

In (2.12) one writes the trace as an integral over momentum space times a 4-volume factor $\mathcal{V}$,

$$\Gamma[\phi_c] = \mathcal{V}\left(V_{\text{cl}}(\phi_c) + \tfrac{1}{2}\int \frac{d^4p}{(2\pi)^4} \log[G(p)^{-1}]\right). \tag{2.17}$$

Dividing out the spacetime volume factor one obtains the one-loop effective potential:

$$V_{\text{eff}}(\phi_c) = \underbrace{V_{\text{cl}}(\phi_c)}_{\text{classical potential}} \underbrace{+\tfrac{1}{2}\int \frac{d^4p}{(2\pi)^4} \log[G(p)^{-1}]}_{\text{zero-point energy density}}. \tag{2.18}$$

For the Lagrangian (1.1) it is

$$V_{\text{eff}}(\phi_c) = V_{\text{cl}}(\phi_c) + I_1(M^2), \tag{2.19}$$

with the definition

$$I_1(M^2) \equiv \tfrac{1}{2}\int \frac{d^4p}{(2\pi)^4} \log(p^2 + M^2). \tag{2.20}$$

Recall that $M^2 = m^2 + \frac{1}{2}\lambda\phi_c^2$ depends on ϕ_c. Note that the leading quantum correction to the classical potential is just the zero-point energy, as one might expect intuitively. In the loop expansion there are further corrections generated by all connected, one-particle-irreducible vacuum diagrams (Feynman diagrams with no external legs that cannot be separated into two pieces by cutting a single line).

At this point it is worth defining the other divergent integrals I_N that we will encounter, and providing some identities that they obey [13, 12, 14]:

$$I_0(M^2) \equiv \int \frac{d^4p}{(2\pi)^4}(p^2+M^2)^{-1}; \qquad I_{-1}(M^2) \equiv 2\int \frac{d^4p}{(2\pi)^4}(p^2+M^2)^{-2}, \tag{2.21}$$

$$I_1(M^2) = I_1(0) + \tfrac{1}{2}M^2 I_0(0) - \tfrac{1}{8}M^4 I_{-1}(M^2) - \frac{3}{128\pi^2}M^4, \tag{2.22}$$

$$I_0(M^2) = I_0(0) - \tfrac{1}{2}M^2 I_{-1}(M^2) - \frac{M^2}{16\pi^2}, \tag{2.23}$$

$$I_{-1}(M^2) - I_{-1}(\mu^2) = -\frac{1}{8\pi^2}\log\left(\frac{M^2}{\mu^2}\right). \tag{2.24}$$

These integrals are all implicitly regularized in some fashion, and the above identities are true up to terms that vanish as the regulator is removed. If the regularization uses an ultraviolet cutoff Λ the explicit forms of the I_{-1} and I_0 integrals are, up to $O\left(\frac{M^2}{\Lambda^2}\right)$,

$$I_{-1}(M^2) = \frac{1}{8\pi^2}\left(\log\frac{\Lambda^2}{M^2} - 1\right), \tag{2.25}$$

$$I_0(M^2) = \frac{M^2}{16\pi^2}\left(\frac{\Lambda^2}{M^2} - \log\frac{\Lambda^2}{M^2}\right). \tag{2.26}$$

2.2 Renormalization and ϕ^4 Triviality (Conventional Approach)

The one-loop effective potential just derived is inconvenient in that the I_1 integral is ultraviolet divergent. Similar problems arise in the order-by-order calculation of scattering matrix elements due to the appearance of divergent integrals. However, the bare parameters that appear in the Lagrangian (such as masses and interaction strengths) are not directly observable and hence need not be finite quantities. The basic idea of renormalization [1] is to reparameterize the theory, eliminating the bare parameters in favor of a new set of renormalized parameters that are manifestly finite and correspond to experimentally measurable quantities. Whenever a calculation of some physical quantity produces a formally infinite result, one must first "regularize" the theory (for example by cutting off the integrals at some scale Λ), and then rearrange terms so that any appearance of Λ can be put together with one of the bare parameters to form a new set of renormalized parameters which one then demands to be finite.

For simplicity we will exhibit this renormalization procedure for the classically scale-invariant case of a massless scalar field. Starting from eqs. (1.1) and (2.19) with $m^2 = 0$ then using the identity (2.22) with eq. (2.25) we obtain the effective potential

$$V_{\text{eff}} = \tfrac{1}{4!}\lambda\phi_c^4 + I_1(0) + \tfrac{1}{2}M^2 I_0(0) + \frac{M^4}{64\pi^2}\left(\log\frac{M^2}{\Lambda^2} - \frac{1}{2}\right), \tag{2.27}$$

where $M^2 \equiv \frac{1}{2}\lambda\phi_c^2$. The constant offset $I_1(0)$, though infinite, is of no consequence so we will drop it. After mass renormalization (which removes terms in ϕ_c^2), we have

$$V_{\text{eff}} = \tfrac{1}{4!}\lambda\phi_c^4 + \frac{\lambda^2\phi_c^4}{256\pi^2}\left(\log\frac{\frac{1}{2}\lambda\phi_c^2}{\Lambda^2} - \frac{1}{2}\right). \tag{2.28}$$

Now we set

$$\lambda_R = \lambda\left(1 - \left(\frac{3}{16\pi^2}\log\tfrac{\Lambda}{\mu} + A\right)\lambda + \cdots\right); \qquad A \equiv \text{numerical constant}, \tag{2.29}$$

where the "$+\cdots$" represents higher-order terms in λ. Inverting this relation perturbatively, substituting the result in the effective potential, and dropping terms in λ_R^3 and higher, we obtain the renormalized effective potential

$$V_{\text{eff}} = \tfrac{1}{4!}\lambda_R\phi_c^4 + \frac{\lambda_R^2\phi_c^4}{256\pi^2}\left(\log\frac{\frac{1}{2}\lambda_R\phi_c^2}{\mu^2} - \frac{1}{2} - A\right). \tag{2.30}$$

It has a nontrivial minimum at $\phi_c = v$, where v is easily calculated in terms of μ and the other parameters, and may be recast as

$$V_{\text{eff}} = \frac{\lambda_R^2\phi_c^4}{256\pi^2}\left(\log\frac{\phi_c^2}{v^2} - \frac{1}{2}\right). \tag{2.31}$$

This "raw" perturbative result is shown in Figure 3.

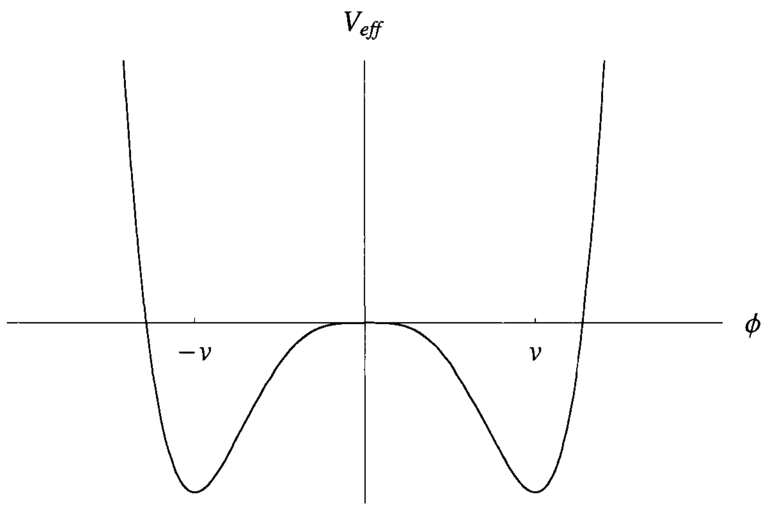


Figure 3: The renormalized effective potential, eq. (2.31)

However, conventionally at this point one invokes the notion of "Renormalization-Group Improvement" [1, 15, 16], where the renormalized coupling λ_R is no longer a constant, but rather depends on the energy scale of interest in a physical process. The "running" of λ_R is characterized by the "beta function", $\beta_\lambda(\lambda_R) = \mu\frac{d\lambda_R(\mu)}{d\mu}$, which can be obtained by differentiating eq. (2.29) and substituting in the perturbative series for λ in terms of λ_R. We then obtain a differential equation which can in principle be solved to obtain the explicit form of $\lambda_R(\mu)$. In the standard analysis we would then use it to obtain the so-called "Renormalization-Group Improved" effective potential by letting ϕ_c determine the scale μ.

In perturbation theory the relationship of the renormalized parameter to the bare parameter is, for a general coupling parameter α,

$$\alpha_R = \alpha\left(1 - \left(b\log\tfrac{\Lambda}{\mu} + A\right)\alpha + \cdots\right); \qquad b, A \equiv \text{numerical constants} \tag{2.32}$$

so that the beta function starts at order α_R^2:

$$\beta_\alpha(\alpha_R) = \mu\frac{d\alpha_R(\mu)}{d\mu} = b\alpha_R^2 + \cdots \tag{2.33}$$

Truncating the series on the right to the one term shown and integrating, one obtains

$$\alpha_R = \frac{1}{-b\log\frac{\mu}{\mu_0}}. \tag{2.34}$$

So α_R has a Landau pole at some scale where $\mu = \mu_0$. In QCD the coefficient b is negative and for high-energy scattering processes we are well to the right of the pole, as shown in Figure 4. At higher energies (hence shorter distances) the coupling gets weaker, which is the famed asymptotic freedom of QCD that allows the use of perturbation theory. At lower energies perturbation theory breaks down, so we never reach the pole. In QED the coefficient b is positive and at typical energy scales we are well to the left of the pole (Figure 5). At higher scales $\alpha(\mu)$ grows in strength, and since at presently accessible energies $\alpha_s(\mu)$ is much stronger than $\alpha(\mu)$, the two are able to meet at some scale (the GUT scale $\approx 10^{16}$ GeV) which is well before the pole, so presumably Nature avoids the $\alpha(\mu)$ pole by transitioning to some different theory in which the strong and electroweak forces are unified.

In the case of scalar theory we also found that the coefficient b is positive, $b = \frac{3}{16\pi^2}$, so that we have a Landau pole at large energies. However, in scalar field theory there is no completely satisfactory way of avoiding the pole. The conventional analysis proceeds by setting the renormalized coupling constant $\lambda_R \sim \frac{1}{\log\frac{\Lambda}{M_h}}$ by hand, where M_h is the Higgs mass. The cutoff Λ is below the pole so as Λ is taken to infinity the pole is pushed out to infinity as well, and the coupling constant at any finite energy scale is driven to zero, hence triviality. Monte Carlo lattice simulations provide

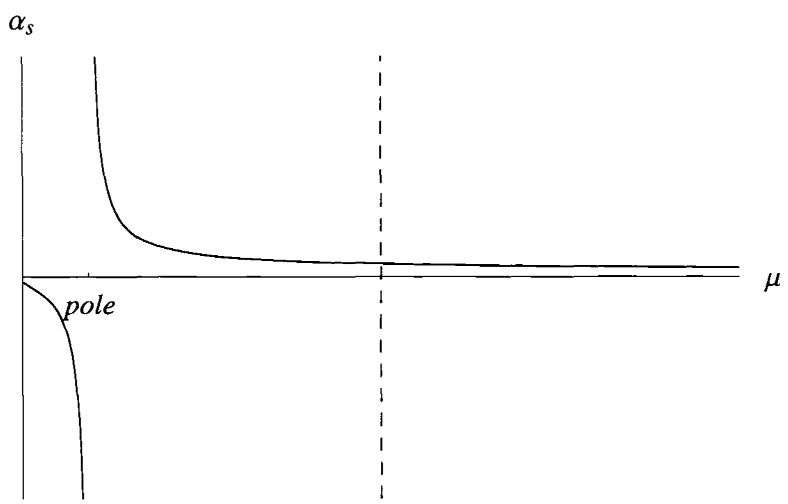


Figure 4: Running of the strong coupling α_s

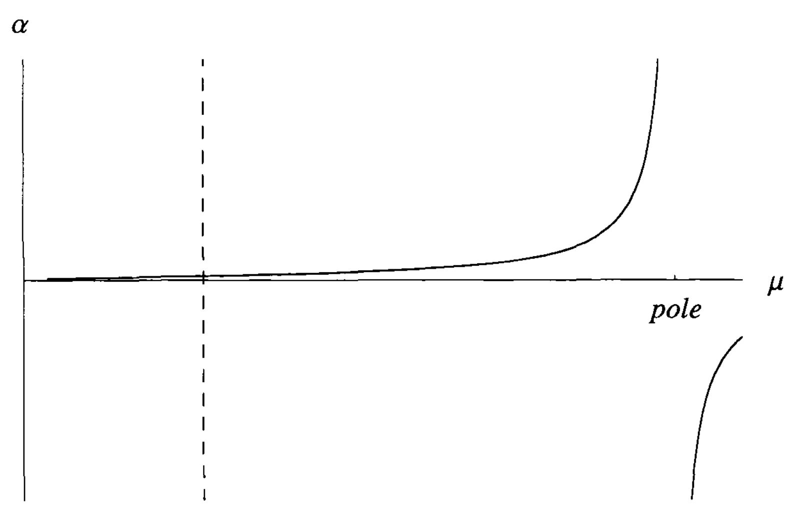


Figure 5: Running of the electromagnetic coupling α

evidence that the renormalized coupling constant does show this $\frac{1}{\log\frac{\Lambda}{M_h}}$ behavior, where the cutoff Λ is proportional to the inverse of the lattice spacing.

Furthermore, the Higgs mass squared is given by $M_h^2 = \frac{1}{2}\lambda_R v^2$. Since v is a constant (measured to be 246 GeV), when the cutoff is sent to infinity the coupling λ_R goes to zero, taking M_h to zero with it. The conventional attitude is that the cutoff Λ must be kept finite so that M_h may remain nonzero. The Standard Model is then only an effective theory, with some unknown new physics present above the cutoff.

In the preceding analysis the renormalized parameter depends on the cutoff, which is not in keeping with basic idea of renormalization, that any physically measurable quantity should be finite and independent of the cutoff in the infinite cutoff limit. The conventional analysis is thus somewhat unappealing, as renormalization-group invariance is one of the cornerstones of field theory.

If we sweep aside these concerns, we may obtain the Renormalization-Group Improved effective potential (to lowest order) by substituting the expression for $\lambda_R(\mu)$ into just the classical term, then setting μ proportional to ϕ_c. [1, 15, 16] (We may absorb the constant of proportionality into a redefinition of μ_0 without loss of generality.) The RGI effective potential thus obtained (Figure 6) is

$$V_{\text{eff}} = \frac{2\pi^2}{9}\frac{\phi_c^4}{\log\frac{\mu_0}{\phi_c}}, \tag{2.35}$$

and there is no longer any symmetry breaking. If we were to repeat the above calculations for the massive case, we would find that the RGI effective potential has a nontrivial minimum only for $m^2 < 0$. This RGI method may be made more precise in higher orders in perturbation theory, but the qualitative result, that no symmetry breaking is found in the massless case, remains.

There is reason to mistrust the RGI result, however, since the expression for $\lambda_R(\mu)$ was obtained by a procedure that amounts to assuming that the higher-order terms we have not calculated should form a geometric series [17]. In the case of scalar field theory, this is analagous to resumming the series $1 - x^2 + x^3 - \cdots$ to obtain $\frac{1}{1+x}$ and then employing this result even when $x = 1$ (which corresponds to $\phi = v$ in eq. (2.31)). In any case, the stark disagreement between the original perturbative result eq. (2.31) and its RGI version eq. (2.35) is worrisome. One motivation for this thesis is to find an escape from this problem.

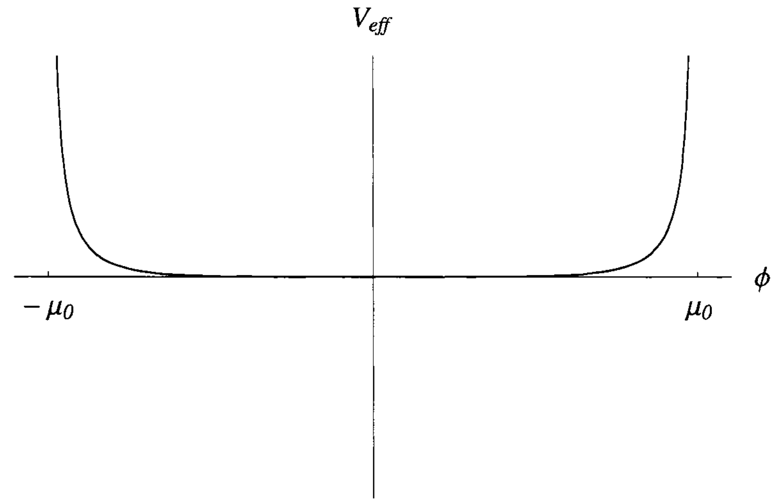


Figure 6: Renormalization-Group Improved Effective Potential, eq. (2.35)

2.3 An Alternative Understanding of Triviality: Autonomous Renormalization

Some authors [6, 7, 8] have proposed a different renormalization procedure that is renormalization-group invariant, the "autonomous" prescription, in which the bare coupling λ is vanishingly small and a renormalized coupling λ_R need not be mentioned. There are still no observable self-interactions, but the Higgs mass is naturally finite in units of the vacuum expectation value v as the cutoff Λ goes to infinity. In this approach a nontrivial vacuum state exists for $m^2 < 0$ and also for $m^2 \geq 0$ but sufficiently small.

We will exhibit the autonomous renormalization procedure for a massless scalar field ($m = 0$ in the Lagrangian (1.1)). We begin with the unrenormalized one-loop effective potential given in eq. (2.28). Since it only consists of the classical potential plus terms proportional to ϕ_c^4 and $\phi_c^4 \log \phi_c^2$, we may express it in terms of the physical mass $M^2 \equiv \alpha\lambda\phi_c^2$ as

$$\begin{aligned} V_{\text{eff}} &= \tfrac{1}{4!}\lambda\phi_c^4 - C_1 M^4 \log \tfrac{\Lambda^2}{M^2} - C_2 M^4 \\ &= M^4\left(\tfrac{1}{24\alpha^2\lambda} - C_1 \log \tfrac{\Lambda^2}{M^2}\right) - C_2 M^4, \end{aligned} \tag{2.36}$$

with $\alpha = \frac{1}{2}$, $C_1 = \frac{1}{64\pi^2}$, and $C_2 = \frac{1}{128\pi^2}$. In later sections we will encounter this same functional form with different coefficients, so we will leave α, C_1, and C_2 as arbitrary constants so that our results found here will be generally applicable.

If we consider the physical mass M to be finite and independent of the cutoff, the only way to make V_{eff} finite is to arrange a cancellation between λ and the $\log\frac{\Lambda^2}{M^2}$ term. Accordingly, we set

$$\lambda = \frac{1}{24\alpha^2 C_1 \log\frac{\Lambda^2}{\mu_1^2}} \tag{2.37}$$

to obtain the finite expression

$$V_{\text{eff}} = M^4\left(C_1 \log\frac{M^2}{\mu_1^2} - C_2\right). \tag{2.38}$$

(The scale parameter μ_1 is at this stage arbitrary.) Now we set

$$\mu_2^2 = \mu_1^2 \exp\left(\frac{C_2}{C_1} - \frac{1}{2}\right) \tag{2.39}$$

to obtain

$$V_{\text{eff}} = C_1 M^4\left(\log\frac{M^2}{\mu_2^2} - \frac{1}{2}\right). \tag{2.40}$$

From which we may immediately see that the scale μ_2 is the physical mass at the symmetry breaking minimum, $\mu_2^2 = M_v^2 \equiv \alpha\lambda v_B^2$, since

$$\frac{dV_{\text{eff}}}{d\phi_c} = 4\alpha C_1 \lambda \phi_c M^2 \log\frac{M^2}{\mu_2^2}. \tag{2.41}$$

Now since the physical particle mass M^2 is finite and nonvanishing while λ is infinitesimal, the relation $M^2 \equiv \alpha\lambda\phi_c^2$ requires us to renormalize the bare field strength ϕ_c as well in order to provide a compensating infinite factor:

$$\phi_c^2 = Z_\phi \phi_R^2; \qquad Z_\phi \equiv \chi \log\frac{\Lambda^2}{\mu_1^2}, \tag{2.42}$$

where χ is a finite constant. Only the constant part of the field (the zero-momentum mode) is rescaled in this way while the finite momentum modes are not. [6, 7, 8] In other words, we have decomposed the field into two parts, $\phi(x) = \phi_c + h(x)$, then renormalized ϕ_c only.

The constant χ can be determined from the condition that the second derivative of the effective potential at the SSB minimum, taken with respect to the renormalized field ϕ_R equals the physical mass squared. We have

$$\left.\frac{d^2V_{\text{eff}}}{d\phi_R^2}\right|_{SSB} = 8\alpha C_1 Z_\phi \lambda M^2\left(1+\frac{3}{2}\log\frac{M^2}{M_v^2}\right)\Bigg|_{SSB} = \frac{\chi}{3\alpha}M_v^2 = M_v^2, \tag{2.43}$$

hence $\chi = 3\alpha$.

