Nuclear Physics B 387 (1992) 373–390
North-Holland

NUCLEAR
PHYSICS B

The standard model effective potential at two loops

C. Ford and I. Jack
DAMTP, University of Liverpool, Liverpool L69 3BX, UK

D.R.T. Jones
ITP, University of California at Santa Barbara, CA 93106, USA
and
DAMTP, University of Liverpool, Liverpool L69 3BX, UK [1]

Received 24 March 1992
(Revised 20 July 1992)
Accepted for publication 22 July 1992

We calculate the standard model effective potential to two loops using minimal subtraction, and use the result to deduce the two-loop beta functions for the scalar m^2 and quartic self-interaction.

1. Introduction

The role of the effective potential $V(\phi)$ in determining the nature of the vacuum in renormalisable field theories was emphasized in the classic paper of Coleman and Weinberg (CW) [1]. Their particular interest was in the special case when the renormalised value of the (mass)^2 parameter, m^2, of the scalar fields was zero. They were able to demonstrate the occurrence of spontaneous symmetry breaking through radiative corrections for this case, as long as gauge fields are present. This scenario, however, is excluded by current experimental limits on m_{H} (the Higgs mass) and m_{t} (the top mass).

In the $m^2 < 0$ case, radiative corrections are still important in determining whether the tree minimum of V corresponds to the true ground state of the theory. (For a review and references, see ref. [2].) The one-loop Yukawa coupling contribution to V tends to destabilise the vacuum, and consequently leads to an upper bound for m_{t} as a function of m_{H} [3].

The analysis described in ref. [3] was based on a renormalisation-group (RG) "improved" form [1,4] of V including one-loop corrections. Here we present the

[1] Permanent address.

results of the two-loop radiative corrections to V, $V^{(2)}$, with a view to possible refinement of the bounds described in ref. [3].

Another motivation for our calculation is as follows: Given the standard model two-loop β-function, the dependence of $V^{(2)}$ on the renormalisation scale μ may be readily inferred from the fact that V satisfies an RG equation. (This calculation was in fact attempted in ref. [5]; we will comment later on this paper.) We shall instead calculate the full $V^{(2)}$, using minimal subtraction (MS) throughout, and use the results to infer the two-loop β-functions for m^2 ($\beta_{m^2}^{(2)}$) and the quartic Higgs coupling λ ($\beta_\lambda^{(2)}$). As we shall see, this procedure will expose some minor errors in the expression for the SM $\beta_\lambda^{(2)}$ given in ref. [6]. As for $\beta_{m^2}^{(2)}$, as far as we know it has not previously appeared in the literature. As already mentioned in ref. [7], it is essential that $V^{(1)}$ be calculated using MS in order that $V^{(2)}$ be consistent with the RG equation with the MS β-functions; the β-functions in a multi-coupling constant theory are scheme dependent at the two-loop level [8]. Thus although there exists [9] (in the special case $h = m^2 = 0$) a calculation of $V^{(2)}$, this cannot be used for comparison with the standard model $\beta_\lambda^{(2)}$, since MS was not employed.

The plan of this paper is as follows. In sect. 2 we describe briefly the effective potential formalism, give the result for the one-loop correction, $V^{(1)}$, and introduce some notation. In sect. 3 we explain our procedure for finding $V^{(2)}$ within the MS scheme, and in sect. 4 we apply a novel method to the evaluation of the pertinent Feynman integral. The method involves the use of differential equations [10] with the added refinement of the method of characteristics. Sect. 5 consists of a summary of the results, while in sect. 6 we show how the fact that $V(\phi)$ satisfies a RG equation leads to a determination of $\beta_{m^2}^{(2)}$ and $\beta_\lambda^{(2)}$. Finally in sect. 7 we discuss applications.

2. The one-loop effective potential $V^{(1)}(\phi)$

The effective potential formalism of CW and the functional refinements introduced by Jackiw [11] are too well known to require any but the briefest review here. In general one shifts scalar fields as follows:

$$\phi(x) \to \phi + \phi_q(x) \tag{2.1}$$

where ϕ is x-independent. Then the effective potential $V(\phi)$ is given by the sum of *vacuum* graphs, with ϕ-dependent propagators. Equivalently, one can calculate graphs with a single ϕ_q external field, which, it is easy to show, leads to a determination of $\partial V/\partial\phi$. The latter method is perhaps simpler at one loop, but not beyond since it leads to more Feynman diagrams. (Here we differ as a matter of opinion from ref. [9].)

In the SM case, one can exploit gauge invariance to perform the shift of (2.1) on one only of the four scalar fields:

$$\Phi(x) \to \begin{pmatrix} 0 \\ \phi \end{pmatrix} + \begin{pmatrix} G^{\pm}(x) \\ (1/\sqrt{2})(H(x) + iG(x)) \end{pmatrix}. \tag{2.2}$$

We must also choose a gauge; the 't Hooft–Landau gauge is the most convenient one. (In fact $V(\phi)$ is gauge invariant only at its extrema; this gauge is a good one in the sense of ref. [12].) In this gauge the W, Z and γ propagators are transverse, and the associated ghosts are massless and couple only to the gauge fields; the "would be Goldstone" bosons $G^{\pm}$, G have a common mass deriving from the scalar potential only.

At the tree level the effective potential is $V^{(0)}(\phi)$, given by

$$V^{(0)}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4. \tag{2.3}$$

(Note that our definition of λ differs by a factor of 3 from that of ref. [6].)

