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Four-loop vacuum energy β function in O(N) symmetric scalar theory

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The β function of the vacuum energy density is computed at the four-loop level in massive O(N) symmetric ϕ^4 theory. Dimensional regularization is used in conjunction with the $\overline{\rm MS}$ scheme and all calculations are in momentum space in the massive theory. The result is $\beta_v=(N/4)\,g+[N(N+2)/96]\,g^3+\{N(N+2)(N+8)\times[12\zeta(3)-25]/1296\}\,g^4+O(g^5)$. [S0556-2821(96)04218-X]

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I. INTRODUCTION

The β function for the coupling constant, β_g, the γ function for the mass parameter, γ_m, and the anomalous dimension γ_ϕ are known to five loops [1] in O(N) symmetric ϕ^4 theory in dimensional regularization (DR) in conjunction with the modified minimal subtraction scheme ($\overline{\rm MS}$). There is, however, another, less well-known, β function β_v related to the vacuum energy density. It was introduced in [2] to facilitate renormalization group improvement of effective potentials in massive theories, which was first performed correctly, but in a less elegant scheme, in [3]. Since then it has become a standard tool for investigations of vacuum stability in massive theories. While in flat space β_v is more a tool of calculational convenience, in curved spacetime it describes the running of the cosmological constant [4]. β_v has never been computed to higher-loop orders in any model. In this paper, we compute the vacuum energy β function to four loops in O(N) symmetric ϕ^4 theory, using DR and $\overline{\rm MS}$. To our knowledge, the highest order to which β_v has been computed in this model is one loop [5].

There are at least two other motivations for computing β_v to high-loop orders.

First, there have been recent claims about a connection between divergences in field theory and invariants of knot theory [6]. Since in any given loop order there are considerably less vacuum diagrams to compute than diagrams for two- and four-point functions, this may be an easier way of tracking the connection between field theory and knot theory. In fact, after absorbing the one-loop mass correction into a modified bare mass, there is only one diagram to compute in each one- to four-loop order in the ϕ^4 model. At five loops there are three, at six loops six diagrams.

Second, when computing the vacuum energy β function to four loops, the postulate that subdivergences are cancelled by the appropriate mass and coupling constant counterterms allows us to get as a by-product γ_m at three loops and β_g at two loops. If one can make this work to higher orders, the rule will be: Computation of β_v to n loops provides γ_m at $(n-1)$ loops and β_g at $(n-2)$ loops.

It is not clear to the present author if there is any connection to the critical theory in three dimensions via the ϵ expansion.

The structure of the paper is as follows. In Sec. II our conventions are established. In Sec. III we provide the relations between the β and γ functions on the one hand and the renormalization constants Z_x $(x=g,m,\phi,v)$ on the other hand and also give recursion relations for the components of the Z_x. Z_g and Z_m are formally reconstructed from β_g and γ_m at two and three loops, respectively. In Sec. IV the one-loop mass correction is absorbed into a modified bare mass to significantly reduce the number of vacuum diagrams to be computed. In Sec. V we finally set out to determine the vacuum energy density counterterms and β_v at the four-loop level, recovering at the same time the two-loop β_g and three-loop γ_m. The appendices are reserved for the computation of the necessary integrals.

II. DEFINITIONS AND CONVENTIONS

We work with the same conventions (except their Z_2, γ_2 are our Z_ϕ, γ_ϕ and their Z_{m^2} is our Z_m) as [1], but extend the usual Lagrange density by a constant term,

$$\mathcal{L}=\tfrac{1}{2}\,\partial_\mu\phi_{Ba}\partial_\mu\phi_{Ba}+\tfrac{1}{2}\,m_B^2\phi_B^2+\tfrac{1}{24}\,(4\pi)^2g_B(\phi_B^2)^2+\frac{m_B^4h_B}{(4\pi)^2g_B},\tag{1}$$

where $\phi_B^2\equiv\phi_{Ba}\phi_{Ba}$, repeated indices are summed over $(\mu=1,\ldots,d,\ a=1,\ldots,N)$ and the subscript B refers to bare quantities. We work in $(d=4-\epsilon)$-dimensional Euclidean space and use DR with $\overline{\rm MS}$. All our loop integrations are in momentum space in the massive theory. The connection between bare and renormalized quantities is given by

$$g_B=\mu^\epsilon Z_g g,\quad m_B^2=Z_m m^2,\quad \phi_B^2=\mu^{-\epsilon}Z_\phi\phi^2,\quad h_B=Z_h h,\tag{2}$$

where μ is the renormalization scale [connected to the $\overline{\rm MS}$ renormalization scale by $\mu^2=\overline{\mu}^2e^{\gamma_E}/(4\pi)$] and the Z's have the form

$$Z_g=1+\sum_{k=1}^{\infty}\frac{Z_{g,k}(g)}{\epsilon^k},\quad Z_{g,k}(g)=\sum_{l=k}^{\infty}Z_{kl}^g g^l,\tag{3}$$

$$Z_m=1+\sum_{k=1}^{\infty}\frac{Z_{m,k}(g)}{\epsilon^k},\quad Z_{m,k}(g)=\sum_{l=k}^{\infty}Z_{kl}^m g^l,$$

$$Z_\phi = 1 + \sum_{k=1}^{\infty} \frac{Z_{\phi,k}(g)}{\epsilon^k}, \quad Z_{\phi,k}(g) = \sum_{l=k}^{\infty} Z_{kl}^{\phi} g^l,$$

$$Z_v \equiv \frac{Z_m^2 Z_h}{Z_g} = 1 + \frac{1}{h}\sum_{k=1}^{\infty} \frac{Z_{v,k}(g)}{\epsilon^k}, \quad Z_{v,k}(g) = \sum_{l=k}^{\infty} Z_{kl}^{v} g^l.$$

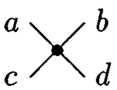
There are different ways to construct the Feynman rules as far as the treatment of counterterms is concerned. For our purposes it is most convenient to choose

propagator:

$$a \;\text{———}\; b = \frac{\delta_{ab} Z_\phi^{-1}}{p^2 + m_B^2}$$

vertices:

$$\bullet \; = - \frac{m_B^4 h_B}{(4\pi)^2 \mu^{-\epsilon} g_B} \tag{4}$$



$$= -[\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}] Z_\phi^2 \frac{(4\pi)^2 \mu^{-\epsilon} g_B}{3}.$$

Then no extra ''countervertices'' have to be considered. After computing a certain diagram, the result has to be reexpanded to the desired order in g. Since we are only interested in graphs without external legs, the wave function renormalization counterterms contained in Z_ϕ cancel from the outset since the total power of Z_ϕ for vacuum graphs is zero. We take the integration measure to be

$$\int_p \equiv \mu^\epsilon \int \frac{d^d p}{(2\pi)^d}, \tag{5}$$

so that all Feynman diagrams have integer dimension even for $\epsilon \neq 0$.

