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Wormholes in a Theory of Scalar Phantom Field

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Abstract We study wormholes in a theory of scalar phantom field coupled to gravity. We consider two cases of this theory corresponding to the complex and real scalar phantom fields.

Keywords Models of gravity · Wormholes · Phantom field · General relativity

In this work we study wormholes [1–3] in a theory of scalar phantom field coupled to gravity. We consider a specific model defined by a certain action and a particular metric tensor and we choose appropriate boundary conditions which provide us the desired wormhole solution. We construct the mathematical formalism for our model and derive the equations of motion of the theory and discuss the physics of wormhole solutions expected from the model.

We consider two cases of this theory named as Model-A and Model-B. In Model-A the scalar field is considered to be complex whereas it is considered as real in the Model-B. These two models require different scalings of field variables and the constant parameters as explained later (at the appropriate place).

The action of the theory under consideration reads [3]:

$$S = \int \left[\frac{R}{16\pi G} + (\nabla_\mu \Psi)^* (\nabla^\mu \Psi) \right] \sqrt{-g} d^4x, \quad g := det(g_{\mu\nu}) \tag{1}$$

Here $G_{\mu\nu}$ is the Einstein tensor, R is the Ricci curvature scalar, G is the Newton's gravitational constant and $g_{\mu\nu}$ is the metric tensor. Ψ is the complex scalar phantom field. Also the asterisk in the above equation denotes complex conjugation. Using the variational principle the equations of motion of the theory are obtained as:

$$G_{\mu\nu} = \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 8\pi G T_{\mu\nu} \tag{2}$$

$$\nabla_\mu \left[\sqrt{-g} \nabla^\mu \Psi \right] = 0, \quad \nabla_\mu \left[\sqrt{-g} \nabla^\mu \Psi \right]^* = 0 \tag{3}$$

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where the energy momentum tensor is given by:

$$T_{\mu\nu} = \left[g_{\mu\nu}(\nabla_\alpha \Psi)^* \nabla_\beta \Psi) g^{\alpha\beta} - (\nabla_\mu \Psi)^* (\nabla_\nu \Psi) - (\nabla_\nu \Psi)^* (\nabla_\mu \Psi)\right] \tag{4}$$

Also, to construct the static spherically symmetric solutions, we work with the metric:

$$ds^2 = \left[-A(r)dt^2 + A^{-1}(r)dr^2 + B^2(r)d\theta^2 + B^2(r)\sin^2\theta d\varphi^2\right]$$

Here $A(r)$ and $B(r)$ are the metric functions and they are assumed to depend only on r. Also, for the scalar phantom field, we make the Ansatz:

$$\Psi(x^\mu) = \psi(r)e^{i\omega t} \tag{5}$$

Where ω is the frequency of the scalar phantom field.

We further introduce new field variables $h(r)$, $b(r)$ and a dimensionless constant parameter β defined as:

$$h(r) = \psi(r)/\omega, \quad b(r) = \omega\, B(r), \quad \beta = 4\pi G\omega^2 \tag{6}$$

By introducing a new dimensionless coordinate $\hat{r}$ defined by $\hat{r} = \omega r$ (implying $\frac{d}{dr} = \omega\frac{d}{d\hat{r}}$), we could write:

$$h(\hat{r}) = \psi(\hat{r})/\omega, \quad b(\hat{r}) = \omega B(\hat{r}) \tag{7}$$

Finally, the equations of motion of the theory expressed in terms of the rescaled dimensionless variables with dimensionless arguments $b(\hat{r})$ and $h(\hat{r})$ (with primes denoting the derivative with respect to $\hat{r}$) read:

$$\left[A^2b^2h'' + 2A^2bb'h' + AA'b^2h' + \omega^2b^2h\right] = 0 \tag{8}$$

$$\left[\frac{2b''A}{b} + \frac{b'^2A}{b^2} + \frac{b'A'}{b} - \frac{1}{b^2}\right] = \beta\left[Ah'^2 + \frac{h^2}{A}\right] \tag{9}$$

$$\left[\frac{b'^2A}{b^2} + \frac{b'A'}{b} - \frac{1}{b^2}\right] = \beta\left[-Ah'^2 - \frac{h^2}{A}\right] \tag{10}$$

$$\left[\frac{b''A}{b} + \frac{b'A'}{b} + \frac{A''}{2}\right] = \beta\left[Ah'^2 - \frac{h^2}{A}\right] \tag{11}$$

where the last three equations are the Einstein equations of motion in the component form. Solving these Einstein equations we obtain:

$$b'' = \left[\beta b(h')^2 + \frac{\beta b h^2}{A^2}\right] \tag{12}$$

$$A'' = \left[-\frac{2A'b'}{b} - \frac{4\beta h^2}{A}\right] \tag{13}$$

The above details describe our Model-A which corresponds to the case of a complex scalar phantom field with the frequency ω.