The result of the one-loop calculation is

$$\kappa V^{(1)} = \frac{H^2}{4}\left(\overline{\ln}\, H - \tfrac{3}{2}\right) + \frac{3G^2}{4}\left(\overline{\ln}\, G - \tfrac{3}{2}\right) - 3T^2\left(\overline{\ln}\, T - \tfrac{3}{2}\right)$$
$$+ \frac{3W^2}{2}\left(\overline{\ln}\, W - \tfrac{5}{6}\right) + \frac{3Z^2}{4}\left(\overline{\ln}\, Z - \tfrac{5}{6}\right), \tag{2.4}$$

where

$$\kappa = 16\pi^2, \qquad H = m^2 + \frac{\lambda}{2}\phi^2, \qquad T = \frac{h^2}{2}\phi^2,$$

$$G = m^2 + \frac{\lambda}{6}\phi^2, \qquad W = \frac{g^2}{4}\phi^2, \qquad Z = \frac{\left(g^2 + g'^2\right)}{4}\phi^2,$$

and

$$\overline{\ln}\, X = \ln\frac{X}{\mu^2} + \gamma - \ln 4\pi,$$

γ being Euler's constant. (Note that the sign of the $\ln 4\pi$ term is rendered incorrectly in ref. [7], eq. (4).) Here h is the top quark Yukawa coupling (we neglect other Yukawa couplings throughout.)

If $m^2 < 0$, then at the minima of $V^{(0)}(\phi)$ we have $G = 0$ and H, T, W, Z become the tree-level (masses)^2 of the Higgs, top quark, W- and Z-bosons

Fig. 1. Two-loop contributions to the effective potential.

respectively. The non-logarithmic terms in eq. (2.4) may be altered (or indeed removed) by a change in subtraction scheme. Since, however, we wish to consider the RG equation with the MS scheme, it is essential that we retain them. (The $\overline{MS}$ scheme corresponds, of course, to simply replacing $\overline{\ln} X$ by $\ln(X/\mu^2)$ throughout; this would not affect the RG analysis since all RG functions are identical in MS and $\overline{MS}$ [13].)

3. The two-loop calculation: preliminaries

In this section we outline our basic strategy for the calculation. There are two types of Feynman diagram, as shown in fig. 1. By elementary manipulation we can reduce each individual Feynman diagram to a sum of integrals of the form of either

$$I(x, y, z) = \frac{\cancel{(\mu^2)}^{2\epsilon}}{(2\pi)^{2d}} \int \frac{\mathrm{d}^d k \, \mathrm{d}^d q}{(k^2+x)(q^2+y)\left((k+q)^2+z\right)} \tag{3.1}$$

or

$$J(x, y) = J(x) J(y), \tag{3.2}$$

where

$$J(x) = \frac{\cancel{(\mu^2)}^{\epsilon}}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{k^2+x} = \frac{\cancel{(\mu^2)}^{\epsilon}}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) x^{\frac{1}{2}d-1} \tag{3.3}$$

and we define $d = 4 - 2\epsilon$. (We work in euclidean space throughout.) The evaluation of $I(x, y, z)$ is non-trivial, and will be the subject of sect. 4.

We turn first to the subject of renormalisation. We choose to work throughout with *renormalised* parameters g_3, g, g', m^2, h, and λ. (g_3, g, g' are the three gauge couplings, with the usual conventions.) Then rather than compute separately a set of one-loop diagrams with counter-term insertions, we subtract (minimally) the sub-divergences *diagram by diagram*. $V^{(2)}$ is then obtained by simply taking the finite parts of the resulting expressions, discarding the $1/\epsilon^2$, $1/\epsilon$ poles which are automatically cancelled by the usual renormalisation constants (which we need not calculate). For graphs not involving vector bosons, this procedure amounts simply to the replacement of $I(x, y, z)$ and $J(x, y)$ by their subtracted values:

$$I(x, y, z) \to \hat{I}(x, y, z) = I(x, y, z) - \frac{(\mu^2)^\epsilon}{\kappa\epsilon}(J(x) + J(y) + J(z)), \quad (3.4)$$

$$J(x, y) \to \hat{J}(x, y) = J(x, y) + \frac{(\mu^2)^\epsilon}{\kappa\epsilon}(xJ(y) + yJ(x)). \quad (3.5)$$

There is one complication however. When vector bosons are present, the algebra involved in reducing the graph to dependence on I and J may produce explicit factors of d. This gives rise to an apparent ambiguity: what is the subtracted form of dI? The answer is that the result depends on which subgraph produced the factor of d. Thus when vector fields are present we *must* explicitly evaluate the contribution of the subtractions to each graph. This is still, however, much easier than explicitly considering counter-term insertions. Because we work in the 't Hooft–Landau gauge, the gauge parameter is unrenormalised and so creates no difficulties.

We conclude this section with the results for J which will be relevant subsequently. It is straightforward to show from (3.3) that

$$\kappa^2 \hat{J}(x, y) = -\frac{xy}{\epsilon^2} + xy(1 - \overline{\ln}\, x - \overline{\ln}\, y + \overline{\ln}\, x\, \overline{\ln}\, y), \quad (3.6)$$

$$\kappa J(x) = -\frac{x}{\epsilon} + x(\overline{\ln}\, x - 1), \quad (3.7)$$

$$\kappa^2 \epsilon J(x, y) = \frac{xy}{\epsilon} + xy(2 - \overline{\ln}\, x - \overline{\ln}\, y). \quad (3.8)$$

It is the finite part of each expression that is substituted in our expressions for $V^{(2)}$ in sect. 5. Terms of the form of eqs. (3.7) and (3.8) appear in connection with the d-dependence discussed above.

4. Evaluation of $I(x, y, z)$

In this section we indicate how the differential equations method in conjunction with the method of characteristics leads to a simple form for the integral $I(x, y, z)$. Of course there has been much effort expended on higher-loop Feynman integrals, and many powerful techniques have been developed (for some examples see ref. [14]); nevertheless we feel that our method here has some interesting features and is worth presenting in detail.

