Slowly rotating scalar field wormholes: the second order approximation

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We discuss rotating wormholes in general relativity with a scalar field with negative kinetic energy. To solve the problem, we use the assumption about slow rotation. The role of a small dimensionless parameter plays the ratio of the linear velocity of rotation of the wormhole's throat and the velocity of light. We construct the rotating wormhole solution in the second order approximation with respect to the small parameter. The analysis shows that the asymptotical mass of the rotating wormhole is greater than that of the non-rotating one, and the NEC violation in the rotating wormhole spacetime is weaker than that in the non-rotating one.

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I. INTRODUCTION

Wormholes are usually defined as topological handles in spacetime linking widely separated regions of a single universe, or "bridges" joining two different spacetimes [1, 2]. As is well-known [3], they can exist only if their throats contain exotic matter which possesses a negative pressure and violates the null energy condition. The search of realistic physical models providing the wormhole existence represents an important direction in wormhole physics. Various models of this kind include scalar fields, wormhole solutions in semiclassical gravity, solutions in Brans-Dicke theory, wormholes on branes, wormholes supported by matter with exotic equations of state, such as phantom energy, the Chaplygin gas, tachyon matter, and others [4, 5].

It is worth being noticed that most of the investigations deal with static spherically symmetric wormholes because of their simplicity and high symmetry. At the same time it would be important and interesting from the physical point of view to study more wide classes of wormholes including non-static and rotating ones. Non-static wormholes whose geometry is depending on time have been discussed in the literature. In 1993 Roman [6] explored the possibility that inflation might provide a mechanism for the enlargement of submicroscopic, i.e., Planck scale wormholes to macroscopic size. He used the line element with the exponential scale factor. Kim [7] generalized the Roman's consideration by using the scale factor in a general form. Various aspects of non-static wormholes conformally related to static wormhole geometries were investigated in [8, 9, 10, 11]; in particular, some issues concerning WEC violation and traversability in these time-dependent geometries were discussed. Kuhfittig [12] considered a spherically symmetric wormhole spacetime with the metric whose components are time-depending. Exact solutions describing cosmological evolution of scalar field wormholes were obtained in [13, 14].

Rotating wormholes have also been an object for study. Some general geometrical properties of stationary rotating wormholes have been first analyzed by Teo [15]. General requirements for the stress-energy tensor necessary to generate the rotating wormhole were discussed in [16]. The WEC violation and traversability in the rotating wormhole spacetime were in details studied in [17]. Kim [18] investigated scalar perturbations in a particular model of the rotating wormhole. In Ref. [19] the authors studied a slowly rotating wormhole surrounded by a cloud of charged particles. Arguments in favour of the possibility of existence of semiclassical rotating wormholes were given in [20].

In previous work [21] we were continuing the study of rotating wormholes. Our aim was to construct an exact solution describing these objects in general relativity with a scalar field. As is well-known (see [22, 23]) a scalar ghost, i.e. a scalar field with negative kinetic energy can support static spherically symmetric wormholes. Moreover, such wormholes are stable against linear spherically symmetric perturbations [24]. In [21] we looked for rotating wormholes supported by the scalar field with negative kinetic energy. To solve the problem we supposed that a wormhole is very slowly rotating and constructed a solution in the first order approximation with respect to a small parameter characterizing the velocity of rotation. In this approximation the only term $\sim \Omega dt d\varphi$ is added to the initial non-rotating wormhole metric, where Ω is the local angular velocity of rotation. This term results in the well-know dragging effect in general relativity. Namely, it was shown that a test particle initially propagating along the radial direction turns

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out to be involving into the wormhole rotation so that after passing through the throat of a wormhole it continues its motion along a spiral trajectory moving away from the throat. Propagation of light exhibits similar behavior. The ray of light after passing through the rotating wormhole throat is propagating along the spiral. At the same time, the first order approximation does not give an answer to a number of interesting and important problems concerning the rotating wormhole mass, the NEC violation and others. For this reason, in this paper we construct and analyze the second order solution describing the rotating wormhole in general relativity with the scalar field with negative kinetic energy.

The paper is organized as follows. In Section II we give some general formulas and write down the field equations. A static spherically symmetric wormhole is briefly discussed in Section III. In Section IV we formulate the condition of slow rotation and introduce a small parameter characterizing the rotation velocity. Then we construct a solution describing a rotating wormhole in the second order approximation with respect to this small parameter. Some properties of the solution is analyzed in Section V. Namely, we discuss how a mass and a value of the NEC violation by a rotating wormhole differ from those of a non-rotating one. A summary of results obtained is given in Section VI. In Appendix we give details of solving field equations in the second order approximation and write their solution in an explicit form.

II. GENERAL FORMULAS

Consider general relativity with a scalar field Φ , describing by the action

$$S = \int d^4x \sqrt{-g} \left[\mathcal{R} + (\nabla \Phi)^2 \right], \tag{1}$$

where $g_{\mu\nu}$ is a metric, $g = \det(g_{\mu\nu})$, \mathcal{R} is the scalar curvature, and $(\nabla\Phi)^2 = g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$ is the kinetic term. Throughout the paper we use units G = c = 1 and the signature (-+++). For this signature the + sign before the kinetic term corresponds to negative kinetic energy, hence Φ is a *ghost*.