Next we define the various β and γ functions in arbitrary dimension:

$$\beta_{g,\epsilon}(g,\epsilon) = \mu^2 \left(\frac{\partial g}{\partial \mu^2}\right)_B,$$

$$\gamma_{m,\epsilon}(g,\epsilon) = \frac{\mu^2}{m^2}\left(\frac{\partial m^2}{\partial \mu^2}\right)_B, \tag{6}$$

$$\gamma_{\phi,\epsilon}(g,\epsilon) = -\frac{\mu^2}{\phi^2}\left(\frac{\partial \phi^2}{\partial \mu^2}\right)_B,$$

$$\beta_{v,\epsilon}(g,\epsilon) = \frac{g\mu^{2+\epsilon}}{m^4}\left[\frac{\partial}{\partial \mu^2}\left(\frac{m^4 h}{\mu^\epsilon g}\right)\right]_B.$$

As an aside let us mention that then the effective potential V_ϵ in $4-\epsilon$ dimensions obeys the renormalization group equation

$$\left\{\mu^2 \frac{\partial}{\partial \mu^2} + \beta_{g,\epsilon}\frac{\partial}{\partial g} + \gamma_{m,\epsilon} m^2 \frac{\partial}{\partial m^2} - \gamma_{\phi,\epsilon}\phi^2 \frac{\partial}{\partial \phi^2} + \left[\beta_{v,\epsilon} - h\left(-\frac{\epsilon}{2} + 2\gamma_{m,\epsilon} - \frac{\beta_{g,\epsilon}}{g}\right)\right]\frac{\partial}{\partial h}\right\} V_\epsilon = 0 . \tag{7}$$

Since the only term in V_ϵ containing h is $m^4 h/[(4\pi)^2 \mu^\epsilon g]$, the last equation can be written as

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta_{g,\epsilon}\frac{\partial}{\partial g} + \gamma_{m,\epsilon} m^2 \frac{\partial}{\partial m^2} - \gamma_{\phi,\epsilon}\phi^2 \frac{\partial}{\partial \phi^2}\right] V_\epsilon(g, m^2, \phi^2, h=0, \mu^2) = -\beta_{v,\epsilon} \frac{m^4}{(4\pi)^2 \mu^\epsilon g}. \tag{8}$$

Note that only the ϕ-independent part of $V_\epsilon(h=0)$ is affected by the inhomogeneous term. Therefore one can get around introducing a constant term into the Lagrange density and using β_v when renormalization group improving the effective potential by considering $V_\epsilon(\phi^2) - V_\epsilon(\phi_0^2)$ where $\partial V_\epsilon/\partial\phi = 0$ at ϕ_0 or by improving $\partial V_\epsilon/\partial\phi$ or $V_\epsilon(\phi^2)'$, since these quantities obey the usual homogeneous renormalization group equation (see [3,7]). However, those methods are less elegant.

III. RELATIONS FOR THE β_x, γ_x, AND Z_x

Using standard methods [8] one arrives at

$$\beta_{g,\epsilon}(g,\epsilon) = -\tfrac{1}{2}\epsilon g + \beta_g(g), \quad \beta_g(g) = \tfrac{1}{2} g^2 Z_{g,1}',$$

$$\gamma_{m,\epsilon}(g,\epsilon) = \gamma_m(g) = \tfrac{1}{2} g Z_{m,1}', \tag{9}$$

$$\gamma_{\phi,\epsilon}(g,\epsilon) = -\tfrac{1}{2}\epsilon + \gamma_\phi(g), \quad \gamma_\phi(g) = -\tfrac{1}{2} g Z_{\phi,1}',$$

$$\beta_{v,\epsilon}(g,\epsilon) = \beta_v(g) = \tfrac{1}{2} g Z_{v,1}',$$

where the functions without index ϵ are the ones for the four-dimensional theory, i.e., $\epsilon \to 0$. Therefore, the β and γ functions have the simple structure

$$\beta_g = \sum_{k=1}^{\infty} \beta_k g^{k+1}, \quad \gamma_m = \sum_{k=1}^{\infty} \alpha_k g^k,$$

$$\gamma_\phi = \sum_{k=1}^{\infty} \gamma_k g^k, \quad \beta_v = \sum_{k=1}^{\infty} \delta_k g^k, \tag{10}$$

where the β_k, α_k, γ_k, and δ_k are just real numbers. In the course of deriving the relations (9) one can also extract the recursion relations

$$Z_{g,k+1}' = Z_{g,1}'(g Z_{g,k})', \tag{11}$$

$$Z_{m,k+1}' = Z_{m,1}' Z_{m,k} + g Z_{g,1}' Z_{m,k}',$$

$$Z_{\phi,k+1}' = Z_{\phi,1}' Z_{\phi,k} + g Z_{g,1}' Z_{\phi,k}',$$

$$Z'_{v,k+1}=(2Z'_{m,1}-Z'_{g,1})Z_{v,k}+gZ'_{g,1}Z'_{v,k},$$

valid for $k\geqslant 1$. We use the first two relations in both Eqs. (9) and (11) together with Eqs. (3) and (10) to formally reconstruct the coupling constant and mass counterterms from β_g and γ_m to the orders needed later. For Z_g we get at the two-loop level

$$Z_g=1+\frac{2\beta_1 g+\beta_2 g^2}{\epsilon}+\frac{4\beta_1^2 g^2}{\epsilon^2}+O(g^3), \qquad (12)$$

while the three-loop approximation of Z_m is

$$Z_m=1+\frac{2\alpha_1 g+\alpha_2 g^2+\frac{2}{3}\alpha_3 g^3}{\epsilon}+\frac{2\alpha_1(\alpha_1+\beta_1)g^2+2[\alpha_1\alpha_2+(2/3)\alpha_1\beta_2+(2/3)\alpha_2\beta_1]g^3}{\epsilon^2}$$
$$+\frac{(4/3)\alpha_1(\alpha_1+\beta_1)(\alpha_1+2\beta_1)g^3}{\epsilon^3}+O(g^4). \qquad (13)$$

IV. ABSORPTION OF ONE-LOOP MASS CORRECTION INTO BARE MASS

Table I shows all vacuum graphs up to four loops, i.e., to order g^3, together with their symmetry factors. To reduce the number of diagrams to be considered, we will now absorb the one-loop mass correction into a modified bare mass term in the Lagrangian. This will get rid of all diagrams carrying a one-loop correction with the exception of the two-loop diagram.

