Numerical Analysis – Problem sheet

Aeson Feehan – Edgar Jouisse

1 Importance sampling

Question 1

After 10^7 iterations, the naive Monte Carlo implementation has an estimated price of 6.1692 and a standard error of 0.0094.

Question 2

By definition of $\tilde{\Pi}$,

$$\begin{aligned}
\tilde{\Pi} &= e^{-rT} \int \phi\left(x+\lambda, y+\tilde{\lambda}\right) \frac{1}{2\pi} \exp\left(-\frac{x^2}{2} - \frac{y^2}{2} - \lambda x - \frac{\lambda^2}{2} - \tilde{\lambda} y - \frac{\tilde{\lambda}^2}{2}\right) \mathrm{d}x\mathrm{d}y \\
&= e^{-rT} \int \phi\left(x+\lambda, y+\tilde{\lambda}\right) \frac{1}{2\pi} \exp\left(-\frac{(x+\lambda)^2}{2} - \frac{\left(y+\tilde{\lambda}\right)^2}{2}\right) \mathrm{d}x\mathrm{d}y \\
&= e^{-rT} \int \phi\left(a, b\right) \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi}} \mathrm{d}a\mathrm{d}b \\
&= e^{-rT} \mathbb{E}\left[\phi\left(G, \tilde{G}\right)\right] \\
\tilde{\Pi} &= \Pi
\end{aligned}$$

The variable change in the third equation is trivial because the jacobian of the linear variable change function is the identity matrix.

Question 3

Running the Monte Carlo simulation for several combinations of λ and $\tilde{\lambda}$, we obtain Figure 1. The minimum error seems to be attained for $\lambda = \tilde{\lambda} \simeq 0.8$.

The optimal value can be determined numerically using a minimization algorithm. Because λ and $\tilde{\lambda}^*$ play symmetrical roles, necessarily $\lambda^* = \tilde{\lambda}^*$.

The `scipy.optimize.minimize` function yields an optimal value for λ and $\tilde{\lambda}$ of 0.8375. For this value of λ, the standard error of the estimator is 0.0032 (about a third of the error obtained without importance sampling).

Question 4

Due to put-call parity, the price of a call is a deterministic function of the price of a put. To price a put, the only difference with the previous questions is the payoff, which is now $(K - S_T)_+$ instead of

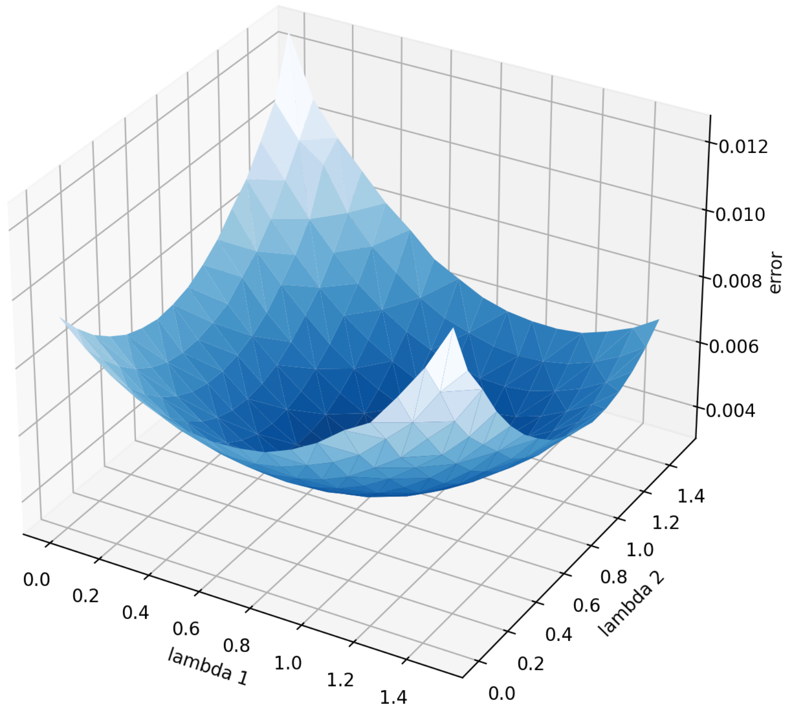

Figure 1: Monte Carlo standard error for different values of $(\lambda, \tilde{\lambda})$

$(S_T - K)_+)$. Using the same minimization procedure as in question 3, we obtain an estimation error of 0.0028 (slightly lower than in question 3), for $\lambda^* = \tilde{\lambda}^* \simeq -0.7875$.