The renormalization prescription and the resulting effective potential are:

$$\lambda = \frac{1}{24\alpha^2 C_1 \log\frac{\Lambda^2}{\mu_1^2}}; \quad \phi_c^2 = \frac{3}{4\alpha}\left(\log\frac{\Lambda^2}{\mu_1^2}\right)\phi_R^2; \quad M^2 = \frac{1}{8C_1}\phi_R^2 \tag{2.44}$$

$$V_{\text{eff}} = \frac{1}{64C_1}\phi_R^4\left(\log\frac{\phi_R^2}{v_R^2} - \frac{1}{2}\right). \tag{2.45}$$

First we note that the constant C_2 has disappeared into the definition of v_R. Second, we note that the relation of the physical mass M to the field value v_R at the symmetry breaking minimum depends only on the coefficient C_1 of the $M^4\log M^2$ term in the unrenormalized effective potential.

Substituting in the particular value $C_1 = \frac{1}{64\pi^2}$, we obtain:

$$V_{\text{eff}} = \pi^2\phi_R^4\left(\log\left(\frac{\phi_R^2}{v_R^2}\right) - \frac{1}{2}\right); \qquad M^2|_{SSB} = 8\pi^2 v_R^2. \tag{2.46}$$

For $\alpha = \frac{1}{2}$ and $C_2 = \frac{1}{128\pi^2}$, the renormalization prescription used can be summarized as:

$$\lambda = \frac{32\pi^2}{3\log\frac{\Lambda^2}{\mu_1^2}}; \quad \phi_c^2 = \frac{3}{2}\left(\log\frac{\Lambda^2}{\mu_1^2}\right)\phi_R^2. \tag{2.47}$$

The key difference from the conventional renormalization prescription is that here we rescale the constant part of the field by an infinite factor. (The resulting theory turns out to be trivial in that all scattering amplitudes vanish as the cutoff Λ is taken to infinity.)

The resulting effective potential has the same form as the raw perturbative result eq. (2.31) shown in Figure 3, and hence possesses spontaneous symmetry breaking even though the bare mass $m^2 = 0$ (by constrast, in the RGI approach the effective potential for $m^2 = 0$ has a single minimum at the origin as shown in Figure 6). The "autonomous" renormalization procedure can easily be generalized to the case of a massive scalar field, in which case symmetry breaking still occurs for $m^2 > 0$ as long as the bare mass m is sufficiently small. In that case we have $M^2 = m^2 + \frac{1}{2}\lambda\phi^2$, with $m^2 \ll \frac{1}{2}\lambda\phi_c^2$.

In the autonomous approach, additional terms in the loop expansion do not affect the form of the effective potential (written in terms of the renormalized field ϕ_R), as they will at most bring in more terms in ϕ_c^4 and/or $\phi_c^4 \log \phi_c^2$ [6, 7, 8, 18]. This may change the renormalization prescription but will not change the functional form of V_{eff}.

2.4 Particle Gas Picture and the Long-Range Induced Interaction

The results of the autonomous renormalization procedure can be understood physically in terms of a "particle gas" picture in which the nonempty vacuum corresponds to a vacuum condensate of scalar particles. The physical Higgs field $h(x)$ then corresponds to quasiparticle excitations of the condensate with mass M. In this subsection we summarize the results of Ref. [19].

One immediate puzzle is why the vacuum condensate should be a lower energy configuration than an empty vacuum when the fundamental interparticle interaction (Figure 7a) is repulsive. The need to provide the rest-mass energy of each particle in the condensate plus the energy to overcome the repulsive interparticle interaction would seem to indicate that a nonzero particle density should correspond to a higher energy state than the empty vacuum. However, we will show that the one loop "fish diagram" (Figure 7b), in which two scalar particles scatter via exchange of two virtual scalar particles, generates a long-range attractive interparticle potential. If the fundamental particle mass is sufficiently small, this attractive potential brings the energy of a nonzero vacuum particle density state down below the energy of an empty vacuum state.

The form of an equivalent interparticle potential for some interaction is just the three dimen-

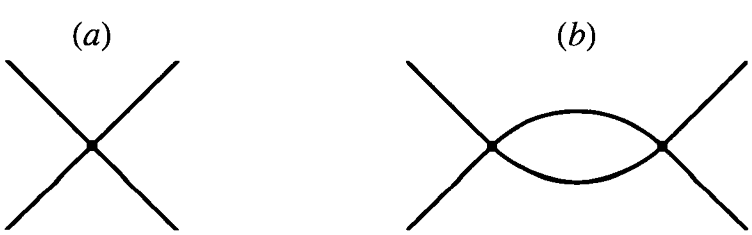


Figure 7: Feynman Diagrams: (a) fundamental interaction, (b) one-loop "fish diagram"

sional Fourier transform of the corresponding scattering matrix element $\mathcal{M}$ [20, 21],

$$V_{\text{equiv}}(r) = \frac{1}{4E^2}\int \frac{d^3q}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\mathcal{M} \tag{2.48}$$

where r is the interparticle distance, $\vec{q}$ is the 3-momentum transfer, and E is the energy of each particle. In the nonrelativistic limit, $E \approx m$, the potential becomes independent of the kinematics. In QED, for example, one can use this method to obtain the Coulomb potential from the scattering matrix element for photon exchange. In the scalar $\lambda\phi^4$ theory the lowest order Feynman diagram (Figure 7a) involves only the fundamental point-like interaction and its scattering matrix element is $\mathcal{M}^{(\text{core})} = \lambda$, hence the equivalent potential is

$$V^{(\text{core})}_{\text{equiv}}(r) = \frac{1}{4m^2}\lambda\delta^{(3)}(r), \tag{2.49}$$

in the nonrelativistic limit. A standard result of nonrelativistic quantum mechanics is that a delta function potential produces no scattering in 3 or more dimensions, which fits with our understanding of triviality outlined earlier. At one loop order, in the massless case, the t-channel "fish diagram" (Figure 7b) matrix element is

$$\mathcal{M}^{(\text{tail,t})} = \frac{\lambda^2}{16\pi^2}\log\frac{q}{\Lambda}. \tag{2.50}$$

Inserting this in eq. (2.48), we obtain

$$\begin{aligned} V^{(\text{tail,t})}_{\text{equiv}}(r) &= \tfrac{\lambda^2}{64\pi^2E^2}\tfrac{1}{(2\pi)^2}\int_0^\infty dq \int_{-1}^1 d(\cos\theta)\; q^2 e^{iqr\cos\theta}\log\frac{q}{\Lambda} \\ &= \tfrac{\lambda^2}{256\pi^4E^2}\int_0^\infty dq\; q^2\frac{2\sin qr}{qr}\log\frac{q}{\Lambda} \\ &= \tfrac{\lambda^2}{128\pi^4E^2}\tfrac{1}{r^3}\int_0^\infty du\; u^2\frac{\sin u}{u}\log\frac{u}{r\Lambda} \\ &= -\tfrac{\lambda^2}{256\pi^3E^2}\tfrac{1}{r^3}. \end{aligned} \tag{2.51}$$

(The last integral should be understood as the limit $\epsilon \to 0$ of an integral including a convergence factor $e^{-\epsilon u}$.[19])

The u-channel diagram produces the same result, while the s-channel diagram produces a delta function potential which can be absorbed into the lowest-order result via a redefinition of λ. The net result is that $V^{(\text{tail,t})}_{\text{equiv}}(r)$ has a $-\frac{1}{r^3}$ form. Including a nonzero mass produces the same potential times a factor $2mrK_1(2mr)$, where K_1 is a familiar modified Bessel function. For small mr this factor tends to one, while for large mr it tends to $\sqrt{\pi mr}e^{-2mr}$. The exponential suppression of the potential at large r due to the mass parallels that of a Yukawa potential, with a factor of e^{-mr} for each exchanged particle. A simple way to incorporate the effect of the particle mass m is to cut off the $-\frac{1}{r^3}$ potential at $r_{\max} \sim \frac{1}{2m}$. However, in a condensate the interaction would be screened by intervening particles. Each such interaction leads to a mass insertion $-i(\frac{1}{2}\lambda\phi_c^2)$ in the propagator, and summing the geometric series of these diagrams shifts the pole in the propagator to $M^2 = m^2 + \frac{1}{2}\lambda\phi_c^2$. As such, the quasiparticle mass M effectively replaces m in the exponential and we have $r_{\max} \sim \frac{1}{2M}$, or equivalently $r_{\max}M \sim 1$.

To complete the picture, we make a simple estimate of the energy density of the condensate. Consider the total energy of the N condensate particles in a volume $\mathcal{V}$. First we must include the rest mass energies of the particles, mN. Particles in the condensate have zero momentum, so there is no kinetic contribution to the energy. We assume that the gas is dilute ($na^3 \ll 1$, where $n \equiv \frac{N}{\mathcal{V}}$ and $a = \frac{\lambda}{8\pi m}$ is the scattering length) so almost all particles are in the condensate and only two-body interaction energies are significant. We will see later that the assumption of diluteness is valid. The number of interacting pairs is $\frac{1}{2}N(N-1) \approx \frac{1}{2}N^2$ and each pair has on average a potential energy $\frac{1}{\mathcal{V}}\int d^3r V_{\text{equiv}}(r)$. The energy density is[5]

$$\begin{aligned}
\mathcal{E} = \tfrac{1}{\mathcal{V}}E_{\text{tot}} &= \tfrac{1}{\mathcal{V}}\left(mN + \tfrac{1}{2}N^2\tfrac{1}{\mathcal{V}}\int d^3r V_{\text{equiv}}(r)\right) \\
&= mn + \tfrac{1}{2}n^2\int d^3r V_{\text{equiv}}^{(\text{core})}(r) + \tfrac{1}{2}n^2\int d^3r V_{\text{equiv}}^{(\text{tail},t)}(r) \\
&= mn + C_1\tfrac{\lambda n^2}{m^2} - C_2\tfrac{\lambda^2 n^2}{m^2}\int\frac{dr}{r}.
\end{aligned} \tag{2.52}$$

The integral must be cut off not only at the upper limit $r_{\max} = \frac{1}{2M}$, due to the screening mechanism, but also at some small distance r_0 at the lower limit. This is natural if we consider the delta function of the core interaction $V_{\text{equiv}}^{(\text{core})}$ to be the $r_0 \to 0$ limit of a step function out to r_0, after which the tail interaction $V_{\text{equiv}}^{(\text{tail},t)}(r)$ kicks in. Hence, we have

$$\mathcal{E} = mn + C_1\frac{\lambda n^2}{m^2} - C_2\frac{\lambda^2 n^2}{m^2}\log\frac{r_{\max}}{r_0} \tag{2.53}$$

[5]For this quick estimate we will concentrate on the functional form without chasing up constant factors, instead inserting unknown constants C_1 and C_2 as placeholders.

where $\frac{1}{r_0}$ plays the role of an ultraviolet cutoff.

Now, to see how the particle density n corresponds to the field strength ϕ_c, we write the field as a plane-wave expansion in particle creation and annihilation operators $a_{\vec{k}}(t)$ and $a^{\dagger}_{\vec{k}}(t)$.

$$\phi(\vec{x},t) = \sum_{\vec{k}} \frac{1}{\sqrt{2\mathcal{V}E_{\vec{k}}}} \left(a_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}}(t)e^{-i\vec{k}\cdot\vec{x}} \right). \tag{2.54}$$

The vacuum expectation value involves only the zero momentum mode

$$\phi_c \sim \langle \phi(\vec{x},t) \rangle = \frac{1}{\sqrt{2\mathcal{V}m}} \left(a_0 + a_0^{\dagger} \right). \tag{2.55}$$

The particle number operator reduces similarly,

$$\widehat{N} = \sum_{k} a^{\dagger}_{\vec{k}}(t) a_{\vec{k}}(t) \tag{2.56}$$

$$N = \langle \widehat{N} \rangle \sim a_0^{\dagger} a_0, \tag{2.57}$$

so that $a_0 \sim \sqrt{N}$ and hence

$$\phi_c \sim \sqrt{\frac{2N}{\mathcal{V}m}} = \sqrt{\frac{2n}{m}} \tag{2.58}$$

$$n \sim \tfrac{1}{2} m \phi_c^2. \tag{2.59}$$

Inserting this in eq. (2.53) we obtain the energy density as a function of ϕ_c, which is the field-theoretic effective potential

$$\mathcal{E} = V_{\text{eff}}(\phi_c) = \tfrac{1}{2} m^2 \phi_c^2 + \tfrac{1}{4} C_1 \lambda \phi_c^4 - \tfrac{1}{4} C_2 \lambda^2 \phi_c^4 \log \frac{r_{\text{max}}}{r_0}. \tag{2.60}$$

This corresponds to the result of the one-loop calculation [eq. (2.19) with use of eqs. (2.22) and (2.25)],

$$V_{\text{eff}}(\phi_c) = \tfrac{1}{2} m^2 \phi_c^2 + \tfrac{1}{4!} \lambda \phi_c^4 + \frac{\lambda^2 \phi_c^4}{256\pi^2} \left[\log\left(\frac{M^2}{\Lambda^2} \right) - \frac{1}{2} \right]. \tag{2.61}$$

These two forms agree since the $\log \frac{r_{\text{max}}}{r_0}$ corresponds to the $\log \frac{\Lambda}{M}$ factor if, as already argued, $r_{\text{max}} \sim \frac{1}{M}$ and $\frac{1}{r_0}$ plays the role of the ultraviolet cutoff Λ.

Now we will verify the consistency of our starting assumption that the condensate is infinitely dilute. First we note that eq. (2.61) has a nonzero minimum only if $m^2 \le \frac{\lambda^2 v_B^2}{128\pi^2} = \frac{\lambda M^2}{64\pi^2}$. Since M^2 is finite while λ is infinitesimal, $\lambda = O\left(\frac{1}{\log\frac{\Lambda}{M}}\right)$, we have $\frac{m}{M} = O\left(\frac{1}{\sqrt{\log\frac{\Lambda}{M}}}\right)$. The condensate is then infinitely dense on the scale of the finite quasiparticle mass M, since $\frac{n}{M^3} = O\left(\sqrt{\log\frac{\Lambda}{M}}\right)$ is infinite. Despite this fact, the gas is infinitely dilute since the scattering length $a \equiv \frac{\lambda}{8\pi m}$ is $O\left(\frac{1}{\sqrt{\log\frac{\Lambda}{M}}}\right)$, giving

$$na^3 \propto \sqrt{\log\frac{\Lambda}{M}}\left(\log\frac{\Lambda}{M}\right)^{-\frac{3}{2}} = \left(\log\frac{\Lambda}{M}\right)^{-1} \to 0, \tag{2.62}$$

in accordance with our initial assumption. This somewhat strange result that the particle gas is both infinitely dense and infinitely dilute is related to the hierarchy of scales that emerges. The proliferation of scales is a typical characterisitic of nearly scale-invariant theories such as this one, and will also be a major issue discussed in the next section.

The particle gas picture provides a means of acquiring some physical intuition about various characteristics of the theory in the broken phase. However, it is not a relativistically covariant framework. The aim of this thesis is to explore the effects of a nonlocal long-range effective interaction from a manifestly covariant approach.

3 Effective Potential of a Nonlocal Scalar Field Theory

3.1 The Nonlocal Model Theory

We will address the calculation of the effective potential of a scalar field theory whose Lagrangian incorporates the standard point-like ϕ^4 interaction as well as a nonlocal ϕ^4 interaction. The latter term corresponds to the interparticle equivalent potential generated by the point-like interaction via a Feynman diagram in which two virtual scalar particles are exchanged.

Consider the Euclideanized Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \int d^4y\,\phi(x)^2 U_{\text{equiv}}(x-y)\phi(y)^2 \tag{3.1}$$

where the interaction kernel has two pieces, a repulsive core and an attractive tail (Figure 8).

$$U_{\text{equiv}}(z) = U_{\text{core}}(z) + U_{\text{tail}}(z) \tag{3.2}$$

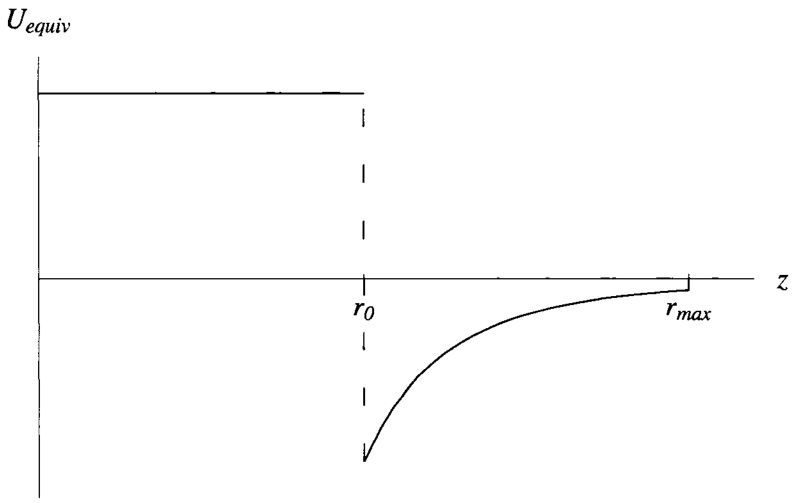


Figure 8: Interaction kernel $U_{\text{equiv}}(z) = U_{\text{core}}(z) + U_{\text{tail}}(z)$

$$U_{\text{core}}(z) = \begin{cases} V_0 & \text{for } |z| < r_0 \\ 0 & \text{for } |z| > r_0 \end{cases} \; ; \qquad U_{\text{tail}}(z) = \begin{cases} 0 & \text{for } |z| < r_0 \\ -\frac{\zeta}{z^4} & \text{for } r_0 < |z| < r_{\max} \\ 0 & \text{for } |z| > r_{\max} \end{cases} \tag{3.3}$$

The norm $|z|$ here is taken using the four-dimensional Euclidean metric and V_0, ζ, r_0 and $r_{\max}$ are free parameters (the latter two are not necessarily to be identified with the r_0 and $r_{\max}$ of the previous subsection.) Note that the covariant interaction kernel $U_{\text{tail}}(z)$ with $z = (t, \vec{r})$ gives rise to a nonrelativistic interparticle potential as

$$\begin{aligned} V_{\text{equiv}}^{(\text{tail})}(r) &= \frac{1}{(2m)^2} \int_{-\infty}^{\infty} dt U_{\text{tail}}(z) = -\frac{\zeta}{(2m)^2} \int_{-\infty}^{\infty} dt \frac{1}{z^4} = -\frac{\zeta}{(2m)^2} \int_{-\infty}^{\infty} dt \frac{1}{(t^2 + r^2)^2} \\ &= -\left(\frac{\zeta\pi}{8m^2}\right) \frac{1}{r^3}. \end{aligned} \tag{3.4}$$

(The $\frac{1}{(2m)^2}$ arises from conversion to the nonrelativistic normalization of the field. [22])

The core piece U_{core} leads to the standard $\lambda\phi^4$ term in the delta-function limit, $r_0 \to 0$ while $V_0 \to \infty$ with

$$\tfrac{1}{4!}\lambda \equiv \int d^4z \, U_{\text{core}}(z) = \frac{\pi^2}{2} V_0 r_0^4 \tag{3.5}$$

remaining constant. The tail piece U_{tail} corresponds to the long-range nonlocal effective interparticle interaction, and we may define a dimensionless coupling analagous to λ,

$$\tfrac{1}{4!}\xi \equiv -\int d^4z\, U_{\text{tail}}(z) = 2\pi^2\zeta \log \frac{r_{\max}}{r_0}. \tag{3.6}$$

The classical potential is then

$$V_{\text{cl}} = \tfrac{1}{4!}\lambda_{\text{eff}}\phi_c^4; \qquad \lambda_{\text{eff}} \equiv \lambda - \xi, \tag{3.7}$$

where λ_{eff} is in this sense an effective coupling constant.

The inverse propagator for the Lagrangian (3.1) is

$$\begin{aligned} G(x,y)^{-1} &= (-\partial^2)\delta^{(4)}(x-y) + \left(\tfrac{\delta^2}{\delta\phi(x)\delta\phi(y)} \int d^4z \int d^4w\, \phi(z)^2 U_{\text{equiv}}(z-w)\phi(w)^2\right)\Big|_{\phi=\phi_c} \\ &= (-\partial^2)\delta^{(4)}(x-y) + 4\left(\tfrac{\delta}{\delta\phi(x)} \int d^4z\, \phi(y) U_{\text{equiv}}(y-z)\phi(z)^2\right)\Big|_{\phi_c} \\ &= \left(-\partial^2 + 4\phi_c^2 \int d^4z\, U_{\text{equiv}}(z)\right)\delta^{(4)}(x-y) + 8\phi_c^2 U_{\text{equiv}}(x-y) \\ &= \left(-\partial^2 + \tfrac{1}{6}(\lambda-\xi)\phi_c^2\right)\delta^{(4)}(x-y) + 8\phi_c^2 U_{\text{equiv}}(x-y). \end{aligned} \tag{3.8}$$

(Note that in the local limit, $U_{\text{equiv}}(x-y) \to \frac{1}{4!}\lambda\delta^{(4)}(x-y)$, $\xi \to 0$, the ϕ_c^2 terms combine to form the physical mass $\frac{1}{2}\lambda\phi_c^2$.)