Suppose we added a term $\frac{1}{2}\delta m_B^2\phi_B^2$ with $\delta m_B^2=O(g)$ to the free part of our Lagrangian (1) and subtracted it again in the interaction part. If then we compute all diagrams to a given order in interaction vertices and, at the end, reexpand in g to that order, we will get the same result as if we never made that manipulation. The changes in the Feynman rules are replace m_B^2 by $\overline{m_B^2}\equiv m_B^2+\delta m_B^2$ in the propagator and introduce an additional interaction vertex a b $=Z_\phi\delta m_B^2\delta_{ab}$.

In Table II we list the additional graphs up to order g^3 introduced by this resummation together with their symmetry factors.

Now choose δm_B^2 such that

a b + a  b $=0.$ (14)

Then $\delta m_B^2=O(g)$ and thus we can carry out the resummation program of the last paragraph. Notice however, that when summing the diagrams of Tables I and II (keeping in mind that the symmetry factors are considered part of the diagrams here and are not multiplying the diagrams), most of them cancel through relation (14). The only remaining diagrams to order g^3 are listed in Table III. We thus have succeeded in eliminating all diagrams with one-loop mass corrections with the exception of the two-loop diagram for which the symmetry factor does not work out, since each of the two bubbles can act as a correction to the other one.

Next we have to solve Eq. (14) for $\overline{m_B^2}$ With

a b $=\delta_{ab}Z_\phi\delta m_B^2=\delta_{ab}Z_\phi(\overline{m_B^2}-m_B^2)$ (15)

and

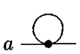
a  b $$=\frac{1}{2}\left(-Z_\phi^2\frac{(4\pi)^2\mu^{-\epsilon}g_B}{3}\right)[\delta_{ab}\delta_{cc}+2\delta_{ac}\delta_{bc}]\int_p\frac{Z_\phi^{-1}}{p^2+\bar m_B^2}$$
$$=-\frac{\delta_{ab}(N+2)(4\pi)^2 I_{1A}}{6}Z_\phi Z_g g\left(\frac{\bar m_B^2}{m^2}\right)^{1-\frac{\epsilon}{2}} \qquad (16)$$

TABLE I. Vacuum graphs up to four loops and their symmetry factors. In the equations in the text the symmetry factor is considered part of each respective diagram. The one-loop graph cannot be constructed by the Feynman rules and has to be dealt with separately. Therefore it does not carry a symmetry factor.

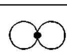
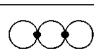
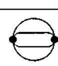
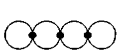
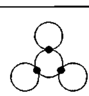
Order	Diagrams and symmetry factors.
0 loops, g^{-1}	1
1 loop, g^0	
2 loops, g^1	$\frac{1}{8}$ 
3 loops, g^2	$\frac{1}{16}$  $\frac{1}{48}$ 
4 loops, g^3	$\frac{1}{32}$  $\frac{1}{48}$  $\frac{1}{24}$  $\frac{1}{48}$ 

TABLE II. Additional vacuum graphs up to order g^3 and their symmetry factors as introduced by a quadratic interaction vertex of order g.

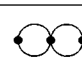
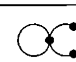
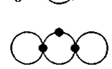
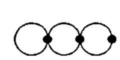
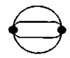
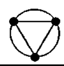
Order in g	Diagrams and symmetry factors.
g^1	$\frac{1}{2}$ 
g^2	$\frac{1}{4}$  $\frac{1}{4}$ 
g^3	$\frac{1}{6}$  $\frac{1}{8}$  $\frac{1}{4}$ 
	$\frac{1}{8}$  $\frac{1}{8}$  $\frac{1}{12}$ 

TABLE III. Remaining diagrams after resummation of the quadratic part of the Lagrangian.

Number of loops	Order in g	Remaining diagrams and revised symmetry factors
0	g^{-1}	1 •
1	g^0	
2	g^1	$-\frac{1}{8}$ 
3	g^2	$\frac{1}{48}$ 
4	g^3	$\frac{1}{48}$ 

with I_{1A} from Eq. (46), we get

$$\overline{m_B^2}=m_B^2+\frac{(N+2)(4\pi)^2 I_{1A}}{6}Z_g g\left(\frac{\overline{m_B^2}}{m^2}\right)^{1-\epsilon/2} \tag{17}$$

as the defining equation for $\overline{m_B^2}$ This cannot be solved explicitly. However, we are only interested in the first few terms of a power series of $\overline{m_B^2}$ in g. Define a_l and $\overline{a_l}$ by

$$m_B^2=Z_m m^2=m^2\left(1+\sum_{l=1}^{\infty} a_l g^l\right), \tag{18}$$

$$\overline{m_B^2}=Z_{\overline{m}} m^2=m^2\left(1+\sum_{l=1}^{\infty} \overline{a_l} g^l\right). \tag{19}$$

The a_l can be read off from Eq. (13) to be

$$a_1=\frac{2\alpha_1}{\epsilon},$$

$$a_2=\frac{\alpha_2}{\epsilon}+\frac{2\alpha_1(\alpha_1+\beta_1)}{\epsilon^2}, \tag{20}$$

$$a_3=\frac{\frac{2}{3}\alpha_3}{\epsilon}+\frac{+2[\alpha_1\alpha_2+(2/3)\alpha_1\beta_2+(2/3)\alpha_2\beta_1]}{\epsilon^2}+\frac{(4/3)\alpha_1(\alpha_1+\beta_1)(\alpha_1+2\beta_1)}{\epsilon^3}.$$

Further, define b_l by

$$\frac{(N+2)(4\pi)^2}{6}\frac{I_{1A}}{m^2}Z_g g=\sum_{l=1}^{\infty} b_l g^l, \tag{21}$$

such that the b_l are given by

$$b_1=\frac{(N+2)(4\pi)^2}{6}\frac{I_{1A}}{m^2},$$

$$b_l=\frac{(N+2)(4\pi)^2}{6}\frac{I_{1A}}{m^2}\sum_{k=1}^{l-1}\frac{Z^g_{k,l-1}}{\epsilon^k}, \quad l>1. \tag{22}$$

That is, with the help of Eqs. (3) and (12) we can write

$$b_1=\frac{(N+2)(4\pi)^2}{6}\frac{I_{1A}}{m^2},$$

$$b_2=\frac{(N+2)(4\pi)^2}{6}\frac{I_{1A}}{m^2}\frac{2\beta_1}{\epsilon}, \tag{23}$$

$$b_3=\frac{(N+2)(4\pi)^2}{6}\frac{I_{1A}}{m^2}\left(\frac{\beta_2}{\epsilon}+\frac{4\beta_1^2}{\epsilon^2}\right).$$

Now we can restate Eq. (17) as

$$\sum_{l=1}^{\infty}\overline{a_l}g^l=\sum_{l=1}^{\infty}a_l g^l+\sum_{l=1}^{\infty}b_l g^l\left(1+\sum_{l=1}^{\infty}\overline{a_l}g^l\right)^{1-\epsilon/2}. \tag{24}$$

Expanding in powers of g and comparing coefficients of powers of g, we get the relations

$$\overline{a_1}=a_1+b_1,$$

$$\overline{a_2}=a_2+b_2+\left(1-\frac{\epsilon}{2}\right)\overline{a_1}b_1, \tag{25}$$

$$\overline{a_3}=a_3+b_3+\left(1-\frac{\epsilon}{2}\right)\left[\overline{a_1}b_2+\overline{a_2}b_1+\left(-\frac{\epsilon}{2}\right)\frac{\overline{a_1^2}}{2}b_1\right],$$

which recursively define the three coefficients needed for a four-loop computation of β_v.

