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ON THE STABILITY OF AXISYMMETRIC SYSTEMS TO AXISYMMETRIC PERTURBATIONS IN GENERAL RELATIVITY. I. THE EQUATIONS GOVERNING NONSTATIONARY, STATIONARY, AND PERTURBED SYSTEMS

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ABSTRACT

Axisymmetric systems in general relativity are considered. The field and the fluid equations that are appropriate to general nonstationary (but axisymmetric) systems are first derived. They are then specialized to yield the equations which govern stationary equilibrium. The equations which determine the evolution of small departures from equilibrium are also obtained. Related matters that are considered include the Landau-Lifshitz complex, the conserved quantities, and the constancy of the baryon number and the angular momentum (per baryon) of a fluid element as it moves. The theory is developed with a view toward establishing criteria for the stability of rotating systems to axisymmetric perturbations.

I. INTRODUCTION

The present paper is the first of a series devoted to the problem of the stability of axisymmetric systems to axisymmetric perturbations in the exact framework of general relativity. While the current lively interest in the possible significance of the Kerr metric for the "wider aspects of cosmogony" would amply justify a systematic treatment of the problem, our initial motivation for the study was more modest.

The examination of the dynamical stability of spherically symmetric configurations to radial pulsations (Fowler 1964; Chandrasekhar 1964*a*, *b*) has shown that general relativity can, under certain circumstances, initiate instabilities when Newtonian considerations would indicate stability. These general-relativistic considerations are, of course, basic to ascertaining the stability of the models for neutron stars that have been constructed. But neutron stars are known to be in a state of slow rotation—i.e., slow for them! Hartle (1967) has derived the equations that govern configurations in slow uniform rotation in general relativity. Based on these equations Hartle and Thorne (1968) have constructed models for slowly rotating neutron stars; but their discussion of the stability of their models was inconclusive.[1] However, it is known (cf. Clement 1964, 1965; Chandrasekhar and Lebovitz 1968; Lebovitz 1970) that in the Newtonian theory, the square of the frequencies of axisymmetric oscillations of slowly rotating configurations can be obtained by a simple quadrature which requires only a knowledge of the Lagrangian displacement associated with the fundamental mode of radial pulsation of the nonrotating spherical configuration and the uniform ($l = 0$)-deformation caused by the rotation. It appeared to us likely that a similar result will obtain in general relativity. More generally, in the Newtonian theory, the characteristic-value problem associated with the axisymmetric pulsations of a uniformly rotating configuration can be cast in a self-adjoint form and leads to a variational principle which shows that instability, when it sets in, must be via a neutral mode of zero frequency (cf. Chandrasekhar and Lebovitz 1968). If similar circumstances prevail in general rela-

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[1] Hartle and Thorne (1972) have meanwhile obtained a criterion for the onset of instability in a slowly rotating configuration independently of the one we have established in the second paper of this series (Chandrasekhar and Friedman 1972).

tivity—and it is not certain that they will in view of the gravitational radiation that will accompany these oscillations—then one should be able to obtain a criterion for the onset of instability by simply ascertaining when a stationary configuration can be deformed quasi-stationarily to an adjacent configuration, having the same mass and the same angular momentum, without violating any of the conditions for stationary equilibrium. Since the consideration of such quasi-stationary deformations can be effected without reference to the questions which the emission of gravitational radiation during the oscillations will entail, it appeared to us that the establishment of a criterion for the onset of instability via a neutral mode of oscillation (assuming that it is possible) would be useful in all events (for a preliminary report on the solution to this problem, see Chandrasekhar and Friedman 1971).

While the foregoing two problems provided the initial motivation for the study, it soon became apparent that there was no need to restrict the objectives of the study exclusively to their solution. A systematic investigation of the entire problem appeared both useful and possible.

The plan of the first three of the papers is the following.

The present paper (divided into three parts) is devoted to deriving the basic equations of the problem. In Part I, the field equations governing general nonstationary axisymmetric systems are obtained. In Part II, the equations of Part I are specialized to yield the equations governing stationary equilibrium; and finally in Part III, the equations of Part I are linearized about equilibrium to obtain the equations governing small departures from equilibrium. While the equations of the theory will be developed with a view toward applications to hydrodynamic systems, they can be readily adapted to wider classes of systems. In particular, the application of the equations of this paper to an examination of the vacuum solutions of Einstein's equations is immediate.

In the second paper of this series, the solutions to the two problems which initiated this study are given. In the third paper, the analysis of Paper II is specialized to vacuum metrics and to a proof of Carter's theorem. And in the fourth paper (now nearing completion) the general problem of the axisymmetric pulsation of a uniformly rotating configuration is treated allowing for the emission of gravitational radiation during the pulsation. Later papers will deal with the same problems related to the vacuum solutions.

In the meantime a different approach to these problems has been initiated by Schutz (1972*a*, *b*).

PART I

EQUATIONS GOVERNING NONSTATIONARY AXISYMMETRIC SYSTEMS

II. A FORM OF THE METRIC TENSOR APPROPRIATE FOR DESCRIBING NONSTATIONARY AXISYMMETRIC SYSTEMS

The form of the metric tensor that is appropriate for considering *stationary* axisymmetric systems has been discussed widely in the literature (e.g., Hartle and Sharp 1967, Hartle 1967, Bardeen 1970; see also the earlier papers by Lewis 1932, van Stockum 1937, Papapetrou 1953, Cohen and Brill 1968, Levy 1968). But it does not appear that the form appropriate for *nonstationary* systems which preserve at all times their axisymmetry has received much attention. We shall accordingly consider this matter in some detail.

We are restricting ourselves to systems which have axial symmetry at all times. Precisely, we are supposing that the metric tensor is independent of one of the coordinates, that coordinate (ϕ) being cyclic in the sense that we regain the same event if we increase it by 2π, the other three coordinates being held fixed.[2]

Let the spacelike coordinates, besides ϕ, be x^2 and x^3; and let t $(=x^0)$ be the timelike coordinate.

[2] Alternatively, we may say that space-time admits a group of motions along the ϕ-lines.

Consider now the *contravariant* form of the metric tensor:

$$ds^2 = g^{ij}dx_i dx_j \,. \tag{1}$$

Since all the components of this metric are (by assumption) independent of x^1 $(=\phi)$, it is clear that by a transformation involving only the remaining coordinates (x^0, x^2, x^3) we can bring the 3×3 matrix g^{ij} $(i, j = 0, 2, 3)$ to its diagonal form; we may suppose then that

$$g^{02} = g^{03} = g^{23} = 0\,; \tag{2}$$

and let

$$g^{00} = -e^{-2\nu}\,, \quad g^{22} = e^{-2\mu_2}\,, \quad \text{and} \quad g^{33} = e^{-2\mu_3}\,, \tag{3}$$

where ν, μ_2, and μ_3 are functions of x^0, x^2, and x^3. Write the remaining coefficients of g^{ij} in the forms

$$g^{01} = -\omega e^{-2\nu}\,, \quad g^{12} = q_2 e^{-2\mu_2}\,, \quad g^{13} = q_3 e^{-2\mu_3}\,,$$

and

$$g^{11} = e^{-2\psi} - \omega^2 e^{-2\nu} + q_2{}^2 e^{-2\mu_2} + q_2{}^2 e^{-2\mu_3}\,, \tag{4}$$

where ω, q_2, q_3, and ψ are further functions of x^0, x^2, and x^3.

It should be noted that in specifying the coefficients g^{ij}, as we have done in equations (2), (3), and (4), we have not restricted the gauge unduly: we have only used the possibility of reducing g^{02}, g^{03}, and g^{23} simultaneously to zero by coordinate transformations involving only x^0, x^2, and x^3.

With the contravariant form of the metric chosen in the manner we have specified, the *covariant* form of the metric becomes

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - q_2 dx^2 - q_3 dx^3 - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2\,. \tag{5}$$

(Note that in using t and x^0 interchangeably, we are adopting the convention of setting $c = 1$.)

It will be observed that the chosen form of the metric involves *seven* functions, namely, ν, ψ, μ_2, μ_3, ω, q_2, and q_3. On the other hand, since the field equations provide only six linearly independent equations, it is clear that these seven functions cannot appear in the field equations as seven independent quantities. Indeed, the fact that there are only six independent quantities can be seen as follows.

Consider the coordinate transformation

$$\phi = \phi' + f(x^{0'}, x^{2'}, x^{3'})\,, \quad x^0 = x^{0'}\,, \quad x^2 = x^{2'}\,, \quad \text{and} \quad x^3 = x^{3'}\,. \tag{6}$$

Then

$$g_{1'1'} = g_{11}\,, \quad g_{i'j'} = g_{ij} \qquad (i, j = 0, 2, 3)\,,$$

and

$$g_{i'1'} = g_{i1} + \frac{\partial f}{\partial x^{i'}} g_{11} \qquad (i = 0, 2, 3)\,. \tag{7}$$

From the last of these relations it follows that the functions q_2, q_3, and ω, as they appear in equation (5), can occur in the field equations only in the combinations

$$h_0 = \frac{\partial q_2}{\partial x^3} - \frac{\partial q_3}{\partial x^2}\,, \quad h_2 = \frac{\partial \omega}{\partial x^2} - \frac{\partial q_2}{\partial t}\,, \quad \text{and} \quad h_3 = \frac{\partial \omega}{\partial x^3} - \frac{\partial q_3}{\partial t}\,. \tag{8}$$

(We shall presently verify that they do, in fact, occur in these combinations.) Among the combinations h_0, h_2, and h_3 we have the identity

$$\frac{\partial h_2}{\partial x^3} - \frac{\partial h_3}{\partial x^2} = -\frac{\partial h_0}{\partial t}\,. \tag{9}$$

Accordingly, the seven functions ν, ψ, μ_2, μ_3, ω, q_2, and q_3 will appear in the field equations only as six independent quantities.

It will be observed that the metric (5) includes the form

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dx^2)^2 + e^{2\mu_3}(dx^3)^2 , \tag{10}$$

which is generally chosen (cf. Hartle and Sharp 1967 and Bardeen 1970) as appropriate for stationary axisymmetric systems. However, in the stationary case, when ν, ψ, μ_2, μ_3, and ω are functions of x^2 and x^3 only, one has the additional freedom of gauge to restrict the functions μ_2 and μ_3 by a coordinate condition. Thus, the conditions

$$\mu_2 = \mu_3 \quad \text{or} \quad e^{\mu_3} = x^2 e^{\mu_2} \tag{11}$$

can be imposed if, in the stationary case, one wishes to use cylindrical-polar or spherical-polar coordinates. In the nonstationary case we do not have this freedom. Thus, if we should consider time-dependent departures from equilibrium of axisymmetric systems, described in cylindrical-polar or spherical-polar coordinates in the stationary case, we must allow that in consequence of the perturbations the Eulerian changes in μ_2 and μ_3 will differ.

A further fact that should be noticed is that in the nonstationary case, we must introduce in addition the functions q_2 and q_3. Since these functions can be set equal to zero in the stationary case, we can suppose that they are quantities of the first order of smallness when we are considering small time-dependent departures from equilibrium. It will then be convenient to replace q_2 and q_3 by $\partial q_2/\partial t$ and $\partial q_3/\partial t$ to emphasize that their emergence in the metric is a consequence of the imposed time-dependence; it will, in fact, appear that these are the variables that carry the information concerning the gravitational radiation that will be emitted when a system, initially stationary, becomes time-dependent.

In concluding this section, we shall make some general remarks on the notation that will be adopted in this series of papers.

We shall have occasion to use both tensor components and tetrad components (with respect to some chosen basis of one-forms); and we shall distinguish them by enclosing the indices that refer to tetrad components in parentheses. Latin letters (as space-time indices) will be allowed the range 0, 1, 2, and 3; and Greek letters will be restricted to 2 and 3 only. Summation over repeated tensor (or tetrad) indices (Latin or Greek) will be assumed and restricted to their ranges. But the summation convention will not apply if one of the indices (such as α in $e^{2\mu_\alpha}$) is not a tensor (or a tetrad) index: e.g., summation over α is intended in combinations such as $u_\alpha \xi^\alpha$ but not in combinations such as $\xi^\alpha e^{2\mu_\alpha}$. Commas and semicolons will signify ordinary and covariant differentiation, respectively, with respect to the index (or indices) that follow. In § III (and only in § III) we write $dx^4 = idx^0$ and use capital Latin letters to take the values 2, 3, and 4; and the same conventions will apply to these indices as to the others. Finally, the convention of setting "$c = G = 1$" will be adopted.