2 Kemna-Vorst control variate and Greeks computation

Question 1

Let $T > 0$, then the integral can be developped as follows:

$$\begin{aligned}
\frac{1}{T}\int_0^T \left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right)u + \sigma W_u\right)\mathrm{d}u &= \log S_0 + \left(r - \frac{\sigma^2}{2}\right)\frac{1}{T}\int_0^T u\mathrm{d}u + \frac{\sigma}{T}\int_0^T W_u \mathrm{d}u \\
&= \log S_0 + \left(r - \frac{\sigma^2}{2}\right)\frac{1}{T}\left[\frac{u^2}{2}\right]_0^T + \frac{\sigma}{T}\int_0^T W_u \mathrm{d}u \\
&= \log S_0 + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \frac{\sigma}{T}\int_0^T W_u \mathrm{d}u
\end{aligned}$$

Because W_u is a brownian motion, $\int_0^T W_u \mathrm{d}u \sim \mathcal{N}\left(0, \frac{T^3}{3}\right)$, hence

$$\frac{1}{T}\int_0^T \left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right)u + W_u\right)\mathrm{d}u \sim \mathcal{N}\left(\alpha, \beta\right)$$

where $\alpha := \log S_0 + T\left(\frac{r}{2} - \frac{\sigma^2}{4}\right)$ and $\beta := \frac{\sigma^2 T}{3}$. As a consequence,

$$\exp\left[\frac{1}{T}\int_0^T \left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right)u + W_u\right)\mathrm{d}u\right] \sim \log\mathcal{N}\left(\alpha, \beta\right)$$

so the expectation of $e^{-rT}Y$ is given by the formula for the expectation of a log-normal variable, *i.e.*

$$\mathbb{E}\left[Y\right] = e^{-rT}\exp\left(\alpha + \frac{\beta^2}{2}\right) = e^{-rT}\exp\left(\log S_0 + T\left(\frac{r}{2} - \frac{\sigma^2}{4}\right) + \frac{\sigma^4 T^2}{18}\right)$$

Question 2

Writing X the naive Monte Carlo estimator for the payoff of an Asian option, $p = \mathbb{E}\left[X\right]$, and $c \in \mathbb{R}$, an unbiased estimator of the payoff using the covariate variable Y is

$$U_c := X + c\left(Y - \mathbb{E}\left[Y\right]\right)$$

with $\mathbb{E}\left[Y\right]$ explicitly calculated in question 1. Furthermore, the variance of U_c is

$$\mathbb{V}\left[U_c\right] = \mathbb{V}\left[X\right] - 2c \times \operatorname{cov}\left(X, Y\right) + c^2\mathbb{V}\left[Y\right]$$

Simulations suggest that the above expression has a local minimum around -0.5 as shown in Figure 2. Using the same minimizing function as in question 4 of exercise 1, we obtain an optimal weight of $c^* \simeq -0.5566$, which yields an optimal standard deviation of $\sqrt{\mathbb{V}\left[U_{c^*}\right]} \simeq 3.296 \times 10^{-3}$. The standard deviation of the naive estimator is $\mathbb{V}\left[U_0\right] = \mathbb{V}\left[X\right] \simeq 6.855 \times 10^{-3}$, so the covariate reduces the standard error of the estimator by about half.