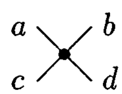
Finally, the effective Feynman rules to be used for our vacuum diagrams are

propagator:

$$a \text{ ------ } b \;=\frac{\delta_{ab}Z_\phi^{-1}}{p^2+\overline{m_B^2}}$$

vertices:

$$\bullet=-\frac{m_B^4 h_B}{(4\pi)^2\mu^{-\epsilon}g_B} \tag{26}$$



$$=-[\delta_{ab}\delta_{cd}+\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}]Z_\phi^2\frac{(4\pi)^2\mu^{-\epsilon}g_B}{3}$$

with $\overline{m_B^2}$ given at the three-loop level by Eqs. (19), (20), (23), (25) and the integral I_{1A} from Eq. (46). Note that the mass has changed only in the propagator, not in the zero-loop diagram represented by the dot. The loop momentum integration measure is again given by Eq. (5). Only the diagrams in Table III are to be calculated. The new symmetry factors are stated there, too. As a general rule, only diagrams without a one-loop mass correction have to be computed with the exception of the two-loop diagram. The two-loop diagram changes sign, while all the other diagrams to be computed have their standard symmetry factor.

V. β_v AND Z_v TO FOUR LOOPS

In order to achieve a four-loop computation of β_v, we have to keep all terms in zero to four loops up to order g^3.

Since we are interested only in the divergent part of diagrams, we will disregard terms of order ϵ^0.

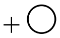
Our strategy will be to use the bare coupling and modified bare mass expressed in terms of the coefficients β_k and α_k and then to postulate the appropriate cancellation of subdivergencies by counterterms, which in practice means the demand that Z_v does not contain any logarithms of the renormalized mass. When determining β_v and Z_v to k loops by using this procedure, we will get as a by-product β_g to $k-2$ loops (since g effectively has its first vacuum loop graph appearance at two loops) and γ_m to $k-1$ loops (since for m^2 this appearance is at one loop).

A. One loop

Using the modified Feynman rules (26), the one-loop diagram is evaluated as

$$=\frac{\delta_{aa}}{2}\int_p \ln\frac{Z_\phi^{-1}}{p^2+\overline{m}_B^2}=-\frac{N}{2}Z_{\overline{m}}^{2-\epsilon/2}I_1, \tag{27}$$

where $I_1 \equiv \int_p \ln(p^2+m^2)$. With $Z_{\overline{m}}$ given by Eqs. (19), (20), (23), and (25), and I_1 from Eq. (45), one gets

$\bullet +$ 

$$=-\frac{m^4}{(4\pi)^2 g}\left[h+\left(Z_{11}^v-\frac{N}{2}\right)\frac{g}{\epsilon}+O(g^2,\epsilon^0)\right]. \tag{28}$$

Demanding this to be finite as $\epsilon \to 0$ gives

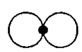
$$Z_{11}^v=\frac{N}{2}. \tag{29}$$

B. Two loops

Using again the modified Feynman rules as (remember that the two-loop diagram now enters with the opposite sign than usual)

$$=\frac{N(N+2)g}{24}Z_g Z_{\overline{m}}^{2-\epsilon}I_{1A}^2 \tag{30}$$

with I_{1A} defined in Eq. (46). Plugging in $Z_{\overline{m}}$ again, using Z_g from Eq. (12) and I_{1A} from Eq. (46) and observing Eq. (29), one gets

$\bullet +$  $+$ 

$$=-\frac{m^4}{(4\pi)^2 g}\left\{h+\left[Z_{12}^v+N\left(\alpha_1-\frac{N+2}{6}\right)\left(\ln\frac{m^2}{\bar{\mu}^2}-1\right)\right]\frac{g^2}{\epsilon}\right.$$

$$\left.+\left[Z_{22}^v-2N\left(\alpha_1-\frac{N+2}{12}\right)\right]\frac{g^2}{\epsilon^2}+O(g^3,\epsilon^0)\right\}. \tag{31}$$

Demanding this to be finite as $\epsilon \to 0$ and that the Z_{kl}^v contain no logarithms gives

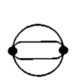
$$\alpha_1=\frac{N+2}{6} \tag{32}$$

and

$$Z_{12}^v=0, \quad Z_{22}^v=\frac{N(N+2)}{6}. \tag{33}$$

C. Three loops

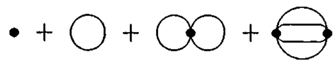
The three-loop diagram is evaluated as



$$=\frac{N(N+2)g^2}{144}Z_g^2 Z_{\overline{m}}^{2-\frac{3}{2}\epsilon}I_2^{cc}, \tag{34}$$

where I_2^{cc} belongs to the class of circle-chain integrals defined in Eq. (55).

Plugging in $Z_{\overline{m}}$ and Z_g and with I_2^{cc} from Eq. (54) and making use of Eqs. (29), (32), and (33), one gets



$$
=-\frac{m^4}{(4\pi)^2 g}\Bigg\{h+\Bigg[Z^v_{13}+\frac{N(N+2)}{8}\Big(\beta_1-\frac{N+8}{6}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)^2+\frac{N}{2}\Big(\alpha_2+\frac{5(N+2)}{36}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)+\frac{N(N+2)}{24}
\times\Big[[1+\zeta(2)]\Big(\beta_1-\frac{N+8}{6}\Big)-\frac{1}{6}\Big]\Bigg]\frac{g^3}{\epsilon}+\Bigg[Z^v_{23}-\frac{N(N+2)}{6}\Big(\beta_1-\frac{N+8}{6}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)-N\Big(\alpha_2+\frac{5(N+2)}{108}\Big)\Bigg]\frac{g^3}{\epsilon^2}
+\Bigg[Z^v_{33}-\frac{N(N+2)(N+4)}{18}\Bigg]\frac{g^3}{\epsilon^3}+O(g^4,\epsilon^0)\Bigg\}. \tag{35}
$$