III. THE COMPONENTS OF THE RIEMANN, THE RICCI, AND THE EINSTEIN TENSORS

The most direct way of obtaining the Einstein field-equations for the metric given by equation (5) is by Cartan's exterior calculus. For this purpose it is convenient to let

$$dt = -idx^4 , \quad \frac{\partial}{\partial t} = i\frac{\partial}{\partial x^4} , \quad \nu = \mu_4 , \quad \text{and} \quad \omega = iq_4 , \tag{12}$$

and to write the metric in the form

$$ds^2 = \Sigma_A\, e^{2\mu_A}(dx^A)^2 + e^{2\psi}(d\phi - \Sigma_A\, q_A dx^A)^2 , \tag{13}$$

where the capital Latin letters (as indices) are restricted to the values 2, 3, and 4. We next introduce a local tetrad frame $\omega^{(i)}$ by the one-forms

$$\omega^{(A)} = e^{\mu_A} dx^A \quad \text{and} \quad \omega^{(1)} = e^{\psi}(d\phi - \Sigma_A\, q_A dx^A) . \tag{14}$$

(Note that there is *no* summation over A in $e^{\mu_A}dx^A$ since A in e^{μ_A} is not a tensor index; also that we have distinguished the tetrad indices from the tensor indices by enclosing the former in parentheses.) The tetrad components of the Riemann tensor then follow from the defining equations

$$d\omega^{(i)} = -\omega^{(i)}{}_{(j)} \wedge \omega^{(j)}$$

and

$$R^{(i)}{}_{(j)(k)(l)}\omega^{(k)} \wedge \omega^{(l)} = d\omega^{(i)}{}_{(j)} + \omega^{(i)}{}_{(a)} \wedge \omega^{(a)}{}_{(j)} \,. \tag{15}$$

We find

$$\begin{aligned}
R_{(A)(B)(A)(B)} &= -\{[\mu_{A,B}\exp(\mu_A-\mu_B)]_{,B} + [\mu_{B,A}\exp(\mu_B-\mu_A)]_{,A}\}\exp(-\mu_A-\mu_B) \\
&\quad -\mu_{A,C}\mu_{B,C}\exp(-2\mu_C) - \tfrac{3}{4}(q_{A,B}-q_{B,A})^2\exp(2\psi-2\mu_A-2\mu_B)\,, \\
R_{(A)(B)(B)(C)} &= [\mu_{B,CA} + \mu_{B,A}(\mu_B-\mu_A)_{,C} - \mu_{B,C}\mu_{C,A}]\exp(-\mu_A-\mu_C) \\
&\quad -\tfrac{3}{4}(q_{A,B}-q_{B,A})(q_{B,C}-q_{C,B})\exp(2\psi-\mu_A-2\mu_B-\mu_C)\,, \\
R_{(1)(A)(1)(A)} &= -[\psi_{,A}\exp(\psi-\mu_A)]_{,A}\exp(-\psi-\mu_A) - \psi_{,B}\mu_{A,B}\exp(-2\mu_B) \\
&\quad -\psi_{,C}\mu_{A,C}\exp(-2\mu_C) + \tfrac{1}{4}[(q_{A,B}-q_{B,A})^2\exp(-2\mu_B) \\
&\quad +(q_{A,C}-q_{C,A})^2\exp(-2\mu_C)]\exp(2\psi-2\mu_A)\,, \\
R_{(1)(A)(1)(B)} &= -[\psi_{,A}\exp(\psi-\mu_A)]_{,B}\exp(-\psi-\mu_B) + \psi_{,B}\mu_{B,A}\exp(-\mu_A-\mu_B) \\
&\quad +\tfrac{1}{4}(q_{C,B}-q_{B,C})(q_{C,A}-q_{A,C})\exp(2\psi-2\mu_C-\mu_A-\mu_B)\,, \\
R_{(1)(A)(A)(B)} &= \tfrac{1}{2}[(q_{A,B}-q_{B,A})_{,A} + (q_{A,B}-q_{B,A})(3\psi-\mu_A-\mu_B)_{,A}] \\
&\quad \times\exp(\psi-2\mu_A-\mu_B) + \tfrac{1}{2}(q_{C,B}-q_{B,C})\mu_{A,C}\exp(\psi-2\mu_C-\mu_B)\,, \\
R_{(1)(A)(B)(C)} &= \tfrac{1}{2}\{[(q_{A,C}-q_{C,A})\exp(\psi-\mu_A)]_{,B} - [(q_{A,B}-q_{B,A})\exp(\psi-\mu_A)]_{,C}\} \\
&\quad \times\exp(-\mu_B-\mu_C) + \tfrac{1}{2}(q_{B,C}-q_{C,A})(2\psi-\mu_B-\mu_C)_{,A} \\
&\quad \times\exp(\psi-\mu_A-\mu_B-\mu_C)\,,
\end{aligned} \tag{16}$$

where $A \neq B \neq C$ and there are no summations.

The components of the Einstein tensor, as appropriate linear combinations of the components of the Riemann tensor, can be readily evaluated with the aid of the foregoing formulae. We find

$$\begin{aligned}
G_{(1)(A)} &= \tfrac{1}{2}\{[(q_{A,B}-q_{B,A})\exp(3\psi-\mu_A-\mu_B+\mu_C)]_{,B} \\
&\quad + [(q_{A,C}-q_{C,A})\exp(3\psi-\mu_A+\mu_B-\mu_C)]_{,C}\}\exp(-2\psi-\mu_B-\mu_C)\,, \\
G_{(A)(B)} &= -(\psi_{,AB} + \psi_{,A}\psi_{,B} - \psi_{,A}\mu_{A,B} - \psi_{,B}\mu_{B,A} + \mu_{C,AB} \\
&\quad + \mu_{C,A}\mu_{C,B} - \mu_{C,A}\mu_{A,B} - \mu_{C,A}\mu_{B,A})\exp(-\mu_A-\mu_B) \\
&\quad - \tfrac{1}{2}(q_{C,A}-q_{A,C})(q_{C,B}-q_{B,C})\exp(-2\psi-\mu_A-\mu_B-2\mu_C)\,, \\
G_{(A)(A)} &= [\psi_{,A}(\mu_C+\mu_B)_{,A} + \mu_{B,A}\mu_{C,A}]\exp(-2\mu_A) \\
&\quad + [\psi_{,BB} + \psi_{,B}(\psi+\mu_C-\mu_B)_{,B} + \mu_{C,BB} + \mu_{C,B}(\mu_C-\mu_B)_{,B}]\exp(-2\mu_B) \\
&\quad + [\psi_{,CC} + \psi_{,C}(\psi+\mu_B-\mu_C)_{,C} + \mu_{B,CC} + \mu_{B,C}(\mu_B-\mu_C)_{,C}]\exp(-2\mu_C) \\
&\quad - \tfrac{1}{4}[(q_{C,A}-q_{A,C})^2\exp(-2\mu_C) + (q_{B,A}-q_{A,B})^2\exp(-2\mu_B)] \\
&\quad \times\exp(2\psi-2\mu_A) + \tfrac{1}{4}(q_{C,B}-q_{B,C})^2\exp(2\psi-2\mu_B-2\mu_C)\,,
\end{aligned}$$

and

$$G_{(1)(1)} = -R_{(A)(B)(A)(B)} - R_{(B)(C)(B)(C)} - R_{(C)(A)(C)(A)} \,. \tag{17}$$

Reverting to the original space-time indices, 0, 1, 2, and 3, we have

$$
\begin{aligned}
G_{(0)(0)} = & -e^{-2\mu_2}[\psi_{,22} + \psi_{,2}(\psi + \mu_3 - \mu_2)_{,2} + \mu_{3,22} + \mu_{3,2}(\mu_3 - \mu_2)_{,2}] \\
& - e^{-2\mu_3}[\psi_{,33} + \psi_{,3}(\psi + \mu_2 - \mu_3)_{,3} + \mu_{2,33} + \mu_{2,3}(\mu_2 - \mu_3)_{,3}] \\
& + e^{-2\nu}[\psi_{,0}(\mu_2 + \mu_3)_{,0} + \mu_{3,0}\mu_{2,0}] - \tfrac{1}{4}e^{2\psi-2\mu_2-2\mu_3}(q_{2,3} - q_{3,2})^2 \\
& - \tfrac{1}{4}e^{2\psi-2\nu}[e^{-2\mu_2}(\omega_{,2} - q_{2,0})^2 + e^{-2\mu_3}(\omega_{,3} - q_{3,0})^2] , \\
G_{(1)(1)} = & \; e^{-2\mu_2}[\nu_{,22} + \nu_{,2}(\nu + \mu_3 - \mu_2)_{,2} + \mu_{3,22} + \mu_{3,2}(\mu_3 - \mu_2)_{,2}] \\
& + e^{-2\mu_3}[\nu_{,33} + \nu_{,3}(\nu + \mu_2 - \mu_3)_{,3} + \mu_{2,33} + \mu_{2,3}(\mu_2 - \mu_3)_{,3}] \\
& - \tfrac{3}{4}e^{2\psi-2\nu}[e^{-2\mu_2}(\omega_{,2} - q_{2,0})^2 + e^{-2\mu_3}(\omega_{,3} - q_{3,0})^2] + \tfrac{3}{4}e^{2\psi-2\mu_2-2\mu_3}(q_{2,3} - q_{3,2})^2 \\
& - e^{-2\nu}[(\mu_2 + \mu_3)_{,00} + \mu_{2,0}(\mu_2 - \nu)_{,0} + \mu_{3,0}(\mu_3 - \nu)_{,0} + \mu_{2,0}\mu_{3,0}] , \\
G_{(2)(2)} = & \; e^{-2\mu_2}[\nu_{,2}(\psi + \mu_3)_{,2} + \psi_{,2}\mu_{3,2}] - \tfrac{1}{4}e^{2\psi-2\mu_2-2\mu_3}(q_{2,3} - q_{3,2})^2 \\
& + e^{-2\mu_3}[\nu_{,33} + \nu_{,3}(\nu - \mu_3)_{,3} + \psi_{,33} + \psi_{,3}(\psi + \nu - \mu_3)_{,3}] \\
& + \tfrac{1}{4}e^{2\psi-2\nu}[e^{-2\mu_2}(\omega_{,2} - q_{2,0})^2 - e^{-2\mu_3}(\omega_{,3} - q_{3,0})^2] \\
& - e^{-2\nu}[\psi_{,00} + \psi_{,0}(\psi + \mu_3 - \nu)_{,0} + \mu_{3,00} + \mu_{3,0}(\mu_3 - \nu)_{,0}] , \\
R_{(0)(0)} = & \; e^{-2\mu_2}[\nu_{,22} + \nu_{,2}(\nu + \psi)_{,2} + \nu_{,2}(\mu_3 - \mu_2)_{,2}] \\
& + e^{-2\mu_3}[\nu_{,33} + \nu_{,3}(\nu + \psi)_{,3} + \nu_{,3}(\mu_2 - \mu_3)_{,3}] \\
& - \tfrac{1}{2}e^{2\psi-2\nu}[e^{-2\mu_2}(\omega_{,2} - q_{2,0})^2 + e^{-2\mu_3}(\omega_{,3} - q_{3,0})^2] \\
& - e^{-2\nu}[\psi_{,00} + \psi_{,0}(\psi - \nu)_{,0} + (\mu_2 + \mu_3)_{,00} + \mu_{2,0}(\mu_2 - \nu)_{,0} + \mu_{3,0}(\mu_3 - \nu)_{,0}] , \\
R_{(1)(1)} = & -e^{-2\mu_2}[\psi_{,22} + \psi_{,2}(\psi + \nu + \mu_3 - \mu_2)_{,2}] - e^{-2\mu_3}[\psi_{,33} + \psi_{,3}(\psi + \nu + \mu_2 - \mu_3)_{,3}] \\
& + e^{-2\nu}[\psi_{,00} + \psi_{,0}(\psi - \nu + \mu_2 + \mu_3)_{,0}] + \tfrac{1}{2}e^{2\psi-2\mu_2-2\mu_3}(q_{2,3} - q_{3,2})^2 \\
& - \tfrac{1}{2}e^{2\psi-2\nu}[e^{-2\mu_2}(\omega_{,2} - q_{2,0})^2 + e^{-2\mu_3}(\omega_{,3} - q_{3,0})^2] , \\
R_{(1)(2)} = & \; \tfrac{1}{2}e^{-2\psi-\nu-\mu_3}\{[e^{3\psi-\nu-\mu_2+\mu_3}(\omega_{,2} - q_{2,0})]_{,0} - [e^{3\psi+\nu-\mu_2-\mu_3}(q_{2,3} - q_{3,2})]_{,3}\} , \\
R_{(0)(2)} = & \; e^{-\nu-\mu_2}[\mu_{2,0}\psi_{,2} - \psi_{,02} + \psi_{,0}\nu_{,2} - \psi_{,0}\psi_{,2} - \mu_{3,20} - \mu_{3,2}(\mu_3 - \mu_2)_{,0} + \mu_{3,0}\nu_{,2}] \\
& - \tfrac{1}{2}e^{2\psi-\mu_2-2\mu_3-\nu}(q_{2,3} - q_{3,2})(\omega_{,3} - q_{3,0}) , \\
R_{(0)(1)} = & \; \tfrac{1}{2}e^{-2\psi-\mu_2-\mu_3}\{[e^{3\psi-\mu_2+\mu_3-\nu}(\omega_{,2} - q_{2,0})]_{,2} + [e^{3\psi-\mu_3+\mu_2-\nu}(\omega_{,3} - q_{3,0})]_{,3}\} , \\
R_{(2)(3)} = & -e^{-\mu_2-\mu_3}[\nu_{,23} + \nu_{,2}\nu_{,3} + \psi_{,23} + \psi_{,2}\psi_{,3} - \mu_{3,2}(\psi + \nu)_{,3} - \mu_{2,3}(\psi + \nu)_{,2}] \\
& + \tfrac{1}{2}e^{2\psi-2\nu-\mu_2-\mu_3}(\omega_{,2} - q_{2,0})(\omega_{,3} - q_{3,0}) .
\end{aligned}
\tag{18}
$$

The expressions for $G_{(3)(3)}$, $R_{(1)(3)}$, and $R_{(0)(3)}$ (not listed) can be obtained from the expressions for $G_{(2)(2)}$, $R_{(1)(2)}$, and $R_{(0)(2)}$ (which are listed) by interchanging the indices (2) and (3).