Varying the action (1) with respect to $g_{\mu\nu}$ and ϕ yields Einstein equations and the equation of motion of the scalar field, respectively:

$$\mathcal{R}_{\mu\nu} = -\Phi_{,\mu}\Phi_{,\nu}, \qquad (2)$$

$$\nabla^{\alpha}\nabla_{\alpha}\Phi = 0. \qquad (3)$$

$$\nabla^{\alpha}\nabla_{\alpha}\Phi = 0. \tag{3}$$

In the paper we will search for solutions of the system (2),(3) describing rotating wormholes. A spacetime with the stationary rotation possesses the axial symmetry. As is known (see, e.g. [25]) a general axially symmetric metric can be given in the following form:

$$ds^{2} = -Adt^{2} + Bdr^{2} + R^{2}[d\theta^{2} + \sin^{2}\theta(d\varphi - \Omega dt)^{2}], \tag{4}$$

where A, B, R, Ω are functions of r, θ . The function Ω has an explicit physical sense; it represents an angular velocity of rotation in a point (r,θ) . The requirement of finiteness of the angular momentum J measured by a distant observer yields the following asymptotical condition for Ω [26]:

$$\Omega = \frac{2J}{r^3} + O(r^{-4}) \quad \text{as} \quad r \to \infty.$$
 (5)

Also, requiring that a spacetime should be asymptotically flat we have $A \to 1$, $B \to 1$, and $R^2 \to r^2$ as $r \to \infty$.

Eqs. (2), (3), written for the metric (4), are second-order partial differential equations for five functions A, B, R, Ω , and Φ . Solving these equations in a general form is a rather complicated mathematical problem. To simplify the problem, in what follows we restrict ourselves by the case of slow rotation.

STATIC SPHERICALLY SYMMETRIC WORMHOLE

To formulate the condition of slow rotation, first of all we will discuss the static spherically symmetric case. A static spherically symmetric solution in general relativity with a ghost scalar field was first found by Ellis [22] and independently by Bronnikov [23]. This solution can be presented as follows (see [13]):

$$ds^{2} = -e^{2u(r)}dt^{2} + e^{-2u(r)}[dr^{2} + (r^{2} + a^{2})(d\theta^{2} + \sin^{2}\theta d\varphi^{2})], \tag{6}$$

$$\Phi(r) = \frac{(m^2 + a^2)^{1/2}}{2\pi^{1/2} m} u(r), \tag{7}$$

where the radial coordinate r varies from $-\infty$ to ∞ , m and a are free parameters, and

$$u(r) = \frac{m}{a} \left(\arctan \frac{r}{a} - \frac{\pi}{2} \right). \tag{8}$$

Taking into account the following asymptotical behavior:

$$e^{2u}|_{r\to\infty} = 1 - \frac{2m}{r} + O(r^{-2}),$$

 $e^{2u}|_{r\to-\infty} = e^{-2\pi m/a} \left(1 - \frac{2m}{r}\right) + O(r^{-2}),$

we can see that the spacetime with the metric (6) possesses two asymptotically flat regions. The parameter m plays a role of the asymptotical mass for a distant observer located at $r = \infty$. We will assume that $m \ge 0$. The asymptotically flat regions are connected by a throat whose radius corresponds to a minimum of the radius of two-dimensional sphere, $R^2(r) = e^{-2u(r)}(r^2 + a^2)$. The minimum of R(r) is achieved at $r_{\rm th} = m$. The value $R_{\rm th} = R(r_{\rm th})$ is called the radius of wormhole throat. It is worth noting that there exist massless wormholes with m = 0. In this case the metric (6) takes the especially simple form:

$$ds^{2} = -dt^{2} + dr^{2} + (r^{2} + a^{2})(d\theta^{2} + \sin^{2}\theta d\varphi^{2}).$$
(9)

It is interesting that the metric (9) was proposed a priori by Morris and Thorne in [1] as a simple example of the wormhole spacetime metric.

IV. ROTATING WORMHOLE

Now let us consider rotating wormholes. For this aim we take the wormhole metric in the form (4) with $r \in (-\infty, +\infty)$. Assume that the throat of a wormhole corresponds to the value $r = r_{\rm th}$. Define the throat's radius as $R_{\rm th} = R|_{r=r_{\rm th},\theta=\pi/2}$. Also we introduce the value $\Omega_{\rm th} = \Omega|_{r=r_{\rm th},\theta=\pi/2}$, being the equatorial angular velocity of rotation of the wormhole throat. Without loss of generality we may suppose that $\Omega_{\rm th} > 0$, i.e. the throat is rotating in the positive direction. Assume now that the following condition is fulfilled:

$$R_{\rm th}\Omega_{\rm th} \ll c,$$
 (10)

where c is the velocity of light. The condition (10) means that the linear velocity of rotation of the throat is much less than c. Further, we will consider an approximation of slow rotation with the small dimensionless parameter $\lambda = R_{\rm th}\Omega_{\rm th}/c$. In this approximation components of the metric (4), describing the rotating wormhole, should just slightly differ from respective components of the static metric (6). Following the procedure given in [25], we represent the metric functions A, B, R, Ω and the field Φ as an expansion in terms of powers of λ :

$$\Omega = \lambda \omega + O(\lambda^3), \tag{11}$$

$$A = A_0(1 + \lambda^2 \alpha) + O(\lambda^4), \tag{12}$$

$$B = B_0(1 + \lambda^2 \beta) + O(\lambda^4), \tag{13}$$

$$R = R_0(1+\lambda^2\rho) + O(\lambda^4), \tag{14}$$

$$\Phi = \Phi_0(1 + \lambda^2 \phi) + O(\lambda^4), \tag{15}$$

where A_0 , B_0 , R_0 , and Φ_0 are zero order solutions corresponding to the unperturbed static spherically symmetric configuration (6), (7):

$$A_0 = e^{2u}, \quad B_0 = e^{-2u}, \quad R_0^2 = e^{-2u}(r^2 + a^2), \quad \Phi_0 = \frac{(m^2 + a^2)^{1/2}}{2\pi^{1/2}m}u(r).$$
 (16)

It is necessary to emphasize that the substitution $\lambda \to -\lambda$ (or, equivalently, $\Omega_{th} \to -\Omega_{th}$) merely corresponds to the rotation in the opposite direction. It is obvious that in this case the angular velocity Ω is also changing its sign, $\Omega \to -\Omega$, while the functions A, B, R, and Φ do not depend on the direction of rotation. Mathematically, this means that Ω is an odd function of λ , i.e. $\Omega(-\lambda) = -\Omega(\lambda)$, and the others are even, i.e. $A(-\lambda) = A(\lambda)$, etc. Therefore, the expansion (11) for Ω contains only odd powers of λ , while the other expansions (12-15) contain only even powers of λ .