Demanding this to be finite as $\epsilon\to 0$ and that the Z^v_{kl} contain no logarithms gives

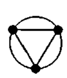
$$
\beta_1=\frac{N+8}{6},\quad \alpha_2=-\frac{5(N+2)}{36}, \tag{36}
$$

and

$$
Z^v_{13}=\frac{N(N+2)}{144},\quad Z^v_{23}=-\frac{5N(N+2)}{54},\quad Z^v_{33}=\frac{N(N+2)(N+4)}{18}. \tag{37}
$$

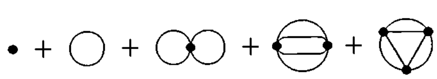
D. Four loops

The four-loop diagram is evaluated as



$$
=\frac{N(N+2)(N+8)g^3}{1296}Z_g^3 Z_{\bar{m}}^{2-2\epsilon} I_3^{cc}, \tag{38}
$$

where $I_3^{\rm cc}$ also belongs to the class of circle-chain integrals defined in Eq. (55). Plugging in $Z_{\bar{m}}$ and Z_g and with $I_3^{\rm cc}$ from Eq. (70) and making use of Eqs. (29), (32), (33), (36), and (37), one gets



$$
=-\frac{m^4}{(4\pi)^2 g}\Bigg\{h+\Bigg[Z^v_{14}+\frac{N(N+2)}{18}\Big(\beta_2+\frac{3N+14}{6}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)^2+\frac{N}{3}\Big(\alpha_3-\frac{(N+2)(5N+37)}{72}\Big)\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)
+\frac{N(N+2)}{72}\Big(\frac{43N}{36}+\frac{71}{9}+\beta_2+\Big(\beta_2+\frac{3N+14}{6}\Big)\zeta(2)-\frac{N+8}{3}\zeta(3)\Big)\Bigg]\frac{g^4}{\epsilon}+\Bigg[Z^v_{24}-\frac{N(N+2)}{18}\Big(\beta_2+\frac{3N+14}{6}\Big)
\times\Big(\ln\frac{m^2}{\bar{\mu}^2}-1\Big)-\frac{2N}{3}\Big(\alpha_3-\frac{(N+2)(N+8)}{72}\Big)\Bigg]\frac{g^4}{\epsilon^2}+\Bigg[Z^v_{34}-\frac{N(N+2)}{18}\Big(\beta_2-\frac{11N+43}{9}\Big)\Bigg]\frac{g^4}{\epsilon^3}
+\Bigg[Z^v_{44}-\frac{N(N+2)(N+4)(N+5)}{54}\Bigg]\frac{g^4}{\epsilon^4}+O(g^5,\epsilon^0)\Bigg\}. \tag{39}
$$

Demanding this to be finite as $\epsilon \to 0$ and that the Z^v_{kl} contain no logarithms gives

$$\beta_2 = -\frac{3N+14}{6}, \quad \alpha_3 = \frac{(N+2)(5N+37)}{72}, \tag{40}$$

and

$$Z^v_{14} = \frac{N(N+2)(N+8)[12\zeta(3)-25]}{2592},$$

$$Z^v_{24} = \frac{N(N+2)(4N+29)}{108},$$

$$Z^v_{34} = -\frac{N(N+2)(31N+128)}{324}, \tag{41}$$

$$Z^v_{44} = \frac{N(N+2)(N+4)(N+5)}{54}.$$

E. Check of recursion relations for the Z^v_{kl}

In this section we check the recursion relations between the Z^v_{kl} we have computed. Putting Eqs. (3) and (11) together and separating into powers of g we get

$$Z^v_{k+1,n+1} = \frac{1}{n+1} \sum_{l=1}^{n-k+1} l[2Z^m_{1l} + (n-l)Z^g_{1l}]Z^v_{k,n-l+1},$$
$$1 \leqslant k \leqslant n. \tag{42}$$

Note that to verify the recursion relations for the $(n+1)$-loop order coefficients $Z^v_{k+1,n+1}$ for all k with $1 \leqslant k \leqslant n$, we need $Z_{m,1}$ to n-loop order and, because of the $(n-l)$ factor, $Z_{g,1}$ only to $(n-1)$-loop order.

The relevant relations are

$$\begin{aligned}
Z^v_{22} &= \tfrac{1}{2} 2Z^m_{11} Z^v_{11},\\
Z^v_{23} &= \tfrac{1}{3}[(2Z^m_{11} + Z^g_{11})Z^v_{12} + 2Z^m_{12}Z^v_{11}],\\
Z^v_{33} &= \tfrac{1}{3}(2Z^m_{11} + Z^g_{11})Z^v_{22},\\
Z^v_{24} &= \tfrac{1}{4}[(2Z^m_{11} + 2Z^g_{11})Z^v_{13} + 2(2Z^m_{12} + Z^g_{12})Z^v_{12} + 3(2Z^m_{13})Z^v_{11}],\\
Z^v_{34} &= \tfrac{1}{4}[(2Z^m_{11} + 2Z^g_{11})Z^v_{23} + 2(2Z^m_{12} + Z^g_{12})Z^v_{22}],\\
Z^v_{44} &= \tfrac{1}{4}(2Z^m_{11} + 2Z^g_{11})Z^v_{33}.
\end{aligned} \tag{43}$$

The Z^v_{kl} involved are given in Eqs. (29), (33), (37), and (41). The necessary Z^g_{kl} and Z^m_{kl} can be constructed with the help of Eqs. (3), (12), and (13), using the β_k and α_k from Eqs. (32), (36), and (40).

It is straightforward to check that all of the above recursion relations hold. Also, the values found for β_1, β_2, α_1, α_2, and α_3 coincide with those in the literature, see, e.g., [1].

F. β_v to four loops

Constructing $Z_{v,1}$ to four loops from Eqs. (29), (33), (37), and (41) and using Eq. (9), we get our final result,

$$\beta_v = \frac{N}{4}g + \frac{N(N+2)}{96}g^3 + \frac{N(N+2)(N+8)[12\zeta(3)-25]}{1296} \times g^4 + O(g^5). \tag{44}$$

It would be worthwhile to continue to higher loops to be able to make meaningful comparisons of divergencies of vacuum diagrams with invariants of knot theory and to try to derive β_g and γ_m to higher loops with this method as well. Also, it would be interesting to investigate possible connections of β_v via the ϵ expansion with the critical theory in three dimensions.