Going to momentum space we may write the inverse propagator of this nonlocal theory as

$$G(p)^{-1} = p^2 + M^2 + g(p), \tag{3.9}$$

where

$$M^2 \equiv \alpha(\lambda-\xi)\phi_c^2; \qquad g(p) \equiv \left(\tfrac{1}{6}-\alpha\right)(\lambda-\xi)\phi_c^2 + 8\phi_c^2 \int d^4x\, e^{ip\cdot x} U_{\text{equiv}}(x). \tag{3.10}$$

Note that this M^2 is not necessarily the physical mass. It depends on a parameter α that cancels between M^2 and $g(p)$. Exact results cannot depend on α, but our approximate calculation introduces dependence on it. The Principle of Minimal Sensitivity [23] can be used to fix an optimal value of α.

3.2 Setting up the Calculation of the Effective Potential

Substituting the above inverse propagator in eq. (2.18) gives us the one-loop effective potential,

$$\begin{aligned} V_{\text{eff}} &= V_{\text{cl}} + \tfrac{1}{2}\int \frac{d^4p}{(2\pi)^4} \log\left(p^2 + M^2 + g(p)\right) \\ &= V_{\text{cl}} + I_1(M^2) + \tfrac{1}{2}\int \frac{d^4p}{(2\pi)^4} \log\left(1 + \frac{g(p)}{p^2+M^2}\right) \\ &= V_{\text{cl}} + I_1(M^2) - \tfrac{1}{2}\textstyle\sum_{j=1}^{\infty}\tfrac{1}{j}\displaystyle\int \frac{d^4p}{(2\pi)^4}\left(-\frac{g(p)}{p^2+M^2}\right)^j. \end{aligned} \tag{3.11}$$

In the derivation of eq. (2.18) we worked only to one-loop order, i.e. up to two powers of the coupling constant. Since each power of $g(p)$ also carries one power of the coupling constants λ and/or ξ (as does M^2) it would not be enlightening to calculate terms in $g(p)^3$ or higher unless we also included interaction terms of the same order, which is beyond the scope of this thesis. We will see shortly from the form of $g(p)$ that the terms up to $g(p)^2$ capture all the ultraviolet sensitive terms. Thus we shall keep just the first two terms of the series expansion of the logarithm,

$$V_{\text{eff}} = V_{\text{cl}} + I_1(M^2) + I_g^{(1)} + I_g^{(2)}, \tag{3.12}$$

where we have defined

$$I_g^{(1)} \equiv \tfrac{1}{2}\int^{\Lambda} \frac{d^4p}{(2\pi)^4}\left(\frac{g(p)}{p^2+M^2}\right); \quad I_g^{(2)} \equiv -\tfrac{1}{4}\int^{\Lambda} \frac{d^4p}{(2\pi)^4}\left(\frac{g(p)}{p^2+M^2}\right)^2. \tag{3.13}$$

For convenience, split $g(p)$ into three pieces:

$$g(p) = g_M + g_{\text{core}}(p) + g_{\text{tail}}(p), \tag{3.14}$$

$$g_M \equiv \left(\tfrac{1}{6} - \alpha\right)(\lambda - \xi)\phi_c^2, \tag{3.15}$$

$$g_{\text{core}}(p) \equiv 8\phi_c^2 \int d^4x\, e^{ip\cdot x} U_{\text{core}}(x); \quad g_{\text{tail}}(p) \equiv 8\phi_c^2 \int d^4x\, e^{ip\cdot x} U_{\text{tail}}(x). \tag{3.16}$$

The explicit forms of $g_{\mathrm{core}}(p)$ and $g_{\mathrm{tail}}(p)$ can be calculated, resulting in

$$\begin{aligned} g_{\mathrm{core}}(p) &= 32\pi V_0\phi_c^2 \int_0^{r_0} dx \int_0^{\pi} d\theta \left(e^{ipx\cos\theta} x^3 \sin^2\theta\right) \\ &= 32\pi^2 V_0 \phi_c^2 \int_0^{r_0} dx \left(x^3 \frac{J_1(px)}{px}\right) \\ &= 32\pi^2 V_0 r_0^2 \phi_c^2 \tfrac{J_2(r_0 p)}{p^2} \\ &= \tfrac{8}{3}\lambda\phi_c^2 \tfrac{J_2(r_0p)}{(r_0p)^2}, \end{aligned} \tag{3.17}$$

where $J_1(u)$ and $J_2(u)$ are familiar Bessel functions, and

$$\begin{aligned} g_{\mathrm{tail}}(p) &= -32\pi\zeta\phi_c^2 \int_{r_0}^{r_{\max}} dx \int_0^{\pi} d\theta \left(e^{ipx\cos\theta} x^{-1} \sin^2\theta\right) \\ &= -32\pi^2\zeta\phi_c^2 \int_{r_0}^{r_{\max}} dx \left(x^{-1}\frac{J_1(px)}{px}\right) \\ &= \pi^2\zeta\phi_c^2 \left(u^2 F^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2)\Big|_{u=r_0p}^{r_{\max}p} - 16\log\tfrac{r_{\max}}{r_0}\right) \\ &= \pi^2\zeta\phi_c^2 \left(u^2 \overline{F}^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2)\right)\Big|_{u=r_0p}^{r_{\max}p}, \end{aligned} \tag{3.18}$$

$$\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right) \equiv F^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right) + \frac{16}{u^2}\left(\frac{1}{2} - \log\frac{u}{\sigma}\right), \tag{3.19}$$

$$\sigma \equiv 2e^{-\gamma_{\mathrm{E}}} = 1.12292\ldots \tag{3.20}$$

where $F^{1,1}_{2,2,3}$ is a hypergeometric function (see Appendix A page 60 for details[6]), and $\overline{F}^{1,1}_{2,2,3}$ is the same function with its leading asymptotic behavior removed (see Table 2). Note that $F^{1,1}_{2,2,3}$ is small for small argument while $\overline{F}^{1,1}_{2,2,3}$ is small for large argument. The use of σ, defined in terms of the Euler constant γ_{E}, as the (otherwise arbitrary) dimensionless scale in $\log\frac{u}{\sigma}$ is convenient because this combination appears naturally in the asymptotic expansion of $F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)$.

The divergences we will encounter will involve $\log\frac{r_{\max}}{r_0}$ or $\log\frac{\Lambda}{M}$, and to understand how these appear it is useful to examine the functions $g_{\mathrm{core}}(p)$ and $g_{\mathrm{tail}}(p)$ in some detail. First, the basic functions $J_2(u)$ and $F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)$ are plotted in Figures 9 and 10, with their small and large argument behavior given in Table 1. The functions $g_{\mathrm{core}}(p)$ and $g_{\mathrm{tail}}(p)$ are plotted in Figures 11 and 12, with their large and small argument behavior shown in Table 2.

For larger values of $\frac{r_{\max}}{r_0}$, $g_{\mathrm{tail}}(p)$ still has much the same form as shown in the figures when

[6]The graphs and asymptotic expansions in the Appendix were generated using Mathematica, as were all of the graphs and many calculations throughout this work.

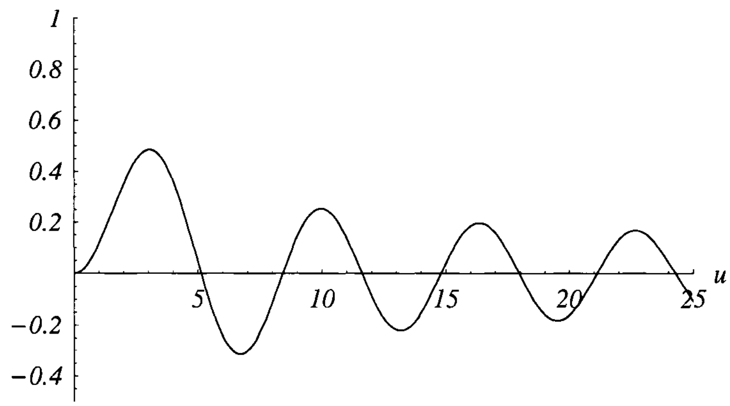


Figure 9: Bessel function $J_2(u)$

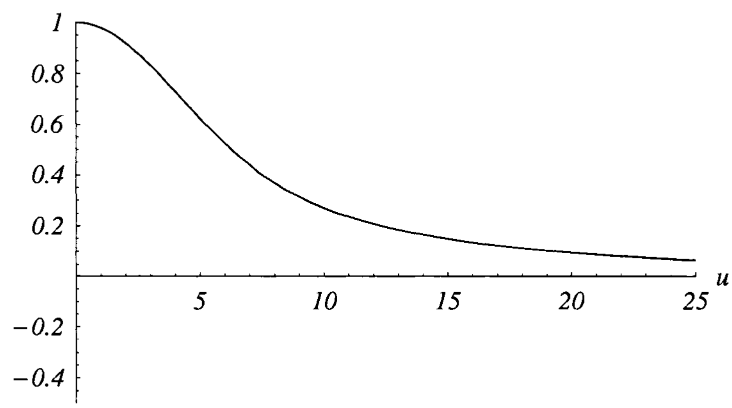


Figure 10: Hypergeometric function $F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)$

	small argument	large argument
$J_2(u)$	$\frac{u^2}{8} - \frac{u^4}{96} + \cdots$	$\sqrt{\frac{2}{\pi}}\frac{\cos(\frac{5\pi}{4}-u)}{u^{\frac{1}{2}}} + \frac{15}{4\sqrt{2\pi}}\frac{\sin(\frac{5\pi}{4}-u)}{u^{\frac{3}{2}}} + \cdots$
$F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)$	$1 - \frac{u^2}{48} + \cdots$	$-\frac{8}{u^2}(1 - 2\log\frac{u}{\sigma}) - 32\sqrt{\frac{2}{\pi}}\frac{\cos(\frac{5\pi}{4}-u)}{u^{\frac{9}{2}}} + \cdots$

Table 1: Small and large argument behavior of $J_2(u)$ and $F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)$

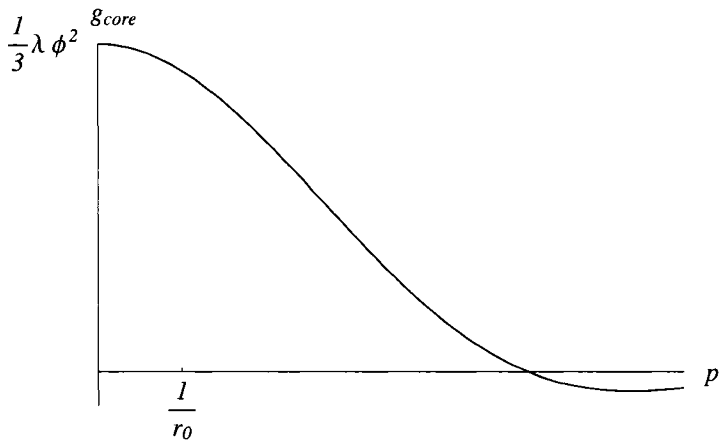


Figure 11: $g_{\rm core}(p)$

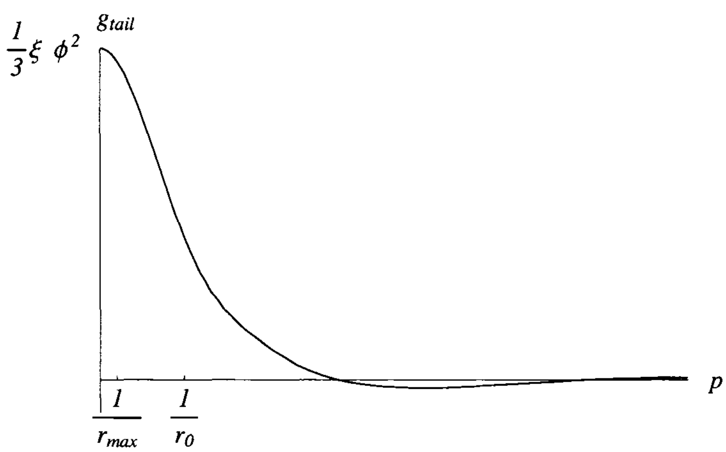


Figure 12: $g_{\rm tail}(p)$ for $\frac{r_{\rm max}}{r_0} = 5$

	small argument	large argument
$g_{\rm core}(p)$	$\frac{1}{3}\lambda\phi_c^2\left(1-\frac{r_0^2p^2}{12}+\cdots\right)$	$-\frac{1}{3}\lambda\phi_c^2\left(\frac{105}{8\sqrt{2\pi}}\frac{\cos(\frac{5\pi}{4}-r_0p)}{(r_0p)^{\frac{9}{2}}}-15\sqrt{\frac{2}{\pi}}\frac{\sin(\frac{5\pi}{4}-r_0p)}{(r_0p)^{\frac{7}{2}}}+\cdots\right)$
$g_{\rm tail}(p)$	$\frac{1}{3}\xi\phi_c^2\left(1-\frac{(r_{\rm max}^2-r_0^2)}{16\log\frac{r_{\rm max}}{r_0}}p^2+\cdots\right)$	$\frac{1}{3}\xi\phi_c^2\left(\frac{2\sqrt{2}}{\sqrt{\pi}\log\frac{r_{\rm max}}{r_0}}\left(\frac{\cos(\frac{5\pi}{4}-r_{\rm max}p)}{(r_{\rm max}p)^{\frac{5}{2}}}-\frac{\cos(\frac{5\pi}{4}-r_0p)}{(r_0p)^{\frac{5}{2}}}\right)+\cdots\right)$

Table 2: Small and large argument behavior of $g_{\rm core}(p)$ and $g_{\rm tail}(p)$

	Λ	M	$\frac{1}{r_0}$
$r_{\max}$	$r_{\max}\Lambda \gg 1$	$r_{\max}M \gtrsim 1$	$\frac{r_{\max}}{r_0} \ll 1$
r_0	$r_0\Lambda$ arbitrary	$r_0 M \ll 1$	
$\frac{1}{M}$	$\frac{\Lambda}{M} \gg 1$		

Table 3: Relationships amongst the various scales

viewed in terms of the natural breakpoints $\frac{1}{r_0}$ and $\frac{1}{r_{\max}}$ (see Appendix B page 61 for additional plots using different values of $\frac{r_{\max}}{r_0}$).

From the particle gas picture presented in section 2.4 it would appear that the most interesting case is where the two scales $\frac{1}{r_0}$ and $\frac{1}{r_{\max}}$ are of the same order as the cutoff Λ and the physical mass M_v, respectively. As such, the combinations $r_0\Lambda$ and $r_{\max}M_v$ may be considered to be of finite order, ~ 1. However, there are other interesting cases to consider. For example the case $r_0 \sim \frac{1}{\log \Lambda}$ corresponds to r_0 of the same order as the scattering length, so that in a nonrelativistic situation the Born approximation would be valid. Another case of interest is where the $r_0 \to 0$ limit is taken first, before the $\Lambda \to \infty$ limit, so that the core term reproduces the standard point-like local $\lambda\phi^4$ interaction. We will also explore the limit $r_{\max}M \to \infty$ in all of these cases.

In practical terms, the above considerations affect which terms we will need to keep as we calculate the effective potential, so that the final result is general enough. We have seen that the functions $g_{\text{core}}(p)$ and $g_{\text{tail}}(p)$ depend on the dimensionless combinations $r_0 p$ and $r_{\max}p$, so we expect to encounter the combinations $r_0\Lambda$ and $r_{\max}\Lambda$. The latter is always large, $r_{\max}\Lambda \gg 1$. We are mostly interested in the case $r_0\Lambda \sim 1$ but may also have $r_0\Lambda \gg 1$ or $r_0\Lambda \ll 1$, so we will keep track of the exact $r_0\Lambda$ dependence until the end of the calculation. We will write our expressions in a form that that is convenient for considering the regime $r_0\Lambda \gtrsim 1$, but will also exhibit the final result in an alternate form suitable for analysing the regime $r_0\Lambda \to 0$. We will routinely drop terms in positive powers of $\frac{M}{\Lambda}$, $r_0 M$ or $\frac{r_0}{r_{\max}}$, and will assume that we always have $r_{\max}M \gtrsim 1$. (Our assumptions are summarized in Table 3.)

We are now in a position to see why contributions to the effective potential involving $g(p)^3$ or higher do not contain any divergences. First, since $g_{\text{core}}(p)$ and $g_{\text{tail}}(p)$ fall off rapidly with p while g_M remains constant, the g_M terms have the worst convergence properties. Comparing eq. (3.11) with the definition of I_N in eqs. (2.20) and (2.21), it is easy to see that a term in g_M^n is proportional to $I_{(1-n)}$. Since I_N is logarithmically divergent for $N=-1$ and convergent for $N<-1$, the terms in $g(p)^n$ are logarithmically divergent for $n=2$ and convergent for $n>2$. Further, observe that $g_{\text{core}}(p)$ and $g_{\text{tail}}(p)$ are approximately constant for small p and fall off when $p \gtrsim \frac{1}{r_0}$. For an estimate of the divergence of an integral in which $g_{\text{core}}(p)$ and/or $g_{\text{tail}}(p)$ appear, we may replace them with a constant and cut off the integral at $p \sim \frac{1}{r_0}$ rather than $p=\Lambda$ (assuming here that $\frac{1}{r_0} \lesssim \Lambda$, i.e. $r_0\Lambda \gtrsim 1$). Picking an appropriate scale for the logarithms, integrals involving $g_{\text{core}}(p)$ thus may diverge approximately as $\log r_0\Lambda$ or $\log r_0 M$, while integrals involving $g_{\text{tail}}(p)$ may diverge as $\log \frac{r_{\max}}{r_0}$. The actual divergences involve polynomials in the logarithms, as we will see by explicit calculation.

Now that we have an intuitive grasp of how $g(p)$ contributes to the effective potential, we are prepared to calculate the one-loop effective potential using eq. (3.12). The integrals $I_g^{(1)}$ and $I_g^{(2)}$ defined in eq. (3.13) fall naturally into pieces corresponding to the pieces of $g(p)$,

$$I_g^{(1)} = I_M^{(1)} + I_{\text{core}}^{(1)} + I_{\text{tail}}^{(1)} \tag{3.21}$$

$$I_M^{(1)} \equiv \tfrac{1}{2} g_M \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{1}{p^2+M^2}; \quad I_{\text{core}}^{(1)} \equiv \tfrac{1}{2} \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{core}}(p)}{p^2+M^2}; \quad I_{\text{tail}}^{(1)} \equiv \tfrac{1}{2} \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{tail}}(p)}{p^2+M^2} \tag{3.22}$$

$$I_g^{(2)} = I_M^{(2)} + I_{M,c}^{(2)} + I_{M,t}^{(2)} + I_{c,c}^{(2)} + I_{c,t}^{(2)} + I_{t,t}^{(2)} \tag{3.23}$$

$$I_M^{(2)} \equiv -\tfrac{1}{4} g_M^2 \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2+M^2)^2} \tag{3.24}$$

$$I_{M,c}^{(2)} \equiv -\tfrac{1}{2} g_M \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{core}}(p)}{(p^2+M^2)^2}; \quad I_{M,t}^{(2)} \equiv -\tfrac{1}{2} g_M \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{tail}}(p)}{(p^2+M^2)^2} \tag{3.25}$$

$$I_{c,c}^{(2)} \equiv -\tfrac{1}{4} \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{core}}(p)^2}{(p^2+M^2)^2} \tag{3.26}$$

$$I_{c,t}^{(2)} \equiv -\tfrac{1}{2} \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{core}}(p) g_{\text{tail}}(p)}{(p^2+M^2)^2}; \quad I_{t,t}^{(2)} \equiv -\tfrac{1}{4} \int^\Lambda \frac{d^4p}{(2\pi)^4} \frac{g_{\text{tail}}(p)^2}{(p^2+M^2)^2}. \tag{3.27}$$

These are all the integrals we will need to calculate explicitly, arranged roughly in ascending order of difficulty at each order in $g(p)$. As we perform these calculations, we will drop any terms that vanish according to our assumptions about the various scales. Constant terms (independent of ϕ_c) are irrelevant so we will drop those when convenient, and mass renormalization subtracts out any terms in ϕ_c^2 (taking into account that $M^2 \propto \phi_c^2$) so we will drop those terms as well.