It will be noticed that, in agreement with what was stated in § II, the functions q_2, q_3, and ω appear in the foregoing equations only in the combinations specified in equation (8).

IV. THE EQUATIONS GOVERNING THE FLUID

In the present study, we shall limit ourselves to the case in which the source of the gravitational field is a perfect fluid described by the energy-momentum tensor

$$T^{ij} = (\epsilon + p)u^i u^j + p g^{ij} , \tag{19}$$

where ϵ and p denote the energy-density and the pressure, respectively. We shall further suppose that there exists an "equation of state" which specifies ϵ uniquely as a function of p and the baryon number N (per unit proper 3-volume):

$$\epsilon \equiv \epsilon(p, N) . \tag{20}$$

The equations of the problem are then provided by the field equations supplemented by the law of the conservation of baryon number,

$$(N u^j \sqrt{-g})_{,j} = 0 . \tag{21}$$

The required field equations can now be written down by equating the expressions for the Einstein tensor given in § III with the corresponding components of $T^{(i)(j)}$; thus

$$G^{(i)(j)} = R^{(i)(j)} - \tfrac{1}{2}g^{(i)(j)}R = 8\pi T^{(i)(j)} , \tag{22}$$

or, equivalently,

$$R^{(i)(j)} = 8\pi[T^{(i)(j)} - \tfrac{1}{2}g^{(i)(j)}T] , \tag{23}$$

where

$$T = -(\epsilon - 3p) \tag{24}$$

is the trace of T^{ij}.

To write out the field equations explicitly, we need expressions for the components of the four-velocity $u^{(i)}$.

With the definitions

$$\frac{d\phi}{dt} = \Omega , \quad \frac{dx^\alpha}{dt} = v^\alpha \qquad (\alpha = 2, 3) ,$$

$$v^{(1)} = e^{\psi-\nu}(\Omega - \omega - q_2 v^2 - q_3 v^3) , \qquad v^{(\alpha)} = e^{\mu_\alpha - \nu} v^\alpha ,$$

and

$$V^2 = [v^{(1)}]^2 + [v^{(2)}]^2 + [v^{(3)}]^2 , \tag{25}$$

we readily find that the contravariant and the covariant components of the four-velocity u^i are given by

$$u^0 = \frac{e^{-\nu}}{(1 - V^2)^{1/2}} , \quad u^1 = \Omega u^0 , \quad u^\alpha = u^0 v^\alpha , \tag{26}$$

and

$$u_0 = -\frac{e^\nu}{(1 - V^2)^{1/2}}[1 + e^{\psi-\nu}\omega v^{(1)}] ,$$

$$u_1 = u^0 e^{2\psi}(\Omega - \omega - q_2 v^2 - q_3 v^3) = \frac{v^{(1)} e^\psi}{(1 - V^2)^{1/2}} ,$$

$$u_\alpha = u^0 e^{2\mu_\alpha} v^\alpha - q_\alpha u_1 . \tag{27}$$

The corresponding tetrad components of the four-velocity can be obtained by the transformation

$$u^{(i)} = e^{(i)}{}_j u^j , \tag{28}$$

where (cf. eq. [14])

$$e^{(i)}{}_j = \begin{vmatrix} e^{\nu} & 0 & 0 & 0 \\ -\omega e^{\psi} & e^{\psi} & -q_2 e^{\psi} & -q_3 e^{\psi} \\ 0 & 0 & e^{\mu_2} & 0 \\ 0 & 0 & 0 & e^{\mu_3} \end{vmatrix} . \tag{29}$$

We find

$$u^{(0)} = \frac{1}{(1 - V^2)^{1/2}}, \quad u^{(1)} = \frac{v^{(1)}}{(1 - V^2)^{1/2}}, \quad u^{(\alpha)} = \frac{v^{(\alpha)}}{(1 - V^2)^{1/2}} . \tag{30}$$

The expressions for the components of $T^{(i)(j)}$ in terms of the components of $u^{(i)}$ given in equations (30) are simple; and the explicit form of the field equations (22) or (23) can be written by using the expressions for the Einstein and the Ricci tensors listed in § III.

a) The Equations of Hydrodynamics

The hydrodynamic equations of motion follow from the identity

$$T^{ij}{}_{;j} = [(\epsilon + p)u^i u^j + g^{ij} p]_{;j} = 0 . \tag{31}$$

A direct consequence of this identity is

$$u^j u_{i;j} = -\frac{1}{\epsilon + p}(\delta^j{}_i + u^j u_i)p_{,j} . \tag{32}$$

Considering the α-component of this last equation and writing out explicitly the expression for $u_{\alpha;j}$ in terms of the Christoffel symbols, we obtain

$$u^0 \frac{du_\alpha}{dt} - \tfrac{1}{2} u^k u^j \frac{\partial g_{kj}}{\partial x^\alpha} = -\frac{1}{(\epsilon + p)}(\delta^j{}_\alpha + u^j u_\alpha) p_{,j} , \tag{33}$$

where we have written

$$\frac{du_\alpha}{dt} = u_{\alpha,0} + v^\beta u_{\alpha,\beta} . \tag{34}$$

Now substituting in equation (33) the expression for the metric coefficients, we find

$$\begin{aligned} & u^0 \frac{d}{dt}(e^{2\mu_\alpha} u^0 v^\alpha - q_\alpha u_1) \\ & - \tfrac{1}{2}(u^0)^2 \Big[\frac{\partial}{\partial x^\alpha}(-e^{2\nu} + \omega^2 e^{2\psi}) + 2\Omega \frac{\partial}{\partial x^\alpha}(-\omega e^{2\psi}) + \Omega^2 \frac{\partial}{\partial x^\alpha}(e^{2\psi}) \\ & \qquad + (v^2)^2 \frac{\partial}{\partial x^\alpha}(e^{2\mu_2} + q_2{}^2 e^{2\psi}) + (v^3)^2 \frac{\partial}{\partial x^\alpha}(e^{2\mu_3} + q_3{}^2 e^{2\psi}) \\ & \qquad + 2v^2 v^3 \frac{\partial}{\partial x^\alpha}(q_2 q_3 e^{2\psi}) + 2\Omega v^\beta \frac{\partial}{\partial x^\alpha}(-q_\beta e^{2\psi}) + 2v^\beta \frac{\partial}{\partial x^\alpha}(q_\beta \omega e^{2\psi}) \Big] \\ & = -\frac{1}{\epsilon + p}\left(\frac{\partial p}{\partial x^\alpha} + u^0 u_\alpha \frac{dp}{dt}\right), \end{aligned} \tag{35}$$

where (cf. eq. [27])

$$u_\alpha = u^0 e^{2\mu_\alpha} v^\alpha - q_\alpha u_1 . \tag{36}$$

Equation (35) is the required hydrodynamic equation of motion.

b) The Equation Governing the Conservation of Angular Momentum

By considering the equations

$$Nu^j{}_{;j} = -N_{,j}u^j \tag{37}$$

and

$$u^j[(\epsilon + p)u_i]_{;j} + (\epsilon + p)u_i u^j{}_{;j} = -p_{,i} \tag{38}$$

(which are alternative forms of eqs. [21] and [31]), we obtain the relation

$$u^j\left(\frac{\epsilon + p}{N}u_i\right)_{;j} = -\frac{1}{N}p_{,i}\,; \tag{39}$$

or writing out fully the expression for the covariant derivative on the left-hand side, we have

$$u^j\left(\frac{\epsilon + p}{N}u_i\right)_{,j} - \frac{1}{2}\frac{\epsilon + p}{N}u^j u^k \frac{\partial g_{jk}}{\partial x^i} = -\frac{1}{N}p_{,i}\,. \tag{40}$$

The 1-component of this last equation is

$$u^j\left(\frac{\epsilon + p}{N}u_1\right)_{,j} = 0\,, \tag{41}$$

since the metric coefficients g_{jk}, as well as the pressure p, are all (by assumption) independent of x^1 $(=\phi)$. An alternative form of equation (41) is

$$\frac{d}{dt}\left(\frac{\epsilon + p}{N}u_1\right) = \left(\frac{\partial}{\partial t} + v^\alpha\frac{\partial}{\partial x^\alpha}\right)\left(\frac{\epsilon + p}{N}u_1\right) = 0\,. \tag{42}$$

This equation expresses the conservation of the angular momentum per baryon.

V. THE LANDAU-LIFSHITZ COMPLEX AND THE CONSERVED QUANTITIES

In our later considerations relating to the emission of gravitational radiation by pulsating objects, we shall find it useful to have the expressions for the Landau-Lifshitz complex in terms of which the various conserved quantities can be defined. We shall not enter here into questions concerning the appropriateness, or otherwise, of using this complex (as in the present context) in a coordinate system which is curvilinear at infinity even if it is assumed (as we shall) that space-time is asymptotically flat. We shall be content to state simply that the prescription given by Cornish (1964) for evaluating and interpreting the Landau-Lifshitz complex (and the others) appears to us valid in the contexts that are contemplated.

In applying Cornish's prescription to our present problem we shall specialize the coordinate system to a spherical-polar system at infinity. We shall write

$$ds^2 = -e^{2\nu}(dt)^2 + r^2e^{2\sigma}\sin^2\theta(d\phi - q_2dr - q_3d\theta - \omega dt)^2 + e^{2\kappa_2}(dr)^2 + r^2e^{2\kappa_3}(d\theta)^2\,, \tag{43}$$

where ν, σ, q_2, q_3, κ_2, and κ_3 are now assumed to be functions of t, r, and θ. We shall suppose that at infinity, the foregoing metric tends to the flat metric

$$ds^2 = -(dt)^2 + r^2\sin^2\theta(d\phi)^2 + (dr)^2 + r^2(d\theta)^2\,, \tag{44}$$

which, following Cornish, we shall call the b-metric.

Cornish's prescription for evaluating the Landau-Lifshitz complex, appropriately for systems described by metrics such as the one we are presently using, is the following. Let

$$\gamma^{ij} = \frac{\sqrt{-g}}{r^2\sin\theta}g^{ij} = e^{\sigma+\nu+\kappa_2+\kappa_3}g^{ij}\,; \tag{45}$$

and define in terms of γ^{ij} the quantities

$$T^{ikjl} = \gamma^{il}\gamma^{kj} - \gamma^{ij}\gamma^{kl} ,$$

and

$$\chi^{ikj} = T^{ikjl}{}_{|l} , \tag{46}$$

where $|l$ signifies covariant differentiation with respect to x^l and the b-metric. Then evaluate

$$\Phi^{ij} = \tfrac{1}{2}\chi^{ikj}{}_{|k} . \tag{47}$$

The Landau-Lifshitz "pseudo-tensor" t^{ij} and complex Θ^{ij} are now given by

$$-8\pi e^{2(\sigma+\nu+\kappa_2+\kappa_3)}t^{ij} = \Phi^{ij} + e^{2(\sigma+\nu+\kappa_2+\kappa_3)}G^{ij} ,$$

and

$$\Theta^{ij} = e^{2(\sigma+\nu+\kappa_2+\kappa_3)}(t^{ij} + T^{ij}) . \tag{48}$$

(Notice that in the foregoing equations G^{ij} and T^{ij} are the coordinate components of these quantities.)

The total energy M, the z-component of the linear momentum $P^{(z)}$, and the angular momentum J (about the z-axis) are given directly in terms of Φ^{ij} by the formulae (cf. Cornish 1964)

$$M = -\frac{1}{8\pi}\iiint_{S\to\infty}\Phi^{00}r^2\sin\theta drd\theta d\phi = -\frac{1}{16\pi}\iint_{S\to\infty}\chi^{020}r^2\sin\theta d\theta d\phi , \tag{49}$$

$$P^{(z)} = -\frac{1}{8\pi}\iiint_{S\to\infty}(\Phi^{02}\cos\theta - r\Phi^{03}\sin\theta)r^2\sin\theta drd\theta d\phi$$

$$= -\frac{1}{16\pi}\iint_{S\to\infty}(\chi^{022}\cos\theta - r\chi^{023}\sin\theta)r^2\sin\theta d\theta d\phi , \tag{50}$$

and

$$J = -\frac{1}{8\pi}\iiint_{S\to\infty}(\Phi^{01}r^2\sin^2\theta)r^2\sin\theta drd\theta d\phi , \tag{51}$$

where by the notation "$S\to\infty$" under the integral signs, it is meant that the integrals are first evaluated within a volume enclosed by a sphere of radius r (in the case of volume integrals) or on the surface of a sphere of radius r (in the case of surface integrals) and then r is allowed to tend to infinity.