ACKNOWLEDGMENTS

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APPENDIX A: I_1, I_{1A}, AND I_{1B}

Using standard methods, I_1, I_{1A}, and I_{1B} are evaluated as

$$I_1 \equiv \int_p \ln(p^2+m^2) = -\frac{m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2}-2\right), \tag{A1}$$

$$I_{1A} \equiv \int_p \frac{1}{p^2+m^2} = \frac{m^2}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2}-1\right), \tag{A2}$$

$$I_{1B} \equiv \int_p \frac{1}{(p^2+m^2)^2} = \frac{1}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right). \tag{A3}$$

APPENDIX B: I_2^{cc}

Since

$$I_2^{\mathrm{cc}} \equiv \int_{pqr} \frac{1}{[(p+q+r)^2+m^2](p^2+m^2)(q^2+m^2)(r^2+m^2)} \propto (m^2)^{(3/2)d-4}, \tag{B1}$$

we can write

$$I_2^{\mathrm{cc}}=\frac{1}{(3/2)d-4}m^2\frac{\partial}{\partial m^2}I_2^{\mathrm{cc}}=-\frac{8m^2}{3d-8}\int_{pqr}\frac{1}{[(p+q+r)^2+m^2]^2(p^2+m^2)(q^2+m^2)(r^2+m^2)}. \tag{B2}$$

Using some simple algebra on the integrand, this can be rewritten as

$$I_2^{\mathrm{cc}}=I_{2a}^{\mathrm{cc}}+I_{2b}^{\mathrm{cc}}+I_{2c}^{\mathrm{cc}}+I_{2d}^{\mathrm{cc}} \tag{B3}$$

with

$$\begin{aligned}
I_{2a}^{\mathrm{cc}}&=\frac{16m^2}{4-3\epsilon}\int_{pqr}\frac{1}{[(p+q+r)^2+m^2]^2p^2q^2r^2},\\
I_{2b}^{\mathrm{cc}}&=-\frac{24m^2}{4-3\epsilon}\int_{pqr}\frac{1}{[(p+q+r)^2+m^2]^2p^2q^2(r^2+m^2)},\\
I_{2c}^{\mathrm{cc}}&=-\frac{24m^6}{4-3\epsilon}\int_{pqr}\frac{1}{[(p+q+r)^2+m^2]^2(p^2+m^2)p^2(q^2+m^2)q^2r^2},\\
I_{2d}^{\mathrm{cc}}&=\frac{8m^8}{4-3\epsilon}\int_{pqr}\frac{1}{[(p+q+r)^2+m^2]^2(p^2+m^2)p^2(q^2+m^2)q^2(r^2+m^2)r^2}.
\end{aligned} \tag{B4}$$

I_{2c}^{cc} and I_{2d}^{cc} are UV finite in four dimensions. The evaluation of I_{2a}^{cc} and I_{2b}^{cc} is straightforward using standard methods. The results are

$$I_{2a}^{\mathrm{cc}}=\frac{2m^4}{(4\pi)^6}\left[\frac{4}{3\epsilon^2}+\frac{1}{\epsilon}\left(5-2\ln\frac{m^2}{\mu^2}\right)\right]+O(\epsilon^0) \tag{B5}$$

and

$$I_{2b}^{\mathrm{cc}}=\frac{m^4}{(4\pi)^6}\left[\frac{16}{\epsilon^3}+\frac{1}{\epsilon^2}\left(28-24\ln\frac{m^2}{\mu^2}\right)+\frac{1}{\epsilon}\left(25-42\ln\frac{m^2}{\mu^2}+18\ln^2\frac{m^2}{\mu^2}+6\zeta(2)\right)\right]+O(\epsilon^0). \tag{B6}$$

Therefore,

$$I_2^{\mathrm{cc}}=\frac{m^4}{(4\pi)^6}\left[\frac{16}{\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{92}{3}-24\ln\frac{m^2}{\mu^2}\right)+\frac{1}{\epsilon}\left(35-46\ln\frac{m^2}{\mu^2}+18\ln^2\frac{m^2}{\mu^2}+6\zeta(2)\right)\right]+I_{2,f}^{\mathrm{cc}}, \tag{B7}$$

where $I_{2,f}^{\mathrm{cc}}=O(\epsilon^0)$. In order to limit the source of π's to phase space factors, we do not evaluate $\zeta(2)=\pi^2/6$ here.

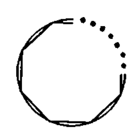
APPENDIX C: GENERAL CIRCLE-CHAIN INTEGRALS I_N^{cc}

In this section we show how to deal with the circle-chain integrals defined by

$$I_n^{\mathrm{cc}}\equiv\int_k\theta(k^2)^n, \tag{C1}$$

$$\theta(k^2)\equiv\int_p\frac{1}{[(k+p)^2+m^2](p^2+m^2)},$$

which are needed for diagrams of the form



$$. \tag{C2}$$

Note that the two-, three-, and four-loop diagrams of Table III are all of this form, and in all higher loop orders there is one diagram of this form, too.

First, separate $\theta(k^2)$ into a divergent part θ_d, independent of k^2, and a finite, k^2-dependent part $\theta_f(k^2)$ according to

$$\theta(k^2)=\theta_d+\theta_f(k^2) \tag{C3}$$

with

$$\theta_d=\int_p\frac{1}{(p^2+m^2)^2}=I_{1B}, \tag{C4}$$

$$\theta_f(k^2)=\int_p\frac{1}{p^2+m^2}\left(\frac{1}{(k+p)^2+m^2}-\frac{1}{p^2+m^2}\right),$$

with I_{1B} from Eq. (A3).

It is useful to establish the recursion relation

$$I_n^{\rm cc}=\int_k \theta_f(k^2)^n+\sum_{k=1}^{n-1}\binom{n}{k}(-1)^{n-k+1}\theta_d^{n-k}I_k^{\rm cc}, \quad \text{(C5)}$$

which follows easily from Eqs. (C1) and (C3) and the fact that $I_0^{\rm cc}=0$. For each loop order we will compute the divergent part of $\int_k \theta_f(k^2)^n$. For $n\geqslant 2$, we will denote the finite part of $I_n^{\rm cc}$ by $I_{n,f}^{\rm cc}$, such that due to the divergent nature of θ_d we will not determine the divergent part of $I_n^{\rm cc}$ completely. The remedy of the situation will be the cancellation of the divergent prefactors of the $I_{n,f}^{\rm cc}$ once the counterterms are properly taken into account. For $n=1$, we will also consider the convergent part to the order needed to avoid writing $I_{1,f}^{\rm cc}$. This is necessary since we are not considering the one- and two-loop subdivergencies in a consistent way that would allow us to avoid the appearing logarithms from the outset. However, this is no problem, since $I_1^{\rm cc}$ is the square of a simple one-loop integral and can be computed to arbitrarily high order in ϵ.