Substituting the expansions (11-15) into the field equations (2),(3) and collecting terms with similar powers of λ yields a sequence of n-th order equations with n corresponding to powers of λ . The zeroth order field equations describe the static spherically symmetric configuration (6), (7). Cutting the sequence of equations on a definite n corresponds to the n-th order approximation of the theory.

A. The first order approximation

Rotating wormholes in the first order approximation have been studied in [21]. As is seen from Eqs. (11)–(15), in this approximation the functions A, B, R, Φ remain to be unperturbed, while the angular velocity Ω , being initially equal to zero, takes the form $\Omega = \lambda \omega$. The only nontrivial equation for ω reads

$$-\frac{1}{\sin^3 \theta} \partial_{\theta} [\sin^3 \theta \, \partial_{\theta} \omega] = (r^2 + a^2) \partial_r^2 \omega + 4(r - m) \partial_r \omega. \tag{17}$$

Its solution, having a natural physical meaning, is

$$\omega(r) = \frac{\omega_0(\mu)}{a} \left[1 - e^{4u(r)} \left(1 + \frac{4m(r+2m)}{r^2 + a^2} \right) \right]. \tag{18}$$

with

$$\omega_0(\mu) = [1 - e^{-2\pi\mu} (1 + 8\mu^2)]^{-1},\tag{19}$$

where $\mu \equiv m/a$ is a dimensionless mass parameter. This solution describes a slowly rotating wormhole with the angular velocity $\Omega = \lambda \omega$ and the angular momenta J_{\pm} :

$$J_{\pm} = \frac{4}{3}\mu\omega_0 q_{\pm}(a^2 + 4m^2), \quad q_{+} = 1, \ q_{-} = e^{-4\pi\mu},$$
 (20)

which are defined from asymptotics $\omega = 2J_{\pm}|r|^{-3} + O(|r|^{-4})$ at $r \to \pm \infty$.

Stress that the first order approximation lets only to determine the angular velocity Ω of the wormhole's rotation and, as a consequence, reveal interesting features in a motion of test particles and a propagation of light in the rotating wormhole spacetime [21]. At the same time, since the metric functions A, B and R and the scalar field Φ remain to be unperturbed in the first order approximation one cannot answer a number of important questions. For example, how does the rotation changes such characteristics of the static wormhole as its mass and throat's radius? How does the value of violation of the null energy condition depend on the wormhole's rotation? To answer these questions, hereafter we will consider the second order approximation.

B. The second order approximation

Substituting the expansions (11)–(15) into the Einstein equations (2) and the scalar field equation (3) and collecting λ^2 -terms yields the following system of equations for α , β , ρ , and ϕ :

$$(r^2 + a^2)\partial_r^2 \alpha + (\partial_\theta^2 \alpha + \cot\theta \,\partial_\theta \alpha) + (2r + m)\partial_r \alpha - m(\partial_r \beta - 4\partial_r \rho) = e^{-4u}\omega'^2(r^2 + a^2)^2\sin^2\theta, \tag{21}$$

$$(r^{2} + a^{2})(\partial_{r}^{2}\alpha + 4\partial_{r}^{2}\rho) + (\partial_{\theta}^{2}\beta + \cot\theta \,\partial_{\theta}\beta) + 3m\partial_{r}\alpha + (2r - m)(4\partial_{r}\rho - \partial_{r}\beta)$$

$$= e^{-4u}\omega'^{2}(r^{2} + a^{2})^{2}\sin^{2}\theta + \frac{8(m^{2} + a^{2})}{m}\partial_{r}(u\phi),$$
(22)

$$(r^{2} + a^{2})(\partial_{r\theta}^{2}\alpha + 2\partial_{r\theta}^{2}\rho) - r(\partial_{\theta}\alpha + \partial_{\theta}\beta) + 2m\partial_{\theta}\alpha = \frac{4(m^{2} + a^{2})}{m}u\partial_{\theta}\phi, \tag{23}$$

$$2(r^2 + a^2)\partial_r^2 \rho + 2(\partial_\theta^2 \rho + \cot\theta \,\partial_\theta \rho) + \partial_\theta^2 \alpha + \partial_\theta^2 \beta + (r - m)(\partial_r \alpha - \partial_r \beta) + 4(2r - m)\partial_r \rho - 2\beta + 4\rho = 0, \tag{24}$$

$$2(r^{2} + a^{2})\partial_{r}^{2}\rho + 2(\partial_{\theta}^{2}\rho + \cot\theta \,\partial_{\theta}\rho) + \cot\theta \,(\partial_{\theta}\alpha + \partial_{\theta}\beta) + (r - m)(\partial_{r}\alpha - \partial_{r}\beta) + 4(2r - m)\partial_{r}\rho -2\beta + 4\rho = -e^{-4u}\omega'^{2}(r^{2} + a^{2})^{2}\sin^{2}\theta,$$
(25)