A further relation that must be satisfied is

$$\frac{dM}{dt} = -\frac{1}{8\pi}\iint_{S\to\infty}\Phi^{02}r^2\sin\theta d\theta d\phi . \tag{52}$$

The evaluation of Φ^{ij} according to the foregoing formulae presents no difficulty. We find

$$2\Phi^{00} = \frac{1}{r^2}(r^2\chi^{020})_{,2} + \frac{1}{\sin\theta}(\chi^{030}\sin\theta)_{,3}$$

$$= \nabla^2 e^{2\sigma+2\kappa_2} + \frac{2}{r^2}[re^{2\sigma}(e^{2\kappa_3} - e^{2\kappa_2})]_{,2}$$

$$+ \operatorname{div}[e^{2\kappa_2}(e^{2\sigma} - e^{2\kappa_3})\operatorname{grad}(\log r\sin\theta) - \tfrac{1}{2}F\operatorname{grad}(r^2\sin^2\theta)] , \tag{53}$$

$$2\Phi^{02} = \frac{1}{r^2}(r^2\chi^{022})_{,2} + \frac{1}{\sin\theta}(\chi^{032}\sin\theta)_{,3} - r\chi^{011}\sin^2\theta - r\chi^{033}$$

$$= -(e^{2\sigma+2\kappa_2})_{,20} - \frac{1}{r}[e^{2\kappa_2}(e^{2\sigma} - e^{2\kappa_3})]_{,0} + rF_{,0}\sin^2\theta$$

$$+ \frac{\sin^2\theta}{r^2}[r^3 q_2\omega e^{2\sigma}(e^{2\kappa_3} - e^{2\kappa_2})]_{,2} + \frac{1}{r^2\sin\theta}(r\mathfrak{M})_{,3}\,, \quad (54)$$

$$2\Phi^{03} = \frac{1}{r^3}(r^3\chi^{023})_{,2} + \frac{1}{\sin\theta}(\chi^{033}\sin\theta)_{,3} + \frac{1}{r}\chi^{032} - \chi^{011}\sin\theta\cos\theta$$

$$= -\frac{1}{r^2}(e^{2\sigma+2\kappa_2})_{,30} - \frac{\cot\theta}{r^2}[e^{2\kappa_2}(e^{2\sigma} - e^{2\kappa_3})]_{,0} + F_{,0}\sin\theta\cos\theta - \frac{1}{r^2\sin\theta}(r\mathfrak{M})_{,2}\,, \quad (55)$$

$$2\Phi^{01}r^2\sin^2\theta = r^2\sin^2\theta\left[\frac{1}{r^3}(r^3\chi^{021})_{,2} + \frac{1}{\sin^2\theta}(\chi^{031}\sin^2\theta)_{,3} + \frac{1}{r}\chi^{012} + \chi^{013}\cot\theta\right]$$

$$= \text{div}\,[r^2\sin^2\theta\,\text{grad}\,(\omega e^{2\sigma+2\kappa_2})]$$

$$+ \frac{\sin^2\theta}{r^2}\{r^3[r\omega e^{2\sigma}(e^{2\kappa_3} - e^{2\kappa_2})]_{,2} - r^4(q_2 e^{2\sigma+2\kappa_3})_{,0}\}_{,2}$$

$$- \frac{1}{\sin\theta}(q_3 e^{2\sigma+2\kappa_2}\sin^3\theta)_{,30}\,, \quad (56)$$

where we have introduced the abbreviations

$$F = e^{2\sigma}\left(q_2{}^2 e^{2\kappa_3} + \frac{q_3{}^2}{r^2}e^{2\kappa_2}\right)$$

and

$$\mathfrak{M} = re^{2\sigma}\omega\left(q_2 e^{2\kappa_3}\cos\theta - \frac{q_3}{r}e^{2\kappa_2}\sin\theta\right)\sin^2\theta\,. \quad (57)$$

Also, in equations (53)–(56) (and in the sequel) "div," "grad," and "∇^2" have their Euclidean meanings in spherical-polar coordinates.

We may also note here the expressions for χ^{020}, χ^{022}, and χ^{023} which occur in the alternative formulae for M and $P^{(z)}$ given in equations (49) and (50). We have

$$\chi^{020} = (e^{2\sigma+2\kappa_2})_{,2} + \frac{1}{r}e^{2\sigma}(2e^{2\kappa_3} - e^{2\kappa_2}) - r\sin^2\theta\left(\frac{e^{2\kappa_2+2\kappa_3}}{r^2\sin^2\theta} + F\right), \quad (58)$$

$$\chi^{022} = -(e^{2\sigma+2\kappa_3})_{,0} + rq_2\omega e^{2\sigma+2\kappa_3}\sin^2\theta\,, \quad (59)$$

and

$$\chi^{023} = \frac{q_3\omega}{r}e^{2\sigma+2\kappa_2}\sin^2\theta\,. \quad (60)$$

The required components of the pseudo-tensor t^{0j} and of the complex Θ^{0j} can be obtained by combining the expressions for Φ^{0j} given in equations (53)–(56) with the corresponding (coordinate) components of the Einstein tensor in accordance with equations (48).

With Φ^{00} and χ^{020} given by equations (53) and (58), we find that the two alternative expressions for M (cf. eq. [49]) give, in agreement with each other,

$$M = -\tfrac{1}{8} \lim_{r\to\infty} \int_0^\pi \Big[\frac{\partial}{\partial r}(e^{2\sigma+2\kappa_2}) + \frac{1}{r}e^{2\kappa_2}(e^{2\sigma} - e^{2\kappa_3}) - rF\sin^2\theta + \frac{2}{r}e^{2\sigma}(e^{2\kappa_3} - e^{2\kappa_2})\Big] r^2 \sin\theta d\theta\,; \tag{61}$$

whereas equations (52) and (54) give

$$\frac{dM}{dt} = +\tfrac{1}{8} \lim_{r\to\infty} \frac{\partial}{\partial t} \int_0^\pi \Big[\frac{\partial}{\partial r}(e^{2\sigma+2\kappa_2}) + \frac{1}{r}e^{2\kappa_2}(e^{2\sigma} - e^{2\kappa_3}) - rF\sin^2\theta\Big] r^2 \sin\theta d\theta - \tfrac{1}{8} \lim_{r\to\infty} \int_0^\pi \sin^3\theta \frac{\partial}{\partial r}[r^3 e^{2\sigma} q_2 \omega (e^{2\kappa_3} - e^{2\kappa_2})] d\theta\,. \tag{62}$$

The agreement of equations (61) and (62) requires only that

$$e^{2\kappa_3} - e^{2\kappa_2} = O(r^{-2}) \quad \text{as} \quad r \to \infty\,, \tag{63}$$

since, as we shall show later, ω is $O(r^{-3})$ as $r \to \infty$.

The alternative expressions for the "conserved" linear momentum $P^{(z)}$ lead to results that are formally different: the expression in terms of Φ^{02} and Φ^{03} gives

$$P^{(z)} = -\tfrac{1}{8} \lim_{r\to\infty} \int_0^\pi [\mathfrak{M} - (e^{2\sigma+2\kappa_2})_{,0}\cos\theta + re^{2\sigma}q_2\omega(e^{2\kappa_3} - e^{2\kappa_2})\sin^2\theta\cos\theta] r^2 \sin\theta d\theta, \tag{64}$$

while the expression in terms of χ^{022} and χ^{023} gives

$$P^{(z)} = -\tfrac{1}{8} \lim_{r\to\infty} \int_0^\pi [\mathfrak{M} - (e^{2\sigma+2\kappa_2})_{,0}\cos\theta] r^2 \sin\theta d\theta\,. \tag{65}$$

And again the agreement of these two results will be assured if equation (63) holds.

Finally, we find with the aid of equation (56) that the expression (51) for the "conserved" angular momentum gives

$$J = -\tfrac{1}{8} \lim_{r\to\infty} \int_0^\pi \Big\{ r^2 \sin^2\theta \Big[\frac{\partial}{\partial r}(\omega e^{2\sigma+2\kappa_2}) - \frac{\partial}{\partial t}(q_2 e^{2\sigma+2\kappa_3})\Big] + r\sin^2\theta \frac{\partial}{\partial r}[re^{2\sigma}\omega(e^{2\kappa_3} - e^{2\kappa_2})]\Big\} r^2 \sin\theta d\theta\,. \tag{66}$$

PART II

EQUATIONS GOVERNING EQUILIBRIUM

VI. THE FIELD EQUATIONS AND THE EQUATION OF HYDRODYNAMIC EQUILIBRIUM

In the stationary case, there can be no motions in the x^2- and the x^3-directions: only rotational motions in the x^1-direction (specified by Ω) can prevail. Also, as we have stated in § II, the functions q_2 and q_3 can be set equal to zero under stationary conditions; and we have a further freedom of gauge to relate μ_2 and μ_3 in any manner that may be convenient.

The components of the four-velocity appropriate under stationary conditions are (cf. eqs. [25]–[27] and [30])

$$v^{(1)} = V = e^{\psi-\nu}(\Omega - \omega)\,, \qquad v^{\alpha} = v^{(\alpha)} = 0\,,$$

$$u^0 = \frac{e^{-\nu}}{(1-V^2)^{1/2}}\,, \qquad u^1 = \Omega u^0\,, \qquad u^{\alpha} = 0\,,$$

$$u_0 = -\frac{e^{\nu}}{(1-V^2)^{1/2}}(1 + e^{\psi-\nu}\omega V)\,, \qquad u_{\alpha} = 0\,,$$

$$u_1 = e^{2\psi-\nu}\frac{\Omega-\omega}{(1-V^2)^{1/2}} = \frac{e^{\psi}V}{(1-V^2)^{1/2}}\,,$$

$$u^{(0)} = \frac{1}{(1-V^2)^{1/2}}\,, \qquad u^{(1)} = \frac{V}{(1-V^2)^{1/2}}\,, \qquad u^{(\alpha)} = 0\,. \tag{67}$$

The components of the Einstein and the Ricci tensors that are appropriate under stationary conditions can be obtained by simply discarding in the expressions listed in § III (eqs. [18]) all terms whose survival depends explicitly on the time-dependence of the system.

In writing the field equations it is convenient to introduce the following abbreviations:

$$X = e^{\mu_3-\mu_2}(\omega_{,2})^2 + e^{\mu_2-\mu_3}(\omega_{,3})^2\,, \qquad Y = e^{\mu_3-\mu_2}(\omega_{,2})^2 - e^{\mu_2-\mu_3}(\omega_{,3})^2$$

$$U = e^{\beta}[e^{\mu_3-\mu_2}(\beta_{,2}\mu_{3,2} + \psi_{,2}\nu_{,2}) + e^{\mu_2-\mu_3}(\beta_{,3}\mu_{2,3} + \psi_{,3}\nu_{,3})]\,,$$

$$W = e^{\beta}[e^{\mu_3-\mu_2}(\beta_{,2}\mu_{3,2} + \psi_{,2}\nu_{,2}) - e^{\mu_2-\mu_3}(\beta_{,3}\mu_{2,3} + \psi_{,3}\nu_{,3})]\,. \tag{68}$$

Also, we shall define

$$\beta = \psi + \nu\,. \tag{69}$$

Indicating in each case the field equation that is involved, we have

$$\begin{aligned} &e^{\mu_3-\mu_2}[\psi_{,22} + \psi_{,2}\psi_{,2} + \mu_{3,22} + (\psi+\mu_3)_{,2}(\mu_3-\mu_2)_{,2}] \\ &+ e^{\mu_2-\mu_3}[\psi_{,33} + \psi_{,3}\psi_{,3} + \mu_{2,33} + (\psi+\mu_2)_{,3}(\mu_2-\mu_3)_{,3}] \\ &+ \tfrac{1}{4}e^{2\psi-2\nu}X = -8\pi e^{\mu_2+\mu_3}\left(\frac{\epsilon+p}{1-V^2} - p\right) \qquad (G^{00})\,, \end{aligned} \tag{70}$$

$$\begin{aligned} &e^{\mu_3-\mu_2}[\nu_{,22} + \nu_{,2}\beta_{,2} + \nu_{,2}(\mu_3-\mu_2)_{,2}] + e^{\mu_2-\mu_3}[\nu_{,33} + \nu_{,3}\beta_{,3} + \nu_{,3}(\mu_2-\mu_3)_{,3}] \\ &- \tfrac{1}{2}e^{2\psi-2\nu}X = 4\pi e^{\mu_2+\mu_3}\left[(\epsilon+p)\frac{1+V^2}{1-V^2} + 2p\right] \qquad (R^{00})\,, \end{aligned} \tag{71}$$

$$\begin{aligned} &e^{\mu_3-\mu_2}[\nu_{,22} + \nu_{,2}\nu_{,2} + (\nu+\mu_3)_{,2}(\mu_3-\mu_2)_{,2} + \mu_{3,22}] \\ &+ e^{\mu_2-\mu_3}[\nu_{,33} + \nu_{,3}\nu_{,3} + (\nu+\mu_2)_{,3}(\mu_2-\mu_3)_{,3} + \mu_{2,33}] \\ &- \tfrac{3}{4}e^{2\psi-2\nu}X = 8\pi e^{\mu_2+\mu_3}\left[(\epsilon+p)\frac{V^2}{1-V^2} + p\right] \qquad (G^{11})\,, \end{aligned} \tag{72}$$

$$(e^{3\psi-\mu_2+\mu_3-\nu}\omega_{,2})_{,2} + (e^{3\psi-\mu_3+\mu_2-\nu}\omega_{,3})_{,3} = -16\pi(\epsilon+p)e^{2\psi+\mu_2+\mu_3}\frac{V}{1-V^2} \qquad (R^{01})\,, \tag{73}$$

and

$$e^{\mu_3-\mu_2}[\beta_{,22}+\beta_{,2}(\beta+\mu_3-\mu_2)_{,2}]$$
$$+\,e^{\mu_2-\mu_3}[\beta_{,33}+\beta_{,3}(\beta+\mu_2-\mu_3)_{,3}]=16\pi p e^{\mu_2+\mu_3}\qquad (G^{22}+G^{33})\,. \tag{74}$$