Now turn to $\theta_f(k^2)$. Define

$$\delta\equiv\frac{4m^2}{k^2+4m^2} \quad \text{(C6)}$$

and use standard methods to write

$$\theta_f(k^2)=2k_\mu\int_0^1 d\alpha\,\alpha\int_p\frac{2p_\mu+k_\mu}{[p^2+2\alpha p\cdot k+\alpha k^2+m^2]^3}=\frac{4\Gamma(\epsilon/2+1)}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2}\theta_\delta \quad \text{(C7)}$$

with

$$\theta_\delta=(1-\delta)\,\delta^{\epsilon/2}\int_0^1\frac{d\alpha\,\alpha(1-2\alpha)}{[4\alpha(1-\alpha)+(1-2\alpha)^2\delta]^{1+\epsilon/2}}=-\frac{(1-\delta)\,\delta^{\epsilon/2}}{4}\int_0^1\frac{d\beta\,\beta^{1/2}}{[1+(\delta-1)\beta]^{1+\epsilon/2}}, \quad \text{(C8)}$$

where in the last step we changed variables according to $\beta=(1-2\alpha)^2$. Use Eqs. (D1)–(D4) to write

$$\begin{aligned}\theta_\delta&=-\frac{(1-\delta)\,\delta^{\epsilon/2}}{6}F\left(1+\frac{\epsilon}{2},\frac{3}{2};\frac{5}{2};1-\delta\right)\\
&=-\frac{(1-\delta)\,\delta^{\epsilon/2}}{6}\Gamma\left(\frac{5}{2}\right)\left[\frac{\Gamma(-\epsilon/2)F(1+\epsilon/2,3/2;1+\epsilon/2;\delta)}{\Gamma(3/2-\epsilon/2)\Gamma(1)}+\frac{\delta^{-\epsilon/2}\Gamma(\epsilon/2)F(3/2-\epsilon/2,1;1-\epsilon/2;\delta)}{\Gamma(1+\epsilon/2)\Gamma(3/2)}\right]\\
&=-\frac{1-\delta}{4}\left[\frac{\Gamma(3/2)\Gamma(-\epsilon/2)}{\Gamma(3/2-\epsilon/2)}\,\delta^{\epsilon/2}(1-\delta)^{-3/2}+\frac{2}{\epsilon}F\left(\frac{3}{2}-\frac{\epsilon}{2},1;1-\frac{\epsilon}{2};\delta\right)\right]\\
&=-\frac{\Gamma(3/2)\Gamma(-\epsilon/2)}{4\Gamma(3/2-\epsilon/2)}\,\delta^{\epsilon/2}(1-\delta)^{-1/2}-\frac{1-\delta}{2\epsilon}F\left(\frac{3}{2}-\frac{\epsilon}{2},1;1-\frac{\epsilon}{2};\delta\right)\\
&=-\frac{\Gamma(3/2)\Gamma(-\epsilon/2)}{4\Gamma(3/2-\epsilon/2)}[\,\delta^{\epsilon/2}+\tfrac{1}{2}\,\delta^{1+\epsilon/2}+\tfrac{3}{8}\,\delta^{2+\epsilon/2}]-\frac{1}{2\epsilon}\left[1+\frac{1}{2-\epsilon}\,\delta+\frac{3-\epsilon}{(2-\epsilon)(4-\epsilon)}\,\delta^2\right]+O(\delta^3,\delta^{3+\epsilon/2}),\end{aligned} \quad \text{(C9)}$$

where $F(\alpha,\beta;\gamma;z)$ is Gauss's hypergeometric function, see Appendix D. Expansion in powers of ϵ shows that this expression is indeed convergent as $\epsilon\rightarrow 0$: Despite the appearance of $\Gamma(\epsilon/2)$ and $2/\epsilon$, no additional UV divergences are introduced and the only UV divergence in $\int_k\theta_f(k^2)^n$ comes from the k integral. Note that a spurious IR divergence appeared in the next-to-last line of Eq. (C9) as $(1-\delta)^{-1/2}$. However, we know that θ_δ is convergent (in fact: zero) as $\delta\rightarrow 1$ and expanding consistently in δ gets rid of this intermediate IR divergence. In other words, there is a cancelling IR divergence in the hypergeometric function on the same line.

With Eq. (C7), we get

$$\theta_f(k^2)=\frac{\Gamma(\epsilon/2)}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2}\left\{\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)}{\Gamma(3/2-\epsilon/2)}\left[\delta^{\epsilon/2}+\frac{\delta^{1+\epsilon/2}}{2}+\frac{3\,\delta^{2+\epsilon/2}}{8}\right]-\left[1+\frac{\delta}{2-\epsilon}+\frac{(3-\epsilon)\,\delta^2}{(2-\epsilon)(4-\epsilon)}\right]\right\}+O(\delta^3,\delta^{3+\epsilon/2}). \quad \text{(C10)}$$

Our strategy for computing the divergent part of $\int_k\theta_f(k^2)^n$ is now very simple: Keep only powers of δ in $\theta_f(k^2)^n$ that are δ^0, δ^1, or δ^2 when $\epsilon\rightarrow 0$ (all higher powers of δ lead to convergent k integrals). Use

$$\int_k\delta^n=\int_k\left(\frac{4m^2}{k^2+4m^2}\right)^n=\frac{16m^4}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2}\frac{2^{-\epsilon}\Gamma(n-2+\epsilon/2)}{\Gamma(n)} \quad \text{(C11)}$$

to do the k integration and expand in powers of ϵ. The terms with negative powers of ϵ give the divergent part.

Now let us check our new method for the two- and three-loop integrals $I_1^{\rm cc}$ and $I_2^{\rm cc}$ and then use it to compute the four-loop integral $I_3^{\rm cc}$. Expanding the two-loop integral

$$I_1^{\rm cc}=\int_{pk}\frac{1}{[(k+p)^2+m^2](p^2+m^2)}=I_{1A}^2=\frac{m^4}{(4\pi)^4}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma\left(\frac{\epsilon}{2}-1\right)^2 \tag{C12}$$

with I_{1A} from Eq. (A2) in powers of ϵ gives the same result as using our new method:

$$\begin{aligned}I_1^{\rm cc}&=\int_k\theta(k^2)=\int_k[\theta_d+\theta_f(k^2)]=\int_k\theta_f(k^2)\\&=\frac{\Gamma(\epsilon/2)}{(4\pi)^2}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon/2}\left\{\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)}{\Gamma(3/2-\epsilon/2)}\int_k\left[\delta^{\epsilon/2}+\frac{\delta^{1+\epsilon/2}}{2}+\frac{3\delta^{2+\epsilon/2}}{8}\right]-\int_k\left[1+\frac{\delta}{2-\epsilon}+\frac{(3-\epsilon)\delta^2}{(2-\epsilon)(4-\epsilon)}\right]\right\}+O(\epsilon^0)\\&=\frac{16m^4\Gamma(\epsilon/2)}{(4\pi)^4}\left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon}2^{-\epsilon}\left\{\frac{\Gamma(3/2)\Gamma(1-\epsilon/2)}{\Gamma(3/2-\epsilon/2)}\left[\frac{\Gamma(\epsilon-2)}{\Gamma(\epsilon/2)}+\frac{\Gamma(\epsilon-1)}{2\Gamma(1+\epsilon/2)}+\frac{3\Gamma(\epsilon)}{8\Gamma(2+\epsilon/2)}\right]\right.\\&\quad\left.-\left[\frac{\Gamma(\epsilon/2-1)}{2-\epsilon}+\frac{(3-\epsilon)\Gamma(\epsilon/2)}{(2-\epsilon)(4-\epsilon)}\right]\right\}+O(\epsilon^0)\\&=\frac{4m^4}{(4\pi)^4}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\left(\ln\frac{m^2}{\mu^2}-1\right)\right]+O(\epsilon^0).\end{aligned} \tag{C13}$$

Of course, at two loops, this method seems awfully contrived.

Using Eq. (C5), we get for the three-loop integral

$$I_2^{\rm cc}=\int_k\theta_f(k^2)^2+2\theta_dI_1^{\rm cc}. \tag{C14}$$

Evaluating the divergent part of $\int_k\theta_f(k^2)^2$ along the lines of the strategy described above and using $\theta_d=I_{1B}$ and Eq. (A3) as well as $I_1^{\rm cc}$ from Eq. (C12), one recovers Eq. (B7), as expected.

Having checked our method for two and three loops, we are now ready to compute $I_3^{\rm cc}$ and, in principle, circle-chain integrals $I_n^{\rm cc}$ to any number of loops $n+1$. Using Eq. (C5) to write

$$I_3^{\rm cc}=\int_k\theta_f(k^2)^3-3\theta_d^2I_1^{\rm cc}+3\theta_dI_2^{\rm cc}, \tag{C15}$$

we can use our strategy to evaluate $\int_k\theta_f(k^2)^3$. Remembering that we do not keep a symbolic finite part of $I_1^{\rm cc}$, but evaluate it to the necessary order in ϵ, the result is

$$\begin{aligned}I_3^{\rm cc}&=\frac{m^4}{(4\pi)^8}\left[\frac{24}{\epsilon^4}+\frac{1}{\epsilon^3}\left(-48\ln\frac{m^2}{\mu^2}+76\right)+\frac{1}{\epsilon^2}\left(48\ln^2\frac{m^2}{\mu^2}-152\ln\frac{m^2}{\mu^2}+134+12\zeta(2)\right)\right.\\&\quad\left.+\frac{1}{\epsilon}\left(22\ln^3\frac{m^2}{\mu^2}-55\ln^2\frac{m^2}{\mu^2}+[47+30\zeta(2)]\ln\frac{m^2}{\mu^2}-\frac{21}{2}-31\zeta(2)+2\zeta(3)\right)\right]+\frac{6I_{2,f}^{\rm cc}}{(4\pi)^2\epsilon}+I_{3,f}^{\rm cc},\end{aligned} \tag{C16}$$

where $I_{3,f}^{\rm cc}=O(\epsilon^0)$.

APPENDIX D: HYPERGEOMETRIC FUNCTION

Here are some formulas for Gauss's hypergeometric function $F(a,b;c;z)$, used for the computation of the circle-chain integrals I_n^{cc} The formulas are taken directly or slightly modified from [9].

$F(\alpha,\beta;\gamma;z)$ is defined by the series

$$F(\alpha,\beta;\gamma;z)=1+\frac{\alpha\beta}{\gamma\times 1}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\times 1\times 2}z^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\times 1\times 2\times 3}z^3+\cdots. \tag{D1}$$

A relevant integral is

$$\int_0^1 \frac{dx x^\mu}{(1+ax)^\nu}=\frac{1}{\mu+1}F(\nu,\mu+1;\mu+2;-a). \tag{D2}$$

A transformation formula is

$$F(\alpha,\beta;\gamma;1-z)=\Gamma(\gamma)\left[\frac{\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\alpha+\beta-\gamma+1;z)+z^{\gamma-\alpha-\beta}\frac{\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;z)\right]. \tag{D3}$$

A representation of an elementary function:

$$F(\alpha,\mu;\alpha;z)=F(\mu,\beta;\beta;z)=(1-z)^{-\mu}. \tag{D4}$$

[1] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K. G. Chetyrkin, and S. A. Larin, Phys. Lett. B **272**, 39 (1991); **319**, 545(E) (1993).

[2] M. Bando, T. Kugo, N. Maekawa, and H. Nakano, Phys. Lett. B **301**, 83 (1993).

[3] B. Kastening, Phys. Lett. B **287**, 287 (1992); ''Renormalization Group Improvement of the Effective Potential in Massive $O(N)$ Symmetric ϕ^4 Theory,'' UCLA Report No. UCLA/92/TEP/26, hep-ph/9207252 (unpublished).

[4] E. Elizalde and S. D. Odintsov, Phys. Lett. B **321**, 199 (1994); Z. Phys. C **64**, 699 (1994).

[5] E. Elizalde, K. Kirsten, and S. D. Odintsov, Phys. Rev. D **50**, 5137 (1994).

[6] D. Kreimer, ''Renormalization and Knot Theory,'' Univ. of Tasmania Report No. UTAS-PHYS-94-25, hep-th/9412045, 1994 (unpublished); Phys. Lett. B **354**, 117 (1995); in *New Computing Techniques in Physics Research IV*, edited by B. Denby and D. Perret-Gallix (World Scientific, Singapore, 1995); J. Knot Theory Ramifications (to be published); D. J. Broadhurst and D. D. Kreimer, Int. J. Mod. Phys. C **6**, 519 (1995); D. J. Broadhurst, J. A. Gracey, and D. Kreimer, ''Beyond the triangle and uniqueness relations: non-zeta counterterms at large *N* from positive knots,'' Open University Report No. OUT-4102-46, hep-th/9607174, 1996 (unpublished).

[7] C. Ford, D. R. T. Jones, P. W. Stevenson, and M. B. Einhorn, Nucl. Phys. **B395**, 17 (1993).

[8] J. C. Collins and A. J. Macfarlane, Phys. Rev. D **10**, 1201 (1974).

[9] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic Press, San Diego, 1994).