We start from the identity

$$0=\int \mathrm{d}^d k\ \mathrm{d}^d q \frac{\partial}{\partial k^{\mu}} \frac{k^{\mu}}{(k^2+x)(q^2+y)\left((q+k)^2+z\right)}. \tag{4.1}$$

From eq. (4.1) it follows that

$$2x\frac{\partial I}{\partial x}+(z+x-y)\frac{\partial I}{\partial y}=(d-3)I+K_1(x,y,z) \tag{4.2}$$

where

$$K_1(x,y,z)=-\frac{\partial J(z)}{\partial z}(J(x)-J(y)).$$

Now from eq. (4.2) and similar equations produced by (x, y, z) permutations, one can eliminate $\partial I/\partial y$ and $\partial I/\partial z$ and then solve the resulting equation in a similar way to that adopted in ref. [7]. It turns out, however, that a more elegant solution follows if we start with the following equation:

$$(y-z)\frac{\partial I}{\partial x}+(z-x)\frac{\partial I}{\partial y}+(x-y)\frac{\partial I}{\partial z}=K(x,y,z) \tag{4.3}$$

where

$$\begin{aligned}K(x,y,z)&=K_1(x,y,z)+K_1(y,z,x)+K_1(z,x,y)\\&=-\Gamma'\left((z-x)(zx)^{-\epsilon}+(x-y)(xy)^{-\epsilon}+(y-z)(yz)^{-\epsilon}\right)\end{aligned} \tag{4.4}$$

and

$$\Gamma'=(\mu^2)^{2\epsilon}\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)(4\pi)^{-d}.$$

The method of characteristics involves solving a system of ordinary differential equations, as follows:

$$\mathrm{d}t=\frac{\mathrm{d}x}{y-z}=\frac{\mathrm{d}y}{z-x}=\frac{\mathrm{d}z}{x-y}=\frac{\mathrm{d}I}{K} \tag{4.5}$$

subject to initial conditions which we shall choose to be $x=X$, $y=Y$ and $z=0$ (at $t=0$), and we will suppose without loss of generality that $X\geq Y$. We then have

$$I(x,y,z)=I(X,Y,0)+\int_0^t \mathrm{d}t'\ K(x(t'),y(t'),z(t')). \tag{4.6}$$

Using eqs. (4.4) and (4.5), we can rewrite eq. (4.6) as

$$I(x,y,z)=I(X,Y,0)-\Gamma'\left[\int_X^x \mathrm{d}x\,(yz)^{-\epsilon}+\int_Y^y \mathrm{d}y\,(zx)^{-\epsilon}+\int_0^z \mathrm{d}z\,(xy)^{-\epsilon}\right]. \tag{4.7}$$

Now from eq. (4.5) it follows that *for all t*,

$$x^2+x^2+z^2=d^2=X^2+Y^2, \tag{4.8}$$

$$x+y+z=c=X+Y, \tag{4.9}$$

where c and d are constants. Therefore

$$xy=z^2-cz+\tfrac{1}{2}(c^2-d^2) \tag{4.10}$$

with similar equations for yz and zx. Hence eq. (4.7) becomes

$$I(x,y,z)=I(X,Y,0)-\Gamma'\left[\left(\int_a^{x-\frac{1}{2}c}+\int_{\frac{1}{2}c-y}^{a}+\int_{\frac{1}{2}c-z}^{\frac{1}{2}c}\right)\mathrm{d}s(s^2-a^2)^{-\epsilon}\right], \tag{4.11}$$

where

$$a=\sqrt{\frac{d^2}{2}-\frac{c^2}{4}}=\tfrac{1}{2}(X-Y)=\tfrac{1}{2}(x^2+y^2+z^2-2xy-2yz-2zx)^{1/2}. \tag{4.12}$$

Now $I(X,Y,0)$ is also tricky to evaluate by elementary methods; but by employing the method of characteristics once again, one can show that

$$I(X,Y,0)=I(X-Y,0,0)+\Gamma'\int_{\frac{1}{2}(X-Y)}^{\frac{1}{2}(X+Y)}\mathrm{d}s\left[s^2-\tfrac{1}{4}(X-Y)^2\right]^{-\epsilon}. \tag{4.13}$$

Substituting eq. (4.13) in eq. (4.11) and using eq. (4.9) and eq. (4.12) we obtain

$$I(x,y,z)=I(2a,0,0)+\Gamma'\left[F\left(\frac{c}{2}-y\right)+F\left(\frac{c}{2}-z\right)-F\left(x-\frac{c}{2}\right)\right], \tag{4.14}$$

where

$$F(w)=\int_a^w \mathrm{d}s(s^2-a^2)^{-\epsilon}. \tag{4.15}$$

Since $I(2a,0,0)$ can be evaluated by elementary methods, we have reduced the problem to a single integral, $F(w)$. In spite of appearances, eq. (4.14) is symmetric with respect to (x, y, z) permutations, since it is easy to show that, for example, $F(\frac{1}{2}c - y) - F(x - \frac{1}{2}c) = F(\frac{1}{2}c - x) - F(y - \frac{1}{2}c)$. However the solution is only valid in the region $a^2 \geq 0$. In the region $a^2 \leq 0$, it is possible to derive the following form of the solution:

$$I(x, y, z) = -I(2b, 0, 0) \sin\frac{\pi d}{2} + \Gamma'\left[G\left(\frac{c}{2} - x\right) + G\left(\frac{c}{2} - y\right) + G\left(\frac{c}{2} - z\right)\right], \tag{4.16}$$

where

$$G(w) = \int_0^w \mathrm{d}s(s^2 + b^2)^{-\epsilon}, \qquad b^2 = -a^2. \tag{4.17}$$

It is a nice exercise to show that for $z = x$, eq. (4.16) can be rewritten as eq. (11a) of ref. [7]. Note that for $a^2 = 0$, which in x, y, z space is a cone with its apex at the origin, the integral is trivial.