An alternative form of equation (74) is

$$[e^{\mu_3-\mu_2}(e^{\beta})_{,2}]_{,2}+[e^{\mu_2-\mu_3}(e^{\beta})_{,3}]_{,3}=16\pi p\sqrt{-g}\,. \tag{75}$$

And finally,

$$[e^{\mu_3-\mu_2}(e^{\beta})_{,2}]_{,2}-[e^{\mu_2-\mu_3}(e^{\beta})_{,3}]_{,3}=2W+\tfrac{1}{2}e^{2\psi-2\nu+\beta}Y\qquad (G^{22}-G^{33})\,. \tag{76}$$

Certain linear combinations of the foregoing equations which we shall find useful are

$$e^{\mu_3-\mu_2}[\nu_{,22}+\nu_{,2}(2\beta-\nu)_{,2}-\mu_{3,22}+(\nu-\mu_3)_{,2}(\mu_3-\mu_2)_{,2}]$$
$$+\,e^{\mu_2-\mu_3}[\nu_{,33}+\nu_{,3}(2\beta-\nu)_{,3}-\mu_{2,33}+(\nu-\mu_2)_{,3}(\mu_2-\mu_3)_{,3}]$$
$$-\,\tfrac{1}{4}e^{2\psi-2\nu}X=8\pi e^{\mu_2+\mu_3}\left(\frac{\epsilon+p}{1-V^2}+p\right) \tag{77}$$

and

$$e^{\mu_3-\mu_2}[(\psi-\mu_3)_{,22}+\psi_{,2}(\psi+2\nu)_{,2}+(\psi-\mu_3)_{,2}(\mu_3-\mu_2)_{,2}]$$
$$+\,e^{\mu_2-\mu_3}[(\psi-\mu_2)_{,33}+\psi_{,3}(\psi+2\nu)_{,3}+(\psi-\mu_2)_{,3}(\mu_2-\mu_3)_{,3}]$$
$$+\,\tfrac{3}{4}e^{2\psi-2\nu}X=-8\pi e^{\mu_2+\mu_3}\left[\frac{\epsilon+pV^2}{1-V^2}-(\epsilon+p)\right]. \tag{78}$$

a) The Equation of Hydrodynamic Equilibrium

In the stationary case, equation (35) reduces to the equation governing hydrodynamic equilibrium. We have

$$\frac{1}{\epsilon+p}\,p_{,\alpha}=-\tfrac{1}{2}(u^0)^2[(e^{2\nu}-\omega^2e^{2\psi})_{,\alpha}+2\Omega(\omega e^{2\psi})_{,\alpha}-\Omega^2(e^{2\psi})_{,\alpha}]\,. \tag{79}$$

If the *rotation is uniform* and Ω *is a constant*, the equation takes a particularly simple form; we have

$$\begin{aligned}\frac{1}{\epsilon+p}\,p_{,\alpha}&=-\tfrac{1}{2}(u^0)^2\{e^{2\nu}[1-e^{2\psi-2\nu}(\Omega-\omega)^2\}_{,\alpha}\\&=-\tfrac{1}{2}(u^0)^2[e^{2\nu}(1-V^2)]_{,\alpha}=-\tfrac{1}{2}(u^0)^2\left[\frac{1}{(u^0)^2}\right]_{,\alpha},\end{aligned} \tag{80}$$

or, alternatively,

$$p_{,\alpha}=(\epsilon+p)(\log u^0)_{,\alpha}\,. \tag{81}$$

VII. THE ASYMPTOTIC BEHAVIOR OF THE POTENTIALS AT INFINITY

For many purposes it is important to know the asymptotic behavior of the potentials ν, ψ, ω, μ_1, and μ_2 at infinity. It is most convenient to consider their behavior at infinity in a system of spherical-polar coordinates. We shall accordingly set (making use of the freedom of gauge now available)

$$x^2=r\,,\qquad x^3=\theta\,,$$

and

$$e^{\psi}=re^{\eta+\zeta}\sin\theta\,,\qquad e^{\mu_2}=e^{\eta-\zeta}\,,\qquad\text{and}\qquad e^{\mu_3}=re^{\eta-\zeta}\,. \tag{82}$$

In these variables the field equations (70), (71), (72), (73), and (78) become

$$\nabla^2\eta + \tfrac{1}{2}|\text{grad}\,(\eta + \zeta)|^2 + \text{grad}\,\zeta\cdot\text{grad}\,[\log\,(r\sin\theta)]$$
$$= -\tfrac{1}{8}r^2e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2\sin^2\theta - 4\pi e^{2\eta-2\zeta}\frac{\epsilon + pV^2}{1 - V^2}\,, \tag{83}$$

$$\nabla^2\nu + \text{grad}\,\nu\cdot\text{grad}\,(\nu + \eta + \zeta) = \tfrac{1}{2}r^2e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2\sin^2\theta$$
$$+ 4\pi e^{2\eta-2\zeta}\left[(\epsilon + p)\frac{1 + V^2}{1 - V^2} + 2p\right]\,, \tag{84}$$

$$\frac{\partial^2}{\partial r^2}(\nu + \eta - \zeta) + \frac{1}{r}\frac{\partial}{\partial r}(\nu + \eta - \zeta) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\nu + \eta - \zeta) + |\text{grad}\,\nu|^2$$
$$= \tfrac{3}{4}r^2e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2\sin^2\theta + 8\pi e^{2\eta-2\zeta}\left[(\epsilon + p)\frac{V^2}{1 - V^2} + p\right]\,, \tag{85}$$

$$\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\zeta}{\partial\theta^2} + \tfrac{1}{2}\,\text{grad}\,(\eta + \zeta)\cdot\text{grad}\,(\eta + \zeta + 2\nu)$$
$$+ \text{grad}\,(\eta + \zeta + \nu)\cdot\text{grad}\,[\log\,(r\sin\theta)]$$
$$= -\tfrac{3}{8}r^2e^{2\eta+2\zeta-2\nu}|\text{grad}\,\omega|^2\sin^2\theta - 4\pi e^{2\eta-2\zeta}\left[\frac{\epsilon + pV^2}{1 - V^2} - (\epsilon + p)\right]\,, \tag{86}$$

and

$$\text{div}\,(e^{3\eta+3\zeta-\nu}r^2\sin^2\theta\,\text{grad}\,\omega) = -16\pi(\epsilon + p)re^{4\eta}\frac{V}{1 - V^2}\sin\theta\,. \tag{87}$$

The foregoing equations are not all linearly independent: equation (86) is, in fact, a linear combination of the preceding three.

We are interested in the asymptotic behavior of the potentials η, ζ, ν, and ω as $r \to \infty$ outside of the fluid sources. In these regions $\epsilon = p = 0$. Also, as we shall show in § VII*a* below, ω is $O(r^{-3})$ as $r \to \infty$; as a result the terms in $r^2|\text{grad}\,\omega|^2$ on the right-hand sides of equations (83)–(86) are of $O(r^{-6})$; and this is an order higher than any that we shall retain on the left-hand sides of these equations. Accordingly, we are effectively concerned only with the solutions of the associated homogeneous equations.

We shall presently verify that

$$\zeta = O(r^{-2}) \quad \text{as} \quad r \to \infty\,. \tag{88}$$

Therefore, in equation (83) the two terms on the left-hand side, besides $\nabla^2\eta$, are at most of $O(r^{-4})$; and we may conclude that

$$\eta = \frac{M}{r} + O(r^{-2})\,, \tag{89}$$

where M is a constant. From equation (84) we may similarly conclude that ν is of $O(r^{-1})$. In equation (85), the terms arising from ζ are of $O(r^{-4})$—on the assumption (88) to be verified—and $|\text{grad}\,\nu|^2$ is also of $O(r^{-4})$. From this last result it follows that

$$\nu + \eta = O(r^{-2})\,; \tag{90}$$

and we infer that (cf. eq. [89])

$$\nu = -M/r + O(r^{-2})\,. \tag{91}$$

Returning to equation (84), we observe that by virtue of equations (88) and (90) the term grad $\nu \cdot$ grad $(\nu + \eta + \zeta)$ is of $O(r^{-5})$. Accordingly, we may sharpen the result (91) to the stronger statement

$$\nu = -M/r + O(r^{-3}) . \tag{92}$$

Again, since the two terms on the left-hand side of equation (83), besides $\nabla^2\eta$, are both of $O(r^{-4})$, we may write

$$\eta = \frac{M}{r} + \frac{N(\theta)}{r^2} + O(r^{-3}) ; \tag{93}$$

and in accordance with equation (88) we may also write

$$\zeta = \frac{Z(\theta)}{r^2} + O(r^{-3}) . \tag{94}$$

Now inserting the forms (92), (93), and (94) for ν, η, and ζ in equation (85), we obtain

$$\left(\frac{d^2}{d\theta^2} + 4\right)(N - Z) = -M^2 ; \tag{95}$$

and the solution of this equation appropriate to the problem on hand is

$$N - Z = A \cos 2\theta - \tfrac{1}{4}M^2 , \tag{96}$$

where A is a constant of integration. (Note that we have not included a term in sin 2θ in the solution in view of the required symmetry of the configuration about the equatorial plane at $\theta = \pi/2$.)

Next, substituting the forms (93) and (94) for η and ζ in equation (83) and making use also of the relation (96), we are left with

$$\frac{d^2N}{d\theta^2} + 2 \cot\theta \frac{dN}{d\theta} + 2A + 4A \cos 2\theta = 0 . \tag{97}$$

The solution of this equation, free of singularity, is given by

$$N = \tfrac{1}{2}A \cos 2\theta + N_0 , \tag{98}$$

where N_0 is a further constant of integration. Equation (96) now gives

$$Z = \tfrac{1}{4}M^2 - \tfrac{1}{2}A + N_0 + A \sin^2\theta . \tag{99}$$

The regularity of the solution at $\theta = 0$ requires that ζ behave like $\sin^2\theta$ at $\theta = 0$. This requirement determines the constant A; we find

$$A = \tfrac{1}{2}M^2 + 2N_0 . \tag{100}$$

The behaviors of η, ζ, and ν to $O(r^{-2})$ becomes determinate. We have

$$\eta = +\frac{M}{r} + \frac{1}{2}\frac{A(1 + \cos 2\theta) - \frac{1}{2}M^2}{r^2} + O(r^{-3}) ,$$

$$\zeta = \frac{1}{2}\frac{A(1 - \cos 2\theta)}{r^2} + O(r^{-3}) ,$$

and

$$\nu = -M/r + O(r^{-3}) . \tag{101}$$

A particular consequence of the foregoing behavior is

$$\nu + \eta + \zeta = (A - \tfrac{1}{4}M^2)\,\frac{1}{r^2} + O(r^{-3})\,. \tag{102}$$

a) The Asymptotic Behavior of ω *as* $r \to \infty$

The behavior of ω as $r \to \infty$ is determined by the equation (cf. eq. [87])

$$\operatorname{div}\,(e^{3\eta+3\zeta-\nu} r^2 \sin^2\theta \operatorname{grad}\omega) = 0\,, \tag{103}$$

or, in view of the behaviors (101), by the equation

$$\operatorname{div}\left[\left(1 + \frac{4M}{r} + \ldots\right) r^2 \sin^2\theta \operatorname{grad}\omega\right] = 0\,. \tag{104}$$

The solution of this equation in "zeroth" order is of the form (cf. Hartle 1967)

$$\omega = \sum_{l=1}^{\infty} \omega^{(0)}{}_l(r)\,\frac{dP_l}{d\mu} = 2\sum_{l=1}^{\infty} \frac{J_l}{r^{l+2}}\,\frac{dP_l}{d\mu} \qquad (\mu = \cos\theta)\,, \tag{105}$$

where a factor 2 has been introduced in the definition of the constants J_l for later convenience; also, P_l denotes the Legendre polynomial.

The solution for ω including the "first-order" term $4M/r$ in equation (104) can be obtained by the substitution

$$\omega = \sum_{l=1}^{\infty} \left[\omega^{(0)}{}_l + \frac{4M}{r}\,\omega^{(1)}{}_l\right] \frac{dP_l}{d\mu}\,. \tag{106}$$

Equation (104) then gives

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 \frac{d\omega^{(1)}{}_l}{dr}\right) - \frac{(l-1)(l+2)}{r^2}\,\omega^{(1)}{}_l = -2\,\frac{l+2}{r^{l+5}}\,J_l\,. \tag{107}$$

We need to consider only the particular integral of equation (107); it is given by

$$\omega^{(1)}{}_l = -\frac{l+2}{l+1}\,\frac{J_l}{r^{l+3}}\,. \tag{108}$$

Therefore the solution for ω to $O(r^{-4})$ is given by

$$\omega = \frac{2J_1}{r^3} - \frac{6J_1 M}{r^4} + O(r^{-5})\,, \tag{109}$$

where a term in $J_2 \cos\theta$ has been omitted to conform to the required symmetry about the equatorial plane.

VIII. THE INERTIAL MASS AND THE ANGULAR MOMENTUM OF THE CONFIGURATION

The inertial mass (M) and the angular momentum (J) that must be associated with a stationary axisymmetric configuration can be ascertained with the aid of the formulae (61) and (66) of § V and the asymptotic behaviors of the potentials established in § VII.