3.3 Calculation of the Integrals

3.3.1 Calculation of $I_M^{(1)}$ and $I_M^{(2)}$

Out of the nine integrals, the two integrals $I_M^{(1)}$ in eq. (3.22) and $I_M^{(2)}$ in eq. (3.24) are easy to perform since they correspond to standard I_0 and I_{-1} integrals defined in eq. (2.21). Using the identities eqs. (2.25) and (2.26) then discarding the ϕ_c^2 term (recalling that we have $M^2 \propto \phi_c^2$ and $g_M \propto \phi_c^2$), we obtain

$$I_M^{(1)} = \tfrac{1}{2} g_M I_0(M^2) = -\frac{g_M M^2}{16\pi^2} \log\frac{\Lambda}{M}, \tag{3.28}$$

$$I_M^{(2)} = -\tfrac{1}{8} g_M^2 I_{-1}(M^2) = -\frac{g_M^2}{32\pi^2}\left(\log\frac{\Lambda}{M} - \frac{1}{2}\right). \tag{3.29}$$

3.3.2 Calculation of $I_{\text{core}}^{(1)}$

The two integrals with one power of $g_{\text{core}}(p)$, $I_{\text{core}}^{(1)}$ in eq. (3.22) and $I_{M,c}^{(2)}$ in eq. (3.25), are also relatively straightforward and may be calculated using a number of different strategies. One simple method is to split the region of integration into two parts,

$$I_{\text{core}}^{(1)} = \tfrac{1}{2}\int_0^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2} - \tfrac{1}{2}\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2}. \tag{3.30}$$

The first integral here is tractable, and in the second integral we may expand the denominator as a series in $\frac{M}{p}$ to obtain a series of tractable integrals (most of which are negligible). The $I_{M,c}^{(2)}$ integral in eq. (3.25) can be calculated by the same method (the details of these two calculations are given in Appendix C.) However if we split the rest of the integrals in this manner, the first integral (from zero to infinity) thus obtained is not tractable so we must resort to other methods of calculation.

Armed with a bit of hindsight we exhibit here a different approach that will be useful for all of the remaining integrals as well. One more trick will be needed to calculate the two integrals involving $g_{\text{core}}(p)g_{\text{tail}}(p)$ and $g_{\text{tail}}(p)^2$, but we will defer its introduction until we begin calculating the simplest integral involving $g_{\text{tail}}(p)$, $I_{\text{tail}}^{(1)}$.

First we change the integration variable in $I^{(1)}_{\text{core}}$ to $u = r_0 p$,

$$\begin{aligned} I^{(1)}_{\text{core}} &= \tfrac{4}{3}\lambda\phi_c^2 \tfrac{2\pi^2}{(2\pi)^4} \int_0^{\Lambda} p^3 dp \frac{J_2(r_0 p)}{(r_0 p)^2 (p^2 + M^2)} \\ &= \tfrac{\lambda\phi_c^2}{6\pi^2 r_0^2} \int_0^{r_0\Lambda} du \frac{u J_2(u)}{u^2 + (r_0 M)^2}. \end{aligned} \tag{3.31}$$

Recall that mass renormalization removes terms with a ϕ_c^2 dependence (including factors of M^2), so it is convenient to separate these out using the identity

$$\frac{1}{u^2 + (r_0 M)^2} = \frac{1}{u^2} - \frac{(r_0 M)^2}{u^2 \left(u^2 + (r_0 M)^2\right)}. \tag{3.32}$$

The integral then becomes

$$I^{(1)}_{\text{core}} = \tfrac{\lambda\phi_c^2}{6\pi^2 r_0^2} \int_0^{r_0\Lambda} du \frac{J_2(u)}{u} - \frac{\lambda\phi_c^2 M^2}{6\pi^2} \int_0^{r_0\Lambda} du \frac{J_2(u)}{u(u^2 + (r_0 M)^2)}. \tag{3.33}$$

Dropping the term proportional to ϕ_c^2 leaves

$$I^{(1)}_{\text{core}} = -\tfrac{\lambda\phi_c^2 M^2}{6\pi^2} \int_0^{r_0\Lambda} du \frac{J_2(u)}{u(u^2 + (r_0 M)^2)}. \tag{3.34}$$

The $(r_0 M)^2$ in the denominator of the integrand cannot simply be sent to zero, since it serves to keep the integral from diverging at the lower limit; where u is small we have $J_2(u) \sim u^2$ so the integrand is $\sim \frac{u}{(r_0 M)^2}$ with the regulator but is $\sim \frac{1}{u}$ without.

Our strategy is to split the region of integration into two parts: a small-u region where the numerator can be expanded in a small argument approximation ($u \ll 1$), and another region where the denominator expansion is valid ($u \gg r_0 M$). Since $r_0 M \ll 1$, if we choose some real number a with $0 < a < 1$, then we have $r_0 M \ll (r_0 M)^a \ll 1$ and thus splitting the integral at $u = (r_0 M)^a$ satisfies the above requirements. Also note that since $0 < (1 - a) < 1$ holds as well, we have $(r_0 M) \ll (r_0 M)^{1-a} \ll 1$.

$$I^{(1)}_{\text{core}} = -\frac{\lambda\phi_c^2 M^2}{6\pi^2} \left(\int_0^{(r_0 M)^a} du \frac{\left(\frac{u^2}{8}\right)}{u\left(u^2 + (r_0 M)^2\right)} + \int_{(r_0 M)^a}^{r_0\Lambda} du \frac{J_2(u)}{u^3} \right) + \cdots. \tag{3.35}$$

Further terms (not explicitly shown here) vanish as $r_0 M \to 0$, so we drop them. The two remaining integrals can be calculated explicitly, and we expect that each one will depend on a but the combination will not. The first integral is

$$\tfrac{1}{8}\int_0^{(r_0M)^a} du\frac{u}{u^2+(r_0M)^2} = \tfrac{1}{8}(a-1)\log(r_0M). \tag{3.36}$$

The second integral is

$$\begin{aligned}\int_{(r_0M)^a}^{r_0\Lambda} du\frac{J_2(u)}{u^3} &= \left(-\tfrac{1}{192}u^2F^{1,1}_{2,2,4}\left(-\tfrac{1}{4}u^2\right)+\tfrac{1}{8}\log u\right)\Big|_{(r_0M)^a}^{r_0\Lambda} \\ &= -\tfrac{1}{192}r_0^2\Lambda^2F^{1,1}_{2,2,4}\left(-\tfrac{1}{4}r_0^2\Lambda^2\right)-\tfrac{1}{8}(a-1)\log(r_0M)+\tfrac{1}{8}\log\tfrac{\Lambda}{M},\end{aligned} \tag{3.37}$$

where in the last line we dropped the term $u^2F^{1,1}_{2,2,4}\left(-\frac{1}{4}u^2\right)$ for $u=(r_0M)^a$ since it vanishes as $r_0M\to 0$. The hypergeometric function $F^{1,1}_{2,2,4}(-\frac{1}{4}u^2)$ is similar to the function $F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)$ that appears in $g_{\text{tail}}(p)$ in that it tends to a constant for small argument and has a similar asymptotic expansion for large argument (see Appendix A for details).

Adding the two integrals, the terms in a cancel and we obtain

$$I^{(1)}_{\text{core}} = \frac{\lambda\phi_c^2M^2}{48\pi^2}\left(\frac{1}{24}r_0^2\Lambda^2F^{1,1}_{2,2,4}\left(-\tfrac{1}{4}r_0^2\Lambda^2\right)-\log\frac{\Lambda}{M}\right). \tag{3.38}$$

As $r_0\Lambda\to 0$, the hypergeometric term vanishes as it must to produce the local theory.

However, $u^2F^{1,1}_{2,2,4}\left(-\frac{1}{4}u^2\right)$ diverges as $u\to\infty$ so to get an expression for $I^{(1)}_{\text{core}}$ that is convenient in the limit $r_0\Lambda\to\infty$ we define

$$\overline{F}^{1,1}_{2,2,4}\left(-\tfrac{1}{4}u^2\right)\equiv F^{1,1}_{2,2,4}\left(-\tfrac{1}{4}u^2\right)+\frac{24}{u^2}\left(\frac{3}{4}-\log\frac{u}{\sigma}\right);\quad \tau_1(u)\equiv\frac{1}{24}u^2\overline{F}^{1,1}_{2,2,4}\left(-\tfrac{1}{4}u^2\right) \tag{3.39}$$

and write eq. (3.38) in terms of $\overline{F}^{1,1}_{2,2,4}$ (which vanishes for large argument):

$$I^{(1)}_{\text{core}} = \frac{\lambda\phi_c^2M^2}{48\pi^2}\left(\tau_1(r_0\Lambda)+\log\frac{r_0M}{\sigma}-\frac{3}{4}\right). \tag{3.40}$$

3.3.3 Calculation of $I_{M,c}^{(2)}$

The other integral with one power of $g_{\text{core}}(p)$, $I_{M,c}^{(2)}$ in eq. (3.25), can be calculated explicitly in the same fashion. First we change the variable to $u = r_0 p$,

$$I_{M,c}^{(2)} = -\frac{\lambda\phi_c^2 g_M}{6\pi^2 r_0^2}\int_0^{r_0\Lambda} du \frac{uJ_2(u)}{(u^2+(r_0M)^2)^2}, \tag{3.41}$$

split it at $u = (r_0M)^a$, expand the two integrals as before, and drop terms in each expansion that vanish as $r_0M \to 0$.

$$I_{M,c}^{(2)} = -\frac{\lambda\phi_c^2 g_M}{6\pi^2}\left(\int_0^{(r_0M)^a} du \frac{u\left(\frac{u^2}{8}\right)}{(u^2+(r_0M)^2)^2} + \int_{(r_0M)^a}^{r_0\Lambda} du \frac{J_2(u)}{u^3}\right) \tag{3.42}$$

The first integral is

$$\begin{aligned}\frac{1}{8}\int_0^{(r_0M)^a} du \frac{u^3}{(u^2+(r_0M)^2)^2} &= \tfrac{1}{16}\left(-\tfrac{1}{1+((r_0M)^{(1-a)})^2} + \log\left(1 + \tfrac{1}{((r_0M)^{(1-a)})^2}\right)\right) \\ &= -\tfrac{1}{16} - \tfrac{1}{8}(1-a)\log(r_0M),\end{aligned} \tag{3.43}$$

where in the last line we dropped terms that vanish as $r_0M \to 0$. The second integral is

$$\begin{aligned}\int_{(r_0M)^a}^{r_0\Lambda} du \frac{J_2(u)}{u^3} &= \left(-\tfrac{1}{192}u^2 F_{2,2,4}^{1,1}\left(-\tfrac{1}{4}u^2\right) + \tfrac{1}{8}\log\tfrac{u}{\sigma}\right)\Big|_{(r_0M)^a}^{r_0\Lambda} \\ &= -\tfrac{1}{192}(r_0\Lambda)^2 F_{2,2,4}^{1,1}\left(-\tfrac{1}{4}r_0^2\Lambda^2\right) + \tfrac{1}{8}\log(r_0\Lambda) - \tfrac{1}{8}a\log(r_0M),\end{aligned} \tag{3.44}$$

where in the last line we have again dropped a vanishing term. Putting the two integrals together, we obtain

$$I_{M,c}^{(2)} = \frac{\lambda\phi_c^2 g_M}{48\pi^2}\left(\tfrac{1}{4!}r_0^2\Lambda^2 F_{2,2,4}^{1,1}\left(-\tfrac{1}{4}r_0^2\Lambda^2\right) - \log\frac{\Lambda}{M} + \frac{1}{2}\right), \tag{3.45}$$

which in terms of τ_1, defined in eq. (3.39), is

$$I_{M,c}^{(2)} = \frac{\lambda\phi_c^2 g_M}{48\pi^2}\left(\tau_1(r_0\Lambda) + \log\frac{r_0M}{\sigma} - \frac{1}{4}\right). \tag{3.46}$$

3.3.4 Calculation of $I^{(2)}_{c,c}$

We now employ the same method to perform the integral with two powers of $g_{\text{core}}(p)$, $I^{(2)}_{c,c}$ in eq. (3.26). First we change the variable to $u = r_0 p$,

$$I^{(2)}_{c,c} = -\frac{2\lambda^2\phi_c^4}{9\pi^2}\int_0^{r_0\Lambda} du \frac{J_2(u)^2}{u(u^2+(r_0M)^2)^2}, \tag{3.47}$$

split it at $u = (r_0M)^a$, expand the two integrals as before, and drop terms in each expansion that vanish as $r_0M \to 0$.

$$I^{(2)}_{c,c} = -\frac{2\lambda^2\phi_c^4}{9\pi^2}\left(\int_0^{(r_0M)^a} du \frac{(\frac{u^2}{8})^2}{u(u^2+(r_0M)^2)^2} + \int_{(r_0M)^a}^{r_0\Lambda} du \frac{J_2(u)^2}{u^5}\right) \tag{3.48}$$

The first integral is

$$\frac{1}{64}\int_0^{(r_0M)^a} du \frac{u^3}{(u^2+(r_0M)^2)^2} = -\frac{1}{128} - \frac{1}{64}(1-a)\log(r_0M), \tag{3.49}$$

where we have dropped terms that vanish as $r_0M \to 0$. The second integral is

$$\begin{aligned}\int_{(r_0M)^a}^{r_0\Lambda} du \frac{J_2(u)^2}{u^5} &= \left(-\tfrac{1}{768}u^2F^{1,1,\frac{7}{2}}_{2,2,4,6}(-u^2) + \tfrac{1}{64}\log\tfrac{u}{\sigma}\right)\Big|_{(r_0M)^a}^{r_0\Lambda} \\ &= -\tfrac{1}{768}(r_0\Lambda)^2F^{1,1,\frac{7}{2}}_{2,2,4,6}(-r_0^2\Lambda^2) + \tfrac{1}{64}\log(r_0\Lambda) - \tfrac{1}{64}a\log(r_0M),\end{aligned} \tag{3.50}$$

where in the last line we have again dropped a vanishing term. Unlike the other hypergeometric functions we have encountered thusfar, $F^{1,1,\frac{7}{2}}_{2,2,4,6}$ does not appear with a factor of $\frac{1}{4}$ in its argument. Putting the two integrals together, we obtain

$$I^{(2)}_{c,c} = \frac{\lambda^2\phi_c^4}{6(48)\pi^2}\left(\frac{1}{12}r_0^2\Lambda^2F^{1,1,\frac{7}{2}}_{2,2,4,6}(-r_0^2\Lambda^2) - \log\frac{\Lambda}{M} + \frac{1}{2}\right). \tag{3.51}$$

Now if we define

$$\overline{F}^{1,1,\frac{7}{2}}_{2,2,4,6}(-u^2) \equiv F^{1,1,\frac{7}{2}}_{2,2,4,6}(-u^2) + \frac{12}{u^2}\left(\frac{11}{24} - \log\frac{u}{\sigma}\right); \quad \tau_2(u) \equiv \frac{1}{12}u^2\overline{F}^{1,1,\frac{7}{2}}_{2,2,4,6}(-u^2) \tag{3.52}$$

this becomes

$$I_{c,c}^{(2)} = \frac{\lambda^2\phi_c^4}{6(48)\pi^2}\left(\tau_2(r_0\Lambda) + \log\frac{r_0 M}{\sigma} + \frac{1}{24}\right). \tag{3.53}$$

3.3.5 Calculation of $I_{\rm tail}^{(1)}$

We now turn to the integrals involving $g_{\rm tail}(p)$, which are complicated by the presence of the scale $\frac{r_{\rm max}}{r_0}$. We start with one of the integrals with one power of $g_{\rm tail}(p)$, $I_{\rm tail}^{(1)}$ in eq. (3.22). Whereas this integral is tractable using the same method we employed to calculate the $g_{\rm core}(p)$ integrals (see Appendix C), two of the later integrals will require an additional trick. As such, we will introduce this new method here in the context of a relatively simpler calculation.

First we write the integral in terms of $u = r_0 p$,

$$I_{\rm tail}^{(1)} = \frac{\zeta\phi_c^2}{16 r_0^2}\left(\int_0^{r_0\Lambda} du \frac{u^5\overline{F}_{2,2,3}^{1,1}(-\frac{1}{4}u^2) - (\frac{r_{\rm max}}{r_0})^2 u^5\overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}(\frac{r_{\rm max}}{r_0}u)^2\right)}{u^2+(r_0M)^2}\right). \tag{3.54}$$

The identity in eq. (3.32) can then be used to separate out the ϕ_c^2 term, which we discard, leaving

$$I_{\rm tail}^{(1)} = -\frac{\zeta\phi_c^2 M^2}{16}\left(\int_0^{r_0\Lambda} du \frac{u^3\overline{F}_{2,2,3}^{1,1}(-\frac{1}{4}u^2) - (\frac{r_{\rm max}}{r_0})^2 u^3\overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}(\frac{r_{\rm max}}{r_0}u)^2\right)}{u^2+(r_0M)^2}\right) \tag{3.55}$$

We then integrate each term of the numerator separately:

$$I_{\rm tail}^{(1)} = -\frac{\zeta\phi_c^2 M^2}{16}\left(\int_0^{r_0\Lambda} du \frac{u^3\overline{F}_{2,2,3}^{1,1}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2} - \int_0^{r_0\Lambda} du \frac{(\frac{r_{\rm max}}{r_0})^2 u^3\overline{F}_{2,2,3}^{1,1}(-\frac{1}{4}(\frac{r_{\rm max}}{r_0}u)^2)}{u^2+(r_0M)^2}\right). \tag{3.56}$$

To evaluate the first integral here, we now introduce a new trick. First we recall the definition of $\overline{F}_{2,2,3}^{1,1}$ in eq. (3.19) on page 25 and the large and small argument expansions of $F_{2,2,3}^{1,1}$ given in Table 1 on page 26. At large u the extra terms of $\overline{F}_{2,2,3}^{1,1}$ remove the leading large-argument behavior of $F_{2,2,3}^{1,1}$ so that the integrand is $\sim u^{-\frac{7}{2}}\cos\left(\frac{5\pi}{4} - u\right)$ and thus the integral converges at the upper limit even if $r_0\Lambda \to \infty$. However, at the lower limit those same terms, $\frac{16}{u^2}\left(\frac{1}{2} - \log\frac{u}{\sigma}\right)$, are the leading terms of the small argument series expansion of $\overline{F}_{2,2,3}^{1,1}$. Simple power counting shows that only those terms cause divergences at small u as $r_0 M \to 0$. As such, we want to write the integrand at large u in terms of $\overline{F}_{2,2,3}^{1,1}$ but at small u we want to write the integrand in terms of $F_{2,2,3}^{1,1}$ and integrate the divergent terms separately. It turns out to be convenient to split the integral at $u = \sigma$

where we recall that σ, defined in eq. (3.20), is the natural dimensionless scale for the logarithms that appear in the asymptotic expansion of the hypergeometric function.

$$\begin{aligned}
\int_0^{r_0\Lambda} du \frac{u^3\overline{F}^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2} &= \int_0^{\sigma} du \frac{u^3F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)+16u\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{u^2+(r_0M)^2} + \int_{\sigma}^{r_0\Lambda} du \frac{u^3\overline{F}^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2} \\
&= \int_0^{\sigma} du \frac{u^3F^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2} + 16\int_0^{\sigma} du \frac{u\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{u^2+(r_0M)^2} \\
&\quad + \int_{\sigma}^{\infty} du \frac{u^3\overline{F}^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2} - \int_{r_0\Lambda}^{\infty} du \frac{u^3\overline{F}^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2}
\end{aligned} \tag{3.57}$$

In the first term here, the integral is well-behaved in the limit $r_0M \to 0$ so we may reduce the denominator. In the third and fourth terms we have $u > \sigma > 1 \gg (r_0M)$ so we may reduce the denominator there as well. Thus,

$$\begin{aligned}
\int_0^{r_0\Lambda} du \frac{u^3\overline{F}^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2+(r_0M)^2} &= \int_0^{\sigma} du\ uF^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2) + 16\int_0^{\sigma} du \frac{u\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{u^2+(r_0M)^2} \\
&\quad + \int_{\sigma}^{\infty} du\ u\overline{F}^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2) - \int_{r_0\Lambda}^{\infty} du\ u\overline{F}^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2).
\end{aligned} \tag{3.58}$$

The first and third terms reduce to pure numbers. Numerical integration gives

$$\int_0^{\sigma} du\ uF^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2) + \int_{\sigma}^{\infty} du\ u\overline{F}^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2) \approx 0.6226(48) + 3.37771 = 4.00000 \tag{3.59}$$

(In fact, an alternative method shows that this number is exactly 4; see Appendix.) In later calculations we will encounter similar pairs of definite integrals that sum to a rational number.

The second term of eq. (3.58) may be integrated analytically as given. The result includes a polylogarithm, of which we need to keep only the leading term,

$$\begin{aligned}
16\int_0^{\sigma} du \frac{u\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{u^2+(r_0M)^2} &= 4\log\left(1+\tfrac{\sigma^2}{r_0^2M^2}\right) - 4\ \mathrm{polylog}\left(2, -\tfrac{\sigma^2}{r_0^2M^2}\right) \\
&= \tfrac{2\pi^2}{3} - 8\log\tfrac{r_0M}{\sigma} + 8\log^2\tfrac{r_0M}{\sigma}.
\end{aligned} \tag{3.60}$$

The last term of eq. (3.58) is a function of $r_0\Lambda$ only, so we define

$$\tau_3(r_0\Lambda) \equiv \frac{1}{8}\int_{r_0\Lambda}^{\infty} du\ u\overline{F}^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2). \tag{3.61}$$

Combining these results, eq. (3.58) becomes

$$\int_0^{r_0\Lambda} du \frac{u^3 \overline{F}^{1,1}_{2,2,3}(-\frac{1}{4}u^2)}{u^2 + (r_0 M)^2} = 4 + \frac{2\pi}{3} - 8\log\frac{r_0 M}{\sigma} + 8\log^2\frac{r_0 M}{\sigma} - 8\tau_3(r_0\Lambda). \tag{3.62}$$

This is the first integral in $I^{(1)}_{\text{tail}}$, eq. (3.56).