$$2u(r^2 + a^2)\partial_r^2 \phi + 2u(\partial_\theta^2 \phi + \cot\theta \,\partial_\theta \phi) + 4(ru + m)\partial_r \phi + m(\partial_r \alpha - \partial_r \beta + 4\partial_r \rho) = 0, \tag{26}$$

where $\partial_r \alpha = \partial \alpha / \partial r$, $\partial_r^2 \alpha = \partial^2 \alpha / \partial r^2$, etc. One may simplify the system (22)–(26) noting that Eq. (23) can be integrated straightforwardly resulting in

$$(r^{2} + a^{2})(\partial_{r}\alpha + 2\partial_{r}\rho) - r(\alpha + \beta) + 2m\alpha = \frac{4(m^{2} + a^{2})}{m}u\phi + f_{1},$$
(27)

where $f_1(r)$ is an arbitrary function of r. Also, it will be convenient to consider their combinations [(24)+(25)] and [(24)-(25)] instead of equations (24) and (25):

$$4(r^{2} + a^{2})\partial_{r}^{2}\rho + 4(\partial_{\theta}^{2}\rho + \cot\theta \partial_{\theta}\rho) + (\partial_{\theta}^{2}\alpha + \cot\theta \partial_{\theta}\alpha) + (\partial_{\theta}^{2}\beta + \cot\theta \partial_{\theta}\beta) + 2(r - m)(\partial_{r}\alpha - \partial_{r}\beta)$$

$$+8(2r - m)\partial_{r}\rho - 4\beta + 8\rho = -e^{-4u}\omega'^{2}(r^{2} + a^{2})^{2}\sin^{2}\theta,$$

$$(28)$$

$$\partial_{\theta}^{2}\alpha + \partial_{\theta}^{2}\beta - \cot\theta \left(\partial_{\theta}\alpha + \partial_{\theta}\beta\right) = e^{-4u}\omega^{\prime 2}(r^{2} + a^{2})^{2}\sin^{2}\theta. \tag{29}$$

Then, integrating Eq. (29) yields

$$\alpha + \beta = \frac{1}{4}e^{-4u}\omega'^2(r^2 + a^2)^2(2\cos^2\theta - 1) + f_3\cos\theta + f_2,\tag{30}$$

where $f_2(r)$ and $f_3(r)$ are arbitrary functions of r. Taking into account that the rotating wormhole configuration should possess the symmetry $\theta \to \pi - \theta$ one should set $f_3(r) \equiv 0$. Finally, the equations (21), (22), (26), (27), (28) and (30) form the system to be solved. Note that only four equations of this system are independent since the Bianchi identity $\nabla_{\mu}G^{\mu}_{\nu} = 0$ and the conservation law $\nabla_{\mu}T^{\mu}_{\nu} = 0$ take place.

To solve the system of field equations we will follow Ref. [25] and expand the functions α , β , ρ , and ϕ in spherical harmonics:¹

$$\alpha(r,\theta) = \alpha_0(r) + \alpha_2(r)P_2(\theta) + \dots, \tag{31a}$$

$$\beta(r,\theta) = \beta_0(r) + \beta_2(r)P_2(\theta) + ...,$$
 (31b)

$$\rho(r,\theta) = \rho_0(r) + \rho_2(r)P_2(\theta) + ..., \tag{31c}$$

$$\phi(r,\theta) = \phi_0(r) + \phi_2(r)P_2(\theta) + \dots$$
(31d)

We would like to stress that the expansions (31) contain only *even* spherical harmonics being symmetric with respect to the equatorial plane $\theta = \frac{\pi}{2}$. Another convenient simplification of the metric may be made here. Transformations of the type $r \to f(r)$ do not change the form of the metric (4). Such a coordinate transformation may, therefore, be used to provide the additional condition

$$\rho_0(r) = 0. \tag{32}$$

This will be assumed in the following. Now substituting Eqs. (31) into (21), (22), (26), (27), (28) and (30) we find

$$(r^{2} + a^{2})\alpha''_{n} + (2r + m)\alpha'_{n} - m(\beta'_{n} - 4\rho'_{n}) - n(n+1)\alpha_{n} = \frac{2}{3}e^{-4u}(r^{2} + a^{2})^{2}\omega'^{2}(\delta_{n0} - \delta_{n2}),$$

$$(r^{2} + a^{2})(\alpha''_{n} + 4\rho''_{n}) + 3m\alpha'_{n} - (2r - m)(\beta'_{n} - 4\rho'_{n}) - n(n+1)\beta_{n} = \frac{2}{3}e^{-4u}(r^{2} + a^{2})^{2}\omega'^{2}(\delta_{n0} - \delta_{n2})$$

$$(33)$$

$$+8\frac{m^2+a^2}{m}(u\phi_n)',$$
 (34)

$$(r^2 + a^2)(\alpha'_n + 2\rho'_n) - (r - 2m)\alpha_n - r\beta_n = 4\frac{m^2 + a^2}{m}u\phi_n + f_1\delta_{n0},$$
 (35)

$$4(r^2 + a^2)\rho_n'' + 8(2r - m)\rho_n' + 2(r - m)(\alpha_n' - \beta_n') + 8\rho_n - 4\beta_n$$

$$-n(n+1)(4\rho_n + \alpha_n + \beta_n) = -\frac{2}{3}e^{-4u}(r^2 + a^2)^2\omega'^2(\delta_{n0} - \delta_{n2}), \quad (36)$$

$$\alpha_n + \beta_n = \frac{1}{3}e^{-4u}(r^2 + a^2)^2\omega'^2\delta_{n2} + f_2\delta_{n0},$$
 (37)

$$2u(r^2 + a^2)\phi_n'' + 4(ur + m)\phi_n' - 2n(n+1)u\phi_n + m(\alpha_n - \beta_n + 4\rho_n)' = 0,$$
(38)

$$\frac{d^2 P_n}{d\theta^2} + \cot \theta \, \frac{dP_n}{d\theta} = -n(n+1)P_n.$$