Writing

$$(s^2 + b^2)^{-\epsilon} = 1 - \epsilon \ln(s^2 + b^2) + \tfrac{1}{2}\epsilon^2 \ln^2(s^2 + b^2) + \dots \tag{4.18}$$

and using

$$(4\pi)^d I(x, 0, 0) = \frac{\Gamma(2 - \frac{1}{2}d)\Gamma(3 - d)\Gamma(\frac{1}{2}d - 1)^2}{\Gamma(\frac{1}{2}d)} x \left(\frac{x}{\mu^2}\right)^{d-4} \tag{4.19}$$

we obtain from eq. (4.16) that

$$\begin{aligned} \kappa^2 I(x, y, z) = & -\frac{c}{2\epsilon^2} - \frac{1}{\epsilon}\left(\frac{3c}{2} - L_1\right) \\ & -\tfrac{1}{2}\big[L_2 - 6L_1 + (y + z - x)\,\overline{\ln}\,y\,\overline{\ln}\,z \\ & + (z + x - y)\,\overline{\ln}\,z\,\overline{\ln}\,x + (y + x - z)\,\overline{\ln}\,y\,\overline{\ln}\,x \\ & + \xi(x, y, z) + c(7 + \zeta(2))\big], \end{aligned} \tag{4.20}$$

where

$$L_m = x\,\overline{\ln}^m x + y\,\overline{\ln}^m y + z\,\overline{\ln}^m z, \tag{4.21}$$

$$\xi(x, y, z) = 8b\left[L(\theta_x) + L(\theta_y) + L(\theta_z) - \frac{\pi}{2}\ln 2\right]. \tag{4.22}$$

$L(t)$ is Lobachevskiy's function [15], defined as

$$L(t) = -\int_0^t \mathrm{d}x \ln \cos x. \tag{4.23}$$

The angles θ_x, θ_y, θ_z are given by

$$\theta_x = \tan^{-1}\left(\frac{\frac{1}{2}c - \underline{x}}{b}\right), \quad \text{etc.} \tag{4.24}$$

Eq. (4.22) is valid only in the region $a^2 \leq 0$ (i.e. inside the cone). For $a^2 > 0$, we obtain from eq. (4.14) a result identical to eq. (4.20) except that now

$$\xi(x, y, z) = 8a\left[-M(-\phi_x) + M(\phi_y) + M(\phi_z)\right], \tag{4.25}$$

where

$$M(t) = -\int_0^t \mathrm{d}x \ln \sinh x$$

and ϕ_x, ϕ_y, ϕ_z are given by

$$\phi_x = \coth^{-1}\left(\frac{\frac{1}{2}c - \underline{x}}{a}\right), \quad \text{etc.}$$

$\xi(x, y, z)$ is μ-independent and therefore plays no role in the RG analysis of sect. 6.

It is interesting to compare the results eq. (4.20) and eq. (4.22) with those of ref. [7] for the special case $z = x$. In that case we have

$$\sin \theta_x = \sin \theta_z = \frac{1}{2}\left(\frac{y}{x}\right)^{1/2}$$

$$\sin \theta_y = 1 - \frac{y}{2x},$$

and using the identity

$$2L\left(\sin^{-1}\left(\frac{\sqrt{t}}{2}\right)\right) + L\left(\sin^{-1}\left(1 - \frac{t}{2}\right)\right) = \frac{\pi}{2}\ln 2 + \int_0^{\sin^{-1}(\frac{1}{2}\sqrt{t})} \mathrm{d}x \ln 2 \sin x \tag{4.26}$$

it is easy to show that eq. (4.20) reduces to eq. (12) of ref. [7]. A similar process works in the $a^2 > 0$ case.

We will make extensive use of the form I takes when its subdivergences are subtracted, $\hat{I}$. From eq. (3.4) we find that

$$\hat{I} = I + \frac{1}{\kappa^2\epsilon^2}\left(c + (c - L_1)\epsilon + \left(\tfrac{1}{2}L_2 - L_1 + c + \tfrac{1}{2}c\zeta(2)\right)\epsilon^2 + \dots\right). \quad (4.27)$$

It is the *finite part* of this expression which we use subsequently.

5. Two-loop results

In this section we present our results for the various contributions to $V^{(2)}$ in (what we hope is) a clear and systematic manner. It is natural to divide the calculation into parts according to the nature of the contributing fields, thus

$$V^{(2)} = V_{\mathrm{S}} + V_{\mathrm{SF}} + V_{\mathrm{SV}} + V_{\mathrm{FV}} + V_{\mathrm{V}}, \quad (5.1)$$

where S, F, V denote scalar, fermion and vector fields, respectively.

The Feynman rules of the standard model are well known; for the calculation of the effective potential we must only remember that we are not calculating at the tree minimum of the potential. Apart from giving the G, $G^{\pm}$ bosons a non-zero mass, this makes little difference. The results are as follows:

$$V_{\mathrm{S}} = \frac{-\lambda^2\phi^2}{12}\left[\hat{I}(H, H, H) + \hat{I}(H, G, G)\right] + \frac{\lambda}{8}\left[\hat{J}(H, H) + 2\hat{J}(H, G) + 5\hat{J}(G, G)\right] \quad (5.2)$$

(in fact this is the special case $N = 4$ of the calculation reported in ref. [7]).