Now to deal with the second integral in $I^{(1)}_{\text{tail}}$, we will first show that the upper limit $r_0\Lambda$ may be taken to ∞ with negligible change to the value of the integral. Then a change of variable $v = \frac{r_{\text{max}}}{r_0}$ transforms the integral into a function of $r_{\text{max}}M$ only.

Since the argument of $\overline{F}^{1,1}_{2,2,3}$ is scaled by the factor $\frac{r_{\text{max}}}{r_0}$, the integrand is highly suppressed unless u is small. In fact, as we now show, the constribution from the region $u > (r_0 M)^a$ $(0 < a < 1)$ is negligible. In that region we have $\frac{r_{\text{max}}}{r_0}u > \frac{r_{\text{max}}M}{(r_0 M)^{(1-a)}} \gg 1$ so we may replace $\overline{F}^{1,1}_{2,2,3}$ with its large-argument form.

$$-\left(\tfrac{r_{\text{max}}}{r_0}\right)^2 \int_{(r_0 M)^a}^{r_0\Lambda} du\ u\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}(\tfrac{r_{\text{max}}}{r_0}u)^2\right) = 32\sqrt{\tfrac{2}{\pi}}\left(\tfrac{r_0}{r_{\text{max}}}\right)^{\frac{5}{2}} \int_{(r_0 M)^a}^{r_0\Lambda} du\ u^{-\frac{7}{2}} \cos\left(\tfrac{5\pi}{4} - \tfrac{r_{\text{max}}}{r_0}u\right) \tag{3.63}$$

The oscillatory factor in the integrand only improves convergence of the integral, so the value of the integral with $\cos\left(\frac{5\pi}{4} - \frac{r_{\text{max}}}{r_0}u\right) \to 1$ sets an upper limit on the absolute value of the actual integral,

$$32\sqrt{\tfrac{2}{\pi}}\left(\tfrac{r_0}{r_{\text{max}}}\right)^{\frac{5}{2}} \int_{(r_0 M)^a}^{r_0\Lambda} du\ u^{-\frac{7}{2}} = \tfrac{64}{5}\sqrt{\tfrac{2}{\pi}}\left(\left(\tfrac{(r_0 M)^{1-a}}{r_{\text{max}}M}\right)^{\frac{5}{2}} - \left(\tfrac{1}{r_{\text{max}}\Lambda}\right)^{\frac{5}{2}}\right). \tag{3.64}$$

This integral then vanishes as $r_0 M \to 0$ and $r_{\text{max}}\Lambda \to \infty$ for $r_{\text{max}}M \gtrsim 1$.

As such, in the second integral in $I^{(1)}_{\text{tail}}$, eq. (3.56), we may change the upper limit of integration from $r_0\Lambda$ to ∞ without changing the value of the integral. We also change the variable of integration to $v = \frac{r_{\text{max}}}{r_0}u$,

$$-\left(\tfrac{r_{\text{max}}}{r_0}\right)^2 \int_0^{r_0\Lambda} du \frac{u^3 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\text{max}}}{r_0}u)^2\right)}{u^2 + (r_0 M)^2} = -\int_0^{\infty} dv \frac{v^3 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)}{v^2 + (r_{\text{max}}M)^2} \equiv -8T_1(r_{\text{max}}M) \tag{3.65}$$

and obtain a function of $r_{\text{max}}M$ only. Putting this and our result for the other integral, eq. (3.62), into eq. (3.56) we obtain

$$I_{\text{tail}}^{(1)} = \frac{1}{2}\zeta\phi_c^2 M^2 \left(\frac{1}{2} + \frac{\pi^2}{12} - \log\frac{r_0 M}{\sigma} + \log^2\frac{r_0 M}{\sigma} - \tau_3(r_0\Lambda) - T_1(r_{\max}M)\right). \tag{3.66}$$

3.3.6 Calculation of $I_{M,t}^{(2)}$

Now we perform the other integral with one power of $g_{\text{tail}}(p)$, $I_{M,t}^{(2)}$ in eq. (3.25). We change the variable of integration to $u = r_0 p$ then integrate the terms in the numerator separately,

$$I_{M,t}^{(2)} = \frac{\zeta\phi_c^2 g_M}{16}\left(\int_0^{r_0\Lambda} du \frac{u^5 \overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} - \int_0^{r_0\Lambda} du \frac{(\frac{r_{\max}}{r_0})^2 u^5 \overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}(\frac{r_{\max}}{r_0}u)^2\right)}{(u^2+(r_0M)^2)^2}\right). \tag{3.67}$$

We split the first integral at $u = \sigma$ then separate terms and reduce the denominators where appropriate, yielding

$$\begin{aligned}\int_0^{r_0\Lambda} du \frac{u^5 \overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} &= \int_0^{\sigma} du \frac{u^5 F_{2,2,3}^{1,1}\left(-\frac{1}{4}u^2\right) + 16u^3\left(\frac{1}{2} - \log\frac{u}{\sigma}\right)}{(u^2+(r_0M)^2)^2} + \int_\sigma^{r_0\Lambda} du \frac{u^5 \overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} \\ &= \int_0^{\sigma} du\ u F_{2,2,3}^{1,1}\left(-\tfrac{1}{4}u^2\right) + 16\int_0^{\sigma} du \frac{u^3\left(\frac{1}{2} - \log\frac{u}{\sigma}\right)}{(u^2+(r_0M)^2)^2} \\ &\quad + \int_\sigma^{\infty} du\ u\overline{F}_{2,2,3}^{1,1}\left(-\tfrac{1}{4}u^2\right) - \int_{r_0\Lambda}^{\infty} du\ u\overline{F}_{2,2,3}^{1,1}\left(-\tfrac{1}{4}u^2\right)\end{aligned} \tag{3.68}$$

Except for the second term, these are the integrals that appeared in eq. (3.58) during the calculation of $I_{\text{tail}}^{(1)}$. The second term is

$$16\int_0^{\sigma} du \frac{u^3\left(\frac{1}{2} - \log\frac{u}{\sigma}\right)}{(u^2+(r_0M)^2)^2} = -4 + \frac{2\pi^2}{3} + 8\log^2\frac{r_0 M}{\sigma}. \tag{3.69}$$

Putting this result and eqs. (3.59) and (3.61) into eq. (3.68), we obtain

$$\int_0^{r_0\Lambda} du \frac{u^5 \overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} = \frac{2\pi^2}{3} + 8\log^2\frac{r_0 M}{\sigma} - 8\tau_3(r_0\Lambda). \tag{3.70}$$

Now we calculate the second integral in $I_{M,t}^{(2)}$, eq. (3.67). Above $u = (r_0M)^a$ it reduces to the same neglible integral we bounded in eqs. (3.63) and (3.64) so as before we may change the upper limit of the integral from $r_0\Lambda$ to ∞. We also change the variable of integration to $v = \frac{r_{\max}}{r_0}u$,

$$-\left(\frac{r_{\max}}{r_0}\right)^2 \int_0^{r_0\Lambda} du \frac{u^5 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\max}}{r_0}u)^2\right)}{(u^2+(r_0M)^2)^2} = -\int_0^{\infty} dv \frac{v^5 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)}{(v^2+(r_{\max}M)^2)^2} \equiv -8T_2(r_{\max}M) \tag{3.71}$$

and are again left with a function of $r_{\max}M$ only. Putting this together with eq. (3.70) in eq. (3.67) we obtain

$$I^{(2)}_{M,t} = \frac{1}{2}\zeta\phi_c^2 g_M \left(\frac{\pi^2}{12} + \log^2\frac{r_0M}{\sigma} - \tau_3(r_0\Lambda) - T_2(r_{\max}M)\right). \tag{3.72}$$

3.3.7 Calculation of $I^{(2)}_{c,t}$

We have two integrals left to calculate. The first is $I^{(2)}_{c,t}$ as defined in eq. (3.27). In terms of $u = r_0M$ it is

$$\begin{aligned} I^{(2)}_{c,t} = \frac{\lambda\zeta\phi_c^4}{6} &\left(\int_0^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2}\right. \\ &\left. -(\tfrac{r_{\max}}{r_0})^2 \int_0^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\max}}{r_0}u)^2\right)}{(u^2+(r_0M)^2)^2}\right) \end{aligned} \tag{3.73}$$

To calculate the first integral, split it at $u = \sigma$ and reduce the denominator in the second piece,

$$\int_0^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} = \int_0^{\sigma} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} + \int_{\sigma}^{r_0\Lambda} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{u}. \tag{3.74}$$

Now we note that in the first piece if we series expand the numerator for small u,

$$J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right) = \left(\tfrac{u^2}{8}+\cdots\right)\left(\tfrac{16}{u^2}\left(\tfrac{1}{2}-\log\tfrac{u}{\sigma}\right)+\cdots\right) = 2\left(\tfrac{1}{2}-\log\tfrac{u}{\sigma}\right) + O(u^2, u^2\log u), \tag{3.75}$$

only the first term of the series creates a divergence at the lower limit as $r_0M \to 0$. As such, we remove that term and integrate it separately. The remaining integrand minus that term no longer requires the regulator so we may send it to zero, so that we have

$$\int_0^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} = \int_0^{\sigma} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right) \; - \; 2\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{u}$$
$$+2\int_0^{\sigma} du \frac{u^3\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{(u^2+(r_0M)^2)^2}$$
$$+\int_{\sigma}^{\infty} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{u} - \int_{r_0\Lambda}^{\infty} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{u}. \tag{3.76}$$

The first and third terms can be numerically integrated, yielding

$$\int_0^{\sigma} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right) \; - \; 2\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{u} + \int_{\sigma}^{\infty} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{u}$$
$$\approx -0.0297234 + 0.363057 = 0.333333 \tag{3.77}$$

We will accept without proof that the result is exactly $\frac{1}{3}$. The second term of eq. (3.76) can be calculated,

$$2\int_0^{\sigma} du \frac{u^3\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)}{(u^2+(r_0M)^2)^2} = -\frac{1}{2}+\frac{\pi^2}{12}+\log^2\frac{r_0M}{\sigma}, \tag{3.78}$$

(dropping terms that vanish) and the fourth term of eq. (3.76) is a function of $r_0\Lambda$ only,

$$\tau_4(r_0\Lambda) \equiv \int_{r_0\Lambda}^{\infty} du \frac{J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{u}. \tag{3.79}$$

Now putting eqs. (3.77), (3.78), and (3.79) into eq. (3.76), we obtain

$$\int_0^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)}{(u^2+(r_0M)^2)^2} = -\frac{1}{6}+\frac{\pi^2}{12}+\log^2\frac{r_0M}{\sigma} - \tau_4(r_0\Lambda) \tag{3.80}$$

for the first integral in $I^{(2)}_{c,t}$, eq. (3.73).

To deal with the second integral in $I^{(2)}_{c,t}$, we proceed as we did for $I^{(1)}_{\text{tail}}$ (see discussion below eq. (3.62).) We consider the contribution above $u = (r_0M)^a$,

$$-\left(\tfrac{r_{\max}}{r_0}\right)^2 \int_{(r_0M)^a}^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}\left(\frac{r_{\max}}{r_0}u\right)^2\right)}{(u^2+(r_0M)^2)^2}$$
$$= 32\sqrt{\tfrac{2}{\pi}}\left(\tfrac{r_0}{r_{\max}}\right)^{\frac{5}{2}} \int_{(r_0M)^a}^{r_0\Lambda} du\; u^{-\frac{7}{2}} \frac{J_2(u)}{u^2}\cos\left(\tfrac{5\pi}{4}-\tfrac{r_{\max}}{r_0}u\right). \tag{3.81}$$

As before we may obtain an upper limit for the absolute value of this integral by making the substitution $\cos\left(\frac{5\pi}{4} - \frac{r_{\rm max}}{r_0}u\right) \to 1$, and also setting $\frac{J_2(u)}{u^2} \to \frac{1}{8}$ since $\frac{J_2(u)}{u^2} < \frac{1}{8}$ for all positive u. Aside from the factor of $\frac{1}{8}$, the resulting integral is the same one we performed in eq. (3.64) so it is negligible and we may change the upper limit of the integral to ∞. Furthermore, we may replace $J_2(u)$ with the first term of its small-argument series expansion $\frac{u^2}{8}$ throughout the whole range of integration. This approximation is clearly justified for $u < (r_0 M)^a$, while the contribution of the region $u > (r_0 M)^a$ is negligible even with $J_2(u)$ replaced by $\frac{u^2}{8}$. Changing the variable of integration to $v = \frac{r_{\rm max}}{r_0}u$,

$$-(\frac{r_{\rm max}}{r_0})^2 \int_0^{r_0\Lambda} du \frac{u^3 J_2(u)\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\rm max}}{r_0}u)^2\right)}{(u^2+(r_0M)^2)^2} = -\frac{1}{8}\int_0^\infty dv \frac{v^5 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)}{(v^2+(r_{\rm max}M)^2)^2} = -T_2(r_{\rm max}M) \tag{3.82}$$

where $T_2(r_{\rm max}M)$ was defined in eq. (3.71) as part of calculating $I^{(2)}_{M,t}$. Putting eqs. (3.80) and (3.82) into eq. (3.73) we obtain the result

$$I^{(2)}_{c,t} = \frac{\lambda\zeta\phi_c^4}{6}\left(-\frac{1}{6}+\frac{\pi^2}{12}+\log^2\frac{r_0M}{\sigma} - \tau_4(r_0\Lambda) - T_2(r_{\rm max}M)\right). \tag{3.83}$$

3.3.8 Calculation of $I^{(2)}_{t,t}$

Finally we calculate $I^{(2)}_{t,t}$ as defined in eq. (3.27). In terms of $u = r_0 M$ it is

$$\begin{aligned} I^{(2)}_{t,t} = & -\frac{\pi^2\zeta^2\phi_c^4}{32}\left(\int_0^{r_0\Lambda} du \frac{u^7 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)^2}{(u^2+(r_0M)^2)^2}\right. \\ & \left. -(\frac{r_{\rm max}}{r_0})^2\int_0^{r_0\Lambda} du \frac{u^7\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\rm max}}{r_0}u)^2\right)\left(2\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right) - (\frac{r_{\rm max}}{r_0})^2\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\rm max}}{r_0}u)^2\right)\right)}{(u^2+(r_0M)^2)^2}\right). \end{aligned} \tag{3.84}$$

To calculate the first integral, split it at $u = \sigma$ and reduce the denominator in the second piece,

$$\int_0^{r_0\Lambda} du \frac{u^7\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)^2}{(u^2+(r_0M)^2)^2} = \int_0^{\sigma} du \frac{u^7\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)^2}{(u^2+(r_0M)^2)^2} + \int_\sigma^{r_0\Lambda} du\; u^3\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)^2. \tag{3.85}$$

Now we note that in the first piece if we series expand the numerator for small u,

$$\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2 = \left(\tfrac{16}{u^2}\left(\tfrac{1}{2}-\log\tfrac{u}{\sigma}\right)+\cdots\right)^2 = \tfrac{256}{u^4}\left(\tfrac{1}{2}-\log\tfrac{u}{\sigma}\right)^2 + O(u^{-2}, u^{-2}\log u), \tag{3.86}$$

only the first term of the series creates a divergence at the lower limit as $r_0M \to 0$. As such, we remove that term and integrate it separately. The remaining integrand minus that term no longer requires the regulator so we may send it to zero, so that we have

$$\begin{aligned}\int_0^{r_0\Lambda} du \frac{u^7 \overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2}{(u^2+(r_0M)^2)^2} = &\int_0^{\sigma} du\ u^3\left(\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2 - \frac{256}{u^4}\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)^2\right) \\ &+256\int_0^{\sigma} du \frac{u^3\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)^2}{(u^2+(r_0M)^2)^2} \\ &+\int_{\sigma}^{\infty} du\ u^3\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2 - \int_{r_0\Lambda}^{\infty} du\ u^3\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2.\end{aligned} \tag{3.87}$$

The first and third terms can be numerically integrated, yielding

$$\begin{aligned}&\int_0^{\sigma} du\ u^3\left(\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2 - \frac{256}{u^4}\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)^2\right) + \int_{\sigma}^{\infty} du\ u^3\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2 \\ &\approx 20.3623\ +\ 21.2815 = 16.0000 + \tfrac{64}{3}\zeta(3)\end{aligned} \tag{3.88}$$

which empirically is an integer plus a rational multiple of a particular value of the Riemann Zeta function. (The number $\zeta(3)$ is known to appear in higher-loop calculations in perturbation theory.) As before, we will take the above result to be exact without proof.

The second term of eq. (3.87) can be calculated,

$$256\int_0^{\sigma} du \frac{u^3\left(\frac{1}{2}-\log\frac{u}{\sigma}\right)^2}{(u^2+(r_0M)^2)^2} = -32\left(1-\left(2-\frac{2\pi^2}{3}\right)\log\frac{r_0M}{\sigma}+\frac{8}{3}\log^3\frac{r_0M}{\sigma}\right), \tag{3.89}$$

(expanding a polylog and dropping terms that vanish) and the fourth term of eq. (3.76) is a function of $r_0\Lambda$ only,

$$\tau_5(r_0\Lambda) \equiv \frac{1}{256}\int_{r_0\Lambda}^{\infty} du\ u^3\overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2. \tag{3.90}$$

Now putting eqs. (3.88), (3.89), and (3.90) into eq. (3.87), we obtain

$$\begin{aligned}\int_0^{r_0\Lambda} du \frac{u^7 \overline{F}^{1,1}_{2,2,3}\left(-\tfrac{1}{4}u^2\right)^2}{(u^2+(r_0M)^2)^2} = &-16\left(1-\tfrac{4}{3}\zeta(3)-4\left(1-\tfrac{\pi^2}{3}\right)\log\tfrac{r_0M}{\sigma}+\tfrac{16}{3}\log^3\tfrac{r_0M}{\sigma}\right) \\ &-256\tau_5(r_0\Lambda)\end{aligned} \tag{3.91}$$

for the first integral in $I^{(2)}_{t,t}$, eq. (3.84).

To calculate the second integral in $I^{(2)}_{t,t}$, we consider the contribution above $u = (r_0M)^a$. As before we may obtain an upper limit for the absolute value of the integral by making the substitution $\cos\left(\frac{5\pi}{4} - \frac{r_{\max}}{r_0}u\right) \to 1$, and also setting $F^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right) \to 1$ since throughout the range of integration $F^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right) < 1$. The resulting integral turns out to be negligible, so we may change the upper limit to ∞. Furthermore, we may replace $\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right)$ with the first term of its small-argument series expansion, $\frac{16}{u^2}\left(\frac{1}{2} - \log\frac{u}{\sigma}\right)$, throughout the whole range of integration. Changing the variable of integration to $v = \frac{r_{\max}}{r_0}u$,

$$
\begin{aligned}
&-\left(\tfrac{r_{\max}}{r_0}\right)^2 \int_0^{r_0\Lambda} du \frac{u^7 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\max}}{r_0}u)^2\right)\left(2\overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}u^2\right) - (\frac{r_{\max}}{r_0})^2 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}(\frac{r_{\max}}{r_0}u)^2\right)\right)}{(u^2 + (r_0 M)^2)^2} \\
&= -\int_0^{\infty} dv \frac{v^7 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)\left(\frac{32}{v^2}\left(\frac{1}{2} - \log\frac{v}{\sigma} + \log\frac{r_{\max}}{r_0}\right) - \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)\right)}{(v^2 + (r_{\max}M)^2)^2} \\
&= -256\left(\log\tfrac{r_{\max}}{r_0}\right) T_2(r_{\max}M) - 256 T_3(r_{\max}M)
\end{aligned}
\tag{3.92}
$$

where we have defined a new function of $r_{\max}M$ only,

$$
T_3(r_{\max}M) \equiv \frac{1}{256}\int_0^{\infty} dv \frac{v^7 \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)\left(\frac{32}{v^2}\left(\frac{1}{2} - \log\frac{v}{\sigma}\right) - \overline{F}^{1,1}_{2,2,3}\left(-\frac{1}{4}v^2\right)\right)}{(v^2 + (r_{\max}M)^2)^2}. \tag{3.93}
$$

Putting eqs. (3.91), (3.92), and (3.93) into eq. (3.84) we obtain the result

$$
\begin{aligned}
I^{(2)}_{t,t} = 8\pi^2\zeta^2\phi_c^4 \quad &\left[\tfrac{1}{16} - \tfrac{\zeta(3)}{12} + \left(\tfrac{\pi^2}{12} - \tfrac{1}{4}\right)\log\tfrac{r_0 M}{\sigma} + \tfrac{1}{3}\log^3\tfrac{r_0 M}{\sigma}\right. \\
&\left. + \tau_5(r_0\Lambda) + \left(\log\tfrac{r_{\max}}{r_0}\right) T_2(r_{\max}M) + T_3(r_{\max}M)\right].
\end{aligned}
\tag{3.94}
$$

3.4 Explicit Form of the Effective Potential

Having explicitly calculated all the integrals set out in eqs. (3.21-3.27), we now collect the results and assemble them to obtain the effective potential to second order in $g_{\text{core}}(p)$ and $g_{\text{tail}}(p)$.