They satisfy the Rodrigues' formula

$$P_n(\theta) = \frac{(-1)^n}{2^n n!} \left(\frac{d}{d\cos\theta}\right)^n \{[1-\cos^2\theta]^n\}.$$

In particular, $P_0(\theta) = 1$, $P_1(\theta) = \cos \theta$, $P_2(\theta) = \frac{3}{2}\cos^2 \theta - \frac{1}{2}$. Note that $P_n(\theta) = (-1)^n P_n(\pi - \theta)$, i.e. even spherical harmonics (with even numbers n) are symmetric with respect to the equatorial plane $\theta = \frac{\pi}{2}$, while odd ones are antisymmetric.

¹ Spherical harmonics or Legendre polynomials $P_n(\theta)$ obey the equation (see, for example, [27])

where a prime means the derivative with respect to r, n = 0, 2, ..., and $\delta_{n\tilde{n}}$ is the Kronecker delta. Note that only the n = 0 and n = 2 equations involve the angular velocity ω . The coefficients in the expansion of α , β , ρ , and ϕ with $n \geq 4$ must, therefore, vanish since they vanish when the wormhole is not rotating, i.e.

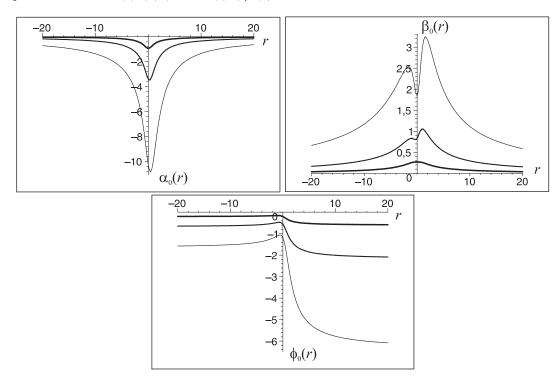
$$\alpha_n = \beta_n = \rho_n = \phi_n \equiv 0, \quad n \ge 4. \tag{39}$$

This reduction in the number of values of n from infinity to 2 is the central simplification of the slow rotation approximation. In place of a system of partial differential equations one now only has ordinary differential equations for the seven unknown functions α_0 , β_0 , ϕ_0 , and α_2 , β_2 , ρ_2 , ϕ_2 . A solution of these equations in an explicit analytic form is given in the appendix. Generally speaking, this solution depends on several constants of integrations. However, their values are fixed if one assumes that the perturbations are everywhere regular and obey the natural boundary conditions:

$$\alpha_0|_{r\to\pm\infty} = 0, \quad \beta_0|_{r\to\pm\infty} = 0, \quad \phi_0|_{r\to\pm\infty} = const,$$

$$\alpha_2|_{r\to\pm\infty} = 0, \quad \beta_2|_{r\to\pm\infty} = 0, \quad \rho_2|_{r\to\pm\infty} = 0, \quad \phi_2|_{r\to\pm\infty} = 0.$$
(40)

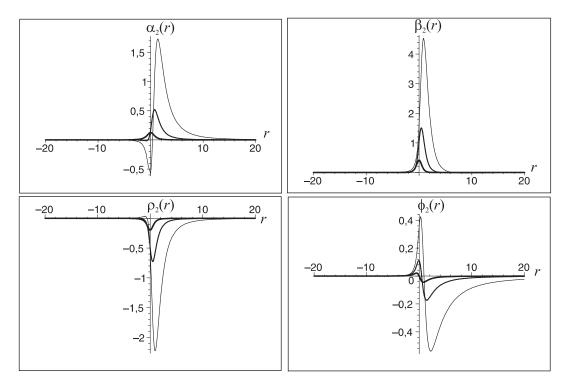
These conditions guarantee that there is no rotation far from the wormhole throat. Note that $\phi_0(r)$ tends to a constant as $r \to \pm \infty$ because the action (1) is invariant with respect to the shift $\Phi \to \Phi + const$. In Figs. 1 and 2 we give graphical representation for α_0 , β_0 , ϕ_0 , and α_2 , β_2 , ρ_2 , ϕ_2 .



Puc. 1: The graphs of $\alpha_0(r)$, $\beta_0(r)$, and $\phi_0(r)$ for a=1. The thick, middle and thin curves correspond to $m=0,\ 0.5,\ 1$, respectively.

V. ANALYSIS OF THE SOLUTION

Properties of the non-rotating wormhole spacetime with the metric (6) are only determined by two parameters m and a, where m represents the wormhole mass measured by a distant observer located at $r=\infty$, and a determines the radius of the wormhole throat. As was shown in the previous section, in addition to m and a the rotating wormhole solution depends on the parameter $\lambda = R_{\rm th}\Omega_{\rm th}/c$ being a dimensionless linear velocity of rotation of the wormhole throat. In this section we will discuss the problem: How do characteristics of the rotating wormhole differ from those of the non-rotating one possessing the same parameters m and a?



Puc. 2: The graphs of $\alpha_2(r)$, $\beta_2(r)$, $\rho_2(r)$ and $\phi_2(r)$ for a=1. The thick, middle and thin curves correspond to $m=0,\ 0.5,\ 1$, respectively.