$$V_{\mathrm{SF}} = 3h^2\left\{(2T - \tfrac{1}{2}H)\hat{I}(T, T, H) - \tfrac{1}{2}G\hat{I}(T, T, G) + (T - G)\hat{I}(T, G, 0) + \hat{J}(T, T) - \hat{J}(T, H) - 2\hat{J}(T, G)\right\}, \quad (5.3)$$

$$\begin{aligned} V_{\mathrm{SV}} = {} & \frac{g^2}{8\cos^2\theta}A(H, G, Z) + \frac{g^2(1 - 2\sin^2\theta)^2}{8\cos^2\theta}A(G, G, Z) + \tfrac{1}{2}e^2 A(G, G, 0) \\ & + \tfrac{1}{4}g^2(A(H, G, W) + A(G, G, W)) - g^2\sin^4\theta\, ZB(Z, W, G) \\ & - e^2 WB(W, 0, G) - \tfrac{1}{2}g^2 WB(W, W, H) - \frac{g^2}{4\cos^2\theta}ZB(Z, Z, H) \\ & + \frac{g^2(1 - 2\sin^2\theta)^2}{4\cos^2\theta}C(Z, G) + \frac{g^2}{8\cos^2\theta}(C(Z, H) + C(Z, G)) \\ & + \tfrac{1}{4}g^4(C(W, H) + 3C(W, G)), \end{aligned} \quad (5.4)$$

$$V_{\mathrm{V}} = -\frac{g^2}{4}\{2\sin^2\theta\, \Delta(W, W, 0) + 2\cos^2\theta\, \Delta(W, W, Z) - 2\Sigma(W, W)$$

$$-4\cos^2\theta\, \Sigma(W, Z) + 2A(0, 0, W) + \cos^2\theta\, A(0, 0, Z)\} \tag{5.5}$$

(the last two terms represent the ghost contribution),

$$V_{\mathrm{FV}} = -3\sum_f \left[\left(v_f^2 + a_f^2\right)D(F, F, Z) + \left(v_f^2 - a_f^2\right)E(F, F, Z)\right]$$

$$-\tfrac{3}{2}g^2(n_{\mathrm{G}} - 1)D(0, 0, W) - \tfrac{3}{2}g^2 D(T, 0, W)$$

$$-\left(4g_3^2 + \tfrac{4}{3}e^2\right)\left[D(T, T, 0) + E(T, T, 0)\right]. \tag{5.6}$$

In eq. (5.6) the sum over f is over all quarks and leptons, and v_f and a_f denote the vector and axial couplings to the Z-boson. Thus, for example,

$$v_{\mathrm{t}} = \frac{g\left(1 - \frac{8}{3}\sin^2\theta\right)}{4\cos\theta} \quad \text{and} \quad a_{\mathrm{t}} = \frac{g}{4\cos\theta}, \tag{5.7}$$

where θ is the usual weak mixing angle. $F = 0$ except for the top quark, when $F = T$. n_{G} is the number of generations; of course since we neglect all Yukawa couplings except h, our calculations are really only applicable for $n_{\mathrm{G}} = 3$. The various functions introduced in eqs. (5.4)–(5.6) are defined as follows:

$$A(x, y, z) = \frac{1}{z}\left\{-4a^2\hat{I}(x, y, z) + (x-y)^2\hat{I}(x, y, 0) + (y-x-z)\hat{J}(x, z)\right.$$

$$\left.+(x-y-z)\hat{J}(y, z) + z\hat{J}(x, y) + 2z\left(x+y-\tfrac{1}{3}z\right)J(z)\right\}, \tag{5.8}$$

$$B(x, y, z) = \frac{1}{4xy}\left\{\left(10xy + z^2 + x^2 + y^2 - 2xz - 2yz\right)\hat{I}(x, y, z)\right.$$

$$+(2zx - x^2 - z^2)\hat{I}(x, z, 0) + (2zy - y^2 - z^2)\hat{I}(y, z, 0)$$

$$+z^2\hat{I}(z, 0, 0) + (x+y-z)\hat{J}(x, y)$$

$$-y\hat{J}(x, z) - x\hat{J}(y, z) + 6xy(J(x) + J(y))$$

$$\left.-8xy\epsilon I(x, y, z)\right\}, \tag{5.9}$$

$$C(x, y) = 3\hat{J}(x, y) - 2\epsilon J(x, y) - 2xJ(y), \tag{5.10}$$

$$
\begin{aligned}
&D(x, y, z)\\
&= \frac{1}{z}\{-(x^2+y^2-2z^2+xz+yz-2xy)\hat{I}(x, y, z) + (x-y)^2\hat{I}(x, y, 0)\\
&-2z\hat{J}(x, y) + (2z+y-x)\hat{J}(x, z)\\
&+(2z+x-y)\hat{J}(y, z) + \tfrac{2}{3}z(2z-3y-3x)J(z)\\
&+2\epsilon z[(x+y-z)I(x, y, z) + J(x, y) - J(y, z) - J(x, z)]\},
\end{aligned} \tag{5.11}
$$

$$
E(x, y, z) = \sqrt{xy}\left(6\hat{I}(x, y, z) + 4J(z) - 4\epsilon I(x, y, z)\right), \tag{5.12}
$$

$$
\Delta(x, y, z) = \hat{\Delta}(x, y, z) + \hat{\Delta}(y, z, x) + \hat{\Delta}(z, x, y) \tag{5.13}
$$

where

$$
\begin{aligned}
&\hat{\Delta}(x, y, z)\\
&= \frac{1}{4xyz}\Big\{(x^4 - 8x^2yz - 2x^2y^2 + 32a^2xy)\hat{I}(x, y, z)\\
&-\left((x^2-y^2)^2 + 8(x-y)^2xy\right)\hat{I}(x, y, 0) + x^4\hat{I}(x, 0, 0)\\
&+z\hat{J}(x, y)(9x^2+9y^2-9xz-9yz+13xy-z^2) - \left(\frac{4}{d}-1\right)xyzJ(x, y)\\
&+8(x-y)^2xy\epsilon I(x, y, 0) + 8z(xz+yz-x^2-y^2-xy)\epsilon J(x, y)\\
&-4xyz\left(\frac{25}{3}x + 6y + 6z\right)J(x) - 32a^2xy\epsilon I(x, y, z)\Big\},
\end{aligned} \tag{5.14}
$$

and

$$
\Sigma(x, y) = \frac{27}{4}\hat{J}(x, y) + \left(\frac{(d-1)^3}{d} - \frac{27}{4}\right)J(x, y) - \tfrac{9}{2}(xJ(y) + yJ(x)). \tag{5.15}
$$

The a^2 that appears in eq. (5.8) and (5.14) is defined in eq. (4.12). The apparent singularities in, for example, $A(x, y, z)$ for $z = 0$ are easily seen to be spurious.