In the following we consider our Model-B which describes the case of the real scalar phantom field and could be easily obtained from Model-A by setting $\omega = 0$ and $\Psi(x^\mu) = \Psi^\star(x^\mu) = \psi(r)$. In Model-B, we introduce the dimensionless field variables through our re-definitions of $\psi(r)$ and $B(r)$ as:

$$B(r) = b(r)\sqrt{G}, \quad \psi(r) = h(r)/\sqrt{G} \tag{14}$$

We then introduce a dimensionless coordinate $\hat{r}$ defined by $\hat{r} = \frac{r}{\sqrt{G}}$ which implies $\frac{d}{dr} = \frac{1}{\sqrt{G}}\frac{d}{d\hat{r}}$.

The equation of motion of Model-B could now be written in terms of $b(\hat{r})$, $h(\hat{r})$ and $A(\hat{r})$. The equation of motion for the matter field in this model is:

$$A\partial_r(AB^2\psi') = 0 \tag{15}$$

which could equivalently be written as:

$$\left[A^2B^2\psi'' + 2A^2BB'\psi' + AA'B^2\psi'\right] = 0 \tag{16}$$

The Einstein equations in the component form for this model (Model-B) reads:

$$\left[\frac{2b''A}{b} + \frac{b'^2 A}{b^2} + \frac{b'A'}{b} - \frac{1}{b^2}\right] = 8\pi A h'^2 \tag{17}$$

$$\left[\frac{b'^2 A}{b^2} + \frac{b'A'}{b} - \frac{1}{b^2}\right] = -8\pi A h^2 \tag{18}$$

$$\left[\frac{b''A}{b} + \frac{b'A'}{b} + \frac{A''}{2}\right] = 8\pi A h'^2 \tag{19}$$

Solving these equations we obtain:

$$b'' = \left[8\pi b h'^2\right], \quad A'' = \left[-\frac{2A'b'}{b}\right] \tag{20}$$

The choice of our boundary conditions is guided by the work of Refs. [2,3]. In order to obtain the wormhole solutions in both the above models, we assume that the metric function $b(\hat{r})$ does not possess any zero and we also assume that $b(\hat{r})$ behaves like $|\hat{r}|$ in the asymptotic regions and possesses at least one minimum [2,3].

We also assume that $b'(0) = 0$ for some $\hat{r}_0$ and $b(\hat{r}_0) = b_0 = \hat{r}_0$. Some of the boundary conditions could be chosen by taking $\hat{r} = 0$ for simplicity which defines the position of the throat. We also assume that the throat of the wormhole is at $\hat{r} = 0$ and b_0 is the radius of the throat and that $b(0) = b_0 = \hat{r}_0 =$ constant and $b'(0) = 0$.

Also, as $(\hat{r}) \longrightarrow (-\hat{r})$, $B(\hat{r}) \longrightarrow B(-\hat{r})$ (implying symmetry under $(\hat{r}) \longrightarrow (-\hat{r})$) and $A(\hat{r} \longrightarrow +\infty) \longrightarrow 1$ and $h(\hat{r} \longrightarrow \pm\infty) \longrightarrow 0$ (where the last condition follows from the requirement of finite energy).

Further the regularity conditions at the throat imply: $b(0) = b_0 > 0$ and $A(0) = A_0 > 0$, where b_0 and A_0 are constants [3]. We also set $b_0 = 1$ and $A_0 = 1$ in our work [3]. The reflection symmetry with respect to the throat also implies: $b'(0) = 0$ and $A'(0) = 0$. Also the shift symmetry of the scalar field $h(\hat{r})$ implies $h(0) = 0$. This gives us five boundary conditions.

All together we need six boundary conditions for solving the theory described by our above two models. The sixth boundary condition (- a condition on $h'(0)$) is determined by requiring that $b''(0)$ stays positive so that the throat is formed (which is in fact, possible for any value of $(h'(0))$). This gives all the required boundary conditions under which the wormhole solutions of the problem could be studied. Under the above boundary conditions we find that

$$b''(0) = \left[\beta (h_0')^2\right] > 0 \tag{21}$$

for Model-A and

$$b''(0) = \left[8\pi (h_0')^2\right] > 0 \tag{22}$$

for Model-B.

This implies in both the cases that $b(\hat{r})$ is a minimum (because $b'(0) = 0$). We find here that the extremum of $b(\hat{r})$ corresponds to $\hat{r} = 0$. In both the models, Model-A and Model-B, our solutions correspond to a wormhole since it describes a surface of minimal area which separates two asymptotically flat regions.

More details of the above work are currently under our investigations and would be published later separately.

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