A. The mass of rotating wormhole

To find the rotating wormhole mass M, we should consider the limit $g_{tt}|_{r\to\infty} \to -(1-2M/r)$. In the second order approximation we have found $g_{tt} = A(r,\theta) = -e^{2u(r)}\{1+\lambda^2[\alpha_0(r)+\alpha_2(r)P_2(\theta)]\}$. Note that $\alpha_2(r)|_{r\to\infty} \sim r^{-2}$, hence the n=2 solution does not give any contribution in M. Taking into account the following asymptotical property:

$$\alpha_0(r)|_{r\to\infty} = -\frac{2\Delta m}{r} + O(r^{-2}),$$
 (41)

with

$$\Delta m(\mu) = a\omega_0^2(\mu) \left[\frac{2(1+10\mu^2)(3\mu\pi + (u(\mu)+1)(e^{-4\mu\pi} - 1))}{3\pi} + \frac{2\mu e^{4u(\mu)}(34\mu^4 - \mu^2 + 1)}{3(1+\mu^2)} - 16\mu^3 \right], \tag{42}$$

where $\mu = m/a$ is a dimensionless mass parameter, we obtain

$$M = m + \lambda^2 \Delta m(\mu), \tag{43}$$

The value of $\lambda^2 \Delta m(\mu)$ characterizes the difference between the rotating and non-rotating wormhole masses M and m, respectively. In Fig. 3 the graph of $\Delta m(\mu)$ versus μ is shown. Note that $\Delta m(\mu)$ is positive for all μ and is the greater the greater μ . It is worth to stress that $\Delta m(\mu)$ is not equal to zero in case $\mu = 0$:

$$\Delta m(0) = \frac{8a}{3\pi}.\tag{44}$$

The case $\mu = 0$ or, equivalently, m = 0 corresponds to the massless non-rotating wormhole. The non-zero value $\Delta m(0)$ means that there do not exist massless rotating wormholes.

B. The NEC violation

A violation of the null energy condition (NEC) in the vicinity of the wormhole throat is an essential feature of wormhole physics [1, 2, 3]. The NEC reads $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, where $T_{\mu\nu}$ is the stress-energy tensor, and k^{μ} is a null vector.

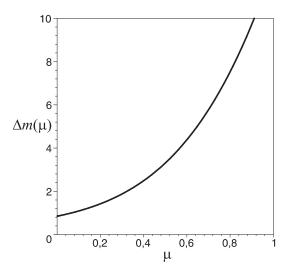


Рис. 3: The graph of $\Delta m(\mu)$ for a=1.

Using the Einstein equations, it can be represented in the geometrical form $R_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, where $R_{\mu\nu}$ is the Ricci tensor. To analyze the NEC in the rotating wormhole spacetime with the metric (4), we choose $k^{\mu} = (A^{-1/2}, B^{-1/2}, 0, \Omega A^{-1/2})$ and introduce the value $\Xi = R_{\mu\nu}k^{\mu}k^{\nu}$. In the second order approximation Ξ takes the following form

$$\Xi(r,\theta) = \Xi_0(r) + \lambda^2 [\xi_0(r) + \xi_2(r) P_2(\theta)], \tag{45}$$

where

$$\Xi_0(r) = -2e^{2u(r)} \frac{m^2 + a^2}{(r^2 + a^2)^2},\tag{46}$$

and

$$\xi_0(r) = \frac{e^{2u(r)}}{(r^2 + a^2)^2} \Big[2\beta_0(m^2 + a^2) + (r - m)(r^2 + a^2)(\alpha_0' + \beta_0') \Big], \tag{47}$$

$$\xi_2(r) = \frac{e^{2u(r)}}{(r^2 + a^2)^2} \Big[2\beta_2(m^2 + a^2) + (r^2 + a^2)[-3(\alpha_2 - \beta_2) + (r - m)(\alpha_2' + \beta_2' - 4\rho_2') - 2\rho_2''(r^2 + a^2)] \Big]. \tag{48}$$

The value Ξ_0 characterizes the configuration without rotation. As is seen, Ξ_0 is everywhere negative, hence the NEC is violated in the whole spacetime of the non-rotating wormhole. As for rotating wormholes, it will be convenient to average the quantity $\Xi(r,\theta)$ over all directions:

$$\Xi(r) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \Xi(r,\theta) \sin\theta d\theta d\varphi$$

$$= \Xi_0(r) + \frac{\lambda^2 e^{2u(r)}}{(r^2 + a^2)^2} \Big[2\beta_0(m^2 + a^2) + (r - m)(r^2 + a^2)(\alpha'_0 + \beta'_0) \Big]. \tag{49}$$

In Fig. (4) the quantities $\Xi(r)$ and $\Xi_0(r)$ are shown together. It is seen that the value of $\Xi(r)$ is everywhere negative but greater than $\Xi_0(r)$. This means that the NEC violation in the rotating wormhole spacetime is weaker than that in the non-rotating one.

VI. SUMMARY

We have constructed a solution describing slow rotating wormholes in general relativity with the scalar field possessing negative kinetic energy. The role of a small dimensionless parameter λ plays the ratio of the linear velocity of rotation of the wormhole's throat and the velocity of light, $\lambda = R_{\rm th}\Omega_{\rm th}/c$. The field equations have been solved in the second order approximation with respect to λ . It is worth noting that we succeeded in finding a solution in

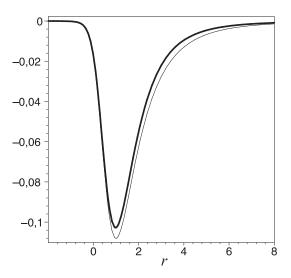


Рис. 4: The graphs of $\Xi(r)$ (thick line) and $\Xi_0(r)$ (thin line) for a=1.

an explicit analytical form. Its analysis has shown that a mass of a rotating wormhole is greater than that of a non-rotating one. As a consequence, this means that rotating wormholes, in contrast to the non-rotating ones, cannot possess a zero mass. The respective analysis of the NEC violation in a rotating wormhole spacetime reveals the fact that it is slightly weaker than that in a non-rotating one.