Question 3

Writing $f_{m,s}$ the density of a normal law with mean m and variance s, N the cumulative distribution function of a standard normal probability law,

$$\begin{aligned}
e^{rT}\mathbb{E}\left[Z\right] &= \int_K^\infty \left(e^x - K\right) f_{\alpha,\beta}(x)\mathrm{d}x \\
&= \int_{\frac{K-\alpha}{\beta}}^\infty \left(e^{\alpha+\beta y} - K\right) f_{0,1}(y)\mathrm{d}y \\
&= e^\alpha \int_{\frac{K-\alpha}{\beta}}^\infty e^{\beta y}\frac{e^{\frac{-y^2}{2}}}{\sqrt{2\pi}}\mathrm{d}y - K\int_{\frac{K-\alpha}{\beta}}^\infty f_{0,1}(y)\mathrm{d}y \\
&= e^{\alpha+\frac{\beta^2}{2}} \int_{\frac{K-\alpha}{\beta}}^\infty \frac{\exp^{-\frac{1}{2}\left(y^2 - 2\beta y + \beta^2\right)}}{\sqrt{2\pi}}\mathrm{d}y - K\int_{\frac{K-\alpha}{\beta}}^\infty f_{0,1}(y)\mathrm{d}y \\
&= e^{\alpha+\frac{\beta^2}{2}} \int_{\frac{K-\alpha+\beta^2}{\beta}}^\infty f_{0,1}(z)\mathrm{d}z - K\int_{\frac{K-\alpha}{\beta}}^\infty f_{0,1}(y)\mathrm{d}y \\
&= e^{\alpha+\frac{\beta^2}{2}}\left(1 - N\left(\frac{K-\alpha+\beta^2}{\beta}\right)\right) - K\left(1 - N\left(\frac{K-\alpha}{\beta}\right)\right) \\
&= e^{\alpha+\frac{\beta^2}{2}} N\left(\frac{\alpha-\beta^2-K}{\beta}\right) - KN\left(\frac{\alpha-K}{\beta}\right) \\
\mathbb{E}\left[Z\right] &= e^{\alpha+\frac{\beta^2}{2}-rT} N\left(\frac{\alpha-\beta^2-K}{\beta}\right) - Ke^{-rT}N\left(\frac{\alpha-K}{\beta}\right)
\end{aligned}$$

Question 4

We proceed exactly as in question 2, replacing Y with Z. As Figure 2 shows, the minimum standard error is much lower than with the previous covariate, with an optimal covariate weight of about -1.

Using the same minimization function as before, the optimal covariate weight is $c^* \simeq 1.039$. For this value of c, the standard error of the estimator is 1.797×10^{-4}, which is almost forty times lower than the naive estimator.

To conclude, the covariate Z is a much better choice than Y for variance reduction, because it is much more correlated with X.

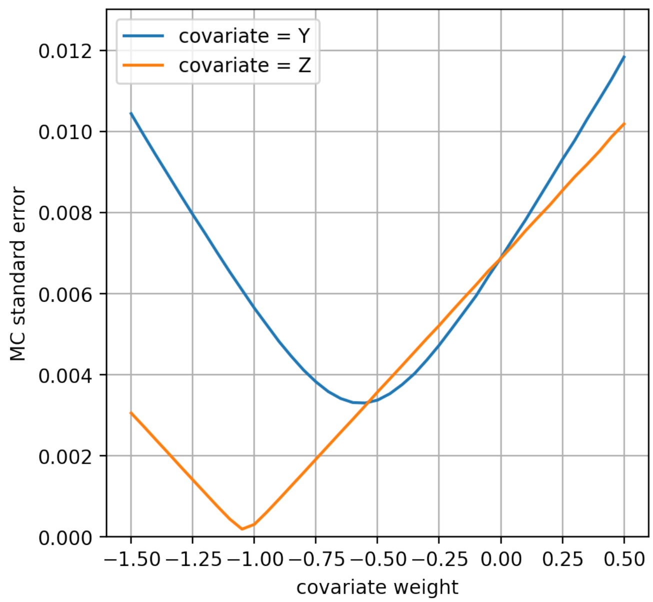

Figure 2: Monte Carlo standard error for different values of c