In applying the formulae (61) and (66) in the present context, we make the identifications

$$\sigma = \eta + \zeta\,, \qquad \kappa_2 = \kappa_3 = \eta - \zeta\,, \qquad \text{and} \qquad q_2 = q_3 = 0\,. \tag{110}$$

Formula (61) now gives

$$M = -\tfrac{1}{8} \operatorname*{limit}_{r\to\infty} \int_0^\pi \left[\frac{\partial}{\partial r} e^{4\eta} + \frac{1}{r} e^{4\eta}(1 - e^{-4\zeta})\right] r^2 \sin\theta d\theta\,. \tag{111}$$

From the asymptotic behavior of ζ as $r \to \infty$, namely, that it is of $O(r^{-2})$, it follows that the second term in brackets in the integrand does not contribute to M; and the first term gives

$$M = -\tfrac{1}{2} \operatorname*{limit}_{r\to\infty} \int_0^\pi \frac{\partial\eta}{\partial r} r^2 \sin\theta d\theta = M\,, \tag{112}$$

in view of the asymptotic behavior of η ($\sim M/r$) as $r \to \infty$. Similarly, formula (66) gives

$$J = -\tfrac{1}{8} \operatorname*{limit}_{r\to\infty} \int_0^\pi r^4 \sin^3\theta \frac{\partial}{\partial r}(e^{4\eta}\omega) d\theta = -\tfrac{1}{8} \operatorname*{limit}_{r\to\infty} \int_0^\pi r^4 \sin^3\theta \frac{\partial\omega}{\partial r} d\theta\,. \tag{113}$$

From the asymptotic behavior of ω given in equation (109), we now find

$$J = \tfrac{3}{4} J_1 \int_0^\pi \sin^3\theta d\theta = J_1\,, \tag{114}$$

in agreement with the interpretation of the leading coefficient in the asymptotic expansion for ω (cf. Hartle 1967).

The expressions (112) and (113) for M and J enable us to express them as volume integrals over suitably defined sources. Thus, rewriting equation (112) in the manner

$$M = -\tfrac{1}{2} \iint_{S\to\infty} \nabla^2\eta r^2 \sin\theta dr d\theta\,, \tag{115}$$

and substituting for $\nabla^2\eta$ from the field equation (83), we obtain

$$M = \tfrac{1}{2} \iint \Big\{4\pi e^{2\eta-2\zeta} \frac{\epsilon + pV^2}{1 - V^2} + \tfrac{1}{8} r^2 e^{2\eta+2\zeta-2\nu} |\operatorname{grad}\omega|^2 \sin^2\theta$$
$$+ \tfrac{1}{2} |\operatorname{grad}(\eta + \zeta)|^2 + \operatorname{grad}\zeta \cdot \operatorname{grad}[\log(r\sin\theta)]\Big\} r^2 \sin\theta dr d\theta\,. \tag{116}$$

It can be verified by an integration by parts that the last term in the integrand does not contribute to the volume integral. We are thus left with

$$M = \iint \Big[2\pi e^{2\eta-2\zeta} \frac{\epsilon + pV^2}{1 - V^2} + \tfrac{1}{16} r^2 e^{2\eta+2\zeta-2\nu} |\operatorname{grad}\omega|^2 \sin^2\theta$$
$$+ \tfrac{1}{4} |\operatorname{grad}(\eta + \zeta)|^2\Big] r^2 \sin\theta dr d\theta\,, \tag{117}$$

in agreement with an earlier result of Bardeen (1970; see also Hartle and Sharp 1967).

Similarly, rewriting equation (113) in the manner

$$J = -\tfrac{1}{8} \iint \operatorname{div}(e^{3\eta+3\zeta-\nu} r^2 \sin^2\theta \operatorname{grad}\omega) r^2 \sin\theta dr d\theta\,, \tag{118}$$

and making use of the field equation (87), we obtain the formula

$$J = 2\pi \iint (\epsilon + p) e^{4\eta} \frac{V}{1 - V^2} r^3 \sin^2\theta dr d\theta \tag{119}$$

which expresses the angular momentum of the configuration as an integral over the sources (cf. Hartle and Sharp 1967).

PART III

EQUATIONS GOVERNING INFINITESIMAL AXISYMMETRIC PERTURBATIONS

IX. THE EULERIAN AND THE LAGRANGIAN CHANGES

We suppose that a configuration that is initially axisymmetric and in a stationary state is subjected to an infinitesimal perturbation; and we shall suppose further that in the nonstationary state that ensues axisymmetry is preserved. In this part, we shall obtain the linear equations which govern such perturbed configurations.

In considering the changes in the various quantities caused by the perturbation, we shall distinguish between the Eulerian and the Lagrangian changes; these are, respectively, the changes that take place at a fixed location and the changes that accompany a fluid element as it moves.[3] It is convenient for this purpose to describe the perturbation by a Lagrangian displacement $\boldsymbol{\xi}$ which is the spatial displacement that an element of the fluid experiences relative to its location in the unperturbed state. Since we have assumed that the perturbation does not affect the axisymmetry of the configuration, it is clear that the components ξ^α ($\alpha = 2, 3$) of $\boldsymbol{\xi}$ should suffice to describe the perturbed configuration. This statement should not be taken to mean that motions in the ϕ-direction do not occur. They do occur; but their role consists principally in altering the distribution of Ω. But as we shall show presently, the changes in Ω that occur can be ascertained by appealing to equation (42) that ensures the conservation of angular momentum per baryon.

We can consider the Eulerian and the Lagrangian changes in a variable as the result of the action of certain operators δ (Eulerian) and Δ (Lagrangian); these operators are clearly related by

$$\Delta = \delta + \xi^\alpha \frac{\partial}{\partial x^\alpha} . \tag{120}$$

In the subsequent analysis the assumption will be made that the changes (Eulerian or Lagrangian) caused by the perturbation are quantities of the "first order" of smallness and that all effects nonlinear in them can be ignored.

In the initial stationary state, there are no motions in the x^2- and the x^3-directions. The motions v^α that ensue in these directions as a result of the perturbation are, therefore, of the first order of smallness and we can write

$$v^\alpha = \xi^\alpha{}_{,0} . \tag{121}$$

X. THE EQUATIONS GOVERNING THE FLUID

We have seen that the q_α's vanish in the stationary state. Their appearance in the metric that describes the perturbed nonstationary state may therefore be thought of as being caused by the perturbation. The q_α's then, like the v^α's, are quantities of the first order of smallness. An important consequence of *both* v^α and q_α being quantities of the first order of smallness is that V and $v^{(1)}$ as *defined* in equation (25) differ from each other and with the *expression* for V in the unperturbed state by quantities of the second order of smallness; they can therefore be ignored in a linear theory such as the present. The relation given in equation (67), namely,

$$V = v^{(1)} = e^{\psi - \nu}(\Omega - \omega) , \tag{122}$$

continues to hold formally in the perturbed state as well—but only formally since ψ, ν, Ω, and ω *are* subject to first-order changes.

[3] For a general account of these concepts, see Chandrasekhar (1969; see especially §§ 13 and 14).

A further consequence of the formal applicability of the definition (122) to the perturbed state is that the expressions for u^0, u^1, u_0, u_1, $u^{(0)}$, and $u^{(1)}$ given in equations (67) also continue to be applicable.

To emphasize that q_α, like v^α, is nonvanishing only in the perturbed state, we shall write

$$q_{\alpha,0} \text{ in place of } q_\alpha \tag{123}$$

in all developments dealing with the perturbation of axisymmetric systems that are initially stationary.[4]

a) The Equations Ensuring the Conservation of Baryon Number and of Entropy

From equation (21) it follows that

$$\Delta(Nu^0\sqrt{-g}) = -(Nu^0\sqrt{-g})\xi^\alpha{}_{,\alpha}\,. \tag{124}$$

Equivalent forms of this relation are

$$\begin{aligned}\frac{\Delta N}{N} &= -\frac{\Delta(u^0\sqrt{-g})}{u^0\sqrt{-g}} - \xi^\alpha{}_{,\alpha} \\ &= -\frac{1}{u^0\sqrt{-g}}(\xi^\alpha u^0\sqrt{-g})_{,\alpha} - \delta[\log(u^0\sqrt{-g})]\,.\end{aligned} \tag{125}$$

From the definitions of u^0 and $\sqrt{-g}$ we find

$$\frac{\delta u^0}{u^0} = -\delta\nu + \frac{V\delta V}{1-V^2} \quad \text{and} \quad \frac{\delta\sqrt{-g}}{\sqrt{-g}} = \delta(\psi + \mu_2 + \mu_3)\,. \tag{126}$$

Inserting these relations in the second form of $\Delta N/N$ given in equation (125), we obtain

$$\frac{\Delta N}{N} = -\frac{V\delta V}{1-V^2} - \delta(\psi + \mu_2 + \mu_3) - \frac{1}{u^0\sqrt{-g}}(\xi^\alpha u^0\sqrt{-g})_{,\alpha}\,. \tag{127}$$

Since we have not included any dissipative mechanisms in the expression for the energy-momentum tensor of the fluid, the fluid elements must conserve their entropy during their motion: in general relativity, the conservation of baryon number and the conservation of entropy are not two independent laws (cf. Chandrasekhar 1969*b*). The changes in the pressure and in the energy density that accompany fluid motion must therefore be adiabatic; and we can write

$$\frac{\Delta p}{p} = \gamma\frac{\Delta N}{N} \quad \text{and} \quad \frac{\Delta\epsilon}{\epsilon + p} = \frac{\Delta N}{N}\,, \tag{128}$$

where γ is a suitably defined adiabatic exponent.

A consequence of the relations (128) which we shall find useful is

$$\Delta\left(\frac{\epsilon + p}{N}\right) = \gamma p\frac{\Delta N}{N^2}\,. \tag{129}$$

b) The Equation Ensuring the Conservation of the Angular Momentum per Baryon

From equation (42) it follows that

$$\Delta\left(\frac{\epsilon + p}{N}u_1\right) = 0\,. \tag{130}$$

[4] Actually, that q_α is expressible as the time-derivative of a function in a linearized perturbation theory can be deduced directly from the $(0, \alpha)$-component of the field equation (see § XI below).

In view of the relation (129), we can rewrite equation (129) in the manner

$$\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N} + \delta(\log u_1) + \xi^\alpha(\log u_1)_{,\alpha} = 0\,. \tag{131}$$

We have already noted that the formula for u_1 given in equations (67), namely,

$$u_1 = \frac{e^\psi V}{(1 - V^2)^{1/2}}\,, \tag{132}$$

is formally applicable to the perturbed state. Evaluating δu_1 according to this formula and inserting the result in equation (131), we obtain

$$\frac{\delta V}{V(1 - V^2)} = -\frac{\gamma p}{\epsilon + p}\frac{\Delta N}{N} - \delta\psi - \xi^\alpha(\log u_1)_{,\alpha}\,. \tag{133}$$

With δV determined by equation (133), the redistribution of Ω that results from the perturbation can be deduced from equation (122); thus

$$\delta(\Omega - \omega) = (\Omega - \omega)\left[\frac{\delta V}{V} - \delta(\psi - \nu)\right]. \tag{134}$$

Finally, we may note the following two identities which can be derived from the foregoing relations:

$$[(\epsilon + p)\xi^\alpha\sqrt{-g}]_{,\alpha} = -\left[\delta\epsilon + (\epsilon + p)\delta(\psi + \mu_2 + \mu_3) + (\epsilon + p)\frac{V\delta V}{1 - V^2}\right]\sqrt{-g}\,,$$

and

$$\delta[(\epsilon + p)u^0u_1\sqrt{-g}] = -[(\epsilon + p)u^0u_1\xi^\alpha\sqrt{-g}]_{,\alpha}\,. \tag{135}$$

c) The Pulsation Equation

The pulsation equation is no more than the equation of motion (35) linearized about the equation of equilibrium (80). In writing out the equation, we shall restrict ourselves to the case when, in the stationary state, Ω is a constant; in this case the equation governing equilibrium, namely (81), is particularly simple.

To obtain the pulsation equation, we apply the Lagrangian operator Δ to equation (35). Noting that in the present context we can interchange the operations of Δ and of partial differentiation (by virtue of the equations governing equilibrium[5]), we obtain

$$\begin{aligned}(u^0)^2e^{2\mu_\alpha}\frac{\partial^2\xi^\alpha}{\partial t^2} - u^0u_1\frac{\partial^2 q_\alpha}{\partial t^2} - 2\frac{\Delta u^0}{u^0}\frac{1}{\epsilon + p}\frac{\partial p}{\partial x^\alpha} + \tfrac{1}{2}(u^0)^2\frac{\partial}{\partial x^\alpha}\Delta\left(\frac{1}{u^0}\right)^2 \\ + (u^0)^2e^{2\psi}(\Omega - \omega)\frac{\partial\Delta\Omega}{\partial x^\alpha} = \frac{\Delta(\epsilon + p)}{(\epsilon + p)^2}\frac{\partial p}{\partial x^\alpha} - \frac{1}{\epsilon + p}\frac{\partial}{\partial x^\alpha}\Delta p\,.\end{aligned} \tag{136}$$

Substituting for Δp and $\Delta(\epsilon + p)$ their values given by the adiabatic conditions (128), we find that equation (136), after some rearranging, becomes

$$\begin{aligned}(\epsilon + p)(u^0)^2e^{2\mu_\alpha}\frac{\partial^2\xi^\alpha}{\partial t^2} = \left[\left(1 + \frac{\gamma p}{\epsilon + p}\right)\frac{\Delta N}{N} + 2\frac{\Delta u^0}{u^0}\right]\frac{\partial p}{\partial x^\alpha} - \frac{\partial}{\partial x^\alpha}\left(\gamma p\frac{\Delta N}{N}\right) \\ - \tfrac{1}{2}(\epsilon + p)(u^0)^2\frac{\partial}{\partial x^\alpha}\Delta\left(\frac{1}{u^0}\right)^2 - (\epsilon + p)u^0u_1\left(\frac{\partial\Delta\Omega}{\partial x^\alpha} - \frac{\partial^2 q_\alpha}{\partial t^2}\right).\end{aligned} \tag{137}$$

[5] For a direct proof of this statement in the Newtonian framework, see Chandrasekhar 1969*b* (§ 14).