Acknowledgments

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Appendix

Here we will solve the system of equations (33)–(38). Note that the equations for different values of n are not coupled together. For this reason, we will consider the equations for n = 0 and n = 2 separately.

The case n = 0. In this case the equations (35) and (37) turn into identities because $f_1(r)$ and $f_2(r)$ are arbitrary functions. As a result, we obtain

$$(r^2 + a^2)\alpha_0'' + 2r\alpha_0' + m(\alpha_0 - \beta_0)' = \frac{2}{3}e^{-4u}(r^2 + a^2)^2\omega^2,$$
(50)

$$(r^2 + a^2)\alpha_0'' + 3m\alpha_0' - (2r - m)\beta_0' = \frac{2}{3}e^{-4u}(r^2 + a^2)^2\omega'^2 + 8\frac{m^2 + a^2}{m}(u\phi_0)', \tag{51}$$

$$(r-m)(\alpha_0 - \beta_0)' - 2\beta_0 = -\frac{1}{3}e^{-4u}(r^2 + a^2)^2\omega'^2, \tag{52}$$

$$(r-m)(\alpha_0 - \beta_0)' - 2\beta_0 = -\frac{1}{3}e^{-4u}(r^2 + a^2)^2\omega'^2,$$

$$2u(r^2 + a^2)\phi_0'' + 4(ur + m)\phi_0' + m(\alpha_0 - \beta_0)' = 0.$$
(52)

Note that the first order equation (52) plays the role of a differential constraint. A general solution to the system

(50)–(53) can be given explicitly as follows:

$$\alpha_{0}(x) = \frac{\omega_{0}^{2}}{6(x-\mu)} \left\{ C_{1} + C_{2}x + C_{3}xu(x) + \frac{8e^{4u(x)}}{x^{2}+1} \left[(1+10\mu^{2})x^{3} + 24\mu^{3}x^{2} + (1+2\mu^{2}+16\mu^{4})x - 4\mu^{3}(1+4\mu^{2}) \right] \right\},$$

$$\beta_{0}(x) = \frac{\omega_{0}^{2}}{6\mu(x-\mu)^{2}} \left\{ -(1+x\mu)(C_{3}\mu u + \mu C_{2} + C_{1}) + C_{3}\mu^{2}(\mu - x) + \frac{8\mu e^{4u(x)}}{x^{2}+1} \left[3\mu(1+2\mu^{2})x^{3} - (1+2\mu^{2}-8\mu^{4})x^{2} + 3\mu(1-2\mu^{2})x - 48\mu^{6} - 28\mu^{4} - 2\mu^{2} - 1 \right] \right\},$$

$$\phi_{0}(x) = \frac{\omega_{0}^{2}}{u(x)} \left\{ \mu C_{4} + u(x)C_{5} + \frac{1}{12(x-\mu)} \left[C_{1}[1+(x-\mu)\arctan x] + C_{2}\mu + C_{3}\mu u(x) \right] + \frac{2\mu e^{4u(x)}}{3(x-\mu)(x^{2}+1)} \left[1 + 16\mu^{4} - 2\mu(1-8\mu^{2})x + (1+4\mu^{2})x^{2} - 2\mu x^{3} \right] \right\}.$$
(56)

where $x \equiv r/a$ is a dimensionless radial coordinate, $\mu = m/a$ is a dimensionless wormhole parameter, $u(x) = \mu(\arctan x - \pi/2)$, and C_k (k = 1, ..., 5) are constants of integration. Stress that values of C_k are not free. They are connected by means of the constraint (52). Also, the constants C_k should be chosen so that to provide an appropriate asymptotical behavior of the solutions:

$$\alpha_0|_{r\to\pm\infty} = 0, \quad \beta_0|_{r\to\pm\infty} = 0, \tag{57}$$

$$\phi_0|_{r\to\pm\infty} = const. \tag{58}$$

Moreover, a choice of C_k should guarantee regularity of the solutions $\alpha_0(x)$, $\beta_0(x)$, $\phi_0(x)$ at the point $x = \mu$. Altogether, these conditions let us fix all values of the constants C_k as follows

$$C_{1} = 8\mu(1+10\mu^{2}) \left[1 - \frac{u(\mu)}{\pi\mu} \left(e^{-4\pi\mu} - 1 \right) \right] - \frac{8\mu e^{4u(\mu)} (34\mu^{4} - \mu^{2} + 1)}{\mu^{2} + 1},$$

$$C_{2} = -8(1+10\mu^{2}), \quad C_{3} = \frac{8}{\pi\mu} (1+10\mu^{2}) (e^{-4\pi\mu} - 1),$$

$$C_{4} = \frac{32\mu^{2} - \pi C_{1}}{24\mu}, \quad C_{5} = \frac{C_{3}\mu^{3} - C_{1}(1+\mu^{2})}{12\mu(1+\mu^{2})}.$$
(59)