It is straightforward (but tedious) to substitute eqs. (5.8)–(5.15) in eqs. (5.2)–(5.6). The resulting expression is not particularly illuminating, however, and so we do not present it. Clearly evaluation by computer will be necessary when we come to applications, in any event.

6. The renormalisation group

$V(\phi)$ obeys the following RG equation:

$$\left(\mu\frac{\partial}{\partial\mu}+\sum_i\beta_i\frac{\partial}{\partial\lambda_i}-\gamma\phi\frac{\partial}{\partial\phi}\right)V=0 \tag{6.1}$$

where $\lambda_i=\{m^2,\lambda,g,g',g_3,h\}$. We begin by verifying eq. (6.1) at leading order. We have

$$\mathscr{D}^{(1)}V^{(0)}=-\mu\frac{\partial}{\partial\mu}V^{(1)} \tag{6.2}$$

where

$$\mathscr{D}^{(n)}=\sum_i\beta_i^{(n)}\frac{\partial}{\partial\lambda_i}-\gamma^{(n)}\phi\frac{\partial}{\partial\phi}.$$

Using eq. (2.4), eq. (6.2) becomes

$$\kappa\mathscr{D}^{(1)}V^{(0)}=\tfrac{1}{2}\left(H^2+3G^2-12T^2+6W^2+3Z^2\right). \tag{6.3}$$

Note that there is a ϕ-independent term on the r.h.s. of eq. (6.3). We therefore need to add a suitable term to $V^{(0)}$ of the form $f(\lambda_i)m^4$. Equivalently, of course, one can simply redefine $V^{(1)}$ by subtracting it at $\phi=0$ [16]. This just shifts the potential by a constant, and has no consequences for the considerations of this section. It will, however, contribute non-trivially to the RG-improved form of the potential [17].

Comparing coefficients of ϕ^4 and ϕ^2, we obtain

$$\kappa\left(\beta_\lambda^{(1)}-4\lambda\gamma^{(1)}\right)=\tfrac{1}{4}\left(16\lambda^2-144h^4+9g'^4+18g^2g'^2+27g^4\right), \tag{6.4}$$

$$\kappa\left(\beta_{m^2}^{(1)}-2m^2\gamma^{(1)}\right)=2m^2\lambda. \tag{6.5}$$

Substituting

$$\kappa\gamma^{(1)}=\tfrac{1}{4}\left(12h^2-9g^2-3g'^2\right) \tag{6.6}$$

we obtain the well-known results

$$\kappa\beta_{\lambda}^{(1)} = 4\lambda^2 + 12\lambda h^2 - 36h^4 - 9\lambda g^2 - 3\lambda g'^2 + \tfrac{9}{4}g'^4 + \tfrac{9}{2}g^2g'^2 + \tfrac{27}{4}g^4, \quad (6.7)$$

$$\kappa\beta_{m^2}^{(1)} = m^2\left(2\lambda + 6h^2 - \tfrac{9}{2}g^2 - \tfrac{3}{2}g'^2\right). \quad (6.8)$$

At the two-loop level we have

$$\mathscr{D}^{(2)}V^{(0)} = -\mu\frac{\partial}{\partial\mu}V^{(2)} - \mathscr{D}^{(1)}V^{(1)}. \quad (6.9)$$

The evaluation of $(\mu\partial/\partial\mu)V^{(2)}$ using the results of sect. 5 is a straightforward application of the following results:

$$\kappa^2\mu\frac{\partial}{\partial\mu}\hat{I}(x, y, z) = 2(L_1 - 2c), \quad (6.10a)$$

$$\kappa^2\mu\frac{\partial}{\partial\mu}\hat{J}(x, y) = 2xy(2 - \overline{\ln}\,x - \overline{\ln}\,y), \quad (6.10b)$$

$$\kappa^2\mu\frac{\partial}{\partial\mu}\{\epsilon I(x, y, z)\} = -2c, \quad (6.10c)$$

$$\kappa^2\mu\frac{\partial}{\partial\mu}\{\epsilon J(x, y)\} = 4xy, \quad (6.10d)$$

$$\kappa\mu\frac{\partial}{\partial\mu}J(x) = -2x. \quad (6.10e)$$

Using eq. (6.10) and eqs. (5.2)–(5.6) we find that

$$\begin{aligned}\kappa^2\mu\frac{\partial}{\partial\mu}V^{(2)} = \phi^4\Big\{&\tfrac{19}{18}\lambda^3 + 2h^2\lambda^2 - 7h^4\lambda - 12h^6 + \tfrac{8}{3}h^4g'^2 + 32g_3^2h^4 + g'^4h^2 \\ &- 3g^2g'^2h^2 + \tfrac{3}{32}\lambda g^4 + \tfrac{1}{32}\lambda g'^4 + \tfrac{7}{16}\lambda g^2g'^2 - \tfrac{3}{2}\lambda^2g^2 - \tfrac{1}{2}\lambda^2g'^2 - \tfrac{139}{16}g^6 \\ &+ \tfrac{5}{4}n_G\left(g^6 + \tfrac{1}{3}g^4g'^2 + \tfrac{5}{9}g^2g'^4 + \tfrac{5}{9}g'^6\right) + \tfrac{23}{48}g^4g'^2 + \tfrac{143}{96}g^2g'^4 + \tfrac{35}{96}g'^6\Big\} \\ &+ m^2\phi^2\Big\{4\lambda^2 + 12\lambda h^2 - 18h^4 - \tfrac{7}{2}\lambda\left(3g^2 + g'^2\right) \\ &- \tfrac{9}{16}g^4 + \tfrac{15}{8}g^2g'^2 - \tfrac{3}{16}g'^4\Big\} + \text{logarithmic terms}. \end{aligned} \quad (6.11)$$

The logarithmic terms in eq. (6.11) are not given explicitly because they simply *cancel* an identical set of terms from $\mathscr{D}^{(1)}V^{(1)}$. This provides an excellent check on the calculation, since there are terms of the form ln H, ln G, ln T, ln W and ln Z which must cancel separately.