Now, it can be verified that

$$-\tfrac{1}{2}(\epsilon + p)(u^0)^2 \frac{\partial}{\partial x^\alpha} \Delta\left(\frac{1}{u^0}\right)^2 = (\epsilon + p)\frac{\partial}{\partial x^\alpha}\frac{\Delta u^0}{u^0} - 2\frac{\Delta u^0}{u^0}\frac{\partial p}{\partial x^\alpha}. \tag{138}$$

Inserting this last relation in equation (137), we find after some further simplifications

$$\begin{aligned}(\epsilon + p)(u^0)^2(\sqrt{-g})e^{2\mu_\alpha}\frac{\partial^2 \xi^\alpha}{\partial t^2} = &-u^0\sqrt{-g}\frac{\partial}{\partial x^\alpha}\left(\frac{\gamma p}{u^0}\frac{\Delta N}{N}\right) + (\epsilon + p)\sqrt{-g}\frac{\partial}{\partial x^\alpha}\frac{\Delta u^0}{u^0} \\ &+ \sqrt{-g}\frac{\Delta N}{N}\frac{\partial p}{\partial x^\alpha} - (\epsilon + p)u^0 u_1\sqrt{-g}\left(\frac{\partial \Delta\Omega}{\partial x^\alpha} - \frac{\partial^2 q_\alpha}{\partial t^2}\right).\end{aligned} \tag{139}$$

Equation (139) is the required *pulsation equation.*

XI. THE LINEARIZED VERSIONS OF THE $(0, \alpha)$- AND THE $(1, \alpha)$-COMPONENTS OF THE FIELD EQUATIONS

The (0, 2)-component of the field equation that is valid quite generally under non-stationary conditions is (cf. eqs. [18], [25], and [30])

$$\begin{aligned}R^{(0)(2)} &= -R_{(0)(2)} \\ &= -e^{-\nu-\mu_2}[\mu_{2,0}\psi_{,2} - \psi_{,02} + \psi_{,0}(\nu - \psi)_{,2} - \mu_{3,20} - \mu_{3,2}(\mu_3 - \mu_2)_{,0} + \mu_{3,0}\nu_{,2}] \\ &\quad + \tfrac{1}{2}e^{2\psi-\nu-\mu_2-2\mu_3}(q_{2,30} - q_{3,20})(\omega_{,3} - q_{3,00}) \\ &= 8\pi(\epsilon + p)u^{(0)}u^{(2)} = 8\pi\frac{\epsilon + p}{1 - V^2}e^{\mu_2-\nu}v^2,\end{aligned} \tag{140}$$

where in accordance with our present convention we have replaced q_α of Part I by $q_{\alpha,0}$. Remembering that in the present linearized theory $v^2 = \xi^2{}_{,0}$, we can directly integrate equation (140) with respect to time if we ignore all terms that are nonlinear in the perturbation. Thus

$$\begin{aligned}&-e^{-\nu-\mu_2}[\psi_{,2}\delta\mu_2 - \delta\psi_{,2} + (\nu - \psi)_{,2}\delta\psi - \delta\mu_{3,2} + \mu_{3,2}\delta(\mu_3 - \mu_2) + \nu_{,2}\delta\mu_3] \\ &\qquad + \tfrac{1}{2}e^{2\psi-\nu-\mu_2-2\mu_3}(q_{2,3} - q_{3,2})\omega_{,3} = 8\pi\frac{\epsilon + p}{1 - V^2}e^{\mu_2-\nu}\xi^2.\end{aligned} \tag{141}$$

Introducing the abbreviation

$$Q = e^{3\psi+\nu-\mu_2-\mu_3}(q_{2,3} - q_{3,2}), \tag{142}$$

we can rewrite equation (141) in the form

$$\begin{aligned}8\pi\sqrt{-g}\frac{\epsilon + p}{1 - V^2}\xi^2 = e^{\psi+\nu+\mu_3-\mu_2}[&\delta\psi_{,2} + (\psi - \nu)_{,2}\delta\psi + \delta\mu_{3,2} - (\nu - \mu_3)_{,2}\delta\mu_3 \\ &- (\psi + \mu_3)_{,2}\delta\mu_2] + \tfrac{1}{2}Q\omega_{,3}.\end{aligned} \tag{143}$$

An alternative form of this equation which we shall find useful is

$$\begin{aligned}8\pi\sqrt{-g}\frac{\epsilon + p}{1 - V^2}\xi^2 = e^{\mu_3-\mu_2}[&(e^\beta\delta\psi)_{,2} - (e^\beta)_{,2}\delta\mu_3 + e^\beta\delta\mu_{3,2} - 2e^\beta\nu_{,2}\delta\psi \\ &+ e^\beta(\psi + \mu_3)_{,2}\delta(\mu_3 - \mu_2)] + \tfrac{1}{2}Q\omega_{,3},\end{aligned} \tag{144}$$

where it will be recalled that $\beta = \psi + \nu$.

The (0, 3)-component of the field equation similarly gives

$$8\pi\sqrt{-g}\,\frac{\epsilon+p}{1-V^2}\,\xi^3 = e^{\mu_2-\mu_3}[(e^{\beta}\delta\psi)_{,3} - (e^{\beta})_{,3}\delta\mu_2 + e^{\beta}\delta\mu_{2,3} - 2e^{\beta}\nu_{,3}\delta\psi + e^{\beta}(\psi+\mu_2)_{,3}\delta(\mu_2-\mu_3)] - \tfrac{1}{2}Q\omega_{,2}\,. \tag{145}$$

At a later stage in the development of the theory, it will be convenient to define the variables

$$\delta\mu = \tfrac{1}{2}\delta(\mu_3+\mu_2) \quad \text{and} \quad \delta\tau = \tfrac{1}{2}\delta(\mu_3-\mu_2)\,. \tag{146}$$

In terms of these variables, the basic relations (144) and (145) take the forms

$$(\delta\psi+\delta\mu)_{,2} - \nu_{,2}(\delta\psi+\delta\mu) + \psi_{,2}(\delta\psi-\delta\mu) = e^{2\mu_2}\left(8\pi\frac{\epsilon+p}{1-V^2}\xi^2 - \frac{Q}{2\sqrt{-g}}\omega_{,3}\right) - \delta\tau_{,2} - (2\mu_3+\psi-\nu)_{,2}\delta\tau\,, \tag{147}$$

and

$$(\delta\psi+\delta\mu)_{,3} - \nu_{,3}(\delta\psi+\delta\mu) + \psi_{,3}(\delta\psi-\delta\mu) = e^{2\mu_3}\left(8\pi\frac{\epsilon+p}{1-V^2}\xi^3 + \frac{Q}{2\sqrt{-g}}\omega_{,2}\right) + \delta\tau_{,3} + (2\mu_2+\psi-\nu)_{,3}\delta\tau\,. \tag{148}$$

Considering next the (1, 2)-component of the field equation, we have quite generally,

$$\begin{aligned} R^{(1)(2)} &= R_{(1)(2)} \\ &= \tfrac{1}{2}e^{-2\psi-\nu-\mu_3}\{[e^{3\psi-\nu-\mu_2+\mu_3}(\omega_{,2}-q_{2,00})]_{,0} + [e^{3\psi+\nu-\mu_2-\mu_3}(q_{2,30}-q_{3,20})]_{,3}\} \\ &= 8\pi(\epsilon+p)u^{(1)}u^{(2)} = 8\pi(\epsilon+p)u^0u_1e^{-\psi+\mu_2}v^2\,. \end{aligned} \tag{149}$$

Again this equation can be integrated with respect to time if we ignore all terms that are nonlinear in the perturbation. Thus,

$$16\pi(\epsilon+p)u^0u_1e^{-2\psi+2\nu+2\mu_2}\xi^2 = (\delta\omega_{,2}-q_{2,00}) + \omega_{,2}(3\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3) + e^{-3\psi+\nu+\mu_2-\mu_3}[e^{3\psi+\nu-\mu_2-\mu_3}(q_{2,3}-q_{3,2})]_{,3}\,, \tag{150}$$

or, alternatively,

$$\delta\omega_{,2} - q_{2,00} = 16\pi(\epsilon+p)u^0u_1e^{-2\psi+2\nu+2\mu_2}\xi^2 - \omega_{,2}(3\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3) - e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3}\,. \tag{151}$$

The (1, 3)-component of the field equation similarly gives

$$\delta\omega_{,3} - q_{3,00} = 16\pi(\epsilon+p)u^0u_1e^{-2\psi+2\nu+2\mu_3}\xi^3 - \omega_{,3}(3\delta\psi-\delta\nu+\delta\mu_2-\delta\mu_3) + e^{-3\psi+\nu-\mu_2+\mu_3}Q_{,2}\,. \tag{152}$$

Eliminating $\delta\omega$ from equations (151) and (152), we obtain

$$\begin{aligned} (e^{-3\psi+\nu-\mu_2+\mu_3}Q_{,2})_{,2} + (e^{-3\psi+\nu+\mu_2-\mu_3}Q_{,3})_{,3} &= (e^{-3\psi-\nu+\mu_2+\mu_3}Q)_{,00} \\ &- [\omega_{,2}(3\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3)]_{,3} + [\omega_{,3}(3\delta\psi-\delta\nu+\delta\mu_2-\delta\mu_3)]_{,2} \\ &+ 16\pi\{[(\epsilon+p)u^0u_1e^{-2\psi+2\nu+2\mu_2}\xi^2]_{,3} - [(\epsilon+p)u^0u_1e^{-2\psi+2\nu+2\mu_3}\xi^3]_{,2}\}\,. \end{aligned} \tag{153}$$

Equation (153) has the form of an inhomogeneous wave-equation. It will appear later that it is this equation which provides information about the emission of gravitational radiation by the perturbed system.

Finally, we shall state in the form of a lemma two formulae that can be derived with the aid of equations (151) and (152) and which are of considerable importance in the reduction of the remaining field equations (see § XII below).

LEMMA:

$$\begin{aligned}\delta\{e^{2\psi-2\nu}[e^{\mu_3-\mu_2}(\omega_{,2}-q_{2,00})^2+e^{\mu_2-\mu_3}(\omega_{,3}-q_{3,00})^2]\}\\ =32\pi(\epsilon+p)u^0u_1e^{\mu_2+\mu_3}(\xi^2\omega_{,2}+\xi^3\omega_{,3})\\ -e^{2\psi-2\nu}[4X\delta\psi+Y\delta(\mu_3-\mu_2)]+2S\,,\end{aligned}\tag{154}$$

and

$$\begin{aligned}\delta\{e^{2\psi-2\nu}[e^{\mu_3-\mu_2}(\omega_{,2}-q_{2,00})^2-e^{\mu_2-\mu_3}(\omega_{,3}-q_{3,00})^2]\}\\ =32\pi(\epsilon+p)u^0u_1e^{\mu_2+\mu_3}(\xi^2\omega_{,2}-\xi^3\omega_{,3})\\ -e^{2\psi-2\nu}[4Y\delta\psi+X\delta(\mu_3-\mu_2)]+2D\,,\end{aligned}\tag{155}$$

where

$$S=e^{-\psi-\nu}(Q_{,2}\omega_{,3}-Q_{,3}\omega_{,2})\,,\quad\text{and}\quad D=-e^{-\psi-\nu}(Q_{,2}\omega_{,3}+Q_{,3}\omega_{,2})\,,\tag{156}$$

and X and Y have the same meanings as in equations (68).