The case n=2. The system (33)-(38) now takes the following form:

$$(r^2 + a^2)\alpha_2'' + (2r + m)\alpha_2' - m(\beta_2' - 4\rho_2') - 6\alpha_2 = -2v, \tag{60}$$

$$(r^2 + a^2)(\alpha_2'' + 4\rho_2'') + 3m\alpha_2' - (2r - m)(\beta_2' - 4\rho_2') - 6\beta_2 = -2v + 8\frac{m^2 + a^2}{m}(u\phi_2)', \tag{61}$$

$$(r^2 + a^2)(\alpha_2' + 2\rho_2') - (r - 2m)\alpha_2 - r\beta_2 = 4\frac{m^2 + a^2}{m}u\phi_2,$$
 (62)

$$4(r^2 + a^2)\rho_2'' + 8(2r - m)\rho_2' + 2(r - m)(\alpha_2' - \beta_2') - 16\rho_2 - 6\alpha_2 - 10\beta_2 = 2v,$$
(63)

$$\alpha_2 + \beta_2 = v, \tag{64}$$

$$2u(r^2 + a^2)\phi_2'' + 4(ur + m)\phi_2' - 12u\phi_2 + m(\alpha_2 - \beta_2 + 4\rho_2)' = 0,$$
(65)

where $v \equiv \frac{1}{3}e^{-4u}(r^2+a^2)^2\omega'^2$. A general solution of the system (60)–(65) is

$$\begin{split} \alpha_2(x) &= -\omega_0^2 \bigg\{ \frac{4e^{4u(x)}}{3(x^2+1)^2} \Big[3x^6 + 12\mu x^5 + x^4(7+22\mu^2) + 24\mu x^3(\mu^2+1) + x^2(5+36\mu^2+16\mu^4) \\ &+ 4\mu x(3+4\mu^2-8\mu^4) - 64\mu^6 + 14\mu^2+1 \Big] + \arctan x \Big[\frac{1}{2}\mu D_2(9x^2+1) - D_4(3x^2+1) \Big] - \\ &+ \mu x^2 D_1 + \frac{\mu x D_2(8x^2+7)}{2(x^2+1)} - D_3(3x^2+1) - 3x D_4 \bigg\}, \end{split} \tag{66} \\ \beta_2(x) &= \omega_0^2 \bigg\{ \frac{4e^{4u(x)}}{3(x^2+1)^2} \Big[3x^6 + 12\mu x^5 + x^4(7+22\mu^2) + 24\mu x^3(\mu^2+1) + x^2(5+36\mu^2+16\mu^4) \\ &+ 4\mu x(3+4\mu^2-8\mu^4) - 64\mu^6 + 14\mu^2+1 + 16\mu^2(1+4\mu^2)^2 \Big] + \arctan x \Big[\frac{1}{2}\mu D_2(9x^2+1) - D_4(3x^2+1) \Big] \\ &+ \mu x^2 D_1 - 3D_4 x - 3D_3 x^2 - D_3 + \frac{7}{2}\mu x D_2 + \frac{\mu x^3 D_2}{x^2+1} \bigg\}, \tag{67} \\ \rho_2(x) &= \omega_0^2 \bigg\{ \frac{2e^{4u(x)}}{3(x^2+1)^2} \Big[3x^6 + 18\mu x^5 + x^4(7+46\mu^2) + 36\mu x^3(2\mu^2+1) + x^2(5+76\mu^2+80\mu^4) \\ &+ 2\mu x(9+40\mu^2+16\mu^4) - 64\mu^6 + 32\mu^4 + 22\mu^2+1 \Big] \\ &+ \frac{1}{4}\arctan x \Big[D_2(9\mu x^2+6x+\mu) - 2D_4(3x^2+1) \Big] \\ &+ \frac{1}{2}x D_1(\mu x+1) + \frac{1}{4}D_2 \Big[9\mu x + 4 - \frac{2x(x-\mu)}{x^2+1} \Big] - \frac{1}{2}D_3(3x^2+1) - \frac{3}{2}x D_4 \bigg\}, \tag{68} \\ \phi_2(x) &= \omega_0^2 \bigg\{ \frac{4\mu^2 e^{4u(x)}}{3(x^2+1)^2u(x)} \Big[x^4 + 4\mu x^3 + 2x^2(1+4\mu^2) + 8\mu x(2\mu^2+1) + 32\mu^4 + 16\mu^2 + 1 \Big] \\ &- \frac{\mu}{4(\mu^2+1)u(x)} \Big[3x^2(3\mu^2 D_2 - D_2 - 2\mu D_4) + \mu^2 D_2 - 3D_2 - 2\mu D_4 \Big] - \\ &- \frac{\mu}{4(\mu^2+1)u(x)} \Big[x^2(2\mu^2 D_1 - D_1 - 6\mu D_3) + x(-3D_2 + 9\mu^2 D_2 - 6\mu D_4) - 2\mu D_3 \\ &- D_1 - \frac{2x D_2(\mu^2+1)}{x^2+1} \Big] \bigg\}, \tag{69} \end{aligned}$$

where D_k , (k = 1, ..., 4) are constants of integration. Taking into account the boundary conditions

$$\alpha_2|_{r\to\pm\infty} = 0, \quad \beta_2|_{r\to\pm\infty} = 0, \quad \rho_2|_{r\to\pm\infty} = 0, \quad \phi_2|_{r\to\pm\infty} = 0,$$
 (70)

and using the two differential constraints following from the system (60)–(65) we can fix the values of D_k as follows

$$D_1 = -4\mu(e^{-4\pi\mu} + 1), \quad D_2 = \frac{8\mu}{\pi}(e^{-4\pi\mu} - 1),$$

$$D_3 = \frac{2}{3}(1 - 2\mu^2)(1 + e^{-4\pi\mu}), \quad D_4 = \frac{4}{3\pi}(1 - 3\mu^2)(1 + e^{-4\pi\mu}). \tag{71}$$

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