The calculation of $\mathscr{D}^{(1)}V^{(1)}$ is straightforward; the necessary one-loop RG functions are given by eqs. (6.6)–(6.8) and

$$\kappa\beta_h^{(1)} = h\left(\tfrac{9}{2}h^2 - \tfrac{17}{12}g'^2 - \tfrac{9}{4}g^2 - 8g_3^2\right),$$

$$\kappa\beta_{g'} = \tfrac{5}{3}g'^3\left(\frac{4}{3}n_G + \tfrac{1}{10}\right), \qquad \kappa\beta_g = g^3\left(\tfrac{4}{3}n_G - \tfrac{43}{6}\right), \tag{6.12}$$

We hence obtain from eq. (6.9) that

$$\begin{aligned}\kappa^2\left(\beta_\lambda^{(2)} - 4\lambda\gamma^{(2)}\right) = {} & -\tfrac{28}{3}\lambda^3 - 24\lambda^2h^2 + 6\lambda^2\left(3g^2 + g'^2\right) + 24\lambda h^4 + \tfrac{99}{4}\lambda g^4 \\ & + \tfrac{15}{2}\lambda g^2g'^2 + \tfrac{33}{4}\lambda g'^4 \\ & + 180h^6 - 192h^4g_3^2 - 16h^4g'^2 - \tfrac{27}{2}h^2g^4 + 63h^2g^2g'^2 \\ & - \tfrac{57}{2}h^2g'^4 + 3\left\{\left(\tfrac{497}{8} - 8n_G\right)g^6 - \left(\tfrac{97}{24} + \tfrac{8}{3}n_G\right)g^4g'^2\right. \\ & \left. - \left(\tfrac{239}{24} + \tfrac{40}{9}n_G\right)g^2g'^4 - \left(\tfrac{59}{24} + \tfrac{40}{9}n_G\right)g'^6\right\},\end{aligned} \tag{6.13}$$

$$\kappa^2\left(\beta_{m^2}^{(2)} - 2m^2\gamma^{(2)}\right) = 2m^2\left\{-\lambda^2 - 6\lambda h^2 + 2\lambda\left(3g^2 + g'^2\right) + \tfrac{63}{16}g^4 + \tfrac{3}{8}g^2g'^2 + \tfrac{21}{16}g'^4\right\}. \tag{6.14}$$

We now require $\gamma^{(2)}$. For a general gauge theory (and in a general covariant gauge) this is given in ref. [18]; we have merely to specialise to the Landau gauge and the SM, where care must be taken to keep track of the factors. The result is

$$\begin{aligned}\kappa^2\gamma^{(2)} = {} & \tfrac{1}{6}\lambda^2 - \tfrac{27}{4}h^4 + 20g_3^2h^2 + \tfrac{45}{8}g^2h^2 + \tfrac{85}{24}g'^2h^2 \\ & + \left(\tfrac{5}{2}n_G - \tfrac{511}{32}\right)g^4 + \tfrac{9}{16}g^2g'^2 + \left(\tfrac{25}{18}n_G + \tfrac{31}{96}\right)g'^4.\end{aligned} \tag{6.15}$$

Substituting eq. (6.15) in eqs. (6.13) and (6.14) we obtain the final results

$$\begin{aligned}
\kappa^2\beta_\lambda^{(2)} = {} & -\tfrac{26}{3}\lambda^3 - 24\lambda^2h^2 + 6\lambda^2(3g^2+g'^2) - 3\lambda h^4 + 80\lambda g_3^2h^2 \\
& + \tfrac{45}{2}\lambda g^2h^2 + \tfrac{85}{6}\lambda g'^2h^2 + (10n_G - \tfrac{313}{8})\lambda g^4 + \tfrac{39}{4}\lambda g^2g'^2 \\
& + (\tfrac{50}{9}n_G + \tfrac{229}{24})\lambda g'^4 + 180h^6 - 192h^4g_3^2 - 16h^4g'^2 - \tfrac{27}{2}h^2g^4 \\
& + 63h^2g^2g'^2 - \tfrac{57}{2}h^2g'^4 \\
& + 3\big((\tfrac{497}{8} - 8n_G)g^6 - (\tfrac{97}{24} + \tfrac{8}{3}n_G)g^4g'^2 \\
& - (\tfrac{239}{24} + \tfrac{40}{9}n_G)g^2g'^4 - (\tfrac{59}{24} + \tfrac{40}{9}n_G)g'^6\big)
\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}
\kappa^2\beta_{m^2}^{(2)} = {} & 2m^2\big\{-\tfrac{5}{6}\lambda^2 - 6\lambda h^2 + 2\lambda(3g^2+g'^2) - \tfrac{27}{4}h^4 + 20g_3^2h^2 + \tfrac{45}{8}g^2h^2 \\
& + \tfrac{85}{24}g'^2h^2 + (\tfrac{5}{2}n_G - \tfrac{385}{32})g^4 + \tfrac{15}{16}g^2g'^2 + \tfrac{157}{96}g'^4\big\}.
\end{aligned} \tag{6.17}$$

We can compare our results for $\beta_\lambda^{(2)}$, eq. (6.16), directly with eq. (B.8) of ref. [6]. Agreement is complete apart from (i) the sign of the $\lambda g^2g'^2$, $\lambda g'^4$ terms; (ii) the magnitude of the $\lambda g'^4$ term.