XII. THE LINEARIZED VERSIONS OF THE REMAINING FIELD EQUATIONS

The linearized versions of the remaining field equations can be obtained by applying the Eulerian operator δ to the general equations of Part I. Thus, the application of the operator δ to the (0, 0)-component of the field equation gives

$$\begin{aligned}e^{\mu_3-\mu_2}\{[\delta\psi_{,22}+2\psi_{,2}\delta\psi_{,2}+\delta\mu_{3,22}+(\delta\psi+\delta\mu_3)_{,2}(\mu_3-\mu_2)_{,2}+(\psi+\mu_3)_{,2}\delta(\mu_3-\mu_2)_{,2}]\\ +\delta(\mu_3-\mu_2)[\psi_{,22}+\psi_{,2}\psi_{,2}+\mu_{3,22}+(\psi+\mu_3)_{,2}(\mu_3-\mu_2)_{,2}]\}\\ +e^{\mu_2-\mu_3}\{2\leftrightarrow 3\}=-8\pi\delta\left(e^{\mu_2+\mu_3}\frac{\epsilon+pV^2}{1-V^2}\right)-8\pi e^{\mu_2+\mu_3}(\epsilon+p)u^0u_1\xi^\alpha\omega_{,\alpha}\\ +e^{2\psi-2\nu}[X\delta\psi+\tfrac{1}{4}Y\delta(\mu_3-\mu_2)]-\tfrac{1}{2}S\,,\end{aligned}\tag{157}$$

where we have used equation (154) to simplify the terms in $(\omega_{,\alpha}-q_{\alpha,00})^2$. Similarly, the other field equations give

$$\begin{aligned}e^{\mu_3-\mu_2}\{[\delta\nu_{,22}+2\nu_{,2}\delta\nu_{,2}+\delta\mu_{3,22}+(\delta\nu+\delta\mu_3)_{,2}(\mu_3-\mu_2)_{,2}+(\nu+\mu_3)_{,2}\delta(\mu_3-\mu_2)_{,2}]\\ +\delta(\mu_3-\mu_2)[\nu_{,22}+\nu_{,2}\nu_{,2}+\mu_{3,22}+(\nu+\mu_3)_{,2}(\mu_3-\mu_2)_{,2}]\}\\ +e^{\mu_2-\mu_3}\{2\leftrightarrow 3\}=8\pi\delta\left\{e^{\mu_2+\mu_3}\left[(\epsilon+p)\frac{V^2}{1-V^2}+p\right]\right\}+e^{-2\nu+\mu_2+\mu_3}\delta(\mu_2+\mu_3)_{,00}\\ +24\pi e^{\mu_2+\mu_3}(\epsilon+p)u^0u_1\xi^\alpha\omega_{,\alpha}-3e^{2\psi-2\nu}[X\delta\psi+\tfrac{1}{4}Y\delta(\mu_3-\mu_2)]+\tfrac{3}{2}S\,,\end{aligned}\tag{158}$$

$$\begin{aligned}\{e^{3\psi-\nu+\mu_3-\mu_2}[\delta\omega_{,2}-q_{2,00}+\omega_{,2}(3\delta\psi-\delta\nu-\delta\mu_2+\delta\mu_3)]\}_{,2}\\ +\{e^{3\psi-\nu+\mu_2-\mu_3}[\delta\omega_{,3}-q_{3,00}+\omega_{,3}(3\delta\psi-\delta\nu+\delta\mu_2-\delta\mu_3)]\}_{,3}\\ =-16\pi\delta[(\epsilon+p)u^0u_1\sqrt{-g}]=16\pi[(\epsilon+p)u^0u_1\xi^\alpha\sqrt{-g}]_{,\alpha}\,,\end{aligned}\tag{159}$$

$$
\begin{aligned}
&e^{\mu_3-\mu_2}\{[\delta\beta_{,22} + 2\beta_{,2}\delta\beta_{,2} + \delta\beta_{,2}(\mu_3 - \mu_2)_{,2} + \beta_{,2}\delta(\mu_3 - \mu_2)_{,2}] \\
&\qquad + \delta(\mu_3 - \mu_2)[\beta_{,22} + \beta_{,2}\beta_{,2} + \beta_{,2}(\mu_3 - \mu_2)_{,2}]\} \\
&+ e^{\mu_2-\mu_3}\{2 \leftrightarrow 3\} = 16\pi\delta(e^{\mu_2+\mu_3}p) + e^{-2\nu+\mu_2+\mu_3}\delta(2\psi + \mu_2 + \mu_3)_{,00}\,. \qquad (160)
\end{aligned}
$$

An alternative form of equation (160) is

$$
\begin{aligned}
&[e^{\mu_3-\mu_2}(e^{\beta}\delta\beta)_{,2}]_{,2} + [e^{\mu_2-\mu_3}(e^{\beta}\delta\beta)_{,3}]_{,3} = -\delta(\mu_3 - \mu_2)(2W + \tfrac{1}{2}e^{2\psi-2\nu+\beta}Y) \\
&\qquad - e^{\beta}[e^{\mu_3-\mu_2}\beta_{,2}\delta(\mu_3 - \mu_2)_{,2} + e^{\mu_2-\mu_3}\beta_{,3}\delta(\mu_2 - \mu_3)_{,3}] \\
&\qquad + 16\pi\delta(p\sqrt{-g}) + e^{-2\nu}\delta(2\psi + \mu_2 + \mu_3)_{,00}\sqrt{-g}\,. \qquad (161)
\end{aligned}
$$

We also have

$$
\begin{aligned}
&[e^{\mu_3-\mu_2}(e^{\beta}\delta\beta)_{,2}]_{,2} - [e^{\mu_2-\mu_3}(e^{\beta}\delta\beta)_{,3}]_{,3} = -\delta(\mu_3 - \mu_2)(16\pi p\sqrt{-g} + \tfrac{1}{2}e^{2\psi-2\nu+\beta}X) \\
&\qquad - e^{\beta}[e^{\mu_3-\mu_2}\beta_{,2}\delta(\mu_3 - \mu_2)_{,2} - e^{\mu_2-\mu_3}\beta_{,3}\delta(\mu_2 - \mu_3)_{,3}] \\
&\qquad + 2\delta W + e^{2\psi-2\nu+\beta}(\tfrac{1}{2}\delta\beta - 2\delta\psi)Y + e^{\beta}D \\
&\qquad + [16\pi(\epsilon + p)u^0u_1(\xi^2\omega_{,2} - \xi^3\omega_{,3}) - e^{-2\nu}\delta(\mu_3 - \mu_2)_{,00}]\sqrt{-g}\,, \qquad (162)
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta[\nu_{,23} + \nu_{,2}\nu_{,3} + \psi_{,23} + \psi_{,2}\psi_{,3} - (\psi + \nu)_{,2}\mu_{2,3} - (\psi + \nu)_{,3}\mu_{3,2}] \\
&\quad = -2e^{2\psi-2\nu}\omega_{,2}\omega_{,3}\delta\psi + \tfrac{1}{2}e^{-\psi-\nu}(e^{\mu_3-\mu_2}Q_{,2}\omega_{,2} - e^{\mu_2-\mu_3}Q_{,3}\omega_{,3}) \\
&\qquad + 8\pi(\epsilon + p)u^0u_1(e^{2\mu_2}\xi^2\omega_{,3} + e^{2\mu_3}\xi^3\omega_{,2})\,. \qquad (163)
\end{aligned}
$$

Two linear combinations of the foregoing equations which we shall find useful are

$$
\begin{aligned}
&e^{\mu_3-\mu_2}\{[\delta\nu_{,22} + \delta\nu_{,2}(2\beta - \nu)_{,2} + \nu_{,2}\delta(2\beta - \nu)_{,2} - \delta\mu_{3,22} + \delta(\nu - \mu_3)_{,2}(\mu_3 - \mu_2)_{,2} \\
&\qquad + (\nu - \mu_3)_{,2}\delta(\mu_3 - \mu_2)_{,2}] + \delta(\mu_3 - \mu_2)[\nu_{,22} + \nu_{,2}(2\beta - \nu)_{,2} - \mu_{3,22} \\
&\qquad + (\nu - \mu_3)_{,2}(\mu_3 - \mu_2)_{,2}]\} \\
&+ e^{\mu_2-\mu_3}\{2 \leftrightarrow 3\} = 8\pi\delta\left[e^{\mu_2+\mu_3}\left(\frac{\epsilon + p}{1 - V^2} + p\right)\right] + e^{-2\nu+\mu_2+\mu_3}\delta(2\psi + \mu_2 + \mu_3)_{,00} \\
&\qquad + 8\pi e^{\mu_2+\mu_3}(\epsilon + p)u^0u_1\xi^{\alpha}\omega_{,\alpha} - e^{2\psi-2\nu}[X\delta\psi + \tfrac{1}{4}Y\delta(\mu_3 - \mu_2)] + \tfrac{1}{2}S\,, \qquad (164)
\end{aligned}
$$

$$
\begin{aligned}
&e^{\mu_3-\mu_2}\{[\delta\nu_{,22} + \beta_{,2}\delta\nu_{,2} + \delta\beta_{,2}\nu_{,2} + \delta\nu_{,2}(\mu_3 - \mu_2)_{,2} + \nu_{,2}\delta(\mu_3 - \mu_2)_{,2}] \\
&\qquad + \delta(\mu_3 - \mu_2)[\nu_{,22} + \beta_{,2}\nu_{,2} + \nu_{,2}(\mu_3 - \mu_2)_{,2}]\} \\
&+ e^{\mu_2-\mu_3}\{2 \leftrightarrow 3\} = 4\pi\delta\left\{e^{\mu_2+\mu_3}\left[(\epsilon + p)\frac{1 + V^2}{1 - V^2} + 2p\right]\right\} + e^{-2\nu+\mu_2+\mu_3}\delta(\psi + \mu_2 + \mu_3)_{,00} \\
&\qquad + 16\pi e^{\mu_2+\mu_3}(\epsilon + p)u^0u_1\xi^{\alpha}\omega_{,\alpha} - 2e^{2\psi-2\nu}[X\delta\psi + \tfrac{1}{4}Y\delta(\mu_3 - \mu_2)] + S\,, \qquad (165)
\end{aligned}
$$

and

$$e^{\mu_3-\mu_2}\{[\delta\psi_{,22} + (\psi + \nu + \mu_3 - \mu_2)_{,2}\delta\psi_{,2} + \psi_{,2}\delta(\psi + \nu + \mu_3 - \mu_2)_{,2}]$$
$$+ \delta(\mu_3 - \mu_2)[\psi_{,22} + \psi_{,2}(\psi + \nu + \mu_3 - \mu_2)_{,2}]\}$$
$$+ e^{\mu_2-\mu_3}\{2 \leftrightarrow 3\} = -8\pi\delta\left\{e^{\mu_2+\mu_3}\left[(\epsilon + p)\frac{V^2}{1 - V^2} + \tfrac{1}{2}(\epsilon - p)\right]\right\} + e^{-2\nu+\mu_2+\mu_3}\delta\psi_{,00}$$
$$- 16\pi e^{\mu_2+\mu_3}(\epsilon + p)u^0 u_1 \xi^\alpha \omega_{,\alpha} + 2e^{2\psi-2\nu}[X\delta\psi + \tfrac{1}{4}Y\delta(\mu_3 - \mu_2)] - S\,. \quad (166)$$

In § XI we considered the linearized versions of the (0, 2)-, (0, 3)-, (1, 2)-, and (1, 3)-components of the field equations. It can now be verified by direct calculation that symbolically the following relations hold:

$$[(0, 2)\text{-Eq.}]_{,2} + [(0, 3)\text{-Eq.}]_{,3} = (0, 0)\text{-Eq.}$$

and

$$[(1, 2)\text{-Eq.}]_{,2} + [(1, 3)\text{-Eq.}]_{,3} = (0, 1)\text{-Eq.}\,, \quad (167)$$

where the (0, 0)- and the (0, 1)-components of the linearized field equations are given by equations (157) and (159), respectively. (The relations [166] must clearly be equivalent to two of the four Bianchi identities.)

Since only six of the 10 field equations can be linearly independent, it is clear that besides the four equations considered in § XI, only two of the remaining six equations need be considered. In view of the relations (167), equations (157) and (159) can be eliminated from further consideration; and this leaves us with equations (158) and (161)–(163). Of these four equations, it appears that equations (161) and (162) derived from the (2, 2)- and the (3, 3)-components of the field equations (by addition and by subtraction) are the most useful; and for later use we shall rewrite them in the following forms:

$$e^{\mu_3-\mu_2}[\delta(\psi + \nu)_{,22} + (2\psi + 2\nu + \mu_3 - \mu_2)_{,2}\delta(\psi + \nu)_{,2}]$$
$$+ e^{\mu_2-\mu_3}[2 \leftrightarrow 3] = 16\pi\delta(e^{\mu_2+\mu_3}p) + 2e^{-2\nu+\mu_2+\mu_3}\delta(\psi + \mu)_{,00}$$
$$- 2(2e^{-\beta}W + \tfrac{1}{2}e^{2\psi-2\nu}Y)\delta\tau - 2e^{\mu_3-\mu_2}\beta_{,2}\delta\tau_{,2} + 2e^{\mu_2-\mu_3}\beta_{,3}\delta\tau_{,3}\,, \quad (168)$$

and

$$e^{\mu_3-\mu_2}[(\delta\psi + \delta\nu)_{,22} + 2\psi_{,2}\delta\psi_{,2} + 2\nu_{,2}\delta\nu_{,2} - \delta(\psi + \nu)_{,2}(\mu_3 + \mu_2)_{,2} - 2(\psi + \nu)_{,2}\delta\mu_{,2}]$$
$$- e^{\mu_2-\mu_3}[2 \leftrightarrow 3] + 2e^{2\psi-2\nu}Y\delta\psi + e^{-\psi-\nu}(Q_{,2}\omega_{,3} + Q_{,3}\omega_{,2})$$
$$- 16\pi e^{\mu_2+\mu_3}(\epsilon + p)u^0 u_1(\xi^2\omega_{,2} - \xi^3\omega_{,3}) = -\delta\tau(e^{2\psi-2\nu}X + 32\pi p e^{\mu_2+\mu_3}) - 2e^{-2\nu+\mu_2+\mu_3}\delta\tau_{,00}, \quad (169)$$

where $\delta\mu$ and $\delta\tau$ have the same meanings as in equations (146), and W is defined in equation (68).

The derivation of the principal equations of the theory is now completed. Applications of the theory to the solution of the two problems outlined in § I will be found in the second paper of this series (Chandrasekhar and Friedman 1972).

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