First we assemble the core pieces and check that in the limit $\xi \to 0$ (hence $g_{\text{tail}}(p) \to 0$ and $\lambda_{\text{eff}} \to \lambda$) and $r_0\Lambda \to 0$ (where the core interaction becomes pointlike) we recover the standard result for the effective potential. We recall eqs. (3.28), (3.29), (3.38), (3.45), and (3.51), taking the limit $r_0\Lambda \to 0$ to obtain

$$
I^{(1)}_M = -\frac{g_M M^2}{16\pi^2}\log\frac{\Lambda}{M}; \qquad I^{(2)}_M = -\frac{g_M^2}{32\pi^2}\left(\log\frac{\Lambda}{M} - \frac{1}{2}\right), \tag{3.95}
$$

$$I^{(1)}_{\text{core}} = -\frac{\lambda\phi_c^2 M^2}{48\pi^2}\log\frac{\Lambda}{M}, \tag{3.96}$$

$$I^{(2)}_{M,c} = -\frac{\lambda\phi_c^2 g_M}{48\pi^2}\left(\log\frac{\Lambda}{M}-\frac{1}{2}\right); \qquad I^{(2)}_{c,c} = -\frac{\lambda^2\phi_c^4}{6(48)\pi^2}\left(\log\frac{\Lambda}{M}-\frac{1}{2}\right). \tag{3.97}$$

In the limit $\xi \to 0$ we have $g_M = \left(\frac{1}{6}-\alpha\right)\lambda\phi_c^2$ and $M^2 = \alpha\lambda\phi_c^2$ so that $g_M + M^2 = \frac{1}{6}\lambda\phi_c^2$, hence some of the alpha dependence cancels when all the terms are combined:

$$I^{(1)}_g + I^{(2)}_g = -\frac{\lambda^2\phi_c^4}{64\pi^2}\left(\alpha-\frac{1}{2}\right)\left[\alpha-\frac{1}{2}+\left(\alpha+\frac{1}{2}\right)\log\frac{\Lambda^2}{M^2}\right]. \tag{3.98}$$

The effective potential then becomes

$$V_{\text{eff}} = \tfrac{1}{4!}\lambda\phi^4 + \frac{\lambda^2\phi_c^4}{128\pi^2}\left(\frac{1}{2}+\alpha(\alpha-2)-\log\frac{\Lambda}{M}\right). \tag{3.99}$$

Now we employ the Principle of Minimal Sensitivity to determine the optimal value of α,

$$\frac{dV_{\text{eff}}}{d\alpha} = \frac{\lambda^2\phi_c^4}{128\pi^2}\left(2\alpha-2+\frac{1}{2\alpha}\right) = 0, \tag{3.100}$$

the solution of which is

$$\alpha = \frac{1}{2}. \tag{3.101}$$

This result corresponds to $M^2 = \frac{1}{2}\lambda\phi_c^2$, which is indeed the physical mass squared. For this value of α the sum of the extra terms shown in eq. (3.98) vanishes and the effective potential becomes

$$V_{\text{eff}} = \tfrac{1}{4!}\lambda\phi^4 + \frac{M^4}{64\pi^2}\left(\log\frac{M^2}{\Lambda^2}-\frac{1}{2}\right), \tag{3.102}$$

which is the standard result.

We now turn to the full effective potential, including the tail terms and written in a form convenient for the regime $r_0\Lambda \gtrsim 1$. First we summarize the results of the previous section, given in eqs. (3.28), (3.29), (3.40), (3.46), (3.53), (3.66), (3.72), (3.83), and (3.94):

$$I^{(1)}_M = -\frac{g_M M^2}{16\pi^2}\log\frac{\Lambda}{M}, \tag{3.103}$$

$$I_M^{(2)} = -\frac{g_M^2}{32\pi^2}\left(\log\frac{\Lambda}{M} - \frac{1}{2}\right), \tag{3.104}$$

$$I_{\rm core}^{(1)} = \frac{\lambda\phi_c^2 M^2}{48\pi^2}\left(\tau_1(r_0\Lambda) + \log\frac{r_0 M}{\sigma} - \frac{3}{4}\right), \tag{3.105}$$

$$I_{M,c}^{(2)} = \frac{\lambda\phi_c^2 g_M}{48\pi^2}\left(\tau_1(r_0\Lambda) + \log\frac{r_0 M}{\sigma} - \frac{1}{4}\right), \tag{3.106}$$

$$I_{c,c}^{(2)} = \frac{\lambda^2\phi_c^4}{6(48)\pi^2}\left(\tau_2(r_0\Lambda) + \log\frac{r_0 M}{\sigma} + \frac{1}{24}\right), \tag{3.107}$$

$$I_{\rm tail}^{(1)} = \frac{1}{2}\zeta\phi_c^2 M^2\left(\frac{1}{2} + \frac{\pi^2}{12} - \log\frac{r_0 M}{\sigma} + \log^2\frac{r_0 M}{\sigma} - \tau_3(r_0\Lambda) - T_1(r_{\max}M)\right), \tag{3.108}$$

$$I_{M,t}^{(2)} = \frac{1}{2}\zeta\phi_c^2 g_M\left(\frac{\pi^2}{12} + \log^2\frac{r_0 M}{\sigma} - \tau_3(r_0\Lambda) - T_2(r_{\max}M)\right), \tag{3.109}$$

$$I_{c,t}^{(2)} = \frac{\lambda\zeta\phi_c^4}{6}\left(-\frac{1}{6} + \frac{\pi^2}{12} + \log^2\frac{r_0 M}{\sigma} - \tau_4(r_0\Lambda) - T_2(r_{\max}M)\right), \tag{3.110}$$

$$\begin{aligned} I_{t,t}^{(2)} = 8\pi^2\zeta^2\phi_c^4 \quad & \Big[\tfrac{1}{16} - \tfrac{\zeta(3)}{12} + \left(\tfrac{\pi^2}{12} - \tfrac{1}{4}\right)\log\tfrac{r_0 M}{\sigma} + \tfrac{1}{3}\log^3\tfrac{r_0 M}{\sigma} \\ & + \tau_5(r_0\Lambda) + \left(\log\tfrac{r_{\max}}{r_0}\right)T_2(r_{\max}M) + T_3(r_{\max}M)\Big]. \end{aligned} \tag{3.111}$$

Some of the above expressions include the special functions defined in eqs. (3.39), (3.52), (3.61), (3.79), (3.90), (3.65), (3.71), (3.93):

$$\tau_1(r_0\Lambda) \equiv \frac{1}{24}r_0^2\Lambda^2\overline{F}_{2,2,4}^{1,1}\left(-\tfrac{1}{4}r_0^2\Lambda^2\right); \qquad \tau_2(r_0\Lambda) \equiv \frac{1}{12}r_0^2\Lambda^2\overline{F}_{2,2,4,6}^{1,1,\frac{7}{2}}(-r_0^2\Lambda^2), \tag{3.112}$$

$$\tau_3(r_0\Lambda) \equiv \frac{1}{8}\int_{r_0\Lambda}^{\infty} du\; u\overline{F}_{2,2,3}^{1,1}(-\tfrac{1}{4}u^2); \qquad \tau_4(r_0\Lambda) \equiv \int_{r_0\Lambda}^{\infty} du\frac{J_2(u)\overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}u^2\right)}{u}, \tag{3.113}$$

$$\tau_5(r_0\Lambda) \equiv \frac{1}{256}\int_{r_0\Lambda}^{\infty} du\; u^3\overline{F}_{2,2,3}^{1,1}\left(-\tfrac{1}{4}u^2\right)^2, \tag{3.114}$$

$$T_1(r_{\max}M) \equiv \frac{1}{8}\int_0^{\infty} dv\frac{v^3\overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}v^2\right)}{v^2 + (r_{\max}M)^2}; \qquad T_2(r_{\max}M) \equiv \frac{1}{8}\int_0^{\infty} dv\frac{v^5\overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}v^2\right)}{(v^2 + (r_{\max}M)^2)^2}, \tag{3.115}$$

$$T_3(r_{\max}M) \equiv \frac{1}{256}\int_0^{\infty} dv\frac{v^7\overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}v^2\right)\left(\frac{32}{v^2}\left(\frac{1}{2} - \log\frac{v}{\sigma}\right) - \overline{F}_{2,2,3}^{1,1}\left(-\frac{1}{4}v^2\right)\right)}{(v^2 + (r_{\max}M)^2)^2}. \tag{3.116}$$

The functions $\tau_i(r_0\Lambda)$ are finite for $r_0\Lambda \gtrsim 1$ and vanish in the limit $r_0\Lambda \to \infty$. The functions $T_i(r_{\max}M)$ are also finite for $r_{\max}M \gtrsim 1$ and vanish in the limit $r_{\max}M \to \infty$.

The full result for the effective potential, expressed in terms of λ_{eff} and ζ, is

$$
\begin{aligned}
V_{\text{eff}} = \tfrac{1}{4!}\phi_c^4 \quad \Big[\lambda_{\text{eff}} & \\
+\tfrac{1}{48\pi^2}\lambda_{\text{eff}}^2 \quad & \Big[-\tfrac{1}{3} + 9\alpha(\alpha-2) \\
& -\log\tfrac{\Lambda}{M} - 8\log\tfrac{r_{\text{max}}}{r_0} + 8\log\tfrac{r_{\text{max}}M}{\sigma} \\
& +4\tau_1(r_0\Lambda) + 4\tau_2(r_0\Lambda)\Big] \\
+\tfrac{2}{3}\lambda_{\text{eff}}\zeta \quad & \Big[-1 + \tfrac{3\pi^2}{4} + 9\alpha \\
& -\log\tfrac{r_{\text{max}}}{r_0} - 9\log^2\tfrac{r_{\text{max}}}{r_0} \\
& -18\alpha\log\tfrac{r_{\text{max}}M}{\sigma} + 9\log^2\tfrac{r_{\text{max}}M}{\sigma} \\
& -18\alpha T_1(r_{\text{max}}M) + 18(\alpha-\tfrac{1}{2})T_2(r_{\text{max}}M) \\
& +6\log\tfrac{r_{\text{max}}}{r_0}\left(\tau_1(r_0\Lambda)+2\tau_2(r_0\Lambda)\right) - 3\tau_3(r_0\Lambda) - 6\tau_4(r_0\Lambda)\Big] \\
+8\pi^2\zeta^2 \quad & \Big[\tfrac{3}{2} - 2\zeta(3) \\
& +2\log\tfrac{r_{\text{max}}}{r_0} + \log^2\tfrac{r_{\text{max}}}{r_0} - 8\log^3\tfrac{r_{\text{max}}}{r_0} \\
& +2(\pi^2-3)\log\tfrac{r_{\text{max}}M}{\sigma} + 8\log^3\tfrac{r_{\text{max}}M}{\sigma} \\
& +24T_3(r_{\text{max}}M) \\
& +24\log^2\tfrac{r_{\text{max}}}{r_0}\tau_2(r_0\Lambda) - 24\log\tfrac{r_{\text{max}}}{r_0}\tau_4(r_0\Lambda) + 24\tau_5(r_0\Lambda)\Big]\Big].
\end{aligned}
\tag{3.117}
$$

It is worth noting that only the λ_{eff}^2 and $\lambda_{\text{eff}}\zeta$ terms depend on α, and that for the special value $\alpha = \frac{1}{2}$ one of the terms in $\lambda_{\text{eff}}\zeta$ drops out.

4 Renormalization of the Effective Potential

We will consider two different cases for the relative magnitudes of the model's parameters. For each of these cases we will investigate both perturbative and autonomous renormalization strategies.

First, we will consider the case where $r_0\Lambda$ and $r_{\text{max}}M_v$ are of finite order. Since M depends on ϕ_c while r_{max} is a constant independent of ϕ_c, what we really mean is that $r_{\text{max}}M_v$ is finite, where M_v is the vacuum value. In this case all of the logarithms may be cast in terms of a divergent part proportional to powers of $\log\frac{r_{\text{max}}}{r_0}$ plus a finite part. We will consider both perturbative and autonomous approaches in this case.

Next we will consider the case where $r_{\text{max}} \to \infty$ first. The motivation for studying this case is that the r_{max} cutoff may not be needed; the screening mechanism seems to be inherent in the calculation of the effective potential. Again, we will renormalize the effective potential using both a perturbative and an autonomous approach.

4.1 Perturbative Approach, with $r_0\Lambda$ and $r_{\max}M_v$ of Finite Order

One possible perturbative renormalization prescription in this case is

$$\begin{aligned}
\lambda_{\text{eff}}^{R} = \; & \lambda_{\text{eff}} \; -\tfrac{1}{48\pi^2}\lambda_{\text{eff}}^2\left[A_1+\log\tfrac{\Lambda}{\mu}+8\log\tfrac{r_{\max}}{r_0}\right] \\
& -\tfrac{2}{3}\lambda_{\text{eff}}\zeta\left[A_2+\log\tfrac{r_{\max}}{r_0}\left(1-6\left(\tau_1(r_0\Lambda)+2\tau_2(r_0\Lambda)\right)\right)+9\log^2\tfrac{r_{\max}}{r_0}\right] \\
& +8\pi^2\zeta^2\left[A_3+2\log\tfrac{r_{\max}}{r_0}\left(1-12\tau_4(r_0\Lambda)\right)+\log^2\tfrac{r_{\max}}{r_0}\left(1+24\tau_2(r_0\Lambda)\right)-8\log^3\tfrac{r_{\max}}{r_0}\right]+\cdots \\
\zeta_R = \; & \zeta \; + \cdots
\end{aligned} \tag{4.1}$$

where λ_{eff}^{R} and ζ_R are finite renormalized parameters, A_i are finite but otherwise arbitrary numerical constants (representing renormalization scheme dependence) and the "$+\cdots$" terms correspond to higher-order terms in λ_{eff} and ξ beyond the order of the present calculation. Other prescriptions are possible, as terms can be shuffled from the series for one parameter to the series for the other parameter, but we will not address that issue here, though we will return to it in Section 4.3.

We may invert these relations perturbatively and hence eliminate λ_{eff} and ξ in favor of λ_{eff}^{R} and ξ_R, then drop terms higher order in λ_{eff}^{R} and ξ_R. The effective potential is now finite when expressed to second order in these renormalized parameters, and it has the form:

$$
\begin{aligned}
V_{\text{eff}} = \tfrac{1}{4!}\phi_c^4 \quad & \Big[\lambda_{\text{eff}}^R \\
& +\tfrac{1}{48\pi^2}\lambda_{\text{eff}}^{R\,2} \quad \Big[-\tfrac{1}{3} + 9\alpha(\alpha-2) \\
& \qquad +\log\tfrac{M}{\mu} + 8\log\tfrac{r_{\max}M}{\sigma} \\
& \qquad +4\tau_1(r_0\Lambda) + 4\tau_2(r_0\Lambda)\Big] \\
& +\tfrac{2}{3}\lambda_{\text{eff}}^R\zeta_R \quad \Big[-1 + \tfrac{3\pi^2}{4} + 9\alpha \\
& \qquad -18\alpha\log\tfrac{r_{\max}M}{\sigma} + 9\log^2\tfrac{r_{\max}M}{\sigma} \\
& \qquad -18\alpha T_1(r_{\max}M) + 18(\alpha - \tfrac{1}{2})T_2(r_{\max}M) \\
& \qquad -3\tau_3(r_0\Lambda) - 6\tau_4(r_0\Lambda)\Big] \\
& +8\pi^2\zeta_R^2 \quad \Big[\tfrac{3}{2} - 2\zeta(3) \\
& \qquad +2(\pi^2-3)\log\tfrac{r_{\max}M}{\sigma} + 8\log^3\tfrac{r_{\max}M}{\sigma} \\
& \qquad +24T_3(r_{\max}M) \\
& \qquad +24\tau_5(r_0\Lambda)\Big]\Big].
\end{aligned} \tag{4.2}
$$

Taking into account the ϕ_c dependence of M, we observe that the effective potential contains terms in ϕ_c^4, $\phi_c^4\log\phi_c^2$, $\phi_c^4\log^2\phi_c^2$, and $\phi_c^4\log^3\phi_c^2$. The special functions T_i also depend on M and hence ϕ_c, since at this order we have $M^2 = \alpha\lambda_{\text{eff}}^R\phi_c^2$.

It is possible at this point to take the derivative with respect to ϕ_c to find the symmetry breaking minimum of the effective potential, if any exists. (This calculation would also involve taking the derivative with respect to M of the functions T_i, which would generate other special functions of $r_{\max}M$.) We would then have to resort to numerical methods to determine where the equation $\frac{dV_{\text{eff}}}{d\phi_c} = 0$ has solutions, which is a necessary condition for spontaneous symmetry breaking. If the transition is second-order (the second derivative of the effective potential at the origin vanishes when the nontrivial minimum first appears) this condition is also sufficient to obtain symmetry breaking. If the transition is first-order we would need to require that the nontrivial local minimum be a global minimum, which would set a stronger, necessary and sufficient, condition on the parameters.

We will not pursue this line of analysis further, as it does not look as promising as the other alternatives. The lack of a simple, clear result in this case may indicate that putting in the $r_{\max}$ cutoff by hand is a poor choice, and that the screening mechanism is indeed already taken into account in the calculations.

4.2 Autonomous Approach, with $r_0\Lambda$ and $r_{\max}M_v$ of Finite Order

Recall that in eq. (3.6) we defined a new parameter $\xi \equiv 48\pi^2\zeta\log\frac{r_{\max}}{r_0}$ that could naturally be viewed as comparable to λ (recalling the definition $\lambda_{\text{eff}} \equiv \lambda - \xi$). If we regard λ and ξ as being of the same order, then any term in ζ that does not also carry at least one factor of $\log\frac{r_{\max}}{r_0}$ for each power of ζ will tend to zero in the limit $\Lambda \to \infty$ ($r_0 \to 0$) and thus may be discarded. The effective potential written in terms of ξ, and using the identity $\log\frac{\Lambda}{M} = \log\frac{r_0\Lambda}{\sigma} - \log\frac{r_{\max}M}{\sigma} + \log\frac{r_{\max}}{r_0}$, then simplifies to

$$\begin{aligned} V_{\text{eff}} = \tfrac{1}{4!}\phi_c^4 \quad & \Big[\lambda_{\text{eff}} \\ & +\tfrac{3}{16\pi^2}\lambda_{\text{eff}}^2 \quad \left[-\tfrac{1}{27} + \alpha(\alpha-2) - \log\tfrac{r_{\max}}{r_0} + \log\tfrac{r_{\max}M}{\sigma} - \tfrac{1}{9}\log\tfrac{r_0\Lambda}{\sigma} + \tfrac{4}{9}\left(\tau_1(r_0\Lambda) + \tau_2(r_0\Lambda)\right)\right] \\ & +\tfrac{1}{8\pi^2}\lambda_{\text{eff}}\xi \quad \left[-\tfrac{1}{9} - \log\tfrac{r_{\max}}{r_0} + \tfrac{2}{3}\left(\tau_1(r_0\Lambda) + 2\tau_2(r_0\Lambda)\right)\right] \\ & +\tfrac{1}{36\pi^2}\xi^2 \quad \left[\tfrac{1}{8} - \log\tfrac{r_{\max}}{r_0} + 3\tau_2(r_0\Lambda)\right]\Big] . \end{aligned} \tag{4.3}$$

We note that the effective potential only contains terms proportional to ϕ_c^4 and $\phi_c^4\log\phi_c^2$, with the latter coming from the term in $\log\frac{r_{\max}M}{\sigma}$. As in the core case, α only appears in the coefficient of ϕ_c^4, so its value is immaterial as it would disappear if we were to express the effective potential in terms of the vacuum value of ϕ_c. Nonetheless, the Principle of Minimal Sensitivity fixes $\alpha = \frac{1}{2}$ just as in eqs. (3.99)-(3.101). The $\phi_c^4\log\phi_c^2$ term can be rewritten as

$$\frac{1}{4!}\phi_c^4\left(\frac{3}{16\pi^2}\lambda_{\text{eff}}^2\right)\log M = \frac{1}{128\pi^2}\left(4M^4\right)\left(\frac{1}{2}\right)\log M^2 = \frac{1}{64\pi^2}M^4\log M^2, \tag{4.4}$$

which we recognize from eq. (2.36) with $C_1 = \frac{1}{64\pi^2}$. The renormalized effective potential is thus of the same form as for the standard $\lambda\phi^4$ Lagrangian. Since in this case we also have $\alpha = \frac{1}{2}$, from eq. (2.44) the relation of the physical mass M to v_R and the renormalization prescription for λ and ϕ_c are also the same as for the standard theory. The constant C_2 is different, however, so from eq. (2.39) with $\mu_2 = M_v$ we can see that the relation of the scale μ_1 to the physical mass M_v is different.