From the result of a *general* gauge theory (eq. (4.3) of ref. [6], or see also ref. [19]) it appears that these discrepancies arise from errors in the reduction to the SM case in ref. [6] rather than in the general result *. Unfortunately the incorrect form of eq. (B.8) has been applied by other authors to running coupling analyses, e.g. ref. [20]. The numerical effect of the error on these calculations is probably small, however. The result for $\beta_{m^2}^{(2)}$ can be verified from the general formulae of ref. [19] and will be needed when we consider the RG improved form of V.

At this point it is appropriate to comment on the work of Alhendi [5]. He essentially reverses the procedure adopted in this section, in order to deduce the μ-dependent terms in $V^{(2)}$ given the MS β-functions. Unfortunately his expressions for $\gamma^{(2)}$ and $\beta_{m^2}^{(2)}$ are incorrect; he assumes for example (in our notation) that the coefficient of the g^4 term in $\gamma^{(2)}$ is just $\frac{1}{4}$ of the coefficient of the λg^4 term in $\beta_\lambda^{(2)}$. This amounts to the assertion that there are no 1PI contributions of this kind to $\beta_\lambda^{(2)}$, which is not correct (even in the Landau gauge). Aside from this comparison would still be difficult since he does not use the MS form of $V^{(1)}$. These problems aside, it is clear that by this method one can derive the form of the potential in the region $\phi^2 \ll |m^2|$ only.

* See note added.

7. Conclusions and outlook

We have presented a calculation of $V^{(2)}(\phi)$ in the standard model, using dimensional regularisation and minimal subtraction. Applying the renormalisation group to the result led to numerous checks and also expressions for $\beta_\lambda^{(2)}$ and $\beta_{m^2}^{(2)}$. We used differential equations and the method of characteristics to find the relevant Feynman integral; an approach which seems to us preferable to more traditional techniques.

Studies of the stability of the electroweak vacuum reviewed in ref. [2] suggest a limit of around $m_t \leq 95$ GeV $+ 0.6 m_H$ for the top and Higgs masses. Our calculation will enable us to probe further the sensitivity of this result to radiative corrections. We will report these calculations elsewhere; the following argument, however, suggests that it will not be affected dramatically: Instead of solving the RG equations to produce an "improved" V, suppose we take the unimproved V and simply set $\mu = \phi$. All couplings (and m^2) then become functions of ϕ; but this choice of μ means that when $\phi^2 \gg |m^2|$, the radiative corrections to V, although non-zero, have no large logarithms. As long as the running couplings remain perturbative, we may therefore expect the radiative corrections to cause only a small change in the above limit. Perturbative confidence in the limit means that should it turn out that m_t and m_H fail to satisfy it, this would provide compelling evidence for physics beyond the standard model.

While part of this work was done, one of us (T.J.) enjoyed the hospitality of the Institute of Theoretical Physics, University of California at Santa Barbara. We thank Marty Einhorn for conversations. This research was supported in part by the National Science Foundation under grant PHY89-04035, by a NATO collaboration research grant, and by the SERC.

Note added

We understand that the authors of ref. [6] now agree with our result for $\beta_\lambda^{(2)}$, eq. (6.16). Our thanks to Mike Vaughn for correspondence.

References

[1] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888
[2] M. Sher, Phys. Rep. 179 (1989) 274
[3] M.J. Duncan, R. Phillippe and M. Sher, Phys. Lett. B153 (1985) 165;
M. Sher and H.W. Zaglauer, Phys. Lett. B206 (1988) 527;
M. Lindner, M. Sher and H.W. Zaglauer, Phys. Lett. B228 (1989) 139;
J. Ellis, A. Linde and M. Sher, Phys. Lett. B252 (1990) 203

[4] K. Yamagishi, Phys. Rev. D23 (1981) 1880; Nucl. Phys. B216 (1983) 508;
M.B. Einhorn and D.R.T. Jones, Nucl. Phys. B211 (1983) 29
[5] H. Alhendi, Phys. Rev. D37 (1988) 3749
[6] M.E. Machacek and M.T. Vaughn, Nucl. Phys. B249 (1985) 70
[7] C. Ford and D.R.T. Jones, Phys. Lett. B274 (1992) 409; (erratum B285 (1992) 399)
[8] D.R.T. Jones, Nucl. Phys. B277 (1986) 153
[9] K.T. Mahanthappa and M. Sher, Phys. Rev. D22 (1980) 1711
[10] A.V. Kotikov, Phys. Lett. B254 (1991) 158; B259 (1991) 314
[11] R. Jackiw, Phys. Rev. D9 (1974) 1686
[12] N.K. Nielsen, Nucl. Phys. B101 (1975) 173;
R. Fukuda and T. Kugo, Phys. Rev. D13 (1976) 3469
[13] E. Braaten and J.P. Leveille, Phys. Rev. D24 (1981) 13
[14] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, Nucl. Phys. B174 (1980) 345;
F.V. Tkachov, Phys. Lett. B100 (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B192 (1981) 159;
D.I. Kazakov, Phys. Lett. B133 (1983) 406; Teor. Math. Fiz. 58 (1984) 210; 62 (1985) 127;
D.T. Barfoot and D.J. Broadhurst, Z. Phys. C41 (1988) 81;
D.J. Broadhurst, Z. Phys. C47 (1990) 115;
N. Gray, D.J. Broadhurst, W. Grafe and K. Schilcher, Z. Phys. C48 (1990) 673
[15] I.S. Gradshteyn and I.M. Ryzhik, Tables of integrals, series and products (Academic Press, New York, 1980)
[16] M.B. Einhorn and D.R.T. Jones, Nucl. Phys. B211 (1983) 29
[17] B. Kastening, Phys. Lett. B283 (1992) 287
[18] M.E. Machacek and M.T. Vaughn, Nucl. Phys. B222 (1983) 83
[19] I. Jack and H. Osborn, Nucl. Phys. B249 (1985) 472
[20] K.S. Babu and E. Ma, Z. Phys. C31 (1986) 45