4.3 Perturbative Approach, with $r_0\Lambda$ of Finite Order and $r_{\max}M \to \infty$

A perturbative renormalization procedure is only suitable for removing ultraviolet divergences, whereas the divergences associated with the limit $r_{\max} \to \infty$ are long-range and hence infrared in nature. As such, we must cancel these divergences by other means before renormalization. This can be done, as in the previous subsection, by letting ζ be infinitesimal so that $\xi \equiv 48\pi^2\zeta \log \frac{r_{\max}}{r_0}$ is finite. We first show that this is indeed the only possibility.

First we collect all the $r_{\max}$ dependence of the effective potential (3.117) into powers of $\log \frac{r_{\max}}{r_0}$, using the relation $\log \frac{r_{\max}M}{\sigma} = \log \frac{r_{\max}}{r_0} + \log \frac{r_0 M}{\sigma}$ when needed. There is then a cancellation of $\log^2 \frac{r_{\max}}{r_0}$ terms in the $\lambda_{\text{eff}}\zeta$ term and of $\log^3 \frac{r_{\max}}{r_0}$ terms in the ζ^2 term. For the effective potential to be infrared finite (considering r_0 as nonvanishing for the time being), we need:

$$\begin{aligned}
\tfrac{2}{3}\lambda_{\text{eff}}\zeta \quad & \Big[-(1+18\alpha)\log \tfrac{r_{\max}}{r_0} + 18\log\tfrac{r_0M}{\sigma}\log\tfrac{r_{\max}}{r_0} \\
& +6\log\tfrac{r_{\max}}{r_0}\left(\tau_1(r_0\Lambda)+2\tau_2(r_0\Lambda)\right)\Big] \\
+8\pi^2\zeta^2 \quad & \Big[2(\pi^2-2)\log\tfrac{r_{\max}}{r_0} \\
& +\log^2\tfrac{r_{\max}}{r_0} + 24\log\tfrac{r_0M}{\sigma}\log^2\tfrac{r_{\max}}{r_0} + 24\log^2\tfrac{r_0M}{\sigma}\log\tfrac{r_{\max}}{r_0} \\
& +24\tau_2(r_0\Lambda)\log^2\tfrac{r_{\max}}{r_0} - 24\tau_4(r_0\Lambda)\log\tfrac{r_{\max}}{r_0}\Big] \\
= \text{finite}, &
\end{aligned} \tag{4.5}$$

as $r_{\max} \to \infty$. This relation simplifies to

$$\frac{2}{3}\zeta \log\frac{r_{\max}}{r_0}\left[C_1\lambda_{\text{eff}} + 12\pi^2\zeta\left(C_2\log\frac{r_{\max}}{r_0} + C_3\right)\right] = \text{finite} \tag{4.6}$$

where the infrared-finite constants C_i are given by

$$C_1 \equiv -1 - 18\alpha + 18\log\frac{r_0M}{\sigma} + 6\left(\tau_1(r_0\Lambda) + 2\tau_2(r_0\Lambda)\right) \tag{4.7}$$

$$C_2 \equiv 1 + 24\log\frac{r_0M}{\sigma} + 24\tau_2(r_0\Lambda) \tag{4.8}$$

$$C_3 \equiv 2(\pi^2-2) + 24\log^2\frac{r_0M}{\sigma} - 24\tau_4(r_0\Lambda). \tag{4.9}$$

The important point is that the logarithms of $r_{\max}$ always appear with a factor of ζ, so we may substitute ξ for ζ and thus absorb all of the infrared divergences. The terms in C_1 and C_2 are the only terms proportional to ζ that are not subleading.

The effective potential in terms of ξ, dropping terms that vanish as $r_{\max} \to \infty$, is

$$
\begin{aligned}
V_{\text{eff}} = \tfrac{1}{4!}\phi_c^4 \Big[\lambda_{\text{eff}} & \\
+\tfrac{1}{48\pi^2}\lambda_{\text{eff}}^2 & \Big[-\tfrac{1}{3} + 9\alpha(\alpha-2) \\
& - \log\tfrac{\Lambda}{M} + 8\log\tfrac{r_0 M}{\sigma} \\
& + 4\tau_1(r_0\Lambda) + 4\tau_2(r_0\Lambda) \Big] \\
+\tfrac{2}{3(48\pi^2)}\lambda_{\text{eff}}\xi & \Big[-1 - 18\alpha + 18\log\tfrac{r_0 M}{\sigma} \\
& + 6\left(\tau_1(r_0\Lambda) + 2\tau_2(r_0\Lambda)\right) \Big] \\
+\tfrac{1}{6(48\pi^2)}\xi^2 & \Big[1 + 24\log\tfrac{r_0 M}{\sigma} + 24\tau_2(r_0\Lambda) \Big]\Big] .
\end{aligned}
\tag{4.10}
$$

This expression is different from eq. (4.3) because different terms are subleading in this limit. The value of α does not affect the perturbative renormalization prescription, as α does not appear in the coefficients of any divergent terms. The Principle of Minimal Sensitivity provides a simple expression for α,

$$\frac{dV_{\text{eff}}}{d\alpha} = \frac{\partial V_{\text{eff}}}{\partial\alpha} + \left(\frac{M}{2\alpha}\right)\frac{\partial V_{\text{eff}}}{\partial M} = \frac{1}{2304\pi^2\alpha}\left(3 - 6\alpha + 2\frac{\xi}{\lambda_{\text{eff}}}\right)^2 \lambda_{\text{eff}}^2\phi_c^4 = 0 \tag{4.11}$$

$$\alpha = \frac{1}{6}(3 + 2\eta); \quad \eta \equiv \frac{\xi}{\lambda_{\text{eff}}}. \tag{4.12}$$

This reduces to $\alpha = \frac{1}{2}$ in the limit $\xi \to 0$.

Though it is not immediately apparent, this value of α corresponds to $M^2 = \alpha\lambda_{\text{eff}}\phi_c^2$ being the physical mass. To see this we must find the pole in the propagator in the limit $r_{\max} \to \infty$. Alternatively we must see where the inverse propagator, eqs. (3.9) and (3.10), vanishes. We will find that the propagator has a discontinuity at zero momentum.

For the zero-momentum mode, $g(p)$ reduces to

$$
\begin{aligned}
g(p=0) &= \left(\tfrac{1}{6} - \alpha\right)(\lambda - \xi)\phi_c^2 + 8\phi_c^2 \int d^4x\, U_{\text{equiv}}(x) \\
&= \left(\tfrac{1}{6} - \alpha\right)(\lambda_{\text{eff}})\phi_c^2 + \tfrac{1}{3}(\lambda_{\text{eff}})\phi_c^2 \\
&= \left(\tfrac{1}{2} - \alpha\right)\lambda_{\text{eff}}\phi_c^2 .
\end{aligned}
\tag{4.13}
$$

The inverse propagator at zero momentum is

$$\begin{aligned} G(p=0)^{-1} &= M^2 + g(p) \\ &= \alpha\lambda_{\text{eff}}\phi_c^2 + \left(\tfrac{1}{2}-\alpha\right)\lambda_{\text{eff}}\phi_c^2 \\ &= \tfrac{1}{2}\lambda_{\text{eff}}\phi_c^2. \end{aligned} \tag{4.14}$$

However, for all other modes $g(p)$ reduces differently if we take the limit $r_{\max} \to \infty$ first. Using eqs. (3.17) and (3.18) then writing the expression in terms of ξ,

$$g(p) \;= \left(\tfrac{1}{6}-\alpha\right)(\lambda-\xi)\phi_c^2 + \tfrac{8}{3}\lambda\phi_c^2\frac{J_2(r_0p)}{(r_0p)^2} + \pi^2\xi\left(\log\frac{r_{\max}}{r_0}\right)^{-1}\phi_c^2\left(u^2\overline{F}^{1,1}_{2,2,3}(-\tfrac{1}{4}u^2)\right)\Big|_{u=r_0p}^{r_{\max}p} \tag{4.15}$$

In the limit $r_{\max} \to \infty$ we have $p \gg \frac{1}{r_{\max}}$ for all modes except the zero-momentum mode, so the last term tends to zero. (Neglecting this term requires $\frac{\log\frac{r_0M}{\sigma}}{\log\frac{r_{\max}M}{\sigma}} \ll 1$, so that we need $\frac{r_{\max}}{r_0}$ to tend to infinity faster than any power of $\frac{\Lambda}{M}$.)

If as a second step we take the $r_0 \to 0$ limit and assume $p \ll \frac{1}{r_0}$, the Bessel function may be approximated by the first term of its series expansion, and we obtain

$$\begin{aligned} g(p) &= \left(\tfrac{1}{6}-\alpha\right)\lambda_{\text{eff}}\phi_c^2 + \tfrac{1}{3}(\lambda_{\text{eff}}+\xi)\phi_c^2 \\ &= \left(\tfrac{1}{2}-\alpha\right)\lambda_{\text{eff}}\phi_c^2 + \tfrac{1}{3}\xi\phi_c^2. \end{aligned} \tag{4.16}$$

The inverse propagator for finite momenta ($p \neq 0$ and $p \ll \frac{1}{r_0}$) is then

$$G(p)^{-1} = p^2 + \frac{1}{6}\left(3+2\eta\right)\lambda_{\text{eff}}\phi_c^2 \tag{4.17}$$

which corresponds to a physical mass M^2 with α given by eq. (4.12). As mentioned earlier, the $p \to 0$ limit of this expression is different from the result found earlier for the zero-momentum mode, so there is a discontinuity in the propagator at zero momentum.

The perturbative renormalization prescription has some ambiguity, which is best seen by looking at the effective potential in terms of λ and ξ. Using the above value for α and writing all the divergent logarithms in terms of $\log\frac{r_0 M}{\sigma}$, it is

$$
\begin{aligned}
V_{\text{eff}} = \tfrac{1}{4!}\phi_c^4 \Big[\lambda - \xi & \\
+\tfrac{1}{48\pi^2}\lambda^2 & \Big[-\tfrac{85}{12} \\
& +9\log\tfrac{r_0 M}{\sigma} - \log\tfrac{r_0\Lambda}{\sigma} \\
& +4\tau_1(r_0\Lambda) + 4\tau_2(r_0\Lambda)\Big] \\
+\tfrac{2}{3(48\pi^2)}\lambda\xi & \Big[+\tfrac{27}{4} - 9\log\tfrac{r_0 M}{\sigma} + 3\log\tfrac{r_0\Lambda}{\sigma} \\
& -6\tau_1(r_0\Lambda)\Big] \\
+\tfrac{1}{6(48\pi^2)}\xi^2 & \Big[-\tfrac{3}{2} + 6\log\tfrac{r_0 M}{\sigma} - 6\log\tfrac{r_0\Lambda}{\sigma}\Big]\Big] .
\end{aligned}
\tag{4.18}
$$

In order to make this expression finite, we must set

$$
\begin{aligned}
\lambda_R - \xi_R = \quad \lambda - \xi + \tfrac{1}{48\pi^2} & \Big[\lambda^2\left(A_1 + 9\log\tfrac{r_0\mu}{\sigma}\right) \\
& \lambda\xi\left(A_2 - 6\log\tfrac{r_0\mu}{\sigma}\right) \\
& \xi^2\left(A_3 + \log\tfrac{r_0\mu}{\sigma}\right)\Big] + \cdots
\end{aligned}
\tag{4.19}
$$

where A_i are finite numerical constants. However, this does not give us a prescription for λ_R and ξ_R separately. We require $\lambda_R = \lambda + \cdots$ and $\xi_R = \xi + \cdots$, where the "$+\cdots$" terms begin at second order in λ and ξ respectively. However, the partitioning of the divergences between the two series is at this stage undetermined. Substituting eq. (4.19) into eq. (4.18) and re-expanding to second order in λ_R and ξ_R, the renormalized effective potential is

$$
\begin{aligned}
V_{\text{eff}} = \tfrac{1}{4!}\phi_c^4 \Big[\lambda_R - \xi_R & \\
+\tfrac{1}{48\pi^2}\lambda_R^2 & \Big[-A_1 - \tfrac{85}{12} + 9\log\tfrac{M}{\mu} - \log\tfrac{r_0\Lambda}{\sigma} + 4\tau_1(r_0\Lambda) + 4\tau_2(r_0\Lambda)\Big] \\
+\tfrac{2}{3(48\pi^2)}\lambda_R\xi_R & \Big[-\tfrac{3}{2}A_2 + \tfrac{27}{4} - 9\log\tfrac{M}{\mu} + 3\log\tfrac{r_0\Lambda}{\sigma} - 6\tau_1(r_0\Lambda)\Big] \\
+\tfrac{1}{6(48\pi^2)}\xi_R^2 & \Big[-6A_3 - \tfrac{3}{2} + 6\log\tfrac{M}{\mu} - 6\log\tfrac{r_0\Lambda}{\sigma}\Big]\Big] .
\end{aligned}
\tag{4.20}
$$

Any one of the arbitrary constants A_i may be absorbed into a redefinition of the scale μ without loss of generality. Since the renormalized effective potential only has terms in ϕ_c^4 and $\phi_c^4\log\phi^2$, it has a nontrivial minimum and hence symmetry breaking.

If we wish to proceed to obtain the Renormalization-Group Improved effective potential, we need to separate out the series eq. (4.19) into a series for each parameter. In order to do this we must make additional assumptions: (i.) Physically we expect the short-range core interaction coupling λ to only be renormalized by divergences associated with the short-range interaction, since a short-range interaction modified by a long-range interaction would no longer be short-range. (ii.) The divergent $\log \Lambda$ part of the scattering amplitude in eq. (2.50) only contributes to the core delta function 3-potential while the $\log q$ part produces a *finite* $\frac{1}{r^3}$ 3-potential [19]. As such, we expect that the core-core interaction should only contribute a finite term to the renormalization of the long-range interaction parameter ξ. To put this another way, if the bare ξ were zero the induced ξ_R would be not be UV divergent; rather it would be finite, proportional to λ^2. The first of these assumptions implies that the series for λ_R has no terms in ξ, while the second assumption implies in that the series for ξ_R the λ^2 term is finite, with no divergence. Making these assumptions, we may rewrite the renormalization prescription as

$$\begin{aligned} \lambda_R = &\ \lambda + \tfrac{1}{48\pi^2} & \left[\lambda^2 \left(B_1 + 9 \log \tfrac{r_0 \mu}{\sigma}\right)\right] + \cdots \\ \xi_R = &\ \xi - \tfrac{1}{48\pi^2} & \left[\lambda^2 \left(B_2\right)\right. \\ & & \lambda\xi \left(B_3 - 6 \log \tfrac{r_0 \mu}{\sigma}\right) \\ & & \left.\xi^2 \left(B_4 + \log \tfrac{r_0 \mu}{\sigma}\right)\right] + \cdots \end{aligned} \tag{4.21}$$

where the finite constants B_i are related to the constants A_i. B_2 is calculable, while the other B_i are arbitrary (renormalization scheme dependent).

These relations produce the beta functions for λ_R and ξ_R,

$$\beta_\lambda(\lambda_R) \equiv \mu \frac{d\lambda_R}{d\mu} = \frac{3}{16\pi^2} \lambda_R^2 + \cdots , \tag{4.22}$$

$$\beta_\xi(\xi_R, \lambda_R) \equiv \mu \frac{d\xi_R}{d\mu} = \frac{1}{48\pi^2} \left(6\lambda_R \xi_R - \xi_R^2\right) + \cdots . \tag{4.23}$$

The first of these is the same as the standard case, and to leading order integrates up to

$$\lambda_R(\mu) = \frac{16\pi^2}{3 \log \frac{\mu_0}{\mu}}, \tag{4.24}$$

where the constant of integration has been cast as μ_0, the scale of the Landau pole (Figure 13).

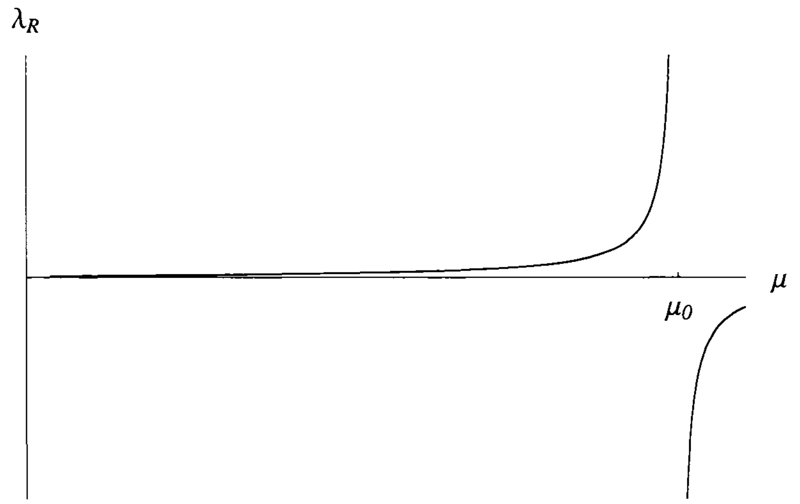


Figure 13: Running of λ_R with the scale μ

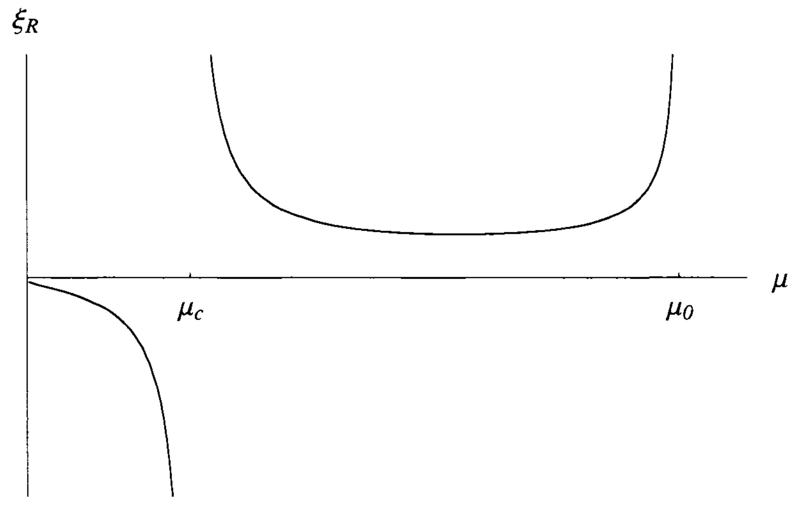


Figure 14: Running of ξ_R with the scale μ

We may now use this relation to obtain a differential equation for ξ_R at leading order,

$$\mu\frac{\partial\xi_R}{\partial\mu} = \frac{2}{3\log\frac{\mu_0}{\mu}}\xi_R - \frac{1}{48\pi^2}\xi_R^2, \tag{4.25}$$

which can be solved using an appropriate integrating factor to obtain

$$\xi_R(\mu) = \frac{16\pi^2}{\log^{\frac{2}{3}}\frac{\mu_0}{\mu}\left(\log^{\frac{1}{3}}\frac{\mu_0}{\mu_c} - \log^{\frac{1}{3}}\frac{\mu_0}{\mu}\right)}. \tag{4.26}$$

The constant of integration has been cast as μ_c, the scale of another pole of $\xi_R(\mu)$ (Figure 14).

Now that we have explicit expressions for how the renormalized parameters "run" with the scale μ, we may obtain the Renormalization-Group Improved effective potential by substituting these expressions into just the classical term, then setting $\mu \propto \phi_c$ (as in the discussion leading up to eq. (2.35)). The RGI effective potential thus obtained (Figure 15) is

$$V_{\text{eff}} = \frac{2\pi^2}{9}\frac{\phi_c^4}{\log\frac{\mu_0}{\phi_c}}\left(1 - \frac{3}{\frac{\log^{\frac{1}{3}}\frac{\mu_0}{\mu_c}}{\log^{\frac{1}{3}}\frac{\mu_0}{\phi_c}} - 1}\right), \tag{4.27}$$

where we have absorbed the constant of proportionality between μ and ϕ_c into the scales μ_0 and μ_c without loss of generality.

Since we need ξ_R to be positive, the physical region is between the two poles (see Figure 14). Also, as with the standard case we must place the cutoff Λ below the pole at μ_0. For perturbation theory to be justifiable we need both λ_R and ξ_R to be small, so we must push the μ_c pole to the left so that the minimum of ξ_R becomes small enough, hence $\mu_c \ll \mu_0$. The pole at μ_c is thus an asymptotically free (infrared) pole, which is probably a signal not of a pathology but rather of complicated physics at small energy scales (cf. the situation with QCD at low energies). We may thus reasonably conclude that this pole is smoothed out by other effects, so the RGI effective potential is physically reasonable and displays true symmetry breaking.

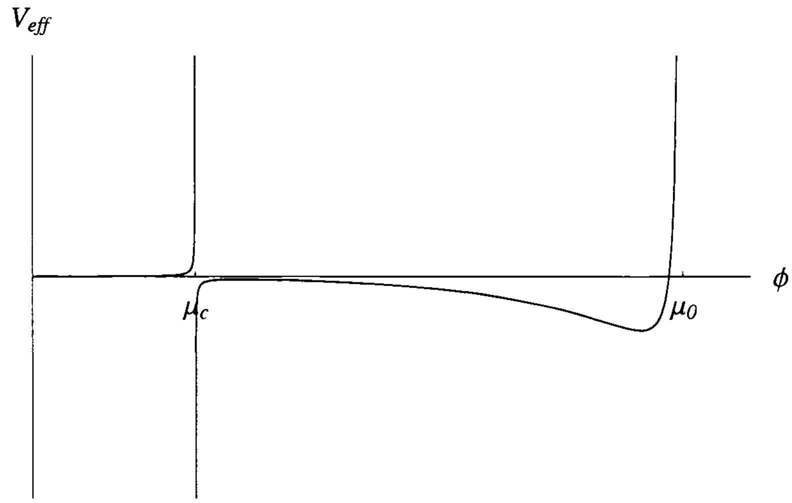


Figure 15: Renormalization-Group Improved Effective Potential ($r_{\max} \to \infty$)

4.4 Autonomous Approach, with $r_0\Lambda$ of Finite Order and $r_{\max}M \to \infty$

The autonomous renormalization procedure for the effective potential in this limit begins with the same analysis as in the perturbative approach, leading to eq. (4.10). The determination of α and of the physical mass is also unchanged. The effective potential is of the typical ϕ_c^4, $\phi_c^4 \log \phi_c^2$ form, and as usual we are interested in the coefficient of the latter term. It turns out to be $\frac{1}{64\pi^2}\left(\frac{1}{2}\lambda_{\text{eff}} + \frac{1}{3}\xi\right)^2 = \frac{1}{64\pi^2}\alpha^2\lambda_{\text{eff}}^2$, so that the $\phi_c^4 \log \phi_c^2$ term may be written as $\frac{1}{64\pi^2}M^4 \log M^2$. As such, the physical predictions of this case are once again identical to those of eq. (2.46), including the form of the renormalized effective potential and the relation of the physical mass to the vacuum value.

5 Conclusions and Outlook

Motivated by the role that an induced long-range interaction plays in the 'particle-gas' picture (Section 2.4) we have considered a model Lagrangian (Section 3.1) that includes both a short-range 'core' and a long-range 'tail' interaction, and we have calculated the effective potential of this nonlocal theory to second order in the couplings λ and ζ. Our calculations show that, at this order, a nonlocal theory of this form is renormalizable, and that the physical predictions in some cases are identical to the standard local $\lambda\phi^4$ theory.

We have just begun to explore the many possibilities the theory presents. For example, we could have considered other relationships between the scale parameters than the two cases examined in the previous section. In particular, the case where $r_0\Lambda \to 0$ and $r_{\max} \to \infty$ might be of interest, perhaps with these limits coupled in some fashion. We could also consider taking $r_0 \to 0$ logarithmically, as $\frac{1}{\sqrt{\log \frac{\Lambda}{M_v}}}$, so that it is of order the scattering length in the particle gas picture. Our explorations so far seem to indicate that taking $r_{\max} \to \infty$ is a more promising choice than putting in a screening mechanism by hand and setting $r_{\max} M_v$ of finite order. That is, it seems that the calculation of V_{eff} inherently includes the screening effect.

Another obvious next step would be to generalize the theory to the case where the fundamental nonlocal Lagrangian contains a nonzero mass term. This would make better contact with the "particle gas" picture, whose intuitive appeal is limited to the massive case. Such an exploration might yield new insight, particularly in determining the best way of dealing with the $r_{\max} \to \infty$ limit since the screening mechanism due to m^2 might then be manifest. The details of such a calculation, while involving more terms, should be tractable using the methods developed here to handle the massless case.

The perturbative renormalization schemes merit further investigation. The main issue is that it is not clear how to exploit the arbitrariness associated with the constants A_i that appear in expression eqs. (4.1) and (4.19) of the renormalized parameters in terms of the bare parameters. This issue might become clearer if the calculation is performed to higher order. Perhaps a better general understanding is needed of the Renormalization Group flow in a nonlocal theory.

Another issue with the simple theory we have considered is that since the tail potential is effectively generated by the core potential, we are in some sense double-counting some terms. In order to establish a precise correspondence to the local theory, we would need to remove some of the λ^2 terms. A direct comparison to existing two and three loop calculations in the local theory should help in this endeavor. Such a comparison also might make apparent the best treatment for the various scale parameters. The present work could be viewed as a stepping stone to get a better approximation to the effective potential of the usual, local $\lambda\phi^4$ theory. In particular, the RGI result of the previous section displays some promise of resolving the apparent conflict between the RGI and autonomous approaches to renormalization.

Finally, the results of the last section indicate that the propagator of this theory can have a discontinuity at zero momentum, in the case where $r_{\max} \to \infty$. The discontinuity becomes large (see eq. (4.17)) if there is a near-cancellation between λ and ξ so that $\lambda_{\text{eff}} \ll \lambda$. The form given for the propagator seems to provide a good fit to data from lattice calculations in the standard $\lambda\phi^4$ theory [24], and it would be worthwhile to explore the possibility of the presently considered nonlocal theory as an "effective theory" for the standard theory on the lattice.

Although our present work does not offer many concrete conclusions, it does offer avenues of future research and novel calculational methods that might be useful in other contexts. In addition, the prediction that the physical mass of the Higgs should be $M^2 = 8\pi^2 v_R^2$ in the autonomous renormalization scheme appears to be fairly robust, as we obtain the same relation in this nonlocal theory in two different limits for the scale parameters.

A Hypergeometric Functions

A.1 Definition

The generalized hypergeometric functions can be defined by their series representation [25]

$$F_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(u) \equiv \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k} \frac{u^k}{k!}; \qquad (a)_k \equiv \frac{\Gamma(a+k)}{\Gamma(a)} \tag{A.1}$$

where $(a)_k$ is the Pochammer symbol, or rising factorial.

Other notations equivalent to ours are used in the literature.

$$F_{b_1,b_2,\cdots,b_q}^{a_1,a_2,\cdots,a_p}(u) = {}_pF_q\left[{a_1,a_2,\cdots,a_p \atop b_1,b_2,\cdots,b_q}; u\right] = {}_pF_q(a_1,a_2,\cdots,a_p;b_1,b_2,\cdots,b_q;u) \tag{A.2}$$

We have adopted the first (nonstandard but perfectly general) notation for the sake of succinctness. The hypergeometric functions may seem esoteric at first glance, but many familiar functions such as trigonometric functions, Bessel functions, and logarithms can be expressed as hypergeometric functions. Several particular hypergeometric functions appear in this thesis, and the following tables and plots show their general characteristics. For convenience, we have adopted the notation $\overline{F} \equiv F-$ (leading large-argument term of F) so that the combination $u^2\overline{F}$ vanishes as $u \to \infty$.

	small argument	large argument
$F_{2,2,3}^{1,1}(-\frac{1}{4}u^2)$	$1 - \frac{1}{48}u^2 + \cdots$	$-\frac{16}{u^2}(\frac{1}{2} - \log\frac{u}{\sigma}) - 32\sqrt{\frac{2}{\pi}}\frac{\cos(\frac{5\pi}{4}-u)}{u^{\frac{9}{2}}} + \cdots$
$F_{2,2,4}^{1,1}(-\frac{1}{4}u^2)$	$1 - \frac{1}{64}u^2 + \cdots$	$-\frac{24}{u^2}(\frac{3}{4} - \log\frac{u}{\sigma}) - 192\sqrt{\frac{2}{\pi}}\frac{\cos(\frac{5\pi}{4}-u)}{u^{\frac{11}{2}}} + \cdots$
$F_{2,2,4,6}^{1,1,\frac{7}{2}}(-u^2)$	$1 - \frac{7}{192}u^2 + \cdots$	$-\frac{12}{u^2}(\frac{11}{24} - \log\frac{u}{\sigma}) + \frac{768}{5\pi u^7} + \frac{384\cos(2u)}{\pi u^8} + \cdots$

Table 4: Small and large argument behavior of selected hypergeometric functions

B Additional Figures

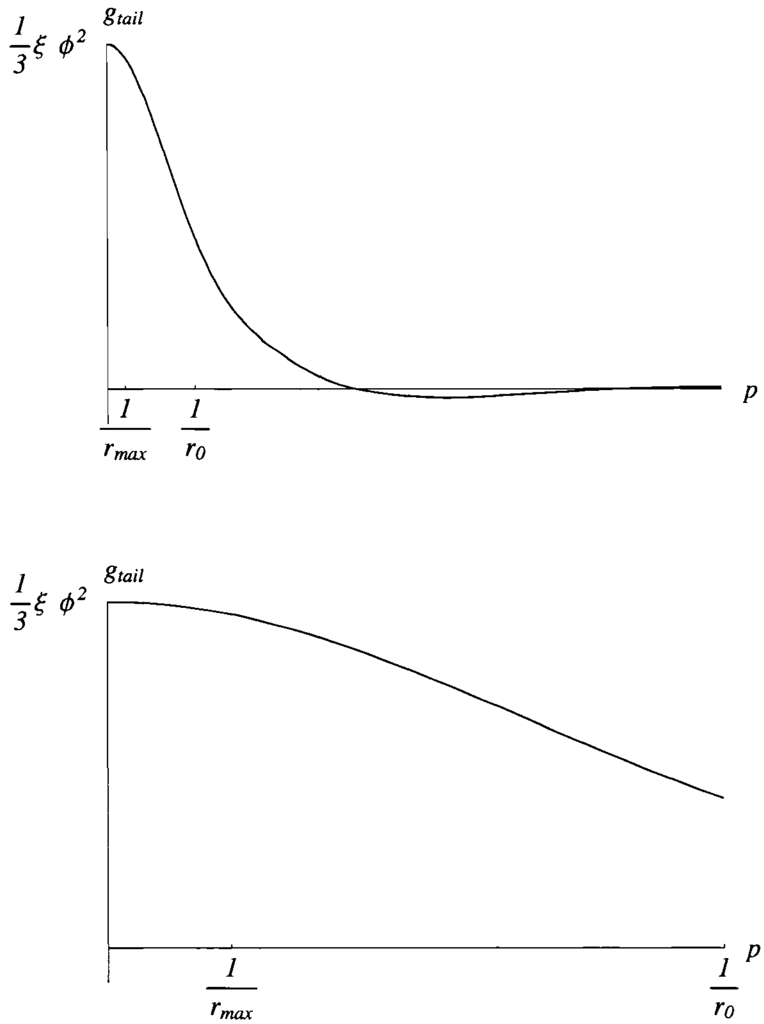


Figure 16: $g_{\text{tail}}(p)$ for $\frac{r_{\text{max}}}{r_0} = 5$

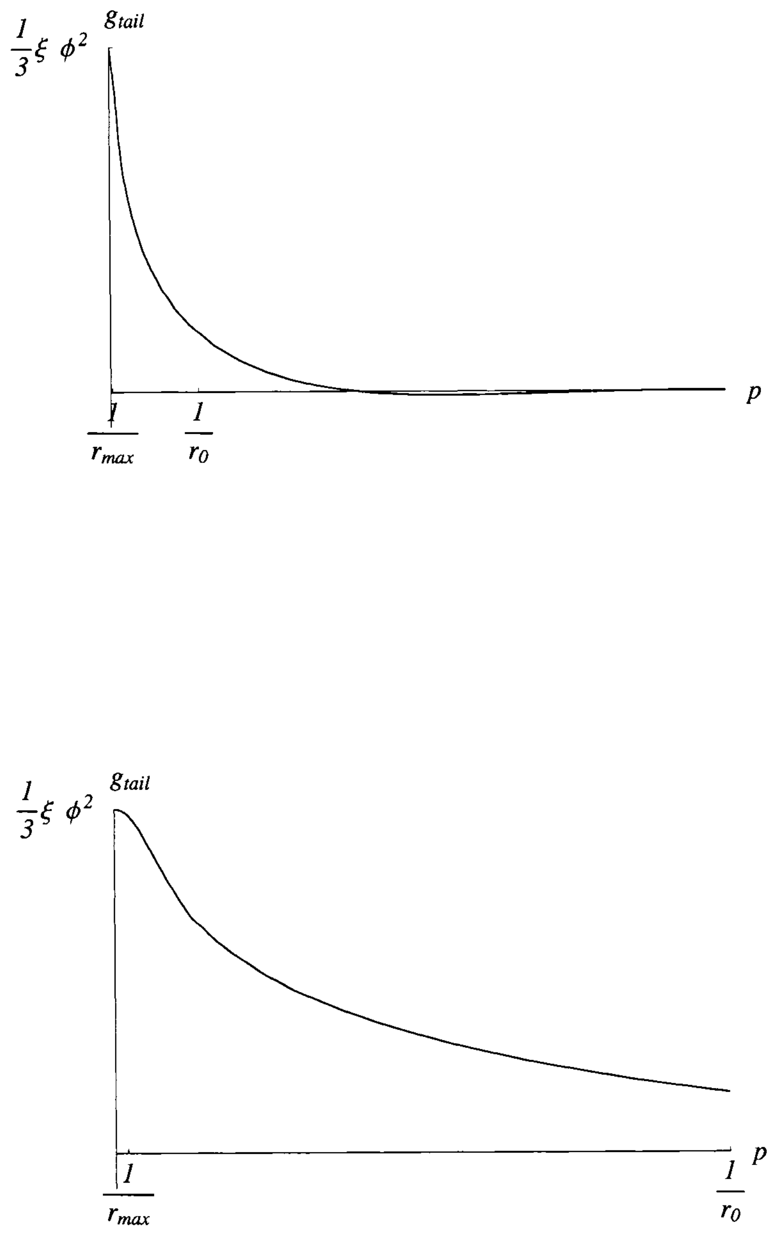
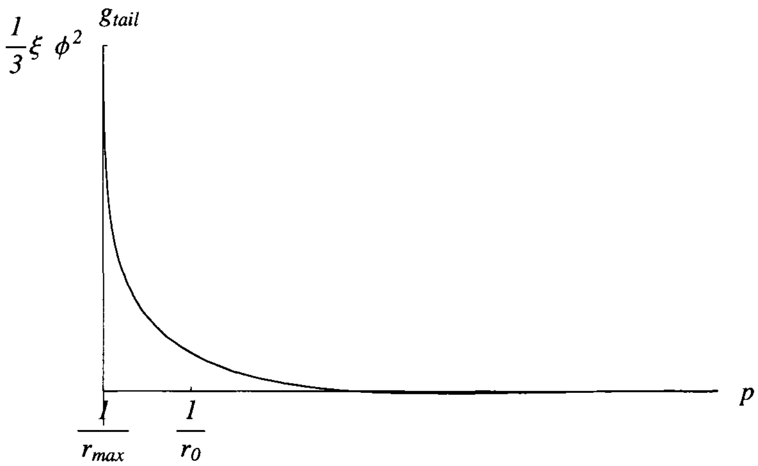


Figure 17: $g_{\text{tail}}(p)$ for $\frac{r_{\text{max}}}{r_0} = 50$



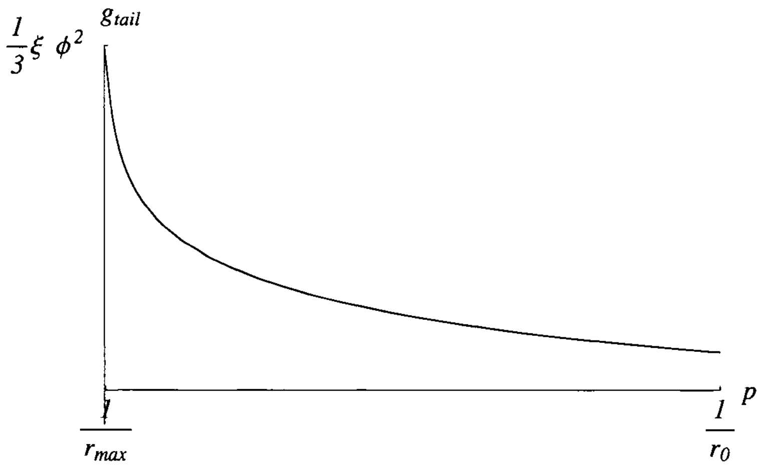


Figure 18: $g_{\mathrm{tail}}(p)$ for $\frac{r_{\mathrm{max}}}{r_0} = 500$

C Supplementary Calculations

C.1 Alternate Calculation of $I^{(1)}_{\text{core}}$

The two integrals with one power of $g_{\text{core}}(p)$, $I^{(1)}_{\text{core}}$ in eq. (3.22) and $I^{(2)}_{M,c}$ in eq. (3.25), may be performed by splitting the region of integration into two parts:

$$I^{(1)}_{\text{core}} = \tfrac{1}{2}\int_0^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2} - \tfrac{1}{2}\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2}. \tag{C.1}$$

The first term can be integrated exactly then expanded in $r_0 M \ll 1$,

$$\begin{aligned}\tfrac{1}{2}\int_0^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2} &= \tfrac{\lambda\phi_c^2}{6\pi^2 r_0^4 M^2}\left(2 - r_0^2 M^2 K_2(r_0 M)\right)\\ &= \tfrac{\lambda\phi_c^2}{6\pi^2 r_0^4 M^2}\left(\tfrac{1}{2}r_0^2 M^2 + \left(\tfrac{1}{8}\log\tfrac{r_0 M}{\sigma} - \tfrac{3}{32}\right) r_0^4 M^4 + \cdots\right).\end{aligned} \tag{C.2}$$

Mass renormalization removes the first term, while terms beyond the second contain positive powers of r_0 and hence vanish, so only the second term remains:

$$\tfrac{1}{2}\int_0^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2} = \frac{\lambda\phi_c^2 M^2}{48\pi^2}\left(\log\frac{r_0 M}{\sigma} - \frac{3}{4}\right). \tag{C.3}$$

In the second term of $I^{(1)}_{\text{core}}$, eq. (C.1), we have $p > \Lambda \gg M$ so we may expand the denominator as a series in $\frac{M}{p}$.

$$-\tfrac{1}{2}\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2} = -\tfrac{1}{2}\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2} + \tfrac{1}{2}M^2\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^4} - \cdots \tag{C.4}$$

The first term in the series is

$$-\tfrac{1}{2}\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2} = -\frac{\lambda\phi_c^2}{6\pi^2 r_0^3\Lambda}J_1(r_0\Lambda) \tag{C.5}$$

and is removed by mass renormalization. The third and further terms in the series vanish by $\left(\frac{M}{\Lambda}\right)^2 \ll 1$, while the second remains finite due to the combination $r_0\Lambda \sim 1$, so we have just the second term,

$$-\tfrac{1}{2}\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{p^2+M^2} = \frac{\lambda\phi_c^2 M^2}{48\pi^2}\left(\tfrac{1}{4!}r_0^2\Lambda^2 F^{1,1}_{2,2,4}(-\tfrac{1}{4}r_0^2\Lambda^2) - \log\frac{r_0\Lambda}{\sigma} + \frac{3}{4}\right). \tag{C.6}$$

Putting eqs. (C.3) and (C.6) in eq. (C.1), most terms cancel and we have

$$I^{(1)}_{\text{core}} = \frac{\lambda\phi_c^2 M^2}{48\pi^2}\left(\tfrac{1}{4!}r_0^2\Lambda^2 F^{1,1}_{2,2,4}(-\tfrac{1}{4}r_0^2\Lambda^2) - \log\frac{\Lambda}{M}\right) \tag{C.7}$$

which is the same as the result in eq. (3.38) obtained by other means.

C.2 Alternate Calculation of $I^{(1)}_{M,c}$

The $I^{(2)}_{M,c}$ integral in eq. (3.25) can be calculated by the same method, keeping or dropping terms for the same reasons. Split the integral into two pieces and calculate each one,

$$I^{(2)}_{M,c} = -\tfrac{1}{2}g_M\int_0^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{(p^2+M^2)^2} + \tfrac{1}{2}g_M\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{(p^2+M^2)^2} \tag{C.8}$$

$$\begin{aligned} -\tfrac{1}{2}g_M\int_0^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{(p^2+M^2)^2} &= \tfrac{\lambda\phi_c^2 g_M^2}{12\pi^2 r_0^4 M^4}\left(-4 + r_0^2M^2(r_0MK_1(r_0M) + 2K_2(r_0M))\right) \\ &= \tfrac{\lambda\phi_c^2 g_M^2}{48\pi^2}\left(\log\tfrac{r_0M}{\sigma} - \tfrac{1}{4}\right) \end{aligned} \tag{C.9}$$

$$\tfrac{1}{2}g_M\int_\Lambda^\infty \frac{d^4p}{(2\pi)^4}\frac{g_{\text{core}}(p)}{(p^2+M^2)^2} = \frac{\lambda\phi_c^2 g_M^2}{48\pi^2}\left(\tfrac{1}{4!}r_0^2\Lambda^2F^{1,1}_{2,2,4}(-\tfrac{1}{4}r_0^2\Lambda^2) - \log\frac{r_0\Lambda}{\sigma} + \frac{3}{4}\right). \tag{C.10}$$

Then putting the pieces together we obtain

$$I^{(2)}_{M,c} = \frac{\lambda\phi_c^2 g_M^2}{48\pi^2}\left(\tfrac{1}{4!}r_0^2\Lambda^2F^{1,1}_{2,2,4}(-\tfrac{1}{4}r_0^2\Lambda^2) - \log\frac{\Lambda}{M} + \frac{1}{2}\right). \tag{C.11}$$

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